MA1508E: LINEAR ALGEBRA FOR ENGINEERING

Lecture 4 Notes

References

1. Elementary Linear Algebra: Application Version, Section 1.4-1.6, 2.1-2.2
2. Linear Algebra with Application, Section 2.4-2.5, 3.1-3.2

2.9 Inverse and Linear system

Theorem (Invertibility and homogeneous sytem)**.** *A square matrix* $\mathbf{A}$ *of order* n *is invertible if and only if the homogeneous system* $\mathbf{Ax} = \mathbf{0}$ *has only the trivial solution.*

Proof. Suppose $\mathbf{A}$ is invertible. Let $\mathbf{u}$ be a solution to the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$. Then $\mathbf{Au} = \mathbf{0}$ and by pre-multiplying both sides of the equation by $\mathbf{A}^{-1}$, we get

$$\mathbf{u} = \mathbf{A}^{-1}\mathbf{0} = \mathbf{0},$$

which shows that the only solution is the trivial one.

Suppose $\mathbf{A}$ is not invertible. Reduce the augmented matrix $\left(\begin{array}{c|c} \mathbf{A} & \mathbf{0} \end{array}\right) \longrightarrow \left(\begin{array}{c|c} \mathbf{R} & \mathbf{0} \end{array}\right)$ to its RREF. Since $\mathbf{A}$ is not invertible, $\mathbf{R}$ is not the identity matrix, and thus must have a non-pivot column. Thus, any general solution of the homogeneous system $\mathbf{Ax} = \mathbf{0}$ must at least 1 parameter, and so the system admits nontrivial solutions. $\square$

Example.

1. We have shown in lecture 3 that $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ is invertible. Thus, $\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$ has only the trivial solution. Indeed,

$$\mathbf{A} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 - R_2} \xrightarrow{R_2 - R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right),$$

and thus $x = 0, y = 0z = 0$ is the only solution.

2. We have shown in lecture 3 that $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ is singular. Thus, $\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$ should admit non-trivial solutions. Indeed,

$$\mathbf{A} = \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

with general solution $x = -s$, $y = 0$, $z = s$.

Next, we will prove the claim that for a square matrix to be invertible, suffice to check that it has either a left or a right inverse.

Theorem. *Let* $\mathbf{A}$ *be a square matrix of order* n*. Suppose* $\mathbf{BA} = \mathbf{I}$ *for some square matrix* $\mathbf{B}$*. Then* $\mathbf{A}$ *is invertible and* $\mathbf{A}^{-1} = \mathbf{B}$.

Proof. Consider the homogeneous system $\mathbf{Ax} = \mathbf{0}$. If $\mathbf{u}$ is a solution to the system, then

$$\mathbf{u} = \mathbf{BAu} = \mathbf{B0} = \mathbf{0}.$$

So the homogeneous system $\mathbf{Ax} = \mathbf{0}$ has only the trivial solution, and thus $\mathbf{A}$ is invertible. Finally,

$$\mathbf{B} = \mathbf{B}(\mathbf{AA}^{-1}) = (\mathbf{BA})\mathbf{A}^{-1} = \mathbf{A}^{-1}.$$

□

The theorem shows that a left inverse of $\mathbf{A}$ is the inverse of $\mathbf{A}$. Applying transpose to both sides, and using the fact that the inverse of a transpose is the transpose of the inverse, we get that right inverse is also the inverse. That is, if $\mathbf{AB} = \mathbf{I}$, then

$$\mathbf{I} = \mathbf{B}^T\mathbf{A}^T,$$

and so applying the theorem to $\mathbf{A}^T$, we have

$$\mathbf{B}^T = (\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T,$$

and hence $\mathbf{B} = \mathbf{A}^{-1}$

Theorem. *A square matrix* $\mathbf{A}$ *of order* n *is invertible if and only if the system* $\mathbf{Ax} = \mathbf{b}$ *has a unique solution for all* $\mathbf{b}$.

Remark. Let $\mathbf{A}$ be a $m \times n$ matrix and $\mathbf{Ax} = \mathbf{b}$ represent a linear system. Consider the augmented matrix $\left(\begin{array}{c|c} \mathbf{A} & \mathbf{b} \end{array} \right)$. Suppose the system has a unique solution. Let

$$\left(\begin{array}{c|c} \mathbf{A} & \mathbf{b} \end{array} \right) \xrightarrow{r_1} \xrightarrow{r_2} \cdots \xrightarrow{r_k} \left(\begin{array}{c|c} \mathbf{R} & \mathbf{u} \end{array} \right),$$

where $\mathbf{R}$ is the RREF of $\mathbf{A}$. Then necessarily $\mathbf{R}$ has the form

$$\mathbf{R} = \begin{pmatrix} \mathbf{I}_n \\ \mathbf{0}_{m-n\times n} \end{pmatrix}.$$

Let $\mathbf{E}_1, \mathbf{E}_2, ..., \mathbf{E}_k$ be the corresponding elementary matrices. Then

$$\mathbf{u} = \mathbf{E}_k \cdots \mathbf{E}_2\mathbf{E}_1\mathbf{b}$$

is the unique solution to $\mathbf{Ax} = \mathbf{b}$.

Example. 1. Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. For any $\mathbf{b} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$,

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 1 & c \end{array} \right) \xrightarrow{R_1 - R_2} \xrightarrow{R_2 - R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & a-b \\ 0 & 1 & 0 & b-c \\ 0 & 0 & 1 & c \end{array} \right).$$

That is, $\begin{pmatrix} a-b \\ b-c \\ c \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is the unique solution to $\mathbf{Ax} = \mathbf{b}$.

2. The matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ is not invertible. Consider $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$,

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_3-R_1} \xrightarrow{R_3+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right).$$

The system is inconsistent, and hence, $\mathbf{Ax} = \mathbf{b}$ has no solution.

We will summarize all the equivalent statements for invertibility of a matrix.

Theorem. *Let* $\mathbf{A}$ *be a square matrix of order* n*. The following statements are equivalent.*

(i) $\mathbf{A}$ *is invertible.*

(ii) $\mathbf{A}$ *has a left inverse.*

(iii) $\mathbf{A}$ *has a right inverse.*

(iv) The reduced row-echelon form of $\mathbf{A}$ *is the identity matrix* $\mathbf{I}_n$*.*

(v) $\mathbf{A}$ *is a product of elementary matrices.*

(vi) The homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ *has only the trivial solution.*

(vii) The linear system $\mathbf{Ax} = \mathbf{b}$ *has a unique solution for every* $\mathbf{b}$*.*

2.10 Determinants

We will define the determinant of $\mathbf{A}$ of order n by induction.

1. For $n = 1$, $\mathbf{A} = (a)$, $\det(\mathbf{A}) = a$.

2. For $n = 2$, $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(\mathbf{A}) = ad - bc$.

Suppose we have defined the determinant of all square matrices of order $\leq n - 1$. Let $\mathbf{A}$ be a square matrix of order n.

- Define $\mathbf{M}_{ij}$, called the (i, j) matrix minor of $\mathbf{A}$, to be the matrix obtained from $\mathbf{A}$ be deleting the i-th row and j-th column.

  Example. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}$. Then

  (i) $\mathbf{M}_{23} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$,

  (ii) $\mathbf{M}_{12} = \begin{pmatrix} -1 & 3 \\ 3 & 1 \end{pmatrix}$, and

  (iii) $\mathbf{M}_{31} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$.

- For a square matrix of order n, the (i,j)-cofactor of $\mathbf{A}$, denoted as A_{ij}, is the (real) number given by
$$A_{ij} = (-1)^{i+j}\det(\mathbf{M}_{ij}).$$
This definition is well-defined since $\mathbf{M}_{ij}$ is a square matrix of order $n-1$, and by induction hypothesis, the determinant is well defined. Take note of the sign of the (i,j)-entry, $(-1)^{i+j}$. Here's a visualization of the sign of the entries of the matrix
$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \ddots & \end{pmatrix}.$$
Example. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}$. Then
  (i) $A_{23} = (-1)^5(2-6) = 4$,
  (ii) $A_{12} = (-1)^3(-1-9) = 10$, and
  (iii) $A_{31} = (-1)^4(6+1) = 7$.

- The determinant of $\mathbf{A}$ is defined to be
$$\det(\mathbf{A}) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} = \sum_{k=1}^{n} a_{1k}A_{1k}.$$
This is called the <u>cofactor expansion</u> along row 1.

The determinant of $\mathbf{A}$ is also denoted as $\det(\mathbf{A}) = |\mathbf{A}|$.

So for order 3 matrices, the determinant is defined to be
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = aei - afh - bdi + bfg + cdh - ceg.$$
There is an easy way to compute determinant for order 3 matrices.
$$\det(\mathbf{A}) = \left|\begin{array}{ccc|cc} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{a} & \mathrm{b} \\ \mathrm{d} & \mathrm{e} & \mathrm{f} & \mathrm{d} & \mathrm{e} \\ \mathrm{g} & \mathrm{h} & \mathrm{i} & \mathrm{g} & \mathrm{h} \end{array}\right|$$

Example. Compute the determinant of $\begin{pmatrix} 1 & 5 & 1 & 2 \\ 0 & 2 & 6 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.
$$\begin{aligned} \begin{vmatrix} 1 & 5 & 1 & 2 \\ 0 & 2 & 6 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} &= \begin{vmatrix} 2 & 6 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} - 5\begin{vmatrix} 0 & 6 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} \\ &= (2) - 5(0) + 0 - 2(0) = 2. \end{aligned}$$

How about if we try compute using cofactor expansion along the first column?

$$\begin{vmatrix} 1 & 5 & 1 & 2 \\ 0 & 2 & 6 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \begin{vmatrix} 2 & 6 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} - 0 \begin{vmatrix} 5 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} + 0 \begin{vmatrix} 5 & 1 & 2 \\ 2 & 6 & 3 \\ 0 & 0 & 1 \end{vmatrix} - 0 \begin{vmatrix} 5 & 1 & 2 \\ 2 & 6 & 3 \\ 0 & 1 & 2 \end{vmatrix} = 2$$

The are the same!

Theorem. *The determinant of a square matrix* $\mathbf{A}$ *of order* n *can be computed by cofactor expansion along any rows or columns,*

$$\det(\mathbf{A}) = \sum_{k=1}^{n} a_{ik}A_{ik} = \sum_{l=1}^{n} a_{lj}A_{lj}.$$

The proof requires knowledge of the symmetric groups, which is beyond the scope of this module.

Example. Compute the determinant of $\begin{pmatrix} 1 & 5 & 1 & 2 \\ 0 & 2 & 6 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

Cofactor expansion along the first column.

$$\begin{vmatrix} 1 & 5 & 1 & 2 \\ 0 & 2 & 6 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 2 & 6 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} - 0 \begin{vmatrix} 5 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} + 0 \begin{vmatrix} 5 & 1 & 2 \\ 2 & 6 & 3 \\ 0 & 1 & 1 \end{vmatrix} - 0 \begin{vmatrix} 5 & 1 & 2 \\ 2 & 6 & 3 \\ 0 & 1 & 2 \end{vmatrix} = 2(1-2) = -2.$$

Corollary. *The determinant of* $\mathbf{A}$ *and* $\mathbf{A}^T$ *are equal,*

$$\det(\mathbf{A}) = \det(\mathbf{A}^T).$$

This statement is proved by induction on the size n of the matrix $\mathbf{A}$, and using fact that cofactor expansion along first row of $\mathbf{A}$ is equal to cofactor expansion along the first column of $\mathbf{A}^T$.

Corollary. *If a square matrix* $\mathbf{A}$ *has a zero row or column, then* $\det(\mathbf{A}) = 0$.

Proof. Cofactor expand along the zero row or column. □

Corollary. *The determinant of a triangular matrix is the multiplication of the diagonal entries. If* $\mathbf{A} = (a_{ij})_n$ *is a triangular matrix, then*

$$\det(\mathbf{A}) = a_{11}a_{22} = \cdots a_{nn} = \Pi_{k=1}^{n} a_{ii}.$$

Sketch of proof:

Upper triangular matrix , cofactor expand along first column,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

Lower triangular matrix, cofactor expand along the first row,

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

In general, we can always reduce a square matrix $\mathbf{A}$ to a triangular matrix $\mathbf{R}$ (for example, REF). Then $\mathbf{R} = \mathbf{E}_k \cdots \mathbf{E}_2\mathbf{E}_1\mathbf{A}$ for some elementary matrices. If there is a relationship between the determinant of $\mathbf{A}$ and $\mathbf{R}$, this will make computation easier, since the determinant of $\mathbf{R}$ is just the product of the diagonals. We will state the results, but omit the proof.

Theorem. *If* $\mathbf{A}$ *has 2 identical row or columns, then* $\det(\mathbf{A}) = 0$.

Theorem (Determinant and elementary row operations)**.** *If* $\mathbf{B}$ *is obtained from* $\mathbf{A}$ *be performing the following elementary row operations*

(i) $\mathbf{A} \xrightarrow{R_i + aR_j} \mathbf{B}$,

(ii) $\mathbf{A} \xrightarrow{R_i \leftrightarrow R_j} \mathbf{B}$,

(iii) $\mathbf{A} \xrightarrow{cR_j} \mathbf{B}$, $c \neq 0$,

then

(i) $\det(\mathbf{B}) = \det(\mathbf{A})$,

(ii) $\det(\mathbf{B}) = -\det(\mathbf{A})$,

(iii) $\det(\mathbf{B}) = c\det(\mathbf{A})$,

respectively.

Theorem (Determinant of elementary matrices)**.** *Suppose* $\mathbf{E}$ *is an elementary matrix corresponding to the row operation*

(i) $R_i + cR_j$, *then* $\det(\mathbf{E}) = 1$.

(ii) $R_i \leftrightarrow R_j$, *then* $\det(\mathbf{E}) = -1$.

(iii) cR_j, *then* $\det(\mathbf{E}) = c$.

Theorem (Determinant of row equivalent matrices)**.** *Let* $\mathbf{A}$ *and* $\mathbf{R}$ *be square matrices such that*

$$\mathbf{R} = \mathbf{E}_k \cdots \mathbf{E}_2\mathbf{E}_1\mathbf{A}$$

for some elementary matrices $\mathbf{E}_1, \mathbf{E}_2, ..., \mathbf{E}_k$. *Then*

$$\det(\mathbf{R}) = \det(\mathbf{E}_k) \cdots \det(\mathbf{E}_2)\det(\mathbf{E}_1)\det(\mathbf{A}).$$

So if $\mathbf{R}$ is a REF of $\mathbf{A}$, that is, $\mathbf{R} = \mathbf{E}_k \cdots \mathbf{E}_2\mathbf{E}_1\mathbf{A}$, then

$$\begin{aligned} \det(\mathbf{R}) &= \det(\mathbf{E}_k) \cdots \det(\mathbf{E}_2) \det(\mathbf{E}_1) \det(\mathbf{A}) \\ \Rightarrow \det(\mathbf{A}) &= \det(\mathbf{E}_1)^{-1} \det(\mathbf{E}_2)^{-1} \cdots \det(\mathbf{E}_k)^{-1} \det(\mathbf{R}) \\ &= \det(\mathbf{E}_1)^{-1} \det(\mathbf{E}_2)^{-1} \cdots \det(\mathbf{E}_k)^{-1} \Pi_{k=1}^{n} r_{ii}, \end{aligned}$$

where for $i = 1, ..., n$, r_{ii} is the i-th diagonal entry of $\mathbf{R}$. Since the determinant of elementary matrices can be obtained easily (recall that inverse of elementary matrix is an elementary matrix), the determinant of $\mathbf{A}$ is thus easily found.

Example.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix} \xrightarrow[\mathbf{E}_2: R_3 - 2R_4]{\mathbf{E}_1: R_1 - 2R_2} \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & -3 & -6 \\ 0 & 1 & 2 & 3 \end{pmatrix} \xrightarrow[\mathbf{E}_4: -\frac{1}{3}R_3]{\mathbf{E}_3: R_4 - R_1} \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 4 \end{pmatrix} \xrightarrow{\mathbf{E}_5: R_4 - R_3}$$

$$\begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix} \xrightarrow{\mathbf{E}_6: R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

So

$$\begin{aligned} 2 &= \det(\mathbf{E}_6) \det(\mathbf{E}_5) \det(\mathbf{E}_4) \det(\mathbf{E}_3) \det(\mathbf{E}_2) \det(\mathbf{E}_1) \det(\mathbf{A}) \\ &= (-1)(1)(1)(-\frac{1}{3})(1)(1) \det(\mathbf{A}) \end{aligned}$$

$\Rightarrow \det(\mathbf{A}) = 6$.

Theorem (Determinant of product of matrices)**.** *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be square matrices of the same size. Then*

$$\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B}).$$

By induction, we get

$$\det(\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_k) = \det(\mathbf{A}_1) \det(\mathbf{A}_2) \cdots \det(\mathbf{A}_k).$$

Corollary (Determinant of inverse)**.** *If* $\mathbf{A}$ *is invertible, then*

$$\det(\mathbf{A}^{-1}) = \det(\mathbf{A})^{-1}.$$

Proof. Since the identity matrix $\mathbf{I}$ is a triangular matrix, $\det(\mathbf{I}) = 1$. Then

$$1 = \det(\mathbf{I}) = \det(\mathbf{AA}^{-1}) = \det(\mathbf{A}) \det(\mathbf{A}^{-1}).$$

So $\det(\mathbf{A})^{-1} = \det(\mathbf{A}^{-1})$. □

Corollary (Equivalence of invertibility and determinant)**.** *A square matrix* $\mathbf{A}$ *is invertible if and only if* $\det(\mathbf{A}) \neq 0$.

Proof. Recall that $\mathbf{A}$ is invertible if and only if its RREF $\mathbf{R}$ is the identity. If $\mathbf{R}$ is not the identity, then it will have at least a zero row and thus $\det(\mathbf{R}) = 0$. Hence

$$\begin{aligned}\det(\mathbf{A}) &= \det(\mathbf{E}_1)^{-1}\det(\mathbf{E}_2)^{-1}\cdots\det(\mathbf{E}_k)^{-1}\det(\mathbf{R})\\ &= \det(\mathbf{E}_1)^{-1}\det(\mathbf{E}_2)^{-1}\cdots\det(\mathbf{E}_k)^{-1}0 = 0.\end{aligned}$$

On the other hand, if $\mathbf{R} = \mathbf{I}$, then $\det(\mathbf{A}) = \det(\mathbf{E}_1)^{-1}\det(\mathbf{E}_2)^{-1}\cdots\det(\mathbf{E}_k)^{-1}$, and since the determinant of elementary matrices are nonzero, $\det(\mathbf{A}) \neq 0$. □

We will add this equivalence of invertibility to list.

Theorem. *Let $\mathbf{A}$ be a square matrix of order n. The following statements are equivalent.*

(i) $\mathbf{A}$ *is invertible.*

(ii) $\mathbf{A}$ *has a left inverse.*

(iii) $\mathbf{A}$ *has a right inverse.*

(iv) The reduced row-echelon form of $\mathbf{A}$ *is the identity matrix* $\mathbf{I}_n$.

(v) $\mathbf{A}$ *is a product of elementary matrices.*

(vi) The homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ *has only the trivial solution.*

(vii) The linear system $\mathbf{Ax} = \mathbf{b}$ *has a unique solution for every* $\mathbf{b}$.

(viii) The determinant of $\mathbf{A}$ *is nonzero,* $\det(\mathbf{A}) \neq 0$.

Corollary (Determinant of scalar multiplication)**.** *For any square matrix* $\mathbf{A}$ *of order* n *and scalar* $c \in \mathbb{R}$,

$$\det(c\mathbf{A}) = c^n \det(\mathbf{A}).$$

We will present two proofs.

Proof 1: $c\mathbf{A} \xrightarrow{\frac{1}{c}R_1}\xrightarrow{\frac{1}{c}R_2} \cdots \xrightarrow{\frac{1}{c}R_k} \mathbf{A}$. The elementary operations are multiplying a row by $\frac{1}{c}$, and since the determinant of the corresponding elementary matrices are $\frac{1}{c}$, $\det(A) = (\frac{1}{c})^n \det c\mathbf{A}$. Hence, $c^n\det(\mathbf{A}) = \det(c\mathbf{A})$.

Proof 2: Observe that $c\mathbf{A} = c\mathbf{IA} = \text{diag}\{c, ..., c\}\mathbf{A}$. So

$$\det(c\mathbf{A}) = \det(\text{diag}\{c, ..., c\})\det(\mathbf{A}) = c^n\det(\mathbf{A}).$$

Example. Let $\mathbf{A} = \begin{pmatrix} 1 & 5 & 1 & 2\\ 0 & 2 & 6 & 3\\ 0 & 0 & 1 & 2\\ 0 & 0 & 1 & 1\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 & 1\\ 0 & 1 & 1 & -1\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 3\end{pmatrix}$.

$$\mathbf{A} \xrightarrow{R_4 - R_3} \begin{pmatrix} 1 & 5 & 1 & 2\\ 0 & 2 & 6 & 3\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & -1\end{pmatrix}.$$

So $\det(\mathbf{A}) = -2$

1. $\det(3\mathbf{A}) = 3^4(-2) = -162$.
2. $\det(3\mathbf{A}\mathbf{B}^{-1}) = 3^4(-2)(-3)^{-1} = -18$.
3. $\det((3\mathbf{B})^{-1}) = (3^4 \times 3)^{-1} = 3^{-5}$.

Appendix for Lecture 4

Theorem. *A square matrix* $\mathbf{A}$ *of order* n *is invertible if and only if the system* $\mathbf{A}\mathbf{x} = \mathbf{b}$ *has a unique solution for all* $\mathbf{b}$*.*

Proof. Suppose $\mathbf{A}$ is invertible. Then for any $\mathbf{b}$, $\mathbf{u} = \mathbf{A}^{-1}b$ is a solution to the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$. Suppose $\mathbf{u}$ and $\mathbf{v}$ are two solution, $\mathbf{A}(\mathbf{u} - \mathbf{v}) = \mathbf{A}\mathbf{u} - \mathbf{A}\mathbf{v} = \mathbf{b} - \mathbf{b} = \mathbf{0}$. This shows that $(\mathbf{u} - \mathbf{v})$ is a solution to the homogeneous system $\mathbf{A}\mathbf{x} = \mathbf{0}$. Since $\mathbf{A}$ is invertible, the homogeneous system has only the trivial solution. Hence, $\mathbf{u} = \mathbf{v}$.

Suppose now $\mathbf{A}\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$. In particular, let $\mathbf{b} = \mathbf{e}_i$ for $i = 1, .., n$, where $\mathbf{e}_i$ is the $n \times 1$ matrix with 1 in the i-th row, and 0 everywhere else. Let $\mathbf{b}_i$ be the solution to $\mathbf{A}\mathbf{x} = \mathbf{e}_i$ for $i = 1, ..., n$, and let $\mathbf{B} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix}$. Then by the block multiplication,

$$\mathbf{AB} = \mathbf{A}\begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix} = \begin{pmatrix} \mathbf{A}\mathbf{b}_1 & \mathbf{A}\mathbf{b}_2 & \cdots & \mathbf{A}\mathbf{b}_n \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{pmatrix} = \mathbf{I}.$$

This shows that $\mathbf{B}$ is a right inverse, and thus the inverse of $\mathbf{A}$. Hence, $\mathbf{A}$ is invertible. □

Observe that the first part of the proof can be adapted to show that for any size $m \times n$ matrix $\mathbf{A}$, if $\mathbf{A}$ has a left inverse, then the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is consistent, the solution is not unique if $m \neq n$.

We will now proof the two claims in lecture 1

Theorem (Uniqueness of RREF)**.** *Suppose* $\mathbf{R}$ *and* $\mathbf{S}$ *are two reduced row-echelon forms of* $\mathbf{A}$*, then* $\mathbf{R} = \mathbf{S}$*.*

Proof. First note that there exists an invertible matrix $\mathbf{P}$ such that

$$\mathbf{PR} = \mathbf{S}. \tag{1}$$

This is because $\mathbf{A}$ is row equivalent to $\mathbf{R}$ and $\mathbf{S}$, and so there are invertible matrices $\mathbf{P}_1, \mathbf{P}_2$ such that $\mathbf{A} = \mathbf{P}_1\mathbf{R}$ and $\mathbf{A} = \mathbf{P}_2\mathbf{S}$. Let $\mathbf{P} = \mathbf{P}_2^{-1}\mathbf{P}_1$. We will prove by induction on the numbers of rows n of $\mathbf{R}$ and $\mathbf{S}$.

Suppose $n = 1$. Then $\mathbf{R}$, $\mathbf{S}$ are row matrices and $\mathbf{P}$ is a nonzero real number. Since the leading entries of $\mathbf{R}$ and $\mathbf{S}$ must be 1, by the equation (1), $\mathbf{P} = 1$. So $\mathbf{R} = \mathbf{S}$.

Now suppose $n > 1$. Write $\mathbf{R} = \begin{pmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \cdots & \mathbf{r}_n \end{pmatrix}$ and $\mathbf{S} = \begin{pmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \cdots & \mathbf{s}_n \end{pmatrix}$. By equation (1), we have

$$\mathbf{P}\mathbf{r}_j = \mathbf{s}_j, \tag{2}$$

for $j = 1, ..., n$. Since $\mathbf{P}$ is invertible, by the theorem above, $\mathbf{R}$ and $\mathbf{S}$ must have the same zero columns. By deleting the zero columns and forming a new matrix, we may assume that $\mathbf{R}$ and $\mathbf{S}$ has no zero columns. With this assumption, and the fact that $\mathbf{R}$ and $\mathbf{S}$ are in RREF, necessarily the first column of both $\mathbf{R}$ and $\mathbf{S}$ must have 1 in the first entry and 0 everywhere else. By the equation (1), the first column of $\mathbf{P}$ also have 1 in the first entry and zero everywhere else. So we write $\mathbf{R}$, $\mathbf{S}$, and $\mathbf{P}$ in is submatrices,

$$\mathbf{P} = \begin{pmatrix} 1 & \mathbf{p}' \\ 0 & \\ \vdots & \mathbf{P}' \\ 0 & \end{pmatrix}, \ \mathbf{R} = \begin{pmatrix} 1 & \mathbf{r}' \\ 0 & \\ \vdots & \mathbf{R}' \\ 0 & \end{pmatrix}, \text{ and } \mathbf{S} = \begin{pmatrix} 1 & \mathbf{s}' \\ 0 & \\ \vdots & \mathbf{S}' \\ 0 & \end{pmatrix},$$

where $\mathbf{p}', \mathbf{r}', \mathbf{s}'$ are row matrices. By the equation (1) and block multiplication, we have $\mathbf{P}'\mathbf{R}' = \mathbf{S}'$. Note that $\mathbf{P}'$ is invertible. Since $\mathbf{R}$ and $\mathbf{S}$ are in RREF, $\mathbf{R}'$ and $\mathbf{S}'$ are in RREF too. Hence, by the induction hypothesis, $\mathbf{R}' = \mathbf{S}'$. We are left to show that $\mathbf{r}' = \mathbf{s}'$. Since $\mathbf{R}' = \mathbf{S}'$, and both $\mathbf{R}$ and $\mathbf{S}$ are in RREF, $\mathbf{R}$ and $\mathbf{S}$ must have the same pivot columns, say columns $i_1, i_2, ..., i_r$. In these columns, the entries of $\mathbf{r}'$ and $\mathbf{s}'$ must be zero. For the nonzero entries, by equation (2), and the fact that the entries of the columns agree from second row onward, the entries in the first row of each column agrees too, that is $\mathbf{r}' = \mathbf{s}'$ too. Thus the inductive step in complete, and the statement is proven. $\square$

Theorem. *Two matrices are row equivalent if and only if they have the same reduced row-echelon form.*

Proof. Suppose $\mathbf{A}$ and $\mathbf{B}$ has the same RREF $\mathbf{R}$. Then there are invertible matrices $\mathbf{P}$ and $\mathbf{Q}$ such that $\mathbf{PA} = \mathbf{R}$ and $\mathbf{QB} = \mathbf{R}$. Then

$$\mathbf{Q}^{-1}\mathbf{PA} = \mathbf{Q}^{-1}\mathbf{R} = \mathbf{B}.$$

Since $\mathbf{Q}^{-1}\mathbf{P}$ is invertible, it can be written as a product of elementary matrices, and so $\mathbf{A}$ is row equivalent to $\mathbf{B}$.

Suppose now $\mathbf{A}$ is row equivalent to $\mathbf{B}$. Let $\mathbf{P}$ be an invertible matrix such that $\mathbf{PA} = \mathbf{B}$. Let $\mathbf{R}$ be the RREF of $\mathbf{A}$ and $\mathbf{S}$ be the RREF of $\mathbf{B}$. Then $\mathbf{R} = \mathbf{UA}$ and $\mathbf{S} = \mathbf{VB}$ for some invertible matrices $\mathbf{U}$ and $\mathbf{V}$. Then

$$\mathbf{VPU}^{-1}\mathbf{R} = \mathbf{VPA} = \mathbf{VB} = \mathbf{S},$$

which shows that $\mathbf{R}$ is row equivalent to $\mathbf{S}$. By the uniqueness of RREF, $\mathbf{R} = \mathbf{S}$. $\square$