NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

AY2021, Semester 1 MA1508E Linear Algebra for Engineering Tutorial 8

Solutions

1. For each of the following matrices $\mathbf{A}$,

   (i) Find a basis for the row space of $\mathbf{A}$.

   (ii) Find a basis for the column space of $\mathbf{A}$.

   (iii) Find a basis for the nullspace of $\mathbf{A}$.

   (iv) Hence determine $\text{rank}(\mathbf{A})$, $\text{nullity}(\mathbf{A})$ and verify the dimension theorem for matrices.

   (v) Is $\mathbf{A}$ full rank?

   (a) $\mathbf{A} = \begin{pmatrix} 1 & 2 & 5 & 3 \\ 1 & -4 & -1 & -9 \\ -1 & 0 & -3 & 1 \\ 2 & 1 & 7 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}$

   $$\mathbf{A} \longrightarrow \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

   (i) A basis for the row space is $\{(1, 0, 3, -1), (0, 1, 1, 2)\}$.

   (ii) A basis for the column space is $\{(1, 1, -1, 2, 0)^T, (2, -4, 0, 1, 1,)^T\}$.

   (iii) A basis for the nullspace is $\{(-3, -1, 1, 0)^T, (1, -2, 0, 1)^T\}$.

   (iv) $\text{rank}(\mathbf{A}) = 2$, $\text{nullity}(\mathbf{A}) = 2$. Since $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = 2 + 2 = 4$ which is the number of columns of $\mathbf{A}$, we have verified the dimension theorem for matrices.

   (v) $\text{rank}(\mathbf{A}) = 2 < \min\{4, 5\}$. $\mathbf{A}$ is not full rank.

   (b) $\mathbf{A} = \begin{pmatrix} 1 & 3 & 7 \\ 2 & 1 & 8 \\ 3 & -5 & -1 \\ 2 & -2 & 2 \\ 1 & 1 & 5 \end{pmatrix}$.

   $$\mathbf{A} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   (i) A basis for the row space is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

   (ii) A basis for the column space is $\{(1, 2, 3, 2, 1)^T, (3, 1, -5, -2, 1)^T, (7, 8, -1, 2, 5)^T\}$.

(iii) The basis for the nullspace is the empty set.

(iv) $\text{rank}(\mathbf{A}) = 3$, $\text{nullity}(\mathbf{A}) = 0$. Since $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = 3 + 0 = 3$ which is the number of columns of $\mathbf{A}$, we have verified the dimension theorem for matrices.

(v) $\text{rank}(\mathbf{A}) = 3 = \min\{3, 5\}$. $\mathbf{A}$ is full rank.

2. Show that for any linear system $\mathbf{Ax} = \mathbf{b}$, the solution set is

$$\{\ \mathbf{x}_p + \mathbf{u} \mid \mathbf{u} \in \text{Null}(\mathbf{A})\ \},$$

where $\mathbf{x}_p$ is a particular solution to the linear system, and $\text{Null}(\mathbf{A})$ is the nullspace of $\mathbf{A}$ (See tutorial 5 question 6).

See tutorial 5 question 6.

3. Suppose $\mathbf{A}$ and $\mathbf{B}$ are two matrices such that $\mathbf{AB} = \mathbf{0}$. Show that the column space of $\mathbf{B}$ is contained in the nullspace of $\mathbf{A}$.

Write $\mathbf{B} = \begin{pmatrix}\mathbf{b}_1 & \mathbf{b}_2 \cdots \mathbf{b}_n\end{pmatrix}$, where $\mathbf{b}_i$ is the i-th column of $\mathbf{B}$. Then

$$\mathbf{0} = \mathbf{AB} = \mathbf{A}\begin{pmatrix}\mathbf{b}_1 & \mathbf{b}_2 \cdots \mathbf{b}_n\end{pmatrix} = \begin{pmatrix}\mathbf{Ab}_1 & \mathbf{Ab}_2 \cdots \mathbf{Ab}_n\end{pmatrix}$$

By comparing the columns, we conclude that $\mathbf{Ab}_i = \mathbf{0}$ for all $i = 1, ..., n$. Hence, $\mathbf{b}_i \in \text{Null}(A)$ for all $i = 1, ..., n$.

4. **(MATLAB)** Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $\mathbf{V} = \begin{pmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{pmatrix}$.

(a) Compute $\mathbf{v}_1 \cdot \mathbf{v}_1$, $\mathbf{v}_1 \cdot \mathbf{v}_2$, $\mathbf{v}_2 \cdot \mathbf{v}_1$, and $\mathbf{v}_2 \cdot \mathbf{v}_2$.

`>> v1=[1 2 -1]';`

`>> v2=[1 0 1]';`

`>> dot(v1,v1)`, or `>> v1'*v1`, ans=6.

Note that $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot \mathbf{v}_1$.

`>> dot(v1,v2)`, or `>> v1'*v2`, ans=0.

`>> dot(v2,v2)`, or `>> v2'*v2`, ans=2.

(b) Compute $\mathbf{V}^T\mathbf{V}$. What does the entries of $\mathbf{V}^T\mathbf{V}$ represent?

`>> V=[v1 v2];`

`>> V'* V`, ans=$\begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$.

Since $\mathbf{V} = (\mathbf{v}_1\ \mathbf{v}_2)$, $\mathbf{V}^T = \begin{pmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{pmatrix}$. So

$$\mathbf{V}^T\mathbf{V} = \begin{pmatrix} \mathbf{v}_1^T\mathbf{v}_1 & \mathbf{v}_1^T\mathbf{v}_2 \\ \mathbf{v}_2^T\mathbf{v}_1 & \mathbf{v}_2^T\mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 \\ \mathbf{v}_2 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 \end{pmatrix},$$

that is, the (i, j)-entry of $\mathbf{V}^T\mathbf{V}$ is $\mathbf{v}_i \cdot \mathbf{v}_j$.

5. Let W be a subspace of $\mathbb{R}^n$. The *orthogonal complement* of W, denoted as $W^\perp$, is defined to be

$$W^\perp := \{\ \mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\ \}.$$

Let $\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{w}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \\ 0 \end{pmatrix}$, and $\mathbf{w}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$, and $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

(a) Show that $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly independent.

Since any orthogonal set of nonzero vectors is linearly independent, suffice to show that S is orthogonal. Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & -2 & -1 \\ 1 & 0 & 0 \end{pmatrix}$. Then

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

shows that S is orthogonal.

(b) Show that S is orthogonal.

Shown in (a).

(c) Show that $W^\perp$ is a subspace of $\mathbb{R}^5$ by showing that it is a span of a set. What is the dimension? (**Hint**: See Tutorial 4 question 6.)

By Tutorial 4 question 6, $W^\perp$ is the nullspace of $\mathbf{A}^T$.

$$\mathbf{A}^T \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -2 & -1/4 \\ 0 & 1 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 2 & 3/4 \end{pmatrix}$$

So the nullspace of $\mathbf{A}^T$ is spanned by $\left\{ \begin{pmatrix} 2 \\ -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -3 \\ 0 \\ 4 \end{pmatrix} \right\}$. This shows that $W^\perp$ is a subspace of $\mathbb{R}^5$. In fact, the fact that $W^\perp$ is a nullspace of some matrix proves that it is a subspace. It has 2 dimensions.

(d) Obtain an orthonormal set T by normalizing $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$.

From (b) we know that $||\mathbf{w}_1||^2 = 5$, $||\mathbf{w}_2||^2 = 10$, and $||\mathbf{w}_3||^2 = 4$. So $T = \left\{ \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \\ 0 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}$.

(e) Let $\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}$. Find the projection of $\mathbf{v}$ onto W.

The projection is

$$\frac{\mathbf{v}\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1}\mathbf{w}_1 + \frac{\mathbf{v}\cdot\mathbf{w}_2}{\mathbf{w}_2\cdot\mathbf{w}_2}\mathbf{w}_2 + \frac{\mathbf{v}\cdot\mathbf{w}_3}{\mathbf{w}_3\cdot\mathbf{w}_3}\mathbf{w}_3 = \frac{1}{10}\begin{pmatrix} 10 \\ -1 \\ 12 \\ 3 \\ 6 \end{pmatrix}.$$

(f) Let $\mathbf{v}_W$ be the projection of $\mathbf{v}$ onto W. Show that $\mathbf{v} - \mathbf{v}_W$ is in $W^\perp$.

$$\mathbf{v} - \mathbf{v}_W = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{10}\begin{pmatrix} 10 \\ -1 \\ 12 \\ 3 \\ 6 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 10 \\ 1 \\ -2 \\ 7 \\ -16 \end{pmatrix}$$

$$\mathbf{A}^T(\mathbf{v} - \mathbf{v}_W) = \frac{1}{10}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -2 & 0 \\ 1 & -1 & 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 10 \\ 1 \\ -2 \\ 7 \\ -16 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This shows that $(\mathbf{v} - \mathbf{v}_W)$ is in the nullspace of $\mathbf{A}^T$, which is $W^\perp$.

This exercise demonstrated the fact that every vector $\mathbf{v}$ in $\mathbb{R}^5$ can be written as $\mathbf{v} = \mathbf{v}_W + \mathbf{v}_W^\perp$, for some $\mathbf{v}_W$ in W and $\mathbf{v}_W^\perp$ in $W^\perp$. In other words, $W + W^\perp = \mathbb{R}^5$ (see Tutorial 7 question 1).

6. Let $S = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}\}$ where

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}, \mathbf{u_2} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{u_3} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \text{ and } \mathbf{u_4} = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 2 \end{pmatrix}.$$

(a) Check that S is an orthogonal set.

Let $\mathbf{U} = \begin{pmatrix} 1 & 1 & -1 & -2 \\ 2 & 1 & 1 & 1 \\ 2 & -1 & -1 & 1 \\ -1 & 1 & -1 & 2 \end{pmatrix}$. Then

$$\mathbf{U}^T\mathbf{U} = \begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

Hence, S is an orthogonal set. Since it is an orthogonal set of nonzero vectors, it is linearly independent, and since it contains 4 vectors, it must be a basis. Alternatively, since the product $\mathbf{U}^T\mathbf{U}$ is invertible, $\mathbf{U}$ is invertible. So the columns form a basis for $\mathbb{R}^4$.

(b) Is S a basis for $\mathbb{R}^4$?

Shown in (a).

(c) Is it possible to find a nonzero vector $\mathbf{w}$ in $\mathbb{R}^4$ such that $S \cup \{\mathbf{w}\}$ is an orthogonal set?

No, since if $\mathbf{w}$ exists, then $S \cup \{\mathbf{w}\}$ will be a linearly independent set in $\mathbb{R}^n$ containing 5 vectors, a contradiction. Alternatively, from Tutorial 4 question 6, $\mathbf{w}$ must be in the nullspace of U. But since U is invertible, it has the trivial nullspace, and so there can be no nonzero vector that is orthogonal to the set S.

(d) Obtain an orthonormal set T by normalizaing $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$.

From (a) we know that $||\mathbf{u}_1||^2 = 10$, $||\mathbf{u}_2||^2 = 4$, $||\mathbf{u}_3||^2 = 4$, and $||\mathbf{u}_4||^2 = 10$. So

$$T = \left\{ \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{10}} \begin{pmatrix} -2 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\}.$$

(e) Let $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$. Find $(\mathbf{v})_S$ and $(\mathbf{v})_T$.

We have

$$\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{v} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{v} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3 + \frac{\mathbf{v} \cdot \mathbf{u}_4}{\mathbf{u}_4 \cdot \mathbf{u}_4}\mathbf{u}_4,$$

which means that

$$(\mathbf{v})_S = \begin{pmatrix} \frac{\mathbf{v} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \\ \frac{\mathbf{v} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \\ \frac{\mathbf{v} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3} \\ \frac{\mathbf{v} \cdot \mathbf{u}_4}{\mathbf{u}_4 \cdot \mathbf{u}_4} \end{pmatrix} = \begin{pmatrix} 3/10 \\ 1/2 \\ -1 \\ 9/10 \end{pmatrix}.$$

Let

$$\mathbf{u}_1' = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}, \mathbf{u}_2' = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{u}_3' = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \mathbf{u}_4' = \frac{1}{\sqrt{10}} \begin{pmatrix} -2 \\ 1 \\ 1 \\ 2 \end{pmatrix}.$$

Then

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{u}_1')\mathbf{u}_1' + (\mathbf{v} \cdot \mathbf{u}_2')\mathbf{u}_2' + (\mathbf{v} \cdot \mathbf{u}_3')\mathbf{u}_3' + (\mathbf{v} \cdot \mathbf{u}_4')\mathbf{u}_4',$$

which means that

$$(\mathbf{v})_T = \begin{pmatrix} \mathbf{v} \cdot \mathbf{u}_1' \\ \mathbf{v} \cdot \mathbf{u}_2' \\ \mathbf{v} \cdot \mathbf{u}_3' \\ \mathbf{v} \cdot \mathbf{u}_4' \end{pmatrix} = \begin{pmatrix} 3/\sqrt{10} \\ 1 \\ -2 \\ 9/\sqrt{10} \end{pmatrix}.$$

(f) Suppose $\mathbf{w}$ is a vector in $\mathbb{R}^4$ such that $(\mathbf{w})_S = \begin{pmatrix}1\\2\\1\\1\end{pmatrix}$. Find $(\mathbf{w})_T$.

Note that $\mathbf{u}_i' = \frac{\mathbf{u}_i}{||\mathbf{u}_i||}$, and so

$$\begin{aligned}
\mathbf{w} &= \frac{\mathbf{w}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{w}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{w}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}\mathbf{u}_3 + \frac{\mathbf{w}\cdot\mathbf{u}_4}{\mathbf{u}_4\cdot\mathbf{u}_4}\mathbf{u}_4 \\
&= \left(\frac{\mathbf{w}\cdot\mathbf{u}_1}{||\mathbf{u}_1||}\right)\frac{\mathbf{u}_1}{||\mathbf{u}_1||} + \left(\frac{\mathbf{w}\cdot\mathbf{u}_2}{||\mathbf{u}_2||}\right)\frac{\mathbf{u}_2}{||\mathbf{u}_2||} + \left(\frac{\mathbf{w}\cdot\mathbf{u}_3}{||\mathbf{u}_3||}\right)\frac{\mathbf{u}_3}{||\mathbf{u}_3||} + \left(\frac{\mathbf{w}\cdot\mathbf{u}_4}{||\mathbf{u}_4||}\right)\frac{\mathbf{u}_4}{||\mathbf{u}_4||} \\
&= \left(\frac{\mathbf{w}\cdot\mathbf{u}_1}{||\mathbf{u}_1||}\right)\mathbf{u}_1' + \left(\frac{\mathbf{w}\cdot\mathbf{u}_2}{||\mathbf{u}_2||}\right)\mathbf{u}_2' + \left(\frac{\mathbf{w}\cdot\mathbf{u}_3}{||\mathbf{u}_3||}\right)\mathbf{u}_3' + \left(\frac{\mathbf{w}\cdot\mathbf{u}_4}{||\mathbf{u}_4||}\right)\mathbf{u}_4'
\end{aligned}$$

Let $(\mathbf{w})_S(i)$ and $(\mathbf{w})_T(i)$ denote the i-th coordinate of $(\mathbf{w})_S$ and $(\mathbf{w})_T$, respectiwely. Then, we hawe

$$(\mathbf{w})_S(i) = \frac{\mathbf{w}\cdot\mathbf{u}_i}{\mathbf{u}_i\cdot\mathbf{u}_i} = \frac{\mathbf{w}\cdot\mathbf{u}_i}{||\mathbf{u}_i||^2} = \frac{1}{||\mathbf{u}_i||}\frac{\mathbf{w}\cdot\mathbf{u}_i}{||\mathbf{u}_i||} = \frac{1}{||\mathbf{u}_i||}(\mathbf{w})_T(i).$$

And so $(\mathbf{w})_T = \begin{pmatrix}\sqrt{10}\\4\\2\\\sqrt{10}\end{pmatrix}$.

Supplementary Problems

7. Recall that a matrix $\mathbf{A}$ is an orthogonal matrix if $\mathbf{A}^T = \mathbf{A}^{-1}$ (see Tutorial 4 question 1(d)).

(a) Show that if $\mathbf{A}$ is an orthogonal matrix of order n, then the columns of $\mathbf{A}$ is an orthonormal basis of $\mathbb{R}^n$.

Write $\mathbf{A} = \begin{pmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n\end{pmatrix}$, where $\mathbf{a}_i$ is the i-th column of $\mathbf{A}$. Then

$$\begin{aligned}
\begin{pmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix} &= \mathbf{A}^{-1}\mathbf{A} = \mathbf{A}^T\mathbf{A} = \begin{pmatrix}\mathbf{a}_1^T\mathbf{a}_1 & \mathbf{a}_1^T\mathbf{a}_2 & \cdots & \mathbf{a}_1^T\mathbf{a}_n\\ \mathbf{a}_2^T\mathbf{a}_1 & \mathbf{a}_2^T\mathbf{a}_2 & \cdots & \mathbf{a}_2^T\mathbf{a}_n\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{a}_n^T\mathbf{a}_1 & \mathbf{a}_n^T\mathbf{a}_2 & \cdots & \mathbf{a}_n^T\mathbf{a}_n\end{pmatrix} \\
&= \begin{pmatrix}\mathbf{a}_1\cdot\mathbf{a}_1 & \mathbf{a}_1\cdot\mathbf{a}_2 & \cdots & \mathbf{a}_1\cdot\mathbf{a}_n\\ \mathbf{a}_2\cdot\mathbf{a}_1 & \mathbf{a}_2\cdot\mathbf{a}_2 & \cdots & \mathbf{a}_2\cdot\mathbf{a}_n\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{a}_n\cdot\mathbf{a}_1 & \mathbf{a}_n\cdot\mathbf{a}_2 & \cdots & \mathbf{a}_n\cdot\mathbf{a}_n\end{pmatrix}.
\end{aligned}$$

This shows that

$$\begin{cases}\mathbf{a}_i\cdot\mathbf{a}_j = 1 & \text{when } i = j,\\ \mathbf{a}_i\cdot\mathbf{a}_j = 0 & \text{when } i \neq j,\end{cases}$$

that is, $\{\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n\}$ is an orthonormal set. Since an orthonormal set is linearly independent, and the set has n vectors in $\mathbb{R}^n$, it must be a basis of $\mathbb{R}^n$.

(b) Show that if $\mathbf{A}$ is an orthogonal matrix of order n, then the rows of $\mathbf{A}$ is an orthonormal basis of $\mathbb{R}^n$.

Write $\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{pmatrix}$, where $\mathbf{a}_i$ is the i-th row of $\mathbf{A}$. Then

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{A}\mathbf{A}^T = \begin{pmatrix} \mathbf{a}_1\mathbf{a}_1^T & \mathbf{a}_1\mathbf{a}_2^T & \cdots & \mathbf{a}_1\mathbf{a}_n^T \\ \mathbf{a}_2\mathbf{a}_1^T & \mathbf{a}_2\mathbf{a}_2^T & \cdots & \mathbf{a}_2\mathbf{a}_n^T \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_n\mathbf{a}_1^T & \mathbf{a}_n\mathbf{a}_2^T & \cdots & \mathbf{a}_n\mathbf{a}_n^T \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{a}_1 \cdot \mathbf{a}_1 & \mathbf{a}_1 \cdot \mathbf{a}_2 & \cdots & \mathbf{a}_1 \cdot \mathbf{a}_n \\ \mathbf{a}_2 \cdot \mathbf{a}_1 & \mathbf{a}_2 \cdot \mathbf{a}_2 & \cdots & \mathbf{a}_2 \cdot \mathbf{a}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_n \cdot \mathbf{a}_1 & \mathbf{a}_n \cdot \mathbf{a}_2 & \cdots & \mathbf{a}_n \cdot \mathbf{a}_n \end{pmatrix}.$$

This shows that

$$\begin{cases} \mathbf{a}_i \cdot \mathbf{a}_j = 1 & \text{when } i = j, \\ \mathbf{a}_i \cdot \mathbf{a}_j = 0 & \text{when } i \neq j, \end{cases}$$

that is, $\{\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n\}$ is an orthonormal set. Since an orthonormal set is linearly independent, and the set has n vectors in $\mathbb{R}^n$, it must be a basis of $\mathbb{R}^n$.

An analogous proof shows that if $\mathbf{A}^T\mathbf{A}$ ($\mathbf{A}$ not necessarily a square matrix) is a diagonal matrix, then the columns of $\mathbf{A}$ is an orthogonal set, and if $\mathbf{A}\mathbf{A}^T$ is a diagonal matrix, then the rows of $\mathbf{A}$ is an orthogonal set.