MATH 502 Notes

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References:

- Atiyah and MacDonald, *Commutative Algebra*.
- J.P. Serre, *Local Algebra*.
- Zariski and Samuel, *Commutative Algebra* Volume 1 and 2.
- Matsumura, *Commutative Algebra*.
- Bourbaki, *Commutative Algebra*.

We always assume a ring R has a multiplicative identity and is commutative.

0 Noetherian, Artinian, and Localization

Proposition 0.1. Let R be a (commutative) ring, and let M be an A-module, then the following are equivalent:

(i) Given an infinite increasing chain of submodules of M

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq M_{n+1} \subseteq \cdots$$

then there exists some $N \in \mathbb{N}$ such that $M_N = M_{N+1} = \cdots$, i.e., for all $n \geqslant N$, $M_n = M_{n+1}$.

(ii) Every non-empty family of submodules has a maximal element.

(iii) Every submodule of M is finitely-generated.

Proof. $(i) \Rightarrow (ii)$: This is a direct result of Zorn's lemma.

$(ii) \Rightarrow (i)$: Obvious.

$(i), (ii) \Rightarrow (iii)$: Take any submodule N of M and take $x_1 \in N$. If $(x_1) \neq N$, then there exists $x_2 \in N \backslash (x_1)$, so $(x_1, x_2) \subseteq N$, now we proceed inductively, but by the given property we know this stops in finite number of steps, hence we have $N = (x_1, \ldots, x_n)$ for some $n \in \mathbb{N}$, thus N is finitely-generated.

$(iii) \Rightarrow (i)$: Note that the property implies M is finitely-generated, but that means the chain of submodules must be finite. $\square$

Definition 0.2 (Noetherian Module)**.** If any of the conditions in Proposition 0.1 holds, then M is said to be a Noetherian module. Alternatively, we say M satisfies the ascending chain condition.

Proposition 0.3. Let R be a (commutative) ring, and let M be an A-module, then the following are equivalent:

(i) Given an infinite decreasing chain of submodules of M

$$M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq M_{n+1} \supseteq \cdots$$

then there exists some $N \in \mathbb{N}$ such that $M_N = M_{N+1} = \cdots$, i.e., for all $n \geqslant N$, $M_n = M_{n+1}$.

(ii) Every non-empty family of submodules has a minimal element.

Proof. Again, Zorn's lemma. ☐

Definition 0.4 (Artinian Module)**.** If any of the conditions in Proposition 0.3 holds, then M is said to be a Artinian module. Alternatively, we say M satisfies the descending chain condition.

Example 0.5.

- $\mathbb{Z}$ is Noetherian.
- $\mathbb{Q}/\mathbb{Z}$ is not Noetherian.
- Let p be a prime. Let $\mathbb{Z}(p^\infty)$ be the union of chains (as direct limits)

$$\left\langle \frac{\bar{1}}{p} \right\rangle \subseteq \left\langle \frac{\bar{1}}{p^2} \right\rangle \subseteq \cdots \subseteq \left\langle \frac{\bar{1}}{p^n} \right\rangle \subseteq \cdots$$

then there is an embedding $\mathbb{Z}(p^\infty) \subseteq \mathbb{Q}/\mathbb{Z}$, where $\bar{a}$ is the image of a in $\mathbb{Q}/\mathbb{Z}$. With this construction, $\mathbb{Z}(p^\infty)$ is Artinian.

Exercise 0.6. Show that $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}(p^\infty)$ where p traverses through all the primes.

Proposition 0.7. Let N be a submodule of M. Suppose M satisfies ascending (respectively, descending) chain condition, then N and M/N also satisfy ascending (respectively, descending) chain condition. If, for some submodule N of M, we know N and M/N satisfy ascending (respectively, descending) chain condition, then M also satisfies ascending (respectively, descending) chain condition.

Proof. Suppose M satisfies ascending (respectively, descending) chain condition, and let N be a submodule of M. Let $\{N_i\}$ be an increasing (respectively, decreasing) sequence of submodules of N, then they can be regarded as submodules of M, therefore by the Noetherian (respectively, Artinian) condition, we know N satisfies ascending (respectively, descending) chain condition. Now let $\bar{M} = M/N$, and take $\{\bar{M}_i\}$ be an increasing (respectively, decreasing) sequence of submodules of $\bar{M}$. Let $\pi : M \to M/N$ be the quotient map, then the preimages give an increasing (respectively, decreasing) sequence $\{M_i\}$ of submodules of M, where $M_i = \pi^{-1}(\bar{M}_i)$, but by the Notherian (respectively, Artinian) condition, we know the sequence stops in finite steps, therefore the original sequence stops in finite steps as well, hence $\bar{M}$ satisfies the ascending (respectively, descending) chain condition.

Suppose a submodule N of M is such that N and M/N both satisfy ascending chain condition. Take a submodule T of M, then we have a short exact sequence

$$0 \longrightarrow T \cap N \hookrightarrow T \longrightarrow T/(T \cap N) \longrightarrow 0$$

Now $T \cap N$ is finitely-generated as N is finitely-generated, therefore we have an embedding $T/T \cap N \hookrightarrow M/N$, thus $T/T \cap N$ is finitely-generated, therefore T is also finitely-generated by a vector space argument.

Suppose we have a decreasing sequence $\{M_n\}$ of M, then we have a decreasing sequence $\{N \cap M_n\}$. Let $\bar{M} = M/N$, then $\bar{M}_n := (M_n + N)/N$ defines a decreasing sequence of submodules in $\bar{M}$, but N satisfies the descending chain condition, so the sequence $\{N \cap M_n\}$ stops in finite number of steps, say n_0. Moreover, the sequence of $\bar{M}_n$'s also stops in finite number of steps, so by definition the sequence of $(M_n + N)/N$ stops in finite number of steps, say m_0, but by the isomorphism theorem this shows that the sequence of $M_n/(N \cap M_n)$ stops in m_0 steps. Therefore, whenever $n \geqslant m_0, n_0$, then $N \cap M_n = N \cap M_{n+1}$, hence $M_n = M_{n+1} = \cdots$ for such n. ☐

Remark 0.8. The final argument should also work in the Noetherian case.

Definition 0.9 (Simple Module)**.** An A-module M is simple if the submodules of M are either 0 or M.

Exercise 0.10. Let A be a commutative ring, and M is an A-module, then M is simple if and only if $M \cong A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of A.

Definition 0.11 (Jordan-Hölder Chain)**.** Let A be a commutative ring and M be an A-module. We say M has a Jordan-Hölder chain if there exists a decreasing chain of submodules $\{M_i\}$ such that

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_{n-1} \supsetneq M_n = 0$$

such that M_i/M_{i+1} is simple. In such a situation, we know n is the length of the Jordan-Hölder chain, and such n is unique. We say M is a module of finite length, and the length is $\ell_A(M) = n$.

Exercise 0.12. Let A be a commutative ring, and let M be an A-module, then M is of finite length if and only if M is both Noetherian and Artinian.

Theorem 0.13. Let A be a commutative ring, then A is Artinian if and only if A is Noetherian and every prime ideal of A is maximal.

Proof. ($\Leftarrow$):

Lemma 0.14. Let A be Noetherian, then every ideal of A contains a product of prime ideals.

Subproof. Suppose, towards contradiction, that there exists some ideal I of A that does not contain a product of prime ideals. Let $\mathcal{J}$ be the set of such ideals of A, then $\mathcal{J} \neq \varnothing$, and we can take a maximal element of $\mathcal{J}$, namely J.[1] By definition, J is not prime, therefore there exists $a, b \in A$ such that $a \notin J$ and $b \notin J$, but $ab \in J$. Now $J \subsetneq J + Aa$ and $J \subsetneq J + Ab$, therefore $J + Aa, J + Ab \notin J$, therefore $J + Aa$ and $J + Ab$ both contain product of prime ideals. But now $(J + Aa)(J + Ab)$ should also contain products of prime ideals, but by distribution this is just $J^2 + Ja + Jb + Aab$, which is contained in J because every term is contained in J, so J contains a product of prime ideals as well, contradiction. ■

In particular, (0) contains a product of prime ideals, in particular (0) equals to this product, but every prime ideal is maximal, therefore $(0) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$ becomes the product of maximal ideals (which may not necessarily be distinct), hence we have a descending chain of ideals

$$A \supseteq \mathfrak{m}_1 \supseteq \mathfrak{m}_1\mathfrak{m}_2 \supseteq \cdots \supseteq \mathfrak{m}_1 \cdots \mathfrak{m}_n = (0),$$

and in particular $(\mathfrak{m}_1 \cdots \mathfrak{m}_{i-1})/(\mathfrak{m}_1 \cdots \mathfrak{m}_i)$ is a finite-dimensional since A is Noetherian, and it has a natural structure as a $A/\mathfrak{m}_i$-vector space. From the short exact sequence

$$0 \longrightarrow \mathfrak{m}_1 \cdots \mathfrak{m}_i \longrightarrow \mathfrak{m}_1 \cdots \mathfrak{m}_{i-1} \longrightarrow (\mathfrak{m}_1 \cdots \mathfrak{m}_{i-1})/(\mathfrak{m}_1 \cdots \mathfrak{m}_i) \longrightarrow 0$$

we know the two sides of the sequence are Artinian, hence the central term is Artinian. Proceeding inductively, we know that $\mathfrak{m}_1$ is Artinian, and $R/\mathfrak{m}_1$ would also be Artinian, hence A is Artinian.

($\Rightarrow$): Now suppose A is Artinian, and we want to show that every prime ideal is maximal, and (0) is a product of maximal ideals. The result then follows from the argument above.

Lemma 0.15. Every Artinian domain is a field.

Subproof. Let $0 \neq a \in A$, then consider the chain

$$(a) \supseteq (a^2) \supseteq \cdots \supseteq (a^n) \supseteq \cdots$$

and by the Artinian property, for some large enough n the descending chain stops. Hence, we have $a^n = \lambda a^{n+1}$ for some large enough n and some $\lambda \in A$. Hence, $a^n(1 - \lambda a) = 0$, by the cancellation property of a domain, since $a \neq 0$, we must have $\lambda a = 1$, therefore a is a unit, as desired. ■

Corollary 0.16. Let A be Artinian, then every prime ideal of A is maximal.

Finally, it suffices to show that $(0) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$. Let $\mathfrak{J}$ be the set of finite products of maximal ideals, then $\mathfrak{J}$ has a minimal element, and it suffices to show that this element is (0). Suppose not, let $I \neq (0)$ be a minimal element of R. For any two ideals α, β of A, let $(\alpha : \beta) = \{a \in A \mid a\beta \subseteq \alpha\}$. Note that this has a natural structure as an ideal of A. Let $J = ((0) : I)$, and suppose $J = A$, then $I = 0$, contradiction, so $J \neq A$ is a proper ideal of A, now consider A/J which is Artinian, then let $\mathfrak{G}$ be the set of all non-zero ideals of A/J, so $\mathfrak{G}$ has a minimal element as well, call it $\bar{H}$. Let $H = \pi^{-1}(\bar{H})$ where $\pi : A \to A/J$, so we have $J \subsetneq H$, thus let $P = (J : H)$.

Claim 0.17. P is a prime ideal.

Subproof. Given $c, d \notin P$, we want to show that $cd \notin P$. Indeed, consider $J \subsetneq J + cH \subseteq H$, then since H is minimal, then $J + cH = H$, and similarly we have that $J + dH = H$. Therefore, we have that $J + cdH = J + c(dH + J) = J + cH = H$, hence we know $cd \notin P$, as desired. ■

[1] The existence of this maximal element is the result of Zorn's lemma and ACC condition.

Now $P = (J : H)$ and $J = (0 : I)$, the by definition we have $PHI = (0)$. Since P is a prime ideal, then P is maximal, and now

$$(0 : PI) \supseteq H \supsetneq J = (0 : I)$$

Therefore $PI \subsetneq I$, where I is a minimal element, contradiction, hence (0) is a product of maximal ideals. □

Definition 0.18 (Short Exact Sequence)**.** Consider the sequence

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} T \longrightarrow 0$$

This is called a short exact sequence if $\ker(f) = 0$, $\operatorname{im}(g) = T$, and $\ker(g) = \operatorname{im}(f)$. In particular, one slot of the sequence is said to be exact if the kernel of the previous map equals to the image of the subsequent map.

Definition 0.19 (Flat Module)**.** Let M be an A-module, then we say M is a flat A-module if for every short exact sequence

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

the tensored sequence

$$0 \longrightarrow M \otimes_A N_1 \longrightarrow M \otimes_A N_2 \longrightarrow M \otimes_A N_3 \longrightarrow 0$$

remains exact.

Remark 0.20. Recall that the properties of modules have the following implications: free ⇒ projective ⇒ flat ⇒ torsion-free, and in the case of finitely-generated modules, torsion-free ⇒ free.

Remark 0.21. We already know that the tensor functor is right exact, namely given the short exact sequence above, then

$$M \otimes_A N_1 \longrightarrow M \otimes_A N_2 \longrightarrow M \otimes_A N_3 \longrightarrow 0$$

is exact.

Exercise 0.22. Let M be an A-module, and if there exists a short exact sequence of A-modules

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

where N_1 and N_2 are finitely-generated as A-modules, and such that tensoring M preserves the short exact sequence, then M is flat.

Definition 0.23 (Multiplicatively Closed Subset)**.** Let A be a commutative ring and M be an A-module. Let $S \subseteq A$ be a subset. We say S is a multiplicatively closed subset of A if $1 \in S$, $0 \notin S$, and whenever $s_1, s_2 \in S$, then $s_1 s_2 \in S$.

Definition 0.24 (Localization)**.** Let $S \subseteq A$ be a multiplicatively closed subset, and let M be an A-module, then $S^{-1}M = (M \times S)/\sim$, where $\sim$ is an equivalence relation defined by the following: $(m_1, s_1) \sim (m_2, s_2)$ if and only if there exists $t \in S$ such that $t(m_1 s_2 - m_2 s_1) = 0$. $S^{-1}M$ is said to be the localization of M at S.

Given $(m, s) \in M \times S$, we write $\overline{(m, s)}$ to be the equivalence class in $S^{-1}M$ represented by (m, s).

Exercise 0.25. Similarly, one can define the localization $S^{-1}A$ of A at S. In fact, $S^{-1}A$ inherits a ring structure from A, namely

- $\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{a_1 s_2 + a_2 s_1}{s_1 s_2}$,
- $\frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{a_1 a_2}{s_1 s_2}$,
- $\frac{1}{s} \cdot \frac{s}{1} = \frac{1}{1} = 1$.

Remark 0.26. Note that a ring structure does not guarantee every element to have a multiplicative inverse. The localization of A at S ensures that every element of S now becomes invertible in the new ring $S^{-1}A$. In particular, this induces a ring homomorphism

$$\begin{aligned} f : A &\to S^{-1}A \\ a &\mapsto \frac{a}{1} \end{aligned}$$

This homomorphism is injective if A is a domain.

Remark 0.27. Let I be an ideal of A.

- Consider the ring homomorphism $f : A \to S^{-1}A$ above, then
$$S^{-1}I = IS^{-1}A = f(I)S^{-1}A.$$
In particular, $f^{-1}(IS^{-1}A) \supseteq I$.
- If $I \cap S \neq \varnothing$, then $IS^{-1}A = S^{-1}A$.
- If P is a prime ideal of A such that $P \cap S = \varnothing$, then $f^{-1}(PS^{-1}A) = P$.
- Let M be an A-module, then if $N \subseteq M$ is a submodule, then $S^{-1}N \subseteq S^{-1}M$. That is, given an exact sequence
$$0 \longrightarrow N \longrightarrow M$$
then we obtain an exact sequence
$$0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M$$
Indeed, given $0 \to N \xrightarrow{f} M$, say we have it sending $\frac{n}{1} \mapsto \frac{f(n)}{1} = 0$, then there exists $s \in S$ such that $sf(n) = 0$, so $f(sn) = 0$, therefore $sn = 0$ by injection, hence $\frac{n}{1} = 0$ in $S^{-1}N$ as well.

Exercise 0.28. The localization functor is exact.

Lemma 0.29. Let A be a commutative ring and S be a multiplicatively closed subset of A, then $S^{-1}A \otimes_A M \cong S^{-1}M$.

Proof. We define

$$\begin{aligned} \varphi : S^{-1}A \otimes_A M &\to S^{-1}M \\ \frac{a}{s} \otimes m &\mapsto \frac{am}{s}. \end{aligned}$$

For any $\frac{m}{s} \in S^{-1}M$, we have $\varphi\left(\frac{1}{s} \otimes m\right) = \frac{m}{s}$, so the map is onto. Now suppose $\varphi\left(\sum\limits_{i=1}^{n} \frac{a_i}{s_i} \otimes m_i\right) = 0$ (since this is a finite sum), then $\varphi\left(\sum\limits_{i=1}^{n} \frac{a_i}{s_i} \otimes m_i\right) = \sum\limits_{i=1}^{n} \frac{a_i m_i}{s_i} = 0$. We make $s = s_1 \cdots s_n$, so

$$\frac{a_i}{s_i} \otimes m_i = \frac{a_i s_1 \cdots s_{i-1} s_{i+1} \cdots s_n}{s} \otimes m_i =: \frac{b_i}{s} \otimes m_i,$$

then $\sum\limits_{i=1}^{n} \frac{a_i}{s_i} \otimes m_i = \sum\limits_{i=1}^{n} \frac{b_i}{s} \otimes m_i$, therefore

$$\varphi\left(\sum_{i=1}^{n} \frac{a_i}{s_i} \otimes m_i\right) = \varphi\left(\sum_{i=1}^{n} \frac{b_i}{s} \otimes m_i\right) = \frac{\sum\limits_{i=1}^{n} b_i m_i}{s} = 0,$$

so there exists $t \in S$ such that $t \sum\limits_{i=1}^{n} b_i m_i = 0$, now

$$\sum_{i=1}^{n} \frac{a_i}{s_i} \otimes m_i = \sum_{i=1}^{n} \frac{b_i}{s} \otimes m_i$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \frac{1}{s} \otimes b_i m_i \\
&= \frac{1}{s} \otimes \sum_{i=1}^{n} b_i m_i \\
&= \frac{t}{ts} \otimes \sum_{i=1}^{n} b_i m_i \\
&= \frac{1}{ts} \otimes t \sum_{i=1}^{n} b_i m_i \\
&= \frac{1}{ts} \otimes 0 \\
&= 0.
\end{aligned}
$$

□

Proposition 0.30. The map $A \to S^{-1}A$ is A-flat, i.e., $S^{-1}A$ is a flat A-module.

Proof. Consider

$$0 \longrightarrow N \longrightarrow M \longrightarrow T \longrightarrow 0$$

By Lemma 0.29 (since the isomorphism is functorial), it suffices to show the exactness of

$$0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M \longrightarrow S^{-1}T \longrightarrow 0$$

and this follows from Exercise 0.28. □

Definition 0.31 (Quasi-local, Local)**.** Let A be a commutative ring. We say A is quasi-local if A has exactly one maximal ideal. In particular, if A is also Noetherian, then we say A is a local ring.

Definition 0.32 (Localization)**.** Let A be a commutative ring and $\mathfrak{p}$ be a prime ideal of A. Note that $S = A \backslash \mathfrak{p}$ is a multiplicatively closed subset, then we write $S^{-1}A = A_{\mathfrak{p}}$ (in general, we have $S^{-1}M = M_{\mathfrak{p}}$, where $M \otimes_A A_{\mathfrak{p}} \cong M_{\mathfrak{p}}$) to denote the localization of A away from the prime ideal $\mathfrak{p}$.

Exercise 0.33. $A_{\mathfrak{p}}$ is quasi-local with unique maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$.

Remark 0.34. Take $x \in M$, then the following are equivalent:

- $x = 0$;
- $\frac{x}{1} = 0$ in $M_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of A;
- $\frac{x}{1} = 0$ in $M_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p}$ of A.

Proof. We will prove the first two are equivalent. The ($\Rightarrow$) direction is obvious. Conversely, let $I = \{a \in A \mid ax = 0\}$ to be the annihilator of x in A. Suppose, towards contradiction, that $I \neq A$, then I is contained in some maximal ideal $\mathfrak{m}$ of A, then consider $M_{\mathfrak{m}}$. Since $\frac{x}{1} = 0$ in $\mathfrak{m}$, then there exists $t \in A \backslash \mathfrak{m}$ such that $tx = 0$, but $I \subseteq \mathfrak{m}$ and $t \notin \mathfrak{m}$, then we reach a contradiction, hence $I = A$, and obviously we are done. □

Exercise 0.35. 1. Given the sequence

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} T \longrightarrow 0$$

the following are equivalent:

- the sequence is exact;
- the sequence
$$0 \longrightarrow M_{\mathfrak{m}} \xrightarrow{f_{\mathfrak{m}}} N_{\mathfrak{m}} \xrightarrow{g_{\mathfrak{m}}} T_{\mathfrak{m}} \longrightarrow 0$$
is exact for all maximal ideals $\mathfrak{m}$ of A;

- the sequence

$$0 \longrightarrow M_{\mathfrak{p}} \xrightarrow{f_{\mathfrak{p}}} N_{\mathfrak{p}} \xrightarrow{g_{\mathfrak{p}}} T_{\mathfrak{p}} \longrightarrow 0$$

is exact for all prime ideals $\mathfrak{p}$ of A.

To see this, apply Remark 0.34.

2. Let A be a commutative ring and M be an A-module, then the following are equivalent:
   - M is A-flat;
   - $M_{\mathfrak{m}}$ is $A_{\mathfrak{m}}$-flat for all maximal ideals $\mathfrak{m}$ of A;
   - $M_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-flat for all prime ideals $\mathfrak{p}$ of A;

Hence, exactness is a local property.

Exercise 0.36. Let A be a commutative ring, then A is Artinian if and only if A as an A-module is of finite length, i.e., $\ell_A(A) < \infty$. Indeed, note that $(0) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$, and write down the Jordan-Hölder series.

1 Primary Decomposition Theorem

Throughout Section 1, the commutative ring A is always Noetherian. In Section 1.1, M is a finitely-generated A-module; in Section 1.2, we drop this assumption.

1.1

Definition 1.1 (Coprimary)**.** We say M is a coprimary module if for all $a \in A$, the left multiplication $m_a : M \to M$ is either injective or nilpotent (i.e., there exists $n > 0$ such that $a^n M = 0$).

Remark 1.2. (i) If M is coprimary, then N is coprimary for all $N \subseteq M$.

(ii) If M is coprimary, let $P = \{a \in A \mid a : M \to M \text{ is nilpotent}\}$, then P is a prime ideal of A.

Proof. For $a, b \notin P$, $a, b : M \to M$ are injective maps, so $ab : M \to M$ is injective, hence $ab \notin P$. □

Hence, we usually say M is P-coprimary.

(iii) Let M be P-coprimary, then there exists an injection (as M-linear map) $A/P \hookrightarrow M$.

Proof. Take any $x \neq 0$ in M, then consider

$$\begin{aligned} a_x : A &\to M \\ 1 &\mapsto x \end{aligned}$$

Let $I = \ker(a_x)$, then we have

$$\begin{aligned} A/I &\hookrightarrow M \\ \bar{1} &\mapsto x \end{aligned}$$

Now $I \subseteq P$ since I already kills x. Since A is Noetherian, P is finitely-generated, thus consider $P = (a_1, \ldots, a_r)$, then $a_i^{t_i} \cdot x = 0$ for all i and some t_i's. Let $t = t_1 + \cdots + t_r$, then $P^t \cdot x = 0$ by binomial theorem, so $P^t \subseteq I \subseteq P$, hence there exists j such that $P^j \subseteq I \subsetneq P^{j-1}$. Take $y \in P^{j-1} \backslash I$, so $\bar{y} \neq 0$ in A/P, taking the injection into M, then $\mathrm{Ann}_A(\bar{y}) = P$. We now have the composition

$$\begin{aligned} A/P \hookrightarrow A/I &\hookrightarrow M \\ \bar{1} &\mapsto \bar{y} \end{aligned}$$

to be injective. □

(iv) Suppose M is P-coprimary, and Q is a prime ideal such that $A/Q \hookrightarrow M$, then $P = Q$.

Proof. By definition of P, $Q \subseteq P$ is obvious: Q kills elements in M, therefore the mapping becomes nilpotent. The other direction is also easy. □

Definition 1.3 (Primary)**.** Let $N \subseteq M$ be a submodule. We say N is a primary submodule of M if M/N is coprimary. If M/N is P-coprimary, we say N is P-primary.

Remark 1.4. Let $\mathfrak{p}$ be a prime ideal of A. We claim that $\mathfrak{p}^t$ is P-primary. Consider

$$m_x : A/\mathfrak{p}^t \to A/\mathfrak{p}^t$$

then $x^t = 0$ on $A/\mathfrak{p}^t$.

Example 1.5. Let $A = k[X, Y, Z]/(Z^2 - XY)$, let $\mathfrak{p} = (x, z)$ where $x = \operatorname{im}(X)$ and $z = \operatorname{im}(Z)$. Now $A/\mathfrak{p} = k[Y]$. $\mathfrak{p}^2$ is not P-primary. Indeed, note that $A/\mathfrak{p}^2 = k[X, Y, Z]/(z^2 - xy, x^2, z^2) \cong k[X, Y, Z]/(X^2, XY, Z^2, XZ)$. Now the mapping given by multiplication by y on this map is injective, so $\mathfrak{p}^2$ is not P-primary.

In particular, the represented surface is not smooth, since the origin $(0, 0, 0)$ is a singularity.

Theorem 1.6 (Primary Decomposition Theorem)**.** By assumption, A is Noetherian and M is finitely-generated. Let $N \subseteq M$ be a submodule, then there exists a decomposition

$$N = \bigcap_{i=1}^{r} N_i$$

where each N_i is P_i-primary, and such that

1. all P_i's are distinct, and
2. this decomposition is irredundant, i.e., minimal. In particular, this means removing any of the N_i's gives a different intersection, i.e., $\bigcap_{j \neq i} N_j \nsubseteq N_i$.

This is called a primary decomposition of N. Moreover, the primary decomposition is unique up to permutation of modules, that is, if there exists another primary decomposition, i.e., $N = \bigcap_{i=1}^{s} N_i'$ where N_i''s are P_i'-primary, then $r = s$ and $\{N_1, \ldots, N_r\} = \{N_1', \ldots, N_s'\}$.

Proof.

Definition 1.7 (Irreducible)**.** A submodule $T \subsetneq M$ is called irreducible if $T \neq T_1 \cap T_2$, where T_1, T_2 are distinct proper submodules of M.

Claim 1.8. Every submodule T of M can be expressed by $T = T_1 \cap \cdots \cap T_l$ where each T_i is irreducible.

Subproof. Suppose, towards contradiction, that there exists some T for which the claim fails, then the set of all such submodules T is a non-empty set $\mathcal{T}$. Since M is Noetherian, then $\mathcal{T}$ has a maximal element W, therefore W is not irreducible. By definition, $W = W_1 \cap W_2$ where W_1, W_2 are distinct proper submodules of M, so $W_1 \notin \mathcal{T}$ and $W_2 \notin \mathcal{T}$, therefore $W_1 = T_1 \cap \cdots \cap T_r$ for irreducible T_i's, and $W_2 = T_1' \cap \cdots \cap T_s'$ where T_i' are irreducible. Therefore, W becomes an intersection of irreducible submodules, a contradiction. ■

Claim 1.9. Suppose T is irreducible in M, then T is a primary submodule of M. That is, we need to show $\bar{M} := M/T$ is coprimary.

Subproof. It suffices to show the following: for all $a \neq 0$ in A, the multiplication map $a : \bar{M} \to \bar{M}$ is either nilpotent or injective. Note that (0) in $\bar{M}$ is irreducible. To see this, we take the ascending chain

$$\ker(a) \subseteq \ker(a^2) \subseteq \ker(a^3) \subseteq \cdots$$

and since A is Noetherian we know $\ker(a^n) = \ker(a^{n+1}) = \cdots$ for some large enough n, therefore for $g = a^n$ we know $\ker(g) = \ker(g^2)$.

Claim 1.10. $\ker(g) \cap \operatorname{im}(g) = (0)$ in $\bar{M}$.

Subproof of Subclaim. Let $x \in \ker(g) \cap \operatorname{im}(g)$, then $g(x) = 0$, and there exists $y \in \bar{M}$ such that $x = g(y)$, so $0 = g(x) = g^2(y)$, but that means $y \in \ker(g^2) = \ker(g)$, so $x = 0$. ■

Therefore, (0) is irreducible in $\bar{M}$, so either $\ker(g) = (0)$ or $\ker(g) = \bar{M}$. If $\ker(g) = (0)$, we have g to be injective, hence multiplication by a is injective; if $\ker(g) = \bar{M}$, we have $a^n\bar{M} = 0$, so a becomes nilpotent. ■

Claim 1.11. If N_1 and N_2 are both P-primary as submodules, then $N_1 \cap N_2$ is also P-primary.

Subproof. By definition, M/N_1 and M/N_2 are both P-coprimary, then it is easy to see that $M/N_1 \oplus M/N_2$ is also P-coprimary. We know there is an obvious inclusion

$$\begin{aligned} M/(N_1 \cap N_2) &\hookrightarrow M/N_1 \oplus M/N_2 \\ \bar{x} &\mapsto (\bar{x}, \bar{x}) \end{aligned}$$

so $M/(N_1 \cap N_2)$ is also coprimary by the inclusion, therefore $N_1 \cap N_2$ is P-primary. ■

Now by Claim 1.8 we have an irreducible decomposition $N = N_1 \cap \cdots \cap N_r$ and without loss of generality let it be of the smallest length, that is, the N_i's are irreducible modules that are irredundant. By Claim 1.9, we know each of the N_i's is primary with respect to some prime ideal. Now for any two P-primary modules N_i and N_j, we know the intersection is still P-primary according to Claim 1.11, therefore we obtain an irredundant intersection $N = N_1' \cap \cdots N_s'$ where each N_i' is P_i-primary (where P_i's are now distinct!), and this proves the existence.

For the uniqueness, suppose we have $N = N_1 \cap \cdots \cap N_r$ where N_i is P_i-primary, where P_i's are distinct, and suppose we have $N = N_1' \cap \cdots \cap N_s'$ where N_i' is P_i'-primary, where all P_i' are distinct as well. It is enough to show the following:

Claim 1.12. For any prime ideal p of A, $p \in \{P_1, \ldots, P_r\}$ if and only if there exists an injection $A/p \hookrightarrow M/N$.

Subproof. Let $p \in \{P_1, \ldots, P_r\}$, without loss of generality denote $p = P_1$, then we have an injection $A/p \hookrightarrow M/N_1$ by Remark 1.2. In $\bar{M} = M/N$, we have $(0) = N_1/N \cap \cdots \cap N_r/N =: \bar{N}_1 \cap \cdots \cap \bar{N}_r$, therefore $\bar{N}_2 \cap \cdots \cap \bar{N}_r \hookrightarrow \bar{M}/\bar{N}_1 = M/N_1$. But $M/N_1 = \bar{M}/\bar{N}_1$, so this gives an injection $\bar{N}_2 \cap \cdots \cap \bar{N}_r \hookrightarrow M/N_1$, but M/N_1 is P_1-coprimary, so $\bar{N}_2 \cap \cdots \cap \bar{N}_r$ is also P_1-coprimary, therefore $A/P_1 \hookrightarrow \bar{N}_2 \cap \cdots \cap \bar{N}_r \hookrightarrow \bar{M} = M/N$ by Remark 1.2.

Now suppose $A/p \hookrightarrow M/N$, to show $p \in \{P_1, \ldots, P_r\}$, it suffices to show $A/p \hookrightarrow M/N_i$ is injective for some $1 \leqslant i \leqslant r$. We have

$$\begin{tikzcd} A/p \arrow[r, hook, "\varphi"] \arrow[rr, bend left, "\varphi_i"] & M/N = \bar{M} \arrow[r, "\eta_i"] & \bar{M}/\bar{N}_i = M/N_i \end{tikzcd}$$

and we want to show there exists some injective φ_i. Suppose not, then $\ker(\varphi_i) \neq 0$ in A/p for all $1 \leqslant i \leqslant r$. But A/p is an integral domain, therefore $\bigcap_{i=1}^{r} \ker(\varphi_i) \neq 0$. Therefore, we have

$$A/p \overset{\varphi}{\hookrightarrow} M/N \overset{(\eta_1, \ldots, \eta_r)}{\hookrightarrow} \bigoplus_{i=1}^{r} M/N_i$$

Thus, the defined composition above is the injection $(\varphi_1, \ldots, \varphi_r)$. This implies $\bigcap_{i=1}^{r} \ker(\varphi_r) = \ker(\varphi_1, \ldots, \varphi_r) = 0$, a contradiction. Thus, there exists some injective φ_i, and therefore $p \in \{P_1, \ldots, P_r\}$. ■

□

Definition 1.13 (Zero-divisor)**.** Let A be Noetherian and M be a finitely-generated A-module. We say $0 \neq a \in A$ is a zero-divisor on M if there exists $0 \neq x \in M$ such that $ax = 0$. Otherwise, we say a is a non-zero-divisor on M.

Definition 1.14 (Essential prime ideal, Associated prime ideal)**.** Given a primary decomposition $N = \bigcap_{i=1}^{r} N_i$, the corresponding prime ideals $\{P_1, \ldots, P_r\}$ are called the essential prime ideals of N. In particular, if $N = (0)$, we say these are the associated prime ideals of M, denoted by $\operatorname{Ass}_A(M) = \{P_1, \ldots, P_r\}$.

Corollary 1.15. Let A be Noetherian and M be a finitely-generated A-module, and let $\mathrm{Ass}_A(M) = \{P_1, \ldots, P_r\}$, then $\bigcup_{i=1}^{r} P_i$ is the set of all zero-divisors on M.

Proof. If $p \in \mathrm{Ass}_A(M)$, then there exists an injection $A/p \hookrightarrow M$ mapping $\bar{1} \mapsto x$ by Claim 1.12. Therefore, $px = 0$, so elements of p are zero-divisors of M. Let a be a zero-divisor on M, i.e., let $0 \neq x \in M$ be such that $ax = 0$. Take the primary decomposition $(0) = N_1 \cap \cdots \cap N_r$ in M, where N_i is P_i-primary, then there exists i such that $x \notin N_i$. Since $\bar{x} \neq 0$ in M/N_i, then $a : M/N_i \to M/N_i$ is such that $a\bar{x} = 0$, so a is nilpotent on M/N_i. Therefore, M/N_i is P_i-coprimary, and by definition $a \in P_i$. □

Exercise 1.16. Let $\mathrm{Ass}_A(M) = \{P_1, \ldots, P_r\}$, then the set of all nilpotent elements of M is $\bigcap_{i=1}^{r} P_i$.

Corollary 1.17. Suppose $N \subseteq M$ is a submodule, then

$$\mathrm{Ass}_A(N) \subseteq \mathrm{Ass}_A(M) \subseteq \mathrm{Ass}_A(N) \cup \mathrm{Ass}_A(M/N).$$

Proof. The first inclusion is obvious by $A/p \hookrightarrow N \hookrightarrow M$. We now show the second inclusion. Let $p \in \mathrm{Ass}_A(M)$, and suppose $p \notin \mathrm{Ass}_A(N)$, and we have an inclusion $i : A/p \to M$.

Claim 1.18. $i(A/p) \cap N = (0)$.

Subproof. Suppose not, then let $0 \neq x \in i(A/p) \cap N$, then $x \in N$ and $x \in i(A/p)$, but A/p is an integral domain and is p-coprimary, so $i(A/p) \cap N$ is p-coprimary. Therefore, we have

$$A/p \hookrightarrow i(A/p) \cap N \hookrightarrow N$$

and so $p \in \mathrm{Ass}_A(N)$, a contradiction. ■

Therefore, we have the composition $A/p \to M \to M/N$ to be injection, thus $p \in \mathrm{Ass}_A(M/N)$. □

1.2

2 Filtered Rings and Modules, Completions

3 Dimension Theory

4 Integral Extensions

5 Noether's Normalization Lemma

6 Homological Algebra