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1. a) $$\begin{bmatrix} 1 & a & 2 \\ 1 & 1 & 1 \\ 1 & 1 & a+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ a \\ 2a \end{bmatrix}$$

$$x_1 + a x_2 + 2x_3 = a$$
$$x_1 + x_2 + x_3 = a$$
$$x_1 + x_2 + (a+1)x_3 = 2a$$

b) $$\left[\begin{array}{ccc|c} 1 & a & 2 & a \\ 1 & 1 & 1 & a \\ 1 & 1 & a+1 & 2a \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & a & 2 & a \\ 0 & 1-a & -1 & 0 \\ 0 & 1-a & a-1 & a \end{array}\right]$$

$$\xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & a & 2 & a \\ 0 & 1-a & -1 & 0 \\ 0 & 0 & a & a \end{array}\right]$$

case 1: $a = 0$

sub in values of a $\rightarrow$ $$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\uparrow$
free variable

system has infinitely many solutions with 1 parameter

$x_3 = s$
$x_2 - x_3 = 0 \Rightarrow x_2 = s$
$x_1 + 2x_3 = 0 \Rightarrow x_1 = -2s$

case 2: $a \neq 0$, $a = 1$

sub in values of a $\rightarrow$ $$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

system is inconsistent

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Case 3 : $a \neq 0$, $a \neq 1$

$$\xrightarrow{\frac{1}{a}R_3} \left[\begin{array}{ccc|c} 1 & a & 2 & a \\ 0 & 1-a & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 1 & a & 2 & a \\ 0 & 1-a & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{1-a}R_2} \left[\begin{array}{ccc|c} 1 & a & 2 & a \\ 0 & 1 & 0 & 1/1-a \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_1-2R_3]{R_1-aR_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{-a^2+2a-2}{1-a} \\ 0 & 1 & 0 & 1/1-a \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$a - \frac{a}{1-a} - 2$$
$$= \frac{a-a^2-a-2+2a}{1-a}$$
$$= \frac{-a^2+2a-2}{1-a}$$

rref $= I$, $r = m = n$

system has a unique solution

i) No solution during case 2 : $a = 1$

system is inconsistent

ii) Unique solution during case 3 : $a \neq 0$, $a \neq 1$

$x_3 = 1$

$x_2 = \frac{1}{1-a}$

$x_1 = \frac{-a^2+2a-2}{1-a}$

iii) Infinitely many solution, during case 1 : $a = 0$

let $x_3 = s$

$x_2 = s$

$x_1 = -2s$

general solution (nullspace) : $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

particular solution : $s=0 \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

nullspace must contain zero vector

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2a) i)

$$\underset{5\times 3}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{5\times 1}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}}$$

$$A \quad x = b$$

ii)

$$A^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$A^TA = \underset{3\times 5}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}} \underset{5\times 3}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} = \underset{3\times 3}{\begin{bmatrix} 5 & 4 & 3 \\ 4 & 8 & 2 \\ 3 & 2 & 3 \end{bmatrix}}$$

A^TA is symmetric

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

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b) $x_1 = -s + 2t$

$x_2 = s - t$

$x_3 = s$

$x_4 = t$

substitute in

$$\left.\begin{aligned} x_1 &= -x_3 + 2x_4 \\ x_2 &= x_3 - x_4 \end{aligned}\right\} \quad \begin{aligned} x_1 \quad + x_3 - 2x_4 &= 0 \\ x_2 - x_3 + x_4 &= 0 \end{aligned}$$

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3) $AB = BA$

$$\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} a+2c & b+2d \\ a-c & b-d \end{pmatrix} = \begin{pmatrix} a+b & 2a-b \\ c+d & 2c-d \end{pmatrix}$$

$$\begin{aligned} a+2c &= a+b \\ b+2d &= 2a-b \\ a-c &= c+d \\ b-d &= 2c-d \end{aligned} \equiv \begin{aligned} -b+2c &= 0 \\ -2a+2b+2d &= 0 \\ a-2c-d &= 0 \\ b-2c &= 0 \end{aligned}$$

$$\begin{bmatrix} 0 & -1 & 2 & 0 \\ -2 & 2 & 0 & 2 \\ 1 & 0 & -2 & -1 \\ 0 & 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\xrightarrow{\substack{R_1 \leftrightarrow R_3 \\ R_2 \leftrightarrow R_4}} \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ -2 & 2 & 0 & 2 & 0 \end{array}\right] \xrightarrow{\substack{R_3+R_2 \\ R_4+2R_1}} \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -4 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_4-2R_2} \left[\begin{array}{cccc|c} 1 & 0 & -2 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

free variables

let $d = s$, $c = t$

$b - 2c = 0 \Rightarrow b = 2t$

$a - 2c - d = 0 \Rightarrow a = 2t + s$

$$B = \begin{pmatrix} 2t+s & 2t \\ t & s \end{pmatrix} \text{ for any } s, t$$