NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

AY2021, Semester 1 MA1508E Linear Algebra for Engineering Tutorial 11

Solutions

1. **(Application)** Two species of fish, species A and species B, live in the same ecosystem (e.g. a pond) and compete with each other for food, water and space. Let the population of species A and B at time t years be given by $a(t)$ and $b(t)$ respectively.

   In the absence of species B, species A's growth rate is $4a(t)$ but when species B are present, the competition slows the growth of species A to $a'(t) = 4a(t) - 2b(t)$. In a similar manner, when species A is absent, species B's growth rate is $3b(t)$ but in the presence of species A, the growth rate reduces to $b'(t) = 3b(t) - a(t)$.

   (a) Write down a system of linear differential equations involving $a(t), b(t), a'(t)$ and $b'(t)$.

   $$\begin{cases} a'(t) &= 4a(t) &- &2b(t) \\ b'(t) &= -a(t) &+ &3b(t) \end{cases}$$

   (b) Represent the system in (i) as $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ where

   $$\mathbf{A} \text{ is a } 2 \times 2 \text{ matrix and } \mathbf{x}(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}, \quad \mathbf{x}'(t) = \begin{pmatrix} a'(t) \\ b'(t) \end{pmatrix}.$$

   Let

   $$\mathbf{x}'(t) = \begin{pmatrix} a'(t) \\ b'(t) \end{pmatrix} = \mathbf{A}\mathbf{x}(t) = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}.$$

   (c) Solve the system using the initial condition $a(0) = 60$, $b(0) = 120$.

   We first find the eigenvalues of $\mathbf{A}$:

   $$\begin{aligned} \begin{vmatrix} \lambda - 4 & 2 \\ 1 & \lambda - 3 \end{vmatrix} &= (\lambda - 4)(\lambda - 3) - 2 \\ &= \lambda^2 - 7\lambda + 12 - 2 \\ &= \lambda^2 - 7\lambda + 10 = (\lambda - 2)(\lambda - 5) \end{aligned}$$

   So $\mathbf{A}$ has two distinct eigenvalues $\lambda = 2$ and $\lambda = 5$.

   Solving $(2\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

   $$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = x_2 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad s \in \mathbb{R}.$$

   So $\{(1,1)^T\}$ is a basis for E_2.

   Solving $(5\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$

   $$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = -2x_2 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad s \in \mathbb{R}.$$

So $\{(-2,1)^T\}$ is a basis for E_5.
A general solution to the given system is

$$\mathbf{x}(t) = A\begin{pmatrix}1\\1\end{pmatrix}e^{2t} + B\begin{pmatrix}-2\\1\end{pmatrix}e^{5t}$$

$$\text{i.e.} \qquad \begin{pmatrix}a(t)\\b(t)\end{pmatrix} = \begin{pmatrix}Ae^{2t} - 2Be^{5t}\\ Ae^{2t} + Be^{5t}\end{pmatrix}$$

Using the given initial conditions:

$$\begin{cases} a(0) = 60 = A - 2B \\ b(0) = 120 = A + B \end{cases}$$

We find that $B = 20$, $A = 100$. So

$$\mathbf{x}(t) = 100\begin{pmatrix}e^{2t}\\e^{2t}\end{pmatrix} + 20\begin{pmatrix}-2e^{5t}\\e^{5t}\end{pmatrix} = \begin{pmatrix}100e^{2t} - 40e^{5t}\\100e^{2t} + 20e^{5t}\end{pmatrix}.$$

2. Instead of a first order system of linear differential equations $\mathbf{Y}' = \mathbf{A}\mathbf{Y}$ (involving n variables $y_1, y_2, \ldots, y_n$), we may encounter a second order system of the form $\mathbf{Y}'' = \mathbf{A_1}\mathbf{Y} + \mathbf{A_2}\mathbf{Y}'$. To solve this second order system, we can translate it into a first order system by introducing n additional new variables $y_{n+1}, y_{n+2}, \ldots, y_{2n}$ as follows:

$$\begin{aligned} y_{n+1}(t) &= y_1'(t) \\ y_{n+2}(t) &= y_2'(t) \\ &\vdots \\ y_{2n}(t) &= y_n'(t) \end{aligned}$$

Suppose we let

$$\mathbf{Y_1} = \mathbf{Y} = \begin{pmatrix}y_1\\y_2\\\vdots\\y_n\end{pmatrix} \quad \text{and} \quad \mathbf{Y_2} = \mathbf{Y}' = \begin{pmatrix}y_1'\\y_2'\\\vdots\\y_n'\end{pmatrix} = \begin{pmatrix}y_{n+1}\\y_{n+2}\\\vdots\\y_{2n}\end{pmatrix}.$$

Then

$$\mathbf{Y_1'} = \mathbf{0}\mathbf{Y_1} + \mathbf{I_n}\mathbf{Y_2} \quad \text{and} \quad \mathbf{Y_2'} = \mathbf{Y_1''} = \mathbf{A_1}\mathbf{Y_1} + \mathbf{A_2}\mathbf{Y_2}$$

The two equations above can be combined to give the first order system with a $2n \times 2n$ matrix as shown:

$$\begin{pmatrix}\mathbf{Y_1'}\\\mathbf{Y_2'}\end{pmatrix} = \begin{pmatrix}\mathbf{0_n} & \mathbf{I_n}\\\mathbf{A_1} & \mathbf{A_2}\end{pmatrix}\begin{pmatrix}\mathbf{Y_1}\\\mathbf{Y_2}\end{pmatrix}.$$

In this way, $\mathbf{Y_1}$ (the original $\mathbf{Y}$) and $\mathbf{Y_2}$ (the first derivatives of $\mathbf{Y}$) can now be solved by solving the first order system.

Use the method described above to solve the following second order linear differential equations:

(a)

$$y'' + 2y' + 5y = 0$$

.

Let $y_1 = y$ and $y_2 = y' = y_1'$. Then $y_2' = y'' = -5y - 2y' \Leftrightarrow y_2' = -5y_1 - 2y_2$. Together with $y_1' = y_2$, we have

$$\begin{cases} y_1' &= & & & y_2 \\ y_2' &= & -5y_1 & - & 2y_2 \end{cases}$$

Let $\mathbf{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, then we have $\mathbf{Y}' = \mathbf{AY}$ where $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix}$. The eigenvalues of $\mathbf{A}$ are $\lambda = -1 + 2i$ and $\bar{\lambda} = -1 - 2i$. Solving

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x} \in \text{span}\left\{\begin{pmatrix} 1+2i \\ -5 \end{pmatrix}\right\}.$$

Thus $E_\lambda = \text{span}\left\{\begin{pmatrix} 1+2i \\ -5 \end{pmatrix}\right\}$. Let $\mathbf{x} = \begin{pmatrix} 1+2i \\ -5 \end{pmatrix}$. Two real solutions to the system of linear differential equations are $\text{Re}(e^{\lambda t}\mathbf{x})$ and $\text{Im}(e^{\lambda t}\mathbf{x})$, where

$$\begin{aligned} e^{\lambda t}\mathbf{y} &= e^{(-1+2i)t}\begin{pmatrix} 1+2i \\ -5 \end{pmatrix} \\ &= e^{-t}(\cos 2t + i\sin 2t)\begin{pmatrix} 1+2i \\ -5 \end{pmatrix} \\ &= e^{-t}\begin{pmatrix} \cos 2t - 2\sin 2t + i(2\cos 2t + \sin 2t) \\ -5\cos 2t - i5\sin 2t \end{pmatrix} \end{aligned}$$

So the two real solutions are

$$\mathbf{x}_r = e^{-t}\begin{pmatrix} \cos 2t - 2\sin 2t \\ -5\cos 2t \end{pmatrix} \text{ and } \mathbf{x}_i = e^{-t}\begin{pmatrix} 2\cos 2t + \sin 2t \\ -5\sin 2t \end{pmatrix}.$$

A general solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathbf{Y} = c_1e^{-t}\begin{pmatrix} \cos 2t - 2\sin 2t \\ -5\cos 2t \end{pmatrix} + c_2e^{-t}\begin{pmatrix} 2\cos 2t + \sin 2t \\ -5\sin 2t \end{pmatrix}$$

and the solution to the original second-order differential equation is $y = c_1e^{-t}(\cos 2t - 2\sin 2t) + c_2e^{-t}(2\cos 2t + \sin 2t)$ where $c_1, c_2 \in \mathbb{R}$.

(b)

$$\begin{aligned} y_1'' &= 2y_1 + y_2 + y_1' + y_2' \\ y_2'' &= -5y_1 + 2y_2 + 5y_1' - y_2' \end{aligned}$$

given the initial condition $y_1(0) = y_2(0) = y_1'(0) = 4$ and $y_2'(0) = -4$.

Set $y_3 = y_1'$ and $y_4 = y_2'$. This gives the first-order system

$$\begin{cases} y_1' &= y_3 \\ y_2' &= y_4 \\ y_3' &= 2y_1 + y_2 + y_3 + y_4 \\ y_4' &= -5y_1 + 2y_2 + 5y_3 - y_4 \end{cases}$$

The coefficient matrix for this system is

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ -5 & 2 & 5 & -1 \end{pmatrix}.$$

Solving for the eigenvalues of $\mathbf{A}$, we find that $\mathbf{A}$ has 4 distinct eigenvalues $\lambda_1 = 1,\ \lambda_2 = -1,\ \lambda_3 = 3,\ \lambda_4 = -3$ and the corresponding eigenvectors

$$\mathbf{x_1} = (1, -1, 1, -1)^T \qquad \mathbf{x_2} = (1, 5, -1, -5)^T$$

$$\mathbf{x_3} = (1, 1, 3, 3)^T \qquad \mathbf{x_4} = (1, -5, -3, 15)^T.$$

Thus, the general solution to the first-order system is of the form

$$c_1\mathbf{x_1}e^t + c_2\mathbf{x_2}e^{-t} + c_3\mathbf{x_3}e^{3t} + c_4\mathbf{x_4}e^{-3t}.$$

Now we use the initial condition provided to find c_1, c_2, c_3, c_4. When $t = 0$, we have

$$c_1\mathbf{x_1} + c_2\mathbf{x_2} + c_3\mathbf{x_3} + c_4\mathbf{x_4} = (4, 4, 4, -4)$$

or equivalently

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 5 & 1 & -5 \\ 1 & -1 & 3 & -3 \\ -1 & -5 & 3 & 15 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \\ -4 \end{pmatrix}.$$

The above system can be solved to give the unique solution $c_1 = 2, c_2 = 1, c_3 = 1, c_4 = 0$. Thus the solution to the initial value problem is

$$\mathbf{Y} = 2\mathbf{x_1}e^t + \mathbf{x_2}e^{-t} + \mathbf{x_3}e^{3t}.$$

Thus

$$\begin{pmatrix} y_1 \\ y_2 \\ y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2e^t + e^{-t} + e^{3t} \\ -2e^t + 5e^{-t} + e^{3t} \\ 2e^t - e^{-t} + 3e^{3t} \\ -2e^t - 5e^{-t} + 3e^{3t} \end{pmatrix}.$$

3. For each of the following homogeneous system of differential equations,

   (i) find a fundamental set of solutions for the system;

   (ii) use Wronskian to verify that your answer in (i) are linearly independent;

   (iii) write down a general solution using the answers in (i);

   (iv) find the solution to the initial value problem.

   (a)

$$\begin{array}{rcl} y_1' & = & y_1 \\ y_2' & = & -3y_2 \end{array}, \qquad y_1(1) = e^1,\ y_2(1) = e^{-3}.$$

The eigenvalues are 1 and -3, with eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_{-3} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(i) Fundamental set of solutions: $\left\{e^t\begin{pmatrix}1\\0\end{pmatrix}, e^{-3t}\begin{pmatrix}0\\1\end{pmatrix}\right\}$.

(ii) Wronskian:

$$\begin{vmatrix}e^t & 0\\ 0 & e^{-3t}\end{vmatrix} = e^{-2t} \neq 0$$

for any $t \in \mathbb{R}$. Hence, the set in (i) is linearly independent.

(iii) General solution: $y_1 = c_1e^t$, $y_2 = c_2e^{-3t}$.

(iv) $c_1e^1 = y_1(1) = e^1 \Rightarrow c_1 = 1$, $c_2e^{-3} = y_2(1) = e^{-3} \Rightarrow c_2 = 1$. Solution:

$$\begin{pmatrix}y_1\\y_2\end{pmatrix} = \begin{pmatrix}e^t\\e^{-3t}\end{pmatrix}.$$

(b)

$$\begin{array}{rcrcr} y_1' &=& y_1 &-& 2y_2\\ y_2' &=& 2y_1 &+& y_2\end{array}, \qquad y_1(0) = 1,\ y_2(0) = -2.$$

The eigenvalues are $\lambda = 1 + 2i$ and $\bar{\lambda} = 1 - 2i$. For $\lambda = 1 + 2i$,

$$\begin{pmatrix}2i & 2\\ -2 & 2i\end{pmatrix} \Rightarrow v = \begin{pmatrix}i\\1\end{pmatrix}.$$

We have $\mathbf{v}_r = \begin{pmatrix}0\\1\end{pmatrix}$ and $\mathbf{v}_i = \begin{pmatrix}1\\0\end{pmatrix}$ The two real solutions are

$$\mathbf{x}_r = e^t\begin{pmatrix}-\sin 2t\\ \cos 2t\end{pmatrix} \quad \text{and} \quad \mathbf{x}_i = e^t\begin{pmatrix}\cos 2t\\ \sin 2t\end{pmatrix}.$$

(i) Fundamental set of solutions: $\left\{e^t\begin{pmatrix}-\sin 2t\\ \cos 2t\end{pmatrix}, e^t\begin{pmatrix}\cos 2t\\ \sin 2t\end{pmatrix}\right\}$.

(ii) Wronskian:

$$\begin{vmatrix}-e^t\sin 2t & e^t\cos 2t\\ e^t\cos 2t & e^t\sin 2t\end{vmatrix} = -e^{2t}(\sin^2 2t + \cos^2 2t) = -e^{2t} \neq 0$$

for any $t \in \mathbb{R}$. Hence, the set in (i) is linearly independent.

(iii) General solution:

$$\mathbf{y} = c_1e^t\begin{pmatrix}-\sin 2t\\ \cos 2t\end{pmatrix} + c_2e^t\begin{pmatrix}\cos 2t\\ \sin 2t\end{pmatrix}.$$

(iv) Using the initial condition

$$\begin{pmatrix}1\\-2\end{pmatrix} = c_1\begin{pmatrix}0\\1\end{pmatrix} + c_2\begin{pmatrix}1\\0\end{pmatrix} \Rightarrow c_1 = -2, c_2 = 1.$$

Thus the solution to the system is

$$\begin{pmatrix}y_1\\y_2\end{pmatrix} = -2e^t\begin{pmatrix}-\sin 2t\\ \cos 2t\end{pmatrix} + e^t\begin{pmatrix}\cos 2t\\ \sin 2t\end{pmatrix}.$$

(c)

$$\begin{array}{rcrcl} y_1' & = & -8y_1 & - & 5y_2 \\ y_2' & = & 5y_1 & + & 2y_2 \end{array}, \qquad y_1(0) = 1, \; = y_2(0) = 3.$$

$$\begin{vmatrix} x+8 & 5 \\ -5 & x-2 \end{vmatrix} = (x+3)^2.$$

Eigenvalue $\lambda = -3$, multiplicity 1.

$$\begin{pmatrix} -3+8 & 5 \\ -5 & -3-2 \end{pmatrix} \Rightarrow \mathbf{v}_{-3} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

We now find a non zero vector $\mathbf{u}$ such that $(\mathbf{A} + 3\mathbf{I})\mathbf{u} = \mathbf{v}_{-3}$.

$$\left(\mathbf{A} + 3\mathbf{I} \,\middle|\, \mathbf{v} \right) = \left(\begin{array}{cc|c} -5 & -5 & -1 \\ 5 & 5 & 1 \end{array} \right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & \frac{1}{5} \\ 0 & 0 & 0 \end{array} \right).$$

So $\mathbf{u} = \begin{pmatrix} \frac{1}{5} - s \\ s \end{pmatrix}$ where $s \in \mathbb{R}$. We may choose $\mathbf{u} = \begin{pmatrix} 0 \\ \frac{1}{5} \end{pmatrix}$.

(i) Fundamental set of solutions: $\left\{ e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, e^{-3t} \left(t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{5} \end{pmatrix} \right) \right\}$.

(ii) Wronskian:

$$\begin{vmatrix} -e^{3t} & -te^{-3t} \\ e^{-3t} & e^{-3t}(t + \frac{1}{5}) \end{vmatrix} = -\frac{1}{5} e^{-6t} \neq 0$$

for any $t \in \mathbb{R}$. Hence, the set in (i) is linearly independent.

(iii) General solution:

$$y_1 = -c_1 e^{-3t} - c_2 t e^{-3t}, \; y_2 = c_1 e^{-3t} + c_2 e^{-3t}(t + \frac{1}{5})$$

(iv) Using the initial conditions, we get

$$-c_1 = y_1(0) = 1 \Rightarrow \; c_1 = -1, \qquad c_1 + c_2 \frac{1}{5} = 3 \Rightarrow \; c_2 = 20.$$

So the particular solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} e^{-3t} - 20te^{-3t} \\ 3e^{-3t} + 20te^{-3t} \end{pmatrix}.$$

(d)

$$\begin{array}{rcrcl} y_1' & = & 3y_1 & + & 2y_2 \\ y_2' & = & -8y_1 & - & 5y_2 \end{array}, \qquad y_1(0) = 3, \; = y_2(0) = 2.$$

$\begin{vmatrix} x-3 & -2 \\ 8 & x+5 \end{vmatrix} = (x+1)^2$. Eigenvalue $\lambda = 1$ with multiplicity 2.

Eigenvector: $\begin{pmatrix} -4 & -2 \\ 8 & 4 \end{pmatrix} \Rightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Solve for $\mathbf{v}_2$: $\left(\begin{array}{cc|c} 4 & 2 & 1 \\ -8 & -4 & -2 \end{array} \right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1/2 & 1/4 \\ 0 & 0 & 0 \end{array} \right)$ and so $\mathbf{v}_2 = \begin{pmatrix} \frac{1}{4} - s\frac{1}{2} \\ s \end{pmatrix}$

for any $s \in \mathbb{R}$. Choose $s = 0$, then $\mathbf{v}_2 = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$.

(i) Fundamental set of solutions: $\left\{ e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}, e^{-t}\left(t\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} \right) \right\}$.

(ii) Wronskian:

$$\begin{vmatrix} e^{-t} & -te^{-t}(t+\frac{1}{4}) \\ -2te^{-t} & -2e^{-t} \end{vmatrix} = \frac{1}{2}e^{-2t} \neq 0$$

for any $t \in \mathbb{R}$. Hence, the set in (i) is linearly independent.

(iii) General solution:

$$y_1 = c_1 e^{-t} + c_2 e^{-t}(t + 1/4), \; y_2 = -2c_1 e^{-t} - c_2 2te^{-t}.$$

(iv) Using the initial conditions, we get

$$c_1 + \frac{1}{4}c_2 = y_1(0) = 3, \; -2c_1 = y_2(0) = 2 \Rightarrow c_1 = -1, \; c_2 = 16.$$

So the particular solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = e^{-t}\begin{pmatrix} 3+16t \\ 2-32t \end{pmatrix}.$$

Supplementary Problems

4. **(MATLAB)** Consider the following system of linear differential equations

$$\begin{array}{rcrcrcrcrcr} y_1' & = & 2y_1 & + & y_2 & + & y_3 & - & 2y_4 & - & 2y_5 \\ y_2' & = & & & y_2 & & & & & & \\ y_3' & = & & & & & 2y_3 & & & & \\ y_4' & = & & & & - & y_3 & + & 2y_4 & & \\ y_5' & = & y_1 & + & y_2 & + & 2y_3 & - & y_4 & & \end{array}$$

with the initial condition $y_1(0) = y_2(0) = y_3(0) = y_4(0) = y_5(0) = 1$.

(a) The `charpoly` function in MATLAB can be used to compute the characteristic polynomial of a matrix. First create a symbolic variable, say x,

```
>> syms x
```

Then we compute the characteristics polynomial of **A**

```
>> p=charpoly(A,x)
```

Then use the `factor` function to factorize the characteristics polynomial,

```
>> factor(p,x)
```

Hence or otherwise, solve the system of linear differential equations.

The coefficient matrix is $\mathbf{A} = \begin{pmatrix} 2 & 1 & 3 & -2 & -2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 \\ 1 & 1 & 2 & -1 & 0 \end{pmatrix}$.

```
>> A=[2 1 3 -2 -2;0 1 0 0 0;0 0 2 0 0;0 0 -1 2 0;1 1 2 -1 0];

>> syms x;

>> p=charpoly(A,x)
```

ans= $x^5 - 7 * x^4 + 20 * x^3 - 30 * x^2 + 24 * x - 8$.

```
>> factor(p,x)
```

ans= $[x - 1, x^2 - 2x + 2, x - 2, x - 2]$

In other words, the characteristics polynomial is

$$p(x) = (x - 1)(x - 2)^2(x^2 - 2x + 2),$$

and thus the eigenvalues are $\lambda = 1$, $\lambda = 2$, $\lambda = 1 + i$, $\lambda = 1 - i$, with algebraic multiplicities $r_1 = 1$, $r_2 = 2$, $r_{1+i} = r_{1-i} = 1$. Now we solve for the eigenspaces.

For $\lambda = 1$.

```
>> rref(eye(5)-A)
```

ans=

$$\begin{matrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0. \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad \text{So let } \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

For $\lambda = 2$.

```
>> rref(2*eye(5)-A)
```

ans=

$$\begin{matrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0\ . \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad \text{So let } \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

The geometric multiplicity is $\dim(E_2) = 1$, which is not equal to the algebraic multiplicity. So, we need to find a generalized eigenvector associated to 2.

```
>> v2=[1 0 0 -1 1]';

>> rref([(A-2*eye(5)) v2])
```

ans=

$$\begin{matrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1. \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad \text{So let } \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

For $\lambda = 1 + i$.

```
>> rref((1+i)*eye(5)-A)
```

ans=

$$\begin{matrix} 1 & 0 & 0 & 0 & -1-i \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}. \qquad \text{So let } \mathbf{v}_4 = \begin{pmatrix} 1+i \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\begin{aligned} e^{(1+i)t}\begin{pmatrix} 1+i \\ 0 \\ 0 \\ 1 \end{pmatrix} &= e^t(\cos(t) + i\sin(t))\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right) \\ &= e^t\left(\begin{pmatrix} \cos(t) - \sin(t) \\ 0 \\ 0 \\ \cos(t) \end{pmatrix} + i\begin{pmatrix} \cos(t) + \sin(t) \\ 0 \\ 0 \\ \sin(t) \end{pmatrix}\right). \end{aligned}$$

So the general solution is

$$c_1e^t\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} + c_3e^{2t}\left(t\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right)$$
$$+c_4e^t\begin{pmatrix} \cos(t) - \sin(t) \\ 0 \\ 0 \\ 0 \\ \cos(t) \end{pmatrix} + c_5e^t\begin{pmatrix} \cos(t) + \sin(t) \\ 0 \\ 0 \\ 0 \\ \sin(t) \end{pmatrix}.$$

When $t = 0$,

$$\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} = c_1\begin{pmatrix}1\\-1\\0\\0\\0\end{pmatrix} + c_2\begin{pmatrix}1\\0\\0\\-1\\1\end{pmatrix} + c_3\begin{pmatrix}0\\0\\1\\1\\0\end{pmatrix} + c_4\begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix} + c_5\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}$$

Solving the system, we get $c_1 = -1, c_2 = 0, c_3 = 1.c_4 = 1.c_5 = 1$. Hence, the solution is

$$-e^t\begin{pmatrix}1\\-1\\0\\0\\0\end{pmatrix} + e^{2t}\left(t\begin{pmatrix}1\\0\\0\\-1\\1\end{pmatrix} + \begin{pmatrix}0\\0\\1\\1\\0\end{pmatrix}\right) + e^t\begin{pmatrix}2\cos(t)\\0\\0\\0\\\cos(t)+\sin(t)\end{pmatrix},$$

that is,

$$\begin{aligned}
y_1(t) &= -e^t + te^{2t} + 2e^t\cos(t)\\
y_2(t) &= e^t\\
y_3(t) &= e^{2t}\\
y_4(t) &= e^{2t}(1-t)\\
y_5(t) &= te^{2t} + e^t(\cos(t)+\sin(t))
\end{aligned}$$

(b) We can use MATLAB command `dsolve` to find the general solution of a system of differential equations $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

```
>> syms y1(t) y2(t) y3(t) y4(t) y4(t);

>> y=[y1; y2; y3; y4; y5];

>> [Sy1 Sy2 Sy3 Sy4 Sy5]=dsolve(diff(y,t)==A*y)
```

If we are solving an initial value problem, we need to input this command line before the last line

```
>> conds=[y1(0)==1, y2(0)==1, y3(0)==1,y4(0)==1,y5(0)==1];
```

and modify the last command line to

```
[Sy1, Sy2, Sy3, Sy4, Sy5]=dsolve(diff(Y,t)==A*Y, conds)
```

Compare the answers obtained to the ones in (a).

```
>> [Sy1 Sy2 Sy3 Sy4 Sy5]=dsolve(diff(y,t)==A*y)
>> [Sy1 Sy2 Sy3 Sy4 Sy5]=dsolve(diff(y,t)==A*y)
```

```
Sy1 =

C5*(exp(2*t) + t*exp(2*t)) - C4*((3*exp(t)*cos(t))/2
+(exp(t)*sin(t))/2) - C3*((exp(t)*cos(t))/2
- (3*exp(t)*sin(t))/2) - C2*exp(t) + C1*exp(2*t)

Sy2 =

C2*exp(t)

Sy3 =

C5*exp(2*t)

Sy4 =

- C1*exp(2*t) - C5*t*exp(2*t)

Sy5 =

C5*(exp(2*t) + t*exp(2*t)) - C4*((exp(t)*cos(t))/2
+ exp(t)*sin(t)) - C3*(exp(t)*cos(t)
- (exp(t)*sin(t))/2) + C1*exp(2*t)
```

Up to renaming the c_i, the solution obtained agree with the general solution in (a).

```
>> conds=[y1(0)==1, y2(0)==1, y3(0)==1,y4(0)==1,y5(0)==1];

>> [Sy1, Sy2, Sy3, Sy4, Sy5]=dsolve(diff(Y,t)==A*Y, conds)

Sy1 =

t*exp(2*t) - exp(t) + 2*exp(t)*cos(t)

Sy2 =

exp(t)

Sy3 =

exp(2*t)

Sy4 =
```

```
exp(2*t) - t*exp(2*t)

Sy5 =

t*exp(2*t) + exp(t)*cos(t) + exp(t)*sin(t)
```

which is exactly what we got in (a).