NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

AY2021, Semester 2 MA1508E Linear Algebra for Engineering Practice 1
Solutions

1. Consider the following augmented matrix

$$\left(\begin{array}{ccc|c} 1 & a & 2 & a \\ 1 & 1 & 1 & a \\ 1 & 1 & a+1 & 2a \end{array}\right).$$

(a) [2 marks] Write down the original linear system that correspond to the augmented matrix above. Use variables x_1, x_2, x_3.

$$\begin{array}{ccccccc} x_1 & + & ax_2 & + & 2x_3 & = & a \\ x_1 & + & x_2 & + & x_3 & = & a \\ x_1 & + & x_2 & + & (a+1)x_3 & = & 2a \end{array}$$

(b) [8 marks]

(i) Find the conditions on a such that the system has no solution.

(ii) Find the conditions on a such that the system has a unique solution, and write down the unique solution.

(iii) Find the conditions on a such that the system has infinitely many solutions, and write down a general solution and a particular solution.

Write down the elementary row operation that you used in each step clearly.

$$\left(\begin{array}{ccc|c} 1 & a & 2 & a \\ 1 & 1 & 1 & a \\ 1 & 1 & a+1 & 2a \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & a & 2 & a \\ 1 & 1 & a+1 & 2a \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & a-1 & 1 & 0 \\ 0 & 0 & a & a \end{array}\right).$$

(i) The system has no solutions when $a = 1$.

(ii) The system has a unique solution when $a \neq 1$ and $a \neq 0$. In this case, the augmented matrix reduces to

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & a-1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right),$$

and the unique solution is $x_1 = \frac{a^2-2a+2}{a-1}, x_2 = \frac{-1}{a-1}, x_3 = 1$.

(iii) The system has infinitely many solutions when $a = 0$. In this case, the augmented matrix reduces to

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

A general solution is $x_1 = -2s, x_2 = s, x_3 = s$, $s \in \mathbb{R}$. Let $s = 1$, a particular solution is $x_1 = -2, x_2 = 1, x_3 = 1$.

2. (a) Consider the following linear system

$$\begin{array}{ccccccc} x_1 & + & x_2 & & & = & 1 \\ x_1 & + & 2x_2 & + & x_3 & = & 0 \\ x_1 & - & x_2 & + & x_3 & = & 1 \\ x_1 & + & x_2 & & & = & -1 \\ x_1 & + & x_2 & + & x_3 & = & -1 \end{array}$$

(i) [1 mark] Write the corresponding matrix equation $\mathbf{Ax} = \mathbf{b}$.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

(ii) [3 marks] Compute $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}^T\mathbf{b}$ in part (i).

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 4 & 3 \\ 4 & 8 & 2 \\ 3 & 2 & 3 \end{pmatrix}$$

$$\mathbf{A}^T\mathbf{b} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}$$

(b) [1 mark] Write down a linear system that has the following general solution.

$$\begin{aligned} x_1 &= -s + 2t \\ x_2 &= s - t \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

$$\begin{array}{ccccccccc} x_1 & + & & & x_3 & - & 2x_4 & = & 0 \\ & & x_2 & - & x_3 & + & x_4 & = & 0 \end{array}$$

3. [5 marks] Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$. Find all the matrices $\mathbf{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\mathbf{AB} = \mathbf{BA}$.

$\mathbf{AB} = \begin{pmatrix} a+2c & b+2d \\ a-c & b-d \end{pmatrix} = \mathbf{BA} = \begin{pmatrix} a+b & 2a-b \\ c+d & 2c-d \end{pmatrix}$ if and only if

$$\begin{array}{rcrcrcrcl} & - & b & + & 2c & & & = & 0 \\ -2a & + & 2b & & & + & 2d & = & 0 \\ a & & & - & 2c & - & d & = & 0 \\ & & b & - & 2c & & & = & 0 \end{array}$$

Solving the system,

$$\left(\begin{array}{cccc|c} 0 & -1 & 2 & 0 & 0 \\ -2 & 2 & 0 & 2 & 0 \\ 1 & 0 & -2 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

we get a general solution $a = 2s+t, b = 2s, c = s, d = t,\ s, t \in \mathbb{R}$. So

$$\mathbf{B} = \begin{pmatrix} 2s+t & 2s \\ s & t \end{pmatrix}$$