NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

AY2021, Semester 1 MA1508E Linear Algebra for Engineering Tutorial 8

1. For each of the following matrices $\mathbf{A}$,

   (i) Find a basis for the row space of $\mathbf{A}$.

   (ii) Find a basis for the column space of $\mathbf{A}$.

   (iii) Find a basis for the nullspace of $\mathbf{A}$.

   (iv) Hence determine rank($\mathbf{A}$), nullity($\mathbf{A}$) and verify the dimension theorem for matrices.

   (v) Is $\mathbf{A}$ full rank?

   (a) $\mathbf{A} = \begin{pmatrix} 1 & 2 & 5 & 3 \\ 1 & -4 & -1 & -9 \\ -1 & 0 & -3 & 1 \\ 2 & 1 & 7 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}$

   (b) $\mathbf{A} = \begin{pmatrix} 1 & 3 & 7 \\ 2 & 1 & 8 \\ 3 & -5 & -1 \\ 2 & -2 & 2 \\ 1 & 1 & 5 \end{pmatrix}$.

2. Show that for any linear system $\mathbf{Ax} = \mathbf{b}$, the solution set is

$$\{\ \mathbf{x}_p + \mathbf{u} \mid \mathbf{u} \in Null(\mathbf{A})\ \},$$

   where $\mathbf{x}_p$ is a particular solution to the linear system, and $Null(\mathbf{A})$ is the nullspace of $\mathbf{A}$ (See tutorial 5 question 6).

3. Suppose $\mathbf{A}$ and $\mathbf{B}$ are two matrices such that $\mathbf{AB} = \mathbf{0}$. Show that the column space of $\mathbf{B}$ is contained in the nullspace of $\mathbf{A}$.

4. **(MATLAB)** Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}$.

   (a) Compute $\mathbf{v}_1 \cdot \mathbf{v}_1$, $\mathbf{v}_1 \cdot \mathbf{v}_2$, $\mathbf{v}_2 \cdot \mathbf{v}_1$, and $\mathbf{v}_2 \cdot \mathbf{v}_2$.

   (b) Compute $\mathbf{V}^T\mathbf{V}$. What does the entries of $\mathbf{V}^T\mathbf{V}$ represent?

5. Let W be a subspace of $\mathbb{R}^n$. The *orthogonal complement* of W, denoted as $W^\perp$, is defined to be

$$W^\perp := \{\ \mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\ \}.$$

   Let $\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{w}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \\ 0 \end{pmatrix}$, and $\mathbf{w}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$, and $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

(a) Show that $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is linearly independent.

(b) Show that S is orthogonal.

(c) Show that $W^\perp$ is a subspace of $\mathbb{R}^5$ by showing that it is a span of a set. What is the dimension? (**Hint**: See Tutorial 4 question 6.)

(d) Obtain an orthonormal set T by normalizing $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$.

(e) Let $\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}$. Find the projection of $\mathbf{v}$ onto W.

(f) Let $\mathbf{v}_W$ be the projection of $\mathbf{v}$ onto W. Show that $\mathbf{v} - \mathbf{v}_W$ is in $W^\perp$.

This exercise demonstrated the fact that every vector $\mathbf{v}$ in $\mathbb{R}^5$ can be written as $\mathbf{v} = \mathbf{v}_W + \mathbf{v}_W^\perp$, for some $\mathbf{v}_W$ in W and $\mathbf{v}_W^\perp$ in $W^\perp$. In other words, $W + W^\perp = \mathbb{R}^5$ (see Tutorial 7 question 1).

6. Let $S = \{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}\}$ where

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}, \mathbf{u_2} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{u_3} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \text{ and } \mathbf{u_4} = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 2 \end{pmatrix}.$$

(a) Check that S is an orthogonal set.

(b) Is S a basis for $\mathbb{R}^4$?

(c) Is it possible to find a nonzero vector $\mathbf{w}$ in $\mathbb{R}^4$ such that $S \cup \{\mathbf{w}\}$ is an orthogonal set?

(d) Obtain an orthonormal set T by normalizaing $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}$.

(e) Let $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$. Find $(\mathbf{v})_S$ and $(\mathbf{v})_T$.

(f) Suppose $\mathbf{w}$ is a vector in $\mathbb{R}^4$ such that $(\mathbf{w})_S = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$. Find $(\mathbf{w})_T$.

Supplementary Problems

7. Recall that a matrix $\mathbf{A}$ is an orthogonal matrix if $\mathbf{A}^T = \mathbf{A}^{-1}$ (see Tutorial 4 question 1(d)).

(a) Show that if $\mathbf{A}$ is an orthogonal matrix of order n, then the columns of $\mathbf{A}$ is an orthonormal basis of $\mathbb{R}^n$.

(b) Show that if $\mathbf{A}$ is an orthogonal matrix of order n, then the rows of $\mathbf{A}$ is an orthonormal basis of $\mathbb{R}^n$.