MATH 518 Notes

Jiantong Liu

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Definition 1.1. Let M be a topological space. An *atlas* on M is a collection $\{\varphi_\alpha : U_\alpha \to W_\alpha\}_{\alpha \in A}$ of homeomorphisms called *coordinate charts*, so that

1. $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M,
2. for all $\alpha \in A$, W_α is an open subset of some $\mathbb{R}^{n_\alpha}$,
3. for all $\alpha, \beta \in A$, the induced map $\varphi_\beta \circ \varphi_\alpha^{-1}|_{U_\alpha \cap U_\beta}$ is C^∞, i.e., smooth.

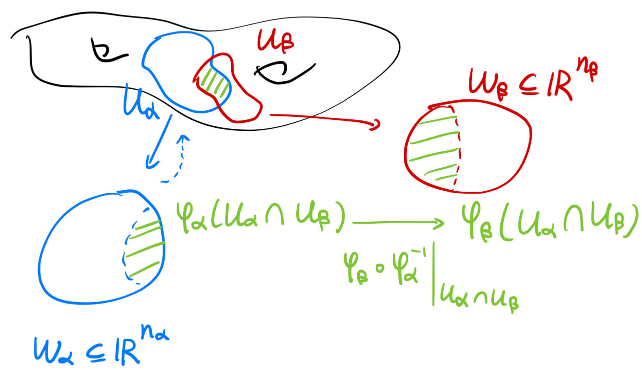

Figure 1: Atlas and Coordinate Chart

Example 1.2. Let $M = \mathbb{R}^n$ be equipped with standard topology, and let $A = \{*\}$, so $U_* = \mathbb{R}^n$ is the open cover of itself. Now the identity map

$$\begin{aligned} \varphi_* : U_* &\to \mathbb{R}^n \\ u &\mapsto u \end{aligned}$$

is an atlas on $\mathbb{R}^n$.

Example 1.3. Let $M = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be equipped with subspace topology. Let $U_\alpha = S^1 \backslash \{(1, 0)\}$ and $U_\beta = S^1 \backslash \{(-1, 0)\}$, and let $A = \{\alpha, \beta\}$. Let $W_\alpha = (0, 2\pi)$ and $W_\beta = (-\pi, \pi)$. We define $\varphi_\alpha^{-1}(\theta) = (\cos(\theta), \sin(\theta))$ and $\varphi_\beta^{-1}(\theta) = (\cos(\theta), \sin(\theta))$, then

$$(\varphi_\beta \circ \varphi_\alpha^{-1})(\theta) = \begin{cases} \theta, 0 < \theta < \pi \\ \theta - 2\pi, \pi < \theta < 2\pi \end{cases}$$

is smooth.

Example 1.4. Let X be a topological space with discrete topology, and let $A = X$, then $\{\varphi_x : \{x\} \to \mathbb{R}^0\}_{x\in X}$ gives an atlas.

Example 1.5. Let V be a finite-dimensional real vector space of dimension n. Pick a basis $\{v_1, \ldots, v_n\}$ of V, then there is a linear bijection φ with inverse

$$\begin{aligned} \varphi^{-1} : \mathbb{R}^n &\to V \\ (x_1, \ldots, x_n) &\mapsto \sum_{i=1}^n x_i v_i. \end{aligned}$$

The topology on V needs to make φ^{-1} a homeomorphism, and the obvious choice is just the collection of preimages, namely

$$\mathcal{T} = \{\varphi^{-1}(W) \mid W \subseteq \mathbb{R}^n \text{ open}\},$$

then $\varphi : V \to \mathbb{R}^n$ becomes an atlas.

Definition 1.6. Two atlases $\{\varphi_\alpha : U_\alpha \to W_\alpha\}_{\alpha\in A}$ and $\{\psi_\beta : V_\beta \to O_\beta\}_{\beta\in B}$ on a topological space M are *equivalent* if for all $\alpha \in A$ and $\beta \in B$,

$$\psi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap V_\beta) \subseteq \mathbb{R}^{n_\alpha} \to \psi_\beta(U_\alpha \cap V_\beta) \subseteq \mathbb{R}^{n_\beta}$$

is always C^∞, with C^∞-inverses. Such continuous maps are called *diffeomorphisms*. Alternatively, the two atlases are equivalent if their union $\{\varphi_\alpha\}_{\alpha\in A} \cup \{\psi_\beta\}_{\beta\in B}$ is always an atlas.

Exercise 1.7. Equivalence of atlases is an equivalence condition.

Definition 1.8. A (smooth) *manifold* is a topological space together with an equivalence class of atlases.

Convention. All manifolds are assumed to be smooth of C^∞, but not necessarily *Haudorff* and/or *second countable*.

Example 1.9. Continuing from Example 1.5, now suppose $\{w_1, \ldots, w_n\}$ gives another basis of V, with

$$\begin{aligned} \psi^{-1} : \mathbb{R}^n &\to V \\ (y_1, \ldots, y_n) &\mapsto \sum_{i=1}^n y_i w_i. \end{aligned}$$

This gives a change-of-basis matrix, so it is automatically C^∞ as a multiplication of invertible matrices. Therefore, the topology here does not depend on the chosen basis.

Recall. A topological space X is *Hausdorff* if for all distinct points $x, y \in X$, there exists open neighborhoods $U \ni x$ and $V \ni y$ such that $U \cap V = \varnothing$.

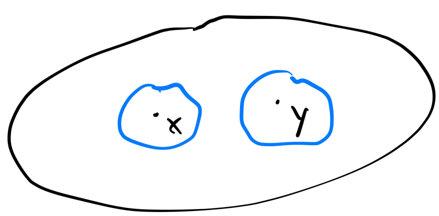

Figure 2: Hausdorff Condition

Convention. Via our definition (Definition 1.8), not all manifolds are Hausdorff.

Example 1.10. Let $Y = \mathbb{R} \times \{0, 1\}$, i.e., a space with two parallel lines, with a fixed topology. Define $\sim$ to be the smallest equivalence relation on Y such that $(x, 0) \sim (x, 1)$ for $x \neq 0$, and define $X = Y/\sim$. X is called the *line with two origins*, and it is second countable but not Hausdorff.

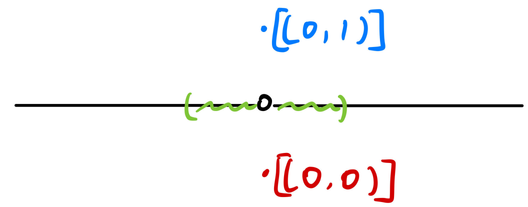

Figure 3: Line with Two Origins

Example 1.11. Take charts

$$\begin{aligned}\{\varphi : M = \mathbb{R} &\to \mathbb{R}\} \\ x &\mapsto x\end{aligned}$$

and

$$\begin{aligned}\{\psi : M = \mathbb{R} &\to \mathbb{R}\} \\ x &\mapsto x^3\end{aligned}$$

on $M = \mathbb{R}$, then

$$\begin{aligned}\varphi \circ \psi^{-1} : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^{\frac{1}{3}}\end{aligned}$$

is not C^∞, so φ and ψ are two different charts, hence give two different manifolds.

Definition 1.12. A map $F : M \to N$ between two manifolds is *smooth* if

1. F is continuous, and
2. for all charts $\varphi : U \to \mathbb{R}^m$ on M and charts $\psi : V \to \mathbb{R}^n$ on N, $\psi^{-1} \circ F \circ \varphi|_{\varphi(U \cap F^{-1}(V))}$ is C^∞.

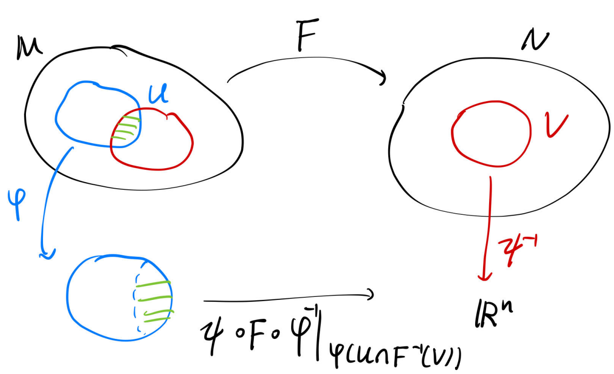

Figure 4: Smooth Map between Manifolds

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Exercise 2.1.

1. $\mathrm{id} : M \to M$ is smooth.
2. If $f : M \to N$ and $g : N \to Q$ are smooth maps between manifolds, then so is $gf : M \to Q$.

Punchline. The manifolds and the smooth maps between manifolds form a category.

Recall. A smooth map $f : M \to N$ is called a *diffeomorphism*, as seen in Definition 1.6, if it has a smooth inverse. This is the notion of an isomorphism in the category of manifolds.

Warning.

1. Following Example 1.11,
$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^3 \end{aligned}$$
has an inverse
$$\begin{aligned} f^{-1} : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^{\frac{1}{3}}, \end{aligned}$$
but f^{-1} is not differentiable at $x = 0$. Hence, f is not a diffeomorphism.

2. Take $\mathbb{R}$ with discrete topology, then all singletons are open sets, then the map
$$\begin{aligned} f : \mathbb{R}_{\text{dis}} &\to \mathbb{R}_{\text{std}} \\ x &\mapsto x \end{aligned}$$
is a smooth bijection, but f^{-1} is not continuous.

Example 2.2. Consider $M = (\mathbb{R}, \{\psi = \text{id} : \mathbb{R} \to \mathbb{R}\})$ and $N = (\mathbb{R}, \{\psi : \mathbb{R} \to \mathbb{R}, x \mapsto x^3\})$ as two manifolds on $\mathbb{R}$ with standard topology. To see that they are equivalent, consider the homeomorphism
$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^{\frac{1}{3}}, \end{aligned}$$
then $(\psi \circ f \circ \varphi^{-1})(x) = \psi(f(x)) = (x^{\frac{1}{3}})^3 = x$, so f is smooth, and $(\psi \circ f \circ \varphi^{-1})^{-1} = \varphi \circ f^{-1} \circ \psi^{-1} = \text{id}$, therefore f^{-1} is also smooth. Hence, f is a diffeomorphism.

We will now consider the real projective space $\mathbb{R}P^{n-1}$ and the quotient map $\pi : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}P^{n-1}$.

Definition 2.3. Define a binary relation on $\mathbb{R}^n \backslash \{0\}$ by $v_1 \sim v_2$ if and only if there exists $\lambda \neq 0$ such that $v_1 = \lambda v_2$. This is an equivalence relation, and we identify the equivalence class $[v]$ of $v \in \mathbb{R}^n \backslash \{0\}$ as a line $\mathbb{R}v = \text{span}_{\mathbb{R}}\{v\}$ through v. Then we define the *real projective space* $\mathbb{R}P^{n-1} = (\mathbb{R}^n \backslash \{0\}) / \sim$.

The natural topology on $\mathbb{R}P^{n-1}$ is the quotient topology, where $\pi : \mathbb{R}^n \backslash \{0\} \twoheadrightarrow \mathbb{R}P^{n-1}$ is surjective and continuous, so we define $U \subseteq \mathbb{R}P^{n-1}$ to be open if and only if $\pi^{-1}(U)$ is open in $\mathbb{R}^n \backslash \{0\}$.

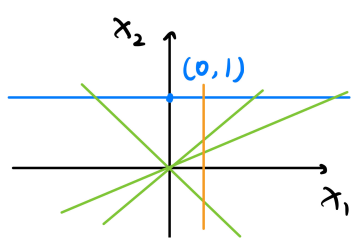

Figure 5: Stereographical Projection

Claim 2.4. $\mathbb{R}P^{n-1}$ is a manifold.

Proof. Define
$$\begin{aligned} \varphi_i : U_i &\to \mathbb{R}^{n-1} \\ [v_1, \ldots, v_n] &\mapsto \left(\frac{v_1}{v_i}, \ldots, \frac{v_{i-1}}{v_i}, \frac{v_{i+1}}{v_i}, \ldots, \frac{v_n}{v_i} \right), \end{aligned}$$

then

$$\begin{aligned}\varphi_i^{-1}:\mathbb{R}^{n-1}&\mapsto U_i\\(x_1,\dots,x_{n-1})&\mapsto[(x_1,\dots,x_{i-1},1,x_i,\dots,x_{n-1})],\end{aligned}$$

therefore

$$\begin{aligned}\varphi_j\circ\varphi_i^{-1}:\varphi_i(U_i\cap U_j)&\to\varphi_j(U_i\cap U_j)\\(x_1,\dots,x_{n-1})&\mapsto\varphi_j([(x_1,\dots,x_{i-1},1,x_i,\dots,x_{n-1})])\\&=\begin{cases}\left(\frac{x_1}{x_j},\dots,\frac{x_{j-1}}{x_j},\frac{x_{j+1}}{x_j},\dots,\frac{x_{n-1}}{x_j}\right), & j<i\\(x_1,\dots,x_{n-1}), & j=i\\\left(\frac{x_1}{x_{j-1}},\dots,\frac{1}{x_{j-1}},\dots,\frac{x_{j-2}}{x_{j-1}},\frac{x_j}{x_{j-1}},\dots,\frac{x_{n-1}}{x_{j-1}}\right), & j>i\end{cases}\end{aligned}$$

Therefore, this is C^∞ as a rational map on $\varphi_i(U_i\cap U_j)$, and so this gives an atlas, hence $\mathbb{R}P^{n-1}$ is a manifold. □

Claim 2.5. $\pi:\mathbb{R}^n\backslash\{0\}\to\mathbb{R}P^{n-1}$ is smooth.

Proof. Note that

$$\begin{aligned}\psi:\mathbb{R}^n\backslash\{0\}&\hookrightarrow\mathbb{R}^n\\x&\mapsto x\end{aligned}$$

is an atlas on $\mathbb{R}^n\backslash\{0\}$, and

$$\begin{aligned}\varphi_i\circ\pi\circ\varphi_i^{-1}:\mathbb{R}^n\backslash\{0\}&\to\mathbb{R}^{n-1}\\(v_1,\dots,v_n)&\mapsto\varphi_i([(v_1,\dots,v_n)])\\&=\left(\frac{v_1}{v_i},\dots,\frac{v_{i-1}}{v_i},\frac{v_{i+1}}{v_i},\dots,\frac{v_n}{v_i}\right).\end{aligned}$$

This is C^∞ on $\pi^{-1}(U_i)=\{(v_1,\dots,v_n)\mid v_i\neq 0\}$, so π is smooth. □

Definition 2.6. A *smooth function* on a manifold M is a function $f:M\to\mathbb{R}$ so that for any coordinate chart $\varphi:U\to\varphi(U)$ open in $\mathbb{R}^m$, the function $f\circ\varphi^{-1}:\varphi(U)\to\mathbb{R}$ is smooth.

Remark 2.7. $f:M\to\mathbb{R}$ is smooth if and only if $f:M\to(\mathbb{R},\{\mathrm{id}:\mathbb{R}\to\mathbb{R}\})$, usually called the *standard manifold structure on* $\mathbb{R}$, is smooth.

Notation. We denote $C^\infty(M)$ to be the set of all smooth functions $f:M\to\mathbb{R}$.

Remark 2.8. $C^\infty(M)$ is a smooth $\mathbb{R}$-vector space, that is, for all $\lambda,\mu\in\mathbb{R}$ and $f,g\in C^\infty(M)$,

- $(\lambda f+\mu g)(x)=\lambda f(x)+\mu g(x)$ for all $x\in M$,
- $(f\cdot g)(x)=f(x)g(x)$ for all $x\in M$.

Therefore, $C^\infty(M)$ becomes a (commutative, associative) $\mathbb{R}$-algebra.

Fact. Connecting manifolds have the notion of dimension. That is, the dimensions of open subsets induced by coordinate charts are the same.

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Definition 3.1. Let M be a manifold, then for every point $q\in M$, there exists a well-defined non-negative integer $\dim_M(q)$, so that for any coordinate chart $\varphi:U\to\mathbb{R}^m$ for $U\ni q$, we have $\dim_M(q)=m$ for some non-negative integer m that only depend on M. Consequently, $\dim_M:M\to\mathbb{Z}^{\geqslant 0}$ is a locally constant function. This integer m is called the *dimension* of M.

Proof. Indeed, say $\psi : V \to \mathbb{R}^n$ is another chart with $U \cap V \ni q$, then $\psi \circ \varphi^{-1}|_{\varphi(U \cap V)} : \varphi(U \cap V) \subseteq \mathbb{R}^m \to \psi(U \cap V) \subseteq \mathbb{R}^n$ is a diffeomorphism, therefore the Jacobian $D(\psi \circ \varphi^{-1})(\varphi(a)) : \mathbb{R}^m \to \mathbb{R}^n$ is a linear isomorphism, thus $m = n$. $\square$

Definition 3.2. Suppose $(M, \{\varphi_\alpha : U_\alpha \to \mathbb{R}^m\}_{\alpha \in A})$ and $(N, \{\psi_\alpha : V_\beta \to \mathbb{R}^n\}_{\beta \in B})$ are two manifolds. One can give a manifold structure to the product set $M \times N$, called the *product manifold*, as follows:

- give $M \times N$ the product topology,
- let $\{\varphi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n\}_{(\alpha,\beta) \in A \times B}$ to be the atlas on $M \times N$. This is well-defined since the transition maps of $\alpha, \alpha' \in A$ and $\beta, \beta' \in B$ are over $(U_\alpha \times V_\beta) \cap U_{\alpha'} \times V_{\beta'} = (U_\alpha \cap U_{\alpha'}) \times (V_\beta \cap V_{\beta'})$ with $(\varphi_{\alpha'} \times \psi_{\beta'}) \circ (\varphi_\alpha \times \psi_\beta)^{-1} = (\varphi_{\alpha'} \circ \varphi_\alpha^{-1}, \psi_{\beta'} \circ \psi_\beta^{-1})$. This is smooth since products of smooth maps are smooth.

Punchline. The product construction of manifolds gives the categorical product in the category of manifolds.

Property.

1. The projection maps

$$\begin{aligned} p_M : M \times N &\to M \\ (m, n) &\mapsto m \end{aligned}$$

and

$$\begin{aligned} p_N : M \times N &\to N \\ (m, n) &\mapsto n \end{aligned}$$

are C^∞.

2. *Universal Property of Product*: for any manifold Q and smooth maps $f_M : Q \to M$ and $f_N : Q \to N$, there exists a unique map

$$\begin{aligned} g : Q &\to M \times N \\ q &\mapsto (f(q), g(q)) \end{aligned}$$

such that $p_M \circ g = f_M$, and $p_N \circ g = f_N$.

$$\begin{tikzcd}
& Q \arrow[d, dashed, "g"', "\exists!"] \arrow[ddl, bend right, "f_M"'] \arrow[ddr, bend left, "f_N"] & \\
& M \times N \arrow[dl, "p_M"] \arrow[dr, "p_N"'] & \\
M & & N
\end{tikzcd}$$

Figure 6: Universal Property of Product

Recall.

- A topological space X is *second countable* if the topology has a countable basis: there exists a collection $\mathcal{B} = \{B_i\}_{i \in \mathbb{N}}$ of open sets so that any open set of X is a union of some B_i's.
- A cover $\{U_\alpha\}_{\alpha \in A}$ of a topological space is *locally finite* if for all $x \in X$, there exists a neighborhood N of X such that $N \cap U_\alpha = \varnothing$ for all but finitely many α's.

Example 3.3. Let $X = \mathbb{R}$, then

- $\{U_n = (-n, n)\}_{n \geqslant 0}$ is an open cover, but is not locally finite,
- $\{U_n = (n, n+2)\}_{n \in \mathbb{Z}}$ is a locally finite open cover of $\mathbb{R}$,
- $\{U_n = (n, n+2]\}_{n \in \mathbb{Z}}$ is a locally finite cover of $\mathbb{R}$, but is not an open cover.

Recall. An (open) cover $\{V_\beta\}_{\beta\in B}$ is a *refinement* of a cover $\{U_\alpha\}_{\alpha\in A}$ if for all β, there exists $\alpha = \alpha(\beta)$ such that $V_\beta \subseteq U_{\alpha(\beta)}$.

Definition 3.4. A Hausdorff topological space is *paracompact* if every open cover has a locally finite open refinement.

Fact. A connected Hausdorff manifold is paracompact if and only if it is second countable.

Corollary 3.5. A Haudorff manifold is paracompact if and only if its connected components are second countable.

Example 3.6. $\mathbb{R}$ with discrete topology is paracompact but not second countable.

Convention. Usually, we assume manifolds are paracompact, except when we need a non-Haudorff manifold. This condition is required for the existence of *partition of unity* (i.e., constant function id).

Recall. If X is a space, and $Y \subseteq X$ is a subset, then the *closure* $\bar{Y}$ of Y is the smallest closed set containing Y.

Definition 3.7. Given a topological space X and a function $f : X \to \mathbb{R}$, the *support* of f over X is

$$\operatorname{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}.$$

Example 3.8. The function

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leqslant 0 \end{cases}$$

is C^∞, with support $\overline{(0,\infty)} = [0,\infty)$.

Definition 3.9. Let M be a topological space and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover. A *partition of unity* subordinate to the cover is a collection of continuous functions $\{\psi_\alpha : M \to [0,1]\}_{\alpha\in A}$ such that

1. $\operatorname{supp}(\psi_\alpha) \subseteq U_\alpha$ for all $\alpha \in A$,
2. $\{\operatorname{supp}(\psi_\alpha)\}_{\alpha\in A}$ is a locally finite closed cover of M,
3. $\sum\limits_{\alpha\in A} \psi_\alpha(x) = 1$ for all $x \in M$.

Remark 3.10. For all $x \in M$, there exists $\alpha_1, \ldots, \alpha_n$ such that $x \in \operatorname{supp}(\psi_{\alpha_i})$. Hence, for $\alpha \neq \alpha_1, \ldots, \alpha_n$, $\psi_\alpha(x) = 0$. Therefore, the summation in Definition 3.9 is finite.

Theorem 3.11. Let M be a paracompact manifold with open cover $\{U_\alpha\}_{\alpha\in A}$, then there exists a partition of unity $\{\psi_\alpha : U_\alpha \to [0,1]\}_{\alpha\in A} \subseteq C^\infty(M)$ subordinate to the cover.

Example 3.12. Let $M = \mathbb{R}$ and consider for $n > 0$ the open sets $\{U_n = (-n, n)\}_{n\in\mathbb{N}}$. This is not locally finite at one point.

Example 3.13. Let $M = \mathbb{R}^n$, then for all $x \in \mathbb{R}^n$ and for $r > 0$, we have $B_r(x) = \{x' \in \mathbb{R}^n \mid ||x - x'|| < r\}$ and so $\{B_r(x)\}_{r>0, x\in\mathbb{R}^n}$ is an open cover, but this is not locally finite everywhere.

4 Aug 28, 2023

We will start to talk about tangent vectors.

Recall. For any point $q \in \mathbb{R}^n$ and any vector $v \in \mathbb{R}^n$, and any $f \in C^\infty(\mathbb{R}^n)$, the *directional derivative* of q in direction v with respect to f is

$$D_v f(q) = \frac{d}{dt}|_0 f(q + tv).$$

This gives a map $D_v(-)(q) : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ which is

- linear, and

- Leibniz rule holds, i.e.,
$$D_v(fg)(q) = D_v(f)(q) \cdot g(q) + f(q)D_v(g)(q).$$

In other words, $D_v(-)(q) : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation.

Definition 4.1. Let q be a point of a manifold M. A *tangent vector* to M at q is an $\mathbb{R}$-linear map $v : C^\infty(M) \to \mathbb{R}$ such that for all $f, g \in C^\infty(M)$,
$$v(fg) = v(f)g(q) + f(q)v(g).$$

Remark 4.2. v gives smooth vector fields over M an $C^\infty(M)$-module structure via evaluation.

Lemma 4.3. The set T_qM of all tangent vectors to M at q is an $\mathbb{R}$-vector space.

Lemma 4.4. Suppose $c \in C^\infty(M)$ is a constant function, then for all q and all $v \in T_qM$, $v(c) = 0$.

Proof. We have $v(1) = v(1 \cdot 1) = 1(q)v(1) + v(1)1(q) = 2v(1)$, so $v(1) = 0$. For a constant function c, we have
$$v(c) = v(c \cdot 1) = cv(1) = c(0) = 0.$$
□

Lemma 4.5 (Hadamard)**.** For any $f \in C^\infty(\mathbb{R}^n)$, there exists $g_1, \ldots, g_n \in C^\infty(\mathbb{R}^n)$ such that

- $f(x) = f(0) + \sum\limits_{i=1}^{n} x_i g_i(x)$, and
- $g_i(0) = \left(\frac{\partial}{\partial x_i} f\right)(0)$.

Proof. We have
$$\begin{aligned} f(x) - f(0) &= \int_0^1 \frac{d}{dt}(f(tx))dt \\ &= \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(tx) \cdot x_i dt \\ &= \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt \\ &= \sum_{i=1}^{n} x_i g_i(x). \end{aligned}$$

Therefore, $g_i(0) = \int_0^1 \frac{\partial f}{\partial x_i}(t \cdot 0)dt = \frac{\partial f}{\partial x_i}(0)$. □

Remark 4.6. For $1 \leqslant i \leqslant n$, we have canonical tangent vectors to $\mathbb{R}^n$ at 0 given by
$$\begin{aligned} \frac{\partial}{\partial x_i}|_0 : C^\infty(\mathbb{R}^n) &\to \mathbb{R} \\ f &\mapsto \frac{\partial f}{\partial x_i}(0). \end{aligned}$$

Lemma 4.7. $\left\{\frac{\partial}{\partial x_1}|_0, \ldots, \frac{\partial}{\partial x_n}|_0\right\}$ is a basis of $T_0\mathbb{R}^n$.

Proof. Suppose $\sum c_i \frac{\partial}{\partial x_i}|_0 = 0$, then
$$0 = \left(\sum_i c_i \frac{\partial}{\partial x_i}|_0\right)(x_j) = \sum_i c_i \delta_{ij} = c_j.$$

Therefore, $c_j = 0$ for all j, thus we have linear independence. For all $v \in T_0\mathbb{R}^n$, i.e., $v : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation, then $v = \sum\limits_i v(x_i)\frac{\partial}{\partial x_i}|_0$. Let $f \in C^\infty(\mathbb{R}^n)$, then $f(X) = f(0) + \sum x_i g_i(x)$, thus

$$\begin{aligned} v(f) &= v(f(0)) + \sum_{i=1}^n v(x_i g_i(x)) \\ &= \sum_{i=1}^n v(x_i g_i(x)) \\ &= \sum_{i=1}^n (v(x_i)g_i(0) + x_i(0)v(g_i)) \\ &= \sum_{i=1}^n v(x_i)g_i(0) \\ &= \sum_{i=1}^n v(x_i)\frac{\partial f}{\partial x_i}(0). \end{aligned}$$

□

Remark 4.8. This shows $\dim(T_0\mathbb{R}^n) = n$ with the basis above.

Now let V be a finite-dimensional vector space with a basis $e_1, \dots, e_n$, then

$$\begin{aligned} \varphi : \mathbb{R}^n &\to V \\ (t_1, \dots, t_n) &\mapsto \sum_{i=1}^n t_i e_i \end{aligned}$$

is a linear bijection, with linear inverse

$$\begin{aligned} \psi : V &\to \mathbb{R}^n \\ v &\mapsto (\psi_1(v), \dots, \psi_n(v)) \end{aligned}$$

where $\psi_i(v)$'s are linear maps. To describe this with a basis, we have $\psi(\sum\limits_i a_i e_i) = (a_1, \dots, a_n)$, i.e., $\psi_i(e_j) = \delta_{ij}$.

Claim 4.9. $\{\psi_1, \dots, \psi_n\}$ is a basis of $V^* = \mathrm{Hom}(V, \mathbb{R})$, called the *dual basis* of $\{e_1, \dots, e_n\}$, denoted $e_j^* = \psi_j$.

Proof. Linear independence follows from $e_j^*(e_i) = \delta_{ij}$. Given $\ell : V \to \mathbb{R}$ to be a linear map, then $\ell = \sum \ell(e_i)e_i^*$ since $\left(\sum\limits_i \ell(e_i)e_i^*\right)(e_j) = \ell(e_j)$. Given $v \in T_0\mathbb{R}^n$, $v(f) = \sum a_i \left(\frac{\partial}{\partial x_i}|_0 f\right)$ for all $f \in C^\infty(\mathbb{R}^n)$. Note that $\frac{\partial}{\partial x_i}|_0(x_j) = \delta_{ij}$, so $v(x_j) = \sum a_i \frac{\partial}{\partial x_i}|_0(x_j) = \sum\limits_i a_i\delta_{ij} = a_j$. Therefore, we have $a_i = v(x_i)$ for all i, thus $v(f) = \sum v(x_i)\left(\frac{\partial}{\partial x_i}|_0 f\right)$. Thus, the dual basis to $\frac{\partial}{\partial x_1}|_0, \dots, \frac{\partial}{\partial x_n}|_0$ is $\{d(x_i)_0\}_{i=1}^n$ where $(dx_i)_0(v) = v(x_i)$ for all i. Hence, we have $v = \sum (dx_i)_0(v)\frac{\partial}{\partial x_i}|_0$. □

Remark 4.10. Via a change of basis, this works at every point q on the local chart, so we can describe the tangent space on any point on a local chart.

5 Aug 30, 2023

Let M be a manifold and $x \in M$. Recall that a tangent vector $v : C^\infty(M) \to \mathbb{R}$ is a derivation, i.e., linear map, and the set of tangent vectors at q gives the tangent space.

Example 5.1. Let $M = \mathbb{R}^n$, and $q = 0$, then $\left\{\frac{\partial}{\partial x_1}|_0, \dots, \frac{\partial}{\partial x_n}|_0\right\}$ is a basis of $T_0\mathbb{R}^n$. Moreover, for all $v \in T_0\mathbb{R}^n$, $v = \sum v(x_i)\frac{\partial}{\partial x_i}|_0$, thus $\{v \mapsto v(x_i)\}_{i=1}^n$ is the dual basis, with $v(x_i) = (dx_i)_0(v)$ for all $1 \leqslant i \leqslant n$.

Remark 5.2. The proof used Hadamard's lemma (Lemma 4.5) and the fact that for all $x \in \mathbb{R}^n$ and all $t \in [0,1]$, $f(tx)$ is defined. Thus, the same argument should work for a version of Hadamard's lemma for star-shaped open subsets $U \subseteq \mathbb{R}^n$.

Definition 5.3. We say an open subset $U \subseteq \mathbb{R}^n$ is a *star-shaped domain* if for all $t \in [0,1]$ and all $x \in U$, $tx \in U$.

Definition 5.4. Let $F : M \to N$ be a smooth map between two manifolds, and $q \in M$ is a point, then

$$\begin{aligned} T_qF : T_qM &\to T_qN \\ v(f) &\mapsto v(f \circ F) \end{aligned}$$

via the pullback.

Exercise 5.5. Check that the definition makes sense, in particular:

(i) $(T_qF)(v)$ is a tangent vector to N of $F(q)$, and

(ii) T_qF is a derivation.

Remark 5.6. (a) It is easy to deduce the *chain rule*. That is, given $M \xrightarrow{F} N \xrightarrow{G} Q$ with $q \in M$, then $T_q(G \circ F) = T_{F(q)}G \circ T_qF$ because for all $f \in C^\infty(Q)$ and all $v \in T_qM$, we have

$$(T_q(G \circ F)(v))(f) = v(f \circ (G \circ F))$$

and

$$(T_{F(q)}G(T_qF(v))) = (T_qF)(v)(f \circ G) = v((f \circ G) \circ F).$$

(b) $T_q(\mathrm{id}_M) = \mathrm{id}_{T_qM}$.

As a result, we know T is a functor from the category of pointed manifolds to the category of $\mathbb{R}$-vector spaces.

Corollary 5.7. If $F : M \to N$ is a diffeomorphism, then for all $q \in M$, $T_qF : T_qM \to T_{F(q)}N$ is an isomorphism.

Proof. Since F is a diffeomorphism, then it has a smooth inverse $G : N \to M$, so

$$\mathrm{id}_{T_qM} = T_q(\mathrm{id}_M) = T_q(G \circ F) = T_{F(q)}G \circ T_qF$$

and

$$\mathrm{id}_{T_{F(q)}N} = T_{F(q)}(\mathrm{id}_N) = T_{F(q)}(F \circ G) = T_{F(q)}F \circ T_{F(q)}G.$$

□

We also need to show that $\dim(T_qM) = \dim_q(M)$, which is a result of Lemma 5.8, whose proof will be postponed till next time.

Lemma 5.8. Let M be a manifold and $q \in M$, and let U be an open neighborhood of q in M, and let $i : U \hookrightarrow M$ be an inclusion, then

$$\begin{aligned} I = T_qi : T_qU &\to T_qM \\ v(f) &\mapsto v(f|_U) \end{aligned}$$

is an isomorphism for all $v \in T_qM$ and all $U \subseteq M$.

Notation. We denote $r_1, \dots, r_n : \mathbb{R}^m \to \mathbb{R}$ to be the standard coordinates on $\mathbb{R}^m$.

Let M be a manifold, $q_0 \in M$, and $\varphi : U \to \mathbb{R}^m$ is a coordinate chart with $q_0 \in U$. Now let $x_i = r_i \circ \varphi$, then $\varphi(q) = (x_1(q), \dots, x_m(q))$.

We may now assume that

- $\varphi(q_0) = 0$, otherwise, we replace $\varphi(q)$ by $\varphi(q) := \varphi(q) - \varphi(q_0)$, and
- $\varphi(U)$ is an open ball $B_R(0) = \{r \in \mathbb{R}^m \mid ||r|| < R\}$ because there exists $R > 0$ such that $B_R(0) \subseteq \varphi(U)$, and we can then replace U with $\varphi^{-1}(B_R(0))$ and restrict the charts φ to $\varphi|_{\varphi^{-1}(B_R(0))}$.

We now define

$$\begin{aligned}\frac{\partial}{\partial x_j}|_{q_0} : C^\infty(U) &\to \mathbb{R}\\ f &\mapsto \frac{\partial}{\partial r_j}|_0(f \circ \varphi^{-1})\end{aligned}$$

Claim 5.9. $\left\{\frac{\partial}{\partial x_j}|_{q_0}\right\}_{j=1}^m$ is a basis of T_qM and for all $v \in T_{q_0}M$, $v = \sum v(x_j)\frac{\partial}{\partial x_j}|_{q_0}$.

Proof. By Hadamard's lemma Lemma 4.5 on $B_R(0)$, for all $f \in C^\infty(U)$, we have $f \circ \varphi^{-1} \in C^\infty(B_R(0))$, so there exists $g_1, \dots, g_m \in C^\infty(B_R(0))$ such that $(f \circ \varphi^{-1})(r) = f(\varphi^{-1}(0)) + \sum r_i g_i(r)$. Therefore, $f(q) = f(q_0) + \sum(r_i \circ \varphi)(q)(g_i \circ \varphi)(q)$, hence $f = f(q_0) + \sum x_i(g_i \circ \varphi)$, and $(g_i \circ \varphi)(q_0) = g_i(0) = \frac{\partial}{\partial r_i}|_0(f \circ \varphi^{-1}) = \frac{\partial}{\partial x_i}|_0(f)$.

Hence, for all $v \in T_{q_0}(U)$, we know

$$\begin{aligned}v(f) &= v(f(q_0)) + v\left(\sum x_i \cdot (g_i \circ \varphi)\right)\\ &= \sum_i v(x_i)(g_i \circ \varphi)(q_0)\\ &= \sum v(x_i)\frac{\partial}{\partial x_i}|_{q_0}(f).\end{aligned}$$

□

Remark 5.10.

1. The linear functionals

$$\begin{aligned}(dx_i)_{q_0} : T_{q_0}U &\to \mathbb{R}\\ v &\mapsto v(x_i)\end{aligned}$$

is the basis of $(T_{q_0}U)^*$ dual to $\left\{\frac{\partial}{\partial x_i}|_{q_0}\right\}$.

2. $(T_0\varphi^{-1})\left(\frac{\partial}{\partial r_i}|_0\right) = \frac{\partial}{\partial x_i}|_{q_0}$ by definition. Since $\left\{\frac{\partial}{\partial x_i}|_0\right\}_{i=1}^n$ is a basis of $T_0(B_R(0))$, then $\left\{\frac{\partial}{\partial x_i}|_{q_0}\right\}$ has to be a basis.

Lemma 5.11. Let M be a manifold and $q \in M$ a point. Let $U \ni q$ be anopen neighborhood, and $f \in C^\infty(M)$ such that $f|_U = 0$, then for all $v \in T_qM$, we have $v(f) = 0$.

Proof. We have shown the existence of a bump function $\rho \in C^\infty(M)$ in homework 1, that is, $0 \leqslant \rho(x) \leqslant 1$, $\mathrm{supp}(\rho) \subseteq U$ and $\rho \equiv 1$ near q.

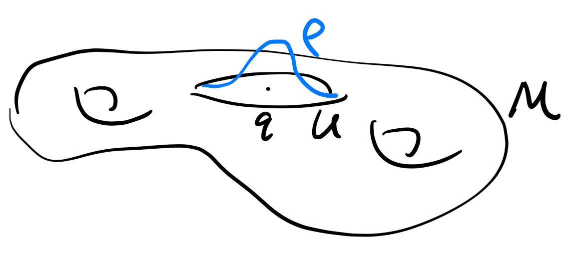


Figure 7: Bump Function

Therefore, $\rho f \equiv 0$, so $v(f) = v(\rho)f(q) + \rho(q)v(f) = v(\rho f) = 0$. □

6 Sept 1, 2023

Recall. Given a coordinate chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$, and $q \in U$ with $f(q) = 0$, we defined $\left\{ \frac{\partial}{\partial x_i}|_q \right\}_{i=1}^m \subseteq T_qU$ by $\frac{\partial}{\partial x_i}|_q f = \frac{\partial}{\partial r_i}(f \circ \varphi^{-1})|_{\varphi(q)}$ where $\frac{\partial}{\partial r_i}$'s are the standard partials on $C^\infty(\mathbb{R}^m)$.

We know this is a basis with dual basis

$$\begin{aligned}(dx_i)_q : T_qM &\to \mathbb{R}\\ v &\mapsto v(x_i)\end{aligned}$$

therefore $v = \sum v(x_i)\frac{\partial}{\partial x_i}|_q$ for all v. Note that

$$\begin{aligned}C^\infty(M) &\to C^\infty(U)\\ f &\mapsto f|_U\end{aligned}$$

is not surjective.

Also, we know $v \in T_qM$ is local, if $f, g \in C^\infty(M)$ agree on a neighborhood of q, then $v(f) = v(g)$.

Finally, given $F : M \to N$, this induces

$$\begin{aligned}T_qF : T_qM &\to T_{F(q)}N\\ v &\mapsto v(f \circ F).\end{aligned}$$

Lemma 6.1. Given a manifold M and $q \in M$, open neighborhood $q \in U \subseteq M$ and $i : U \hookrightarrow M$ inclusion, then

$$I \equiv T_qi : T_qU \to T_qM$$

is an isomorphism with $(I(v))(f) = v(f|_U)$ for all $f \in C^\infty(M)$.

Proof. Suppose $v \in \ker(I)$, then $v(f|_U) = 0$ for all $f \in C^\infty(M)$. We want $v(h) = 0$ for all $h \in C^\infty(U)$. We first choose bump function $\rho : M \to [0, 1]$ that is C^∞, and $\rho \equiv 1$ near q, and suppose $\mathrm{supp}(\rho) \subseteq U$, hence $\rho|_{M\setminus U} \equiv 0$. Then define $\rho h \in C^\infty(M)$ via

$$\rho h(x) = \begin{cases}\rho(x)h(x), & x \in U\\ 0, & x \notin U\end{cases}$$

Now $\rho h|_U \equiv h$ near q, i.e., identically 1. Therefore, $v(h) = v(\rho h|_U) = 0$, so $v \equiv 0$.

It remains to show that for all $w \in T_qM$, there exists $v \in T_qU$ such that $I(v) = w$, i.e., for all $f \in C^\infty(M)$, $w(f) = v(f|_U)$. Take the same $\rho \in C^\infty(M, [0.1])$ as above, define $v(h) = w(\rho h)$ for all $h \in C^\infty(M)$, and we can check that

- $v \in T_qM$, and
- for all $f \in C^\infty(M)$, $v(f|_U) = w(f)$.

Note that v is $\mathbb{R}$-linear, and for all $f, g \in C^\infty(W)$ we have $v(fg) = w(\rho fg) = w(\rho^2 fg)$ since $\rho fg = \rho^2 fg$ near q, then we have

$$\begin{aligned}v(fg) &= w(\rho^2 fg)\\ &= w((\rho f)(\rho g))\\ &= v(\rho f) \cdot (\rho g)(g) + \rho(f)(q) \cdot v(\rho g)\\ &= v(f)g(q) + f(q)v(g).\end{aligned}$$

Finally, for all $f \in C^\infty(M)$, we have $v(f|_U) = w(\rho f) = w(f)$ since $\rho f = f$ near q. □

Notation. We now suppress the isomorphisms $I : T_qU \to T_qM$. In particular, given a chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$, we view $\left\{ \frac{\partial}{\partial x_i}|_q \right\}_{i=1}^m$ as a basis of T_qM.

Lemma 6.2. Let V be a finite-dimensional vector space with $q \in V$, then

$$\begin{aligned}\varphi : V &\to T_qV \\ v(f) &\mapsto \frac{d}{dt}|_0 f(q+tv)\end{aligned}$$

for all $f \in C^\infty(V)$, is an isomorphism.

Proof. One can see this is linear, so it suffices to show injectivity. We have

$$\ker(\varphi) = \{v \in V \mid \frac{d}{dt}|_0(q+tv) = 0 \, \forall f \in C^\infty(V)\}.$$

If $0 \neq v \in \ker(\varphi)$, then there exists $\ell : V \to \mathbb{R}$ such that $\ell(V) \neq 0$, so

$$0 \neq \frac{d}{dt}|_0(\ell(q+tv)) = \frac{d}{dt}|_0(\ell(q) + t\ell(v)) = \ell(v).$$

$\square$

Definition 6.3. A *curve* through a point $q \in M$ on a manifold M is a C^∞-map $\gamma : (a,b) \to M$ with $0 \in (a,b)$ such that $\gamma(0) = q$.

Definition 6.4. Given $\gamma : (a,b) \to M$ with $\gamma(0) = q$, we define $\dot\gamma(0) \in T_qM$ by $\dot\gamma(0)f = \frac{d}{dt}|_0 f(\gamma(t)) = \frac{d}{dt}|_0(f \circ \gamma)$ for all $f \in C^\infty(M)$.

Remark 6.5.

$$\begin{aligned}t : (a,b) &\to \mathbb{R} \\ x &\mapsto x\end{aligned}$$

is a coordinate chart on (a,b), where $\frac{d}{dt}|_0 \in T_0(a,b)$ is a basis vector. Since γ is C^∞,

$$\begin{aligned}T_0\gamma : T_0(a,b) &\to T_{\gamma(0)}M \equiv T_qM \\ ((T_0\gamma)(\frac{d}{dt}|_0))f &= \frac{d}{dt}|_0(f \circ \gamma) = \dot\gamma(0),\end{aligned}$$

so $\dot\gamma(0) = (T_0\gamma)\left(\frac{d}{dt}|_0\right)$.

Let $\mathscr{C} = \{\gamma : I \to M \mid \gamma(0) = q, I \text{ interval depending on } \gamma\}$, then we have a map

$$\begin{aligned}\Phi : \mathscr{C} &\to T_qM \\ \gamma &\mapsto \dot\gamma(0)\end{aligned}$$

Note that Φ is not injective. However, there is an equivalence relation $\sim$ on $\mathscr{C}$ defined by $\gamma \sim \sigma$ if and only if $\Phi(\gamma) = \Phi(\sigma)$, so this gives an injection

$$\begin{aligned}\tilde\Phi : \mathscr{C}/\sim &\to T_qM \\ [\gamma] &\mapsto \dot\gamma(0).\end{aligned}$$

Claim 6.6. $\tilde\Phi$ is onto.

Proof. Choose coordinates $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$ near q such that $(x_1, \dots, x_m)(q) = 0$. Now, for all $v \in T_qM$, we have $v = \sum v(x_i)\frac{\partial}{\partial x_i}|_q$. Consider $\gamma(t) = \varphi^{-1}(tv(x_1), \dots, tv(x_m))$, then $\gamma(0) = \varphi^{-1}(0) = q$ and for any $f \in C^\infty(M)$, we have

$$\begin{aligned}\dot\gamma(0)f &= \frac{d}{dt}|_0(f \circ \varphi^{-1})(tv(x_1), \dots, tv(x_m)) \\ &= \sum \frac{\partial}{\partial r_i}(f \circ \varphi^{-1})|_0 \cdot v(x_i) \\ &= \sum v(x_i)\frac{\partial}{\partial x_i}|_q f \\ &= v(f).\end{aligned}$$

$\square$

Lemma 6.7. For any smooth map $F : M \to N$ between manifolds, for all $q \in M$, we have

$$T_qF(\dot{\gamma}(0)) = (F \circ \gamma)^{\cdot}(0).$$

Proof.

$$\begin{aligned} T_qF(\dot{\gamma}(0)) &= T_qF(T_0\gamma\left(\frac{d}{dt}|_0\right)) \\ &= T_0(F \circ \gamma)\left(\frac{d}{dt}|_0\right) \\ &= (F \circ \gamma)^{\cdot}(0). \end{aligned}$$

□

Example 6.8. Let $M = N = \mathbb{C}$ and $F(z) = e^z$. We claim that $(T_zF)(v) = e^z v$, which uses $\mathbb{C} \cong T_w\mathbb{C}$ for all $w \in \mathbb{C}$. Indeed, since $\frac{d}{dt}|_0 e^{tv} = v$, then

$$\begin{aligned} (T_zF)(v) &= \frac{d}{dt}|_0 F(z + tv) \\ &= \frac{d}{dt}|_0 e^{z+tv} \\ &= \frac{d}{dt}|_0 (e^z e^{tv}) \\ &= e^z v. \end{aligned}$$

Note that T_zF is an isomorphism for all z, given by

$$\begin{tikzcd} T_z\mathbb{C} \arrow[r, "T_zF"] \arrow[d, "\cong"] & T_{F(z)}\mathbb{C} \arrow[d, "\cong"] \\ \mathbb{C} \arrow[r, "e^z \cdot -"'] & \mathbb{C} \end{tikzcd}$$

Also note that this is not a diffeomorphism, since the inverse is the complex logarithm function, which is only well-behaved on the principal branches.

7 Sept 6, 2023

Definition 7.1. Given a manifold M, $q \in M$, and $f \in C^\infty(M)$, we define the *exact differential* to be a linear map

$$\begin{aligned} df_q : T_qM &\to \mathbb{R} \\ v &\mapsto v(f) \end{aligned}$$

in $\mathrm{Hom}(T_qM, \mathbb{R}) =: T_q^*M$, the cotangent space.

Exercise 7.2.

- df_q is linear,
- $f \equiv g$ near q, then $df_q = dg_q$.

We have seen differentials before: given a coordinate chart $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ is a coordinate chart, then $\{(dx_i)_q\}_{i=1}^m$ is a basis of T_q^*M dual to $\{\frac{\partial}{\partial x_i}|_q\}_{i=1}^m$. Note that for all $\eta \in T_q^*M \equiv (T_qM)^*$, then $\eta = \sum \eta\left(\frac{\partial}{\partial x_i}|_q\right)(dx_i)_q$.

Lemma 7.3. Let M be a manifold, $q \in M$, and $f \in C^\infty(M)$, then the derivative

$$(T_qf)(v) = df_q(v)\frac{d}{dt}|_{f(q)}.$$

Proof. Note that $\{dt_{f(q)}\}$ is a basis of $T^*_{f(q)}\mathbb{R}$, then

$$dt_{f(q)}(T_q f(v)) = (T_q f(v))t = v(t \circ f) = v(f) = df_q(v),$$

so $(T_q f)(v) = df_q(v)\frac{d}{dt}|_{f(q)}$. □

Recall. Let $T : V \to W$ be a linear map, and let $\{e_1, \dots, e_n\}$ be a basis of V, and let $\{f_1, \dots, f_n\}$ be a basis of W, with dual basis $\{f_1^*, \dots, f_n^*\}$ in W^*. Then let $t_{ij} = f_i^*(Te_j)$, then

$$T(e_j) = \sum_i f_i^*(Te_j)f_i = \sum_i t_{ij} f_i.$$

For all $F : \mathbb{R}^m \to \mathbb{R}^n$, consider the coordinates $(x_1, \dots, x_m) : \mathbb{R}^m \to \mathbb{R}$ and $(y_1, \dots, y_n) : \mathbb{R}^n \to \mathbb{R}$, which gives coordinates $\{(\frac{\partial}{\partial x_i}|_q)\}$ and $\{(\frac{\partial}{\partial y_i}|_{F(q)})\}$, respectively. With $T = T_q F$, we have

$$t_{ij} = (dy_i)_{F(q)}(T_q F(\frac{\partial}{\partial x_j}|_q)) = (T_q F(\frac{\partial}{\partial x_j}|_q))y_i = \frac{\partial}{\partial x_j}|_q(y_i \circ F).$$

If we denote $F = (F_1, \dots, F_n)$ where $F_i = y_i \circ F$ then this is just $\frac{\partial F_i}{\partial x_j}(q)$, so $\left(\frac{\partial F_i}{\partial x_j}(q)\right)$ is the matrix of $T_q F$.

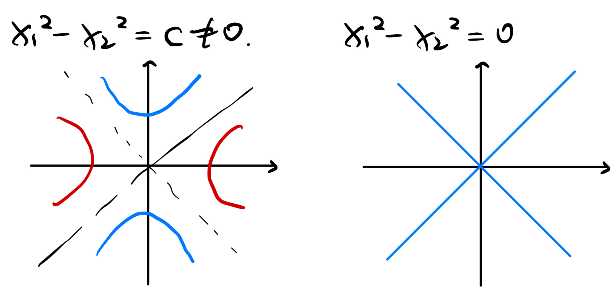
Definition 7.4. Let $F : M \to N$ be a smooth map, we say $c \in N$ is a *regular value* of F if either $F^{-1}(c) = \varnothing$, or for all $q \in F^{-1}(c)$, $T_q F : T_q M \to T_{F(q)}N = T_c N$ is onto.

We say $c \in N$ is a *singular value* if it is not a regular value.

Example 7.5. Consider

$$F : \mathbb{R}^2 \to \mathbb{R}$$
$$(x_1, x_2) \mapsto x_1 - x_2^2$$

for all $q = (x_1, x_2) \in \mathbb{R}^2$, then $T_q F$ is the matrix $\left(\frac{\partial F}{\partial x_1}(q), \frac{\partial F}{\partial x_1}(q)\right) = (2x_1, 2x_2)$. Hence, $c \neq 0$ is a regular value, and $c = 0$ is a singular value.



Definition 7.6. An *embedded submanifold* (of dimension k) of a manifold M is a subspace $Z \subseteq M$ such that for all $q \in Z$ there exists a coordinate chart $\varphi = (x_1, \dots, x_k, x_{k+1}, \dots, x_m) : U \to \mathbb{R}^m$ with $\varphi(U \cap Z) = \{(r_1, \dots, r_m) \in \varphi(U) \mid r_k = \dots = r_m = 0\}$.

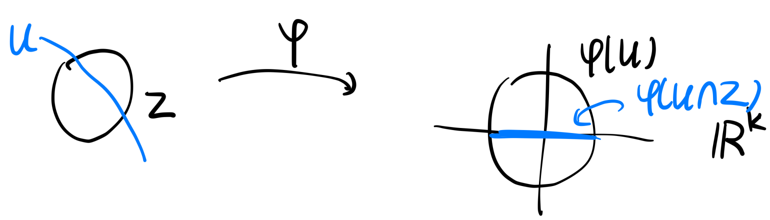


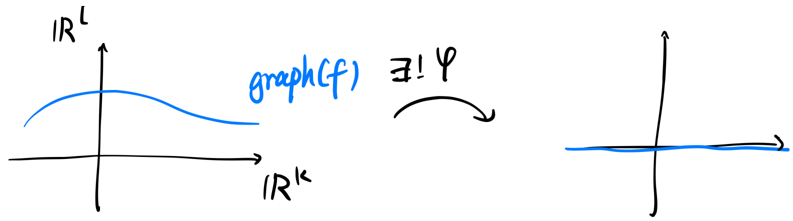
Figure 8: Embedded Submanifold

Remark 7.7.
- Any open subset $U \subseteq M$ is an embedded submanifold.
- Any singleton in M is an embedded submanifold.

Example 7.8. Let $f : \mathbb{R}^k \to \mathbb{R}^l$ be C^∞, then the graph of f is

$$\mathrm{graph}(f) = \{(x, f(x) \in \mathbb{R}^k \times \mathbb{R}^l \mid x \in \mathbb{R}^k\}$$

is an embedded submanifold of $\mathbb{R}^k \times \mathbb{R}^l$.



Here $\varphi(x, y) = (x, y - f(x))$ is a coordinate chart of $\mathbb{R}^k \times \mathbb{R}^l$ with inverse $\varphi^{-1}(x, y') = (x, y' + f(x))$.

Theorem 7.9 (Regular Value Theorem)**.** Let $c \in N$ be a regular value of smooth function $F : M \to N$. If $F^{-1}(c) = \varnothing$, then for all $q \in F^{-1}(c)$, $T_qF : T_qM \to T_qN$ is onto, so $F^{-1}(c)$ is an embedded submanifold of M. Moreover, $T_qF^{-1}(c) = \ker(T_qF)$ and $\dim(F^{-1}(c)) = \dim(M) - \dim(N)$.

Example 7.10. Consider

$$\begin{aligned} F : \mathbb{R}^m &\to \mathbb{R} \\ x &\mapsto \sum x_i^2 = ||x||^2 \end{aligned}$$

Now T_qF gives a local chart with $(2x_1, \dots, 2x_m)$. Any $c \neq 0$ is a regular value. We have $F^{-1}(c) = \{x \mid ||x||^2 = c\}$ is the sphere of radius $\sqrt{c}$ for $c > 0$. Moreover, $F^{-1}(0) = \{0\}$, an embedded submanifold, but $\dim(\{0\}) \neq \dim(\mathbb{R}^m) - \dim(\mathbb{R})$.

8 Sept 8, 2023

Recall. A subset Z of a manifold M is an embedded submanifold (of dimension k and codimension $m - k$ for $m = \dim(M)$) if for all $z \in Z$, there exists a coordinate chart $\varphi : U \to \mathbb{R}^m$ and $z \in U$ which is adapted to Z, i.e., $\varphi(U \cap Z) = \varphi(U) \cap (\mathbb{R}^k \times \{0\})$.

Remark 8.1.
- Submanifolds of codimension 0 are open subsets.
- Submanifolds of codimension $m = \dim(M)$ are discrete sets of points.

We will proceed to prove Theorem 7.9.

Remark 8.2. Once we proved $F^{-1}(c)$ is embedded and $\dim(F^{-1}(c)) = \dim(M) - \dim(N)$, then the last statement follows. Indeed, given $v \in T_q(F^{-1}(c))$, there exists $\gamma : (a, b) \to F^{-1}(c)$ such that $\gamma(0) = q, \gamma'(0) = v$, and $F(\gamma(t)) = c$ for all t. Therefore,

$$0 = \frac{d}{dt}|_0 F(\gamma(t)) = T_qF(\gamma'(0)) = T_qFv,$$

so $v \in \ker(T_qF)$, and so $T_qF^{-1}(c) \subseteq \ker(T_qF)$. By dimension argument, we have equality.

We will introduce inverse function theorem and implicit function theorem.

Theorem 8.3 (Inverse Function Theorem)**.** Let $U \subseteq \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^n$ be C^∞ with $q \in U$ such that $T_qf = Df(q) : T_qU = \mathbb{R}^n \to \mathbb{R}^n = T_{F(q)}\mathbb{R}^n$ is an isomorphism. Then there exists an open neighborhood $q \in V \subseteq U$ and $f(q) \in W$ such that $f : V \to W$ is a diffeomorphism.

Notation. Given $F:\mathbb{R}^k\times\mathbb{R}^l\to\mathbb{R}^m$ for $(a,b)\in\mathbb{R}^k\times\mathbb{R}^l$, then we denote

- $\frac{\partial F}{\partial x}(a,b)=T_{(a,b)}F|_{\mathbb{R}^k\times\{0\}}=DF(a,b)|_{\mathbb{R}^k\times\{0\}}$,
- $\frac{\partial F}{\partial y}(a,b)=T_{(a,b)}F|_{\{0\}\times\mathbb{R}^l}=DF(a,b)|_{\{0\}\times\mathbb{R}^l}$.

Theorem 8.4 (Implicit Function Theorem). Let $F:\mathbb{R}^k\times\mathbb{R}^n\to\mathbb{R}^n$ be C^∞, let $(a,b)\in\mathbb{R}^k\times\mathbb{R}^l$. Suppose $\frac{\partial F}{\partial y}(a,b):\mathbb{R}^n\to\mathbb{R}^n$ is an isomorphism, then there exists a neighborhood $W\ni(a,b)$ and $U\ni a$ in $\mathbb{R}^k$, as well as C^∞-map $g:U\to\mathbb{R}^n$ such that $F^{-1}(c)\cap W=\text{graph}(g)\cap W$.

Remark 8.5. inverse function theorem and implicit function theorem are equivalent.

Proof. Consider

$$\begin{aligned}H:\mathbb{R}^k\times\mathbb{R}^n&\to\mathbb{R}^k\times\mathbb{R}^n\\(x,y)&\mapsto(x,F(x,y))\end{aligned}$$

then $H(a,b)=(a,F(a,b))=(a,c)$. The partials give

$$DH(a,b)=\begin{pmatrix}I&0\\\frac{\partial F}{\partial x}(a,b)&\frac{\partial F}{\partial y}(a,b)\end{pmatrix}$$

As $\frac{\partial F}{\partial y}(a,b)$ is invertible, so is $DH(a,b)$, so there exists neighborhoods $(a,b)\in W\subseteq\mathbb{R}^n\times\mathbb{R}^k$ and $a\in U\subseteq\mathbb{R}^k$, $c\in V\subseteq\mathbb{R}^n$, such that $H:W\to U\times V$ is a diffeomorphism. Consider

$$\begin{aligned}G=H^{-1}:U\times V&\to W\subseteq\mathbb{R}^n\times\mathbb{R}^l\\(u,v)&\mapsto(G_1(u,v),G_2(u,v))\end{aligned}$$

therefore

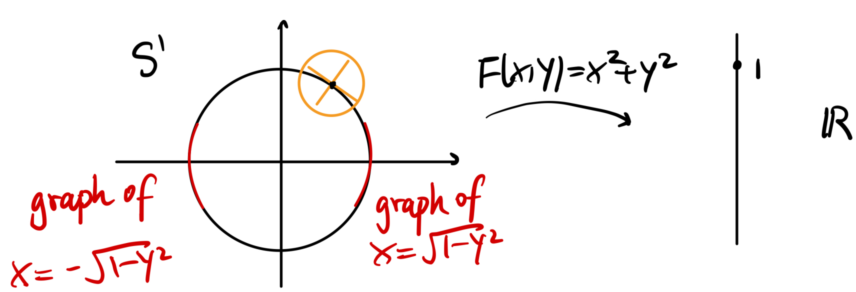
$$(u,v)=H(H^{-1}(u,v))=H(G_1(u,v),G_2(u,v))=(G_1(u,v),F(G_1(u,v),G_2(u,v))$$

so $G_1(u,v)=u$, and $v=F(u,G_2(u,v))$ for all u,v, hence $c=F(u,G_2(u,c))$ for all u. Now let $g(u)=G_2(u,c)$, then $F(u,g(u))=c$ for all u. Hence, $\text{graph}(g)\subseteq F^{-1}(c)$. □

Proof of Regular Value Theorem. Let $F:M\to N$, $c\in N$, $F^{-1}(c)\neq\varnothing$. Now for all $q\in F^{-1}(c)$, then $T_qF:T_qM\to T_qN$ is onto. Given $q\in F^{-1}(c)$, we want a chart T from a neighborhood of q to $\mathbb{R}^m$, adapted to $F^{-1}(c)$. Let $\varphi:U\to\mathbb{R}^m$ and $\psi:V\to\mathbb{R}^m$ be charts such that $q\in U$, $c\in V$, then

$$\tilde{F}=\psi\circ F\circ\varphi^{-1}|_{\varphi(F^{-1}(V)\cap U}:\varphi(F^{-1}(V)\cap U)\subseteq\mathbb{R}^m\to\mathbb{R}^n$$

is C^∞. Now $\psi(c)$ is a regular value in $\tilde{F}$, Let $r=\varphi(q)$, then we have $D\tilde{F}(r):\mathbb{R}^m\twoheadrightarrow\mathbb{R}^n$. Let $X=\ker(D\tilde{F}(r))$ and Y be a complement in $\mathbb{R}^m$. So $\mathbb{R}^m=X\otimes Y$ and $D\tilde{F}(r)|_Y:Y\to\mathbb{R}^n$ is an isomorphism. Apply inverse function theorem to $\tilde{F}$ from the intersection of $X\times Y$ and the open subset to $\mathbb{R}^n$.



□

Example 8.6. Let $\mathrm{Sym}^2(\mathbb{R}^n)$ be the $n \times n$ symmetric real matrices, also known as $\mathbb{R}^{\frac{n^2-n}{2}+n}$. There is

$$\begin{aligned} F : \mathrm{GL}_(n, \mathbb{R}) &\to \mathrm{Sym}^2(\mathbb{R}^n) \\ A &\mapsto A^T A \\ F^{-1}I = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T A = I\} &\leftarrow I \end{aligned}$$

Remark 8.7. We have $F = F \circ L_A$ for all $A \in O(U)$, then for all A, we have $T_A F$ onto.

Claim 8.8. 1 is a regular value of F, so $O(n)$ is an embedded submanifold of $\mathrm{GL}(n, \mathbb{R})$.

Proof.

$$\begin{aligned} (T_I F)(v) &= \frac{d}{dt}|_0 (I + tv)^T (I + tv) \\ &= \frac{d}{dt}|_0 (I^2 + tv^T + tv + t^2 v^T v) \\ &= v^T + v \end{aligned}$$

and this is surjective since for all $Y \in \mathrm{Sym}^2(\mathbb{R})$, we have $Y = \frac{1}{2}(Y^T + Y)$, so $Y = (T_I F)(\frac{1}{2}Y)$. $\square$

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Recall. Let $F : M \to N$ be C^∞, let $c \in N$ be a regular value such that $F^{-1}(c) \neq \varnothing$. (For all $q \in F^{-1}(c)$, $T_q F : T_q M \to T_q N$ is onto.) Then:

i $F^{-1}(c)$ is an embedded submanifold of M.

ii $\dim(M) = \dim(F^{-1}(c)) = \dim(N)$.

iii for all $q \in F^{-1}(c)$, $T_q F^{-1}(c) = \ker(T_q F)$.

The proof uses inverse function theorem and/or implicit function theorem, and the key is to note that locally $f^{-1}(c)$ is a graph.

Also, $O(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T A = I\}$ is an embedded submanifold.

Definition 9.1. A *Lie group* G is a group and a manifold so that

i the multiplication map

$$\begin{aligned} m : G \times G &\to G \\ (a, b) &\mapsto (a, b) \end{aligned}$$

is C^∞.

ii the inverse map

$$\begin{aligned} \mathrm{inv} : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

is C^∞.

Notation. $e_G = 1_G$ is the identity element.

Example 9.2. $G = \mathbb{R}^n$ with $m(v, w) = v + w$, and $\mathrm{inv}(v) = -v$ gives a Lie group.

Example 9.3. Let $G = \mathrm{GL}(n, \mathbb{R})$ be with $e_G = \mathrm{diag}(1, \ldots, 1) = I$, with maps $m(A, B) = AB$ and $\mathrm{inv}(A) = A^{-1}$.

Remark 9.4. One can think of a Lie group G as four pieces of data:

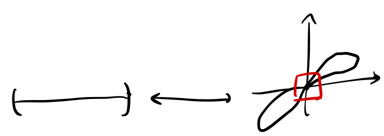
- manifold G,
- map $m : G \times G \to G$,
- map $\operatorname{inv} : G \to G$,
- $e_G \in G$.

Note that a subgroup H of a Lie group G is not necessarily a Lie group. The sufficient condition would be H is an embedded submanifold of G, i.e.,

- $m|_{H\times H} : H \times H \to H$ are C^∞,
- $\operatorname{inv}|_H : H \to H$

are C^∞. Note $m|_{H\times H} : H \times H \to G$ is C^∞ since $i : H \hookrightarrow G$ is C^∞ and $m|_{H\times H} = m(i \times i)$.

Example 9.5. For example, think of the embedding

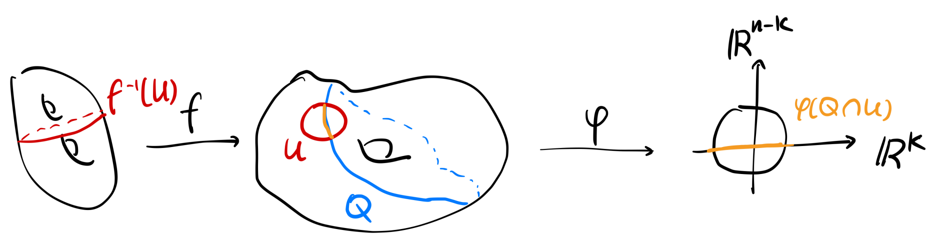


but at the origin the preimage is split into three pieces, because the inverse is not continuous, which does not embed into a submanifold.

Lemma 9.6. If $i : Q \hookrightarrow M$ is an embedded submanifold, and $f : N \to M$ is a smooth map such that $f(N) \subseteq Q$, then $g : N \to Q$ with $g(n) = f(n)$ is C^∞.

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[dr, dashed, "g"'] & M \\ & Q \arrow[u, hook, "i"'] \end{tikzcd}$$

Proof. Since $Q \hookrightarrow M$ is embedded, for all $q \in Q$, there exists an adapted chart $\varphi = (x_1, \dots, x_n, x_{k+1}, \dots, x_m) : U \to \mathbb{R}^m$ such that $Q \cap U = \{x_k = \cdots = x_n = 0\}$. Consider $\varphi \circ f|_{f^{-1}(U)} : f^{-1}(U) \to \mathbb{R}^m$, then $f(f^{-1}(U)) \subseteq Q \cap U$.



Then $\varphi \circ f|_{f^{-1}(U)} = \varphi(U \cap Q) = \{(r_1, \dots, r_k, r_{k+1}, \dots, r_m) \mid r_{k+1} = \cdots = r_n = 0\}$, so $\varphi \circ f = (h_1, \dots, h_k, 0, \dots, 0)$ where $h_1, \dots, h_k \in C^\infty(f^{-1}(U))$. Therefore, $\varphi|_{U\cap Q}\, g|_{f^{-1}(U)} = (h_1, \dots, h_k)$. $\square$

Example 9.7. $O(n) \subseteq \mathrm{GL}(n, \mathbb{R})$ is embedded, thus a Lie group.

Example 9.8. $\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid \det(A) = 1\}$ is also a Lie group.

Claim 9.9. $1 \in \mathbb{R}$ is a regular value of $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$.

Proof. The key fact is that $T_I(\det) : \mathbb{R}^{n^2} \to \mathbb{R}$ is an $(n \times n)$-matrix given by $A \mapsto \operatorname{tr}(A)$. Indeed, note that the trace is the differential of the determinant. $\square$

Definition 9.10. A (real) *Lie algebra* is a (real) vector space $\mathfrak{g}$ with an $\mathbb{R}$-bilinear map

$$\begin{aligned}[\cdot,\cdot]: \mathfrak{g}\times\mathfrak{g} &\to \mathfrak{g}\\ (X,Y) &\mapsto [X,Y]\end{aligned}$$

such that for all $X, Y, Z \in \mathfrak{g}$,

- $[Y,X] = -[X,Y]$,
- $[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$.

Example 9.11. Let $\mathfrak{g} = M_n(\mathbb{R})$, $[X,Y] = XY - YX$ is the anti-commutator.

Example 9.12. Let M be a manifold, $\mathfrak{g} = \mathrm{Der}(C^\infty(M)) = \{X : C^\infty(M) \to C^\infty(M) \mid X(fg) = X(f)\cdot g + f\cdot X(g)\}$. Therefore, $\mathfrak{g}$ is a Lie algebra with the bracket $[X,Y](f) = X(Y(f)) - Y(X(f))$ for all $f \in C^\infty(M)$. This is the Lie algebra of vector fields on M.

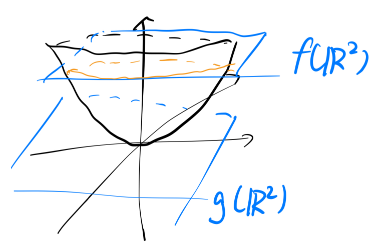
Example 9.13. Let $\mathfrak{g} = \mathbb{R}^3$, then $[v,w] := v \times w$ is a Lie algebra with cross product.

We will see that for all Lie group G, $\mathfrak{g} = \mathrm{Lie}(G) = T_eG$ is naturally a Lie algebra.

Definition 9.14. Let $F : M \to N$ be a C^∞-map, $Z \subseteq N$ be an embedded submanifold. We say F is *transverse* to Z, denoted $F \pitchfork Z$, if for all $x \in F^{-1}(Z)$, $T_xF(T_xM) + T_{F(x)}Z = T_{F(x)}N$.

Example 9.15. If $Z = \{c\}$, then $F \pitchfork c$ if and only if for all $q \in F^{-1}(c)$, $(T_xF)(T_xN) + T_cc = T_cN$, if and only if for all $q \in F^{-1}(c)$, $(T_xF)(T_xN) = T_cN$, if and only if c is a regular value of F.

Example 9.16. Let $M = \mathbb{R}^2$, $N = \mathbb{R}^3$, $Z = \{(x,y,z) \mid z = x^2 + y^2\}$, with $f(x,y) = (x,y,1)$ and $g(x,y) = (x,y,0)$, then $f \pitchfork Z$ but $g \not\pitchfork Z$.



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Theorem 10.1. Suppose $f : M \to N$ is transverse to an embedded submanifold $Z \subseteq N$, then

(i) $f^{-1}(z)$ is an embedded submanifold of M.

(ii) If $f^{-1}(z) \neq \varnothing$, then $\dim(M) - \dim(f^{-1}(z)) = \dim(N) - \dim(Z)$, i.e., $\mathrm{codim}(f^{-1}(Z)) = \mathrm{codim}(Z)$.

Proof. Fix $z_0 \in Z$ with $f^{-1}(z_0) \neq \varnothing$, let $\psi : V \to \mathbb{R}^n$ be a coordinate chart on N, adapted to Z such that $\psi(V \cap Z) = \psi(V) \cap (\mathbb{R}^k \backslash \{0\})$. Let $\pi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ be the canonical projection, then

$$(\pi\circ\psi)^{-1}(0) = \psi^{-1}(\pi^{-1}(0)) = \psi^{-1}(\psi(V) \cap (\mathbb{R}^k \times \{0\})) = Z \cap V,$$

therefore

$$(\pi\circ\psi\circ f)^{-1}(0) = f^{-1}(Z\cap V) = f^{-1}(Z) \cap f^{-1}(V).$$

Claim 10.2. 0 is a regular value of $\pi \circ \psi \circ f|_{f^{-1}}(V)$.

Subproof. Take arbitrary $x \in (\pi \circ \psi \circ f)^{-1}(0) = f^{-1}(V) \cap f^{-1}(Z)$, then $T_x f(T_x M) + T_{f(x)} Z = T_{f(x)} N$. Note that $T_x M = T_x(f^{-1}(V))$. Therefore,

$$\mathbb{R}^k \times \mathbb{R}^{n-k} = T_{f(x)}\psi(T_{f(x)}N) = T_{f(x)}\psi(T_x f(T_x f^{-1}(V))) + T_{f(x)}\psi(T_{f(x)}Z)$$

by applying $T_{f(x)}\psi$ on both sides. Now apply $T_{\psi(f(x))}\psi$ on both sides, then $T_{f(x)}\psi(T_{f(x)}Z)$ vanishes, so we get

$$\begin{aligned} \mathbb{R}^{n-k} &= T_{\psi(f(x))}\pi(T_{f(x)}\psi(T_x f(T_x f^{-1}(V)))) \\ &= T_x(\pi \circ \psi \circ f)(T_x f^{-1}(V)). \end{aligned}$$

$\blacksquare$

$\square$

Definition 10.3. A C^∞-map $f : Q \to M$ is an *embedding* if

(i) $f(Q) \subseteq M$ is an embedded submanifold, and

(ii) $f : Q \to f(Q)$ is a diffeomorphism.

Remark 10.4. We know $f : Q \to f(Q)$ is C^∞ since $f(Q) \subseteq M$ is embedded and $f : Q \to M$ is given by the composition of $i : f(Q) \hookrightarrow M$ and $f : Q \to f(Q)$.

Remark 10.5.

1. Since $f : Q \to f(Q)$ is a diffeomorphism, then it is a homeomorphism. Thus $f : Q \to M$ is a topological embedding.
2. For all $q \in Q$, then $T_q f : T_q Q \to T_{f(q)} M$ is injective, i.e., $T_q f(T_q Q) = T_{f(q)} f(Q)$.

Example 10.6 (Non-example)**.** Let $Q = \mathbb{R}$ with discrete topology, then Q is a paracompact but not second countable as a 0-dimensional manifold. Consider

$$\begin{aligned} f : Q &\to \mathbb{R}^2 \\ x &\mapsto (x, 0) \end{aligned}$$

be a C^∞-map, then this is not an embedding.

Example 10.7. Let M be a manifold with $f \in C^\infty(M)$, then

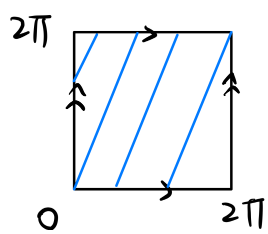
$$\begin{aligned} g : M &\to M \times \mathbb{R} \\ q &\mapsto (q, f(q)) \end{aligned}$$

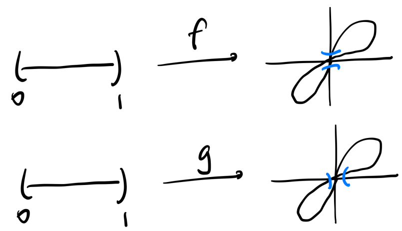
gives an embedding of M into $R \times \mathbb{R}$, as the graph of f.

Definition 10.8. A C^∞-map $f : Q \to M$ is an *immersion* if for all $q \in Q$, $T_q f : T_q Q \to T_{f(q)} M$ is injective.

Example 10.9. Consider

$$\begin{aligned} f : \mathbb{R} &\to S^1 \times S^1 \\ \theta &\mapsto (e^{i\theta}, e^{i\sqrt{2}\theta}) \end{aligned}$$





Example 10.10. Now $g \circ f^{-1} : (0,1) \to (0,1)$ is not an embedding, as it is not continuous.

Definition 10.11. The *rank* of a C^∞-map $f : M \to N$ at a point $q \in M$ is the rank of the linear map $T_q f : T_q M \to T_{f(q)} N$, i.e., $\operatorname{rank}_q(f) = \dim(T_q f(T_q M))$.

Example 10.12. If $f : M \to N$ is an immersion, then $\operatorname{rank}_q(f) = \dim_q(M)$.

Remark 10.13. Immersions are embeddings.

Theorem 10.14 (Rank Theorem)**.** Let $F : M \to N$ be a C^∞-map of constant rank k. Then for all $q \in M$, there exists coordinates $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ on M with $q \in U$, and $\psi = (y_1, \ldots, y_n) : V \to \mathbb{R}^n$ with $F(q) \in V$ such that $(\psi \circ F \circ \varphi^{-1})(r_1, \ldots, r_m) = (r_1, \ldots, r_k, 0, \ldots, 0)$ for all $r = (r_1, \ldots, r_m) \in \varphi(F^{-1}(V) \cap U)$.

Notation. Given a collection of sets $\{S_\alpha\}_{\alpha \in A}$, $\coprod\limits_{\alpha \in A} S_\alpha$ is the disjoint union of the collection.

We will give the following construction of a tangent bundle.

Remark 10.15. Given a manifold M, we form a set $TM = \coprod\limits_{q \in M} T_q M$. Given a chart $\varphi = (x_1, \ldots, x_n) : U \to \mathbb{R}^m$ on M, the corresponding candidate chart is $\tilde{\varphi} : TU = \coprod\limits_{q \in U} T_q M \to \varphi(U) \times \mathbb{R}^m$. One can check that if $\varphi : U \to \mathbb{R}^m$ and $\psi : V \to \mathbb{R}^m$ are charts on M with $U \cap V \neq \varnothing$, then $\tilde{\psi}\tilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^m \to \psi(U \cap V) \times \mathbb{R}^m$ is C^∞. Now we give TM the topology making $\tilde{\varphi}$'s homeomorphic onto their images, then $\{\tilde{\varphi} : TU \to \varphi(U) \times \mathbb{R}^m\}$ will be an atlas on TM.

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Definition 11.1. A map $f : M \to N$ is a *submersion* if for all $p \in M$, the differential $T_q f : T_q M \to T_{f(q)} N$ is onto.

Remark 11.2. Every value over a submersion is regular.

Recall. For a manifold M, we defined the set $TM = \coprod\limits_{q \in M} T_q M = \bigcup (\{q\} \times T_q M)$, which is a called a tangent bundle, with additional structures. We will show that TM is a manifold, and

$$\begin{aligned} \pi : TM &\to M \\ (q, v) &\mapsto q \end{aligned}$$

is C^∞ and a submersion.

Proof. Let $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ be a coordinate chart on M. For any $q \in U$, let $\left\{ \frac{\partial}{\partial x_1}\Big|_q, \ldots, \frac{\partial}{\partial x_m}\Big|_q \right\}$ be a basis of $T_q M$. The dual basis is $\{(dx_1)_q, \ldots, (dx_m)_q\}$. For any $v \in T_q M$, we have $v = \sum v(x_i) \left.\frac{\partial}{\partial x_i}\right|_q := \sum (dx_i)_q(v) \left.\frac{\partial}{\partial x_i}\right|_q$, and

$$\begin{aligned} T_q M &\to \mathbb{R} \\ v &\mapsto ((dx_1)_q(v), \ldots, (dx_m)_q(v)) \end{aligned}$$

is a linear isomorphism. Define

$$\begin{aligned}\tilde{\varphi} : TU = \coprod_{q \in M} T_q M &\to \mathbb{R}^m \times \mathbb{R}^m \\ (q, v) &\mapsto (x_1(q), \ldots, x_m(q), (dx_1)_q(v), \ldots, (dx_m)_q(v)).\end{aligned}$$

Suppose $\psi = (y_1, \ldots, y_m) : V \to \mathbb{R}^m$ is another chart, we then have

$$\begin{aligned}\tilde{\psi} : TV &\to \mathbb{R}^m \times \mathbb{R}^m \\ (q, v) &\mapsto (y_1(q), \ldots, y_m(q), (dy_1)_q(v), \ldots, (dy_m)_q(v)).\end{aligned}$$

Claim 11.3. For any $(r, w) \in \varphi(U \cap V) \times \mathbb{R}^m$, we have

$$\begin{aligned}(\tilde{\psi} \circ \tilde{\varphi}^{-1})(r, w) &= ((\psi \circ \varphi^{-1})(r), \sum_j \frac{\partial y_1}{\partial x_j}(\varphi^{-1}(r))w_i, \ldots, \sum_j \frac{\partial y_m}{\partial x_j}(\varphi^{-1}(r))w_i) \\ &= \left((\psi \circ \varphi^{-1})(r), \left(\frac{\partial y_i}{\partial x_j}(\varphi^{-1}(r))\right)\begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}\right)\end{aligned}$$

Subproof.

Recall. If $T : A \to B$ is a linear map, with $\{e_1, \ldots, e_n\}$ basis of A, $\{f_1, \ldots, f_n\}$ is a basis of B, with dual basis $\{f_1^*, \ldots, f_n^*\}$, then we set $t_{ij} = f_u^*(Te_j)$, i.e.,

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "(t_{ij})"] \arrow[d, "(v_1,\ldots,v_n)\mapsto\sum v_i e_i"'] & \mathbb{R}^n \arrow[d] \\ A \arrow[r, "T"'] & B \end{tikzcd}$$

In our case, we have $A = B = T_q M$ with $T = \mathrm{id}$, with basis $\left\{ \frac{\partial}{\partial x_i}\Big|_q \right\}$ of A, $\{f_1, \ldots, f_n\} = \left\{ \frac{\partial}{\partial y_1}\Big|_q, \ldots, \frac{\partial}{\partial y_m}\Big|_q \right\}$ and dual basis $\{f_1^*, \ldots, f_m^*\} = \{(dy_1)_q, \ldots, (dy_m)_q\}$, then

$$\begin{aligned}t_{ij} &= (dy_i)_q \left(\frac{\partial}{\partial x_j}\Big|_q \right) \\ &= \frac{\partial}{\partial x_j}(y_i)(q) \\ &= \frac{\partial y_i}{\partial x_j}(\varphi^{-1}(\gamma)).\end{aligned}$$

■

We define the topology on TM to be the topology generated by the sets of form $\tilde{\varphi}^{-1}(W)$ where $\varphi : U \to \mathbb{R}^m$ is a coordinate chart with open subset $W \subseteq \mathbb{R}^m \times \mathbb{R}^m$. Given an atlas $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^m\}$ on M, we get an induced atlas $\{\tilde{\varphi}_\alpha : TU_\alpha \to \mathbb{R}^m \times \mathbb{R}^m\}$ on TM. One can check that the choice of an atlas on M does not matter. □

Exercise 11.4.

- If M is Hausdorff, then so is TM.
- If M is second countable, then so is TM.

Lemma 11.5. The canonical projection $\pi : TM \to M$ is C^∞ and is a submersion.

Proof. Let $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ be a coordinate chart, $\tilde{\varphi} : TU \to \mathbb{R}^m \times \mathbb{R}^m$ be the induced chart on TM, then

$$\begin{aligned}(\varphi \circ \pi \circ \tilde{\varphi}^{-1})(r, w) &= \varphi \circ \pi \left(\varphi^{-1}(r), \sum_i w_i \left.\frac{\partial}{\partial x_i}\right|_q \right) \\ &= \varphi(\varphi^{-1}(r)) \\ &= r.\end{aligned}$$

Moreover,

$$\left(T_{(r,w)}(\varphi \circ \pi \circ \tilde{\varphi}^{-1})\right)(v, w') = v$$

where $(v, w') \in T_{(r,w)}(\varphi(U) \times \mathbb{R}^m) \cong \mathbb{R}^n \times \mathbb{R}^m$. Therefore, $T_{(q,v)}\pi : T_{(q,v)}TM \to T_qM$ is onto, hence a submersion. □

Definition 11.6. A *(algebraic) vector field* on a manifold M is a derivation $v : C^\infty(M) \to C^\infty(M)$, i.e., v is $\mathbb{R}$-linear and $v(fg) = v(f)g + fv(g)$ for all $f, g \in C^\infty(M)$.

Definition 11.7. A *(geometric) vector field* on a manifold M is a section of the tangent bundle TM of M, i.e., $X : M \to TM$ is C^∞ with $\pi \circ X = \mathrm{id}_M$. Geometrically, this depicts tangent vectors over a point with directions in $X(q)$.

Notation.
- $\mathrm{Der}(C^\infty(M))$ is the set of all derivations of $C^\infty(M)$.
- $\mathfrak{X}(M) = \Gamma(TM)$ is the set of sections of $\pi : TM \to M$.

Proposition 11.8. Given a section $v : M \to TM$ in $\mathfrak{X}(M)$, we can try and define

$$\begin{aligned}D_v : C^\infty(M) &\to C^\infty(M) \\ (D_v(f))(q) &\mapsto v(q)f\end{aligned}$$

and this assignment $v \mapsto D_v$ is a linear isomorphism.

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Recall. $TM = \coprod_{q \in M} T_qM$ is a manifold. To show this, given chart $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ on M, we set

$$\begin{aligned}\tilde{\varphi} = (x_1, \ldots, x_m, dx_1, \ldots, dx_m) : TU \equiv \coprod_{q \in U} T_qM &\to \mathbb{R}^m \to \mathbb{R}^m \\ (q, v) &\mapsto (\varphi(q), (dx_1)_q(v), \ldots, (dx_m)_q(v))\end{aligned}$$

with inverse

$$\tilde{\varphi}^{-1}(r, u) = (\varphi^{-1}(r), \sum w_i \left.\frac{\partial}{\partial q_i}\right|_{\varphi(r)}.$$

Also,

$$\begin{aligned}\pi : TM &\to M \\ (q, v) &\mapsto q\end{aligned}$$

is a C^∞-submersion.

We defined vector fields in two ways,

- as sections of tangent bundle $\pi : TM \to M$, i.e., as C^∞-maps $X : M \to TM$ such that $\pi X = \mathrm{id}$, i.e., $X(q) \in T_qM$, and
- as derivations $c : C^\infty(M) \to C^\infty(M)$, i.e., as $\mathbb{R}$-linear maps such that $v(fg) = fv(g) + v(f)g$ for all $f, g \in C^\infty(M)$.

Remark 12.1. Both $\Gamma(TM)$ and $\mathfrak{X}(M)$ are $\mathbb{R}$-vector spaces, and $C^\infty(M)$-modules.

We now prove Proposition 11.8.

Proof. Given $v \in \Gamma(TM)$ and $f \in C^\infty(M)$, consider a function

$$\begin{aligned} D_v f &: M \to \mathbb{R} \\ (D_v(f))(q) &= v(q)f \end{aligned}$$

To go back, given $X \in \mathrm{Der}(C^\infty(M))$, for any $q \in M$, we have $\mathrm{ev}_q : C^\infty(M) \to \mathbb{R}$, and then $\mathrm{ev}_q \circ X : C^\infty(M) \to \mathbb{R}$ is a tangent vector. Define $v_X(q) = \mathrm{ev} \circ X$, and we can check other requirements like C^∞ and so on.

Claim 12.2. $D_v f$ is C^∞.

Subproof. Given a chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$, we have

$$\begin{aligned} \tilde{\varphi} &: TU \to \mathbb{R}^m \times \mathbb{R}^m \\ (q, v) &\mapsto (\varphi(q), dx_1(v), \dots, dx_m(v)) \end{aligned}$$

Since v is C^∞, the map $\tilde{\varphi} \circ v|_U : U \to \mathbb{R}^m \times \mathbb{R}^m$, defined by $(\tilde{\varphi} \circ v)(q) = (\varphi(q), (dx_1)_q(v(q)), \dots, (dx_m)_q(v(q)))$, is C^∞. Therefore, the assignment $q \mapsto (dx_i)_q(v(q))$ are C^∞ on U. Hence, $v = \sum v_i \frac{\partial}{\partial x_i}$ where $v_i(q) = (dx_i)_q(v(q))$ for all i. So $(D_v f)|_U = \left(\sum v_i \frac{\partial}{\partial x_i}\right) f = \sum v_i \frac{\partial f}{\partial x_i}$. This concludes the proof. ■

Also, for all $f, g \in C^\infty(M)$ and all q, we have

$$\begin{aligned} (D_v(fg))(q) &= v(q)(fg) \\ &= (v(q)f)g(q) + f(q)(v(q)g) \\ &= ((D_v f)g + f(D_v g))(q). \end{aligned}$$

Recall that derivations are local, i.e., for $X \in \mathrm{Der}(C^\infty(M))$ and $f \in C^\infty(M)$ and $f|_U \equiv 0$, then $Xf|_U \equiv 0$. As a consequence, for $U \subseteq M$ open, define $X|_U : C^\infty(U) \to C^\infty(U)$ such that $(X|_U)(f|_U) = (Xf)|_U$ for all $f \in C^\infty(M)$. Now given a chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$, we know x_i's are in $C^\infty(U)$, then $(X|_U)(x_i)$ is a smooth function on U. Therefore,

$$\begin{aligned} v_X|_U &= \sum (dx_i)(v_X) \frac{\partial}{\partial x_i} \\ &= \sum v_X X(x_i) \frac{\partial}{\partial x_i} \\ &= \sum X|_U (x_i) \frac{\partial}{\partial x_i}, \end{aligned}$$

and thus $v_X|_U : U \to TU$ is C^∞, and since U is arbitrary, then $v_X \in \Gamma(TM)$. □

Recall. For any $X, Y \in \mathrm{Der}(C^\infty(M))$, $[X, Y] \in \mathrm{Der}(C^\infty(M))$. Therefore, $\mathrm{Der}(C^\infty(M))$ is a real Lie algebra with bracket $(X, Y) \mapsto [X, Y]$. Note that $\mathrm{Der}(C^\infty(M)) \subseteq \mathrm{Hom}_{\mathbb{R}}(C^\infty(M), C^\infty(M))$.

Recall. If $(A, \circ)$ is a real associative algebra, then $[a, b] := a \circ b - b \circ a$ gives A the structure of a Lie algebra, and $\mathrm{Der}(C^\infty(M)) \subseteq \mathrm{Hom}_{\mathbb{R}}(C^\infty(M), C^\infty(M))$.

Now given a C^∞-map $f : M \to N$ of manifolds, we get a map

$$\begin{aligned} Tf &: TM \to TN \\ (q, v) &\mapsto (f(q), T_q f v) \end{aligned}$$

Exercise 12.3. Tf is C^∞.

Remark 12.4. Given $f : M \to N$ and $v \in \Gamma(TM)$, we may not have a commutative diagram:

$$\begin{tikzcd} TM \arrow[r, "Tf"] & TN \\ M \arrow[u, "v"] \arrow[r, "f"'] & N \arrow[u, dashed, "?"'] \end{tikzcd}$$

Definition 12.5. Let $f : M \to N$ be a smooth map on manifolds, then $v \in \Gamma(TM)$ and $w \in \Gamma(TN)$ are *f-related* if we have a commutative diagram

$$\begin{tikzcd} TM \arrow[r, "Tf"] & TN \\ M \arrow[u, "v"] \arrow[r, "f"'] & N \arrow[u, "w"'] \end{tikzcd}$$

That is, for any $q \in M$, $w(f(q)) = (f(q), T_q f(v(q)))$.

Equivalently, for $f : M \to N$, we say $X \in \mathrm{Der}(C^\infty(M))$ is *f-related* to $Y \in \mathrm{Der}(C^\infty(N))$ if for all $h \in C^\infty(N)$, we have $Y(h) \circ f = X(h \circ f)$ in $C^\infty(M)$.