NATIONAL UNIVERSITY OF SINGAPORE

MA1508E - LINEAR ALGEBRA FOR ENGINEERING

(Semester 2 : AY2017/2018)

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. Please write your matriculation/student number only. Do not write your name.
2. This examination paper contains **FIVE** questions and comprises **FOUR** printed pages.
3. Answer **ALL** questions.
4. Please start each question on a new page.
5. This is a CLOSED BOOK examination.
6. Candidates may use scientific calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [20 marks]

(a) Let $V = \{(a+b-c,\ b+2c,\ 2a-b+c,\ c,\ a-3b) \mid a,b,c \in \mathbb{R}\}$ be a subset of $\mathbb{R}^5$.

(i) Show that V is a subspace of $\mathbb{R}^5$ by writing it as a linear span.

(ii) Use your answer in (i) to find a basis for V. Hence, state the dimension of V.

(iii) Find all vectors orthogonal to V. Express your answer as a linear span.

(iv) Show that $(1,1,1,0,1)$ does not belong to V.

(v) Find a subspace W of $\mathbb{R}^5$ such that $\dim(W) = 4$ and $V \subseteq W$. Justify your answer.

(b) Let

$$\boldsymbol{B} = \begin{pmatrix} s & 0 & 1 \\ -t & -1 & 0 \\ 0 & t & -s \end{pmatrix}$$

where s,t are real numbers. By computing the determinant of $\boldsymbol{B}$, find all values of s and t such that $\boldsymbol{Bx} = \boldsymbol{0}$ has infinitely many solutions.

Question 2 [20 marks]

(a) For the following matrix $\boldsymbol{A}$, compute $\boldsymbol{A}^{-1}$ and express $\boldsymbol{A}^{-1}$ as a product of exactly 5 elementary matrices.

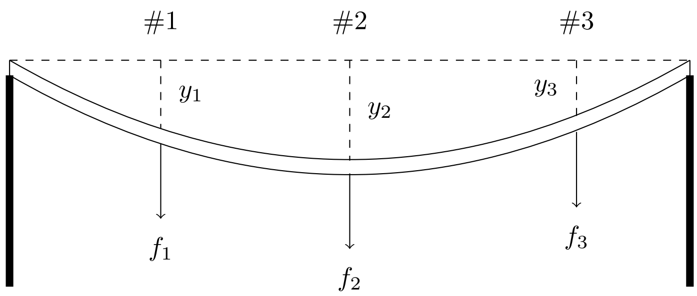
$$\boldsymbol{A} = \begin{pmatrix} 1 & 3 & 0 \\ -1 & -2 & 1 \\ 0 & 4 & 3 \end{pmatrix}.$$

(b) (i) Let

$$\boldsymbol{S} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{T} = \frac{1}{8}\begin{pmatrix} a & 2 & 1 \\ 2 & b & 2 \\ 1 & 2 & c \end{pmatrix}.$$

Suppose we know that $\boldsymbol{ST} = \boldsymbol{I}$. Find the values of a, b, c.

(ii) We would like to study the flexibility of a horizontal elastic beam shown in the figure below, which shows that when forces of f_1, f_2, f_3 (in the direction specified by the arrows) are applied at the 3 points #1, #2 and #3, the amount of deflection of the beam observed at the 3 points are y_1, y_2, y_3 respectively.



A series of experiments were conducted to determine the forces required to be applied at the 3 points in order to observe various desired deflections at the 3 points. The results of the experiments are given in the table below.

	Force applied			Deflection observed		
	f_1	f_2	f_3	y_1	y_2	y_3
Experiment 1	0.25	-0.125	0	1	0	0
Experiment 2	-0.125	0.375	-0.125	0	1	0
Experiment 3	0	-0.125	0.25	0	0	1

For example, in order to produce 1 unit of deflection at point #1 and no deflection and points #2 and #3, we need to apply a force of 0.25 units downwards at point #1, a force of 0.125 units **upwards** at point #2 and no force at point #3.

Find the flexibility matrix $\boldsymbol{D}$ of the elastic beam.

(**Hint:** Use part (i).)

(iii) Determine f_1, f_2, f_3 if we wish to observe the deflection $y_1 = y_2 = y_3 = 1$.

(iv) The beam will snap if $\max\{y_1, y_2, y_3\} \geq 3$ or $y_1 + y_2 + y_3 \geq 5$. Determine whether the beam snaps when the forces applied to the 3 points is given by

$$\boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 0.5 \\ -0.2 \\ 0.3 \end{pmatrix}.$$

Question 3 [20 marks]

(a) Let $S = \{(1, -1, 0, 2), (0, 2, 1, 1), (2, -2, -3, 1)\}$.

(i) Show that S is a basis, but not an orthogonal basis for $\text{span}(S)$.

(ii) Use Gram-Schmidt Process to find an orthogonal basis for $\text{span}(S)$. You may let $\boldsymbol{u_1} = (1, -1, 0, 2)$, $\boldsymbol{u_2} = (0, 2, 1, 1)$ and $\boldsymbol{u_3} = (2, -2, -3, 1)$.

(iii) Compute the orthogonal projection of $\boldsymbol{w} = (1, 1, 1, 1)$ onto $\text{span}(S)$.

(b) It is believed that two physical quantities x and y are linearly related according to the equation $y = cx + d$ where c and d are constants. A series of experiments are conducted to determine the values of c and d. Different values of x were used in the experiments and the corresponding values of y were observed. The results of the experiments are shown in the table below.

x	1	2	3	4
y	1	2	3	3

(i) Based on the data set obtained from the experiments, write down a (over-determined) linear system $\boldsymbol{Ax} = \boldsymbol{b}$ with 4 equations and the 2 unknowns c and d.

(ii) Show that $\boldsymbol{Ax} = \boldsymbol{b}$ is inconsistent. Does $\boldsymbol{b}$ belong to the column space of $\boldsymbol{A}$?

(iii) Find a least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$. Using your least squares solution, predict the value of y when $x = 5$.

(iv) Compute the orthogonal projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$.

Question 4 [20 marks]

(a) Let $S = \{(1, -3, 2), (0, 1, -3), (-2, 4, 4)\}$.

(i) Show that S is a linearly independent set.

(ii) Let $\boldsymbol{w} = (-1, 0, 4)$. Using Cramer's Rule or otherwise, find $(\boldsymbol{w})_S$, the coordinate vector of $\boldsymbol{w}$ relative to S.

(b) Let $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}$ be 4 points in $\mathbb{R}^3$.

(i) Suppose for each $j = 1, 2, 3, 4$, an object with mass m_j is located at point $\boldsymbol{v_j}$ as given in the table below. Show that the center of gravity (or *center of mass*) of the system is located at $\boldsymbol{v} = (9/5, 1, 3/5)$.

Point	Mass
$\boldsymbol{v_1} = (2, 1, 0)$	2g
$\boldsymbol{v_2} = (1, 1, -1)$	1g
$\boldsymbol{v_3} = (3, 1, 1)$	1g
$\boldsymbol{v_4} = (1, 1, 3)$	1g

(ii) Suppose we are allowed to change the values of m_j, $j = 1, 2, 3, 4$ to any **positive integer** values such that $m_1 + m_2 + m_3 + m_4 \leq 10$. Find **all** (m_1, m_2, m_3, m_4) such that the location of the center of gravity remains unchanged.

Question 5 [20 marks]

(a) Let

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 0 & 8 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 12 \\ 0 & 7 & 5 & 4 \end{pmatrix}.$$

(i) Find a basis for the row space of $\boldsymbol{A}$. What is the rank of $\boldsymbol{A}$?

(ii) Find a basis for the column space of $\boldsymbol{A}$.

(iii) Find a basis for the nullspace of $\boldsymbol{A}$. What is the nullity of $\boldsymbol{A}$?

(iv) For each of the following, determine if such a matrix $\boldsymbol{B}$ exists. If it does, find one such $\boldsymbol{B}$ and if it does not, explain why.

(I) $\boldsymbol{B}$ is a 3×3 matrix and the nullspace of $\boldsymbol{B}$ is equal to the column space of $\boldsymbol{A}$.

(II) $\boldsymbol{B}$ is a 4×4 matrix and the nullspace of $\boldsymbol{B}$ is equal to the row space of $\boldsymbol{A}$.

(b) Two species of fish, species A and species B, live in the same ecosystem (e.g. a pond) and compete with each other for food, water and space. Let the population of species A and B at time t years be given by $a(t)$ and $b(t)$ respectively.

In the absence of species B, species A's growth rate is $4a(t)$ but when species B are present, the competition slows the growth of species A to $a'(t) = 4a(t) - 2b(t)$. In a similar manner, when species A is absent, species B's growth rate is $3b(t)$ but in the presence of species A, the growth rate reduces to $b'(t) = 3b(t) - a(t)$.

(i) Write down a system of linear differential equations involving $a(t), b(t), a'(t)$ and $b'(t)$.

(ii) Represent the system in (i) as $\boldsymbol{x'}(t) = \boldsymbol{A}\boldsymbol{x}(t)$ where

$$\boldsymbol{A} \text{ is a } 2 \times 2 \text{ matrix and } \boldsymbol{x}(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}, \quad \boldsymbol{x'}(t) = \begin{pmatrix} a'(t) \\ b'(t) \end{pmatrix}.$$

(iii) Solve the system using the initial condition $a(0) = 60$, $b(0) = 120$.

End of Paper