Stick Numbers in the Simple Hexagonal Lattice

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Overview

Preliminary Knowledge

Definition (Simple Hexagonal Lattice)

For vectors $x =< 1, 0, 0 >$, $y =< \frac{1}{2}, \frac{\sqrt{3}}{2}, 0 >$, and $w =< 0, 0, 1 >$, we define the Simple Hexagonal (sh) Lattice as all linear combinations of x, y, and w. In other words,
sh $= \{a < 1, 0, 0 > +b < \frac{1}{2}, \frac{\sqrt{3}}{2}, 0 > +c < 0, 0, 1 > \mid a, b, c \in \mathbb{Z}\}$.
For convenience, we also define $z =< -\frac{1}{2}, \frac{\sqrt{3}}{2}, 0 >$. Note that $z = y - x$.

Preliminary Knowledge

Definition (Stick)

An α-stick in the α direction is a maximal segment in polygon $\mathcal{P}$.

The number of x-, y-, z-, and w-sticks in a polygon $\mathcal{P}$ will be denoted $|\mathcal{P}|_x$, $|\mathcal{P}|_y$, $|\mathcal{P}|_z$, and $|\mathcal{P}|_w$, respectively, and the total number of sticks used will be $|\mathcal{P}|$. Such a polygon, when closed and non-intersecting, is an *sh* lattice knot. The stick number of a knot type K in the lattice, denoted $s[K]$, is the minimum number of sticks required to form a polygon of type K.

Lower Bound for Stick Numbers

Theorem (Lower Bound for sh-Lattice Stick Numbers)

For any knot K in the simple hexagonal lattice, $s[K] \geq 5b[K]$.

Lower Bound for Stick Numbers

Proposition (Lower Bound for Cubic Lattice Stick Numbers)

For any knot K in the cubic lattice, $s[K] \geq 6b[K]$.

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Proof.

See *[Van Rensburg, EJ Janse, and S. D. Promislow, 1999]* for a detailed proof. Recall that the bridge number $b[K]$ in the lattice of a knot K is defined to be "the minimum number of local maxima of any projection of a knot onto any single vector", so we conclude that there are at least $b[K]$ (sticks taking value of) local maximums in each of the three directions. Note that, for example, a local maximum in the z-direction (i.e. parallel to the xy-plane) is connected to two z-sticks. Hence, there are at least $2b[K]$ z-sticks in K. Similarly, we know that there are at least $2b[K]$ x-sticks and $2b[K]$ y-sticks in K. In particular, the lower bound for the number of sticks in a cubic lattice is $6b[K]$, i.e. $s[K] \geq 6b[K]$. □

Lower Bound for Stick Numbers

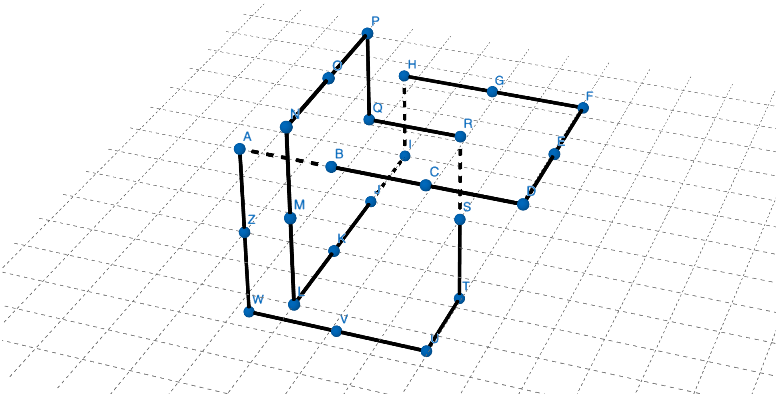

Figure: Trefoil Knot in Cubic Lattice

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Proof.

The proof is similar to the previous proposition. Suppose we have a stick that takes maximum on the w-direction (i.e. a stick in the xy-plane), then it must have two w-sticks attached at the ends of it. Because there are at least $b[K]$ local maxima, we have $|\mathcal{P}|_w \geq 2b[K]$.
Now suppose we have a maxima occurred in an xw-plane, then the sticks attached to this maxima have to be y-sticks or z-sticks. Again, because of the limitation of number of local maximas, we have $|\mathcal{P}|_y + |\mathcal{P}|_z \geq 2b[K]$.
Similarly, we have the same restriction on the yw-plane and the zw-plane.
We thereby conclude the following system of equations:

Lower Bound for Stick Numbers

Theorem (Lower Bound for sh-Lattice Stick Numbers)

For any knot K in the simple hexagonal lattice, $s[K] \geq 5b[K]$.

Proof. (Cont.)

$$\begin{cases} |\mathcal{P}|_w & \geq 2b[K], \\ |\mathcal{P}|_y + |\mathcal{P}|_z & \geq 2b[K], \\ |\mathcal{P}|_x + |\mathcal{P}|_z & \geq 2b[K], \\ |\mathcal{P}|_x + |\mathcal{P}|_y & \geq 2b[K]. \end{cases} \tag{1}$$

In particular, we have

$$s[K] = |\mathcal{P}|_x + |\mathcal{P}|_y + |\mathcal{P}|_z + |\mathcal{P}|_w \geq 5b[K].$$

Lower Bound for Stick Numbers

Corollary

If $|\mathcal{P}| = 5b[K]$, *then* $|\mathcal{P}|_x = |\mathcal{P}|_y = |\mathcal{P}|_z = \frac{1}{2}|\mathcal{P}|_w = b[K]$.

Lower Bound for Stick Numbers

Corollary

If $|\mathcal{P}| = 5b[K]$, *then* $|\mathcal{P}|_x = |\mathcal{P}|_y = |\mathcal{P}|_z = \frac{1}{2}|\mathcal{P}|_w = b[K]$.

Proof.

By the theorem, the stick numbers must satisfy *(1)*:

$$\begin{cases} |\mathcal{P}|_w & \geq 2b[K], \\ |\mathcal{P}|_y + |\mathcal{P}|_z & \geq 2b[K], \\ |\mathcal{P}|_x + |\mathcal{P}|_z & \geq 2b[K], \\ |\mathcal{P}|_x + |\mathcal{P}|_y & \geq 2b[K]. \end{cases}$$

Suppose the corollary is false, then one of $|\mathcal{P}|_w \neq 2b[K]$, $|\mathcal{P}|_x \neq b[K]$, $|\mathcal{P}|_y \neq b[K]$, and $|\mathcal{P}|_z \neq b[K]$ must hold. Suppose $|\mathcal{P}|_w \neq 2b[K]$, then by *(1)* we know $|\mathcal{P}|_w > 2b[K]$ and $|\mathcal{P}|_x + |\mathcal{P}|_y + |\mathcal{P}|_z \geq 3b[K]$.

Lower Bound for Stick Numbers

Corollary

If $|\mathcal{P}| = 5b[K]$*, then* $|\mathcal{P}|_x = |\mathcal{P}|_y = |\mathcal{P}|_z = \frac{1}{2}|\mathcal{P}|_w = b[K]$*.*

Proof. (Cont.)

However, that means $|\mathcal{P}| > 5b[K]$, contradiction. Hence, $|\mathcal{P}|_w = 2b[K]$, so $|\mathcal{P}|_x + |\mathcal{P}|_y + |\mathcal{P}|_z = 3b[K]$ and one of $|\mathcal{P}|_x \neq b[K]$, $|\mathcal{P}|_y \neq b[K]$, and $|\mathcal{P}|_z \neq b[K]$ must hold. In particular, one of the three stick numbers must be strictly less than $b[K]$. Without loss of generality, say $|\mathcal{P}|_x < b[K]$, so $|\mathcal{P}|_x = b[K] - n$ for some integer $n > 0$. By *(1)* we conclude that $|\mathcal{P}|_y \geq b[K] + n$ and $|\mathcal{P}|_z \geq b[K] + n$. But then

$$|\mathcal{P}| = |\mathcal{P}|_x + |\mathcal{P}|_y + |\mathcal{P}|_z + |\mathcal{P}|_w = 5b[K] + n > 5b[K],$$

contradiction. This concludes the proof. □

Stick Number of the Lattice

Corollary

In the simple hexagonal lattice, the stick number of any nontrivial knot is at least 10*.*

Proof.

Recall that any nontrivial knot K has stick number $b[K] \geq 2$, then by the theorem we conclude that $s[K] \geq 5 \times 2 = 10$ for any nontrivial knot K. □

Stick Number of the Lattice

Theorem (Minimum Stick Number in the Simple Hexagonal Lattice)

In the simple hexagonal lattice, the stick number of any nontrivial knot is at least 11.

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Proof.

By the previous corollary and the fact that we can draw a trefoil knot with 11 sticks, it suffices to show that any knot K with $s[K] = 10$ is an unknot. Take a knot K with $s[K] = 10$. By the theorem, we know $b[K] \leq 2$. Suppose K is not an unknot, then we have $b[K] = 2$. In particular, that means $s[K] = 10 = 5 \times 2 = 5b[K]$, then by the first corollary we proved, we know this knot must have 2 x-sticks, 2 y-sticks, 2 z-sticks and 4 w-sticks. We now consider the projection of K onto the xy-plane, which should be a projection of 6 sticks. Recall that a projection with less than 3 crossings is equivalent to an unknot, so the projection has at least 3 crossings.

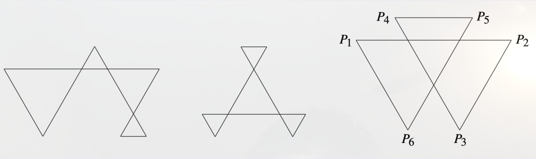
Stick Number of the Lattice

Theorem (Minimum Stick Number in the Simple Hexagonal Lattice)

In the simple hexagonal lattice, the stick number of any nontrivial knot is at least 11*.*

Proof. (Cont.)

Note that up to permutation and rotation, the sticks must be in one of the following two orders: $\diagup\diagdown - \diagup\diagdown -$ or $\diagup\diagdown\diagup - \diagdown -$. By drawing out all possible such projections, we reach the conclusion that all such projections must have 3 crossing, and are drawn below:



Stick Number of the Lattice

Theorem (Minimum Stick Number in the Simple Hexagonal Lattice)

In the simple hexagonal lattice, the stick number of any nontrivial knot is at least 11.

Proof. (Cont.)

Note that the first and second projections are equivalent to the unknot obviously. As for the last projection, it suffices to show it does not have alternating crossings. We use the labelling as in the figure. Without loss of generality, suppose that P_1P_2 on level i crosses over P_3P_4 on level j, then that means $i > j$. Because the crossings are alternating, we know that P_3P_4 on level j crosses over P_5P_6 on level k and P_5P_6 crosses over P_1P_2. Again, by the interpretation of crossing on levels, we conclude that $i > j > k > i$, contradiction. Therefore, all knots of 10 sticks in the simple hexagonal lattice must be an unknot. □

Stick Number of the Lattice

Corollary

The trefoil knot is the knot that requires the least number of sticks in the sh-lattice while being nontrivial.

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Proof.

The following projection of the trefoil knot takes exactly 11 sticks in the sh-lattice, then the corollary follows from the theorem.

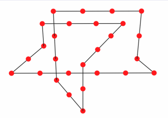

Figure: Trefoil Knot Projection in sh-Lattice

□

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