**Model(Aiyagari, 1994)**

- Preference: $\mathbb{E}_0 \sum_{i=0}^{\infty} \beta^t u(C_t)$
- Budget constraint: $c_t + a_{t+1} = (1+r)a_t + w_t l_t$ where $l_t$ follows the AR(1) process: $l_t = \rho l_{t-1} + \sigma \epsilon_t$ and $\epsilon_t \sim N(0,1)$.
- $a_t \geq -B$ and $c_t \geq 0$
- Bellman equation is $v(a,s) = \max_{a_t \geq -B, c_t \geq 0} u((1+r_t)a_t + w_t l_t - a_{t+1}) + \beta \mathbb{E}[v(a', s') \mid s]$.
- C-D production function: $Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$ which could drive the prices from FOCs $r = \alpha A_t \left(\frac{K}{L}\right)^{\alpha-1}$ and $w = (1-\alpha) A_t \left(\frac{L}{K}\right)^{\alpha}$.
- Market clearing conditions: $Y = \int c d\mu + K' - (1-\delta)K$ and $L = \int l d\mu$ and capital $K = \int k d\mu$.
- Calibration: set $\beta = 0.96, \gamma = 2, B = 0, \delta = 0.1, \alpha = 0.33$.

**Tauchen method** Here we use the Tauchen method to discretize the AR(1) process

$$\log l_t = \rho \log l_{t-1} + \sigma_t \varepsilon_t$$

here $\sigma_z = \frac{1}{\sqrt{1-\rho^2}}$. First, we set $n$, the number of realizations (usually 5 or 6). Second, we pick $m$ and set $z_{\max} = \sigma_z m$ and $z_{\min} = -\sigma_z m$ where usual values of $m$ 2 or 3. Hence, we could write down the discretization $z_i = z_{\min} + \frac{z_{\max} - z_{\min}}{n-1}(i-1)$ for $i = 1, \ldots, n$ for equal distance and construct the midpoint $\{\tilde{z}\}_{i=1}^{n-1}$, which are given by $\tilde{z}_i = \frac{z_{i+1} + z_i}{2}$. Lastly, we calculate the transition probabilities $\pi_{z,z'}$ by the following formula:

$$\begin{aligned}
\pi_{ij} &= \Phi\left(\frac{\tilde{z}_j - \rho z_i}{\sigma}\right) - \Phi\left(\frac{\tilde{z}_{j-1} - \rho z_i}{\sigma}\right) \qquad j = 2, 3, \cdots, n-1 \\
\pi_{i1} &= \Phi\left(\frac{\tilde{z}_1 - \rho z_i}{\sigma}\right) \\
\pi_{in} &= 1 - \Phi\left(\frac{\tilde{z}_{n-1} - \rho z_i}{\sigma}\right)
\end{aligned} \tag{1}$$

where $\Phi(\cdot)$ denotes a CDF of a $\mathcal{N}(0,1)$.

**Endogenous Grid Method(Carroll, 2006)** Start from at the end of period level of capital, using the Euler equation, one may recover the beginning-of-period consumption and level of capital without using a root-finding algorithm.

Cash-on-hand is $m_t = (1+r_t)a_t + w_t l_t$ and the Euler equation is $u'(c_t) = \beta(1 + r_{t+1}) u'(c_{t+1})$.

Consider the Euler equation

$$u'(c(M)) = \beta R \mathbb{E}_y u'(c(RA + \tilde{y}))$$

if policy function $c(M)$ is optimal, then it satisfies the above equation with $A = M - c(M)$. given any policy function $c(M)$ an updated policy function $c'(M')$ with *parametrized curve*

$$\begin{cases} c' = (u')^{-1}\left(\beta R \mathbb{E}_y u'(c(RA+\tilde{y}))\right) \\ M' = A + c' \end{cases}$$

where $A$ is the parameter ranges from 0 to $M$.

Recall Coleman-Reffet operator $K(c)(M) : \mathcal{P} \to \mathcal{P}$ is defined as

- takes as input policy function $c(M) \in \mathcal{P}$
- returns the updated policy function $c'(M) \in \mathcal{P}$ that for every $M$ satisfies the Euler equation. $u'(c'(M)) = \beta R \mathbb{E}_y u'(c[R(M - c'(M)) + \tilde{y}])$

Standard implementation: fix grid over $M$; with given $c(M)$ solve the equation for $c$ in each point $M$ on the grid. EGM implementation of Coleman-Reffet operator:

1. Fix grid over $A$;
2. With given $c(M)$ for each point on the grid compute

$$c' = (u')^{-1}\left(\beta R \mathbb{E}_y u'(c(RA+\tilde{y}))\right) \quad M' = A + c'$$

3. Build the return policy function as interpolation over $(M', c)$

$$M \to c(M) \to A = M - c(M) \to M' = R(M - c(M)) + \tilde{y} = RA + \tilde{y}$$

A contains all the information about calculation of $M'$ and $c'$, and $M'$ contains all the information about the calculation of $c'$.

**BKM(Boppart et al., 2018)**

**SSJ(Auclert et al., 2021)**

## References


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