MA1508E: LINEAR ALGEBRA FOR ENGINEERING

Lecture 8 Notes

References

1. Elementary Linear Algebra: Application Version, Section 4.5, 4.7
2. Linear Algebra with Application, Section 5.2, 5.4, 6.3

3.10 Dimension

Recall that $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is a basis for a subspace $V \subseteq \mathbb{R}^n$ if

(i) $\text{span}(S) = V$, and

(ii) S is linearly independent.

Now even though the choice of a basis for a subspace is not unique, the number of vectors in any basis must be the same. We will state it as a theorem. The proof of the theorem is beyond the scope of the course; interested readers can refer to the appendix.

Theorem. *Suppose* $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ *and* $T = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ *are bases for a subspace* $V \subseteq \mathbb{R}^n$. *Then* $k = m$.

We shall give a name to this intrinsic property of a subspace. The dimension of a subspace $V \subseteq \mathbb{R}^n$ is the number of vectors in any basis, denoted as $\dim(V)$.

Example. 1. The dimension of $\mathbb{R}^n$ is n, since the standard basis $E = \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$ has n vectors.

2. $\mathbb{R}^3$ is called the 3-dimensional Euclidean space, and $\mathbb{R}^2$ is called the 2-dimensional Euclidean space.

3. The xy-plane in $\mathbb{R}^3$ is a 2-dimensional subspace of $\mathbb{R}^3$.

4. Any k-dimensional subspace of $\mathbb{R}^n$ is an embedding of $\mathbb{R}^k$ into $\mathbb{R}^n$ for $k \leq n$.

Recall that the zero space, $\{\mathbf{0}\}$ is a subspace. What is its dimension? Although the zero space is spanned by $\{\mathbf{0}\}$, it cannot be a basis since it is linearly dependent. It turns out that the basis for the zero space is the empty set, and hence, the dimension of the zero space is 0. Intuitively, the dimension is the least number of directions you will need to be able to travel to any point of the subspace. For example, plane has 2 dimensions, since to be able to go to any point on a plane, you need at least 2 direction. A line is 1-dimensional, since we only need one direction to get to any point along the line, moving forward along the direction or backwards in the negative direction. Since the zero space is a point, we cannot travel anywhere, and hence, we need 0 directions. Readers may refer to the appendix for the formal proof.

Theorem. *The zero space is 0-dimensional. The empty set* $\varnothing = \{\}$ *is the basis for the zero space.*

Remark. Note that the empty set is not $\{\mathbf{0}\}$, and $\{\mathbf{0}\}$ is not the empty set since it contains the origin $\mathbf{0}$.

Recall that any set containing more than n vectors in $\mathbb{R}^n$ must be linearly dependent, and any set containing less than n vectors cannot span $\mathbb{R}^n$. We have similar results for subspaces.

Theorem. *Let $V \subseteq \mathbb{R}^n$ be a k-dimensional subspace. Any subset of V containing more than k vectors must be linearly dependent.*

Theorem. *Let $V \subseteq \mathbb{R}^n$ be a k-dimensional subspace. Any subset of V containing less than k vectors cannot span V.*

Theorem. *Let $\mathbf{A}$ be a $m \times n$ matrix. The number of non-pivot columns in the reduced row-echelon form of $\mathbf{A}$ is the dimension of the solution space*

$$\left\{\ \mathbf{u} \in \mathbb{R}^n \ \middle|\ \mathbf{A}\mathbf{u} = \mathbf{0}\ \right\}.$$

Example. Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle|\ 2x - 3y + z = 0 \right\}$ and

$$S = \left\{ \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right\}.$$

Let $x = s, y = t$, then $V = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\}$. So $T = \left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\}$ is a basis for V. This shows that V has 2 dimensions. Hence, S must be linearly dependent since it is a subset of V containing $3 > 2$ vectors. Indeed,

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ -5 & 1 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}.$$

3.11 Equivalent Criteria for Basis

Lemma. *Suppose $U, V \subseteq \mathbb{R}^n$ are subspaces of $\mathbb{R}^n$ such that U is a subset of V, $U \subseteq V$. Then $\dim(U) \leq \dim(V)$, with equality if and only if $U = V$.*

The intuitive explanation is as such. Suppose U is a subspace contained in V. Let $\{\mathbf{u}_1, ..., \mathbf{u}_k\}$ be a minimum set of directions we need to travel the whole U. Then since $U \subseteq V$, we are allowed to travel these directions $\mathbf{u}_1, ..., \mathbf{u}_k$ in V too. If $U = V$, then the directions $\mathbf{u}_1, ..., \mathbf{u}_k$ are sufficient to travel the entire V too, hence, U and V has the same dimension. Otherwise, we need more directions to travel the whole V, and hence the dimension of V is strictly larger than the dimension of U.

Theorem. *Let $V \subseteq \mathbb{R}^n$ be a k-dimensional subspace. If $S = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is a subset of V containing k number of linearly independent vectors, then S is a basis of V.*

Proof. We need to show that S spans V. Since $S \subseteq V$, we have $\text{span}(S) \subseteq V$. By the previous lemma, they must be equal, for since S is linearly independent, $k = \dim(\text{span}(S)) = \dim(V)$, and thus $\text{span}(S) = V$. $\square$

Theorem. *Let $V \subseteq \mathbb{R}^n$ be a k-dimensional subspace. Suppose $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k\}$ is a set containing k number of vectors and $V \subseteq$ span(S). Then S is a basis for V.*

Proof. We need to show that S is linearly independent. This follows from the previous lemma, since $V \subseteq \text{span}(S)$ and hence $k = \dim(V) \leq \text{span}(S)$. Hence, S must be linearly independent, for otherwise, the subspace spanned by S must have less than k dimensions, which is a contradiction. $\square$

Hence, we have arrived at the equivalent criteria for a set S to be a basis for a subspace V.

Definition	(**B1**)	(**B2**)
(1) $\text{span}(S) = V$ (2) S is L.I.	(1) $\lvert S\rvert = \dim(V)$ (2) $S \subseteq V$ and S is L.I.	(1) $V \subseteq \text{span}(S)$ (2) $\lvert S\rvert = \dim(V)$

By using (B1), we avoid the need to check that S is a spaning set, and by using (B2), we avoid the need to check that S is linearly independent. The other significance of the results above is that (B1) shows that any subset of V containing the maximal number of linearly independent vectors must be a basis, and (B2) shows that any set that spans V containing the minimum number of vectors to do so must be a basis.

Example. 1. Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, x + y - z = 0 \right\}$ and $S = \left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$. Then $\dim(V) = 2 = |S|$. By inspection, since the 2 vectors in S are not a multiple of each other, S is linearly independent. Finally, but substituting the vectors in S into the equation defining V, we can see that $S \subseteq V$. Hence, by (B1), S is a basis for V.

2. $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \middle| \, x_1 - 2x_2 + x_3 = 0, x_2 + x_3 - 2x_4 = 0 \right\}$ and $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}$.
By inspection, S is linearly independent and $S \subseteq V$. Since the 2 equations defining V are not a multiple of each other and there are 4 variables, any general solution of V must have 2 parameters, and thus $\dim(V) = 2$. Since $|S| = 2$, S is a basis for V.

3. Let $S = \left\{ \begin{pmatrix} 0 \\ 2 \\ 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ 1 \\ 4 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ and $T = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$.
We will show that S and T are both bases for the same subspace. Firstly, $\text{span}(T) \subseteq$

span(S) follows from

$$\left(\begin{array}{cccc|c|c|c|c} 0 & 0 & 4 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 6 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 4 & 1 & 0 & 0 & 0 & 1 \\ -1 & -1 & -3 & -1 & 0 & 0 & 1 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c|c|c|c} 1 & 0 & 0 & 0 & -1/4 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 0 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/2 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Now clearly T is linearly independent, and since $|T| = |S|$, by (B2), both S and T are bases for the same subspace.

4 Vector Spaces Associated the a Matrix

4.1 Column and Row Spaces

Let $\mathbf{A}$ be an $m \times n$ matrix,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

The rows of $\mathbf{A}$,

$$\mathbf{r}_1 = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{pmatrix}, \mathbf{r}_2 = \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix}, ..., \mathbf{r}_m = \begin{pmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

are vectors in $\mathbb{R}^n$, and the columns of $\mathbf{A}$,

$$\mathbf{c}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \mathbf{c}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, ..., \mathbf{c}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix},$$

are vectors in $\mathbb{R}^m$.

Define the row space of $\mathbf{A}$, denoted as Row(A), to be the subspace of $\mathbb{R}^n$ spanned by the rows of $\mathbf{A}$,

$$\text{Row}(\mathbf{A}) = \text{span}\left\{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{pmatrix}, \begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix}, ..., \begin{pmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}\right\},$$

and the column space of $\mathbf{A}$, denoted as Col(A), to be the subspace of $\mathbb{R}^m$ spanned by the columns of $\mathbf{A}$,

$$\text{Col}(\mathbf{A}) = \text{span}\left\{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, ..., \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}\right\}.$$

Example. $\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix}$. Then the column space of $\mathbf{A}$ is

$$\mathrm{Col}(\mathbf{A}) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \right\} = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\},$$

and the row space is

$$\begin{aligned} \mathrm{Row}(\mathbf{A}) &= \mathrm{span}\{\begin{pmatrix} 1 & 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 & 2 \end{pmatrix}\} \\ &= \mathrm{span}\{\begin{pmatrix} 1 & 0 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 2 \end{pmatrix}\}. \end{aligned}$$

Remark. Note that even though the rows of a $m \times n$ matrix is a $1 \times n$ matrix, one may still use a column vector, that is a $n \times 1$ matrix, to represent it. For example, one may write the row space of the matrix $\mathbf{A}$ in the above example as

$$\mathrm{Row}(\mathbf{A}) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} \right\}.$$

Similarly for the column vectors and the column space.

In the example above, we are able to find a basis for the column and row space easily by inspection. We will now discuss how we can find bases for column and row space of a matrix in general. We need the following results.

Theorem (Row operations preserve row space)**.** *Suppose* $\mathbf{A}$ *and* $\mathbf{B}$ *are row equivalent matrices. Then* $\mathrm{Row}(\mathbf{A}) = \mathrm{Row}(\mathbf{B})$.

In words, the theorem says that row operations preserve the row space of a matrix. We will demonstrate this theorem with an example.

Example. Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 4 & 1 & 0 \\ 1 & 5 & -4 & -9 \end{pmatrix}$. Let $\mathbf{B}$ be obtained from $\mathbf{A}$ via the following row operations.

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 4 & 1 & 0 \\ 1 & 5 & -4 & -9 \end{pmatrix} \xrightarrow{R_2-2R_1} \xrightarrow{R_3-R_1} \xrightarrow{R_3-2R_2} \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 2 & -3 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{B}.$$

We will show that $\mathrm{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ -4 \\ -9 \end{pmatrix} \right\} = \mathrm{Row}(\mathbf{A}) = \mathrm{Row}(\mathbf{B}) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -3 \\ -6 \end{pmatrix} \right\}$.

Indeed,

$$\left(\begin{array}{ccc|c|c} 1 & 2 & 1 & 1 & 0 \\ 1 & 4 & 5 & 1 & 2 \\ 2 & 1 & -4 & 2 & -3 \\ 3 & 0 & -9 & 3 & -6 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c|c} 1 & 0 & -3 & 1 & -2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

shows that $\text{Row}(\mathbf{B}) \subseteq \text{Row}(\mathbf{A})$, and

$$\left(\begin{array}{cc|c|c|c} 1 & 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 4 & 5 \\ 2 & -3 & 2 & 1 & -4 \\ 3 & -6 & 3 & 0 & -9 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c|c|c} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

shows that $\text{Row}(\mathbf{A}) \subseteq \text{Row}(\mathbf{B})$. Furthermore, this shows that $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -3 \\ -6 \end{pmatrix} \right\}$ is a basis for both row spaces, since it is clearly linearly independent.

Now suppose $\mathbf{R}$ is the reduced row-echelon form of a matrix $\mathbf{A}$. By the theorem above, $\text{Row}(\mathbf{A}) = \text{Row}(\mathbf{R})$. Observe that since $\mathbf{R}$ is in reduced row-echelon form, necessarily the nonzero rows of $\mathbf{R}$ are linearly independent. This is because the leading entries in the nonzero rows are the only entries that are nonzero in its column, and hence the nonzero rows cannot be written as a linear combination of the other rows. Clearly the nonzero rows of $\mathbf{R}$ spans the row space of $\mathbf{R}$. Hence, the nonzero rows of $\mathbf{R}$ form a basis for the row space of $\mathbf{R}$, and thus form a basis for the row space of $\mathbf{A}$.

Theorem (Finding a basis for row space)**.** *For any matrix* $\mathbf{A}$*, the nonzero rows of the reduced row-echelon form of* $\mathbf{A}$ *form a basis for* $\text{Row}(\mathbf{A})$*.*

Theorem (Row operations preserve linear relations)**.** *Let* $\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix}$ *and* $\mathbf{B} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix}$ *be row equivalent* $m \times n$ *matrices, where* $\mathbf{a}_i$ *and* $\mathbf{b}_i$ *are the* i*-th column of* $\mathbf{A}$ *and* $\mathbf{B}$*, respectively, for* $i = 1, ..., n$*. Then for any* $c_1, c_2, ..., c_n \in \mathbb{R}^n$*, if*

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$$

then (for the same coefficients $c_1, c_2, ..., c_n \in \mathbb{R}$*),*

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n = \mathbf{0}.$$

Proof. Tutorial 7. □

In words, the theorem says that row operations preserve the linear relations of the columns of a matrix. That is, if one of the column in $\mathbf{A}$ is a linear combination of the other columns, then the corresponding column in $\mathbf{B}$ linearly depends on the other columns with exactly the same coefficients.

Now suppose $\mathbf{R}$ is the reduced row-echelon form of a matrix $\mathbf{A}$. Then the non-pivot columns of $\mathbf{R}$ can be written as a linear combination as the pivot columns. Hence, we have the same linear dependency in the corresponding columns of $\mathbf{A}$. Also, the pivot columns of $\mathbf{R}$ form a linearly independent set. Thus, the pivot columns of $\mathbf{R}$ form a basis for the column space of $\mathbf{R}$. Thus, the corresponding columns of $\mathbf{A}$ form a basis for the column space of $\mathbf{A}$.

Theorem (Finding basis for column space)**.** *Suppose* $\mathbf{R}$ *is the reduced row-echelon form of a matrix* $\mathbf{A}$*. Then the columns of* $\mathbf{A}$ *corresponding to the pivot columns in* $\mathbf{R}$ *form a basis for the column space of* $\mathbf{A}$*.*

Example. $\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 4 & 2 & 2 & 3 & 2 \\ 2 & 1 & -2 & 2 & 0 \\ 6 & 3 & 6 & 4 & 4 \end{pmatrix} \longrightarrow \mathbf{R} = \begin{pmatrix} 1 & 1/2 & 0 & 5/6 & 1/3 \\ 0 & 0 & 1 & -1/6 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Let $\mathbf{r}_i$ and $\mathbf{a}_i$ be the i-th column of $\mathbf{R}$ and $\mathbf{A}$, respectively, for $i = 1, ..., 5$. Then observe that

$$\begin{aligned} \mathbf{r}_2 &= \tfrac{1}{2}\mathbf{r}_1, & \mathbf{r}_4 &= \tfrac{1}{6}(5\mathbf{r}_1 - \mathbf{r}_3), & \mathbf{r}_5 &= \tfrac{1}{3}(\mathbf{r}_1 + \mathbf{r}_3), \quad \text{and} \\ \mathbf{a}_2 &= \tfrac{1}{2}\mathbf{a}_1, & \mathbf{a}_4 &= \tfrac{1}{6}(5\mathbf{a}_1 - \mathbf{a}_3), & \mathbf{a}_5 &= \tfrac{1}{3}(\mathbf{a}_1 + \mathbf{a}_3). \end{aligned}$$

Hence, $\left\{ \begin{pmatrix} 2 \\ 4 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ -2 \\ 6 \end{pmatrix} \right\}$ form a basis for $\mathrm{Col}(\mathbf{A})$.

Next, clearly $\left\{ \begin{pmatrix} 1 & 1/2 & 0 & 5/6 & 1/3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & -1/6 & 1/3 \end{pmatrix} \right\}$ is a linearly independent set. Hence, it form a basis for $\mathrm{Row}(\mathbf{A})$.

Remark.

1. Observe that in all the examples above, the dimension of the column space of $\mathbf{A}$ is equal to the dimension of the row space of $\mathbf{A}$.

2. Row operations do not preserve column space. Consider

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then column space of $\mathbf{A}$ is the x-axis in $\mathbb{R}^2$ while the row space of $\mathbf{B}$ is the y-axis in $\mathbb{R}^2$.

3. Row operations do no preserve linear relations of the rows. Consider

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then the second row or $\mathbf{A}$ is twice the first row, but the second row of $\mathbf{B}$ is zero times the first row.

The theorems above are not only useful in finding bases of the row space and column space of a matrix $\mathbf{A}$, but also to find basis of subspaces.

1. Let $S = \left\{ \mathbf{u}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 3 \\ 6 \\ 6 \\ 3 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 4 \\ 9 \\ 9 \\ 5 \end{pmatrix}, \mathbf{u}_4 = \begin{pmatrix} -2 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{u}_5 = \begin{pmatrix} 5 \\ 8 \\ 9 \\ 4 \end{pmatrix}, \mathbf{u}_6 = \begin{pmatrix} 4 \\ 2 \\ 7 \\ 3 \end{pmatrix} \right\}$ and $V = \mathrm{span}(S)$.

   Suppose we want to find a subset of S that forms a basis for the subspace V, we use the column method. That is, we arrange the vectors as columns of a matrix $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_6 \end{pmatrix}$ and find a basis for the column space of $\mathbf{A}$,

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 4 & -2 & 5 & 4 \\ 2 & 6 & 9 & -1 & 8 & 2 \\ 2 & 6 & 9 & -1 & 9 & 7 \\ 1 & 3 & 5 & 1 & 4 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 0 & -14 & 0 & -37 \\ 0 & 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

   Hence, a basis for $V = \mathrm{Col}(\mathbf{A})$ is $\{\mathbf{u}_1, \mathbf{u}_3, \mathbf{u}_5\}$.

2. Let S and V be as defined in the previous example. Suppose we want a basis of V such that it is easier to find the coordinates of a vectors in V relative to this basis, we use the row method. That is, we arrange the vectors in S (letting them be row vectors) as rows of a matrix $\mathbf{A} = \begin{pmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_6^T \end{pmatrix}$ and find a basis for the row space of $\mathbf{A}$,

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 3 & 6 & 6 & 3 \\ 4 & 9 & 9 & 5 \\ -2 & -1 & -1 & 1 \\ 5 & 8 & 9 & 4 \\ 4 & 2 & 7 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ form a basis for V. Now we can easily write any vector in V relative to this basis, for example,

$$\mathbf{u}_6 = 4 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 7 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

3. Let $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$. Suppose we want extend S to form a basis for $\mathbb{R}^4$. We will use the row method.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix} \longrightarrow \mathbf{R} = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -2 \end{pmatrix}.$$

Observe that if we add the rows $\begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$ to the bottom of the matrix $\mathbf{R}$, we get

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which is clearly invertible, and thus the rows span $\mathbb{R}^4$. Hence, if we add the same two rows into the original matrix $\mathbf{A}$, then the row space of the resultant matrix is also the whole $\mathbb{R}^4$. Therefore $S \cup \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ form a basis for $\mathbb{R}^4$.

Appendix for Lecture 8

Recall that for each $\mathbf{v} \in V$, the coordinates of $\mathbf{v}$ relative to the basis S is the unique vector

$$(\mathbf{v})_S = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} \in \mathbb{R}^k,$$

where $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k$.

Lemma. *Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ be a basis for a subspace $V \subseteq \mathbb{R}^n$. For any $\mathbf{u}, \mathbf{v} \in V$ and $\alpha, \beta \in \mathbb{R}$*

$$(\alpha\mathbf{u} + \beta\mathbf{v})_S = \alpha(\mathbf{u})_S + \beta(\mathbf{v})_S$$

Proof. Let $(\mathbf{u})_S = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$ and $(\mathbf{v})_S = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_k \end{pmatrix}$. Then

$$\begin{aligned} \alpha\mathbf{u} + \beta\mathbf{v} &= \alpha(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k) + \beta(d_1\mathbf{u}_1 + d_c\mathbf{u}_2 + \cdots + d_k\mathbf{u}_k) \\ &= (\alpha c_1 + \beta d_1)\mathbf{u}_1 + (\alpha c_2 + \beta d_2)\mathbf{u}_2 + \cdots + (\alpha c_k + \beta d_k)\mathbf{u}_k. \end{aligned}$$

So

$$(\alpha\mathbf{u} + \beta\mathbf{v})_S = \begin{pmatrix} \alpha c_1 + \beta d_1 \\ \alpha c_2 + \beta d_2 \\ \vdots \\ \alpha c_k + \beta d_k \end{pmatrix} = \alpha \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix} + \beta \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_k \end{pmatrix} = \alpha(\mathbf{u})_S + \beta(\mathbf{v})_S.$$

□

Lemma. *Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is a basis for a subspace $V \subseteq \mathbb{R}^n$. Then for $\mathbf{0}_{n\times 1} \in \mathbb{R}^n$,*

$$(\mathbf{0}_{n\times 1})_S = \mathbf{0}_{k\times 1}.$$

Proof. Exercise. □

Theorem (Criteria for linear independence of a subset in a subspace)**.** *Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is a basis for a subspace $V \subseteq \mathbb{R}^n$. A subset*

$$\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\} \subseteq V$$

is linearly independent if and only if

$$\{(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, ..., (\mathbf{v}_m)_S\} \subseteq \mathbb{R}^k$$

is linearly independent.

Proof. The theorem follows from the following equivalence derived from two lemmas above,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = \mathbf{0}_{n\times 1} \Leftrightarrow c_1(\mathbf{v}_1)_S + c_2(\mathbf{v}_2)_S + \cdots + c_m(\mathbf{v}_m)_S = \mathbf{0}_{k\times 1}.$$

Suppose $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ is linearly independent. Then if $c_1(\mathbf{v}_1)_S + c_2(\mathbf{v}_2)_S + \cdots + c_m(\mathbf{v}_m)_S = \mathbf{0}_{k\times 1}$, the left side of the equivalence above tells us that necessarily $c_1 = c_2 = \cdots = c_m = 0$. So $\{(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, ..., (\mathbf{v}_m)_S\}$ is a linearly independent subset of $\mathbb{R}^k$.

Conversely, suppose $\{(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, ..., (\mathbf{v}_m)_S\}$ is linearly independent. Then if $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = \mathbf{0}_{n\times 1}$, the right side of the equivalence tells us that necessarily $c_1 = c_2 = \cdots = c_m = 0$. So $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ is a linearly independent subset of $\mathbb{R}^n$ □

Corollary. *Let $V \subseteq \mathbb{R}^n$ be a k-dimensional subspace. Any subset of V containing more than k vectors must be linearly dependent.*

Proof. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is a basis for a subspace $V \subseteq \mathbb{R}^n$. By the previous theorem, a subset $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ of V is linearly independent if and only if $\{(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, ..., (\mathbf{v}_m)_S\}$ is linearly independent in $\mathbb{R}^k$. Thus if $m > k$, then $\{(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, ..., (\mathbf{v}_m)_S\}$ cannot be linearly independent in $\mathbb{R}^k$. □

Theorem. *Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ and $T = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ are bases for a subspace $V \subseteq \mathbb{R}^n$. Then $k = m$.*

Proof. For $i = 1, ..., m$, write the vectors $\mathbf{v}_i$ in coordinate relative to basis S,

$$T' = \{(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, ..., (\mathbf{v}_m)_S\} \subseteq \mathbb{R}^k.$$

Since T is linearly independent, by the previous theorem, T' is a linearly independent set in $\mathbb{R}^k$. This means that necessarily $l \leq k$.

We now exchange the roles of S and T. That is, for $i = 1, ..., k$, write the vectors $\mathbf{u}_i$ in S in coordinates relative to the basis T,

$$S' = \{(\mathbf{u}_1)_T, (\mathbf{u}_2)_T, ..., (\mathbf{u}_k)_T\} \subseteq \mathbb{R}^l.$$

Since S is linearly independent, by the previous theorem, S' is a linearly independent set in $\mathbb{R}^k$. This means that necessarily $k \leq l$.

Therefore, we have equality, $k = l$. □

By the criteria of linear independence of a subset in a subspace, we can conclude that if $V \subseteq \mathbb{R}^n$ is a k-dimensional subspace, then any set containing more than k number of vectors in V must be linearly independent. We have a similar criteria for whether a subset spans V.

Theorem (Criteria for a subset to span a subspace)**.** *Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is a basis for a subspace $V \subseteq \mathbb{R}^n$. A subset*

$$\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\} \subseteq V$$

spans V if and only if

$$\{(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, ..., (\mathbf{v}_m)_S\} \subseteq \mathbb{R}^k$$

spans $\mathbb{R}^k$.

Proof. ($\Rightarrow$) Suppose $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ spans V. Given any $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_k \end{pmatrix} \in \mathbb{R}^k$, let

$$\mathbf{v} = w_1\mathbf{u}_1 + w_2\mathbf{u}_2 + \cdots + w_k\mathbf{u}_k \in V.$$

By construction, $(\mathbf{v})_S = \mathbf{w}$. Since $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ spans V, we can find $d_1, d_2, ..., d_m \in \mathbb{R}$ such that $\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_m\mathbf{v}_m$. Then by a lemma above,

$$d_1(\mathbf{v}_1)_S + d_2(\mathbf{v}_2)_S + \cdots + d_m(\mathbf{v}_m)_S = (\mathbf{v})_S = \mathbf{w}.$$

This shows that any $\mathbf{w} \in \mathbb{R}^k$ can be written as a linear combination of $\{(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, ..., (\mathbf{v}_m)_S\}$, and thus the set spans $\mathbb{R}^k$.

($\Leftarrow$) Suppose $\{(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, ..., (\mathbf{v}_m)_S\}$ spans $\mathbb{R}^k$. Now given any $\mathbf{v} \in V$, let

$$\mathbf{v} = w_1\mathbf{u}_1 + w_2\mathbf{u}_2 + \cdots + w_k\mathbf{u}_k \in V,$$

that is $(\mathbf{v})_S = \mathbf{w} = (w_i) \in \mathbb{R}^k$. Then since $\{(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, ..., (\mathbf{v}_m)_S\}$ spans $\mathbb{R}^k$, we can find $d_1, d_2, ..., d_m \in \mathbb{R}$ such that $(\mathbf{v})_S = \mathbf{w} = d_1(\mathbf{v}_1)_S + d_2(\mathbf{v}_2)_S + \cdots + d_m(\mathbf{v}_m)_S$. Then by the same lemma above,

$$d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_m\mathbf{v}_m = (\mathbf{v})_S = \mathbf{v}.$$

This shows that any $\mathbf{v} \in V$ can be written as a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$, and thus the set spans V.

$\square$

Corollary. *Let $V \subseteq \mathbb{R}^n$ be a k-dimensional subspace. Any subset of V containing less than k vectors cannot span V.*

Proof. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is a basis for a subspace $V \subseteq \mathbb{R}^n$. Given a subset $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\} \subseteq V$, $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ spans V if and only if $\{(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, ..., (\mathbf{v}_m)_S\}$ spans $\mathbb{R}^k$. But if $m < k$, then if $\{(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, ..., (\mathbf{v}_m)_S\}$ cannot span $\mathbb{R}^k$, and hence $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ cannot span V. $\square$

Combining all the results we have thus far, we obtain the following statement.

Corollary. *Let V be a k-dimensional subspace of $\mathbb{R}^n$. Then $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m\}$ is a basis for V if and only if $m = k$ and the order k square matrix $\mathbf{A} = \begin{pmatrix} (\mathbf{v}_1)_S & (\mathbf{v}_2)_S & \cdots & (\mathbf{v}_m)_S \end{pmatrix}$ is invertible.*

Lemma. *Suppose $U, V \subseteq \mathbb{R}^n$ are subspaces of $\mathbb{R}^n$ such that U is a subset of V, $U \subseteq V$. Then $\dim(U) \leq \dim(V)$, with equality if and only if $U = V$.*

Proof. Suppose $\dim(U) = k$. Let $S = \{\mathbf{u}_1, ..., \mathbf{u}_k\}$ be a basis for U. Since $U \subseteq V$, S is a linearly independent set in V. Suppose S spans V too. Then $U = \text{span}(S) = V$ and thus $\dim(U) = \dim(V)$. Otherwise, $\text{span}(S) \subsetneqq V$ is not the whole V. Then we can find a $v \in V \backslash \text{span}(S)$ that is in V but not in the span of S. This would mean that $\{\mathbf{u}_1, ..., \mathbf{u}_k, \mathbf{v}\}$ is a linearly independent subset of V, since v is not a linearly combination of the vectors in S. Thus, $\dim(V) \geq k + 1 > k = \dim(U)$. $\square$

Theorem. *The zero space is 0-dimensional. The empty set $\varnothing = \{\}$ is the basis for the zero space.*

Proof. (i) It is vacuously true that the empty set is linearly independent.

(ii) The more general (accurate) definition of the span of a set S is the smallest subspace containing S. It turns out that if S is a finite set, then the span is indeed all possible linear combination of the vectors in S. Now since every subspace contains the empty set as a subset, the smallest one to do so is the zero space. Hence, by definition, the zero space is the span of the empty set.

$\square$

Theorem (Row operations preserve row space)**.** *Suppose* $\mathbf{A}$ *and* $\mathbf{B}$ *are row equivalent matrices. Then* $\mathrm{Row}(\mathbf{A}) = \mathrm{Row}(\mathbf{B})$*.*

Proof. Suffice to show for the case that $\mathbf{B} = \mathbf{EA}$ where $\mathbf{E}$ is an elementary matrix. This is because if it was so, then in general, if $\mathbf{B} = \mathbf{E}_k \cdots \mathbf{E}_2\mathbf{E}_1\mathbf{A}$, then $\mathrm{Row}(\mathbf{B}) = \mathrm{Row}(\mathbf{E}_{k-1} \cdots \mathbf{E}_2\mathbf{E}_1\mathbf{A}) = \cdots = \mathrm{Row}(\mathbf{E}_2\mathbf{E}_1\mathbf{A}) = \mathrm{Row}(\mathbf{E}_1\mathbf{A}) = \mathrm{Row}(\mathbf{A})$.

Let $\mathbf{A}$ and $\mathbf{B}$ be $m \times n$ matrices. Write $\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{pmatrix}$, where $\mathbf{a}_i$ is the i-th row of $\mathbf{A}$ for $i = 1, ..., m$.

Suppose $\mathbf{E}$ corresponds to multiplying row i by a nonzero constant $c \in \mathbb{R}$, $c \neq 0$,

$$\mathbf{E} \leftrightarrow cR_i.$$

Since $c \neq 0$, it is clear that

$$\begin{aligned} \mathrm{Row}(\mathbf{A}) &= \mathrm{span}\{\mathbf{a}_1, ..., \mathbf{a}_{i-1}, \mathbf{a}_i, \mathbf{a}_{i+1}, ..., \mathbf{a}_m\} \\ &= \mathrm{span}\{\mathbf{a}_1, ..., \mathbf{a}_{i-1}, c\mathbf{a}_i, \mathbf{a}_{i+1}, ..., \mathbf{a}_m\} = \mathrm{Row}(\mathbf{B}). \end{aligned}$$

Suppose $\mathbf{E}$ corresponds to a row swap between the i-th row and the j-th row,

$$\mathbf{E} \leftrightarrow R_i \leftrightarrow R_j.$$

Then it is obvious that the row spaces of $\mathbf{A}$ and $\mathbf{B}$ are equal.

Suppose $\mathbf{E}$ corresponds to adding a times of the j-th row to the i-th row for some $a\mathbb{R}$,

$$\mathbf{E} \leftrightarrow R_i + aR_j.$$

Then clearly

$$\mathrm{Row}(\mathbf{B}) = \mathrm{span}\{\mathbf{a}_1, ..., \mathbf{a}_{i-1}, \mathbf{a}_i + a\mathbf{a}_j, \mathbf{a}_{i+1} ..., \mathbf{a}_m\} \subseteq \mathrm{span}\{\mathbf{a}_1, ..., \mathbf{a}_{i-1}, \mathbf{a}_i, \mathbf{a}_{i+1}, ..., \mathbf{a}_m\}.$$

Note that the j-th row of $\mathbf{A}$ and $\mathbf{B}$ are both $\mathbf{a}_j$, and since $\mathbf{a}_i = (\mathbf{a}_i + a\mathbf{a}_j) - a\mathbf{a}_j$, it shows that $\mathbf{a}_i \in \mathrm{Row}(\mathbf{B})$. Hence, $\mathrm{Row}(\mathbf{A}) \subseteq \mathrm{Row}(\mathbf{B})$ too. $\square$