# Motivic Homotopy Theory Notes

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These notes were taken from a course on Motivic Homotopy Theory taught by Dr. P. Du in Spring 2024 at BIMSA. Any mistakes and inaccuracies would be my own. References for this course include [BH21], [EH23], [Lur18], [Lur09], and others mentioned in the references.

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## 1 Commutative Monoids and Commutative Semirings as Functors

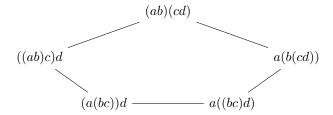
The materials from this section can be found in [EH23], Chapter 1.1-1.2.

#### 1.1 Spans and Monoids

**Definition 1.1.** A commutative monoid  $(M, \times, 1)$  has a multiplication operation

$$\times: M \times M \to M$$
  
 $(a,b) \mapsto a \times b =: ab$ 

that satisfies ab = ba, as well as the associativity by the pentagon axiom



**Definition 1.2.** Denote  $\mathbb{F} = \mathbf{FinSet}$  to be the finite category of finite sets, then a commutative monoid M induces a contravariant functor

$$\bar{M}: \mathbb{F}^{\text{op}} \to \mathbf{Set}$$
 $I \mapsto M^I$ 

$$(I \stackrel{f}{\leftarrow} S) \mapsto (M^I \stackrel{f^*}{\longrightarrow} M^S)$$

$$(a_i)_{i \in I} \mapsto (a_{f(s)})_{s \in S}$$

and similarly a covariant functor

$$\bar{M}' : \mathbb{F} \to \mathbf{Set}$$
 $I \mapsto M^I$ 

$$(s \xrightarrow{g} I) \mapsto (M^S \xrightarrow{g \otimes} M^I)$$

$$(b_s)_{s \in S} \mapsto \left(\prod_{s \in g^{-1}(j)} b_s\right)_{j \in J}$$

Now given the construction in Definition 1.2 above, suppose we have a zigzag

we can use  $\bar{M}$  and  $\bar{M}'$  and obtain  $f^*$  and  $g_{\otimes}$ . One can map Diagram 1.3 to a morphism  $g_{\otimes}f^*:M^I\to M^J$ .

**Remark 1.4.** To define a functor precisely, we need to specify what category Diagram 1.3 lies in. As we will see later, we want a category with the same objects as  $\mathbb{F}$ , and morphisms are the zigzags of the form Diagram 1.3, which are called spans (or correspondences).

To define the composition of spans as morphisms, we should think of a diagram

The two zigzags give rise to  $g_{\otimes}f^*$  and  $v_{\otimes}u^*$ . For compositions to be well-defined, we should map this diagram to  $v_{\otimes}u^*g_{\otimes}f^*$ . In order to obtain functoriality, we would hope

$$v_{\otimes}u^*g_{\otimes}f^* = v_{\otimes}g_{\otimes}u^*f^* = (vg)_{\otimes}(fu)^*.$$

This is certainly not true. As a remedy, we complete Diagram 1.5 to

as we obtain  $u^*g_{\otimes}:M^S\to M^T$  defined by the composition

$$(b_s)_{s \in S} \mapsto \left(\prod_{s \in g^{-1}(j)} b_s\right)_{j \in J} \mapsto \left(\prod_{s \in g^{-1}(u(t))} b_s\right)_{t \in T}.$$

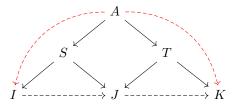
Remark 1.7. If Diagram 1.6 is a commutative diagram, then there is a restriction of u' given by  $u': g'^{-1}(t) \to g^{-1}(u(t))$ . In particular, if Diagram 1.6 is a pullback diagram, then this restriction map is a bijection. In this setting, the map  $u^*g_{\otimes}$  sends  $(b_s)_{s\in S}$  to

$$\left(\prod_{s \in g^{-1}(u(t))} b_s\right)_{t \in T} = \left(\prod_{a \in g'^{-1}(t)} b_{u'(a)}\right)_{t \in T} = g'_{\bigotimes} u'^*(b_s)_{s \in S}.$$

Therefore,

$$v_{\otimes}u^*g_{\otimes}f^* = v_{\otimes}g'_{\otimes}u'^*f^* = (vg')_{\otimes}(fu')^*.$$

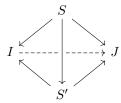
**Definition 1.8.** We define  $\mathbf{Span}(\mathbb{F})$  to be the category of span of  $\mathbb{F}$ , where objects are finite sets as in  $\mathbb{F}$ , and morphisms of the form  $I \to J$  are the zigzag of the form  $I \leftarrow S \to J$ . The composition of morphisms  $I \to J \to K$  on the zigzag is now defined by  $I \leftarrow A \to K$  using the diagram



whenever A is constructed as the pullback, otherwise known as the outer span  $S \times_K T$ .

**Remark 1.9.** One issue that persists from this construction is the fact that the pullback A is not unique. (This may be unique up to unique isomorphism.) With this in mind,  $\mathbf{Span}(\mathbb{F})$  admits a (2,1)-category structure instead of an ordinary category.

The 2-morphisms of **Span**( $\mathbb{F}$ ) are defined by  $S \to S'$  via



Moreover, these 2-morphisms are isomorphisms (of spans) and hence invertible, therefore admitting the (2,1)-category structure.

Remark 1.10. The functors we defined in Definition 1.2 can be extended to a functor

$$\tilde{M}: \mathbf{Span}(\mathbb{F}) \to \mathbf{Set}$$

such that  $\tilde{M}\Big|_{\mathbb{R}^{op}} \in \mathbf{Fun}^{\times}(\mathbb{F}^{op}, \mathbf{Set})$ . To see this, recall that there is a natural inclusion

$$\mathbb{F}^{\text{op}} \hookrightarrow \mathbf{Span}(\mathbb{F})$$

$$A \mapsto A$$

$$(I \leftarrow S) \mapsto (I \leftarrow S \xrightarrow{=} S)$$

then the extension  $\tilde{M}$  is the functor we want, as the product and coproduct of the 2-category  $\mathbf{Span}(\mathbb{F})$  are both the coproduct on  $\mathbf{FinSet}$ , i.e., the disjoint union.

**Remark 1.11.** In fact, given any category  $\mathscr C$  with finite products, then there is an identification of commutative monoids on  $\mathscr C$  with product-preserving functors  $\mathbf{Span}(\mathbb F) \to \mathscr C$ . Moreover, this is true homotopically, c.f., [Cra09] and [Cra11].

This is the story of how we induce functors from commutative monoids, and we will see below that there is a similar one for commutative semirings.

#### 1.2 BISPANS AND SEMIRINGS

**Definition 1.12.** A commutative semiring  $(R, +, \times, 0, 1)$  is a set R equipped with operations + and  $\times$  as well as additive identity 0 and multiplicative identity 1. However, we do not assume the existence of additive inverse and/or multiplicative inverse. Therefore, R is both an additive monoid and a multiplicative monoid.

Using the same construction in Definition 1.2, we have a functor

$$\mathbb{F} \to \mathbf{Set}$$
$$I \mapsto R^I$$

which induces a functor

$$\tilde{R}_{\times}:$$
 "Span( $\mathbb{F}$ )"  $\to$  Set
$$(I \stackrel{f}{\leftarrow} S \stackrel{g}{\to} J) \mapsto g_{\boxtimes} f^*$$

Now note that we still have an additive monoidal structure on  $R_0$ , so we would hope to define a functor of the form

$$\tilde{R}_+:$$
 "Span( $\mathbb{F}$ )"  $\to$  Set 
$$? \mapsto g_{\oplus}f^*$$

for some unknown category "**Span**( $\mathbb{F}$ )". These two functors altogether shall define a desired functor  $\tilde{R}$ : "**Span**( $\mathbb{F}$ )"  $\to$  **Set**. In particular, admitting two different structures here already tells us that the spans are no longer suitable, and a natural adaptation would be bispans.

**Definition 1.13.** A bispan (or a polynomial diagram) from I to J is given by a diagram

$$I \xrightarrow{p} X \xrightarrow{f} Y$$

The category of bispans, denoted  $\mathbf{Bispan}(\mathbb{F})$ , has objects (again) the same with objects of  $\mathbb{F}$ , and morphisms are bispans.

Given a semiring R, we would want to construct a functor

$$\begin{aligned} \mathbf{Bispan}(R) &\to \mathbf{Set} \\ I &\mapsto R^I \\ (I \xleftarrow{p} X \xrightarrow{f} Y \xrightarrow{q} J) &\mapsto q_{\bigoplus} f_{\bigotimes} p^* \end{aligned}$$

where

$$p^*: R^I \to R^X$$
  
 $p^*(\varphi)(x) = \varphi(px),$ 

$$f_{\otimes}: R^X \to R^Y$$
  
$$f_{\otimes}(\varphi)(y) = \prod_{x \in f^{-1}(y)} \varphi(x),$$

and

$$q_{\bigoplus}: R^Y \to R^J$$
  
$$q_{\bigoplus}(\varphi)(j) = \sum_{y \in q^{-1}(j)} \varphi(y),$$

which represent composition (as pullback), fiberwise multiplication (as pushforward), and fiberwise addition (as pushforward), respectively. Altogether, this gives

$$q_{\bigoplus} f_{\bigotimes} p^* : M^I \to M^J$$

$$(a_i)_{i \in I} \mapsto \left( \sum_{y \in q^{-1}(j)} \prod_{x \in f^{-1}(y)} a_{p(x)} \right)_{j \in J}.$$

Again, to construct such a functor, we need to consider the composition of bispans:

$$I \xrightarrow{p} X \xrightarrow{f} Y \xrightarrow{q} X' \xrightarrow{g} Y' \xrightarrow{v} K$$

As we have seen previously, we need to study the pullback structure so that we can resolve  $v_{\oplus}g_{\otimes}u^*q_{\oplus}f_{\otimes}p^*$ . Using similar construction, we have

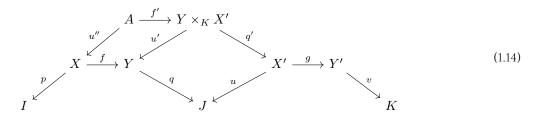
$$v_{\oplus}g_{\otimes}u^*q_{\oplus}f_{\otimes}p^* = v_{\oplus}g_{\otimes}q'_{\oplus}u'^*f_{\otimes}p^*$$

$$= v_{\oplus} q''_{\oplus} g'_{\otimes} u'^* f_{\otimes} p^*$$

$$= v_{\oplus} q''_{\oplus} g'_{\otimes} f'_{\otimes} u''^* p^*$$

$$= (vq'')_{\oplus} (g'f')_{\otimes} (pu'')^*$$

assuming we can construct  $g'_{\otimes}$  and  $q''_{\oplus}$  such that  $g_{\otimes}q'_{\oplus}=q''_{\oplus}g'_{\otimes}$ . That is, we have constructed two pullback squares



To deal with this, recall that addition distributes over multiplication, therefore given any

$$I \xrightarrow{u} J \xrightarrow{v} K$$

we know  $v_{\otimes}u_{\oplus}:R^I\to R^K$  is the mapping defined by

$$(a_i)_{i \in I} \mapsto \left( \prod_{j \in v^{-1}(k)} \sum_{i \in u^{-1}(j)} a_j \right)_{k \in K} = \left( \sum_{\substack{(i_j) \in \prod_{j \in v^{-1}(k)} u^{-1}(j)}} \prod_{t \in v^{-1}(k)} a_{i_t} \right)_{k \in K}.$$
(1.15)

The goal is to identify the said image from Equation (1.15). Recall that the slice categories  $\mathbf{FinSet}/\mathbf{K}$  and  $\mathbf{FinSet}/\mathbf{J}$  are involved in a pullback/pushforward adjunction

$$\mathbf{FinSet/K} \xrightarrow{\cong} \mathbf{Fun}(\mathbf{K}, \mathbf{Set})$$

$$v^* \downarrow \uparrow v_*$$

$$\mathbf{FinSet/J} \xrightarrow{\cong} \mathbf{Fun}(\mathbf{J}, \mathbf{Set})$$
(1.16)

where

•  $\mathbf{FinSet}/\mathbf{J} \cong \mathbf{Fun}(\mathbf{J}, \mathbf{Set})$  is a Grothendieck correspondence, where given  $u: I \to J$ , we obtain a functor

$$J \to \mathbf{FinSet}$$
  
 $j \mapsto u^{-1}(j)$ 

• FinSet/K  $\cong$  Fun(K, Set) is a Grothendieck correspondence, where given  $v: J \to K$ , we obtain a functor

$$K \to \mathbf{FinSet}$$
  
 $k \mapsto v^{-1}(k)$ 

•  $h = v_* u \in \mathbf{Set}/\mathbf{K}$  is a functor, and by the correspondence we obtain a functor

$$h': K \to \mathbf{Set}$$
 $k \mapsto \prod_{j \in v^{-1}(k)} u^{-1}(j) = \prod_{j \in h^{-1}(k)} u^{-1}(j)$ 

•  $v^*$  is the pullback along  $v: J \to K$ . In particular, consider the counit  $\varepsilon: v^*v_*I \to I$  of the adjunction, then for  $X = v_*I$ , the pullback  $v^*X = v^*v_*I$  gives a counit  $\varepsilon$  in the diagram

$$\begin{array}{c|c}
v^*v_*I & \xrightarrow{\tilde{v}} v_*I \\
\downarrow I & \downarrow h \\
\downarrow J & \longrightarrow K
\end{array}$$
(1.17)

We now make an effort to show that Diagram 1.17 actually commutes.

For any  $k \in K$ , we pullback  $\alpha \in X$  such that  $h(\alpha) = k$ , but by the correspondence we know  $\alpha$  is in the image of k along h'. Similarly, for  $k \in K$ , we pullback  $j \in J$ , and using the same argument we would then conclude that the pullback element in  $v^*X$  is just a pair  $(\alpha, j)$ .

Now let  $i = \varepsilon(\alpha, j)$ , but one can identify i to be the image of  $\alpha$  under the projection  $h^{-1}(k) \to \prod_{j' \in v^{-1}(k)} u^{-1}(j') \to u^{-1}(j)$ . Therefore,  $\alpha = (i_j)_{j \in v^{-1}(k)}$ . For any fixed  $\alpha$ , one can then identify

$$\prod_{t \in v^{-1}(k)} a_{it} = \prod_{j \in v^{-1}(k)} a_{\varepsilon(\alpha,j)}.$$

Therefore, the image of Equation (1.15) is

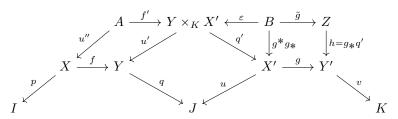
$$\left(\sum_{\alpha \in h^{-1}(k)} \prod_{j \in v^{-1}(k)} a_{\varepsilon(v,j)}\right)_{k \in K} = h_{\oplus} \tilde{v}_{\otimes} \varepsilon^*(a_i)_{i \in I}.$$

In particular, we obtain

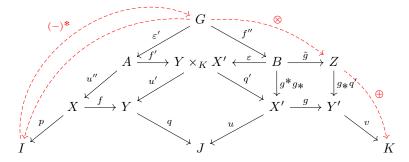
$$v_{\otimes}u_{\oplus}=h_{\oplus}\tilde{v}_{\otimes}\varepsilon^*,$$

i.e., Diagram 1.17 commutes, which describes the distributivity.

Let us go back to Diagram 1.14. Using Diagram 1.17, we extend the diagram to



which can be extended by taking one last pullback



and we define the composition to be the outer bispan in this diagram.

**Remark 1.18.** An explicit construction of this (2,1)-category  $\mathbf{Bispan}(\mathbb{F})$  can be found in [Cra09], where it is proven that the category has a product structure given by coproducts of  $\mathbf{FinSet}$ . In this sense, commutative semirings in a category  $\mathscr S$  correspond to functors  $\mathbf{Bispan}(\mathbb{F}) \to \mathscr S$  that preserve finite products.

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