NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

AY2021, Semester 1 MA1508E Linear Algebra for Engineering Tutorial 4

1. (a) Suppose $\mathbf{A} = \mathbf{PDP}^{-1}$ for some invertible matrix $\mathbf{P}$. Show that $\det(\mathbf{A}) = \det(\mathbf{D})$.
   (b) Suppose $\mathbf{A} = \mathbf{PDP}^{-1}$ for some invertible matrix $\mathbf{P}$ and $\mathbf{D}$ is a diagonal matrix. Show that $\mathbf{A}$ is invertible if and only if all the diagonal entries of $\mathbf{D}$ is nonzero.
   (c) Recall that a square matrix $\mathbf{A}$ is nilpotent if there is a positive integer k such that $\mathbf{A}^k = \mathbf{0}$. Show that if $\mathbf{A}$ is nilpotent, then $\det(\mathbf{A}) = 0$.
   (d) A square matrix is an *orthogonal* matrix if $\mathbf{A}^T = \mathbf{A}^{-1}$. Show that if $\mathbf{A}$ is orthogonal, then $\det(\mathbf{A}) = \pm 1$.

2. Let $\mathbf{A}$ be a $k \times k$ matrix and let $\mathbf{B}$ be a $(n-k) \times (n-k)$ matrix. Let

$$\mathbf{E} = \begin{pmatrix} \mathbf{I_k} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_{n-k}} \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix},$$

   where $\mathbf{I_k}$ and $\mathbf{I_{n-k}}$ are the $k \times k$ and $(n-k) \times (n-k)$ identity matrices respectively.
   (a) Show that $\det(\mathbf{E}) = \det(\mathbf{B})$.
   (b) Show that $\det(\mathbf{F}) = \det(\mathbf{A})$.
   (c) Show that $\det(\mathbf{C}) = \det(\mathbf{A})\det(\mathbf{B})$.

   Hint: For (a) and (b) use cofactor expansions. For (c), try to write the matrix $\mathbf{C}$ as a product of (block) matrices.

3. Let $\mathbf{A} = \begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$ (cf: Tutorial 3 question 1(b)).
   (a) What is $\det(\mathbf{A})$?
   (b) Suppose $\mathbf{B}$ is an order 3 square matrix. Show that the homogeneous linear system $\mathbf{ABx} = \mathbf{0}$ have infinitely many solutions.

4. (a) Consider the follow linear system (cf: Tutorial 1 question 1(b))

$$\left\{ \begin{array}{rcrcrcrcr} a & + & b & - & c & - & 2d & = & 0 \\ 2a & + & b & - & c & + & d & = & -2 \\ -a & + & b & - & 3c & + & d & = & 4 \end{array} \right.$$

   Express the solutions in the set notation.
   (b) Suppose a linear system has reduced row-echelon form

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & -2 \\ 0 & 1 & 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

   Express the solutions in the set notation.

5. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be an orthonormal set. Suppose

$$\mathbf{x} = \mathbf{v}_1 - 2\mathbf{v}_2 - 2\mathbf{v}_3 \quad \text{and} \quad \mathbf{y} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3.$$

Determine the value for each of the following (you may use your calculators for this question.)

(a) $\mathbf{x} \cdot \mathbf{y}$.

(b) $||\mathbf{x}||$ and $||\mathbf{y}||$.

(c) The angle θ between $\mathbf{x}$ and $\mathbf{y}$.

6. (a) Let $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ be a linear equation. Express this linear system as $\mathbf{a} \cdot \mathbf{x} = b$ for some (column) vectors $\mathbf{a}$ and $\mathbf{x}$.

(b) Find the solution set of the linear system

$$\begin{array}{rcrcrcrl} x_1 & + & 3x_2 & - & 2x_3 & & & = 0 \\ 2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & = 0 \\ & & & + & 5x_3 & + & 10x_4 & = 0 \end{array}$$

(c) Find a nonzero vector $\mathbf{v} \in \mathbb{R}^3$ such that $\mathbf{a}_1 \cdot \mathbf{v} = 0$, $\mathbf{a}_2 \cdot \mathbf{v} = 0$, and $\mathbf{a}_3 \cdot \mathbf{v} = 0$, where

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 3 \\ -2 \\ 0 \end{pmatrix}, \qquad \mathbf{a}_2 = \begin{pmatrix} 2 \\ 6 \\ -5 \\ -2 \end{pmatrix}, \qquad \mathbf{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 5 \\ 10 \end{pmatrix}$$

This exercise demonstrates the fact that if $\mathbf{A}$ is a $m \times n$ matrix, then the solution set of the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ consist of all the vectors in $\mathbb{R}^n$ that are orthogonal to every row vector of $\mathbf{A}$.

Supplementary Problems

7. **(Application)** (Statistics)

Suppose in a math test, the results of a class of n students are $x_1, x_2, ..., x_n$. We can represent the result as a *sample vector*

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

The sample *mean*, $\overline{x}$ is defined by

$$x = \frac{1}{n}(x_1 + x_2 + \cdots + x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i$$

The *centred sample vector* $\mathbf{x}_c$ is define as

$$\mathbf{x}_c = \begin{pmatrix} x_1 - \overline{x} \\ x_2 - \overline{x} \\ \vdots \\ x_n - \overline{x} \end{pmatrix}$$

The sample *variance* $\sigma_{\mathbf{x}}^2$ is defined as

$$\sigma_{\mathbf{x}}^2 = \frac{1}{n-1}((x_1-\overline{x})^2 + (x_2-\overline{x})^2 + \cdots + (x_n-\overline{x})^2) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\overline{x})^2$$

The square root of the variance $\sigma_{\mathbf{x}}$ is called the *sample standard deviation.* Let

$$\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

denote the vector with enties equal to 1. Express the

(a) mean,

(b) centred sample vector,

(c) variance, and

(d) standard deviation

using the vector $\mathbf{1}$, dot product, and norm.

(**MATLAB**) The vector $\mathbf{1}$ can be obtained via

```
>> ones(n,1)
```

the dot product between $\mathbf{u}$ and $\mathbf{v}$ can be computed via

```
>> dot(u,v)
```

and the norm of $\mathbf{v}$ is

```
>> norm(v)
```

Suppose the results of a math test of 10 students are $51, 35, 62, 78, 84, 55, 68, 92, 55, 69$. Use MATLAB to compute the

(e) mean,

(f) centred sample vector,

(g) variance, and

(h) standard deviation

of the simulated results you obtained. To calculate a percentile of the sample $\mathbf{x}$, use

```
>> prctile(x,p)
```

where p is the percentile to be computed.

(i) Calculate the 75-th percentile of the results.

(j) Suppose to obtain an A grade in the math test a student needs to be above the 80th-percentile. How many students will get A in the math test?