NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

AY2021, Semester 1 MA1508E Linear Algebra for Engineering Tutorial 4

Solutions

1. (a) Suppose $\mathbf{A} = \mathbf{PDP}^{-1}$ for some invertible matrix $\mathbf{P}$. Show that $\det(\mathbf{A}) = \det(\mathbf{D})$.

   $\det(\mathbf{A}) = \det(\mathbf{PDP}^{-1}) = \det(\mathbf{P})\det(\mathbf{D})\det(\mathbf{P}^{-1}) = \det(\mathbf{P})\det(\mathbf{P})^{-1}\det(\mathbf{D}) = \det(\mathbf{D})$. The second equality follows from commutativity of multiplication of real numbers, and that $\det(\mathbf{P}^{-1}) = \det(\mathbf{P})^{-1}$.

   (b) Suppose $\mathbf{A} = \mathbf{PDP}^{-1}$ for some invertible matrix $\mathbf{P}$ and $\mathbf{D}$ is a diagonal matrix. Show that $\mathbf{A}$ is invertible if and only if all the diagonal entries of $\mathbf{D}$ is nonzero.

   From (a), we have $\det(\mathbf{A}) = \det(\mathbf{D}) = d_{11}d_{22}\cdots d_{nn}$, where d_{ii} is the i-th diagonal entry of $\mathbf{D}$. Thus $\det(\mathbf{A})$ is nonzero if any only if $d_{ii} \neq 0$ for all $i = 1, ..., n$.

   (c) Recall that a square matrix $\mathbf{A}$ is nilpotent if there is a positive integer k such that $\mathbf{A}^k = \mathbf{0}$. Show that if $\mathbf{A}$ is nilpotent, then $\det(\mathbf{A}) = 0$.

   $0 = \det(\mathbf{A}^k) = \det(\mathbf{A})^k \Rightarrow \det(\mathbf{A}) = 0$ since $\det(\mathbf{A})$ is a real number.

   (d) A square matrix is an *orthogonal* matrix if $\mathbf{A}^T = \mathbf{A}^{-1}$. Show that if $\mathbf{A}$ is orthogonal, then $\det(\mathbf{A}) = \pm 1$.

   Follows from $1 = \det(\mathbf{A}^{-1}\mathbf{A}) = \det(\mathbf{A}^T)\det(\mathbf{A}) = \det(\mathbf{A})^2$, since $\det(\mathbf{A}^T) = \det(\mathbf{A})$. Alternatively, $\det(\mathbf{A})^{-1} = \det(\mathbf{A}^{-1}) = \det(\mathbf{A}^T) = \det(\mathbf{A})$ tells us that $\det(\mathbf{A}) = \pm 1$.

2. Let $\mathbf{A}$ be a $k \times k$ matrix and let $\mathbf{B}$ be a $(n-k) \times (n-k)$ matrix. Let

$$\mathbf{E} = \begin{pmatrix} \mathbf{I_k} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_{n-k}} \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix},$$

where $\mathbf{I_k}$ and $\mathbf{I_{n-k}}$ are the $k \times k$ and $(n-k) \times (n-k)$ identity matrices respectively.

   (a) Show that $\det(\mathbf{E}) = \det(\mathbf{B})$.

   By cofactor expansion along the first row,

$$\begin{vmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{vmatrix} = \begin{vmatrix} \mathbf{I}_{k-1\times k-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{vmatrix}.$$

   Again, by cofactor expansion along the first row,

$$\begin{vmatrix} \mathbf{I}_{k-1\times k-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{vmatrix} = \begin{vmatrix} \mathbf{I}_{k-2\times k-2} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{vmatrix}.$$

   Continuing this way, (by performing cofactor expansion along the first row each time), we have the desired result that $\det(\mathbf{E}) = \det(\mathbf{B})$.

   (b) Show that $\det(\mathbf{F}) = \det(\mathbf{A})$.

By cofactor expansion along the last row,

$$\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-k\times n-k} \end{vmatrix} = \begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-k-1\times n-k-1} \end{vmatrix}.$$

Again, by cofactor expansion along the last row,

$$\begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-k-1\times n-k-1} \end{vmatrix} = \begin{vmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-k-2\times n-k-2} \end{vmatrix}.$$

Continuing this way, (by performing cofactor expansion along the last row each time), we have the desired result that $\det(\mathbf{F}) = \det(\mathbf{A})$.

(c) Show that $\det(\mathbf{C}) = \det(\mathbf{A})\det(\mathbf{B})$.

Observe that

$$\mathbf{C} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} = \mathbf{EF}.$$

So $\det(\mathbf{C}) = \det(\mathbf{EF}) = \det(\mathbf{E})\det(\mathbf{F}) = \det(\mathbf{A})\det(\mathbf{B})$.

Hint: For (a) and (b) use cofactor expansions. For (c), try to write the matrix $\mathbf{C}$ as a product of (block) matrices.

3. Let $\mathbf{A} = \begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$ (cf: Tutorial 3 question 1(b)).

(a) What is $\det(\mathbf{A})$?

$\det(\mathbf{A}) = 0$ since we have shown in tutorial 3 that $\mathbf{A}$ is singular.

(b) Suppose $\mathbf{B}$ is an order 3 square matrix. Show that the homogeneous linear system $\mathbf{ABx} = \mathbf{0}$ have infinitely many solutions.

$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}) = 0\det(\mathbf{B}) = 0$. So $\mathbf{AB}$ is also singular, hence the the homogeneous linear system $\mathbf{ABx} = \mathbf{0}$ must have infinitely many solutions.

Remark: This proof wont work if $\mathbf{B}$ is not a square matrix. For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$$

only has the trivial solution.

4. (a) Consider the follow linear system (cf: Tutorial 1 question 1(b))

$$\begin{cases} a + b - c - 2d = 0 \\ 2a + b - c + d = -2 \\ -a + b - 3c + d = 4 \end{cases}$$

Express the solutions in the set notation.

From tutorial 1, we have found that the general solution is $a = -2 - 3s, b = 2 + \frac{19s}{2}, c = \frac{9s}{2}, d = s$, $s \in \mathbb{R}$. So the solution in set notation is

$$\left\{ \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3 \\ \frac{19}{2} \\ \frac{9}{2} \end{pmatrix} \middle| \; s \in \mathbb{R} \right\}.$$

(b) Suppose a linear system has reduced row-echelon form

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & -2 \\ 0 & 1 & 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Express the solutions in the set notation.

The solution set is

$$\left\{ \begin{pmatrix} -2 \\ 3 \\ -2 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} \frac{1}{2} \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{1}{2} \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}.$$

5. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthonormal set. Suppose

$$\mathbf{x} = \mathbf{v}_1 - 2\mathbf{v}_2 - 2\mathbf{v}_3 \quad \text{and} \quad \mathbf{y} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3.$$

Determine the value for each of the following (you may use your calculators for this question.)

(a) $\mathbf{x} \cdot \mathbf{y}$.

Note that $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$. Furthermore, since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal set, $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ for $i = 1, 2, 3$.

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &= (\mathbf{v}_1 - 2\mathbf{v}_2 - 2\mathbf{v}_3) \cdot (2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3) \\ &= 2(\mathbf{v}_1 \cdot \mathbf{v}_1) + 6(\mathbf{v}_2 \cdot \mathbf{v}_2) - 2(\mathbf{v}_3 \cdot \mathbf{v}_3) \\ &= 2 + 6 - 2 = 6. \end{aligned}$$

(b) $||\mathbf{x}||$ and $||\mathbf{y}||$.

$$\begin{aligned} ||\mathbf{x}|| &= \sqrt{\mathbf{x} \cdot \mathbf{x}} \\ &= \sqrt{(\mathbf{v}_1 \cdot \mathbf{v}_1) + 4(\mathbf{v}_2 \cdot \mathbf{v}_2) + 4(\mathbf{v}_3 \cdot \mathbf{v}_3)} \\ &= \sqrt{1 + 4 + 4} = 3 \\ ||\mathbf{y}|| &= \sqrt{\mathbf{y} \cdot \mathbf{y}} \\ &= \sqrt{4(\mathbf{v}_1 \cdot \mathbf{v}_1) + 9(\mathbf{v}_2 \cdot \mathbf{v}_2) + (\mathbf{v}_3 \cdot \mathbf{v}_3)} \\ &= \sqrt{4 + 9 + 1} = \sqrt{14} \end{aligned}$$

(c) The angle θ between $\mathbf{x}$ and $\mathbf{y}$.

$$\cos(\theta) = \frac{6}{3\sqrt{14}} \Rightarrow \theta = \cos^{-1}\frac{2}{\sqrt{14}} = 57.69^o$$

6. (a) Let $a_1x_1+a_2x_2+\cdots+a_nx_n=b$ be a linear equation. Express this linear system as $\mathbf{a}\cdot\mathbf{x}=b$ for some (column) vectors $\mathbf{a}$ and $\mathbf{x}$.

$$\mathbf{a}=\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix},\ \mathbf{x}=\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}.$$

(b) Find the solution set of the linear system

$$\begin{array}{rcrcrcrl} x_1 & + & 3x_2 & - & 2x_3 & & & =0\\ 2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & =0\\ & & & + & 5x_3 & + & 10x_4 & =0 \end{array}$$

The RREF of the matrix coefficient is

$$\begin{pmatrix}1&3&0&4\\0&0&1&2\\0&0&0&0\end{pmatrix}$$

So the solution set is

$$\left\{\ s\begin{pmatrix}-3\\1\\0\\0\end{pmatrix}+t\begin{pmatrix}-4\\0\\-2\\1\end{pmatrix}\ \middle|\ s,t\in\mathbb{R}\ \right\}$$

(c) Find a nonzero vector $\mathbf{v}\in\mathbb{R}^3$ such that $\mathbf{a}_1\cdot\mathbf{v}=0$, $\mathbf{a}_2\cdot\mathbf{v}=0$, and $\mathbf{a}_3\cdot\mathbf{v}=0$, where

$$\mathbf{a}_1=\begin{pmatrix}1\\3\\-2\\0\end{pmatrix},\qquad \mathbf{a}_2=\begin{pmatrix}2\\6\\-5\\-2\end{pmatrix},\qquad \mathbf{a}_3=\begin{pmatrix}0\\0\\5\\10\end{pmatrix}$$

Write $\mathbf{v}=\begin{pmatrix}v_1\\v_2\\v_3\\v_4\end{pmatrix}$. Then $\mathbf{a}_1\cdot\mathbf{v}=0$, $\mathbf{a}_2\cdot\mathbf{v}=0$, and $\mathbf{a}_3\cdot\mathbf{v}=0$ is equivalent to solving the following linear system

$$\begin{array}{rcrcrcrl} v_1 & + & 3v_2 & - & 2v_3 & & & =0\\ 2v_1 & + & 6v_2 & - & 5v_3 & - & 2v_4 & =0\\ & & & + & 5v_3 & + & 10v_4 & =0 \end{array}$$

From (b), we may choose $s=1$ and $t=0$, that is, $\mathbf{v}=\begin{pmatrix}-3\\1\\0\\0\end{pmatrix}$.

This exercise demonstrates the fact that if $\mathbf{A}$ is a $m\times n$ matrix, then the solution set of the homogeneous linear system $\mathbf{Ax}=\mathbf{0}$ consist of all the vectors in $\mathbb{R}^n$ that are orthogonal to every row vector of $\mathbf{A}$.

Supplementary Problems

7. **(Application)** (Statistics)
Suppose in a math test, the results of a class of n students are $x_1, x_2, ..., x_n$. We can represent the result as a *sample vector*

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

The sample *mean*, $\overline{x}$ is defined by

$$x = \frac{1}{n}(x_1 + x_2 + \cdots + x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i$$

The *centred sample vector* $\mathbf{x}_c$ is define as

$$\mathbf{x}_c = \begin{pmatrix} x_1 - \overline{x} \\ x_2 - \overline{x} \\ \vdots \\ x_n - \overline{x} \end{pmatrix}$$

The sample *variance* $\sigma^2_{\mathbf{x}}$ is defined as

$$\sigma^2_{\mathbf{x}} = \frac{1}{n-1}((x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + \cdots + (x_n - \overline{x})^2) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$$

The square root of the variance $\sigma_{\mathbf{x}}$ is called the *sample standard deviation*. Let

$$\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

denote the vector with enties equal to 1. Express the

(a) mean,
$\overline{x} = \frac{1}{n}\mathbf{x} \cdot \mathbf{1}$.

(b) centred sample vector,
$\mathbf{x}_c = \mathbf{x} - \overline{x}\mathbf{1}$.

(c) variance, and
$\sigma^2_{\mathbf{x}} = \frac{1}{n-1}||\mathbf{x}_c||^2 = \frac{1}{n-1}||\mathbf{x} - \overline{x}\mathbf{1}||^2$.

(d) standard deviation
$\frac{1}{\sqrt{n-1}}||\mathbf{x}_c||$.

using the vector **1**, dot product, and norm.

(**MATLAB**) The vector **1** can be obtained via

```
>> ones(n,1)
```

the dot product between **u** and **v** can be computed via

```
>> dot(u,v)
```

and the norm of **v** is

```
>> norm(v)
```

Suppose the results of a math test of 10 students are $51, 35, 62, 78, 84, 55, 68, 92, 55, 69$. Use MATLAB to compute the

(e) mean,

```
>> x=[51;35;62;78;84;55;68;92;55;69];
```

We can sort the entries of a vector **x** in asscending order via

```
>> x=sort(x);
```

```
>> xmean=(1/10)*dot(x,ones(10,1))
```

ans= 64.9.

We can compute the mean directly in MATLAB using `>> mean(x)`.

(f) centred sample vector,

```
>> xcenter=x-xmean*ones(10,1);
```

$$\text{xcenter}=\begin{pmatrix}-29.9\\-13.9\\-9.9\\-9.\\-2.9\\3.1\\4.1\\13.1\\19.1\\27.1\end{pmatrix}$$

(g) variance, and

```
>> xvar=(1/9)*norm(xcenter)^2
```

ans≈ 287.7.
We can compute the variance directly in MATLAB using `>> var(x)`.

(h) standard deviation

```
>> xstd=(1/sqrt(9))*norm(xcenter)
```

ans≈ 16.96.
We can compute the standard deviation directly in MATLAB using `>> std(x)`.

of the simulated results you obtained. To calculate a percentile of the sample **x**, use

```
>> prctile(x,p)
```

where p is the percentile to be computed.

(i) Calculate the 75-th percentile of the results.

`>> prctile(x,75)` ans= 78

(j) Suppose to obtain an The 80th percentile is 81 marks. 2 students obtained above 81 marks. A grade in the math test a student needs to be above the 80th-percentile. How many students will get A in the math test?