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1.
$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 2 \\ 3 & a & 9 & 0 & 6 \\ 2 & 0 & a+6 & a & b+2 \\ 2 & 0 & 6 & b & b+2 \end{array}\right) \xrightarrow[\substack{R_3-2R_1 \\ R_4-2R_1}]{R_2-3R_1} \left(\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 2 \\ 0 & a & 0 & -3 & 0 \\ 0 & 0 & a & a-2 & b-2 \\ 0 & 0 & 0 & b-2 & b-2 \end{array}\right)$$

If $a=0$ & $b=2$:
$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x_4 = 0$
$x_3 = s$
$x_2 = t$
$x_1 = 2 - 3s$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

general solution

iv) system has infinitely many solutions with 2 parameters.

If $a=0$ & $b \neq 2$:
$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -2 & b-2 \\ 0 & 0 & 0 & b-2 & b-2 \end{array}\right) \xrightarrow{\frac{1}{b-2}R_4} \left(\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -2 & b-2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & b \end{array}\right)$$

inconsistent

i) system has no solution

If $a \neq 0$ & $b=2$:
$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 2 \\ 0 & a & 0 & -3 & 0 \\ 0 & 0 & a & a-2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 0 & -\frac{3}{a} & 0 \\ 0 & 0 & 1 & \frac{a-2}{a} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

system has infinitely many solution with 1 parameter

$d = s$ $\quad b = \frac{3}{a}s$

$c = \frac{2-a}{a}s$ $\quad a = 2 - s - \frac{6-3a}{a}s$

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iii)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} \frac{2a-6}{a} \\ \frac{3}{a} \\ \frac{2-a}{a} \\ 1 \end{pmatrix}$$

If $a \neq 0$ & $b \neq 0$:

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 2 \\ 0 & a & 0 & -3 & 0 \\ 0 & 0 & a & a-2 & b-2 \\ 0 & 0 & 0 & b-2 & b-2 \end{array}\right) \xrightarrow[\frac{1}{b-2}R_4]{\frac{1}{a}R_2,\ \frac{1}{a}R_1} \left(\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 0 & -\frac{3}{a} & 0 \\ 0 & 0 & 1 & \frac{a-2}{a} & \frac{b-2}{a} \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{R_3 - \frac{a-2}{a}R_4 \\ R_1 - R_4}]{R_2 + \frac{3}{a}R_4} \left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 & \frac{3}{a} \\ 0 & 0 & 1 & 0 & \frac{b-2}{a} - \frac{a-2}{a} \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{R_1 - 3R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 - \frac{3(b-a)}{a} \\ 0 & 1 & 0 & 0 & \frac{3}{a} \\ 0 & 0 & 1 & 0 & \frac{b-a}{a} \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

ii) unique solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 - \frac{3(b-a)}{a} \\ \frac{3}{a} \\ \frac{b-a}{a} \\ 1 \end{pmatrix}$$

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2. a)

$$
\begin{aligned}
-722 &= a_5(-3)^5 + a_4(-3)^4 + a_3(-3)^3 + a_2(-3)^2 + a_1(-3) + a_0 \\
-103 &= a_5(-2)^5 + a_4(-2)^4 + a_3(-2)^3 + a_2(-2)^2 + a_1(-2) + a_0 \\
-2 &= a_5(-1)^5 + a_4(-1)^4 + a_3(-1)^3 + a_2(-1)^2 + a_1(-1) + a_0 \\
-10 &= a_5(1)^5 + a_4(1)^4 + a_3(1)^3 + a_2(1)^2 + a_1(1) + a_0 \\
13 &= a_5(2)^5 + a_4(2)^4 + a_3(2)^3 + a_2(2)^2 + a_1(2) + a_0 \\
262 &= a_5(3)^5 + a_4(3)^4 + a_3(3)^3 + a_2(3)^2 + a_1(3) + a_0 \\
1343 &= a_5(4)^5 + a_4(4)^4 + a_3(4)^3 + a_2(4)^2 + a_1(4) + a_0
\end{aligned}
$$

$$
\left(\begin{array}{cccccc|c}
-243 & 81 & -27 & 9 & -3 & 1 & -722 \\
-32 & 16 & -8 & 4 & -2 & 1 & -103 \\
-1 & 1 & -1 & 1 & -1 & 1 & -2 \\
1 & 1 & 1 & 1 & 1 & 1 & -10 \\
32 & 16 & 8 & 4 & 2 & 1 & 13 \\
243 & 81 & 27 & 9 & 3 & 1 & 262 \\
1024 & 256 & 64 & 16 & 4 & 1 & 1343
\end{array}\right)
$$

$$
\xrightarrow{\text{rref}}
\left(\begin{array}{cccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 & 0 & 0 & -3 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 & -7 \\
0 & 0 & 0 & 0 & 0 & 1 & -5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\qquad
\begin{aligned}
a_5 &= 2 \\
a_4 &= -3 \\
a_3 &= 1 \\
a_2 &= 2 \\
a_1 &= -7 \\
a_0 &= -5
\end{aligned}
$$

$$p(x) = 2x^5 - 3x^4 + x^3 + 2x^2 - 7x - 5$$

b) i)

$$\left(\begin{array}{ccccc|c} 81 & -27 & 9 & -3 & 1 & -722 \\ 16 & -8 & 4 & -2 & 1 & -103 \\ 1 & -1 & 1 & -1 & 1 & -2 \\ 1 & 1 & 1 & 1 & 1 & -10 \\ 16 & 8 & 4 & 2 & 1 & 13 \\ 81 & 27 & 9 & 3 & 1 & 262 \\ 256 & 64 & 16 & 4 & 1 & 1343 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

system is inconsistent $\Rightarrow$ no solution for b_4, b_3, b_2, b_1, b_0

ii) let $A = \begin{pmatrix} 81 & -27 & 9 & -3 & 1 \\ 16 & -8 & 4 & -2 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 81 & 27 & 9 & 3 & 1 \\ 256 & 64 & 16 & 4 & 1 \end{pmatrix}$ $b = \begin{pmatrix} -722 \\ -103 \\ -2 \\ -10 \\ 13 \\ 262 \\ 1343 \end{pmatrix}$

To minimize equation, we find a projection of b onto the column space of A
coefficients are the solution to

$$A^T A x = A^T b$$

$$\left(A^TA \mid A^Tb\right) \longrightarrow \left(\begin{array}{ccccc|c} 79172 & 16384 & 5684 & 1024 & 452 & 305096 \\ 16384 & 5684 & 1024 & 452 & 64 & 113440 \\ 5684 & 1024 & 452 & 64 & 44 & 16976 \\ 1024 & 452 & 64 & 44 & 4 & 8548 \\ 452 & 64 & 44 & 4 & 7 & 781 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 2.6585 \\ 0 & 1 & 0 & 0 & 0 & 26.9348 \\ 0 & 0 & 1 & 0 & 0 & -53.6093 \\ 0 & 0 & 0 & 1 & 0 & -72.8864 \\ 0 & 0 & 0 & 0 & 1 & 72.2683 \end{array}\right)$$

$b_4 = 2.6585$
$b_3 = 26.9348$
$b_2 = -53.6098$
$b_1 = -72.8864$
$b_0 = 72.2683$

3 a) $x_1 = -2x_2 + x_3 - x_4 + x_5$

$$V_1 = \text{span}\left\{\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix}\right\}$$

$x_1 = 2x_2 - x_3 - 2x_4 + x_5$

$$V_2 = \text{span}\left\{\begin{pmatrix}2\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix}\right\}$$

$V = V_1 \cap V_2$

$w \in V_1 \cap V_2$

$$w = \alpha_1\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix} + \alpha_2\begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix} + \alpha_3\begin{pmatrix}-1\\0\\0\\1\\0\end{pmatrix} + \alpha_4\begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix}$$

$$= \beta_1\begin{pmatrix}2\\1\\0\\0\\0\end{pmatrix} + \beta_2\begin{pmatrix}-1\\0\\1\\0\\0\end{pmatrix} + \beta_3\begin{pmatrix}-2\\0\\0\\1\\0\end{pmatrix} + \beta_4\begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}-2 & 1 & -1 & 1 & -2 & 1 & 2 & -1\\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \beta_1\\ \beta_2\\ \beta_3\\ \beta_4\end{pmatrix} = 0$$

$\beta_4 = s$
$\beta_3 = t$
$\beta_2 = r$
$\beta_1 = \frac{1}{4}t + \frac{1}{2}r$

$\alpha_4 = s$
$\alpha_3 = t$
$\alpha_2 = r$
$\alpha_1 = \frac{1}{4}t + \frac{1}{2}r$

$$w = \left(\frac{1}{4}t + \frac{1}{2}r\right)\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix} + r\begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix} + t\begin{pmatrix}-1\\0\\0\\1\\0\end{pmatrix} + s\begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix}$$

$$= \text{span}\left\{\begin{pmatrix}-3/2\\1/4\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1/2\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\\1\end{pmatrix}\right\}$$

$\dim(V) = 3$

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b)

$|S| = \dim(V)$

To show $V \subseteq \text{span}(S)$

$$\left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & -1/2 & 0 & 1 \\ 0 & 1 & 0 & 1/4 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 4 & -2 & 1 & 0 & 0 \\ 1 & 6 & -3 & 0 & 0 & 1 \end{array}\right) \longrightarrow \text{consistent}$$

$$\left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 0 & 4 & -2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 6 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ S is a basis for V

c)

$$\left(\begin{array}{ccc|c|c|c} -1/2 & 0 & 1 & 3 & 3 & 4 \\ 1/4 & 1/2 & 0 & -2 & 2 & 2 \\ 0 & 1 & 0 & -5 & 4 & 2 \\ 1 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 1 & 6 & 10 & 10 \end{array}\right)$$

$\longrightarrow$ consistent $\Rightarrow$ belongs in V

$$\left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 2 & 0 & 4 \\ 0 & 1 & 0 & -5 & 0 & 2 \\ 0 & 0 & 1 & 6 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$(u_1)_S = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix}$

$(u_3)_S = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$

$u_1 \in V$ $\quad$ $u_2 \notin V$

$u_3 \in V$

d) $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$$v_2 = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 4 \\ 6 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 4 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 4 \\ 3 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \\ -3 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 0 \\ 0 \\ 1 \\ -2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 0 \\ 4 \\ 3 \end{pmatrix}}{35} \begin{pmatrix} -3 \\ 1 \\ 0 \\ 4 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3/70 \\ 3/7 \\ 1 \\ -4/7 \\ -3/70 \end{pmatrix}$$

orthonormal basis = $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{35}} \begin{pmatrix} -3 \\ 1 \\ 0 \\ 4 \\ 3 \end{pmatrix}, \frac{1}{\sqrt{\left(\frac{3}{70}\right)^2 + \left(\frac{3}{7}\right)^2 + 1 + \left(\frac{4}{7}\right)^2 + \left(\frac{3}{70}\right)^2}} \begin{pmatrix} 3/70 \\ 3/7 \\ 1 \\ -4/7 \\ -3/70 \end{pmatrix} \right\}$

e)

orthogonal to $V \Rightarrow \in \text{Null}(A^T)$

where $A =$ column of S

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 & 6 \\ 0 & 0 & 1 & -2 & -3 \end{pmatrix}$$

$e = s$
$d = t$
$c = 3s + 2t$
$b = -6s - 4t$
$a = -s$

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = s \underbrace{\begin{pmatrix} -1 \\ -6 \\ 3 \\ 0 \\ 1 \end{pmatrix}}_{u_1} + t \underbrace{\begin{pmatrix} 0 \\ -4 \\ 2 \\ 1 \\ 0 \end{pmatrix}}_{u_2}$$

4 a)

$$A = \begin{pmatrix} 6 & 2 & 1 & -1 \\ -3 & 4 & 5 & 1 \\ 6 & 5 & 4 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Homogeneous system $Ax=0$ does not only contain trivial solutions

columns of A are linearly dependent

Left inverse of A does not exist

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

columns of are linearly independent

B has a left inverse $(B^T B)^{-1} B^T$

$$(B^T B)^{-1} B^T = \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 3 & 1 & -1 & -2 \\ -3 & 1 & 2 & 1 \end{pmatrix}$$

b) From (a), rows of A are linearly independent $\Rightarrow$ A has a right inverse

right inverse: $A^T(AA^T)^{-1} = \frac{1}{3} \begin{pmatrix} 3 & 1 & -2 \\ -5 & -2 & 4 \\ 4 & 2 & -3 \\ 9 & 4 & -7 \end{pmatrix}$

From (a) rows of B are not linearly independent $\Rightarrow$ B does not have a right inverse

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c)

$$AB = \begin{pmatrix} 6 & 3 & 3 \\ 3 & 3 & 9 \\ 9 & 6 & 9 \end{pmatrix}$$

$$\det(AB) = -27$$

d) column space of B $= \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

From (a), $d = s$, $c = -s$, $b = s$, $a = 0$

$\text{Null}(A) = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}$

$\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$ is orthogonal to every vector in $\text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Basis for Intersection of $\text{Col}(B) \cap \text{Null}(A) = \{\}$ empty set

5. a) $A = \begin{pmatrix} 5 & 2 & -4 \\ 2 & 5 & -4 \\ 4 & 4 & -5 \end{pmatrix}$

$\det(xI - A) = \begin{vmatrix} x-5 & -2 & 4 \\ -2 & x-5 & 4 \\ -4 & -4 & x+5 \end{vmatrix} = (x+1)(x-3)^2$

$\lambda = -1, 3$

$\lambda = -1$: $\begin{pmatrix} -6 & -2 & 4 \\ -2 & -6 & 4 \\ -4 & -4 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$

$c = s$
$b = \frac{1}{2}s$
$a = \frac{1}{2}s$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{2}s \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

Basis for $E_{-1} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$

$\lambda = 3$: $\begin{pmatrix} -2 & -2 & 4 \\ -2 & -2 & 4 \\ -4 & -4 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$c = s$
$b = t$
$a = 2s - t$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = s\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Basis for $E_3 = \left\{ \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$

$P = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$A = PDP^{-1}$

b) $\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 5 & 2 & -4 \\ 2 & 5 & -4 \\ 4 & 4 & -5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$A = \begin{pmatrix} 5 & 2 & -4 \\ 2 & 5 & -4 \\ 4 & 4 & -5 \end{pmatrix} = PDP^{-1}$ (from (a))

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fundamental set of solutions

$$\left\{ e^{-t}\begin{pmatrix}1\\1\\2\end{pmatrix},\ e^{3t}\begin{pmatrix}2\\0\\1\end{pmatrix},\ e^{3t}\begin{pmatrix}-1\\1\\0\end{pmatrix} \right\}$$

general solution $$\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix} = c_1 e^{-t}\begin{pmatrix}1\\1\\2\end{pmatrix} + c_2 e^{3t}\begin{pmatrix}2\\0\\1\end{pmatrix} + c_3 e^{3t}\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$$y_1(0) = 2 = c_1 + 2c_2 - c_3$$
$$y_2(0) = 2 = c_1 + c_3$$
$$y_3(0) = 1 = 2c_1 + c_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$c_1 = -1$$
$$c_2 = 3$$
$$c_3 = 3$$

$$\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix} = -e^{-t}\begin{pmatrix}1\\1\\2\end{pmatrix} + 3e^{3t}\begin{pmatrix}2\\0\\1\end{pmatrix} + 3e^{3t}\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

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6. a) $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 6 & -4 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$A = \begin{pmatrix} 6 & -4 \\ 4 & -2 \end{pmatrix}$

$\det(xI - A) = \begin{vmatrix} x-6 & 4 \\ -4 & x+2 \end{vmatrix} = x^2 - 4x + 4$

$= (x-2)^2$

$\lambda = 2 : \begin{pmatrix} -4 & 4 \\ -4 & 4 \end{pmatrix} \to \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad b = s$, $a = s$

Basis for $E_2 = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ $\quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Find a non zero vector v_2 such that $(A - 2I) v_2 = v_1$

$\left(\begin{array}{cc|c} 4 & -4 & 1 \\ 4 & -4 & 1 \end{array} \right) \to \left(\begin{array}{cc|c} 1 & -1 & 1/4 \\ 0 & 0 & 0 \end{array} \right)$ $\quad b = s$, $a = \frac{1}{4} + s$

$v_2 = \begin{pmatrix} \frac{1}{4} + s \\ s \end{pmatrix}$ let $s = 0$ $\quad v_2 = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix}$

Fundamental set of solutions : $\left\{ e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, e^{2t} \left(t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} \right) \right\}$

b) Wronskian: $\begin{vmatrix} e^{2t} & te^{2t} + \frac{1}{4} e^{2t} \\ e^{2t} & te^{2t} \end{vmatrix} = e^{4t} \begin{vmatrix} 1 & t + \frac{1}{4} \\ 1 & t \end{vmatrix}$

$= -\frac{1}{4} e^{4t} \neq 0$

Hence, fundamental set of solutions is linearly independent

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c) $y_1 = c_1 e^{2t} + c_2 (t + \frac{1}{4}) e^{2t}$

$y_2 = c_1 e^{2t} + c_2 t e^{2t}$

d) $y_1(1) = e = c_1 e^2 + \frac{5}{4} c_2 e^2 = (c_1 + \frac{5}{4} c_2) e^2$

$y_2(1) = e = (c_1 + c_2) e^2$

$c_1 + \frac{5}{4} c_2 = \frac{1}{e}$

$c_1 + c_2 = \frac{1}{e}$

$\frac{1}{4} c_2 = 0$

$c_2 = 0$

$c_1 = \frac{1}{e}$

$y_1 = \frac{1}{e} e^{2t}$ ~~+~~

$y_2 = \frac{1}{e} e^{2t}$