Model(Aiyagari, 1994)

- Preference: $\mathbb{E}_0 \sum_{i=0}^{\infty} \beta^t u(C_t)$
- Budget constraint: $c_t + a_{t+1} = (1+r)a_t + w_t l_t$ where l_t follows the AR(1) process: $l_t = \rho l_{t-1} + \sigma \epsilon_t$ and $\epsilon_t \sim N(0,1)$.
- $a_t \geq -B$ and $c_t \geq 0$
- Bellman equation is $V(a,s) = \max_{a_t \geq -B, c_t \geq 0} u((1+r_t)a_t + w_t l_t - a_{t+1}) + \beta \mathbb{E}[V(a', s') \mid s]$.
- C-D production function: $Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$ which could drive the prices from FOCs $r = \alpha A_t \left(\frac{K}{L}\right)^{\alpha-1}$ and $w = (1-\alpha) A_t \left(\frac{L}{K}\right)^{\alpha}$.
- Market clearing conditions: $Y = \int c d\mu + K' - (1-\delta)K$ and $L = \int l d\mu$ and capital $K = \int k d\mu$.
- Calibration: set $\beta = 0.96, \gamma = 2, B = 0, \delta = 0.1, \alpha = 0.33$.

Tauchen method Here we use the Tauchen method to discretize the AR(1) process

$$\log l_t = \rho \log l_{t-1} + \sigma_t \varepsilon_t$$

here $\sigma_z = \frac{1}{\sqrt{1-\rho^2}}$. First, we set n, the number of realizations (usually 5 or 6). Second, we pick m and set $z_{\max} = \sigma_z m$ and $z_{\min} = -\sigma_z m$ where usual values of m 2 or 3. Hence, we could write down the discretization $z_i = z_{\min} + \frac{z_{\max} - z_{\min}}{n-1}(i-1)$ for $i = 1, \ldots, n$ for equal distance and construct the midpoint $\{\tilde{z}\}_{i=1}^{n-1}$, which are given by $\tilde{z}_i = \frac{z_{i+1} + z_i}{2}$. Lastly, we calculate the transition probabilities $\pi_{z,z'}$ by the following formula:

$$\begin{aligned}
\pi_{ij} &= \Phi\left(\frac{\tilde{z}_j - \rho z_i}{\sigma}\right) - \Phi\left(\frac{\tilde{z}_{j-1} - \rho z_i}{\sigma}\right) \qquad j = 2, 3, \cdots, n-1 \\
\pi_{i1} &= \Phi\left(\frac{\tilde{z}_1 - \rho z_i}{\sigma}\right) \\
\pi_{in} &= 1 - \Phi\left(\frac{\tilde{z}_{n-1} - \rho z_i}{\sigma}\right)
\end{aligned} \tag{1}$$

where $\Phi(\cdot)$ denotes a CDF of a $\mathcal{N}(0,1)$.

Endogenous Grid Method(Carroll, 2006) Start from at the end of period level of capital, using the Euler equation, one may recover the beginning-of-period consumption and level of capital without using a root-finding algorithm.

Cash-on-hand is $m_t = (1+r_t)a_t + w_t l_t$ and the Euler equation is $u'(c_t) = \beta(1 + r_{t+1})u'(c_{t+1})$.

1. Fix a grid over a;

2. Given current guess of policy function $c(M)$, (and given current grid M), directly compute $c'(M')$ for each point on the grid M by the following formula:

$$\begin{cases} c_j = (u')^{-1}\left(\beta R E_y u'(c((1+r)a_j + w_t l_t))\right) \\ M_j = (1+r)a_j + c_j \end{cases}$$

   Iterating on all points $a_j \in A$

3. The policy function c' is the interpolation of points, i.e. $c'(M') \equiv \{(M_i, c_i)\}_{i=1}^{n}$

Extension: endogenous labor supply The value function will be

$$V(a_t, l_t) = \max_{c_t, h_t, a_{t+1}} u(c_t) - v(h_t) + \beta E_t[V(a_{t+1}, l_{t+1})] \tag{2}$$

subject to

$$c_t + a_{t+1} = (1+r)a_t + h_t l_t w_t$$

$$a_{t+1} \geq -B, c_t \geq 0 \text{ and } h_t \geq 0$$

The **optimality conditions** for every period t are

$$\begin{aligned} c_t: \quad & 0 = u'(c_t) - \lambda_t \\ h_t: \quad & 0 = -v'(h_t) + \lambda_t l_t w_t \\ a_{t+1}: \quad & 0 = \beta \mathrm{E}_t\left[\frac{\partial V(a_{t+1}, y_{t+1})}{\partial a_{t+1}}\right] - \lambda_t + \mu_t, \quad \mu_t(a_{t+1}+B) = 0, \mu_t \geq 0 \\ \text{Envelope condition:} \quad & \frac{\partial V(a_t, l_t)}{\partial a_t} = \lambda_t(1+r_t) \\ \text{Budget constraint:} \quad & 0 = (1+r_t)a_t + h_t l_t w_t - c_t - a_{t+1} \end{aligned}$$

Eliminating the Lagrange multiplier λ, and the Envelope condition yields the set of equations

$$\begin{aligned} v'(h_t) &= u'(c_t) l_t w_t \\ u'(c_t) &= \beta \mathrm{E}_t\left[u'(c_{t+1})(1+r_t)\right] + \mu_t \\ \mu_t(a_{t+1}+B) &= 0, \mu_t \geq 0 \\ a_{t+1} &= (1+r_t)a_t + h_t l_t w_t - c_t \end{aligned}$$

that characterizes the optimality of the household's choice.

BKM(Boppart et al., 2018)

SSJ(Auclert et al., 2021)

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