NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

**AY2021, Semester 1 MA1508E Linear Algebra for Engineering Tutorial 2**

**Solutions**

1. (a) Suppose $\mathbf{A}$ is a square matrix such that $\mathbf{A}^2 = \mathbf{0}$. Show that $\mathbf{I} - \mathbf{A}$ is invertible, with inverse $\mathbf{I} + \mathbf{A}$.

   (b) Suppose $\mathbf{A}^3 = \mathbf{0}$. Is $\mathbf{I} - \mathbf{A}$ invertible?

   (c) A square matrix $\mathbf{A}$ is said to be *nilpotent* if there is a positive integer $n$ such that $\mathbf{A}^n = \mathbf{0}$. Show that if $\mathbf{A}$ is nilpotent, then $\mathbf{I} - \mathbf{A}$ is invertible.

   (a) To show that $\mathbf{I} - \mathbf{A}$, suffice to check that it has a left inverse. Indeed,

$$(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A}) = \mathbf{I}^2 - \mathbf{A}^2 = \mathbf{I}.$$

   (b) Substituting $\mathbf{A}$ into the polynomial identity $(1 - x)(1 + x + x^2) = 1 - x^3$, we get

$$(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A} + \mathbf{A}^2) = \mathbf{I} - \mathbf{A}^3 = \mathbf{I}.$$

   (c) Substituting $\mathbf{A}$ into the polynomial identity $(1 - x)(1 + x + x^2 + \cdots + x^{n-1}) = 1 - x^n$, we get

$$(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A} + \cdots + \mathbf{A}^{n-1}) = \mathbf{I} - \mathbf{A}^3 = \mathbf{I}.$$

   So the inverse of $\mathbf{I} - \mathbf{A}$ is $(\mathbf{I} + \mathbf{A} + \cdots + \mathbf{A}^{n-1})$.
   Remark: The inverse could be derived from the formula for the sum of a geometric progression,

$$\sum_{k=1}^{n} x^{k-1} = \frac{1 - x^n}{1 - x},$$

   which is equivalent to $(1 - x)\sum_{k=1}^{n} x^{k-1} = 1 - x^n$.
   Extra: Show that every strictly upper or lower triangular matrix is nilpotent.

2. (i) Reduce the following matrices $\mathbf{A}$ to its reduced row-echelon form $\mathbf{R}$.

   (ii) For each of the elementary row operation, write the corresponding elementary matrix.

   (iii) Write the matrices $\mathbf{A}$ in the form $\mathbf{E}_1\mathbf{E}_2 \ldots \mathbf{E}_n\mathbf{R}$ where $\mathbf{E}_1, \mathbf{E_2}, \ldots, \mathbf{E}_n$ are elementary matrices and $\mathbf{R}$ is the reduced row-echelon form of $\mathbf{A}$.

   (a) $\mathbf{A} = \begin{pmatrix} 5 & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{pmatrix}$.

   (b) $\mathbf{A} = \begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$.

   (c) $\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$.

(a) (i) $\mathbf{A} \xrightarrow{r_1: R_2 + \frac{2}{5}R_1} \xrightarrow{r_2: \frac{1}{5}R_1} \xrightarrow{r_3: 5R_2} \xrightarrow{r_4: R_1 + \frac{2}{5}R_2} \mathbf{R}$

(ii) $\mathbf{E}_1 = \begin{pmatrix} 1 & 0 \\ \frac{2}{5} & 1 \end{pmatrix}$, $\mathbf{E}_2 = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{E}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$, $\mathbf{E}_4 = \begin{pmatrix} 1 & \frac{2}{5} \\ 0 & 1 \end{pmatrix}$.

(iii)

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{5} & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 1 & -\frac{2}{5} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{pmatrix}.$$

(b) (i) $\mathbf{A} \xrightarrow{r_1: R_2 + 2R_1} \xrightarrow{r_2: R_3 - 4R_1} \xrightarrow{r_3: R_3 + R_2} \xrightarrow{r_4: -R_1} \xrightarrow{r_5: \frac{1}{10}R_2} \xrightarrow{r_6: R_1 + 3R_2} \mathbf{R}$

(ii) $\mathbf{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}$, $\mathbf{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $\mathbf{E}_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$\mathbf{E}_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{E}_6 = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(iii)

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{19}{10} \\ 0 & 1 & -\frac{7}{10} \\ 0 & 0 & 0 \end{pmatrix}.$$

(c) (i) $\mathbf{A} \xrightarrow{r_1: R_2 - 2R_1} \xrightarrow{r_2: R_3 - R_1} \xrightarrow{r_3: R_2 \leftrightarrow R_3} \xrightarrow{r_4: \frac{1}{3}R_2} \xrightarrow{r_5: R_2 - R_3} \xrightarrow{r_6: R_1 + R_2} \mathbf{R}$

(ii) $\mathbf{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$, $\mathbf{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\mathbf{E}_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$\mathbf{E}_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{E}_6 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(iii)

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3. **(Application)** (Polynomial Interpolation)
Given any $n$ points in the $xy$-plane that has distinct $x$-coordinates, it is known that there is a unique polynomial of degree $n-1$ or less whose graph passes through those point. A degree $n-1$ polynomial has the following expression

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

Suppose its graph passes through the points $(x_1, y_1)$, $(x_2, y_2)$,...,$(x_n, y_n)$, it follows that the coordinates of the points must satisfy

$$\begin{array}{ccccccccc} a_0 & + & a_1x_1 & + & a_2x_1^2 & +\cdots+ & a_{n-1}x_1^{n-1} & = y_1 \\ a_0 & + & a_1x_2 & + & a_2x_2^2 & +\cdots+ & a_{n-1}x_2^{n-1} & = y_2 \\ \vdots & & \vdots & & \vdots & & \vdots & \vdots \\ a_0 & + & a_1x_n & + & a_2x_n^2 & +\cdots+ & a_{n-1}x_n^{n-1} & = y_n \end{array}$$

This is a linear system in the unknowns $a_0, a_1, ..., a_{n-1}$. The augmented matrix for the system is

$$\left(\begin{array}{ccccc|c} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & y_1 \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} & y_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & y_n \end{array}\right) \tag{V}$$

which has a unique solution whenever $x_1, x_2, ..., x_n$ are distinct.

(a) Find a cubic polynomial whose graph passes through the points

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $y$ | 3 | $-2$ | $-5$ | 0 |

The augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 & -2 \\ 1 & 3 & 9 & 27 & -5 \\ 1 & 4 & 16 & 64 & 0 \end{array}\right)$$

Its RREF is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

So the cubic polynomial is $x^3 - 5x^2 + 3x + 4$.

(b) **(MATLAB)** The coefficient matrix of the linear system ($\boxed{\text{V}}$) is called a *Vandermonde Matrix*. The function fliplr(vander(v)) returns the Vandermonde matrix such that its rows are powers of the vector $v$. For example,

```
>> v=[1;2;3;4;5;6;7;8];

>> A=fliplr(vander(v))
```

will generate the following matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 & 2^7 \\ 1 & 3 & 3^2 & 3^3 & 3^4 & 3^5 & 3^6 & 3^7 \\ 1 & 4 & 4^2 & 4^3 & 4^4 & 4^5 & 4^6 & 4^7 \\ 1 & 5 & 5^2 & 5^3 & 5^4 & 5^5 & 5^6 & 5^7 \\ 1 & 6 & 6^2 & 6^3 & 6^4 & 6^5 & 6^6 & 6^7 \\ 1 & 7 & 7^2 & 7^3 & 7^4 & 7^5 & 7^6 & 7^7 \\ 1 & 8 & 8^2 & 8^3 & 8^4 & 8^5 & 8^6 & 8^7 \end{pmatrix}$$

Use the Vandermonde matrix function to find a degree 7 polynomial that passes through

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $y$ | 12 | 70 | 1244 | 10500 | 54268 | 205682 | 630540 | 1657024 |

```
>> v=[1;2;3;4;5;6;7;8];
```

```
>> A=fliplr(vander(v))
```

```
>> b=[12;70;1244;10500;54268;205682;630540;1657024];
```

`>> A\b` OR `>> rref([A b])`

which gives $a_0 = 8$, $a_1 = 7$, $a_2 = -6$, $a_3 = 5$, $a_4 = -4$, $a_5 = 3$, $a_6 = -2$, $a_7 = 1$. Thus, the polynomial is

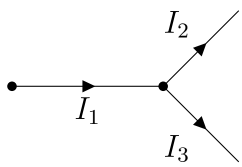
$$x^7 - 2x^6 + 3x^5 - 4x^4 + 5x^3 - 6x^2 + 7x + 8$$

4. **(Application)** Electrical networks provides information about power sources, such as batteries, and devices powered by these sources, such as light bulbs or motors. A power source 'forces' a current of electrons to flow through the network, where it encounters various resistors, each of which requires that a certain amount of force be applied in order for the current to flow through it.

   The fundamental law of electricity is Ohm's law, which states exactly how much force $E$ is needed to drive a current $I$ through a resistor with resistance $R$. Ohm's law states $E = IR$, in other words, force = current $\times$ resistance. Here, force is measured in volts, resistance in ohms and current in amperes.

   The following two laws (discovery due to Kirchhoff), govern electrical networks. The first is a 'conservation of flow' law at each node; the second is a 'balancing of votage' law around each loop.

   **(Kirchoff's Current Law (KCL))** At each node, the sum of the currents flowing into any node is equal to the sum of the currents flowing out of that node. For example, in the diagram below, by KCL, we have $I_1 = I_2 + I_3$.



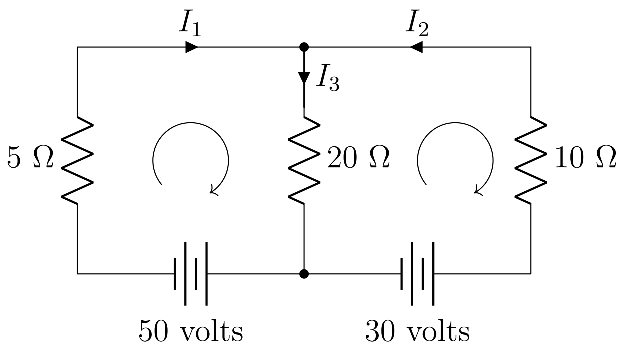

   **(Kirchoff's Voltage Law (KVL))** In one traversal of any closed loop, the sum of the voltage rises equals to the sum of the voltage drops.

   In circuits with multiple loops and batteries there is usually no way to tell in advance which way the currents are flowing, so the usual procedure in circuit analysis is to assign *arbitrary* directions to the current flows in the branches and let the mathematical computations determine whether the assignments are correct. In addition to assigning directions to the current flows, Kirchoff's Voltage Law requires a direction of travel for each closed loop. The choice is arbitrary, but for the sake of consistency we will always take this direction to be *clockwise*. We will also make the following conventions:

- A voltage drop occurs at a resistor if the direction assigned to the current through the resistor is the same as the direction assigned in the loop, and a voltage rise occurs at a resistor if the direction assigned to the current through the resistor is the opposite to that assigned in the loop.
- A voltage rise occurs at a battery if the direction assigned to the loop is from $-$ to $+$ through the battery, and a voltage drop occurs at a battery if the direction assigned to the loop is from $+$ to $-$ through the battery.

If we follow these conventions when calculating currents, then those currents whose directions were assigned correctly will have positive values and those whose direction were assigned incorrectly will have negative values.

For each of the following circuits, use KCL and KVL to write down a linear system with equations involving variables $I_1, I_2, \ldots$. Solve the linear system by Gaussian Elimination.

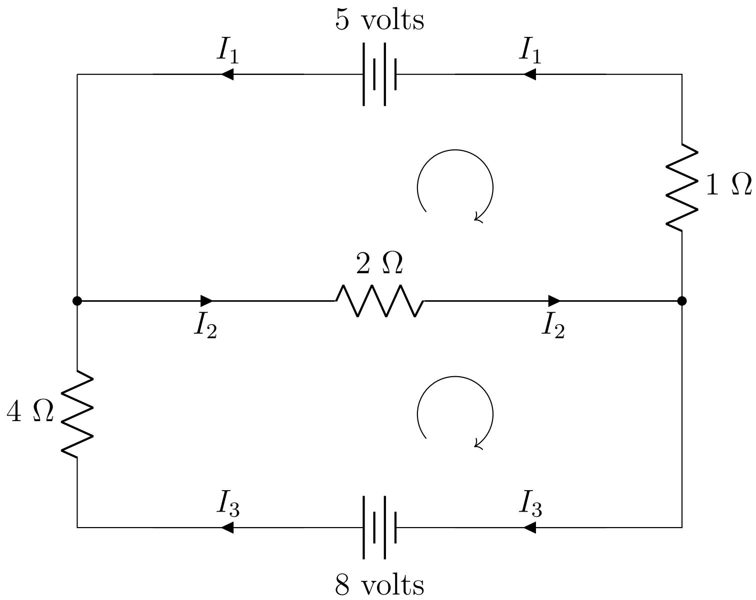



(a)

The linear system is

$$\begin{cases} I_1 + I_2 - I_3 = 0 \\ 5I_1 + 20I_3 = 50 \\ 10I_2 + 20I_3 = -30 \end{cases}$$

Solving by Gaussian elimination, we have $I_1 = 6$, $I = 2 = -5$ and $I_3 = 1$. So the actual direction of the current $I_2$ is in the opposite direction from what is shown in the figure.



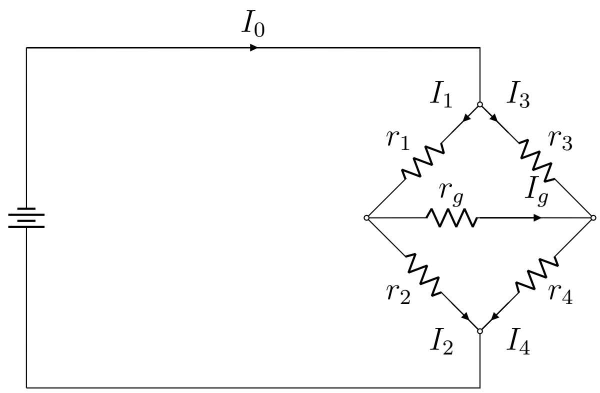

(b)

The linear system is

$$\begin{cases} I_1 - I_2 + I_3 = 0 \\ I_1 + 2I_2 = -5 \\ 2I_2 + 4I_3 = -8 \end{cases}$$

Solving by Gaussian elimination, we have $I_1 = -1, I_2 = -2, I_3 = -1$. So the actual direction of all the currents are in the opposite direction from what is shown in the figure.

5. **(MATLAB)** A *Wheatstone bridge* is a special type of electrical circuit that can be used to measure resistance. One such circuit is illustrated below.




(a) The main application of Wheatstone bridges is in determining an unknown resistance. In the diagram above, the resistance $r_4$ is usually unknown, while the resistances $r_1, r_2$, and $r_3$ are known. If the current flowing through the resistor $r_g$ is zero—that is, $I_g = 0$ amperes—find an expression for $r_4$ in terms of $r_1, r_2$, and $r_3$. (**Hint:** Begin by setting up a linear system in the unknowns $I_0, I_1, I_2, I_3, I_4$, and $I_g$.)

(b) Suppose that, in the Wheatstone bridge illustrated above, the battery supplies 10 volts to the circuit. Moreover, suppose that the five resistance values are known: $r_1 = 5\,\Omega$, $r_2 = 10\,\Omega$, $r_3 = 2\,\Omega$, $r_4 = 4\,\Omega$, and $r_g = 50\,\Omega$. Determine the currents $I_0, \ldots, I_4$, and show that $I_g = 0$ amperes. What can you say about the relationship between $r_1, r_2, r_3$ and $r_4$?

(a) We set up a linear system in the unknown currents for a general Wheatstone bridge. Applying Kirchhoff's Current Law at the four nodes, we obtain the equations

$$\begin{array}{rcrcrcrcrcrcl} I_0 & - & I_1 & & & - & I_3 & & & & & = & 0, \\ & & I_1 & - & I_2 & & & & & - & I_g & = & 0, \\ & & & & & & I_3 & - & I_4 & + & I_g & = & 0, \\ -I_0 & & & + & I_2 & & & + & I_4 & & & = & 0. \end{array}$$

Applying Kirchhoff's Voltage Law to the two larger loops (involving $I_0$), as well as the two smaller loops on the right, we have

$$\begin{array}{rcrcrcrcrcl} r_1I_1 & + & r_2I_2 & & & & & & & = & V, \\ & & & + & r_3I_3 & + & r_4I_4 & & & = & V, \\ r_1I_1 & & & - & r_3I_3 & & & + & r_gI_g & = & 0, \\ & - & r_2I_2 & & & & r_4I_4 & + & r_gI_g & = & 0, \end{array}$$

where $V$ is the voltage due to the battery.

We now consider the case when $I_g = 0$. It suffices to consider the equations involving $I_g$, which yields the system

$$\left\{ \begin{array}{rcrcrcrcl} I_1 & - & I_2 & & & & & = & 0 \\ & & & & I_3 & - & I_4 & = & 0 \\ r_1I_1 & & & - & r_3I_3 & & & = & 0 \\ & - & r_2I_2 & & & & r_4I_4 & = & 0 \end{array} \right. .$$

The four equations above give us four equalities:

$$I_1 = I_2, \quad I_3 = I_4, \quad r_1I_1 = r_3I_3, \quad r_2I_2 = r_4I_4.$$

In particular, we have

$$r_4 = \frac{r_2I_2}{I_4} = \frac{r_2I_1}{I_3} = \frac{r_2r_3}{r_1}.$$

(b) We now consider the entire linear system, substituting in the given resistance and voltage values:

$$\left\{ \begin{array}{rcrcrcrcrcrcl} I_0 & - & I_1 & & & - & I_3 & & & & & = & 0 \\ & & I_1 & - & I_2 & & & & & - & I_g & = & 0 \\ & & & & & & I_3 & - & I_4 & + & I_g & = & 0 \\ -I_0 & & & + & I_2 & & & + & I_4 & & & = & 0 \\ & & 5I_1 & + & 10I_2 & & & & & & & = & 10 \\ & & & & & & 2I_3 & + & 4I_4 & & & = & 10 \\ & & 5I_1 & & & - & 2I_3 & & & + & 50I_g & = & 0 \\ & & & - & 10I_2 & & & & 4I_4 & + & 50I_g & = & 0 \end{array} \right. .$$

Rewriting the linear system as a matrix equation $\boldsymbol{Ax} = \boldsymbol{b}$, we have

$$\begin{pmatrix} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 5 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 & 0 \\ 0 & 5 & 0 & -2 & 0 & 50 \\ 0 & 0 & -10 & 0 & 4 & 50 \end{pmatrix} \begin{pmatrix} I_0 \\ I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 10 \\ 10 \\ 0 \\ 0 \end{pmatrix}.$$

We can use MATLAB to find the reduced row-echelon form of this system:

```
>> A = [1 -1 0 -1 0 0; 0 1 -1 0 0 -1; 0 0 0 1 -1 1;
-1 0 1 0 1 0; 0 5 10 0 0 0; 0 0 0 2 4 0; 0 5 0 -2 0 50;
0 0 -10 0 4 50]
>> b = [0; 0; 0; 0; 10; 10; 0; 0]
>> rref([A b])
```

This yields the augmented matrix

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 2.3333 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0.6667 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0.6667 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1.6667 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1.6667 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

In particular, we find that $I_0 = 7/3$, $I_1 = I_2 = 2/3$ and $I_3 = I_4 = 5/3$. Furthermore, we have $I_g = 0$; since the current that goes through the central resistor is 0 amperes, we expect our resistances to be related by the equation we derived previously:

$$r_4 = \frac{r_2 r_3}{r_1}.$$

Indeed, our resistance values satisfy this equation: we have $4 = (10 \cdot 2) \div 5$.

## Supplementary Problems

6. **(Application, MATLAB)**(Approximate integration)
There are some integral that are not possible to solve by finding the antiderivate. The integral can be approximated by using an interpolating polynomial to approximate the integrand and integrating the approximating polynomial. For example, suppose we want to evaluate the integral

$$\int_0^1 e^{-x^2} dx$$

We begin by picking a few points between the limits, say

$$x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1$$

and evaluate the integrand $f(x) = e^{-x^2}$ at these points. They are approximately

$$f(0) = 1, f(0.25) = 0.9394, f(0.5) = 0.7788, f(0.75) = 0.5698, f(1) = 0.3679$$

The interpolating polynomial is (check it!)

$$p(x) = 0.0416x^4 + 0.4882x^3 - 1.1846x^2 + 0.0226x + 1$$

and

$$\int_0^1 p(x)dx \approx 0.7468$$

which, up to the fourth decimal place, is exactly the integral $\int_0^1 e^{-x^2}dx$. The more points we pick, and thus the higher the degree of the polynomial, the more accurate the approximation becomes.

By using MATLAB, we can easily approximate the integral using interpolating polynomial. Suppose we want an interpolating polynomial of degree $n$. First create a vector whose entries are $n+1$ regular steps between $a$ to $b$

```
>> v=[a:(b-a)/n:b]';
```

Then we create the Vandermonde matrix

```
>> A=fliplr(vander(v));
```

Let $\mathbf{b}$ be the vector whose entries are the evaluation of $e^{-x^2}$ at the points of $\mathbf{v}$

```
>> b=exp(-v.^2);
```

Then we obtain the coefficient of the interpolating polynomial of degree $n$

```
>> A\b
```

Use an interpolating polynomial of degree 10 to approximate the integral $\int_0^1 e^{-x^2}dx$.

```
>> v=[0:1/10:1]';
```

The interpolating polynomial is

$$\begin{aligned}&0.0043x^{10} - 0.0345x^9 + 0.0893x^8 - 0.0405x^7\\ &- 0.1440x^6 - 0.0084x^5 + 0.502x^4 - 0.0003x^3 - x^2 + 1\end{aligned}$$

whose antiderivative is

$$\frac{0.0043}{11}x^{11} - \frac{0.0345}{10}x^10 + \frac{0.0893}{9}x^9 - \frac{0.0405}{8}x^8 - \frac{0.1440}{7}x^7$$
$$- \frac{0.0084}{6}x^6 + \frac{0.502}{5}x^5 - \frac{0.0003}{4}x^4 - \frac{1}{3}x^3 + x$$

Thus, the integral is approximately

$$\frac{0.0043}{11} - \frac{0.0345}{10} + \frac{0.0893}{9} - \frac{0.0405}{8} - \frac{0.1440}{7} - \frac{0.0084}{6} + \frac{0.502}{5} - \frac{0.0003}{4} - \frac{1}{3} + 1 = 0.7468$$