NATIONAL UNIVERSITY OF SINGAPORE

MA1508E - LINEAR ALGEBRA FOR ENGINEERING

(Semester 1 : AY2018/2019)

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. Please write your matriculation/student number only. Do not write your name.
2. This examination paper contains **SIX** questions and comprises **FOUR** printed pages.
3. Answer **ALL** questions.
4. Please start each question on a new page.
5. This is a CLOSED BOOK (with helpsheet) examination.
6. Students are allowed to use one A4 size helpsheet.
7. Candidates may use scientific calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [20 marks]

(a) Consider the following linear system, where a is a real number.

$$\begin{cases} 2x & - & ay & + & 3z & = & 0 \\ 4x & - & 2y & + & 5z & = & -a \\ -2x & + & ay & - & 2z & = & a \end{cases}$$

(i) If $a = 1$, show that the linear system has infinitely many solutions. Write down two different solutions to the linear system.

(ii) For **<u>all other</u>** values of a, show that the linear system has a unique solution. Find the unique solution in terms of a.

(b) Let $W = \text{span}\{\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3}\}$ where

$$\boldsymbol{w_1} = (1, 1, 1, 1), \qquad \boldsymbol{w_2} = (-3, 1, -3, 1), \qquad \boldsymbol{w_3} = (-5, -2, 1, 4).$$

(i) Show that $\{\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3}\}$ is a basis for W.

(ii) Apply Gram-Schmidt Process to find an orthogonal basis $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ for W.

(iii) Find the vector in W that is closest to $\boldsymbol{w} = (1, 0, 3, 0)$. Use your answer to extend $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ to an orthogonal basis for $\mathbb{R}^4$.

Question 2 [20 marks]

(a) Find a least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$ where

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 \\ 4 & -3 \\ 2 & 1 \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix}.$$

Hence or otherwise, compute the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$.

(b) Consider each of the following subsets S (of the respective $\mathbb{R}^n$). If S is a subspace, find a basis for and state the dimension of S. If S is not a subspace, explain why.

(i) $S = \{\boldsymbol{x} \in \mathbb{R}^2 \mid ||\boldsymbol{x}|| \le 1\}$.

(ii) $S = \{(x, y, z) \mid x = y = 2z\}$.

(iii) $S = \{(a, b, c, d) \mid a = b \text{ or } c = d\}$.

(iv) $S = \{(c_1, c_2, c_3) \mid c_1(1, 1, 1) + c_2(2, 2, 2) + c_3(1, -1, -1) = (0, 0, 0)\}$.

Question 3 [20 marks]

(a) Use Gauss-Jordan elimination to find the inverse of the following matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ -1 & 1 & 2 \end{pmatrix}.$$

Show all the elementary row operations used. Hence or otherwise, solve the following equation:

$$\begin{pmatrix} 1 & 3 & -1 \\ 0 & -1 & 1 \\ 2 & 1 & 2 \end{pmatrix} \boldsymbol{x} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

(b) Let $\boldsymbol{A} = \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}$.

(i) Find a non trivial solution to $(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$. Hence deduce that 1 is an eigenvalue of $\boldsymbol{A}$.

(ii) Show that 0.3 is another eigenvalue of $\boldsymbol{A}$. Is $\boldsymbol{A}$ diagonalizable? Justify your answer.

(iii) In a city, every resident is either a supporter of soda drink A or soda drink B (but not both). Due to the constant marketing efforts by both brands of soda, residents change their support frequently. It is estimated that every month, 40% of brand A supporters will switch to brand B while 30% of brand B supporters will switch to brand A.

For $i = 1, 2, 3, \cdots$, let $\boldsymbol{x_i} = \begin{pmatrix} a_i \\ b_i \end{pmatrix}$ where a_i (resp. b_i) is the number of supporters of brand A (resp. B) in the i-th month.

Find a 2×2 matrix $\boldsymbol{B}$ such that $\boldsymbol{x_{i+1}} = \boldsymbol{B}\boldsymbol{x_i}$ for $i = 1, 2, 3, \cdots$.

Suppose in the first month, there were 5000 supporters for each brand. Find the proportion of residents in the city that would support brand A in the long run. You may assume that the city's population remains constant at 10000.

Question 4 [15 marks]

(a) Let

$$\boldsymbol{u_1} = (-1, 6, -5, 3), \quad \boldsymbol{u_2} = (1, -2, 3, 1), \quad \boldsymbol{u_3} = (2, -6, 7, 0), \quad \boldsymbol{u_4} = (0, 2, -1, 2).$$

Suppose $V = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$. Find a basis for V and find all vectors orthogonal to V.

(b) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be matrices (not necessarily square) of appropriate size such that

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix}$$

is an $m \times n$ matrix.

(i) Prove that

$$\text{rank}(\boldsymbol{X}) = \text{rank}(\boldsymbol{A}) + \text{rank}(\boldsymbol{B}).$$

(**Hint:** Find a basis for the column space of $\boldsymbol{X}$.)

(ii) Let $\boldsymbol{C}$ be a matrix of appropriate size such that

$$\boldsymbol{Y} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{B} \end{pmatrix}$$

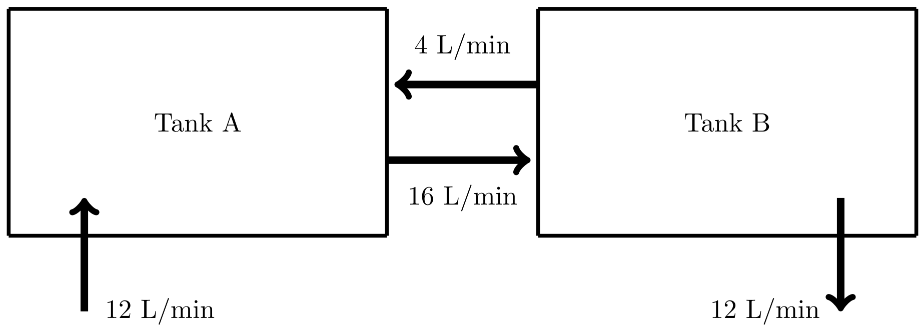
is an $m \times n$ matrix. Is it true that

$$\text{rank}(\boldsymbol{Y}) = \text{rank}(\boldsymbol{A}) + \text{rank}(\boldsymbol{B})?$$

Justify your answer.

Question 5 [15 marks]

Consider two large tanks that are connected as shown in the figure below.



Tank A is initially filled with 100 L (litres) of water and 40 g (grams) of salt was dissolved in it. Tank B is initially filled with 100 L of water and 20 g of salt was dissolved in it. The well-mixed solution from Tank A is constantly pumped into Tank B at the rate of 16 L per minute while the solution in Tank B is pumped back into Tank A at the rate of 4 L per minute. Pure water is constantly pumped into Tank A at the rate of 12 L per minute while water exits the system from Tank B at the rate of 12 L per minute.

At t minutes after the start of the mixing, let $a(t)$ and $b(t)$ be the amount of salt in Tanks A and B respectively. Construct a system of linear first order differential equations to evaluate $a(t)$ and $b(t)$ for each t.

Hence deduce that the amount of salt in Tank B will always be less than twice the amount of salt in Tank A.

Question 6 [10 marks]

(a) Let $\theta \in [0, 2\pi]$, and define $\boldsymbol{R_\theta}$ to be the matrix

$$\boldsymbol{R_\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Show that $\boldsymbol{R_\theta}$ is an orthogonal matrix.

(b) Hence, or otherwise, explain why the set

$$\mathcal{B} = \left\{ \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix}, \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

forms an orthonormal basis for $\mathbb{R}^3$.

(c) Describe the geometric transformation that converts the standard basis to the basis $\mathcal{B}$.

(d) A charged particle P is confined to travelling along a wire given by the line $x = y$ on the plane $z = 0$ in $\mathbb{R}^3$. The particle P is drawn to a positive electric charge M that is situated at the point $(\boldsymbol{m})_{\mathcal{B}} = (20, 20, 10)$.

(i) Find the coordinates of M relative to the standard basis.

(ii) P will stop travelling once it is at the point closest to M. Find the coordinates of P relative to $\mathcal{B}$ at the point when it stops travelling. How far is P away from M when it stops traveling?

End of Paper