NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

AY2021, Semester 1 MA1508E Linear Algebra for Engineering Tutorial 10

1. (Cayley-Hamilton theorem)
   Consider
   $$p(\mathbf{X}) = \mathbf{X}^3 - 4\mathbf{X}^2 - \mathbf{X} + 4\mathbf{I}.$$

   (a) Compute $p(\mathbf{X})$ for $\mathbf{X} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$.

   (b) Find the characteristic polynomial of $\mathbf{X}$.

   (c) Show that $\mathbf{X}$ invertible. Express the inverse of $\mathbf{X}$ as a function of $\mathbf{X}$.

   This question demonstrated the *Cayley-Hamilton theorem*, which states that if $p(x)$ is the characteristic polynomial of $\mathbf{X}$, then $p(\mathbf{X}) = 0$. This also show that if 0 is not an eigenvalue of $\mathbf{X}$, then the constant term of the characteristic polynomial $p(x)$ is nonzero, and we can use that to compute the inverse of $\mathbf{X}$.

2. (Markov chain)
   A *Markov chain* is an evolving system wherein the state to which it will go next depends only on its preset state and does not depend on the earlier history of the system. The probabilities that the current state does to the various states at the next stage are called the *transition probabilities* for the chain. Consider the following simple example.
   A population of ants is put into a maze with 3 compartments labeled a, b, and c. If the ant is in compartment a, an hour later, there is a 20% chance it will go to compartment b, and a 40% change it will go to compartment c. If it is in compartment b, an hour later, there is a 10% chance it will go to compartment a, and a 30% chance it will go to compartment c. If it is in compartment c, an hour later, there is a 50% chance it will go to compartment a, and a 20% chance it will go to compartment b.

   Suppose 100 ants has been placed in compartment a. The state vectors for the Markov chain are $\mathbf{x}_n = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$, where a_n, b_n, and c_n is the number of ants in compartment a, b, and c, respectively, after n hours. The relationship between the state vectors is given by $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$, where $\mathbf{A}$ is the transition probability matrix. The *initial state vector* is then $\mathbf{x}_0 = \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix}$.

   (a) Find the transition probability matrix $\mathbf{A}$. Show that it is a regular stochastic matrix.

   (b) Find the number of ants in each compartment after 3 hours.

   (c) By diagonalizing $\mathbf{A}$, find the number of ants in each compartment after 3 hours.

(d) **(MATLAB)** We can use MATLAB to diagonalize the matrix $\mathbf{A}$. Type

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>> [P D]=eig(A)
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The matrix $\mathbf{P}$ will be an invertible matrix, and $\mathbf{D}$ will be a diagonal matrix. Compare the answer with what you have obtained in (b). Explain the differences, if there is any.

(e) In the long run (assuming no ants died), where will the majority of the ants be?

(f) A vector $\mathbf{v}$ is an *equilibrium distribution* of $\mathbf{P}$ if it is an eigenvector associated to 1. Suppose the initial state vector is $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. What is the population distribution in the long run (assuming no ants died)? How is this related to the equilibrium distribution?

(g) A *non-equilibrium distribution* eigenvector $\mathbf{v}$ of $\mathbf{P}$ is an eigenvector that is not associated to 1. Is it possible to have an initial state vector of $\mathbf{A}$ to be a non-equilibrium distribution eigenvector (**Hint:** See question 6(c))?

3. (a) Show that the only diagonalizable nilpotent matrix is the zero matrix. (**Hint:** See Tutorial 9 question 7(d))

   (b) Show that the only diagonalizable matrix with 1 eigenvalue λ is the scalar matrix $\lambda\mathbf{I}$.

4. (a) Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. Is $\mathbf{A}$ diagonalizable? Is it invertible?

   (b) By diagonalizing $\mathbf{A} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$, find a matrix $\mathbf{B}$ such that $\mathbf{B}^2 = \mathbf{A}$.

5. For each of the following symmetric matrices $\mathbf{A}$, find an orthogonal matrix $\mathbf{P}$ that orthogonally diagonalizes $\mathbf{A}$.

   (a) $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

   (b) $\mathbf{A} = \begin{pmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{pmatrix}$.

   (c) $\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 1 \end{pmatrix}$.

Supplementary Problems

6. (Stochastic matrices)
   Recall that a stochastic matrix is square matrix $\mathbf{P} = (p_{ij})$ such that the sum of

each column is equal to 1 (see Tutorial 3 question 6)

$$p_{1j} + p_{2j} + \cdots + p_{nj} = 1 \text{ for all } j = 1, ..., n.$$

It is called *regular* if all its entries are non-negative. Let $\mathbf{P}$ be a regular stochastic matrix.

(a) Show that $\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ is always an eigenvector of $\mathbf{P}^T$ associated with the eigenvalue 1. This shows that 1 is always an eigenvalue of a stochastic matrix (see Tutorial 3 question 6(b)).

(b) Show that if λ is an eigenvalue of $\mathbf{P}$, then $|\lambda| \leq 1$. (**Hint:** Pick an eigenvector $\mathbf{v}$ of $\mathbf{P}^T$ associated with λ. Let $k \in \{1, 2, ..., n\}$ be a coordinate of $\mathbf{v}$ with the maximum absolute value, $|v_k| \geq |v_i|$ for all $i = 1, ..., n$. Consider the k-th coordinate of the equation $\mathbf{P}^T\mathbf{v} = \lambda\mathbf{v}^T$.

(c) A vector $\mathbf{v}$ is an *equilibrium distribution* of $\mathbf{P}$ if it is an eigenvector associated to 1. For any vector $\mathbf{v}$, let $\mathbf{v}^{(k)} = \mathbf{P}^k\mathbf{v}$. Show that if $\mathbf{v}$ is an eigenvector of $\mathbf{P}$ that is not an equilibrium distribution, than $\mathbf{v}^{(k)} \to 0$ as $k \to \infty$.

7. **(Application)** I have a supply of seven kind of tiles:

(1) 1×1 red-colored (square) tiles;

(2) 1×2 blue-colored (rectangular) tiles;

(3) 1×2 green-colored (rectangular) tiles;

(4) 1×2 purple-colored (rectangular) tiles;

(5) 1×2 silver-colored (rectangular) tiles;

(6) 1×2 orange-colored (rectangular) tiles;

(7) 1×2 yellow-colored (rectanguakr) tiles.

We represent each red tile by $(1R)$ (R for red, 1 since it is 1×1), each blue tile by $(2B)$ (B for blue, 2 since it is 1×2) and each green tile by $(2G)$ (G for green, 2 since it is 1×2). Similarly for the other colors $(2P)$, $(2S)$, $(2O)$ and $(2Y)$.

I intend to tile a pavement that is $1 \times n$ units long and would like to know how many ways are there to tile the entire pavement with the colored tiles. Let b_n represent the number of differnent ways to tile a $1 \times n$ pavement. For example, $b_1 = 1$ since I can only tile it using $(1R)$.

(a) The value of b_2 is 7. Write down all the 7 ways of tiling a 1×2 pavement.

(b) Determine the value of b_3.

(c) Observe that if we want to tile a $1 \times n$ units long pavement, we could choose to lay the 1×1 red tile first, than any of the choice of $1 \times n - 1$ tiles arrangements, or to lay any of the choice of 1×2 tiles and then any of the choice of $1 \times n - 2$ tiles arrangements. Write down a linear recurrence relation involving 3 consecutive terms (b_n, b_{n-1} and b_{n-2}) in the sequence (b_n).

(d) Solve the linear recurrence relation you obtained in part (c) and use it to find the number of ways to tile a 1×100 pavement.

8. Let $\mathbf{A}$ be a symmetric matrix. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors of $\mathbf{A}$ associated to the eigenvalues λ_1 and λ_2, respectively. Suppose $\lambda_1 \neq \lambda_2$. Show that v_1 and v_2 are orthogonal. This shows that the eigenspaces of a symmetric matrix are orthogonal to each other. (See Tutorial 9 question 8.)