MA1508E: LINEAR ALGEBRA FOR ENGINEERING

Lecture 1 Notes

References

1. Elementary Linear Algebra: Application Version, Section 1.1-1.2
2. Linear Algebra with Application, Section 1.1-1.3

1 Linear Systems

1.1 Introduction to Linear Systems

An equation is linear if the variables (unknowns) are only acted upon by multiplying by constants and adding them up. A linear equation in n variables has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

Here $a_1, a_2, ..., a_n$ are constants (fixed real numbers), called the coefficients, b is called the constant, and $x_1, x_2, ..., x_n$ are the variables.

A system of linear equations, or a linear system consists of a finite number of linear equations. In general, a linear system with n variables and m equations has the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

A linear system can be expressed uniquely as an augmented matrix

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right).$$

The linear system is homogeneous if there are no constant terms

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array}$$

Then the corresponding augemented matrix is

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right).$$

Given a linear system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

the homogeneous system associated to it is

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array}$$

Example. (Nonhomogeneous) Linear system:

$$\begin{array}{ccccccc} 3x & + & 2y & - & z & = & 1 \\ & & 5y & + & z & = & 3 \\ x & & & + & z & = & 2 \end{array}$$

The corresponding augmented matrix is

$$\left(\begin{array}{ccc|c} 3 & 2 & -1 & 1 \\ 0 & 5 & 2 & 3 \\ 1 & 0 & 1 & 2 \end{array}\right)$$

The associated homogeneous system is

$$\begin{array}{ccccccc} 3x & + & 2y & - & z & = & 0 \\ & & 5y & + & z & = & 0 \\ x & & & + & z & = & 0 \end{array}$$

with augmented matrix

$$\left(\begin{array}{ccc|c} 3 & 2 & -1 & 0 \\ 0 & 5 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

1.2 Solutions to a Linear System

Given a linear system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

we say that

$$x_1 = c_1,\ x_2 = c_2,\ \ldots,\ x_n = c_n$$

is a solution to the linear system if the equations are simultaneously satisfied after making the substitution, that is,

$$\begin{array}{ccccccccc} a_{11}c_1 & + & a_{12}c_2 & + & \cdots & + & a_{1n}c_n & = & b_1 \\ a_{21}c_1 & + & a_{22}c_2 & + & \cdots & + & a_{2n}c_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}c_1 & + & a_{m2}c_2 & + & \cdots & + & a_{mn}c_n & = & b_m \end{array}$$

Example. $x = 1 = y$ is a solution to

$$\begin{array}{rcrcr} 3x & - & 2y & = & 1 \\ x & + & y & = & 2 \end{array}$$

Note that solutions may not be unique.

Example.

$$\begin{array}{rcrcr} x & + & 2y & = & 5 \\ 2x & + & 4y & = & 10 \end{array},$$

solutions: $x = 1,\ y = 2$, or $x = 3,\ y = 1$, etc.

If the solution is not unique, we need to introduce parameters, usually denoted by r, s, t, or $s_1, s_2, ..., s_k$. This means that any choice of a real number for each of the parameter is a solution to the linear system. A general solution to a linear system captures all possible solutions to the linear system.

Example.

$$\begin{array}{rcrcr} x & + & 2y & = & 5 \\ 2x & + & 4y & = & 10 \end{array},$$

General solutions: $x = 5 - 2s,\ y = s$, or $x = s,\ y = \frac{1}{2}(5 - s)$, etc.

We are able to obtain solutions of a linear system by reducing the augmented matrix to some special form known as row-echelon form or reduced row-echelon form.

In the augmented matrix, a zero row is a row with all entries 0. A leading entry of a row is the first nonzero entry of the row counting from the left.

An augmented matrix is in row-echelon form (REF) if

1. All zero rows are at the bottom of the matrix.
2. The leading entries are further to the right as we move down the rows.

An augmented matrix in REF has the form

$$\left(\begin{array}{ccccccccc|c} * & & & & & & & & & * \\ 0 & \cdots & 0 & * & & & & & & * \\ 0 & \cdots & 0 & 0 & \cdots & 0 & * & & & * \\ 0 & & & & & & & & 0 & 0 \\ \vdots & & & & & & & & \vdots & \vdots \\ 0 & \cdots & & & & & & \cdots & 0 & 0 \end{array}\right).$$

In the REF, a pivot column is a column containing a leading entry. The augmented matrix is in reduced row-echelon form (RREF) if further

3. The leading entries are 1.
4. In each pivot column, all entries except the leading entry is 0.

A matrix in RREF has the form

$$\left(\begin{array}{cccccccc|c} 1 & & * & 0 & & * & 0 & & * \\ 0 & \cdots & 0 & 1 & & * & 0 & & * \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & & * \\ 0 & & & 0 & & & 0 & 0 & 0 \\ \vdots & & & \vdots & & & \vdots & \vdots & \vdots \\ 0 & \cdots & & 0 & & \cdots & 0 & 0 & 0 \end{array}\right).$$

Note that REF of an augmented matrix is not unique, but RREF is unique.

Once the augmented matrix is in REF or RREF, we can use back substitution to find a general solution.

Example.

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 3 \end{array}\right)$$

tells us that $x + y = 2$ and $y = 3$. So $y = 3$ and $x = 2 - y = -1$.

Once in RREF, we say that the linear system is <u>inconsistent</u> if the last column (after the vertical line) is a pivot column.

$$\left(\begin{array}{ccc|c} * & \cdots & * & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{array}\right).$$

This means that the system has no solution. For if the leading entry is in the last column, we will solving for

$$0x_1 + 0x_2 + \cdots + 0x_1 = 1,$$

which is impossible. Otherwise, the linear system is <u>consistent</u>, that is, there will be solutions to the linear system. In this case,

$$\#\text{ of parameters } = \#\text{ of nonpivot columns on the left of the vertical line}$$

Meaning that if there are at least 1 nonpivot columns before the vertical line, the linear system will have infinitely many solutions, that is, the solution is not unique.

Example.

1. The system corresponding to

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -3 & 1 \end{array}\right)$$

is consistent since the last column is not a pivot column. The system has 1 parameter since the third column is also a nonpivot column.

2. The system corresponding to

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{array}\right)$$

is inconsistent since the last column is a pivot column.

Hence, given a linear system, to find the solutions (if it exists) our task is to reduce the augmented matrix to REF or RREF. This will be accomplished through elementary row operations.

1.3 Elementary Row Operations

There are 3 types of elementary row operations.

1. Exchanging 2 row, $R_i \leftrightarrow R_j$,
2. Adding a multiple of a row to another, $R_i + cR_j$, $c \in \mathbb{R}$,
3. Multiplying a row by a nonzero constant, aR_j, $a \neq 0$.

Remark. 1. Note that we cannot multiply a row by 0, as it may change the linear system. For example, consider

$$\begin{array}{ccccc} x & + & y & = & 2 \\ x & - & y & = & 0 \end{array}$$

It has a unique solution $x = 1,\ y = 1$. Suppose in the augmented matrix we multiply row 2 by 0,

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right) \xrightarrow{0R_2} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

then the system now has a general solution $x = 2 - s$, $y = s$.

2. Elementary row operations may not commute. For example,

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{2R_1} \xrightarrow{R_2 \leftrightarrow R_1} \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 2 & 0 & 0 \end{array}\right)$$

is not the same as

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_1} \xrightarrow{2R_1} \left(\begin{array}{cc|c} 0 & 2 & 0 \\ 1 & 0 & 0 \end{array}\right)$$

But if the elementary row operations do commute, we can stack them

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow[2R_1]{2R_2} \left(\begin{array}{cc|c} 2 & 0 & 0 \\ 0 & 2 & 0 \end{array}\right)$$

3. For the second type of elementary row operation, the row we put first is the row we are performing the operation upon,

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1 + 2R_2} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

instead of

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{2R_2 + R_1} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 2 & 0 \end{array}\right)$$

In fact, the $2R_2 + R_1$ is not an elementary row operation, but a combination of 2 operations, $2R_2$ then $R_2 + R_1$. Here's another example,

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1 + R_2} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right)$$

and

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{R_2 + R_1} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 0 \end{array}\right)$$

Two augmented matrices are <u>row equivalent</u> if one can be obtained from the other by elementary row operations.

Theorem. *Two augmented matrices are row equivalent if and only have they have the same RREF.*

We will give a proof to this theorem in lecture 3.

Observe that from the RREF we are able to uniquely obtain the solution set, and from the solution set, if we know the number of equations the linear system has, we are able to reconstruct the RREF uniquely. Hence, the previous theorem gives us the following statement.

Theorem. *Two linear systems have the same solution set if and only if their augmented matrices are row equivalent.*

Example. From the REF or RREF, we are able to read off a general solution.

1.
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right)$$
This is in REF. We let the third variable be the parameter s, then we get $y = -s$ from the second row, and $x = 1 - s - (-s) = 1$ from the first row. So a general solution is $x = 1, y = -s, z = s$.

2.
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right)$$
This is in RREF. General solution: $x = 1, y = s, z = s$.

Example. We will now reconstruct the RREF of the augmented matrix of a linear system given a general solution.

1. $x = 1 - 2s + t,\ y = s,\ z = t$, the linear system has 3 equations. Then substituting back $y = s$ and $z = t$ into $x = 1 - 2s + t$, we get $x + 2y - z = 1$. So the RREF of the augmented matrix is
$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

2. $x = 3 - 5s,\ y = 2 + 2s,\ z = s$, the linear system has 3 equations. Substituting back, we get,
$$\begin{array}{ccccccc} x & & & + & 5z & = & 3 \\ & & y & - & 2z & = & 2 \end{array}$$
and thus the RREF of the augmented matrix is
$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

3. $x = 3,\ y = 2,\ z = 1$, 3 equations. RREF:
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

1.4 Gaussian Elimination and Gauss-Jordan Elimination

Step 1: Locate the leftmost column that does not consist entirely of zeros.

Step 2: Interchange the top row with another row, if necessay, to bring a nonzero entry to the top of the column found in Step 1.

Step 3: For each row below the top row, add a suitable multiple of the top row to it so that the entry below the leading entry of the top row becomes zero.

Step 4: Now cover the top row in the augmented matrix and begin again with Step 1 applied to the submatrix that remains. Continue this way until the entire matrix is in row-echelon form.

Once the above process is completed, we will end up with a REF. The following steps continue the process to reduce it to its RREF.

Step 5: Multiply a suitable constant to each row so that all the leading entries become 1.

Step 6: Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading entries.

Remark. The Gaussian elimination and Gauss-Jordan elimination may not be the fastest way to obtain the RREF of an augmented matrix. We do not have to follow the algorithm strictly when reducing the augmented matrix.

Example.

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ -1 & 2 & -1 & 1 \\ 2 & 0 & 3 & -2 \end{array}\right) \xrightarrow[R_3-2R_1]{R_2+R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 0 & -2 & -1 & -10 \end{array}\right) \xrightarrow{R_3+\frac{2}{3}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & -1/3 & -20/3 \end{array}\right)$$

The augmented matrix is now in REF. By back substitution, we have

$$z = 20,\ y = \frac{1}{3}(5-z) = -5,\ x = 4 - y - 2z = -31.$$

Alternatively, we can continue to reduce it to its RREF and read off the solution.

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & -1/3 & -20/3 \end{array}\right) \xrightarrow{-3R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 1 & 20 \end{array}\right) \xrightarrow[R_1-2R_3]{R_2-R_3} \left(\begin{array}{ccc|c} 1 & 1 & 0 & -36 \\ 0 & 3 & 0 & -15 \\ 0 & 0 & 1 & 20 \end{array}\right)$$

$$\xrightarrow{\frac{1}{3}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & -36 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 20 \end{array}\right) \xrightarrow{R_1-R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -31 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 20 \end{array}\right)$$

Indeed, the system is consistent, with unique solution $x = -31, y = -5, z = 20$.

1.5 Solving Linear Systems: Examples

1. Solve the following linear system

$$\begin{array}{rcrcrcrcrcrcr} x_1 & + & 3x_2 & - & 2x_3 & + & & + & 2x_5 & & & = & 0 \\ 2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & + & 4x_5 & - & 3x_6 & = & -1 \\ & + & & + & 5x_3 & + & 10x_4 & & & + & 15x_6 & = & 5 \\ 2x_1 & + & 6x_2 & & & + & 8x_4 & + & 4x_5 & + & 18x_6 & = & 6 \end{array}$$

The augmented matrix is

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right)$$

We will begin the Gaussian elimination.

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right) \xrightarrow{R_2-2R_1} \xrightarrow{R_4-2R_1} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

$$\xrightarrow{R_3+5R_2} \xrightarrow{R_4+4R_2} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right) \xrightarrow{R_3 \leftrightarrow R_4} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The augmented matrix is now in REF. We may use back substitution to obtain the solution, or continue to reduce to its RREF.

$$\xrightarrow{-R_2} \xrightarrow{(1/6)R_3} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2-3R_3} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1+2R_2} \left(\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This corresponds to the following linear system

$$\begin{array}{rcrcrcrcrcrcr} x_1 & + & 3x_2 & & & + & 4x_4 & + & 2x_5 & & & = & 0 \\ & + & & & x_3 & + & 2x_4 & & & & & = & 0 \\ & & & & & & & & & + & x_6 & = & 1/3 \end{array}$$

Letting $x_2 = r, x_4 = s, x_5 = t$, a general solution is

$$x_1 = -3r - 4s - 2t,\ x_2 = r,\ x_3 = -2s,\ x_4 = s,\ x_5 = t,\ x_6 = 1/3.$$

2. Solving linear system,

$$\begin{array}{rcrcrcr} x_1 & + & ax_2 & + & 2x_3 & = & 0 \\ x_1 & & & + & x_3 & = & 1 \\ x_1 & & & + & ax_3 & = & 2 \end{array}$$

for some fixed real number a.

$$\left(\begin{array}{ccc|c} 1 & a & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & a & 2 \end{array}\right) \xrightarrow{R_2-R_1} \xrightarrow{R_3-R_1} \left(\begin{array}{ccc|c} 1 & a & 2 & 0 \\ 0 & -a & -1 & 1 \\ 0 & -a & a-2 & 2 \end{array}\right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} 1 & a & 2 & 0 \\ 0 & -a & -1 & 1 \\ 0 & 0 & a-1 & 1 \end{array}\right)$$

The augmented matrix is almost a REF. To proceed to reduce it further, we have to consider cases. If $a = 1$,

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

the system is inconsistent. Next, consider the case where $a = 0$,

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right) \xrightarrow{R_3-R_2} \xrightarrow{R_1+2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Then a general solution is

$$x_1 = 2,\ x_2 = s,\ x_3 = -1.$$

Otherwise, if $a \neq 0, 1$, then the system has a unique solution

$$x_3 = \frac{1}{a-1},\ x_2 = \frac{-1}{a}(1 + x_3) = \frac{-1}{a-1},\ x_1 = -ax_2 - 2x_3 = \frac{a-2}{a-1}.$$