NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

AY2021, Semester 1 MA1508E Linear Algebra for Engineering Tutorial 11

1. **(Application)** Two species of fish, species A and species B, live in the same ecosystem (e.g. a pond) and compete with each other for food, water and space. Let the population of species A and B at time t years be given by $a(t)$ and $b(t)$ respectively.

   In the absence of species B, species A's growth rate is $4a(t)$ but when species B are present, the competition slows the growth of species A to $a'(t) = 4a(t) - 2b(t)$. In a similar manner, when species A is absent, species B's growth rate is $3b(t)$ but in the presence of species A, the growth rate reduces to $b'(t) = 3b(t) - a(t)$.

   (a) Write down a system of linear differential equations involving $a(t), b(t), a'(t)$ and $b'(t)$.

   (b) Represent the system in (i) as $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$ where

   $$\mathbf{A} \text{ is a } 2 \times 2 \text{ matrix and } \mathbf{x}(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}, \quad \mathbf{x}'(t) = \begin{pmatrix} a'(t) \\ b'(t) \end{pmatrix}.$$

   (c) Solve the system using the initial condition $a(0) = 60$, $b(0) = 120$.

2. Instead of a first order system of linear differential equations $\mathbf{Y}' = \mathbf{AY}$ (involving n variables $y_1, y_2, \ldots, y_n$), we may encounter a second order system of the form $\mathbf{Y}'' = \mathbf{A_1Y} + \mathbf{A_2Y}'$. To solve this second order system, we can translate it into a first order system by introducing n additional new variables $y_{n+1}, y_{n+2}, \ldots, y_{2n}$ as follows:

   $$\begin{array}{rcl} y_{n+1}(t) & = & y_1'(t) \\ y_{n+2}(t) & = & y_2'(t) \\ \vdots & & \vdots \\ y_{2n}(t) & = & y_n'(t) \end{array}$$

   Suppose we let

   $$\mathbf{Y_1} = \mathbf{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad \text{and} \quad \mathbf{Y_2} = \mathbf{Y}' = \begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ \vdots \\ y_{2n} \end{pmatrix}.$$

   Then

   $$\mathbf{Y_1'} = \mathbf{0Y_1} + \mathbf{I_nY_2} \quad \text{and} \quad \mathbf{Y_2'} = \mathbf{Y_1''} = \mathbf{A_1Y_1} + \mathbf{A_2Y_2}$$

   The two equations above can be combined to give the first order system with a $2n \times 2n$ matrix as shown:

   $$\begin{pmatrix} \mathbf{Y_1'} \\ \mathbf{Y_2'} \end{pmatrix} = \begin{pmatrix} \mathbf{0_n} & \mathbf{I_n} \\ \mathbf{A_1} & \mathbf{A_2} \end{pmatrix} \begin{pmatrix} \mathbf{Y_1} \\ \mathbf{Y_2} \end{pmatrix}.$$

   In this way, $\mathbf{Y_1}$ (the original $\mathbf{Y}$) and $\mathbf{Y_2}$ (the first derivatives of $\mathbf{Y}$) can now be solved by solving the first order system.

Use the method described above to solve the following second order linear differential equations:

(a)

$$y'' + 2y' + 5y = 0.$$

(b)

$$\begin{array}{rcrcrcrcr} y_1'' &=& 2y_1 &+& y_2 &+& y_1' &+& y_2' \\ y_2'' &=& -5y_1 &+& 2y_2 &+& 5y_1' &-& y_2' \end{array}$$

given the initial condition $y_1(0) = y_2(0) = y_1'(0) = 4$ and $y_2'(0) = -4$.

3. For each of the following homogeneous system of differential equations,

   (i) find a fundamental set of solutions for the system;

   (ii) use Wronskian to verify that your answer in (i) are linearly independent;

   (iii) write down a general solution using the answers in (i);

   (iv) find the solution to the initial value problem.

(a)

$$\begin{array}{rcl} y_1' &=& y_1 \\ y_2' &=& -3y_2 \end{array}, \qquad y_1(1) = e^1,\ y_2(1) = e^{-3}.$$

(b)

$$\begin{array}{rcrcr} y_1' &=& y_1 &-& 2y_2 \\ y_2' &=& 2y_1 &+& y_2 \end{array}, \qquad y_1(0) = 1,\ y_2(0) = -2.$$

(c)

$$\begin{array}{rcrcr} y_1' &=& -8y_1 &-& 5y_2 \\ y_2' &=& 5y_1 &+& 2y_2 \end{array}, \qquad y_1(0) = 1,\ = y_2(0) = 3.$$

(d)

$$\begin{array}{rcrcr} y_1' &=& 3y_1 &+& 2y_2 \\ y_2' &=& -8y_1 &-& 5y_2 \end{array}, \qquad y_1(0) = 3,\ = y_2(0) = 2.$$

Supplementary Problems

4. **(MATLAB)** Consider the following system of linear differential equations

$$\begin{array}{rcrcrcrcrcr} y_1' &=& 2y_1 &+& y_2 &+& y_3 &-& 2y_4 &-& 2y_5 \\ y_2' &=& && y_2 &&&&&& \\ y_3' &=& &&&& 2y_3 &&&& \\ y_4' &=& &&&-& y_3 &+& 2y_4 && \\ y_5' &=& y_1 &+& y_2 &+& 2y_3 &-& y_4 && \end{array}$$

with the initial condition $y_1(0) = y_2(0) = y_3(0) = y_4(0) = y_5(0) = 1$.

(a) The `charpoly` function in MATLAB can be used to compute the characteristic polynomial of a matrix. First create a symbolic variable, say x,

```
>> syms x
```

Then we compute the characteristics polynomial of $\mathbf{A}$

```
>> p=charpoly(A,x)
```

Then use the `factor` function to factorize the characteristics polynomial,

```
>> factor(p,x)
```

Hence or otherwise, solve the system of linear differential equations.

(b) We can use MATLAB command `dsolve` to find the general solution of a system of differential equations $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

```
>> syms y1(t) y2(t) y3(t) y4(t) y4(t);

>> y=[y1; y2; y3; y4; y5];

>> [Sy1 Sy2 Sy3 Sy4 Sy5]=dsolve(diff(y,t)==A*y)
```

If we are solving an initial value problem, we need to input this command line before the last line

```
>> conds=[y1(0)==1, y2(0)==1, y3(0)==1,y4(0)==1,y5(0)==1];
```

and modify the last command line to

```
[Sy1, Sy2, Sy3, Sy4, Sy5]=dsolve(diff(Y,t)==A*Y, conds)
```

Compare the answers obtained to the ones in (a).