NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

AY2021, Semester 1 MA1508E Linear Algebra for Engineering Tutorial 7

1. Let U and V be subspaces of $\mathbb{R}^n$. We define the sum $U+V$ to be the set

$$\{\ \mathbf{u}+\mathbf{v} \mid \mathbf{u}\in U \text{ and } \mathbf{v}\in V\ \}.$$

Suppose $U = \text{span}\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\2\\1\end{pmatrix}\right\}$, $V = \text{span}\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\2\\-1\end{pmatrix}\right\}$.

(a) Is $U\cup V$ a subspace of $\mathbb{R}^4$?

(b) Show that $U+V$ a subspace by showing that it can be written as a span of a set. What is the dimension?

(c) Show that $U+V$ contains U and V. This shows that $U+V$ is a subspace containing $U\cup V$.

(d) What are the dimensions of U and V?

(e) Show that $U\cap V$ a subspace by showing that it can be written as a span of a set. What is the dimension?

(f) Verify that $\dim(U+V) = \dim(U)+\dim(V)-\dim(U\cap V)$.

2. Let $V = \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \,\middle|\, x+2y-z=0 \right\}$. Let $\mathbf{u}_1 = \begin{pmatrix}1\\1\\3\end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix}1\\0\\1\end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix}0\\1\\2\end{pmatrix}$.

(a) Show that $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ belongs to V.

(b) Is the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ linearly independent?

(c) Find a basis for V.

3. (a) Let $\mathbf{A} = \begin{pmatrix}1 & -1 & 1\\ 1 & 1 & -1\\ -1 & -1 & 1\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}2\\1\\0\end{pmatrix}$. Is $\mathbf{b}$ in the column space of $\mathbf{A}$? If it is, express it as a linear combination of the rows of $\mathbf{A}$.

(b) Let $\mathbf{A} = \begin{pmatrix}1 & 9 & 1\\ -1 & 3 & 1\\ 1 & 1 & 1\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}5, 1, -1\end{pmatrix}$. Is $\mathbf{b}$ in the row space of $\mathbf{A}$? If it is, express it as a linear combination of the columns of $\mathbf{A}$.

(c) Let $\mathbf{A} = \begin{pmatrix}1 & 2 & 0 & 1\\ 0 & 1 & 2 & 1\\ 1 & 2 & 1 & 3\\ 0 & 1 & 2 & 2\end{pmatrix}$. Is the row space and column space of $\mathbf{A}$ the whole $\mathbb{R}^4$?

4. (a) Find a subset of the vectors

$$\mathbf{v}_1 = (1, -2, 0, 3), \ \ \mathbf{v}_2 = (2, -5, -3, 6), \ \ \mathbf{v}_3 = (0, 1, 3, 0)$$
$$\mathbf{v}_4 = (2, -1, 4, -7), \ \ \mathbf{v_5} = (5, -8, 1, 2)$$

that forms a basis for span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v_5}\}$.

(b) Express each vector not in the basis as a linear combination of the basis vectors.

(c) Extend the basis in (a) to a basis for $\mathbb{R}^4$

Supplementary Problems

5. **(MATLAB)** In the last tutorial, we introduced the "binary n-space" $\mathbb{B}^n$, which is governed by the special addition rule: $1 + 1 = 0$. Our goal in this problem is to investigate how working with binary vectors can help us detect and correct errors in information transmission.

When a computer transmits a piece of information (a binary string) to another device, there is always a possibility of an error—that is, the receiving device might receive an incorrect binary string, perhaps due to external interference or noise in the communication channel. One way that a computer might "protect" its message is by adding extra information to the binary string so that the receiving device can detect—and ideally, correct—any errors that may have occured in transmission.

Consider the following scenario: your friend Annette wants to send you a message $\boldsymbol{u}$. For the sake of simplicity, let's assume that Annette's message contains four bits—that is, $\boldsymbol{u}$ is a vector in $\mathbb{B}^4$ (as opposed to a standard *byte*, which is a vector in $\mathbb{B}^8$). Annette, however, is afraid that a transmission error might send you the wrong message—say, by accidentally changing a 1 to a 0, or a 0 to a 1.

(a) Rather than just sending you the message $\boldsymbol{u}$, Annette instead sends you the 8-vector that results when each bit in $\boldsymbol{u}$ is repeated twice. You receive the vector

$$\left(\begin{array}{cc|cc|cc|cc} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right),$$

where divider bars have been used to split the string up into segments, each representing one bit in Annette's original message. Do you have enough information to decode Annette's original message?

As the above situation suggests, a simple—perhaps naïve—error-correcting code would employ *repetition*: sending each bit repeatedly, with the hope that the recipient will be able to spot any errors. Observe, however, that this method significantly increases the required amount of data to be transmitted. In the above example, Annette needed to send out twice as much data—this may be problematic for longer messages, specially since bandwidth is expensive!

(b) Annette, who is running out of mobile data, attempts a different error-correcting code, invented by the 20th Century mathematician Richard Hamming. Recall that, taking the non-zero vectors in $\mathbb{B}^3$ as columns, we can create the *Hamming matrix*

$$\boldsymbol{H} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

From this, we formed the matrix $\boldsymbol{M}$ by taking the basis vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ for $S = \{\boldsymbol{x} \in \mathbb{B}^7 \mid \boldsymbol{H}\boldsymbol{x} = \boldsymbol{0}\}$ (i.e., the basis for the null space of $\boldsymbol{H}$) as its columns:

$$\boldsymbol{M} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

i. Explain why the set S is identical to the column space of $\boldsymbol{M}$. Hence, without explicitly calcuating any matrix products, explain why $\boldsymbol{H}(\boldsymbol{M}\boldsymbol{x}) = \boldsymbol{0}$ for all vectors $\boldsymbol{x} \in \mathbb{B}^4$.

ii. Consider the vector $\boldsymbol{v} = \boldsymbol{M}\boldsymbol{u}$, which is a vector in $\mathbb{B}^7$. The first three entries of $\boldsymbol{v}$ will later be used to detect errors. What are the last four entries of $\boldsymbol{v}$?

iii. Instead of transmitting the vector $\boldsymbol{u}$, Annette's computer instead sends out $\boldsymbol{v}$. Assume that *at most* one error can occur during transmission.

- What vector will you receive if no errors occur during transmission?
- Let $\boldsymbol{e}_i$ denote the standard unit vector whose i-th entry is 1 and remaining entries are all 0's. Explain why you would receive a vector of the form $\boldsymbol{v} + \boldsymbol{e}_i$, for some $i \in \{1, \dots, 7\}$, if an error has occured during transmission.

iv. Let $\boldsymbol{w}$ be the vector you receive on your device. Explain how calculating the matrix product $\boldsymbol{H}\boldsymbol{w}$ will allow you to detect and correct a potential transmission error. [*Hint:* First consider what the vector $\boldsymbol{w}$ would look like if no error has occured, then consider the possibility that an error has occured during transmission.]

v. Your device receives the vector

$$\boldsymbol{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Has an error been made during transmission? Do you have enough information to deduce Annette's original message?

6. Let $\mathbf{A}$ and $\mathbf{B}$ be row equivalent matrices. Prove that any linear relationship between the columns of $\mathbf{A}$ will be preserved (that is, remain the same) between the columns of $\mathbf{B}$. (**Hint:** Relate a linear relationship between the columns of $\mathbf{A}$ with a homogeneous linear system.)
This, in particular, shows that if $\mathbf{A}$ and $\mathbf{B}$ are row equivalent, then a set of columns of $\mathbf{A}$ forms a basis for the column space of $\mathbf{A}$ if any only if the set of corresponding columns of $\mathbf{B}$ forms a basis for the column space of $\mathbf{B}$.