

FPSAC 2020 Online:

Crystal for stable Grothendieck polynomials

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Stable Grothendieck Polynomials for w

Definition

Stable Grothendieck polynomial (or K -Stanley symmetric function):

$$\mathfrak{G}_w(\mathbf{x}, \beta) = \sum_{h^m \dots h^2 h^1 \in \mathcal{H}_w^m} \beta^{\ell(h^1) + \dots + \ell(h^m) - \ell(w)} x_1^{\ell(h^1)} \dots x_m^{\ell(h^m)}$$

where $\ell(w)$ is the length of any reduced word of w .

\mathcal{H}_w^m : Decreasing factorizations of 0-Hecke Monoid

Example

$w = 132 \in \mathcal{H}_0(4)$

Reduced Hecke words 132, 312

Decreasing factorizations for constant term:

(31)(2), (1)(32) (3)(1)(2), (1)(3)(2)

$$\beta^0 : (x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2) + 2x_1 x_2 x_3 = s_{21}$$

Schur positivity

Schur positivity (Fomin, Greene 1998)

$$\mathfrak{G}_w(\mathbf{x}, \beta) = \sum_{\lambda} \beta^{|\lambda| - \ell(w)} g_w^{\lambda} s_{\lambda}(\mathbf{x})$$

$$g_w^{\lambda} = |\{T \in SST^n(\lambda') \mid \text{column reading of } T \equiv w\}|$$

Example

$$\mathfrak{G}_{132}(\mathbf{x}, \beta) = s_{21} + \beta(2s_{211} + s_{22}) + \beta^2(3s_{2111} + 2s_{221}) + \cdots$$

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Crystal for F_w or \mathfrak{G}_w ?

Idea

Fix $w \in S_n$, create Graph $B(w)$

- ① vertices are decreasing factorizations of w
- ② edges are imposed and colored by f_i, e_i
- ③ highest weights are vertices with no unpaired entries

Theorem (Morse, Schilling 2016)

$B(w)$ is a *crystal graph* of type A_ℓ

Motivation: Schubert Calculus

Polynomial Representatives for Schubert Cells

	Grassmannian $\mathbb{G}_{m,n}$	Flag Varieties Fl_n
cohomology	s_λ	$\mathfrak{S}_w \rightarrow F_w$
k-theory	\mathcal{G}_λ	$\mathcal{G}_w \rightarrow \mathfrak{G}_w$

Grassmannian Grothendieck polynomials: \mathfrak{G}_λ Lascoux, Schützenberger 1982

Stable Grothendieck polynomials: \mathfrak{G}_w Fomin, Kirillov 1994

Combinatorial Approach?

- Crystal Structure on F_w
(Morse & Schilling 2015)
- Nonlocal crystal structure on \mathfrak{G}_w
(Monical & Pechenik & Scrimshaw 2018)

★-Crystal Structure on $\mathcal{H}^{m,\star}$ (Morse, Pan, Poh, Schilling 2019)

Bracketing rule on $h^m \dots h^{i+1} h^i \dots h^1$

- 1 Start with the **largest** letter b in h^{i+1} , pair it with the smallest $a \geq b$ in h^i . If there is no such a , then b is unpaired.
- 2 Proceed in decreasing order in h^{i+1} , ignore previously paired letters.

Crystal operator f_i^\star , x : largest unpaired letter in h^i

- If $x + 1 \in h^i \cap h^{i+1}$, then remove $x + 1$ from h^i , add x to h^{i+1} .
- Otherwise, remove x from h^i and add x to h^{i+1} .

Example

- $(1)(32) \xrightarrow{\text{bracket}} (\textcolor{red}{1})(\textcolor{red}{3}2) \xrightarrow{f_1^\star} (31)(2)$
- $(7532)(621) \xrightarrow{\text{bracket}} (7\textcolor{red}{5}3\textcolor{green}{2})(\textcolor{red}{6}21) \xrightarrow{f_1^\star} (75321)(61)$

Vertices and Edges

$$w = 132, \beta^1$$

$$\textcircled{1} (3, 1)(3, 2)()$$

$$\textcircled{2} (3, 1)(1)(2)$$

$$\textcircled{3} (3, 1)(2)(2)$$

$$\textcircled{4} (3, 1)(3)(2)$$

$$\textcircled{5} (1)(3, 1)(2)$$

$$\textcircled{6} (1)(3, 2)(2)$$

$$\textcircled{7} (3)(3, 1)(2)$$

$$\textcircled{8} (3, 1)()(3, 2)$$

$$\textcircled{9} (1)(1)(3, 2)$$

$$\textcircled{10} (1)(3)(3, 2)$$

$$\textcircled{11} (3)(1)(3, 2)$$

$$\textcircled{12} ()(3, 1)(3, 2)$$

$$\mathfrak{G}_{132}(\mathbf{x}, \beta) = s_{21} + \beta(2s_{211} + s_{22}) + \beta^2(3s_{2111} + 2s_{221}) + \dots$$

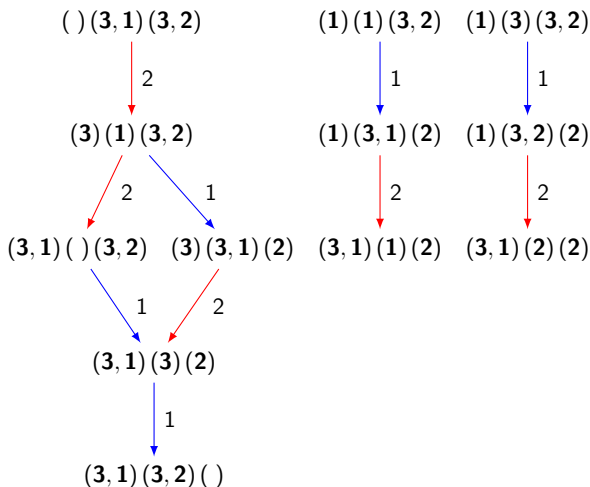


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The Residue Map

$\mathcal{H}^{m,\star}$ = set of 321-avoiding decreasing factorizations with m factors

Definition: $\text{res}(T)$

- $\text{res} : \text{SVT}^m(\lambda/\mu) \rightarrow \mathcal{H}^{m,\star}$
- Associate cell (i, j) with $\ell(\lambda) + j - i$
- Form the i th factor h^i by taking the labels of all cells in T containing i in decreasing order

Example ($m=5$)

$$\begin{array}{|c|c|} \hline 34_1 & 45_2 \\ \hline & 12_3 \quad 25_4 \\ \hline \end{array} \xrightarrow{\text{res}} (42)(21)(1)(43)(3) \in \mathcal{H}^5, \mathcal{H}^{5,\star}$$

Grothendieck polynomials for skew shapes

$$\mathfrak{G}_{\nu/\lambda}(\mathbf{x}; \beta) = \sum_{T \in \text{SVT}(\nu/\lambda)} \beta^{\text{ex}(T)} x_1^{\# \text{of } 1\text{'s}} x_2^{\# \text{of } 2\text{'s}} \dots \quad (\text{Buch 2002})$$

$\text{SVT}(\nu/\lambda)$ = set of semistandard set-valued tableaux of shape ν/λ

Excess in T is $\text{ex}(T)$

Semistandard set-valued tableaux $\text{SVT}(\nu/\lambda)$

Fill boxes of skew shape ν/λ with nonempty sets. **Semistandardness:**

C	
A	B

 $\max(A) \leq \min(B), \max(A) < \min(C)$

Example (Which one is a valid filling?)

✓	<table border="1"><tr><td>34</td><td>45</td></tr><tr><td>12</td><td>25</td></tr></table>	34	45	12	25
34	45				
12	25				

34	35	
	12	456

2	35	
	14	56

The Residue as a Crystal Isomorphism

Theorem (Morse, Pan, Poh, Schilling 2019)

The crystal on SVT defined by [MPS18] and the crystal on decreasing factorizations $\mathcal{H}^{m,*}$ intertwine under the residue map. That is, the following diagram commutes:

$$\begin{array}{ccc} \text{SVT}^m(\lambda/\mu) & \xrightarrow{\text{res}} & \mathcal{H}^{m,*} \\ \downarrow f_k & & \downarrow f_k^* \\ \text{SVT}^m(\lambda/\mu) & \xrightarrow{\text{res}} & \mathcal{H}^{m,*} \end{array}$$

Example

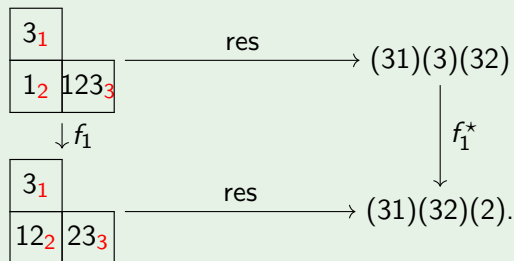


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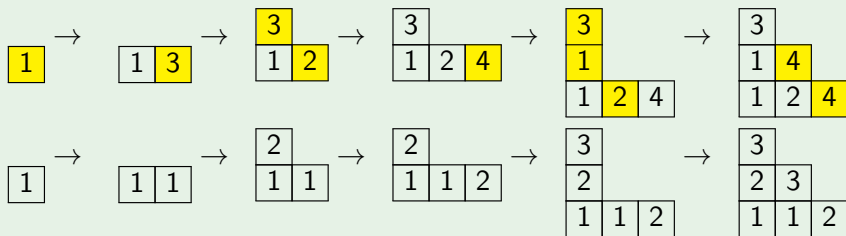
★-Insertion Algorithm (Morse, Pan, Poh, Schilling 2019)

Insert x into row R of a transpose of a semistandard tableau

- ① Try to append x to the right of R (terminate and record)
- ② $x \notin R$, bump the minimal $z > x$ (proceed to the next row)
- ③ $x \in R$, proceed to next row with y minimal such that $[y, x] \subseteq R$

Example

$$\mathbf{h} = (42)(42)(31) = \begin{bmatrix} 3 & 3 & 2 & 2 & 1 & 1 \\ 4 & 2 & 4 & 2 & 3 & 1 \end{bmatrix}$$



Association with \star -crystal

Theorem (Morse, Pan, Poh, Schilling 2019)

Let $\mathbf{h} \in \mathcal{H}^{m,\star}$ and $(P^\star(\mathbf{h}), Q^\star(\mathbf{h})) = \star(\mathbf{h})$.

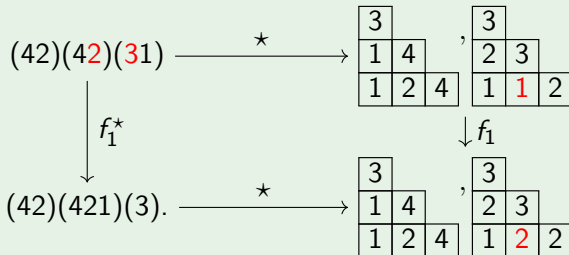
① $f_i^\star(\mathbf{h}) \neq 0$ if and only if $f_i(Q^\star(\mathbf{h})) \neq 0$.

② If $f_i^\star(\mathbf{h}) \neq 0$, then $Q^\star(f_i^\star(\mathbf{h})) = f_i Q^\star(\mathbf{h})$.

In other words, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}^{m,\star} & \xrightarrow{Q^\star} & \text{SSYT}^m \\ \downarrow f_i^\star & & \downarrow f_i \\ \mathcal{H}^{m,\star} & \xrightarrow{Q^\star} & \text{SSYT}^m. \end{array}$$

Example



The Hecke Insertion and the Residue Map

Theorem (Morse, Pan, Poh, Schilling 2019)

Let $T \in \text{SVT}(\lambda)$ and $[\mathbf{k}, \mathbf{h}]^t = \text{res}(T)$. Apply Hecke row insertion from the right on $[\mathbf{k}, \mathbf{h}]^t$ to obtain the pair of tableaux (P, Q) . Then $Q = T$.

Example

$$T = \begin{array}{|c|c|} \hline 2_1 & 4_2 \\ \hline 1_2 & 23_3 \\ \hline \end{array} \xrightarrow{\text{res}} (2)(3)(31)(2) = \begin{bmatrix} 4 & 3 & 2 & 2 & 1 \\ 2 & 3 & 3 & 1 & 2 \end{bmatrix}$$

$$\begin{array}{ccccccc} \boxed{2} & \rightarrow & \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array} & \rightarrow & \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array} & \rightarrow & \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array} & \rightarrow & \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & 2 \\ \hline \end{array} = P. \\ \\ \boxed{1} & \rightarrow & \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array} & \rightarrow & \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 2 \\ \hline \end{array} & \rightarrow & \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 23 \\ \hline \end{array} & \rightarrow & \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 23 \\ \hline \end{array} = Q. \end{array}$$

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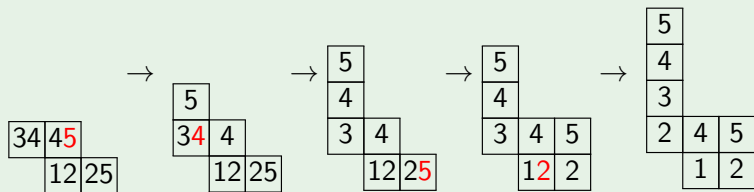
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Uncrowding Algorithm

Uncrowding Operator (Lenart 2000; Buch 2002; Bandlow, Morse 2012; Patrias 2016; Reiner, Tenner, Yong 2018)

- Identify the topmost row in T containing a multicell.
- Let x be the largest letter in that row which lies in a multicell.
- Delete this x and perform RSK algorithm into the rows above.
- Result is a single-valued skew tableau.

Example

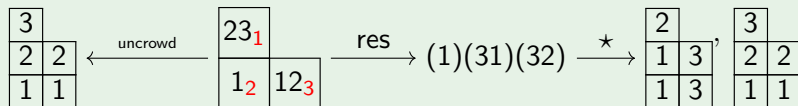


Connection to the Uncrowding map

Theorem (Morse, Pan, Poh, Schilling, 2019)

Let $T \in \text{SVT}^m(\lambda)$, $(\tilde{P}, \tilde{Q}) = \text{uncrowd}(T)$, and $(P, Q) = \star \circ \text{res}(T)$. Then $Q = \tilde{P}$.

Example



Thank you!



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0-Hecke Monoid

Definition

The **0-Hecke monoid**, denoted by $\mathcal{H}_0(n)$, where $n \in \mathbb{N}$, is the monoid of all finite words in $[n-1] := \{1, 2, \dots, n-1\}$, such that for all $p, q \in [n]$,

$$pp \equiv p, \quad pqp \equiv qpq,$$

if $|p - q| > 1$ we also have $pq \equiv qp$.

Examples

- $2112 \equiv 212 \equiv 121$
- $2121 \equiv 1211 \equiv 121 \equiv 212$
- $31312 \equiv 3132 \equiv 312 \equiv 132$

Back to definition of stable Grothendieck polynomials

Decreasing factorizations in $\mathcal{H}_0(n)$

Definition

A **decreasing factorization** of $w \in \mathcal{H}_0(n)$ into m **factors** is a product of decreasing factors

$$\mathbf{h} = h^m \dots h^2 h^1$$

such that $\mathbf{h} \equiv w$ in $\mathcal{H}_0(n)$.

\mathcal{H}_w^m = set of decreasing factorizations of w in $\mathcal{H}_0(n)$ with m factors

Example

Decreasing factorizations for $132 \in \mathcal{H}_0(3)$ of length 5 with 3 factors:

$$\begin{array}{ccc} (31)(31)(2) & (31)(32)(2) & (31)(1)(32) \\ (31)(3)(32) & (1)(31)(32) & (3)(31)(32) \end{array}$$

Back to definition of stable Grothendieck polynomials

321-avoiding Hecke words (braid-free, fully-commutative)

Definition

An element $w \in \mathcal{H}_0(n)$ is **321-avoiding** if none of the reduced expressions for w contain a consecutive subword of the form $i \ i + 1 \ i$ for any $i \in [n - 1] = \{1, 2, \dots, n - 1\}$.

Examples

- $121 \equiv 212$ is not 321-avoiding
- $132 \equiv 312$ is 321-avoiding
- $22132 \equiv 2132 \equiv 2312$ is 321-avoiding

Denote $\mathcal{H}^{m,\star}(n)$ as the set of all 321-avoiding decreasing factorizations of $\mathcal{H}_0(n)$ with m factors.

Examples

- $(\) (1) (21) \in \mathcal{H}^3, \notin \mathcal{H}^{3,\star}.$
- $(31) (2) \in \mathcal{H}^{2,\star}$

Crystal Structure on SVT (Monical & Pechenik & Scrimshaw 2018)

A Signature Rule

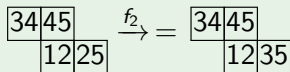
Assign $-$ to every column of T containing an i but not an $i + 1$. Similarly, assign $+$ to every column of T containing an $i + 1$ but not an i . Then successively pair each $+$ that is adjacent to a $-$.

Crystal Operator f_i

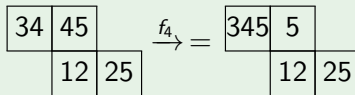
- changes the rightmost unpaired $i -$ to $i + 1$, except
- if its right neighbor contains both $i, i + 1$, then *move* the i over and turn it to be $i + 1$

Example

$+$ $-$ $-$



$-$ $+$



The Hecke Insertion Algorithm (Buch 2008, Patrias, Pylyavskyy 2016)

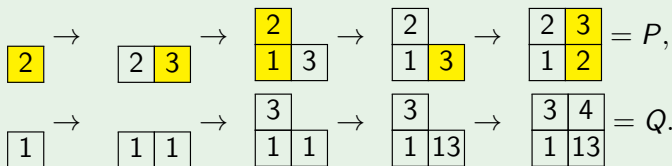
Insert x to row R of an **increasing tableau**

- Try to append x to the right of R (record and terminate)
- Try to bump the smallest letter that is bigger than x (proceed to the next row)

$$\mathcal{H}^m \longleftrightarrow (P, Q)$$

Example

$$\mathbf{h} = (2)(31)(\quad)(32) = \begin{bmatrix} 4 & 3 & 3 & 1 & 1 \\ 2 & 3 & 1 & 3 & 2 \end{bmatrix}.$$



A Solution to $\mathcal{H}^m(3)$

$(2, 1, 21)$



$(2, 21, 2)$



$(21, 1, 2)$

$(\quad, 21, 21)$

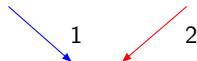


$(1, 2, 21)$



$(21, \quad, 21)$

$(1, 21, 1)$



$(21, 2, 1)$



$(21, 21, \quad)$

$(1, 1, 21)$



$(1, 21, 2)$



$(21, 2, 2)$

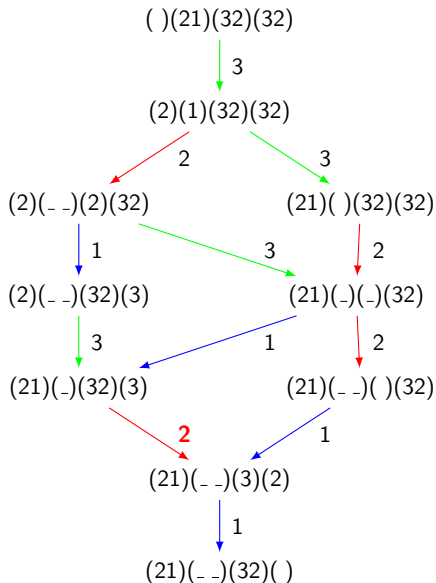
A Counterexample

Desired properties:

- ① f_i only changes the i -th and $(i + 1)$ -st factors;
- ② f_i is determined by the first $(i + 1)$ factors;
- ③ f_i does not change w and excess.

Our crystal on $\mathcal{H}^m(3)$ has these properties.

However, we found a counter example in $\mathcal{H}^4(4)$.



Pop Motivation: Why crystals?

- Irreducible representations V_λ and V_μ

-

$$V_\lambda \otimes V_\mu \cong \bigoplus_{\nu} c_{\lambda\mu}^{\nu} V_{\nu}$$

Question: How to count multiplicities $c_{\lambda\mu}^{\nu}$?

- Crystals $\mathcal{B}_\lambda \longleftrightarrow V_\lambda$, $\mathcal{B}_\mu \longleftrightarrow V_\mu$

-

$$\mathcal{B}_\lambda \otimes \mathcal{B}_\mu = \bigoplus_{\nu} c_{\lambda\mu}^{\nu} \mathcal{B}_{\nu}$$

- $c_{\lambda\mu}^{\nu} = \#\{\text{Yamanouchi tableaux of shape } \nu/\lambda \text{ and content } \mu\}$

Littlewood-Richardson Coefficients

- Character of crystal $\mathcal{B}_\lambda = s_\lambda$

An Illustration on Lie Algebra \mathfrak{sl}_3

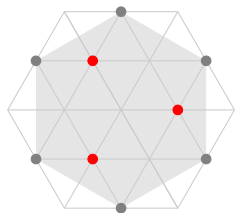


Figure: Std Rep of $\mathfrak{sl}_3 : V_{(1,0)}$

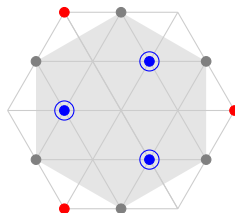


Figure: Tensor Product

Characters:

- $x_1^2 + x_1x_2 + x_2^2 + x_1x_3 + x_2x_3 + x_3^2 = s_2$

- $x_2x_1 + x_3x_1 + x_3x_2 = s_{11}$

Tensor Product Using Crystals

$$1 \xrightarrow{1} 2 \xrightarrow{2} 3$$

1	$1 \otimes 1$	$\xrightarrow{1}$	$1 \otimes 2$	$\xrightarrow{2}$	$1 \otimes 3$
$\downarrow 1$			$\downarrow 1$		$\downarrow 1$
2	$2 \otimes 1$		$2 \otimes 2$	$\xrightarrow{2}$	$2 \otimes 3$
$\downarrow 2$			$\downarrow 2$		$\downarrow 2$
3	$3 \otimes 1$	$\xrightarrow{1}$	$3 \otimes 2$		$3 \otimes 3$