NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

AY2021, Semester 1 MA1508E Linear Algebra for Engineering Tutorial 9

1. Apply Gram-Schmidt Process to convert

   (a) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$ into an orthonormal basis for $\mathbb{R}^4$.

   (b) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$ into an orthonormal set. Is the set obtained an orthonormal basis? Why?

2. Let $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ 3 \\ -1 \\ 1 \end{pmatrix}$.

   (a) Show that the linear system $\mathbf{Ax} = \mathbf{b}$ is inconsistent.

   (b) Find a least squares solution to the system. Is the solution unique? Why?

   (c) Use your answer in (b), compute the projection of $\mathbf{b}$ onto the column space of $\mathbf{A}$. Is the solution unique? Why?

3. **(MATLAB)** Let W be the nullspace of $\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$.

   (a) Find a basis S for W.

   (b) Find the projection of the i-th vector in the standard basis $\mathbf{e}_i$ of $\mathbb{R}^5$ onto W for $i = 1, ..., 5$. (**Hint:** Let $\mathbf{N}$ be a matrix whose columns are vectors in S. Consider the equation $\mathbf{N}^T\mathbf{N} = \mathbf{N}^T\mathbf{b}$ for some $\mathbf{b}$.)

   (c) Find the projection of $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ onto W.

4. **(Application)** A line
$$p(x) = a_1x + a_0$$
is said to be the *least squares approximating line* for a given a set of data points (x_1, y_1), (x_2, y_2), ..., (x_m, y_m) if the sum
$$S = [y_1 - p(x_1)]^2 + [y_2 - p(x_2)]^2 + \cdots + [y_m - p(x_m)]^2$$

is minimized. Writing

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \ \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}, \ \text{and } p(\mathbf{x}) = \begin{pmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_m) \end{pmatrix} = \begin{pmatrix} a_1x_1 + a_0 \\ a_1x_2 + a_0 \\ \vdots \\ a_1x_m + a_0 \end{pmatrix}$$

the problem is now rephrased as finding a_0, a_1 such that

$$S = ||\mathbf{y} - p(\mathbf{x})||^2$$

is minimized. Observe that if we let

$$\mathbf{N} = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_m \end{pmatrix} \text{ and } \mathbf{a} = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix},$$

then $\mathbf{Na} = p(\mathbf{x})$. And so our aim is to find $\mathbf{a}$ that minimizes $||\mathbf{y} - \mathbf{Na}||^2$.

It is known the equation representing the dependency of the resistance of a cylindrically shaped conductor (a wire) at 20^oC is given by

$$R = \rho \frac{L}{A},$$

where R is the resistance measured in Ohms Ω, L is the length of the material in meters m, A is the cross-sectional area of the material in meter squared m^2, and ρ is the resistivity of the material in Ohm meters Ωm. A student wants to measure the resistivity of a certain material. Keeping the cross-sectional area constant at $0.002m^2$, he connected the power sources along the material at varies length and measured the resistance and obtained the following data.

L	0.01	0.012	0.015	0.02
R	2.75×10^{-4}	3.31×10^{-4}	3.92×10^{-4}	4.95×10^{-4}

It is known that the Ohm meter might not be calibrated. Taking that into account, the student wants to find a linear graph $R = \frac{\rho}{0.002}L + R_0$ from the data obtained to compute the resistivity of the material.

(a) Relabeling, we let $R = y$, $\frac{\rho}{0.002} = a_1$ and $R_0 = a_0$. Is it possible to find a graph $y = a_1x + a_0$ satisfying the points?

(b) Find the least square approximating line for the data points and hence find the resistivity of the material. Would this material make a good wire?

5. **(Application, MATLAB)** Suppose the equation governing the relation between data pairs is not known. We may want to then find a polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

of degree n, $n \leq m-1$, that best approximates the data pairs (x_1, y_1), (x_2, y_2), ..., (x_m, y_m). A *least square approximating polynomial* of degree n is such that

$$||\mathbf{y} - p(\mathbf{x})||^2$$

is minimized. If we write

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \ \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}, \ \mathbf{N} = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^n \end{pmatrix} \text{ and } \mathbf{a} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix},$$

then $p(\mathbf{x}) = \mathbf{Na}$, and the task is to find $\mathbf{a}$ such that $||\mathbf{y} - \mathbf{Na}||^2$ is minimized. Observe that $\mathbf{N}$ is a matrix minor of the Vandermonde matrix. If at least $n+1$ of the x-values $x_1, x_2, ..., x_m$ are distinct, the columns of $\mathbf{N}$ are linearly independent, and thus $\mathbf{a}$ is uniquely determined by

$$\mathbf{a} = (\mathbf{N}^T\mathbf{N})^{-1}\mathbf{N}^T\mathbf{y}.$$

We shall now find a quartic polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

that is a least square approximating polynomial for the following data points

x	4	4.5	5	5.5	6	6.5	7	8	8.5
y	0.8651	0.4828	2.590	-4.389	-7.858	3.103	7.456	0.0965	4.326

Enter the data points.

```
>> x=[4 4.5 5 5.5 6 6.5 7 8 8.5]';

>> y=[0.8651 0.4828 2.590 -4.389 -7.858 3.103 7.456 0.0965 4.326]';
```

Next, we will generate the 10×10 Vandermonde matrix.

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>> N=fliplr(vander(x));
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We only want the matrix minor up to the 4-th power, that is, up to the the 5-th column,

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>> N=N(:,1:5);
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Use this to find the least square approximating polynomial of degree 4.

6. Compute the eigenvalues of the following matrices $\mathbf{A}$.

(a) $\mathbf{A} = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$.

(b) $\mathbf{A} = \begin{pmatrix} 9 & 8 & 6 & 3 \\ 0 & -1 & 3 & -4 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

7. (a) Show that λ is an eigenvalue of $\mathbf{A}$ if and only if it is an eigenvalue of $\mathbf{A}^T$.
   (b) Suppose $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ associated to eigenvalue λ. Show that $\mathbf{v}$ is an eigenvector of $\mathbf{A}^k$ associated to eigenvalue λ^k for any positive integer k.
   (c) If $\mathbf{A}$ is invertible, show that $\mathbf{v}$ is an eigenvector of $\mathbf{A}^k$ associated to eigenvalue λ^k for any negative integer k.
   (d) Recall that a matrix is nilpotent if there is a positive integer k such that $\mathbf{A}^k = \mathbf{0}$. Show that if $\mathbf{A}$ is nilpotent, then 0 is the only eigenvalue.

Supplementary Problems

8. (**QR-factorisation**) Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{pmatrix}$, where $\mathbf{u}_i$ is the i-th column of $\mathbf{A}$ for $i = 1, 2, 3$.
   (a) Use Gram-Schmidt Process to transform $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ into an orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ for the column space of $\mathbf{A}$. (Do not change the order of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ when applying the Gram-Schmidt Process.)
   (b) Write each of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ as a linear combination of $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$.
   (c) Hence or otherwise, write $\mathbf{A} = \mathbf{QR}$ where $\mathbf{Q}$ is a 4×3 matrix with orthonormal columns and $\mathbf{R}$ is a 3×3 upper triangular matrix with positive entries along its diagonal. (**Hint:** Recall that if $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is a solution to $\mathbf{Mx} = \mathbf{b}$, then $\mathbf{b} = a\mathbf{m}_1 + b\mathbf{m}_2 + \mathbf{m}_3$, where $\mathbf{m}_i$ is the i-th column of $\mathbf{M}$.)

In general, we have

Theorem. *If* $\mathbf{A}$ *is an* $m \times n$ *matrix with linearly independent columns, then* $\mathbf{A}$ *can be factorised into* $\mathbf{A} = \mathbf{QR}$ *where* $\mathbf{Q}$ *is an* $m \times n$ *matrix whose columns form an orthonormal basis for the column space of* $\mathbf{A}$ *and* $\mathbf{R}$ *is an* $n \times n$ *invertible upper triangular matrix.*

Remark: **QR**-factorisation is widely used in computer algorithms for various computations concerning matrices. We can show easily that in general, the matrix $\mathbf{R}$ is always invertible. Let $\mathbf{x}$ be a solution to the linear system $\mathbf{Rx} = \mathbf{0}$. Pre-multiplying $\mathbf{Q}$ on both sides, we have

$$\mathbf{Q}(\mathbf{Rx}) = \mathbf{Q0} \Rightarrow (\mathbf{QR})\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{Ax} = \mathbf{0}.$$

Since the columns are $\mathbf{A}$ are linearly independent, the rank of $\mathbf{A}$ is equal to the number of columns, and thus the nullity of $\mathbf{A}$ zero. Hence, the nullspace is trivial, and so necessarily $\mathbf{x} = \mathbf{0}$. This means the trivial solution is the only solution to $\mathbf{Rx} = \mathbf{0}$. Thus $\mathbf{R}$ must be invertible.

9. Let $\mathbf{v}_1$ be an eigenvector of $\mathbf{A}$ associated to the eigenvalue λ_1 and $\mathbf{v}_2$ an eigenvector of $\mathbf{A}^T$ associated to eigenvalue λ_2. Suppose $\lambda_1 \neq \lambda_2$. Show that v_1 and v_2 are orthogonal.