A0214561M

Zhuang Jianning

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1. i) $\left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 1 \\ 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & -2 & 2 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 1 \\ 0 & -1 & -2 & 3 & 0 \\ 0 & 1 & 2 & -2 & 2 \end{array}\right]$

$\xrightarrow{R_3 + R_2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 1 \\ 0 & -1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 1 \\ 0 & -1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$

$x_4 = 2$

let $x_3 = s$, $s \in \mathbb{R}$

$-x_2 - 2x_3 + 3x_4 = 0 \Rightarrow x_2 = 6 - 2s$

$x_1 + 2x_4 = 1 \Rightarrow x_1 = -3$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \quad s \in \mathbb{R}$$

ii) $\left[\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 0 \\ 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 2 & -2 & 1 \end{array}\right] \xrightarrow[\substack{R_3 + R_2 \\ R_1 + R_2}]{\substack{\text{same steps as i)} \\ R_2 - R_1}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -1 \\ 0 & -1 & -2 & 3 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

$x_4 = 0$

let $x_3 = s$, $s \in \mathbb{R}$

$-x_2 - 2x_3 + 3x_4 = -1 \Rightarrow x_2 = 1 - 2s$

$x_1 + 2x_4 = -1 \Rightarrow x_1 = -1$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \quad s \in \mathbb{R}$$

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2. a)

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 3 & -6 & -3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - 3R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & -3 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & -3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & -3 & -3 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1/3 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -3 & 2/3 & 1 \\ 0 & 0 & 1 & 1 & -1/3 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3/2 & 1/3 & 1/2 \\ 0 & 0 & 1 & 1 & -1/3 & 0 \end{array}\right] \xrightarrow{R_1 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2/3 & 1 \\ 0 & 1 & 0 & -3/2 & 1/3 & 1/2 \\ 0 & 0 & 1 & 1 & -1/3 & 0 \end{array}\right]$$

A^{-1} $[A \mid I] \rightarrow [I \mid A^{-1}]$

$$A^{-1} = \begin{bmatrix} -2 & 2/3 & 1 \\ -3/2 & 1/3 & 1/2 \\ 1 & -1/3 & 0 \end{bmatrix}$$

2b)

$$\overset{③}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}} \overset{②}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{①}{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 3 & 0 \end{bmatrix} \xrightarrow[]{\overset{④}{R_2 \leftrightarrow R_3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \xrightarrow[2R_3 \; ⑥]{⑤ \; \frac{1}{3}R_2} \underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$A = \underset{E_1}{\overset{①^{-1}}{\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}} \underset{E_2}{\overset{②^{-1}}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}} \underset{E_3}{\overset{③^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}} \underset{E_4}{\overset{④^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}} \underset{E_5}{\overset{⑤^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}}} \underset{E_6}{\overset{⑥^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}}} I$$

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2 c) $\det(A) = \det(E_1)\ \det(E_2)\ \det(E_3)\ \det(E_4)\ \det(E_5)\ \det(E_6)\ \det(I)$

$= 1 \times 1 \times 1 \times 1 \times 3 \times \frac{1}{2} \times 1$

$= 3/2$

3. a)

$$\det(A) = 1\begin{vmatrix} 2 & 3 & 1 \\ 2 & 0 & 0 \\ 2 & 5 & 7 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 & 1 \\ 0 & 0 & 0 \\ 1 & 5 & 7 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 7 \end{vmatrix} - 0\begin{vmatrix} 2 & 2 & 3 \\ 0 & 2 & 0 \\ 1 & 2 & 5 \end{vmatrix}$$

$$= 1\left(-2\begin{vmatrix} 3 & 1 \\ 5 & 7 \end{vmatrix} + 0\begin{vmatrix} 2 & 1 \\ 2 & 7 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix}\right)$$

$$-2\left(-0\begin{vmatrix} 3 & 1 \\ 5 & 7 \end{vmatrix} + 0\begin{vmatrix} 2 & 1 \\ 1 & 7 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix}\right)$$

$$-1\left(-0\begin{vmatrix} 2 & 1 \\ 2 & 7 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 1 & 7 \end{vmatrix} - 0\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}\right) - 0$$

expansion along 2nd row

$$= 1(-2 \times 16) - 2(0) - 1(2 \times 13)$$

$$= -58$$

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3 b) As the determinant of A is non zero, A is invertible and the rank of the matrix = 4. Hence, all the columns are independent and the 4 independent columns fill $\mathbb{R}^4$ space, thus the linear system $Ax=b$ has a unique solution for every b

As the reduced row echelon form of A is the identity matrix I_4

$[A \mid b] \longrightarrow [I \mid A^{-1}b]$ so $\therefore$ $Ax=b$ is consistent for all values of a.

3 c) i) $\det\left(\frac{1}{2}A^T\right) = \frac{1}{2}^4 \det(A^T)$

$= \frac{1}{2}^4 \det(A)$

$= -\frac{58}{16} = -\frac{29}{8}$

ii) $\det(AB^{-1}) = \det(A)\det(B^{-1})$

$= \det(A)\det(B)^{-1}$

$= -58 \times \frac{1}{3}$

$= -\frac{58}{3}$

iii) $\det((3B)^{-1}) = (3^4 \times \det(B))^{-1}$

$= 3^{-5}$

$= \frac{1}{243}$