MA1508E: LINEAR ALGEBRA FOR ENGINEERING

Lecture 12 Notes

References

1. Elementary Linear Algebra: Application Version, Section 5.3-5.4, Appendix B
2. Linear Algebra with Application, Section 3.5, 6.6, 8.7, Appendix A

7 System of Linear Differential Equations

A $m \times n$ matrix with function entries, or function-valued matrix (with variable t) has the form

$$\mathbf{A}(t) = (a_{ij}(t))_{m\times n},$$

where for each $i = 1, ..., m$, $j = 1, ..., n$, $a_{ij}(t)$ is a function. The domain of the function-valued matrix $\mathbf{A}(t)$ is the intersection of all the domains of the functions $a_{ij}(t)$, $i = 1, ..., m$, $j = 1, ..., n$.

Example. Consider the following function-valued matrix

$$\mathbf{A}(t) = \begin{pmatrix} 1/t & t^2 \\ t & \sqrt{t} \end{pmatrix}.$$

The domain of $\mathbf{A}(t)$ is all the positive t, $t > 0$.

An n-vector with function entries, or function-valued n-vector is a $n \times 1$ matrix with function entries.

Example. $\mathbf{v}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$. Domain of $\mathbf{v}(t)$ is all the real numbers, $t \in \mathbb{R}$.

If there is no need to specify the variable, or if it is clear in the context, we may just use $\mathbf{v}$ to denote a function-valued vector. Similarly for function-valued matrices.

A function-valued vector $v(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_n(t) \end{pmatrix}$ is differentiable if each $v_i(t)$ is differentiable for $i = 1, ..., n$. For a differentiable vector v, the derivative is defined as

$$\mathbf{v}'(t) = \begin{pmatrix} v_1'(t) \\ v_2'(t) \\ \vdots \\ v_n'(t) \end{pmatrix}, \text{ where } v_i'(t) = \frac{d}{dt}v_i(t), i = 1, ..., n.$$

Example. The derivative of $\mathbf{v}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ is

$$\mathbf{v}'(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}.$$

7.1 First Order Linear System of Differential Equations

A first order linear system of differential equations (with variable t) can be written as

$$\begin{cases} y_1'(t) &= a_{11}(t)y_1(t) &+ \cdots &+ a_{1n}(t)y_n(t) &+ g_1(t) \\ y_2'(t) &= a_{21}(t)y_1(t) &+ \cdots &+ a_{2n}(t)y_n(t) &+ g_2(t) \\ &\vdots \\ y_n'(t) &= a_{n1}(t)y_1(t) &+ \cdots &+ a_{nn}(t)y_n(t) &+ g_n(t) \end{cases},$$

where $y_i(t)$, $g_i(t)$, $a_{ij}(t)$ are all functions, for $i, j = 1, ..., n$. The above system is equivalent to the following function-valued matrix equation

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_n'(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & \cdots & a_{2n}(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix},$$

or

$$\mathbf{y}'(t) = \mathbf{A}(t)\mathbf{y}(t) + \mathbf{g}(t),$$

where $\mathbf{A}(t) = (a_{ij}(t))$, $\mathbf{y} = (y_i(t))$, and $\mathbf{g} = (g_i(t))$. Typically, our task is given the functions a_{ij} and g_i, $i, j = 1, ..., n$, solve for the unknown functions $y_i(t)$, $i = 1, ..., n$ such that the above system is satisfied. A differential equation is linear when the unknown functions are acted upon by multiplying by the known functions and adding them up. It is first order if the highest derivative is the first derivative. It is an ordinary differential equation if the derivative is taken with respect to only one variable (as opposed to partial differential equation). A system of first order linear ordinary differential equations (ODE) is a finite collection of first order linear ODE.

A first order linear system of differential equations is homogeneous if $g_i(t) = 0$ for all $i = 1, .., n$, that is, if it has the form

$$\mathbf{y}'(t) = \mathbf{A}(t)\mathbf{y}(t),$$

and it is of constant coefficient if $a_{ij} \in \mathbb{R}$ are constants, that is, $\mathbf{A}$ is a real-valued matrix. So, a first order homogeneous linear system of differential equations with constant coefficient is of the form

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t).$$

Finally, an initial condition for the linear system of differential equations is

$$\mathbf{y}(t_0) = \begin{pmatrix} y_1(t_0) \\ y_2(t_0) \\ \vdots \\ y_n(t_0) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n.$$

Example. 1. $\begin{cases} y_1' &= \cos(t)y_1 &- \sin(t)y_2 &+ t^2 \\ y_2' &= \sin(t)y_1 &+ \cos(t)y_2 &+ 2t \end{cases}$, or

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} t^2 \\ 2t \end{pmatrix}.$$

This is a first order non-homogeneous linear system of differential equations with non-constant coefficient. The associated homogeneous first order linear system of differential equations is $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

2. $\left\{ \begin{array}{rcrcr} y_1' & = & y_1 & + & y_2 \\ y_2' & = & 2y_1 & - & y_2 \end{array} \right.$, or

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

This is a first order homogeneous linear system of differential equations with constant coefficient.

In this module, we will only be discussing techniques in solving homogeneous first order linear system of differential equations with constant coefficients. Throughout the rest of this section, the coefficient matrix $\mathbf{A}$ will be constant.

7.2 Solutions to System of Different Equations

A function-valued vector $\mathbf{x}(t)$ is a solution to the differential system

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{g}(t)$$

if

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{g}(t).$$

If further $\mathbf{y}(t_0) = \mathbf{a} \in \mathbb{R}^n$ is an initial condition for the system, then $\mathbf{x}(t)$ is a solution to the initial value problem if

$$\mathbf{x}(t_0) = \mathbf{a}.$$

Consider the differential equation $\frac{dy}{dt} = \lambda y$. It is known that the function $y = e^{\lambda t}$ is a solution. Consider now the following system of differential equations $y_1' = \lambda_1 y_1$, $y_2' = \lambda_2 y_2$,...,$y_n' = \lambda_n y_n$, or

$$\mathbf{y}' = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \mathbf{y},$$

where $\mathbf{y} = (y_i)$. It has a solution $y_1 = e^{\lambda_1 t}$, $y_2 = e^{\lambda_2 t}$,..., $y_n = e^{\lambda_n t}$. That is

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} e^{\lambda_1 t}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} e^{\lambda_2 t}, ..., \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} e^{\lambda_n t}$$

are solutions to the system of differential equation. Notice that λ_i are the eigenvalues of the matrix $\mathbf{A} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$ with associated eigenvector $\mathbf{e}_i$, the i-th vector in the standard basis. This is also true in general for any first order homogeneous linear system of differential equations with constant coefficient.

Theorem. *Suppose* $\mathbf{v} \in \mathbb{R}^n$ *is an eigenvector associated to the eigenvalue* λ *of a matrix* $\mathbf{A}$. *Then* $\mathbf{v}e^{\lambda t}$ *is a solution to the first order homogeneous linear ODE with constant coefficient*

$$\mathbf{y}' = \mathbf{A}\mathbf{y}.$$

Proof. Since λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{v}$ is an associated eigenvector, we have $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$. Let $\mathbf{y} = \mathbf{v}e^{\lambda t}$. Then

$$\mathbf{y}' = \frac{d}{dt}\mathbf{v}e^{\lambda t} = \lambda\mathbf{v}e^{\lambda t} = \mathbf{A}\mathbf{v}e^{\lambda t} = \mathbf{A}\mathbf{y}.$$

Hence, $\mathbf{y} = \mathbf{v}e^{\lambda t}$ is indeed a solution to the differential system. □

Theorem (Superposition principle)**.** *Suppose* $\mathbf{x}_1(t)$ *and* $\mathbf{x}_2(t)$ *are solutions to* $\mathbf{y}' = \mathbf{A}\mathbf{y}$. *For any* $\alpha, \beta \in \mathbb{R}$,

$$\alpha\mathbf{x}_1(t) + \beta\mathbf{x}_2(t)$$

is also a solution to the system of differential equations. That is, linear combinations of solutions to a first order homogeneous linear system of differential equations with constant coefficient is a solution.

A function-valued vector $\mathbf{v}(t)$ is zero if it is the constant $\mathbf{0}$ vector, that is, for every t in its domain, $\mathbf{v}(t) = \mathbf{0} \in \mathbb{R}^n$. A set $\{\mathbf{x}_1(t), \mathbf{x}_2(t), ..., \mathbf{x}_k(t)\}$ of function-valued vector is linearly independent if whenever $c_1, c_2, ..., c_k \in \mathbb{R}$ are real numbers such that

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_k\mathbf{x}_k(t) = \mathbf{0}$$

for all t in the domain, necessarily $c_1 = c_2 = \cdots = c_k = 0$. That is, the only linear combination to obtain the constant zero is the trivial one. Otherwise, we say that it is linearly dependent.

Let $S = \{\mathbf{x}_1(t), \mathbf{x}_2(t), ..., \mathbf{x}_n(t)\}$ be a set containing n functioned-valued vectors with n coordinates. Define the Wronskian of S to be

$$W(\mathbf{x}_1(t), \mathbf{x}_2(t), ..., \mathbf{x}_n(t)) = \det(\begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \cdots & \mathbf{x}_n(t) \end{pmatrix}).$$

This is a real-valued function. Then $S = \{\mathbf{x}_1(t), \mathbf{x}_2(t), ..., \mathbf{x}_n(t)\}$ is linearly independent if the Wronskian is not the constant zero function, $W(\mathbf{x}_1(t), \mathbf{x}_2(t), ..., \mathbf{x}_n(t)) \neq 0$ (as functions).

Example.

1. Let $S = \left\{ \begin{pmatrix} \cos(t) \\ t \end{pmatrix}, \begin{pmatrix} \sin(t) \\ t \end{pmatrix} \right\}$. Then the Wronskian is

$$W\left(\begin{pmatrix} \cos(t) \\ t \end{pmatrix}, \begin{pmatrix} \sin(t) \\ t \end{pmatrix} \right) = \begin{vmatrix} \cos(t) & \sin(t) \\ t & t \end{vmatrix} = t(\cos(t) - \sin(t)),$$

which is not the constant zero function. Hence, S is linearly independent.

2. Let $S = \left\{ \begin{pmatrix} e^t \\ 2e^t \\ -e^t \end{pmatrix}, \begin{pmatrix} -e^t \\ 0 \\ e^t \end{pmatrix}, \begin{pmatrix} e^{2t} \\ e^{2t} \\ 0 \end{pmatrix} \right\}$. Then the Wronskian is

$$W\left(\begin{pmatrix} e^t \\ 2e^t \\ -e^t \end{pmatrix}, \begin{pmatrix} -e^t \\ 0 \\ e^t \end{pmatrix}, \begin{pmatrix} e^{2t} \\ e^{2t} \\ 0 \end{pmatrix} \right) = \begin{vmatrix} e^t & -e^t & e^{2t} \\ 2e^t & 0 & e^{2t} \\ -e^t & e^t & 0 \end{vmatrix} = (e^t)(e^t)(e^{2t}) \begin{vmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix} = 2e^{4t},$$

which is not the constant zero function. Hence, S is linearly independent. Note that we may factorize out the common factors in the columns in computing the determinant.

Remark. Warning: A set $S = \{\mathbf{x}_1(t), \mathbf{x}_2(t), ..., \mathbf{x}_n(t)\}$ of function-valued vectors can be linearly independent even though the Wronskian is the constant zero function. For example, consider

$$S = \left\{ \begin{pmatrix} t \\ t \end{pmatrix}, \begin{pmatrix} t^2 \\ t^2 \end{pmatrix} \right\}.$$

Then S is linearly independent since

$$c_1 \begin{pmatrix} t \\ t \end{pmatrix} + c_2 \begin{pmatrix} t^2 \\ t^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ for all } t \Leftrightarrow t(c_1 + c_2 t) = 0 \text{ for all } t \Leftrightarrow c_1 = 0 = c_2.$$

But the Wronskian is

$$W\left(\begin{pmatrix} t \\ t \end{pmatrix}, \begin{pmatrix} t^2 \\ t^2 \end{pmatrix} \right) = \begin{vmatrix} t & t^2 \\ t & t^2 \end{vmatrix} = t^3 - t^3 = 0$$

for all t.

A set $S = \{\mathbf{x}_1(t), \mathbf{x}_2(t), ..., \mathbf{x}_n(t)\}$ of solutions to the differential system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ is called a <u>fundamental set of solutions</u> if its Wronskian is nonzero. A <u>general solution</u> to $\mathbf{y}' = \mathbf{A}\mathbf{y}$ captures all possible linear combinations of the function-valued vectors in a fundamental set of solutions, that is, it is of the form

$$\mathbf{x} = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t), c_1, c_2, ..., c_n \in \mathbb{R}.$$

Suppose $\mathbf{y}(t_0) = \mathbf{a} \in \mathbb{R}^n$ is an initial condition for the system, then we are able to find (the) solution to the initial value problem by solving for $c_1, c_2, ..., c_n$ in the equation

$$c_1\mathbf{x}_1(t_0) + c_2\mathbf{x}_2(t_0) + \cdots + c_n\mathbf{x}_n(t_0) = \mathbf{a}.$$

Theorem. *Suppose* $\mathbf{A}$ *is diagonalizable. Let* $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ *be* n *linearly independent eigenvectors associated to (real) eigenvalues* $\lambda_1, \lambda_2, ..., \lambda_n$ *(not necessarily distinct). Then*

$$\left\{ \mathbf{v}_1 e^{\lambda_1 t}, \mathbf{v}_2 e^{\lambda_2 t}, ..., \mathbf{v}_n e^{\lambda_n t} \right\}$$

is a fundamental set of solutions to the first order homogeneous system of differential equations $\mathbf{y}' = \mathbf{A}\mathbf{y}$ *with constant coefficients, and*

$$\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{v}_n e^{\lambda_n t}$$

is a general solution.

Proof. We have already shown that if $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ associated to eigenvalue λ, then $\mathbf{v}e^{\lambda t}$ is a solution to the differential system $\mathbf{y}' = \mathbf{A}\mathbf{y}$. Hence, $\mathbf{v}_i e^{\lambda_i t}$ is a solution for all $i = 1, ..., n$. The Wronskian of the set is

$$W(\mathbf{v}_1 e^{\lambda_1 t}, \mathbf{v}_2 e^{\lambda_2 t}, ..., \mathbf{v}_n e^{\lambda_n t}) = e^{\lambda_1 t} e^{\lambda_2 t} \cdots e^{\lambda_n t} \begin{vmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{vmatrix} \neq 0$$

since $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is linearly independent, and thus $\begin{vmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{vmatrix} \neq 0$ □

Example. Solve the differential system

$$\begin{cases} y_1' &= y_1 & \\ y_2' &= y_1 &+ 2y_2 \end{cases}$$

with initial conditions: $y_1(0) = 1$, $y_2(0) = 1$. Write

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

The eigenvalues of $\mathbf{A}$ are $\lambda = 1, 2$. Since $\mathbf{A}$ has 2 distinct eigenvalues, it is diagonalizable. We will now compute the eigenvectors.

$\lambda = 1$: $\mathbf{I} - \mathbf{A} = \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}$. So $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an associated eigenvector.

$\lambda = 2$: $2\mathbf{I} - \mathbf{A} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$. So $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an associated eigenvector.

By the theorem above,

$$\left\{ e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}, e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

is a fundamental set of solutions. Indeed, its Wronskian is

$$W\left(\begin{pmatrix} e^t \\ -e^t \end{pmatrix}, \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix} \right) = \begin{vmatrix} e^t & 0 \\ -e^t & e^{2t} \end{vmatrix} = e^{3t} \neq 0.$$

So a general solution of the differential system is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

that is, $y_1 = c_1 e^t$ and $y_2 = -c_1 e^t + c_2 e^{2t}$.

We will now find the solution satisfying the initial condition. Substituting the initial conditions, we have

$$\begin{matrix} 1 &=& y_1(0) &=& c_1 & & \\ 1 &=& y_2(0) &=& -c_1 &+& c_2 \end{matrix}$$

Solving the system, we get

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ -1 & 1 & 1 \end{array} \right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array} \right),$$

that is, $c_1 = 1$ and $c_2 = 2$. So the (unique) solution to the initial value problem is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} e^t \\ -e^t + 2e^{2t} \end{pmatrix},$$

or $y_1 = e^t$ and $y_2 = 2e^{2t} - e^t$.

Recall that there are two obstructions to diagonalization. Roughly speaking, either there are not enough (real) eigenvalues, or not enough linearly independent eigenvectors. To be precise, either

(i) the characteristic polynomial does not factorize into real linear factors,

(ii) the geometric multiplicity of an eigenvalue is strictly less than the algebraic multiplicity.

The solutions are

(i) allow complex eigenvalues,

(ii) use generalized eigenvectors.

In the next few sections, we will discuss the two solutions above.

7.3 Complex Eigenvalues and Eigenvetors

Readers may refer to the appendix for an introduction to complex numbers.

An n-complex vector is a collection of n ordered complex numbers,

$$\mathbf{v} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}, z_j \in \mathbb{C} \text{ for all } j = 1, ..., n.$$

The collection of all n-complex vectors is denoted as $\mathbb{C}^n$. Given any complex vector $\mathbf{v} \in \mathbb{C}^n$, we can split it into its real and imaginary parts,

$$\mathbf{v} = \begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ \vdots \\ x_n + iy_n \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + i \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = Re(\mathbf{v}) + iIm(\mathbf{v}),$$

where $Re(\mathbf{v}), Im(\mathbf{v}) \in \mathbb{R}^n$.

Let $\mathbf{A}$ be an order n square matrix. A complex number $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathbf{A}$ if there is a nonzero vector $\mathbf{v} \in \mathbb{C}^n$, $\mathbf{v} \neq \mathbf{0}$ such that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}.$$

In this case, $\mathbf{v}$ is called an eigenvector associated to λ.

Theorem. *Let $\mathbf{A}$ be an order n square matrix with real entries.*

(i) Then complex eigenvalues of $\mathbf{A}$ comes in conjugate pairs, that is, if $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathbf{A}$, then $\overline{\lambda}$ is also an eigenvalue of $\mathbf{A}$.

(ii) If $\mathbf{v} \in \mathbb{C}^n$ is an eigenvector associated to eigenvalue λ, then $\overline{\mathbf{v}}$ is an eigenvectors associated to eigenvalue $\overline{\lambda}$.

Proof. Suppose $\lambda \in \mathbb{C}$ is a complex eigenvalue of $\mathbf{A}$. Let $\mathbf{v} \in \mathbb{C}^n$ be an eigenvector associated to eigenvalue λ, that is $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$. Then since $\mathbf{A}$ has real entries, $\overline{\mathbf{A}} = \mathbf{A}$ and so

$$\mathbf{A}\overline{\mathbf{v}} = \overline{\mathbf{A}\mathbf{v}} = \overline{\lambda\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}.$$

Since $\overline{\mathbf{v}} \neq \mathbf{0}$, $\overline{\mathbf{v}}$ is a witness to $\overline{\lambda}$ being and eigenvalue of $\mathbf{A}$. $\square$

In words, this means that complex eigenvalues and complex eigenvectors come in conjugate pairs.

The algorithm to find the complex eigenvalues and eigenvectors are analogous to that of their real counterparts.

Example. Let $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The characteristic polynomial is

$$\begin{vmatrix} x & 1 \\ -1 & x \end{vmatrix} = x^2 + 1 = (x+i)(x-i).$$

The matrix $\mathbf{A}$ does not have real eigenvalues, its complex eigenvalues are $\lambda = \pm i$. The eigenvalues are indeed conjugate pairs. We will now compute the eigenvectors. Since the eigenvectors come in conjugate pairs, suffice to compute an eigenvector associated to one of the eigenvalue, say $\lambda = i$.

$$i\mathbf{I} - \mathbf{A} = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}$$

Since the two rows are necessarily linearly dependent, because $(i\mathbf{I} - \mathbf{A})$ cannot be invertible, we may use any of the rows to read of a general solution of the homogeneous system. Reading off the second row, we obtain that $\mathbf{v} = \begin{pmatrix} i \\ 1 \end{pmatrix}$ is an eigenvector associated to eigenvalue i. Indeed,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix} = i \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Then $\overline{\mathbf{v}} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ is an eigenvector associated to eigenvalue $\lambda = \overline{i} = -i$. Indeed,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -i \end{pmatrix} = -i \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Consider the first order homogeneous system of differential equations $\mathbf{y}' = \mathbf{A}\mathbf{y}$ with constant coefficient. Suppose $\lambda \in \mathbb{C}$ is a complex eigenvalue of $\mathbf{A}$ associated to complex eigenvector $\mathbf{v} \in \mathbb{C}^n$. Decompose λ and $\mathbf{v}$ into their real and imaginary parts,

$$\lambda = \lambda_r + i\lambda_i, \mathbf{v} = \mathbf{v}_r + i\mathbf{v}_i, \lambda_r, \lambda_i \in \mathbb{R}, \mathbf{v}_r, \mathbf{v}_i \in \mathbb{R}^n.$$

Then

$$\begin{aligned} e^{\lambda t}\mathbf{v} &= e^{(\lambda_r + i\lambda_i)t}(\mathbf{v}_r + i\mathbf{v}_i) \\ &= e^{\lambda_r t}(\cos(\lambda_i t) + i\sin(\lambda_i t))(\mathbf{v}_r + i\mathbf{v}_i) \\ &= e^{\lambda_r t}(\cos(\lambda_i t)\mathbf{v}_r - \sin(\lambda_i t)\mathbf{v}_i) + ie^{\lambda_r t}(\sin(\lambda_i t)\mathbf{v}_r + \cos(\lambda_i t)\mathbf{v}_i) \\ &= \mathbf{x}_r(t) + i\mathbf{x}_i(t), \end{aligned}$$

where

$$Re(e^{\lambda t}\mathbf{v}) = \mathbf{x}_r(t) = e^{\lambda_r t}(\cos(\lambda_i t)\mathbf{v}_r - \sin(\lambda_i t)\mathbf{v}_i)$$

and

$$Im(e^{\lambda t}\mathbf{v}) = \mathbf{x}_i(t) = e^{\lambda_r t}(\sin(\lambda_i t)\mathbf{v}_r + \cos(\lambda_i t)\mathbf{v}_i)$$

are real function-valued vectors.

Theorem. *Both $\mathbf{x}_r(t)$ and $\mathbf{x}_i(t)$ are linearly independent real solutions to the first order homogeneous system of differential equations $\mathbf{y}' = \mathbf{A}\mathbf{y}$ with constant coefficient.*

Corollary. *Suppose $\mathbf{A}$ is an order 2 square matrix and $\lambda \in \mathbb{C}$ is a nonreal complex eigenvalue $\lambda \notin \mathbb{R}$. Let $\mathbf{v} \in \mathbb{C}^2$ be an associated eigenvector. Write $e^{\lambda t}\mathbf{v} = \mathbf{x}_r(t) + i\mathbf{x}_i(t)$. Then $\{\mathbf{x}_r(t), \mathbf{x}_i(t)\}$ is a fundamental set of solutions for the first order homogeneous system of differential equations $\mathbf{y}' = \mathbf{A}\mathbf{y}$ with constant coefficients.*

Example. Solve the following differential system

$$\begin{aligned} y_1 &= \quad - y_2 \\ y_2 &= y_1 \end{aligned}$$

with initial conditions $y_1(0) = 1 = y_2(0)$.

Let $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We have computed that $\lambda = i$ is an eigenvalue with eigenvector $\begin{pmatrix} i \\ 1 \end{pmatrix}$. Then we say that λ

$$\lambda_r = 0, \lambda_i = 1, \mathbf{v}_r = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{v}_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and so

$$\begin{aligned} \mathbf{x}_r(t) &= e^{0t}(\cos(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sin(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} \\ \mathbf{x}_i(t) &= e^{0t}(\sin(t)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \cos(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \end{aligned}$$

are solutions to the differential equations. The Wronskian is

$$\begin{vmatrix} -\sin(t) & \cos(t) \\ \cos(t) & \sin(t) \end{vmatrix} = -\sin^2(t) - \cos^2(t) \neq 0$$

for any $t \in \mathbb{R}$. Hence

$$\left\{ \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}, \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \right\}$$

is a fundamental set of solution, and a general solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} + c_2 \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}.$$

Now substituting the initial conditions into the general solution, we have

$$1 = y_1(0) = c_2, 1 = y_2(0) = c_1.$$

Therefore, the solution to the initial value problem is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} + \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix},$$

or

$$\begin{aligned} y_1 &= \cos(t) - \sin(t) \\ y_2 &= \cos(t) + \sin(t). \end{aligned}$$

7.4 Repeated Eigenvalue and Generalized Eigenvector

In this section, we will restriction our attention only to order 2 square matrices. Readers may refer to the appendix for the discussion for general square matrices.

Suppose $\mathbf{A}$ is an order 2 square matrix and λ is an eigenvalue of $\mathbf{A}$ with algebraic multiplicity $r_\lambda = 2$ and geometric multiplicity $\dim(E_\lambda) = 1$. Then we say that λ is a repeated eigenvalue. Let $\mathbf{v}_1 \in \mathbb{R}^2$ be an eigenvector assocaited to λ. A vector $\mathbf{v}_2 \in \mathbb{R}^2$ is a generalized eigenvector if it is a solution to

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{v}_1.$$

Let

$$\begin{aligned} \mathbf{x}_1(t) &= e^{\lambda t}\mathbf{v}_1 \\ \mathbf{x}_2(t) &= e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2). \end{aligned}$$

Then $\{\mathbf{x}_1(t), \mathbf{x}_2\}$ is a linearly independent set of real solutions to the differential equations $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

Remark. Note that when we are computing eigenvectors, it does not matter if we are solving for $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ or $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$. However, in solving for generalized eigenvectors, we are looking for solutions to $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{v}_1$. If we exchange the position of $\lambda\mathbf{I}$ and $\mathbf{A}$, then we are solving for $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = -\mathbf{v}_1$ instead. Readers may refer to the appendix for details.

Theorem. *Let $\mathbf{A}$ be an order 2 square matrix and λ be a repeated eigenvalue. Suppose $\mathbf{v}_1$ is an eigenvector and $\mathbf{v}_2$ a generalized eigenvector associated to λ. Then $\{\mathbf{x}_1(t), \mathbf{x}_2(t)\}$ is a fundamental set of solutions for the first order homogeneous system of differential equations $\mathbf{y}' = \mathbf{A}\mathbf{y}$ with constant coefficients.*

Example. Solve the differential system

$$\begin{aligned} y_1' &= -y_1 + y_2 \\ y_2' &= -4y_1 + 3y_2 \end{aligned}$$

with initial conditions $y_1(0) = 2$, $y_2(0) = 1$.

Let $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}$. Compute the eigenvalues.

$$\begin{vmatrix} x+1 & -1 \\ 4 & x-3 \end{vmatrix} = (x-1)^2.$$

The eigenvalue is $\lambda = 1$ with algebraic multiplicity $r_1 = 2$. Compute the eigenspace.

$$\mathbf{I} - \mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix},$$

reading off the first row, we have $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. This shows that the geometric multiplicity is 1. We will need to compute the generalized eigenvector, that is, to solve for $(\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{v}_1$.

$$\left(\begin{array}{cc|c} -2 & 1 & 1 \\ -4 & 2 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -1/2 & -1/2 \\ 0 & 0 & 0 \end{array}\right).$$

A general solution is $\begin{pmatrix}(1/2)(s-1)\\ s\end{pmatrix}$, $s \in \mathbb{R}$. Choose $s = 1$, then $\mathbf{v}_2 = \begin{pmatrix}0\\1\end{pmatrix}$ is a generalized eigenvector. So

$$\begin{aligned}\mathbf{x}_1(t) &= e^t\begin{pmatrix}1\\2\end{pmatrix}\\ \mathbf{x}_2(t) &= e^t\left(t\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\right)\end{aligned}$$

are solutions to the differential equations. The Wronskian is

$$\begin{vmatrix}e^2 & t\\ 2e^t & e^t(2t+1)\end{vmatrix} = e^{2t}\begin{vmatrix}1 & t\\ 2 & 2t+1\end{vmatrix} = e^{2t} \neq 0$$

for any $t \in \mathbb{R}$. Hence,

$$\left\{e^t\begin{pmatrix}1\\2\end{pmatrix}, e^t\left(t\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\right)\right\}$$

is a fundamental set of solutions, and a general solution is

$$\begin{pmatrix}y_1\\y_2\end{pmatrix} = c_1e^t\begin{pmatrix}1\\2\end{pmatrix} + c_2e^t\left(t\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\right), c_1, c_2 \in \mathbb{R}.$$

Now substitute the initial condition into the general solution, we have

$$2 = y_1(0) = c_1, 1 = y_2(0) = 2c_1 + c_2$$

and so $c_1 = 0$ and $c_2 = -1$. Therefore, the solution to the initial value problem is

$$\begin{pmatrix}y_1\\y_2\end{pmatrix} = 2e^t\begin{pmatrix}1\\2\end{pmatrix} - 3e^t\left(t\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\right) = e^t\begin{pmatrix}2-3t\\1-6t\end{pmatrix},$$

or

$$\begin{aligned}y_1 &= e^t(2-3t)\\ y_2 &= e^t(1-6t)\end{aligned}$$

Remark. Note that when solving for generalized eigenvector, we will get a general solution. In the example above, the general solution is $\begin{pmatrix}(1/2)(s-1)\\ s\end{pmatrix}$. In finding the fundamental set of solutions or general solution for the differential system, we are not to include the parameter s; the function-valued vectors are functions in variable t. We must always pick a particular solution for the generalized eigenvector when forming the fundamental set of solutions or general solution to the differential system.

Appendix to Lecture 12

Solutions to First order homogeneous differential system with constant coefficient

Theorem. *Suppose* $\mathbf{v} \in \mathbb{R}^n$ *is an eigenvector associated to the eigenvalue* λ *of a matrix* $\mathbf{A}$. *Then* $\mathbf{v}e^{\lambda t}$ *is a solution to the first order homogeneous linear ODE with constant coefficient*

$$\mathbf{y}' = \mathbf{A}\mathbf{y}.$$

Proof. Since λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{v}$ is an associated eigenvector, we have $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$. Let $\mathbf{y} = \mathbf{v}e^{\lambda t}$. Then

$$\mathbf{y}' = \frac{d}{dt}\mathbf{v}e^{\lambda t} = \lambda\mathbf{v}e^{\lambda t} = \mathbf{A}\mathbf{v}e^{\lambda t} = \mathbf{A}\mathbf{y}.$$

Hence, $\mathbf{y} = \mathbf{v}e^{\lambda t}$ is indeed a solution to the differential system. □

Theorem (Superposition principle)**.** *Suppose* $\mathbf{x}_1(t)$ *and* $\mathbf{x}_2(t)$ *are solutions to* $\mathbf{y}' = \mathbf{A}\mathbf{y}$. *For any* $\alpha, \beta \in \mathbb{R}$,

$$\alpha\mathbf{x}_1(t) + \beta\mathbf{x}_2(t)$$

is also a solution to the system of differential equations. That is, linear combinations of solutions to a first order homogeneous linear system of differential equations with constant coefficient is a solution.

Proof.

$$\begin{aligned}
\frac{d}{dt}(\alpha\mathbf{x}_1(t) + \beta\mathbf{x}_2(t)) &= \alpha\frac{d}{dt}\mathbf{x}_1(t) + \beta\frac{d}{dt}\mathbf{x}_2(t) \\
&= \alpha\mathbf{A}\mathbf{x}_1(t) + \beta\mathbf{A}\mathbf{x}_2(t) \\
&= \mathbf{A}(\alpha\mathbf{x}_1(t) + \beta\mathbf{x}_2(t)).
\end{aligned}$$

□

Introduction to Complex Numbers

Define the imaginary number i to be such that $i^2 = -1$, or $i = \sqrt{-1}$. It is not a real number, $i \notin \mathbb{R}$. A <u>complex number</u> can be written as

$$z = x + iy,$$

for some $x, y \in \mathbb{R}$. The collection of all complex numbers is denoted as

$$\mathbb{C} = \{\ z = x + iy \mid x, y \in \mathbb{R}\ \}.$$

We can think of the set of real numbers as a subset of the set of all complex numbers, $\mathbb{R} \subseteq \mathbb{C}$, via

$$x \mapsto x + i0.$$

For a complex number $z = x + iy$, $Re(z) = x$ is called the <u>real part</u> and $Im(z) = y$ is called the <u>imaginary part</u>. A complex number z is <u>purely imaginary</u> if $z = 0 + iy$, that is $Re(z) = 0$, denoted as $z \in i\mathbb{R}$; it is non-real if the imaginary part is nonzero, $z = x + iy \in \mathbb{C}\backslash\mathbb{R}$, $y \neq 0$.

Here are some operations for complex numbers

(i) Addition: $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

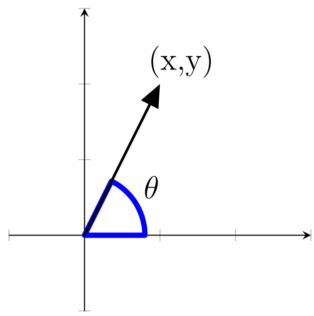
(ii) Multiplication: $(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

(iii) Conjugation: $\overline{x + iy} = x - iy$.

(iv) Norm: $|x + iy| = \sqrt{x^2 + y^2} = \sqrt{(x + iy)\overline{(x + iy)}}$

Exercise: Write $\overline{i(1 + 2i)}$ as $x + iy$ for some $x, y \in \mathbb{R}$.

Every complex number $z = x + iy$ can be represented by a point (x, y) on the xy-plane, $\mathbb{R}^2$. The argument of a complex number $z = x + iy$ is the value θ such that $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$. Equivalently, if $z = x + iy$ corresponds to (x, y) on $\mathbb{R}^2$, then the argument θ of z is such that $(x, y) = (r\cos\theta, r\sin\theta)$. The expression $z = r(\cos\theta + i\sin\theta)$ is called the polar form of z.



Recall that we have the Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Thus every complex number z can be written as $z = re^{i\theta}$, where $r = |z|$ and θ is the argument of z. This is called the exponential form of z.

Complex Eigenvalues and Eigenvectors

Recall that if λ is an eigenvalue of a matrix $\mathbf{A}$ with associated eigenvector $\mathbf{v}$, then $e^{\lambda t}\mathbf{v}$ is a solution to the first order homogeneous system of differential equations $\mathbf{y}' = \mathbf{A}\mathbf{y}$ with constant coefficient. This is true also for complex eigenvalues and eigenvectors. The proof is similar to that for the theorem for real eigenvalues and eigenvectors.

Theorem. *Suppose $\lambda \in \mathbb{C}$ is a complex eigenvalue of $\mathbf{A}$ and $\mathbf{v} \in \mathbb{C}^n$ is an associated eigenvector. Then $e^{\lambda t}\mathbf{v}$ is a complex solution to the first order homogeneous system of differential equations $\mathbf{y}' = \mathbf{A}\mathbf{y}$ with constant coefficient.*

It turns out that the real and imaginary parts of the complex solutions are real solution to the differential system. Write $Re(e^{\lambda t}\mathbf{v}) = \mathbf{x}_r(t)$ and $Im(e^{\lambda t}\mathbf{v}) = \mathbf{x}_i(t)$. Then we have the following theorem.

Theorem. *Both $\mathbf{x}_r(t)$ and $\mathbf{x}_i(t)$ are linearly independent real solutions to the first order homogeneous system of differential equations $\mathbf{y}' = \mathbf{A}\mathbf{y}$ with constant coefficient.*

Proof. Since $e^{\lambda t}\mathbf{v}$ is a (complex) solution to the differential system,

$$\begin{aligned}\mathbf{A}\mathbf{x}_r(t) + i\mathbf{A}\mathbf{x}_i(t) &= \mathbf{A}(\mathbf{x}_r(t) + i\mathbf{x}_i(t)) = \mathbf{A}(e^{\lambda t}\mathbf{v}) \\ &= \frac{d}{dt}(e^{\lambda t}\mathbf{v}) = \frac{d}{dt}\mathbf{x}_r(t) + i\frac{d}{dt}\mathbf{x}_i(t)\end{aligned}$$

and since $\mathbf{A}$ has real entries and $\mathbf{x}_i(t)$ and $\mathbf{x}_i(t)$ are real function-valued vectors, by compairing the real and imaginary parts, we have

$$\mathbf{A}\mathbf{x}_r(t) = \mathbf{x}_r'(t) \text{ and } \mathbf{A}\mathbf{x}_i(t) = \mathbf{x}_i'(t).$$

Hence, $\mathbf{x}_r(t)$ and $\mathbf{x}_i(t)$ are real solutions to $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

Next, we need to show that $\{\mathbf{x}_r(t), \mathbf{x}_i(t)\}$ is linearly independent. Suppose $c_1, c_2 \in \mathbb{R}$ are such that

$$c_1 e^{\lambda_r t}(\cos(\lambda_i t)\mathbf{v}_r - \sin(\lambda_i t)\mathbf{v}_i) + c_2 e^{\lambda_r t}(\sin(\lambda_i t)\mathbf{v}_r + \cos(\lambda_i t)\mathbf{v}_i) = \mathbf{0}.$$

Let $t = 0$ and $t = \frac{2\pi}{\lambda_i}$, we have

$$\begin{array}{rcrcl} c_1\mathbf{v}_r & + & c_2\mathbf{v}_i & = & \mathbf{0} \\ -c_1\mathbf{v}_i & + & c_2\mathbf{v}_r & = & \mathbf{0} \end{array}$$

Since it cannnot be that both $\mathbf{v}_r = \mathbf{0} = \mathbf{v}_i$, it is clear that if $\mathbf{v}_r = \mathbf{0}$ or $\mathbf{v}_i = \mathbf{0}$, necessarily $c_1 = c_2 = 0$. If $\mathbf{v}_r$ is a multiple of $\mathbf{v}_i$, then again it is clear that necessarily $c_1 = c_2 = 0$. Otherwise, $\{\mathbf{v}_r, \mathbf{v}_i\}$ is linearly independent, and thus $c_1 = c_2 = 0$. This shows that $\{\mathbf{x}_r(t), \mathbf{x}_i(t)\}$ is linearly independent. $\square$

Now suppose $\lambda \in \mathbb{C}$ is a complex eigenvalue of $\mathbf{A}$ and $\mathbf{v} \in \mathbb{C}^n$ is a complex eigenvector. From the complex solution $e^{\lambda t}\mathbf{v}$ to the differential system $\mathbf{y}' = \mathbf{A}\mathbf{y}$, we obtain the real solutions $Re(e^{\lambda t}\mathbf{v}) = \mathbf{x}_r(t)$ and $Im(e^{\lambda t}\mathbf{v}) = \mathbf{x}_i(t)$. Now consider the conjugate pair of complex eigenvalue and eigenvector $\overline{\lambda}$ and $\overline{\mathbf{v}}$. From the complex solution $e^{\overline{\lambda}t}\overline{\mathbf{v}}$, we obtain another set of real solutions $Re(e^{\overline{\lambda}t}\overline{\mathbf{v}}) = \mathbf{w}_r(t)$ and $Im(e^{\overline{\lambda}t}\overline{\mathbf{v}}) = \mathbf{w}_i(t)$. However, observe that

$$\begin{aligned}\mathbf{w}_r(t) &= Re(e^{\overline{\lambda}t}\overline{\mathbf{v}}) = Re(\overline{e^{\lambda t}\mathbf{v}}) = Re(e^{\lambda t}\mathbf{v}) = \mathbf{x}_r(t) \\ \mathbf{w}_i(t) &= Im(e^{\overline{\lambda}t}\overline{\mathbf{v}}) = Im(\overline{e^{\lambda t}\mathbf{v}}) = -Im(e^{\lambda t}\mathbf{v}) = -\mathbf{x}_i(t)\end{aligned}$$

and thus $\text{span}\{\mathbf{x}_r(t), \mathbf{x}_i(t)\} = \text{span}\{\mathbf{w}_r(t), \mathbf{w}_i(t)\}$. Therefore, it suffice to extract the real solutions from either the pair λ and $\mathbf{v}$ or $\overline{\lambda}$ and $\overline{\mathbf{v}}$.

Generalized Eigenvectors

Let $\mathbf{A}$ be an order n square matrix and λ an eigenvalue of $\mathbf{A}$. A nonzero vector $\mathbf{v} \in \mathbb{R}^n$ is a generalized eigenvector of $\mathbf{A}$ associated to eigenvalue λ if there is a positive integer $k \in \mathbb{Z}$, $k > 0$, such that

$$(\mathbf{A} - \lambda\mathbf{I})^k\mathbf{v} = \mathbf{0}.$$

If furthermore $(\mathbf{A} - \lambda\mathbf{I})^{k-1}\mathbf{v} \neq \mathbf{0}$, we say that the generalized eigenvector has rank k. Note that rank 1 generalized eigenvectors are the usual eigenvectors. Let $\mathbf{v}_k$ be a rank k generalized eigenvector associated to eigenvalue λ. Define inductively

$$\mathbf{v}_i = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_{i+1}$$

for $i = 1, ..., k-1$.

Lemma. $\mathbf{v}_i$ *is a generalized eigenvector of rank* i.

Proof. We will prove be induction on $i \geq 1$. For $i = 1$, $\mathbf{v}_1$ is a usual eigenvector and thus $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$. Also $(\mathbf{A} - \lambda\mathbf{I})^{1-1}\mathbf{v} = \mathbf{I}\mathbf{v} \neq \mathbf{0}$ since $\mathbf{v}$ is an eigenvector. This proves the base case.

Now suppose the statement is true for $i - 1$. Then

$$(\mathbf{A} - \lambda\mathbf{I})^i\mathbf{v}_i = (\mathbf{A} - \lambda\mathbf{I})^{i-1}(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_i = (\mathbf{A} - \lambda\mathbf{I})^{i-1}\mathbf{v}_{i-1} = \mathbf{0}$$

by definition of $\mathbf{v}_{i-1}$and the induction hypothesis. Also

$$(\mathbf{A} - \lambda\mathbf{I})^{i-1}\mathbf{v}_i = (\mathbf{A} - \lambda\mathbf{I})^{i-2}\mathbf{v}_{i-1} \neq \mathbf{0}$$

by the induction hypothesis. Hence, $\mathbf{v}_i$ is indeed a generalized eigenvector of rank i. □

The set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is called a <u>Jordan chain</u>.

Lemma. *The set* $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ *is linearly independent.*

Proof. First observe that $\mathrm{Null}((\mathbf{A} - \lambda\mathbf{I})^i) \subseteq \mathrm{Null}((\mathbf{A} - \lambda\mathbf{I})^{i+1})$. Next, by definition of $\mathbf{v}_i$, we have

$$\mathbf{v}_i \in \mathrm{Null}((\mathbf{A} - \lambda\mathbf{I})^i) \setminus \mathrm{Null}((\mathbf{A} - \lambda\mathbf{I})^{i-1}).$$

This shows that $\mathbf{v}_i \notin \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{i-1}\} \subseteq \mathrm{Null}((\mathbf{A} - \lambda\mathbf{I})^{i-1})$ for all $i = 1, ..., k$. And so $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is linearly independent. □

Let $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ be a Jordan chain of generalized eigenvectors of $\mathbf{A}$ associated to eigenvalue λ. Define

$$\begin{aligned}
\mathbf{x}_1(t) &= e^{\lambda t}\mathbf{v}_1, \\
\mathbf{x}_2(t) &= e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2), \\
\mathbf{x}_3(t) &= e^{\lambda t}\left(\frac{t^2}{2}\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{v}_2\right) \\
&\vdots \\
\mathbf{x}_k(t) &= e^{\lambda t}\left(\frac{t^{k-1}}{(k-1)!}\mathbf{v}_1 + \cdots + \frac{t^2}{2}v_{k-2} + t\mathbf{v}_{k-1} + \mathbf{v}_k\right).
\end{aligned}$$

Lemma. *The set* $\{\mathbf{x}_1(t), \mathbf{x}_2(t), ..., \mathbf{x}_k(t)\}$ *form a linearly independent set of solutions to the differential system* $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

Proof. Write $\mathbf{x}_l(t) = e^{\lambda t}\sum_{i=1}^{l}\frac{t^{l-i}}{(l-i)!}\mathbf{v}_i$ for $l = 1, ..., k$. Then

$$\begin{aligned}
\mathbf{x}_l'(t) &= \lambda e^{\lambda t}\sum_{i=1}^{l}\frac{t^{l-i}}{(l-i)!}\mathbf{v}_i + e^{\lambda t}\sum_{i=1}^{l-1}(l-i)\frac{t^{l-i-1}}{(l-i)!}\mathbf{v}_i \\
&= \lambda e^{\lambda t}\sum_{i=1}^{l}\frac{t^{l-i}}{(l-i)!}\mathbf{v}_i + e^{\lambda t}\sum_{i=1}^{l}\frac{t^{l-i}}{(l-i)!}\mathbf{v}_{i-1},
\end{aligned}$$

where $\mathbf{v}_0 = \mathbf{0}$. Since $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_i = \mathbf{v}_{i-1}$, or $\mathbf{A}\mathbf{v}_i = \mathbf{v}_{i-1} + \lambda\mathbf{v}_i$,

$$\begin{aligned}
\mathbf{A}\mathbf{x}_l(t) &= e^{\lambda t}\sum_{i=1}^{l}\frac{t^{l-i}}{(l-i)!}\mathbf{A}\mathbf{v}_i \\
&= e^{\lambda t}\sum_{i=1}^{l}\frac{t^{l-i}}{(l-i)!}\mathbf{v}_{i-1} + \lambda e^{\lambda t}\sum_{i=1}^{l}\frac{t^{l-i}}{(l-i)!}\mathbf{v}_i \\
&= \mathbf{x}_l'(t).
\end{aligned}$$

This shows that indeed $\mathbf{x}_l(t)$ is a solution, for $l = 1, ..., k$.

Next, we will check that $\{\mathbf{x}_1(t), \mathbf{x}_2(t), ..., \mathbf{x}_k(t)\}$ is indeed linearly independent. Suppose

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_k\mathbf{x}_k(t) = \mathbf{0}.$$

Let $t = 0$, we have $\mathbf{x}_1(0) = \mathbf{v}_1, \mathbf{x}_2(0) = \mathbf{v}_2, ..., \mathbf{x}_k(0) = \mathbf{v}_k$, and since they are linearly independent, necessarily $c_1 = c_2 = \cdots = c_k = 0$. Hence, the set of solutions $\{\mathbf{x}_1(t), \mathbf{x}_2(t), ..., \mathbf{x}_k(t)\}$ is indeed linearly independent. $\square$

Now suppose $\mathbf{A}$ is an order 2 square matrix and λ is an eigenvalue. Then we can at most have a Jordan chain with 2 generalized eigenvectors, they form a linearly independent set. Here $\mathbf{v}_1$ is an eigenvector λ, and $\mathbf{v}_2$ is a solution to

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{v}_1.$$

Then $\{\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1, \mathbf{x}_2(t) = e^{\lambda t}(t\mathbf{v}_1 + \mathbf{v}_2)\}$ is a linearly independent set of solutions to the differential system $\mathbf{y}' = \mathbf{A}\mathbf{y}$, and thus is a fundamental set of solutions. Thus, we obtained the following theorem.

Theorem. *Let $\mathbf{A}$ be an order 2 square matrix and λ be a repeated eigenvalue. Suppose $\mathbf{v}_1$ is an eigenvector and $\mathbf{v}_2$ a generalized eigenvector associated to λ. Then $\{\mathbf{x}_1(t), \mathbf{x}_2(t)\}$ is a fundamental set of solutions for the first order homogeneous system of differential equations $\mathbf{y}' = \mathbf{A}\mathbf{y}$ with constant coefficients.*