ST2334 (2021/22 Semester 1)
Solution to Tutorial 5

Question 1

X	2	3	4	5	6
$f_X(x)$	0.01	0.25	0.40	0.30	0.04

(a) First moment: $E(X) = \sum x f_X(x) = 2(0.01) + \cdots + 6(0.04) = 4.11$.
Second moment: $E(X^2) = \sum x^2 f_X(x) = 2^2(0.01) + \cdots + 6^2(0.04) = 17.63$.

(b) (i) Definition: $V(X) = \sum (x-\mu)^2 f_X(x)$. Hence, $V(X) = (2-4.11)^2 0.01 + \cdots + (6-4.11)^2 0.04 = 0.7379$.
(ii) Computation formula: $V(X) = E(X^2) - [E(X)]^2 = 17.63 - 4.11^2 = 0.7379$.

(c) $E(Z) = E(3X-2) = 3E(X) - 2 = 3(4.11) - 2 = 10.33$.
$V(Z) = V(3X-2) = 3^2 V(X) = 9(0.7379) = 6.6411$.

(d) The probability function of Z is given by

X	2	3	4	5	6
$Z(=3X-2)$	4	7	10	13	16
$f_Z(z)$	0.01	0.25	0.40	0.30	0.04

Mean: $E(Z) = \sum z f_Z(z) = 4(0.01) + \cdots + 16(0.04) = 10.33$.
Variance: $V(Z) = \sum (z-\mu)^2 \, f_Z(z) = (4-10.33)^2 0.01 + \cdots + (16 - 10.33)^2 0.04 = 6.6411$.

(e) $W = aZ + b$
Mean: $E(W) = aE(Z) + b = 10.33a + b$
Variance: $V(W) = a^2 V(Z) = 6.6411a^2$

Question 2

X	0	1	2	3	4	5
$f_X(x)$	1/15	2/15	2/15	3/15	4/15	3/15

$E(X) = \sum x \, f_X(x) = 0(1/15) + \cdots + 5(3/15) = 46/15 = 3.0667$.
Profit = revenue – cost
$= 1.65X + \frac{3}{4}(1.20)(5-X) - 5(1.20) = 0.75X - 1.50$.
Expected Profit, E(Profit) $= E(0.75X - 1.50) = 0.75E(X) - 1.50 = 0.75(46/15) - 1.50 = \$0.80$.

Question 3

(a) Since

$$
\begin{aligned}
\Pr(X \geq 1) &= \Pr(X=1) + \Pr(X=2) + \Pr(X=3) + \Pr(X=4) + \cdots \\
\Pr(X \geq 2) &= + \Pr(X=2) + \Pr(X=3) + \Pr(X=4) + \cdots \\
\Pr(X \geq 3) &= + \Pr(X=3) + \Pr(X=4) + \cdots \\
\vdots \quad &= \qquad \vdots
\end{aligned}
$$

Adding these equalities, we have

$$
\begin{aligned}
\sum_{k=1}^{\infty} \Pr(X \geq k) &= 1\Pr(X=1) + 2\Pr(X=2) + 3\Pr(X=3) + \cdots \\
&= \sum_{k=1}^{\infty} k \Pr(X=k) = E(X).
\end{aligned}
$$

(b) Let X_1, X_2 and X_3 denote respectively the number obtained in the first, second and third die. Then $M = \min\{X_1, X_2, X_3\}$. For $k = 1, 2, \cdots, 6$,

$$\begin{aligned}\Pr(M \geq k) &= \Pr(X_1 \geq k, X_2 \geq k, X_3 \geq k)\\ &= \Pr(X_1 \geq k)\Pr(X_2 \geq k)\Pr(X_3 \geq k) \text{ since } X_i's \text{ are independent}\\ &= \left(\frac{6-(k-1)}{6}\right)\left(\frac{6-(k-1)}{6}\right)\left(\frac{6-(k-1)}{6}\right)\\ &= \left(\frac{7-k}{6}\right)^3.\end{aligned}$$

In other words,

$$\Pr(M \geq 1) = 1, \Pr(M \geq 2) = \frac{5^3}{216}, \Pr(M \geq 3) = \frac{4^3}{216},$$
$$\Pr(M \geq 4) = \frac{3^3}{216}, \Pr(M \geq 5) = \frac{2^3}{216}, \Pr(M \geq 6) = \frac{1^3}{216},$$

and $\Pr(M \geq k) = 0$ for $k = 7, 8, 9, \cdots$.

It follows that

$$\begin{aligned}E(M) &= \sum_{k=1}^{\infty} \Pr(M \geq k) = \sum_{k=1}^{6} \Pr(M \geq k)\\ &= \sum_{i=1}^{6}\left(\frac{7-k}{6}\right)^3\\ &= \frac{1^3 + 2^3 + \cdots + 6^3}{6^3} = 2.0417\end{aligned}$$

<u>Question 4</u>

$$f_X(x) = \begin{cases} 2(1-x), & 0 < x < 1,\\ 0, & \text{otherwise.}\end{cases}$$

(a) The mean of X is given by

$$\begin{aligned}E(X) &= \int_{-\infty}^{\infty} x\, f_X(x)dx = \int_{-\infty}^{0} x\, 0\, dx + \int_0^1 x\, 2(1-x)dx + \int_1^{\infty} x\, 0\, dx\\ &= 2\int_0^1 (x - x^2)dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{3}.\end{aligned}$$

The second moment is given by

$$\begin{aligned}E(X^2) &= \int_{-\infty}^{\infty} x^2\, f_X(x)dx = \int_0^1 x^2\, f_X(x)\, dx = 2\int_0^1 x^2(1-x)dx\\ &= 2\int_0^1 (x^2 - x^3)dx = 2\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\left[\frac{1}{3} - \frac{1}{4}\right] = \frac{1}{6}.\end{aligned}$$

Thus,

$$V(X) = E(X^2) - [E(X)]^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}.$$

(b) $Y = 3X - 2$

The mean of Y, $E(Y) = 3E(X) - 2 = 3\left(\frac{1}{3}\right) - 2 = -1.$

The variance of Y, $V(Y) = 3^2V(X) = 9\left(\frac{1}{18}\right) = \frac{1}{2}.$

Question 5

To solve for the two unknowns, a and b, we need two equations which come from the two conditions: (1) $\int_{-\infty}^{\infty} f_x(x) = 1$ and (2) $E(X) = 3/5$.

$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 (a + bx^2)\,dx = \left[ax + \frac{bx^3}{3}\right]_0^1 = a + \frac{b}{3}$. Therefore $\int_0^1 f_X(x)\,dx = 1$ gives

$$a + \frac{b}{3} = 1 \qquad (1)$$

$E(X) = \int_0^1 x(a + bx^2)\,dx = \left[\frac{ax^2}{2} + \frac{bx^4}{4}\right]_0^1 = \frac{a}{2} + \frac{b}{4}$. Therefore, $E(X) = \frac{3}{5}$ gives

$$\frac{a}{2} + \frac{b}{4} = \frac{3}{5} \qquad (2)$$

Solving these 2 equations, we have

$$a = \frac{3}{5} \text{ and } b = \frac{6}{5}.$$

Question 6

$E[(X-1)^2] = E[X^2 - 2X + 1] = E(X^2) - 2E(X) + 1$.

Hence, $E[(X-1)^2] = 10$ implies

$$E(X^2) - 2E(X) + 1 = 10 \qquad (1)$$

$E[(X-2)^2] = E[X^2 - 4X + 4] = E(X^2) - 4E(X) + 4$

Hence, $E[(X-2)^2] = 6$ implies

$$E(X^2) - 4E(X) + 4 = 6 \qquad (2)$$

Subtracting Equation (2) from Equation (1), we have

$$2E(X) - 3 = 4 \text{ or } E(X) = 7/2.$$

Substitute $E(X) = 7/2$ into Equation (1), we have $E(X^2) = 16$.

Hence $V(X) = E(X^2) - [E(X)]^2 = 16 - (7/2)^2 = 15/4$.

Question 7

We write the probabilities in the form of $\Pr(|X - \mu| \geq k\sigma)$, where $\mu = 10$ and $\sigma^2 = 4$. We then apply Chebyshev's Inequality.

(a) $\Pr(5 < X < 15) = \Pr\left[10 - \left(\frac{5}{2}\right)(2) < X < 10 + \left(\frac{5}{2}\right)(2)\right] = \Pr\left(|X - 10| < \left(\frac{5}{2}\right)(2)\right)$

Applying Chebyshev's Inequality with $k = 5/2$, we have

$$\Pr\left(|X - 10| < \left(\frac{5}{2}\right)(2)\right) \geq 1 - \frac{1}{(5/2)^2} = \frac{21}{25}.$$

(b) $\Pr(6 < X < 14) = \Pr[10 - 2(2) < X < 10 + 2(2)] = \Pr\big(|X - 10| < 2(2)\big)$

Applying Chebyshev's Inequality with $k = 2$, we have

$$\Pr\big(|X - 10| < 2(2)\big) \geq 1 - \frac{1}{2^2} = \frac{3}{4}.$$

Hence,

$$\Pr(5 < X < 14) \geq \Pr(6 < X < 14) \geq \frac{3}{4}.$$

(c) $\Pr(|X - 10| < 3) = \Pr\left(|X - 10| < \left(\frac{3}{2}\right)2\right)$

Applying Chebyshev's Inequality with $k = 3/2$, we have

$$\Pr\left[10 - \left(\frac{3}{2}\right)(2) < X < 10 + \left(\frac{3}{2}\right)(2)\right] \geq 1 - \frac{1}{(3/2)^2} = \frac{5}{9}.$$

(d) $\Pr(|X-10|\geq 3)=\Pr\left(|X-10|\geq\left(\frac{3}{2}\right)(2)\right)$

Applying Chebyshev's Inequality with $k=3/2$, we have

$$\Pr\left(|X-10|\geq\left(\frac{3}{2}\right)(2)\right)\leq\frac{1}{(3/2)^2}=\frac{4}{9}$$

(e) We apply Chebyshev's Inequality to obtain

$$\Pr(|X-10|\geq c)\leq\frac{4}{c^2}.$$

In order to determine a c satisfying the required inequality, we impose

$$\frac{4}{c^2}\leq 0.04.$$

leading to $c\geq 10$. Choose $c=10$ will ensure the probability at most 0.04.

Question 8

(a)

$$f_X(x)=\begin{cases}6x(1-x), & 0< x<1,\\ 0, & \text{otherwise.}\end{cases}$$

$$E(X)=\int_{-\infty}^{\infty}x\,f_X(x)\,dx=\int_0^1 6x^2(1-x)dx=\left[2x^3-\frac{3}{2}x^4\right]_0^1=0.5$$

$$E(X^2)=\int_{-\infty}^{\infty}x^2f_X(x)\,dx=\int_0^1 6x^3(1-x)\,dx=\left[\frac{3}{2}x^4-\frac{6}{5}x^5\right]_0^1=0.3$$

$$V(X)=E(X^2)-[E(X)]^2=0.3-0.5^2=0.05$$

Hence, $\sigma=\sqrt{0.05}=0.2236$.

(b) To compute the exact value, we proceed as follows

$$\begin{aligned}\Pr(\mu-2\sigma<X<\mu+2\sigma)&=\Pr(0.5-2\sqrt{0.05}<X<0.5+\sqrt{0.05}))\\&=\Pr(0.0528<X<0.9472)\\&=\int_{0.0528}^{0.9472}6x(1-x)dx\\&=[3x^2-2x^3]_{0.0528}^{0.9472}=0.9839.\end{aligned}$$

(c) Applying Chebyshev's Inequality to

$$\Pr(\mu-2\sigma<X<\mu+2\sigma)=1-\Pr(|X-\mu|\geq 2\sigma)\geq 1-\frac{\sigma^2}{(2\sigma)^2}=\frac{3}{4}=0.75.$$

(d) The answer in (c) states that the probability of X lies between two standard deviation above the mean and two standard deviation below the mean is at least 0.75, which is consistent with the actual probability 0.9839.

Question 9

Given that $\mu=900$ and $\sigma=50$, hence, 700 is 4 standard deviation below the mean. Furthermore, since the distribution is symmetric about the mean implies that $\Pr(X\leq 700)=\Pr(X\leq 900-200)=\Pr(X\geq 900+200)$. Therefore,

$$\begin{aligned}\Pr(X\leq 700)&=\frac{1}{2}[\Pr(X\leq 900-200)+\Pr(X\geq 900+200)]\\&=\frac{1}{2}[\,\Pr(X\leq 700\text{ or }X\geq 1100)]=\frac{1}{2}\Pr\big(|X-900|\geq 4(50)\big)\\&=\frac{1}{2}\Pr(|X-\mu|\geq 4\sigma)\leq\frac{1}{2}\left(\frac{1}{4^2}\right)=0.03125,\end{aligned}$$