# MATH 595 (Group Cohomology) Notes

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## 1 Aug 21, 2023: Introduction

Group cohomology works over different settings of groups, like finite groups, profinite groups, and topological groups. The course will develop towards

- duality in $H^*(G, -)$, and
- focus on computations, e.g., spectral sequences.

We first establish some notations.

- Let $G$ be a group. If $G$ has a topology, that would also be part of the information of $G$.
- A (left) $G$-module is an abelian group $M$ with an action map

$$\begin{aligned} G \times M &\to M \\ (g, m) &\mapsto g \cdot m = gm \end{aligned}$$

  satisfying

  - $1 \cdot m = m$,
  - $(gh) \cdot m = g \cdot (hm)$,
  - $g(m + m') = gm + gm'$.

**Remark 1.1.** If $G$ is a finite group, then the associated (non-commutative) group ring $\mathbb{Z}[G] := \bigoplus_{g \in G} \mathbb{Z}e_g$, where the multiplication is determined by $e_g e_h = e_{gh}$. Therefore, a $G$-module is just a $\mathbb{Z}[G]$-module.

**Example 1.2.**

- Trivial module $\mathbb{Z}$, or any abelian group with the trivial action $g \cdot a = a$.
- $C_2$, or any group with $f : G \twoheadrightarrow C_2$, then $G$ with $C_2$ as a quotient gives the sign representation $\mathbb{Z}_{\text{sgn}}$, with $g \cdot (a) = (-1)^{\rho(g)} a$.
- $\mathbb{Z}[G]$ is a $G$-module via the left multiplication action, and/or the conjugation action.

**Definition 1.3** (Fixed points/Invariants)**.** The set of fixed points of $M$ over $G$ is $M^G = \{m \in M \mid gm = m \; \forall g \in G\}$.

**Definition 1.4** (Orbits/Coinvariants)**.** The set of orbits of $M$ over $G$ is $M_G = M/(gm - m)$.

**Example 1.5.** If $M = \mathbb{Z}_{\text{sgn}}$, then everything gets multiplied by $-1$, so there are no fixed points. The orbits of $M$ over $G$ would be $\mathbb{Z}_{\text{sgn}}/(-2) \cong \mathbb{Z}/2\mathbb{Z}$.

**Example 1.6.** If $M = \mathbb{Z}[G]$, then the fixed points are $\mathbb{Z}\left\{\sum_{g \in G} e_g\right\}$.

Thinking in a categorical setting, we have a trivial action function $\mathbb{Z}$-**Mod** $\to G$-**Mod**, sending $ga \mapsto a$ for all $g \in G$ and $a \in A$. This gives an exact functor from **Ab** to $G$-**Mod**. Then this functor has a right adjoint $(\ )^G : G$-**Mod** $\to$ **Ab**, and a left adjoint $(\ )_G :$ **Ab** $\to G$-**Mod**. More specifically, $M^G$ becomes the maximal trivial action submodule of $M$, namely $\mathrm{Hom}_G(\mathbb{Z}, M)$; $M_G$ becomes the largest quotient of $M$ with trivial action, namely $\mathbb{Z} \otimes_{\mathbb{Z}[G]} M$. This simplifies to the tensor-hom adjunction in some sense. For a more detailed derivation of this, see Chapter 6.1 of Weibel.

**Remark 1.7.** In general, as in the category of $G$-sets, we have the orbit functor $X \mapsto X/G$ and the fixed point functor $X \mapsto X^G$. The orbit functor is left adjoint to the free $G$-set functor, and the fixed point functor is the right adjoint of the trivial $G$-set functor.

**Remark 1.8.** Read more about the setting in profinite groups with their topologies in Neukirch-Schmidt-Wingberg.

**Definition 1.9** (Profinite Group)**.** A profinite group of a collection of groups is $G = \varprojlim_i G_i$ as an inverse limit, where each $G_i$ is a finite group of the form $G/U_i$ for some open $U_i$. This gives a topology to the profinite group.

**Remark 1.10.** The groups rings $\mathbb{Z}[[G]] = \varprojlim_i \mathbb{Z}[G_i]$. For instance, let $G = \hat{\mathbb{Z}_p} = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$, then $\mathbb{Z}_p[[G]] = \varprojlim_n \mathbb{Z}_p[\mathbb{Z}/p^n\mathbb{Z}]$, where each $\mathbb{Z}[\mathbb{Z}/p^n\mathbb{Z}] \cong \bigoplus_{i=0}^{n-1} \mathbb{Z}\{e_i\}$ where $e_i \cdot e_j = e_{ij}$. Therefore, $\mathbb{Z}_p[[G]]$ is now equivalent to $\varprojlim_n \mathbb{Z}_p[t]/(t^{p^n} - 1_e)$, and hence becomes a power series.

**Remark 1.11.** By a change of variables, this becomes $\varprojlim_n \mathbb{Z}_p[x]/(x^{p^n})$, but this only works in the finite group $\mathbb{Z}_p$ case, and not in general for $\mathbb{Z}$.

**Example 1.12.** $\mathbb{Z}[C_n] \cong \mathbb{Z}\{e\} \oplus \mathbb{Z}\{g\} \oplus \mathbb{Z}\{g^2\} \oplus \cdots \oplus \mathbb{Z}\{g^{n-1}\} \cong \mathbb{Z}[g]/(g^n - 1_e)$.

## 2 Aug 23, 2021: Cohomology of groups

**Definition 2.1.** Let $G$ be a group, then we have a diagram

$$EG^{\cdot} : \cdots \Rrightarrow G \times G \rightrightarrows G$$

where the arrows are given by

$$EG^n = G^{n+1} \xrightarrow{d_i} G^n$$

for all $0 \leqslant i \leqslant n$. In the sense of simplicial sets, we have $d_i(g_0, \ldots, g_n) = (g_0, \ldots, \hat{g}_i, \ldots, g_n)$.

Now let $M$ be a $G$-module, then we define $X^n = X^n(G, M) = \mathrm{Map}_{\mathbf{Set}}(G^{n+1}, M)$. $G$ now has an action on this set, given by

$$(g \circ f)(g_0, \ldots, g_n) = gf(g^{-1}g_0, \ldots, g^{-1}g_n).$$

The action on $d^i$'s are contravariant, namely we obtain $d_i^* : X_n \to X^{n+1}$ with an inherited structure. Note that $M$ sits inside $X^0$, therefore we have a complex $(*)$:

$$0 \longrightarrow M \xhookrightarrow{\partial_0} X^0 \xrightarrow{\partial_1} X^1 \xrightarrow{\partial_2} X^2 \xrightarrow{\partial_3} \cdots$$

Here $\partial_0$ includes $M$ as the constant functions into $X$, namely $\partial_0(m) = f$ for $f(g) = m$, and so on. In general, for $n > 0$, we have

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i^*.$$

**Lemma 2.2.** The complex $(*) : M \to X^{\cdot}$ is an exact complex of $G$-modules, i.e., $\partial^2 = 0$ and $\ker(\partial_{n+1}) = \mathrm{im}(\partial_n)$, and the $\partial_i$'s preserves the $G$-action. This is called the standard resolution of $M$ as a $G$-module.

*Proof.* Exercise. □

**Definition 2.3.** The $G$-fixed points of the $X^n$'s are defined by $C^n(G,M) = (X^n(G,M))^G$, called the homogeneous $n$-cochains of $G$ with coefficients in $M$. Because the complex preserves $G$-actions, then we obtain a complex of $C^n(G,M)$'s, given by

$$0 \longrightarrow C^0(G,M) \xrightarrow{\partial_0} C^1(G,M) \xrightarrow{\partial_1} \cdots$$

**Remark 2.4.** To see what the induced mapping is, suppose $A \to B$ is a $G$-module map, then there is an induced map of fixed points $A^G \to B^G$ by the restriction. In particular, let $a \in A$ be fixed with $ga = a$ for all $g \in G$, then $f(a) = f(ga) = gf(a)$.

**Remark 2.5.** In the complex of Definition 2.3, $\partial^2 = 0$ as well, but in general this is not an exact sequence.

**Definition 2.6** (Group Cohomology)**.** The group cohomology of $G$ with coefficients in $M$ is the collection

$$\{H^n(G,M)\}_{n \geqslant 0},$$

where $H^n(G,M) := H^n(C^\cdot(G,M)) = \ker(\partial : C^n \to C^{n+1})/\operatorname{im}(\partial : C^{n-1} \to C^n)$. We usually use the notion of cocycles $Z^n(G,M) = \ker(\partial : C^n \to C^{n+1})$ and coboundaries $B^n(G,M) = \operatorname{im}(\partial : C^{n-1} \to C^n)$.

**Exercise 2.7.** Show that $H^0(G,M)$ is isomorphic to $M^G$.

**Definition 2.8.** The inhomogeneous cochains $C_i^\cdot(G,M)$ are given by

- $C_i^0 = M$, and
- for $n > 0$, $C_i^n = \operatorname{Map}(G^n, M)$,

with coboundary maps $\partial^{n+1} : C_i^n \to C_i^{n+1}$, given by

- $\partial^1(m)(g) = gm - m$,
- $\partial^2(f)(g_1,g_2) = g_1 f(g_2) - f(g_1g_2) + f(g_1)$, and so on, with
- $\partial^{n+1}(f)(g_1,\dots,g_{n+1}) = g_1 f(g_2,\dots,g_{n+1}) + \sum_{i=1}^{n}(-1)^i f(g_1,\dots,g_ig_{i+1},\dots,g_{n+1}) + (-1)^{n+1} f(g_1,\dots,g_n)$.

This gives the inhomogeneous setting of this cochain.

**Lemma 2.9.** The maps

$$\begin{aligned} C^n(G,M) &\to C_i^n(G,M) \\ (\varphi : G^{n+1} \to M) &\mapsto (f : G^n \to M) \\ & \qquad f(g_1,\dots,g_n) := \varphi(1, g_1, g_1g_2, \dots, g_1g_2\cdots g_n) \end{aligned}$$

give a cochain homotopy equivalence $C^\cdot(G,M) \xrightarrow{\sim} C_i^\cdot(G,M)$, and hence this is a quasi-isomorphism.

**Corollary 2.10.** The cohomology $H^*(C_i^\cdot(G,M)) \cong H^*(G,M)$.

**Remark 2.11.** Any cohomology class can be represented by a normalized inhomogeneous cocycle $f : G^n \to M$, i.e., $f(g_1,\dots,g_n) = 0$ where $g_i = 1$ for some $i$.

**Remark 2.12.** Even for $G = C_2$, $C_i^n$ or $C^n$ get large as $n$ grows.

**Remark 2.13.**

- Using homological algebra, we can find other cochain complexes which computes group cohomology $H^*(G,M)$.
- We would also understand $H^*(G,M)$ as the failure of exactness of $(\ )^G : G\text{-}\mathbf{Mod} \to \mathbf{Ab}$. Therefore, when taking the fixed points, the exact sequence may not be mapped to another exact sequence. In particular, if we take an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of $G$-modules, the induced sequence

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G$$

do not give a surjection at $B^G \to C^G$. One needs to take higher cohomology to obtain a long exact sequence. Hence, $(\ )^G : G\text{-}\mathbf{Mod} \to \mathrm{Ab}$ is a left exact functor, but not necessarily right exact.

## 3 Aug 25, 2021: Cohomology of groups, continued

**Example 3.1.** Let $G$ be $C_2$, or any group with a surjection $p$ onto $C_2$, then it has an action on $\mathbb{Z}_{\text{sgn}}$ given by $g \cdot a = (-1)^{p(g)} a$, therefore we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}_{\text{sgn}} \xrightarrow{\times 2} \mathbb{Z}_{\text{sgn}} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

and taking the fixed point functor we have

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

**Remark 3.2.** Higher homologies measure the failure of exactness.

**Remark 3.3.** The collection $\{H^n(G,-)\}_{n\in\mathbb{Z}}$ satisfies

- $H^n(G,-) = 0$ for $n < 0$;
- for short exact sequence $0 \to A \to B \to C \to 0$ in $G$-**Mod**, we have a long exact sequence

  $$0 \longrightarrow H^0(G,A) \longrightarrow H^1(G,B) \longrightarrow H^1(G,C) \xrightarrow{\delta} H^1(G,A) \longrightarrow \cdots$$

  where $\delta$ is the connecting homomorphism.
- the connecting homomorphisms $\delta$ are natural, i.e., given a commutating diagram

  $$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

  the induced diagram

  $$\begin{tikzcd} H^n(G,C) \arrow[r, "\delta"] \arrow[d] & H^{n+1}(G,A) \arrow[d] \\ H^n(G,C') \arrow[r, "\delta"'] & H^{n+1}(G,A') \end{tikzcd}$$

  also commutes, and $\{H^n(G,-)\}_{n\in\mathbb{Z}}$ is a cohomological $\delta$-functor. Note that a $\delta$-functor is additive, and usually occurs in abelian categories.

**Definition 3.4** ($\delta$-functor)**.** A map of $\delta$-functors $T^* \to F^*$ is a collection of natural transformations $T^n \to F^n$, commuting with the $\delta$'s, i.e.,

$$\begin{tikzcd} T^n \arrow[r] \arrow[d, "\delta_T"'] & F^n \arrow[d, "\delta_F"] \\ T^{n+1} \arrow[r] & F^{n+1} \end{tikzcd}$$

A $\delta$-functor $T^*$ is universal if, given any other $\delta$-functor $F^*$, a map $T^* \to F^*$ is uniquely determined by $T^0 \to F^0$.

**Proposition 3.5.** $H^*(G,-) : G$-**Mod** $\to$ **Ab** is a $\delta$-functor.

*Proof.* We need to show:

- each $H^n(G,-)$ is a well-defined functor,
- the connecting homomorphisms $\delta$'s gives a long exact sequence,
- the naturality of $\delta$.

First, let $f: A \to B$ be in $G$-**Mod**, then $C^*(G,A) \to C^*(G,B)$ is equivalent to $\mathrm{Map}(G^{*+1},A)^G \to \mathrm{Map}(G^{*+1},B)^G$ by composition with $f$. One can show that this is equivariant, i.e., respects the $G$-action, so it is well-defined to take the fixed points, and thus commutes with $\partial$'s.

Second, we need to apply the snake lemma. Given a short exact sequence $0 \to A \to B \to C \to 0$, we claim:

**Claim 3.6.** $0 \longrightarrow C^*(G,A) \longrightarrow C^*(G,B) \longrightarrow C^*(G,C) \longrightarrow 0$ is a short exact sequence of cochain complexes, i.e., $C^*(G,-): G\text{-}\mathbf{Mod} \to \mathbf{coCh}$ is an exact functor.

*Subproof.* Exercise. ■

Now take the complex

$$\begin{tikzcd}
0 \arrow[r] & C^n(G,A) \arrow[r] \arrow[d, "\partial"] & C^n(G,B) \arrow[r] \arrow[d, "\partial"] & C^n(G,C) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C^{n+1}(G,A) \arrow[r] & C^{n+1}(G,B) \arrow[r] & C^{n+1}(G,C) \arrow[r] & 0
\end{tikzcd}$$

and quotient the boundaries everywhere (and thus lose the injectivity/surjectivity when applicable)

$$\begin{tikzcd}
 & C^n(G,A)/B^n(G,A) \arrow[r] \arrow[d, "\partial"] & C^n(G,B)/B^n(G,B) \arrow[r] \arrow[d, "\partial"] & C^n(G,C)/B^n(G,C) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & Z^{n+1}(G,A) \arrow[r] & Z^{n+1}(G,B) \arrow[r] & Z^{n+1}(G,C) &
\end{tikzcd}$$

Taking the kernels and cokernels on $\partial$'s, we obtain a complex

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & H^n(G,A) \arrow[r] \arrow[d] & H^n(G,B) \arrow[r] \arrow[d] & H^n(G,C) \arrow[d] & \\
 & C^n(G,A)/B^n(G,A) \arrow[r] \arrow[d, "\partial"] & C^n(G,B)/B^n(G,B) \arrow[r] \arrow[d, "\partial"] & C^n(G,C)/B^n(G,C) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & Z^{n+1}(G,A) \arrow[r] \arrow[d] & Z^{n+1}(G,B) \arrow[r] \arrow[d] & Z^{n+1}(G,C) \arrow[d] & \\
 & H^{n+1}(G,A) \arrow[r] & H^{n+1}(G,B) \arrow[r] & H^{n+1}(G,C) &
\end{tikzcd}$$

By the snake lemma, we obtain the long exact sequence. □

**Proposition 3.7.** If $0 \to A \to B \to C \to 0$ is a short exact sequence such that $H^*(G,B) = 0$ for $* > 0$ (or at least $H^n(G,B) = 0 = H^{n+1}(G,B)$), then $\delta: H^n(G,C) \to H^{n+1}(G,A)$ is an isomorphism.

**Definition 3.8** (Acyclic, Cohomologically Trivial)**.** A $G$-module $M$ is

- acyclic if $H^*(G,M) = 0$ for $* > 0$,
- cohomologically trivial if $H^*(H,M) = 0$ for $* > 0$ and any (closed) subgroup $H \subseteq G$.

**Definition 3.9** (Induced Module)**.** Given any $G$-module $M$, the induced module $\mathrm{ind}_G(M) = \mathrm{Map}(G,M) = X^0(G,M)$.

**Example 3.10.** $M$ could have the trivial action.

**Exercise 3.11.** For any $M$, the induced module of $M$ over $G$ is isomorphic (under the $G$-action) to the induced module of module given by forgetful action over $G$.

**Remark 3.12.**

- $\mathrm{Ind}_G(-): G\text{-}\mathbf{Mod} \to G\text{-}\mathbf{Mod}$ is exact.

- We say $A$ is an induced module if $A \cong \mathrm{Ind}_G(M)$ for some module $M$. If $A$ is an induced $G$-module, then $A$ is induced as an $H$-module for any subgroup $H \subseteq G$.

**Lemma 3.13.** Induced modules are cohomologically trivial.

*Proof.* There is an isomorphism
$$C^*(G, \mathrm{Ind}_G(M)) \cong X^*(G, M).$$
□

**Remark 3.14.** We have an equivariant inclusion of fixed points
$$M \hookrightarrow \mathrm{Ind}_G(M)$$
which is an embedding, and we take $Q \cong \mathrm{Ind}_G(M)/M$, then this extends to a short exact sequence
$$0 \longrightarrow M \hookrightarrow \mathrm{Ind}_G(M) \longrightarrow Q \longrightarrow 0$$
then $H^{n+1}(G, M) \cong H^n(G, Q)$. One say that $H^*(G, -)$ is effaceable. By Tohoku, an effaceable is universal.

# 4 Aug 28, First Cohomology of Groups

There are three ways to think about $H^1(G, M)$.

## 4.1 Crossed Homomorphims

Recall that $H^1(G, M) = Z_i^1(G, M)/B_i^1(G, M)$ as inhomogeneous cochains, where

- $Z_i^1(G, M) = \ker(\mathrm{Map}(G, M) \to \mathrm{Map}(G \times G, M)$ where the map sends $f \mapsto (g, h) \mapsto gf(h) - f(gh) + f(g)$. The kernel of this is exactly the maps $f$ such that $f(gh) = gf(h) + f(g)$, and note that this is not a group homomorphism.
- $B_i(G, M) = \mathrm{im}(M \to \mathrm{Map}(G, M))$ given by $m \mapsto (g \mapsto gm - m)$, where the image is called a principal crossed homomorphism.

**Exercise 4.1.** $B_i^1(G, M) \cong M/M^G$ as an isomorphism of $\mathbb{Z}[G]$-modules.

**Remark 4.2.** If the $G$-action is trivial, then $H^1(G, M) = \mathrm{Hom}_{\mathrm{Grp}}(G, M)$.

**Corollary 4.3.** If $G$ is a finite group with trivial action, then $H^1(G, \mathbb{Z}) = 0$.

**Theorem 4.4** (Hilbert's Theorem 90)**.** Let $L/K$ be a Galois extension with (finite or profinite) Galois group $G$, then $H^1(G, L^\times) = 0$.

*Proof.* Let $f : G \to L^\times$ be a crossed homomorphism. We know the addition is given by $f(gh) = gf(h) + f(g)$, and the multiplication is given by $f(gh) = (g \cdot f(h))f(g)$, where $\cdot$ represents the group action. Now for any $l \in L^\times$, the multiplication with respect to $l$ is given by $m_l = \sum_{h \in G} f(h)(h \cdot l)$. We can first choose $l$ so that $m_l \neq 0$, since the Galois conjugates $h \cdot l$ over $l \in L$ are linearly independent. For $g \in G$, we have
$$\begin{aligned} g \cdot m_l &= \sum_{h \in G} (g \cdot f(h))(gh \cdot l) \\ &= \sum_{h \in G} \frac{f(gh)}{f(g)}(gh \cdot l) \\ &= \frac{1}{f(g)} \sum_{h \in G} f(gh)(gh \cdot l) \\ &= \frac{1}{f(g)} m_l. \end{aligned}$$
Therefore, $f(g) = \frac{m_l}{g \cdot m_l}$. For any crossed homomorphism, there exists $m \in L^\times$ such that $f(g) = \frac{gm}{m}$, so every crossed homomorphism is principal. □

**Exercise 4.5.** Let $G$ acts over a commutative ring $R$, then $H^1(G, R^\times)$ classifies invariant $R$-modules with a compatible $G$-action.

### 4.2 Non-abelian $H^1$ and Torsors

Let $A$ be a group with $G$-action, so let the action $g \cdot a = {}^g a$. Hence, $g \cdot (ab) = {}^g a {}^g b$. Define the $G$-cocycles to be $f : G \to A$ such that $f(gh) = f(g)^g f(h)$. Two cocycles $f$ and $f'$ are said to be cohomologous as $f \sim f'$ if there exists $a \in A$ such that for all $g \in G$, $f'(g) = a^{-1} f(g)^g a$. This becomes an equivalence relation on the set of $G$-cocycles with coefficients in $A$, then $H^1(G, A)$ is the set of equivalence classes of $G$-cocycles. Now the first cohomology $H^1(G, A)$ has only a pointed set structure with distinguished point $f \equiv 1$, the constant function at $1$.

**Exercise 4.6.** This definition is equivalent to the inhomogeneous cochain definition in the abelian case.

**Definition 4.7.** An $A$-torsor is a $G$-set $X$ with action

$$\begin{aligned} X \times A &\to A \\ (x, a) &\mapsto xa \end{aligned}$$

that is free and transitive, i.e., for any $x, y \in G$, there exists a unique $a \in A$ such that $y = xa$. Moreover, the action $X \times A \to X$ respects the $G$-action, i.e., ${}^g(xa) = {}^g x {}^g a$.

**Remark 4.8.**

- $A$ is an $A$-torsor.
- An isomorphism of $A$-torsors is a bijection that respects the $G$- and $A$- action.
- If $A \subseteq B$ is a sub-$G$-group, then $bA$ is an $A$-torsor.
- An $A$-torsor is a principal $A$-bundle on the classifying space $BG$.

**Theorem 4.9.** There is a canonical bijection of pointed sets

$$H^1(G, A) \cong \mathrm{Torsor}(G, A)$$

*Proof.*

- The backwards map $\lambda : \mathrm{Torsor}(G, A) \to H^1(G, A)$ is defined as follows: for $x \in \mathrm{Torsor}(G, A)$, we want to define a cocycle $f(X) : G \to A$. For arbitrary $x \in X$, note that for any $g \in G$, there exists a unique $f_x(g) \in A$ such that ${}^g x = x f_x(g)$ by the simple transitivity of the $A$-action on $X$. To see this is well-defined, if we have another $y \in X$, then $y = xb$ for some $b \in A$, then $f_y(g) = b^{-1} f_x(g)^g b$, so $f_x$ and $f_y$ are cohomologous and define the same class in $H^1(G, A)$, which is defined to be the image $\lambda(X)$.
- To define $\mu : H^1(G, A) \to \mathrm{Torsor}(G, A)$, given a cocycle $f : G \to A$, let $X_f$ be the group $A$, then the action of $A$ on $X_f$ is by multiplication on the right, and one can twist the $G$-action on it using cocycle $f : G \to A$ with ${}^{\bar{g}} x = f(g) g x$, which defines an $A$-torsor. This is well-defined.

$\square$

**Remark 4.10.** Suppose

$$1 \longrightarrow A \longrightarrow B \xrightarrow{p} C \longrightarrow 1$$

is a short exact sequence of $G$-groups, i.e., $A$ is a sub-$G$-group and $C \cong B/A$, then there is a long exact sequence

$$1 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \xrightarrow{\delta} H^1(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, C)$$

where $\delta$ is given by $\delta(c) = p^{-1}(c)$. For the exactness in the sense of pointed sets to work, the kernel is the subset mapping to the distinguished element.

### 4.3 Extension Splitting

Consider the a split extension

$$1 \longrightarrow A \longrightarrow E \xrightarrow{p} G \longrightarrow 1$$

That is, $E$ is the direct product $A \times G$ with group action $(a, g)(a', g') = (a^g a', gg')$, and by definition $E$ is the semidirect product $A \rtimes G$. Equivalently, there exists a section (as group homomorphism) $s : G \to E$.

There is an equivalence relation on the set of sections to the projection $p : E \to G$, where the sections $s, s' : G \to E$ are conjugates if there exists $a \in A$ such that $s'(g) = a^{-1} s(g) a$. We denote $\sec(E \to G)$ to be the conjugacy class of sections of $p$. Note that the class of trivial section $s : g \mapsto (1, g) \in E$ is the distinguished element.

**Proposition 4.11.** The pointed set $H^1(G, A)$ is isomorphic to $\sec(E \to G)$.

*Proof.* Take $\varphi \in \sec(E \to G)$, then the composition $G \xrightarrow{\varphi} E \xrightarrow{\pi_1} A$, where $\pi_1$ is the set-theoretic projection to the first component, defines a cocycle $G \to A$. Conversely, given a cocycle $f : G \to A$, the section is given by $g \mapsto (f(g), g)$. $\square$

**Exercise 4.12.** Expand the proof above.

**Exercise 4.13.** Describe $\mathbb{Z} \rtimes C_2$ where $C_2$ acts on $\mathbb{Z}$ by inversion. How many sections are there of $\mathbb{Z} \rtimes C_2 \to C_2$?

**Exercise 4.14.** How many sections are there to the projection $D_{2n} \to C_2$?