NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

AY2021, Semester 2 MA1508E Linear Algebra for Engineering Practice 2 Solutions

1. [3 marks] Solve the following linear system

$$\begin{array}{ccccccccc} x_1 & + & x_2 & + & 2x_3 & - & x_4 & = & a \\ x_1 & & & & & + & 2x_4 & = & b \\ & & x_2 & + & 2x_3 & - & 2x_4 & = & c \end{array}$$

for

(i) $a = 1, b = 1, c = 2$,

(ii) $a = 0, b = -1, c = 1$.

$$\left(\begin{array}{cccc|c|c} 1 & 1 & 2 & -1 & 1 & 0 \\ 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 2 & -2 & 2 & 1 \end{array}\right) \xrightarrow{R_1-R_3} \left(\begin{array}{cccc|c|c} 1 & 0 & 0 & 1 & -1 & -1 \\ 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 2 & -2 & 2 & 1 \end{array}\right) \xrightarrow{R_2-R_1}$$

$$\left(\begin{array}{cccc|c|c} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & -2 & 2 & 1 \end{array}\right) \xrightarrow{R_1-R_2} \xrightarrow{R_3+2R_2} \left(\begin{array}{cccc|c|c} 1 & 0 & 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 & 6 & 1 \end{array}\right)$$

(i) General solution: $x_1 = -3, x_2 = 6 - 2s, x_3 = s, x_4 = 2,\ s \in \mathbb{R}$.

(ii) General solution: $x_1 = -1, x_2 = 1 - 2s, x_3 = s, x_4 = 0,\ s \in \mathbb{R}$.

2. (a) [4 marks] Let

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 0 \\ 3 & -6 & -3 \\ 1 & 0 & 2 \end{pmatrix}.$$

Compute the inverse of $\mathbf{A}$ by performing elementary row operations. Write down the elementary row operation that you used in each step clearly.

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 3 & -6 & -3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2-3R_1} \xrightarrow{R_3-R_1} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & -3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & -3 & -3 & 1 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}R_2} \xrightarrow{-\frac{1}{3}R_3} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 1 & -1/3 & 0 \end{array}\right)$$

$$\xrightarrow{R_2-R_3} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3/2 & 1/3 & 1/2 \\ 0 & 0 & 1 & 1 & -1/3 & 0 \end{array}\right) \xrightarrow{R_1+2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2/3 & 1 \\ 0 & 1 & 0 & -3/2 & 1/3 & 1/2 \\ 0 & 0 & 1 & 1 & -1/3 & 0 \end{array}\right)$$

(b) [3 marks] Suppose

$$\mathbf{A} \xrightarrow{R_1+2R_3} \xrightarrow{R_1 \leftrightarrow R_2} \xrightarrow{R_2-R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 3 & 0 \end{pmatrix}.$$

Write $\mathbf{A}$ as a product of 6 elementary matrices, $\mathbf{A} = \mathbf{E}_1\mathbf{E}_2\mathbf{E}_3\mathbf{E}_4\mathbf{E}_5\mathbf{E}_6$.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$$

(c) [2 marks] Compute the determinant of $\mathbf{A}$ from $\boxed{2b}$.

$$\det(\mathbf{A}) = (1)(-1)(1)(-\frac{3}{2}) = \frac{3}{2}.$$

3. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 2 & 3 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 2 & 5 & 7 \end{pmatrix}$.

(a) [3 marks] Compute the determinant of $\mathbf{A}$ by cofactor expansion along the first row.

$$\begin{aligned} \begin{vmatrix} 1 & 2 & -1 & 0 \\ 2 & 2 & 3 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 2 & 5 & 7 \end{vmatrix} &= \begin{vmatrix} 2 & 3 & 1 \\ 2 & 0 & 0 \\ 2 & 5 & 7 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 & 1 \\ 0 & 0 & 0 \\ 1 & 5 & 7 \end{vmatrix} - \begin{vmatrix} 2 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 7 \end{vmatrix} \\ &= (10-42) - (28-2) = -58. \end{aligned}$$

(b) [2 marks] Let $\mathbf{b} = \begin{pmatrix} 1 \\ a \\ 0 \\ -2 \end{pmatrix}$. For which value of a is $\mathbf{Ax} = \mathbf{b}$ consistent? Why?

For any $a \in \mathbb{R}$, $\mathbf{Ax} = \mathbf{b}$ is consistent since $\det(\mathbf{A}) \neq 0$, and thus $\mathbf{A}$ is invertible.

(c) [3 marks] Suppose $\mathbf{B}$ is an order 4 square matrix such that $\det(B) = 3$. Find

(i) $\det(\frac{1}{2}\mathbf{A}^T)$,

(ii) $\det(\mathbf{AB}^{-1})$,

(iii) $\det((3\mathbf{B})^{-1})$.

(i) $\det(\frac{1}{2}\mathbf{A}^T) = \frac{1}{2^4}(-58) = -29/8$,

(ii) $\det(\mathbf{AB}^{-1}) = -58/3$,

(iii) $\det((3\mathbf{B})^{-1}) = \frac{1}{3^4}\det(\mathbf{B})^{-1} = 1/243$.