MA1508E: LINEAR ALGEBRA FOR ENGINEERING

Lecture 6 Notes

References

1. Elementary Linear Algebra: Application Version, Section 4.2
2. Linear Algebra with Application, Section 5.1, 6.2

3.5 Linear Combination and Linear Span

A linear combination of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k \in \mathbb{R}^n$ is

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k, \text{ for some } c_1, c_2, ..., c_k \in \mathbb{R}.$$

Remark. One can think of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$ as the possible directions, and $c_1, c_2, ..., c_k$ as the amount of units to walk in the respective directions.

The collection of all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k \in \mathbb{R}^n$ is call the span,

$$\text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} = \{\ c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k \mid c_1, c_2, ..., c_k \in \mathbb{R}\ \}.$$

It is straight forward to compute a linear combination of a vectors. We may ask the reverse question. Is a given vector $\mathbf{v}$ a linear combination of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$? Equivalently, whether $\mathbf{v} \in \text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$. This is asking for whether we are able to find real numbers $c_1, c_2, ..., c_k$ such that

$$\mathbf{u} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k.$$

Example. Is $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ a linear combination of $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$?

We are looking for $a, b, c \in \mathbb{R}$, if possible, such that

$$a\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Comparing the entries, we end up with a linear system

$$\begin{cases} a + b + c = 1 \\ a - b + 2c = 2 \\ a + c = 3 \end{cases}$$

Then

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 2 \\ 1 & 0 & 1 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{array}\right).$$

Observe that the constant is the vector $\mathbf{v}$, and the columns of the coefficient matrix (the left hand side) are the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

So the system is consistent, and we have

$$6\begin{pmatrix}1\\1\\1\end{pmatrix}-2\begin{pmatrix}1\\-1\\0\end{pmatrix}-3\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}.$$

Theorem. *Let $\mathbf{u},\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_k$ be vectors in $\mathbb{R}^n$. Then $\mathbf{v}\in span\{\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_k\}$ if and only if the linear system associated to the augmented matrix*

$$\left(\begin{array}{cccc|c}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v}\end{array}\right)$$

is consistent.

So this shows that a vector $\mathbf{v}\in\mathbb{R}^n$ is in $\text{span}\{\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_k\}$ if and only if we can find a vector $\mathbf{u}=\begin{pmatrix}c_1\\c_2\\\vdots\\c_k\end{pmatrix}\in\mathbb{R}^k$ such that

$$\begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k\end{pmatrix}\begin{pmatrix}c_1\\c_2\\\vdots\\c_k\end{pmatrix}=c_1\mathbf{u}_1+c_2\mathbf{u}_2+\cdots+c_k\mathbf{u}_k=\mathbf{v}.$$

Summarising, we have the following statement.

Corollary. *Let $S=\{\mathbf{u}_1,\mathbf{u}_2,...,\mathbf{u}_k\}\subseteq\mathbb{R}^n$ be a subset and $\mathbf{A}=\begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k\end{pmatrix}$ be a $n\times k$ matrix whose columns are the vectors in S. Then $\mathbf{v}\in\mathbb{R}^n$ if and only if there is a $\mathbf{u}\in\mathbb{R}^k$ such that $\mathbf{A}\mathbf{u}=\mathbf{v}$, that is, the system $\mathbf{A}\mathbf{x}=\mathbf{v}$ is consistent.*

Example.

1. Let $\mathbf{u}=\begin{pmatrix}4\\-3\\0\end{pmatrix}$, $\mathbf{u}_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$, $\mathbf{u}_2=\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $\mathbf{u}_3=\begin{pmatrix}1\\2\\1\end{pmatrix}$.

$$\left(\begin{array}{ccc|c}1&1&1&4\\1&-1&2&-3\\1&0&1&0\end{array}\right)\longrightarrow\left(\begin{array}{ccc|c}1&0&0&-1\\0&1&0&4\\0&0&1&1\end{array}\right).$$

   This means that $\begin{pmatrix}1&1&1\\1&-1&2\\1&0&1\end{pmatrix}\begin{pmatrix}-1\\4\\1\end{pmatrix}=\begin{pmatrix}4\\-3\\0\end{pmatrix}=-\begin{pmatrix}1\\1\\1\end{pmatrix}+4\begin{pmatrix}1\\-1\\0\end{pmatrix}+\begin{pmatrix}1\\2\\1\end{pmatrix}=\begin{pmatrix}4\\-3\\0\end{pmatrix}$, and hence $\mathbf{u}\in\text{span}\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$.

2. Let $\mathbf{v}=\begin{pmatrix}1\\2\\3\end{pmatrix}$, $\mathbf{u}_1=\begin{pmatrix}1\\0\\1\end{pmatrix}$, $\mathbf{u}_2=\begin{pmatrix}0\\1\\-1\end{pmatrix}$.

$$\left(\begin{array}{cc|c}1&0&1\\0&1&2\\1&-1&3\end{array}\right)\longrightarrow\left(\begin{array}{cc|c}1&0&1\\0&1&2\\0&0&4\end{array}\right).$$

   Hence, $\mathbf{v}\notin\text{span}\{\mathbf{u}_1,\mathbf{u}_2\}$.

3. Let $\mathbf{e}_i$ be the i-th column of the order n identity matrix $\mathbf{I}_n$ for $i = 1, ..., n$. Then for any $\mathbf{w} = (w_i)$,
$$\mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + \cdots + w_n\mathbf{e}_n.$$
This shows that $\text{span}\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\} = \mathbb{R}^n$. This set is called the standard basis of $\mathbb{R}^n$.

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$ be a subset. Now instead of asking if a specific vector $\mathbf{v} \in \mathbb{R}^n$ is in the span, we may ask if all the vectors in $\mathbb{R}^n$ is in the span, that is, whether $\text{span}(S) = \mathbb{R}^n$. This is equivalent to asking if for every $\mathbf{v}$ in $\mathbb{R}^n$, the system $\mathbf{Ax} = \mathbf{v}$ is consistent, where $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$ is the $n \times k$ matrix whose columns are the vectors in S.

Example. 1. $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \right\}$.
$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & x \\ 1 & 2 & 3 & y \\ 1 & 1 & 2 & z \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2x - y \\ 0 & 1 & 1 & -x + y \\ 0 & 0 & 0 & -x + z \end{array}\right)$$

So $\text{span}(S) \neq \mathbb{R}^3$, since for example, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \notin \text{span}(S)$ or $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \notin \text{span}(S)$.

2. Let $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$.
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & x \\ 1 & -1 & 2 & y \\ 1 & 0 & 1 & z \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -x - y + 3z \\ 0 & 1 & 0 & x - z \\ 0 & 0 & 1 & x + y - 2z \end{array}\right)$$

So for any $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$,
$$(-x - y + 3z)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (x - z)\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + (x + y - 2z)\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

From the examples above, observe that we can always pick a $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that the last entry in the right hand side of the RREF of the augmented matrix is nonzero. This seems to indicate that S spans $\mathbb{R}^n$ if and only if the RREF of $\mathbf{A}$ does not have zero rows. This is indeed the case. For the general proof, readers may refer to the appendix.

Corollary. *Let* $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$ *be a subset and* $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$ *be a* $n \times k$ *matrix whose columns are the vectors in* S*. Then span*$(S) = \mathbb{R}^n$ *if and only if the reduced row-echelon form* $\mathbf{R}$ *of* $\mathbf{A}$ *has no zero rows.*

Observe that if $k < n$, then $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ cannot span $\mathbb{R}^n$, since the reduced row-echelon form $\mathbf{R}$ of $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$ can have at most k pivot columns, and hence, it can at most have k nonzero rows. So $\mathbf{R}$ must have $n-k$ number of zero rows,

$$\mathbf{R} = \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & * \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

We will state this as a result.

Theorem. *A subset $S \subseteq \mathbb{R}^n$ containing less than n vectors cannot span $\mathbb{R}^n$.*

So n is the lower bound on the number of vectors needed to span $\mathbb{R}^n$. Is this lower bound achieved? Yes, for example, the standard basis $E = \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ spans $\mathbb{R}^n$. So n is the "most efficient" number of vectors needed to span $\mathbb{R}^n$.

Remark. We will learn later that the above corollary is equivalent to

$$\text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} = \mathbb{R}^n \Leftrightarrow \text{rank}(\mathbf{A}) = n,$$

where $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$.

We will see eventually that spanning sets are not only ubiquitous, they are fundamental as they produce geometrical objects call subspaces. Here we present their fundamental properties that make them important.

Theorem. *Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$. Then*

(i) (Contains the origin) $\mathbf{0} \in span(S)$, and

(ii) (Closed under linear combination) for any $\mathbf{u}, \mathbf{v} \in span(S)$ and $\alpha, \beta \in \mathbb{R}$,

$$\alpha\mathbf{u} + \beta\mathbf{v} \in span(S).$$

Proof. (i) Take the trivial combination,

$$0\mathbf{u}_1 + 0\mathbf{u}_2 + \cdots + 0\mathbf{u}_k = \mathbf{0}.$$

(ii) Suppose $\mathbf{u}, \mathbf{v} \in \text{span}(S)$. Then we can find real numbers $c_1, c_2, ..., c_k, d_1, d_2, ..., d_k \in \mathbb{R}$ such that

$$\mathbf{u} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k \text{ and } \mathbf{v} = d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + \cdots + d_k\mathbf{u}_k.$$

Then

$$\begin{aligned} \alpha\mathbf{u} + \beta\mathbf{v} &= \alpha(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k) + \beta(d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + \cdots + d_k\mathbf{u}_k) \\ &= (\alpha c_1 + \beta d_1)\mathbf{u}_1 + (\alpha c_2 + \beta d_2)\mathbf{u}_2 + \cdots + (\alpha c_k + \beta d_k)\mathbf{u}_k \end{aligned}$$

tells us that $\alpha\mathbf{u} + \beta\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$, and thus it is in the span.

$\square$

By induction using property (ii), we have property (ii'), that if $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m \in \text{span}(S)$, then any linear combination of $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ is also in $\text{span}(S)$. This is an important result and we will state it as a theorem.

Theorem. *Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$. If $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m \in span(S)$, then for any $c_1, c_2, ..., c_m \in \mathbb{R}$,*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m \in span(S).$$

That is, $span\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\} \subseteq span(S)$.

Observe that the vectors used to span a set is not unique, for example, both $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ and $T = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$ span the whole x, y plane in $\mathbb{R}^3$. So given two spanning sets S and T, the theorem above gives us a way to tell if they span the same set, or even if one of them is contained in the other.

For suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$, and $T = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$, both subsets of $\mathbb{R}^n$. Then if $\mathbf{v}_i \in \text{span}(S)$ for every $i = 1, ..., m$, the theorem above tells us that $\text{span}(T) \subseteq \text{span}(S)$. If further $\mathbf{u}_i \in \text{span}(T)$ for every $i = 1, ..., k$, then we too have $\text{span}(S) \subseteq \text{span}(T)$. Thus, equality $\text{span}(S) = \text{span}(T)$ holds. So rephrasing the theorem, we get the following statement.

Theorem. *Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$, both subsets of $\mathbb{R}^n$. Then $span(T) \subseteq span(S)$ if and only if $\mathbf{v}_i \in span(S)$ for every $i = 1, ..., m$.*

The algorithm to check if $\text{span}(T) \subseteq \text{span}(S)$ is thus as follows. We want to check if the system associated to the augmented matrix

$$\left(\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v}_i \end{array} \right)$$

is consistent for all $i = 1, ..., m$. Recall from lecture 2 that we can do this simultaneously, that is, check if the systems associated to the augmented matrix

$$\left(\begin{array}{cccc|c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_m \end{array} \right)$$

is consistent.

Example. 1. $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}, T = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$.

To check if $\text{span}(T) \subseteq \text{span}(S)$,

$$\left(\begin{array}{cc|c|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

is consistent.

To check if $\text{span}(S) \subseteq \text{span}(T)$,

$$\left(\begin{array}{cc|c|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right) \longrightarrow \left(\begin{array}{cc|c|c} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

is consistent. Hence, $\text{span}(S) = \text{span}(T)$.

2. $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \right\}$, $T = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$.

To check if $\text{span}(S) \subseteq \text{span}(T)$,

$$\left(\begin{array}{ccc|c|c|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 1 & 2 & 3 \\ 1 & 0 & 1 & 1 & 1 & 2 \end{array} \right) \longrightarrow \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array} \right)$$

To check if $\text{span}(T) \subseteq \text{span}(S)$,

$$\left(\begin{array}{ccc|c|c|c} 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 3 & 1 & -1 & 2 \\ 1 & 1 & 2 & 1 & 0 & 1 \end{array} \right) \longrightarrow \left(\begin{array}{ccc|c|c|c} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right)$$

This shows that $\text{span}(T) \not\subseteq \text{span}(S)$. In particular, $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \notin \text{span}(S)$.

Remark. We have seen that the standard basis $E = \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ spans $\mathbb{R}^n$. Since

$$\left(\begin{array}{cccc|c|c|c|c} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n & \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{array} \right)$$

is always consistent, if we want to check if a set $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ spans $\mathbb{R}^n$, it suffice to check that

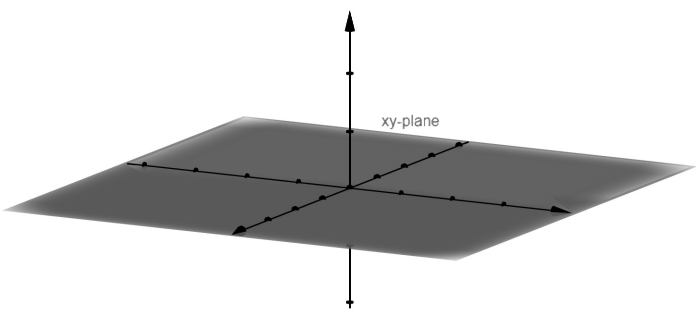
$$\left(\begin{array}{cccc|c|c|c|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{array} \right)$$

is consistent. But we have a theorem above saying that $\text{span}(S) = \mathbb{R}^n$ if and only if the reduced row-echelon form of $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$ has no zero rows. Are these two different algorithms?

This also show that $\mathbf{v} \in \text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ if and only if $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k, \mathbf{v}\}$. That is, a vector in the spanning set is a linear combination of the others is and only if it is "reduntant" in the spanning set. This means that if we want the most efficient/no redundancy spanning set, we have to make sure that none of the vectors in the spanning set can be written as a linear combination of the others. We will discuss this in details in lecture 7.

3.6 Subspace

A subspace is a vector space that is contained in another vector space. Here we are restriction ourselves to only subspaces in Euclidean spaces $\mathbb{R}^n$. We will give a precise definition in a while. Consider the xy-plane in $\mathbb{R}^3$.



It looks exactly like $\mathbb{R}^2$. So can we say that $\mathbb{R}^2$ is a subset of $\mathbb{R}^3$? For $\mathbb{R}^2$ to be a subset of $\mathbb{R}^3$, every vector in $\mathbb{R}^2$ must also be a vector in $\mathbb{R}^3$. So for $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, to be a vector in $\mathbb{R}^3$, it needs 3 coordinates,

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \square \\ \square \\ \square \end{pmatrix}?$$

Naturally, we can let $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$. Then we can say that we have an image of $\mathbb{R}^2$ in $\mathbb{R}^3$, $\mathbb{R}^2 \hookrightarrow \mathbb{R}^3$ given by this mapping. However, the mapping is not unique. The following mappings also produce images of $\mathbb{R}^2 \hookrightarrow \mathbb{R}^3$,

$$1.\ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ x \\ y \end{pmatrix}, 2.\ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ 0 \\ y \end{pmatrix}, 3.\ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ x-y \\ 0 \end{pmatrix}, 4.\ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

However, we do not just want the image to look like $\mathbb{R}^2$. We want the image to have the same behaviour as $\mathbb{R}^2$, that is, in $\mathbb{R}^2$, we have the origin, we can add vectors, and we can take scalar multiple, etc. So one can see for example the image under mapping 4.,

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

does not contain the origin. Moreover, it does not respect vector addition,

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) \mapsto \begin{pmatrix} (x_1+x_2) \\ (y_1+y_2) \\ 1 \end{pmatrix} \neq \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix}.$$

It turns out that if a subset $V \subseteq \mathbb{R}^n$ contains the origin and is closed under linear combinations, then it looks exactly like the image of $\mathbb{R}^k$ in $\mathbb{R}^n$ for some $k \leq n$, and have all the desired properties $\mathbb{R}^k$ has. So we have the following definition.

A subset $V \subseteq \mathbb{R}^n$ is a <u>subspace</u> if

(i) (Contains the origin) $\mathbf{0} \in V$, and

(ii) (Closed under linear combination) for any $\mathbf{u}, \mathbf{v} \in V$ and $\alpha, \beta \in \mathbb{R}$,

$$\alpha\mathbf{u} + \beta\mathbf{v} \in V.$$

Remark. In some textbooks, condition (i) is replaced by the condition that V is nonempty. This is equivalent for if V is nonempty, then by (ii), pick any vector $\mathbf{v} \in V$, then $\mathbf{0} = 0\mathbf{v} \in V$, and conversely, certainly if V contains the origin, it is nonempty.

We have seen that a spanning set satisfies these 2 properties, and hence is a subspace. It turns out that for Euclidean spaces, they are actually equivalent. The proof that every subspaces can be written as a spanning set will be given in lecture 8. We will state it as a theorem.

Theorem. *A subset $V \subseteq \mathbb{R}^n$ is a subspace if and only if there exists a finite set $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$ such that $V = span(S)$.*

Remark. Hence, to show that $V \subseteq \mathbb{R}^n$ is a subspace, we can either check that the definition is satisfied, that is, it contains the origin and is closed under linear combination, or find a finite subset $S \subseteq \mathbb{R}^n$ such that $V = \text{span}(S)$. However, if a subset V is not a subspace, it is impossible to directly show that there exists no $S \subseteq \mathbb{R}^n$ such that $V = \text{span}(S)$, since there are infinitely many S to check.

To show that $V \subseteq \mathbb{R}^n$ is not a subspace, we have to show that it does not satisfies some of the conditions in the definition. That is, show that either

(i) it does not contain the origin $\mathbf{0} \notin V$,

(ii) there is a vector $\mathbf{v} \in V$ and a $\alpha \in \mathbb{R}$ such that $\alpha\mathbf{v} \notin V$, or

(iii) there are vectors $\mathbf{u}, \mathbf{v} \in V$ such that $\mathbf{u} + \mathbf{v} \notin V$.

Example. 1. $V = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$ is a subspace spanned by $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

2. $V = \left\{ \begin{pmatrix} x + y \\ x - y \\ 0 \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$ is a subspace spanned by $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$.

3. The set $\left\{ \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$ is not a subspace since it does not contain the origin. It is not possible to write it as a span of some vectors in $\mathbb{R}^3$.

4. The subset $V = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \,\middle|\, ab = cd \right\}$ is not a subspace since $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \in V$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \in V$, but

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} \notin V.$$

Thus, it is not possible to write it as a span of some vectors in $\mathbb{R}^4$.

5. The subset $V = \left\{ \begin{pmatrix} s \\ s^2 \\ t \end{pmatrix} \,\middle|\, s, t \in \mathbb{R} \right\}$ is not a subspace since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in V$ but $2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \notin V$, since $2^2 \neq 2$. Thus, It is not possible to write it as a span of some vectors in $\mathbb{R}^3$.

By the theorem above, the set containing only the origin $\{\mathbf{0}\}$ in $\mathbb{R}^n$ is a subspace, since $\{\mathbf{0}\} = \text{span}\{\mathbf{0}\}$. We can also check that it satisfies the conditions of a subspace,

(i) it contains the origin, $\mathbf{0} \in \{\mathbf{0}\}$, and

(ii) the only vector in $\{\mathbf{0}\}$ is $\mathbf{0}$, and for any $\alpha, \beta \in \mathbb{R}$,

$$\alpha\mathbf{0} + \beta\mathbf{0} = \mathbf{0} \in \{\mathbf{0}\}.$$

This space is called the zero space. It is the only subspace that has finitely many (one) vector. Any subspaces V besides the zero space must have infinitely many vectors, since if $\mathbf{v} \in V$ and $\mathbf{v} \neq \mathbf{0}$, then $t\mathbf{v} \in V$ for all $t \in \mathbb{R}$, and they are distinct for different choices of $t \in \mathbb{R}$.

Exercise: Is the set $\left\{ \begin{pmatrix} s^3 \\ t^3 \\ 0 \end{pmatrix} \middle| \; s, t \in \mathbb{R} \right\}$ a subspace?

Subspaces of $\mathbb{R}^2$

(i) Zero space: $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$.

(ii) Image of $\mathbb{R}$: lines, $L = \text{span}\left\{ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \right\}$ for some fixed $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(iii) Whole $\mathbb{R}^2$.

Subspaces of $\mathbb{R}^3$

(i) Zero space: $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$.

(ii) Image of $\mathbb{R}$: Lines, $L = \text{span}\left\{ \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \right\}$ for some fixed $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

(iii) Image of $\mathbb{R}^2$: Planes, $P = \text{span}\left\{ \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \right\}$ for some $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ that are not a scalar multiple of each other.

(iv) Whole $\mathbb{R}^3$.

3.7 Solution Set of Linear Systems and Subspaces

Theorem. *The solution set* $\{\ \mathbf{u} \mid \mathbf{A}\mathbf{u} = \mathbf{b}\ \}$ *to a linear system* $\mathbf{A}\mathbf{x} = \mathbf{b}$ *is a subspace if and only if* $\mathbf{b} = \mathbf{0}$*, that is, the system is homogeneous.*

Proof. Suppose the solution $\{ \mathbf{u} \mid \mathbf{Au} = \mathbf{b} \}$ is a subspace. Then it must contain the origin. Hence, $\mathbf{b} = \mathbf{A0} = \mathbf{0}$.

Conversely, let $V = \{ \mathbf{u} \mid \mathbf{Au} = \mathbf{0} \}$ be the solution set to a homogeneous system $\mathbf{Ax} = \mathbf{0}$. We will show that V contains the origin and is closed under linear combinations. Clearly it contains the origin since $\mathbf{A0} = \mathbf{0}$. Now suppose $\mathbf{u}$ and $\mathbf{v}$ are in V, that is, they are solutions to the homogeneous system, $\mathbf{Au} = \mathbf{0} = \mathbf{Av}$. Then for any $\alpha, \beta \in \mathbb{R}$,

$$A(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha(\mathbf{Au}) + \beta(\mathbf{Av}) = \alpha\mathbf{0} + \beta\mathbf{0} = \mathbf{0}.$$

Hence, $\alpha\mathbf{u} + \beta\mathbf{v}$ is a solution to the homogeneous system, and thus is in V too. So V is closed under linear combination. $\square$

In general, the solution set of any linear system is known as an affine subspace. A set $W \subseteq \mathbb{R}^n$ is an <u>affine subspace</u> if there is a vector $\mathbf{u} \in \mathbb{R}^n$ and subspace $V \subseteq \mathbb{R}^n$ such that $W = \mathbf{u} + V$, that is, for every $\mathbf{w} \in W$, we can write $\mathbf{w} = \mathbf{u} + \mathbf{v}$ for some $\mathbf{v} \in V$.

Given a solution set

$$W = \{ \mathbf{w} \mid \mathbf{Aw} = \mathbf{b} \}$$

of a linear system $\mathbf{Ax} = \mathbf{b}$, we claim that $W = \mathbf{u} + V$, where

$$V = \{ \mathbf{v} \mid \mathbf{Av} = \mathbf{0} \}$$

is the solution set to the homogeneous system, and $\mathbf{u}$ is a particular solution, $\mathbf{Au} = \mathbf{b}$. This is because for any $\mathbf{v} \in V$, a solution to the homogeneous system, we have

$$\mathbf{A}(\mathbf{u} + \mathbf{v}) = \mathbf{Au} + \mathbf{Av} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

This shows that $\mathbf{u} + V \subseteq W$.

Conversely, suppose $\mathbf{w} \in W$ is a solution to the system $\mathbf{Ax} = \mathbf{b}$. Let $\mathbf{v} = \mathbf{w} - \mathbf{u}$. Then

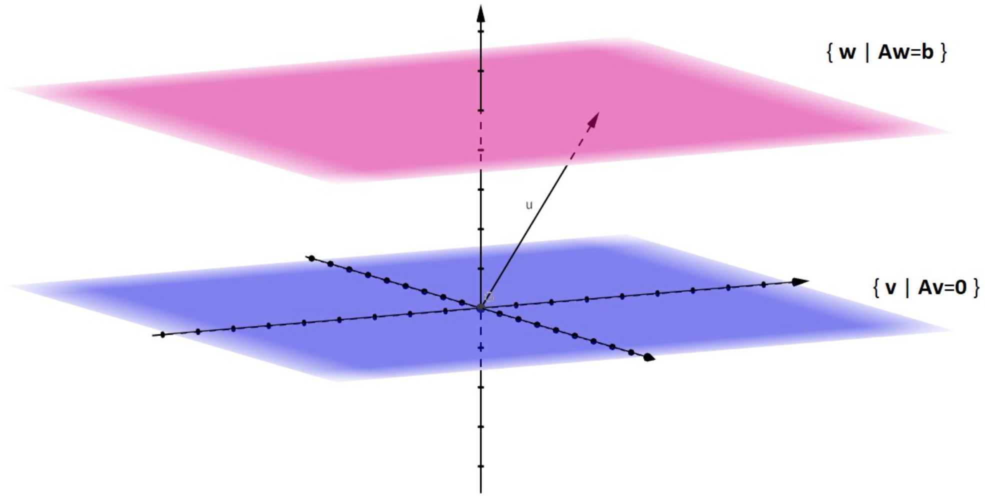
$$\mathbf{Av} = \mathbf{A}(\mathbf{w} - \mathbf{u}) = \mathbf{Aw} - \mathbf{Au} = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

tells us that $\mathbf{v} \in V$ is a solution to the homogeneous system. So we can write $\mathbf{w} = \mathbf{u} + (\mathbf{w} - \mathbf{u}) = \mathbf{u} + \mathbf{v}$ for some $\mathbf{v} \in V$. This shows that every $\mathbf{w} \in W$ can be written as $\mathbf{u} + \mathbf{v}$, and hence $W \subseteq \mathbf{u} + V$.

Hence, we have equality. Thus we have the following theorem.

Theorem. *The solution set* $W = \{ \mathbf{w} \mid \mathbf{Aw} = \mathbf{b} \}$ *of a linear system* $\mathbf{Ax} = \mathbf{b}$ *is given by* $\mathbf{u} + V$*, where* $V = \{ \mathbf{v} \mid \mathbf{Av} = \mathbf{0} \}$ *is the solution space to the associated homogeneous system and* $\mathbf{u}$ *is a particular solution,* $\mathbf{Au} = \mathbf{b}$.

That is, the solution set of a linear system is an affine subspace.


{ w | Aw=b }
u
{ v | Av=0 }

Appendix for Lecture 6

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$ be a subset. Now instead of asking if a specific vector $\mathbf{v} \in \mathbb{R}^n$ is in the span, we may ask if all the vectors in $\mathbb{R}^n$ is in the span, that is, whether $\text{span}(S) = \mathbb{R}^n$. This is equivalent to asking if for every $\mathbf{v}$ in $\mathbb{R}^n$, the system $\mathbf{Ax} = \mathbf{v}$ is consistent, where $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$ is the $n \times k$ matrix whose columns are the vectors in S. Let $\mathbf{R}$ be the reduced row-echelon form of $\mathbf{A}$. Then $\mathbf{PA} = \mathbf{R}$, where $\mathbf{P} = \mathbf{E}_r \cdots \mathbf{E}_2\mathbf{E}_1$ is an order n invertible matrix and $\mathbf{E}_i$, $i = 1, ..., r$ are the elementary matrices that reduce $\mathbf{A}$ to $\mathbf{R}$. Let us consider whether $\mathbf{R}$ has zero rows.

(i) Case 1: $\mathbf{R}$ has a zero row, that is, $\mathbf{R}$ is of the form

$$\mathbf{R} = \begin{pmatrix} \mathbf{Q} \\ \mathbf{0}_{1\times n} \end{pmatrix}$$

for some $n-1 \times k$ matrix $\mathbf{Q}$. Let $\mathbf{v} = \mathbf{P}^{-1}\mathbf{e}_n$, where $\mathbf{e}_n$ is the n-th column of the order n identity matrix. Then

$$\mathbf{Ax} = \mathbf{v} = \mathbf{P}^{-1}\mathbf{e}_n \Rightarrow \mathbf{Rx} = \mathbf{PAx} = \mathbf{e}_n$$

is inconsistent since the augmented matrix of $\mathbf{Rx} = \mathbf{e}_n$ is

$$\left(\begin{array}{ccc|c} & \mathbf{Q} & & 0 \\ & & & \vdots \\ 0 & \cdots & 0 & 1 \end{array}\right).$$

This means that $\text{span}(S) \neq \mathbb{R}^n$, since $\mathbf{v} = \mathbf{P}^{-1}\mathbf{e}_n \notin \text{span}(S)$.

(ii) Case 2: If $\mathbf{R}$ has no zero rows, then for any $\mathbf{v} \in \mathbb{R}^n$,

$$\left(\begin{array}{c|c} \mathbf{A} & \mathbf{v} \end{array}\right) \longrightarrow \left(\begin{array}{c|c} \mathbf{R} & \mathbf{v}' \end{array}\right)$$

is consistent since the pivot columns are always in the left hand side of the augmented matrix. Hence, for any $\mathbf{v} \in \mathbb{R}^n$, the system $\mathbf{Ax} = \mathbf{v}$ is consistent, and therefore $\text{span}(S) = \mathbb{R}^n$.

In summary, we have the following statement.

Corollary. *Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$ be a subset and $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$ be a $n \times k$ matrix whose columns are the vectors in S. Then span$(S) = \mathbb{R}^n$ if and only if the reduced row-echelon form $\mathbf{R}$ of $\mathbf{A}$ has no zero rows.*