ST2334 (2021/22 Semester 1)
Solution to Tutorial 6

Question 1

$f_{(X,Y)}(x,y)$		x			$\boldsymbol{f_Y(y)}$
		1	2	3	
y	1	0.05	0.05	0.10	**0.20**
	2	0.05	0.10	0.35	**0.50**
	3	0	0.20	0.10	**0.30**
$\boldsymbol{f_X(x)}$		**0.10**	**0.35**	**0.55**	1

(a)

x	1	2	3
$f_X(x)$	0.10	0.35	0.55

(b)

y	1	2	3
$f_Y(y)$	0.20	0.50	0.30

(c) Find $\Pr(Y = 3 \mid X = 2)$.

$$f_{Y|X}(y = 3|x = 2) = \frac{f_{X,Y}(2,3)}{f_X(2)} = \frac{0.20}{0.35} = \frac{4}{7} = 0.57143$$

(d) $f_{Y|X}(y|x = 2) = f_{X,Y}(2, y)/f_X(2)$

y	1	2	3
$f_{Y\|X}(y\|x = 2)$	0.05/0.35 = 1/7 = 0.14286	0.10/0.35 = 2/7 = 0.28571	0.20/0.35 = 4/7 = 0.57143

(e) X and Y are dependent if $f_{X,Y}(x, y) \neq f_X(x)f_Y(y)$ for some values of x and y.
$f_{X,Y}(1,1) = 0.05$
$f_X(1)f_Y(1) = (0.10)(0.20) = 0.02$
Since $f_{X,Y}(1, 1) \neq f_X(1)f_Y(1)$, therefore X and Y are dependent.

Question 2

(a) First, random variable X can only take values in 0, 1, 2, 3; Y in 0, 1, 2. As only 4 pieces of fruit is selected, therefore $x + y \leq 4$. Since there are only three bananas, one piece of the selected fruit must be either an orange or an apple, that is, $x + y \geq 1$.

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{\binom{3}{x}\binom{2}{y}\binom{3}{4-x-y}}{\binom{8}{4}}, & x = 0,1,2,3;\ y = 0,1,2;\ 1 \leq x + y \leq 4; \\ 0, & \text{otherwise.} \end{cases}$$

(b) $\Pr(X = 1, Y = 1) = f_{X,Y}(1,1) = \dfrac{\binom{3}{1}\binom{2}{1}\binom{3}{2}}{\binom{8}{4}} = 0.2571$

(c) $\Pr(X + Y \leq 2) = f_{X,Y}(0,1) + f_{X,Y}(0,2) + f_{X,Y}(1,0) + f_{X,Y}(1,1) + f_{X,Y}(2,0) = 0.5$

(d) Recall the possible values of X are 0, 1, 2, 3. Since 4 pieces of fruit are selected, $(4 - X)$ pieces of fruit must be selected from 5 pieces of apples and bananas. That is,

$$f_X(x) = \begin{cases} \dfrac{\binom{3}{x}\binom{5}{4-x}}{\binom{8}{4}}, & x = 0,1,2,3; \\ 0, & \text{otherwise.} \end{cases}$$

(e) For $x = 2$,

$$f_{Y|X}(y|2) = \begin{cases} \dfrac{\binom{2}{y}\binom{3}{4-2-y}}{\binom{5}{4-2}}, & y = 0,1,2; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$f_{Y|X}(y|2) = \begin{cases} \dfrac{\binom{2}{y}\binom{3}{2-y}}{\binom{5}{2}} = \dfrac{1}{10}\binom{2}{y}\binom{3}{2-y}, & y = 0,1,2; \\ 0\,, & \text{otherwise.} \end{cases}$$

and $\Pr(Y = 0|X = 2) = \frac{1}{10}\binom{2}{0}\binom{3}{2} = \frac{1}{10}(1)(3) = 0.3.$

Question 3

Let D_1 and D_2 denote the number obtained by the first die and the second die respectively. The entries of the table below correspond to the values of (x, y) as defined in the question:

d_2	d_1 = 1	2	3	4	5	6
1	(0, 0)	(0, 0)	(0, 0)	(1, 0)	(0, 1)	(0, 0)
2	(0, 0)	(0, 0)	(0, 0)	(1, 0)	(0, 1)	(0, 0)
3	(0, 0)	(0, 0)	(0, 0)	(1, 0)	(0, 1)	(0, 0)
4	(1, 0)	(1, 0)	(1, 0)	(2, 0)	(1, 1)	(1, 0)
5	(0, 1)	(0, 1)	(0, 1)	(1, 1)	(0, 2)	(0, 1)
6	(0, 0)	(0, 0)	(0, 0)	(1, 0)	(0, 1)	(0, 0)

(a) From the table above, we have

$f_{(X,Y)}(x,y)$		x = 0	1	2	$\boldsymbol{f_Y(y)}$
y	0	16/36 = 4/9	8/36 = 2/9	1/36	**25/36**
	1	8/36 = 2/9	2/36 = 1/18	0	**5/18**
	2	1/36	0	0	**1/36**
$\boldsymbol{f_X(x)}$		**25/36**	**5/18**	**1/36**	1

(b) $\Pr(2X + Y < 3) = f_{X,Y}(0,0) + f_{X,Y}(0,1) + f_{X,Y}(0,2) + f_{X,Y}(1,0)$

$$= \frac{4}{9} + \frac{2}{9} + \frac{1}{36} + \frac{2}{9} = \frac{11}{12} = 0.91667.$$

(c) X and Y are dependent if $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$ for some values of X and Y.
$f_X(2)f_Y(2) = (1/36)(1/36) = 1/1296 \neq f_{X,Y}(2,2)\ (= 0)$
Since $f_{X,Y}(2,2) \neq f_X(2)f_Y(2)$, therefore X and Y are dependent.

Question 4

$$f_{X,Y}(x,y) = \begin{cases} k(x^2+y^2), & 3 \le x \le 5; \quad 3 \le y \le 5; \\ 0, & \text{otherwise} \end{cases}$$

$$k\int_3^5\int_3^5 (x^2+y^2)\,dy\,dx = k\int_3^5 \left[yx^2 + \frac{y^3}{3}\right]_3^5 dx$$

$$= \frac{2}{3}k\int_3^5 3x^2 + 49\,dx = \frac{2}{3}k[x^3 + 49x]_3^5 = \frac{392}{3}k$$

Hence, $k\int_3^5\int_3^5 (x^2+y^2)\,dydx = 1$ implies $\frac{392}{3}k = 1$ or $k = \frac{3}{392}$

(b) $\Pr(3 \le X \le 4 \text{ and } 4 \le Y < 5)$

$$= \frac{3}{392}\int_3^4\int_4^5 (x^2+y^2)\,dydx = \frac{3}{392}\int_3^4\left[yx^2 + \frac{y^3}{3}\right]_4^5 dx$$

$$= \frac{3}{392}\int_3^4\left(x^2 + \frac{61}{3}\right)dx = \frac{1}{392}[x^3 + 61x]_3^4 = \frac{1}{392}(98) = \frac{1}{4} = 0.25$$

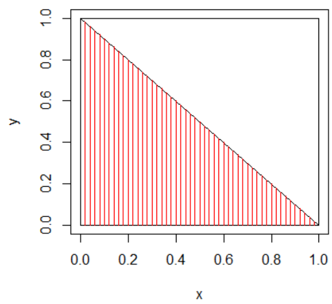
(c) $f_X(x) = \frac{3}{392}\int_3^5 (x^2+y^2)dy = \frac{3}{392}\left[x^{2y} + \frac{y^3}{3}\right]_3^5 = \frac{3}{392}\left(2x^2 + \frac{98}{3}\right) = \frac{1}{196}(3x^2+49)$, for $3 \le x \le 5$

$$\Pr(3.5 < X < 4) = \frac{1}{196}\int_{3.5}^4 (3x^2+49)\,dx$$

$$= \frac{1}{196}[x^3 + 49x]_{3.5}^4 = \frac{1}{196}\frac{365}{8} = \frac{365}{1568} = 0.2328$$

<u>Question 5</u>

$$f_{X,Y}(x,y) = \begin{cases} 24xy, & 0 \le x \le 1, \quad 0 \le y \le 1, \quad x + y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$



(a) $f_X(x) = \int_0^{1-x}(24xy)\,dy = [12xy^2]_0^{1-x} = 12x(1-x)^2$, for $0 \le x \le 1$

$f_Y(y) = \int_0^{1-y}(24xy)\,dx = [12x^2y]_0^{1-x} = 12y(1-y)^2$, for $0 \le y \le 1$

(b) $f_{X,Y}(x,y) = 24xy \ne f_X(x)f_Y(y)(= 12x(1-x)^2 12y(1-y)^2)$, for $0 \le x \le 1$ and $0 \le y \le 1$. Hence X and Y are not independent.

Alternatively, we may consider a point, let say, $(x,y) = \left(\frac{2}{3},\frac{1}{2}\right)$. Then $f_X\left(\frac{2}{3}\right) = \frac{8}{9}$ and $f_Y\left(\frac{1}{2}\right) = \frac{3}{2}$, while $f_{X,Y}\left(\frac{2}{3},\frac{1}{2}\right) = 0$

(c) For $0 \le x \le 1$,

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{24xy}{12x(1-x)^2} = \frac{2y}{(1-x)^2}, \text{ for } 0 \le y \le 1-x.$$

$$f_{Y|X}\left(y|x = \frac{3}{4}\right) = \frac{2y}{\left(1-\frac{3}{4}\right)^2} = 32y, \text{ for } 0 \le y \le \frac{1}{4}$$

$$\Pr\left(Y < \frac{1}{8}|x = \frac{3}{4}\right) = \int_0^{\frac{1}{8}} f_{Y|X}\left(y|x=\frac{3}{4}\right)dy = \int_0^{\frac{1}{8}}(32y)dy = [16y^2]_0^{\frac{1}{8}} = \frac{1}{4}$$

$$= 0.25.$$