NATIONAL UNIVERSITY OF SINGAPORE
Department of Statistics and Applied Probability

(2021/22) Semester 1 ST2334 Probability and Statistics Tutorial 7

1. Suppose that X and Y are random variables having the joint probability distribution below.

$f_{X,Y}(x,y)$		x = 2	x = 4
y	1	0.10	0.15
	3	0.20	0.30
	5	0.10	0.15

   (a) Determine whether or not X and Y are independent.
   (b) Find $E(Y|X=2)$.
   (c) Find $E(X|Y=3)$.
   (d) Find $E(2X-3Y)$.
   (e) Find $E(XY)$.
   (f) Find $V(X)$ and $V(Y)$.
   (g) Find $\sigma_{X,Y}$ and $\rho_{X,Y}$.

2. Consider a ferry that can carry both buses and cars on a trip across a waterway. Each trip costs the owner approximately $10. The fee for cars is $3 and the fee for buses is $8. Let X and Y denote the number of buses and cars, respectively, carried on a given trip. The joint distribution of X and Y is given below.

$f_{X,Y}(x,y)$		x = 0	x = 1	x = 2
y	0	0.01	0.01	0.03
	1	0.03	0.08	0.07
	2	0.03	0.06	0.06
	3	0.07	0.07	0.13
	4	0.12	0.04	0.03
	5	0.08	0.06	0.02

   Compute the expected value and variance of profit for the ferry trip.

3. A store operates both a drive-in facility and a walk-in facility. On a randomly selected day, let X and Y respectively, be the proportions of times that the drive-in and walk-in facilities are in use, and suppose that the joint density function of these random variables is given below.

$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{3}(x+2y), & 0 \le x \le 1, 0 \le y \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Determine whether X and Y are independent.
   (b) Find σ_X^2
   (c) Find σ_Y^2
   (d) Find $\sigma_{X,Y}$

4. A service facility operates with two service lines. On a randomly selected day, let X be the proportion of time that the first line is in use whereas Y is the proportion of time that the second line is in use. Suppose that the joint probability density function for (X, Y) is given below.

$$f_{X,Y}(x,y) = \begin{cases} \frac{3}{2}(x^2 + y^2), & 0 \leq x \leq 1, 0 \leq y \leq 1; \\ 0, & \text{otherwise} \end{cases}$$

(a) Determine whether X and Y are independent.
(b) Find the mean and variance of X.
(c) Find the mean and variance of Y.
(d) Find the covariance of X and Y.
(e) Find the mean and variance of $X + Y$.

5. The random variables X and Y have the joint probability density function below.

$$f_{X,Y}(x,y) = \begin{cases} x + y, & 0 \leq x \leq 1, 0 \leq y \leq 1; \\ 0, & \text{otherwise}. \end{cases}$$

Find
(a) $\sigma_{X,Y}$
(b) $E(Y|X = 0.2)$
(c) $E(X|Y = 0.5)$

6. Given that $Var(X) = 5$ and $Var(Y) = 3$, and Z is defined as $Z = -2X + 4Y - 3$.
(a) Find the variance of Z if X and Y are independent.
(b) If $Cov(X, Y) = 1$, find the variance of Z.
(c) If $Cov(X, Y) = 1$, compute the correlation of X and Y.

7. An employee is selected from a staff of 10 to supervise a certain project by selecting a tag at random from a box containing 10 tags numbered from 1 to 10.
(a) Find the formula for the probability distribution of X representing the number on the tag that is drawn
(b) What is the probability that the number drawn is less than 4?
(c) Find the mean and variance of X.

Answers to selected problems

1. (a) $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for <u>all</u> (x,y). X and Y are independent
   (b) 3
   (c) 3.2
   (d) -2.6
   (e) 9.6
   (f) 0.96; 2
   (g) 0; 0

2. 6.55; 44.4275

3. (a) $f_X(x) = \frac{2}{3}(x+1)$, $0 \le x \le 1$. $f_Y(y) = \frac{1}{3}(4y+1)$, $0 \le y \le 1$. $f_{X,Y}(x,y) \ne f_X(x)f_Y(y)$. Therefore, X and Y are dependent.
   (b) 0.08024 (= 13/162)
   (c) 0.07099 (= 23/324)
   (d) -0.00617 (=$-1/162$)

4. (a) $f_X(x) = \frac{3}{2}\left(x^2 + \frac{1}{3}\right)$, $0 \le x \le 1$. $f_Y(y) = \frac{3}{2}(\frac{1}{3} + y^2)$, $0 \le y \le 1$. $f_{X,Y}(x,y) \ne f_X(x)f_Y(y)$. Therefore, X and Y are dependent.
   (b) 0.625 (= 5/8); 0.07604 (= 73/960)
   (c) 0.625 (= 5/8); 0.07604 (= 73/960)
   (d) -0.01563 (= $-1/64$)
   (e) $E(X+Y) = 1.25\ (= 5/4)$; $V(X+Y) = V(X) + V(Y) + 2Cov(X,Y) = 0.12083\ (= 29/240)$

5. (a) $f_X(x) = x + 1/2,$ for $0 \le x \le 1,$ $f_Y(y) = y + 1/2,$ for $0 \le y \le 1.$ $E(XY) = 1/3,$ $Cov(X,Y) = -0.00694\ (= -1/144)$
   (b) $f_{Y|X}(y|0.2) = (2+10y)/7$, for $0 \le y \le 1$. $E(Y|X=0.2) = 0.61905\ (= 13/21)$
   (c) $f_{X|Y}(x|0.5) = x + 1/2$, for $0 \le x \le 1$. $E(X|Y=0.5) = 0.58333\ (= 7/12)$

6. (a) 68
   (b) 52
   (c) 0.2582

7. (a) $f_{X,Y}(x,y) = \begin{cases} \frac{1}{10}, & x = 1, 2, \cdots, 10\ ; \\ 0, & \text{otherwise}\ . \end{cases}$
   (b) 0.3
   (c) $E(X) = 5.5$; $V(X) = 8.25$