Overview This two papers Hopenhayn (1992) and Hopenhayn and Rogerson (1993) are important papers in firm dynamics.

- DRS production function.
- Perfect competition in product and labor market
- No aggregate shocks
- Idiosyncratic productivity shocks
- Entry-exit dynamics: Fixed cost to *entry* and fixed cost to *operate* firm each period.
- Introduce adjustment costs $\Rightarrow$ induce misallocation of resource across heterogeneous producers.

Model Preferences:

$$\max_{C_t, N_t} \sum_{t=0}^{\infty} \beta^t \ln C_t + AN_t \quad \text{s.t.} \quad p_t C_t = w_t N_t + \Pi_t + T_t$$

Note that if we plug C_t into objective function, we will transfer the problem to a static problem. Production function:

$$p_t z f(n_t) - n_t - p_t c_f - g(n_t, n_{t-1}), \quad z \sim F(z) \tag{1}$$

where p_t is the price of output, $f(n_t)$ is the production function, n_t is the number of workers, c_f is the fixed cost to operate the firm, and $g(n_t, n_{t-1}) = \tau[n_{t-1} - n_t]^+$ is the adjustment cost. Note that n_{t-1} is the state variable.

The timing of decision: exit: receive $-g(0, n_{t-1})$ this period zero in all future periods. Entry: make employment decision n_t and receive $p_t z f(n_t) - n_t - g(n_t, n_{t-1}) - p_t c_f$.

Value function of incumbent firm:

$$\begin{aligned} V(z, n) = \max_{n' \geq 0} \Big\{ & pzf(n') - n' - g(n', n) - pc_f \\ & + \beta \max[-g(0, n'), \int V(z', n') dF(z' \mid z)] \Big\} \end{aligned} \tag{2}$$

and the policy functions are $n' = n^d(z, n; p)$ and $\chi(z, n; p) \in \{0, 1\}$.

Entrants and Free entry condition Potential entrants are ex-ante identical. An entrants firm must pay the entry cost $c_e > 0$ to set-up the plant and draw $z \sim G(z)$ starting producing next period with $n_{t-1} = 0$. $M \geq 0$ mass of entrants. *Free entry condition*:

$$\beta \int V(z, 0; p) \mathrm{d}G(z) \leq c_e$$

with strict equality if $M > 0$ entrants.

Equilibrium Consider the stationary equilibrium: $\mu(z,n)$ denotes the distribution of firms across the state.

$$\mu_{t+1}(z',n') = \underbrace{\int F(z' \mid z)(1-\chi(z,n))\mathbf{1}_{[n'=n^d(z,n)]}\mathrm{d}\mu_t(z,n)}_{\text{Incumbents don't exit}} + \underbrace{M_{t+1}G(z')\mathbf{1}_{[n'=0]}}_{\text{Entrants}} \tag{3}$$

In stationary distribution, we have $\mu_{t+1} = \mu_t = \mu$. Again, linearity implies that $\mu(p,M) = M \times \mu(p,1)$.

Aggregation Aggregate production and labor demand:

$$Y(p,M) = \int (zf(n^d(z,n;p)) - c_f)\mathrm{d}\mu, \quad \text{and} \quad N^d(p,M) = \int n^d(z,n;p)\mathrm{d}\mu + Mc_e$$

Expected firing tax revenue for a firm with state (z,n) is

$$r(z,n;p) = [1-\chi(z,n;p)]E_{z'|z}[g(n^d(z',n'(z,n)),n^d(z,n))] + \chi(z,n)g(0,n') \tag{4}$$

and aggregate tax revenue $T(p,M) = \int r(z,n;p)\mathrm{d}\mu$.

Aggregate profits:

$$\Pi(p,M) = pY - N^d - T = \int \pi(z,n;p)d\mu - Mc_e \tag{5}$$

Solving Equilibrium Guess price p^*, and solve for the DPP of the incumbents. Check if p^* satisfies the free entry $\beta \int V(z,0;p)\mathrm{d}G(z) = c_e$. If not, update p^* and repeat.

Given p^*, and the policy functions, assume $M = 1$ and solve for the stationary distribution $\mu(p^*,1)$. Using either the goods market or labor market clearing conditions[1] to solve for M:

$$Y(p^*,M) = M\int (zf(n^d(z,n;p)) - c_f)d\mu(p^*,1) = C(p^*)$$

Then, calculate the aggregate variables.

Implications The adjustment cost implies that adjustment is lumpy. If the adjustment cost is quadratic, firms will adjust slowly (no inaction region). The linear adjustment cost induces the inaction region.

Adjustment costs are less important if shocks are very persistent.

The tax also prevents inefficient firms from exiting.

Note that misallocation here is induced by an aggregate friction. In more sophisticated models, the misallocation can be firm-specific.

[1]Actually, using the market clearing conditions is to connect firms and household's problems

Continuous-time version

The problem of incumbent firm is

$$\begin{aligned}
V(z) = & \max_{\{n_t\}_{t\geq 0},\tau} \left\{ \mathbb{E}_0 \int_0^\tau e^{-\rho t}(pf(z,n) - wn_t - c_f)\mathrm{d}t + e^{-\rho\tau}V^* \right\} \\
& \mathrm{d}z_t = \mu_z \mathrm{d}t + \sigma_z \mathrm{d}W_t, \quad z_0 = z.
\end{aligned} \tag{6}$$

We can rewrite the problem using HJBVI equation and KFE.

$$\min\left\{\rho V(z) - V'(z)\mu(z) - \frac{1}{2}V''(z)\sigma^2(z) - \pi(z), V(z) - V^*\right\} = 0, \quad \text{all } z \in (0,1) \tag{7}$$

$$\partial_t g(z,t) = \partial_z(\mu(z)g(z,t)) + \frac{1}{2}\partial_{zz}(\sigma^2(z)g(z,t)) + m(t)\psi(z), \quad \text{all } z \in \mathcal{Z} \tag{8}$$

where $\mathcal{Z} = [x,1]$ denotes the set of productivities such that firms remain in the industry. $m(t)$ denotes the exit rate (assume a constant mass of firms) which pinned down by the requirement that the total mass of firms is constant $\int_0^1 g(z,t)\mathrm{d}z = 1$.

If there is a simple threshold rule for exit, i.e. exit whenever z falls below x, then

$$m(t) = \frac{1}{2}\partial_z(\sigma^2(x)g(x,t)) \tag{9}$$

Using the fact that $\int_x^1 g(z,t)\mathrm{d}z = 1 \Rightarrow \int_x^1 \partial_t g(z,t) = 0$. to get that

$$\begin{aligned}
0 &= -\int_x^1 \partial_z(\mu(z)g(z,t))\mathrm{d}z + \frac{1}{2}\int_x^1 \partial_{zz}(\sigma^2(z)g(z,t))\mathrm{d}z + m(t)\int_x^1 \psi(z)\mathrm{d}z \\
&= \mu(x)g(x,t) - \frac{1}{2}\partial_z(\sigma^2(x)g(x,t))
\end{aligned} \tag{10}$$

Further, using the diffusion process $\mathcal{A}V = \mu(z)\partial_z V + \frac{1}{2}\partial_{zz}V$ and $\mathcal{A}^*$ its adjoint $\mathcal{A}^*g = -\partial_z(\mu(z)g) + \frac{1}{2}\partial_{zz}(\sigma^2_{zz}g)$ so that the KFE is $\partial_t g = \mathcal{A}^*g + m(t)\psi(z)$. We then have

$$m(t) = \int_{\mathcal{Z}} (\mathcal{A}^*g)(z,t)\mathrm{d}z \tag{11}$$

We can characterize the equilibrium by the following system of equations:

$$\begin{aligned}
& 0 = \min\left\{\rho V(z) - V'(z)\mu(z) - \frac{1}{2}V''(z)\sigma^2(z) - \pi(z), V(z) - V^*\right\}, \quad \text{all } z \in (0,1) \\
& 0 = -(\mu(z)g(z))' + \frac{1}{2}\left(\sigma^2(z)g(z)\right)'' + m\psi(z), \quad \text{all } z \in \mathcal{Z} \\
& m = -\int_{\mathcal{Z}} \left(\mathcal{A}^*g\right)(z)dz \\
& p = D(Q), \quad w = W(N), \quad Q = \int_{\mathcal{Z}} q(z)g(z)dz, \quad N = \int_{\mathcal{Z}} n(z)g(z)\mathrm{d}z
\end{aligned} \tag{12}$$

References

Hopenhayn, Hugo A, "Entry, exit, and firm dynamics in long run equilibrium," *Econometrica: Journal of the Econometric Society*, 1992, pp. 1127–1150.

Hopenhayn, Hugo and Richard Rogerson, "Job turnover and policy evaluation: A general equilibrium analysis," *Journal of political Economy*, 1993, *101* (5), 915–938.