NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

AY2021, Semester 1 MA1508E Linear Algebra for Engineering Tutorial 6

1. For each of the following sets of vectors S,

   (i) Determine if S is linearly independent.

   (ii) If S is linearly dependent, express one of the vectors in S as a linear combination of the others.

   (a) $S = \left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 6 \end{pmatrix} \right\}$.

   (b) $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} \right\}$.

   (c) $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$.

   (d) $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$.

2. For each of the following subspaces V, write down a basis for V.

   (a) $V = \left\{ \begin{pmatrix} a+b \\ a+c \\ c+d \\ b+d \end{pmatrix} \middle| \ a, b, c, d \in \mathbb{R} \right\}$.

   (b) $V = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$.

   (c) V is the solution space of the following homogeneous linear system

$$\left\{ \begin{array}{rcrcrcrcrcl} a_1 & & & + & a_3 & + & a_4 & - & a_5 & = & 0 \\ & & a_2 & + & a_3 & + & 2a_4 & + & a_5 & = & 0 \\ a_1 & + & a_2 & + & 2a_3 & + & a_4 & - & 2a_5 & = & 0 \end{array} \right.$$

3. Let $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{v}_4 = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 2 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} 4 \\ 4 \\ 3 \\ 2 \end{pmatrix}$.

   (a) Express $\mathbf{u}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in two distinct ways.

   (b) Is it possible to express $\mathbf{u}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in two distinct ways?

This question demonstrates that a vector $\mathbf{u} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ has a unique linear combination expression if and only if the set $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is linearly independent.

4. (a) For what values of a will $\mathbf{u}_1 = \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -1 \\ a \\ 1 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 1 \\ -1 \\ a \end{pmatrix}$ form a basis for $\mathbb{R}^3$?

   (b) For what values of a will the determinant of $\begin{pmatrix} a & -1 & 1 \\ 1 & a & -1 \\ -1 & 1 & a \end{pmatrix}$ be nonzero?

   (c) Base on your results in (a) and (b), make a conjecture.

5. For each of the following cases, find the coordinate vector of $\mathbf{v}$ relative to the basis $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

   (a) $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix}, \mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}.$

   (b) $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}.$

Supplementary Problems

6. In computers, information is stored and processed in the form of strings of binary digits, 0 and 1. For this exercise, we will work in the "world" of binary digits

$$\mathbb{B} = \{0, 1\}.$$

Addition in $\mathbb{B}$ works just as it does in $\mathbb{R}$, save for one special rule:

$$1 + 1 = 0.$$

We can similarly perform scalar multiplication in $\mathbb{B}$—however, note that in our "binary world", we only have two possible scalars: 0 and 1 (as opposed to any real number).

Remark. The special rule for binary addition is equivalent to performing our standard operations **modulo 2**. That is, in our "binary world," we evaluate a sum according to its remainder when divided by 2: if the remainder is 0 (i.e., when a number is even), then it corresponds to the binary digit 0, and if the remainder is 1 (i.e., when a number is odd), then it corresponds to the binary digit 1.

   (a) Using the rules on the basic operations in $\mathbb{B}$, complete the addition and multiplication tables below.

+	0	1
0		
1		

×	0	1
0		
1		

(b) Recall that we created the Euclidean space $\mathbb{R}^n$ by taking the set of all n-vectors with real components (i.e., with components in $\mathbb{R}$). We can create the set $\mathbb{B}^n$ in a similar fashion, by taking the set of all n-vectors whose components are binary digits, 0 or 1. Observe, then, that the basic properties of addition and scalar multiplication in $\mathbb{R}^n$ directly apply to $\mathbb{B}^n$, as long as we remember that $1+1=0$ and the only scalars we are allowed to multiply by are 0 and 1.

i. Consider the Euclidean 3-space $\mathbb{R}^3$, which has infinitely many vectors. How many vectors does $\mathbb{B}^3$ have?

ii. A *byte*—the fundamental unit of data used by many computers—is a string of 8 binary digits. Observe that we can treat each byte as a vector in $\mathbb{B}^8$. How many distinct bytes exist; that is, how many vectors are there in $\mathbb{B}^8$? How does this compare to Euclidean 8-space $\mathbb{R}^8$?

iii. The Euclidean n-space $\mathbb{R}^n$ has infinitely many vectors. More generally, how many vectors are there in $\mathbb{B}^n$?

For the purposes of this exercise, you may assume that $\mathbb{B}^n$ has all the properties of a subspace—that is, $\mathbb{B}^n$ is closed under addition and scalar multiplication. (Try to prove this yourself!)

(c) To get a sense of how vectors work in $\mathbb{B}^n$, we take a simple example. Let's begin by working in $\mathbb{B}^3$—the set of all 3-vectors whose components are binary digits.

i. Let $S = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$ be the set of standard unit vectors in $\mathbb{R}^3$. Show that S forms a basis for $\mathbb{B}^3$.

ii. Show that the set $T = \left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\}$ forms a basis for $\mathbb{R}^3$. Does T form a basis for $\mathbb{B}^3$?

(d) The *Hamming matrix* with n rows, $\boldsymbol{H}_n$, is formed by collecting all the nonzero vectors in $\mathbb{B}^n$ as columns of a matrix.

i. How many columns does $\boldsymbol{H}_n$ have?

ii. Suppose we write $\boldsymbol{H}_3$ as

$$\boldsymbol{H}_3 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Show that the solution set S of $\boldsymbol{H}_3\boldsymbol{x} = \boldsymbol{0}$ is a subspace of $\mathbb{B}^7$ by expressing it as a linear span $S = \operatorname{span}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}$, where $\boldsymbol{v}_i \in \mathbb{B}^7$ for $i = 1, \ldots, k$. Hence, find a basis for S.

(e) Create the matrix $\boldsymbol{M}$ by taking the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$ as its columns.

i. What is the size of $\boldsymbol{M}$?

ii. For an arbitrary vector $\boldsymbol{x} \in \mathbb{B}^4$, what can you say about $\boldsymbol{H}_3(\boldsymbol{M}\boldsymbol{x})$? *Hint:* MATLAB can take a number x and calculate its value modulo 2. To do this, we may simply key in

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>> mod(x,2)
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What might happen if we replace the number x with an entire matrix?

In next week's tutorial, we will see how working with binary vectors can help us detect—and potentially, correct—errors in information transmitted between computers.

7. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. Let $\mathbf{u}$ be a vector in V and let c be a scalar. Prove the following:

   (a) $(\mathbf{u} + \mathbf{v})_S = (\mathbf{u})_S + (\mathbf{v})_S$.

   (b) $(c\mathbf{u})_S = c\,(\mathbf{u})_S$.

   Suppose $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are vectors in V. Note that for each $i = 1, 2, \ldots, k$, $(\mathbf{u}_i)_S$ is a vector in $\mathbb{R}^n$. By induction and using (a) and (b), it follows that if $c_1, c_2, \ldots, c_k \in \mathbb{R}$, then

$$(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k)_S = c_1\,(\mathbf{u}_1)_S + c_2\,(\mathbf{u}_2)_S + \ldots + c_k\,(\mathbf{u}_k)_S\,.$$

   Prove the following:

   (c) $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is linearly independent in V if and only if $\{(\mathbf{u}_1)_S, (\mathbf{u}_2)_S, \ldots, (\mathbf{u}_k)_S\}$ is linearly independent in $\mathbb{R}^n$.