ST2334 (2021/2022 Semester 1) Solutions to Questions in Tutorial 2

Question 1

Let A = {The factory will be set up in City A}, B = {The factory will be set up in City B}.
It is given that $\Pr(A) = 0.7, \Pr(B) = 0.4$ and $\Pr(A \cup B) = 0.8$.

(a) $\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B) = 0.7 + 0.4 - 0.8 = 0.3$.

(b) $\Pr(A' \cap B') = \Pr((A \cup B)') = 1 - \Pr(A \cup B) = 1 - 0.8 = 0.2$.

Question 2

(a) Number of ways to choose 5 out of 25 qualified applicants $= {}_{25}C_5 = 25!/(5!\,20!) = 53130$.

(b) Number of ways to choose 5 out of 25 qualified applicants such that none of the minority is hired $= {}_6C_0 \times {}_{19}C_5 = 1 \times 19!/(5!\,14!) = 11628$. Therefore the desired probability is $11628/53130 = 0.2189$.

(c) Number of ways to choose 5 out of 25 qualified applicants such that one minority is hired $= {}_6C_1 \times {}_{19}C_4 = 6 \times [19!/(4!\,15!)] = 23256$.
Let A_0 and A_1 denote the events that no minority and one minority is hired respectively.
Hence $\Pr(A_1) = 23256/53130 = 0.4377$.
From part (b), $\Pr(A_0) = 0.2189$.
Therefore $\Pr(\text{at most one minority is hired}) = \Pr(A_0) + \Pr(A_1) = 0.6566$.

Question 3

Number of possible hands of 5 cards is ${}_{52}C_5 = 52(51)(50)(49)(48)/5! = 2598960$.

(a) Number of spade flush hands is ${}_{13}C_5 \times {}_{13}C_0 \times {}_{13}C_0 \times {}_{13}C_0 = 1287$.
Similarly, the number of heart flush hands is ${}_{13}C_0 \times {}_{13}C_5 \times {}_{13}C_0 \times {}_{13}C_0 = 1287$ and so on.
$\Pr(\text{a flush hand}) = 4(1287)/2598960 = 5148/2598960 = 0.001981$.

(b) Number of straight hands with 1 as the smallest card is $({}_4C_1)^5 \times ({}_4C_0)^8 - 4 = 1020$.
Similarly, the number of straight hands with 2 as the smallest card is ${}_4C_0 \times ({}_4C_1)^5 \times ({}_4C_0)^7 - 4 = 1020$ and so on. The smallest card can be any one from 1 to 10.
$\Pr(\text{a straight hand}) = 10(1020)/2598960 = 10200/2598960 = 0.003925$.

Question 4

Let $A_i, i = 1, 2$ denote the event that the motorist stops at light i.
We have $\Pr(A_1) = 0.4, \Pr(A_2) = 0.5$ and $\Pr(A_1 \cup A_2) = 0.6$.

(a) $\Pr(A_1 \cap A_2) = \Pr(A_1) + \Pr(A_2) - \Pr(A_1 \cup A_2) = 0.4 + 0.5 - 0.6 = 0.3$.

(b) Stops at exactly one light $= (A_1 \cap A_2') \cup (A_1' \cap A_2)$
But $\Pr(A_1 \cap A_2') = \Pr(A_1) - \Pr(A_1 \cap A_2) = 0.4 - 0.3 = 0.1$ and
$\Pr(A_1' \cap A_2) = \Pr(A_2) - \Pr(A_1 \cap A_2) = 0.5 - 0.3 = 0.2$.
Hence $\Pr(\text{Stops at exactly one light}) = 0.1 + 0.2 = 0.3$.

(c) $\Pr(A_1' \cap A_2') = \Pr((A_1 \cup A_2)') = 1 - \Pr(A_1 \cup A_2) = 1 - 0.6 = 0.4$.

(d) $\Pr(A_2 \mid A_1) = \Pr(A_1 \cap A_2) / \Pr(A_1) = 0.3/0.4 = 0.75$.

Question 5

Number of possible 9-digit numbers with no restriction $= {}_9C_1 \times ({}_{10}C_1)^8 = 9(10)^8$.

(a) There are 9 ways to choose the first digit, and also 9 ways (in order not repeat the number chosen for the first/previous digit) to choose the second digit, and so on until the ninth digit. Hence, the number of 9-digit numbers with no two consecutive digits are the same $= ({}_9C_1)^9 = 387420489$.
The probability that no two consecutive digits are the same in a randomly selected 9-digit number $= ({}_9C_1)^9/[{}_9C_1 \times ({}_{10}C_1)^8] = 387420489/[9(10)^8] = 0.4305$.

(b) In a 9-digit number, there are 8 places (except the first digit) where we can place the three zeros, number of ways of doing so $= {}_8C_3 = 56$. For the other places, there are 6 of them, we have 9 choices (except the choice of zero), number of ways doing so $= ({}_9C_1)^6 = 9^6$. So the number of 9-digit numbers with 0 appears as a digit for a total of 3 times $= {}_8C_3 \times ({}_9C_1)^6 = 29760696$.
The probability that a 9-digit number with 0 appears as a digit for a total of 3 times being selected $= {}_8C_3 \times ({}_9C_1)^6/[{}_9C_1 \times ({}_{10}C_1)^8] = 29760696/[9(10)^8] = 0.03307$.

Question 6

Let $A = \{\text{Player A wins the game}\}$ and $B = \{\text{Player B enters the game}\}$.
It is given that $\Pr(A|B) = 1/6, \Pr(A|B') = 3/4$ and $\Pr(B) = 1/3$.
Hence $\Pr(B') = 1 - \Pr(B) = 2/3$.
Applying the total probability law, $\Pr(A) = \Pr(A|B)\Pr(B) + \Pr(A|B')\Pr(B') = (1/6)(1/3) + (3/4)(2/3) = 5/9$.

Question 7

Let $M_1 = \{\text{the selected bottle was filled on machine I}\}$, $M_2 = \{\text{the selected bottle was filled on machine II}\}$ and $N = \{\text{a nonconforming bottle was selected}\}$
It is given that $\Pr(N \cap M_1) = 0.01, \Pr(N \cap M_2) = 0.025, \Pr(M_1) = \Pr(M_2) = 0.5$.

(a) $\Pr(N) = \Pr((N \cap M_1) \cup (N \cap M_2)) = 0.01 + 0.025 = 0.035$.
(b) $\Pr(M_2) = 0.5$.
(c) $\Pr(M_2 \cap N') = \Pr(M_2) - \Pr(M_2 \cap N) = 0.5 - 0.025 = 0.475$.
(d) $\Pr(M_1 \cup N') = \Pr(M_1) + \Pr(N') - \Pr(M_1 \cap N')$.
But $\Pr(N') = 1 - \Pr(N) = 1 - 0.035 = 0.965$, $\Pr(M_1 \cap N') = \Pr(M_1) - \Pr(M_1 \cap N) = 0.5 - 0.01 = 0.49$, therefore $\Pr(M_1 \cup N') = 0.5 + 0.965 - 0.49 = 0.975$.
(e) $\Pr(N|M_1) = \Pr(N \cap M_1) / \Pr(M_1) = 0.01/0.5 = 0.02$.
(f) $\Pr(M_1|N) = \Pr(N \cap M_1) / \Pr(N) = 0.01/0.035 = 0.2857$.
(g) The events are different and the conditions are different. The answer in part (e) is the probability of having a nonconforming item given the condition that the item was from machine I (i.e. $\Pr(N|M_1)$). The answer in part (f) is the probability of having an item from machine I given that it was a nonconforming item (i.e. $\Pr(M_1|N)$).

Question 8

Let $P = \{\text{the women is pregnant}\}$ and $T = \{\text{test result is positive}\}$. Hence, $P' = \{\text{the woman is not pregnant}\}$ and $T' = \{\text{test result is negative}\}$.
We have $\Pr(P) = 0.75, \Pr(T|P) = 0.99, \Pr(T|P') = 0.02$.
Hence $\Pr(T) = \Pr(P)\Pr(T|P) + \Pr(P')\Pr(T|P') = 0.75(0.99) + 0.25(0.02) = 0.7475$.
(a) $\Pr(P|T) = \Pr(P \cap T) / \Pr(T) = 0.75(0.99)/0.7475 = 0.9933$.
(b) $\Pr(P'|T') = \Pr(P' \cap T') / \Pr(T') = (1 - 0.02)(1 - 0.75)/(1 - 0.7475) = 0.9703$.