Sample Space and Sample Points (Continued)

1.1.1 Sample Space (Continued)

- **Sample Space:** The set of all possible outcomes of a statistical experiment is called the **sample space** and it is represented by the symbol S.

One important point here one should pay attention is that the "Sample Space" depends on not only how the experiment is carried out, but also the "problem of interest". Also refer to page 7 for an example.

Examples

1. Consider an experiment of tossing a die.
   - If we are interested in the number that shows on the top face, then the sample space would be
     $$S = \{1, 2, 3, 4, 5, 6\}.$$
   - If we are interested only in whether the number is even or odd, then the sample space is simply
     $$S = \{\text{even}, \text{odd}\}.$$

Examples (Continued)

2. Consider an experiment of tossing two dice.
   – If we are interested in the numbers that show on the top faces, then the sample space would be

$$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), \cdots\cdots, (6,5), (6,6)\}.$$

If the dice are labelled, i.e., whether the first die shows 1 or the second shows 1 matters, the sample space is the one given in the slides; each observation in this sample space is equally likely to appear. If the two dice are not labelled, the sample space is

$$\begin{aligned} S = \{&\{1,1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\} \\ &\{2,2\}, \{2,3\}, \{2,4\}, \{2,5\}, \{2,6\} \\ &\{3,3\}, \{3,4\}, \{3,5\}, \{3,6\} \\ &\{4,4\}, \{4,5\}, \{4,6\} \\ &\{5,5\}, \{5,6\} \\ &\{6,6\}\} \end{aligned}$$

But now, take note that each observation in the sample space is not equally likely; in other words, their chance of appearance are not the same. In particular, "{1,1}" is less likely to appear than "{1,2}".

1.1.3 Events

An **event** is a subset of a sample space.

Examples

1. (a) $S = \{1, 2, 3, 4, 5, 6\}$.

An event that an odd number occurs $= \{1, 3, 5\}$

An event that a number greater than 4 occurs $= \{5, 6\}$

(b) $S = \{\text{even}, \text{odd}\}$.

An event that an odd number occurs $= \{\text{odd}\}$

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In principle, any arbitrary subset of the sample space can be an event. So, for a sample space with n number of possible sample points, there are 2^n number of possible events (we shall be able to verify this rigorously after we have learned “combinations”), including the empty event (null event), denoted by Ø by convention. Here empty event is the event which contains no sample point. So, if S={1,2,3,4,5,6}, the number of possible event is $2^6 = 64$, which is quite a big number, and it is not feasible for us to list them all. In this slide, it gives some frequently used events in practice.

Here is one take home question to practice: if S = {1,2,3} so that we have 8 possible events, please try to write down all the possible events relating to this sample space.

For this case, the list of events are $\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$.

Examples (Continued)

$S = \{1, 2, 3, 4, 5, 6\}$. $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$ and $C = \{2, 4, 6\}$.

Then

- $A' = \{4, 5, 6\}$
- $B' = \{2, 4, 6\} = C$
- $A' \cap B' = \{4, 5, 6\} \cap \{2, 4, 6\} = \{4, 6\}$
- $(A \cup B)' = \{1, 2, 3, 5\}' = \{4, 6\}$

Notice that both $A' \cap B'$ and $(A \cup B)'$ equal $\{4, 6\}$ in this example. Is it true that $A' \cap B' = (A \cup B)'$ in general?

$(\mathrm{A} \cup \mathrm{B})' = \mathrm{A}' \cap \mathrm{B}'$ is in fact the special case of the De Morgan's Law. So you may read this caculation jointly with page 36:

1.2.7 De Morgan's Law

For any n events $A_1, A_2, \cdots, A_n$,

1. $$(A_1 \cup A_2 \cup \cdots \cup A_n)' = A_1' \cap A_2' \cap \cdots \cap A_n'$$

2. $$(A_1 \cap A_2 \cap \cdots \cap A_n)' = A_1' \cup A_2' \cup \cdots \cup A_n'$$

Also, you can refer to page 34 and 35, points 5 and 6.

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1.2.6 Some Basic Properties of Operations of Events

1. $A \cap A' = \emptyset$.
2. $A \cap \emptyset = \emptyset$.
3. $A \cup A' = S$.
4. $(A')' = A$
5. $(A \cap B)' = A' \cup B'$

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Some Basic Properties of Operations of Events (Continued)

6. $(A \cup B)' = A' \cap B'$
7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
8. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
9. $A \cup B = A \cup (B \cap A')$
10. $A = (A \cap B) \cup (A \cap B')$

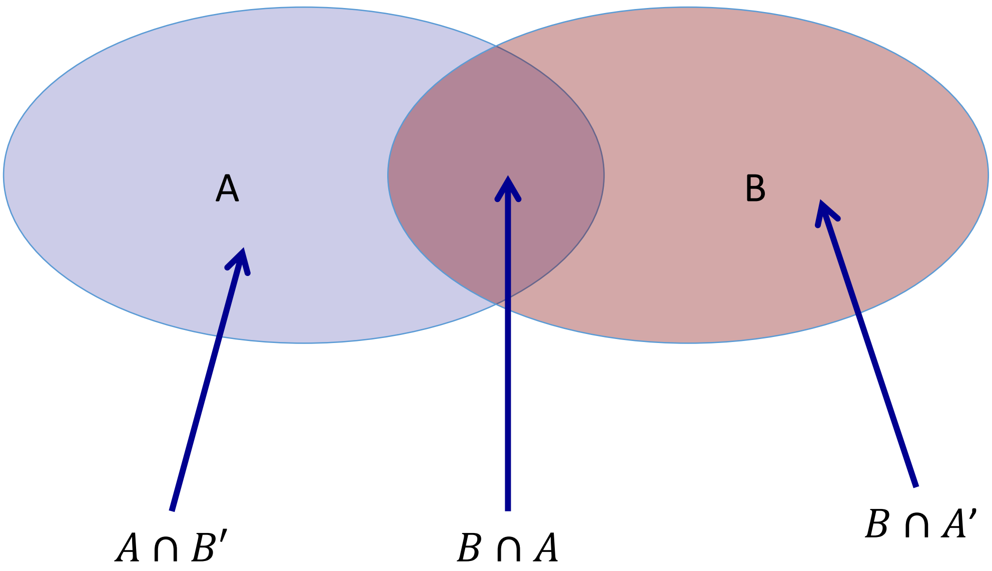
- These are very useful formulae.
- Besides, you may also add the formulae discussed in pages 27 and 28:

$$A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C)$$

$$A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$$

- Also take note that $A \cap (B \cup C) \neq (A \cap B) \cup C$; in other words, when the operations are not of the same type, which operation being done first matters.

- Property 9 in the above slides essentially gives one mutually exclusive partition for $A \cup B$, in other words, besides $A \cup B = A \cup (B \cap A')$, we should also observe $A \cap (B \cap A') = \emptyset$. So, is there any other mutually exclusive partition on $A \cup B$? With the help of the venn diagram, one can indeed identify at least two other possible partitions. Here we give one example of such a partition: $A \cap B, A \cap B', B \cap A'$; the other is left as an excise.
- Property 10 in the above slides have given a mutually exclusive partition for A, which means besides $A = (A \cap B) \cup (A \cap B')$, we also have $(A \cap B) \cap (A \cap B') = \emptyset$. Can you also come up with a similar partition for B?



Example 7

In how many ways can 4 boys and 5 girls sit in a row if the boys and girls must alternate?

If 5 boys and 5 girls are to sit in a row, such that boys and girls are sitting alternatively, how many ways can we make the arrangement?

There are two sitting plans:

Plan 1: BGBGBGBGBG

Plan 2: GBGBGBGBGB

Therefore, the total number of ways is the number of ways for Plan 1 plus the number of ways for Plan 2.

For Plan 1, the number of ways is $5 \times 5 \times 4 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1 = 5! \times 5!$; and likewise, for Plan 2, the number of ways is $5! \times 5!$ as well. So the total number of ways is $2 \times 5! \times 5! = 28800$.

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1.3.2 Addition Principle

- Suppose that a procedure, designated by 1 can be performed in n_1 ways.
- Assume that a procedure, designated by 2 can be performed in n_2 ways.
- Suppose furthermore that it is **NOT possible** that both procedures 1 and 2 are **performed together**.
- Then the number of ways in which we can perform 1 **or** 2 is

$$n_1 + n_2$$

ways.

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If we are doing parallel counting, “addition principle” takes effect; if we are doing sequential counting, “multiplication rule” applies. The example in the last page in fact has jointly used both principles. In particular, “Plan 1” and “Plan 2” parallel, so “addition principle” should be used; therefore, we add the number ways based on “Plan 1” and those based on “Plan 2”. Then, when computing the number of ways for each Plan, Plan 1 say, “multiplication principle” has been applied, we assume that kids are sitting from left to right sequentially.

Binomial Coefficient

- The quantity $\binom{n}{r}$ is called a **binomial coefficient** because it is the coefficient of the term $a^r b^{(n-r)}$ in the binomial expansion of $(a+b)^n$.
- It can be verified that the following hold:
  1. $\binom{n}{r} = \binom{n}{n-r}$ for $r = 0, 1, \cdots, n$
  2. $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$ for $1 \le r \le n$
  3. $\binom{n}{r} = 0$ for $r < 0$ or $r > n$

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The first statement in this page means:

$(a+b)^n = \binom{n}{n}a^n + \binom{n}{n-1}a^{n-1}b + \binom{n}{n-2}a^{n-2}b^2 + \cdots + \binom{n}{1}ab^{n-1} + \binom{n}{0}b^n$. (1)

A direct consequence of this formula is, by setting $a = b = 1$,

$$2^n = \binom{n}{n} + \binom{n}{n-1} + \binom{n}{n-2} + \cdots + \binom{n}{1} + \binom{n}{0}.$$

Note: This last formula is able to be used to verify the statement we claimed in the 3rd page of this document: "for a sample space with n number of possible sample points, there are 2^n number of possible events".

An approach to achieve the expansion in (1) is as follows:

- Let's think $(a+b)^n = (a+b)(a+b)\cdots(a+b) = Factor1 \times Factor2 \times \cdots \times Factorn$, i.e., the multiplication of n number of "$(a+b)$"s. When expanding this formula directly with algebra, we shall get 2^n expanded terms: $(a+b)(a+b)\cdots(a+b) = E1 + E2 + \cdots + E2^n$. Here Ei is a product of n factors, i.e., $Ei = T1 \times T2 \times \cdots \times Tn$ with $T1(= a \text{ or } b)$ from $Factor1$, $T2(= a \text{ or } b)$ from $Factor2$, and so on.
- Now, we can count that among all these $Ei's$:

- the number of Ei's that are equal to a^n should be $\binom{n}{n}$: all the corresponding $T1, T2, \cdots, Tn$ must be a. So the coefficient for a^n is $\binom{n}{n}$.
- the number of Ei's that are equal to $a^{n-1}b$ should be $\binom{n}{n-1}$: out of all the $T1, T2, \cdots, Tn$, $n-1$ of them must be a, and the rest must be b. So the coefficient for $a^{n-1}b$ is $\binom{n}{n-1}$.
- the number of Ei's that are equal to $a^{n-2}b^2$ should be $\binom{n}{n-2}$: out of all the $T1, T2, \cdots, Tn$, $n-2$ of them must be a, and the rest must be b. So the coefficient for $a^{n-2}b^2$ is $\binom{n}{n-2}$.
- …..
- the number of Ei's that are equal to b^n should be $\binom{n}{0}$: out of all the $T1, T2, \cdots, Tn$, 0 of them must be a, and the rest must be b. So the coefficient for b^n is $\binom{n}{0}$.