Chapter 1

Basic Concepts of Probability

Overview

- Basic probability concepts and definitions
  - Sample space and events
- Operations of events
  - Complement events, mutually exclusive events
  - Union of events, Intersection of events
- Counting methods
  - Multiplication principle
  - Addition principle

Overview (Continued)

- Permutation
- Combination
- Approaches to probability:
  - Classical, Relative frequency, and Subjective
- Axioms of Probability
- Basic properties of probability
- Conditional probability
- Multiplicative rule of probability

Overview (Continued)

- The Law of Total Probability
- Bayes' Theorem
- Independent events

1.1 Sample Space and Sample Points

1.1.1 Sample Space

- **Observation:** We refer to any recording of information, whether it is *numerical or categorical*, as an observation.

- **Statistical Experiment:** Any procedure that generates a set of data (observations).

Sample Space and Sample Points (Continued)

1.1.1 Sample Space (Continued)

- **Sample Space:** The set of all possible outcomes of a statistical experiment is called the **sample space** and it is represented by the symbol S.

Examples

1. Consider an experiment of tossing a die.
   - If we are interested in the number that shows on the top face, then the sample space would be

     $$S = \{1, 2, 3, 4, 5, 6\}.$$

   - If we are interested only in whether the number is even or odd, then the sample space is simply

     $$S = \{\text{even}, \text{odd}\}.$$

Examples (Continued)

2. Consider an experiment of tossing two dice.
   - If we are interested in the numbers that show on the top faces, then the sample space would be

$$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), \cdots\cdots, (6,5), (6,6)\}.$$

Examples (Continued)

3. An experiment consists of flipping a coin and then flipping it a second time if a head occurs. If a tail occurs on the first flip, then a die is tossed once.

   The sample space is

   $$S = \{(H,H),(H,T),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}.$$

4. Recording the Straits Times Index.

   $$S = \{\cdots, 3328.95, 3286.32, 3265.73 \cdots\}.$$

Examples (Continued)

5. An experiment consists of drawing two balls from a box containing a blue, a white and a red ball.

   If we are interested in the colours of the two balls drawn, then the sample space is

$$S = \{(B,W),(B,R),(W,B),(W,R),(R,B),(R,W)\}.$$

1.1.2 Sample Points

Sample Points

- Every outcome in a **sample space** is called an element of the sample space or simply a **sample point**.

Examples (Continued)

1. $S = \{1, 2, 3, 4, 5, 6\}$.

   Sample point: 1 or 2 or 3 or 4 or 5 or 6.

   $S = \{\text{even}, \text{odd}\}$.

   Sample point: even or odd

2. $S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), \cdots, (6,5), (6,6)\}$.

   Sample point: (1,1) or (1,2) or $\cdots$ or (6,5) or (6,6).

Examples (Continued)

3. $S = \{(H,H),(H,T),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$

   Sample point: (H,H), or (H,T) or $(T,1)$ or $(T,2)$ or $(T,3)$ or $(T,4)$ or $(T,5)$ or $(T,6)$.

Examples (Continued)

4. $S = \{\cdots, 3328.95, 3286.32, 3265.73, \ \cdots\}$.

   Sample point: $\cdots$ or 3328.95 or 3286.32 or 3265.73 or $\cdots$

5. $S = \{(B,W),(B,R),(W,B),(W,R),(R,B),(R,W)\}$.

   Sample point: (B,W) or (B,R) or (W,B) or (W,R) or (R,B) or (R,W)

1.1.3 Events

An **event** is a subset of a sample space.

Examples

1. (a) $S = \{1, 2, 3, 4, 5, 6\}$.

   An event that an odd number occurs $= \{1, 3, 5\}$

   An event that a number greater than 4 occurs $= \{5, 6\}$

   (b) $S = \{\text{even}, \text{odd}\}$.

   An event that an odd number occurs $= \{\text{odd}\}$

Events (Continued)

Examples (Continued)

2. In rolling a pair of dice, if event $A = \{\text{the sum of the dice equals 7}\}$, then

$$A = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}.$$

3. $S = \{(H,H),(H,T),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$

   An event that no die is thrown

$$\{(H,H),(H,T)\}$$

Events (Continued)

Examples (Continued)

4. $S = \{(B,W),(B,R),(W,B),(W,R),(R,B),(R,W)\}$.

   If event $A = \{$ a white ball is chosen $\}$, then

   $$A = \{(W,B),(W,R),(B,W),(R,W)\}.$$

5. In tossing two coins, $S = \{(H,H),(H,T),(T,H),(T,T)\}$.

   Getting no head or one head or two heads $=$

   $$\{(T,T),(H,T),(T,H),(H,H)\}.$$

   Getting no head or one head $= \{(T,T),(H,T),(T,H)\}$.

   Getting two heads or two tails $= \{(T,T),(H,H)\}$.

1.1.4 Simple and Compound Events

- **Simple Event:** An event is said to be **simple** if it consists of exactly **one** outcome (i.e. one sample point)

- **Compound Event:** An event is said to be **compound** if it consists of more than one outcomes (or sample points).

Examples (Continued)

1. (a) $S = \{1, 2, 3, 4, 5, 6\}$.

   Compound events:

   (a) An odd number occurs $= \{1, 3, 5\}$.

   (b) $\{\text{Obtain a number} > 4\} = \{5, 6\}$.

   Simple event:

   Obtain a "six" $= \{6\}$.

   (b) $S = \{\text{even}, \text{odd}\}$.

   Simple event: An odd number occurs $= \{\text{odd}\}$

Examples (Continued)

2. In rolling a pair of dice,

   Compound event:

   $$A = \{\text{the sum of the dice equals } 7\}$$
   $$= \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}.$$

   Simple event:

   $$B = \{\text{the sum of the dice equals } 2\} = \{(1,1)\}.$$

Examples (Continued)

3. $S = \{(H,H),(H,T),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$

   Compound event:

   No die is thrown $= \{(H,H),(H,T)\}$

   Simple event:

   Obtain a "one" $= \{(T,1)\}$

More Example

- Let $S = \{t: t \geq 0\}$, where t is the life in years of a certain electronic component.
- Find the event A that the component fails before the end of the fifth year.
- $A = \{t: 0 \leq t < 5\}$.

Remarks

1. The sample space is itself an event and is usually called a ***sure event***.

2. A subset of S that contains no elements at all is the empty set, denoted by $\emptyset$, and is usually called a ***null event***.

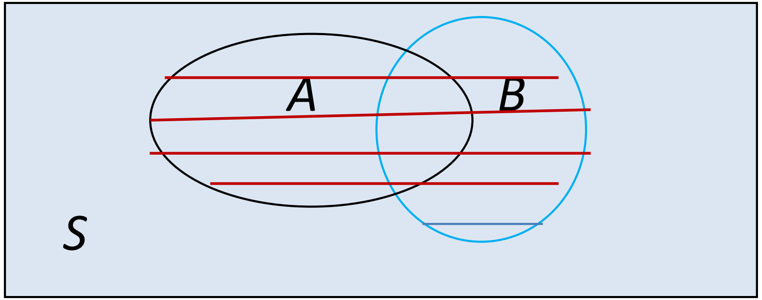
1.2 Operations with Events

1.2.1. Union and Intersection Events

Let S denote a sample space, A and B are any two events of S.

- **Union:** The **Union** of two events A and B, denoted by $A \cup B$, is the event containing all the elements that belong to A or B or to both. That is,

$$A \cup B = \{x: x \in A \textbf{ or } x \in B\}$$



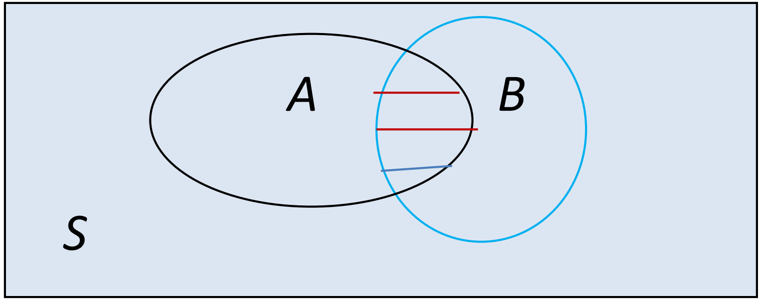
Operations with Events (Continued)

Union and Intersection Events (Continued)

Let S denote a sample space, A and B are any two events of S.

- **Intersection:** The **intersection** of two events A and B, denoted by $A \cap B$ or simply AB, is the event containing all elements that are common to A and B. That is

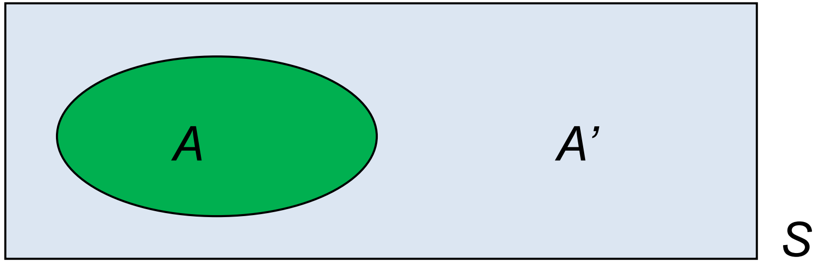
$$A \cap B = \{x: x \in A \textbf{ and } x \in B\}$$



1.2.2 Complement Event

- **Complement:** The **complement** of event A with respect to S, denoted by A' or A^C, is the set of all elements of S that are not in A. That is

$$A' = \{x : x \in S \text{ and } \boldsymbol{x} \notin \boldsymbol{A}\}$$



Examples

$S = \{1, 2, 3, 4, 5, 6\}$. $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$ and $C = \{2, 4, 6\}$.

Then

- $A \cup B = \{1, 2, 3, 5\}$
- $A \cup C = \{1, 2, 3, 4, 6\}$
- $B \cup C = \{1, 2, 3, 4, 5, 6\} = S$
- $A \cup B \cup C = A \cup (B \cup C) = A \cup S = S$
- $A \cup B \cup C = (A \cup B) \cup C = \{1, 2, 3, 5\} \cup \{2, 4, 6\} = S$

Examples (Continued)

$S = \{1, 2, 3, 4, 5, 6\}$. $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$ and $C = \{2, 4, 6\}$.

Then

- $A \cap B = \{1, 3\}$, $A \cap C = \{2\}$, $B \cap C = \emptyset$
- $A \cap B \cap C = (A \cap B) \cap C = \{1, 3\} \cap \{2, 4, 6\} = \emptyset$
- $A \cap (B \cap C) = \{1, 2, 3\} \cap \emptyset = \emptyset$
- $(A \cap B) \cup C = \{1, 3\} \cup \{2, 4, 6\} = \{1, 2, 3, 4, 6\}$
- $A \cap (B \cup C) = A \cap S = A = \{1, 2, 3\}$

Notice that $(A \cap B) \cup C \neq A \cap (B \cup C)$

Examples (Continued)

$S = \{1,2,3,4,5,6\}$. $A = \{1,2,3\}$, $B = \{1,3,5\}$ and $C = \{2,4,6\}$.

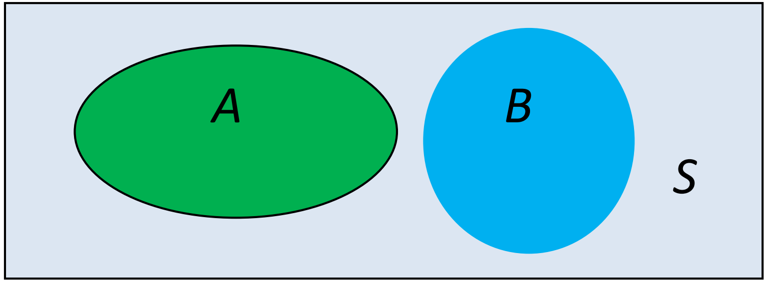
Then

- $A' = \{4,5,6\}$
- $B' = \{2,4,6\} = C$
- $A' \cap B' = \{4,5,6\} \cap \{2,4,6\} = \{4,6\}$
- $(A \cup B)' = \{1,2,3,5\}' = \{4,6\}$

Notice that both $A' \cap B'$ and $(A \cup B)'$ equal $\{4,6\}$ in this example. Is it true that $A' \cap B' = (A \cup B)'$ in general?

1.2.3 Mutually Exclusive Events

Two events A and B are said to be **mutually exclusive** or **mutually disjoint** if $A \cap B = \emptyset$, that is, if A and B have no elements in common.



Examples

$S = \{1,2,3,4,5,6\}$. $A = \{1,2,3\}$, $B = \{1,3,5\}$ and $C = \{2,4,6\}$.

Then

- Events B and C are mutually exclusive events since $B \cap C = \emptyset$.
- Events A and B are not mutually exclusive events since $A \cap B = \{1,3\}$.
- Having a "one" and having a "six" are mutually exclusive events since $\{1\} \cap \{6\} = \emptyset$.

Note:

- Events A and A' are mutually exclusive.

1.2.4 Union of n events

Union:

The **Union** of n events $A_1, A_2, \cdots, A_n$, denoted by

$$A_1 \cup A_2 \cup \cdots \cup A_n,$$

is the event containing all the elements that belong to one or more of the events A_1, or A_2, or $\cdots$, or A_n. That is,

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n = \{x: x \in A_1 \text{ or} \cdots \text{or } x \in A_n\}$$

1.2.5 Intersection of n events

Intersection:

The **intersection** of n events $A_1, A_2, \cdots, A_n$, denoted by

$$A_1 \cap A_2 \cap \cdots \cap A_n,$$

is the event containing all the elements that are common to all the events A_1, and A_2, and $\cdots$, and A_n,. That is,

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n = \{x: x \in A_1 \text{ and} \cdots \text{and } x \in A_n\}$$

1.2.6 Some Basic Properties of Operations of Events

1. $A \cap A' = \emptyset$.
2. $A \cap \emptyset = \emptyset$.
3. $A \cup A' = S$.
4. $(A')' = A$
5. $(A \cap B)' = A' \cup B'$

Some Basic Properties of Operations of Events (Continued)

6. $(A \cup B)' = A' \cap B'$
7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
8. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
9. $A \cup B = A \cup (B \cap A')$
10. $A = (A \cap B) \cup (A \cap B')$

1.2.7 De Morgan's Law

For any n events $A_1, A_2, \cdots, A_n$,

1.
$$(A_1 \cup A_2 \cup \cdots \cup A_n)' = A_1' \cap A_2' \cap \cdots \cap A_n'$$

2.
$$(A_1 \cap A_2 \cap \cdots \cap A_n)' = A_1' \cup A_2' \cup \cdots \cup A_n'$$

Example

$S = \{1,2,3,4,5,6\}$. $A = \{1,2,3\}$, $B = \{1,3,5\}$ and $C = \{2,4,6\}$.

Then

$A' = \{4,5,6\}$, $B' = \{2,4,6\} = C$ and $C' = \{1,3,5\} = B$.

$$\begin{aligned}(A \cup B \cup C)' &= (\{1,2,3\} \cup \{1,3,5\} \cup \{2,4,6\})' \\ &= \{1,2,3,4,5,6\}' = \emptyset.\end{aligned}$$

On the other hand,

$$A' \cap B' \cap C' = \{4,5,6\} \cap \{2,4,6\} \cap \{1,3,5\} = \emptyset.$$

Example (Continued)

$S = \{1,2,3,4,5,6\}$. $A = \{1,2,3\}$, $B = \{1,3,5\}$ and $C = \{2,4,6\}$.

Then

$$A' = \{4,5,6\}, B' = \{2,4,6\} = C \text{ and } C' = \{1,3,5\} = B.$$

$$\begin{aligned}(A \cap B \cap C)' &= (\{1,2,3\} \cap \{1,3,5\} \cap \{2,4,6\})' \\ &= \emptyset' = \{1,2,3,4,5,6\}.\end{aligned}$$

On the other hand,

$$A' \cup B' \cup C' = \{4,5,6\} \cup \{2,4,6\} \cup \{1,3,5\} = \{1,2,3,4,5,6\}.$$

1.2.8 Contained ($\subset$)

- If **all** of the elements in event A are also in event B, then event A is contained in event B, denoted by

$$A \subset B.$$

(or equivalently, $B \supset A$).

- If $A \subset B$ and $B \subset A$, then $A = B$.

(i.e. Event A is equivalent with event B).

1.2.9 More Examples

Tossing a die and then flipping a coin if the number on the die is even and twice if the number on the die is odd.

$$S = \{1HH, 1HT, 1TH, 1TT, 2H, 2T, 3HH, 3HT, 3TH, 3TT, 4H, 4T, 5HH, 5HT, 5TH, 5TT, 6H, 6T\}$$

Let $A = \{\text{Number shown on the die} < 3\}$
$= \{1HH, 1HT, 1TH, 1TT, 2H, 2T\}$,

$B = \{\text{2 tails occur}\} = \{1TT, 3TT, 5TT\}$

$C = \{\text{Even number shown on the die}\}$
$= \{2H, 2T, 4H, 4T, 6H, 6T\}$

More Examples (Continued)

- $A' = \{1HH, 1HT, 1TH, 1TT, 2H, 2T\}'$
  $= \{3HH, 3HT, 3TH, 3TT, 4H, 4T, 5HH, 5HT, 5TH, 5TT, 6H, 6T\}$
- $B' = \{1TT, 3TT, 5TT\}'$
  $= \{1HH, 1HT, 1TH, 2H, 2T, 3HH, 3HT, 3TH, 4H, 4T, 5HH, 5HT, 5TH, 6H, 6T\}$
- $C' = \{2H, 2T, 4H, 4T, 6H, 6T\}'$
  $= \{1HH, 1HT, 1TH, 1TT, 3HH, 3HT, 3TH, 3TT, 5HH, 5HT, 5TH, 5TT\}$

More Examples (Continued)

- $A \cap B = \{1TT\}$
- $A \cap C = \{2H, 2T\}$
- $B \cap C = \emptyset$
- $A' \cap B' = \{3HH, 3HT, 3TH, 4H, 4T, 5HH, 5HT, 5TH, 6H, 6T\}$
- $A' \cap C' = \{3HH, 3HT, 3TH, 3TT, 5HH, 5HT, 5TH, 5TT\}$
- $B' \cap C' = \{1HH, 1HT, 1TH, 3HH, 3HT, 3TH, 5HH, 5HT, 5TH\}$

More Examples (Continued)

$$
\begin{aligned}
A \cup B &= \{1HH, 1HT, 1TH, 1TT, 2H, 2T, 3TT, 5TT\} \\
&= (A' \cap B')' \\
A \cup C &= \{1HH, 1HT, 1TH, 1TT, 2H, 2T, 4H, 4T, 6H, 6T\} \\
&= (A' \cap C')' \\
B \cup C &= \{1TT, 2H, 2T, 3TT, 4H, 4T, 5TT, 6H, 6T\} \\
&= (B' \cap C')' \\
A' \cup B' &= \{1HH, 1HT, 1TH, 2H, 2T, 3HH, 3HT, 3TH, 3TT, \\
&\quad 4H, 4T, 5HH, 5HT, 5TH, 5TT, 6H, 6T\} \\
&= (A \cap B)'
\end{aligned}
$$

1.3 Counting Methods

1.3.1 Multiplication Principle

- If an operation can be performed in $\boldsymbol{n_1}$ ways, and
- if for **each of these ways** a second operation can be performed in $\boldsymbol{n_2}$ ways, then
- the two operations can be performed together in

$$\boldsymbol{n_1\, n_2}$$

ways.

Counting Methods (Continued)

Multiplication Principle (Continued)

An alternative statement:

- Suppose that two experiments are to be performed.
- If the experiment 1 can result in any one of the $\boldsymbol{n_1}$ possible outcomes and
- if for **each outcome of experiment 1**, there are $\boldsymbol{n_2}$ possible outcomes of experiment 2, then
- together there are $\boldsymbol{n_1 n_2}$ possible outcomes of the two experiments.

Examples

1. How many sample points are there in the sample space when a die and a coin are thrown together?

Solution

- There are 6 possible outcomes $\{1, 2, 3, 4, 5, 6\}$ for throwing a die.
- For each outcome of the die, there are two possible outcomes for throwing a coin $\{H, T\}$.
- Hence the sample space is given by

$$S = \{(x, y)\colon\ x = 1, 2, 3, 4, 5, \text{or } 6;\, y = H \text{ or } T\}.$$

Examples (Continued)

- There are 6 possible outcomes in the first operation (throwing a die).
- For ***each outcome in the first operation***, there are 2 outcomes in the second operation (throwing a coin).
- Hence there are altogether $6 \times 2 = 12$ elements in the sample space.

Examples (Continued)

2. A small community consists of 10 men, each of whom has 3 sons. If one man and one of his sons are to be chosen as father and son of the year, how many different choices are possible?

<u>Solution</u>

There are 10 possible men to be chosen.

For each of the men, there are 3 sons to be chosen.

Therefore the number of choices is $10 \times 3 = 30$.

Remark

Note

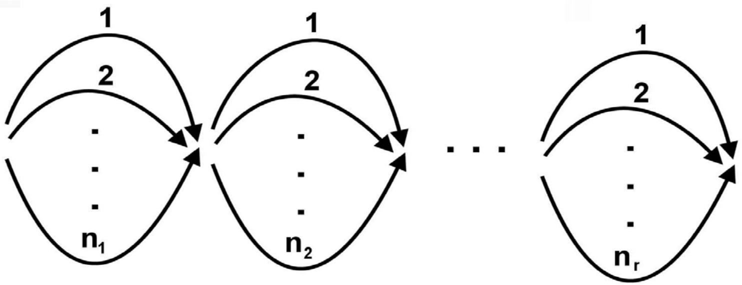
The above principle can be generalized to a sequence of **more than two** operations.

Generalized Multiplication Rule

- If an operation can be performed in n_1 ways, and
- if for each of these ways, a second operation can be performed in n_2 ways, and
- for each of the first two ways, a third operation can be performed in n_3 ways, and so forth, then
- the sequence of k operations can be performed in

$$\boldsymbol{n_1\, n_2 \cdots n_k}$$

ways.



Generalized Multiplication Rule (continued)

An alternative statement:

- If k experiments that are to be performed are such that the first one may result in n_1 possible outcomes, and
- if for **each** of these n_1 possible outcomes there are n_2 possible outcomes of the second experiment, and
- if for **each** of the possible outcomes of the first two experiments there are n_3 possible outcomes of the third experiment, and so on, then there are a total of

$$\boldsymbol{n_1\, n_2 \cdots n_k}$$

possible outcomes of the k experiments.

Example 1

How many **even three-digit numbers** can be formed from the digits 1, 2, 5, 6, and 9 if each digit can be used **only once**?

Solution

For the ones, we can use digits 2 and 6 only. (why?)

For tens and hundreds, we can freely use whatever digits left.

	Hundreds	Tens	Ones
Number of ways	3	4	2
Possible digits			{ 2or 6}

Hence the number of even three-digit numbers is $3 \times 4 \times 2 = 24$.

Example 2

A test consists of 10 multiple choice questions with each has 4 possible answers.

(a) How many possible ways are there in which a student can choose one answer to each question?

(b) Among all these cases, how many are such that all answers are wrong?

Solution to Example 2

(a) There are 4 possible answers for the first question.

For each of these answers, there are 4 possible answers for the second question.

For each of the ways in answering the first and second questions, there are 4 possible answers for the third questions and so on.

Therefore there are $4 \times 4 \times \cdots \times 4 = 4^{10} = 1048576$ possible ways in answering the test.

Solution to Example 2 (Continued)

(b) There are 3 wrong answers for the first question.

For each of these answers, there are 3 wrong answers for the second question.

For each of the ways in getting wrong answers for the first and second questions, there are 3 wrong answers for the third questions and so on.

Therefore there are $3 \times 3 \times \cdots \times 3 = 3^{10} = 59049$ possible ways in getting all wrong answers in the test.

Example 3

A college freshman must take a science course, a humanity course, and a mathematics course.

If she may select any of 6 science courses, any of 4 humanity courses, and any of 4 mathematics courses, in how many ways can she arrange her program?

Solution

The number of ways that the student can arrange her program is $6 \times 4 \times 4 = 96$.

Example 4

A person can travel from Singapore to San Francisco by 4 different airlines and each airlines can go via 3 different routes.

Solution

The number of different routes that the person can travel from Singapore to San Francisco is

$$4 \times 3 = 12.$$

Example 5

Find the number of even 3-digit numbers to be formed from digits 1, 2, 5, 6, 7 and 8.

(a) if each digit can be used once; and

(b) if no restriction on how many times a digit is used.

Example 5 (Continued)

<u>Solution</u>

(a) There are 3 choices for the ones place, namely $\{2, 6, 8\}$.

For each choice of ones place, we have $6 - 1 = 5$ choices for the hundreds places

For each choice of ones and hundreds places, we have $6 - 2 = 4$ choices for the tens place.

Hence, there are $5 \times 4 \times 3 = 60$ different even numbers.

Example 5 (Continued)

<u>Solution</u>

(b) There are 3 choices for the ones place, namely $\{2, 6, 8\}$.

For each choice of ones place, we have 6 choices, $\{1, 2, 5, 6, 7, 8\}$ for the hundreds place as any of the 6 digits can be used again.

For each choice of ones and hundreds places, we have 6 choices for the tens place.

Hence, there are $6 \times 6 \times 3 = 108$ different even numbers.

Example 6

How many set-lunches consisting of a bowl of soup, a sandwich, a dessert and a drink are possible if we can select from 4 choices of soup, 3 choices of sandwich, 5 choices of dessert and 4 choices of drink?

<u>Solution</u>

There are $4 \times 3 \times 5 \times 4 = 240$ different set-lunches.

Example 7

In how many ways can 4 boys and 5 girls sit in a row if the boys and girls must alternate?

Example 7 (Continued)

Solution

Arrangement of boys and girls: G B G B G B G B G

The number of ways:

$$5(4)(4)(3)(3)(2)(2)(1)(1)$$
$$= 5(4)(3)(2)(1)(4)(3)(2)(1) = 5!\,4! = 2880$$

where $n! = n(n-1)(n-2)\cdots(2)(1).$

Question : What happens if there are 5 boys and 5 girls?

1.3.2 Addition Principle

- Suppose that a procedure, designated by 1 can be performed in n_1 ways.
- Assume that a procedure, designated by 2 can be performed in n_2 ways.
- Suppose furthermore that it is **NOT possible** that both procedures 1 and 2 are **performed together**.
- Then the number of ways in which we can perform 1 **or** 2 is

$$n_1 + n_2$$

ways.

Addition Principle (Continued)

It may be generalized as follows.

- If there are k procedures and the i^{th} procedure may be performed in n_i ways, $i = 1, 2, \cdots, k$,
- then the number of ways in which we may perform procedure 1 **or** procedure 2 **or** $\cdots$ **or** procedure k is given by

$$n_1 + n_2 + \cdots + n_k,$$

assuming that **no** two procedures may be performed together.

Example 1

- We may take MRT **or** bus from home to Orchard Road.
- If there are three bus routes and two MRT routes, then there are $3 + 2 = 5$ different routes available for the trip.

Bus Route	MRT
1 2 3	1 2

Example 2

How many **even** three-digit numbers can be formed from the digits 0, 1, 2, 5, 6, and 9 if each digit can be used only once?

Solution

We have to consider two cases.

Case A: 0 is used for the ones.

- There are $5 \times 4 = 20$ ways to arrange the hundreds and tens places.
- Hence the number of even numbers formed with 0 at the ones place is 20.

Example 2 (Continued)

Case B: 0 is not used for the ones.

- We have to use either 2 or 6 in the ones.
- Next, we *deal with the hundreds*. As we cannot put 0 in the hundreds, there are only 4 eligible digits to be put in the hundreds.
- Then we can put whatever digits left in the tens. Therefore the number of even 3-digit numbers is $4 \times 4 \times 2 = 32$.

Example 2 (Continued)

Since both Cases A and B lead to an even 3-digit number and they cannot occur together, by applying the addition rule, we have $20 + 32 = 52$ even 3-digit numbers.

Example 3

Consider the digits 0, 1, 2, 3, 4, 5 and 6.

(a) If each digit can be used only once,

(i) how many 3-digit numbers are **even**?

(ii) how many of these are **greater than 420**?

Example 3 (Continued)

Consider the digits 0, 1, 2, 3, 4, 5 and 6.

(b) If each digit can be used more than once,

(i) how many 3-digit numbers can be formed?

(ii) how many 3-digit numbers formed are **odd**? even?

(iii) How many 3-digit numbers formed are **greater than 420**?

Solution to Example 3

(a) (i)

Case 1: Number of 3-digit even numbers with last digit being 0:

$$6(5)(1) = 30.$$

Case 2: Number of 3-digit even numbers with last digit other than 0 :

$$5(5)(3) = 75.$$

Solution to Example 3 (Continued)

(a) (i)

The required 3-digit even number either comes from Case 1 **or** Case 2

Hence, the number of 3-digit even numbers $= 30 + 75 = 105$.

Solution to Example 3 (Continued)

(a) (ii)

Case 1: Hundreds is 4. Tens is 2.

Number of 3-digit numbers = 1(1)(4) = 4.

Case 2: Hundreds is 4. Tens is 3, 5 or 6.

Number of 3-digit numbers = 1(3)(5) = 15.

Case 3: Hundreds is 5 or 6.

Number of 3-digit numbers =2(6)(5) = 60.

Solution to Example 3 (Continued)

(a) (ii)

The required 3 digit number comes from Case 1, Case 2 **or** Case 3

Hence, the number of 3-digit numbers formed greater than $420 = 4 + 15 + 60 = 79$.

Solution to Example 3 (Continued)

(b) (i)

6 choices for hundreds place (1 to 6), 7 choices for tens place (0 to 6) and 7 choices for ones place (0 to 6)

Hence, the number of 3-digit numbers is given by

$$6(7)(7) = 294.$$

Solution to Example 3 (Continued)

(b) (ii)

3 digit **odd** number

3 choices for the ones place (1, 3, & 5), 6 choices for the hundreds place (1 to 6) and 7 choices for tens place (0 to 6)

Number of odd 3-digit numbers is given by

$$6(7)(3) = 126.$$

Solution to Example 3 (Continued)

(b) (ii)

3 digit **even** number

4 choices for the units (0, 2, 4, & 6), 6 choices for the hundreds (1 to 6) and 7 choices for tens (0 to 6)

Number of 3-digit even numbers is given by

$$6(7)(4) = 168.$$

Solution to Example 3 (Continued)

(b) (iii)

Case 1: Hundreds is 4. Tens is 2.

Number of 3-digit numbers $= 1(1)(6) = 6.$

Case 2 : Hundreds is 4. Tens is 3, 4, 5 or 6.

Number of 3-digit numbers $= 1(4)(7) = 28.$

Case 3 : Hundreds is 5 or 6.

Number of 3-digit numbers $= 2(7)(7) = 98.$

Solution to Example 3 (Continued)

(b) (iii)

The required 3-digit number comes from Case 1, Case 2 **or** Case 3

Hence, the number of 3-digit numbers formed greater than 420 =

$$6 + 28 + 98 = 132.$$

1.3.3 Permutation

A **permutation** is an arrangement of r objects from a set of n objects, where $r \leq n$.

(Note that the **order** is taken into consideration in permutation.)

Permutation (Continued)

Example

Given a set $\{a, b, c\}$, the possible arrangements or permutations are:

$$abc \quad acb \quad bac \quad bca \quad cab \quad cba$$

Permutation (Continued)

- There are 3 choices for the first place.
- For each of these choices, there are two choices for the second place.
- For each of these choices for the first and second places, there is only 1 choice left for the last place.
- Hence the number of possible arrangements is

$$3 \times 2 \times 1 = 6.$$

Permutation (Continued)

- In general n distinct objects can be arranged in

$$n(n-1)(n-2)\cdots \times 2 \times 1 = n!$$

ways which is read as ***n factorial*** ways.

Note: $1! = 1$ and $0! = 1$.

1.3.3.1 Permutations of n distinct objects taken r at a time

Number of **permutations** of n distinct objects taken r at a time is denoted by

$$_nP_r = n(n-1)(n-2)\cdots(n-(r-1)) = n!/(n-r)!$$

We may consider by putting n distinct objects in r compartments:

r compartments

n ways	$n - 1$ ways	$n - 2$ ways	....	$n - (r - 1)$ ways

By the multiplication principle, there are

$$n(n-1)(n-2)\cdots(n-(r-1)) \text{ ways.}$$

Permutations of *n* distinct objects

A special case is when $r = n$.

- The number of permutations of n distinct objects taken all together is

$$ {}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n! $$

Example 1

Find the number of all possible four-letter code words in which all letters are different.

Solution

There are 26 distinct letters. $n = 26$

We want a 4 letter code word. $r = 4$

The number of all possible four-letter code words is

$$_{26}P_4 = \frac{26!}{(26-4)!} = \frac{26!}{22!} = 26(25)(24)(23) = 358800$$

Example 2

Twenty horses take part in a horse race.

How many possible ways to arrange the first, second and third places?

Example 2 (Continued)

Solution

There are 20 distinct horses. $n = 20$

We want first 3 places. $r = 3$

The number of ways to arrange the first, second and third places is given by

$$ {}_{20}P_3 = \frac{20!}{17!} = 20(19)(18) = 6840 $$

Example 3

(a) How many ways can 6 persons line up to get on a bus?

(b) If certain 3 persons insisting on following each other, how many ways can these 6 persons line up?

(c) If 2 persons refuse to follow each other, how many ways of line up are possible?

Example 3 (continued)

(a) How many ways can 6 persons line up to get on a bus?

Solution

(a) There are 6 persons (objects). Hence $n = 6$.

There are 6 places in the queue to permute. Hence $r = 6$.

Therefore there are

$$ {}_6P_6 = 6! = 720 \text{ ways}. $$

Example 3 (continued)

(b) If certain 3 persons insisting on following each other, how many ways can these 6 persons line up?

Solution

(b) Let a, b, c, d, e, f denote the 6 persons.

- Without loss of generality, we assume a, b, c insisting on following each other.
- Group a, b and c into one group. Denote this group by A. i.e. $A = \{a, b, c\}$
- Then we permute 4 objects, A, d, e, f. Hence the number of permutations is ${}_4P_4 = 4! = 24$.

Example 3 (continued)

(b) If certain 3 persons insisting on following each other, how many ways can these 6 persons line up?

Solution (Continued)

(b) However, for each permutation such as (A, d, e, f), we can permute a, b, c within the group A.

The number of permutations within the group A is

$$ {}_3P_3 = 3! = 6. $$

Therefore, applying the multiplicative rule, the number of ways to form the line up is

$$ 24 \times 6 = 144. $$

Example 3 (Continued)

(c) If 2 persons refuse to follow each other, how many ways of line up are possible?

Solution

(c) Firstly, find the number of ways of line up with these 2 persons follow each other.

Similar to part (b), the number of line up with these 2 persons follow each other is

$$_5P_5 \times {}_2P_2 = 5! \times 2! = 240.$$

Example 3 (Continued)

(c) If 2 persons refuse to follow each other, how many ways of line up are possible?

Solution (Continued)

(c) From part (a), the total number of line up is 720.

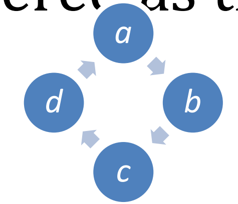
Therefore there are $720 - 240 = 480$ possible ways of line up with the 2 persons not following each other.

1.3.3.2 Permutations of n distinct objects arranged in a circle

- The number of permutations of n distinct objects arranged in **a circle** is $(n-1)!$.
- By considering 1 person in a fixed position and arranging the other $n-1$ persons, therefore there are $(n-1)!$ ways.
- Consider the following 4 different ways of arrangement in a line

$$abcd \qquad bcda \qquad cdab \qquad dabc$$

And these arrangements are considered as the same arrangement in a circle



Permutations of n distinct objects arranged in a circle (Continued)

Example

How many different arrangements are possible for sitting 4 persons around a round table?

Solution

There are $(4-1)! = 6$ ways for sitting 4 persons around a round table.

1.3.3.3 Permutations when not all n objects are distinct

- Suppose we have n objects such that there are n_1 of one kind, n_2 of second kind, $\cdots$, n_k of a k^{th} kind, where

$$n_1 + n_2 + \cdots + n_k = n\,.$$

- Then the number of distinct permutations of these n objects taken all together is given by

$${}_nP_{n_1,n_2,\cdots,n_k} = \frac{n!}{n_1!\,n_2!\cdots n_k!}$$

Example 1

In how many ways can 3 red, 4 yellow and 2 blue bulbs be arranged in a string of Christmas tree lights with 9 sockets?

Solution

There are $3 + 4 + 2 = 9$ objects. i.e. $n = 9$

There are 3 red, 4 yellow and 2 blue bulbs.

i.e. $n_1 = 3, n_2 = 4$ and $n_3 = 2$

The total number of distinct arrangements is

$$\frac{9!}{3!\,4!\,2!} = 1260$$

Example 2

In how many ways can 4 Mathematics books, 3 Physics books, and 5 Chemistry books are arranged in a bookshelf ?

Solution

There are 12 books. i.e. $n = 12$

There are 4 Maths, 3 Physics and 5 Chemistry books.

i.e. $n_1 = 4, n_2 = 3$ and $n_3 = 5$

The total number of distinct arrangements is

$$\frac{12!}{4!\,3!\,5!} = 27720$$

Example 3

How many ways can 7 people be assigned to 1 triple and 2 double rooms? Assume that the double rooms are different

Solution

There are 7 people. i.e. $n = 7$

There are 1 group of 3 persons, 2 groups of 2 persons.

i.e. $n_1 = 3, n_2 = 2$ and $n_3 = 2$

Number of ways of assignments is given by

$$\frac{7!}{3!\,2!\,2!} = 210$$

1.3.4 Combination

- In many problems we are interested in the number of ways of selecting r objects from n objects **without regard to the order**.
- These selections are called **combinations.**
- A combination creates a partition with 2 groups, one group containing the r objects selected and the other group containing the $n - r$ objects that are left.
- The number of such combinations is denoted by

$$\binom{n}{r} \quad \text{or} \quad {}_nC_r \quad \text{or} \quad C_r^n$$

Combination (Continued)

- The number of **combinations** of n **distinct** objects taken r at a time is

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Combination (Continued)

- To develop the formula, let us first examine the following ${}_4P_3$ permutations of the 4 letters a, b, c, d taken 3 at a time:

abc	acb	bac	bca	cab	cba
abd	adb	bad	bda	dab	dba
acd	adc	cad	cda	dac	dca
bcd	bdc	cbd	cdb	dbc	dcb

Combination (Continued)

- If we are not concerned with the order in which 3 letters are chosen from the 4 letters a, b, c, d, there are only 4 ways in which the selection can be made.
- They are those shown in the first column.
- Each row of the table merely contains the $3! = 6$ different permutations of the letters shown in the first column. Hence

$$4 \times 3! = {}_4P_3.$$

Combination (Continued)

- In general, there are $r!$ permutations of any r objects we select from a set of n distinct objects.
- Let $\binom{n}{r}$ or ${}_nC_r$ denote the number of ways of choosing r out of n distinct objects, disregarding order.
- Then we have

$$ {}_nC_r \, r! = {}_nP_r $$

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)!}\frac{1}{r!} = \frac{n!}{r!\,(n-r)!} $$

Binomial Coefficient

- The quantity $\binom{n}{r}$ is called a **binomial coefficient** because it is the coefficient of the term $a^r b^{(n-r)}$ in the binomial expansion of $(a+b)^n$.
- It can be verified that the following hold:

1. $\binom{n}{r} = \binom{n}{n-r}$ for $r = 0, 1, \cdots, n$
2. $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$ for $1 \leq r \leq n$
3. $\binom{n}{r} = 0$ for $r < 0$ or $r > n$

Example 1

A box contains 25 good apples and 5 bad apples. If a sample of 5 apples are selected from the box (**without** replacement).

(a) How many samples can be selected?

(b) How many of the samples in (a) involve exactly one bad apple?

(c) How many of the samples in (a) involve at least 2 bad apples?

Solution to Example 1

(a) There are $25 + 5$ distinct apples. i.e. $n = 30$

We select a group of 5 apples. i.e. $r = 5$

The number of samples can be selected is given by

$$ {}_{30}C_5 = \frac{30!}{5!\,25!} = 142506. $$

Solution to Example 1 (Continued)

(b) The number of ways a bad apple being selected from the 5 bad ones is ${}_5C_1 = 5$.

The number of ways 4 good apples being selected from 25 good ones is ${}_{25}C_4 = 25!/(4!\,21!) = 12650$.

There are 2 groups in the sample: One group of bad apple and another group of good apples.

Solution to Example 1 (Continued)

(b) (Continued)

For each choice of the first group of bad apple, there are $_{25}C_4$ ways of selecting 4 good apples.

By the **multiplication principle**, the number of samples involving exactly one bad apple is

$$_{5}C_1 \times {}_{25}C_4 = 5(12650) = 63250$$

Solution to Example 1 (Continued)

(c) Number of samples with exactly two bad apples is

$$ {}_5C_2 \times {}_{25}C_3 = 23000. $$

Number of samples with exactly three bad apples is

$$ {}_5C_3 \times {}_{25}C_2 = 3000. $$

Number of samples with exactly four bad apples is

$$ {}_5C_4 \times {}_{25}C_1 = 125. $$

Number of samples with exactly five bad apples is

$$ {}_5C_5 \times {}_{25}C_0 = 1. $$

Solution to Example 1 (Continued)

(c) Since **at least** two bad apples in the sample implies there are two or three or four or five bad apples in the sample, therefore by **the addition principle**, the number of samples with at least two bad apples is given by

$$\begin{aligned} {}_5C_2 \times {}_{25}C_3 + {}_5C_3 \times {}_{25}C_2 + {}_5C_4 \times {}_{25}C_1 + {}_5C_5 \times {}_{25}C_0 \\ = 23000 + 3000 + 125 + 1 \\ = 26126. \end{aligned}$$

Example 2

From 4 women and 3 men, find the number of committees of 3 that can be formed with 2 women and 1 man.

Solution to Example 2

Number of ways of selecting 2 women from 4 is given by

$$_4C_2 = 6$$

Number of ways of selecting 1 man from 3 is given by

$$_3C_1 = 3$$

By the **multiplication principle**, the number of possible committees formed with 2 women and 1 man is given by

$$_4C_2 \times {_3C_1} = 6(3) = 18.$$

Example 3

From a group of 4 men and 5 women, how many committees of size 3 are possible

(a) with **no restriction**?

(b) with 2 men and 1 woman if a ***certain man*** *must be on the committee*?

(c) with 2 men and 1 woman if ***2 of the men*** are feuding and ***refuse*** *to serve* on the committee **together**?

Solution to Example 3

(a) Number of committees $= {}_9C_3 = 9!/(3!\,6!) = 84$.

(b) Since a particular man must be on the committee, therefore we can only choose one man from the remaining 3 men to the committee.

Hence the number of ways to choose 3 men to fill in the remaining place of the committee with a certain man in the committee is ${}_1C_1 \times {}_3C_1 = 3$.

Therefore the number of committees is

$$({}_1C_1 \times {}_3C_1) \times {}_5C_1 = 15.$$

Solution to Example 3 (continued)

(c) There are ${}_2C_2 \times {}_5C_1 = 5$ ways to form a committee on which these 2 “particular” men serve together.

As such a case is undesirable, thus the number of desirable ways of forming a committee is given by

$${}_4C_2 \times {}_5C_1 - {}_2C_2 \times {}_5C_1 = 30 - 5 = 25.$$

Example 4

Shortly after being put into service, some buses manufactured by a certain company have developed cracks on the underside of the main frame.

Suppose a particular bus company has 20 of these buses, and the cracks have actually appeared in 8 of them.

(a) How many ways are there to select a sample of 5 buses from the 20 for a thorough inspection?

(b) In how many ways can a sample of 5 buses contain **exactly** 4 buses with visible cracks?

Solution to Example 4

(a) Number of ways selecting 5 buses out of 20 buses

$$_{20}C_5 = 20!/(5!\,15!) = 15504.$$

Solution to Example 4 (Continued)

(b) Number of ways selecting 4 buses from the 8 buses with visible cracks is

$$_8C_4 = 8!/(4!\,4!) = 70.$$

Number of ways selecting 1 bus from the remaining 12 buses with no cracks is

$$_{12}C_1 = 12!/(1!\,11!) = 12.$$

Number of samples of 5 buses that contains exactly 4 buses with visible cracks

$$_8C_4 \times {}_{12}C_1 = 840.$$

Example 5

How many bridge hands are possible containing 4 spades, 6 diamonds, 1 club and 2 hearts?

Solution

Number of possible hands:

$$
\begin{aligned}
&{}_{13}C_4 \times {}_{13}C_6 \times {}_{13}C_1 \times {}_{13}C_2 \\
&= 715(1716)(13)(78) \\
&= 1{,}244{,}117{,}160.
\end{aligned}
$$

1.4 Relative frequency and definition of probability

1.4.1 Introduction

- In an experiment, we do not know which particular outcome will occur when the experiment is performed.
- If A is an event associated with the experiment, then we cannot state with certainty that A will or will not occur.
- Hence it becomes very important to try to associate a number with the event A which will **measure how likely** the event A occurs.

Relative frequency and definition of probability

(Continued)

Introduction (Continued)

- This task leads us to the theory of probability.
- You may refer to p.1-164 to p1-165 on computing probability based on the sample space.

1.4.2 Relative Frequency

- Alternatively, probability can be derived based on the relative frequency
- Suppose we repeat the experiment E for n times and let A be an event associated with E.
- We let n_A be the number of times that the event A has occurred among the n repetitions respectively.
- Then $f_A = \frac{n_A}{n}$ is called the ***relative frequency*** of the event A in the n repetitions of E.

Properties of Relative Frequency

The relative frequency f_A has the following important properties:

(1) $0 \leq f_A \leq 1$.

(2) $f_A = 1$ if and only if A occurs every time among the n repetitions.

(3) $f_A = 0$ if and only if A never occurs among the n repetitions.

Properties of Relative Frequency (Continued)

(4) If A and B are two **mutually exclusive** events and if $f_{A \cup B}$ is the relative frequency associated with the event $A \cup B$, then $f_{A \cup B} = f_A + f_B$.

(5) f_A "**stabilizes**" near some definite numerical value as the experiment is repeated more and more times.

For example, when tossing a die, the relative frequency of having a "1" stabilizes to $1/6$ when the number of experiments is very large.

1.4.3 Axioms of Probability

- Consider an experiment whose sample space is S.
- The objective of probability is to assign to each event A, a number $\Pr(A)$, called the **probability** of the event A, which will give a precise measure of the chance that A will occur.
- Consider the collection of all events and denote it by $\mathcal{P}$.
- For each event A of the sample space S we assume that a number $\Pr(A)$, which is called the **probability** of the event A, is defined and satisfies the following three axioms:

Axioms of Probability (Continued)

Axiom 1: $0 \leq \Pr(A) \leq 1$.

Axiom 2: $\Pr(S) = 1$.

Axiom 3: If $A_1, A_2, \cdots$ are **mutually exclusive** (disjoint) events (that is, $A_i \cap A_j = \emptyset$ when $i \neq j$), then

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(A_i)$$

In particular, if A and B are two mutually exclusive events then $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

Example 1

Consider an experiment of tossing a coin. Let H denote the event of having a head.

Find $\Pr(H)$ if

(a) the coin is fair;

(b) if the coin is biased and a head is twice as likely to appear as a tail.

Solution to Example 1

<u>Solution</u>

(Refer to p.1-164 to p.1-165 on computing probability based on the sample space)

Sample space $= \{H, T\}$

(a) $\Pr(H) + \Pr(T) = 1$

Since the coin is fair, therefore $\Pr(H) = \Pr(T)$

Solving the 2 equations, we have $\Pr(H) = 1/2$.

Hence $\Pr(H) = 1/2$ for a fair coin.

Solution to Example 1 (Continued)

Solution (Continued)

(b) If a head is twice as likely to appear as a tail, then $\Pr(H) = 2\Pr(T)$ and $\Pr(H) + \Pr(T) = 1$.

Solving the 2 equations for $\Pr(H)$, we have

$$\Pr(H) = \frac{2}{3}.$$

Example 2

- A fair die is tossed.
- Let A be the event that an even number turns up and let B be the event that either an “1” or a “3” occurs.
- Find $\Pr(A)$, $\Pr(B)$ and $\Pr(A \cup B)$.

Solution to Example 2

Sample space = {1, 2, 3, 4, 5, 6}

- A = {2, 4, 6}. Hence $\Pr(A) = 3/6 = 1/2$.
- B = {1, 3}. Hence $\Pr(B) = 2/6 = 1/3$.
- Since $A \cap B = \emptyset$, hence A and B are mutually exclusive.
- Thus,

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) \\ &= 1/2 + 1/3 = 5/6.\end{aligned}$$

Example 3

Refer to example 2.

- Let C be the event that a number divisible by 3 occurs.
- Find $\Pr(C)$ and $\Pr(A \cup C)$.

Solution to Example 3

Sample space = {1, 2, 3, 4, 5, 6} and C = {3, 6}

Hence $\Pr(C) = 2/6$.

$A \cup C = \{2, 3, 4, 6\}$. Hence $\Pr(A \cup C) = 2/3$.

Notice that

$$\Pr(A) + \Pr(C) = 1/2 + 1/3 = 5/6 \neq \Pr(A \cup C) = 2/3$$

This is because A and C are **not mutually exclusive** events as $A \cap C = \{6\} \neq \emptyset$.