ST2334 (2021/2022 Semester 1) Solutions to Questions in Tutorial 11

Question 1

$X =$ lifetime. $X \sim N(\mu, 40^2)$

(a) Test $H_0: \mu = 800$ against $H_1: \mu \neq 800$

z= $\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{788-800}{40/\sqrt{30}} = -1.64$

Since $|z_{obs}| = 1.64 < z_{0.025}$ $(= 1.96)$, therefore we do not reject H0.

Alternatively, p-value $= 2\min\{\Pr(Z < -1.64), \Pr(Z > -1.64)\} = 2(0.0505) = 0.1010$. Since p-value $> \alpha$ $(= 0.05)$, we do not reject H_0.

(b) 95% confidence interval for μ: $\bar{X} \pm Z\frac{\sigma}{\sqrt{n}} = 788 \pm 1.96\frac{40}{\sqrt{30}} = (773.69, 802.31)$.

Yes, 800 is plausible.

(c) Under H0, H0 is not rejected if $-1.96 < Z < 1.96$ or $\mu - 1.96\frac{\sigma}{\sqrt{n}} < \bar{X} < \mu + 1.96\frac{\sigma}{\sqrt{n}}$ or $785.69 < \bar{X} < 814.31$.

When $\mu = 790$ (i.e. H0 is false), $\bar{X} \sim N\left(790, \frac{40^2}{30}\right)$.

$\Pr(\text{Do not reject } H_0 | \mu = 790) = \Pr(785.69 < \bar{X} < 814.31 \mid \mu = 790) =$

$\Pr\left(\frac{785.69-790}{40/\sqrt{30}} < \frac{\bar{X}-790}{40/\sqrt{30}} < \frac{814.31-790}{40/\sqrt{30}}\right) = \Pr(-0.590 < Z < 3.329) = 0.9996 - 0.2774 = 0.7222.$

(d) When $\mu = 790$, Power $= 1 - \Pr(\text{Type II error } | \mu = 790) = 1 - 0.7222 = 0.2778$.

Question 2

$X =$ content of lubricant. $X \sim N(\mu, \sigma^2)$

(a) Test H_0: $\mu = 10$ against H_1: $\mu \neq 10$

From the data, $\bar{x} = 10.06$, $s = 0.24585$. Hence, $t_{obs} = \frac{\bar{x}-10}{s/\sqrt{10}} = \frac{10.06-10}{0.246/\sqrt{10}} = 0.772$.

Since $|t_{obs}| = 0.772 < t_{9;0.005}(= 3.25)$, therefore we do not reject H0

Alternatively, p-value $= 2\min\{\Pr(T < 0.772), \Pr(T > 0.772)\} = 0.4600$ (from statistical software). Since p-value > $\alpha(= 0.01)$, therefore we do not reject H0.

(b) Test $H_0: \sigma^2 = 0.03$ against H_1: $\sigma^2 \neq 0.03$

$\chi^2_{obs} = \frac{(n-1)s^2}{\sigma^2} = \frac{(9)(0.246)^2}{0.03} = 18.13$ which falls between $\chi^2_{9;0.975}$ $(= 2.70)$ and $\chi^2_{9;0.025}(= 19.023)$. Hence, we do not reject H0.

p-value $= 2\min\{Pr(\chi^2_9 > 18.13), Pr(\chi^2_9 < 18.13)\} = 0.0673$ from statistical software. Since the p-value > 0.05. We do not reject H0.

(c) 99% confidence interval for $\sigma^2 = \left(\frac{(n-1)s^2}{\chi^2{}_{9,0.025}}, \frac{(n-1)s^2}{\chi^2{}_{9,0.975}}\right) = \left(\frac{9(0.246)^2}{19.023}, \frac{9(0.246)^2}{2.7}\right) = (0.0286, 0.2014)$. Note: $\chi^2{}_{9,0.025}$ satisfies $\Pr(W > \chi^2{}_{9,0.025}) = 0.025$ with $W \sim \chi^2(9)$.

Question 3

$X =$ amount of soft drink dispensed. $X \sim N(\mu, \sigma^2)$.

Test $H_0: \sigma^2 = 1.15$ against H_1: $\sigma^2 > 1.15$

From the data, we have $n = 25$, $s^2 = 2.03$. Hence $\chi^2_{obs} = \frac{(n-1)s^2}{\sigma^2} = \frac{(24)(2.03)}{1.15} = 42.37$

Since the observed test statistic $> \chi^2_{24;0.05}$ $(= 36.415)$, we reject H0 at 5% significance level.

Alternatively, p-value is between 0.01 and 0.025 as $\Pr(\chi^2_{24} > 39.364) = 0.025$ and $\Pr(\chi^2_{24} > 42.98) = 0.01$ [Exact p-value = 0.0117]

Question 4

X_A = tensile strength of thread A. $E(X_A) = \mu_A$ and $V(X_A) = 6.28^2$

X_B = tensile strength of thread B. $E(X_B) = \mu_B$ and $V(X_B) = 5.61^2$

(a) Test H$_0$: $\mu_A - \mu_B = 12$ against H$_1$: $\mu_A - \mu_B > 12$

Let $Z = \frac{(\bar{X}_A - \bar{X}_B) - (\mu_A - \mu_B)}{\sqrt{\sigma_A^2/50 + \sigma_B^2/50}}$. $Z\ approx \sim N(0,1)$ by CLT since both n_A and n_B are large.

From the data, we have $n_A = 50$, $\bar{x}_A = 86.7$,$n_B = 50$, $\bar{x}_B = 77.8$. Hence

$z_{obs} = \frac{(86.7-77.8)-(12)}{\sqrt{\frac{6.28^2}{50}+\frac{5.61^2}{50}}} = -2.60$

Since $z_{obs} < z_{0.05}$ $(= 1.645)$, we do not reject H$_0$.

Alternatively, p-value $= \Pr(Z > -2.60) = 1 - 0.0047 = 0.9953$.

Since p-value $> \alpha$ $(= 0.05)$. We do not reject H$_0$.

(b) We committed an error if our decision of not rejecting H$_0$ is wrong. Hence it is Type II error. (Type I error is committed if our decision of rejecting H$_0$ is wrong.)

Question 5

X_A = grades of students in the 3-semester-hour course $\sim N(\mu_A, \sigma^2)$

X_B = grades of students in the 4-semester-hour course $\sim N(\mu_B, \sigma^2)$

From the data, $n_A = 18$, $\bar{x}_A = 77$, $s_A = 6$; n_B =12, $\bar{x}_B = 84$, $s_B = 4$. Hence,

$s_p = \sqrt{\frac{(n_A-1){s_A}^2+(n_B-1){s_B}^2}{n_A+n_B-2}} = 5.3050$

(a) 99% confidence interval for $\mu_B - \mu_A = (\bar{X}_B - \bar{X}_A) \pm t_{28,0.005} s_p\sqrt{\frac{1}{n_B} + \frac{1}{n_A}} =$

$(84 - 77) \pm (2.763)(5.304)\sqrt{\frac{1}{12} + \frac{1}{18}} = (1.537, 12.463)$.

Or 99% confidence interval for $\mu_A - \mu_B = (\bar{X}_A - \bar{X}_B) \pm t_{28,0.005} s_p\sqrt{\frac{1}{n_B} + \frac{1}{n_A}} =$

$(77 - 84) \pm (2.763)(5.304)\sqrt{\frac{1}{12} + \frac{1}{18}} = (-12.463, -1.537)$.

(b) Test H_0: $\mu_A - \mu_B = 0$ against H_1: $\mu_A - \mu_B > 0$

$t_{obs} = \frac{\bar{x}_A - \bar{x}_B}{s_p\sqrt{\frac{1}{n_B}+\frac{1}{n_A}}} = \frac{77-84}{(5.304)\sqrt{\frac{1}{12}+\frac{1}{18}}} = -3.541$

Since $t_{obs} = -3.541 < t_{28;0.05}(= 1.701)$, therefore, we do not reject H$_0$.

[Note: Exact p-value= $\Pr(T > -3.541) = 0.9993$ (from statistical software)]

Question 6

X_R = gasoline consumption by radial tires

X_B = gasoline consumption by belted tires

$d = X_R - X_B$. $d \sim N(\mu_d, \sigma_d^2)$

From the data, $n_d = 12$, $\bar{x}_d = 0.1417$, $s_d = 0.1975$

(a) 95% confidence interval for $\mu_d = \bar{x}_d \pm t_{11,0.025} \frac{s_d}{\sqrt{n_d}} = 0.1417 \pm 2.201 \frac{0.1975}{\sqrt{12}} =$

$(0.0162, 0.2672)$

(b) Test H_0: $\mu_d = 0$ against H_1: $\mu_d > 0$

$t_{obs} = \frac{\bar{x}_d}{s_d/\sqrt{n}} = \frac{0.14167}{0.1975/\sqrt{12}} = 2.485 > t_{11;\ 0.05}$ $(= 1.796)$. Reject H$_0$

Alternatively, p-value $= 2\min\{\Pr(T > 2.485), \Pr(T < 2.485)\} = 2(0.01515) =$ $0.0303 < 0.05$. Reject H$_0$.

Question 7
X_M= the length of time taken to assemble a product by men $\sim N(\mu_M, \sigma_M^2)$
X_W= the length of time taken to assemble a product by women $\sim N(\mu_W, \sigma_W^2)$
Test H_0: $\sigma_M^2 = \sigma_W^2$ against H_1: $\sigma_M^2 > \sigma_W^2$
From the data, $n_M = 11$, $s_M = 6.1$, $n_W = 14$, $s_W = 5.3$
Hence, $F_{obs} = \frac{s_M{}^2}{s_W{}^2} = \frac{6.1^2}{5.3^2} = 1.325$
Since $F_{obs} = 1.325 < F_{10,13;0.05} (= 2.67)$, therefore, we do not reject H_0.
[Note: Exact p-value= $\Pr(F > 1.325) = 0.3117$ (from statistical software)]
At α = 0.05, we do not have enough evidence to conclude that the variance of the times for women is less than that for men.

Question 8
X_1= the running times of film produced by company 1. Assume $X_1 \sim N(\mu_1, \sigma_1^2)$
X_2= the running times of film produced by company 2. Assume $X_2 \sim N(\mu_2, \sigma_2^2)$
(a) Test H_0: $\sigma_1^2 = \sigma_2^2$ against H_1: $\sigma_1^2 \neq \sigma_2^2$
From the data, $n_1 = 5$, $s_1{}^2 = 78.8$, $n_2 = 7$, $s_2{}^2 = 913.3333$
Hence, $F_{obs} = \frac{s_1^2}{s_2^2} = \frac{78.8}{913.3333} = 0.0863 < F_{4,6;0.975}$ $(= 1/F_{6,4;0.025} = 1/9.20 = 0.1087)$. Reject H_0.
Alternatively, p-value = $2\min\{\Pr(F < 0.086), \Pr(F > 0.086)\} = 2\min\{0.01639, 0.98361\} = 2(0.01639) = 0.0328 < 0.05$. Reject H_0.
(b) 95% confidence interval for $\frac{\sigma_1{}^2}{\sigma_2{}^2} = \left(\frac{s_1{}^2}{s_2{}^2}\frac{1}{F_{4,6,0.025}}, \frac{s_1{}^2}{s_2{}^2}F_{6,4,0.025}\right) = \left(\frac{78.8}{913.33}\frac{1}{6.23}, \frac{78.8}{913.33}(9.20)\right) = (0.01385, 0.79375)$
(c) 95% confidence interval for $\frac{\sigma_1}{\sigma_2} = \left(\sqrt{0.01385}, \sqrt{0.79375}\right) = (0.1177, 0.8909)$

Question 9
We have $E(W) = E(a_1X_1 + \cdots + a_nX_n) = a_1E(X_1) + \cdots + a_nE(X_n) = a_1\mu_1 + \cdots + a_n\,\mu_n$.
Also recall variance of sum of independent random variables is the sum of their variances. Therefore, $V(W) = V(a_1X_1 + \cdots + a_nX_n) = V(a_1X_1) + \cdots + V(a_nX_n) = a_1^2\sigma_1^2 + \cdots + a_n^2\sigma_n^2$.