18/19 sem 2

1. a)

$$\left(\begin{array}{cccc|c} 1 & 2 & -3 & -1 & 2 \\ 2 & 5 & -6 & 2 & 3 \\ 3 & 4 & -8 & -2 & -1 \end{array}\right) \xrightarrow[R_3-3R_1]{R_2-2R_1} \left(\begin{array}{cccc|c} 1 & 2 & -3 & -1 & 2 \\ 0 & 1 & 0 & 4 & -1 \\ 0 & -2 & 1 & 1 & -7 \end{array}\right)$$

$$\xrightarrow{R_3+2R_2} \left(\begin{array}{cccc|c} 1 & 2 & -3 & -1 & 2 \\ 0 & 1 & 0 & 4 & -1 \\ 0 & 0 & 1 & 9 & -9 \end{array}\right) \xrightarrow[R_1+3R_3]{R_1-2R_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 18 & -23 \\ 0 & 1 & 0 & 4 & -1 \\ 0 & 0 & 1 & 9 & -9 \end{array}\right)$$

$z = s$
$y = -9 - 9s$
$x = -1 - 4s$
$w = -23 - 18s$

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -23 \\ -1 \\ -9 \\ 0 \end{pmatrix} + s \begin{pmatrix} -18 \\ -4 \\ -9 \\ 1 \end{pmatrix}, \quad s \in \mathbb{R}$$

b)

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & b_1 \\ -1 & -3 & 1 & b_2 \\ 1 & 8 & -2 & b_3 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 8 & -2 & b_3 \\ -1 & -3 & 1 & b_2 \\ 2 & 1 & -1 & b_1 \end{array}\right) \xrightarrow[R_3-2R_1]{R_2+R_1} \left(\begin{array}{ccc|c} 1 & 8 & -2 & b_3 \\ 0 & 5 & -1 & b_2+b_3 \\ 0 & -15 & 3 & b_1-2b_3 \end{array}\right)$$

$$\xrightarrow{R_3+3R_2} \left(\begin{array}{ccc|c} 1 & 8 & -2 & b_3 \\ 0 & 5 & -1 & b_2+b_3 \\ 0 & 0 & 0 & b_1+3b_2+b_3 \end{array}\right)$$

For the linear system to be consistent

$b_1 + 3b_2 + b_3 = 0$

c)

$$\begin{pmatrix} 1 \\ 0 \\ 5 \\ -11 \\ 5 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} -2 \\ 0 \\ -10 \\ 22 \\ 10 \end{pmatrix}$$

$$\text{solution} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix}$$

2 a)

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 0 & -3 \\ 0 & 2 & 0 \end{pmatrix} \xrightarrow[R_2+2R_1]{E_1} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \xrightarrow[R_1-2R_2]{E_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

$$\xrightarrow[\frac{1}{2}R_3]{E_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow[R_2 \leftrightarrow R_3]{E_4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$I = E_4 E_3 E_2 E_1 A$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} I$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\det(A) = 1 \cdot 1 \cdot 2 \cdot -1$$
$$= -2$$

b) i)

$$a\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \quad a,b,c \in \mathbb{R}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

col 3 is a LC of col 1 and col 2

$$\cancel{\text{span}(v) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}}$$

$$V = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 2 \end{pmatrix} \right\}$$

ii) Basis for V by taking first 2 columns

$$\text{Basis} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\} \qquad \dim(V) = 2$$

3) $y = ax + b$

$$\left.\begin{aligned} 0 &= a + b \\ 1 &= 2a + b \\ 3 &= 3a + b \end{aligned}\right\} \quad \begin{array}{cc} A & b \\ \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 1 & 3 \end{array}\right) \end{array} \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 3 \end{array}\right)$$

$$\xrightarrow{R_3 - 2R_2} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{array}\right)$$

inconsistent

$\Rightarrow$ no solution for a, b

$\Rightarrow$ no straight line that passes through all 3 points

$$A^T A x = A^T b$$

$$\left(A^T A \mid A^T b\right) = \left(\begin{array}{cc|c} 14 & 6 & 11 \\ 6 & 3 & 4 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & -5/3 \end{array}\right)$$

$a = \frac{3}{2}$

$b = -\frac{5}{3}$

line of best fit is $y = \frac{3}{2}x - \frac{5}{3}$

when $x = 10$, $y = \frac{3}{2} \times 10 - \frac{5}{3} = \frac{40}{3}$

4) i)

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 3 \\ -1 & 1 & 0 \end{pmatrix}}_{\underset{\sim}{A}} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$$

ii) $\det(xI - A) = \begin{vmatrix} x-1 & 1 & 0 \\ 0 & x+1 & -3 \\ 1 & -1 & x \end{vmatrix} = (x-1)(x^2+x-3) + 1(-3)$

$= x^3 - 4x = x(x+2)(x-2)$

$\lambda = 0, 2, -2$

$\lambda = 0$: $\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & -3 \\ 1 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$ $v_0 = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$

$\lambda = 2$: $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -3 \\ 1 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ $v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda = -2$: $\begin{pmatrix} -3 & 1 & 0 \\ 0 & -1 & -3 \\ 1 & -1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ $v_{-2} = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$

general solution $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{-2t} \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$

$y_1(0) = 0 = 3c_1 - c_2 - c_3$

$y_2(0) = 0 = 3c_1 + c_2 - 3c_3$

$y_3(0) = 1000 = c_1 + c_2 + c_3$

$c_1 = 250$

$c_2 = 375$

$c_3 = 375$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 250 \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} + 375 e^{2t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + 375 e^{-2t} \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$$

5 i)

$$c_x = \frac{(-1\cdot 2)+(1\cdot 2)+(4\cdot 2)+(1\cdot 2)}{2+2+2+2} = 1$$

$$c_y = \frac{(2\cdot 2)+(1\cdot 2)+(1\cdot 2)+(1\cdot 2)}{8} = 1$$

$$c_z = \frac{(2\cdot 2)+(1\cdot 2)+(3\cdot 2)+(-1\cdot 2)}{8} = 1$$

$$c = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

ii)

$$\cancel{8}\begin{pmatrix} 1 & 1 & 4 & 0 \\ 3/2 & 1 & 0 & 1 \\ 4 & 0 & 3 & -1 \end{pmatrix}\begin{pmatrix} \cancel{m_1}\ 2 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\cdot(2+m_2+m_3+m_4)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 4 & 0 & 8 \\ 3/2 & 1 & 0 & 1 & 8 \\ 4 & 0 & 3 & -1 & 8 \end{array}\right) \xrightarrow[R_3-4R_1]{R_2-\frac{3}{2}R_1} \left(\begin{array}{cccc|c} 1 & 1 & 4 & 0 & 8 \\ 0 & -1/2 & -6 & 1 & -4 \\ 0 & -4 & -13 & -1 & -24 \end{array}\right)$$

Cannot change m_1!

$$\xrightarrow{R_3-8R_2} \left(\begin{array}{cccc|c} 1 & 1 & 4 & 0 & 8 \\ 0 & -1/2 & -6 & 1 & -4 \\ 0 & 0 & 35 & -9 & 8 \end{array}\right)$$

$m_4 = s$

$m_3 = (8+9s)/35$

$m_2 = (184-38s)/35$

$m_1 = (64+2s)/35$

$s = 3$

$m_4 = 3$

$m_3 = 1$

$m_2 = 2$

$m_1 = 2$

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \frac{1}{35}\begin{pmatrix} 64 \\ 184 \\ 8 \\ 0 \end{pmatrix} + \frac{1}{35}\begin{pmatrix} 2 \\ -38 \\ 9 \\ 35 \end{pmatrix}s, \quad s\in\mathbb{R}$$

no other integer values, restricted by $m_2 = \frac{184-38s}{35}$

iii)

$$\cancel{\left(\begin{array}{cccc|c} d_1 & 1 & 4 & 0 & 8 \\ d_2 & 1 & 0 & 1 & 8 \\ d_3 & \underbrace{0 & 3 & -1}_{} & 8 \end{array}\right)}$$

~~rref = I~~

~~adding~~ $\cancel{\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}}$ ~~makes set linearly dependent~~

~~$\Rightarrow$ nonpivot column~~

~~$\Rightarrow$ redistribution matrices more than 1 way~~

$$\boxed{\text{Cannot change } m_1 \text{ !}}$$

$$\begin{pmatrix} d_1 & 1 & 4 & 0 \\ d_2 & 1 & 0 & 1 \\ d_3 & 0 & 3 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = (2+m_2+m_3+m_4)\cdot\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

6 i) $\det(xI - A) = \begin{vmatrix} x-1 & -2 & -2 \\ -2 & x-1 & -2 \\ 0 & 0 & x+1 \end{vmatrix} = x+1\left((x-1)^2-4\right)$

$= (x+1)^2(x-3)$

$\lambda = -1, 3$

ii) $\lambda = -1$: $\begin{pmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Basis of E_{-1} = $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\lambda = 3$: $\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ 0 & 0 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Basis for E_3 = $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

$P = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$A = PDP^{-1}$

iii) $4A = P(4D)P^{-1}$

$Q = P = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

7. i)

$$\begin{pmatrix}1&0&1\\1&1&2\\1&1&3\end{pmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{pmatrix}1&0&1\\0&1&1\\0&1&2\end{pmatrix} \xrightarrow{R_3-R_2} \begin{pmatrix}1&0&1\\0&1&1\\0&0&1\end{pmatrix}$$

$$\xrightarrow[R_2-R_3]{R_1-R_3} \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$A = (w_1 \; w_2 \; w_3)$ is invertible and has order 3 ~~LI vectors~~

$\Rightarrow S$ is a basis for $\mathbb{R}^3$

ii)

$$v_1 = w_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$v_2 = \begin{pmatrix}0\\1\\1\end{pmatrix} - \frac{\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}-2\\1\\1\end{pmatrix}$$

$$v_3 = \begin{pmatrix}1\\2\\3\end{pmatrix} - \frac{\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix}}{6}\begin{pmatrix}-2\\1\\1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}0\\-1\\1\end{pmatrix}$$

orthogonal basis $T = \left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}-2\\1\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix} \right\}$

iii)

$$\left(\begin{array}{ccc|c}1&-2&0&1\\1&1&-1&-1\\1&1&1&3\end{array}\right) \longrightarrow \left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&0\\0&0&1&2\end{array}\right)$$

$$(u)_T = \begin{pmatrix}1\\0\\2\end{pmatrix}$$

8 i)

$$\begin{pmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 10 \\ 0 & -5 & -5 & -8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Basis for row space $= \left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

ii) Basis for column space includes 1st, 2nd and 4th non pivot columns

$= \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 10 \\ -8 \end{pmatrix} \right\}$

$$\left(\begin{array}{ccc|c} 1 & 3 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & -5 & -8 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$ inconsistent

$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \notin col(B)$

iii) nullspace $\perp$ row space

from i)
$d = 0$
$c = s$
$b = -s$
$a = 2s$

$s\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}$

Basis for nullspace $= \left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

nullity $= 1 = 4-3$