NATIONAL UNIVERSITY OF SINGAPORE

MA1508E - LINEAR ALGEBRA FOR ENGINEERING

(Semester 1 : AY2019/2020)

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. Please write your student number only. Do not write your name.
2. This examination paper contains **SEVEN** questions and comprises **FOUR** printed pages.
3. Answer **ALL** questions.
4. Please start each question on a new page.
5. This is a CLOSED BOOK (with helpsheet) examination.
6. Students are allowed to use one A4 size helpsheet.
7. Candidates may use scientific calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [15 marks]

(a) Find the reduced row-echelon form of the following matrix

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{pmatrix}.$$

Hence, or otherwise, solve the following linear system

$$\begin{cases} a + 2b - c = 1 \\ 2a + 4b - 3c = 0 \\ a + 2b + c = 5. \end{cases}$$

(b) Consider the following linear system where k is a real number. Determine the values of k such that the linear system

(i) is inconsistent;

(ii) has a unique solution;

(iii) has infinitely many solutions.

$$\begin{cases} x - y + kz = 1 \\ kx + z = 1 \\ 2kx - 2ky + z = 2 \end{cases}$$

(c) For each of the following, write down the reduced row-echelon form of the augmented matrix for the linear system with the stated properties. If such a matrix is not possible, write down **not possible**. No justification is required.

(i) Linear system has 4 equations, 3 unknowns x, y, z and the unique solution $x = 1, y = -1, z = 4$.

(ii) Linear system has 3 equations, 4 unknowns w, x, y, z and the unique solution $w = 0, x = 0, y = 0, z = 0$.

(iii) Linear system has 4 equations, 3 unknowns x, y, z and the general solution

$$\begin{cases} x = 2 - s + 3t, \\ y = s, \\ z = t, \qquad s, t \in \mathbb{R}. \end{cases}$$

(iv) Linear system has 3 equations, 5 unknowns x_1, x_2, x_3, x_4, x_5 and the general solution

$$\begin{cases} x_1 = 2 \\ x_2 = s + t, \\ x_3 = s, \\ x_4 = 1 - t, \\ x_5 = t, \qquad s, t \in \mathbb{R}. \end{cases}$$

Question 2 [15 marks]

I am trying to fit a straight line $y = ax + b$, where $a, b \in \mathbb{R}$, through the following 4 points: $(1, 2)$, $(2, 3)$, $(-1, 1)$ and $(0, 1)$.

(i) Show that there is no straight line that passes through all the 4 points simultaneously.

(ii) Find the line of best fit using the least squares method.

(iii) Using your line of best fit obtained in (ii), predict the value of y when $x = 3$.

(iv) Compute the projection of $\boldsymbol{b}$ onto $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$ where

$$\boldsymbol{u_1} = (1, 2, -1, 0)^T, \quad \boldsymbol{u_2} = (1, 1, 1, 1)^T, \quad \boldsymbol{b} = (2, 3, 1, 1)^T.$$

Question 3 [20 marks]

(a) You may assume that the following matrix $\boldsymbol{A}$ is invertible. Express $\boldsymbol{A}^{-1}$ as a product of elementary matrices. Write down all your elementary matrices explicitly.

$$\boldsymbol{A} = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 6 \\ 1 & -1 & 0 \end{pmatrix}.$$

(b) For each of the following, if V is a subspace of $\mathbb{R}^4$, write V as a linear span and determine the dimension of V. If V is not a subspace, explain why.

(i) $V = \{(w, x, y, z) \mid w + x + y + z = 1\}$.

(ii) $V = \{(a, b, c, d) \mid a + 1 = b + 1 \text{ and } c + 2 = d + 2\}$.

(iii) $V = \{(a + b + d,\ a - b,\ a - b,\ c) \mid a, b, c, d \in \mathbb{R}\}$.

(c) Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

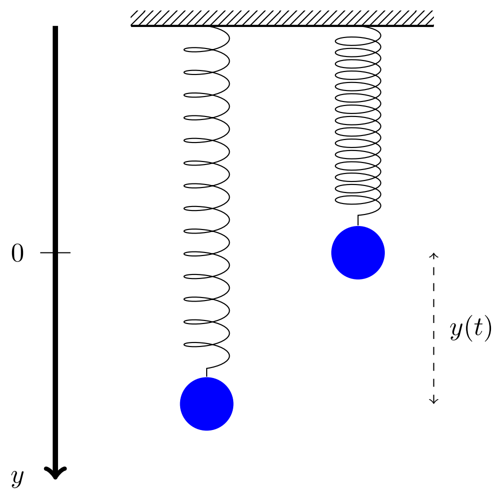
(i) Find a vector $\boldsymbol{b} \in \mathbb{R}^3$ such that $\boldsymbol{b}$ does not belong to the column space of $\boldsymbol{A}$. Justify your answer.

(ii) Is it possible to find a vector $\boldsymbol{c} \in \mathbb{R}^3$ such that $\boldsymbol{c}$ does not belong to the column space of $\boldsymbol{A}$ **but** belongs to the row space of $\boldsymbol{A}$? Justify your answer.

Question 4 [16 marks]

(a) Consider a body of mass m (measured by kilogram, Kg) that is suspended from a spring (see Figure below). Suppose that the body is moved from its resting position and set in motion. Let $y(t)$ denote the distance of the body from its resting position at time t, where positive distances are measured **downward**. If k is the **spring constant** (measured by Newtons per meter, N/m), and b is the **damping constant** (measured by Newtons second per meter, Ns/m), then the motion of the body (in a free and unforced system) satisfies the differential equation

$$my''(t) + by'(t) + ky(t) = 0.$$



Find a general solution to the simple harmonic oscillator equation when a mass of $m = 500$ grams is moving in the system with spring constant $k = 1.5$ N/m and damping constant $b = 2$ Ns/m.

(b) Solve the system of linear differential equations $\boldsymbol{Y'} = \boldsymbol{AY}$ where

$$\boldsymbol{Y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{Y}(0) = \begin{pmatrix} -4 \\ 8 \end{pmatrix}.$$

Question 5 [10 marks]

In the city of UNPREDICTABILITY, the weather each day is either **dry** or **wet**. Through a series of studies, it has been concluded that whenever the weather is dry on a particular day, it will be dry again the next day with probability 0.2. On the other hand, if the weather is wet on a particular day, it will be wet again the next day with probability 0.4.

For $n = 1, 2, \ldots$, let d_n be the probability that the weather is dry on day n and w_n be the probability that the weather is wet on day n. Let $\boldsymbol{X}_n = \begin{pmatrix} d_n \\ w_n \end{pmatrix}$ for $n = 1, 2, \ldots$. Assume that it is January 1 in year 2020 today and there is equal probability that the weather today will be dry or wet. Thus, we let $d_1 = 0.5$ and $w_1 = 0.5$.

(i) Compute d_2 and w_2. Note that these would give us the probabilities of having dry or wet weather on January 2.

(ii) Find a 2×2 matrix $\boldsymbol{A}$ such that $\boldsymbol{X}_{n+1} = \boldsymbol{A}\boldsymbol{X}_n$ for $n = 1, 2, \ldots$.

(iii) Find all the eigenvalues of $\boldsymbol{A}$ and explain why $\boldsymbol{A}$ is diagonalizable. By evaluating d_n as n tends to infinity, estimate the probability of having dry weather on a particular day in the city in the long run.

Question 6 [12 marks]

We are familiar with the three-dimensional environment that we live in, where each point in space can be described by a vector $(x, y, z) \in \mathbb{R}^3$. It is often said that **time** is the fourth dimension, so in addition to the location, we also indicate at which point in time are we at that particular location. Such a vector (x, y, z, t) is thus a vector in $\mathbb{R}^4$.

Suppose you are lost in space (and time) and have only one chance to get back to reality. Your current space/time location is described by $\boldsymbol{w} = (1, -1, 1, 1)$. To get back to reality, you need to find the space/time location (called the portal) from the set

$$V = \{(x, y, z, t) \mid x - y + z - t = 0\}$$

that is **nearest** to your current space/time location. Only through this location will you be able to be transported back to reality.

(i) Where is this location?

(ii) Even if you manage to find this location, in order to get there before the portal closes, your current location cannot be more than 1 unit length away from the portal location. Are you too far away to get there in time?

Question 7 [12 marks]

Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} & \boldsymbol{a_4} & \boldsymbol{a_5} \end{pmatrix}$ where $\boldsymbol{a_i}$ is the i-th column of $\boldsymbol{A}$. Suppose

$$\boldsymbol{a_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \quad \boldsymbol{a_2} = \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{a_4} = \begin{pmatrix} 1 \\ 1 \\ -3 \\ 2 \end{pmatrix}.$$

We also know that the reduced row-echelon form of $\boldsymbol{A}$ is

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(i) Find $\boldsymbol{A}$.

(ii) Find rank($\boldsymbol{A}$), nullity($\boldsymbol{A}$), rank($\boldsymbol{A}^T$) and nullity($\boldsymbol{A}^T$).

(iii) Find a basis for the nullspace of $\boldsymbol{A}$.

(iv) It is known that $\begin{pmatrix} 0 & 1 & 1 & 1 \end{pmatrix} \boldsymbol{A} = \boldsymbol{0}$. Find a basis for the nullspace of $\boldsymbol{A}^T$.

(v) Is every vector in $\mathbb{R}^4$ in the column space of $\boldsymbol{A}$? Justify your answer.

End of Paper