NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

AY2021, Semester 1 MA1508E Linear Algebra for Engineering Tutorial 5

Solutions

1. Let $\mathbf{u}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 3 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 3 \\ -1 \\ 5 \\ 2 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$.

   (a) If possible, express each of the following vectors as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

   (i) $\begin{pmatrix} 2 \\ 3 \\ -7 \\ 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ (iv) $\begin{pmatrix} -4 \\ 6 \\ -13 \\ 4 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & x_1 \\ 1 & -1 & 0 & x_2 \\ 0 & 5 & 2 & x_3 \\ 3 & 2 & 1 & x_4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & x_2 \\ 0 & 5 & -1 & x_1 - 2x_2 \\ 0 & 0 & 3 & -x_1 + 2x_2 + x_3 \\ 0 & 0 & 0 & x_1 + 7x_2 + 2x_3 - 3x_4 \end{array}\right)$$

   Suppose $x_1 + 7x_2 + 2x_3 - 3x_4 = 0$. We may proceed

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2x_1+11x_2+x_3}{15} \\ 0 & 1 & 0 & \frac{2x_1-4x_2+x_3}{15} \\ 0 & 0 & 1 & \frac{-x_1+2x_2+x_3}{3} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

   So a vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ if and only if it satisfies $x_1 + 7x_2 + 2x_3 - 3x_4 = 0$. If that is true, then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3$$

   where

$$a = \frac{2x_1 + 11x_2 + x_3}{15}, \qquad b = \frac{2x_1 - 4x_2 + x_3}{15}, \qquad c = \frac{-x_1 + 2x_2 + x_3}{3}$$

   (i) $2 + 7(3) + 2(-7) - 3(3) = 0$. It is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, $\begin{pmatrix} 2 \\ 3 \\ -7 \\ 3 \end{pmatrix} = 2\mathbf{u}_1 - \mathbf{u}_2 - \mathbf{u}_3$.

   (ii) This is clearly a linear combination with $a = b = c = 0$.

(iii) $1+7+2-3=7$. It is not a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

(iv) $-4+7(6)+2(-13)-3(4)=0$. It is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, $\begin{pmatrix} -4 \\ 6 \\ -13 \\ 4 \end{pmatrix} = 3\mathbf{u}_1 - 3\mathbf{u}_2 + \mathbf{u}_3$.

(b) Is it possible to find 2 vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that they are not a multiple of each other, and both are not a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$?

Yes, for example, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

2. Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; x-y-z=0 \right\}$ be a subset of $\mathbb{R}^3$.

(a) Let $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \right\}$. Show that $\text{span}(S) = V$.

Since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$ satisfy the equation $x-y-z=0$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \in V$ and hence $\text{span}(S) \subseteq V$.

Note that a general solution of $x-y-z=0$ is $x=s+t$, $y=s$, $z=t$ where $s,t \in \mathbb{R}$. Let $(s+t,s,t)$ be any vector in V. Consider the following equation:

$$a \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} s+t \\ s \\ t \end{pmatrix} \quad \Leftrightarrow \quad \begin{cases} a+5b = s+t \\ a+2b = s \\ \quad\; 3b = t. \end{cases}$$

Since

$$\left(\begin{array}{cc|c} 1 & 5 & s+t \\ 1 & 2 & s \\ 0 & 3 & t \end{array} \right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{cc|c} 1 & 5 & s+t \\ 0 & 3 & t \\ 0 & 0 & 0 \end{array} \right),$$

the system is consistent for all $s,t \in \mathbb{R}$. So $V \subseteq \text{span}(S)$.

We have shown that $\text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \right\} = V$.

(b) Let $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. Find a vector $\mathbf{y}$ such that $\text{span}\{\mathbf{x}, \mathbf{y}\} = V$.

Note that V is a plane in $\mathbb{R}^3$ and $\mathbf{x}$ belongs to the plane since $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ satisfies $x-y-z=0$. To span V, we just need another vector on the plane that is not a multiple of $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. For example $\mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(c) Let $T = S \cup \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. Show that $\text{span}(T) = \mathbb{R}^3$.

Consider the row-echelon form of the matrix:

$$\begin{pmatrix} 1 & 5 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 5 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{R}.$$

Since there are no zero rows in $\mathbf{R}$, we conclude that T spans $\mathbb{R}^3$.

3. (a) Which of the following sets S spans $\mathbb{R}^4$?

(i) $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

(ii) $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

(iii) $S = \left\{ \begin{pmatrix} 6 \\ 4 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ -2 \\ -1 \end{pmatrix} \right\}$.

(iv) $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}$.

(b) For those sets that does not span $\mathbb{R}^4$, find a vector $\mathbf{x}$ in $\mathbb{R}^4$ that does not belong to $\text{span}(S)$. Does $S \cup \{\mathbf{x}\}$ span $\mathbb{R}^4$?

(c) For those sets that spans $\mathbb{R}^4$, find a vector $\mathbf{y}$, if possible, in the set S such that $\text{span}(S) = \mathbb{R}^4 = \text{span}(S - \{\mathbf{y}\})$.

(i)

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

So S spans $\mathbb{R}^4$. It is not possible to find $\mathbf{y}$ such that $\text{span}(S - \{\mathbf{y}\}) = \mathbb{R}^4$ since we need at least 4 vectors to span $\mathbb{R}^4$..

(ii) S does not span $\mathbb{R}^4$ since it has only 3 vectors. To span $\mathbb{R}^4$ we find a $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ such that the reduced row-echelon form of $\begin{pmatrix} 1 & 1 & 0 & x_1 \\ 2 & 1 & 0 & x_2 \\ 1 & -1 & 0 & x_3 \\ 0 & 0 & 1 & x_4 \end{pmatrix}$ has no zero rows,

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & x_1 \\ 2 & 1 & 0 & x_2 \\ 1 & -1 & 0 & x_3 \\ 0 & 0 & 1 & x_4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & x_1 \\ 0 & -1 & 0 & x_2 - 2x_1 \\ 0 & 0 & 1 & x_4 \\ 0 & 0 & 0 & x_3 + 2x_2 - 5x_1 \end{array}\right).$$

Any choice such that $x_3 + 2x_2 - 5x_1 \neq 0$ will work, say $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. Indeed, $S \cup \{\mathbf{x}\}$ will span $\mathbb{R}^4$.

Remark: In the next tutorial we will learn a more efficient way, that is, to use the row space method.

(iii)

$$\begin{pmatrix} 6 & 2 & 3 & 5 & 0 \\ 4 & 0 & 2 & 6 & 4 \\ -2 & 0 & -1 & -3 & -2 \\ 4 & 1 & 2 & 2 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{R}$$

Since there is a row of zeros in $\mathbf{R}$, S does not span $\mathbb{R}^4$. Following the method in (ii), we can find $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ that does not belong to span(S). Again $S \cup \{\mathbf{x}\}$ spans $\mathbb{R}^4$.

(iv)

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 & 2 \\ 0 & -1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} = \mathbf{R}.$$

So S spans $\mathbb{R}^4$. From the reduced row-echelon form of $\mathbf{A}$, we see that

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}.$$

So we can choose $\mathbf{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ and $S - \{\mathbf{y}\}$ will span $\mathbb{R}^4$.

4. (a) Determine whether span$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subseteq$ span$\{\mathbf{v}_1, \mathbf{v}_2\}$ and/or span$\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq$ span$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ if

(i) $\mathbf{u}_1 = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}, \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$

(ii) $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}, \mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 8 \\ 9 \end{pmatrix}.$

(b) In each of the above, describe $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ geometrically. If $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ or $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane, find the equation of the plane.

(i)

$$\left(\begin{array}{ccc|cc} 2 & -1 & 0 & 1 & 0 \\ -2 & 1 & 0 & -1 & 1 \\ 0 & -1 & 9 & -5 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|cc} 1 & 0 & -\frac{9}{2} & 3 & 0 \\ 0 & 1 & -9 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

So $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \not\subseteq \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

$$\left(\begin{array}{cc|ccc} 1 & 0 & 2 & -1 & 0 \\ -1 & 1 & -2 & 1 & 0 \\ -5 & 1 & 0 & -1 & 9 \end{array}\right) \longrightarrow \left(\begin{array}{cc|ccc} 1 & 0 & 0 & \frac{1}{5} & -\frac{9}{5} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{5} & \frac{9}{10} \end{array}\right).$$

So $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \not\subseteq \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. From the first RREF above, we can see that $\mathbf{u}_3 = -\frac{9}{2}\mathbf{u}_1 - 9\mathbf{u}_2$ so $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$, which is a plane in $\mathbb{R}^3$. To find the equation of the plane $ax+by+cz = 0$, we substitute $(2, -2, 0)$ and $(-1, 1, -1)$ into $ax + by + cz = 0$ and solve for a, b, c:

$$\begin{cases} 2a - 2b = 0 \\ -a + b - c = 0 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ -1 & 1 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

So a solution to the linear system is $a = 1, b = 1, c = 0$ and the equation of the plane spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$ is $x + y + 0z = 0$. Similarly $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is also a plane and the equation of the plane is found to be $4x - y + z = 0$. Note that in this case, $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ are both planes in $\mathbb{R}^3$ that intersect in a line that passes through the origin.

(ii)

$$\left(\begin{array}{ccc|cc} 1 & 2 & -1 & 1 & 0 \\ 6 & 4 & 2 & -2 & 8 \\ 4 & -1 & 5 & -5 & 9 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|cc} 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

So $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

$$\left(\begin{array}{cc|ccc} 1 & 0 & 1 & 2 & -1 \\ -2 & 8 & 6 & 4 & 2 \\ -5 & 9 & 4 & -1 & 5 \end{array}\right) \longrightarrow \left(\begin{array}{cc|ccc} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

So $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subseteq \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. We conclude that the two linear spans are equal. To find the equation of the plane $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, we substitute $(1, -2, -5)$ and $(0, 8, 9)$ into $ax + by + cz = 0$ to find a, b, c:

$$\begin{cases} a - 2b - 5c = 0 \\ 8b + 9c = 0 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -5 & 0 \\ 0 & 8 & 9 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -\frac{11}{4} & 0 \\ 0 & 1 & \frac{9}{8} & 0 \end{array}\right)$$

So a solution to the linear system is $a = 22, b = -9, c = 8$ and the equation of the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ is $22x - 9y + 8z = 0$.

5. Determine which of the following sets are subspaces. For those sets that are subspaces, express the set as a linear span. For those sets that are not, explain why.

(a) $S = \left\{ \begin{pmatrix} p \\ q \\ p \\ q \end{pmatrix} \middle| \, p, q \in \mathbb{R} \right\}$.

$$S = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(b) $S = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \, a \geq b \text{ or } b \geq c \right\}$.

S is not a linear span (thus not a subspace) since $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ is in S but $(-1)\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ is not.

(c) $S = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \middle| \, 4x = 3y \text{ and } 2x = -3w \right\}$.

$$S = \left\{ \begin{pmatrix} x \\ \frac{4x}{3} \\ z \\ -\frac{2x}{3} \end{pmatrix} \middle| \, x, z \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ \frac{4}{3} \\ 0 \\ -\frac{2}{3} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

(d) $S = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \middle| \, \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ a & b & c & d \end{vmatrix} = 0 \right\}$.

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ a & b & c & d \end{vmatrix} = a - c - d.$$

So the set S can be rewritten as

$$\begin{aligned} S &= \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \middle| \, a - c - d = 0 \right\} = \left\{ \begin{pmatrix} s+t \\ u \\ s \\ t \end{pmatrix} \middle| \, s, t, u \in \mathbb{R} \right\} \\ &= \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \end{aligned}$$

(e) $S = \left\{ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \middle| \; w + x = y + z \right\}$.

S can be rewritten as $S = \left\{ \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \middle| \; w + x - y - z = 0 \right\}$. Solving the equation $w + x - y - z = 0$, we have

$$S = \left\{ \begin{pmatrix} -s+t+u \\ s \\ t \\ u \end{pmatrix} \middle| \; s, t, u \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(f) $S = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \middle| \; ab = cd \right\}$.

S is not a linear span (thus not a subspace) since $\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$ are vectors in S but $\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix}$ is not.

(g) S is the solution set of $\mathbf{Ax} = \mathbf{0}$ where $\mathbf{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix}$.

Solving $\mathbf{Ax} = \mathbf{0}$, we have

$$\left(\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \end{array} \right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

So an arbitrary vector in the solution set of $\mathbf{Ax} = \mathbf{0}$ is

$$\left\{ \begin{pmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{pmatrix} \middle| \; s, t \in \mathbb{R} \right\}.$$

So we can rewrite S as

$$\text{span}\left\{\begin{pmatrix}-1\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}-1\\0\\-1\\0\\1\end{pmatrix}\right\}.$$

(h) $S = \left\{\ \mathbf{u}+\mathbf{v}\ \middle|\ \mathbf{v} \in \text{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}\ \right\}$ and $\mathbf{u} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ is a fixed vector.

Let $T = \text{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$. We claim that $S = T$. Suppose $\mathbf{y}$ is a vector in S, then $\mathbf{y} = \begin{pmatrix}1\\1\\1\end{pmatrix} + a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}0\\0\\1\end{pmatrix}$ for some real numbers a, b. But notice that $\mathbf{y} = (a+1)\begin{pmatrix}1\\1\\0\end{pmatrix} + (b+1)\begin{pmatrix}0\\0\\1\end{pmatrix}$ which is also a vector in T, thus $S \subseteq T$. On the other hand, suppose $\mathbf{x}$ is a vector in T. Then $\mathbf{x} = c\begin{pmatrix}1\\1\\0\end{pmatrix} + d\begin{pmatrix}0\\0\\1\end{pmatrix}$ for some real numbers c, d. Then $\mathbf{x}$ can be rewritten as

$$\mathbf{x} = c\begin{pmatrix}1\\1\\0\end{pmatrix} + d\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} + (c-1)\begin{pmatrix}1\\1\\0\end{pmatrix} + (d-1)\begin{pmatrix}0\\0\\1\end{pmatrix}$$

which is a vector in S. Thus $T \subseteq S$. Combining, we have $S = T = \text{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$.

Supplementary Problems

6. (a) Show that the solution set to any homogeneous linear system

$$S = \{\ \mathbf{v} \in \mathbb{R}^n \ |\ \mathbf{A}\mathbf{v} = \mathbf{0}\ \}$$

is a subspace.

We will show that the solution set $S = \{\ \mathbf{v} \in \mathbb{R}^n \ |\ \mathbf{A}\mathbf{v} = \mathbf{0}\ \}$ is nonempty and is closed under linear combinations. It is obviously nonempty, since it contains the trivial solution $\mathbf{A}\mathbf{0} = \mathbf{0}$. Suppose $\mathbf{u}, \mathbf{v} \in S$. Then for any $\alpha, \beta \in \mathbb{R}$, $\mathbf{A}(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha\mathbf{A}\mathbf{u} + \beta\mathbf{A}\mathbf{v} = \alpha\mathbf{0} + \beta\mathbf{0} = \mathbf{0}$. So $(\alpha\mathbf{u} + \beta\mathbf{v}) \in S$. Hence, S is a subspace of $\mathbb{R}^n$.

(b) Suppose the homogeneous linear system $\mathbf{A}\mathbf{x} = 0$ has a nontrivial solution. Prove that if the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is consistent, it must have infinitely many solutions.

Let $\mathbf{x}_0$ be a nontrivial solution to $\mathbf{Ax} = \mathbf{0}$. Let $\mathbf{x}_p$ be a particular solution of $\mathbf{Ax} = \mathbf{b}$. Then for any $t \in \mathbb{R}$, $\mathbf{A}(\mathbf{x}_p + t\mathbf{x}_0) = \mathbf{A}\mathbf{x}_p + t\mathbf{A}\mathbf{x}_0 = \mathbf{b} + t\mathbf{0} = \mathbf{b}$. Since $\mathbf{x}_0$ is nontrivial, the set $\left\{ \mathbf{x}_p + t\mathbf{x}_0 \in \mathbb{R}^n \;\middle|\; t \in \mathbb{R} \right\}$ is infinite and is a subset of the solution set of $\mathbf{Ax} = \mathbf{b}$.

(c) Prove that if the linear system $\mathbf{Ax} = \mathbf{b}$ has two distinct solutions, then it must have infinitely many solution.

Suppose $\mathbf{u}$ and $\mathbf{v}$ are distinct solutions to $\mathbf{Ax} = \mathbf{b}$. Then $\mathbf{A}(\mathbf{u}-\mathbf{v}) = \mathbf{Au}-\mathbf{Av} = \mathbf{b} - \mathbf{b} = 0$. Since $\mathbf{u} \neq \mathbf{v}$, $(\mathbf{u} - \mathbf{v})$ is a nontrivial solution to the homogenous linear system $\mathbf{Ax} = \mathbf{0}$. The statement now follows from (b).

(d) **(MATLAB)** Let $\mathbf{A}$ be the 10×10 *magic square*, and let $\mathbf{b}$ be the 10-vector of all 1's. We may key these special matrices into MATLAB fairly quickly.

```
>> A=magic(10);
```

```
>> b=ones(10,1);
```

i. Express the solution set of $\mathbf{Ax} = \mathbf{b}$ as

$$\left\{ \mathbf{x}_p + s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k \;\middle|\; s_1, s_2, \ldots, s_k \in \mathbb{R} \right\}.$$

The RREF of the augmented matrix of $\mathbf{Ax} = \mathbf{b}$ is

```
>> rref([A b])
```

$$\operatorname{rref}([\boldsymbol{A}\ \boldsymbol{b}]) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{253} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & \frac{1}{101} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{253} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{252} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & \frac{1}{253} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -2 & -1 & \frac{-1}{168} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We assign arbitrary parameters $x_8 = r, x_9 = s, x_{10} = t$. Then, the solution set is given by

$$\left\{ \begin{pmatrix} \frac{1}{253} \\ \frac{1}{101} \\ \frac{1}{253} \\ \frac{1}{253} \\ 0 \\ \frac{-1}{168} \\ 0 \\ 0 \\ 0 \end{pmatrix} + s_1 \begin{pmatrix} -1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s_2 \begin{pmatrix} 0 \\ -2 \\ 0 \\ -1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s_3 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; s_1, s_2, s_3 \in \mathbb{R} \right\}.$$

ii. Pick a few $s_1, s_2, ..., s_k \in \mathbb{R}$ and compute $\mathbf{A}(s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \cdots + s_k\mathbf{u}_k)$. What is the set $S = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$?
Whatever the choice of $s_1, s_2, s_3 \in \mathbb{R}$, $\mathbf{A}(s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + s_3\mathbf{u}_2) = \mathbf{0}$. S is the solution set to the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$.

iii. Running the `null` command outputs a collection of column vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell$.

```
>> null(A)
```

Show that span $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k\}$ = span $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell\}$. What does this say about the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell$? In particular, what is the output of the `null` command in relation to the matrix $\boldsymbol{A}$?
Let N represent the matrix `null(A)`

```
>> N=null(A);
```

Let $\mathbf{U}$ represent the matrix whose columns are $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$

```
>> U=[-1 -1 -1 0 0 1 1 1 0 0;0 -2 0 -1 0 0 2 0 1 0;0 -1 0 0 -1
0 1 0 0 1]';
```

Find the RREF of $\left(\mathbf{U} \mid \mathbf{N} \right)$ and $\left(\mathbf{N} \mid \mathbf{U} \right)$

```
>>rref([U N])
```

```
>>rref([N U])
```

and check that both systems are consistent. Hence, the columns of the output of the `null` command spans the solution set to the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$.

A subset of $\mathbb{R}^n$ is called an *affine space* if it is of the form $\{\ \mathbf{u} + \mathbf{v} \mid \mathbf{v} \in V\ \}$ for some subspace $V \subseteq \mathbb{R}^n$. Geometrically, an affine space is a subset of $\mathbb{R}^n$ that is parallel to a subspace. This exercise shows that the solution set to the linear system $\mathbf{Ax} = \mathbf{b}$ is an affine space $\{\ \mathbf{x}_p + \mathbf{v} \mid \mathbf{v} \in S\ \}$, where S is the solutions to homogeneous linear system $\mathbf{Ax} = \mathbf{0}$, and $\mathbf{x}_p$ is any particular solution.