NATIONAL UNIVERSITY OF SINGAPORE

MA1508E - LINEAR ALGEBRA FOR ENGINEERING

(Semester 2 : AY2020/2021)

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. On every page of your answer script, please write your matriculation/student number only. Do not write your name.
2. Please write the page number on the top right corner of each page of your answer script.
3. This examination paper contains **6** questions and comprises **7** printed pages.
4. The total mark for the paper is **80 marks**.
5. Answer **ALL** questions.
6. Please start each question on a new page.
7. This is an OPEN BOOK examination.
8. This exam is proctored by Zoom.
9. Candidates may use MATLAB or any scientific or graphical calculator. However, they should lay out systematically the various steps in the calculations.
10. At the end of the exam, give yourself sufficient time to:
    (a) scan or take pictures of your work (it is your responsibility to make sure the images can be read clearly);
    (b) merge all your images into one pdf file (arrange them in the order: Q1, Q2, ... in their page sequence);
    (c) name the pdf file by Matric No_Module Code (e.g. **A123456R_MA1508E**);
    (d) upload your pdf into the LumiNUS folder "Exam Submission".
    (e) you have 15 minutes after the end time of the exam to do the above.

Question 1 [12 marks]

Consider the following linear system

$$\begin{cases} x_1 & & + & 3x_3 & + & x_4 & = & 2 \\ 3x_1 & + \; ax_2 & + & 9x_3 & & & = & 6 \\ 2x_1 & & + & (a+6)x_3 & + & ax_4 & = & b+2 \\ 2x_1 & & + & 6x_3 & + & bx_4 & = & b+2 \end{cases}$$

where a and b are some constants.

(i) [3 marks] Find the conditions on a and b such that the system has no solution.

(ii) [3 marks] Find the conditions on a and b such that the system has a unique solution, and write down the unique solution.

(iii) [3 marks] Find the conditions on a and b such that the system has infinitely many solutions with 1 parameter, and write down a general solution.

(iv) [3 marks] Find the conditions on a and b such that the system has infinitely many solutions with 2 parameters, and write down a general solution.

You should show workings clearly. You do not need to state the elementary row operations used.

Question 2 [15 marks]

Let $(x_1, y_1) = (-3, -722)$, $(x_2, y_2) = (-2, -103)$, $(x_3, y_3) = (-1, -2)$, $(x_4, y_4) = (1, -10)$, $(x_5, y_5) = (2, 13)$, $(x_6, y_6) = (3, 262)$, and $(x_7, y_7) = (4, 1343)$.

(a) [5 marks] Find a polynomial of degree 5,

$$p(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

such that $p(x_i) = y_i$ for all $i = 1, ..., 7$. Show your workings clearly.

(b) (i) [3 marks] Is it possible to find a degee 4 polynomial

$$q(x) = b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0$$

such that $q(x_i) = y_i$ for all $i = 1, ..., 7$? Why?

(ii) [7 marks] Find a degree 4 polynomial $q_0(x)$ such that

$$\sum_{i=1}^{7}(q_0(x_i) - y_i)^2 = (q_0(x_1) - y_1)^2 + (q_0(x_2) - y_2)^2 + \cdots + (q_0(x_7) - y_7)^2$$

is minimized, that is,

$$\sum_{i=1}^{7}(q_0(x_i) - y_i)^2 \leq \sum_{i=1}^{7}(q(x_i) - y_i)^2$$

for all degree 4 polynomial $q(x)$. Write the coefficients of $q_0(x)$ as rational numbers. Show your workings clearly.

Question 3 [15 marks]

Let

$$V_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \middle| \; x_1 + 2x_2 - x_3 + x_4 - x_5 = 0 \right\}, \quad V_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \middle| \; x_1 - 2x_2 + x_3 + 2x_4 - x_5 = 0 \right\},$$

and let $V = V_1 \cap V_2$.

(a) [3 marks] Show that V is a subspace by finding a spanning set for V.

(b) [3 marks] Show that $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \\ -3 \end{pmatrix} \right\}$ is a basis for V. Show your workings clearly

(c) [3 marks] Which of the following vectors $\mathbf{u}_1 = \begin{pmatrix} 3 \\ -2 \\ -5 \\ 2 \\ 6 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 3 \\ 2 \\ 4 \\ 1 \\ 10 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 4 \\ 10 \end{pmatrix}$ belong to V?
For the vectors that belong to V, find their coordinates relative to the basis S in (b).

(d) [3 marks] Use Gram-Schmidt process to convert S in (b) to an orthonormal basis for V. Show your workings clearly.

(e) [3 marks] Find a linearly independent set $T = \{\mathbf{w}_1, \mathbf{w}_2\}$ such that $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal to V. You need to show that T is orthogonal to V.

Question 4 [10 marks]

Let $\mathbf{A} = \begin{pmatrix} 6 & 2 & 1 & -1 \\ -3 & 4 & 5 & 1 \\ 6 & 5 & 4 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

(a) [3 marks] For matrices $\mathbf{A}$ and $\mathbf{B}$, find a left inverse, if it exists.

(b) [3 marks] For matrices $\mathbf{A}$ and $\mathbf{B}$, find a right inverse, if it exists.

(c) [2 marks] Compute $\det(\mathbf{AB})$.

(d) [2 marks] Find a basis for the intersection of the column space of $\mathbf{B}$ and the nullspace of $\mathbf{A}$, $\mathrm{Col}(\mathbf{B}) \cap \mathrm{Null}(\mathbf{A})$.

Question 5 [13 marks]

Let $\mathbf{A} = \begin{pmatrix} 5 & 2 & -4 \\ 2 & 5 & -4 \\ 4 & 4 & -5 \end{pmatrix}$.

(a) [9 marks] Find an invertible matrix $\mathbf{P}$ **with integer entries** (the entries of $\mathbf{P} = (p_{ij})$ are integers, $p_{ij} \in \mathbb{Z}$ for all $i, j = 1, ..., 3$), and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$. You should show all your workings clearly.

(b) [4 marks] Solve the following system of linear differential equations

$$\begin{cases} y_1'(t) &=& 5y_1(t) &+& 2y_2(t) &-& 4y_3(t) \\ y_2'(t) &=& 2y_1(t) &+& 5y_2(t) &-& 4y_3(t) \\ y_3'(t) &=& 4y_1(t) &+& 4y_2(t) &-& 5y_3(t) \end{cases}$$

with initial condition $y_1(0) = 2$, $y_2(0) = 2$, $y_3(0) = 1$. Show your workings clearly.

Question 6 [15 marks]

Consider the following system of linear differential equations

$$\begin{cases} y_1'(t) & = & 6y_1(t) & - & 4y_2(t) \\ y_2'(t) & = & 4y_1(t) & - & 2y_2(t) \end{cases}$$

with initial condition $y_1(1) = e$, $y_2(1) = e$ (here e is the natural number, $\ln(e) = 1$).

(a) [7 marks] Find a fundamental set of solutions for the system.

(b) [3 marks] Use Wronskian to verify that your answer in (a) is linearly independent.

(c) [2 marks] Write down a general solution using the answer in (a).

(d) [3 marks] Find the solution to the initial value problem.

End of Paper