MA1508 E

AY 19/20 sem 1

1. a)

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{pmatrix} \xrightarrow[R_3-R_1]{R_2-2R_1} \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 2 & 4 \end{pmatrix}$$

$$\xrightarrow{R_3+2R_2} \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{-R_2} \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1+R_2} \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$c = 2$
$b = s$
$a = 3-2s$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + s\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \quad s \in \mathbb{R}$$

b)

$$\left(\begin{array}{ccc|c} 1 & -1 & k & 1 \\ k & 0 & 1 & 1 \\ 2k & -2k & 1 & 2 \end{array}\right) \xrightarrow[R_3-2kR_1]{R_2-kR_1} \left(\begin{array}{ccc|c} 1 & -1 & k & 1 \\ 0 & k & 1-k^2 & 1-k \\ 0 & 0 & 1-2k^2 & 2-2k \end{array}\right)$$

i) If $k=0$:

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \text{ inconsistent}$$

If $k \neq 0$ and $1-2k^2 = 0$:
$k = \pm\frac{1}{\sqrt{2}}$

$$\left(\begin{array}{ccc|c} 1 & -1 & \pm\frac{1}{\sqrt{2}} & 1 \\ 0 & \pm\frac{1}{\sqrt{2}} & \frac{1}{2} & 1-\pm\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 2-\pm\frac{2}{\sqrt{2}} \end{array}\right) \text{ inconsistent}$$

ii) If $k \neq 0$, $k \neq \pm\frac{1}{\sqrt{2}}$:

$$\left(\begin{array}{ccc|c} 1 & -1 & k & 1 \\ 0 & k & 1-k^2 & 1-k \\ 0 & 0 & 1 & \frac{2(1-k)}{1-2k^2} \end{array}\right) \xrightarrow[R_2-(1-k^2)R_3]{R_1-kR_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1-\frac{2k(1-k)}{1-2k^2} \\ 0 & k & 0 & 1-k-\frac{2(1-k)(1-k^2)}{1-2k^2} \\ 0 & 0 & 1 & \frac{2(1-k)}{1-2k^2} \end{array}\right)$$

$\xrightarrow{\frac{1}{k}R_2} \xrightarrow{R_1+R_2}$ unique solution

iii) no value of k for infinitely many solutions?

c) i)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

ii) not possible

at least 1 non pivot column

iii) $x + y - 3z = 2$

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

iv) $x_2 - x_3 - x_5 = 0$

$x_4 + x_5 = 1$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

2. i)

$$\left.\begin{array}{l} 2 = a + b \\ 3 = 2a + b \\ 1 = -a + b \\ 1 = b \end{array}\right\} \quad \begin{array}{cc} A & b \\ \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 1 & 3 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right) & \end{array} \xrightarrow[R_3 + R_1]{R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow[R_1 + R_2]{R_3 + 2R_2,\ R_4 + R_2} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

inconsistent $\Rightarrow$ no solution for a, b

$\Rightarrow$ no straight line that passes through all 4 points ii)

$$A^TAu = A^Tb$$

$$\left(A^TA \,\middle|\, A^Tb\right) = \left(\begin{array}{cc|c} 6 & 2 & 7 \\ 2 & 4 & 7 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 0.7 \\ 0 & 1 & 1.4 \end{array}\right)$$

line of best fit is $y = 0.7x + 1.4$

iii) when $x = 3$, $y = 0.7 \times 3 + 1.4$
$= 3.5$

iv) projection of b onto span$\{u_1, u_2\}$ $= A(A^TA)^{-1}A^Tb$

linearly independent
A^TA is invertible

$$= \begin{pmatrix} 2.1 \\ 2.8 \\ 0.7 \\ 1.4 \end{pmatrix}$$

3 a)

$$A = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 6 \\ 1 & -1 & 0 \end{pmatrix} \xrightarrow[R_1 \leftrightarrow R_3]{E_1} \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 6 \\ 0 & -2 & 0 \end{pmatrix} \xrightarrow[R_2 - 2R_1]{E_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 6 \\ 0 & -2 & 0 \end{pmatrix}$$

$$\xrightarrow[R_3 + R_2]{E_3} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 6 \end{pmatrix} \xrightarrow[\frac{1}{6}R_3]{E_4} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_2 - 6R_3]{E_5} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[\frac{1}{2}R_2]{E_6} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_1 + R_2]{E_7} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$I = \overbrace{E_7 E_6 E_5 E_4 E_3 E_2 E_1}^{A^{-1}} A$$

$$A^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

b) i) origin $\notin V \Rightarrow$ not a subspace

ii) $a = b$, $c = d$

$$V = \left\{ s\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + t\begin{pmatrix}0\\0\\1\\1\end{pmatrix} \;\middle|\; s,t \in \mathbb{R} \right\}$$

$$= \text{span}\left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right\}$$

$\dim(V) = 2$

iii)

$$\begin{pmatrix}a+b+d\\a-b\\a-b\\c\end{pmatrix} = a\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + b\begin{pmatrix}1\\-1\\-1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\0\\1\end{pmatrix} + d\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

L.C. of

$$V = \text{span}\left\{\begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right\}$$

$\dim(V) = \cancel{4}$ 3

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\-1\\-1\\0\end{pmatrix}$$

c) i)

$$b = \begin{pmatrix}0\\0\\1\end{pmatrix} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1 \end{array}\right)$$

$Ax = b$ is inconsistent

$b \notin \text{col}(A)$

ii)

$$\text{Row}(A) = \text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$$

$\begin{pmatrix}1\\0\\1\end{pmatrix}$ is not in $\text{col}(A)$ but belongs to $\text{Row}(A)$

4. a)

$$0.5y'' + 2y' + 1.5y = 0$$

$y_1 = y$

$y_2 = y' = y_1'$

$y_2' = y'' = -3y - 4y'$

$= -3y_1 - 4y_2$

$y_1' = y_2$

$y_2' = -3y_1 - 4y_2$

$A = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}$

$\det(xI - A) = \begin{vmatrix} x & -1 \\ 3 & x+4 \end{vmatrix} = x^2 + 4x + 3$

$(x+3)(x+1)$

$\lambda = -1$: $\begin{pmatrix} -1 & -1 \\ 3 & 3 \end{pmatrix}$ $v_{-1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda = -3$: $\begin{pmatrix} -3 & -1 \\ 3 & 1 \end{pmatrix}$ $v_{-3} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

general solution $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

solution to the original 2nd order DE is

$y = -c_1 e^{-t} - c_2 e^{-3t}$

b) $\det(xI - A) = \begin{vmatrix} x & -1 \\ 1 & x \end{vmatrix} = x^2 + 1$

$\lambda = i$: $\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}$ $v = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

$\lambda_r = 0 \quad \lambda_i = 1$

$V_r = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad V_i = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$$x_r(t) = e^{0t}\left(\cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sin t \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$$x_i(t) = e^{0t}\left(\sin t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \cos t \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

$-4 = c_2$

$8 = c_1$

$y_1 = 8\sin t - 4\cos t$

$y_2 = 8\cos t - 4\sin t$

$y_1(0) = -4 \Rightarrow -c_2 \cos(0) = -4 \Rightarrow c_2 = 4$

$y_2(0) = 8 \Rightarrow c_1 \cos(0) = 8 \Rightarrow c_1 = 8$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 8\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + 4\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

5 i)

$$d_n = 0.2 d_{n-1} + 0.6 w_{n-1}$$

$$w_n = 0.8 d_{n-1} + 0.4 w_{n-1}$$

$$\begin{pmatrix} d_n \\ w_n \end{pmatrix} = \begin{pmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} d_{n-1} \\ w_{n-1} \end{pmatrix}$$

$$\begin{pmatrix} d_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

$$= \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}$$

ii) $$X_{n+1} = \begin{pmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{pmatrix} X_n$$

iii) $$\det(xI - A) = \begin{vmatrix} x-0.2 & -0.6 \\ -0.8 & x-0.4 \end{vmatrix}$$

$$= x^2 - 0.6x - 0.4$$

$$= \frac{1}{5}(5x+2)(x-1)$$

$\lambda = 1$, $\lambda = -\frac{2}{5}$

square matrix of order 2 with 2 distinct eigenvalues $\Rightarrow$ diagonalizable

$$\lambda = 1: \begin{pmatrix} 0.8 & -0.6 \\ -0.8 & 0.6 \end{pmatrix} \rightarrow \quad v_1 = \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}$$

$$\lambda = -0.4: \begin{pmatrix} -0.6 & -0.6 \\ -0.8 & -0.8 \end{pmatrix} \rightarrow \quad v_{-0.4} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0.6 & -1 \\ 0.8 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -0.4 \end{pmatrix} \begin{pmatrix} 0.6 & -1 \\ 0.8 & 1 \end{pmatrix}^{-1}$$

$$A^n \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.6 & -1 \\ 0.8 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0.6 & -1 \\ 0.8 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} \underline{0.4286} \\ \underline{0.5714} \end{pmatrix}$$

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$$\underline{d_n = 0.4286}$$

6, i)

projection of w onto V

$x = y - z + t$

$y = y$

$z = z$

$t = t$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = y\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$w' = A(A^T A)^{-1} A^T w$$

$$= \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 3 \end{pmatrix}$$ ✓

ii) $$\| w - w' \| = \sqrt{\left(1-\tfrac{1}{2}\right)^2 + \left(-1+\tfrac{1}{2}\right)^2 + \left(1-\tfrac{1}{2}\right)^2 + \left(1-\tfrac{3}{2}\right)^2}$$

$$= 1$$

Just nice ✓

7. i)

$$a_3 = 3a_1 + a_2$$

$$= 3\begin{pmatrix}1\\0\\-1\\1\end{pmatrix} + \begin{pmatrix}-2\\-1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\-2\\3\end{pmatrix}$$

$$a_5 = 4a_1 + 2a_2 + a_4$$

$$= 4\begin{pmatrix}1\\0\\-1\\1\end{pmatrix} + 2\begin{pmatrix}-2\\-1\\1\\0\end{pmatrix} + \begin{pmatrix}1\\1\\-3\\2\end{pmatrix} = \begin{pmatrix}1\\-1\\-5\\6\end{pmatrix}$$

$$A = \begin{pmatrix}1 & -2 & 1 & 1 & 1\\ 0 & -1 & -1 & 1 & -1\\ -1 & 1 & -2 & -3 & -5\\ 1 & 0 & 3 & 2 & 6\end{pmatrix}$$

ii) $\text{rank}(A) = 3$

$\text{nullity}(A) = 5-3 = 2$

$\text{rank}(A^T) =$ $\text{rank}(\underset{\sim}{A}) =$ 3

$\text{nullity}(A^T) = 4-3 = 1$

iii)

$x_5 = s$

$x_4 = -s$

$x_3 = t$

$x_2 = -2s - t$

$x_1 = -4s - 3t$

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} = s\begin{pmatrix}-4\\-2\\0\\-1\\1\end{pmatrix} + t\begin{pmatrix}-3\\-1\\1\\0\\0\end{pmatrix}, \quad s,t \in \mathbb{R}$$

$$\text{basis} = \left\{\begin{pmatrix}-4\\-2\\0\\-1\\1\end{pmatrix}, \begin{pmatrix}-3\\-1\\1\\0\\0\end{pmatrix}\right\}$$

iv) $$\left((0\ 1\ 1\ 1)A\right)^T = A^T\begin{pmatrix}0\\1\\1\\1\end{pmatrix} = 0$$

$$\text{basis} = \left\{\begin{pmatrix}0\\1\\1\\1\end{pmatrix}\right\}$$

v) No, 3 linearly independent vectors do not span $\mathbb{R}^4$