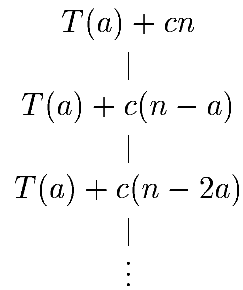
Homework 2: Solutions by your TA

Problem 1

4.2-4 $T(n) = T(n-a) + T(a) + cn$



The tree expands until its height h is such that $(n - ha) \leq a \Rightarrow h = \lceil n/a - 1 \rceil$.

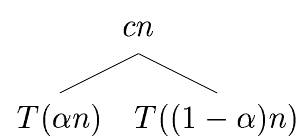
$$\begin{aligned} T(n) &= \textstyle\sum_{i=0}^{h} \{T(a) + c(n - ia)\} \\ &= (h+1)T(a) + cn(h+1) - c\textstyle\sum_{i=0}^{h} i \end{aligned}$$

Say n is a multiple of a. Then $h = n/a - 1$. We also know that since a is constant $T(a)$ is a constant. Therefore,

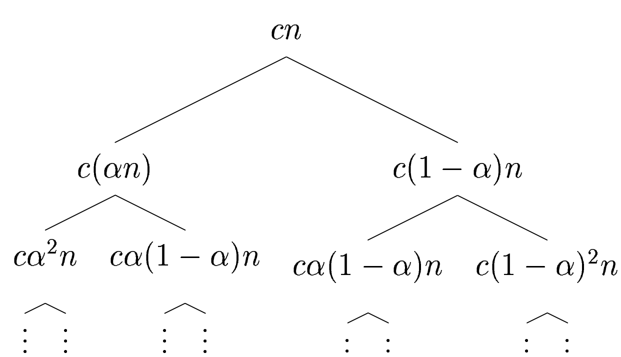
$$\begin{aligned} T(n) &= (n/a)T(a) + cn(n/a) - c(n/a)(n/a-1)/2 \\ &= c(1/a - 1/2a^2)\mathbf{n^2} + O(n) & a \geq 1 \Rightarrow (1/a - 1/2a^2) > 0 \\ &= \Theta(n^2) \end{aligned}$$

4.2-5 $T(n) = T(\alpha n) + T((1-\alpha)n) + cn$

Expanding the tree by one stage we get



Expanding further we have



The height of the tree, $h = \Theta(\lg n)$ Each level contributes cn to $T(n)$. Therefore, $T(n) = hcn = \Theta(n \lg n)$.

Problem 2

Problem 4-1 from page 85 in [CLRS].

(a) $T(n) = 2T(\frac{n}{2}) + n^3$

Let $n^3 = f(n), 2 = a = b, \epsilon = 2 > 0$. $\lg(2) = 1$.

Now $f(n) = \Omega(n^{\log_b(a)+\epsilon})$; that is, $n^3 = \Omega(n^{1+2}) = \Omega(n^3)$ obviously.

Let $c = \frac{1}{4} < 1$; now $af(\frac{n}{b}) \leq cf(n)$; that is, $2(\frac{n}{2})^3 = 2(\frac{n^3}{8}) = \frac{n^3}{4} \leq (\frac{1}{4})n^3 = \frac{n^3}{4}$ for all n.

Thus, by case 3 of the Master Theorem, $T(n) = \Theta(n^3)$.

(b) $T(n) = T(\frac{9}{10}n) + n$

Let $n = f(n), 1 = a, \frac{10}{9} = b, \epsilon = 1 > 0$. $\log_{\frac{10}{9}}(1) = 0$.

Now $f(n) = \Omega(n^{\log_b(a)+\epsilon})$; that is, $n = \Omega(n^{0+1}) = \Omega(n)$ obviously.

Let $c = \frac{9}{10} < 1$; now $af(\frac{n}{b}) \leq cf(n)$; that is, $1(\frac{9}{10}n) = \frac{9n}{10} \leq (\frac{9}{10})n = \frac{9n}{10}$ for all n.

Thus, by case 3 of the Master Theorem, $T(n) = \Theta(n)$.

(c) $T(n) = 16T(\frac{n}{4}) + n^2$

Let $n^2 = f(n), 16 = a, 4 = b$. $\log_4(16) = 2$.

Now $f(n) = \Theta(n^{\log_b(a)})$; that is, $n^2 = \Theta(n^2)$ obviously.

Thus, by case 2 of the Master Theorem, $T(n) = \Theta(n^2 \lg n)$.

(d) $T(n) = 7T(\frac{n}{3}) + n^2$

Let $n^2 = f(n), 7 = a, 3 = b, \epsilon = 0.1 > 0$. $\log_3(7) = 1.777$.

Now $f(n) = \Omega(n^{\log_b(a)+\epsilon})$; that is, $n^2 = \Omega(n^{1.777+0.1}) = \Omega(n^{1.877})$ obviously.

Let $c = \frac{7}{9} < 1$; now $af(\frac{n}{b}) \leq cf(n)$; that is, $7(\frac{n}{3})^2 = 7(\frac{n^2}{9}) = \frac{7}{9}(n^2) \leq (\frac{7}{9})n^2$ for all n.

Thus, by case 3 of the Master Theorem, $T(n) = \Theta(n^2)$.

(e) $T(n) = 7T(\frac{n}{2}) + n^2$

Let $n^2 = f(n), 7 = a, 2 = b, \epsilon = 0.5 > 0$. $\lg(7) = 2.85$.

Now $f(n) = O(n^{log_b(a)-\epsilon})$; that is, $n^2 = O(n^{2.85-0.5}) = O(n^{2.35})$ obviously.

Thus, by case 1 of the Master Theorem, $T(n) = \Theta(n^{2.85})$.

(f) $T(n) = 2T(\frac{n}{4}) + n^{\frac{1}{2}}$

Let $n^{\frac{1}{2}} = f(n), 2 = a, 4 = b$. $\log_4(2) = \frac{1}{2}$.

Now $f(n) = \Theta(n^{\log_b(a)})$; that is, $n^{\frac{1}{2}} = \Theta(n^{\frac{1}{2}})$ obviously.

Thus, by case 2 of the Master Theorem, $T(n) = \Theta(n^{\frac{1}{2}} \lg n)$.

(g) $T(n) = T(n-1) + n$

$T(n) = n + T(n-1) = n + (n-1) + T(n-2) = \ldots$

$= n + (n-1) + (n-2) + \ldots + 3 + T(2)$

$= \sum_{i=3}^{n} i + \Theta(1) = \sum_{i=1}^{n} i - 2 - 1 + \Theta(1)$

$= \frac{n(n+1)}{2} + \Theta(1) = \Theta(n^2)$, by iteration.

(h) $T(n) = T(n^{\frac{1}{2}}) + 1$

Let $k = \lg n$, then $T(2^k) = T(2^{k/2}) + 1$.

Renaming $S(k) = T(2^k)$, we have $S(k) = S(k/2) + 1$.

Using the recurrence tree, $S(k) = \sum_{i=1}^{h} 1 = h$, where h is the height of the recurrence-tree. $h = \Theta(\lg k)$. Therefore, $S(k) = \Theta(\lg k) = T(2^k) = T(n)$.

$\Rightarrow T(n) = \Theta(\lg k) = \Theta(\lg \lg n)$.

Problem 3

Problem 4-4 from page 86 in [CLRS].

(a) $T(n) = 3T(n/2) + n \lg n$

For this recurrence, we have $a = 3, b = 2, f(n) = n \lg n$, and thus we have that $n^{\log_b a} = n^{\log_2 3} \approx n^{1.585}$. Since $f(n) = O(n^{\log_2 3 - \epsilon})$, where $\epsilon \approx 0.5$, we can apply case 1 of the Master's Theorem and conclude that $T(n) = \Theta(n^{\log_2 3})$.

(b) $T(n) = 5T(n/5) + n/\lg n$

Say $n = 5^k$ or $k = \log_5 n$. Then we have,

$$T(5^k) = 5T(5^{k-1}) + \frac{5^k}{k}$$

Dividing by $n = 5^k$ on both sides we get

$$\frac{T(5^k)}{5^k} = \frac{5T(5^{k-1})}{5^k} + \frac{1}{k}$$

$$\Rightarrow \frac{T(5^k)}{5^k} = \frac{T(5^{k-1})}{5^{k-1}} + \frac{1}{k}$$

Renaming $S(k) = T(5^k)/5^k$, we get

$S(k) = S(k-1) + 1/k$.

This is the sum of harmonic series. We know that $S(k) = \Theta(\lg k)$ (same as problem (g)). Therefore, we have, $T(5^k)/5^k = \Theta(\lg k) \Rightarrow T(n)/n = \Theta(\lg \log_5 n) \Rightarrow T(n) = \Theta(n \lg \lg n)$.

(c) $T(n) = 4T(n/2) + n^2\sqrt{n}$

In this recurrence we have $a = 4, b = 2, f(n) = n^{2.5}$, and thus we have that $n^{\log_b a} = n^{\log_2 4} = n^2$. Since $f(n) = O(n^{\log_2 4+\epsilon})$, where $\epsilon = 0.5$, we can apply case 3 of the Master's Theorem if we can show that the regularity condition holds for $f(n)$. $af(n/b) = 4(n/2)^{2.5} = cf(n)$ for $c = (1/\sqrt{2}) < 1$. Consequently by case 3, the solution of the recurrence is $T(n) = \Theta(n^{2.5})$.

(d) $T(n) = 3T(n/3 + 5) + n/2$

If we ignore the $+5$ in the recurrence, we have $T(n) = 3T(n/3) + n/2$. Plugging in values $a = 3, b = 3, f(n) = n/2$ and since, $n^{\log_3 3} = n = \Theta(f(n))$, we can apply the case 2 of the master's theorem to get $T(n) = \Theta(n \lg n)$.

So let's guess that $T(n) = \Theta(n \lg n)$ and try to prove by induction. For establishing the lower bound, say $cm \leq m \leq T(m)$ for all $m < n$. Then

$$\begin{aligned}
T(n) &\geq 3c(n/3+5)\lg(n/3+5) + n/2 \\
&\geq 3c(n/3)\lg(n/3) + n/2 \\
&\geq cn\lg n - cn\lg 3 + n/2 \\
&\geq cn\lg n
\end{aligned}$$

for $c = 1$. Hence $T(n) = \Omega(n \lg n)$.

For establishing the upper bound, say $T(m) \leq cm\lg m$ for any $m < n$. Then

$$\begin{aligned}
T(n) &\leq 3c(n/3+5)\lg(n/3+5) + n/2 \\
&= cn\lg(n/3+5) + 15c\lg(n/3+5) + n/2
\end{aligned}$$

for $n/3 \geq 5$, we have $\lg(n/3+5) \leq \lg(2n/3)$. Therefore,

$$\begin{aligned}
T(n) &\leq cn\lg(2n/3) + 15c\lg(2n/3) + n/2 \\
&= cn\lg n - cn\lg(3/2) + n/2 + 15c\lg n - 15c\lg(3/2) \\
&\leq cn\lg n
\end{aligned}$$

for $c > 1$ and $n > 15$.

Hence, $T(n) = O(n \lg n)$. And since we already showed that $T(n) = \Omega(n \lg n)$, we have $T(n) = \Theta(n \lg n)$.

(e) $T(n) = 2T(n/2) + n/\lg n$

Following exactly the same approach as in problem (b) above, we can show that

$T(n) = \Theta(n \lg\lg n)$.

(f) $T(n) = T(n/2) + T(n/4) + T(n/8) + n$

$T(n) \geq n$. Therefore $T(n) = \Omega(n)$.

By the recurrence tree, we have
$T(n) \leq n + (7/8)n + (7/8)^2 n + \ldots + (7/8)^h n = n \sum_{i=0}^{h} (7/8)^i$, where h is the height of the tree. Therefore $T(n) \leq n \sum_{i=0}^{\infty} (7/8)^i = n \frac{1}{1-(7/8)} \Rightarrow T(n) = O(n)$.

Thus $T(n) = \Theta(n)$.

(g) $T(n) = T(n-1) + 1/n$

By expanding the recurrence, we have $T(n) = T(n-2) + 1/(n-1) + 1/n = \sum(1/i)$. This is the sum of the harmonic series. Therefore, $T(n) = \Theta(\lg n)$. (see page 1067 in [CLRS]).

(h) $T(n) = T(n-1) + \lg n$

By expanding the recurrence, we have
$T(n) = \lg n + \lg(n-1) + \ldots + \lg 1 = \lg(n!) = \Theta(n \lg n)$.

(i) $T(n) = T(n-2) + 2\lg n$

By expanding the recurrence we have
$T(n) = 2\{\lg n + \lg(n-2) + \lg(n-4) + \ldots\}$.

To prove the upper bound,

$$\begin{array}{rcl} T(n) & = & 2\{\lg n + \lg(n-2) + \lg(n-4) + \ldots\} \\ & \leq & 2\{\lg n + \lg(n-1) + \lg(n-2) + \lg(n-3) + \lg(n-4) + \ldots\} \\ & = & 2\lg(n!) \end{array}$$

Therefore $T(n) = O(\lg(n!))$.

To prove the lower bound,

$$\begin{array}{rcl} T(n) & = & \lg n + \lg n + \lg(n-2) + \lg(n-2) + \ldots \\ & \geq & \lg n + \lg(n-1) + \lg(n-2) + \lg(n-3) + \ldots \\ & = & \lg(n!) \end{array}$$

Therefore, $T(n) = \Omega(\lg(n!))$. Since $T(n) = O(\lg(n!))$ and $T(n) = \Omega(\lg(n!))$, we have $T(n) = \Theta(\lg(n!)) = \Theta(n \lg n)$.

(j) $T(n) = \sqrt{n}T(\sqrt{n}) + n$

Diving by n on both sides we get,
$\frac{T(n)}{n} = \frac{T(\sqrt{n})}{\sqrt{n}} + 1$.

Renaming $S(n) = T(n)/n$, we get

$S(n) = S(\sqrt{n}) + 1$.

We know from the first problem in the homework (problem 4-1(h)) that

$S(n) = \Theta(\lg\lg n)$.

$\Rightarrow T(n) = nS(n) = \Theta(n \lg\lg n)$