

# Chapter 2

## Concepts of Random Variables

# Overview

- Random variables
- Discrete probability distributions
  - Probability function
- Continuous probability distributions
  - Probability density function
- Cumulative distribution function

# Overview (Continued)

- Expectation
  - Mean and variance
  - Expectation of functions of random variables
  - Properties of expectation
- Chebychev's Inequality

## 2.1 Introduction

- It is frequently the case that, when an experiment is performed, we are mainly interested in **some function of the outcome** as opposed to the actual outcome itself.
- For instance, in testing 100 electronic components, we are often concerned with the number of defectives that occur.

# Introduction (Continued)

- Also, in coin-flipping, we may be interested in the total number of heads that occur and do not care at all about the actual head-tail sequence that result.
- These values are, of course, **random** quantities determined by the outcomes of the experiment.

# Example 1

- Let  $S = \{HH, HT, TH, TT\}$  be a sample space associated with the experiment of tossing two coins.
- Define the random variable (a function)

$X$  = number of heads obtained.

$X : S \rightarrow \mathbb{R}$ , where  $\mathbb{R}$  is the set of all real numbers

such that  $X(HH) = 2$ ,  $X(HT) = 1$ ,  $X(TH) = 1$  and  $X(TT) = 0$ .

- In fact the range space,  $R_X$ , for the random variable  $X$  is  
 $\{0, 1, 2\}$ .

# Example 2

Consider tossing a pair of fair dice.

- Let  $X$  be the sum of the upturned faces.

$$S = \{(x, y) | x = 1, 2, \dots, 6; y = 1, 2, \dots, 6\}.$$

$$X: S \rightarrow \mathbb{R}$$

such that  $X((x, y)) = x + y$

for  $x = 1, 2, \dots, 6; y = 1, 2, \dots, 6$ .

- e.g.  $X((1, 1)) = 2; X((1, 2)) = 3; X((3, 6)) = 9$
- $R_X = \{2, 3, 4, 5, \dots, 11, 12\}.$

# Example 3

- A coin is thrown until a “Head” occurs.
- Let  $X$  be the number of trials required.

$$S = \{H, TH, TTH, TTTH, TTTTH, \dots\}.$$

$$X: S \rightarrow \mathbb{R}$$

such that  $X(H) = 1, X(TH) = 2, X(TTH) = 3$  and so on.

- $R_X = \{1, 2, 3, 4, 5, \dots\}$ , the set of positive integers.



## 2.1.1 Random Variable

### Definition 2.1

- Let  $S$  be a sample space associated with the experiment,  $E$ .
- A *function*  $X$ , which assigns a number to every element  $s \in S$ , is called a **random variable**.

# Random Variable (Continued)

## Notes:

1.  $X$  is a **real-valued function**.
2. The range space of  $X$  is the set of real numbers

$$R_X = \{x \mid x = X(s), s \in S\}.$$

Each possible value  $x$  of  $X$  represents an event that is a subset of the sample space  $S$ .

3. If  $S$  has elements that are themselves real numbers, we take  **$X(s) = s$** . In this case  **$R_X = S$** .

## 2.1.2 Equivalent Events

### Definition 2.2

- Let  $E$  be an experiment and  $S$  its sample space.
- Let  $X$  be a random variable defined on  $S$  and  $R_X$  be its range space.

That is,  $X : S \rightarrow \mathbb{R}$

- Let  $B$  be an event with respect to  $R_X$ ;

That is  $B \subset R_X$ .

# Equivalent Events (Continued)

## Definition 2.2 (Continued)

- Suppose that  $A$  is defined as

$$A = \{s \in S \mid X(s) \in B\}.$$

In words:  $A$  consists of all sample points,  $s$ , in  $S$  for which  $X(s) \in B$ .

- In this case we say that  $A$  and  $B$  are **equivalent events** and  $\Pr(B) = \Pr(A)$ .

# Example 1 (Continued)

- Consider tossing a coin twice.
- Then  $S = \{HH, HT, TH, TT\}$ .
- Let  $X$  be the number of heads obtained.
- Then the possible values for  $X(s)$ , (usually we just write  $X$ ) are 0, 1, 2 hence and  $R_X = \{0, 1, 2\}$ .

# Example 1 (Continued)

- Therefore

$A_1 = \{HH\}$  is equivalent to  $B_1 = \{2\}$

$A_2 = \{HT, TH\}$  is equivalent to  $B_2 = \{1\}$

$A_3 = \{TT\}$  is equivalent to  $B_3 = \{0\}$

$A_4 = \{HH, HT, TH\}$  is equivalent to  $B_4 = \{2, 1\}$

# Example 1 (Continued)

$$\Pr(A_1) = \Pr(B_1) = 1/4$$

$$\Pr(A_2) = \Pr(B_2) = 2/4 = 1/2$$

$$\Pr(A_3) = \Pr(B_3) = 1/4$$

$$\Pr(A_4) = \Pr(B_4) = 3/4.$$

**Note:** Event  $\{HH, HT\}$  does not have an equivalent event based on random variable  $X$  defined above.

# Example 1 (Continued)

We can summarize the probabilities of the random variable  $X$  as follows.

| Number of heads, $x$ | 0     | 1     | 2     |
|----------------------|-------|-------|-------|
| $\Pr(X = x)$         | $1/4$ | $1/2$ | $1/4$ |



## Example 2

- When a pair of fair dice is tossed, what is the probability that a sum of 3 is obtained?

$$S = \{(x_1, x_2) | x_1 = 1, 2, 3, 4, 5, 6; x_2 = 1, 2, 3, 4, 5, 6\}.$$

- Let  $X((x_1, x_2)) = x_1 + x_2$ , then

$$\mathbf{R}_X = \{2, 3, 4, \dots, 12\}.$$

- Hence the event  $B = \{3\}$  in  $\mathbf{R}_X$  is equivalent to the event  $A = \{(1, 2), (2, 1)\}$  in  $S$ .

## Example 2 (Continued)

- Therefore

$$\begin{aligned}\Pr(X = 3) &= \Pr(\{(1, 2), (2, 1)\}) \\ &= \Pr(\{(1, 2)\}) + \Pr(\{(2, 1)\}) \\ &= 1/36 + 1/36 = 1/18.\end{aligned}$$

- The probabilities of all other possible sums can be found in a similar manner. They are given in the table in the next slide.

## Example 2 (Continued)

- The probabilities of the random variable  $X$  are given in the following table.

|              |      |      |      |      |      |      |
|--------------|------|------|------|------|------|------|
| $x$          | 2    | 3    | 4    | 5    | 6    | 7    |
| $\Pr(X = x)$ | 1/36 | 2/36 | 3/36 | 4/36 | 5/36 | 6/36 |

|              |      |      |      |      |      |
|--------------|------|------|------|------|------|
| $x$          | 8    | 9    | 10   | 11   | 12   |
| $\Pr(X = x)$ | 5/36 | 4/36 | 3/36 | 2/36 | 1/36 |

## 2.2 Discrete Probability Distributions

### 2.2.1 Discrete Random Variable

#### Definition 2.3

- Let  $X$  be a random variable.
- If the number of possible values of  $X$  (i.e.,  $R_X$ , the range space) is **finite or countable infinite**, we call  $X$  a **discrete** random variable.
- That is, the possible values of  $X$  may be listed as  $x_1, x_2, x_3, \dots$ .

## 2.2.2 Probability Function

- For a discrete random variable, each value of  $X$  has a certain probability  $f(x)$ .
- Such a function  $f(x)$  is called the **probability function**, p.f. (or **probability mass function**, p.m.f.).
- The collection of pairs  $(x_i, f(x_i))$  is called the **probability distribution of  $X$** .

# Probability Function (Continued)

The probability of  $X = x_i$  denoted by  $f(x_i)$  (i.e.  $f(x_i) = \Pr(X = x_i)$ ), must satisfy the following two conditions.

(1)  $f(x_i) \geq 0$  for all  $x_i$ .

(2)  $\sum_{i=1}^{\infty} f(x_i) = 1$ .

# Example 1

- Consider tossing a coin twice.
- Let  $X$  be the number of heads obtained.
- Then we have

| $x$                 | 0   | 1   | 2   |
|---------------------|-----|-----|-----|
| $f(x) = \Pr(X = x)$ | 1/4 | 1/2 | 1/4 |

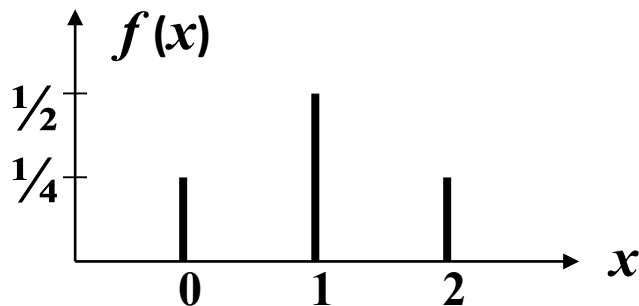
# Example 1 (Continued)

- Hence, we can see the condition (1) is satisfied since
$$f(0) = 1/4 > 0, f(1) = 1/2 > 0 \text{ and } f(2) = 1/4 > 0.$$
- Condition (2) is also satisfied since
$$f(0) + f(1) + f(2) = 1/4 + 1/2 + 1/4 = 1.$$



# Example 1 (Continued)

- One might plot the points  $(x_i, f(x_i))$  of the Example 1 so that the probability distribution is displayed in a probability line graph as follows.

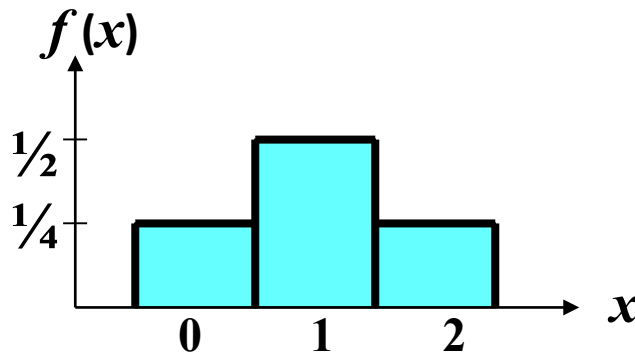


# Example 1 (Continued)

- Instead of using lines, we can use rectangles for the plot and the resulting plot is called a **probability histogram**.
- The rectangles are constructed so that their bases of equal width are centered at each value of  $X$ , and their heights are equal to the corresponding probabilities.

# Example 1 (Continued)

- If the **base of the rectangle has unit width**, then the probability  $\Pr(X = x_i)$  is equal to the area of the rectangle centered at  $x_i$ .
- The total area of all the rectangles is 1.



# Example 2

- Consider throwing a pair of fair dice.
- Let  $X$  be the sum of the two dice. Then we have

| $x$          | 2    | 3    | 4    | 5    | 6    | 7    |
|--------------|------|------|------|------|------|------|
| $\Pr(X = x)$ | 1/36 | 2/36 | 3/36 | 4/36 | 5/36 | 6/36 |

| $x$          | 8    | 9    | 10   | 11   | 12   |
|--------------|------|------|------|------|------|
| $\Pr(X = x)$ | 5/36 | 4/36 | 3/36 | 2/36 | 1/36 |

## Example 2 (Continued)

- Notice that **all**  $f(x) > 0$  for  $x = 2, 3, 4, 5, \dots, 11, 12$  and
- $f(2) + f(3) + f(4) + \dots + f(11) + f(12) = 1.$

# Example 3

Six lots of components are ready to be shipped by a certain supplier. The number of defective components in each lot is as follows:

| Lot                  | 1 | 2 | 3 | 4 | 5 | 6 |
|----------------------|---|---|---|---|---|---|
| Number of defectives | 0 | 2 | 0 | 1 | 2 | 0 |

- One of these lots is to be randomly selected for the shipment to a particular customer.
- Let  $X$  be the number of defectives in the selected lot.
- The three possible  $X$  values are 0, 1, and 2.

## Example 3 (Continued)

- Of the six equally likely choices of one lot, three result in  $X = 0$ , one in  $X = 1$ , and the other two in  $X = 2$ .
- Let  $f(x)$  denote the probability that  $X = x$  for  $x = 0, 1, 2$ .
- Then

$$f(0) = \Pr(X = 0) = \Pr(\text{lot 1 or 3 or 6 is selected}) = 3/6.$$

$$f(1) = \Pr(X = 1) = \Pr(\text{lot 4 is selected}) = 1/6.$$

$$f(2) = \Pr(X = 2) = \Pr(\text{lot 2 or 5 is selected}) = 2/6.$$

## Example 3 (Continued)

- Therefore, the probability function of  $X$  is given by

| $x$                 | 0   | 1   | 2   |
|---------------------|-----|-----|-----|
| $f(x) = \Pr(X = x)$ | 1/2 | 1/6 | 1/3 |

- Hence,  $f(x_i) > 0$  for  $x_1 = 0$ ,  $x_2 = 1$  and  $x_3 = 2$ .
- Also  $f(0) + f(1) + f(2) = 1$ .



# Example 4

- Find the constant  $c$  so that

$$f(x) = cx, \text{ for } x = 1, 2, 3, 4,$$

and  $0$  otherwise, is a **probability function** of a random variable  $X$ .

- Hence compute  $\Pr(X \geq 3)$ .

## Example 4 (Continued)

### Solution

- By the property that  $\sum_{i=1}^{\infty} f(x_i) = 1$ , we have

$$f(x_1) + f(x_2) + f(x_3) + f(x_4) = 1$$

$$c + 2c + 3c + 4c = 1.$$

- Therefore  $c = 1/10$ .

- Hence

$$\Pr(X \geq 3) = f(3) + f(4) = 3/10 + 4/10 = 7/10.$$

# Example 5

- Consider a group of five potential blood donors — A, B, C, D and E — of whom **only A and B have type O+ blood**.
- Five blood samples, one from each individual, will be typed in random order until an O+ individual is identified.

## Example 5 (Continued)

- Let the random variable  $Y$  = the number of typing necessary to identify an O+ individual.
- Let  $O_i$  and  $O'_i$  be the event that an O+ and a non-O+ individual is typed in the  $i$ -th typing

$$f(1) = \Pr(Y = 1) = \Pr(O_1) = 2/5 = 0.4.$$

$$\begin{aligned} f(2) &= \Pr(Y = 2) \\ &= \Pr(O'_1)\Pr(O_2|O'_1) \\ &= \left(\frac{3}{5}\right)\left(\frac{2}{4}\right) = \frac{3}{10} = 0.3. \end{aligned}$$

## Example 5 (Continued)

$$\begin{aligned}
 f(3) &= \Pr(Y = 3) = \Pr(O'_1) \Pr(O'_2|O'_1) \Pr(O_3|O'_1 \cap O'_2) \\
 &= \binom{3}{\frac{3}{5}} \binom{2}{\frac{2}{4}} \binom{2}{\frac{2}{3}} = \frac{1}{5} = 0.2.
 \end{aligned}$$

$$\begin{aligned}
 f(4) &= \Pr(Y = 4) \\
 &= \Pr(O'_1) \Pr(O'_2|O'_1) \Pr(O'_3|O'_1 \cap O'_2) \Pr(O_4|O'_1 \cap O'_2 \cap O'_3) \\
 &= \binom{3}{\frac{3}{5}} \binom{2}{\frac{2}{4}} \binom{1}{\frac{1}{3}} \binom{2}{\frac{2}{2}} = \frac{1}{10} = 0.1.
 \end{aligned}$$

$$f(y) = 0 \text{ if } y \neq 1, 2, 3, 4.$$

## Example 5 (Continued)

- Then the probability function of  $Y$  is

|        |     |     |     |     |
|--------|-----|-----|-----|-----|
| $y$    | 1   | 2   | 3   | 4   |
| $f(y)$ | 0.4 | 0.3 | 0.2 | 0.1 |

## 2.2.3 Another View of Probability Function

It is often to think of a **probability function** as specifying a **mathematical model for a finite population**.

### Example

- Consider selecting at random a student who is among the 35,000 registered for the current semester in NUS.
- Let  $X$  = the number of modules for which the selected student is registered and suppose that  $X$  has the probability function.

# Another View of Probability Function (Continued)

|        |      |      |      |      |      |      |      |
|--------|------|------|------|------|------|------|------|
| $x$    | 1    | 2    | 3    | 4    | 5    | 6    | 7    |
| $f(x)$ | 0.01 | 0.03 | 0.13 | 0.25 | 0.39 | 0.17 | 0.02 |

- One way to view this situation is to think of the population as consisting 35,000 individuals, each having his or her own  $X$  value; the proportion with each  $X$  value is given by  $f(x)$  above.



# Another View of Probability Function (Continued)

- An alternative viewpoint is to forget about the students and think of the population itself as consisting of the  $X$  values.
- There are some 1's in the population, some 2's,  $\dots$  and finally some 7's.
- The population then consists of the numbers 1, 2,  $\dots$ , 7 (so are discrete values), and  $f(x)$  gives a model for the distribution of population values.
- Once we have such a mathematical model for a population, we will use it to compute values of population characteristics (such as the mean).

## 2.3 Continuous Probability Distributions

### 2.3.1 Continuous Random Variable

#### Definition 2.4

- Suppose that  $R_X$ , the range space of a random variable,  $X$ , is an **interval or a collection of intervals**.
- Then we say that  $X$  is a **continuous random variable**.

## 2.3.2 Probability Density Function

### Definition 2.5

- Let  $X$  be a continuous random variable.
- The **probability density function (p.d.f.)**  $f(x)$ , is a function,  $f(x)$ , satisfying the following conditions:
  1.  $f(x) \geq 0$  for all  $x \in \mathbf{R}_X$ ,
  2.  $\int_{\mathbf{R}_X} f(x) dx = 1$  or  $\int_{-\infty}^{\infty} f(x) dx = 1$   
since  $f(x) = 0$  for  $x$  not in  $\mathbf{R}_X$ .

# Probability Density Function (Continued)

## Definition 2.5 (Continued)

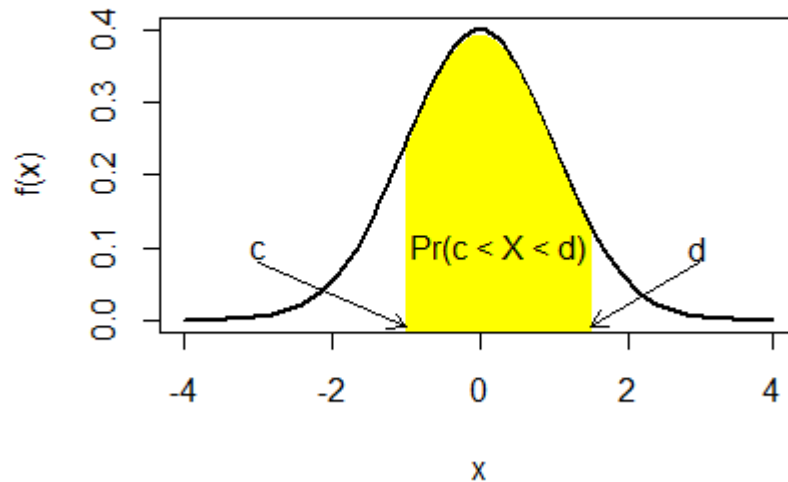
3. For any  $c$  and  $d$  such that  $c < d$ , (i.e.  $(c, d) \subset \mathbf{R}_X$ ),

$$\Pr(c \leq X \leq d) = \int_c^d f(x) dx$$

# Remarks

$$1. \quad \Pr(c \leq X \leq d) = \int_c^d f(x) dx$$

represents the area under the graph of the p.d.f.  $f(x)$  between  $x = c$  and  $x = d$ .



# Remarks (Continued)

2. For any specified value of  $X$ , say  $x_0$ , we have

$$\Pr(X = x_0) = \int_{x_0}^{x_0} f(x)dx = 0$$

Hence in the **continuous** case, the probability of  $X$  equals to a fixed value is 0 and

$$\Pr(c \leq X \leq d) = \Pr(c \leq X < d) = \Pr(c < X \leq d) = \Pr(c < X < d).$$

Therefore in the continuous case,  $\leq$  **and**  $<$  can be used **interchangeably** in a probability statement.

# Remarks (Continued)

3.  $\Pr(A) = 0$  does **not** necessary imply  $A = \emptyset$ .
4. If  $X$  assumes values only in some interval  $[a, b]$ , we may simply set  $f(x) = 0$  for all  $X$  outside  $[a, b]$ .

# Example 1

- Suppose that the random variable  $X$  is continuous.
- Let the p.d.f.  $f(x)$  be given by

$$f(x) = \begin{cases} cx, & \text{for } 0 < x < 1; \\ 0, & \text{otherwise.} \end{cases}$$



# Example 1 (Continued)

- (a) Find the value  $c$ .
- (b) Find  $\Pr(X \leq 1/2)$ .
- (c) Find  $\Pr(1/3 \leq X \leq 2/3)$ .
- (d) Find  $\Pr(X \geq 3/4)$ .

# Solution to Example 1

(a)

$$\int_0^1 cx \, dx = c \left[ \frac{x^2}{2} \right]_0^1 = \frac{c}{2}$$

- Hence  $\int_0^1 cx \, dx = 1$  implies that  $c/2 = 1$ .
- Therefore  $c = 2$ .

# Solution to Example 1 (Continued)

(b)

$$\begin{aligned}\Pr\left(X \leq \frac{1}{2}\right) &= \int_0^{1/2} f(x) dx \\ &= \int_0^{1/2} 2x \, dx = [x^2]_0^{1/2} \\ &= \frac{1}{4}\end{aligned}$$

# Solution to Example 1 (Continued)

(c)

$$\begin{aligned}\Pr\left(\frac{1}{3} \leq X \leq \frac{2}{3}\right) &= \int_{1/3}^{2/3} 2x \, dx \\ &= [x^2]_{1/3}^{2/3} \\ &= \left(\frac{4}{9} - \frac{1}{9}\right) = \frac{1}{3}\end{aligned}$$

# Solution to Example 1 (Continued)

(d)

$$\begin{aligned}\Pr\left(X \geq \frac{3}{4}\right) &= \int_{3/4}^1 2x \, dx \\ &= [x^2]_{3/4}^1 \\ &= 1 - \frac{9}{16} = \frac{7}{16}\end{aligned}$$

## Example 2

- “Time headway” in traffic flow is the elapsed time between the time that one car finishes passing a fixed point and the instant that the next car begins to pass that point.
- Let  $X = \text{the time headway}$  for two randomly chosen consecutive cars on a highway during a period of heavy flow.

## Example 2 (Continued)

- The following p.d.f. of  $X$  was suggested:

$$f(x) = \begin{cases} 0.15e^{-0.15(x-0.5)}, & \text{for } x \geq 0.5; \\ 0, & \text{otherwise.} \end{cases}$$

Note:  $f(x) \geq 0$  for all  $x$ .

## Example 2 (Continued)

- Clearly,  $f(x) \geq 0$  and

$$\begin{aligned}
 \int_{-\infty}^{\infty} f(x) dx &= \int_{-\infty}^{0.5} 0 dx = \int_{0.5}^{\infty} 0.15 e^{-0.15(x-0.5)} dx \\
 &= 0.15 e^{0.075} \int_{0.5}^{\infty} e^{-0.15x} dx \\
 &= 0.15 e^{0.075} \left[ -\frac{1}{0.15} e^{-0.15x} \right]_{0.5}^{\infty} \\
 &= 0.15 e^{0.075} \left( 0 - \left( -\frac{1}{0.15} e^{-0.15(0.5)} \right) \right) = 1
 \end{aligned}$$



## Example 2 (Continued)

- Hence the given function  $f(x)$  is a legitimate probability density function.
- What is the probability that headway time is **at most** 5 sec?  
i.e. What is  $\Pr(X \leq 5)$ ?

# Solution to Example 2

$$\begin{aligned}
 \Pr(X \leq 5) &= \int_{-\infty}^5 f(x) dx \\
 &= \int_{-\infty}^{0.5} 0 dx + \int_{0.5}^5 0.15 e^{-0.15(x-0.5)} dx \\
 &= 0.15 e^{0.075} \int_{0.5}^5 e^{-0.15x} dx \\
 &= 0.15 e^{0.075} \left[ -\frac{1}{0.15} e^{-0.15x} \right]_{0.5}^5 \\
 &= e^{0.075} (-e^{-0.75} + e^{-0.075}) = 0.4908.
 \end{aligned}$$

## 2.4 Cumulative Distribution Function

### Definition 2.6

- Let  $X$  be a random variable, discrete or continuous.
- We define  $F(x)$  to be the **cumulative distribution function** of the random variable  $X$  (abbreviated as c.d.f.) where

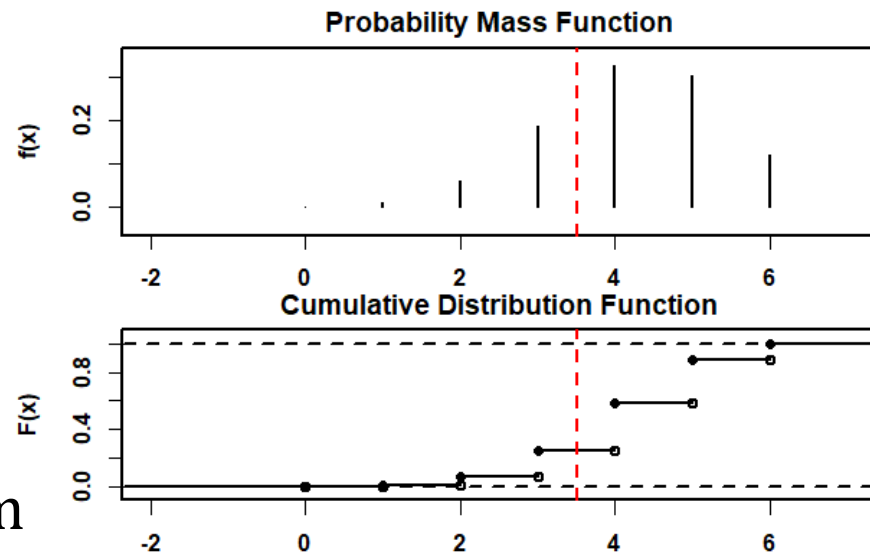
$$F(x) = \Pr(X \leq x).$$

## 2.4.1 CDF for Discrete Random Variables

- If  $X$  is a **discrete random variable**, then

$$\begin{aligned}
 F(x) &= \sum_{t \leq x} f(t) \\
 &= \sum_{t \leq x} \Pr(X = t)
 \end{aligned}$$

- The c.d.f. of a discrete random variable is a step function.



# CDF for Discrete Random Variables (Continued)

- For any two numbers  $a$  and  $b$  with  $a \leq b$ .

$$\begin{aligned}\Pr(a \leq X \leq b) &= \Pr(X \leq b) - \Pr(X < a) \\ &= F(b) - F(a^-)\end{aligned}$$

where “ $a^-$ ” represents the largest possible value of  $X$  value that is strictly less than  $a$ .

# CDF for Discrete Random Variables (Continued)

- In particular, if the only possible values are integers and if  $a$  and  $b$  are integers, then

$$\Pr(a \leq X \leq b) = \Pr(X = a \text{ or } a + 1 \text{ or } \cdots \text{ or } b)$$

Also  $\Pr(a \leq X \leq b) = F(b) - F(a - 1)$

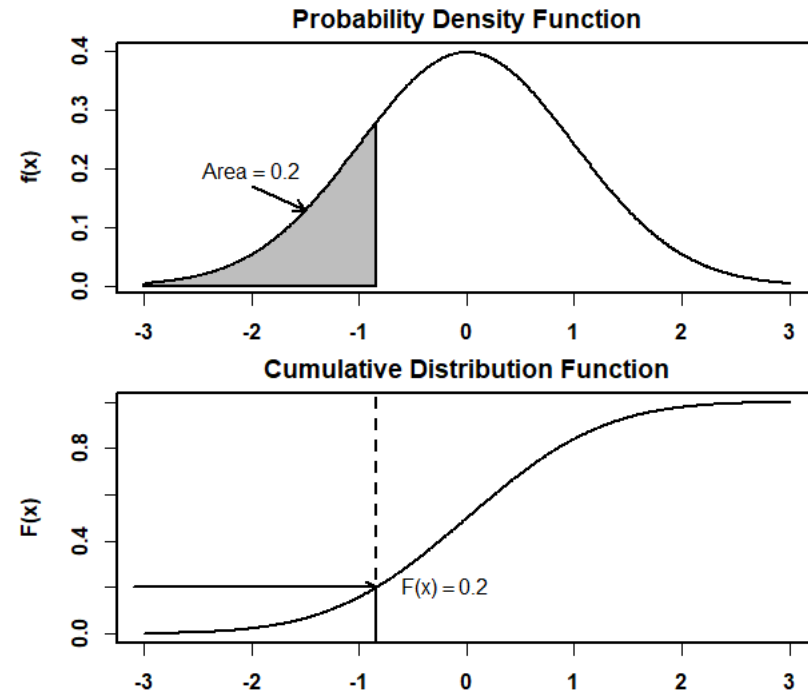
- Taking  $a = b$  yields

$$\Pr(X = a) = F(a) - F(a - 1).$$

## 2.4.2 CDF for Continuous Random Variables

- If  $X$  is a **continuous random variable**, then

$$F(x) = \int_{-\infty}^x f(t) dt$$



# CDF for Continuous Random Variables (Continued)

- For a **continuous random variable**  $X$ ,

$$f(x) = \frac{d F(x)}{dx}$$

if the derivative exists.

- Also,

$$\begin{aligned}\Pr(a \leq X \leq b) &= \Pr(a < X \leq b) \\ &= F(b) - F(a).\end{aligned}$$



# CDF for Continuous Random Variables (Continued)

## Remarks:

- $F(x)$  is a **non-decreasing** function.

That is, if  $x_1 < x_2$ , then  $F(x_1) \leq F(x_2)$ .

- $0 \leq F(x) \leq 1$ .

# Example 1

The p.f. of  $X$  is given as follows

$$f(x) = \begin{cases} (1-p)^{x-1}p, & \text{if } x = 1, 2, 3, \dots; \\ 0, & \text{otherwise.} \end{cases}$$

# Example 1 (Continued)

For any positive integers  $x$ ,

$$\begin{aligned}
 F(x) &= \sum_{t \leq x} f(t) = \sum_{t=1}^x (1-p)^{t-1} p \\
 &= p \sum_{s=0}^{x-1} (1-p)^s, \quad \text{where } s = t - 1 \\
 &= \frac{p(1 - (1-p)^x)}{1 - (1-p)} \\
 &= 1 - (1-p)^x, \quad \text{for } x = 1, 2, 3, \dots
 \end{aligned}$$

# Example 1 (Continued)

## Remark

- Since  $f(x) = 0$  between positive integers, hence  $F(x)$  is constant between positive integers and

$$F(x) = \begin{cases} 0, & \text{if } x < 1, \\ 1 - (1 - p)^{[x]}, & \text{if } x \geq 1 \end{cases}$$

where  $[x]$  is the largest integer  $\leq x$ .

(e.g.,  $[2.7] = 2$ ,  $[3] = 3$ ).

## Example 2

- Let  $X$  = the number of days of sick leave taken by a randomly selected employee of a large company during a particular year.
- If the maximum number of allowable sick leave days per year is 14, possible values of  $X$  are 0, 1, 2,  $\dots$ , 14.
- Suppose it is given that  $F(0) = 0.58, F(1) = 0.72, F(2) = 0.76, F(3) = 0.81, F(4) = 0.88$ , and  $F(5) = 0.94$ .

## Example 2 (Continued)

- Then

$$\begin{aligned}\Pr(2 \leq X \leq 5) &= F(5) - F(2^-) \\ &= F(5) - F(1) \\ &= 0.94 - 0.72 = 0.22.\end{aligned}$$

and

$$\begin{aligned}\Pr(X = 3) &= F(3) - F(3^-) \\ &= F(3) - F(2) \\ &= 0.81 - 0.76 = 0.05.\end{aligned}$$

# Example 3

- Many manufacturers have quality control programs that include inspection of incoming materials for defects.
- Suppose a computer manufacturer receives computer boards in lots of five.
- Two boards are selected from each lot for inspection.

## Example 3 (Continued)

- (a) List all possible inspections.
- (b) Suppose that **boards 1 and 2 are the only defectives** in a lot of five.

Two boards are chosen randomly.

Define  **$X$  to be the number of defective boards** observed among those inspected.

Find the probability distribution of  $X$ .

- (c) Let  $F(x)$  denote the c.d.f. of  $X$ . Obtain  $F(x)$  for **all**  $x$ .



# Solution to Example 3

(a)  $\#(S) = {}_5C_2 = 5!/(2! 3!) = 10.$

The possible selections are

$\{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}.$

(b)  $X$  takes values 0, 1, and 2.

$$f(0) = \Pr(X = 0) = \Pr(\{(3, 4), (3, 5), (4, 5)\}) = 3/10,$$

$$f(2) = \Pr(X = 2) = \Pr(\{(1, 2)\}) = 1/10,$$

$$f(1) = \Pr(X = 1) = 1 - (f(0) + f(2)) = 6/10,$$

and  $f(x) = 0$  for  $x \neq 0, 1, 2.$

# Solution to Example 3 (Continued)

(c)

$$F(0) = \Pr(X \leq 0) = \Pr(X = 0) = 0.3,$$

$$\begin{aligned} F(1) &= \Pr(X \leq 1) = \Pr(X = 0 \text{ or } 1) \\ &= \Pr(X = 0) + \Pr(X = 1) \\ &= 0.3 + 0.6 = 0.9, \end{aligned}$$

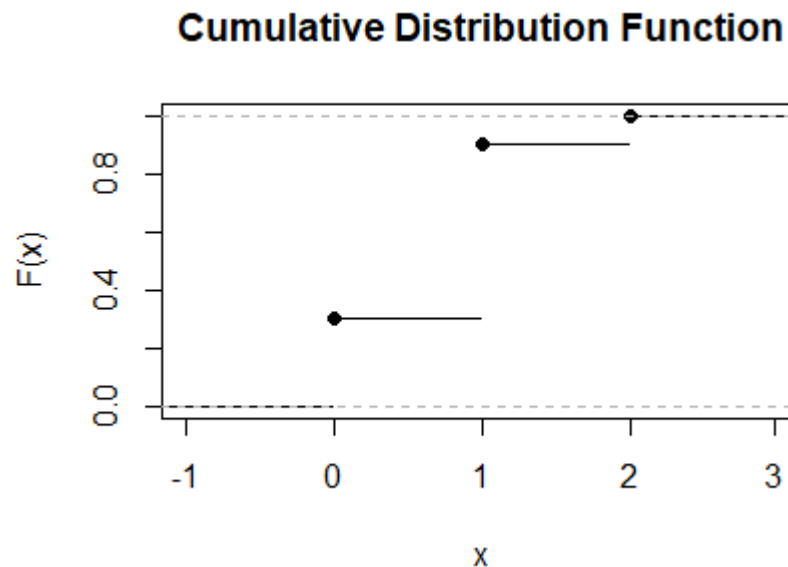
$$F(2) = \Pr(X \leq 2) = 1.$$

# Solution to Example 3 (Continued)

(c)

The c.d.f. is thus given by

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ 0.3, & \text{if } 0 \leq x < 1, \\ 0.9, & \text{if } 1 \leq x < 2, \\ 1, & \text{if } 2 \leq x. \end{cases}$$



## Example 4

- The p.d.f. of a random variable  $X$  is given by

$$f(x) = \begin{cases} 2x, & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

- Find the c.d.f of  $X$ .

# Solution to Example 4

$$\begin{aligned}
 F(x) &= \int_{-\infty}^x f(t) dt \\
 &= \begin{cases} \int_{-\infty}^x 0 dt, & \text{for } x < 0, \\ \int_{-\infty}^0 0 dt + \int_0^x 2t dt, & \text{for } 0 \leq x < 1, \\ \int_{-\infty}^0 0 dt + \int_0^1 2t dt + \int_1^x 0 dt, & \text{for } x \geq 1. \end{cases}
 \end{aligned}$$

# Solution to Example 4 (Continued)

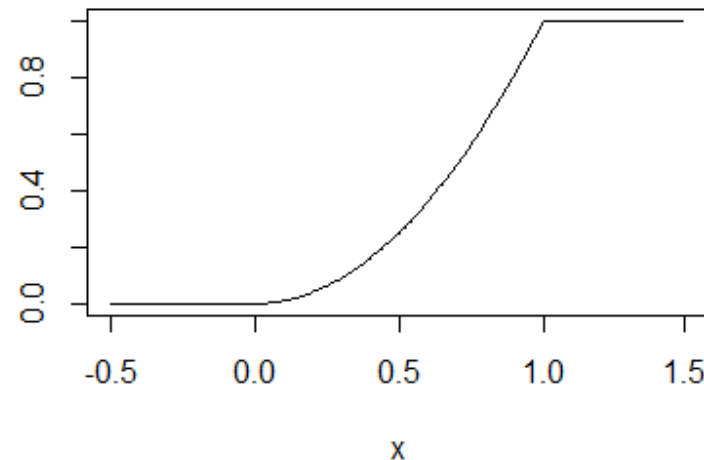
$$F(x) = \int_{-\infty}^x f(t) dt$$

= ...

$$= \begin{cases} 0, & \text{for } x < 0, \\ x^2, & \text{for } 0 \leq x < 1, \\ 1, & \text{for } x \geq 1. \end{cases}$$

$F(x)$

**Cumulative Distribution Function**



# Example 5

- Let  $X$  denote the vibratory stress (psi) on a wind turbine blade at a particular wind tunnel.
- The following p.d.f. for  $X$  is proposed

$$f(x; \theta) = \begin{cases} \frac{x}{\theta^2} e^{-x^2/(2\theta^2)}, & \text{for } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

## Example 5 (Continued)

(a) Verify that  $f(x; \theta)$  is a legitimate p.d.f.

(b) Suppose that  $\theta = 100$ .

What is the probability that  $X$  is at most 200? Less than 200?

At least 200?

(c) Give an expression for  $\Pr(X \leq x)$ .

(d) What is the probability that  $X$  is between 100 and 200?



# Solution to Example 5

(a) It is obvious that  $f(x) > 0$  for  $x > 0$ .

$$\begin{aligned}\int_{-\infty}^{\infty} f(x; \theta) dx &= \int_0^{\infty} \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} dx \\&= \int_0^{\infty} d\left(e^{-\frac{x^2}{2\theta^2}}\right) \\&= \left[e^{-\frac{x^2}{2\theta^2}}\right]_0^{\infty} \\&= 0 - (-1) = 1.\end{aligned}$$

# Solution to Example 5 (Continued)

(b)

$$\begin{aligned}
 \Pr(X \leq 200; \theta = 100) &= \int_{-\infty}^{200} f(x; \theta) dx \\
 &= \int_0^{200} \frac{x}{100^2} e^{-\left(\frac{x^2}{2(100)^2}\right)} dx \\
 &= \left[ -e^{-\left(\frac{x^2}{20000}\right)} \right]_0^{200} = -e^{-2} + e^0 \\
 &= -0.1353 + 1 = 0.8647.
 \end{aligned}$$

# Solution to Example 5 (Continued)

(b)

- $\Pr(X < 200) = \Pr(X \leq 200) = 0.8647$  since  $X$  is a continuous random variable.
- $\Pr(X \geq 200) = 1 - \Pr(X \leq 200) = 0.1353.$

# Solution to Example 5 (Continued)

(c) For  $x > 0$ ,

$$\begin{aligned} F(x; \theta) &= \Pr(X \leq x; \theta) = \int_{-\infty}^x f(t; \theta) dt \\ &= \int_0^x \frac{t}{\theta^2} e^{-\left(\frac{t^2}{2\theta^2}\right)} dt \\ &= \left[ -e^{-\left(\frac{t^2}{2\theta^2}\right)} \right]_0^x \\ &= 1 - e^{-\left(\frac{x^2}{2\theta^2}\right)}. \end{aligned}$$

# Solution to Example 5 (Continued)

(d)

$$\begin{aligned}\Pr(100 \leq X \leq 200) &= F(200) - F(100) \\ &= 1 - \exp(-(200)^2/20000) \\ &\quad - [1 - \exp(-(100)^2/20000)] \\ &= e^{-1/2} - e^{-2} \\ &= 0.4712.\end{aligned}$$

# Solution to Example 5 (Continued)

(d)

Alternatively,

$$\begin{aligned}\Pr(100 \leq X \leq 200) &= \int_{100}^{200} \frac{x}{100^2} e^{-\left(\frac{x^2}{2(100)^2}\right)} dx \\ &= \left[ -e^{-\left(\frac{x^2}{20000}\right)} \right]_{100}^{200} = 0.4712.\end{aligned}$$

# 2.5 Mean and Variance of a Random Variable

## 2.5.1 Expected Values

### Definition 2.7a

- If  $X$  is a **discrete** random variable. taking on values  $x_1, x_2, \dots$  with probability function  $f_X(x)$ ,
- then the **mean** or **expected value** (or **mathematical expectation**) of  $X$ , denoted by  $E(X)$  as well as by  $\mu_X$ , is defined by

$$\mu_X = E(X) = \sum_i x_i f_X(x_i) = \sum_x x f_X(x)$$

# Mean and Variance of a Random Variable (Continued)

## 2.5.1 Expected Values (Continued)

Note : The expected value is not necessarily a possible value of the random variable  $X$ .



# Expected Values (Continued)

## Definition 2.7b

- If  $X$  is a **continuous** random variable with probability density function  $f_X(x)$ , the mean of  $X$  is defined by

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x f_X(x) dx .$$

- Roughly speaking, the mathematical expectation is an “average” (or more precisely, a “weighted average”).

# Remarks

1. The expected value exists provided the sum or the integral in the above definitions exists.
2. In the discrete case, if  $f_X(x) = 1/N$  for each of the  $N$  values of  $x$ , hence the mean,

$$E(X) = \sum_i x_i f(x_i) = \frac{1}{N} \sum_i x_i ,$$

becomes the average of the  $N$  items.

# Example 1

- In a gambling game, a man gains 5 if he gets all heads or all tails in tossing a fair coin 3 times,
- and he pays out 3 if either 1 or 2 heads show.
- What is his expected gain?

# Solution to Example 1

- Let  $X$  be the amount he can gain.
- Then  $X = 5$  or  $-3$  with the following probabilities:  
 $\Pr(X = 5) = \Pr(\{HHH, TTT\}) = 1/8 + 1/8 = 1/4$   
 $\Pr(X = -3) = 1 - \Pr(X = 5) = 3/4.$
- Therefore  $E(X) = 5 \left(\frac{1}{4}\right) + (-3) \left(\frac{3}{4}\right) = -1.$
- Hence, he will lose 1 per toss **in a long run.**

## Example 2

- Suppose a game consists of rolling a balanced die.
- We pay  $c$  to play the game and we get  $i$  if number  $i$  occurs.
- How much should we pay if the game is fair?  
(We say a game is “fair” if  $E(\text{gain}) = 0$ .)

# Solution to Example 2

- Let  $X$  denote the **amount that one gets** when rolling a die.
- Then clearly  $\Pr(X = 1) = \dots = \Pr(X = 6) = 1/6$ .
- $E(X) = (1 + 2 + \dots + 6) \left(\frac{1}{6}\right) = 3.5$ .
- To be a fair game,  $E(\text{paying}) = E(\text{getting})$ , and hence for a fair game, the admission fee should be  **$c = E(X) = 3.5$** .

# Solution to Example 2 (Continued)

## Alternate Solution,

- Let  $Y$  denote the amount that **one gains** when rolling a die.
- Then  $Y = i - c$ , where  $i = 1, 2, \dots, 6$
- Then clearly  $\Pr(Y = 1 - c) = \dots = \Pr(Y = 6 - c) = 1/6$ .
- $E(Y) = ((1 - c) + \dots + (6 - c)) \left(\frac{1}{6}\right) = 3.5 - c$ .
- To be a fair game,  $E(Y) = 0$ , and hence the admission fee,  $c$ , satisfies

$$3.5 - c = 0 \quad \text{or} \quad c = 3.5$$

## Example 3

- A private pilot wishes to insure his airline for 1,000,000.
- The insurance company estimates that a total loss may occur with probability 0.0002, a 50% loss with probability 0.001, and a 25% loss with probability 0.01, and a 10% loss with probability 0.01.
- Ignoring all other partial losses, what premium should the insurance company charge each year to realize an average profit of 5,000?



# Solution to Example 3

- The **expected loss** is given by
$$1000000(0.0002) + 500000(0.001) + 250000(0.01) + 100000(0.01) + 0(1 - 0.0002 - 0.001 - 0.01 - 0.01) = 4200.$$
- Therefore, the insurance company should charge the **premium 9200** so as to realize an average profit of 5000.

# Example 4

The p.d.f. of weekly gravel sales  $X$  is

$$f_X(x) = \begin{cases} \frac{3}{2}(1 - x^2), & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find  $E(X)$ .

# Solution to Example 4

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x) dx = \int_0^1 x \frac{3}{2} (1 - x^2) dx \\ &= \frac{3}{2} \int_0^1 (x - x^3) dx = \frac{3}{2} \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 \\ &= \frac{3}{8}. \end{aligned}$$

# Example 5

- The probability density function of a continuous random variable  $X$ , the total number of hours, in units of 100 hours, that a family runs a vacuum cleaner over a period of one year, was given as follows.

$$f_X(x) = \begin{cases} x, & \text{for } 0 < x < 1, \\ 2 - x, & \text{for } 1 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

- Find the average number of hours per year that families run their vacuum cleaners.

# Solution to Example 5

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f_X(x) dx \\ &= \int_{-\infty}^0 x(0) dx + \int_0^1 x^2 dx + \int_1^2 x(2-x) dx + \int_2^{\infty} x(0) dx \\ &= \left[ \frac{x^3}{3} \right]_0^1 + \left[ x^2 - \frac{x^3}{3} \right]_1^2 \end{aligned}$$

# Solution to Example 5 (Continued)

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f_X(x) dx = \dots \\ &= \left[ \frac{x^3}{3} \right]_0^1 + \left[ x^2 - \frac{x^3}{3} \right]_1^2 \\ &= \left[ \frac{1}{3} - 0 \right] + \left[ \left( 4 - \frac{8}{3} \right) - \left( 1 - \frac{1}{3} \right) \right] = 1. \end{aligned}$$

- Families run their vacuum cleaners 100 hours per year on average.

## 2.5.2 Expectation of a Function of a RV

### Definition 2.8

For any function  $g(X)$  of a random variable  $X$  with p.f. (or p.d.f.)  $f_X(x)$ ,

(a)  $E[g(X)] = \sum_x g(x)f_X(x)$

if  $X$  is a **discrete** r.v. providing the sum exists; and

(b)  $E[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x) dx$

if  $X$  is a **continuous** r.v. providing the integral exists.

# Some Special Cases

1.  $g(x) = (x - \mu_X)^2.$

This leads to the definition of variance of a given random variable  $X$ .



# Some Special Cases (Continued)

## Definition 2.9

- Let  $X$  be a random variable with p.f. (or p.d.f.)  $f(x)$ , then the **variance** of  $X$  is defined as

$$\sigma_X^2 = V(X) = E[(X - \mu_X)^2]$$

$$= \begin{cases} \sum_x (x - \mu_X)^2 f_X(x), & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) dx, & \text{if } X \text{ is continuous.} \end{cases}$$

# Some Special Cases (Continued)

## Remarks:

(a)  $V(X) \geq 0$ .

(b)  $V(X) = E(X^2) - [E(X)]^2$ .

- The positive square root of the variance is called the **standard deviation** of  $X$ . That is

$$\sigma_X = \sqrt{V(X)}$$

# Some Special Cases (Continued)

2.  $g(x) = x^k.$

Then  $E(X^k)$  is called the  **$k$ -th moment of  $X$** .

# Example 1

Let the p.d.f. of  $X$  be given by:

|          |       |       |       |       |
|----------|-------|-------|-------|-------|
| $x$      | $-1$  | $0$   | $1$   | $2$   |
| $f_X(x)$ | $1/8$ | $2/8$ | $1/8$ | $4/8$ |

- (a) Find  $E(X)$  and  $V(X)$ .
- (b) Define  $Y = X^2 + 2$ . Find  $E(Y)$  and  $V(Y)$ .

# Solution to Example 1

(a)

$$\begin{aligned} E(X) &= \sum_x x f_X(x) \\ &= (-1) \left(\frac{1}{8}\right) + 0 \left(\frac{2}{8}\right) + 1 \left(\frac{1}{8}\right) + 2 \left(\frac{4}{8}\right) = 1. \end{aligned}$$

# Solution to Example 1 (Continued)

(a) (Continued)

$$\begin{aligned} V(X) &= \sum_x (x - 1)^2 f_X(x), & \text{since } \mu_X = 1 \\ &= (-1 - 1)^2 \left(\frac{1}{8}\right) + (0 - 1)^2 \left(\frac{2}{8}\right) \\ &\quad + (1 - 1)^2 \left(\frac{1}{8}\right) + (2 - 1)^2 \left(\frac{4}{8}\right) = \frac{5}{4}. \end{aligned}$$

# Solution to Example 1 (Continued)

(a) (Continued)

Alternatively, we can find  $E(X^2)$  first and then obtain  $V(X)$  using the formula  $V(X) = E(X^2) - [E(X)]^2$ .

$$\begin{aligned}
 E(X^2) &= \sum_x x^2 f_X(x) \\
 &= (-1)^2 \left(\frac{1}{8}\right) + 0^2 \left(\frac{2}{8}\right) + 1^2 \left(\frac{1}{8}\right) + 2^2 \left(\frac{4}{8}\right) = \frac{9}{4}. \\
 V(X) &= E(X^2) - [E(X)]^2 = \frac{9}{4} - 1^2 = \frac{5}{4}.
 \end{aligned}$$

# Solution to Example 1 (Continued)

(b)

$$\begin{aligned}
 E(Y) &= E(X^2 + 2) = \sum_x (x^2 + 2)f_X(x) \\
 &= ((-1)^2 + 2) \left(\frac{1}{8}\right) + (0^2 + 2) \left(\frac{2}{8}\right) + (1^2 + 2) \left(\frac{1}{8}\right) \\
 &\quad + (2^2 + 2) \left(\frac{4}{8}\right) = \frac{17}{4}.
 \end{aligned}$$

Alternatively,  $E(Y) = E(X^2) + 2 = \frac{9}{4} + 2 = \frac{17}{4}.$



# Solution to Example 1 (Continued)

(b)

$$\begin{aligned}
 V(Y) &= \sum_x \left( x^2 + 2 - \frac{17}{4} \right)^2 f_X(x) \\
 &= \left( 3 - \frac{17}{4} \right)^2 \left( \frac{1}{8} \right) + \left( 2 - \frac{17}{4} \right)^2 \left( \frac{2}{8} \right) \\
 &\quad + \left( 3 - \frac{17}{4} \right)^2 \left( \frac{1}{8} \right) + \left( 6 - \frac{17}{4} \right)^2 \left( \frac{4}{8} \right) = \frac{51}{16}.
 \end{aligned}$$

# Solution to Example 1 (Continued)

(b) (Continued)

Alternatively, we can find  $E(Y^2)$  first and then obtain  $V(Y)$  using the formula

$$V(Y) = E(Y^2) - [E(Y)]^2,$$

where

$$E(Y^2) = 3^2 \left(\frac{1}{8}\right) + 2^2 \left(\frac{2}{8}\right) + 3^2 \left(\frac{1}{8}\right) + 6^2 \left(\frac{4}{8}\right) = \frac{170}{8}$$

$$V(Y) = \frac{170}{8} - \left(\frac{17}{4}\right)^2 = \frac{51}{16}.$$

## Example 2

- Suppose that the random variable  $X$  is continuous with the following p.d.f.:

$$f_X(x) = \begin{cases} \frac{x}{225}, & \text{for } 0 < x < 15, \\ \frac{30 - x}{225}, & \text{for } 15 \leq x \leq 30, \\ 0, & \text{otherwise.} \end{cases}$$

- Find  $E(X)$  and  $V(X)$ .

# Solution to Example 2

$$\begin{aligned}
 E(X) &= \int_0^{15} x \left( \frac{x}{225} \right) dx + \int_{15}^{30} x \left( \frac{30-x}{225} \right) dx \\
 &= \frac{1}{225} \left\{ \left[ \frac{x^3}{3} \right]_0^{15} + \left[ 15x^2 - \frac{x^3}{3} \right]_{15}^{30} \right\} \\
 &= \frac{1}{225} \left\{ \frac{15^3}{3} + \left[ 15(30)^2 - \frac{30^3}{3} - 15(15)^2 + \frac{15^3}{3} \right] \right\} \\
 &= 15.
 \end{aligned}$$

## Solution to Example 2 (Continued)

$$\begin{aligned} E(X^2) &= \int_0^{15} x^2 \left( \frac{x}{225} \right) dx + \int_{15}^{30} x^2 \left( \frac{30-x}{225} \right) dx \\ &= \frac{1}{225} \left\{ \left[ \frac{x^4}{4} \right]_0^{15} + \left[ 10x^3 - \frac{x^4}{4} \right]_{15}^{30} \right\} = \frac{525}{2} = 262.5. \end{aligned}$$

Therefore,

$$\begin{aligned} V(X) &= E(X^2) - [E(X)]^2 \\ &= 262.5 - 15^2 = 37.5. \end{aligned}$$

## Example 3

- Let  $X$  denote the amount of time for which a book on 2-hour reserve at the science library is checked out by a randomly selected student and suppose  $X$  has the probability density function

$$f_X(x) = \begin{cases} \frac{x}{2}, & \text{for } 0 \leq x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

## Example 3 (Continued)

- (a) Compute  $E(X)$ .
- (b) Compute  $V(X)$  and  $\sigma_X$ .
- (c) If the borrower is charged an amount  $h(X) = X^2$  when checkout duration is  $X$ , compute the expected charge  $E[h(X)]$ .

# Solution to Example 3

(a)

$$\begin{aligned} E(X) &= \int_0^2 x \left(\frac{x}{2}\right) dx \\ &= \left[\frac{x^3}{6}\right]_0^2 = \frac{4}{3}. \end{aligned}$$



# Solution to Example 3 (Continued)

(b)

$$E(X^2) = \int_0^2 x^2 \left(\frac{x}{2}\right) dx = \left[\frac{x^4}{8}\right]_0^2 = 2.$$

$$V(X) = E(X^2) - [E(X)]^2 = 2 - \left(\frac{4}{3}\right)^2 = \frac{2}{9}.$$

$$\sigma_X = \sqrt{2/9}$$

(c)  $E[h(X)] = E[X^2] = 2.$

## 2.5.3 Properties of Expectation

### Property 1

$$E(aX + b) = a E(X) + b,$$

where  $a$  and  $b$  are constants.

# Properties of Expectation (Continued)

## Property 1 (Continued)

- Proof: For discrete case,

$$\begin{aligned} E(aX + b) &= \sum_x (ax + b)f_X(x) \\ &= \sum_x ax f_X(x) + \sum_x b f_X(x) \\ &= a \left[ \sum_x x f_X(x) \right] + b \left[ \sum_x f_X(x) \right] = aE(X) + b. \end{aligned}$$

# Properties of Expectation (Continued)

Two special cases:

(a) Put  $b = 0$ , we have  $E(aX) = a E(X)$ .

(b) Put  $a = 1$ , we have  $E(X + b) = E(X) + b$ .

In general,

$$\begin{aligned} &E[a_1 g_1(X) + a_2 g_2(X) + \cdots + a_k g_k(X)] \\ &= a_1 E[g_1(X)] + a_2 E[g_2(X)] + \cdots + a_k E[g_k(X)] \end{aligned}$$

where  $a_1, a_2, \cdots, a_k$  are constants.

# Properties of Expectation (Continued)

## Property 2

$$V(X) = E(X^2) - [E(X)]^2.$$

# Properties of Expectation (Continued)

## Property 2 (Continued)

Proof:

$$\begin{aligned} V(X) &= E[(X - \mu_X)^2] \\ &= E[X^2 - 2X\mu_X + \mu_X^2] \\ &= E(X^2) - E(2X\mu_X) + E(\mu_X^2) \\ &= E(X^2) - 2\mu_X E(X) + (\mu_X^2) \\ &= E(X^2) - 2\mu_X^2 + \mu_X^2 = E(X^2) - \mu_X^2 \end{aligned}$$

Notice that  $\mu_X = E(X)$  is a constant.

# Properties of Expectation (Continued)

## Property 3

$$V(aX + b) = a^2 V(X),$$

where  $a$  and  $b$  are constants

# Properties of Expectation (Continued)

## Property 3 (Continued)

Proof:

$$\begin{aligned} V(aX + b) &= E[(aX + b)^2] - [E(aX + b)]^2 \\ &= E(a^2X^2 + 2abX + b^2) - (a\mu_X + b)^2 \\ &= a^2E(X^2) + 2abE(X) + b^2 - (a^2\mu_X^2 + 2ab\mu_X + b^2) \\ &= a^2E(X^2) - a^2\mu_X^2 \\ &= a^2[E(X^2) - \mu_X^2] \\ &= a^2V(X). \end{aligned}$$



# Example 4

- A jewelry shop purchased three necklaces of a certain type at \$500 a piece.
- It will sell them for \$1000 a piece. The designer has agreed to repurchase any necklace still unsold after a specified period at \$200 a piece.
- Let  $X$  denote the number of necklaces sold and suppose  $X$  follows the following probability distribution.

|          |     |     |     |     |
|----------|-----|-----|-----|-----|
| $x$      | 0   | 1   | 2   | 3   |
| $f_X(x)$ | 0.1 | 0.2 | 0.3 | 0.4 |

- Find the expected gain and the variance of the gain.

# Solution to Example 4

- With  $g(X) = \text{revenue} - \text{cost} = 1000X + 200(3 - X) - 3(500) = 800X - 900$ .
- $$\begin{aligned} E(g(X)) &= g(0)f_X(0) + g(1)f_X(1) + g(2)f_X(2) + g(3)f_X(3) \\ &= (-900)(0.1) + (-100)(0.2) + (700)(0.3) + 1500(0.4) \\ &= 700. \end{aligned}$$
- Hence the expected profit is \$700.

# Solution to Example 4 (Continued)

- $E \left[ (g(X))^2 \right] = (-900)^2(0.1) + (-100)^2(0.2) + (700)^2(0.3) + 1500^2(0.4) = 1130000.$

- Hence

$$V(g(X)) = 1130000 - 700^2 = 640000$$

and

$$\sqrt{V(g(X))} = \sqrt{640000} = 800$$

# Example 5

- The distribution of the amount of gravel (in tons) sold by a particular construction supply company in a given week is a continuous random variable  $X$  with p.d.f.

$$f_X(x) = \begin{cases} \frac{3}{2}(1 - x^2), & \text{for } 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

- Find  $E(X)$  and  $V(X)$ .

# Solution to Example 5

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 x \frac{3}{2} (1 - x^2) dx \\ &= \frac{3}{2} \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{3}{2} \left( \frac{1}{4} \right) = \frac{3}{8}. \end{aligned}$$

# Solution to Example 5 (Continued)

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_0^1 x^2 \frac{3}{2} (1 - x^2) dx$$

$$= \frac{3}{2} \left[ \frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = \frac{3}{2} \left( \frac{2}{15} \right) = \frac{1}{5}.$$

$$V(X) = \frac{1}{5} - \left( \frac{3}{8} \right)^2 = \frac{19}{320} = 0.0594.$$

# Example 6

- The hospital period, in days, for patients following treatment for a certain type of kidney disorder is a random variable  $Y = X + 4$ , where  $X$  has the probability density function

$$f_X(x) = \begin{cases} \frac{32}{(x+4)^3}, & \text{for } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

- Find the average number of days that a person is hospitalized following treatment for this disorder.

# Solution to Example 6

$$\begin{aligned} E(Y) &= \int_0^{\infty} (x+4) \frac{32}{(x+4)^3} dx \\ &= \int_0^{\infty} \frac{32}{(x+4)^2} dx \\ &= \left[ -\frac{32}{x+4} \right]_0^{\infty} = 8. \end{aligned}$$



## 2.6 Chebyshev's Inequality

- If we know the probability distribution of a random variable  $X$ , we may then compute  $E(X)$  and  $V(X)$ .
- However, the converse is not true. From the knowledge of  $E(X)$  and  $V(X)$  we cannot reconstruct the probability distribution of  $X$  and
- hence, we cannot compute quantities such as

$$\Pr(|X - E(X)| \leq c),$$

where  $c$  is a positive constant.

[Note:  $\Pr(|Y| \leq a)$  is equivalent to  $\Pr(-a \leq Y \leq a)$ .

# Chebyshev's Inequality (Continued)

- Nevertheless, the Russian mathematician Chebyshev gave a very **useful upper (or lower) bound** to such probability.
- This result is known as **Chebyshev's inequality**.

# Chebyshev's Inequality (Continued)

- Let  $X$  be a random variable (discrete or continuous) with  $E(X) = \mu$  and  $V(X) = \sigma^2$ .
- Then for **any positive number  $k$**  we have

$$\Pr(|X - \mu| \geq k\sigma) \leq 1/k^2.$$

- That is, the probability that the value of  $X$  lies at least  $k$  standard deviation from its mean is at most  $1/k^2$ .
- Alternatively,

$$\Pr(|X - \mu| < k\sigma) \geq 1 - 1/k^2.$$

# Remarks

1. The quantity  $k$  in Chebyshev's Inequality can be any positive number.
2. This inequality is true for **all** distributions with finite mean and variance.
3. The theorem gives a **lower bound** on the probability that  $|X - \mu| < k\sigma$ . No guarantee that this lower bound is close to the exact probability.

# Example 1

- Number of telephone calls  $X$  in a day has  $\mu_X = 14$ ,  $\sigma_X = 3.5$ .
- What can you say about  $\Pr(7 < X < 21)$ ?

# Solution to Example 1

$$\begin{aligned}\Pr(7 < X < 21) &= \Pr(14 - 2(3.5) < X < 14 + 2(3.5)) \\ &= \Pr(-2(3.5) < X - 14 < 2(3.5)) \\ &= \Pr(|X - 14| < 2(3.5)) \\ &\geq 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}.\end{aligned}$$

where we apply the Chebyshev in quality with  $k = 2$

## Example 2

The p.f.  $X$  is given as follows.

|          |     |      |     |      |     |      |      |
|----------|-----|------|-----|------|-----|------|------|
| $x$      | 0   | 1    | 2   | 3    | 4   | 5    | 6    |
| $f_X(x)$ | 0.1 | 0.15 | 0.2 | 0.25 | 0.2 | 0.06 | 0.04 |

It can be shown that  $E(X) = 2.64$  and  $\sigma^2 = 2.37$ .

Hence  $\sigma = 1.54$ .

# Solution to Example 2

By Chebyshev's Inequality,

$$\Pr(|X - \mu| \geq 2\sigma) \leq 1/2^2 = 0.25.$$

But

$$\begin{aligned} & \Pr(|X - \mu| \geq 2\sigma) \\ &= \Pr(X - \mu \leq -2\sigma \text{ or } X - \mu \geq 2\sigma) \\ &= \Pr(X \leq \mu - 2\sigma \text{ or } X \geq \mu + 2\sigma) \\ &= \Pr(X \leq -0.44 \text{ or } X \geq 5.72) \\ &= \Pr(X = 6) = 0.04. \quad (\text{From the given p.f.}) \end{aligned}$$

Chebyshev's bound of 0.25 is too conservative.