MA1508E: LINEAR ALGEBRA FOR ENGINEERING

Lecture 2 Notes

References

1. Elementary Linear Algebra: Application Version, Section 1.3, 1.7
2. Linear Algebra with Application, Section 2.1-2.3

2 Matrices

2.1 Introduction to Matrices

A matrix is a rectangular array of number

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

were $a_{ij} \in \mathbb{R}$ are real numbers.

The size of a matrix is given by $m \times n$, where m is the number of rows and n is the number of columns. The (i,j)-entry of the matrix is the number a_{ij} in the i-th row and j-th column, for $i = 1, ..., m$, $j = 1, ..., n$. A matrix can also be denoted as

$$\mathbf{A} = (a_{ij})_{m \times n} = (a_{ij})_{i=1}^{m} {}_{j=1}^{n}.$$

Matrices are usually denoted by upper case bolded letters.

Example. 1. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & -1 \end{pmatrix}$ is a 3×2 matrix. The $(2,1)$-entry is 3.

2. $\begin{pmatrix} 2 & 1 & 0 \end{pmatrix}$ is a 1×3 matrix. The $(1,2)$-entry is 2.

3. $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a 3×1 matrix. The $(3,1)$-entry is 3.

4. $\begin{pmatrix} 4 \end{pmatrix}$ is a 1×1 matrix.

The last example shows that all real numbers can be thought of as 1×1 matrices.

Remark. 1. To be precise, the above examples are called real-valued matrices, or matrices with real number entries. Later we will be introduced to complex-valued and even matrices with function entries.

2. The choice of using round or square brackets is a matter of taste.

Example. 1. $\mathbf{A} = (a_{ij})_{2\times 3}$, $a_{ij} = i + j$.

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

2. $\mathbf{B} = (b_{ij})_{3\times 2}$, $b_{ij} = (-1)^{i+j}$.

$$\mathbf{B} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{pmatrix}$$

3. $\mathbf{C} = (c_{ij})_{3\times 3}$, $c_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$.

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.2 Special types of Matrices

Here are some special types of matrices.

Column and row matrices. A $1 \times n$ matrix is called a row matrix, and a $n \times 1$ matrix is called a column matrix. However, we seldom use these terms. We usually call them row vectors and column vectors, respectively, which will be introduced in lecture 5.

Square matrices. A $m \times n$ matrix is a square matrix if the number of columns is equal to the number of rows $m = n$. A square matrix of size $n \times n$ is called an order n square matrix. It is usually denoted by $\mathbf{A} = (a_{ij})_n$.

Example. Order 2: $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ Order 3: $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ 5 & 6 & 6 \end{pmatrix}$

The i-th diagonal entry of a square matrix is its (i, i)-entry. The diagonal entries of a square matrix $\mathbf{A} = (a_{ij})_n$ of order n is the collection $\{a_{11}, a_{22}, ..., a_{nn}\}$.

Diagonal matrices. A square matrix with all the non diagonal entries equal 0 is called a diagonal matrix, $\mathbf{D} = (d_{ij})_n$ with $d_{ij} = 0$ for all $i \neq j$. It is usually denoted by $\mathbf{D} = \text{diag}\{d_1, d_2, ..., d_n\}$.

Example.

$$\text{diag}\{1,1\} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{diag}\{0,0,0\} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{diag}\{1,2,3,4\} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Scalar matrices. A diagonal matrix $\mathbf{A} = \text{diag}\{a_1, a_2, ..., a_n\}$ such that all the diagonal entries are equal $a_1 = a_2 = ... = a_n$ is called a scalar matrix.

Example. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Identity matrices. A scalar matrix with all diagonal entries equal 1 is called an identity matrix. An identity matrix of order n is denoted as $\mathbf{I}_n$. If there is no confusion with the order of the matrix, we will write $\mathbf{I}$ instead. So a scalar matrix can be written as $c\mathbf{I}$.

Zero matrices. A matrix (of any size) with all entries equal 0 is called a zero matrix. Usually denoted as $\mathbf{0}_{m\times n}$ for the size $m \times n$ zero matrix, and $\mathbf{0}_n$ for the zero square matrix of order n. If it is clear in the context, we will just denote it as $\mathbf{0}$.

Triangular matrices. A square matrix $\mathbf{A} = (a_{ij})_n$ with all entries below (above) the diagonal equal 0, that is, $a_{ij} = 0$ for all $i > j$ $(i < j)$, is called an upper (lower) triangular matrix. It is a strictly upper or lower matrix if the diagonals are equal to zero too, that is, $a_{ij} = 0$ for all $i \geq j$ $(i \leq j)$.

Upper triangular: $\begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{pmatrix}$ Strictly upper triangular: $\begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$

Lower triangular: $\begin{pmatrix} * & 0 & \cdots & 0 \\ * & * & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{pmatrix}$ Strictly lower triangular: $\begin{pmatrix} 0 & 0 & \cdots & 0 \\ * & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & 0 \end{pmatrix}$

Exercise: Is it true that every matrix is row equivalent to an upper triangular matrix? How about strictly upper matrix?

Symmetrix matrices. A square matrix $\mathbf{A} = (a_{ij})_n$ such that $a_{ij} = a_{ji}$ for all $i, j = 1, ..., n$ is called a symmetric matrix, that is, the entries are diagonally reflected along the diagonal of $\mathbf{A}$.

Example. $\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & 5 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{pmatrix}$.

2.3 Matrix Operations

Two matrices are equal if they are of the same size and all the entries are equal.

Example. 1.

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \neq \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

for any choice of a, b, c, d, e, f.

2.

$$\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

if and only if $a = 1$, $b = 1$, $c = 3$, $d = 2$.

Matrix Addition and Scalar Multiplication

Let $\mathbf{A} = (a_{ij})_{m\times n}$ and $\mathbf{B} = (b_{ij})_{m\times n}$ be two matrices and $c \in \mathbb{R}$ a real number. We define the following operations as such.

1. (Scalar multiplication) $c\mathbf{A} = (ca_{ij})$.
2. (Matrix addition) $\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})_{m\times n}$

Remark. 1. Matrix addition is only defined between matrices of the same size.

2. $-\mathbf{A} = (-1)\mathbf{A}$.

3. Matrix substraction is defined to be the addition of a negative multiple of another matrix,
$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B}.$$

Theorem (Properties of matrix addition and scalar multiplication)**.** *For matrices* $\mathbf{A} = (a_{ij})_{m\times n}$*,* $\mathbf{B} = (b_{ij})_{m\times n}$*,* $\mathbf{C} = (c_{ij})_{m\times n}$*, and real numbers* $a, b \in \mathbb{R}$*,*

(i) (Commutative) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$*,*

(ii) (Associative) $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$*,*

(iii) (Additive identitiy) $\mathbf{0}_{m\times n} + \mathbf{A} = \mathbf{A}$*,*

(iv) (Additive inverse) $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}_{m\times n}$*,*

(v) (Distributive law) $a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B}$*,*

(vi) (Scalar addition) $(a + b)\mathbf{A} = a\mathbf{A} + b\mathbf{A}$*,*

(vii) $(ab)\mathbf{A} = a(b\mathbf{A})$*,*

(viii) if $a\mathbf{A} = \mathbf{0}_{m\times n}$*, then either* $a = 0$ *or* $\mathbf{A} = \mathbf{0}$*.*

Proof. To show equality, we have to show that the matrices on the left and right of the equality have the same size, and that the corresponding entries are equal. It is clear that the matrices on both sides has the same size, so we will only check that the entries agree.

(i) $a_{ij} + b_{ij} = b_{ij} + a_{ij}$ follows directly from commutativity of addition of real numbers.

(ii) $a_{ij} + (b_{ij} + c_{ij}) = (a_{ij} + b_{ij}) + c_{ij}$ follows directly from associativity of addition of real numbers.

(iii) $0 + a_{ij} = a_{ij}$ follows directly from the additive identity property of real numbers.

(iv) $a_{ij} + (-a_{ij}) = 0$ follows directly from additive inverse property of real numbers.

(v) $a(a_{ij} + b_{ij}) = aa_{ij} + ab_{ij}$ follows directly from distributive property of addition of real numbers.

(vi) $(a+b)a_{ij} = aa_{ij} + ba_{ij}$ follows directly from distributive property of addition of real numbers.

(vii) $(ab)a_{ij} = a(ba_{ij})$ follows directly from associativity of multiplication of real numbers.

(viii) If $aa_{ij} = 0$, then $a = 0$ or $a_{ij} = 0$. Suppose $a \neq 0$, then $a_{ij} = 0$ for all i, j. So $\mathbf{A} = \mathbf{0}$.

$\square$

Remark. 1. Since addition is associative, we will not write the parentheses when adding multiple matrices.

2. Property (iii) and (i) imply that $\mathbf{A} + \mathbf{0} = \mathbf{A}$.

3. Property (iv) and (i) imply that $-\mathbf{A} + \mathbf{A} = \mathbf{0}$.

Matrix Multiplication

Let $\mathbf{A} = (a_{ij})_{m\times p}$ and $\mathbf{B} = (b_{ij})_{p\times n}$. The product $\mathbf{AB}$ is defined to be a $m \times n$ matrix whose (i, j)-entry is

$$\sum_{k=1}^{p} a_{ik}b_{kj} = a_{i1}b_{ij} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj}.$$

Example.

$$\underset{(2\times 3)}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}} \underset{(3\times 2)}{\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix}} = \underset{(2\times 2)}{\begin{pmatrix} 1+4-3=2 & 1+6-6=1 \\ 4+10-6=12 & 4+15-12=7 \end{pmatrix}}$$

Remark. 1. For $\mathbf{AB}$ to be defined, the number of columns of $\mathbf{A}$ must agree with the number of rows of $\mathbf{B}$. The resultant matrix has the same number of rows as $\mathbf{A}$, and the same number of columns as $\mathbf{B}$.

$$(m \times p)(p \times n) = (m \times n).$$

2. Matrix multiplication is not commutative, that is $\mathbf{AB} \neq \mathbf{BA}$ in general. For example, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

3. If we are multiplying $\mathbf{A}$ to the left of $\mathbf{B}$, we are <u>pre-multiplying</u> $\mathbf{A}$ to $\mathbf{B}$, $\mathbf{AB}$. If we multiply $\mathbf{A}$ to the right of $\mathbf{B}$, we are <u>post-multiplying</u> $\mathbf{A}$ to $\mathbf{B}$, $\mathbf{BA}$. Pre-multiplying $\mathbf{A}$ to $\mathbf{B}$ is the same as post-multiplying $\mathbf{B}$ to $\mathbf{A}$.

Theorem (Properties of matrix multiplication)**.** *(i) (Associative) For matrices* $\mathbf{A} = (a_{ij})_{m\times p}$, $\mathbf{B} = (b_{ij})_{p\times q}$, *and* $\mathbf{C} = (c_{ij})_{q\times n}$ $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

(ii) (Left distributive law) For matrices $\mathbf{A} = (a_{ij})_{m\times p}$, $\mathbf{B} = (b_{ij})_{p\times n}$, *and* $\mathbf{C} = (c_{ij})_{p\times n}$, $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.

(iii) (Right distributive law) For matrices $\mathbf{A} = (a_{ij})_{m\times p}$, $\mathbf{B} = (b_{ij})_{m\times p}$, *and* $\mathbf{C} = (c_{ij})_{p\times n}$, $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$.

(iv) (Commute with scalar multiplication) For any real number $c \in \mathbb{R}$, and matrices $\mathbf{A} = (a_{ij})_{m\times p}$, $\mathbf{B} = (b_{ij})_{p\times n}$, $c(\mathbf{AB}) = (c\mathbf{A})\mathbf{B} = \mathbf{A}(c\mathbf{B})$.

(v) (Multiplicative identity) For any $m \times n$ matrix $\mathbf{A}$, $\mathbf{I}_m\mathbf{A} = \mathbf{A} = \mathbf{A}\mathbf{I}_n$.

(vi) (Zero divisor) There exists $\mathbf{A} \neq \mathbf{0}_{m\times p}$ and $\mathbf{B} \neq \mathbf{0}_{p\times n}$ such that $\mathbf{AB} = \mathbf{0}_{m\times n}$.

(vii) (Zero matrix) For any $m \times n$ matrix $\mathbf{A}$, $\mathbf{A}\mathbf{0}_{n\times p} = \mathbf{0}_{m\times p}$ and $\mathbf{0}_{p\times m}\mathbf{A} = \mathbf{0}_{p\times n}$.

The proof is beyond the scope of this course. Interested readers may refer to the appendix.

Remark. 1. For square matrices, we define $\mathbf{A}^2 = \mathbf{AA}$, and define inductively, $\mathbf{A}^n = \mathbf{A}\mathbf{A}^{n-1}$, for $n \geq 2$. It follows that $\mathbf{A}^n\mathbf{A}^m = \mathbf{A}^{n+m}$.

2. In general $(\mathbf{AB})^n \neq \mathbf{A}^n\mathbf{B}^n$. (Why?)

Tranpose

For a $m \times n$ matrix $\mathbf{A}$, the transpose of $\mathbf{A}$, written as $\mathbf{A}^T$, is a $n \times m$ matrix whose (i, j)-entry is the (j, i)-entry of $\mathbf{A}$, that is, if $\mathbf{A}^T = (b_{ij})_{n\times m}$, then

$$b_{ij} = a_{ji}$$

for all $i = 1, ..., n$, $j = 1, ..., m$. Equivalently, the rows of $\mathbf{A}$ are the columns of $\mathbf{A}^T$ and vice versa.

Example. 1. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 5 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$ 2. $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$ 3. $\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$

This gives us an alternative way to define symmetric matrices. A square matrix $\mathbf{A}$ is symmetric if and only if $\mathbf{A}^T = \mathbf{A}$.

Theorem (Properties of transpose)**.** *(i) For any matrix $\mathbf{A}$, $(\mathbf{A}^T)^T = \mathbf{A}$.*

(ii) For any matrix $\mathbf{A}$, and real number $c \in \mathbb{R}$, $(c\mathbf{A})^T = c\mathbf{A}^T$.

(iii) For matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size, $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.

(iv) For matrices $\mathbf{A} = (a_{ij})_{m\times p}$ and $\mathbf{B} = (b_{ij})_{p\times n}$, $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Refer to the appendix for the proof.

Example.

$$\begin{aligned}
\left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix}\right)^T &= \begin{pmatrix} 2 & 1 \\ 12 & 7 \end{pmatrix}^T = \begin{pmatrix} 2 & 12 \\ 1 & 7 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 1 & 3 & -2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & -2 \end{pmatrix}^T\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T
\end{aligned}$$

Which of the following statements are true? Justify.

(a) If $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices of the same size, then so is $\mathbf{A} + \mathbf{B}$.

(b) If $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices (with the appropriate sizes), then so is $\mathbf{AB}$.

2.4 Revisit Linear System

Given a linear system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

We can represent it as a matrix equation $\mathbf{Ax} = \mathbf{b}$, or

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

or as vector equation

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

A homogeneous system is thus written as $\mathbf{Ax} = \mathbf{0}$. From property (vii) of matrix multiplication, we have $\mathbf{x} = \mathbf{0}$ as a solution to any homogeneous system. This is called the trivial solution. If there is a solution $\mathbf{x} \neq \mathbf{0}$, then we say that the homogeneous linear system admits nontrivial solutions.

Example.

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

admits a nontrivial solution $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Appendix for Lecture 2

Theorem (Properties of matrix multiplication). *(i) (Associative) For matrices* $\mathbf{A} = (a_{ij})_{m\times p}$*,* $\mathbf{B} = (b_{ij})_{p\times q}$*, and* $\mathbf{C} = (c_{ij})_{q\times n}$ $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$*.*

(ii) (Left distributive law) For matrices $\mathbf{A} = (a_{ij})_{m\times p}$*,* $\mathbf{B} = (b_{ij})_{p\times n}$*, and* $\mathbf{C} = (c_{ij})_{p\times n}$*,* $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$*.*

(iii) (Right distributive law) For matrices $\mathbf{A} = (a_{ij})_{m\times p}$*,* $\mathbf{B} = (b_{ij})_{m\times p}$*, and* $\mathbf{C} = (c_{ij})_{p\times n}$*,* $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$*.*

(iv) (Commute with scalar multiplication) For any real number $c \in \mathbb{R}$*, and matrices* $\mathbf{A} = (a_{ij})_{m\times p}$*,* $\mathbf{B} = (b_{ij})_{p\times n}$*,* $c(\mathbf{AB}) = (c\mathbf{A})\mathbf{B} = \mathbf{A}(c\mathbf{B})$*.*

(v) (Multiplicative identity) For any $m\times n$ *matrix* $\mathbf{A}$*,* $\mathbf{I}_m\mathbf{A} = \mathbf{A} = \mathbf{A}\mathbf{I}_n$*.*

(vi) (Zero divisor) There exists $\mathbf{A} \neq \mathbf{0}_{m\times p}$ *and* $\mathbf{B} \neq \mathbf{0}_{p\times n}$ *such that* $\mathbf{AB} = \mathbf{0}_{m\times n}$*.*

(vii) (Zero matrix) For any $m\times n$ *matrix* $\mathbf{A}$*,* $\mathbf{A}\mathbf{0}_{n\times p} = \mathbf{0}_{m\times p}$ *and* $\mathbf{0}_{p\times m}\mathbf{A} = \mathbf{0}_{p\times n}$*.*

Proof. We will check that the corresponding entries on each side agrees. The check for the size of matrices agree is trivial and is left to the reader.

(i) The (i,j)-entry of $(\mathbf{AB})\mathbf{C}$ is

$$\sum_{l=1}^{q}(\sum_{k=1}^{p} a_{ik}b_{kl})c_{lj} = \sum_{l=1}^{q}\sum_{k=1}^{p} a_{ik}b_{kl}c_{lj}.$$

The (i,j)-entry of $\mathbf{A}(\mathbf{BC})$ is

$$\sum_{k=1}^{p} a_{ik}(\sum_{l=1}^{q} b_{kl}c_{lj}) = \sum_{k=1}^{p}\sum_{l=1}^{q} a_{ik}b_{kl}c_{lj}.$$

Since both sums has finitely many terms, the sums commute and thus the (i,j)-entry of $(\mathbf{AB})\mathbf{C}$ is equal to the (i,j)-entry of $\mathbf{A}(\mathbf{BC})$.

(ii) The (i,j)-entry of $\mathbf{A}(\mathbf{B}+\mathbf{C})$ is $\sum_{k=1}^{p} a_{ik}(b_{kj}+c_{kj}) = \sum_{k=1}^{p}(a_{ik}b_{kj}+a_{ik}c_{kj}) = \sum_{k=1}^{p} a_{ik}b_{kj} + \sum_{k=1}^{p} a_{ik}c_{kj}$, which is the (i,j)-entry of $\mathbf{AB}+\mathbf{AC}$.

(iii) The proof is analogous to left distributive law.

(iv) Left to reader.

(v) Note that $I = (\delta_{ij})$, where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. So the (i,j)-entry of $\mathbf{I}_m\mathbf{A}$ is

$$\delta_{i1}a_{1j} + \cdots + \delta_{ii}a_{ij} + \cdots + \delta_{im}a_{mj} = 0a_{1j} + \cdots + 1a_{ij} + \cdots + 0a_{mj} = a_{ij}.$$

The proof for $\mathbf{A} = \mathbf{A}\mathbf{I}_n$ is analogous.

(vi) Consider for example $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(vii) Left to reader, if you have read till this far, surely this proof is trivial to you.

$\square$

Theorem (Properties of transpose). *(i) For any matrix* $\mathbf{A}$, $(\mathbf{A}^T)^T = \mathbf{A}$.

(ii) For any matrix $\mathbf{A}$*, and real number* $c \in \mathbb{R}$, $(c\mathbf{A})^T = c\mathbf{A}^T$.

(iii) For matrices $\mathbf{A}$ *and* $\mathbf{B}$ *of the same size,* $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.

(iv) For matrices $\mathbf{A} = (a_{ij})_{m\times p}$ *and* $\mathbf{B} = (b_{ij})_{p\times n}$, $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Proof. We will only proof (iv). The rest is left to the reader. The (j,i)-entry of $\mathbf{AB}$ is

$$\sum_{k=1}^{p} a_{jk}b_{ki},$$

which is the (i,j)-entry of $(\mathbf{AB})^T$. The (i,j)-entry of $\mathbf{B}^T\mathbf{A}^T$ is

$$\sum_{k=1}^{p} b_{ki}a_{jk} = \sum_{k=1}^{p} a_{jk}b_{ki},$$

which is exactly the (i,j)-entry of $(\mathbf{AB})^T$. $\square$