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TE6

1. a) A is symmetric, hence it is orthogonally diagonalizable

$$\det(xI - A) = \begin{vmatrix} x-3 & -1 & -1 & -1 \\ -1 & x-3 & -1 & -1 \\ -1 & -1 & x-3 & -1 \\ -1 & -1 & -1 & x-3 \end{vmatrix}$$

$$= x-3\left[(x-3)(x-4)(x-2) - 2(x-2)\right]$$
$$+1\left[-(x-4)(x-2) - 2(x-2)\right]$$
$$-1\left[.\ (x-2)^2\right]$$
$$+1\left[-(x-2)^2\right]$$

$$= x^4 - 12x^3 + 48x^2 - 80x + 48$$
$$= (x-2)^3(x-6)$$

A has eigenvalues 2 and 6 with multiplicity $r_2 = 3$, $r_6 = 2$

$$\lambda = 2 : \begin{pmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$d = s$
$c = t$
$b = r$
$a = -r - t - s$

basis for E_2 is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$\lambda = 6 : \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$d = s$
$c = s$
$b = s$
$a = s$

basis for E_6 is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

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E_2 and E_6 are orthogonal

Apply Gram-Schmidt to basis for E_6

$$v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix}$$

$\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix}\right\}$ is an orthogonal basis

Normalizing, $\left\{\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{12}}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix}\right\}$

$$A = \begin{pmatrix} 1/2 & -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} \\ 1/2 & 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} \\ 1/2 & 0 & -2/\sqrt{6} & 1/\sqrt{12} \\ 1/2 & 0 & 0 & -3/\sqrt{12} \end{pmatrix}\begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1/2 & -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} \\ 1/2 & 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} \\ 1/2 & 0 & -2/\sqrt{6} & 1/\sqrt{12} \\ 1/2 & 0 & 0 & -3/\sqrt{12} \end{pmatrix}^T$$

let $$P = \begin{pmatrix} 1/2 & -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} \\ 1/2 & 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} \\ 1/2 & 0 & -2/\sqrt{6} & 1/\sqrt{12} \\ 1/2 & 0 & 0 & -3/\sqrt{12} \end{pmatrix}$$

let $$D = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

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h) $$\frac{1}{6}A = \frac{1}{6}PDP^T$$
$$= P\frac{1}{6}DP^T$$
$$= P\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 \end{pmatrix}P^T$$

$$\left(\frac{1}{6}A\right)^n = P\begin{pmatrix} 1^n & & & 0 \\ & 1/3^n & & \\ & & 1/3^n & \\ 0 & & & 1/3^n \end{pmatrix}P^T$$
$$= P\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}P^T \quad \text{as } n \to \infty$$
$$\left(\frac{1}{6}A\right)^n\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$
$$= \frac{1}{4}\begin{pmatrix} a+b+c+d \\ a+b+c+d \\ a+b+c+d \\ a+b+c+d \end{pmatrix}$$

2. a) $$y' = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{pmatrix} y$$

$$\begin{vmatrix} \lambda-1 & -1 & 1 \\ 0 & \lambda & -1 \\ 0 & 2 & \lambda+3 \end{vmatrix} = (\lambda-1)\left[\lambda^2+3\lambda+2\right]$$
$$= (\lambda-1)(\lambda+1)(\lambda+2)$$

3 distinct eigen values $\lambda=1$, $\lambda=-1$ and $\lambda=-2$

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$\lambda = 1$: $\begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $c=0$, $b=0$, $a=s$

basis for E_1 is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

$\lambda = -1$: $\begin{pmatrix} -2 & -1 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $c=s$, $b=-s$, $a=s$

basis for E_{-1} is $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$

$\lambda = -2$: $\begin{pmatrix} -3 & -1 & 1 \\ 0 & -2 & -1 \\ 0 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}$ $c=s$, $b=-\frac{1}{2}s$, $a=\frac{1}{2}s$

basis for E_{-2} is $\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\}$

general solution to given system is

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = c_1 e^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{-2t} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

using initial condition,

$$\left.\begin{aligned} y_1(0) &= 1 = c_1 + c_2 + c_3 \\ y_2(0) &= 2 = -c_2 - c_3 \\ y_3(0) &= 3 = c_2 + 2c_3 \end{aligned}\right\} \quad \begin{aligned} c_1 &= 3 \\ c_3 &= 5 \\ c_2 &= -7 \end{aligned}$$

$$y(t) = 3e^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 7e^{-t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 5e^{-2t} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

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3 a) $Av = \lambda v$

$$\begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} i-3 \\ 3i+1 \end{pmatrix} = 1+3i \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$\lambda = 1+3i$

Just to confirm

$$\begin{vmatrix} \lambda-1 & 3 \\ -3 & \lambda-1 \end{vmatrix} = \lambda^2 - 2\lambda + 10$$

$$\lambda = \frac{2 \pm \sqrt{4-40}}{2} = 1 \pm 3i$$

$$\lambda = 1+3i : \begin{pmatrix} 3i & 3 \\ -3 & 3i \end{pmatrix} \rightarrow v = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$\lambda = 1-3i : \begin{pmatrix} -3i & 3 \\ -3 & -3i \end{pmatrix} \rightarrow v = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

b) $\lambda_r = 1 \quad \lambda_i = 3$

$$v_r = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad v_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$y_r(t) = e^t \left(\cos 3t \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sin 3t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = e^t \begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix}$$

$$y_i(t) = e^t \left(\sin 3t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \cos 3t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = e^t \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix}$$

Fundamental set of solutions $\left\{ e^t \begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix}, e^t \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} \right\}$