NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

AY2021, Semester 1 MA1508E Linear Algebra for Engineering Tutorial 1

Solutions

1. Solve the following linear systems by Gaussian Elimination or Gauss-Jordan Elimination. (Make sure you are able to perform the necessary elementary row operations without the help of MATLAB.)

(a)

$$\left\{\begin{array}{rcrcrcr} 3x_1 & + & 2x_2 & - & 4x_3 & = & 3 \\ 2x_1 & + & 3x_2 & + & 3x_3 & = & 15 \\ 5x_1 & - & 3x_2 & + & x_3 & = & 14 \end{array}\right.$$

(b)

$$\left\{\begin{array}{rcrcrcrcr} a & + & b & - & c & - & 2d & = & 0 \\ 2a & + & b & - & c & + & d & = & -2 \\ -a & + & b & - & 3c & + & d & = & 4 \end{array}\right.$$

(c)

$$\left\{\begin{array}{rcrcrcr} x & - & 4y & + & 2z & = & -2 \\ x & + & 2y & - & 2z & = & -3 \\ x & - & y & & & = & 4 \end{array}\right.$$

(a)

$$\left(\begin{array}{ccc|c} 3 & 2 & -4 & 3 \\ 2 & 3 & 3 & 15 \\ 5 & -3 & 1 & 14 \end{array}\right) \xrightarrow[R_2-\frac{2}{3}R_1]{R_3-\frac{5}{3}R_1} \left(\begin{array}{ccc|c} 3 & 2 & -4 & 3 \\ 0 & \frac{5}{3} & \frac{17}{3} & 13 \\ 0 & -\frac{19}{3} & \frac{23}{3} & 9 \end{array}\right) \xrightarrow{R_3+\frac{19}{5}R_1} \left(\begin{array}{ccc|c} 3 & 2 & -4 & 3 \\ 0 & \frac{5}{3} & \frac{17}{3} & 13 \\ 0 & 0 & \frac{146}{5} & \frac{292}{5} \end{array}\right)$$

$$\xrightarrow[3R_2]{\frac{5}{146}R_3} \left(\begin{array}{ccc|c} 3 & 2 & -4 & 3 \\ 0 & 5 & 17 & 39 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{R_2+17R_3} \left(\begin{array}{ccc|c} 3 & 2 & -4 & 3 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{\frac{1}{5}R_2} \xrightarrow{R_1-2R_2} \xrightarrow{R_1+4R_2}$$

$$\left(\begin{array}{ccc|c} 3 & 0 & 0 & 9 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{\frac{1}{3}R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

System has a unique solution $x_1 = 3, x_2 = 1, x_3 = 2$.

(b) The reduced row-echelon form of the augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & -\frac{19}{2} & 2 \\ 0 & 0 & 1 & -\frac{9}{2} & 0 \end{array}\right).$$

So a general solution to the system is $a = -2 - 3s, b = 2 + \frac{19s}{2}, c = \frac{9s}{2}, d = s$, $s \in \mathbb{R}$.

(c) The linear system is inconsistent. Its reduced row-echelon form is

$$\left(\begin{array}{ccc|c} 1 & 0 & -2/3 & 0 \\ 1 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

2. Reduce the following matrix to its reduced row echelon form using Gaussian and Gauss-Jordan elimination.
$$\left(\begin{array}{ccc|c} 2 & 6 & 5 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 4 & 5 & 0 \end{array}\right)$$
Could the number of operations be reduced if we do not insist on using Gaussian or Gauss-Jordan elimination?

$$\left(\begin{array}{ccc|c} 2 & 6 & 5 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 4 & 5 & 0 \end{array}\right) \xrightarrow[R_2-\frac{1}{2}R_1]{R_3-\frac{1}{2}R_1} \left(\begin{array}{ccc|c} 2 & 6 & 5 & 0 \\ 0 & -3 & \frac{3}{2} & 0 \\ 0 & 1 & \frac{5}{2} & 0 \end{array}\right) \xrightarrow{R_3+\frac{1}{3}R_2} \left(\begin{array}{ccc|c} 2 & 6 & 10 & 0 \\ 0 & -3 & \frac{3}{2} & 0 \\ 0 & 0 & 3 & 0 \end{array}\right) \xrightarrow[R_2-\frac{1}{2}R_3]{R_1-\frac{5}{3}R_3}$$
$$\left(\begin{array}{ccc|c} 2 & 6 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right) \xrightarrow{R_1+2R_2} \left(\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 \end{array}\right) \xrightarrow[\substack{-\frac{1}{3}R_2 \\ \frac{1}{3}R_3}]{\frac{1}{2}R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

We can reduce the number of operations. For example,

$$\left(\begin{array}{ccc|c} 2 & 6 & 5 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 4 & 5 & 0 \end{array}\right) \xrightarrow[R_3-R_2]{R_1-2R_2} \left(\begin{array}{ccc|c} 0 & 6 & -3 & 0 \\ 1 & 0 & 4 & 0 \\ 0 & 4 & 1 & 0 \end{array}\right) \xrightarrow{R_1+3R_3} \left(\begin{array}{ccc|c} 0 & 18 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 0 & 4 & 1 & 0 \end{array}\right)$$
$$\xrightarrow{\frac{1}{18}R_1} \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 0 & 4 & 1 & 0 \end{array}\right) \xrightarrow{R_3-4R_1} \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_2-4R_3} \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$
$$\xrightarrow{R_1\leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

Especially since we solving a linear system, we could stop after the 6^{th} operation.

3. Determine the values of a and b so that the linear system
$$\begin{cases} ax & & + & bz & = & 2 \\ ax & + \; ay & + & 4z & = & 4 \\ & \;\;\; ay & + & 2z & = & b \end{cases}$$
   (a) has no solution;
   (b) has only one solution;
   (c) has infinitely many solutions and a general solution has one arbitrary parameter;
   (d) has inifinitely many solutions and a general solution has two arbitrary parameters.

$$\left(\begin{array}{ccc|c} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{array}\right) \to \left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & b-2 & b-2 \end{array}\right).$$

Case 1: $b \neq 2$.
$$\to \left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} a & 0 & 0 & 2-b \\ 0 & a & 0 & b-2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

If $a = 0$, system is inconsistent. If $a \neq 0$, system has a unique solution.
Case 2: $b = 2$.

$$\rightarrow \left(\begin{array}{ccc|c} a & 0 & 2 & 2 \\ 0 & a & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

If $a = 0$, then system has infinitely many solutions with two parameter. If $a \neq 0$, then system has infinitely many solution with one parameter.

(a) $a = 0$ and $b \neq 2$;
(b) $a \neq 0$ and $b \neq 2$;
(c) $a \neq 0$ and $b = 2$;
(d) $a = 0$ and $b = 2$.

4. **(Application)** When chemical compounds are combined under the right conditions, the atoms in their molecules rearrange to form new compounds. This is represented by a chemical equation. For example, the when methane burns, the methane (CH_4) and stable oxygen (O_2) react to form carbon dioxide (CO_2) and water (H_2O). The chemical equation is

$$CH_4 + O_2 \longrightarrow CO_2 + H_2O$$

A chemical equation is said to be balanced if for each type of atom in the reaction, the same number of atoms appears on each side of the arrow. We can balance the equation by letting x_1, x_2, x_3, and x_4 to be the number of methane, stable oxygen, carbon dioxide, and water molecule, that is,

$$x_1CH_4 + x_2O_2 \longrightarrow x_3CO_2 + x_4H_2O$$

From it we obtain the following homogeneous linear system

$$\begin{array}{rcrcrcrl} x_1 & & & - & x_3 & & & = 0 \\ 4x_1 & & & & & - & 2x_4 & = 0 \\ & & 2x_2 & - & 2x_3 & - & x_4 & = 0 \end{array}$$

The smallest positive integer values solution of the system will give us the balanced equation for combustion of methane

$$CH_4 + 2O_2 \longrightarrow CO_2 + 2H_2O$$

Note that the system will always have infinitely many solutions. Why?

Write the balanced equation for the given chemical reactions

(a) $CO_2 + H_2O \longrightarrow C_6H_{12}O_6 + O_2$
From

$$x_1CO_2 + x_2H_2O \longrightarrow x_3C_6H_{12}O_6 + x_4O_2$$

we obtain the following homogeneous linear system

$$\begin{array}{rcrcrcrl} x_1 & & & - & 6x_3 & & & = 0 \\ 2x_1 & + & x_2 & - & 6x_3 & - & 2x_4 & = 0 \\ & & 2x_2 & - & 12x_3 & & & = 0 \end{array}$$

The reduced row-echelon form is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1/6 & 0 \end{array}\right).$$

The general solutions is $x_1 = t$, $x_2 = t$, $x_3 = \frac{1}{6}t$, $x_4 = t$. The smallest possible integer values occur when $t = 6$. Hence, $x_1 = 6$, $x_2 = 6$, $x_3 = 1$, $x_4 = 6$.

(b) $Fe_2O_3 + Al \longrightarrow Al_2O_3 + Fe$

From

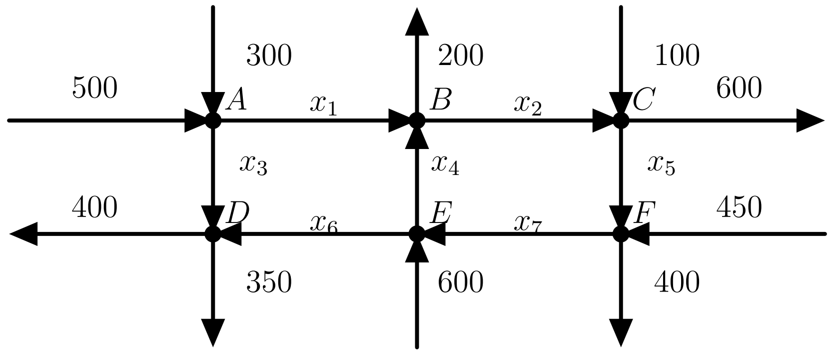
$$x_1 Fe_2O_3 + x_2 Al \longrightarrow x_3 Al_2O_3 + x_4 Fe$$

we obtain the following homogeneous linear system

$$\begin{array}{rcrcrcrl} 2x_1 & & & & & - & x_4 & = 0 \\ 3x_1 & & & - & 3x_3 & & & = 0 \\ & & x_2 & - & 2x_3 & & & = 0 \end{array}$$

The general solutions is $x_1 = \frac{1}{2}t$, $x_2 = t$, $x_3 = \frac{1}{2}t$, $x_4 = t$. The smallest possible integer values occur when $t = 2$. Hence, $x_1 = 1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 2$.

5. **(Application, MATLAB)** A network of one-way streets of a downtown section can be represented by the diagram below, with traffic flowing in the direction indicated. The average hourly volume of traffic entering and leaving this section during rush hour is given in the diagram.



(a) Do we have enough information to find the traffic volumes x_1, x_2, x_3, x_4, x_5, x_6, and x_7?

No, there are 6 equations (junctions), but 7 unknowns.

$$\begin{array}{rcrcrcrcrcrcrcl} x_1 & & & + & x_3 & & & & & & & & & = 800 \\ x_1 & - & x_2 & & & + & x_4 & & & & & & & = 200 \\ & & x_2 & & & & & - & x_5 & & & & & = 500 \\ & & & & x_3 & & & & & + & x_6 & & & = 750 \\ & & & & & - & x_4 & & & - & x_6 & + & x_7 & = -600 \\ & & & & & & & & x_5 & & & - & x_7 & = -50 \end{array}$$

In fact, the RREF is $\left(\begin{array}{ccccccc|c} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 50 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 450 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 750 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 & 600 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & -50 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$, which shows that we need 2 parameters.

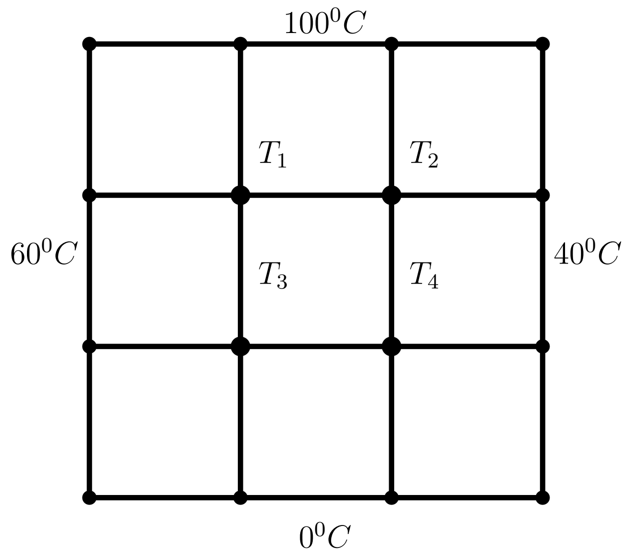
The general solution is $x_1 = 50+s$, $x_2 = 450+t$, $x_3 = 750-s$, $x_4 = 600-s+t$, $x_5 = t-50$, $x_6 = s$, $x_7 = t$.

(b) Suppose $x_6 = 50$ and $x_7 = 100$. What is x_1, x_2, x_3, x_4, and x_5?
$x_1 = 100$, $x_2 = 550$, $x_3 = 700$, $x_4 = 650$, $x_5 = 50$.

(c) Can the road between junction A and B be closed for construction while still keeping the traffic flowing on the other streets? Explain.
No, for in that case, $x_6 = -50$, a contradiction.

6. **(Application)** A simple model for estimating the temperature distribution on a square plate gives rise to a linear system of equations. To construct the appropriate linear system, we use the following information: The square plate is perfectly insulated on its top and bottom so that the only heat flow is through the plate itself. The four edges are held at various temperatures. To estimate the temperature at an interior point on the plate, we use the rule that it is the average of the temperature at its four compass-point neighbours, to the west, north, east and south.

Suppose we wish to estimate the temperatures T_i, $i = 1, 2, 3, 4$, at the four equispaced interior points on the plate as shown in the figure below.



We now construct the linear system to estimate the temperatures. The points at which we need the temperatures of the plate for this model are indicated by the dots in the figure above. To obtain linear equations involving the unknowns T_i, $i = 1, 2, 3, 4$, we use our averaging rule, for example,

$$T_1 = \frac{60 + 100 + T_2 + T_3}{4} \Rightarrow 4T_1 - T_2 - T_3 = 160.$$

(a) Write down three other linear equations, by considering T_2, T_3 and T_4.

(b) Solve the linear system. Is it possible to have more than one solution?

Food for thought: Is it possible for the system to be inconsistent?

(a)

$$
\begin{aligned}
T_2 &= \frac{T_1 + 100 + 40 + T_4}{4} \Rightarrow -T_1 + 4T_2 - T_4 = 140 \\
T_3 &= \frac{60 + T_1 + T_4 + 0}{4} \Rightarrow -T_1 + 4T_3 - T_4 = 60 \\
T_4 &= \frac{T_3 + T_2 + 40 + 0}{4} \Rightarrow -T_2 - T_3 + 4T_4 = 40.
\end{aligned}
$$

(b) The augmented matrix for this linear system is

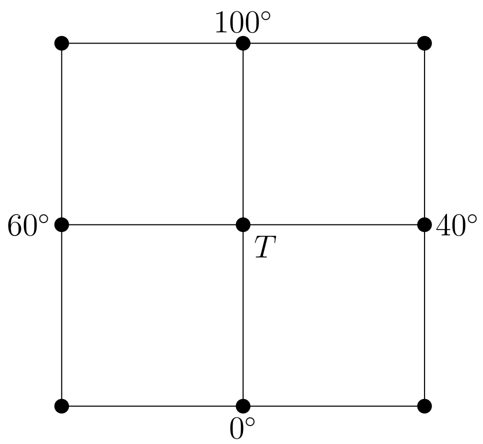
$$
\left(\begin{array}{cccc|c}
4 & -1 & -1 & 0 & 160 \\
-1 & 4 & 0 & -1 & 140 \\
-1 & 0 & 4 & -1 & 60 \\
0 & -1 & -1 & 4 & 40
\end{array}\right)
$$

Solving the system above, we have $T_1 = 65^o$, $T_2 = 60^o$, $T_3 = 40^o$ and $T_4 = 35^o$. The system has a unique solution.

Supplementary Problems

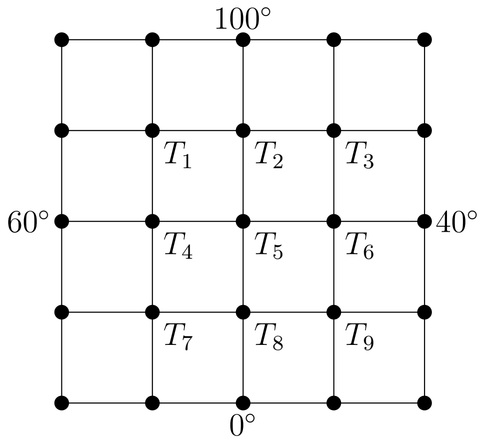
7. **(MATLAB)**

(a) Consider, once again, the perfectly insulated square plate from Problem 6, with its interior mesh altered, as shown below.



Directly applying the averaging rule from Problem 5, estimate the temperature T of the central node.

(b) We might notice that our temperature values vary according to how finely or coarsely we dissect the metal plate into its interior nodes. To more accurately estimate the temperature at precise points on the plate, we produce a finer interior mesh, as shown below.



i. Set up a linear system in nine equations that will allow us to find the temperatures T_1 through T_9 of the interior nodes. Express your answer as a matrix equation $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{x}$ is the column matrix whose entries are given by $T_1, \ldots, T_9$.

ii. Use MATLAB to solve the linear system. Note that T_5 corresponds to the temperature at the central node of the plate. How does this compare to the temperature at the central node you obtained from part (a)?

Directly applying the averaging rule, the temperature T is given by

$$T = \frac{100 + 60 + 40 + 0}{4} = \frac{200}{4} = 50°.$$

We now set up an equation for each of the temperatures T_1 through T_9 :

$$
\begin{aligned}
T_1 &= \frac{100 + 60 + T_2 + T_4}{4} &&\Rightarrow\ 4T_1 - T_2 - T_4 = 160,\\
T_2 &= \frac{100 + T_1 + T_3 + T_5}{4} &&\Rightarrow\ -T_1 + 4T_2 - T_3 - T_5 = 100,\\
T_3 &= \frac{100 + T_2 + 40 + T_6}{4} &&\Rightarrow\ -T_2 + 4T_3 - T_6 = 140,\\
T_4 &= \frac{T_1 + 60 + T_5 + T_7}{4} &&\Rightarrow\ -T_1 + 4T_4 - T_5 - T_7 = 60,\\
T_5 &= \frac{T_2 + T_4 + T_6 + T_8}{4} &&\Rightarrow\ -T_2 - T_4 + 4T_5 - T_6 - T_8 = 0,\\
T_6 &= \frac{T_3 + T_5 + 40 + T_9}{4} &&\Rightarrow\ -T_3 - T_5 + 4T_6 - T_9 = 40,\\
T_7 &= \frac{T_4 + 60 + T_8 + 0}{4} &&\Rightarrow\ -T_4 + 4T_7 - T_8 = 60,\\
T_8 &= \frac{T_5 + T_7 + T_9 + 0}{4} &&\Rightarrow\ -T_5 - T_7 + 4T_8 - T_9 = 0,\\
T_9 &= \frac{T_6 + T_8 + 40 + 0}{4} &&\Rightarrow\ -T_6 - T_8 + 4T_9 = 40.
\end{aligned}
$$

We may rewrite this system as a matrix equation $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{x}$ is the column matrix whose entries are given by $T_1, \ldots, T_9$.

$$\begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{pmatrix} = \begin{pmatrix} 160 \\ 100 \\ 140 \\ 60 \\ 0 \\ 40 \\ 60 \\ 0 \\ 40 \end{pmatrix}.$$

We can use MATLAB to solve this massive linear system. To begin, we create the matrix $\boldsymbol{A}$—typing in each of the 81 digits is fairly taxing, so we will try to use some of MATLAB's commands to help us create the matrix. We observe that we can decompose the matrix into three, more manageable matrices according to the diagonal / off-diagonal entries of $\boldsymbol{A}$:

- To create a square matrix of order 9 whose main diagonal only contains 4's,

  ```
  >> X = diag(4*ones(1,9))
  ```

- To create a square matrix of order 9 whose off-diagonal entries immediately above and below the main diagonal consist of the string $-1, -1, 0, -1, -1, 0, -1, -1$,

  ```
  >> Y1 = diag([-1,-1,0,-1,-1,0,-1,-1],1)
  >> Y2 = diag([-1,-1,0,-1,-1,0,-1,-1],-1)
  >> Y = Y1 + Y2
  ```

- To create a square matrix of order 9 whose entries on the third diagonals above and below the main diagonal only contain -1's,

  ```
  >> Z = diag(-1*ones(1,6),3)+ diag(-1*ones(1,6),-3)
  ```

Thus, the desired matrix $\boldsymbol{A}$ is given by the sum of the matrices we've just created, after which we can create the column matrix $\boldsymbol{b}$ as well:

```
>> A = X + Y + Z
>> b = [160; 100; 140; 60; 0; 40; 60; 0; 40]
```

We can thus solve the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ by row-reducing the augmented matrix $(\boldsymbol{A} \mid \boldsymbol{b})$, which we can do by concatenating the two matrices we've just created.

```
>> rref([A b])
```

We find that the left-hand side of the augmented matrix reduces to the identity matrix of order 9; in particular,

$$\operatorname{rref}(\boldsymbol{A} \mid \boldsymbol{b}) = \left(\begin{array}{c|c} & 71.4286 \\ & 71.4286 \\ & 64.2857 \\ & 54.2857 \\ \boldsymbol{I}_9 & 50.0000 \\ & 45.7143 \\ & 35.7143 \\ & 28.5714 \\ & 28.5714 \end{array}\right) \implies \begin{array}{c} T_1 = 71.4286^\circ \\ T_2 = 71.4286^\circ \\ T_3 = 64.2857^\circ \\ T_4 = 54.2857^\circ \\ T_5 = 50.0000^\circ \\ T_6 = 45.7143^\circ \\ T_7 = 35.7143^\circ \\ T_8 = 28.5714^\circ \\ T_9 = 28.5714^\circ \end{array}.$$

Note that in the discrete formulation of the temperature distribution problem, the temperature at the central node remains identical to the answer we arrived at in the previous part.