

$$1. \quad i) \quad \left[ \begin{array}{cccc|c} 1 & 1 & 2 & -1 & 1 \\ 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & -2 & 2 \end{array} \right] \xrightarrow{R_2 - R_1} \left[ \begin{array}{cccc|c} 1 & 1 & 2 & -1 & 1 \\ 0 & -1 & -2 & 3 & 0 \\ 0 & 1 & 2 & -2 & 2 \end{array} \right]$$

$$\xrightarrow{R_1 + R_2} \left[ \begin{array}{cccc|c} 1 & 1 & 2 & -1 & 1 \\ 0 & -1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right] \xrightarrow{R_1 + R_2} \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 2 & 1 \\ 0 & -1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right]$$

$$x_4 = 2$$

$$\text{let } x_3 = s, s \in \mathbb{R}$$

$$-x_2 - 2x_3 + 3x_4 = 0 \Rightarrow x_2 = 6 - 2s$$

$$x_1 + 2x_4 = 1 \Rightarrow x_1 = -3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, s \in \mathbb{R}$$

$$ii) \quad \left[ \begin{array}{cccc|c} 1 & 1 & 2 & -1 & 0 \\ 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 2 & -2 & 1 \end{array} \right] \xrightarrow[\substack{R_2 - R_1 \\ R_3 + R_2}]{\text{same as w i)}} \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 2 & -1 \\ 0 & -1 & -2 & 3 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

$$x_4 = 0$$

$$\text{let } x_3 = s, s \in \mathbb{R}$$

$$-x_2 - 2x_3 - 1 \cdot x_4 = -1 \Rightarrow x_2 = 1 - 2s$$

$$x_1 + 2x_4 = -1 \Rightarrow x_1 = -1$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, s \in \mathbb{R}$$

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$$2. a) \left[ \begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 3 & -6 & -3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_2 - 3R_1} \left[ \begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & -3 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{R_3 - R_1} \left[ \begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & -3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3} \left[ \begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & -3 & -3 & 1 & 0 \end{array} \right]$$

$$\xrightarrow{-\frac{1}{3}R_3} \left[ \begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -\frac{1}{3} & 0 \end{array} \right] \xrightarrow{R_2 - 2R_3} \left[ \begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -3 & \frac{2}{3} & 1 \\ 0 & 0 & 1 & 1 & -\frac{1}{3} & 0 \end{array} \right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[ \begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & -\frac{1}{3} & 0 \end{array} \right] \xrightarrow{R_1 + 2R_2} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & \frac{2}{3} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & -\frac{1}{3} & 0 \end{array} \right]$$

$$A^{-1} [A | I] \rightarrow [I | A^{-1}]$$

$$A^{-1} = \begin{bmatrix} -2 & \frac{2}{3} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{3} & \frac{1}{2} \\ 1 & -\frac{1}{3} & 0 \end{bmatrix}$$

$$2b) \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array} \right] \left[ \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right] \left[ \begin{array}{ccc} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 3 & 0 \end{bmatrix}$$

$$\left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 3 & 0 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3} \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1/2 \end{array} \right] \xrightarrow{\frac{1}{3}R_2} \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]$$

$$A = \left[ \begin{array}{ccc} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right]_{E_1} \left[ \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right]_{E_2} \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right]_{E_3} \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right]_{E_4} \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{array} \right]_{E_5} \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{array} \right]_{E_6}$$

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$$\begin{aligned}
 2 \text{ c) } \det(A) &= \det(E_1) \det(E_2) \det(E_3) \det(E_4) \det(E_5) \det(E_6) \det(E_7) \\
 &= 1 \times -1 \times 1 \times -1 \times 3 \times \frac{1}{2} \times 1 \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 3. \text{ a) } \det(A) &= 1 \begin{vmatrix} 2 & 3 & 1 \\ 2 & 0 & 0 \\ 2 & 5 & 7 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 & 1 \\ 0 & 0 & 0 \\ 1 & 5 & 7 \end{vmatrix} + (-1) \begin{vmatrix} 2 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 7 \end{vmatrix} \\
 &\quad - 0 \begin{vmatrix} 2 & 2 & 3 \\ 0 & 2 & 0 \\ 1 & 2 & 5 \end{vmatrix} \\
 &= 1 \left( -2 \begin{vmatrix} 3 & 1 \\ 5 & 7 \end{vmatrix} + 0 \begin{vmatrix} 2 & 1 \\ 2 & 7 \end{vmatrix} - 0 \begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix} \right) \quad \begin{array}{l} \text{expansion along} \\ \text{2nd row} \end{array} \\
 &\quad - 2 \left( -0 \begin{vmatrix} 3 & 1 \\ 5 & 7 \end{vmatrix} + 0 \begin{vmatrix} 2 & 1 \\ 1 & 7 \end{vmatrix} - 0 \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} \right) \\
 &\quad - 1 \left( -0 \begin{vmatrix} 2 & 1 \\ 2 & 7 \end{vmatrix} + 2 \begin{vmatrix} 2 & 1 \\ 1 & 7 \end{vmatrix} - 0 \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} \right) - 0 \\
 &= 1 (-2 \times 16) - 2 (0) - 1 (2 \times 13) \\
 &= -58 //
 \end{aligned}$$

3 b) As the determinant of  $A$  is non-zero,  $A$  is invertible and the rank of the matrix  $= 4$ . Hence, all the columns are independent and the 4 independent columns fill  $\mathbb{R}^4$  space, thus the linear system  $Ax=b$  has a unique solution for every  $b$ .

As the reduced row echelon form of  $A$  is the identity matrix  $I_4$   
 $[A | b] \rightarrow [I | A^{-1}b]$  so  $Ax=b$  is consistent for all values of  $a$ .

$$\begin{aligned} 3 c) \quad i) \quad \det\left(\frac{1}{2}A^T\right) &= \frac{1}{2}^4 \det(A^T) \\ &= \frac{1}{2}^4 \det(A) \\ &= -\frac{58}{16} = -\frac{29}{8} \end{aligned}$$

$$\begin{aligned} ii) \quad \det(AB^{-1}) &= \det(A) \det(B^{-1}) \\ &= \det(A) \det(B)^{-1} \\ &= -58 \times \frac{1}{3} \\ &= -\frac{58}{3} \end{aligned}$$

$$\begin{aligned} iii) \quad \det((3B)^{-1}) &= (3^4 \times \det(B))^{-1} \\ &= 3^{-5} \\ &= \frac{1}{243} \end{aligned}$$