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1. a) Given $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 3 & 1 & 1 & -1 \\ 2 & 1 & 0 & 1 \\ a & b & c & d \end{bmatrix}$ is singular, its determinant $= 0$

By cofactor expansion along the last row

$$-a\begin{vmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} + b\begin{vmatrix} 1 & -1 & 1 \\ 3 & 1 & -1 \\ 2 & 0 & 1 \end{vmatrix} - c\begin{vmatrix} 1 & 0 & 1 \\ 3 & 1 & -1 \\ 2 & 1 & 1 \end{vmatrix} + d\begin{vmatrix} 1 & 0 & -1 \\ 3 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 0$$

$$-a(1) + b(4) - c(3) + d(-2) = 0$$

$$-a + 4b - 3c - 2d = 0$$

$$\left.\begin{aligned} a &= 4b - 3c - 2d \\ b &= b \\ c &= c \\ d &= d \end{aligned}\right\} \quad \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = b\begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$V = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = b\begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; b, c, d \in \mathbb{R} \right\}$$

V is a subspace of $\mathbb{R}^4$ spanned by $\underline{\left\{ \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}}$

$\boxed{\text{Alternatively}}$, V contains origin $-(0) + 4(0) - 3(0) - 2(0) = 0$ //

and is closed under linear combination for any $u, v \in V$ and $\alpha, \beta \in \mathbb{R}$

let $u = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{pmatrix}$ and $v = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{pmatrix}$

$$\alpha u + \beta v = \begin{pmatrix} \alpha a_1 + \beta a_2 \\ \alpha b_1 + \beta b_2 \\ \alpha c_1 + \beta c_2 \\ \alpha d_1 + \beta d_2 \end{pmatrix} \Rightarrow -(\alpha a_1 + \beta a_2) + 4(\alpha b_1 + \beta b_2) - 3(\alpha c_1 + \beta c_2) - 2(\alpha d_1 + \beta d_2)$$

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$$= \alpha(-a_1 + 4b_1 - 3c_1 - 2d_1) + \beta(-a_2 + 4b_2 - 3c_2 - 2d_2)$$

$$= \alpha(0) + \beta(0)$$

$$= 0$$

Hence, $\alpha u + \beta v \in V$ for any $u, v \in V$ and $\alpha, \beta \in \mathbb{R}$

V is a subspace of $\mathbb{R}^4$ as it contains the origin and is closed under linear combi

b) From a) $V = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = b\begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \middle| \; b, c, d \in \mathbb{R} \right\}$

V is spanned by $\left\{ \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$\left[\begin{array}{ccc|c} 4 & -3 & -2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_1 + 2R_4]{\substack{R_1 - 4R_2 \\ R_1 + 3R_3}} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{swap rows}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

→ take first 3 col as basis also

since the homogeneous system $Ax = 0$ has only the trivial solution where $A = \begin{bmatrix} 4 & -3 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, the columns are linearly independent

let $T = \left\{ \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

since $\text{span}(T) = V$ and T is linearly independent, T is a basis for V

$V = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = b\begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \middle| \; b, c, d \in \mathbb{R} \right\}$

explicit form

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c) From b) $T = \left\{ \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

To show S is a basis for V, $\text{span}(S) = V = \text{span}(T)$

and S is linearly independent

$\text{span}(T) \subseteq \text{span}(S)$

$$\left[\begin{array}{ccc|c|c|c} 3 & 2 & 0 & 4 & -3 & -2 \\ 5 & 3 & 2 & 1 & 0 & 0 \\ 5 & 4 & 2 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 2 & -5/3 & -2/3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -3 & 8/3 & 5/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

is consistent

$\text{span}(S) \subseteq \text{span}(T)$

$$\left[\begin{array}{ccc|c|c|c} 4 & -3 & -2 & 3 & 2 & 0 \\ 1 & 0 & 0 & 5 & 3 & 2 \\ 0 & 1 & 0 & 5 & 4 & 2 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 5 & 3 & 2 \\ 0 & 1 & 0 & 5 & 4 & 2 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

is consistent

$\Rightarrow \text{span}(S) = \text{span}(T)$

S is linearly independent

$$\text{rref}\begin{bmatrix} 3 & 2 & 0 \\ 5 & 3 & 2 \\ 5 & 4 & 2 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow S \text{ is linearly independent}$$

S is a basis for V

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d)
$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 1 \\ 5 & 3 & 2 & 1 \\ 5 & 4 & 2 & 1 \\ 1 & -1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$(v)_S = \frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

e) To span $\mathbb{R}^4$, we find a $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ such that the ref of $\begin{bmatrix} 3 & 2 & 0 & x_1 \\ 5 & 3 & 2 & x_2 \\ 5 & 4 & 2 & x_3 \\ 1 & -1 & 1 & x_4 \end{bmatrix}$ has no zero rows

$$\begin{bmatrix} 3 & 2 & 0 & x_1 \\ 5 & 3 & 2 & x_2 \\ 5 & 4 & 2 & x_3 \\ 1 & -1 & 1 & x_4 \end{bmatrix} \xrightarrow{R_4 \leftrightarrow R_1} \begin{bmatrix} 1 & -1 & 1 & x_4 \\ 5 & 3 & 2 & x_2 \\ 5 & 4 & 2 & x_3 \\ 3 & 2 & 0 & x_1 \end{bmatrix} \xrightarrow[R_4 - 3R_1]{R_2 - 5R_1,\ R_3 - 5R_1} \begin{bmatrix} 1 & -1 & 1 & x_4 \\ 0 & 8 & -3 & x_2 - 5x_4 \\ 0 & 9 & -3 & x_3 - 5x_4 \\ 0 & 5 & -3 & x_1 - 3x_4 \end{bmatrix}$$

$$\xrightarrow[R_3 - 9R_2,\ R_4 - 5R_2]{\frac{1}{8}R_2} \begin{bmatrix} 1 & -1 & 1 & x_4 \\ 0 & 1 & -3/8 & \frac{x_2 - 5x_4}{8} \\ 0 & 0 & 3/8 & x_3 - \frac{9}{8}x_2 + \frac{5}{8}x_4 \\ 0 & 0 & -9/8 & x_1 - \frac{5}{8}x_2 + \frac{1}{8}x_4 \end{bmatrix} \xrightarrow{R_4 + 3R_3} \begin{bmatrix} 1 & -1 & 1 & x_4 \\ 0 & 1 & -3/8 & \frac{x_2 - 5x_4}{8} \\ 0 & 0 & 3/8 & x_3 - \frac{9}{8}x_2 + \frac{5}{8}x_4 \\ 0 & 0 & 0 & x_1 - 4x_2 + 3x_3 + 2x_4 \end{bmatrix}$$

Any choice such that $x_1 - 4x_2 + 3x_3 + 2x_4 \neq 0$ works

e.g. $x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$S \cup \{x\}$ will span $\mathbb{R}^4$ $\left\{ \begin{pmatrix} 3 \\ 5 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

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2. a) $||v_1 - v_2|| = \left|\left| \begin{pmatrix} 4-(-1) \\ 1-5 \\ -2-3 \\ 5-1 \end{pmatrix} \right|\right|$

$= \sqrt{5^2 + (-4)^2 + (-5)^2 + 4^2}$

$= \sqrt{82}$

b) $||v_3|| = \sqrt{v_3 \cdot v_3} = \sqrt{\begin{pmatrix} 8 \\ 3 \\ 0 \\ -7 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 3 \\ 0 \\ -7 \end{pmatrix}}$

$= \sqrt{64+9+0+49}$

$= \sqrt{122}$

c) $v_1 \cdot v_2 = \begin{pmatrix} 4 \\ 1 \\ -2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ 3 \\ 1 \end{pmatrix} = -4+5-6+5 = 0$ //

$v_1 \cdot v_3 = \begin{pmatrix} 4 \\ 1 \\ -2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 3 \\ 0 \\ -7 \end{pmatrix} = 32+3-0-35 = 0$ //

$v_2 \cdot v_3 = \begin{pmatrix} -1 \\ 5 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 3 \\ 0 \\ -7 \end{pmatrix} = -8+15+0-7 = 0$ //

vectors are pairwise orthogonal hence $\{v_1, v_2, v_3\}$ is an orthogonal set

It is not orthonormal since the vectors are not of unit length / unit vectors

e.g. $||v_3|| = \sqrt{122}$ from b)