NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

AY2021, Semester 2 MA1508E Linear Algebra for Engineering Practice 3
Solutions

1. Let

$$V = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \middle| \begin{pmatrix} 1 & 0 & -1 & 1 \\ 3 & 1 & 1 & -1 \\ 2 & 1 & 0 & 1 \\ a & b & c & d \end{pmatrix} \text{ is singular} \right\}.$$

(a) [4 marks] Show that V is a subspace of $\mathbb{R}^3$.

$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 3 & 1 & 1 & -1 \\ 2 & 1 & 0 & 1 \\ a & b & c & d \end{pmatrix}$ is singular if and only if the determinant is 0.

$$0 = \begin{vmatrix} 1 & 0 & -1 & 1 \\ 3 & 1 & 1 & -1 \\ 2 & 1 & 0 & 1 \\ a & b & c & d \end{vmatrix} = -a + 4b - 3c - 2d.$$

This is a homogeneous linear system, and thus V is a subspace.

(b) [3 marks] Find a basis for V and express V in explicit set form.

A general solution for the above homogeneous system is $a = 4r - 3s - 2t, b = r, c = s, d = t$. Thus

$$\left\{ \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for V.

$$V = \left\{ r\begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \middle| r, s, t \in \mathbb{R} \right\}.$$

(c) [4 marks] Show that $S = \left\{ \begin{pmatrix} 3 \\ 5 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \\ 1 \end{pmatrix} \right\}$ is a basis for V.

Let T be the basis for V found above.

$$\left(\begin{array}{ccc|c|c|c} 3 & 2 & 0 & 4 & -3 & -2 \\ 5 & 3 & 2 & 1 & 0 & 0 \\ 5 & 4 & 2 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 2 & -5/3 & -2/3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -3 & 8/3 & 5/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

shows that $T \subseteq \text{span}(S)$, and thus $V = \text{span}(T) \subseteq \text{span}(S)$. Then since $|S| = 3 = \dim(V)$, S is a basis for V.

(d) [2 marks] Let $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$. Find $(\mathbf{v})_S$.

$$\left(\begin{array}{ccc|c} 3 & 2 & 0 & 1 \\ 5 & 3 & 2 & 1 \\ 5 & 4 & 2 & 1 \\ 1 & -1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

$(\mathbf{v})_S = \frac{1}{3} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

(e) [2 marks] Extend S to a basis for $\mathbb{R}^4$.

$$\begin{pmatrix} 3 & 5 & 5 & 1 \\ 2 & 3 & 4 & -1 \\ 0 & 2 & 2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

So we may include $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ to S to extend it to a basis for $\mathbb{R}^4$.

2. Let

$$\mathbf{v}_1 = \begin{pmatrix} 4 \\ 1 \\ -2 \\ 5 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 5 \\ 3 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 8 \\ 3 \\ 0 \\ -7 \end{pmatrix}.$$

(a) [1 mark] Find the distance between $\mathbf{v}_1$ and $\mathbf{v}_2$. Leave your answer as surds.

$$d(\mathbf{v}_1, \mathbf{v}_2) = \left\| \begin{pmatrix} 4 \\ 1 \\ -2 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 5 \\ 3 \\ 1 \end{pmatrix} \right\| = \sqrt{5^2 + (-4)^2 + (-5)^2 + 4^2} = \sqrt{82}.$$

(b) [1 mark] Find the norm of $\mathbf{v}_3$. Leave your answer as surds.

$$\left\| \begin{pmatrix} 8 \\ 3 \\ 0 \\ -7 \end{pmatrix} \right\| = \sqrt{8^2 + 3^2 + (-7)^2} = \sqrt{122}.$$

(c) [3 marks] Check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal set. Is it orthonormal?

Let $\mathbf{A} = \begin{pmatrix} 4 & -1 & 8 \\ 1 & 5 & 3 \\ -2 & 3 & 0 \\ 5 & 1 & -7 \end{pmatrix}$,

$$\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 46 & 0 & 0 \\ 0 & 36 & 0 \\ 0 & 0 & 122 \end{pmatrix}.$$

Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is orthogonal if and only if $\mathbf{A}^T\mathbf{A}$ is a diagonal matrix, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is orthonormal if and only if $\mathbf{A}^T\mathbf{A}$ is the identity matrix.