NATIONAL UNIVERSITY OF SINGAPORE

MA1508E - LINEAR ALGEBRA FOR ENGINEERING

(Semester 2 : AY2018/2019)

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. Please write your matriculation/student number only. Do not write your name.
2. This examination paper contains **EIGHT** questions and comprises **FOUR** printed pages.
3. Answer **ALL** questions.
4. Please start each question on a new page.
5. This is a CLOSED BOOK (with helpsheet) examination.
6. Students are allowed to use one A4 size helpsheet.
7. Candidates may use scientific (non-programmable) calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [**12 marks**]

(a) Solve the following linear system using Gaussian or Gauss-Jordan Elimination:

$$\left\{\begin{array}{rcrcrcrcr} w & + & 2x & - & 3y & - & z & = & 2 \\ 2w & + & 5x & - & 6y & + & 2z & = & 3 \\ 3w & + & 4x & - & 8y & - & 2z & = & -1 \end{array}\right.$$

(b) Let $S = \{(b_1, b_2, b_3) \mid \text{linear system below is consistent}\}$. Determine the set S by finding all relation(s) that must be satisfied by b_1, b_2, b_3 such that (b_1, b_2, b_3) belongs to S.

$$\left\{\begin{array}{rcrcrcr} 2x & + & y & - & z & = & b_1 \\ -x & - & 3y & + & z & = & b_2 \\ x & + & 8y & - & 2z & = & b_3 \end{array}\right.$$

(c) Consider the following linear system

$$\left\{\begin{array}{rcrcrcrcrcr} 6x_1 & - & 9x_2 & + & 31x_3 & + & 5x_4 & - & 2x_5 & = & 1 \\ 3x_1 & + & 20x_2 & + & 42x_3 & + & 2x_4 & & & = & 0 \\ 26x_1 & + & 9x_2 & + & 92x_3 & + & 20x_4 & - & 10x_5 & = & 5 \\ x_1 & + & 2x_2 & + & 81x_3 & + & 4x_4 & + & 22x_5 & = & -11 \\ 4x_1 & - & x_2 & + & 3x_3 & + & x_4 & - & 10x_5 & = & 5 \end{array}\right.$$

Suppose it is known that the linear system has exactly one solution. Show that x_2 must be zero in the (only) solution for the linear system.

Warning: Do not attempt to solve the system directly.

Question 2 [**12 marks**]

(a) Write the following matrix $\boldsymbol{A}$ as a product of exactly **four** elementary matrices. Hence compute $\det(\boldsymbol{A})$.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 0 & -3 \\ 0 & 2 & 0 \end{pmatrix}.$$

(b) Consider the following subset of $\mathbb{R}^5$.

$$V = \{(a-b,\ a+b+2c,\ b+c,\ a-b,\ a+b+2c) \mid a, b, c \in \mathbb{R}\}.$$

(i) Show that V is a subspace of $\mathbb{R}^5$ by writing it as a linear span.

(ii) Find a basis for V and determine its dimension.

Question 3 [**12 marks**]

Two physical quantities x and y are known to be related linearly according to the equation $y = ax + b$, where a, b are real constants. A series of experiments are conducted to determine the values of a and b. Different values of x were used in the experiments and the corresponding values of y were observed. The results of the experiments are shown in the table below.

x	1	2	3
y	0	1	3

Show that it is not possible to find a straight line that fits the data obtained from the three experiments. Using the method of least squares, find the line of best fit $y = \hat{a}x + \hat{b}$ and use it to predict the value of y when $x = 10$.

Question 4 [15 marks]

There are three species of organisms co-existing in a common habitat (e.g a pond). The number of each species present at any time t, are given by $y_1(t)$, $y_2(t)$ and $y_3(t)$ respectively. Arising from reproduction, death, competition for resources and other factors, the rate of change in the number of each species depends on the number of each species present at any particular time t. These rates of change are given below:

$$\begin{array}{rcrcr} y_1'(t) & = & y_1(t) & - & y_2(t) \\ y_2'(t) & = & -y_2(t) & + & 3y_3(t) \\ y_3'(t) & = & -y_1(t) & + & y_2(t) \end{array}.$$

(i) For any fixed t, let $\boldsymbol{Y'} = \boldsymbol{Y'}(t) = \begin{pmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{pmatrix}$ and $\boldsymbol{Y} = \boldsymbol{Y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$.

Represent the given system of linear differential equations by finding a matrix $\boldsymbol{A}$ such that $\boldsymbol{Y'} = \boldsymbol{AY}$.

(ii) Suppose at $t = 0$, all the 1000 organisms in the habitat comprises of only the third species, that is, $y_1(0) = 0$, $y_2(0) = 0$, $y_3(0) = 1000$. Solve the system $\boldsymbol{Y'} = \boldsymbol{AY}$ using the initial conditions given.

Question 5 [15 marks]

Let $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}$ be 4 points in $\mathbb{R}^3$.

(i) Suppose for each $j = 1, 2, 3, 4$, an object of mass m_j is located at point $\boldsymbol{u_j}$ as given in the table below. Show that the center of gravity (or *center of mass*) of the system is located at $\boldsymbol{c} = (1, 1, 1)$.

Point	Mass m_j
$\boldsymbol{u_1} = (-1, 2, 2)$	2g
$\boldsymbol{u_2} = (1, 1, 0)$	2g
$\boldsymbol{u_3} = (4, 0, 3)$	2g
$\boldsymbol{u_4} = (0, 1, -1)$	2g

(ii) If the object at point $\boldsymbol{u_1}$ is now moved to another location at $(1, \frac{3}{2}, 4)$, determine if it is possible to redistribute the masses at $\boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}$ (without altering the locations of these three points) such that the center of gravity remains unchanged at $\boldsymbol{c}$. If it is possible, find all the ways where the masses can be redistributed. If it is not possible, explain why.

Note: For this part, when redistributing the masses, we can only allow for positive integer values of m_2, m_3 and m_4.

(iii) If the object at the point $\boldsymbol{u_1}$ is now moved to another location $\boldsymbol{d} = (d_1, d_2, d_3)$ in $\mathbb{R}^3$, and if we allow the masses at $\boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}$ (again without altering the locations of these three points) to take on **any positive real values** (not necessarily integers), is it possible to redistribute the masses at $\boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}$ in **more than one way** such that the center of gravity remains unchanged at $\boldsymbol{c}$? Explain your answer.

Question 6 [12 marks]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$.

(i) Show that $\lambda_1 = -1$ and $\lambda_2 = 3$ are the only two eigenvalues of $\boldsymbol{A}$.

(ii) Show that $\boldsymbol{A}$ is diagonalizable by finding an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{AP} = \boldsymbol{D}$ where $\boldsymbol{D}$ is a diagonal matrix. Write down your diagonal matrix $\boldsymbol{D}$.

(iii) Hence or otherwise, find an invertible matrix $\boldsymbol{Q}$ that diagonalizes $4\boldsymbol{A}$.

Question 7 [12 marks]

Let $\boldsymbol{w_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\boldsymbol{w_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\boldsymbol{w_3} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

(i) Show that $S = \{\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3}\}$ is a basis for $\mathbb{R}^3$.

(ii) Apply Gram-Schmidt Process on the set S to obtain an orthogonal basis $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ for $\mathbb{R}^3$.

(iii) Find $(\boldsymbol{u})_T$ where $\boldsymbol{u} = (1, -1, 3)$.

Question 8 [10 marks]

Let $\boldsymbol{B} = \begin{pmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 10 \\ 0 & -5 & -5 & -8 \end{pmatrix}$.

(i) Find a basis for the row space of $\boldsymbol{B}$.

(ii) Find a basis for the column space of $\boldsymbol{B}$ and thus show that the column space of $\boldsymbol{B}$ **does not** contain the vector $(0, 0, 0, 1)^T$.

(iii) Find a basis for the nullspace of $\boldsymbol{B}$ and state the nullity of $\boldsymbol{B}$.

End of Paper