

Statistic

- A **statistic** is a function of the random sample which does not depend on any unknown parameters.
- For example

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$$

or

$$X_{(n)} = \max(X_1, X_2, \dots, X_n)$$

are some examples of a statistic.

- ✓ A statistic/estimator must be a function of a certain sample, X_1, X_2, \dots, X_n say.
- ✓ It does not depend on any unknown parameter. For example
 - ★ $\sum_{i=1}^n (X_i - \mu)^2$ is NOT a statistic if μ is unknown; but it is a statistic if μ is assumed to be a known value.
 - ★ $(\bar{X} - 1)/5$ is a statistic; but $(\bar{X} - \mu)/\sigma$ is not if either μ or σ is unknown.
 - ★ $\min\{X_1, X_2, \dots, X_n\}$ is a statistic. $\min\{X_1, X_n\}$ is a statistic. For each i , X_i is also a statistic.
- ✓ A statistic/estimator can be viewed either as a random variable or as a computational rule, but not a computed value. They should be distinguished from their realized value based on the observed sample. For example, \bar{X} is an estimator, but \bar{x} is not. Question: whether a specific value, e.g., 1, can be viewed as a statistic/estimator?

Point Estimate of Mean

- Suppose μ is the population mean.
- The statistic that one uses to obtain a point estimate is called **an estimator**,
 For example, \bar{X} is an estimator of μ .
 The value of \bar{X} , denoted by \bar{x} , is an estimate of μ .

We should clearly distinguish these three concepts: an estimator/statistic (e.g., \bar{X}), an estimate (e.g., \bar{x}), a (population) parameter (e.g., μ).

- ✓ An estimator/statistic is a computational rule. It is also a random variable. When the data (random sample) are available, it tells how to compute. For example, \bar{X} is a random variable, and when we have $X_1 = x_1, X_2 = x_2, \dots, X_n = x_n$ available, it tells that we should compute the mean value of them.
- ✓ An estimate is a computed value of the estimator based on the observed data (random sample). It is NOT a random variable.
- ✓ A population parameter is something about the population, but unknown.
- ✓ Think about which of the following probability statements is/are valid.

★ $Pr(\bar{X} \leq 1) = 0.5.$

★ $Pr(0 < \bar{x} < 2) \leq 0.8.$

★ $Pr(\bar{x} - 4 < \mu < \bar{x} + 4) = 0.95.$

★ $Pr(\bar{X} - \bar{x} < \mu - 2) = 0.90.$

★ $Pr(\bar{X} = \bar{x}) > 0.2.$

Interval Estimation

- Interval estimation is to define two statistics, say,

$$\hat{\Theta}_L \text{ and } \hat{\Theta}_U, \quad \text{where } \hat{\Theta}_L < \hat{\Theta}_U$$

so that $(\hat{\Theta}_L, \hat{\Theta}_U)$ constitutes a random interval for which the probability of containing the unknown parameter θ can be determined.

- ✓ Both interval estimation and point estimation are used to estimate a specific parameter, e.g., μ , of a population.
- ✓ Point estimator uses one single statistic to estimate; but interval estimator uses two statistics, which form an interval, to estimate.
- ✓
- ✓ In analogous to the point estimator, the interval estimator a RANDOM interval, so that we can assert that $Pr(\hat{\theta}_L < \theta < \hat{\theta}_U) = 1 - \alpha$.
- ✓ Similarly to the point estimation, we should distinguish the terminologies “interval estimator” and “interval estimate”.

6.1.3 Unbiased Estimator

Definition 6.1 (Unbiased estimator)

- A statistic $\hat{\Theta}$ is said to be an **unbiased estimator** of the parameter θ if

$$E(\hat{\Theta}) = \theta.$$

We look at two criteria when comparing different estimators for the same parameter.

- ✓ First, we shall consider only unbiased estimators. The definition of unbiased estimator is given on this page of the lecture slide. Note that $E(\hat{\theta}) = \theta$ needs to be true for any arbitrary θ such that the probability density function $f_X(x; \theta)$ is correctly defined.
- ✓ Second, among the unbiased estimators, we shall recommend the one that leads to the smallest variance.

For example, consider the sample X_1, X_2, \dots, X_n with $n > 2$. The following are candidate estimators for estimating μ : $\hat{\mu}_1 = \bar{X}$, $\hat{\mu}_2 = X_1$, $\hat{\mu}_3 = \frac{X_1 + X_2}{2}$, $\hat{\mu}_4 = 2$, $\hat{\mu}_5 = X_n - X_1$.

- ✓ Based on the unbiasedness criterion, $\hat{\mu}_1$, $\hat{\mu}_2$, and $\hat{\mu}_3$ are all unbiased estimators; but $E(\hat{\mu}_4) = 2 \neq \mu$ and $E(X_n - X_1) = \mu - \mu = 0 \neq \mu$ indicate that they are biased. As a consequence, we should drop $\hat{\mu}_4$ and $\hat{\mu}_5$.

- ✓ Based on the second criteria, we need to compare the variances of $\hat{\mu}_1$, $\hat{\mu}_2$, and $\hat{\mu}_3$. $V(\hat{\mu}_1) = \sigma^2/n$, $V(\hat{\mu}_2) = \sigma^2$, and $V(\hat{\mu}_3) = \sigma^2/2$. So when $n > 2$, $\hat{\mu}_1$ is unbiased and has the minimum variance.

Unbiased Estimator (Continued)

Example 1

\bar{X} is an unbiased estimator of μ . That is, $E(\bar{X}) = \mu$.

Example 2

$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$ is an **unbiased** estimator of σ^2 .

That is,

$$E(S^2) = \sigma^2$$

Here is a derivation for $E(S^2) = \sigma^2$.

✓ Recall the formula:

$$\sum_{i=1}^n (X_i - \bar{X})^2 = \sum_{i=1}^n X_i^2 - n\bar{X}^2,$$

which is an algebraic formula, and is valid no matter whether X_i 's are random variables and no matter what are the values for X_i .

✓ Set $Y_i = X_i - \mu$. Then Y_1, \dots, Y_n are i.i.d. with $E(Y_i) = 0$ and $V(Y_i) = V(X_i) = \sigma^2$.

Furthermore $\bar{Y} = \bar{X} - \mu$, thus $E(\bar{Y}) = 0$ and $V(\bar{Y}) = V(\bar{X}) = \sigma^2/n$. We have

$$\begin{aligned} E\left(\sum_{i=1}^n (X_i - \bar{X})^2\right) &= E\left(\sum_{i=1}^n (Y_i - \bar{Y})^2\right) = E\left(\sum_{i=1}^n Y_i^2 - n\bar{Y}^2\right) \\ &= \sum_{i=1}^n EY_i^2 - nE(\bar{Y}^2) = \sum_{i=1}^n V(Y_i) - nV(\bar{Y}) = n\sigma^2 - n\sigma^2/n = (n-1)\sigma^2. \end{aligned}$$

Interval Estimation (Continued)

- We shall seek a random interval $(\hat{\theta}_L, \hat{\theta}_U)$ containing θ with a given probability $1 - \alpha$.
- That is

$$\Pr(\hat{\theta}_L < \theta < \hat{\theta}_U) = 1 - \alpha.$$

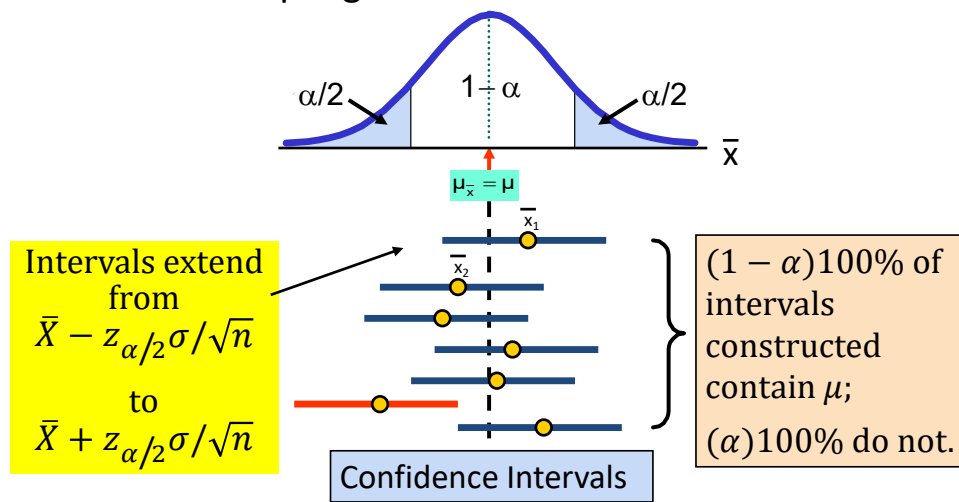
ST2334 Probability and Statistics

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Estimation based on Normal Distribution 6-18

Intervals and Level of Confidence

Sampling Distribution of the Mean



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Estimation based on Normal Distribution 6-21

There are two valid ways to interpret the meaning of level $1 - \alpha$ confidence interval.

- ✓ The first is given on page 6-18 of the lecture slide; it is also how the $1 - \alpha$ confidence interval is defined. Importantly, the upper and lower bounds $\hat{\theta}_L$ and $\hat{\theta}_U$ are random variable! For

example we can claim $Pr(\bar{X} - z_{\alpha/2}\sigma/\sqrt{n} < \mu < \bar{X} + z_{\alpha/2}\sigma/\sqrt{n}) = 1 - \alpha$, where \bar{X} is the random variable which supplies the randomness such that we can talk about “probability”.

But if you have the observed data $X_1 = x_1, \dots, X_n = x_n$, then you apply the formula to get the computed value of the confidence interval $(\bar{x} - z_{\alpha/2}\sigma/\sqrt{n}, \bar{x} + z_{\alpha/2}\sigma/\sqrt{n})$, it is no longer valid to talk about probability. In other words, $Pr(\bar{x} - z_{\alpha/2}\sigma/\sqrt{n} < \mu < \bar{x} + z_{\alpha/2}\sigma/\sqrt{n}) = 1 - \alpha$ makes nonsense, as there is no random variable in the $Pr(\cdot)$ statement. More specifically, if your computed confidence interval is $(3, 6)$, it is invalid to report “the probability that μ is contained in $(3, 6)$ is 95%.”

- ✓ The second is given on page 6-21. Imaging that we can sample the data infinitely many times from the population

- ★ Get the first sample $(X_1^{(1)}, X_2^{(1)}, \dots, X_n^{(1)})$ from the distribution $f_X(x; \theta)$, and compute the CI $(\hat{\theta}_{L,1}, \hat{\theta}_{U,1})$.
- ★ Get the second sample $(X_1^{(2)}, X_2^{(2)}, \dots, X_n^{(2)})$ from the distribution $f_X(x; \theta)$, and compute the CI $(\hat{\theta}_{L,2}, \hat{\theta}_{U,2})$.
- ★ Continue with this procedure \dots, \dots, \dots
- ★ Get the K th sample $(X_1^{(K)}, X_2^{(K)}, \dots, X_n^{(K)})$ from the distribution $f_X(x; \theta)$, and compute the CI $(\hat{\theta}_{L,K}, \hat{\theta}_{U,K})$.
- ★ For a sufficiently large K , the proportion of these intervals that contain the true value of θ is $1 - \alpha$.

- ✓ Practically, for a computed CI, we can only claim that with a certain confidence the interval will cover the true value. For example, if the computed 95% CI for μ is $(3, 6)$, we can only claim that we have 95% “confidence” that the true value of μ will be contained in $(3, 6)$.

Here is a general strategy (formula) for constructing mean related confidence intervals. Suppose we are to construct a $1 - \alpha$ confidence interval for mean related parameter θ (e.g., θ could be μ , $\mu_1 - \mu_2$, or other possible combinations of the population means)

- ✓ Step 1: look for an estimator $\hat{\theta}$ for θ , e.g., \bar{X} for θ , $\bar{X}_1 - \bar{X}_2$ for $\mu_1 - \mu_2$.
- ✓ Step 2: Derive the variance $V(\hat{\theta})$.
- ✓ Step 3: Construct $(1 - \alpha)$ CI to be $\hat{\theta} \pm M\sqrt{V}$. M is called the multiplier, and V is related to $V(\hat{\theta})$. The following is how they are determined.

- ★ If $V(\hat{\theta})$ does not depend on any other parameter (e.g., in the case σ^2 is known, $V(\bar{X}) = \sigma^2/n$), $V = V(\hat{\theta})$, and $M = z_{\alpha/2}$. Here we may need the condition that the data are normal or/and the sample size n is big.

The CIs given on pages 6-25 and 6-46 belong to this situation.

- ★ If the derived $V(\hat{\theta})$ contains some other unknown parameter, e.g., σ^2 , we replace the parameter with its estimator, e.g., we use S^2 to replace σ^2 ; this result in $\hat{V}(\hat{\theta})$. Then, we use $V = \hat{V}(\hat{\theta})$, however M has two possibilities (it has more in the literature):

- (1) if the sample size n is sufficiently large, $M = z_{\alpha/2}$; the CIs given on pages 6-36, 6-52, and 6-73 belong to this situation;
- (2) if the sample size n is not large, but the data are normally distributed, $M = t(df, \alpha/2)$. Here df = degrees of freedom, which is the df of the estimator for the parameter contained in $V(\hat{\theta})$. The CIs given on pages 6-35, 6-62, and 6-72 belong to this situation.

Note: this strategy does not apply to construct variance related CIs in general. See Pages from 6-78 to 6-100 for the development of the variance related CIs.

Unknown but Equal Variances (Continued)

- σ^2 can be estimated by the pooled sample variance

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2},$$

with S_1^2 and S_2^2 being the sample variances of the first and second samples respectively.

Let $X_{1,1}, X_{1,2}, \dots, X_{1,n_1}$ be observations from population 1 and let $X_{2,1}, X_{2,2}, \dots, X_{2,n_2}$ be observations from population 2. Then $(n_1 - 1)S_1^2 = \sum_{i=1}^{n_1} (X_{1,i} - \bar{X}_1)^2$ and $(n_2 - 1)S_2^2 = \sum_{j=1}^{n_2} (X_{2,j} - \bar{X}_2)^2$.

We therefore have

$$S_p^2 = \frac{\sum_{i=1}^{n_1} (X_{1,i} - \bar{X}_1)^2 + \sum_{j=1}^{n_2} (X_{2,j} - \bar{X}_2)^2}{n_1 + n_2 - 2}.$$

Note that this formula intuitively makes sense: every observation contributes equally in the estimation of their common variance σ^2 .

In terms of samples, it is the weighted average of the two sample variances with the weights being one less than the sample sizes.

Unknown but Equal Variances (Continued)

- Note that if the two populations are normal with the same variance σ^2 , then

$$\frac{(n_1 - 1)S_1^2}{\sigma^2} \sim \chi_{n_1-1}^2 \quad \text{and} \quad \frac{(n_2 - 1)S_2^2}{\sigma^2} \sim \chi_{n_2-1}^2,$$

Hence

$$\frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{\sigma^2} \sim \chi_{n_1+n_2-2}^2.$$

The derivation on page used the property of chi-square distribution: if $Y_1 \sim \chi^2(n_1)$, $Y_2 \sim \chi^2(n_2)$, and Y_1 and Y_2 are independent, then $Y_1 + Y_2 \sim \chi^2(n_1 + n_2)$.