

1.4.2 Relative Frequency

- Alternatively, probability can be derived based on the relative frequency
- Suppose we repeat the experiment E for n times and let A be an event associated with E .
- We let n_A be the number of times that the event A has occurred among the n repetitions respectively.
- Then $f_A = \frac{n_A}{n}$ is called the **relative frequency** of the event A in the n repetitions of E .

1.4.3 Axioms of Probability

- Consider an experiment whose sample space is S .
- The objective of probability is to assign to each event A , a number $\Pr(A)$, called the **probability** of the event A , which will give a precise measure of the chance that A will occur.
- Consider the **collection of all events** and denote it by \mathcal{P} .
- For each event A of the sample space S we assume that a number $\Pr(A)$, which is called the **probability** of the event A , is defined and satisfies the following three axioms:

Relative frequency and Probability are very close, but different terminologies. One can view the former as the practical explanation of the latter in some sense. More rigorously, we can assert that $f_A = n_A/n$ approaches $\Pr(A)$ when n approaches infinity, or described more mathematically:

$$\Pr(A) = \lim_{n \rightarrow \infty} f_A$$

Axioms of Probability (Continued)

Axiom 1: $0 \leq \Pr(A) \leq 1$.

Axiom 2: $\Pr(S) = 1$.

Axiom 3: If A_1, A_2, \dots are **mutually exclusive** (disjoint) events (that is, $A_i \cap A_j = \emptyset$ when $i \neq j$), then

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(A_i)$$

In particular, if A and B are **two mutually exclusive events** then $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

These Axioms, in many literatures, are used as the definition of probability.

One should take note that

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(A_i)$$

and that in page 127: $f_{A \cup B} = f_A + f_B$ are correct ONLY IF the events for taking the unions are **MUTUALLY EXCLUSIVE!**

1.5 Basic Properties of Probability

1.5.1 Some Basic properties of probability

1. $\Pr(\emptyset) = 0$.

Take $A_1 = \emptyset, A_2 = \emptyset, A_3 = \emptyset, \dots \dots$

then $A_1 \cap A_2 \cap A_3 \cap \dots \dots = \emptyset$ and $A_1 \cup A_2 \cup A_3 \cup \dots \dots = \emptyset$.

Hence, by Axiom 3

$$\Pr(\emptyset) = \Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(A_i) = \sum_{i=1}^{\infty} \Pr(\emptyset)$$

which can hold only if $\Pr(\emptyset) = 0$.

This can be derived slightly simpler (but the same idea): set $A_1 = \emptyset$ and $A_2 = \emptyset$. Then

$\Pr(\emptyset) = \Pr(A_1 \cup A_2) = \Pr(A_1) + \Pr(A_2) = \Pr(\emptyset) + \Pr(\emptyset)$, which leads to $\Pr(\emptyset) = 0$.

Note that the inverse is not necessarily true: $\Pr(A) = 0$ does not imply $A = \emptyset$. A can contain sample points that has zero probability. For example, consider the outcome of rolling a die and $A = \{7,8,9\}$, then $\Pr(A) = 0$, but A is not empty. Here is another example, randomly select a whole number, i.e., all whole numbers are equally likely to be chosen, then S is the set of whole numbers. $A = \{1,2,3,4,5,6\}$, then $\Pr(A) = 0$; but A is not empty.

Basic Properties of Probability (Continued)

7. If $A \subset B$, then $\Pr(A) \leq \Pr(B)$.

Since $B = (B \cap A) \cup (B \cap A')$ and $B \cap A = A$,

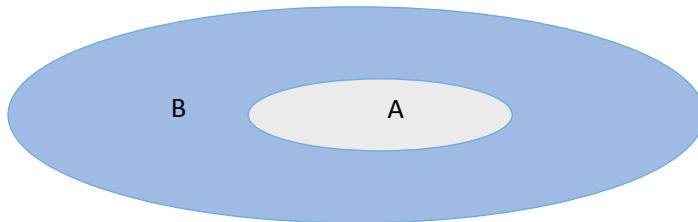
so $B = A \cup (B \cap A')$ and $A \cap (B \cap A') = \emptyset$.

Therefore

$$\Pr(B) = \Pr(A) + \Pr(B \cap A') \geq \Pr(A).$$

The result follows by noting that $\Pr(B \cap A') \geq 0$.

This derivation can be illustrated by the following diagram:



Here, B includes both the blue and gray areas; A is the gray area, $B \cap A'$ is the blue area. So we immediately observe $B = A \cup (B \cap A')$, and $\emptyset = A \cap (B \cap A')$. Therefore,

$$\Pr(B) = \Pr(A) + \Pr(B \cap A') \geq \Pr(A).$$

Based on this derivation, we can answer more involved questions: Can $\Pr(B) = \Pr(A)$? If yes, when? When can we conclude $\Pr(B) > \Pr(A)$, i.e., we can remove the possibility of “=”? The answer is fully due to the probability event $B \cap A'$ and its probability.

Sometimes, it is not so obvious to conclude that two events are mutually exclusive, then we may need to justify with some mathematical arguments. To conclude two events, namely C and D, are mutually exclusive, we need to

- Assume an arbitrary $x \in C$, then verify $x \notin D$;
- And assume an arbitrary $x \in D$, then verify $x \notin C$.

Therefore, we can check that A and $B \cap A'$ are mutually exclusive by

- Assume an arbitrary $x \in A$, then clearly $x \notin A'$ based on the definition of A' ; therefore we conclude $x \notin B \cap A'$ based on the defintion of intersection.
- Assume an arbirary $x \in B \cap A'$, then we must have $x \in A'$, so $x \notin A$.

Note that we can extend such arguments to check other relationships of sets.

To check $A = B$, we need to

- Assume an arbitrary $x \in A$, then verify $x \in B$;
- And assume an arbitrary $x \in B$, then verify $x \in A$.

To check $A \subset B$, i.e., A is a subset of B , we only need to assume arbitrary $x \in A$, then verify $x \in B$.

To check that $A \not\subset B$, we only need to find an $x \in A$, and verify that $x \notin B$.

Solution to Example 6

- The probability that n persons all have different birthdays from you is $\left(\frac{364}{365}\right)^n$.
 - So we need n such that $1 - \left(\frac{364}{365}\right)^n \geq 0.5$.
 - Solving, we obtain
- $$n \geq \frac{\log(0.5)}{\log(364/365)} = 252.7.$$
- We need at least 253 people (excluding yourself).

The “blue formula” in this lecture slide can be derived in the same manner as the “Birthday problem”:

- In total, there are 365 possible choices as the birthday of each person. So if there are n persons, the total number of possibilities of their birthdays would be 365^n .
- If we require that no body would be of the same birthday as you. The number of possible choices as the birthday of each person would be 364. So there are 364^n possibilities.

This leads to the “blue formula” in the slide.

To solve the inequality $1 - \left(\frac{364}{365}\right)^n \geq 0.5$,

- Moving terms around, we get $\left(\frac{364}{365}\right)^n \leq 0.5$;
- Taking log on both sides of the inequality leads to $n \log\left(\frac{364}{365}\right) \leq \log(0.5)$, since for any positive number x, y , $x \leq y$ if and only if $\log(x) \leq \log(y)$.
- Note that $\log\left(\frac{364}{365}\right) < 0$, as $\frac{364}{365} < 1$. Then divide a negative number on both sides of the inequality, the direction of the inequality needs to be changed. Therefore $n \geq \log(0.5)/\log(364/365)$.

1.5.2 Sample Spaces Having Finite Outcomes

Consider the sample space S which contains a finite number of k outcomes. That is,

$$S = \{a_1, a_2, \dots, a_k\}$$

Let $\Pr(a_i) = p_i$ be the probability of $\{a_i\}$ and

- (1) $0 \leq p_i \leq 1$, for $i = 1, 2, \dots, k$.
- (2) $p_1 + p_2 + \dots + p_k = 1$.

$p_1 + p_2 + \dots + p_k = 1$ can be derived as follows. Define probability events: $A_1 = \{a_1\}, A_2 = \{a_2\}, \dots, A_k = \{a_k\}$. Then clearly all these events are mutually exclusive as each only carries one sample point, which are distinct to each other, and $S = A_1 \cup A_2 \cup \dots \cup A_k$. Therefore

$$\begin{aligned} 1 &= \Pr(S) = \Pr(A_1 \cup A_2 \cup \dots \cup A_k) = \Pr(A_1) + \Pr(A_2) + \dots + \Pr(A_k) \\ &= \Pr(a_1) + \Pr(a_2) + \dots + \Pr(a_k) = p_1 + p_2 + \dots + p_k. \end{aligned}$$

Example 3

- A box contains 50 bolts and 150 nuts.
- Half of the bolts and half of the nuts are rusted.
- If one item is chosen at random, what is the probability that it is rusted or is a bolt?

Solution

- Let $A = \{\text{the item is rusted}\}$ and $B = \{\text{the item is a bolt}\}$.
- Then

$$\begin{aligned} \Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B) \\ &= 100/200 + 50/200 - 25/200 = 5/8. \end{aligned}$$

This example can also be worked out directly by the formula given on page 172:

$$\Pr(A) = \frac{\text{Number of Sample Points in } A}{\text{Number of Sample points in } S}$$

Here, we define $A = \{\text{the item is rusted or a bolt}\}$, which is the collection of all the bolts and the rusted nuts; so in total, we have $50+75 = 125$ sample points in A ; we have 200 sample points in S . Therefore $\Pr(A) = \frac{125}{200} = 5/8$, which is the same as the answer in the slide.

Example 5

- If 2 balls are “randomly drawn” from an urn containing 6 white and 5 black balls,
- what is the probability that one of the drawn balls is white and the other black?

Page 177 gives the way to solve this problem using combinations, where which ball is drawn first does not matter. To solve this problem but considering that the order of drawing the balls matter, we can think this way:

- The total number of ways to draw is: for the first draw, there are 11 possibilities, and 10 for the second; so based on the multiplication principle, we have $(11)(10) = 110$, which is the number of sample points in the sample space.
- To get a white ball and a black ball, there are two parallel possibilities: WB (the first ball is white, and the second is black) or BW. To get WB, there are $(6)(5)=30$ possibilities: draw the first ball from those 6 white balls, and draw the second from the 5 black balls. Likewise, to get BW, there are $(5)(6) = 30$ possibilities. So in total, we have 60 possible ways to get one white ball and one black ball.

As a consequence, the probability of getting a white ball and a black ball is $60/110 = 6/11$.

Example 9

- In a class of 200 students, 108 study economics, 138 study chemistry and 70 study both chemistry and economics.
- If a student is selected at random, what is the probability that the student
 - takes economics **or** chemistry;
 - doesn't take **neither** of these subjects;
 - takes chemistry **but not** economics.

Part (b) needs to be corrected to be “take neither of these subjects”.

Solution to Example 9 (Continued)

- (c) Since $C = (E' \cap C) \cup (E \cap C)$ and $(E' \cap C) \cap (E \cap C) = \emptyset$, therefore

$$\Pr(C) = \Pr(E' \cap C) + \Pr(E \cap C).$$

$$\begin{aligned} \text{Hence } \Pr(E' \cap C) &= \Pr(C) - \Pr(E \cap C) \\ &= 138/200 - 70/200 \\ &= 0.34. \end{aligned}$$

$C = (E' \cap C) \cup (E \cap C)$ can be derived using formulae of sets computations:

$$C = S \cap C = (E' \cup E) \cap C = (E' \cap C) \cup (E \cap C).$$

Besides, think about whether there is another way to solve (c).

An Illustrative Example (Continued)

- Hence the conditional probability of A given B can be obtained by

$$\Pr(A|B) = \frac{\#(A \cap B)}{\#(B)}$$

- If we divide both the numerator and denominator by $\#(S)$, then

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

- Similarly,

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}.$$

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- From this page, we can clearly observe $\Pr(A|B) \neq \Pr(B|A)$, unless $\Pr(B) = \Pr(A)$. Also take note that these two conditional probabilities have totally different meanings. $\Pr(A|B)$ is confining the probability space to B (namely B is given), whereas $\Pr(B|A)$ is confining probability space to A (namely A is given).
- We also have $\Pr(A|B') \neq 1 - \Pr(A|B)$ in general. But $\Pr(A'|B) = 1 - \Pr(A|B)$.
- The conclusion $\Pr(A|B) = \frac{\#(A \cap B)}{\#B}$ is applicable only when the sample points in the sample space are equally likely. The example on page 198 gives an example that this formula is not applicable:

Example 2

- Roll an unbalanced die.
(The sample space is $S = \{1, 2, 3, 4, 5, 6\}$.)
- Suppose that the respective probabilities are $1/12, 1/12, 1/6, 1/6, 1/6$, and $1/3$.

Note: The sum of the probabilities equals 1

- (a) If the number obtained is even, what is the probability that it is a 6?
- (b) What is the probability that the number obtained is a perfect square number given that a number greater than 3 has obtained?

The following page gives another example that it is essential to ensure that the sample points are equally likely in order to apply this formula; refer to the corresponding lecture video for this example for details.

Example 3

- A couple has 2 children.
- What is the probability that both are boys if it is known that they have at least one son?