1.5 Basic Properties of Probability

1.5.1 Some Basic properties of probability

1. $\Pr(\emptyset) = 0$.

   Take $A_1 = \emptyset, A_2 = \emptyset, A_3 = \emptyset, \cdots\cdots$

   then $A_1 \cap A_2 \cap A_3 \cap \cdots\cdots = \emptyset$ and $A_1 \cup A_2 \cup A_3 \cup \cdots\cdots = \emptyset$.

   Hence, by Axiom 3

$$\Pr(\emptyset) = \Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(A_i) = \sum_{i=1}^{\infty} \Pr(\emptyset)$$

   which can hold only if $\Pr(\emptyset) = 0$.

Basic Properties of Probability (Continued)

2. If $A_1, A_2, \cdots, A_n$ are **mutually exclusive events**, then

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \Pr(A_i)$$

Let $A_{n+1} = \emptyset, A_{n+2} = \emptyset, \cdots\cdots$, then

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{\infty} A_i \quad \text{and}$$

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(A_i) = \sum_{i=1}^{n} \Pr(A_i)$$

Basic Properties of Probability (Continued)

3. For any event A,
$$\Pr(A') = 1 - \Pr(A) \, .$$

Since $S = A \cup A'$ and $A \cap A' = \emptyset$, therefore
$$\Pr(S) = \Pr(A \cup A') = \Pr(A) + \Pr(A').$$
But $\Pr(S) = 1$ by Axiom 2, the result follows.

Basic Properties of Probability (Continued)

4. For any two events A and B,
$$\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B').$$

Since
$$A = (A \cap B) \cup (A \cap B')$$

and
$$(A \cap B) \cap (A \cap B') = \emptyset,$$

therefore
$$\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B').$$

Basic Properties of Probability (Continued)

5. For any two events A and B,

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B).$$

Since $A \cup B = B \cup (A \cap B')$ and $B \cap (A \cap B') = \emptyset$, therefore

$$\Pr(A \cup B) = \Pr(B) + \Pr(A \cap B')$$

But from Property (4), we have

$$\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B')$$

Hence $\Pr(A \cup B) = \Pr(B) + \Pr(A) - \Pr(A \cap B).$

Basic Properties of Probability (Continued)

6. For any three events A, B, C,

$$\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B)$$
$$- \Pr(A \cap C) - \Pr(B \cap C) + \Pr(A \cap B \cap C).$$

Write $A \cup B \cup C = A \cup (B \cup C)$ and then by applying Property (5), we have

$$\Pr\big(A \cup (B \cup C)\big)$$
$$= \Pr(A) + \Pr(B \cup C) - \Pr(A \cap (B \cup C))$$
$$= \Pr(A) + \Pr(B) + \Pr(C) - \Pr(B \cap C) - \Pr(A \cap (B \cup C)).$$

Basic Properties of Probability (Continued)

6.   (Continued)

But
$$\Pr\big(A \cap (B \cup C)\big) = \Pr\big((A \cap B) \cup (A \cap C)\big)$$
$$= \Pr(A \cap B) + \Pr(A \cap C) - \Pr\big((A \cap B) \cap (A \cap C)\big)$$
$$= \Pr(A \cap B) + \Pr(A \cap C) - \Pr(A \cap B \cap C).$$

Hence
$$\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B)$$
$$- \Pr(A \cap C) - \Pr(B \cap C) + \Pr(A \cap B \cap C).$$

Basic Properties of Probability (Continued)

The above property can be extended to n events

$$\Pr(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} \Pr(A_i) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr(A_i \cap A_j)$$

$$+ \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \Pr(A_i \cap A_j \cap A_k) - \cdots\cdots$$

$$+ (-1)^{n+1} \Pr(A_1 \cap A_2 \cap \cdots \cap A_n)$$

It can be proved by mathematical induction.

The above identity is also known as **"The Inclusion-Exclusion Principle"**.

Basic Properties of Probability (Continued)

7. If $A \subset B$, then $\Pr(A) \leq \Pr(B)$.

Since $B = (B \cap A) \cup (B \cap A')$ and $B \cap A = A$,

so $B = A \cup (B \cap A')$ and $A \cap (B \cap A') = \emptyset$.

Therefore

$$\Pr(B) = \Pr(A) + \Pr(B \cap A') \geq \Pr(A).$$

The result follows by noting that $\Pr(B \cap A') \geq 0$.

Example 1

- A retail establishment accepts either the American Express or the VISA credit card.

- A total of 24% of its customers carry an American Express card, 61% carry a VISA card, and 11% carry both.

- What percentage of its customers carries a credit card that the establishment will accept?

Solution to Example 1

- Let A and V represent the events that a customer carries an American Express and Visa card respectively.

- $\Pr(A) = 0.24, \Pr(V) = 0.61$ and $\Pr(A \cap V) = 0.11$.

- The desired probability that customers carry a credit card that the establishment will accept is given by

$$\Pr(A \cup V) = \Pr(A) + \Pr(V) - \Pr(A \cap V)$$
$$= 0.24 + 0.61 - 0.11$$
$$= 0.74.$$

Example 2

- A poll of statisticians in USA was conducted to ascertain their professional responsibilities.

- An analysis of their responses gave the following distribution of professional responsibilities :

A = {Research}   40%

B = {Professional consultation}   64%

C = {Data collection and analysis}   36%

Example 2 (Continued)

- Suppose that 10% are involved in all three activities; 15% are involved in both A and C; and 17% are involved in both A and B.

- Use this information to find the percentage of all statisticians in USA that are involved in both B and C.

- Assume that $\Pr(A \cup B \cup C) = 1$.

Solution to Example 2

- $\Pr(A \cap B \cap C) = 0.1$, $\Pr(A \cap C) = 0.15$ and $\Pr(A \cap B) = 0.17$.

- Since $\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) + \Pr(A \cap B \cap C)$, therefore
$$1 = 0.4 + 0.64 + 0.36 - 0.17 - 0.15 - \Pr(B \cap C) + 0.1$$

- Hence,
$$\Pr(B \cap C) = 0.4 + 0.64 + 0.36 - 0.17 - 0.15 + 0.1 - 1$$
$$= 0.18.$$

Example 3

- The probabilities that a gas station pumps gas into 0, 1, 2, 3, 4, or 5 or more cars during a certain 30-minute period are 0.03, 0.18, 0.24, 0.28, 0.10 and 0.17 respectively.

- Find the probability that in a 30-minute period

  (a) more than 2 cars receive gas;

  (b) at most 4 cars receive gas.

Solution to Example 3

- Let A_i be the event of pumping i cars, $i = 0, 1, 2, 3, 4$ and A_5 be the event of pumping 5 or more cars.

- Note that all A_i's are mutually exclusive events.

(a)   Pr(more than 2 cars receive gas)
$$= \Pr(A_3 \cup A_4 \cup A_5)$$
$$= \Pr(A_3) + \Pr(A_4) + \Pr(A_5)$$
$$= 0.28 + 0.1 + 0.17$$
$$= 0.55.$$

Solution to Example 3 (Continued)

(b)   Pr(at most 4 cars receive gas)
$$= \Pr(A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4)$$
$$= \Pr(A_0) + \Pr(A_1) + \Pr(A_2) + \Pr(A_3) + \Pr(A_4)$$
$$= 0.03 + 0.18 + 0.24 + 0.28 + 0.1$$
$$= 0.83.$$

Alternatively,

Pr(at most 4 cars receive gas)
$$= 1 - \Pr(5 \text{ or more cars receive gas})$$
$$= 1 - \Pr(A_5) = 1 - 0.17 = 0.83.$$

Example 4

(Hall Pageant)

- Audrey is taking part in her hall's pageant.

- The probability that she will win the crown is 0.14.

- The probability that she will win Miss Photogenic is 0.3.

- The probability that she will win both is 0.11.

    (a) What is the probability that she wins at least one of the two?

    (b) What is the probability that she wins only one of two?

Solution to Example 4

Let A be the event that she wins the crown, and B that she wins Miss Photogenic.

(a) The probability that she wins at least one of the two titles

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$
$$= 0.14 + 0.3 - 0.11 = 0.33.$$

(b) The probability that she wins **only** one of the two titles

$$\Pr\big((A \cup B) \backslash (A \cap B)\big) = \Pr(A \cup B) - \Pr(A \cap B)$$
$$= 0.33 - 0.11 = 0.22.$$

Example 5

(Birthday Problem)

- Here's a useful party trick: walk into a room or bar with at least 50 people.

- Boldly claim that you sense two people sharing the same birthday. Act awesome afterwards. How often are you right?

- We can cast this as a probability question:

- There are *n* people in a room, what is the probability that there are at least two people with the same birthday?

Solution to Example 5

We assume each day is **equally likely** to be a birthday of everyone, and there is no leap year.

A person can have his birthday on any of the 365 days. So there are a total of $(365)^n$ possible outcomes, i.e. $\#(S) = (365)^n$

Let A denote the event that there are at least two people among the n people sharing the same birthday.

We will work out the event that no two people sharing the same birthday which is denoted A'.

Solution to Example 5 (Continued)

To count A', note that
$$\#(A') = 365(364) \cdots [365 - (n-1)].$$

Therefore,
$$\Pr(A') = \frac{365(364) \cdots (365 - n + 1)}{365^n}.$$

Hence
$$\Pr(A) = 1 - \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right) \cdots \left(1 - \frac{n-1}{365}\right).$$

Remarks on Example 5

Let $q_n = \Pr(A')$ when there are n people, and

$p_n = \Pr(A) = 1 - q_n.$

The values of p_n and q_n for selected values of n are tabulated on the right.

n	q_n	p_n
1	1	0
2	0.99726	0.00274
3	0.99180	0.00820
10	0.88305	0.11695
15	0.74710	0.25290
20	0.58856	0.41144
21	0.55631	0.44369
22	0.52430	0.47570
23	0.49270	0.50730
30	0.29368	0.70632
40	0.10877	0.89123
50	0.029626	0.979374
100	$3.0725(10)^{-7}$	1
253	$6.9854(10)^{-53}$	1

Remarks on Example 5 (Continued)

- We see that for 50 people, 98% of the time you will be able to find two people with the same birthday.

- Take note that the probability of having two people sharing the same birthday exceeds 1/2 once you have 23 people.

Example 6

(Inverse Birthday Problem)

- How large does a group of (randomly selected) people have to be such that the probability that someone is sharing his or her birthday **with you** is larger that 0.5?

Solution to Example 6

- The probability that n persons all have different birthdays **from you** is $\left(\dfrac{364}{365}\right)^{n}$.

- So we need n such that $1 - \left(\dfrac{364}{365}\right)^{n} \geq 0.5$.

- Solving, we obtain

$$n \geq \frac{\log(0.5)}{\log(364/365)} = 252.7.$$

- We need at least 253 people (excluding yourself).

Remarks on Birthday Problems

Why there is a big difference in the answers between the two birthday problems?

- The inverse birthday problem requires the sharing of **a particular day** as the common birthday

- The birthday problem allows that **any day** is the shared birthday.

1.5.2 Sample Spaces Having Finite Outcomes

Consider the sample space S which contains a finite number of k outcomes. That is,

$$S = \{a_1, a_2, \cdots, a_k\}$$

Let $\Pr(a_i) = p_i$ be the probability of $\{a_i\}$ and

(1) $0 \le p_i \le 1$, for $i = 1, 2, \cdots, k$.

(2) $p_1 + p_2 + \cdots + p_k = 1$.

Sample Spaces Having Finite Outcomes (Continued)

Let an event A consists of r outcomes, $1 \leq r \leq k$, say
$$A = \{a_{j_1}, a_{j_2}, \cdots, a_{j_r}\}$$
where $j_1, j_2, \cdots, j_r$ represent any r indices from $1, 2, \cdots, k$.
Then
$$\mathrm{Pr}(A) = p_{j_1} + p_{j_2} + \cdots + p_{j_r},$$
where $\mathrm{Pr}(a_{j_l}) = p_{j_l}, \ l = 1, \cdots, r$.

That is, the probability of an event A equals the sum of the probabilities of the various individual outcomes making up the event A.

Example 1

- Let S be the sample space of the sum of the numbers when a pair of dice is tossed. The outcomes of tossing 2 dice are

(1,1)   (1,2)   (1,3)   (1,4)   (1,5)   (1,6)
(2,1)   (2,2)   (2,3)   (2,4)   (2,5)   (2,6)
(3,1)   (3,2)   (3,3)   (3,4)   (3,5)   (3,6)
(4,1)   (4,2)   (4,3)   (4,4)   (4,5)   (4,6)
(5,1)   (5,2)   (5,3)   (5,4)   (5,5)   (5,6)
(6,1)   (6,2)   (6,3)   (6,4)   (6,5)   (6,6)

- Hence $S = \{2, 3, 4, \cdots, 11, 12\}$, and

Example 1 (Continued)

- $\Pr(\{2\}) = p_2 = 1/36, \Pr(\{3\}) = p_3 = 2/36,$
  $\Pr(\{4\}) = p_4 = 3/36, \Pr(\{5\}) = p_5 = 4/36,$
  $\Pr(\{6\}) = p_6 = 5/36, \Pr(\{7\}) = p_7 = 6/36,$
  $\Pr(\{8\}) = p_8 = 5/36, \Pr(\{9\}) = p_9 = 4/36,$
  $\Pr(\{10\}) = p_{10} = 3/36, \Pr(\{11\}) = p_{11} = 2/36,$
  $\Pr(\{12\}) = p_{12} = 1/36.$

- Let $A = \{5, 7, 8, 11\}$. Then
  $$\Pr(A) = p_5 + p_7 + p_8 + p_{11}$$
  $$= 4/36 + 6/36 + 5/36 + 2/36 = 17/36.$$

Example 2

A die is loaded in such a way that an even number is twice likely to occur as an odd number.

(a) If E is the event that a number is less than 4 on a single toss, find $\Pr(E)$.

(b) Let A = {even numbers} and

B = {numbers divisible by 3}.

What is $\Pr(A \cap B)$ and $\Pr(A \cup B)$?

Solution to Example 2

(a) Since $\Pr(S) = 1$, so $\Pr(1) + \Pr(2) + \Pr(3) + \Pr(4) + \Pr(5) + \Pr(6) = 1$. We also have and $\Pr(\text{even number}) = 2 \times \Pr(\text{odd number})$. Hence
$$p + 2p + p + 2p + p + 2p = 1.$$

Therefore,
$$p = 1/9.$$

$E = \{1, 2, 3\}$. So
$$\Pr(E) = \Pr(1) + \Pr(2) + \Pr(3)$$
$$= p + 2p + p = 1/9 + 2/9 + 1/9 = 4/9.$$

Solution to Example 2 (Continued)

(b) $A = \{2, 4, 6\}$ and $B = \{3, 6\}$.

Therefore $A \cap B = \{6\}$ and $A \cup B = \{2, 3, 4, 6\}$.

$\Pr(A \cap B) = \Pr(6) = 2p = 2/9$.

$\Pr(A \cup B) = \Pr(\{2, 3, 4, 6\})$
$$= \Pr(2) + \Pr(3) + \Pr(4) + \Pr(6)$$
$$= 2p + p + 2p + 2p = 7/9.$$

Alternatively,
$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$
$$= 6/9 + 3/9 - 2/9 = 7/9.$$

1.5.3 Sample Spaces Having Equally Likely Outcomes

- Consider an experiment whose sample space S is a finite set, say, $S = \{a_1, a_2, \cdots, a_k\}$.

- Assume that all outcomes in the sample space are **<u>equally likely</u>** to occur.

- That is

$$\Pr(a_1) = \Pr(a_2) = \cdots = \Pr(a_k).$$

- Since $\Pr(a_1) + \Pr(a_2) + \cdots + \Pr(a_k) = 1$ and $\Pr(a_i), i = 1, 2, \cdots, k$ are the same, therefore,

$$\Pr(a_i) = 1/k, \qquad i = 1, 2, \cdots, k.$$

Sample Spaces Having Equally Likely Outcomes

(Continued)

- Obviously, if an experiment can result in any one of the k different **equally likely** outcomes, then for any event A

$$\Pr(A) = \frac{\text{Number of sample points in } A}{\text{Number of sample points in } S}$$

Examples

1. For outcomes which are equally likely
    (a) Tossing a fair coin, then
    $$\text{Pr(Head)} = \text{Pr(Tail)} = 1/2.$$
    (b) Tossing a fair die,
    let event A = {numbers greater than 4} = {5, 6},
    then $\text{Pr}(A) = 2/6$.
2. For outcomes which are not equally likely
    Tossing a **bias coin** such that a Head is twice as likely as a Tail to occur, then
    $$\text{Pr(Head)} = 2/3 \text{ and Pr(Tail)} = 1/3.$$

Example 3

- A box contains 50 bolts and 150 nuts.
- Half of the bolts and half of the nuts are rusted.
- If one item is chosen at random, what is the probability that it is rusted or is a bolt?

Solution

- Let A = {the item is rusted} and B = {the item is a bolt}.
- Then

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$
$$= 100/200 + 50/200 - 25/200 = 5/8.$$

Example 4

- If two fair dice are rolled, what is the probability that the sum of the upturned faces will equal 8?

Solution

- Let A be the event that the sum of the upturned faces is 8.
- Then $A = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$.
- Since each outcome (•,•) are equally likely with probability 1/36, therefore

$$\Pr(A) = 5/36.$$

Example 5

- If 2 balls are "randomly drawn" from an urn containing 6 white and 5 black balls,

- what is the probability that one of the drawn balls is white and the other black?

Solution to Example 5

- Number of elements in the sample space (i.e. all possible outcomes) is given by $_{11}C_2 = 11(10)/2 = 55$.

- Number of sample points that have one white and one black ball is given by $_6C_1 \times {}_5C_1 = 6(5) = 30$.

- So, the probability that one of the drawn balls is white and the other black is given by

$$\frac{{}_6C_1 \times {}_5C_1}{{}_{11}C_2} = \frac{30}{55}.$$

Example 6

- In a poker hand consisting of 5 cards, find the probability of holding 2 aces and 3 jacks.

Solution

- Number of all possible outcomes of drawing 5 cards is given by $_{52}C_5 = 52!/(5!\,47!) = 2598960$.

Example 6 (continued)

<u>Solution</u> (Continued)

- Number of outcomes that have 2 aces and 3 jacks is given by

$$_4C_2 \times \ _4C_3 \times \ _{44}C_0 = \left(4!\,/(2!\,2!)\right) \times \left(4!\,/(3!\,1!)\right) \times 1 = 24.$$

- So, the probability that there are 2 aces and 3 jacks in a poker hand is given by

$$\frac{_4C_2 \times \ _4C_3 \times \ _{44}C_0}{_{52}C_5} = \frac{24}{2598960}.$$

Example 7

If 3 books are picked at random from a shelf containing 5 novels, 3 books of poems and a dictionary, what is the probability that

(a)   a dictionary is selected?

(b)   2 novels and 1 book of poems are selected?

Solution to Example 7

(a) Number of all possible selections of 3 books is given by

$$_9C_3 = 9!/(3!6!) = 84.$$

Number of ways selecting a dictionary and 2 other books is

$$_1C_1 \times {}_8C_2 = 1 \times \left(8!/(2!\,6!)\right) = 28.$$

The probability that a dictionary is selected is given by

$$\frac{\text{\# of ways selecting a dictionary and 2 other books}}{\text{\# of all possible selections of 3 books}}$$

$$= \frac{28}{84} = \frac{1}{3}.$$

Solution to Example 7 (Continued)

(b) Number of selecting 2 novels and 1 book of poem is

$$_5C_2 \times\ _3C_1 \times\ _1C_0 = \frac{5!}{(2!\ 3!)} \times \frac{3!}{2!\ 1!} \times \frac{1!}{1!\ 0!} = 10(3) = 30.$$

• The required probability is given by

$$\frac{30}{84} = \frac{5}{14}.$$

Example 8

- A professor hands out a list of 10 topics, 5 of which will appear in a test.

- One student has enough time to prepare only 7 of them.

- If the professor chooses the 5 topics at random, then what is the probability that the student will be prepared for

  (a) all five topics that appear in the test?

  (b) less than 3 topics?

  (c) exactly 4 topics?

Solution to Example 8

(a) Let S be the sample space and

A = {Prepared for all five topics in the test}.

Number of elements in $S = \#(S) = {}_{10}C_5 = 10!\,/(5!\,5!)$.

$\#(A) = {}_7C_5 \times {}_3C_0 = [7!\,/(5!\,2!)] \times [3!\,/(3!\,0!)]$.

$\Pr(A) = \#(A)/\#(S) = [7!\,/(5!\,2!)]/[10!\,/(5!\,5!)] = 1/12$.

(b) B = {Prepared for less than 3 topics in the test}.

$\#(B) = {}_7C_2 \times {}_3C_3 = [7!\,/(2!\,5!)] \times [3!/(3!\,0!)]$.

$\Pr(B) = \#(B)/\#(S) = [7!\,/(2!\,5!)]/[10!\,/(5!\,5!)] = 1/12$.

(c) C = {Prepared for exactly four topics in the test}.

$$\#(C) = {}_7C_4 \times {}_3C_1 = \frac{7!}{4!\,3!} \times \frac{3!}{2!\,1!}.$$

$$\Pr(C) = \frac{\#(C)}{\#(S)} = \frac{\frac{7!}{4!\,3!} \times \frac{3!}{2!\,1!}}{\frac{10!}{5!\,5!}} = \frac{5}{12}.$$

Example 9

- In a class of 200 students, 108 study economics, 138 study chemistry and 70 study both chemistry and economics.

- If a student is selected at random, what is the probability that the student

  (a) takes economics **or** chemistry;

  (b) doesn't take **neither** of these subjects;

  (c) takes chemistry **but not** economics.

Solution to Example 9

- Let E = {The student takes economics} and
  C = {The student takes chemistry}.

- From the info given, we have $\Pr(E) = 108/200$,
  $\Pr(C) = 138/200$ and $\Pr(E \cap C) = 70/200$.

(a) $\Pr(E \cup C) = \Pr(E) + \Pr(C) - \Pr(E \cap C)$
$$= 108/200 + 138/200 - 70/200 = 0.88.$$

(b) $\Pr\big((E \cup C)'\big) = 1 - \Pr(E \cup C) = 1 - 0.88 = 0.12.$

Solution to Example 9 (Continued)

(c) Since $C = (E' \cap C) \cup (E \cap C)$ and $(E' \cap C) \cap (E \cap C) = \emptyset$, therefore

$$\Pr(C) = \Pr(E' \cap C) + \Pr(E \cap C).$$

Hence $\Pr(E' \cap C) = \Pr(C) - \Pr(E \cap C)$
$$= 138/200 - 70/200$$
$$= 0.34.$$

1.6 Conditional probability

1.6.1 Introduction

- We sometimes encounter in calculating probabilities of events when some **partial information** concerning the result of the experiment is available; in such a situation the desired probabilities are **conditional ones**.

- Let A and B be two events associated with an experiment E. We denote

$$\Pr(A|B)$$

  the **conditional probability** of the event A, given that event B has occurred.

An Illustrative Example

- Two fair dice are tossed, the outcome being recorded as (x_1, x_2), where x_i is the outcome of the ith die, $i = 1, 2$.

- Hence the sample space is
$$S = \{(1, 1), (1, 2), (1, 3), \cdots\cdots, (6, 5), (6, 6)\}.$$

- Let $A = \{(x_1, x_2) \mid x_1 + x_2 = 10\} = \{(5, 5), (4, 6), (6, 4)\}$

- $B = \{(x_1, x_2) \mid x_1 > x_2\} = \{(2, 1), (3, 1), (3, 2), \cdots, (6, 4), (6, 5)\}$

- $\#(S) = 36, \#(A) = 3, \#(B) = 15.$

- $\Pr(A) = 3/36 = 1/12, \Pr(B) = 15/36 = 5/12.$

An Illustrative Example (Continued)

- Suppose that we know the outcome satisfies $x_1 > x_2$.
- What is the probability that $x_1 + x_2 = 10$?

  That is, knowing that event B has occurred, what is the chance that event A occurs? (i.e. $\Pr(A|B) = ?$)

- Since the event B has occurred, instead of considering the set of all possible outcomes (i.e. S), we consider the set of sample points for event B.

- Among the sample points in B, figure out those that favor A.

- That is, find out the elements in $A \cap B$.

An Illustrative Example (Continued)

- Hence the conditional probability of A given B can be obtained by

$$\Pr(A|B) = \frac{\#(A \cap B)}{\#(B)}$$

- If we divide both the numerator and denominator by $\#(S)$, then

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

- Similarly,

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}.$$

An Illustrative Example (Continued)

- Since $A \cap B = \{(6, 4)\}$, hence $\#(A \cap B) = 1$.

- Therefore

$$\Pr(A|B) = \frac{\#(A \cap B)}{\#(B)} = \frac{1}{15}.$$

- Similarly,

$$\Pr(B|A) = \frac{\#(A \cap B)}{\#(A)} = \frac{1}{3}.$$

1.6.2 Conditional Probability

Definition

- The **conditional probability of B given A**, is defined as

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}, \qquad \text{if } \Pr(A) \neq 0.$$

- Intuitively $\Pr(B|A)$ means that probability of the occurrence of the event B, under the assumption that event A has already occurred.

Remarks

Note: For fixed A, $\Pr(B|A)$ satisfies the various postulates of probability.

- That is, we have

  1. $0 \leq \Pr(B|A) \leq 1$.

  2. $\Pr(S|A) = 1$.

  3. If $B_1, B_2, \cdots$ are **mutually exclusive** (disjoint) events (that is, $B_i \cap B_j = \emptyset$ when $i \neq j$), then
     $$\Pr(\cup_{i=1}^{\infty} B_i \,|A) = \sum_{i=1}^{\infty} \Pr(B_i|A).$$
     In particular, if B_1 and B_2 are disjoint events, then
     $$\Pr(B_1 \cup B_2|A) = \Pr(B_1|A) + \Pr(B_2|A).$$

Example 1

- Roll 2 balanced dice.

- Suppose that the first die is a 3.

- What is the probability that the sum of the 2 dice equals 8?

Solution to Example 1

- Let $A = \{(3, y) : 1 \leq y \leq 6\}$, which is the event that the first die is a 3.

- Let $B = \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$, the event that the sum of the 2 dice equals 8.

- Hence, $A \cap B = \{(3, 5)\}$.

- We would like to find the **conditional** probability $\Pr(B|A)$, which is given by

$$\frac{\Pr(A \cap B)}{\Pr(A)} = \frac{1/36}{6/36} = \frac{1}{6}.$$

Example 2

- Roll an unbalanced die.

  (The sample space is $S = \{1, 2, 3, 4, 5, 6\}$.)

- Suppose that the respective probabilities are 1/12, 1/12, 1/6, 1/6, 1/6, and 1/3.

  Note: The sum of the probabilities equals 1

  (a) If the number obtained is even, what is the probability that it is a 6?

  (b) What is the probability that the number obtained is a <u>perfect square number</u> given that a number greater than 3 has obtained?

Solution to Example 2

(a) Let $A = \{2, 4, 6\}$, the event that the number obtained is even.
$$\Pr(A) = 1/12 + 1/6 + 1/3 = 7/12$$

$B = \{6\}$ and $A \cap B = \{6\}.$

$$\Pr(A \cap B) = \Pr(\{6\}) = 1/3.$$

Hence,

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{1/3}{7/12} = \frac{4}{7}.$$

Solution to Example 2 (Continued)

(b) Let C = {Number > 3} = {4, 5, 6} and

D = {A perfect square number} = {1, 4}.

Hence, $C \cap D$ = {4} and

$$\Pr(C) = \Pr(\{4\}) + \Pr(\{5\}) + \Pr(\{6\})$$
$$= 1/6 + 1/6 + 1/3 = 2/3$$

$\Pr(C \cap D) = \Pr(\{4\}) = 1/6.$

Hence

$$\Pr(D|C) = \frac{\Pr(C \cap D)}{\Pr(C)} = \frac{1/6}{2/3} = \frac{1}{4}.$$

Example 3

- A couple has 2 children.

- What is the probability that both are boys if it is known that they have at least one son?

Solution to Example 3

- Let us, say, assume it is **equally likely** to be a boy or a girl for every new born baby.

- Put $E = \{\text{Both are boys}\}$ and
  $F = \{\text{At least one son}\}$.

- Then $E = \{(B, B)\}$, $F = \{(B, G), (G, B), (B, B)\}$
  and $E \cap F = E$.

- Hence

$$\Pr(E|F) = \frac{\#(E \cap F)}{\#(F)} = \frac{1}{3}.$$

Example 4

- The probability that a regularly scheduled flight departs on time is $0.83 \ (= \Pr(D))$;

- the probability that it arrives on time is $0.82 \ (= \Pr(A))$; and

- the probability that it departs and arrives on time is $0.78 \ (= \Pr(D \cap A))$.

- Find the probability that a plane arrives on time given that it departed on time.

Solution

$$\Pr(A|D) = \frac{\Pr(D \cap A)}{\Pr(D)} = \frac{0.78}{0.83} = 0.9398.$$

Example 5

- The following data were obtained in a study

	Non-smokers	Moderate smokers	Heavy smokers	Row Total
Hypertension	21	36	30	87
No hyper-tension	48	26	19	93
Column total	69	62	49	180

Example 5 (Continued)

If one of these individuals is selected at random, find the probability that the person is

(a) experiencing hypertension;

(b) experiencing hypertension given that the person is a heavy smoker;

(c) a non-smoker, given that the person is experiencing no hypertension.

Solution to Example 5

(a) Let H and H' denote the events that the selected person is experiencing hypertension and no hypertension, respectively.

Out of 180 persons, there are 87 of them with hypertension. Therefore

$$\Pr(H) = 87/180$$

Out of 180 persons, there are 93 of them with no hypertension. Hence

$$\Pr(H') = 93/180.$$

Alternatively, $\Pr(H') = 1 - \Pr(H) = 1 - \dfrac{87}{180} = \dfrac{93}{180}.$

Solution to Example 5 (Continued)

(b) Let A denote the event that the selected person is a heavy smoker.

There are 49 heavy smokers among 180 persons. That is,
$$\Pr(A) = 49/180.$$

Also there are 30 heavy smokers with hypertension among the 180 persons. Hence,
$$\Pr(H \cap A) = 30/180.$$

Hence the required conditional probability is given by
$$\Pr(H|A) = \frac{\Pr(H \cap A)}{\Pr(A)} = \frac{(30/180)}{49/180} = \frac{30}{49}.$$

Solution to Example 5 (Continued)

(c) Let B denote the event that the selected person is a non-smoker.

There are 48 non-smokers with no hypertension among the 180 persons

$$\Pr(B \cap H') = 48/180$$

From part (a), we have $\Pr(H') = 93/180$

Therefore

$$\Pr(B|H') = \frac{\Pr(B \cap H')}{\Pr(H')} = \frac{48/180}{93/180} = \frac{48}{93} = \frac{16}{31}.$$

Example 6

- For married couples living in a certain district, the probability that the husband will vote in a referendum is 0.21,

- the probability that his wife will vote is 0.28, and

- the probability that both will vote is 0.15.

- What is the probability that

  (a) at least one member of a married couple will vote?

  (b) a wife will vote, given that her husband votes?

  (c) a husband will vote, given that his wife does not vote?

Solution to Example 6

- Let H represent the event that the husband votes and W represent the event that the wife votes.

- It is given that $\Pr(H) = 0.21$, $\Pr(W) = 0.28$, and $\Pr(H \cap W) = 0.15$.

  (a) $\Pr(H \cup W) = \Pr(H) + \Pr(W) - \Pr(H \cap W)$
  $$= 0.21 + 0.28 - 0.15 = 0.34.$$

  (b) $\Pr(W|H) = \Pr(H \cap W) / \Pr(H)$
  $$= 0.15/0.21 = 5/7.$$

Solution to Example 6 (continued)

(c) $\Pr(H|W') = \Pr(H \cap W')/\Pr(W')$

$\quad\quad = [\Pr(H) - \Pr(H \cap W)]/(1 - \Pr(W))$

$\quad\quad = (0.21 - 0.15)/(1 - 0.28) = 1/12.$

- Notice that $H = (H \cap W) \cup (H \cap W')$ and $(H \cap W) \cap (H \cap W') = \emptyset$.

- Hence $\Pr(H) = \Pr(H \cap W) + \Pr(H \cap W')$ or $\Pr(H \cap W') = \Pr(H) - \Pr(H \cap W)$.

Example 7

- A statistics professor is teaching both a morning and an afternoon sections of introductory statistics.

- Let A = {the professor gives a bad morning lecture}

  and B = {the professor gives a bad afternoon lecture}.

- Suppose that

$$\Pr(A) = 0.3, \Pr(B) = 0.2 \text{ and } \Pr(A \cap B) = 0.1,$$

Example 7 (Continued)

Calculate the following probabilities.

(a) $\Pr(B|A)$.

(b) $\Pr(B'|A)$.

(c) $\Pr(B|A')$.

(d) $\Pr(B'|A')$.

(e) If at the conclusion of the afternoon class, the professor is heard to mutter "what a rotten lecture", what is the probability that the morning lecture was also bad?

Solution to Example 7

(a) $\Pr(B|A) = \Pr(A \cap B) / \Pr(A) = 0.1/0.3 = 1/3.$

(b) $\Pr(B'|A) = 1 - \Pr(B \mid A) = 1 - 1/3 = 2/3.$

(c) $\Pr(B|A') = \Pr(A' \cap B) / \Pr(A')$

$\qquad\qquad = [\Pr(B) - \Pr(B \cap A)]/(1 - \Pr(A))$

$\qquad\quad = (0.2 - 0.1)/(1 - 0.3) = 1/7.$

(d) $\Pr(B'|A') = 1 - \Pr(B|A') = 1 - 1/7 = 6/7.$

(e) $\Pr(A \mid B) = \Pr(A \cap B)/\Pr(B) = 0.1/0.2 = 1/2.$

1.6.3 Multiplication Rule of Probability

$$\Pr(A \cap B) = \Pr(A)\Pr(B|A) \quad \text{or}$$

$$\Pr(A \cap B) = \Pr(B)\Pr(A|B)\,,$$

providing $\Pr(A) > 0, \Pr(B) > 0$.

- This rule enables us to calculate the probability that two events will both occur.

- The probability that both events occur is the product of the probability of one event occurs and the conditional probability that the other event occurs given that the first event has occurred.

Multiplication Rule of Probability (Continued)

- It can be extended to more than 2 events:
$$\Pr(A \cap B \cap C) = \Pr(A)\Pr(B|A)\Pr(C|A \cap B),$$
providing that $\Pr(A \cap B) > 0$.

- In general
$$\Pr(A_1 \cap \cdots \cap A_n) = \Pr(A_1)\Pr(A_2 \mid A_1)$$
$$\Pr(A_3 \mid A_1 \cap A_2) \cdots \Pr(A_n \mid A_1 \cap \cdots \cap A_{n-1}),$$
providing that $\Pr(A_1 \cap \cdots \cap A_{n-1}) > 0$.

Example 1

- Suppose that among 12 shirts, 3 are white.

- Two shirts are chosen randomly one by one **without replacement**.

  (a) What is the probability that **both shirts** that being picked are white?

  (b) What is the probability that there is **only one** white shirt being picked?

  (c) If 3 shirts are chosen at random, what is the probability that they are **all** white?

Solution to Example 1

Let A_1 = {the first shirt is white} and

A_2 = {the second shirt is white}.

$$\Pr(A_1) = 3/12$$

If the first shirt is white, then there are 2 white shirts among the remaining 11 shirts.

$$\Pr(A_2|A_1) = 2/11$$

(a) $\Pr(A_1 \cap A_2) = \Pr(A_1) \Pr(A_2|A_1)$

$$= \left(\frac{3}{12}\right)\left(\frac{2}{11}\right) = \frac{1}{22}.$$

Solution to Example 1 (Continued)

(b) We have $\Pr(A_1) = 3/12$ and $\Pr(A_1') = 1 - 3/12 = 9/12$.
$\Pr(A_2|A_1) = 2/11$ and $\Pr(A_2'|A_1) = 1 - 2/11 = 9/11$.
If the first shirt is not white, then there are 3 white shirts among the remaining 11 shirts. Hence
$$\Pr(A_2|A_1') = 3/11.$$

$$\Pr((A_1 \cap A_2') \cup (A_1' \cap A_2))$$
$$= \Pr(A_1 \cap A_2') + \Pr(A_1' \cap A_2)$$
$$= \Pr(A_1)\Pr(A_2'|A_1) + \Pr(A_1')\Pr(A_2|A_1')$$
$$= \left(\frac{3}{12}\right)\left(\frac{9}{11}\right) + \left(\frac{9}{12}\right)\left(\frac{3}{11}\right) = \frac{9}{22}.$$

Solution to Example 1 (Continued)

(c) Let $A_1 = \{(W, *, *)\}$,

$\quad\quad A_2 = \{(*, W, *)\}$ and

$\quad\quad A_3 = \{(*, *, W)\}$

$$\Pr(A_1 \cap A_2 \cap A_3) = \Pr(A_1) \Pr(A_2|A_1) \Pr(A_3|A_1 \cap A_2)$$
$$= \left(\frac{3}{12}\right)\left(\frac{2}{11}\right)\left(\frac{1}{10}\right) = \frac{1}{220}.$$

Example 2

- If 2 balls are "randomly drawn" **<u>without replacement</u>** from an urn containing 6 white and 5 black balls,

- what is the probability that the first drawn ball is black and the second one is white?

Solution to Example 2

- Let B_1 represent the event that a black ball is drawn in the first draw and

- W_2 represent the event that a white ball is drawn in the second draw.

$$\Pr(B_1 \cap W_2) = \Pr(B_1)\Pr(W_2|B_1)$$
$$= \left(\frac{5}{11}\right)\left(\frac{6}{10}\right) = \frac{3}{11}.$$

Example 3

- Two cards are drawn in succession from an ordinary deck of 52 playing cards **without replacement**.

- What is the probability that both cards are greater than 3 but less than 8?

Solution to Example 3

- There are 16 cards that are greater than 3 but less than 8. (i.e. There are 16 cards with numbers 4, 5, 6 and 7.)

- Let A_i represent the event the ith card drawn is greater than 3 but less than 8.

$$\Pr(A_1 \cap A_2) = \Pr(A_1)\Pr(A_2|A_1)$$
$$= \left(\frac{16}{52}\right)\left(\frac{15}{51}\right) = \frac{20}{221}.$$

Example 4

- The probability that a doctor correctly diagnoses a particular illness is 0.7.

- Given that the doctor makes an incorrect diagnosis, the patient will enter a law suit is 0.9.

- What is the probability that the doctor makes an incorrect diagnosis and the patient sues?

Solution to Example 4

- Let D = {Correct diagnosis} and U = {Patient sues}.

- It is given that $\Pr(D) = 0.7$ and $\Pr(U|D') = 0.9$

$$\Pr(D' \cap U) = \Pr(D')\,\Pr(U|D')$$
$$= (1 - 0.7)(0.9)$$
$$= 0.27.$$

Example 5

- Four individuals have responded to a request by a blood bank for blood donations.

- All of them forget their blood types.

- Suppose only type A+ is desired and only one of the four actually has this type.

- If the potential donors are selected at random order for typing, what is the probability that

  (a) at least three individuals must be typed to obtain the desired type?

  (b) the third donor's blood type is A+?

Solution to Example 5

(a) Let A = {First donor not A+} and

$\qquad$ B = {Second donor not A+}.

$\qquad$ $\Pr(A) = 3/4$ and $\Pr(B|A) = 2/3$. Hence

$\qquad$ {At least 3 individuals are typed } $\equiv$ {the first 2 donors are not A+ type}

$\qquad$ $\Pr$(at least 3 individuals are typed) = $\Pr(A \cap B)$

$$= \Pr(A)\,\Pr(B|A) = \left(\frac{3}{4}\right)\left(\frac{2}{3}\right) = \frac{1}{2}.$$

(b) Pr(Third donor is A+) = Pr(First donor isn't A+) $\times$

   Pr(Second donor isn't A+ | First donor isn't A+) $\times$

   Pr(Third donor is A+ | First 2 donors are not A+)

$$= \left(\frac{3}{4}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{4}.$$

1.6.4 The Law of Total Probability

- Let $A_1, A_2, \cdots, A_n$ be a **partition** of the sample space S.

- That is $A_1, A_2, \cdots, A_n$ are **mutually exclusive and exhaustive events** such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{n} A_i = S$.

- Then for any event B

$$\Pr(B) = \sum_{i=1}^{n} \Pr(B \cap A_i) = \sum_{i=1}^{n} \Pr(A_i)\Pr(B|A_i)$$

The Law of Total Probability (Continued)
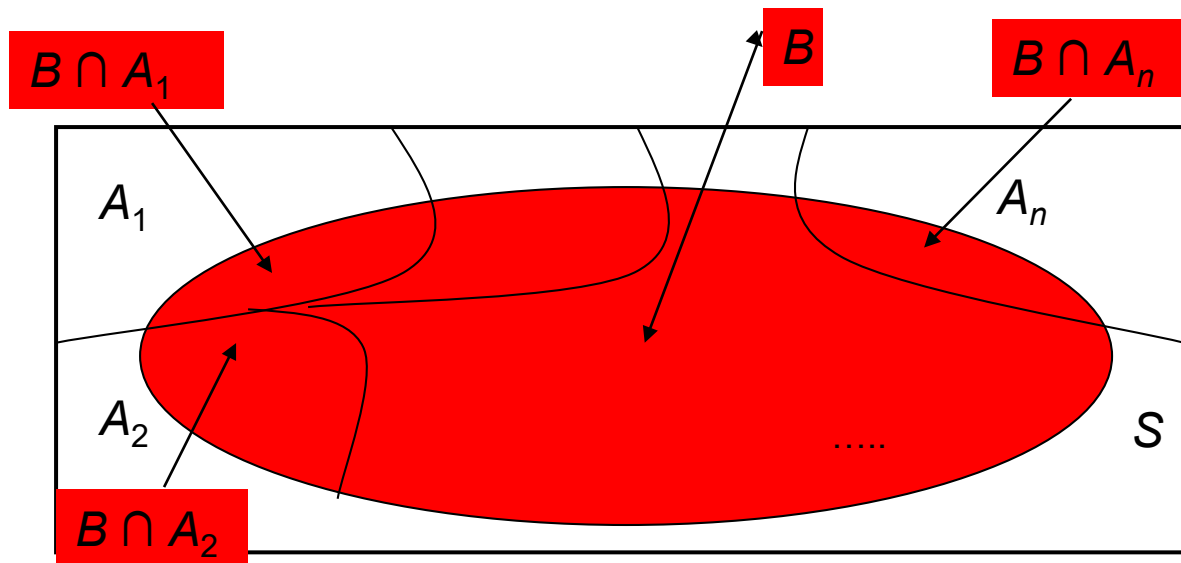
Example 1

- In a certain factory, Machines A, B, and C produce 2%, 1% and 3% defective screws, respectively.

- Of the total production, Machines A, B, and C produce 35%, 25% and 40% respectively.

- If a screw is selected randomly,

  (a) what is the probability that it is a defective?

  (b) what is the probability that it is produced by the Machine C given that it is defective?

Solution to Example 1

Let D be the event that a defective screw is selected.

Let A, B and C be the event that the selected screw come from Machine A, B and C respectively.

A, B and C are mutually exclusive and exhaustive events

(a) $\Pr(D) = \Pr(A)\Pr(D|A) + \Pr(B)\Pr(D|B) + \Pr(C)\Pr(D|C)$
$= (0.35)(0.02) + (0.25)(0.01) + (0.4)(0.03) = 0.0215.$

(b) $\Pr(C|D) = \Pr(C \cap D)/\Pr(D)$
$= \Pr(C)\Pr(D|C)/\Pr(D)$
$= (0.4)(0.03)/0.0215 = 0.5581.$

Example 2

- One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls.

- One ball is drawn from the first bag and <u>placed unseen in the second bag</u>.

- What is the probability that a ball now drawn from the second bag is black?

Solution to Example 2

- Let A_i = {a white ball is drawn from the i-th bag} and
  A_i' = {a black ball is drawn from the i-th bag}.

- $\Pr(A_1) = 4/7$ and
- $\Pr(A_1') = 1 - 4/7 = 3/7$

Solution to Example 2 (Continued)

- There are 4 white balls among 9 balls in the 2nd bag after a white ball was drawn from the 1st bag and put in the 2nd bag. Hence

$$\Pr(A_2|A_1) = 4/9 \text{ or } \Pr(A_2'|A_1) = 5/9.$$

- Similarly, $\Pr(A_2'|A_1') = 6/9.$

- Use the Law of Total Probability.

$$\Pr(A_2') = \Pr(A_1)\Pr(A_2'|A_1) + \Pr(A_1')\Pr(A_2'|A_1')$$

$$= \left(\frac{4}{7}\right)\left(\frac{5}{9}\right) + \left(\frac{3}{7}\right)\left(\frac{6}{9}\right) = \frac{38}{63}.$$

Example 3

- Traffic police plan to enforce speed limits by using speed cameras at 4 different locations of the expressway.

- The speed cameras at each of the locations L_1, L_2, L_3 and L_4 are operated 40%, 30%, 20% and 30% of the time respectively.

- If a driver who is speeding on his way to work has probabilities 0.2, 0.1, 0.5 and 0.2 respectively, of passing through these locations, what is the probability that

  (a) he will receive a speeding ticket?

  (b) he passed through the radar trap at location L_2 if he received a speeding ticket?

Solution to Example 3

Let B = {The driver receives a speeding ticket}

A_i = {He past through the location L_i} for $i = 1, \cdots, 4$

(a) $\Pr(B) = \Pr(A_1)\Pr(B|A_1) + \Pr(A_2)\Pr(B|A_2) +$

$\Pr(A_3)\Pr(B|A_3) + \Pr(A_4)\Pr(B|A_4)$

$= 0.2(0.4) + 0.1(0.3) + 0.5(0.2) + 0.2(0.3) = 0.27.$

(b) $\Pr(A_2|B) = \Pr(A_2 \cap B)/\Pr(B)$

$= \Pr(A_2)\Pr(B|A_2)/\Pr(B)$

$= [0.1(0.3)]/0.27 = 1/9.$

Example 4

- A bag contains one 50-cent, one 20-cent and one 10-cent coins.

- The 10-cent coin is a fake coin with two heads.

- A coin is chosen at random from the bag and tossed four times in succession.

- If the result is 4 heads, what is the probability that the fake 10-cent coin is used?

Solution to Example 4

- Let A = {The 10-cent coin is chosen},

  B = {The 20-cent coin is chosen},

  C = {The 50-cent coin is chosen},

  $4H$ = {Having 4 heads in 4 tosses}.

- If the 10-cent coin is chosen, $\Pr(H) = 1$ and

  $\Pr(4H) = 1^4 = 1$.

- That is, $\Pr(4H|A) = 1^4 = 1$.

Solution to Example 4 (Continued)

- If the 20-cent soin is chosen, then $\Pr(H) = 1/2$ and
  $\Pr(4H) = (1/2)^4 = 1/16.$
  That is $\Pr(4H|B) = 1/16$.

- Similarly, $\Pr(4H|C) = 1/16$.

- A coin is chosen at random implies that
  $$\Pr(A) = \Pr(B) = \Pr(C) = 1/3$$

Solution to Example 4 (Continued)

- Applying the Law of Total Probability, we have

$$\Pr(4H) = \Pr(A) \Pr(4H|A) + \Pr(B) \Pr(4H|B)$$
$$+ \Pr(C) \Pr(4H|C)$$
$$= (1/3)(1 + 1/16 + 1/16) = 3/8.$$

$$\textcolor{red}{\Pr(A|4H) = \Pr(A \cap 4H)/\Pr(4H)}$$
$$= \Pr(A) \Pr(4H|A)/\Pr(4H)$$
$$= [(1/3)(1)]/[3/8] = \textcolor{red}{8/9}.$$

1.6.5 Bayes' Theorem

<u>Bayes' Theorem</u>

- Let $A_1, A_2, \cdots, A_n$ be **a partition** of the sample space S. Then

$$\Pr(A_k \mid B) = \frac{\Pr(A_k)\Pr(B \mid A_k)}{\sum_{i=1}^{n}\Pr(A_i)\Pr(B \mid A_i)}$$

    for $k = 1, \cdots, n$.

Note : The denominator is just $\Pr(B)$. That is,

$$\Pr(A_k \mid B) = \frac{\Pr(A_k)\Pr(B \mid A_k)}{\Pr(B)}$$

Example 1

- Suppose that there is a chance for a newly constructed house to collapse whether the design is faulty or not.

- The chance that the design is faulty is 1%.

- The chance that the house collapses if the design is faulty is 75% and otherwise it is 0.01%.

- It is seen that the house collapsed.

- What is the probability that it is due to faulty design?

Solution to Example 1

- Let B = {The design is faulty},

  A = {The house collapses}.

- Hence

$$\Pr(B) = 0.01,$$
$$\Pr(A|B) = 0.75,$$

and

$$\Pr(A|B') = 0.0001$$

Solution to Example 1 (Continued)

- Applying the Law of Total Probability, we have

$$\Pr(A) = \Pr(A|B)\Pr(B) + \Pr(A|B')\Pr(B')$$
$$= 0.75(0.01) + (0.0001)0.99 = 0.007599$$

Therefore,

$$\Pr(B|A) = \frac{\Pr(A|B)\Pr(B)}{\Pr(A)}$$

$$= \frac{0.75(0.01)}{0.007599} = 0.9870.$$

Example 2

- An insurance company believes that people can be divided into two classes — those who are accident prone and those who are not.

- Their statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability 0.04, whereas this probability decreases to 0.02 for a non-accident-prone person.

- Assume that 30% of the population is accident prone.

Example 2 (Continued)

(a) What is the probability that a new policyholder will have an accident within a year of purchasing a policy?

(b) If a new policyholder has an accident within a year of purchasing a policy, what is the probability that he or she is accident prone?

Solution to Example 2

- Let B be the event that a new policy holder has an accident within a year of purchasing a policy.

- Let A_1 and A_1' be the events that a (randomly selected) new policy holder is accident prone and non-accident prone respectively.

- It is given that $\Pr(A_1) = 0.3$, and hence $\Pr(A_1') = 0.7$.

- Also, $\Pr(B|A_1) = 0.04$, and $\Pr(B|A_1') = 0.02$.

Solution to Example 2 (Continued)

- Using the Law of Total Probability, we obtain
$$\Pr(B) = \Pr(A_1)\Pr(B|A_1) + \Pr(A_1')\Pr(B|A_1')$$
$$= 0.3(0.04) + 0.7(0.02) = 0.26.$$

- Using the Bayes' Rule, we have
$$\Pr(A_1|B) = \frac{\Pr(A_1)\Pr(B|A_1)}{\Pr(A_1)\Pr(B|A_1) + \Pr(A_1')\Pr(B|A_1')}$$
$$= \frac{0.3(0.04)}{0.3(0.04) + 0.7(0.02)} = \frac{6}{13}.$$

Example 3

- Suppose we have 3 cards identical in form except that both sides of the first card are colored red, both sides of the second card are colored black, and one side of the third card is colored red and the other side black.

- The 3 cards are mixed up in a box, and 1 card is randomly selected and put down on the table.

- If the upper side of the chosen card is colored red,

- what is the probability that the other side is colored black?

Solution to Example 3

- Let A_i denote the event that the card i is selected for $i = 1, 2, 3$ and

- let B denote the event that upper side of the chosen card is colored red.

- From the given info, we have,

$$\Pr(A_1) = \Pr(A_2) = \Pr(A_3) = \frac{1}{3},$$

$$\Pr(B|A_1) = 1, \Pr(B|A_2) = 0, \text{ and } \Pr(B|A_3) = \frac{1}{2}.$$

Solution to Example 3 (Continued)

- The desired probability is the conditional probability of the card 3 being selected, given that B occurs.

- That is, we are to evaluate $\Pr(A_3|B)$.

- Applying the Bayes' rule, we have

$$\Pr(A_3\,|\,B) = \frac{\Pr(A_3)\Pr(B\,|\,A_3)}{\sum_{i=1}^{3}\Pr(A_i)\Pr(B\,|\,A_i)}$$

$$= \frac{(1/3)(1/2)}{1/3\,(1 + 0 + 1/2)} = \frac{1}{3}$$

Example 4

- Only 1 in 1000 adults is afflicted with a rare disease for which a diagnostic test has been developed.

- The test is such that, when an individual actually has the disease, a positive result will occur 99% of the time,

- while an individual without the disease will show a positive test only 2% of the time.

- If a randomly selected individual is tested and the result is **positive**, what is the probability that the individual has the disease?
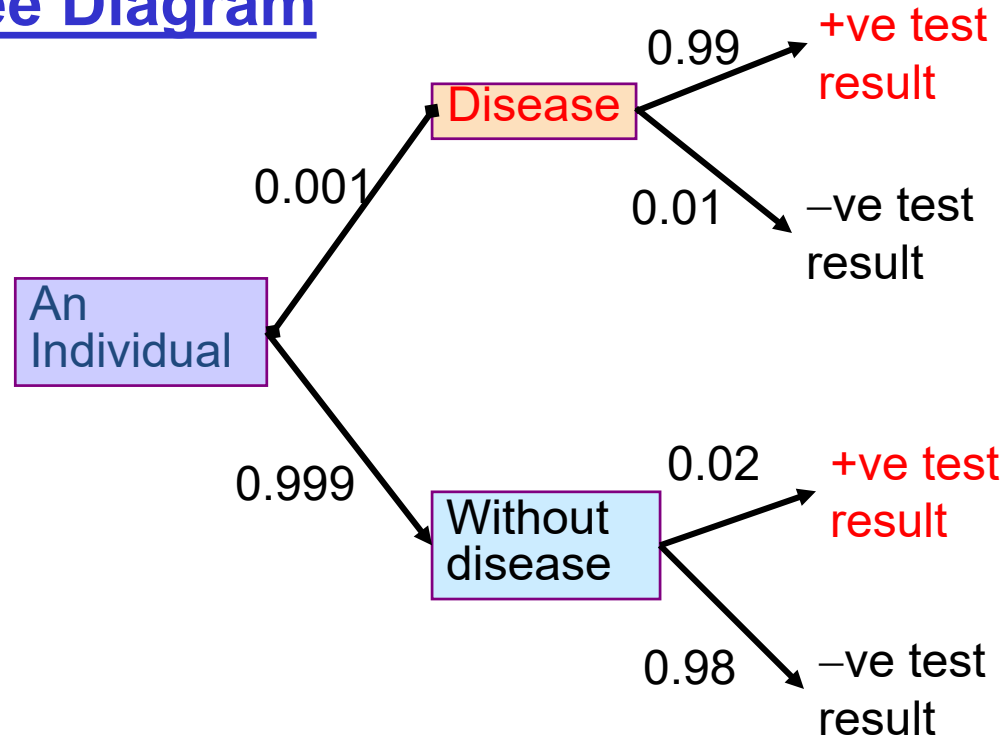
Solution to Example 4

- Let A_1 = {An individual has the disease},

  A_1' = {An individual doesn't have the disease},

  B = {positive test result}

- Using the Bayes' Theorem, we have

$$\Pr(A_1|B) = \frac{\Pr(A_1)\Pr(B|A_1)}{\Pr(A_1)\Pr(B|A_1) + \Pr(A_1')\Pr(B|A_1')}$$

$$= \frac{0.001(0.99)}{0.001(0.99) + 0.999(0.02)}$$

$$= \frac{0.00099}{0.00099 + 0.01998} = 0.0472.$$

Solution to Example 4 (Continued)

Tree Diagram

Pr (+ve test result and with disease) = 0.001(0.99) = 0.00099

Pr (+) = 0.00099 + 0.01998 = 0.02097

Pr (+ve test result and without disease) = 0.999(0.02) = 0.01998

Pr (D|+) = 0.00099/(0.00099 + 0.01998) = 0.047

Example 5

- A student answers a multiple-choice examination question that offers four possible answers.

- Suppose that the probability that the student knows the answer to the question is 0.8 and

- the probability that the student will guess is 0.2.

- Assume that if the student guesses, the probability of selecting the correct answer is 0.25.

- If the student correctly answers a question, what is the probability that the student really knew the correct answer?

Solution to Example 5

- Let A_1 = {A student knows the answer},

   A_1' = {A student doesn't know the answer},

   B = {A student answers correctly}

- Using the Bayes' Theorem, we have

$$\Pr(A_1 \mid B) = \frac{\Pr(A_1)\Pr(B|A_1)}{\Pr(A_1)\Pr(B|A_1) + \Pr(A_1')\Pr(B|A_1')}$$

$$= \frac{0.8(1)}{0.8(1) + 0.2(0.25)} = 0.9412.$$

Example 6

- Components of a certain type are shipped to a supplier in batches of ten.

- Suppose that 50% of all such batches contain no defective components,

  30% contain one defective component, and

  20% contain two defective components.

- Two components from a batch are randomly selected and tested.

- What are the probabilities associated with 0, 1 and 2 defective components being in the batch if neither tested component is defective?

Solution to Example 6

- Let $A_i = \{i \text{ defectives in batch}\}$, for $i = 0, 1, 2$ and

  $D_i = \{i \text{ defectives in sample}\}$ for $i = 0, 1, 2$.

- From the given info, we have

  $\Pr(A_0) = 0.5, \Pr(A_1) = 0.3, \Pr(A_2) = 0.2$.

  The number of ways to select 2 items out of a batch of 10 items $= {}_{10}C_2 = 45$.

- Given A_0, the number of ways to select 2 nondefective items $= {}_{10}C_2 = 45$.

Solution to Example 6 (Continued)

- Given A_1, the no. of ways to select 2 nondefective items $= {_9}C_2 \times {_1}C_0$.

- Given A_2, the no. of ways to select 2 nondefective items $= {_8}C_2 \times {_2}C_0$.

- Therefore $\Pr(D_0|A_0) = {_{10}}C_2 / {_{10}}C_2 = 1$.

- Similarly, $\Pr(D_0|A_1) = ({_9}C_2 \times {_1}C_0) / {_{10}}C_2 = 0.8$ and $\Pr(D_0|A_2) = ({_8}C_2 \times {_2}C_0) / {_{10}}C_2 = 0.6222$.

  Applying the Law of Total Probability, we have
  $\Pr(D_0) = \Pr(A_0)\Pr(D_0|A_0) + \Pr(A_1)\Pr(D_0|A_1) + \Pr(A_2)\Pr(D_0|A_2)$
  $= 0.5(1) + 0.3(0.8) + 0.2(0.6222) = 0.86444.$

Solution to Example 6 (continued)

- Applying the Bayes' Theorem, we have

$$\Pr(A_0|D_0) = \frac{\Pr(A_0)\,\Pr(D_0|A_0)}{\Pr(D_0)} = \frac{0.5(1)}{0.86444} = 0.5784.$$

Similarly,

$$\Pr(A_1|D_0) = \frac{0.3(0.8)}{0.86444} = 0.2776.$$

$$\Pr(A_2|D_0) = \frac{0.2(0.622)}{0.86444} = 0.1440.$$

Example 7

(The Monty Hall Problem)

- Suppose you are on a game show, and you are given the choice of three doors: behind one door is a car; behind the others, goats.

- You pick a door, say No. 1, and the host Monty, who knows what is behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?"

- Is it to your advantage to switch your choice?

Solution to Example 7

Let's look at the probabilities of winning for the two strategies.

(a) **Stick strategy**

I choose a door with the car initially with probability 1/3. Monty will always open a door with a goat, so if I don't switch, I'll win.

I choose a door with a goat initially with probability 2/3. Monty will always open a door with a goat, so if I don't switch, I'll lose.

That is, the probability of winning is $\left(\frac{1}{3}\right) 1 + \frac{2}{3}(0) = \frac{1}{3}$.

Solution to Example 7 (Continued)

(b) Switch strategy

I choose a door with the car initially with probability 1/3. Monty will always open a door with a goat, so if I switch, I'll lose.

I choose a door with the goat initially with probability 2/3. Monty will always open a door with a goat, so if I switch, I'll win.

That is, the probability of winning is $\frac{1}{3}(0) + \frac{2}{3}(1) = \frac{2}{3}$.

Therefore, the "switch" strategy will bear a better chance of winning.

Remark on Monty Hall Problem

Still confused? Watch the following videos:

http://www.youtube.com/watch?v=mhlc7peGlGg

http://www.youtube.com/watch?v=P9WFKmLK0dc

1.7 Independent Events

1.7.1 Introduction

- Suppose that a fair die is tossed twice.

- Define the events A and B as follows:

  A = {the first toss shows an even number}

  B = {the second toss shows a 5 or a 6}.

- The number of outcomes in tossing a die twice is given by
$$_6C_1 \times {}_6C_1 = 36.$$

1.7.1 Introduction (Continued)

- The number of outcomes in the event $A = {}_3C_1 \times {}_6C_1$

$$\Pr(A) = \frac{{}_3C_1 \times {}_6C_1}{{}_6C_1 \times {}_6C_1} = \frac{3}{6} = \frac{1}{2}.$$

Similarly,

$$\Pr(B) = \frac{{}_6C_1 \times {}_2C_1}{{}_6C_1 \times {}_6C_1} = \frac{2}{6} = \frac{1}{3}.$$

$$\Pr(A \cap B) = \frac{{}_3C_1 \times {}_2C_1}{{}_6C_1 \times {}_6C_1} = \frac{6}{36} = \frac{1}{6}.$$

Introduction (Continued)

- Therefore

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{1/6}{1/3} = \frac{1}{2}.$$

On the other hand, $\Pr(A) = 1/2.$ Hence $\Pr(A|B) = \Pr(A).$

- Similarly,

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)} = \frac{1/6}{1/2} = \frac{1}{3}.$$

On the other hand, $\Pr(B) = 1/3.$ Hence $\Pr(B|A) = \Pr(B).$

Introduction (Continued)

- Generally speaking, $\Pr(B|A)$ is not equal to $\Pr(B)$.

- In other words, knowing that event A has occurred generally gives a different view on the chance of event B's occurrence.

- However, the above example shows that there are some special cases where $\Pr(B|A)$ does in fact equal $\Pr(B)$.

1.7.2 Independent Events

<u>Definition</u>:

- Two events A and B are said to be ***<u>independent</u>*** if and only if

$$\Pr(A \cap B) = \Pr(A)\Pr(B).$$

- Two events A and B that are not independent are said to be **dependent**.

Independent Events (Continued)

- Hence the events A and B are independent if the occurrence of one event does not in any way influence (or associate with) the occurrence of the other event.

- Therefore

  A is independent of B

  $\Leftrightarrow B$ is independent of A

  $\Leftrightarrow \Pr(A \cap B) = \Pr(A)\,\Pr(B).$

1.7.3 Properties of Independent Events

1. Suppose $\Pr(A) > 0, \Pr(B) > 0$.

   If A and B are independent, then
   $$\Pr(B|A) = \Pr(B) \text{ and } \Pr(A|B) = \Pr(A).$$
   The above equalities are sometimes used as the definition of two independent events.

   That is, the conditional probability of event B given event A has occurred is the same as the unconditional probability of event B.

Properties of Independent Events (continued)

2. Suppose $\Pr(A) > 0, \Pr(B) > 0$.

   If A and B are **independent** events, then events A and B **cannot be mutually exclusive**.

   Proof: Since A and B are independent events, $\Pr(A) > 0$ and $\Pr(B) > 0$, therefore
   $$\Pr(A \cap B) = \Pr(A)\Pr(B) > 0.$$

   Since $\Pr(A \cap B) \neq 0$, therefore $A \cap B \neq \emptyset$.

   Hence A and B are not mutually exclusive events.

Properties of Independent Events (Continued)

3. Suppose $\Pr(A) > 0, \Pr(B) > 0$.

   If A and B are **mutually exclusive**, then A and B <u>**cannot be independent**</u>.

   <u>Proof</u>: A and B are **mutually exclusive** events implies
   $$\Pr(A \cap B) = 0.$$

   On the other hand, $\Pr(A) > 0$ and $\Pr(B) > 0$ implies
   $$\Pr(A) \Pr(B) > 0$$

   Therefore $\Pr(A \cap B) \neq \Pr(A) \Pr(B)$ and hence events A and B are not independent.

Properties of Independent Events (Continued)

4. The sample space S as well as the empty set $\emptyset$ are independent of any event.

    <u>Proof</u>: For any event A, $A \cap S = A$, and $A \cap \emptyset = \emptyset$.
    $$\mathrm{Pr}(A \cap S) = \mathrm{Pr}(A) = \mathrm{Pr}(A)\,\mathrm{Pr}(S)$$
    since $\mathrm{Pr}(S) = 1$.

    $$\mathrm{Pr}(A \cap \emptyset) = \mathrm{Pr}(\emptyset) = \mathrm{Pr}(\emptyset)\,\mathrm{Pr}(A)$$
    since $\mathrm{Pr}(\emptyset) = 0$.

Properties of Independent Events (Continued)

5. If $A \subset B$, then A and B are dependent unless $B = S$.

    Proof: $A \subset B$ implies that $A \cap B = A$.

    $\Pr(A \cap B) = \Pr(A) \neq \Pr(A) \Pr(B)$ unless $\Pr(B) = 1$ which implies that $B = S$.
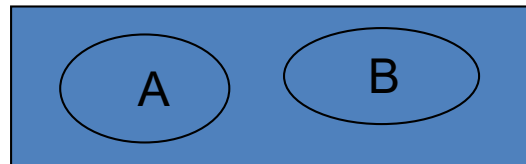
Some remarks about independent events

- The properties of independence, unlike the mutually exclusive property, **cannot** be shown on a Venn diagram.

  This means you can't trust your intuition.

  In general, <u>the only way to check for independence is by checking with the definition for independence,</u> namely $\Pr(A \cap B) = \Pr(A) \Pr(B)$.

- Mutually exclusive events are dependent events since $\Pr(A) \Pr(B) \neq \Pr(A \cap B) = 0$.

Some remarks about independent events (Continued)

Mutually exclusive and not independent events

Consider throwing a die.

A = event of odd numbers = {1, 3, 5}

D = event of even numbers = {2, 4, 6}

A and D are **mutually exclusive** events

- $\Pr(A) = 1/2$, $\Pr(D) = 1/2$, $\Pr(A \cap D) = 0$.

Since $\Pr(A)\Pr(D) \neq \Pr(A \cap D)$, therefore A and D **are not independent** events.

Some remarks about independent events (Continued)

Not mutually exclusive and not independent events

Consider throwing a die.

A = event of odd numbers = {1, 3, 5}

B = event of numbers ≤ 3 = {1, 2, 3}

- $A \cap B = \{1, 3\}$, hence A and B are **not mutually exclusive** events

- $\Pr(A) = 1/2, \Pr(B) = 1/2, \Pr(A \cap B) = 1/3.$

Since $\Pr(A) \Pr(B) \neq \Pr(A \cap B)$, hence A and B **are not independent** events.

Some remarks about independent events (Continued)

Not mutually exclusive and independent events

Consider throwing a die.

A = event of odd numbers = {1, 3, 5}

C = event of numbers ≤ 4 = {1, 2, 3, 4}

- $A \cap C = \{1, 3\}$, hence A and C are not mutually exclusive events

- $\Pr(A) = 1/2$, $\Pr(C) = 2/3$, $\Pr(A \cap C) = 1/3$.

Since $\Pr(A)\Pr(C) = 1/3 = \Pr(A \cap C)$, hence A and C **are independent** events.

Example 1

- If the probability is 0.1 that a person will make a mistake on his state income tax return,

- find the probability that two totally unrelated persons each make a mistake.

Solution

- Let M_i = {Person i makes a mistake in his tax return} for $i = 1, 2$.

- Since M_1 and M_2 are independent events, therefore
$$\Pr(M_1 \cap M_2) = \Pr(M_1) \Pr(M_2) = (0.1)(0.1) = 0.01.$$

Theorem

- If A and B are **independent**, then so are A and B', A' and B, A' and B'.

Example 2:

- The probability that Tom will be alive in 20 years is 0.7, and

- the probability that Jack will be alive in 20 years is 0.9.

- What is the probability that neither will be alive in 20 years?

Solution to Example 2

- Let A and B respectively to be events that Tom and Jack would be alive in 20 years.

- Note that these two events are independent.

- Hence A' and B' are independent, too.

- The desired probability is given by
$$\Pr(A' \cap B') = \Pr(A')\Pr(B')$$
$$= (1 - 0.7)(1 - 0.9) = 0.03.$$

1.7.4 n Independent Events

<u>Pairwise Independent Events</u>

Definition :

- A set of events $A_1, A_2, \cdots, A_n$ are said to be **pairwise independent** if and only if
$$\Pr(A_i \cap A_j) = \Pr(A_i)\Pr(A_j)$$
for $i \neq j$ and $i, j = 1, \cdots, n$.

n Mutually Independent Events

- The events $A_1, A_2, \cdots, A_n$ are called **mutually independent** (or simply **independent**)

  if and only if for **any subset** $\{A_{i_1}, A_{i_2}, \cdots, A_{i_k}\}$ of $A_1, A_2, \cdots, A_n$,

$$\Pr\left(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}\right) = \Pr\left(A_{i_1}\right)\Pr\left(A_{i_2}\right)\cdots\Pr\left(A_{i_k}\right)$$

Remarks

1. When one says that events $A_1, A_2, \cdots, A_n$ are mutually independent, it means that

   - firstly, for any pair of events A_j, A_k where $j \neq k$, the multiplication rule holds, and

   - secondly, for any three events A_i, A_j and A_k for distinct i, j, k, the multiplication rule holds, and so on.

   - Of course, the following multiplication rule also holds:
   $$\Pr(A_1 \cap A_2 \cap \cdots \cap A_n) = \Pr(A_1)\Pr(A_2)\cdots\Pr(A_n)$$

   - There are in total $2^n - n - 1$ different cases.

Remarks (Continued)

2. Mutually independence implies pairwise independence, but pairwise independence does **not** imply mutually independence.

3. Suppose $A_1, A_2, \cdots, A_n$ are mutually independent events. Let

$$B_i = A_i \text{ or } A_i', \ \ i = 1, 2, \cdots, n.$$

Then $B_1, B_2, \cdots, B_n$ are also mutually independent events.

Example 1

- A fair die is tossed three times.

- What is the probability that the first toss gives an odd number, the second toss gives an even number and the third gives a value greater than 4?

Solution

- Let

A = {odd number obtained in 1st toss}

B = {even number obtained in 2nd toss}

C = {5 or 6 obtained in 3rd toss}

Example 1 (Continued)

- $\Pr(A) = ({}_3C_1 \times {}_6C_1 \times {}_6C_1)/({}_6C_1 \times {}_6C_1 \times {}_6C_1) = 1/2.$
- $\Pr(B) = ({}_6C_1 \times {}_3C_1 \times {}_6C_1)/({}_6C_1 \times {}_6C_1 \times {}_6C_1) = 1/2.$
- $\Pr(C) = ({}_6C_1 \times {}_6C_1 \times {}_2C_1)/({}_6C_1 \times {}_6C_1 \times {}_6C_1) = 1/3.$
- $\Pr(A \cap B) = ({}_3C_1 \times {}_3C_1 \times {}_6C_1)/({}_6C_1 \times {}_6C_1 \times {}_6C_1) = 1/4$
  $= (1/2)(1/2) = \Pr(A)\Pr(B).$
- $\Pr(A \cap C) = ({}_3C_1 \times {}_6C_1 \times {}_2C_1)/({}_6C_1 \times {}_6C_1 \times {}_6C_1) = 1/6$
  $= (1/2)(1/3) = \Pr(A)\Pr(C).$
- $\Pr(B \cap C) = ({}_6C_1 \times {}_3C_1 \times {}_2C_1)/({}_6C_1 \times {}_6C_1 \times {}_6C_1) = 1/6$
  $= (1/2)(1/3) = \Pr(B)\Pr(C).$

Example 1 (Continued)

- $\Pr(A \cap B \cap C) = (_3C_1 \times {}_3C_1 \times {}_2C_1)/(_6C_1 \times {}_6C_1 \times {}_6C_1)$
  $= 1/12 = (1/2)(1/2)(1/3) = \Pr(A)\,\Pr(B)\,\Pr(C)$.

- Since $\Pr(A \cap B) = \Pr(A)\,\Pr(B)$,

  $\Pr(A \cap C) = \Pr(A)\,\Pr(C)$,

  $\Pr(B \cap C) = \Pr(B)\,\Pr(C)$ and

  $\Pr(A \cap B \cap C) = \Pr(A)\,\Pr(B)\,\Pr(C)$,

  therefore events A, B and C are mutually independent.

Example 2

Events that are pairwise independent but not mutually independent

- Suppose that we toss 2 dice. Define

  A = {1st die shows an even number}

  B = {2nd die shows an odd number}

  C = {the 2 dice show both odd or both even numbers}

- $\Pr(A) = (\,_3C_1 \times\,_6C_1)\,/(\,_6C_1 \times\,_6C_1) = 1/2$.

- $\Pr(B) = (\,_6C_1 \times\,_3C_1)\,/(\,_6C_1 \times\,_6C_1) = 1/2$.

- $\Pr(C) = \Pr(\text{both dice are odd}) + \Pr(\text{both dice are even})$

  $= (\,_3C_1 \times\,_3C_1)/(\,_6C_1 \times\,_6C_1) + (\,_3C_1 \times\,_3C_1)/(\,_6C_1 \times\,_6C_1) = 1/2$.

Example 2 (Continued)

- $\Pr(A \cap B) = ({}_3C_1 \times {}_3C_1)/({}_6C_1 \times {}_6C_1) = 1/4 = \Pr(A)\Pr(B)$.
- $\Pr(A \cap C) = ({}_3C_1 \times {}_3C_1)/({}_6C_1 \times {}_6C_1) = 1/4 = \Pr(A)\Pr(C)$.
- $\Pr(B \cap C) = ({}_3C_1 \times {}_3C_1)/({}_6C_1 \times {}_6C_1) = 1/4 = \Pr(B)\Pr(C)$.

- Hence A, B and C are pairwise independent.
- However, A, B and C are not mutually independent since $\Pr(A \cap B \cap C) = 0$ and
  $\Pr(A)\Pr(B)\Pr(C) = (1/2)^3 = 1/8 \neq \Pr(A \cap B \cap C) = 0$.

Example 3

- Three fire engines operate independently.

- The probability that a specific engine is available when needed is 0.9.

- What is the probability that no engine is available when needed?

Solution to Example 3

- Let E_i = {i-th engine available when needed}.

- {No engine is available} = $E_1' \cap E_2' \cap E_3'$.

- Since E_1, E_2 and E_3 are independent, therefore E_1', E_2' and E_3' are also **independent.**

    $$\Pr(E_i') = 1 - \Pr(E_i) = 1 - 0.9 = 0.1 \text{ for } i = 1, 2, 3$$

    Hence

    $$\Pr(E_1' \cap E_2' \cap E_3') = \Pr(E_1')\Pr(E_2')\Pr(E_3')$$
    $$= (0.1)^3 = 0.001.$$

Example 4

- The probability that a grader will make a marking error on any particular question of a multiple-choice question exam is 0.05.

- If there are ten questions and questions are marked **independently,**

  (a) what is the probability that <u>no errors</u> are made?

  (b) That <u>at least one error</u> is made?

  (c) If there are n questions and the probability of marking an error is p rather than 0.05, give expressions for these two probabilities.

Solution to Example 4

Let E_i denote the event of making an error in marking the i-th question.

(a) Since $E_1, E_2, \cdots$, and E_{10} are independent, therefore $E'_1, E'_2, \cdots$, and E'_{10} are also **independent.**

Pr(no error in any of the 10 questions)
$= \text{Pr}(E'_1 \cap E'_2 \cap \cdots \cap E'_{10})$

$= \text{Pr}(E'_1)\, \text{Pr}(E'_2) \cdots \text{Pr}(E'_{10})$
$= (1 - 0.05)^{10} = 0.95^{10} = 0.5987.$

Solution to Example 4 (Continued)

(b) Pr(at least one error)

$= 1 -$ Pr(no error in 10 questions)

$= 1 - (0.95)^{10}$

$= 0.4013.$

(c) For p replacing 0.05 and n replacing 10, the two probabilities are

$$(1-p)^n \quad \text{and} \quad 1 - (1-p)^n.$$