NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

AY2021, Semester 1 MA1508E Linear Algebra for Engineering Tutorial 3

Solutions

1. Use the method of Gaussian elimination to determine if the following matrices are invertible. If the matrix is invertible, find its inverse.

   (a) $\begin{pmatrix} -1 & 3 \\ 3 & -2 \end{pmatrix}$.

   (b) $\begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$.

   (a)
$$\left(\begin{array}{cc|cc} -1 & 3 & 1 & 0 \\ 3 & -2 & 0 & 1 \end{array}\right) \xrightarrow{R_2+3R_1,\ -R_1,\ \frac{1}{7}R_2,\ R_1+3R_2} \left(\begin{array}{cc|cc} 1 & 0 & \frac{2}{7} & \frac{3}{7} \\ 0 & 1 & \frac{3}{7} & \frac{1}{7} \end{array}\right).$$

   So the matrix is invertible and its inverse is $\frac{1}{7}\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}$.

   (b)
$$\left(\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2+2R_1,\ R_3-4R_1,\ R_3+R_2} \left(\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right).$$

   The matrix is not invertible.

2. (a) Use the method of Gaussian elimination to write down the conditions so that the matrix $\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$ is invertible.

$$\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} \xrightarrow[R_3-a^2R_1]{R_2-aR_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{pmatrix} \xrightarrow{R_3-(b+a)R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c-a)(c-b) \end{pmatrix}$$

   So we need $c \neq a$ and $c \neq b$ for the last row to be nonzero. Suppose so, we proceed,

$$\xrightarrow{\frac{1}{(c-a)(b-a)}R_3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_1-R_3]{R_2-(c-a)R_3} \begin{pmatrix} 1 & 1 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   If $b \neq a$, then it is clear that the matrix can be reduced to the identity matrix. Thus the conditions are $a \neq b$, $b \neq c$, $c \neq a$, that is, they are distinct points.

   Alternative: One can stop after the third ERO and note that the determinant of the resultant matrix is $(b-a)(c-a)(c-b)$, which is nonzero if and only if the 3 points are distinct.

(b) Notice that the above matrix is the transpose of the order 3 Vandermonde matrix. By (a), what are the conditions needed for the 3 points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ on the xy-plane to ensure that there is a unique polynomial of degree 2 whose graph passes through those points.

Recall that a matrix is invertible if and only its transpose is. So if we pick the 3 points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ on the xy-plane to have distinct x-coordinates, then the Vandermonde matrix is invertible. This means that the linear system corresponding to the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & x_1 & x_1^2 & y_1 \\ 1 & x_2 & x_2^2 & y_2 \\ 1 & x_1 & x_1^2 & y_3 \end{array}\right)$$

has a unique solution, and thus we obtain a unique polynomial of degree 2 interpolating the 3 points.

3. (a) Solve the matrix equation $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 2 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 1 & 0 & 3 & 7 \\ 2 & 1 & 1 & 2 \end{pmatrix}$.

(b) Hence, solve for $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$. (Hint: look at the columns of the matrix on the right.)

(a)

$$\left(\begin{array}{ccc|cccc} 2 & 1 & 1 & 2 & 3 & 4 & 1 \\ 0 & 1 & 2 & 1 & 0 & 3 & 7 \\ 1 & 3 & 2 & 2 & 1 & 1 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & \frac{5}{7} & \frac{11}{7} & \frac{12}{7} & -\frac{5}{7} \\ 0 & 1 & 0 & \frac{1}{7} & -\frac{2}{7} & -\frac{13}{7} & -\frac{15}{7} \\ 0 & 0 & 1 & \frac{3}{7} & \frac{1}{7} & \frac{17}{7} & \frac{32}{7} \end{array}\right)$$

So $\mathbf{X} = \frac{1}{7}\begin{pmatrix} 5 & 11 & 12 & -5 \\ 1 & -2 & -13 & -15 \\ 3 & 1 & 17 & 32 \end{pmatrix}$

(b) Notice that $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ is the subtraction of the first column from the second of the matrix on the right. So let $\mathbf{x} = \frac{1}{7}\left(\begin{pmatrix} 11 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}\right) = \frac{1}{7}\begin{pmatrix} 6 \\ -3 \\ -2 \end{pmatrix}$ works.

Remarks: One may choose to compute the inverse of the matrix on the left, then premultiply it to the matrix on the right to find $\mathbf{X}$ in (a). This approach involves more computation than the approach above. However, the inverse can be used directly to compute part (b). Note that because the matrix on the left is invertible, the answers in (a) and (b) is unique.

4. (Cramer's Rule)

(a) Compute the determinant of the following matrices.

(i) $\mathbf{A} = \begin{pmatrix} 1 & 5 & 3 \\ 0 & 2 & -2 \\ 0 & 1 & 3 \end{pmatrix}$

(ii) $\mathbf{A}_1 = \begin{pmatrix} 1 & 5 & 3 \\ 2 & 2 & -2 \\ 0 & 1 & 3 \end{pmatrix}$

(iii) $\mathbf{A}_2 = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{pmatrix}$

(iv) $\mathbf{A}_3 = \begin{pmatrix} 1 & 5 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 0 \end{pmatrix}$

(b) Solve the matrix equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

(c) Compute $\frac{1}{\det(\mathbf{A})} \begin{pmatrix} \det(\mathbf{A}_1) \\ \det(\mathbf{A}_2) \\ \det(\mathbf{A}_3) \end{pmatrix}$. How is this related to the answer in (b)?

Observe that the matrix $\mathbf{A}_k$ is obtained by replacing the k-th column of $\mathbf{A}$ by b. Cramer's rule state that if A is an invertible matrix of order n and $\mathbf{A}_k$ is the matrix obtained from $\mathbf{A}$ by replacing the k-th column of $\mathbf{A}$ by b, then the matrix equation $\mathbf{Ax} = \mathbf{b}$ has a unique solution

$$\mathbf{x} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} \det(\mathbf{A}_1) \\ \det(\mathbf{A}_2) \\ \vdots \\ \det(\mathbf{A}_n) \end{pmatrix}.$$

(a) (i) 8 (ii) -16 (iii) 6 (iv) -2

(b) $\mathbf{x} = \frac{1}{4} \begin{pmatrix} -8 \\ 3 \\ -1 \end{pmatrix}$.

(c) Equal to the solution in (b).

5. Let $\mathbf{A} = \begin{pmatrix} -x & 1 & 0 \\ 0 & -x & 1 \\ 2 & -5 & 4-x \end{pmatrix}$. Find all values of x such $\det(\mathbf{A}) = 0$. For each of the x found, solve the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$.

$$\begin{vmatrix} -x & 1 & 0 \\ 0 & -x & 1 \\ 2 & -5 & 4-x \end{vmatrix} = -x^3 + 4x^2 - 5x + 2 = -(x-1)^2(x-2)$$

So the values of x such that $\det(\mathbf{A}) = 0$ are $x = 1$ and $x = 2$. When $x = 1$, solving $\mathbf{Ax} = \mathbf{0}$:

$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & -5 & 3 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \left\{\begin{array}{lcl} x_1 & = & s \\ x_2 & = & s \\ x_3 & = & s, \quad s \in \mathbb{R} \end{array}\right.$$

When $x = 2$, solving $\mathbf{Ax} = \mathbf{0}$:

$$\left(\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 2 & -5 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \left\{\begin{array}{lcl} x_1 & = & \frac{s}{4} \\ x_2 & = & \frac{s}{2} \\ x_3 & = & s, \quad s \in \mathbb{R} \end{array}\right.$$

6. A square matrix $\mathbf{P} = (p_{ij})$ of order n is a *stochastic matrix*, or a *Markov matrix* if the sum of each column vector is equal to 1,

$$p_{1j} + p_{2j} + \cdots + p_{nj} = 1$$

for every $j = 1, ..., n$.

(a) Give an example of an invertible stochastic matrix, and a singular one.

The identity matrix $\mathbf{I}$ is an obvious but uninteresting invertible stochastic matrix. $\begin{pmatrix} 0.2 & 0.3 \\ 0.8 & 0.7 \end{pmatrix}$ is an invertible stochastic matrix. The stochastic matrix P in (c) is also invertible. $\begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$ is a singular stochastic matrix.

(b) Show that if $\mathbf{P}$ is a stochastic matrix, then $\mathbf{I} - \mathbf{P}$ is singular.

$$\mathbf{I} - \mathbf{P} = \begin{pmatrix} 1 - p_{11} & -p_{12} & \cdots & -p_{1n} \\ -p_{21} & 1 - p_{22} & \cdots & -p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -p_{n1} & -p_{n2} & \cdots & 1 - p_{nn} \end{pmatrix} \xrightarrow{R_1 + R_2} \xrightarrow{R_1 + R_3} \cdots \xrightarrow{R_1 + R_n}$$

$$\begin{pmatrix} 1 - \sum_{k=1}^{n} p_{k1} & 1 - \sum_{k=1}^{n} p_{k2} & \cdots & 1 - \sum_{k=1}^{n} p_{kn} \\ -p_{21} & 1 - p_{22} & \cdots & -p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -p_{n1} & -p_{n2} & \cdots & 1 - p_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -p_{21} & 1 - p_{22} & \cdots & -p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -p_{n1} & -p_{n2} & \cdots & 1 - p_{nn} \end{pmatrix}$$

and so its RREF cannot be the identity matrix.

(c) Check that $\mathbf{P} = \begin{pmatrix} 0.2 & 0.8 & 0.4 \\ 0.3 & 0.2 & 0.4 \\ 0.5 & 0 & 0.2 \end{pmatrix}$ is a stochastic matrix. Solve the homogeneous system $(\mathbf{I} - \mathbf{P})\mathbf{x} = \mathbf{0}$.

$$\mathbf{I}-\mathbf{P}=\begin{pmatrix}0.8 & -0.8 & -0.4\\ -0.3 & 0.8 & -0.4\\ -0.5 & 0 & 0.8\end{pmatrix}\longrightarrow\begin{pmatrix}1 & 0 & -\frac{8}{5}\\ 0 & 1 & -\frac{11}{10}\\ 0 & 0 & 0\end{pmatrix}\Rightarrow\begin{cases}x_1 &= \frac{8}{5}s,\\ x_2 &= \frac{11}{10}s,\\ x_3 &= s, \quad s\in\mathbb{R}\end{cases}$$

7. Show that $\begin{vmatrix}a+px & b+qx & c+rx\\ p+ux & q+vx & r+wx\\ u+ax & v+bx & w+cx\end{vmatrix}=(1+x^3)\begin{vmatrix}a & b & c\\ p & q & r\\ u & v & w\end{vmatrix}$.

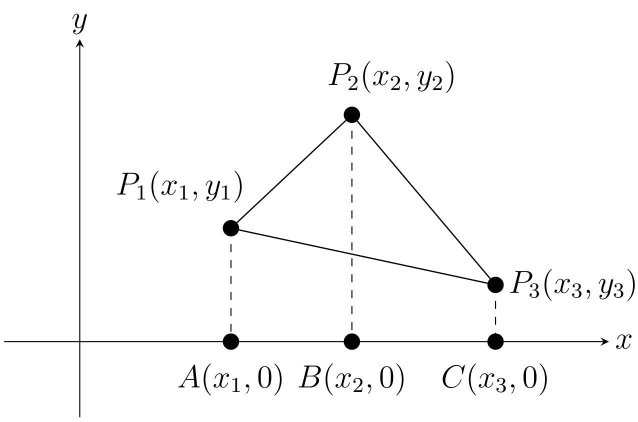
$$\begin{pmatrix}a+px & b+qx & c+rx\\ p+ux & q+vx & r+wx\\ u+ax & v+bx & w+cx\end{pmatrix}\xrightarrow{R_2-xR_3}\begin{pmatrix}a+px & b+qx & c+rx\\ p-ax^2 & q-bx^2 & r-cx^2\\ u+ax & v+bx & w+cx\end{pmatrix}\xrightarrow{R_1-xR_2}$$
$$\begin{pmatrix}a(1+x^3) & b(1+x^3) & c(1+x^3)\\ p-ax^2 & q-bx^2 & r-cx^2\\ u+ax & v+bx & w+cx\end{pmatrix}$$

If $x=-1$, then the first row is a zero row and so the determinant is 0, which agrees with the right hand side too. Otherwise, we continue

$$\xrightarrow{\frac{1}{1+x^3}R_1}\begin{pmatrix}a & b & c\\ p-ax^2 & q-bx^2 & r-cx^2\\ u+ax & v+bx & w+cx\end{pmatrix}\xrightarrow[R_3-xR_1]{R_2+x^2R_1}\begin{pmatrix}a & b & c\\ p & q & r\\ u & v & w\end{pmatrix}$$

Since all the elementary row operation except $\frac{1}{1+x^3}R_1$ are adding a multiple of one row to another, and thus the determinant of the elementary matrices are 1, and the determinant of the elementary matrix corresponding to $\frac{1}{1+x^3}R_1$ is $\frac{1}{1+x^3}$. Hence, $\begin{vmatrix}a & b & c\\ p & q & r\\ u & v & w\end{vmatrix}=\frac{1}{1+x^3}\begin{vmatrix}a+px & b+qx & c+rx\\ p+ux & q+vx & r+wx\\ u+ax & v+bx & w+cx\end{vmatrix}$.

8. (**Application of determinants to computing areas.**) Consider the triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) as shown in the figure below.



We may compute the area of the triangle as

(area of trapezoid AP_1P_2B)+(area of trapezoid BP_2P_3C)−(area of trapezoid AP_1P_3C)

(a) Recall that the area of a trapezoid is $\frac{1}{2}$ the distance between the parallel sides of the trapezoid times the sum of the lengths of the parallel sides. Use this fact to show that the area of the triangle $P_1P_2P_3$ is

$$-\frac{1}{2}\left[(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)\right].$$

The area of triangle $P_1P_2P_3$ is

$$\begin{aligned}
&= \frac{1}{2}(x_2 - x_1)(y_1 + y_2) + \frac{1}{2}(x_3 - x_2)(y_2 + y_3) - \frac{1}{2}(x_3 - x_1)(y_1 + y_3)\\
&= \frac{1}{2}x_2y_1 - \frac{1}{2}x_1y_2 + \frac{1}{2}x_3y_2 - \frac{1}{2}x_2y_3 - \frac{1}{2}x_3y_1 + \frac{1}{2}x_1y_3\\
&= -\frac{1}{2}\left[(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)\right]
\end{aligned}$$

(b) Show that the expression in the square brackets obtained in part (a) is the determinant of the following matrix

$$\mathbf{A} = \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}.$$

By cofactor expansion along the third column of $\mathbf{A}$, we have

$$\det(\mathbf{A}) = \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} - \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = (x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)$$

(c) Explain why we need to take the absolute value of $\det(\mathbf{A})$ before concluding that the area of the triangle is

$$\frac{1}{2}|\det(\mathbf{A})|.$$

The determinant may be positive or negative, depending on the location of the points and how they are labeled. For example, compare the determinant of

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x_2 & y_2 & 1 \end{pmatrix}.$$

(d) Find the area of the following quadrilaterals with the given vertices.

(i) P with vertices $(2, 3)$, $(5, 3)$, $(4, 5)$, $(7, 5)$.

(ii) Q with vertices $(-2, 3)$, $(1, 4)$, $(3, 0)$, $(-1, -3)$.

(i) The quadrilateral is divided into two triangles P_1 and P_2 with vertices $(2, 3), (5, 3), (4, 5)$ and $(4, 5), (5, 3), (7, 5)$ respectively. The area of the quadrilatral is:

$$\frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ 5 & 3 & 1 \\ 4 & 5 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} 4 & 5 & 1 \\ 5 & 3 & 1 \\ 7 & 5 & 1 \end{vmatrix} = \frac{1}{2}(6) + \frac{1}{2}(6) = 6.$$

(ii) The quadrilateral is divided into two triangles P_1 and P_2 with vertices $(-2,3),(1,4),(3,0)$ and $(-2,3),(-1,-3),(3,0)$ respectively. The area of the quadrilateral is:

$$\frac{1}{2}\begin{vmatrix} 1 & 4 & 1 \\ -2 & 3 & 1 \\ 3 & 0 & 1 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} -2 & 3 & 1 \\ -1 & -3 & 1 \\ 3 & 0 & 1 \end{vmatrix} = \frac{1}{2}(14) + \frac{1}{2}(27) = 20.5.$$