**Subspace of R4 can be generated by 3 vectors. BUT 3 vectors cannot form a basis in the vector space R4.*
**Col. Space is set of all possible outputs for Ax = v*
**Nullspace is the space that all vectors become null (the zero vector) Ax = 0*

Gaussian Elimination:

1. Locate the leftmost column that does not consist of entirely of zeros.
2. Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.
3. For each row below the top row, add a suitable multiple of the top row to it so that the entry below the leading entry of the top row becomes zero.
4. Now cover the top row in the matrix and begin again with Step 1 applied to the submatrix that remains. Continue in this way until REF.

Gauss-Jordan Elimination:

5. Multiply a suitable constant to each row so that all leading entries become 1.
6. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading entries.

Check linear system consistency:

1. Inconsistent → last column of REF is a pivot column
2. Consistent with 1 soln → Every column except last column in REF is a pivot column
3. Consistent with infinite solns → At least 1 non-pivot column

Find inverse of matrix A:

1. Form [A l I]. Do ERO until [I l B] is achieved. B is A^{-1}.
Square matrix is invertible ⇔ RREF is Identity.

2. $A^{-1} = \frac{1}{\det(A)} adj(A),\ adj(A) = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}^T =$
$\begin{pmatrix} A_{11} & \cdots & A_{n1} \\ \vdots & \ddots & \vdots \\ A_{1n} & \cdots & A_{nn} \end{pmatrix}, where\ A_{ij}\ is\ the\ (i,j)-cofactor\ of\ A.$

Find a set of vectors that spans a given subspace:

Eg. Show $span\{(1,0,1),(1,1,0),(0,1,1)\} = \mathbb{R}^3$.
a(1,0,1) + b(1,1,0) + c(0,1,1) = (x,y,z)
a + b = x
b + c = y
a + c = z
Make a matrix using the vectors as columns. Solve to see if matrix is consistent. If no zero rows, then linear system is always consistent. Because if there is a zero row, the last entry on the last column can be a random number such that the last column becomes a pivot column, making the system inconsistent.

If the subspace is not $\mathbb{R}^n$(eg. V), need to check $span(S) \subseteq V \wedge span(S) \supseteq V$.

Determine whether given vectors are linearly independent:

1. Vectors are linearly independent iff the homogeneous linear system only has the trivial soln.
2. If one vector can be written as a linear combination of other vectors, the set is linearly dependent.

Check that a given set is a subspace (Let V = span(S) where V is a subspace and S is a finite of vectors in $\mathbb{R}^n$):

1. Check that the zero vector is in span(S).
2. Linear combinations of vectors in the subspace should still exist in the subspace.

The solution set of a homogeneous system of linear equations in n variables is a subspace of R^n.
V1 ={(x,y,z)lx+2y+3z=0andx+y−z=0}.
V2 ={(x,y,z)lx+2y+3z=0}.
V1 and V2 are homogeneous linear systems of two and one linear equations respectively, so they are subspaces of R^n.

Find a basis for and determine the dimension of a given subspace V:

1. Check if S is linearly independent.
2. Check if S spans V.

Find the transition matrix (must be square) from one basis to another:

1. Let S = {u1, u2, …, uk} and T = {v1, v2, …, vk} be 2 bases for a vector space V.
2. For any vector w in V, suppose w = c1u1 + c2u2 + … + ckuk, so, [w]s = (c1 c2 … ck)T.
3. Since T is a basis for V, we can write each ui as a linear combination of v1, v2, …, vk.
4. $Let\ \boldsymbol{P} = ([u_1]_T\ [u_2]_{T\dots}\ [u_k]_T). Then\ [w]_T = \boldsymbol{P}[w]_s\ for\ all\ \boldsymbol{w} \in V.$
5. **P** is called the transition matrix from S to T. $\boldsymbol{P}^{-1}$ is called the transition matrix from T to S.

Gram-Schmidt process:

Theorem 5.2.19 (Gram-Schmidt Process) Let $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k}\}$ be a basis for a vector space V. Let

$$\begin{aligned} \boldsymbol{v_1} &= \boldsymbol{u_1}, \\ \boldsymbol{v_2} &= \boldsymbol{u_2} - \frac{\boldsymbol{u_2}\cdot\boldsymbol{v_1}}{\boldsymbol{v_1}\cdot\boldsymbol{v_1}}\boldsymbol{v_1}, \\ \boldsymbol{v_3} &= \boldsymbol{u_3} - \frac{\boldsymbol{u_3}\cdot\boldsymbol{v_1}}{\boldsymbol{v_1}\cdot\boldsymbol{v_1}}\boldsymbol{v_1} - \frac{\boldsymbol{u_3}\cdot\boldsymbol{v_2}}{\boldsymbol{v_2}\cdot\boldsymbol{v_2}}\boldsymbol{v_2}, \\ &\vdots \\ \boldsymbol{v_k} &= \boldsymbol{u_k} - \frac{\boldsymbol{u_k}\cdot\boldsymbol{v_1}}{\boldsymbol{v_1}\cdot\boldsymbol{v_1}}\boldsymbol{v_1} - \frac{\boldsymbol{u_k}\cdot\boldsymbol{v_2}}{\boldsymbol{v_2}\cdot\boldsymbol{v_2}}\boldsymbol{v_2} - \cdots - \frac{\boldsymbol{u_k}\cdot\boldsymbol{v_{k-1}}}{\boldsymbol{v_{k-1}}\cdot\boldsymbol{v_{k-1}}}\boldsymbol{v_{k-1}}. \end{aligned}$$

Then $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_k}\}$ is an orthogonal basis for V. Furthermore, let

$$\boldsymbol{w_1} = \frac{1}{\|\boldsymbol{v_1}\|}\boldsymbol{v_1},\quad \boldsymbol{w_2} = \frac{1}{\|\boldsymbol{v_2}\|}\boldsymbol{v_2},\quad \dots,\quad \boldsymbol{w_k} = \frac{1}{\|\boldsymbol{v_k}\|}\boldsymbol{v_k}.$$

Then $\{\boldsymbol{w_1}, \boldsymbol{w_2}, \dots, \boldsymbol{w_k}\}$ is an orthonormal basis for V.

Find a least square solution to a linear system:

$Let\ \boldsymbol{Ax} = \boldsymbol{b}\ be\ a\ linear\ system. Then\ \boldsymbol{u}\ is\ a\ least\ squares\ solution\ to\ \boldsymbol{Ax} = \boldsymbol{b}\ iff\ \boldsymbol{u}\ is\ a\ solution\ to\ \boldsymbol{A^T A x} = \boldsymbol{A^T b}.$

Find the projection of a vector on to a subspace:

1. $Let\ V\ be\ a\ subspace\ of\ R^n. Every\ vector\ \boldsymbol{u} \in R^n\ can\ be\ written\ uniquely\ as$

$$\boldsymbol{u} = \boldsymbol{n} + \boldsymbol{p}$$

$such\ that\ \boldsymbol{n}\ is\ a\ vector\ orthogonal\ to\ V\ and\ \boldsymbol{p}\ is\ a\ vector\ in\ V.$
$This\ vector\ \boldsymbol{p}\ is\ called\ the\ orthogonal\ project\ of\ \boldsymbol{u}\ onto\ V.$

Theorem 5.2.15 Let V be a subspace of $\mathbb{R}^n$ and $\boldsymbol{w}$ a vector in $\mathbb{R}^n$.

1. If $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k}\}$ is an orthogonal basis for V, then
$$\frac{\boldsymbol{w}\cdot\boldsymbol{u_1}}{\boldsymbol{u_1}\cdot\boldsymbol{u_1}}\boldsymbol{u_1} + \frac{\boldsymbol{w}\cdot\boldsymbol{u_2}}{\boldsymbol{u_2}\cdot\boldsymbol{u_2}}\boldsymbol{u_2} + \cdots + \frac{\boldsymbol{w}\cdot\boldsymbol{u_k}}{\boldsymbol{u_k}\cdot\boldsymbol{u_k}}\boldsymbol{u_k},$$
is the projection of $\boldsymbol{w}$ onto V.
2. If $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_k}\}$ is an orthonormal basis for V, then
$$(\boldsymbol{w}\cdot\boldsymbol{v_1})\boldsymbol{v_1} + (\boldsymbol{w}\cdot\boldsymbol{v_2})\boldsymbol{v_2} + \cdots + (\boldsymbol{w}\cdot\boldsymbol{v_k})\boldsymbol{v_k}$$

2. is the projection of $\boldsymbol{w}$ onto V.

Diagonalize a square matrix:

Definition 6.2.1: A square matrix **A** is called *diagonalizable* if there exists and invertible matrix **P** such that $\boldsymbol{P^{-1}AP}$ is a diagonal matrix. Hence the matrix **P** is said to *diagonalize* **A**.

Algorithm 6.2.4 Given a square matrix $\boldsymbol{A}$ of order n, we want to determine whether $\boldsymbol{A}$ is diagonalizable. Also, if $\boldsymbol{A}$ is diagonalizable, find an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{AP}$ is a diagonal matrix.

Step 1: Find all distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$. (By Remark 6.1.5, eigenvalues can be obtained by solving the characteristic equation $\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$.)

Step 2: For each eigenvalue λ_i, find a basis S_{λ_i} for the eigenspace E_{λ_i}.

Step 3: Let $S = S_{\lambda_1} \cup S_{\lambda_2} \cup \cdots \cup S_{\lambda_k}$.

(a) If $|S| < n$, then $\boldsymbol{A}$ is not diagonalizable.
(b) If $|S| = n$, say $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_n}\}$, then $\boldsymbol{P} = (\boldsymbol{u_1}\ \ \boldsymbol{u_2}\ \ \cdots\ \ \boldsymbol{u_n})$ is an invertible matrix that diagonalizes $\boldsymbol{A}$.

Orthogonally diagonalize a symmetric matrix:

A square matrix **A** is called *orthogonally diagonalizable* if there exists an orthogonal matrix **P** such that $\boldsymbol{P^T AP}$ is a diagonal matrix. Here the matrix **P** is said to orthogonally diagonalize A.

Theorem 6.3.4: A square matrix is orthogonally diagonalizable iff it is symmetric.

Follow the "**Diagonalize a square matrix**" steps but after finding each eigenspace. Apply the Gram-Schmidt Process to transform each basis to an orthonormal basis.

Use the standard matrix of a linear transformation:

Think of a linear transformation like a function. Taking an input and returning an output. Think of it as left-multiplying a matrix by the input.

Discussion 7.1.8 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with the standard matrix $\boldsymbol{A}$. Let $\{\boldsymbol{e_1}, \boldsymbol{e_2}, \dots, \boldsymbol{e_n}\}$ be the standard basis for $\mathbb{R}^n$. By Discussion 7.1.6, we know that the images $T(\boldsymbol{e_1}), T(\boldsymbol{e_2}), \dots, T(\boldsymbol{e_n})$ completely define T. Furthermore, since for each $\boldsymbol{e_i}$,

$$T(\boldsymbol{e_i}) = \boldsymbol{Ae_i} = \text{the } i\text{th column of } \boldsymbol{A},$$

we have $\boldsymbol{A} = (T(\boldsymbol{e_1})\ \ T(\boldsymbol{e_2})\ \ \cdots\ \ T(\boldsymbol{e_n}))$.

$$T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right) = \begin{pmatrix}1\\3\end{pmatrix},\quad T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right) = \begin{pmatrix}-1\\2\end{pmatrix},\quad T\left(\begin{pmatrix}2\\0\\-1\end{pmatrix}\right) = \begin{pmatrix}4\\-1\end{pmatrix}.$$

By
$$\left(\begin{array}{ccc|c|c|c} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & -1 & 3 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right),$$

$$\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}0\\1\\1\end{pmatrix},\quad \begin{pmatrix}0\\1\\0\end{pmatrix} = -2\begin{pmatrix}1\\1\\1\end{pmatrix} + 3\begin{pmatrix}0\\1\\1\end{pmatrix} + \begin{pmatrix}2\\0\\-1\end{pmatrix},\quad \begin{pmatrix}0\\0\\1\end{pmatrix} = 2\begin{pmatrix}1\\1\\1\end{pmatrix} - 2\begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}2\\0\\-1\end{pmatrix}.$$

Then we have $T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) = T\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right) - T\left(\begin{pmatrix}0\\1\\1\end{pmatrix}\right) = \begin{pmatrix}1\\3\end{pmatrix} - \begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$ and similarly,

$T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) = \begin{pmatrix}-1\\-1\end{pmatrix},\quad T\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \begin{pmatrix}0\\3\end{pmatrix}$. So the standard matrix for T is

$$\left(T\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right)\ \ T\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right)\ \ T\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)\right) = \begin{pmatrix}2 & -1 & 0\\1 & -1 & 3\end{pmatrix}.$$

Theorem 3.5.11: Let S be a basis for a vector space V where |S| = k. Let $v_1, v_2, \ldots, v_r$ be vectors in V. Then
1. $v_1, v_2, \ldots, v_r$ are linearly dependent vectors in V iff $(v_1)_s, (v_2)_s, \ldots, (v_r)_s$ are linearly dependent vectors in $\mathbb{R}^k$.
2. $span\{v_1, v_2, \ldots, v_r\} = V \; iff \; span\{(v_1)_s, (v_2)_s, \ldots, (v_r)_s\} = \mathbb{R}^k$.

Theorem 3.6.7: Let V be a vector space of dimension k and S a subset of V. The following are equivalent:
1. S is a basis for V.
2. S is linearly independent and |S| = k.
3. S spans V and |S| = k.

Definition 4.1.2: Row space & Column space of a *m x n* matrix A
Row space of A is the subspace of $\mathbb{R}^n$ spanned by the rows of A. Column space of A is the subspace of $\mathbb{R}^m$ spanned by the columns of A.

ERO preserve row space but not column space. Nonzero rows in REF(A) is a basis for the row space of A. Corresponding pivot columns in REF(A) is a basis for the column space of A.
To extend a set of vectors to a basis for larger subspace.
1. Obtain REF(A).
2. Identify the non-pivot columns in REF(A).
3. For each non-pivot column, get a vector such that the leading entry of the vector is at that column. Eg. If 3rd column is a non-pivot, add (0,0,1,…). Repeat for each non-pivot column.
If asked to extend to an orthogonal/orthonormal basis, make use of projections.

Theorem 4.2.1: Row space and column space of a matrix have the same dimension.
Rank(A) is the dimension of its row space, which means that rank(A) = number of nonzero rows = number of pivot columns in REF(A)

$$rank(\boldsymbol{A}) = rank(\boldsymbol{A^T})$$

Theorem 4.2.8: Let A and B be *m x n* and *n x p* matrices.
Then $rank(\boldsymbol{AB}) \leq \min\{rank(\boldsymbol{A}), rank(\boldsymbol{B})\}$

Definition 4.3.2: Let A be an *m x n* matrix. The solution space of the homogeneous system of linear equations Ax = 0 is known as the nullspace of A. The dimension of the nullspace of A is called the nullity of A. nullity(A) ≤ n since the nullspace is a subspace of $\mathbb{R}^n$.

Theorem 4.3.4: Dimension Theorem for Matrices
Let **A** be a matrix with n columns. Then rank(**A**) + nullity(**A**) = n.
Rank is number of pivot columns while nullity is number of non-pivot columns.

Theorem 4.3.6: System of linear equations revisited
Suppose the system of linear equations **Ax = b** has a solution **v**. Then the solution set of the system is given by
M = { **u** + **v** | **u** is an element of the nullspace of **A** }
That is, **Ax = b** has a general solution
x = (a general solution for **Ax = 0**) + (one particular solution to **Ax = b**)
→ a consistent linear system **Ax = b** has only 1 solution iff nullspace of **A** = **{0}**

Theorem 5.2.4: (Orthogonality is stricter than linear independence)
Let S be an orthogonal set of nonzero vectors in a vector space. Then S is linearly independent.

To determine whether a set S of nonzero vectors in a vector space of dimension k is an orthogonal basis, we only need to check (i) S is orthogonal and (ii) |S| = k.

Definition 5.2.13: $\boldsymbol{Let\ V\ be\ a\ subspace\ of\ R^n. Every\ vector\ u \in R^n\ can\ be\ written\ uniquely\ as\ u = n + p, such\ that\ n\ is\ a\ vector\ orthogonal\ to\ V\ and\ p\ is\ a\ vector\ in\ V.}$

The vector p is called the (orthogonal) project of u onto V.

Theorem 5.3.2: Let V be a subspace in $\mathbb{R}^n$. If w is a vector in $\mathbb{R}^n$ and p is the projection of u onto V, then d(u,p) < d(u,v) for all v∈V, i.e. p is the best approximation of w in V.

This means that the projection can be given by the matrix multiplied by the least squares solution. So, p = Ax (where x is the least squares solution)

Theorem 5.3.8:
Let **Ax = b** be a linear system, where **A** is an *m x n* matrix, and let **p** be the projection of **b** onto the column space of **A**. Then
||**b – p**|| ≤ ||**b – Av**|| for all **v** in $\mathbb{R}^n$.
.i.e **u** *is a least squares solution to* **Ax = b** iff **Au = p**.

Definition 5.4.3: A square matrix A is called *orthogonal* if $A^{-1} = A^T$. (If matrix is orthogonal, matrix is square. (Converse untrue))

Theorem 5.4.6: Let A be a square matrix of order n. Then,
1. A is orthogonal.
2. Rows of **A** form an orthonormal basis for $\mathbb{R}^n$.
3. Columns of **A** form an orthonormal basis for $\mathbb{R}^n$.

Definition 6.1.3: Let A be a square matrix of order n. A nonzero column vector u in $\mathbb{R}^n$ is called an eigenvector of A if Au = λu for some scalar λ. Λ is called an eigenvalue of A and u is said to be an eigenvector of A associated with the eigenvalue λ.

Definition 6.1.6: Let A be a square matrix of order n.
det(λ**I – A**) = 0 is called the characteristic eqn of **A** and
det(λ**I – A**) is called the characteristic polynomial of **A**

Theorem 6.1.8: The Main Theorem on Invertible Matrices
Let A be a n x n matrix.
1. A is invertible. (Note: If invertible, matrix **MUST** be square)
2. The linear system Ax = 0 has only the trivial solution.
3. RREF(A) is an identity matrix.
4. A can be expressed as a product of elementary matrices.
5. det(A) =/= 0.
6. The rows of A form a basis for $\mathbb{R}^n$.
7. The columns of A form a basis for $\mathbb{R}^n$.
8. rank(A) = n.
9. 0 is not an eigenvalue of A.

R(T) = column space of A, the standard matrix for T. Dimension of R(T) is called rank(T) = rank(A).
Ker(T) = nullspace of A. Dimension of Ker(T) is called nullity(T) = nullity(A).
Eigenspace can be seen as the span of the corresponding eigenvectors.

Theorem 6.1.9: If A is a triangular matrix, the eigenvalues of A are the diagonal entries of A.

Definition 6.1.11: Let A be a square matrix of order n and λ an eigenvalue of A.
The solution space of (**λI – A**)**x** = **0** is called the eigenspace of A associated with the eigenvalue λ.

Theorem 6.2.3: Let A be a square matrix of order n.
Then **A** is diagonalizable iff **A** has n linearly independent eigenvectors.

Theorem 6.2.7: Let A be a square matrix of order n.
If **A** has n distinct eigenvalues, then **A** is diagonalizable. (Converse is untrue)
Theorem 7.1.4: Let $T: R^n \to R^m$ be a linear transformation.
1. T(**0**) = **0**
2. T(c**u** + d**v**) = cT(**u**) + dT(**v**)

Theorem 7.1.11:
$\boldsymbol{If\ A\ and\ B\ are\ standard\ matrices\ for\ linear\ transformations\ S\ and\ T\ respectively, then\ the\ standard\ matrix\ for\ the\ composition\ T^\circ S\ is\ BA.}$

Definition 7.2.1: $\boldsymbol{Let\ T: R^n \to R^m\ be\ a\ linear\ transformation.}$
R(T) = Range(T) = $\{T(u) | u \in R^n\} \subseteq R^m$

Theorem 7.2.4: R(T) = the column space of standard matrix A

Definition 7.2.7: $\boldsymbol{Let\ T: R^n \to R^m\ be\ a\ linear\ transformation.}$
Kernel is set of vectors in R^n whose image is the zero vector in R^m.
Ker(T) = $\{u \mid T(u) = 0\} \subseteq R^n$

Theorem 7.2.9: Ker(T) = nullspace of standard matrix A

Theorem 7.2.12: If $T: R^n \to R^m$, rank(T) + nullity(n) = n

How does ERO change determinant (From Matrix A to Matrix B)	
cR_i	**det(B) = c * det(A)**
$R_i \leftrightarrow R_j$	**det(B) = -det(A)**
$R_i + cR_j$	**det(B) = det(A)**

<u>Some useful laws</u>
1. A(BC) = (AB)C
2. A(B + B') = AB + AB'
3. a(AB) = (aA)B = A(aB)
4. A0 = 0 and 0A = 0
5. det(AB) = det(A)det(B)
6. det(aA) = $\boldsymbol{a^n det(A), n\ is\ the\ order\ of\ A}$
7. $\boldsymbol{det(A^{-1}) = \frac{1}{det(A)}}$
8. $\boldsymbol{(A+B)^T = A^T + B^T}$
9. $\boldsymbol{(AB)^T = B^T A^T}$
10. $\boldsymbol{det(A^T) = det(A)}$
11. $\boldsymbol{(aA)^{-1} = \left(\frac{1}{a}\right) A^{-1}}$
12. $\boldsymbol{(AB)^{-1} = B^{-1} A^{-1}}$
13. $\boldsymbol{(A^T)^{-1} = (A^{-1})^T}$
14.