UCLA Algebra Seminar

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Preliminary Information

This is a summary document for the UCLA Participating Algebra Seminar (i.e., MATH 290C, *Participating Seminar: Current Literature in Algebra*) in Winter 2023. The seminar is organized by Logan Hyslop, focusing on semisimple and reductive groups. Most of the information in this section are taken from the syllabus document.

Resources:

- Chapter 6 of *The Book of Involutions* ([boi])
- Milne's Notes on *Reductive Groups* ([rd])
- Milne's Book *Algebraic Groups: the theory of group schemes of finite type over a field* [(ag)], with a preliminary version available.

Seminar Lineup:

1. **Introduction.** Group schemes, subgroups, connected component of the identity, examples in group theory, c.f. section 20 of [boi].
2. **Specific Kinds of Groups and Lie Algebras.** Diagonalizable group schemes, groups of multiplicative type, Lie Algebra and smoothness, c.f. section 20 and 21 of [boi].
3. **Factor Groups.** Factor groups, representations, representations of diagonizable group schemes, c.f. section 22 of [boi].
4. **Root Systems, Split Semisimple and Reductive Groups.** Root systems, semisimple, reductive, split groups, c.f. section 25 of [boi], section 14-15 of [rd].

5. **Root Systems, Split Semisimple and Reductive Groups.** Split semisimple groups, root systems for split semisimple and split reductive groups, c.f. section 25 of [boi], section 18-19 of [rd].

6. **Semisimple and Reductive Groups over Arbitrary Fields.** Classification of semisimple groups over an arbitrary field, classification of reductive groups, c.f. section 24 and 26 of [rd].

1 Affine Group Scheme: Feb. 8, 2023

Definition 1.1 (Group Scheme)**.** An S-scheme G together with the unit map $u : S \times G \to G$, the inverse map $i : G \times G \to G$, and the multiplication map $m : G \otimes_S G \to G$, is a (affine) *group scheme* over S if the following diagrams commutes:

$$\begin{tikzcd}
G \times G \times G \arrow[r, "\mathrm{id}\times m"] \arrow[d, "m\times\mathrm{id}"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r, "\cong"] & G \times S \arrow[r, "\mathrm{id}\times u"] \arrow[d, "\cong"'] \arrow[ddr, "\mathrm{id}"] & G \times G \arrow[dd] \\
& S \times G \arrow[d, "u\times\mathrm{id}"'] & \\
& G \times G \arrow[r] & G
\end{tikzcd}$$

$$\begin{tikzcd}
& G \times G \arrow[rr, "i\times\mathrm{id}"] & & G \times G \arrow[dr, "m"] & \\
G \arrow[ur, "\Delta"] \arrow[rr] \arrow[dr] & & S \arrow[rr, "u"] & & G \\
& G \times G \arrow[rr, "\mathrm{id}\times u"'] & & G \times G \arrow[ur, "m"'] &
\end{tikzcd}$$

Proposition 1.2. If T is a S-scheme, then

$$G(T) := \mathbf{Hom}(T, G)$$

is a group.

Remark 1.3. A *group scheme homomorphism* $\rho : G \to H$ over S is a morphism of schemes respecting the multiplication map. The group schemes and group scheme homomorphisms form a category, the category of group schemes over S.

Definition 1.4 (Hopf Algebra)**.** Let F be a field and A be a commutative F-algebra. Take $G = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(F)$, then there are dual maps co-unit $u : A \to F$, co-inverse $i : A \to A$, and co-multiplication $c : A \to A \otimes_F A$. Suppose the dual diagrams of Definition 1.1 commutes, then A is a *Hopf algebra* over F. In particular, $\mathbf{Hom}_{F\text{-}\mathbf{Alg}}(A, -)$ are groups.

Example 1.5. Let $\mathbf{1} = \mathrm{Spec}(F)$. Note that F is trivially an F-Hopf algebra. In particular, $\mathbf{1}(T) = \{e\}$. We have $c : F[x] \to F[y, z]$ via $x \mapsto y + z$, $u : x \mapsto 0$, and $i : x \mapsto -x$ over $G_a = \mathrm{Spec}(F[x])$, where $G_a(A)$ is the additive group $(A, +)$.

Let $G_m = \operatorname{Spec}(F[x, x^{-1}])$ with $c : x \mapsto x \otimes x$, we have $G_m(A)$ as the multiplicative group $(A^\times, \cdot)$. Now G_a and G_m correspond to $\mathbb{A}^1$ and $\mathbb{A}^1 \backslash \{0\}$, respectively.

Example 1.6. In general, let V be a F-vector space, then the symmetric algebra $S(V^*)$ of the dual space V^* satisfies

$$\mathbf{Hom}_{\mathbf{Alg}_F}(S(V^*), R) = \mathbf{Hom}_F(V^*, R) = V \otimes_F R,$$

and $S(V^*)$ represents $R \mapsto (R \otimes_F V, +)$, and so $\operatorname{Spec}(S(V^*))$ is a group scheme.

Example 1.7. Given an F-algebra A, $\mathrm{GL}_1(A)$ is described $\mathrm{GL}_1(A)(R) = (A \otimes R)^\times$, represented by $S(A^*)\left[\frac{1}{N}\right]$. In general, $\mathrm{GL}_n(F)$ is given by $F\left[X_{ij} \mid 1 \leq i, j \leq n\right]\left[\frac{1}{\det(X)}\right]$.

Remark 1.8. An (affine) group scheme G over F is a functor $G : \mathbf{Alg}_F \to \mathbf{Grp}$ isomorphic to $\mathbf{Hom}_{\mathbf{Alg}_F}(A, -)$ for some Hopf algebra A over F. By the Yoneda Lemma, the Hopf algebra A is uniquely determined by G (up to isomorphism), and is therefore denoted $F[G]$. In this sense, a group scheme homomorphism is a natural transformation of functors.

Remark 1.9 (Correspondence)**.**

$$\begin{aligned} \text{Group Schemes over } F &\Longleftrightarrow \text{Commutative Hopf Algebras over } F \\ G &\mapsto F[G] \\ G^A &\mapsfrom A \end{aligned}$$

defines an equivalence of categories.

Definition 1.10 (Algebraic Group Scheme)**.** An (affine) group scheme A is algebraic if its coordinate ring is a finitely-generated F-algebra.

Remark 1.11. If L/K is a field extension, then any L-algebra is a K-algebra, and so we say $A \otimes L$ is the restriction of A to L.

Definition 1.12 (Closed Subgroup, Normal)**.** A *closed subgroup* H of a group scheme G is just a closed subscheme such that u, i, m restrict down to H. In particular, if G is affine, i.e., $G = \operatorname{Spec}(A)$, then $H = \operatorname{Spec}(A/J)$ where J is a Hopf ideal, i.e., $c(J) \subseteq J \otimes A + A \otimes J$.

We now denote $H \subseteq G$. We say $H \subseteq G$ is *normal* if $H(T) \subseteq G(T)$ is normal for all spectrums of F-algebras T.

Remark 1.13 (Trivial Subgroup)**.** Every Hopf algebra has an augmented ideal $I = \ker(u : A \to F)$, which is a Hopf ideal. Therefore, $\mathbf{1} \to \operatorname{Spec}(A)$ always gives a trivial normal subgroup.

Remark 1.14 (Kernel)**.** Let $\varphi : G \to H$ be a morphism of group schemes (over discrete groups). The kernel is constructed as the pullback:

$$\begin{tikzcd} \ker(\varphi) \arrow[r] \arrow[d] & G \arrow[d] \\ \mathbf{1} \arrow[r] & H \end{tikzcd}$$

Example 1.15. Consider $F[x, x^{-1}] \to F[x, x^{-1}]$ given by $x \mapsto x^k$, then by applying the spectrum functor, we obtain a morphism of group schemes $\exp(k) : G_m \to G_m$. The kernel of this map is the unit group μ_k.

Now $\mathcal{O}(\mu_k) = F[x, x^{-1}] \otimes_{F[x^k, x^{-k}]} F = F[x]/(x^k - 1)$.

Recall if G is a Lie group, then $\pi_0(G)$ is a group and there is a surjection $G \twoheadrightarrow \pi_0(G)$ whose kernel is the connected component of identity (in the Zariski sense). There is a similar story for group schemes.

Definition 1.16 (étale)**.** A finitely-generated F-algebra A is *étale* if $A \cong L_1 \times \cdots \times L_n$ where L_i/F is separable.

Example 1.17. $A \otimes_F F^{\text{sep}} \cong (F^{\text{sep}})^n$.

Proposition 1.18. If A is a finitely-generated F-algebra, then A has a unique maximal étale subgroup $\pi_0(A)$.

If $B \subseteq A$ is étale, then $\dim_F(B)$ is bounded by the number of essential idempotents, and the compositum of two étale subalgebras is again étale. Therefore, there is a maximal étale subalgebra.

Further, if A is a Hopf algebra, then $\pi_0(A)$ is also a Hopf algebra, then the identity of an affine group scheme is the kernel of the map

$$G \to \operatorname{Spec}(\pi_0 \mathcal{O}(G)) =: \pi_0(G).$$

Proposition 1.19. If G is a finite group and $A := \mathbf{Hom}(G, F) \cong F[e_g \mid g \in G]$, then A has a Hopf algebra structure given by $A \ni e_g \mapsto \sum\limits_{hk=g} e_h \otimes e_k$. Furthermore, $\operatorname{Spec}(A(T)) \cong G$ for Zariski connected T.

2 Groups and Lie Algebras: Feb. 15, 2023

In this section, we construct all concepts upon a based field F.

2.1 Diagonalizable Groups

Definition 2.1 (Diagonalizable Group Schemes)**.** Let H be an Abelian group, then the functor $R \to \mathbf{Hom}(H, R^\times)$ is representable by H_{diag}, for $R \in \mathbf{Alg}_F$. That is, we have $H_{\text{diag}}(R) = \mathbf{Hom}(H, R^\times)$. In particular, H_{diag} is the group scheme representing the group algebra $F\langle H\rangle$ over F to be the the group algebra of H, given by $c(h) = h \otimes h$, $i(h) = h^{-1}$, and $u(h) = 1$. Therefore, this gives a Hopf algebra structure on the group algebra. Group schemes of the form H_{diag} are called diagonalizable.

Remark 2.2. Note that the elements in $F\langle H\rangle$ are given of the form $h \otimes h$ for $h \in H$. Therefore, we have a natural isomorphism from $(H_{\text{diag}})^*$ and H.

Example 2.3. $\mathbb{Z}_{\text{diag}} = G_m$ and $(\mathbb{Z}/n\mathbb{Z})_{\text{diag}} = \mu_n$.

Definition 2.4 (Multiplicative Type)**.** We say a group scheme G is of multiplicative type if $G_{\text{sep}} := G_{F_{\text{sep}}}$ is diagonalizable.

Remark 2.5. In particular, diagonalizable group schemes are of multiplicative type.

Let $\Gamma = \text{Gal}(F^{\text{sep}}/F)$ and H be an Abelian group equipped with a continuous Γ-action, then $R \to \mathbf{Hom}_\Gamma(H, (R \otimes_F F^{\text{sep}})^\times)$ is represented by H_{mult}. In cash, $H_{\text{mult}}(R) = \mathbf{Hom}_\Gamma(H, (R \otimes_F F_{\text{sep}})^\times)$.

Proposition 2.6. There is an equivalence of categories between

- group schemes of multiplicative type over F, and
- Abelian groups with a continuous Γ-action,

defined by two contravariant functors, with $G \mapsto (G_{\text{sep}})^* = \mathbf{Hom}_{\mathbf{GrpSch}}(G_{\text{sep}}, G_m)$ for group scheme G, and with $H \mapsto H_{\text{mult}}$ for Abelian group H.

Remark 2.7. In particular, diagonalizable group schemes correspond to Abelian groups with trivial Γ-action.

2.2 Lie Algebra

Let G be an algebraic group scheme.

Definition 2.8 (Lie Algebra)**.** The Lie algebra of G, denoted $\mathrm{Lie}(G)$ is the tangent space of G at the identity.

Proposition 2.9. $\mathrm{Lie}(G) = (I/I^2)^*$ where I is the augmentation ideal of A, the Hopf Algebra of G, i.e., $G = \mathrm{Spec}(A)$.

Proof. A_I is the local ring at the identity, so $\mathrm{Lie}(A) = (IA_I/I^2A_I)^* = (I/I^2)^*$. □

Definition 2.10 (Derivator)**.** If A is an F-algebra and M is an A-module, then a derivator D from A to M is a F-linear map $A \to M$ satisfying $D(fg) = fD(g) + gD(f)$.

Proposition 2.11. There are natural isomorphisms between

1. Lie group G,
2. $\mathrm{der}(A, F)$, where F is an A-algebra using $u : A \to F$ with $D(fg) = u(f)D(g) + u(g)D(f)$,
3. The left-invariant derivations, given by $\{D \in \mathrm{Der}(A, A) \mid c \circ D = (\mathrm{id} \otimes D) \circ c\}$,
4. $\ker(G(F[\varepsilon]) \to G(F))$ where $F[\varepsilon]$ is the dual numbers over F given by $\varepsilon^2 = 0$. This is a kernel of groups induced by $F[\varepsilon] \to F$. In particular, the kernel carries a natural F-vector space structure: the addition operation is the multiplication on $G(F[\varepsilon])$, and the scalar multiplication operation is defined by the following action: for $a \in F$ and g in the kernel, $a \cdot g = G(l_a)g$, where $l_a : F[\varepsilon] \to F[\varepsilon]$ is the multiplication map defined by $\varepsilon \mapsto a\varepsilon$.

Proposition 2.12. $\mathrm{Lie}(G)$ has finite dimension over F.

Proof. Note that A is Noetherian, so I is finite-dimensional, so I/I^2 is finite-dimensional over $A/I = F$, so $(I/I^2)^*$ is finite-dimensional over F. □

Remark 2.13. The Lie group has the following properties.

1. $\mathrm{Lie}(G_1 \times G_2) = \mathrm{Lie}(G_1) \times \mathrm{Lie}(G_2)$.
2. $\mathrm{Lie}(G^0) = \mathrm{Lie}(G)$.
3. Given a field extension L/F, $\mathrm{Lie}(G_L) = \mathrm{Lie}(G) \otimes_F L$.

Remark 2.14. In particular, the Lie bracket is induced via the structure on vector space and the bracket on deriviators given by $[D_1, D_2] = D_1D_2 - D_2D_1$.

Example 2.15.

1. For the general linear group $\mathrm{GL}_n(R)$, the associated Lie algebra is $\mathfrak{gl}_n = M_n(F)$.
2. For the standard linear group $\mathrm{SL}_n(R)$, the associated Lie algebra is $\mathfrak{sl}_n = \{M \in M_n(F) \mid \mathrm{tr}(M) = 0\}$.
3. For the orthogonal group $\mathrm{O}_n(R)$, the associated Lie algebra is $\mathfrak{O}_n = \{M \in M_n(F) \mid M + M^T = 0\}$.

   In particular, for even $n = 2m$, we have the special linear group $\mathfrak{sp}_n(R) = \{A \in M_n(R) \mid A^T\Omega A = \Omega\}$ for $\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, then the associated Lie algebra is given by $\mathfrak{sp}_n = \{M \in M_n(F) \mid \Omega M + M^T\Omega = 0\}$.

2.3 Dimension and Smoothness

Definition 2.16 (Dimension)**.** If G is a connected group, then the reduced structure $F[G]_{\text{red}}$ is a domain. In particular, the dimension $\dim(G)$ of G is the transcendence degree of F over the field of fractions of $F[G]_{\text{red}}$. If G is not connected, we define $\dim(G) = \dim(G^0)$.

Example 2.17.

- $\dim(V) = \dim_F(V)$.
- $\dim(\mathrm{GL}_n(R)) = n^2$.
- $\dim(\mathrm{SL}_n(R)) = n^2 - 1$.
- $\dim(G_m) = \dim(G_a) = 1$.
- $\dim(\mu_n) = 0$.

Remark 2.18. The dimensions satisfy many important properties.

1. $\dim(G) = \dim(F[G])$, given by the Krull dimension.
2. G is finite if and only if $\dim(G) = 0$.
3. Given a field extension L/F, $\dim(G_L) = \dim(G)$.
4. $\dim(G_1 \times G_2) = \dim(G_1) + \dim(G_2)$.

Recall that we say a commutative local ring R we have $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \geq \dim(R)$. In particular, we say R is regular if the equality holds.

Definition 2.19 (Smooth)**.** We say a group G is smooth if $\dim(\mathrm{Lie}(G)) = \dim(G)$. Equivalently, $F[G]_L$ is reduced for any field extension L/F.

Proposition 2.20. If F is a perfect field, then G is smooth if and only if $F[G]$ is reduced.

Remark 2.21. Suppose F has characteristic 0, every algebraic group is smooth.

Example 2.22. Suppose F has characteristic $p > 0$, then $\mu_p = \mathrm{Spec}(k[x]/x^p - 1) = k[x]/(x - 1)^p$, which is not smooth.

3 Factor Schemes and Representations: Fed. 22, 2023

3.1 Criteria of Injectivity and Surjectivity

Recall that a homomorphism of group schemes $f: F \to G$ is a natural transformation. We now try and describe what it means for this morphism to be injective or surjective.

Definition 3.1 (Injective)**.** f is injective if $\ker(f) = 1$, or equivalently, all the maps $F(R) \to G(R)$ are injective.

Proposition 3.2 (Injectivity Criterion)**.** Let $f: G \to H$ be a homomorphism, then the following are equivalent:

1. f is injective,
2. f is a closed embedding, i.e., $f^*: F[h] \to F[G]$ is surjective,
3. $f_{\text{alg}}: G(F_{\text{alg}}) \to H(F_{\text{alg}})$ is injective, and $df: \text{Lie}(G) \to \text{Lie}(H)$ is injective.

Remark 3.3 (Facts about Hopf Algebras)**.** Suppose $A \subseteq B$ are Hopf algebras over F, then B is *faithfully flat* over A:

- $A \to B$ is flat, and $\text{Spec}(B) \to \text{Spec}(A)$ is surjective.
- M, N are A-modules, $M \to N$ is an injecitve map if and only if $M \otimes_A B \to N \otimes_A B$ is.

Let us see how we may apply this fact. Look at the maps $B \rightrightarrows B \otimes_A B$ defined by $b \mapsto b \otimes 1$ and $b \mapsto 1 \otimes b$, respectively, then the elements of B having the same images under $B \rightrightarrows B \otimes_A B$ are precisely the elements of A.

Proof. We only prove (1) $\Rightarrow$ (2) here.

By restricting the codomain, we can make sure f^* is injective. The elements of $G(F[G] \otimes_{F[H]} F[G])$ corresponding to $F[G] \rightrightarrows F[G] \otimes_{F[H]} F[G]$ have the same image in $H(F[G] \otimes_{F[H]} F[G])$, and therefore they have to be the same. Thus, for $F[G] \rightrightarrows F[G] \otimes_{F[H]} F[G]$, both maps have the same image, thus $F[G] = F[H]$ as a consequence of the remark above. □

Example 3.4.

- $\mu_n \to G_m$.
- $G_m \to \text{GL}_n$.

Definition 3.5 (Surjective)**.** $f: G \to H$ is surjective if $f^*: F[H] \to F[G]$ is injective.

Proposition 3.6 (Surjectivity Criterion)**.** For $f: G \to H$ a homomorphism of algebraic group schemes, and suppose H is smooth, then f is surjective if and only if $f_{\text{alg}}: G(F_{\text{alg}}) \to H(F_{\text{alg}})$ is surjective.

Example 3.7. The $n$the power map $G_m \to G_m$.

3.2 Construction of Quotients

The quotient structure behaves weirdly over group schemes. Consider the exact sequence of group schemes

$$1 \longrightarrow \mu_n \longrightarrow G_m \xrightarrow{.n} G_m \longrightarrow 1$$

but on points we only have

$$1 \longrightarrow \mu_n(R) \longrightarrow G_m(R) \xrightarrow{.n} G_m(R)$$

and we note that $(G/N)(R) \ncong G(R)/N(R)$ in general.

Let us have GL_2 acting on $k^2 = ke_1 \oplus ke_2$. Consider the subgroup scheme B of upper-triangulars, then B is the stabilizer of ke_1, and GL_2 acts transitively on the one-dimensional subspaces of k^2. Therefore, the coset space is P^1, which is not affine.

3.3 Representations

A representation of a group scheme G is $\rho : G \to \mathrm{GL}(V)$ for V a finite-dimensional vector space over F.

On points: R is a F-algebra, then $G(R)$ acts on $V \otimes_F R$ by R-linear automorphsims, for $g \in G(R)$, $v \in V \otimes_F R$, $g \cdot v = \rho_R(g)(v)$.

On the Hopf side, note that for $A = F[G]$ we have $\bar{\rho} : V \to V \otimes_F A$ such that $\bar{\rho}(v) = \mathrm{id}_A \cdot v$.

Example 3.8. For $\dim(V) = 1$, $\mathrm{GL}(V) = G_m$ (characters of $G : G \to G_m$).

There i san adjoint representation from having $G(R)$ acts on $\mathrm{Lie}(G) \otimes_F R$ (with $G = \mathrm{GL}(v)$ conjugation).

Theorem 3.9. If $G \to H$ has kernel N and is surjective, then any group scheme map $f' : G \to H'$ which vanishes on N factors uniquely through f.

If N is a normal subgroup scheme of G, then there is a surjection $G \to H$ with kernel N. Here every structure is affine.

4 Root Systems, Split Semisimple and Reductive Groups: Mar. 1, 2023

4.1 Root Systems and Semisimple Lie Algebra

All concepts in this subsection are over $\mathbb{C}$. Recall that $\mathbf{SL}_n$ is the kernel of the determinant map, and $d(\det)$ is the trace. Therefore, we have SL_n is the kernel of the trace, therefore as a subset of $\mathfrak{gl}_n$. We mainly discuss two examples, $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$.

In $\mathfrak{sl}_2$, there is a basis given by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and in particular this is a h-eigenbasis, with

$$[h, e] = 2e, [h, f] = -2f, [h, h] = 0.$$

Let V be a finite-dimensional irreducible representation of $\mathfrak{sl}_2$, so $V = \bigoplus_{\alpha \in \mathbb{C}} V_\alpha$ where $V_\alpha = \{v \in V : hv = \alpha v\}$. In this sense,

$$\begin{aligned} e &: V_\alpha \to V_{\alpha+2} \\ f &: V_\alpha \to V_{\alpha-2} \end{aligned}$$

We also need to make use of several facts:

- All h-eigenvalues of V are integers, and
- if we let n be the maximal h-eigenvalue with $v \in V_n \backslash \{0\}$, then $\{v, f(v), \dots, f^n(v)\}$ is a basis for V. In particular, there is
$$0 \underset{f}{\overset{e}{\rightleftarrows}} V_{-n} \underset{f}{\overset{e}{\rightleftarrows}} \cdots \underset{f}{\overset{e}{\rightleftarrows}} V_{n-2} \underset{f}{\overset{e}{\rightleftarrows}} V_n \underset{f}{\overset{e}{\rightleftarrows}} 0$$
and therefore V_n is the heghest weight.

Theorem 4.1. Finite-dimensional irreducible representations of $\mathfrak{sl}_2$ are totally and uniquely determined by the largest h-eigenvalue.

Remark 4.2. Note that $\mathfrak{sl}_2$ is semisimple.

We now think of the embeddings $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_3$, and in particular there are three types of them:

(i)

$$\begin{pmatrix} \cdot & \cdot & \\ \cdot & \cdot & \\ & & \ddots \end{pmatrix}$$

$$e \mapsto e_{12}$$

$$f \mapsto f_{21}$$

$$h \mapsto h_{12}$$

(ii)

$$\begin{pmatrix} \cdot & & \cdot \\ & \ddots & \\ \cdot & & \cdot \end{pmatrix}$$

$$e \mapsto e_{13}$$

$$f \mapsto f_{31}$$

$$h \mapsto h_{13}$$

(iii)

$$\begin{pmatrix} \ddots & & \\ & \cdot & \cdot \\ & \cdot & \cdot \end{pmatrix}$$

$$e \mapsto e_{23}$$

$$f \mapsto f_{32}$$

$$h \mapsto h_{23}$$

Definition 4.3. Let $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{h} = \{x \in \mathfrak{g} \mid x \text{ diagonal}\} \subseteq \mathfrak{g}$ the maximal Abelian subalgebra.

A few other crucial facts include:

- The adjoint matrix of $(h_{ij})_{i,j}$ is diagonalizable,
- v is an eigenvector of h, and

- consider the map

$$\begin{aligned} \mathfrak{h} &\to \mathbb{C} \\ h &\mapsto \text{eigenvalue of } h \text{ on } V \end{aligned}$$

then $\mathfrak{h}^* = \mathbb{C}\langle L_1, L_2, L_3\rangle / (L_1 + L_2 + L_3 = 0)$ where $L_i(x) = x_{ii}$. Then

$$\begin{cases} \operatorname{adj}(h)(e_{ij}) = (L_i - L_j)(h)e_{ij} \\ \operatorname{adj}(h)(f_{ji}) = (L_j - L_i)(h)f_{ji} \end{cases}$$

for $i < j$ and all $h \in \mathfrak{h}$.

Proposition 4.4. $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} g_\alpha$ where R is the set $\{L_i - L_j \mid 1 \leq i \neq j \leq 3\}$ (as roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$), and $\mathfrak{g}_\alpha = \{v \in \mathfrak{g} \mid hv = \alpha(h)v \;\; \forall h \in \mathfrak{h}\}$. For instance, $g_{L_1 - L_2} = \operatorname{span}(e_{12})$.

Definition 4.5 (Killing Form)**.** The Killing form is defined by

$$\begin{aligned} k : \mathfrak{g} \otimes \mathfrak{g} &\to \mathbb{C} \\ x \otimes y &\mapsto \operatorname{tr}(\operatorname{adj}(x)\operatorname{adj}(y)) \end{aligned}$$

A few facts to know include:

- k is non-degerneate, then $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^k$.
- For $h \in \mathfrak{h}$, $k(h,h) = \sum_{\alpha \in R} \alpha(h)^2$ where $\alpha(h_\alpha) = 2$. This is the inner product on $\mathfrak{h}$.

Definition 4.6. $E = \mathbb{R}\langle R\rangle$ is the inner product space on the Killing form.

Given a root α, let S_α be (orthonormal) reflection in E about $\operatorname{span}(\alpha)^\perp$.

Definition 4.7 (Weil Group)**.** We define $W(\mathfrak{g}) = \langle S_\alpha \mid \alpha \in R\rangle$ to be the Weil group.

Example 4.8. For $\mathfrak{g} = \mathfrak{sl}_3$, then $W(\mathfrak{g}) = \cong S_3$.

Theorem 4.9. Now let E be a finite-dimensional real vector space, K is an inner product space, and $R \subseteq E$ be a finite subset, such that

(i) for all $\alpha \in R$, $n\alpha \in R$ if and only if $n = \pm 1$,

(ii) S_α preserves R,

(iii) $\alpha, \beta \in R$, then $K(\alpha,\beta)/K(\alpha,\alpha) \in \frac{1}{2}\mathbb{Z}$,

(iv) $E = \operatorname{span}(R)$,

then (E, K, R) is a root system.

4.2 Reductive Groups and Root Datum

Let k be a field in this subsection.

Definition 4.10. An algebraic group is called unipotent if all of its non-zero representations admit a non-zero fixed point.

Definition 4.11. Let G be a smooth, connected algebraic group over k. $R_n(G)$ is the maximal smooth connected unipotent normal group, usually called the unipotent radical.

We say that G is reductive if $R_n(G_{\bar{k}})$ is trivial.

Example 4.12. G_a is unipotent. $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{PGL}_n$ are reductive.

Let G be a reductive group, and let $T \subseteq G$ be the maximal torus.

Definition 4.13.

- The lattice is defined by $X^{\cdot}(T) = \mathbf{Hom}(T, G_n)$.
- Define the set of roots $R \subseteq X^{\cdot}(T)$ as the non-zero characters arising from the roots of $\mathrm{adj}(A)$ on $\mathfrak{g}$.
- The colattice is defined by $X_{\cdot}(T) = \mathbf{Hom}(G_m, T)$.
- The coroots $R^v \subseteq X_{\cdot}(T)$ are the cocharacters arising from the following proposition.

Proposition 4.14. For $\alpha \in R$, there exists maps

$$\begin{aligned} \mathbf{SL}_2 &\mapsto G \\ \mathfrak{sl}_2 &\mapsto \mathfrak{g} \\ e &\mapsto \mathfrak{g}_\alpha \backslash \{0\} \end{aligned}$$

restrict this to $T \cong G_m \subseteq \mathrm{SL}_2$ then gives

$$\begin{tikzcd} G_m \arrow[r] \arrow[d, hook] & T \arrow[d, hook] \\ \mathrm{SL}_2 \arrow[r] & G \end{tikzcd}$$

Now there is a table of root datum:

	SL_2	SL_3
torus T	$\begin{pmatrix}\alpha & \\ & \alpha^{-1}\end{pmatrix}$	$\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}$
$X^{\cdot}(T)$	$\mathbb{Z} \xrightarrow{\sim} \mathbf{Hom}(T, G_m)$ $n \mapsto \left(\begin{pmatrix}\alpha & \\ & \alpha^{-1}\end{pmatrix} \mapsto \alpha^n\right)$	$\mathbb{Z} \xrightarrow{\sim} \mathbf{Hom}(T, G_m)$ $n \mapsto \left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix} \mapsto \alpha^n\right)$
R	$\begin{pmatrix}\alpha & \\ & \alpha^{-1}\end{pmatrix}$ with $\alpha \mapsto \alpha^2$, $\alpha^{-1} \mapsto \alpha^{-2}$, represented by $\{\pm 2\}$	$\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}$ with $\alpha \mapsto \alpha^2$, $1 \mapsto \alpha^{-1}$, represented by $\{\pm 1\}$
$X.(T)$	$\mathbb{Z}$	$\mathbb{Z}$
R^v	$\begin{pmatrix}\alpha & \\ & \alpha^{-1}\end{pmatrix}$ with $\alpha \mapsto \begin{pmatrix}\alpha & \\ & \alpha^{-1}\end{pmatrix}$, $\alpha^{-1} \mapsto \begin{pmatrix}\alpha^{-1} & \\ & \alpha\end{pmatrix}$ represented by $\{\pm 1\}$	$\begin{pmatrix}\alpha & \\ & \alpha^{-1}\end{pmatrix}$ with $\alpha \mapsto \begin{pmatrix}\alpha^2 & \\ & 1\end{pmatrix} \sim \begin{pmatrix}\alpha & \\ & \alpha^{-1}\end{pmatrix}$, $\alpha^{-1} \mapsto \begin{pmatrix}\alpha^{-2} & \\ & 1\end{pmatrix}$ represented by $\{\pm 2\}$

Example 4.15.

- $\mathrm{Lie}(T) = A = \mathfrak{h}$,
- $\mathrm{Lie}(G_m) = G_a$.

Remark 4.16. There is a correspondence between reductive groups over $k = \bar{k}$ and the roots datum above, given by the map $G \mapsto G^L$, where $(-)^L$ is the Langlands dual.