FPSAC 2020 Online: Crystal for stable Grothendieck polynomials

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Stable Grothendieck Polynomials for w

Definition

Stable Grothendieck polynomial (or K-Stanley symmetric function):

$$\mathfrak{G}_w(\mathbf{x}, \beta) = \sum_{h^m \ldots h^2 h^1 \in \mathcal{H}_w^m} \beta^{\ell(h^1)+\cdots+\ell(h^m)-\ell(w)} x_1^{\ell(h^1)} \ldots x_m^{\ell(h^m)}$$

where $\ell(w)$ is the length of any reduced word of w.

$\mathcal{H}_w^m$: Decreasing factorizations of 0-Hecke Monoid

Example

$w = 132 \in \mathcal{H}_0(4)$
Reduced Hecke words $132, 312$
Decreasing factorizations for constant term:
$(31)(2), (1)(32)$ $(3)(1)(2), (1)(3)(2)$

$$\beta^0: \ (x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2) + 2x_1 x_2 x_3 = s_{21}$$

Schur positivity

Schur positivity (Fomin, Greene 1998)

$$\mathfrak{G}_w(\mathbf{x}, \beta) = \sum_{\lambda} \beta^{|\lambda|-\ell(w)} g_w^{\lambda} s_{\lambda}(x)$$

$$g_w^{\lambda} = |\{T \in SST^n(\lambda') | \text{ column reading of } T \equiv w\}|$$

Example

$$\mathfrak{G}_{132}(\mathbf{x}, \beta) = s_{21} + \beta(2s_{211} + s_{22}) + \beta^2(3s_{2111} + 2s_{221}) + \cdots$$

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Crystal for F_w or $\mathfrak{S}_w$?

Idea

Fix $w \in S_n$, create Graph $B(w)$

1. vertices are decreasing factorizations of w
2. edges are imposed and colored by f_i, e_i
3. highest weights are vertices with no unpaired entries

Theorem (Morse, Schilling 2015)

$B(w)$ is a crystal graph of type A_ℓ

Motivation: Schubert Calculus

Polynomial Representatives for Schubert Cells

	Grassmannian $\mathbb{G}_{m,n}$	**Flag Varieties** Fl_n
cohomology	s_λ	$\mathfrak{S}_w \to F_w$
k-theory	$\mathcal{G}_\lambda$	$\mathcal{G}_w \to \mathfrak{G}_w$

Grassmannian Grothendieck polynomials: $\mathfrak{G}_\lambda$ Lascoux, Schützenberger 1982
Stable Grothendieck polynomials: $\mathfrak{G}_w$ Fomin, Kirillov 1994

Combinatorial Approach?

- Crystal Structure on F_w
  (Morse & Schilling 2015)
- Nonlocal crystal structure on $\mathfrak{G}_w$
  (Monical & Pechenik & Scrimshaw 2018)

$\star$-Crystal Structure on $\mathcal{H}^{m,\star}$ (Morse, Pan, Poh, Schilling 2019)

Bracketing rule on $h^m \dots h^{i+1}\, h^i \dots h^1$

1. Start with the **largest** letter b in h^{i+1}, pair it with the smallest $a \geqslant b$ in h^i. If there is no such a, then b is unpaired.
2. Proceed in decreasing order in h^{i+1}, ignore previously paired letters.

Crystal operator $f_i^\star$, x : largest unpaired letter in h^i

- If $x+1 \in h^i \cap h^{i+1}$, then remove $x+1$ from h^i, add x to h^{i+1}.
- Otherwise, remove x from h^i and add x to h^{i+1}.

Example

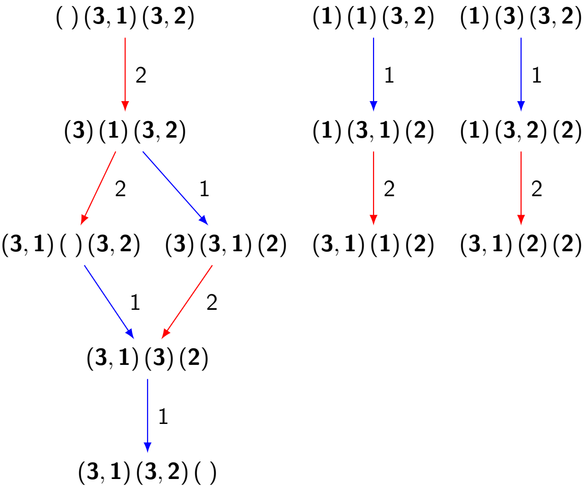
- $(1)(32) \xrightarrow{\text{bracket}} (1)(32) \xrightarrow{f_1^\star} (31)(2)$
- $(7532)(621) \xrightarrow{\text{bracket}} (7532)(621) \xrightarrow{f_1^\star} (75321)(61)$

Vertices and Edges

$w = 132, \ \beta^1$

1. $(3,1)(3,2)(\,)$
2. $(3,1)(1)(2)$
3. $(3,1)(2)(2)$
4. $(3,1)(3)(2)$
5. $(1)(3,1)(2)$
6. $(1)(3,2)(2)$
7. $(3)(3,1)(2)$
8. $(3,1)(\,)(3,2)$
9. $(1)(1)(3,2)$
10. $(1)(3)(3,2)$
11. $(3)(1)(3,2)$
12. $(\,)(3,1)(3,2)$

$$\mathfrak{G}_{132}(\mathbf{x},\beta) = s_{21} + \beta(2s_{211} + s_{22}) + \beta^2(3s_{2111} + 2s_{221}) + \cdots$$



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The Residue Map

$\mathcal{H}^{m,\star}$ = set of 321-avoiding decreasing factorizations with m factors

Definition: res(T)

- res : $\mathrm{SVT}^m(\lambda/\mu) \to \mathcal{H}^{m,\star}$
- Associate cell (i,j) with $\ell(\lambda) + j - i$
- Form the ith factor h^i by taking the labels of all cells in T containing i in decreasing order

Example (m=5)

$$\begin{array}{|c|c|c} \hline 34_{1} & 45_{2} & \multicolumn{1}{c}{} \\ \hline \multicolumn{1}{c|}{} & 12_{3} & \multicolumn{1}{c|}{25_{4}} \\ \cline{2-3} \end{array} \xrightarrow{\text{res}} (42)(21)(1)(43)(3) \in \mathcal{H}^5, \mathcal{H}^{5,\star}$$

The Residue as an Crystal Isomorphism

Theorem (Morse, Pan, Poh, Schilling 2019)

The crystal on SVT *defined by [MPS18] and the crystal on decreasing factorizations* $\mathcal{H}^{m,\star}$ *intertwine under the residue map. That is, the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{SVT}^m(\lambda/\mu) & \xrightarrow{\text{res}} & \mathcal{H}^{m,\star} \\ \downarrow f_k & & \downarrow f_k^\star \\ \mathrm{SVT}^m(\lambda/\mu) & \xrightarrow{\text{res}} & \mathcal{H}^{m,\star}. \end{array}$$

Example

$$\begin{array}{ccc} \begin{array}{|c|c} \hline 3_{1} & \\ \hline 1_{2} & \multicolumn{1}{c|}{123_{3}} \\ \hline \end{array} & \xrightarrow{\text{res}} & (31)(3)(32) \\ \downarrow f_1 & & \downarrow f_1^\star \\ \begin{array}{|c|c} \hline 3_{1} & \\ \hline 12_{2} & \multicolumn{1}{c|}{23_{3}} \\ \hline \end{array} & \xrightarrow{\text{res}} & (31)(32)(2). \end{array}$$

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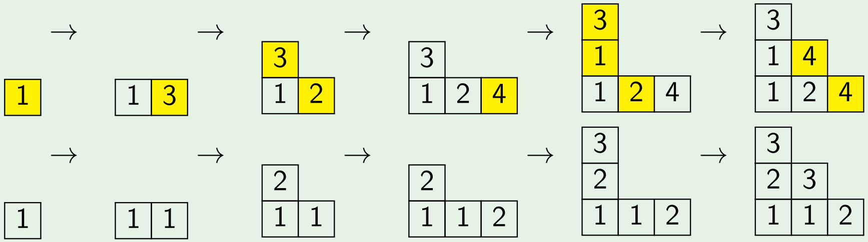
$\star$-Insertion Algorithm (Morse, Pan, Poh, Schilling 2019)

Insert x into row R of a transpose of a semistandard tableau

1. Try to append x to the right of R (terminate and record)
2. $x \notin R$, bump the minimal $z > x$ (proceed to the next row)
3. $x \in R$, proceed to next row with y minimal such that $[y, x] \subseteq R$

Example

$$\mathbf{h} = (42)(42)(31) = \begin{bmatrix} \mathbf{3} & \mathbf{3} & \mathbf{2} & \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{4} & \mathbf{2} & \mathbf{4} & \mathbf{2} & \mathbf{3} & \mathbf{1} \end{bmatrix}$$



Association with $\star$-crystal

Theorem (Morse, Pan, Poh, Schilling 2019)

Let $\mathbf{h} \in \mathcal{H}^{m,\star}$ *and* $(P^\star(\mathbf{h}), Q^\star(\mathbf{h})) = \star(\mathbf{h})$.

1. $f_i^\star(\mathbf{h}) \neq 0$ *if and only if* $f_i(Q^\star(\mathbf{h})) \neq 0$.
2. *If* $f_i^\star(\mathbf{h}) \neq 0$, *then* $Q^\star(f_i^\star(\mathbf{h})) = f_i Q^\star(\mathbf{h})$.

In other words, the following diagram commutes:

$$\begin{tikzcd} \mathcal{H}^{m,\star} \arrow[r, "Q^\star"] \arrow[d, "f_i^\star"] & \mathrm{SSYT}^m \arrow[d, "f_i"] \\ \mathcal{H}^{m,\star} \arrow[r, "Q^\star"] & \mathrm{SSYT}^m. \end{tikzcd}$$

Example

$$\begin{tikzcd} (42)(4{\color{red}2})({\color{red}3}1) \arrow[r, "\star"] \arrow[d, "f_1^\star"] & \begin{array}{|c|c|c|}\hline 3 \\ \cline{1-1} 1 & 4 \\ \cline{1-2} 1 & 2 & 4 \\ \hline\end{array}\ , \begin{array}{|c|c|c|}\hline 3 \\ \cline{1-1} 2 & 3 \\ \cline{1-2} 1 & {\color{red}1} & 2 \\ \hline\end{array} \arrow[d, "f_1"] \\ (42)(421)(3). \arrow[r, "\star"] & \begin{array}{|c|c|c|}\hline 3 \\ \cline{1-1} 1 & 4 \\ \cline{1-2} 1 & 2 & 4 \\ \hline\end{array}\ , \begin{array}{|c|c|c|}\hline 3 \\ \cline{1-1} 2 & 3 \\ \cline{1-2} 1 & {\color{red}2} & 2 \\ \hline\end{array} \end{tikzcd}$$

The Hecke Insertion and the Residue Map

Theorem (Morse, Pan, Poh, Schilling 2019)

Let $T \in \mathrm{SVT}(\lambda)$ and $[\mathbf{k}, \mathbf{h}]^t = \mathrm{res}(T)$. Apply Hecke row insertion from the right on $[\mathbf{k}, \mathbf{h}]^t$ to obtain the pair of tableaux (P, Q). Then $Q = T$.

Example

$$T = \begin{array}{|c|c|}\hline 2_1 & 4_2 \\ \hline 1_2 & 23_3 \\ \hline \end{array} \xrightarrow{\text{res}} (2)(3)(31)(2) = \begin{bmatrix} \mathbf{4} & \mathbf{3} & \mathbf{2} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{3} & \mathbf{3} & \mathbf{1} & \mathbf{2} \end{bmatrix}$$

$$\begin{array}{|c|}\hline 2 \\ \hline \end{array} \to \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array} \to \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \to \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \to \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 2 \\ \hline \end{array} = P.$$

$$\begin{array}{|c|}\hline 1 \\ \hline \end{array} \to \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array} \to \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \to \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{23} \\ \hline \end{array} \to \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 23 \\ \hline \end{array} = Q.$$

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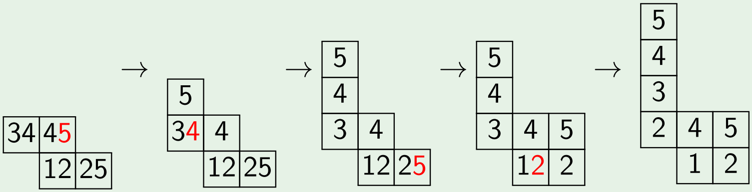
Uncrowding Algorithm

Uncrowding Operator (Lenart 2000; Buch 2002; Bandlow, Morse 2012; Patrias 2016; Reiner, Tenner, Yong 2018)

- Identify the topmost row in T containing a multicell.
- Let x be the largest letter in that row which lies in a multicell.
- Delete this x and perform RSK algorithm into the rows above.
- Result is a single-valued skew tableau.

Example



Connection to the Uncrowding map

Theorem (Morse, Pan, Poh, Schilling, 2019)

Let $T \in \mathrm{SVT}^m(\lambda)$, $(\tilde{P}, \tilde{Q}) = \mathrm{uncrowd}(T)$, *and* $(P, Q) = \star \circ \mathrm{res}(T)$. *Then* $Q = \tilde{P}$.

Example

$$\begin{array}{|c|c} \hline 3 & \multicolumn{1}{|c}{} \\ \hline 2 & \multicolumn{1}{|c|}{2} \\ \hline 1 & \multicolumn{1}{|c|}{1} \\ \hline \end{array} \xleftarrow{\text{uncrowd}} \begin{array}{|c|c} \hline 23_{\color{red}1} & \multicolumn{1}{|c}{} \\ \hline 1_{\color{red}2} & \multicolumn{1}{|c|}{12_{\color{red}3}} \\ \hline \end{array} \xrightarrow{\text{res}} (1)(31)(32) \xrightarrow{\star} \begin{array}{|c|c} \hline 2 & \multicolumn{1}{|c}{} \\ \hline 1 & \multicolumn{1}{|c|}{3} \\ \hline 1 & \multicolumn{1}{|c|}{3} \\ \hline \end{array}, \begin{array}{|c|c} \hline 3 & \multicolumn{1}{|c}{} \\ \hline 2 & \multicolumn{1}{|c|}{2} \\ \hline 1 & \multicolumn{1}{|c|}{1} \\ \hline \end{array}$$

Thank you!

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0-Hecke Monoid

Definition

The 0-Hecke monoid, denoted by $\mathcal{H}_0(n)$, where $n \in \mathbb{N}$, is the monoid of all finite words in $[n-1] := \{1, 2, \ldots, n-1\}$, such that for all $p, q \in [n]$,

$$pp \equiv p, \quad pqp \equiv qpq,$$

if $|p - q| > 1$ we also have $pq \equiv qp$.

Examples

- $2112 \equiv 212 \equiv 121$
- $2121 \equiv 1211 \equiv 121 \equiv 212$
- $31312 \equiv 3132 \equiv 312 \equiv 132$

Back to definition of stable Grothendieck polynomials

Decreasing factorizations in $\mathcal{H}_0(n)$

Definition

A decreasing factorization of $w \in \mathcal{H}_0(n)$ into m factors is a product of decreasing factors

$$\mathbf{h} = h^m \dots h^2 h^1$$

such that $\mathbf{h} \equiv w$ in $\mathcal{H}_0(n)$.

$\mathcal{H}_w^m$ = set of decreasing factorizations of w in $\mathcal{H}_0(n)$ with m factors

Example

Decreasing factorizations for $132 \in \mathcal{H}_0(3)$ of length 5 with 3 factors:

$$\begin{array}{ccc} (31)(31)(2) & (31)(32)(2) & (31)(1)(32) \\ (31)(3)(32) & (1)(31)(32) & (3)(31)(32) \end{array}$$

Back to definition of stable Grothendieck polynomials

321-avoiding Hecke words (braid-free, fully-commutative)

Definition

An element $w \in \mathcal{H}_0(n)$ is 321-avoiding if none of the reduced expressions for w contain a consecutive subword of the form $i\ i+1\ i$ for any $i \in [n-1] = \{1, 2, \ldots, n-1\}$.

Examples

- $121 \equiv 212$ is not 321-avoiding
- $132 \equiv 312$ is 321-avoiding
- $22132 \equiv 2132 \equiv 2312$ is 321-avoiding

Denote $\mathcal{H}^{m,\star}(n)$ as the set of all 321-avoiding decreasing factorizations of $\mathcal{H}_0(n)$ with m factors.

Examples

- $(\ )(1)(21) \in \mathcal{H}^3, \notin \mathcal{H}^{3,\star}$.
- $(31)(2) \in \mathcal{H}^{2,\star}$

Stable Grothendieck polynomials for skew shapes

$$\mathfrak{G}_{\nu/\lambda}(\mathbf{x};\beta) = \sum_{T \in \mathrm{SVT}(\nu/\lambda)} \beta^{\mathrm{ex}(T)} x_1^{\#\text{of 1's}} x_2^{\#\text{of 2's}} \cdots \qquad \text{(Buch 2002)}$$

SVT(ν/λ) = set of semistandard set-valued tableaux of shape ν/λ
Excess in T is ex(T)

Semistandard set-valued tableaux SVT(ν/λ)

Fill boxes of skew shape ν/λ with nonempty sets. Semistandardness:

$$\begin{array}{|c|c} \hline C & \\ \hline A & \multicolumn{1}{c|}{B} \\ \hline \end{array} \quad \max(A) \leqslant \min(B),\ \max(A) < \min(C)$$

Example (Which one is a valid filling?)

$$\checkmark\ \begin{array}{|c|c|c|} \hline 34 & 45 & \\ \hline & 12 & 25 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|} \hline 34 & 35 & \\ \hline & 12 & 456 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|} \hline 2 & 35 & \\ \hline & 14 & 56 \\ \hline \end{array}$$

Crystal Structure on SVT (Monical & Pechenik & Scrimshaw 2018)

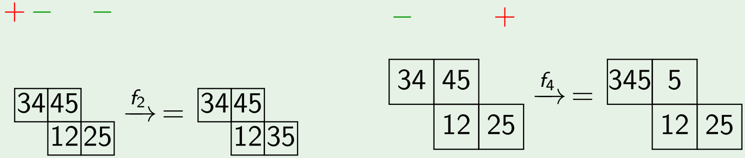
A Signature Rule

Assign $-$ to every column of T containing an i but not an $i+1$. Similarly, assign $+$ to every column of T containing an $i+1$ but not an i. Then successively pair each $+$ that is adjacent to a $-$.

Crystal Operator f_i

- changes the rightmost unpaired i $-$to $i+1$, except
- if its right neighbor contains both $i, i+1$, then *move* the i over and turn it to be $i+1$

Example



The Hecke Insertion Algorithm (Buch 2008, Patrias, Pylyavskyy 2016)

Insert x to row R of an increasing tableau

- Try to append x to the right of R (record and terminate)
- Try to bump the smallest letter that is bigger than x (proceed to the next row)

$$\mathcal{H}^m \longleftrightarrow (P, Q)$$

Example

$$\mathbf{h} = (2)(31)(\,)(32) = \begin{bmatrix} \mathbf{4} & \mathbf{3} & \mathbf{3} & \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{3} & \mathbf{1} & \mathbf{3} & \mathbf{2} \end{bmatrix}.$$

$$\begin{array}{|c|}\hline 2 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array} \to \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \to \begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 2 \\ \hline\end{array} = P,$$

$$\begin{array}{|c|}\hline 1 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array} \to \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} \to \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{13} \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 13 \\ \hline\end{array} = Q.$$

A Solution to $\mathcal{H}^m(3)$

$$\begin{tikzcd}
(2,1,21) \arrow[d, "1"] \\
(2,21,2) \arrow[d, "2"] \\
(21,1,2)
\end{tikzcd}$$

$$\begin{tikzcd}
 & (\quad ,21,21) \arrow[d, "2"] & \\
 & (1,2,21) \arrow[dl, "2"] \arrow[dr, "1"] & \\
(21,\quad ,21) \arrow[dr, "1"] & & (1,21,1) \arrow[dl, "2"] \\
 & (21,2,1) \arrow[d, "1"] & \\
 & (21,21,\quad) &
\end{tikzcd}$$

$$\begin{tikzcd}
(1,1,21) \arrow[d, "1"] \\
(1,21,2) \arrow[d, "2"] \\
(21,2,2)
\end{tikzcd}$$

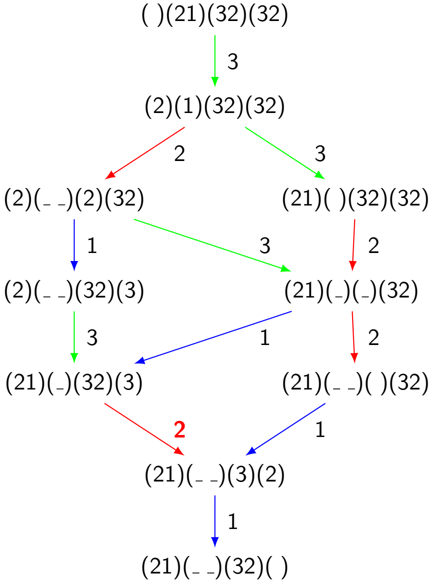
A Counterexample

Desired properties:

1. f_i only changes the i-th and $(i+1)$-st factors;
2. f_i is determined by the first $(i+1)$ factors;
3. f_i does not change w and excess.

Our crystal on $\mathcal{H}^m(3)$ has these properties.

However, we found a counter examle in $\mathcal{H}^4(4)$.



Pop Motivation: Why crystals?

- Irreducible representations V_λ and V_μ
- $$V_\lambda \otimes V_\mu \cong \bigoplus_\nu c^\nu_{\lambda\mu} V_\nu$$

Question: How to count multiplicities $c^\nu_{\lambda\mu}$?

- Crystals $\mathcal{B}_\lambda \longleftrightarrow V_\lambda$, $\mathcal{B}_\mu \longleftrightarrow V_\mu$
- $$\mathcal{B}_\lambda \otimes \mathcal{B}_\mu = \bigoplus_\nu c^\nu_{\lambda\mu} \mathcal{B}_\nu$$
- $c^\nu_{\lambda\mu} = \#\{$Yamanouchi tableaux of shape ν/λ and content $\mu\}$
  Littlewood-Richardson Coefficients
- Character of crystal $\mathcal{B}_\lambda = s_\lambda$

An Illustration on Lie Algebra $\mathfrak{sl}_3$

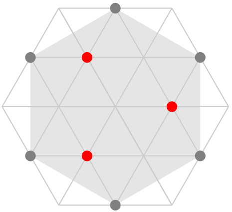


Figure: Std Rep of $\mathfrak{sl}_3$: $V_{(1,0)}$

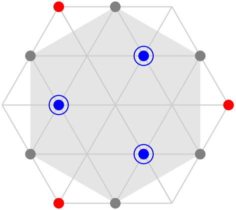


Figure: Tensor Product

Characters:

- $x_1^2 + x_1x_2 + x_2^2 + x_1x_3 + x_2x_3 + x_3^2 = s_2$
- $x_2x_1 + x_3x_1 + x_3x_2 = s_{11}$

Tensor Product Using Crystals

$$1 \xrightarrow{1} 2 \xrightarrow{2} 3$$

$$\begin{tikzcd}
1 \arrow[d, "1"] & 1 \otimes 1 \arrow[r, "1"] & 1 \otimes 2 \arrow[r, "2"] \arrow[d, "1"] & 1 \otimes 3 \arrow[d, "1"] \\
2 \arrow[d, "2"] & 2 \otimes 1 \arrow[d, "2"] & 2 \otimes 2 \arrow[r, "2"] & 2 \otimes 3 \arrow[d, "2"] \\
3 & 3 \otimes 1 \arrow[r, "1"] & 3 \otimes 2 & 3 \otimes 3
\end{tikzcd}$$