NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

AY2021, Semester 2 MA1508E Linear Algebra for Engineering Practice 4 Solutions

1. Let

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix}.$$

(a) [10 marks] Orthogonally diagonalize $\mathbf{A}$.

$$\begin{vmatrix} x-3 & -1 & -1 & -1 \\ -1 & x-3 & -1 & -1 \\ -1 & -1 & x-3 & -1 \\ -1 & -1 & -1 & x-3 \end{vmatrix} = (x-2)^3(x-6).$$ Eigenvalues are $\lambda = 2, 6$.

$\lambda = 6$: $\begin{pmatrix} 6-3 & -1 & -1 & -1 \\ -1 & 6-3 & -1 & -1 \\ -1 & -1 & 6-3 & -1 \\ -1 & -1 & -1 & 6-3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Basis of E_6: $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

$\lambda = 2$: $\begin{pmatrix} 2-3 & -1 & -1 & -1 \\ -1 & 2-3 & -1 & -1 \\ -1 & -1 & 2-3 & -1 \\ -1 & -1 & -1 & 2-3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Basis of E_2: $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\}$.

Basis for E_2 is not orthogonal. Apply Gram-Schmidt process.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \text{ let } \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}$$

$$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix}, \text{ let } \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix}$$

Orthonormal basis for E_2: $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix} \right\}$

Orthonormal basis for E_6: $\left\{ \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

Let $\mathbf{P} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/2\sqrt{3} & 1/2 \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/2\sqrt{3} & 1/2 \\ 0 & -2/\sqrt{6} & 1/2\sqrt{3} & 1/2 \\ 0 & 0 & -3/2\sqrt{3} & 1/2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}$,

$$\mathbf{A} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/2\sqrt{3} & 1/2 \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/2\sqrt{3} & 1/2 \\ 0 & -2/\sqrt{6} & 1/2\sqrt{3} & 1/2 \\ 0 & 0 & -3/2\sqrt{3} & 1/2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} & 0 \\ 1/2\sqrt{3} & 1/2\sqrt{3} & 1/2\sqrt{3} & -3/2\sqrt{3} \\ 1/2 & 1/2 & 1/2 & 1/2 \end{pmatrix}.$$

(b) [2 marks] Find the limit of $\left(\frac{1}{6}\mathbf{A}\right)^n \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ as $n \to \infty$, for $a, b, c, d \in \mathbb{R}$.

$$\left(\frac{1}{6}\mathbf{A}\right)^n = \mathbf{P} \begin{pmatrix} (1/3)^n & 0 & 0 & 0 \\ 0 & (1/3)^n & 0 & 0 \\ 0 & 0 & (1/3)^n & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{P}^T$$

$$\to \mathbf{P} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{P}^T = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

So $\left(\frac{1}{6}\mathbf{A}\right)^n \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \to \frac{a+b+c+d}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

2. [5 marks] Solve the initial value problem

$$\begin{aligned} y_1' &= y_1 + y_2 - y_3 \\ y_2' &= y_3 \\ y_3' &= -2y_2 - 3y_3 \end{aligned}$$

with initial conditions $y_1(0) = 1$, $y_2(0) = 2$, $y_3(0) = 3$.

Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{pmatrix}$. $\det(x\mathbf{I} - \mathbf{A}) = (x-1)(x+1)(x+2) \Rightarrow$ eigenvalues are $\lambda = 1, -1, -2$. $\mathbf{A}$ has 3 distinct eigenvalues, and hence, it is diagonalizable. The eigenvectors are

$$\lambda = 1\text{: } \mathbf{I} - \mathbf{A} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$\lambda = -1\text{: } -\mathbf{I} - \mathbf{A} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathbf{v}_{-1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

$$\lambda = -2\text{: } -2\mathbf{I} - \mathbf{A} \to \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathbf{v}_{-2} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

General solution:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{-2t} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Initial conditions: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \Rightarrow c_1 = 3$, $c_2 = -7$, $c_3 = 5$. So the solution to the initial value problem is

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 3e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 7e^{-t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 5e^{-2t} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

3. Let

$$\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}.$$

(a) [1 mark] Verify that $\begin{pmatrix} i \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$.

$$\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} i-3 \\ 3i+1 \end{pmatrix} = (1+3i) \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

(b) [2 marks] Find a fundamental set of real solutions for the differential system

$$\begin{aligned} y_1' &= y_1 - 3y_2 \\ y_2' &= 3y_1 + y_2 \end{aligned}$$

From above, we know that $\lambda = 1 + 3i$ is an eigenvalue with eigenvector $\begin{pmatrix} i \\ 1 \end{pmatrix}$. Let

$$\lambda_r = 1,\ \lambda_i = 3,\ \mathbf{v}_r = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \mathbf{v}_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and

$$\begin{aligned} \mathbf{x}_r &= e^t \left(\cos(3t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sin(3t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = e^t \begin{pmatrix} -\sin(3t) \\ \cos(3t) \end{pmatrix}, \\ \mathbf{x}_i &= e^t \left(\cos(3t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin(3t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = e^t \begin{pmatrix} \cos(3t) \\ \sin(3t) \end{pmatrix}. \end{aligned}$$

So, a fundamental set of real solutions is

$$\left\{ e^t \begin{pmatrix} -\sin(3t) \\ \cos(3t) \end{pmatrix}, e^t \begin{pmatrix} \cos(3t) \\ \sin(3t) \end{pmatrix} \right\}.$$