NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

AY2021, Semester 1 MA1508E Linear Algebra for Engineering Tutorial 3

1. Use the method of Gaussian elimination to determine if the following matrices are invertible. If the matrix is invertible, find its inverse.

   (a) $\begin{pmatrix} -1 & 3 \\ 3 & -2 \end{pmatrix}$.

   (c) $\begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$.

2. (a) Use the method of Gaussian elimination to write down the conditions so that the matrix $\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$ is invertible.

   (b) Notice that the above matrix is the transpose of the order 3 Vandermonde matrix. By (a), what are the conditions needed for the 3 points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ on the xy-plane to ensure that there is a unique polynomial of degree 2 whose graph passes through those points.

3. (a) Solve the matrix equation $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 2 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 1 & 0 & 3 & 7 \\ 2 & 1 & 1 & 2 \end{pmatrix}$.

   (b) Hence, solve for $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$. (Hint: look at the columns of the matrix on the right.)

4. (Cramer's Rule)

   (a) Compute the determinant of the following matrices.

      (i) $\mathbf{A} = \begin{pmatrix} 1 & 5 & 3 \\ 0 & 2 & -2 \\ 0 & 1 & 3 \end{pmatrix}$

      (ii) $\mathbf{A}_1 = \begin{pmatrix} 1 & 5 & 3 \\ 2 & 2 & -2 \\ 0 & 1 & 3 \end{pmatrix}$

      (iii) $\mathbf{A}_2 = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{pmatrix}$

      (iv) $\mathbf{A}_3 = \begin{pmatrix} 1 & 5 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 0 \end{pmatrix}$

(b) Solve the matrix equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

(c) Compute $\frac{1}{\det(\mathbf{A})}\begin{pmatrix} \det(\mathbf{A}_1) \\ \det(\mathbf{A}_2) \\ \det(\mathbf{A}_3) \end{pmatrix}$. How is this related to the answer in (b)?

Observe that the matrix $\mathbf{A}_k$ is obtained by replacing the k-th column of $\mathbf{A}$ by b. Cramer's rule state that if A is an invertible matrix of order n and $\mathbf{A}_k$ is the matrix obtained from $\mathbf{A}$ by replacing the k-th column of $\mathbf{A}$ by b, then the matrix equation $\mathbf{Ax} = \mathbf{b}$ has a unique solution

$$\mathbf{x} = \frac{1}{\det(\mathbf{A})}\begin{pmatrix} \det(\mathbf{A}_1) \\ \det(\mathbf{A}_2) \\ \vdots \\ \det(\mathbf{A}_n) \end{pmatrix}.$$

5. Let $\mathbf{A} = \begin{pmatrix} -x & 1 & 0 \\ 0 & -x & 1 \\ 2 & -5 & 4-x \end{pmatrix}$. Find all values of x such $\det(\mathbf{A}) = 0$. For each of the x found, solve the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$.

6. A square matrix $\mathbf{P} = (p_{ij})$ of order n is a *stochastic matrix*, or a *Markov matrix* if the sum of each column vector is equal to 1,

$$p_{1j} + p_{2j} + \cdots + p_{nj} = 1$$

for every $j = 1, ..., n$.

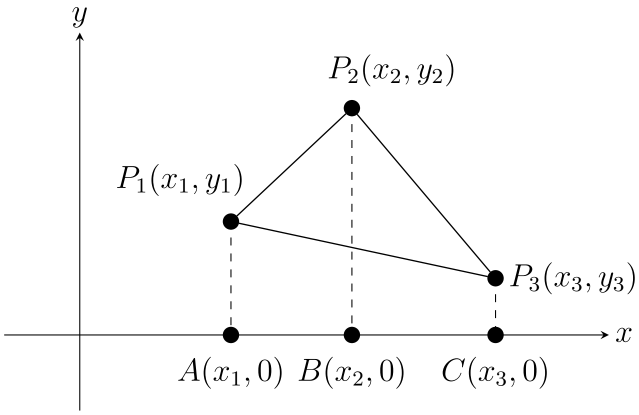
(a) Give an example of an invertible stochastic matrix, and a singular one.

(b) Show that if $\mathbf{P}$ is a stochastic matrix, then $\mathbf{I} - \mathbf{P}$ is singular.

(c) Check that $\mathbf{P} = \begin{pmatrix} 0.2 & 0.8 & 0.4 \\ 0.3 & 0.2 & 0.4 \\ 0.5 & 0 & 0.2 \end{pmatrix}$ is a stochastic matrix. Solve the homogeneous system $(\mathbf{I} - \mathbf{P})\mathbf{x} = \mathbf{0}$.

7. Show that $\begin{vmatrix} a+px & b+qx & c+rx \\ p+ux & q+vx & r+wx \\ u+ax & v+bx & w+cx \end{vmatrix} = (1+x^3)\begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix}$.

8. (**Application of determinants to computing areas.**) Consider the triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) as shown in the figure below.



We may compute the area of the triangle as

(area of trapezoid AP_1P_2B)+(area of trapezoid BP_2P_3C)−(area of trapezoid AP_1P_3C)

(a) Recall that the area of a trapezoid is $\frac{1}{2}$ the distance between the parallel sides of the trapezoid times the sum of the lengths of the parallel sides. Use this fact to show that the area of the triangle $P_1P_2P_3$ is

$$-\frac{1}{2}\left[(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)\right].$$

(b) Show that the expression in the square brackets obtained in part (a) is the determinant of the following matrix

$$\mathbf{A} = \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}.$$

(c) Explain why we need to take the absolute value of $\det(\mathbf{A})$ before concluding that the area of the triangle is

$$\frac{1}{2}|\det(\mathbf{A})|.$$

(d) Find the area of the following quadrilaterals with the given vertices.

(i) P with vertices $(2, 3)$, $(5, 3)$, $(4, 5)$, $(7, 5)$.

(ii) Q with vertices $(-2, 3)$, $(1, 4)$, $(3, 0)$, $(-1, -3)$.