# NATIONAL UNIVERSITY OF SINGAPORE
## Department of Mathematics

**AY2021, Semester 1   MA1508E Linear Algebra for Engineering   Tutorial 5**

1. Let $\mathbf{u}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 3 \\ -1 \\ 5 \\ 2 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$.

   (a) If possible, express each of the following vectors as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

   (i) $\begin{pmatrix} 2 \\ 3 \\ -7 \\ 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ (iv) $\begin{pmatrix} -4 \\ 6 \\ -13 \\ 4 \end{pmatrix}$

   (b) Is it possible to find 2 vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that they are not a multiple of each other, and both are not a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$?

2. Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle|\ x - y - z = 0 \right\}$ be a subset of $\mathbb{R}^3$.

   (a) Let $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \right\}$. Show that $\text{span}(S) = V$.

   (b) Let $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. Find a vector $\mathbf{y}$ such that $\text{span}\{\mathbf{x}, \mathbf{y}\} = V$.

   (c) Let $T = S \cup \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. Show that $\text{span}(T) = \mathbb{R}^3$.

3. (a) Which of the following sets $S$ spans $\mathbb{R}^4$?

   (i) $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

   (ii) $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

   (iii) $S = \left\{ \begin{pmatrix} 6 \\ 4 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ -2 \\ -1 \end{pmatrix} \right\}$.

(iv) $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}$.

(b) For those sets that does not span $\mathbb{R}^4$, find a vector $\mathbf{x}$ in $\mathbb{R}^4$ that does not belong to span$(S)$. Does $S \cup \{\mathbf{x}\}$ span $\mathbb{R}^4$?

(c) For those sets that spans $\mathbb{R}^4$, find a vector $\mathbf{y}$, if possible, in the set $S$ such that span$(S) = \mathbb{R}^4 = \text{span}(S - \{\mathbf{y}\})$.

4. (a) Determine whether $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subseteq \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and/or $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ if

(i) $\mathbf{u}_1 = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}, \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

(ii) $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}, \mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 8 \\ 9 \end{pmatrix}$.

(b) In each of the above, describe $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ geometrically. If $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ or $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane, find the equation of the plane.

5. Determine which of the following sets are subspaces. For those sets that are subspaces, express the set as a linear span. For those sets that are not, explain why.

(a) $S = \left\{ \begin{pmatrix} p \\ q \\ p \\ q \end{pmatrix} \middle| \; p, q \in \mathbb{R} \right\}$.

(b) $S = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \; a \geq b \text{ or } b \geq c \right\}$.

(c) $S = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \middle| \; 4x = 3y \text{ and } 2x = -3w \right\}$.

(d) $S = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \middle| \; \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ a & b & c & d \end{vmatrix} = 0 \right\}$.

(e) $S = \left\{ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \middle| \; w + x = y + z \right\}$.

(f) $S = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \middle| \; ab = cd \right\}$.

(g) $S$ is the solution set of $\mathbf{Ax} = \mathbf{0}$ where $\mathbf{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix}$.

(h) $S = \left\{ \mathbf{u} + \mathbf{v} \;\middle|\; \mathbf{v} \in \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \right\}$ and $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a fixed vector.

## Supplementary Problems

6. (a) Show that the solution set to any homogeneous linear system

$$S = \{ \ \mathbf{v} \in \mathbb{R}^n \mid \mathbf{Av} = \mathbf{0} \ \}$$

is a subspace.

(b) Suppose the homogeneous linear system $\mathbf{Ax} = 0$ has a nontrivial solution. Prove that if the linear system $\mathbf{Ax} = \mathbf{b}$ is consistent, it must have infinitely many solutions.

(c) Prove that if the linear system $\mathbf{Ax} = \mathbf{b}$ has two distinct solutions, then it must have infinitely many solution.

(d) **(MATLAB)** Let $\mathbf{A}$ be the $10 \times 10$ *magic square*, and let $\mathbf{b}$ be the 10-vector of all 1's. We may key these special matrices into MATLAB fairly quickly.

```
>> A=magic(10);

>> b=ones(10,1);
```

i. Express the solution set of $\mathbf{Ax} = \mathbf{b}$ as

$$\{ \ \mathbf{x}_p + s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k \mid s_1, s_2, \ldots, s_k \in \mathbb{R} \ \}.$$

ii. Pick a few $s_1, s_2, ..., s_k \in \mathbb{R}$ and compute $\mathbf{A}(s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k)$. What is the set $S = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$?

iii. Running the `null` command outputs a collection of column vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell$.
`>> null(A)`
Show that $\operatorname{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\} = \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell\}$. What does this say about the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell$? In particular, what is the output of the `null` command in relation to the matrix $\boldsymbol{A}$?

A subset of $\mathbb{R}^n$ is called an *affine space* if it is of the form $\{ \ \mathbf{u} + \mathbf{v} \mid \mathbf{v} \in V \ \}$ for some subspace $V \subseteq \mathbb{R}^n$. Geometrically, an affine space is a subset of $\mathbb{R}^n$ that is parallel to a subspace. This exercise shows that the solution set to the linear system $\mathbf{Ax} = \mathbf{b}$ is an affine space $\{ \ \mathbf{x}_p + \mathbf{v} \mid \mathbf{v} \in S \ \}$, where $S$ is the solutions to homogeneous linear system $\mathbf{Ax} = \mathbf{0}$, and $\mathbf{x}_p$ is any particular solution.