NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

AY2021, Semester 1 MA1508E Linear Algebra for Engineering Tutorial 7

Solutions

1. Let U and V be subspaces of $\mathbb{R}^n$. We define the sum $U+V$ to be the set

$$\left\{ \mathbf{u}+\mathbf{v} \;\middle|\; \mathbf{u}\in U \text{ and } \mathbf{v}\in V \right\}.$$

Suppose $U = \operatorname{span}\left\{ \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\2\\1\end{pmatrix} \right\}$, $V = \operatorname{span}\left\{ \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\2\\-1\end{pmatrix} \right\}$.

Let $\mathbf{u}_1 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix}1\\2\\2\\1\end{pmatrix}$, $\mathbf{v}_1 = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}$, and $\mathbf{v}_2 = \begin{pmatrix}1\\0\\2\\-1\end{pmatrix}$.

(a) Is $U\cup V$ a subspace of $\mathbb{R}^4$?

No. We will show that it is not closed under linear combinations. We have $\mathbf{u}_1 \in U\cap V$ and $\mathbf{v}_1 \in U\cap V$. The sum is

$$\mathbf{u}_1+\mathbf{v}_1 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\2\\1\end{pmatrix}$$

We will check that this is neither in U nor in V, and thus it cannot be in the union $U\cup V$.

$$\left(\begin{array}{cc|c} 1 & 1 & 2\\ 1 & 2 & 1\\ 1 & 2 & 2\\ 1 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0 \end{array}\right)$$

shows that $\mathbf{u}_1+\mathbf{v}_1 \notin U$, and

$$\left(\begin{array}{cc|c} 1 & 1 & 2\\ 0 & 0 & 1\\ 1 & 2 & 2\\ 0 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0 \end{array}\right)$$

shows that $\mathbf{u}_1+\mathbf{v}_1 \notin V$. So $\mathbf{u}_1+\mathbf{v}_1 \notin U\cup V$.

(b) Show that $U+V$ a subspace by showing that it can be written as a span of a set. What is the dimension?

Any vector in U can be written as $\alpha_1\mathbf{u}_1+\alpha_2\mathbf{u}_2$, and any vector in V can be written as $\beta_1\mathbf{v}_1+\beta_2\mathbf{v}_2$. So a vector in $U+V$ has the form $\alpha_1\mathbf{u}_1+\alpha_2\mathbf{u}_2+\beta_1\mathbf{v}_1+\beta_2\mathbf{v}_2$. That is, $U+V = \operatorname{span}\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{v}_1,\mathbf{v}_2\}$. This shows that $U+V$ is a subspace.

We will also give a general proof that for any subspaces U and V, $U+V$ is a subspace. Clearly $\mathbf{0} \in U+V$. Suppose now $\mathbf{w}_1, \mathbf{w}_2 \in U+V$. Then by definition of $U+V$, we can find $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $\mathbf{v}_1, \mathbf{v}_2 \in V$ such that $\mathbf{w}_1 = \mathbf{u}_1 + \mathbf{v}_1$ and $\mathbf{w}_2 = \mathbf{u}_2 + \mathbf{v}_2$. Then for any $\alpha, \beta \in \mathbb{R}$,

$$\alpha\mathbf{w}_1 + \beta\mathbf{w}_2 = \alpha(\mathbf{u}_1 + \mathbf{v}_1) + \beta(\mathbf{u}_2 + \mathbf{v}_2) = (\alpha\mathbf{u}_1 + \beta\mathbf{u}_2) + (\alpha\mathbf{v}_1 + \beta\mathbf{v}_2).$$

Since U and V are subspaces, $(\alpha\mathbf{u}_1 + \beta\mathbf{u}_2) \in U$ and $(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) \in V$. So $\alpha\mathbf{w}_1 + \beta\mathbf{w}_2 \in U+V$.

Since $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2\}$ is a spanning set, suffice to find a linearly independent subset of it to form a basis.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This shows that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1\}$ is a basis for $U+V$, and hence $\dim(U+V) = 3$.

(c) Show that $U+V$ contains U and V. This shows that $U+V$ is a subspace containing $U \cup V$.

This is clear from (a) since $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ are subsets of $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2\}$. In fact, $U+V$ is the smallest subspace that contains $U \cup V$.

(d) What are the dimensions of U and V?

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence, $\dim(U) = 2$.

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 2 \\ 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence, $\dim(V) = 2$.

(e) Show that $U \cap V$ a subspace by showing that it can be written as a span of a set. What is the dimension?

A vector in $\mathbf{w} \in U \cap V$ must be able to be written as a linear combination of $\mathbf{u}_1, \mathbf{u}_2$, and as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. In other words, we must be able to find $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\mathbf{w} = \alpha_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} = \beta_1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix},$$

in other words, we are solving the homogeneous linear system

$$\alpha_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix} - \beta_1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \beta_2 \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 2 & -1 & -2 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

So for any choice of $s \in \mathbb{R}$, $\alpha_1 = -2s$ and $\alpha_2 = 2s$, or $\beta_1 = -2s$ and $\beta_2 = s$ will work, that is, $\mathbf{w} = 2s(-\mathbf{u}_1 + \mathbf{u}_2) = s(-2\mathbf{v}_1 + \mathbf{v}_2)$. Hence, $U \cap V = \text{span}\{-\mathbf{u}_1 + \mathbf{u}_2\} = \text{span}\{-2\mathbf{v}_1 + \mathbf{v}_2\}$, and this shows that $U \cap V$ is a subspace, with $\dim(U \cap V) = 1$.

In fact, we can show in general that for any subspaces U and V in $\mathbb{R}^n$, $U \cap V$ is a subspace of $\mathbb{R}^n$. Clearly $\mathbf{0} \in U \cap V$, so it is nonempty. Suppose now $\mathbf{w}_1, \mathbf{w}_2 \in U \cap V$ and $\alpha, \beta \in \mathbb{R}$. Then since U and V are subspace, $\alpha\mathbf{w}_1 + \beta\mathbf{w}_2$ belongs to U and V. Hence, it is in the intersection, $\alpha\mathbf{w}_1 + \beta\mathbf{w}_2 \in U \cap V$.

(f) Verify that $\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)$.
Indeed, $3 = 2 + 2 - 1$.

2. Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; x + 2y - z = 0 \right\}$. Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

(a) Show that $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ belongs to V.

(b) Is the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ linearly independent?

(c) Find a basis for V.

(a) Check directly $(1)+2(1)-(3) = 0$, $(1)+2(0)-(1) = 0$, and $(0)+2(1)-(2) = 0$.

(b) No. Since V is a plane, if they are linearly independent, then necessarily $2 = \dim(V) \geq 3$, which is a contradiction.

(c) It is obvious that any 2 of them are linearly independent, and since $\dim(V) = 2$, any 2 of them will form a basis for V.

3. (a) Let $\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Is $\mathbf{b}$ in the column space of $\mathbf{A}$? If it is, express it as a linear combination of the rows of $\mathbf{A}$.

(b) Let $\mathbf{A} = \begin{pmatrix} 1 & 9 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5, 1, -1 \end{pmatrix}$. Is $\mathbf{b}$ in the row space of $\mathbf{A}$? If it is, express it as a linear combination of the columns of $\mathbf{A}$.

(c) Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{pmatrix}$. Is the row space and column space of $\mathbf{A}$ the whole $\mathbb{R}^4$?

(a)

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

So $\mathbf{b}$ is not a linear combination of the columns of $\mathbf{A}$.

(b) Note that $\mathbf{b}$ is in the row space of $\mathbf{A}$ if and only if $\mathbf{b}^T$ is in the column space of $\mathbf{A}^T$. So we are solving for

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 9 & 3 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

So $\mathbf{b} = (5, 1, -1) = (1, 9, 1) - 3(-1, 3, 1) + (1, 1, 1)$.

(c)

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So the column space is the whole $\mathbb{R}^4$. Since $\mathbf{A}$ is invertible if and only if $\mathbf{A}^T$ is, the row space must also be the whole $\mathbb{R}^4$.

4. (a) Find a subset of the vectors

$$\mathbf{v}_1 = (1, -2, 0, 3), \quad \mathbf{v}_2 = (2, -5, -3, 6), \quad \mathbf{v}_3 = (0, 1, 3, 0)$$

$$\mathbf{v}_4 = (2, -1, 4, -7), \quad \mathbf{v}_5 = (5, -8, 1, 2)$$

that forms a basis for $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$.

(b) Express each vector not in the basis as a linear combination of the basis vectors.

(c) Extend the basis in (a) to a basis for $\mathbb{R}^4$

(a) Put the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_5$ as columns of a matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \mathbf{R}.$$

So a basis for $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$.

(b) From the third column of $\mathbf{R}$, we see that $\mathbf{v}_3 = 2\mathbf{v}_1 - \mathbf{v}_2$. From the last column of $\mathbf{R}$, we see that $\mathbf{v}_5 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_4$.

(c)

$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 2 & -5 & -3 & 6 \\ 2 & -1 & 4 & -7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -63/5 \\ 0 & 1 & 0 & -39/5 \\ 0 & 0 & 1 & 13/5 \end{pmatrix}.$$

So adding $\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$ to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ forms a basis for $\mathbb{R}^4$.

Supplementary Problems

5. **(MATLAB)** In the last tutorial, we introduced the "binary n-space" $\mathbb{B}^n$, which is governed by the special addition rule: $1 + 1 = 0$. Our goal in this problem is to investigate how working with binary vectors can help us detect and correct errors in information transmission.

When a computer transmits a piece of information (a binary string) to another device, there is always a possibility of an error—that is, the receiving device might receive an incorrect binary string, perhaps due to external interference or noise in the communication channel. One way that a computer might "protect" its message is by adding extra information to the binary string so that the receiving device can detect—and ideally, correct—any errors that may have occured in transmission.

Consider the following scenario: your friend Annette wants to send you a message $\boldsymbol{u}$. For the sake of simplicity, let's assume that Annette's message contains four bits—that is, $\boldsymbol{u}$ is a vector in $\mathbb{B}^4$ (as opposed to a standard *byte*, which is a vector in $\mathbb{B}^8$). Annette, however, is afraid that a transmission error might send you the wrong message—say, by accidentally changing a 1 to a 0, or a 0 to a 1.

(a) Rather than just sending you the message $\boldsymbol{u}$, Annette instead sends you the 8-vector that results when each bit in $\boldsymbol{u}$ is repeated twice. You receive the vector

$$\left(\begin{array}{cc|cc|cc|cc} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right),$$

where divider bars have been used to split the string up into segments, each representing one bit in Annette's original message. Do you have enough information to decode Annette's original message?

We can detect that an error was made in transmitting the fourth bit of Annette's original message. However, we do not have information to recover actual message—it could easily be either $\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}$.

As the above situation suggests, a simple—perhaps naïve—error-correcting code would employ *repetition*: sending each bit repeatedly, with the hope that the recipient will be able to spot any errors. Observe, however, that this method significantly increases the required amount of data to be transmitted. In the above example, Annette needed to send out twice as much data—this may be problematic for longer messages, specially since bandwidth is expensive!

(b) Annette, who is running out of mobile data, attempts a different error-correcting code, invented by the 20th Century mathematician Richard Hamming. Recall that, taking the non-zero vectors in $\mathbb{B}^3$ as columns, we can create the *Hamming matrix*

$$\boldsymbol{H} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

From this, we formed the matrix $\boldsymbol{M}$ by taking the basis vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ for $S = \{\boldsymbol{x} \in \mathbb{B}^7 \mid \boldsymbol{H}\boldsymbol{x} = \boldsymbol{0}\}$ (i.e., the basis for the null space of $\boldsymbol{H}$) as its columns:

$$\boldsymbol{M} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

i. Explain why the set S is identical to the column space of $\boldsymbol{M}$. Hence, without explicitly calcuating any matrix products, explain why $\boldsymbol{H}(\boldsymbol{Mx}) = \boldsymbol{0}$ for all vectors $\boldsymbol{x} \in \mathbb{B}^4$.

The column space of $\boldsymbol{M}$ is simply the linear span of its columns, which is precisely the set $S = \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$. Observe that for an arbitrary vector $\boldsymbol{x} \in \mathbb{B}^4$, we may write the matrix product $\boldsymbol{Mx}$ as

$$\boldsymbol{Mx} = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \boldsymbol{v}_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + x_3\boldsymbol{v}_3 + x_4\boldsymbol{v}_4.$$

That is, the vector $\boldsymbol{Mx}$ is a linear combination of the columns of $\boldsymbol{M}$ and must hence lie in the column space of $\boldsymbol{M}$. Since the column space of $\boldsymbol{M}$ and the set S are identical, $\boldsymbol{Mx}$ must lie in $S = \{\boldsymbol{x} \in \mathbb{B}^7 \mid \boldsymbol{Hx} = \boldsymbol{0}\}$ as well—that is, $\boldsymbol{Mx}$ is a solution to the homogeneous linear system with coefficient matrix $\boldsymbol{H}$. Thus, $\boldsymbol{H}(\boldsymbol{Mx}) = \boldsymbol{0}$.

ii. Consider the vector $\boldsymbol{v} = \boldsymbol{Mu}$, which is a vector in $\mathbb{B}^7$. The first three entries of $\boldsymbol{v}$ will later be used to detect errors. What are the last four entries of $\boldsymbol{v}$?

Observe that the last four rows of the matrix $\boldsymbol{M}$ form the identity matrix of order 4. We can partition the matrix $\boldsymbol{M}$ into two blocks—the first three rows forming the matrix $\boldsymbol{A}$, and the last four rows forming $\boldsymbol{I}_4$. Thus,

$$\boldsymbol{v} = \boldsymbol{Mu} = \left(\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \boldsymbol{u} = \left(\begin{array}{c} \uparrow \\ \boldsymbol{Au} \\ \downarrow \\ \hline \uparrow \\ \boldsymbol{I}_4\boldsymbol{u} \\ \downarrow \end{array}\right) = \left(\begin{array}{c} * \\ * \\ * \\ \hline \uparrow \\ \boldsymbol{u} \\ \downarrow \end{array}\right),$$

and the last four entries of the vector $\boldsymbol{v}$ contains the original message $\boldsymbol{u}$.

iii. Instead of transmitting the vector $\boldsymbol{u}$, Annette's computer instead sends out $\boldsymbol{v} = \boldsymbol{Mu}$. Assume that *at most* one error can occur during transmission.

- What vector will you receive if no errors occur during transmission?
  If no error occurs during transmission, then all the entries of the vector $\boldsymbol{v}$ will remain unchanged—thus, we will receive the vector $\boldsymbol{v}$.
- Let $\boldsymbol{e}_i$ denote the standard unit vector whose i-th entry is 1 and remaining entries are all 0's. Explain why you would receive a vector of

the form $\boldsymbol{v}+\boldsymbol{e}_i$, for some $i \in \{1, \ldots, 7\}$, if an error has occured during transmission.

If an error has occurred during transmission, then one component of the vector $\boldsymbol{v}$ would have been altered. That is, for one of the vector's components—say, the i-th component—a 0 would have been turned into a 1, or a 1 to a 0. Since $0+1=1$ and $1+1=0$, the error would be akin to adding a 1 to the i-th component of $\boldsymbol{v}$. Thus, we would receive $\boldsymbol{v}+\boldsymbol{e}_i$: the vector $\boldsymbol{v}$ whose i-th entry has been altered.

iv. Let $\boldsymbol{w}$ be the vector you receive on your device. Explain how calculating the matrix product $\boldsymbol{H}\boldsymbol{w}$ will allow you to detect and correct a potential transmission error. [*Hint:* First consider what the vector $\boldsymbol{w}$ would look like if no error has occured, then consider the possibility that an error has occured during transmission.]

There are two possibilities for $\boldsymbol{w}$, depending on whether or not an error occurs during transmission.

- If no error has occurred, the vector $\boldsymbol{w}$ we receive would simply be the vector $\boldsymbol{v}$ that Annette's computer sends out. Since $\boldsymbol{w}=\boldsymbol{v}=\boldsymbol{M}\boldsymbol{u}$, calculating the matrix product $\boldsymbol{H}\boldsymbol{w}$ will yield

$$\boldsymbol{H}\boldsymbol{w}=\boldsymbol{H}\boldsymbol{v}=\boldsymbol{H}(\boldsymbol{M}\boldsymbol{u})=\boldsymbol{0}.$$

  Since the last four entries of the vector $\boldsymbol{v}$ contains Annette's original message $\boldsymbol{u}$, we are easily able to decode Annette's message.

- If an error has occured, then the vector $\boldsymbol{w}$ we receive would be of the form $\boldsymbol{w}=\boldsymbol{v}+\boldsymbol{e}_i$. Calculating the matrix product $\boldsymbol{H}\boldsymbol{w}$ yields

$$\boldsymbol{H}\boldsymbol{w}=\boldsymbol{H}(\boldsymbol{M}\boldsymbol{u}+\boldsymbol{e}_i)=\boldsymbol{H}(\boldsymbol{M}\boldsymbol{u})+\boldsymbol{H}\boldsymbol{e}_i=\boldsymbol{0}+\boldsymbol{H}\boldsymbol{e}_i=\boldsymbol{H}\boldsymbol{e}_i.$$

  In particular, the product $\boldsymbol{H}\boldsymbol{w}=\boldsymbol{H}\boldsymbol{e}_i$ will give us the i-th column of $\boldsymbol{H}$. Since all the columns of $\boldsymbol{H}$ are distinct, we can deduce the vector $\boldsymbol{e}_i$ by finding the corresponding column in $\boldsymbol{H}$. This tells us which component of the vector $\boldsymbol{v}$ contains an error.

In summary, if $\boldsymbol{H}\boldsymbol{w}$ is the zero vector, then no error has occurred; if $\boldsymbol{H}\boldsymbol{w}$ yields the i-th column of $\boldsymbol{H}$, then an error has occurred in transmitting the i-th component of $\boldsymbol{v}$.

v. Your device receives the vector

$$\boldsymbol{w}=\begin{pmatrix}1\\0\\1\\0\\1\\0\\0\end{pmatrix}.$$

Has an error been made during transmission? Do you have enough information to deduce Annette's original message?

We calculate $\boldsymbol{H}\boldsymbol{w}$ on MATLAB, performing our operations modulo 2:

```
>> H=[1 0 0 1 0 1 1; 0 1 0 1 1 0 1; 0 0 1 1 1 1 0];

>> w=[1 0 1 0 1 0 0]';

>> mod(H*w,2)
```

We find that $\boldsymbol{Hw} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Since $\boldsymbol{Hw} \neq \mathbf{0}$, an error has been made. $\boldsymbol{Hw}$ corresponds to the 7th column of the matrix $\boldsymbol{H}$, and we deduce that we received the vector $\boldsymbol{w} = \boldsymbol{v} + \boldsymbol{e}_7$, and an error has been made in transmitting the 7th component of $\boldsymbol{v}$. Thus, we recover the transmission sent from Annette's computer to be

$$\boldsymbol{v} = \boldsymbol{w} + \boldsymbol{e}_7 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Since the last four entries of $\boldsymbol{v}$ is the vector $\boldsymbol{u}$, we find that Annette's original message is $\boldsymbol{u} = (0, 1, 0, 1)^T$.

6. Let $\mathbf{A}$ and $\mathbf{B}$ be row equivalent matrices. Prove that any linear relationship between the columns of $\mathbf{A}$ will be preserved (that is, remain the same) between the columns of $\mathbf{B}$. (**Hint:** Relate a linear relationship between the columns of $\mathbf{A}$ with a homogeneous linear system.)
This, in particular, shows that if $\mathbf{A}$ and $\mathbf{B}$ are row equivalent, then a set of columns of $\mathbf{A}$ forms a basis for the column space of $\mathbf{A}$ if any only if the set of corresponding columns of $\mathbf{B}$ forms a basis for the column space of $\mathbf{B}$.

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be the columns of $\mathbf{A}$. Suppose the linear relationship between the columns is:

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \ldots + c_n\mathbf{a}_n = \mathbf{0},$$

where $c_1, c_2, \ldots, c_n$ are real numbers, not all zero. Then the above equation can be rewritten as

$$\begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{0} \Leftrightarrow \mathbf{A} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{0}.$$

In other words, $c_1, c_2, \ldots, c_n$ is a (non-trivial) solution to $\mathbf{Ax} = \mathbf{0}$. Now since $\mathbf{B}$ is row equivalent to $\mathbf{A}$, there are elementary matrices $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$ such that

$$\mathbf{E}_k \ldots \mathbf{E}_2\mathbf{E}_1\mathbf{A} = \mathbf{B}.$$

This implies

$$\mathbf{E}_k \dots \mathbf{E}_2\mathbf{E}_1\mathbf{A}\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{E}_k \dots \mathbf{E}_2\mathbf{E}_1\mathbf{0} \Rightarrow \mathbf{B}\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{0}$$

Thus if $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$ are the columns of $\mathbf{B}$, we have

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \dots + c_n\mathbf{b}_n = \mathbf{0},$$

which is the same linear relationship between the columns of $\mathbf{A}$.

So if a given set of columns of $\mathbf{A}$ form a basis for the column space of $\mathbf{A}$, then this set is linearly independent and the other columns linearly depends on this set. Then the proof above shows that the corresponding set of columns of $\mathbf{B}$ is also linearly independent, and the other columns linearly depend on this set. Hence, the corresponding set of columns forms a basis for $\mathbf{B}$.