

ST2334 (2021/22 Semester 1) Solution to Tutorial 9

Question 1

Let X be the time between two successive arrivals at the drive-up window of a fast-food restaurant. Then $X \sim \text{Exp}(1)$.

- (a) $E(X) = 1/\lambda = 1$.
- (b) $\sigma = 1/\lambda = 1$.
- (c) $\Pr(X \leq 4) = 1 - e^{-1(4)} = 0.9817$.
 $\Pr(2 \leq X \leq 5) = 1 - e^{-5} - (1 - e^{-2}) = 0.1286$.
 (Excel: “=expon.dist(4,1,true)” ; R: “pexp(4,1,lower=T)”)

Question 2

Let X be the time until failure for the fan. Then $X \sim \text{Exp}(1/25000)$

- (a) $\Pr(X > 20000) = e^{-20000/25000} = 0.4493$.
 $\Pr(X \leq 30000) = 1 - e^{-30000/25000} = 0.6988$.
 $\Pr(20000 \leq X \leq 30000) = 0.6988 - (1 - 0.4493) = 0.1481$.
- (b) $\sigma = 1/\lambda = 25000$. Therefore $\Pr(X > \mu + 2\sigma) = \Pr(X > 75000) = e^{-75000/25000} = 0.0498$.

Question 3

X = length of time to fail, in years

$X \sim \text{Exp}(1/2)$

- (a) $V(X) = [E(X)]^2 = [2]^2 = 4$
- (b) $\Pr(X < 1) = 1 - e^{-(1/2)(1)} = 0.39347$
 Y = number of electrical switches out of 100 switches that fail during the first year.
 $Y \sim \text{Binomial}(n = 100, p = 0.39347)$
 $E(Y) = np = 39.35$, $V(Y) = np(1 - p) = 23.87$

$$\Pr(Y \leq 30) = \Pr(Y \leq 30.5) = \Pr\left(Z \leq \frac{30.5 - 39.35}{\sqrt{23.87}}\right)$$

$$= \Pr(Z < -1.81097) = 0.03507$$

(Excel: “=1-norm.dist(30.5,39.35,sqrt(23.87),true)” ; R: “1-

pnorm(30.5,39.35,sqrt(23.87),lower=T)”)

[Exact probability: $\Pr(Y \leq 30) = 0.03347$.]

(Excel: “=binom.dist(30,100,0.39347,false)” ; R: “dbinom(30,100,0.39347,lower=T)”)

Question 4

$$\Pr(\mu - 3\sigma < X < \mu + 3\sigma) = \Pr(-3 < Z < 3) = \Pr(Z < 3) - \Pr(Z < -3)$$

$$= 0.99865 - 0.00135 = 0.9973$$

[Compare with $\Pr(\mu - 3\sigma < X < \mu + 3\sigma) \geq 8/9$ using Chebyshev's Inequality]

Question 5

X = amount of the soft drink

$X \sim \text{Normal}(\mu = 200; \sigma^2 = 15^2)$

- (a) $\Pr(X > 224) = \Pr(Z > 1.60) = 0.05480$, where $Z \sim N(0,1)$
 (Excel: “=1-norm.dist(224,200,15,true)” ; R: “1-pnorm(224,200,15,lower=T)”)
- (b) $\Pr(191 < X < 209) = \Pr(-0.60 < Z < 0.60) = 0.4515$
- (c) $\Pr(X > 230) = \Pr(Z > 2.00) = 0.02275 = p$
 Y = number of cups out of 100 cups that overflow
 $Y \sim \text{Binomial}(n = 1000, p = 0.02275)$
 $E(Y) = np = 1000(0.02275) = 22.75 \approx 23$

- (d) Let $z_{0.25}$ denote the 25th percentile of the standard normal distribution. That is, $\Pr(Z < z_{0.25}) = 0.25$, where $Z \sim N(0,1)$. Hence, $z_{0.25} = -0.6745$.
 Note $Z = \frac{X-\mu}{\sigma}$ or $X = \mu + \sigma Z$. Therefore, $x_{0.25} = \mu + z_{0.25}\sigma = 200 + (-0.6745)(15) = 189.883$, where $x_{0.25}$ denotes the 25th percentile of the distribution for X (i.e., $\Pr(X < x_{0.25}) = 0.25$).
 (Excel: “=norm.inv(0.25,200,15)” ; R: “qnorm(0.25,200,15)”)

Question 6

X = commute time from home to office

$X \sim \text{Normal}(\mu = 24; \sigma^2 = 3.8^2)$

- (a) $\Pr(X > 30) = \Pr(Z > 1.57895) = 0.057174$
 (b) $\Pr(X > 15) = \Pr(Z > -2.36842) = 1 - 0.0089321 = 0.991068 = 99.11\%$
 (c) Y = number of trips out of 3 trips that take at least half an hour
 $Y \sim \text{Binomial}(n = 3, p = 0.057174)$

$$\Pr(Y = 2) = \binom{3}{2} (0.057174)^2 (1 - 0.057174)^1 = 0.0092459$$

(Excel: “=binom.dist(2,3,0.057174,false)” ; R: “dbinom(2,3,0.057174)”)

Question 7

Y = number of head in 400 tosses of a coin

$Y \sim \text{Binomial}(n = 400, p = 0.5)$

$E(Y) = np = 400(0.5) = 200$. $V(Y) = np(1 - p) = 400(0.5)(0.5) = 100$

$Y \sim \text{Normal}(\mu = 200; \sigma^2 = 100)$

- (a) $\Pr(185 \leq Y \leq 210) = \Pr(184.5 < Y < 210.5) = \Pr(-1.55 < Z < 1.05)$
 $= \Pr(Z < 1.05) - \Pr(Z < -1.55) = 0.853141 - 0.060571 = 0.79257$
 (b) $\Pr(Y = 205) = \Pr(204.5 < Y < 205.5) = \Pr(0.45 < Z < 0.55)$
 $= \Pr(Z < 0.55) - \Pr(Z < 0.45) = 0.70884 - 0.67364 = 0.03519$
 (c) $\Pr(Y < 176 \text{ or } Y > 227) = \Pr(Y < 175.5) + \Pr(Y > 227.5)$
 $= \Pr(Z < -2.45) + \Pr(Z > 2.75) = 0.007143 + 0.002980 = 0.010123$

Question 8

Y = number of drunk driver

$Y \sim \text{Binomial}(n = 400, p = 0.1)$

$E(Y) = np = 400(0.1) = 40$. $V(Y) = np(1 - p) = 400(0.1)(0.9) = 36$

$Y \sim \text{Normal}(\mu = 40; \sigma^2 = 6^2)$

- (a) $\Pr(Y < 32) = \Pr(Y < 31.5) = \Pr(Z < -1.41667) = 0.07829$
 (b) $\Pr(Y > 49) = \Pr(Y > 49.5) = \Pr(Z > 1.58333) = 0.056673$
 (c) $\Pr(35 \leq Y < 47) = \Pr(34.5 < Y < 46.5) = \Pr(-0.91667 < Z < 1.08333)$
 $= \Pr(Z < 1.08333) - \Pr(Z < -0.91667) = 0.860669 - 0.179658 = 0.681011$

Question 9

Y = number of defective parts

$Y \sim \text{Binomial}(n = 100, p = 0.05)$

$E(Y) = np = 100(0.05) = 5$. $V(Y) = np(1 - p) = 100(0.05)(0.95) = 4.75$

$Y \sim \text{Normal}(\mu = 5; \sigma^2 = 4.75)$

- (a) $\Pr(Y > 2) = \Pr(Y > 2.5) = \Pr(Z > -1.14708) = 0.87433$.
 (b) $\Pr(Y > 10) = \Pr(Y > 10.5) = \Pr(Z > 2.52357) = 0.0058085$

Question 10

- (a) $\mu = \sum x f_X(x) = 4(0.2) + 5(0.4) + 6(0.3) + 7(0.1) = 5.3$
 $\sigma^2 = \sum (x - \mu)^2 f(x) = (4 - 5.3)^2(0.2) + (5 - 5.3)^2(0.4) + (6 - 5.3)^2(0.3) + (7 - 5.3)^2(0.1) = 0.81$
- (b) With $n = 36$, $\mu_{\bar{X}} = \mu = 5.3$; $\sigma_{\bar{X}}^2 = \frac{\sigma^2}{n} = \frac{0.81}{36} = 0.0225$
- (c) Applying the Central Limit Theorem, $\bar{X} \text{ approx } \sim N(5.3, 0.0225)$
 $\Pr(\bar{X} < 5.5) = \Pr\left(Z < \frac{5.5 - 5.3}{\sqrt{0.0225}}\right) = \Pr(Z < 1.33333) = 0.90879$

Question 11

X = amount of benzene. $E(X) = \mu$ and $V(X) = 100^2$

- (a) $n = 25$. By the CLT, $\bar{X} \sim N\left(\mu, \frac{100^2}{25}\right)$. $\Pr(\bar{X} > 7950 | \mu = 7950) = \Pr(\bar{X} > \mu) = 0.5$
- (b) $X \sim N(\mu, 100^2)$. Hence $\bar{X} \sim N\left(\mu, \frac{100^2}{25}\right)$ approximately.
 $\Pr(\bar{X} \geq 7960 | \mu = 7950) = \Pr\left(Z > \frac{7960 - 7950}{100/\sqrt{25}}\right) = \Pr(Z > 0.5) = 0.30854$

No, there is no strong evidence that the population mean exceeds the government limit as it is likely to see a sample mean is equal to or larger than 7960 if the population mean equals to the government limit 7950.