NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

AY2021, Semester 1   MA1508E Linear Algebra for Engineering   Tutorial 1

1. Solve the following linear systems by Gaussian Elimination or Gauss-Jordan Elimination. Provide the general form and the solution set. (Make sure you are able to perform the necessary elementary row operations without the help of MATLAB.)

   (a)
$$\begin{cases} 3x_1 + 2x_2 - 4x_3 = 3 \\ 2x_1 + 3x_2 + 3x_3 = 15 \\ 5x_1 - 3x_2 + x_3 = 14 \end{cases}$$

   (b)
$$\begin{cases} a + b - c - 2d = 0 \\ 2a + b - c + d = -2 \\ -a + b - 3c + d = 4 \end{cases}$$

   (c)
$$\begin{cases} x - 4y + 2z = -2 \\ x + 2y - 2z = -3 \\ x - y = 4 \end{cases}$$

2. Reduce the following augmented matrix to its reduced row echelon form using Gaussian and Gauss-Jordan elimination.

$$\left(\begin{array}{ccc|c} 2 & 6 & 5 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 4 & 5 & 0 \end{array}\right)$$

   Could the number of operations be reduced if we do not insist on using Gaussian or Gauss-Jordan elimination?

3. Determine the values of a and b so that the linear system

$$\begin{cases} ax + bz = 2 \\ ax + ay + 4z = 4 \\ ay + 2z = b \end{cases}$$

   (a) has no solution;

   (b) has only one solution;

   (c) has infinitely many solutions and a general solution has one arbitrary parameter;

   (d) has inifinitely many solutions and a general solution has two arbitrary parameters.

4. **(Application)** When chemical compounds are combined under the right conditions, the atoms in their molecules rearrange to form new compounds. This is represented by a chemical equation. For example, when methane burns, the methane

(CH_4) and stable oxygen (O_2) react to form carbon dioxide (CO_2) and water (H_2O). The chemical equation is

$$\mathrm{CH_4} + \mathrm{O_2} \longrightarrow \mathrm{CO_2} + \mathrm{H_2O}$$

A chemical equation is said to be balanced if for each type of atom in the reaction, the same number of atoms appears on each side of the arrow. We can balance the equation by letting x_1, x_2, x_3, and x_4 to be the number of methane, stable oxygen, carbon dioxide, and water molecule, that is,

$$x_1\mathrm{CH_4} + x_2\mathrm{O_2} \longrightarrow x_3\mathrm{CO_2} + x_4\mathrm{H_2O}$$

From it we obtain the following homogeneous linear system

$$\begin{array}{rcrcrcrl} x_1 & & & - & x_3 & & & = 0 \\ 4x_1 & & & & & - & 2x_4 & = 0 \\ & & 2x_2 & - & 2x_3 & - & x_4 & = 0 \end{array}$$

The smallest positive integer values solution of the system will give us the balanced equation for combustion of methane

$$\mathrm{CH_4} + 2\mathrm{O_2} \longrightarrow \mathrm{CO_2} + 2\mathrm{H_2O}$$

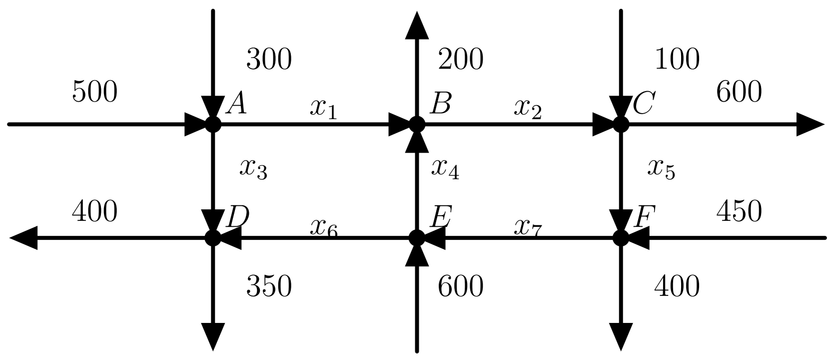
Note that the system will always have infinitely many solutions. Why?

Write the balanced equation for the given chemical reactions

(a) $\mathrm{CO_2} + \mathrm{H_2O} \longrightarrow \mathrm{C_6H_{12}O_6} + \mathrm{O_2}$

(b) $\mathrm{Fe_2O_3} + \mathrm{Al} \longrightarrow \mathrm{Al_2O_3} + \mathrm{Fe}$

5. **(Application, MATLAB)** A network of one-way streets of a downtown section can be represented by the diagram below, with traffic flowing in the direction indicated. The average hourly volume of traffic entering and leaving this section during rush hour is given in the diagram.



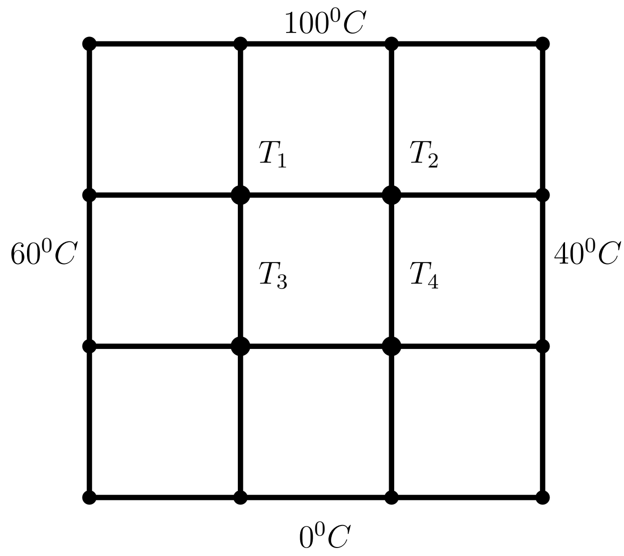
(a) Do we have enough information to find the traffic volumes x_1, x_2, x_3, x_4, x_5, x_6, and x_7?

(b) Suppose $x_6 = 50$ and $x_7 = 100$. What is x_1, x_2, x_3, x_4, and x_5?

(c) Can the road between junction A and B be closed for construction while still keeping the traffic flowing on the other streets? Explain.

6. **(Application)** A simple model for estimating the temperature distribution on a square plate gives rise to a linear system of equations. To construct the appropriate linear system, we use the following information: The square plate is perfectly insulated on its top and bottom so that the only heat flow is through the plate itself. The four edges are held at various temperatures. To estimate the temperature at an interior point on the plate, we use the rule that it is the average of the temperature at its four compass-point neighbours, to the west, north, east and south.

   Suppose we wish to estimate the temperatures T_i, $i = 1, 2, 3, 4$, at the four equispaced interior points on the plate as shown in the figure below.



   We now construct the linear system to estimate the temperatures. The points at which we need the temperatures of the plate for this model are indicated by the dots in the figure above. To obtain linear equations involving the unknowns T_i, $i = 1, 2, 3, 4$, we use our averaging rule, for example,

$$T_1 = \frac{60 + 100 + T_2 + T_3}{4} \Rightarrow 4T_1 - T_2 - T_3 = 160.$$

   (a) Write down three other linear equations, by considering T_2, T_3 and T_4.

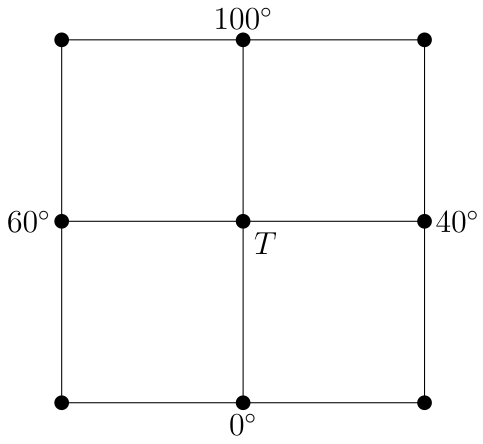
   (b) Solve the linear system. Is it possible to have more than one solution?

   Food for thought: Is it possible for the system to be inconsistent? Is it possible for the system to have infinitely many solutions?

Supplementary Problems

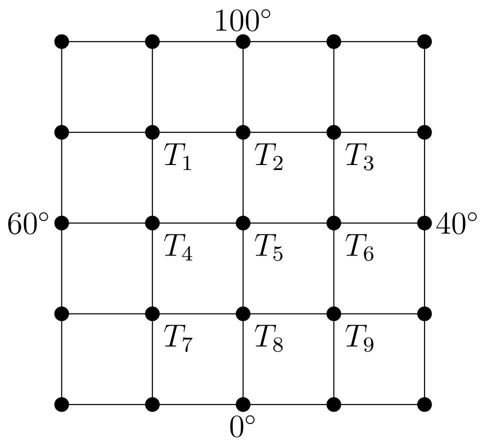
7. **(MATLAB)**

(a) Consider, once again, the perfectly insulated square plate from Problem 6, with its interior mesh altered, as shown below.



Directly applying the averaging rule from Problem 5, estimate the temperature T of the central node.

(b) We might notice that our temperature values vary according to how finely or coarsely we dissect the metal plate into its interior nodes. To more accurately estimate the temperature at precise points on the plate, we produce a finer interior mesh, as shown below.



i. Set up a linear system in nine equations that will allow us to find the temperatures T_1 through T_9 of the interior nodes. Express your answer as a matrix equation $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{x}$ is the column matrix whose entries are given by $T_1, \ldots, T_9$.

ii. Use MATLAB to solve the linear system. Note that T_5 corresponds to the temperature at the central node of the plate. How does this compare to the temperature at the central node you obtained from part (a)?