MA1508E: LINEAR ALGEBRA FOR ENGINEERING

Lecture 7 Notes

References

1. Elementary Linear Algebra: Application Version, Section 4.3-4.4
2. Linear Algebra with Application, Section 5.2, 6.3

3.8 Linear Independence

Consider $\mathbf{u}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix}1\\1\\0\end{pmatrix}, \mathbf{u}_4 = \begin{pmatrix}1\\-1\\0\end{pmatrix}$. Then

$$\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \text{span}\{\mathbf{u}_1, \mathbf{u}_2\} = V$$

where V is the xy-plane in $\mathbb{R}^3$. But span$\{\mathbf{u}_1\}$ is the x-axis, which is not V. So it seems like we are allowed to remove some vectors and still span the same subspace. It is then natural to ask if given a set S such that span$(S) = V$, can we find a smallest subset of S that can still do the job of spanning V. In the above example, one can say that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a smallest set. But note that the choice of smallest set is not unique, we can also choose $\{\mathbf{u}_3, \mathbf{u}_4\}$; in fact, in this case, any choice of two vectors will give us a smallest set to span the xy-plane.

A vector $\mathbf{u}$ is linearly dependent on $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$ if it is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$,

$$\mathbf{u} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$$

for some $c_1, c_2, ..., c_k \in \mathbb{R}$.

Example.

$$\begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}2\\3\\0\end{pmatrix}$$

So $\begin{pmatrix}2\\3\\0\end{pmatrix}$ is linearly dependent on $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\2\\-1\end{pmatrix}$. But observe that

$$\begin{pmatrix}2\\3\\0\end{pmatrix} - \begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \text{ and } \begin{pmatrix}2\\3\\0\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\2\\-1\end{pmatrix}.$$

So we may also say that $\begin{pmatrix}1\\1\\1\end{pmatrix}$ is linearly dependent on $\begin{pmatrix}2\\3\\0\end{pmatrix}$ and $\begin{pmatrix}1\\2\\-1\end{pmatrix}$, and $\begin{pmatrix}1\\2\\-1\end{pmatrix}$ is linearly dependent on $\begin{pmatrix}2\\3\\0\end{pmatrix}$ and $\begin{pmatrix}1\\1\\1\end{pmatrix}$.

So it is more accurate to describe the linearly dependency of a set rather than individual vectors. So we might say that a set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is linearly dependent if one of the $\mathbf{u}_i$ is a linear combination of the others. This is easy to understand, but very tedious to check, since by this definition, we have to check for consistency of

$$\left(\begin{array}{cccccc|c} \mathbf{u}_1 & \cdots & \mathbf{u}_{i-1} & \mathbf{u}_{i+1} & \cdots & \mathbf{u}_k & \mathbf{u}_i \end{array}\right)$$

for every single $i = 1, ..., n$. However, observe that if say, without lost of generality,

$$\mathbf{u}_k = c_1\mathbf{u}_1 + \mathbf{u}_2 + \cdots + c_{k-1}\mathbf{u}_{k-1}, \tag{3}$$

then

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_{k-1}\mathbf{u}_{k-1} - \mathbf{u}_k = \mathbf{0},$$

that is, we are able to find a nontrivial (because the coefficient of $\mathbf{u}_k$ is $-1 \neq 0$) linear combination of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$ to give $\mathbf{0}$. We may replace $\mathbf{u}_k$ in (3) with any of the $\mathbf{u}_i$ and still have the same conclusion that there is a nontrivial combination of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$ to give us $\mathbf{0}$. Hence, we now have a useful definition of linear dependency.

A set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is <u>linearly dependent</u> if there exists $c_1, c_2, ..., c_k \in \mathbb{R}$, not all zero such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}.$$

A set is <u>linearly independent</u> otherwise. Explicitly, a set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is linearly independent if whenever $c_1, c_2, ..., c_k \in \mathbb{R}$ is such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0},$$

necessarily $c_1 = c_2 = \cdots = c_k = 0$. In words, the only linear combination of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$ to give $\mathbf{0}$ is the trivial one.

Theorem. *A set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is linearly dependent if and only if there is a $i = 1, ..., k$ such that*

$$\mathbf{u}_i = c_1\mathbf{u}_1 + \cdots + c_{i-1}\mathbf{u}_{i-1} + c_{i+1}\mathbf{u}_{i+1} + \cdots + c_k\mathbf{u}_k,$$

that is, $\mathbf{u}_i$ is a linear combination of the rest of the vectors in the set S.

Thus to check if a set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is linearly dependent, we are asking if the (homogeneous) system associated to

$$\left(\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{0} \end{array}\right)$$

has nontrivial solution. We will state it as a theorem.

Theorem. *A set $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$ is linearly independent if and only if the homogeneous system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution, where $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$ is the matrix whose column is formed by the vectors in S.*

But recall that the system $\mathbf{Ax} = \mathbf{0}$ has nontrivial solutions if and only if the reduced row-echelon form of $\mathbf{A}$ has non-pivot columns (since the number of non-pivot column is equal to the number of parameters required in a general solution).

Theorem. *A set $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$ is linearly dependent if and only if any row-echelon form of $\mathbf{A}$ has non-pivot columns, where $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$ is the matrix whose column is formed by the vectors in S.*

We will learn later that the theorem above is equivalent to $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$ is linearly independent if and only if $\text{rank}(\mathbf{A}) = k$, where $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$.

Corollary. *Any subset of* $\mathbb{R}^n$ *containing more than* n *vectors must be linearly dependent.*

Proof. If $k > n$ and $S \subseteq \mathbb{R}^n$ has k vectors, the matrix whose columns are vectors in S has size $n \times k$, and must have at least $k - n$ non-pivot columns. $\square$

Example. 1. $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$.

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is already in reduced row-echelon form, and has 2 non-pivot columns. Hence S is linearly dependent. Alternatively, since S contains 4 vectors in $\mathbb{R}^3$, it must be linearly dependent.

2. $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right\}$ is linearly dependent since

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & -1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

3. $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ is linearly independent since

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

A set $S = \{\mathbf{u}\} \subseteq \mathbb{R}^n$ containing one nonzero vector, $\mathbf{u} \neq \mathbf{0}$, is linearly independent. For the only linear combination of $\mathbf{u}$ is taking scalar multiple, and since $\mathbf{u} \neq \mathbf{0}$, $c\mathbf{u} = \mathbf{0}$ if and only if $c = 0$. However, the set containing the origin $S = \{\mathbf{0}\} \subseteq \mathbb{R}^n$ is linearly dependent, since we can take $c = 1$ and

$$c\mathbf{0} = \mathbf{0}$$

is a nontrivial linear combination (scalar multiple) of $\mathbf{0}$ to get $\mathbf{0}$.

Lemma. *A set* $S = \{\mathbf{u}, \mathbf{v}\}$ *containing two vectors is linearly independent if and only if the vectors are not a multiple of each other.*

Proof. S is linearly dependent if and only if $\alpha\mathbf{u} + \beta\mathbf{v} = 0$ for some $\alpha, \beta \in \mathbb{R}$ not both zero. If $\alpha \neq 0$, then $\mathbf{u} = (-\beta/\alpha)\mathbf{v}$, and if $\beta \neq 0$, $\mathbf{v} = (-\alpha/\beta)\mathbf{u}$, that is the vectors are multiple of each other. $\square$

Theorem. *Suppose* $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$ *is linearly dependent. Then for any* $\mathbf{u} \in \mathbb{R}^n$*, the set*

$$\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k, \mathbf{u}\}$$

is linearly dependent.

That is, adding vectors to a linearly dependent set will not make it linearly independent.

This means that any set containing the origin $\mathbf{0}$ must be linearly independent. We can also see this from the fact that a matrix containing a zero column must have non-pivot column in its reduced row-echelon form; the zero column is a non-pivot column.

Converse to the previous theorem, we are allowed to add vectors to linearly independent set and still have a linearly independent set if the added vector is not linearly dependent of the vectors in the original set. We will state it precisely as a theorem.

Theorem. *Suppose* $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$ *is linearly independent and* $\mathbf{u}$ *is not a linearly combination of* $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$*,* $\mathbf{u} \notin span\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$*. Then the set* $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k, \mathbf{u}\}$ *is linearly independent.*

Example. 1. Let $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$. Then

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & 0 \\ 1 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which shows that S is linearly dependent. So for any $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, the set

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\}$$

will be linearly dependent,

$$\begin{pmatrix} 1 & 1 & 0 & x \\ 2 & 3 & 0 & y \\ 1 & 1 & 0 & z \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 3x - y \\ 0 & 1 & 0 & -2x + y \\ 0 & 0 & 0 & -x + z \end{pmatrix}.$$

2. Let $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$. Then S is linearly independent,

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

For any $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$,

$$\begin{pmatrix} 1 & 1 & 0 & x_1 \\ 2 & 0 & 1 & x_2 \\ 1 & 0 & 1 & x_3 \\ -1 & 1 & 0 & x_4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & \frac{x_1-x_4}{2} \\ 0 & 1 & 0 & \frac{x_1+x_4}{2} \\ 0 & 0 & 1 & 2x_3-x_2 \\ 0 & 0 & 0 & \frac{-x_1+2x_2-2x_3+x_4}{2} \end{pmatrix}.$$

Hence, $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \right\}$ is linearly independent if and only if $-x_1 + 2x_2 - 2x_3 + x_4 \neq 0$, if and only if $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \notin \text{span}(S)$.

3.9 Basis

Recall that a subset $V \subseteq \mathbb{R}^n$ is a subspace if and only if $V = \text{span}(S)$ for some finite subset $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$. This would mean that every vector $\mathbf{v} \in V$ can be expressed as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$. Suppose further that the set S is also linearly independent. Then we claim that the coefficient of the linear combination is unique. Indeed, suppose $\mathbf{v} \in V$ is such that

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k \text{ and } \mathbf{v} = d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + \cdots + d_k\mathbf{u}_k$$

for some $c_1, ..., c_k, d_1, ..., d_k \in \mathbb{R}$. Then

$$(c_1 - d_1)\mathbf{u}_1 + (c_2 - d_2)\mathbf{u}_2 + \cdots + (c_k - d_k)\mathbf{u}_k = \mathbf{v} - \mathbf{v} = \mathbf{0}.$$

Since S is linearly independent, necessary the coefficients are zero, $c_1 - d_1 = c_2 - d_2 = \cdots = c_k - d_k = 0$, that is

$$c_1 = d_1, \ c_2 = d_2, ..., \ c_k = d_k.$$

This means that the coefficients of $\mathbf{v}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$ are unique. So a set S that both spans V and is linearly independent is special in the sense that every vectors $\mathbf{v} \in V$ can be written as a linear combination of the vectors in S uniquely. We call such a special sets a basis.

Let $V \subseteq \mathbb{R}^n$ be a subspace. A set $S \subseteq V$ is a <u>basis</u> for V if

(i) $\text{span}(S) = V$, and

(ii) S is linearly independent.

Note that basis is not unique. We have already seen that both $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ and $T = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$ are bases for the xy-plane.

Recall that the solution set to a homogeneous system $\mathbf{Ax} = \mathbf{0}$ is a subspace. Suppose

$$s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k,\ s_1, s_2, ..., s_k \in \mathbb{R}$$

is a general solution to the homogeneous system such that the parameters correspond to the non-pivot columns in the reduced row-echelon form of $\mathbf{A}$. Then observe that in each $\mathbf{u}_i$, there must be a entry where only $\mathbf{u}_i$ has 1 in the entry and $\mathbf{u}_j$ has 0 in this entry for all $j \neq i$. Thus the set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is linearly independent. Clearly the set spans the solution space. We thus arrive at the following theorem.

Theorem. *If $V = \left\{ s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k \;\middle|\; s_1, s_2, ..., s_k \in \mathbb{R} \right\}$ is a solution space to a homogeneous system $\mathbf{Ax} = \mathbf{0}$ such that the parameters correspond to the non-pivot columns in the reduce row-echelon form of $\mathbf{A}$, then $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is a basis for V.*

Example. 1. Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x + y - z = 0 \right\}$. It is the solution space to a homogeneous system, so it is a subspace of $\mathbb{R}^3$. Explicitly, $V = \left\{ s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; s, t \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$. Let $T = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$. To show that T is a basis, suffice to show that it is independent.

$$\begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Since the reduced row-echelon form has no non-pivot columns, T is linearly independent. Alternatively, one can just observe that the vectors in T cannot be a multiple of each other. Hence, T is a basis.

2. Let V and T as defined in 1. Let $S = \left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$. We will show that S is a basis for V. First we show that span$(S) = V$. To show that span$(S) \subseteq V$, suffice to show that $S \subseteq V$. Indeed,

$$(-1) + (2) - (1) = 0 \text{ and } (1) + (1) - 2 = 0.$$

To show that $V \subseteq$ span(S), we show that span$(T) \subseteq$ span(S), that is, suffice to show that $T \subseteq$ span(S). Indeed,

$$\left(\begin{array}{cc|c|c} -1 & 1 & -1 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c|c} 1 & 0 & 2/3 & 1/3 \\ 0 & 1 & -1/3 & 2/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

is consistent. This shows that $\text{span}(S) = V$. Next, we need to show that S is linearly independent. Indeed, it is clear $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ cannot be a multiple of each other.

Question: We have seen that the zero space $\{\mathbf{0}\} \subseteq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$. What is a basis for the zero space?

We will now discuss when is a subset $S \subseteq \mathbb{R}^n$ a basis for the whole $\mathbb{R}^n$. Recall that if the set S contains less than n vectors, it cannot span $\mathbb{R}^n$ and if it has more than n vectors, it must be linearly dependent. So for S to span $\mathbb{R}^n$ and be linearly independent, it must have exactly n vectors, $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n\}$. Let $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{pmatrix}$ be the square matrix of order n whose column is formed by the vectors in S. Then for S to span n, the reduced row-echelon form of $\mathbf{A}$ cannot have zero rows, and for S to be linearly independent, the reduced row-echelon form of $\mathbf{A}$ cannot have non-pivot column. But since $\mathbf{A}$ is a square matrix, either of the condition is equivalent to the reduced row-echelon form of $\mathbf{A}$ being the identity matrix. That is, S is a basis for $\mathbb{R}^n$ if and only if $\mathbf{A}$ is invertible. We thus have another equivalent criteria for invertibility.

Theorem. *A subset $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$ is a basis for $\mathbb{R}^n$ if and only if $k = n$ and $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{pmatrix}$ is an invertible matrix.*

Another way of phrasing the theorem.

Theorem. *A square matrix $\mathbf{A}$ of order n is invertible if and only if the columns of $\mathbf{A}$ form a basis for $\mathbb{R}^n$.*

Taking the transpose and recalling that $\mathbf{A}$ is invertible if and only if its transpose is, we get the following statement.

Theorem. *A square matrix $\mathbf{A}$ of order n is invertible if and only if the rows of $\mathbf{A}$ form a basis for $\mathbb{R}^n$.*

We will add these statements into the list of equivalence of invertibility.

Theorem. *Let $\mathbf{A}$ be a square matrix of order n. The following statements are equivalent.*

(i) $\mathbf{A}$ *is invertible.*

(ii) $\mathbf{A}$ *has a left inverse.*

(iii) $\mathbf{A}$ *has a right inverse.*

(iv) The reduced row-echelon form of $\mathbf{A}$ *is the identity matrix* $\mathbf{I}_n$.

(v) $\mathbf{A}$ *is a product of elementary matrices.*

(vi) The homogeneous linear system $\mathbf{A}\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(vii) The linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ *has a unique solution for every* $\mathbf{b}$.

(viii) The determinant of $\mathbf{A}$ *is nonzero,* $\det(\mathbf{A}) \neq 0$.

(ix) The columns of $\mathbf{A}$ *form a basis for* $\mathbb{R}^n$.

(x) The rows of $\mathbf{A}$ form a basis for $\mathbb{R}^n$.

Recall that we said that subspaces of $\mathbb{R}^n$ are like copies of $\mathbb{R}^k$ inside $\mathbb{R}^n$ for some $k \le n$. Here we can make this statement precise. We have shown the following statement.

Theorem. *Suppose S is a basis for V. then every vectors $\mathbf{v} \in V$ can be written as a linear combination of vectors in S uniquely.*

So now let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a basis for a subspace $V \subseteq \mathbb{R}^n$. Then given any $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} \in \mathbb{R}^k$, we get a unique vector

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k \in V.$$

Conversely, for any $\mathbf{v} \in V$, we can find a unique $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} \in \mathbb{R}^k$ such that

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k \in V.$$

This means that a basis S for V corresponds to an embedding (a copy of) $\mathbb{R}^k$ into $\mathbb{R}^n$. That is,

$$\mathbb{R}^k \xleftrightarrow{\text{via } S} V \subseteq \mathbb{R}^n, \qquad \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} \leftrightarrow c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k.$$

We call the unique vector in $\mathbb{R}^k$ corresponding to $\mathbf{v} \in V$ via S the <u>coordinates of $\mathbf{v}$ relative to basis S</u>, and denote it as

$$(\mathbf{v})_S = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}.$$

Example. 1. Let $E = \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ be the standard basis. For any vector $\mathbf{v} = (v_i) \in \mathbb{R}^n$, the relative coordinate of $\mathbf{v}$ relative to E is itself, $(\mathbf{v})_E = \mathbf{v}$. That is, the coordinates of a vector is its coordinates relative to the standard basis E.

2. $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for V, the xy-plane in $\mathbb{R}^3$. Then given any $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$,

$$\begin{pmatrix} x \\ y \end{pmatrix} \leftrightarrow x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

Hence, for any $\mathbf{v} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \in V$,

$$(\mathbf{v})_S = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Remark. 1. Even though $\mathbf{v} \in V \subseteq \mathbb{R}^n$ has n coordinates, $(\mathbf{v})_S$ has k coordinates if the basis S has k vectors.

2. Note that the correspondence is unique only if S is a basis. If S is not linearly independent, a few vectors in $\mathbb{R}^k$ can map to the same $\mathbf{v} \in V$.

3. The relative coordinates depend to the ordering of the basis, if $S = \left\{ \mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ and $T = \left\{ \mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$, then for $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$,

$$(\mathbf{v})_S = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 1 \end{pmatrix} = (\mathbf{v})_T.$$

4. The relative coordinates are different for different bases. If $S = \left\{ \mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ and $T = \left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$, then for $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$,

$$(\mathbf{v})_S = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (\mathbf{v})_T.$$

We will now give an algorithm to find coordinates relative to a basis. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a basis for a subspace $V \subseteq \mathbb{R}^n$. For any $\mathbf{v} \in V$, we are solving for $c_1, c_2, ..., c_k \in \mathbb{R}$ such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{v}.$$

That is, we are solving for

$$\left(\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v} \end{array} \right).$$

Theorem (Algorithm to finding coordinates relative to a basis)**.** *Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a basis for a subspace $V \subseteq \mathbb{R}^n$. For any $\mathbf{v} \in V$, $(\mathbf{v})_S$ is the unique solution to $\mathbf{Ax} = \mathbf{v}$, where $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{pmatrix}$.*

Example. 1. We have seen that $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$ is a basis for the xy-plane V.

For any $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \in V$,

$$\left(\begin{array}{cc|c} 1 & 1 & x \\ 1 & -1 & y \\ 0 & 0 & 0 \end{array} \right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & (x+y)/2 \\ 0 & 1 & (x-y)/2 \\ 0 & 0 & 0 \end{array} \right).$$

So $(\mathbf{v})_S = \begin{pmatrix} (x+y)/2 \\ (x-y)/2 \end{pmatrix}$.

2. $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \middle| \; x_1 - 2x_2 + x_3 = 0, x_2 + x_3 - 2x_4 = 0 \right\}$. We will leave it to the readers to check that $S = \left\{ \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for V. For $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \in V$,

$$\left(\begin{array}{cc|c} -3 & 4 & 1 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{array} \right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right).$$

So $(\mathbf{v})_S = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Exercise: Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^n$ is a subspace. Let $\mathbf{v} \in V$.

(i) Suppose there is a non-pivot column in the left side of the reduced row-echelon form of

$$\left(\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v} \end{array} \right).$$

What can you conclude?

(ii) Suppose

$$\left(\begin{array}{cccc|c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{v} \end{array} \right)$$

is inconsistent. What can you conclude?

Appendix to Lecture 7

Theorem. *Suppose* $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$ *is linearly dependent. Then for any* $\mathbf{u} \in \mathbb{R}^n$*, the set*

$$\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k, \mathbf{u}\}$$

is linearly dependent.

That is, adding vectors to a linearly dependent set will not make it linearly independent.

Proof. Since S is linearly dependent, we can find $c_1, c_2, ..., c_k \in \mathbb{R}$ not all zero such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}.$$

To show that $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k, \mathbf{u}\}$ is linearly dependent, we let $c = 0$ and add $c\mathbf{u}$ to the above equation and obtain

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k + c\mathbf{u} = \mathbf{0}.$$

This is a nontrivial linear combination of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k, \mathbf{u}$ to give us $\mathbf{0}$, since one of the c_i, $i = 1, ..., k$ is not zero.. Hence, $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k, \mathbf{u}\}$ is also linearly dependent.

Alternatively, since the RREF of $\mathbf{A} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k\end{pmatrix}$ has non-pivot columns, the RREF of $\begin{pmatrix}\mathbf{A} & \mathbf{u}\end{pmatrix} = \begin{pmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k & \mathbf{u}\end{pmatrix}$ must also have non-pivot columns (not obvious, details are left to readers). $\square$

Theorem. *Suppose* $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq \mathbb{R}^n$ *is linearly independent and* $\mathbf{u}$ *is not a linearly combination of* $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$*,* $\mathbf{u} \notin span\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$*. Then the set* $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k, \mathbf{u}\}$ *is linearly independent.*

Proof. Suppose to the contrary that $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k, \mathbf{u}\}$ is linearly dependent, that is we can find $c_1, c_2, ..., c_k, c \in \mathbb{R}$ not all zero such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k + c\mathbf{u} = \mathbf{0}.$$

Case 1: $c \neq 0$. Then

$$\mathbf{u} = -\frac{1}{c}(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k)$$

which contradict the hypothesis that $\mathbf{u}$ is not a linear combination of $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$. So this case cannot be true. Case 2: $c = 0$, and thus one of the $c_i \neq 0$. Then we have

$$\mathbf{0} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k + c\mathbf{u} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k.$$

But this would mean that $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is linearly dependent, a contradiction. So this cannot happen too.

Hence, $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k, \mathbf{u}\}$ must be linearly independent. $\square$

We are now ready to prove that every subspace $V \subseteq \mathbb{R}^n$ can be written a as span of some vectors, $V = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$. Since V is a subspace, it is not empty. If $V = \{\mathbf{0}\}$, then $V = \text{span}\{\mathbf{0}\}$ and we are done. Suppose not. Then pick a nonzero vector $\mathbf{u}_1 \in V$. Suppose $\text{span}\{\mathbf{u}_1\} = V$, we are done. Otherwise, we can pick a vector in V

but not in span$\{\mathbf{u}_1\}$, $\mathbf{u}_2 \in V \backslash \text{span}\{\mathbf{u}_1\}$. Then since $\mathbf{u}_2$ is not a linear combination of $\mathbf{u}_1$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a linearly independent set and span$\{\mathbf{u}_1\} \subsetneqq$ span$\{\mathbf{u}_1, \mathbf{u}_2\}$, span$\{\mathbf{u}_1, \mathbf{u}_2\}$ is a strictly bigger set.

Proceeding inductively, suppose now we have constructed a linearly independent set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} \subseteq V$. If $V = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$, we are done. Otherwise, by the same argument as above, we can find a $\mathbf{u}_{k+1} \in V \backslash \text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$. Then $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k, \mathbf{u}_{k+1}\}$ is a linearly independent subset of V and its span is strictly bigger than span$\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$.

But the process have to stop, since we can at most have n linearly independent vectors in $\mathbb{R}^n$. Hence, there must be a $m \leq n$ such that the constructed set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_m\}$ spans V. Hence, $V = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_m\}$ is a linear span of some vectors. In fact, by the construction, $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_m\}$ will be a basis for V.