NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

AY2021, Semester 1 MA1508E Linear Algebra for Engineering Tutorial 10

Solutions

1. (Cayley-Hamilton theorem)
   Consider
$$p(\mathbf{X}) = \mathbf{X}^3 - 4\mathbf{X}^2 - \mathbf{X} + 4\mathbf{I}.$$

   (a) Compute $p(\mathbf{X})$ for $\mathbf{X} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$.

   $p(\mathbf{X}) = \mathbf{0}$ the order 3 zero square matrix.

   (b) Find the characteristic polynomial of $\mathbf{X}$.

$$\begin{vmatrix} x-1 & -1 & -2 \\ -1 & x-2 & -1 \\ -2 & -1 & x-1 \end{vmatrix} = x^3 - 4x^2 - x + 4 = p(x).$$

   (c) Show that $\mathbf{X}$ invertible. Express the inverse of $\mathbf{X}$ as a function of $\mathbf{X}$.

   $\mathbf{X}^{-1} = \frac{1}{4}(\mathbf{X}^2 - 4\mathbf{X} - \mathbf{I})$.

   This question demonstrated the *Cayley-Hamilton theorem*, which states that if $p(x)$ is the characteristic polynomial of $\mathbf{X}$, then $p(\mathbf{X}) = 0$. This also show that if 0 is not an eigenvalue of $\mathbf{X}$, then the constant term of the characteristic polynomial $p(x)$ is nonzero, and we can use that to compute the inverse of $\mathbf{X}$.

2. (Markov chain)
   A *Markov chain* is an evolving system wherein the state to which it will go next depends only on its preset state and does not depend on the earlier history of the system. The probabilities that the current state does to the various states at the next stage are called the *transition probabilities* for the chain. Consider the following simple example.
   A population of ants is put into a maze with 3 compartments labeled a, b, and c. If the ant is in compartment a, an hour later, there is a 20% chance it will go to compartment b, and a 40% change it will go to compartment c. If it is in compartment b, an hour later, there is a 10% chance it will go to compartment a, and a 30% chance it will go to compartment c. If it is in compartment c, an hour later, there is a 50% chance it will go to compartment a, and a 20% chance it will go to compartment b.
   Suppose 100 ants has been placed in compartment a. The state vectors for the Markov chain are $\mathbf{x}_n = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$, where a_n, b_n, and c_n is the number of ants in compartment a, b, and c, respectively, after n hours. The relationship between the state vectors is given by $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$, where $\mathbf{A}$ is the transition probability matrix. The *initial state vector* is then $\mathbf{x}_0 = \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix}$.

(a) Find the transition probability matrix $\mathbf{A}$. Show that it is a regular stochastic matrix (See question 6).

$\begin{pmatrix} 0.4 & 0.1 & 0.5 \\ 0.2 & 0.6 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{pmatrix}$. In fact, it is a regular doubly stochastic matrix, that is, the sum of the rows are also equal to 1.

(b) By diagonalizing $\mathbf{A}$, find the number of ants in each compartment after 3 hours.

$$\begin{aligned} \begin{vmatrix} x-0.4 & -0.1 & -0.5 \\ -0.2 & x-0.6 & -0.2 \\ -0.4 & -0.3 & x-0.3 \end{vmatrix} &= x^3 - 1.3x + 0.26x + 0.04 \\ &= (x-1)(x+0.1)(x-0.4). \end{aligned}$$

The eigenvalues are $\lambda = 1$, $\lambda = -0.1$, $\lambda = 0.4$.

Eigenspace E_1: $\begin{pmatrix} 1-0.4 & -0.1 & -0.5 \\ -0.2 & 1-0.6 & -0.2 \\ -0.4 & -0.3 & 1-0.3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow E_1 = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

Eigenspace $E_{-0.1}$: $\begin{pmatrix} -0.1-0.4 & -0.1 & -0.5 \\ -0.2 & -0.1-0.6 & -0.2 \\ -0.4 & -0.3 & -0.1-0.3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow E_{-0.1} = \text{span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Eigenspace $E_{0.4}$: $\begin{pmatrix} 0.4-0.4 & -0.1 & -0.5 \\ -0.2 & 0.4-0.6 & -0.2 \\ -0.4 & -0.3 & 0.4-0.3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow E_{-0.1} = \text{span}\left\{ \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix} \right\}$.

So $\mathbf{A} = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & 0.4 \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$. Then

$$\begin{aligned} \mathbf{x}_3 &= \mathbf{A}^3\mathbf{x}_0 = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.1^3 & 0 \\ 0 & 0 & 0.4^3 \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 35 \\ 31.2 \\ 33.8 \end{pmatrix}. \end{aligned}$$

(c) **(MATLAB)** We can use MATLAB to diagonalize the matrix $\mathbf{A}$. Type

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>> [P D]=eig(A)
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The matrix $\mathbf{P}$ will be an invertible matrix, and $\mathbf{D}$ will be a diagonal matrix. Compare the answer with what you have obtained in (b). Explain the differences, if there is any.

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>> A=[0.4 0.1 0.5; 0.2 0.6 0.2;0.4 0.3 0.3];

>> [P D]=eig(A)
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$$P = \begin{pmatrix} -0.5774 & -0.7071 & 0.6172 \\ -0.5774 & 0.0000 & -0.7715 \\ -0.5774 & 0.7071 & 0.1543 \end{pmatrix}$$

and

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & 0.4 \end{pmatrix}.$$

The diagonal matrix obtained must defer from above only by rearranging the diagonal entries. The invertible matrix then defer in that, up to a factor of -1, the columns of the matrix generated by MATLAB that corresponds to the eigenvalues are unit eigenvectors. For example, $\begin{pmatrix} -0.5774 \\ -0.5774 \\ -0.5774 \end{pmatrix} = -\frac{1}{||\mathbf{v}_1||}\mathbf{v}_1$, where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(d) In the long run (assuming no ants died), where will the majority of the ants be?

As $n \to \infty$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & (-0.1)^n & 0 \\ 0 & 0 & 0.4^n \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. So in the long run,

$$\begin{aligned} \mathbf{x}_\infty &= \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 33.33 \\ 33.33 \\ 33.33 \end{pmatrix}. \end{aligned}$$

(e) A vector $\mathbf{v}$ is an *equilibrium distribution* of $\mathbf{P}$ if it is an eigenvector associated to 1. Suppose the initial state vector is $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. What is the population distribution in the long run (assuming no ants died)? How is this related to the equilibrium distribution?

$$\begin{pmatrix}1 & -1 & 4\\ 1 & 0 & -5\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & -1 & 4\\ 1 & 0 & -5\\ 1 & 1 & 1\end{pmatrix}^{-1}\begin{pmatrix}a\\ b\\ c\end{pmatrix} \tag{1}$$

$$= \frac{1}{3}\begin{pmatrix}a+b+c\\ a+b+c\\ a+b+c\end{pmatrix}. \tag{2}$$

This is always an equilibrium distribution if $a+b+c \neq 0$.

(f) A *non-equilibrium distribution* eigenvector $\mathbf{v}$ of $\mathbf{P}$ is an eigenvector that is not associated to 1. Is it possible to have an initial state vector of $\mathbf{A}$ to be a non-equilibrium distribution eigenvector (**Hint:** See question 6(c))?

It is not possible since the other eigenvectors of $\mathbf{A}$ must have at least one negative component. Also, this is impossible since we are assuming that the time is not long enough that any ants died, but as seen in question 6, in the long run, the distribution will be zero in all 3 compartments if the initial condition is a non-equilibrium distribution.

3. (a) Show that the only diagonalizable nilpotent matrix is the zero matrix. (**Hint:** See Tutorial 9 question 7(d))

Let $\mathbf{A}$ be a nilpontent matrix. By Tutorial 9 question 7(d), 0 is the only eigenvalue. So if $\mathbf{A}$ is diagonalizable, then $\mathbf{A} = \mathbf{P}\,\mathrm{diag}(0,0,...,0)\mathbf{P}^{-1} = \mathbf{0}$ for some invertible matrix P.

(b) Show that the only diagonalizable matrix with 1 eigenvalue λ is the scalar matrix $\lambda\mathbf{I}$.

$\mathbf{A} = \mathbf{P}\,\mathrm{diag}(\lambda,\lambda,...,\lambda)\mathbf{P}^{-1} = \mathbf{P}\lambda\mathbf{I}\mathbf{P}^{-1} = \lambda\mathbf{P}\mathbf{P}^{-1} = \lambda\mathbf{I}$.

4. (a) Let $\mathbf{A} = \begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 1\\ 0 & 0 & 2\end{pmatrix}$. Is $\mathbf{A}$ diagonalizable? Is it invertible?

$\begin{vmatrix}x-1 & -1 & 0\\ -1 & x-1 & -1\\ 0 & 0 & x-2\end{vmatrix} = x(x-2)^2$. It has eigenvalues $\lambda = 0$ and $\lambda = 2$, with algebraic multiplicities $r_0 = 1$, and $r_2 = 2$.

Eigenspace E_2: $\begin{pmatrix}2-1 & -1 & 0\\ -1 & 2-1 & -1\\ 0 & 0 & 2-2\end{pmatrix} \longrightarrow \begin{pmatrix}1 & -1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$

$\Rightarrow E_2 = \mathrm{span}\left\{\begin{pmatrix}1\\ 1\\ 0\end{pmatrix}\right\}$.

$\dim(E_2) = 1 \neq r_2 = 2$. So $\mathbf{A}$ is not diagonalizable. It is not invertible since 0 as an eigenvalue.

(b) By diagonalizing $\mathbf{A} = \begin{pmatrix}1 & 0 & 3\\ 0 & 4 & 0\\ 0 & 0 & 4\end{pmatrix}$, find a matrix $\mathbf{B}$ such that $\mathbf{B}^2 = \mathbf{A}$.

$\mathbf{A}$ is a triangular matrix, so its eigenvalues are 1 and 4 with algebraic multiplicities $r_1 = 1$ and $r_4 = 2$.

Eigenspace E_4: $\begin{pmatrix} 3 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow E_2 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

$\dim(E_4) = 2 = r_4$. So $\mathbf{A}$ is diagonalizable.

Eigenspace E_1: $\begin{pmatrix} 0 & 0 & -3 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow E_2 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

So $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1}$.

Consider any of the 8 choices of $\mathbf{C} = \begin{pmatrix} \pm 2 & 0 & 0 \\ 0 & \pm 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $\mathbf{C}^2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

So any choice of $\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \pm 2 & 0 & 0 \\ 0 & \pm 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1}$ will work.

5. For each of the following symmetric matrices $\mathbf{A}$, find an orthogonal matrix $\mathbf{P}$ that orthogonally diagonalizes $\mathbf{A}$.

   (a) $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

   $\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, then $\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$.

   (b) $\mathbf{A} = \begin{pmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{pmatrix}$.

   $\mathbf{P} = \begin{pmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{pmatrix}$, then $\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

   (c) $\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 1 \end{pmatrix}$.

   $\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$, then $\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

Supplementary Problems

6. (Stochastic matrices)
Recall that a stochastic matrix is square matrix $\mathbf{P} = (p_{ij})$ such that the sum of each column is equal to 1 (see Tutorial 3 question 6)

$$p_{1j} + p_{2j} + \cdots + p_{nj} = 1 \text{ for all } j = 1, ..., n.$$

It is called *regular* if all its entries are non-negative. Let $\mathbf{P}$ be a regular stochastic matrix.

(a) Show that $\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ is always an eigenvector of $\mathbf{P}^T$ associated with the eigenvalue 1. This shows that 1 is always an eigenvalue of a stochastic matrix (see Tutorial 3 question 6(b)).

$$\mathbf{P}^T\mathbf{1} = \begin{pmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & \cdots & p_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} p_{11} + p_{21} + \cdots + p_{n1} \\ p_{12} + p_{22} + \cdots + p_{n2} \\ \vdots \\ p_{1n} + p_{2n} + \cdots + p_{nn} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

(b) Show that if λ is an eigenvalue of $\mathbf{P}$, then $|\lambda| \leq 1$. (**Hint:** Pick an eigenvector $\mathbf{v}$ of $\mathbf{P}^T$ associated with λ. Let $k \in \{1, 2, ..., n\}$ be a coordinate of $\mathbf{v}$ with the maximum absolute value, $|v_k| \geq |v_i|$ for all $i = 1, ..., n$. Consider the k-th coordinate of the equation $\mathbf{P}^T\mathbf{v} = \lambda\mathbf{v}^T$.

Let $\mathbf{v}$ and v_k be chosen according to the hint. Taking the absolute value of the k-th coordinate of the equation $\mathbf{P}^T\mathbf{v} = \lambda\mathbf{v}^T$, we have

$$\begin{aligned} |\lambda v_k| &= |p_{1k}v_1 + p_{2k}v_2 + \cdots + p_{nk}v_n| \\ &\leq p_{1k}|v_1| + p_{2k}|v_2| + \cdots + p_{nk}|v_n| \\ &\leq p_{1k}|v_k| + p_{2k}|v_k| + \cdots + p_{nk}|v_k| \\ &\leq (p_{1k} + p_{2k} + \cdots + p_{nk})|v_k| \\ &= |v_k|, \end{aligned}$$

which shows that $|\lambda||v_k| \leq |v_k|$. The second line follow from the fact that $p_{ij} \geq 0$ for all $i, j = 1, ..., n$. Since $\mathbf{v}$ is an eigenvector, necessarily $v_k \neq 0$. Hence, $|\lambda| \leq 1$.

(c) A vector $\mathbf{v}$ is an *equilibrium distribution* of $\mathbf{P}$ if it is an eigenvector asscoiated to 1. For any vector $\mathbf{v}$, let $\mathbf{v}^{(k)} = \mathbf{P}^k\mathbf{v}$. Show that if $\mathbf{v}$ is an eigenvector of $\mathbf{P}$ that is not an equilibrium distribution, than $\mathbf{v}^{(k)} \to 0$ as $k \to \infty$.

Let λ be the eigenvalue associated to $\mathbf{v}$. Since $\lambda \neq 1$, by (b), $|\lambda| < 1$.

Fact. *If $|\lambda| < 1$, then $\lambda^k \to 0$ as $k \to \infty$.*

So by Tutorial 9 question 7(b), $\mathbf{v}^{(k)} = \mathbf{P}^k\mathbf{v} = \lambda^k\mathbf{v} \to 0\mathbf{v} = \mathbf{0}$.

7. **(Application)** I have a supply of seven kind of tiles:

(1) 1×1 red-colored (square) tiles;

(2) 1×2 blue-colored (rectangular) tiles;

(3) 1×2 green-colored (rectangular) tiles;

(4) 1×2 purple-colored (rectangular) tiles;

(5) 1×2 silver-colored (rectangular) tiles;

(6) 1×2 orange-colored (rectangular) tiles;

(7) 1×2 yellow-colored (rectangular) tiles.

We represent each red tile by $(1R)$ (R for red, 1 since it is 1×1), each blue tile by $(2B)$ (B for blue, 2 since it is 1×2) and each green tile by $(2G)$ (G for green, 2 since it is 1×2). Similarly for the other colors $(2P)$, $(2S)$, $(2O)$ and $(2Y)$.

I intend to tile a pavement that is $1 \times n$ units long and would like to know how many ways are there to tile the entire pavement with the colored tiles. Let b_n represent the number of different ways to tile a $1 \times n$ pavement. For example, $b_1 = 1$ since I can only tile it using $(1R)$.

(a) The value of b_2 is 7. Write down all the 7 ways of tiling a 1×2 pavement.
$(1R)(1R)$, $(2B)$, $(2G)$, $(2P)$, $(2S)$, $(2O)$, $(2Y)$.

(b) Determine the value of b_3.
b_3 is obtained from putting down a red tile first, then any choice of 2 tiles arrangement, $(1R)(2X)$, or put down any the 6 choices of 1×2 tiles arrangement first, then put down the 1 red tile, $(2X)(1R)$. So in total, it is $b_2 + 6 \times b_1 = 13$.

(c) Observe that if we want to tile a $1 \times n$ units long pavement, we could choose to lay the 1×1 red tile first, than any of the choice of $1 \times n-1$ tiles arrangements, or to lay any of the choice of 1×2 tiles and then any of the choice of $1 \times n-2$ tiles arrangements. Write down a linear recurrence relation involving 3 consecutive terms (b_n, b_{n-1} and b_{n-2}) in the sequence (b_n).
b_n = Total choices = (choices for $1 \times n - 1$ arrangement) + (choices for 1×2 arrangement) $\times$ (choices for $1 \times n - 2$ arrangement) $= b_{n-1} + 6b_{n-2}$.

(d) Solve the linear recurrence relation you obtained in part (c) and use it to find the number of ways to tile a 1×100 pavement.
Let $\mathbf{x_1} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$ and in general $\mathbf{x_n} = \begin{pmatrix} b_n \\ b_{n+1} \end{pmatrix}$. Then

$$\mathbf{x_n} = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix} \mathbf{x_{n-1}} \Rightarrow \mathbf{x_n} = \mathbf{A}^{n-1}\mathbf{x_1}$$

where $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}$. $\mathbf{A}$ has eigenvalues -2 and 3, with eigenvectors corresponding eigenvectors $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Let $\mathbf{P} = \begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix}$. We have

$\mathbf{P}^{-1}\mathbf{x}_1 = \begin{pmatrix} 4/5 \\ 9/5 \end{pmatrix}$. Then

$$\begin{pmatrix} b_n \\ b_{n+1} \end{pmatrix} = \frac{4}{5}(-2)^{n-1}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{9}{5}(3)^{n-1}\begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Hence, $b_n = \frac{1}{5}[(-1)^n 2^{n+1} + 3^{n+1}]$.
$b_{100} = \frac{1}{5}[2^{101} + 3^{101}]$.

8. Let $\mathbf{A}$ be a symmetric matrix. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors of $\mathbf{A}$ associated to the eigenvalues λ_1 and λ_2, respectively. Suppose $\lambda_1 \neq \lambda_2$. Show that v_1 and v_2 are orthogonal. This shows that the eigenspaces of a symmetric matrix are orthogonal to each other. (See Tutorial 9 question 8.)

$$\lambda_2\mathbf{v}_2 \cdot \mathbf{v}_1 = \lambda_2\mathbf{v}_2^T\mathbf{v}_1 = (\mathbf{A}\mathbf{v}_2)^T\mathbf{v}_1 = \mathbf{v}_2^T\mathbf{A}^T\mathbf{v}_1 = \lambda_1\mathbf{v}_2^T(\mathbf{A}\mathbf{v}_1) = \lambda_1\mathbf{v}_2^T\mathbf{v}_1 = \lambda_1\mathbf{v}_2 \cdot \mathbf{v}_1.$$

In other words, $(\lambda_2 - \lambda_1)\mathbf{v}_2 \cdot \mathbf{v}_1 = \mathbf{0}$. Since $(\lambda_2 - \lambda_1) \neq 0$, necessarily $\mathbf{v}_2 \cdot \mathbf{v}_1 = \mathbf{0}$.