Abstract Algebra

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- Group Theory
- Category Theory in Group Context
- Ring Theory
- Module Theory
- Field Theory
- Hilbert's Nullstellensatz
- Dedekind Domain
- Representation Theory

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Contents

Introduction

This book contains a compilation of lecture notes based on the graduate algebra (and some topology) courses at UCLA. The project was initiated on June 21, 2021, containing a few solution files. Specifically, the book contains the following materials:

- The numbered chapters, which are the materials taught in UCLA's Graduate Algebra series (210A/B/C) by Professor Alexander Merkurjev in the 2021-2022 academic year. The notes will be revised in 2023 for clarity and details.

  210A (covers Chapter 1 and 2) and 210B (covers Chapter 3 to 5) revisit materials taught in the Undergraduate Honors Algebra series (110AH/BH/C) and formalize them in the context of categories. 210C (covers Chapter 6 to 8) is a survey of advanced algebra, by considering its connections with representation theory, algebraic number theory, and many other fields.

- Algebra Qualification Exam Problems (from UCLA) introduced in the corresponding discussion sections. Discussions held in the 2021-2022 academic year were led by Matthew Gherman, and other discussions were led by Jung Joo Suh.

- Homework Problems assigned weekly. The solutions provided may contain mistakes since only a few problems from each homework set were graded.

1 Group Theory

1.1 Introduction

Definition 1.1.1 (Group)**.** *A group G is a set G with a binary operation $\cdot : G \times G \to G$ that $(x, y) \mapsto xy = x \cdot y$ such that:*

1. *Associativity:* $\forall x, y, z \in G$, $(xy)z = x(yz)$.
2. *Existence of Unit:* $\exists e \in G$ *such that* $ex = xe = x$ $\forall x \in G$.
3. *Existence of Inverses:* $\forall x \in G$, $\exists y \in G$ *such that* $xy = yx = e$.

Remark 1.1.2.

1. *Element $e \in G$ given by 2) is unique. Indeed, suppose we also have $e' \in G$ as the unit, then $xe' = e'x = x$ and so $e' = e'e = e$.*
2. *Element $y \in G$ given in 3) is uniquely determined by $x \in G$. Consider $xy' = y'x = e$ for some other $y' \in G$, then $y' = e \cdot y' = (yx)y' = y(xy') = y \cdot e = y$. In particular, we write $y \in G$ as $y = x^{-1} \in G$.*
3. *Note that $xyz = (xy)z = x(yz)$ and $xyzt = ((xy)z)t = (x(yz))t = (yx)(zt) = x(y(zt)) = x((yz)t)$. This can be generalized by induction.*

Definition 1.1.3 (Abelian/Commutative Group)**.** *If 4) commutativity: $xy = yx$ $\forall x, y \in G$ holds for a group G, then G is abelian (communitative).*

Remark 1.1.4. *If a group is abelian, we use $+$ to denote the binary operation. In particular, we can rewrite the group definition as:*

1. $(x + y) + z = x + (y + z)$.
2. $\exists 0 \in G$ *such that* $0 + x = x + 0 = x$ *for all* $x \in G$.
3. $\forall x \in G$, $\exists y = -x \in G$ *such that* $x + (-x) = 0 \in G$.
4. *We also denote* $x - y = x + (-y)$.

Remark 1.1.5. *Groups also have cancellation laws.*

1. *Left cancellation:* $xy = xz$ *indicates* $y = z$ *for all* $x, y, z \in G$*. Indeed,* $x^{-1}(xy) = x^{-1}(xy)$*, and therefore* $y = z$*.*
2. *Right cancellation:* $yx = zx$ *indicates* $y = z$ *for all* $x, y, z \in G$*.*
3. *Usually* $xy = zx$ *does not indicate* $y = z$*.*
4. *We also have* $(xy)^{-1} = y^{-1}x^{-1}$ *and* $(x^{-1})^{-1} = x$*.*

Example 1.1.6. 1. *Trivial Group:* $G = \{e\}$*.*

2. *Addition Group of Integers* $\mathbb{Z}$*.*
3. *For positive integer* n*,* $\mathbb{Z}/n\mathbb{Z} = \{[a]_n\}$ *for* $a \in \mathbb{Z}$ *where* $[a]_n = \{b \in \mathbb{Z} : b \equiv a \pmod{n}\}$*. The operation is defined as* $[a]_n + [b]_n = [a+b]_n$*. The unit of the group is* $[0]_n$*. The inverse is* $-[a]_n = [-a]_n$ *for all* $[a]_n \in \mathbb{Z}/n\mathbb{Z}$*.*
4. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ *are groups with respect to addition. Notice that the operation is part of a group's definition. Moreover, these structures are not groups with respect to multiplication since there is the zero element.*
5. *Multiplication groups* $\mathbb{Q}^* = \mathbb{Q}\backslash\{0\}$*,* $\mathbb{R}^*\backslash\{0\}$*,* $\mathbb{C}^*\backslash\{0\}$*.*
6. *Klein-4 Group* $G = \{e, a, b, c\}$*.*

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

7. *Symmetric Group* $\Sigma(X)$ *of set* X*. Define* $\Sigma(X) = \{f : X \to X \text{ bijection}\}$*. For* $f, g \in \Sigma(X)$*, we define* $f \circ g = f(g(x))$*. Similarly* $(f \circ g) \circ h = f \circ (g \circ h)$ *for all* $f, g, h \in \Sigma(X)$*.*
   - *Notice that if* X *is a finite set, then* $card(\Sigma(X)) = card(X)!$*.*
   - $\Sigma(X)$ *is not abelian if* $card(X) > 2$*.*
8. *Consider ring* R*, e.g.* $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$*. For positive integer* n*, consider* $GL_n(R) = \{n \times n$ *invertible matrix with entri as a group. This is called the general linear group of* R*.*
   - *We say* A *is invertible if there exists* B *such that* $AB = BA = I_n$*.*
9. *Let* G *and* H *be groups. Then* $G \times H = \{(g, h) : g \in G, h \in H\}$ *where* $(g, h) \cdot (g', h') = (gg', hh')$ *and* $e_{G \times H} = (e_G, e_H)$*.*
10. *We say a group* G *is finite if the order of the group* $|G| = card(G) < \infty$*.*

Algebra studies the relations between different algebraic structures in general. Relations between groups are given by homomorphisms.

1.2 Homomorphism

Definition 1.2.1 (Group Homomorphism)**.** *For groups G, H, a map $f : G \to H$ is called a homomorphism if $f(x \cdot_G y) = f(x) \cdot_H f(y)$ for all $x, y \in G$.*

Example 1.2.2. *1. Identity $id : G \to G$ that maps every element $g \in G$ to itself.*

2. Trivial homomorphism $f : G \to H$ that maps every element $g \in G$ to $e_H \in H$.

Property 1.2.3. *1. $f(e_G) = e_H$. Note that $f(e_G) = f(e_G \cdot e_G) = f(e_G) \cdot f(e_G)$, and therefore $e_H \cdot f(e_G) = f(e_G) \cdot f(e_G)$. By cancellation law, $f(e_G) = e_H$.*

2. $f(x^{-1}) = f(x)^{-1}$. Note that $e_H = f(e_G) = f(x \cdot x^{-1}) = f(x) \cdot f(x^{-1})$, then $f(x^{-1}) = f(x)^{-1}$ by definition.

Remark 1.2.4. *Composition of homomorphisms is a homomorphism.*

Definition 1.2.5 (Isomorphism)**.** *A homomorphism $f : G \to H$ is an isomorphism if f is a bijection. Two groups G and H are isomorphic if there exists an isomorphism $f : G \to H$, denoted $G \cong H$.*

Remark 1.2.6. *1. $id : G \to G$ is an isomorphism.*

2. If there is an isomorphism $f : G \to H$, then $f^{-1} : H \to G$ is also an isomorphism.

3. Let $h = f(g)$, $h' = f(g')$ for some $g, g' \in G$. Then $hh' = f(g)f(g') = f(gg')$, and so $f'(hh') = gg' = f^{-1}(h)f^{-1}(h')$.

4. If f, g are isomorphisms, then $g \circ f$ is an isomorphism.

Claim 1.2.7. *$\cong$ is an equivalence relation.*

Proof. This is a direct result of remark 1.2.6, we can conclude reflexivity, symmetry and transitivity respectively. $\square$

Example 1.2.8. *1. If $|G| = |H| = 1$, then $G \cong H$.*

2. Two finite groups are isomorphic if they have the same multiplication table.

3. Every two groups of order 2 are isomorphic. Moreover, they are all isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

4. $\mathbb{C} \cong \mathbb{R} \times \mathbb{R}$. (We can obviously construct it by $\mathbb{R}[x]/(x^2 + 1)$.) Furthermore, we have $f : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ by mapping $f(a, b) = a + bi$ for arbitrary $a, b \in \mathbb{R}$.

5. $\mathbb{R}^+ \cong \mathbb{R}^{x,>0}$. Consider $f(x) = e^x$ with $f(x + y) = f(x) \cdot f(y)$.

1.3 Cyclic Group

Definition 1.3.1 (Order, Generator, Cyclic Group)**.** *Consider arbitrary group G with $x \in G$ and some $n > 0$. We define x^n as the n-term multiplication of x, and $x^0 = e$ with $x^{-n} = (x^{-1})^n = (x^n)^{-1}$.*

For $x \in G$, we say the smallest $n > 0$ such that $x^n = e$ is the order of x. If such n does not exist, we say the order is ∞.

For a group G, $x \in G$ is a generator of G if $\forall y \in G$, $y = x^n$ for some $n \in \mathbb{Z}$.

A group G is cyclic if G has a generator.

Remark 1.3.2. *For abelian groups, we write nx as the n-term summation of x, and $0 \cdot x = 0 \in G$, with $(-n)x = -(nx) = n \cdot (-x)$.*

Example 1.3.3.

1. *$\mathbb{Z}$ is a cyclic group with generators 1 and -1.*
2. *Take $0 < n \in \mathbb{Z}$, then $\mathbb{Z}/n\mathbb{Z}$ is cyclic. The generators are $[a]_n \in \mathbb{Z}/n\mathbb{Z}$ with some $1 \leq a \leq n-1$ such that $\gcd(a, n) = 1$. Moreover, the number of generators is exactly $\varphi(n)$, where φ is the Euler Function.*

Theorem 1.3.4. *Every cyclic group is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ for some $n > 0$.*

Proof. Case 1: suppose $|G| = \infty$. Let $g \in G$ be a generator. Define $f : \mathbb{Z} \to G$ with $f(m) = g^m$. Obviously f is onto because g is a generator. Now suppose $g^k = g^m$ for some $k > m$. Then $g^{k-m} = e$ with $k - m > 0$. Hence, the order of g has to be finite. Then G cannot have infinite cardinality, contradiction. Hence f is a bijection. Therefore, f is an isomorphism, $\mathbb{Z} \cong G$.

Case 2: suppose $|G| = n$ finite. Let $g \in G$ be a generator. Obviously $\mathrm{ord}(g) < \infty$. We claim that $\mathrm{ord}(g) = n$. Suppose $\mathrm{ord}(g) = m$ for some $m > 0$. We can check that $g^0 = e, g, g^2, \cdots, g^{m-1}$ are all the elements in G. Indeed, $g^m = e$, and suppose $g^i = g^j$ for some $0 \leq j < i \leq m-1$, then $g^{i-j} = e$ for $0 < i - j < m$. Since m is the order, we have a contradiction. Hence, $|G| = m = n$.

Now take $f : \mathbb{Z}/n\mathbb{Z} \to G$ with $f([a]_n) = g^a$. We check that $[a]_n = [b]_n$ indicates $g^a = g^b$. Indeed, $b \equiv a \pmod{n}$ indicates $b = a + nc$, which means $g^b = g^{a+nc} = g^a$. This concludes the proof. ☐

Remark 1.3.5.

1. *If G is cyclic with generator $g \in G$, then $|G| = ord(g)$.*
2. *Let G, H be cyclic. Then $G \cong H$ if and only if $|G| = |H|$.*
3. *The number of generators in a cyclic group of order n is $\begin{cases} 2 \text{ if } n = \infty \\ \varphi(n) \text{ if } n < \infty \end{cases}$.*
4. *Consider a finite group G with the isomorphism $f : \mathbb{Z}/n\mathbb{Z} \to G$ that maps $[1]_n \mapsto f([1]_n)$, with $[2]_n \mapsto f([1]_n)^2$. Note that such maps must preserve generators. i.e. $f([1]_n)$ is always a generator, and can be any generator of G. In particular, f is uniquely determined by $f([1]_n)$. There are $\varphi(n)$ isomorphisms between $\mathbb{Z}/n\mathbb{Z}$ and G (or any two cyclic groups of order n).*

1.4 Subgroup

Definition 1.4.1 (Subgroup)**.** *Consider group G with subset $H \subseteq G$. Assume $\forall h, h' \in H$ we have $h \cdot_G h' \in H$. Then H is a subgroup of G if H is a group with respect to $\cdot_G$.*

Proposition 1.4.2. *Let G be a group and $H \subseteq G$ is a subset. Then H is a subgroup if and only if the following holds:*

1. *$\forall h, h' \in H$, $hh' \in H$.*
2. *$e \in H$, i.e. H is not empty.*
3. *$\forall h \in H$, $h^{-1} \in H$.*

Proof. If the three properties hold, then H is a group, and so H is a subgroup of G.

Suppose H is a subgroup then it is obviously closed. Let $e' \in H$ be the unit, then $e' \cdot h = e \cdot h = h$ for all $h \in H$. Then $e' = e \in H$ by cancellation. Take $h \in H$, then there is $h' \in H$ such that $hh' = e$. Moreover, $h^{-1}hh' = h^{-1}e = h^{-1}$. Hence, $h' = h^{-1} \in H$. □

Example 1.4.3.

1. *$\{e\}, G \subseteq G$ are subgroups.*
2. *There exists a sequence of subgroups: $n\mathbb{Z} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.*
3. *There is also a list of subgroups $\mathbb{Q}^\times \subseteq \mathbb{R}^\times \subseteq \mathbb{C}^\times$. Note that $\mathbb{Q}^\times$ is not a subgroup of $\mathbb{Q}$ since they hold different operations.*
4. *Let $(H_i)_{i \in I}$ be a family of subgroups of G. Then $\bigcap_{i \in I} H_i$ is a subgroup of G. In general, $\bigcup_{i \in I} H_i$ is not a subgroup.*

Definition 1.4.4. *(Kernel, Image) Let $f : G \to H$ be a group homomorphism, with $f(gg') = f(g)f(g')$. Then $\ker(f) = \{g \in G : f(g) = e_H\}$ and $im(f) = \{h \in H : h = f(g) \text{ for some } g \in G\}$.*

Proposition 1.4.5. *$\ker(f)$ is a subgroup of G and $im(f)$ is a subgroup of H.*

Proof. We prove the first claim.

Note that for all $g, g' \in \ker(f)$, we have $f(g) = f(g') = e$, which means $f(gg') = f(g)f(g') = e$. Hence, $gg' \in \ker(f)$.

Since $f(e_G) = e_H$, then $e_G \in \ker(f)$.

If $g \in \ker(f)$, then $f(g^{-1}) = f(g)^{-1} = e^{-1} = e$. □

Proposition 1.4.6. *Let $f : G \to H$ be a group homomorphism. Then:*

1. *f is surjective if and only if $im(f) = H$.*
2. *f is injective if and only if $\ker(f) = \{e_G\}$.*

3. *f is an isomorphism if $im(f) = H$ and $\ker(f) = \{e_G\}$.*

Proof. Part 1 and part 3 are obvious. We only have to prove part 2.

If f is injective, take $g \in \ker(f)$, then $f(g) = e_H$. Therefore, $f(g) = e_H = f(e_G)$, then $g = e_G$ by injection.

If $\ker(f) = \{e_G\}$, then consider $f(g) = f(g')$. Then $f(g^{-1}) \cdot f(g) = f(g^{-1}) \cdot f(g')$ and so $e_H = f(g^{-1}) \cdot f(g') = f(g^{-1}g')$. That means $g^{-1}g' = e_G$ and so $g' = g$. □

Example 1.4.7. 1. *Suppose $H \subseteq G$ is a subgroup. Then the inclusion map $inc : H \to G$ is injective defined as $inc(h) = h$ for all $h \in H$.*

2. *Consider an injective homomorphsim $f : H \to G$ for groups G, H, then $f' : H \to im(F)$ with $f'(h) = f(h)$ defined is an isomorphism. Then H is isomorphic to a subgroup of G, i.e. $H \cong im(f) \subseteq G$.*

3. *Let G be a group with $g \in G$. Then consider $f_g : G \to G$ with $f_g(x) = gx$. Note that $f_g \circ f_{g'} = f_{gg'}$ for all $g, g' \in G$. Moreover, $f_e = id_G$.*

   Note that f_g is a bijection because $f_g \circ f_{g^{-1}} = f_e = id_G = f_{g^{-1}} \circ f_g$ and so $f_{g^{-1}} = (f_g)^{-1}$. Therefore, $f_g \in \sum(G)$. Notice that f_g may not be a homomorphism.

   However, consider $f : G \to \sum(G)$ by $f(g) = f_g$, then f is a homomorphism. Furthermore, f is injective: if $g \in \ker(f)$, then $f_g = id_G$, hence $f_g(x) = x$ for all $x \in G$. Therefore, by definition $gx = x$ for all $x \in G$, which means $g = e_G$. Thus, f is injective. Following from the argument above, we know G is isomorphic to a subgroup of $\sum(G)$.

   Note that if $|G| = n$, then $\sum(G) = S_n$, the n-th symmetric group. Every finite group is contained in some symmetric group.

Definition 1.4.8. *(Coset) Suppose S, T to be subsets of a group G. Then define $S \cdot T = \{s \cdot t : s \in S, t \in T\} \subseteq G$.*

Note that if $S = \{s\}$, then $ST = sT$. Similarly if $T = \{t\}$ then $ST = St$.

Let $H \subseteq G$ be a subgroup, with $x \in G$. Then xH is the left coset of H in G, and Hx is the right coset of H in G.

Property 1.4.9. 1. *$(S \cdot T) \cdot V = S \cdot (T \cdot V)$.*

2. *If $H \subseteq G$ is a subgroup, then $H \cdot H = H$.*

   - *$\forall h, h' \in H$, $hh' \in H$, and so $H \cdot H \subseteq H$.*

   - *$\forall h \in H$, we have $h = h \cdot e \in H \cdot H$, therefore $H \subseteq H$.*

Lemma 1.4.10. *$xH = H \iff x \in H \iff Hx = H$.*

Proof. We prove the equivalence of the first two statements.

If $xH = H$, then $x = x \cdot e \in xH = H$. If $x \in H$, then $xH \subseteq H \cdot H = H$, and for all $h \in H$, $h = x \cdot (x^{-1} \cdot h) \in xH$. Hence, $xH = H$.

Note that $xH = yH$ if and only if $(y^{-1}x)H = H$ if and only if $y^{-1}x \in H$. Similarly $Hx = Hy$ if and only if $yx^{-1} \in H$. □

Remark 1.4.11. *Note that* $xH = yH \iff (y^{-1}x)H = H \iff y^{-1}x \in H$. *Similarly* $Hx = Hy \iff yx^{-1} \in H$.

Proposition 1.4.12. *Let $H \subseteq G$ be a subgroup, then xH and yH are either disjoint or equal.*

Proof. Consider xH and yH that are not disjoint. Then there is $z \in xH \cap yH$, which means $z \in xH$ and $z \in yH$. By definition, since $z \in xH$, then $zH = xH$, and similarly we have $zH = yH$, hence $xH = yH$. Therefore, they are equal. □

Remark 1.4.13. *Note that G is the disjoint union of left (right) cosets.*

Definition 1.4.14 (Index)**.** *Let G be a group with subgroup $H \subseteq G$. Index of H in G, denoted as $[G : H]$, is the number of left/right cosets of H in G.*

Theorem 1.4.15 (Lagrange)**.** *Let G be a finite group with subgroup $H \subseteq G$. Then $|G| = |H| \cdot [G : H]$. In particular, $|H|$ divides $|G|$.*

Proof. It suffices to show that $\mathrm{card}(xH) = \mathrm{card}(yH)$ for all $x, y \in G$. Notice that $H \to xH$ given by $h \mapsto xh$ is a bijection, therefore the cardinalities all equal to the cardinality of H. Hence, the cardinalities agree. □

Corollary 1.4.16. *Let G be a finite group with $x \in G$. Then 1) $ord(x) \mid G$ and 2) $x^{|G|} = e$.*

Proof.

1. Let $\mathrm{ord}(x) = n$, then $\langle x \rangle = \{e, x, x^2, \cdots, x^{n-1}\}$ is a cyclic subgroup of G with order n. Therefore $|\langle x \rangle| \mid |G|$, hence the order of a divides the order of G.
2. We write $|G| = nk$ with $\mathrm{ord}(x) = n$. Then $x^{|G|} = (x^n)^k = e^k = e$.

□

Example 1.4.17. *Let $n > 0$. Then $(\mathbb{Z}/n\mathbb{Z})^\times = \{[a]_n : \gcd(a, n) = 1\}$ is a group of order $\varphi(n)$. Since $\gcd(a, n) = 1$, then $[a]_n^{\varphi(n)} = [1]_n$ and so $a^{\varphi(n)} \equiv 1 \pmod{n}$.*

Corollary 1.4.18. *Every group of prime order is cyclic.*

Proof. Take $|G| = p$, then $\exists e \neq x \in G$. As $\mathrm{ord}(x) \mid |G|$ then $\mathrm{ord}(G)$ is either 1 or p. However, since $x \neq e$, then $\mathrm{ord}(x) = p$. Therefore $G = \langle x \rangle$. □

Proposition 1.4.19. *Let G be a group of order $2n$; then G contains an element of order 2. If n is odd and G Abelian, there is only one element of order 2.*

Proof. Suppose not, then for every $e \neq g \in G$, we have $g \neq g^{-1}$, so we group pairs of elements by g and g^{-1}. Note that there is one element left. In particular, this element does not have a distinct inverse, which means it has order 2, contradiction.

We now show that this element is unique if n is odd and G is Abelian. Suppose not, then we have h_1, h_2 with order 2. But now $\{e, h_1, h_2, h_1h_2\}$ is a group of order 4. By Lagrange's Theorem, we have a contradiction. ☐

Definition 1.4.20 (Normal)**.** *Let $H \subseteq G$ be a subgroup. We say H is normal in G or $H \lhd G$ if $xH = Hx$ for all $x \in G$.*

Example 1.4.21. *1. If G is abelian, every subgroup H is normal.*

2. $\{e\}, G \lhd G$.

Proposition 1.4.22. *Let $H \subseteq G$ be a subgroup, then $H \lhd G$ if and only if $xHx^{-1} \subseteq H$ for all $x \in G$.*

Proof. If $H \lhd G$, then $xH = Hx$ and so $xHx^{-1} = Hxx^{-1} = H \subseteq H$.

Suppose $xHx^{-1} \subseteq H$, then $xHx^{-1}x \subseteq Hx$, hence $xH \subseteq Hx$. Similarly as $x^{-1}Hx \subseteq H$, then $Hx \subseteq xH$, and so $xH = Hx$, so H is normal in G. ☐

Example 1.4.23. *1. $SL_n(\mathbb{R}) \lhd GL_n(\mathbb{R})$, with $SL_n(\mathbb{R})$ as the set of $n \times n$ matrices with determinant 1. Indeed, take $A \in SL_n(\mathbb{R})$ and $B \in GL_n(\mathbb{R})$, we have $\det(BAB^{-1}) = \det(B)\det(A)\det(B)^{-1} = 1$.*

2. Note that if $H \lhd G$, then $(xH) \cdot (yH) = x(Hy)H = x(yH)H = (xy)H$. Let G/H be the set of all cosets $xH = Hx$. Operation $(xH) \cdot (yH) = (xy)H$ is well-defined if and only if $H \lhd G$.

Proposition 1.4.24. *Suppose G is a group and K and H are subgroups, satisfying $K \subseteq H \subseteq G$ and $H \lhd G$. Show that if H is cyclic, then $K \subseteq G$.*

Proof. Since H is cyclic, one can write $H = \langle h \rangle$ for some $h \in G$. In particular, since K is a subgroup of H, it must have the form $K = \left\langle h^k \right\rangle$ as a cyclic group as well.

Take arbitrary $g \in G$, and it suffices to show that $gKg^{-1} \subseteq K$. Since $H \lhd G$, there is $ghg^{-1} = h^n$ for some integer n. We then have $(ghg^{-1})^k = (h^n)^k$, which is just $gh^kg^{-1} = h^{nk}$. Now take arbitrary element $(h^k)^a \in K$, then for $g \in G$, we have $g(h^k)^a g^{-1} = (gh^kg^{-1})^a = h^{nka} = (h^n)^{ka} \in K$. By definition, $gKg^{-1} \subseteq K$ for all $g \in G$. Therefore, $K \subseteq G$. ☐

Example 1.4.25. *Consider $G = D_8$, and let a be of order 2 and b be of order 4, satisfying $abab = e$. Then $H = \left\langle a, b^2 \right\rangle$ has order 4, which is normal in G. Also, $K = \langle a \rangle$ has order 2 so it is normal in H. But one can show that K is not normal in G, otherwise $bab^{-1} \in \langle a \rangle$, which means $bab^{-1} = a$, so $ba = ab$, contradiction.*

This is an example of subgroups $K \lhd H \lhd G$, where K is not normal in G.

Claim 1.4.26. *If $H \lhd G$, G/H is a group.*

Proof. 1. $(xH \cdot yH) \cdot zH = (xyH) \cdot zH = (xy)zH = x(yz)H = xH \cdot (yH \cdot zH)$.

2. $e_{G/H} = H$, then $xH \cdot H = xH$, $H \cdot xH = e_H \cdot xH = xH$.

3. $(xH)(x^{-1}H) = eH = H = (x^{-1}H)(xH)$.

$\square$

Remark 1.4.27. *The group G/H called the factor group of G by H.*

Property 1.4.28. *Consider $f : G \to G/H$ such that $x \mapsto xH$. Observe that $f(xy) = (xy)H = xH \cdot yH = f(x) \cdot f(y)$. Also note that f is surjective. Furthermore, $x \in \ker(f) \iff f(x) = e_{G/H} = H \iff xH = H \iff x \in H$. Therefore, $\ker(f) = H$.*

Remark 1.4.29. *The group homomorphism $f : G \to G/H$ defined in 1.4.28 is called the canonical homomorphism.*

Example 1.4.30. *1. $\mathbb{Z}/n\mathbb{Z} = \{a + n\mathbb{Z} = [a]_n\}$.*

2. $\mathbb{C}/\mathbb{R}$. For $z \in \mathbb{C}$, $z + \mathbb{R}$ is the set of horizontal lines on $\mathbb{R}$-$\mathbb{C}$ plane.

3. $\mathbb{C}^\times/U$ for $U = \{z \in \mathbb{C}, |z| = 1\}$. For $z \in \mathbb{C}^\times$, $z \cdot U$ are the circles on the plane.

Proposition 1.4.31 (Universal Property)**.** *Let $f : G \to H$ be a group homomorphism, and $N \lhd G$ such that $N \subseteq \ker(f)$. Then $\exists!$ group homomorphism $\bar{f} : G/N \to H$ such that $f = \bar{f} \circ \pi$, where $\pi : G \to G/N$ is the canonical homomorphism.*

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "\pi"'] & & H \\ & G/N \arrow[ur, dashed, "\exists! \bar{f}"'] & \end{tikzcd}$$

Figure 1.1: Universal Property of Group Homomorphism

Proof. Uniqueness: Suppose there exists $\bar{f}$ such that $f = \bar{f} \circ \pi$. For $x \in G$, $f(x) = \bar{f}(\pi(x)) = \bar{f}(xN)$. Therefore defining $\bar{f}(xN) = f(x)$ is unique.

Existence: We show $\bar{f}(xN) = f(x)$ is well-defined. For $xN = yN$ show $f(x) = f(y)$. $N = x^{-1}yN$ and so $x^{-1}y \in N \subseteq \ker(f)$. Hence, $f(x^{-1}y) = e_H$, so $f(x) = f(y)$.

We can also show that $\bar{f}$ is a group homomorphism. $\bar{f}(xN \cdot yN) = \bar{f}(xyN) = f(xy) = f(x)f(y) = \bar{f}(xN) \cdot \bar{f}(yN)$. Therefore, $\bar{f}$ is homomorphsim. $\square$

1.5 Isomorphism Theorems

Lemma 1.5.1. *Let $f : G \to H$ be a group homomorphism. Then $\ker(f) \lhd G$.*

Proof. For $x \in \ker(f)$, $y \in G$, then $f(yxy^{-1}) = f(y)f(x)f(y)^{-1} = f(y)f(y)^{-1} = e$. So $yxy^{-1} \in \ker(f)$, hence $\ker(f) \lhd G$.

Note if $\pi : G \to G/H$ for group $H \subseteq G$ (not normal), with $\ker(f) = H$, then $H \lhd G$. □

Remark 1.5.2. *Let $f : G \to H$ be a homomorphism with $N = \ker(f) \lhd G$. By the universal property 1.4.31, $\exists! \bar{f} : G/N \to H$ such that the universal property holds with $\bar{f}(xH) = f(x)$. Then $\bar{f} : G/N \to im(f)$ is surjective.*

Theorem 1.5.3 (First Isomorphism Theorem)**.** *$\bar{f} : G/N \to im(f)$ is an isomorphism.*

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"] & H \\ G/N \arrow[r, "\bar{f}"] & \mathrm{im}(f) \arrow[u, hook, "inc"'] \end{tikzcd}$$

Figure 1.2: First Isomorphism Theorem

Proof. It suffices to show that $\ker(\bar{f}) = e_{G/N}$.

Take $xN \in \ker(\bar{f})$ for some $x \in G$. Note $f(x) = \bar{f}(xN) = e_H$, so $x \in \ker(f) = N$, hence $xN = N$. □

Remark 1.5.4. *Note that for $N \lhd G$, if homomorphism $f : G \to H$ is surjective, then $G/N \cong H$.*

Example 1.5.5.

1. *$\mathbb{C}/\mathbb{R} \cong \mathbb{R}$. $f : \mathbb{C} \to \mathbb{C}$ for $f(x + yi) = y$ is surjective with $\ker(f) = \mathbb{R}$.*

2. *$\mathbb{C}^\times/U$ where $U = \{z \in \mathbb{C} : |z| = 1\}$ has the property $\mathbb{C}^\times/U \cong \mathbb{R}^{\times,>0}$. Here $f : \mathbb{C}^\times \to \mathbb{R}^{\times,>0}$ with $f(z) = |z|$.*

Theorem 1.5.6 (Second Isomorphism Theorem)**.** *Let K, N be two subgroups of G with $N \lhd G$. Then:*

1. *KN is a subgroup of G.*

2. *$N \lhd KN$ and $K \cap N \lhd K$, and $KN/N \cong K/(K \cap N)$.*

Proof.

1. $e_G = e_K \cdot e_N \in KN$.

   $(k_1n_1)(k_2n_2) = (k_1k_2)[(k_2^{-1}n_1k_2)n_2] \in KN$.

   $(kn)^{-1} = n^{-1}k^{-1} = k^{-1}(kn^{-1}k^{-1}) \in KN$.

2. $n = e \cdot n \in KN$, $N \subseteq KN$, then since $N \lhd G$, we have $N \lhd KN$.

   Consider $K \xhookrightarrow{f} KN \xrightarrow{\pi} KN/N$ defined by $f(k) = kN$ and π as the canonical homomorphism.

   Note that f is surjective: $(k \cdot n) \cdot N = k \cdot N = f(k)$.

   Now $k \in \ker(f) \iff f(k) = e_{k \cdot N/N} = N \iff k \cdot N = N \iff k \in K \cap N$. Then $\ker(f) = L \cap N \lhd K$.

   By the first isomorphism theorem 1.5.3, $K/(K \cap N) \cong KN/N$.

$\square$

Theorem 1.5.7 (Third Isomorphism Theorem). *Let K and H be two normal subgroups of a group G such that $K \subseteq H$. Then:*

1. *$H/K \lhd G/K$.*
2. *$(G/K)/(H/K) \cong G/H$.*

Proof. Consider $G \xrightarrow{\pi_1} G/K \xrightarrow{\pi_2} (G/K)/(H/K)$.

1. First note $h \in H, g \in G$, then $(gK) \cdot (hK) \cdot (gK)^{-1} = (ghg^{-1})K \in H/K$. Hence $H/K \lhd G/K$.
2. $x \in \ker(f) \iff \pi_1(x) \in \ker(\pi_2) = H/L \iff x \in H$. Therefore, $\ker(f) = H$. By the first isomorphism theorem, $(G/K)/(H/K) \cong G/H$.

$\square$

Example 1.5.8. *Consider $n, m > 0$, $nm\mathbb{Z} \subseteq n\mathbb{Z} \subseteq \mathbb{Z}$, then $(\mathbb{Z}/nm\mathbb{Z})/(n\mathbb{Z}/nm\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$.*

1.6 Group Actions

Definition 1.6.1 (Group Action). *Let G be a group and X be a set. A G-action on X is a map $G \times X \to X$ by $(g, x) \mapsto gx = g \cdot x$, called the action on x, such that:*

1. *$e \cdot x = x$ for all $x \in X$.*
2. *$g_1(g_2x) = (g_1g_2)x$ $\forall g_1, g_2 \in G, x \in X$.*

Example 1.6.2.
1. *Trivial Action $g \cdot x = x$.*
2. *$\sum(X)$ acts on X, for $g \in \sum(X)$ we have $g : X \xrightarrow{\cong} X$, so for $x \in X$ we have $g \cdot x = g(x)$.*
3. *If H acts on X and $f : G \to H$ is a homomorphism, then G acts on X by $g \cdot x = f(g) \cdot x$. This is the pullback action with respect to f.*

4. G acts on G as the left translation: $g \cdot x = gx$. Then $G \to \sum(G)$ is injective, so $G \cong A$ for some subgroup $A \subseteq \sum(G)$.

5. G acts on G by conjugation: $g * x = gxg^{-1}$.

6. If $H \subseteq G$ is a subgroup, take $X = G/H$ as the set of left cosets, then G acts on X with $g \cdot (aH) = g(aH) = gaH$.

Remark 1.6.3. *For G acts on X, $g \in G$, consider $f_g : X \to X$ defined as $f_g(x) = gx$, $f_e = id$ as $f_e(x) = x$ and $f_{g_1} \circ f_{g_2} = f_{g_1 g_2}$, and $f_g \circ f_{g^{-1}} = id = f_{g^{-1}} \circ f_g$. Then f_g is a bijection, which means $f_g \in \sum(X)$. Then $f : G \to \sum(X)$ defined by $g \mapsto f_g$ is a homomorphism.*

In particular, there exists a bijective correspondence between G-actions on X and $Hom(G, \sum(X))$, the set of homomorphisms from G to $\sum(X)$. This map takes $g \cdot x = f(g)(x)$, the pullback of the natural $\sum(X)$-action on X (universal action) with respect to f, to $f : G \to \sum(X)$. Moreover, there is a correspondence between the trivial action and the trivial homomorphism.

Example 1.6.4. 1. An automorphism of G is an isomorphism $G \xrightarrow{\cong} G$. $Aut(G)$ is the automorphism group of G.

For arbitrary $x \in G$, consider $f_x : G \to G$ defined by $f_x(g) = xgx^{-1}$, then f_x is homomorphism. Furthermore, f_x is a bijection, then f_x is an isomorphism. Note $id_G = f_{x^{-1}} \circ f_x = f_x \circ f_{x^{-1}}$ and $f_x(gg') = xgg'g^{-1} = xgx^{-1}xg'x^{-1} = f_x(g)f_x(g')$.

Here, we say $f_x \in Aut(G)$ is an inner automorphism of G.

Consider G acts on $X = G$ by $x * g = xgx^{-1}$, defined by $G \to \sum(X)$, $x \mapsto f_x \in Aut(G)$, $G \xrightarrow{f} Aut(G) \hookrightarrow \sum(G)$. Image of f is the subgroup of all inner automorphisms of G. Note that $Inn(G) \subseteq Aut(G)$ is a subgroup in particular. On the other hand, $\ker(f) = \{x \in G : f_x = id\} = Z(G) \subseteq G$, is exactly the center of G. Indeed, $g = f_x(g) = xgx^{-1}$ for all $g \in G$, so $gx = xg$ for all $g \in G$. In particular, the center is a normal subgroup of G, i.e. $Z(G) \lhd G$.

By the first isomorphism theorem, $G/Z(G) \cong Inn(G)$. Note that the group $Inn(G)$ is trivial if and only if G is an Abelian group, and $Inn(G)$ is cyclic if and only if it is trivial.

2. Consider $Aut(\mathbb{Z}/n\mathbb{Z})$. Note that an automorphism $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is determined by image of the identity: $[1]_n \mapsto [a]_n$ where $\gcd(a, n) = 1$, then $[k]_n \mapsto [ka]_n$.

   Note $Aut(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$: if $f([1]) = [a]$, $g([1]) = [b]$, then $(g \circ f)([1]) = [ab]$.

   In particular, $Aut(\mathbb{Z}) = \{\pm 1\}$.

3. Let X be the set of all subsets of G. Consider G acts on X by conjugation $g * H = gHg^{-1} = f_g(H)$.

Definition 1.6.5 (Orbit, Stabilizer, Transitive)**.** *Consider G acts on X. Define a relation on X: $\forall x, x' \in X$, $x \sim x'$ if $x' = gx$ for some $g \in G$. Note that $\sim$ is an equivalence relation. Now X is a disjoint union of equivalence classes, called orbits (G-orbits).*

More generally, for $x \in X$, $G \cdot x = \{g \cdot x \mid g \in G\}$ is the orbit of x. Note $Gx_1 = Gx_2$ if and only if x_1, x_2 belong to the same orbit.

The group action is transitive if there is exactly one orbit. Note that G acts transitively on X if $X \neq \varnothing$ and for all $x, x' \in X$, $\exists g \in X$ such that $x' = g \cdot x$. So $G \cdot x = X$ for all $x \in X$.

For G acts on X and $x \in X$, $Stab(x) = \{g \in G : g \cdot x = x\}$ is a subgroup of G, called the stabilizer of G.

Example 1.6.6. *1. If G acts trivially on X, then orbit $G \cdot x = \{x\}$ and stabilizer is G.*

2. Suppose G acts on itself by conjugation $*$. For $x \in G$, $G * x = \{gxg^{-1}, g \in G\}$, the orbit is the conjugacy classes of x in G.

Note that $x \in Z(G) \iff gxg^{-1} = x \; \forall g \in G \iff$ conjugacy class of x is $\{x\}$. The stabilizer is $\{g \in G : gxg^{-1} = x$ i.e. $gx = xg\}$, which is called the centralizer of x. Moreover, a centralizer of a subgroup H of G, denoted $C_G(H)$, is the set of elements in G that acts as centralizers on all elements in H. In particular, $C_G(H) \cap H = Z(H)$.

3. Suppose subgroup $H \subseteq G$, then let $X = G/H$ be the set of left cosets xH. Consider G acts on X by left translation. This action is transitive, as $xH = (xy^{-1})yH$ for all $x, y \in G$. Note that $H = eH \in X$, then the stabilizer of H is $Stab(H) = \{g \in G : gH = H\} = H$.

4. Take subgroup $H \subseteq G$. Let X be the set of subgroups of G. Suppose G acts on X by conjugation: $g * H = gHg^{-1} = f_g(H)$. Now the orbit of H is the set of all subgroups gHg^{-1}. $Stab(H) = \{g \in G : gHg^{-1} = H\} = N_G(H)$, the normalizer of H in G. In particular, $H \lhd N_G(H)$. Hence, $H \lhd G \iff N_G(H) = G$.

Theorem 1.6.7 (Orbit-Stabilizer Theorem)**.** *Let group G act on a set X. Take $x \in X$. Then $card(G \cdot x) = [G : Stab(X)]$. In particular, if G is finite, then $card(G \cdot x) = \frac{|G|}{|Stab(x)|}$.*

Proof. Consider $f : G/\text{Stab}(x) \to G \cdot x$, $f(g\text{Stab}(x)) = gx$. We need to show that $g \cdot \text{Stab}(x) = g' \cdot \text{Stab}(x)$ then $gx = g'x$. In particular, if $g^{-1}g' \in \text{Stab}(x)$, $g^{-1}g'x = x$, hence $gx = g'x$. Therefore, the function is well-defined.

We claim that f is injective, note that if $gx = g'x$, then $g^{-1}g'x = x$, then $g^{-1}g' \in \text{Stab}(x)$. Therefore, $g \cdot \text{Stab}(x) = g' \cdot \text{Stab}(x)$ as desired. Note f is also surjective, hence it is a bijection. □

Example 1.6.8. *Let $H \subseteq G$ be a subgroup. The number of subgroups of G conjugate to H is $[G : N_G(H)]$.*

Theorem 1.6.9. *Let G be a finite group, and p is the smallest prime divisor of $|G|$. Then every subgroup $H \subseteq G$ with $[G : H] = p$ is normal.*

Proof. Take $X = G/H$ with $\text{card}(X) = p$. Consider G acts on X by left translation.

Define $f : G \to \sum(X) = S_p$, then $N = \ker(f) \lhd G$. We claim that $H = N$.

Note $f(g)(xH) = gxH$. When $g \in N$, $x = e$, $f(g)(H) = H$. Therefore, $N \subseteq H$.

Consider $\mathrm{im}(f) \subseteq S_p$ as subgroup. Then $|\mathrm{im}(f)| \mid p!$. Note $\mathrm{im}(f) \cong G/N$. Then $|\mathrm{im}(f)| = [G : N] \mid |G|$. Therefore, $|\mathrm{im}(f)| = 1$ or p. However, $[G : N] \leq p$, so $[G : H] = p$, hence $H \subseteq N$.

Therefore, $[G : N] = p$, and so $H = N$. □

Proposition 1.6.10 (Class Equation of a Group Action)**.** *Let G be a group and X be a finite set. Suppose we are given a group action of G on X.*

- *Let S_0 be the set of points in S that is fixed by the action of all elements of G.*
- *Let $O_1, \cdots, O_r$ be the orbits of size greater than 1 under this action. For each orbit O_i, take $s_i \in O_i$ and let $G_i = \mathbf{stab}(s_i)$.*

The class equation of this action is given by

$$|S| = |S_0| + \sum_{i=1}^{r} \frac{|G|}{|G_i|}.$$

Corollary 1.6.11 (Class Equation of a Group)**.** *Suppose G is a finite group, $Z(G)$ is the center of G, and $C_1, C_2, \cdots, C_r$ are all the conjugacy classes in G comprising the elements outside the center. Let g_i be an element in C_i for each $1 \leq i \leq r$. Then, we have $|G| = |Z(G)| + \sum_{i=1}^{r} |G : C(g_i)|$, where $C(g_i)$ is the centralizer of g_i.*

Remark 1.6.12. *This is a particular case of class equation of a group action, when we consider the action to be G acting on itself by conjugation.*

Lemma 1.6.13 (Burnside's Lemma)**.** *Let G be a finite group that acts on a set X. For each g in G let X^g denote the set of elements in X that are fixed by g (also said to be left invariant by g), i.e. $X^g = \{x \in X | g \cdot x = x\}$. Then the number of orbits, denoted as $|X/G|$, satisfies $|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$.*

Proof. First of all, observe that $\sum_{g \in G} |X^g| = \{(g, x) \in (G, X) : g \cdot x = x\} = \sum_{x \in X} \mathrm{Stab}(x) = |G| \cdot \sum_{x \in X} \frac{1}{\mathrm{Orb}(x)}$. Therefore, $\sum_{x \in X} \frac{1}{|\mathrm{Orb}(x)|} = \frac{1}{|G|} \sum_{g \in G} |X^g|$. However, by splitting the elements into individual orbits, we may derive $\sum_{x \in X} \frac{1}{|\mathrm{Orb}(x)|} = \sum_{A \in X/G} \sum_{x \in A} \frac{1}{|\mathrm{Orb}(x)|} = \sum_{A \in X/G} 1 = |X/G|$. Therefore, $|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$. □

Remark 1.6.14. *The proof of class equations above should be very similar to the proof we provided for Burnside's Lemma, so we only listed this particular proof here.*

1.7 Sylow Theorems

Definition 1.7.1 (p-group, Fixed Set)**.** *Let p be a prime integer. A group G is called a p-group if $|G| = p^n$ for some $n > 0$. A subgroup $H \subseteq G$ is a p-subgroup if H is a p-group.*

Suppose G acts on X, then we define $X^G = \{x \in X : gx = x \ \forall g \in G\} \subseteq X$.

Lemma 1.7.2. *Let a group H act on a set X. If H is a p-group and X is finite, then $|X^H| \equiv |X| \pmod{p}$.*

Proof. Suppose $X^H = \{x_1, \cdots, x_n\}$ with $|X^H| = n$.

Note that $\mathrm{Orb}(X_i) = H \cdot x_i = \{x_i\}$ with size 1.

Now consider $X = \coprod_{i=1}^{n+m} \mathrm{Orb}(x_i)$ as the disjoint union of orbits. For $i \leq n$, $|\mathrm{Orb}(x_i)| = 1$. For $i > n$, $\mathrm{Orb}(x_i) = \frac{|H|}{|\mathrm{Stab}(x_i)|}$ where $H = p^k$ for some k. In particular, $p \mid \mathrm{Orb}(x_i)$ since their orbit sizes are greater than 1.

Therefore, $|X| = n \times 1 + \sum_{i=n+1}^{n+m} |\mathrm{Orb}(x_i)|$, which means $|X| \equiv n \pmod{p}$. Therefore, $|X| \equiv |X^H| \pmod{p}$. $\square$

Theorem 1.7.3 (Cauchy)**.** *Let G be a finite group and p be a prime divisor of $|G|$. Then G has an element of order p.*

Proof. Consider the set $X = \{(g_1, g_2, \cdots, g_p), g_i \in G, g_1 g_2 \cdots g_p = e\}$. Note that $g_p = (g_1 g_2 \cdots g_{p-1})^{-1}$. So $|X| = |G|^{p-1}$ and is divisible by p. Also note that $g_p g_1 g_2 \cdots g_{p-1} = e$ as well. Then if $(g_1, g_2, \cdots, g_p) \in X$, then $(g_p, g_1, \cdots, g_{p-1}) \in X$. By shifting p times, we are back to the start. Hence, there is a cyclic group H of order p with generator $\sigma \in H$, then H acts on X by $\sigma(g_1, \cdots, g_p) = (g_p, g_1, \cdots, g_{p-1})$.

Since H is a p-group, by lemma, $|X^H| \equiv |X| \pmod{p}$.

Observe that $(e, e, \cdots, e) \in X^H$, then $|X^H| > 0$, therefore $|X^H| \geq p > 1$, then there exists a non-trivial tuple $(g_1, g_2, \cdots, g_p) \in X^H$. By definition, this tuple must have the form $(g, g, \cdots, g)$ for some $e \neq g \in G$. Recall that $g^p = e$ by definition, then $\mathrm{ord}(g) = p$. $\square$

Proposition 1.7.4. *Let G be a p-group, then $Z(G) \neq \{e\}$.*

Proof. Consider G acts on $X = G$ by conjugation. Then $X^G = \{x \in G : gxg^{-1} = x \ \forall x \in G\} = Z(G)$.

By lemma 1.7.2, $|X^G| \equiv |X| \pmod{p}$. Then $|Z(G)| \equiv |G| \pmod{p}$. Hence, $p \mid |Z(G)|$. In particular, $Z(G) \neq \{e\}$. $\square$

Remark 1.7.5. *The center of a group is the set of fixed points on the group action of self-conjugation on G.*

Lemma 1.7.6. *Let H be a p-subgroup of a finite group G. Then $[N_G(H) : H] \equiv [G : H] \pmod{p}$.*

Remark 1.7.7. *Note that the normalizer of H, $N_G(H)$, is the largest subgroup of G that satisfies $H \lhd N_G(H)$.*

Proof. Consider H acts on the set $X = G/H$ of left cosets by left translation. Note that $|X| = [G : H]$. We want to show that $|X^H| = [N_G(H) : H] = |N_G(H)/H|$. For $g \in G$, note that $gH \in X^H \iff hgH = gH \ \forall gH \iff g^{-1}hgH = H \ \forall h \in H \iff g^{-1}hg \in H \ \forall h \in$

$H \iff g^{-1}Hg \subseteq H \iff g^{-1} \in N_G(H) \iff g \in N_G(H) \iff gH \in N_G(H)/H$. Therefore, $X^H = N_G(H)/H$. We conclude the proof by applying the lemma 1.7.2. □

Theorem 1.7.8 (First Sylow Theorem)**.** *Let G be a finite group of order $p^n \cdot m$ for prime p and $n > 0$, and $\gcd(p, m) = 1$. So p^n is the highest power of p dividing $|G|$.*

1. *For every $k = 0, 1, \cdots, n-1$, every subgroup of G of order p^k is a normal subgroup of a subgroup of order p^{k+1}.*
2. *G has subgroups of order $1, p, p^2, \cdots, p^n$.*

Remark 1.7.9. *It is not true that if $a \mid |G|$ then G has a subgroup of order a.*

Proof. It suffices to prove the first statement. If 1) is true, then $\{e\} \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n$ can be found where H_i has order p^i.

Consider $|H| = p^k$ for $k = 0, 1, \cdots, n-1$. As H is a p-subgroup, by lemma 1.7.6, $[N_G(H) : H] \equiv [G : H] \pmod p$. Therefore, $[N_G(H) : H] \equiv \frac{p^n \cdot m}{p^k} \pmod p$. Since $k < n$, then $p \mid [N_G(H) : H]$. Recall that $H \lhd N_G(H)$. Then $N_G(H)/H$ is a factor group of order divisible by p.

By Cauchy's Theorem, $\exists F \subseteq N_G(H)/H$ such that $|F| = p$. Then we have

$$\begin{tikzcd} N_G(H) \arrow[r, two heads, "\pi"] & N_G(H)/H \\ \pi^{-1}(F) \arrow[u, hook, "inc"] \arrow[r, two heads, "\pi'"] & F \arrow[u, hook, "inc"'] \end{tikzcd}$$

Recall that $H = \pi^{-1}(e)$, then $H \subseteq N_G(H) \cap \pi^{-1}(F)$. In particular, $H \lhd \pi^{-1}(F)$. By the first isomorphism theorem, $\pi^{-1}(F)/H \cong F$, so $\frac{|\pi^{-1}(F)|}{|H|} = |\pi^{-1}(F)/H| = |F| = p$, which means $|\pi^{-1}(F)| = |H| \cdot p = p^k \cdot p = p^{k+1}$. This concludes the proof. □

Remark 1.7.10. *Consider $|G| = p^n \cdot m$ where $\gcd(m, p) = 1$ and $n > 0$. By the First Sylow theorem, there exists a subgroup $P \subseteq G$ of order p^n.*

Definition 1.7.11 (Sylow p-group)**.** *The group P defined in remark 1.7.10 is called a Sylow p-group of G.*

Remark 1.7.12. *For $g \in G$ and a Sylow p-group P, gPg^{-1} is also a Sylow p-group. In particular, $gPg^{-1} \cong P$.*

Take $|G| = p^n \cdot m$, $\gcd(m, p) = 1$ with $n > 0$. Let $P \subseteq G$ be a Sylow p-subgroup, i.e. $|P| = p^n$.

Theorem 1.7.13 (Second Sylow Theorem)**.** *Let G be a finite group, $|G|$ is divisible by prime p, let $P \subseteq G$ be a Sylow p-subgroup. Then*

1. *For every p-subgroup $H \subseteq G$ there is $g \in G$ such that $H \subseteq gPg^{-1}$, and*
2. *Every two Sylow p-subgroup of G are conjugate.*

Proof.

1. Consider H acts on $X = G/P$ by left translations.
   $|X| = [G : P] = \frac{|G|}{|P|} = \frac{p^n \cdot m}{p^n} = m$. By lemma **1.7.2**, $|X^H| \equiv |X| = m \not\equiv 0 \pmod{p}$. Therefore, $X^H \neq \varnothing$, so $\exists gP \in X$ such that $hgP = gP$ for all $h \in H$. Then $g^{-1}hgP = P$, and so $g^{-1}hg \in P \in P$, then $h \in gPg^{-1}$ for all $h \in H$, hence $H \subseteq gPg^{-1}$.

2. Let Q be another Sylow p-subgroup of G. By 1), $Q \subseteq gPg^{-1}$ for some $g \in G$. Therefore, $p^n = |Q| = |P| = |gPg^{-1}|$, and so $Q = gPg^{-1}$.

□

Corollary 1.7.14. *A Sylow p-subgroup P in G is normal in G if and only if P is the only Sylow p-subgroup of G.*

Proof. $\Rightarrow$: If Q is Sylow p-subgroup, then $Q = gPg^{-1} = P$.
$\Leftarrow$: Suppose gPg^{-1} is a Sylow p-subgroup, then $gPg^{-1} = P$ for all $g \in G$, therefore $P \lhd G$. □

Theorem 1.7.15 (Third Sylow Theorem)**.** *Let G be a finite group, with $|G| = p^n \cdot m$, and suppose* $\gcd(m, p) = 1$ *and $n > 0$. Then the number of Sylow p-subgroup divides m and is congruent to* 1 *mod p.*

Proof. Note that the number of Sylow p-subgroup is the number of subgroups in the conjugacy class of a fixed Sylow p-subgroup $P \subseteq G$. Therefore, the number is equivalent to $[G : N_G(P)] = \frac{|G|}{|N_G(P)|}$ divides $\frac{|G|}{|P|} = m$.

Let X be the set of all Sylow p-subgroups of G, then P acts on X by conjugation. By lemma, $|X^P| \equiv |X| \pmod{p}$. Take $Q \in X^P$, then $pQp^{-1} = Q$ for all $p \in P$. Now P and Q are both subgroups of $N_G(Q)$. Also note that since P is Sylow in G, and $P \subseteq N_G(Q) \subseteq G$, then P is a Sylow p-subgroup in $N_G(Q)$. On the other hand, by definition Q is a Sylow p-subgroup of G as well, then similarly Q is a Sylow p-subgroup in $N_G(Q)$ since $Q \subseteq N_G(Q) \subseteq G$. Furthermore, recall that $Q \lhd N_G(Q)$, then by the previous corollary **1.7.14**, Q is the only Sylow p-subgroup of $N_G(Q)$. Therefore, $P = Q$, and so $X^P = \{P\}$, which means $|X^P| = 1$, then $|X| \equiv 1 \pmod{p}$. □

Proposition 1.7.16. *Let P denote a Sylow p-subgroup of a finite group G. Let $N_G(P)$ denote the normalizer of P in G.*

1. *Show that P is the unique Sylow-p subgroup of $N_G(P)$.*

2. *Let $\varphi \in \mathbf{Aut}(G)$, then $\varphi(P)$ is also a Sylow p-subgroup of G.*

3. *$N_G(N_G(P)) = N_G(P)$.*

Proof.

1. Since P is a Sylow p-subgroup of G, then it is also a Sylow p-subgroup in $N_G(P)$. By definition, $P \lhd N_G(P)$, so P is the unique Sylow p-subgroup in $N_G(P)$.

2. Since φ is an automorphism, the image of the map has the same order as P. In particular, the image is also a subgroup of G by definition, so $\varphi(P)$ is a Sylow p-subgroup of G.

3. Suppose $g \in N_G(N_G(P))$. We show that $g \in N_G(P)$. By definition, $gN_G(P)g^{-1} \subseteq N_G(P)$. Since P is the unique Sylow p-subgroup in $N_G(P)$, then any automorphism would preserve P. In particular, Therefore, g is a normalizer of P, i.e. $g \in N_G(P)$. We then conclude that $N_G(N_G(P)) = N_G(P)$.

$\square$

Example 1.7.17. *Every group G of order $380 = 2^2 \cdot 5 \cdot 19$ is not simple.*

Suppose otherwise, that G does not have a non-trivial normal subgroup.

By Sylow Theorem, $n_{19} \equiv 1 \pmod 19$ and $n_{19} \mid 20$, so n_{19} has to be 20, otherwise we have a normal subgroup. Similarly, $n_5 \equiv 1 \pmod 5$ and $n_5 \mid 76$, so n_5 has to be 1 or 76, and we must have 76 Sylow 5-subgroups. This gives us $20 \cdot (19-1) + 76 \cdot (5-1) = 664$ elements because the two types of Sylow subgroups would not intersect non-trivially. Therefore, contradiction, and G must have be non-simple.

Proposition 1.7.18. *If $p > q$ are primes, a group of order pq has at most one subgroup of order p.*

Proof. Suppose a subgroup H of order p in group G of order pq exists. Note that H must have index q, which is the smallest prime dividing $|G|$, then by **theorem 1.6.9**, H must be normal. In particular, H having order p means it is a Sylow p-subgroup of G. Therefore, H must be unique.

Therefore, if a subgroup of order p exists, it must be unique. That means G would have at most one subgroup of order p. $\square$

1.8 Product

Definition 1.8.1 (External Product, Internal Product). *For groups $G_1, G_2, \cdots, G_n$, $G = G_1 \times G_2 \times \cdots \times G_n$ is the external product.*

For group G and subgroups $H_1, \cdots, H_n \subseteq G$, we say that G is the internal product of $H_1, \cdots, H_N$, i.e. $G = H_1 \times H_2 \times \cdots \times H_n$ if:

1. *$H_i \lhd G$ for all i, and*

2. *Every $g \in G$ can be uniquely written as $g = h_1 \cdots h_n$ with $h_i \in H_i$.*

Remark 1.8.2. 1. *Both external product and internal product are groups.*

2. *For $G = H_1 \times \cdots \times H_n$, $H_i \cap H_j = \{e\}\ \forall i \neq j$.*

   Indeed, take $g \in H_i \cap H_j$, then $g = e_1 \cdots e_{i-1} g e_{i+1} \cdots e_n = e_1 \cdots e_{j-1} g e_{j+1} \cdots e_n$. However, since g has to be uniquely expressed, then $g = e$.

3. *For $x \in H_i$ and $y \in H_j$ and $i \neq j$, we have $xy = yx$.*

   Let $[x, y] = xyx^{-1}y^{-1}$ be the commutator of x and y.

We claim that $[x,y]=e$. *Indeed,* $H_i \ni x(yx^{-1}y^{-1}) = [x,y] = (xyx^{-1})y^{-1} \in H_j$. *Therefore,* $[x,y] \in H_i \cap H_j = \{e\}$. *In particular,* $xy = yx$.

Proposition 1.8.3. 1. *If* G *is the internal product of subgroups, then* $G \cong H_1 \times H_2 \times \cdots \times H_n$ *as an external product.*

2. *If* G *is the external product, then by definition we have* $G = H_1 \times H_2 \times \cdots \times H_n$*, then* $H_i' = \{(e_1, e_2, \cdots, e_{i-1}, h_i, e_{i+1}, \cdots, e_n)\} \subseteq G$. *Then* $H_i' \lhd G$, $H_i' \cong H_i$*, and* $G = H_1' \times H_2' \times \cdots \times H_n'$ *as the internal product.*

Proof. 1. Define $f : H_1 \times H_2 \times \cdots \times H_n \to G$ as the map from the defined external product to G, where $f(h_1, h_2, \cdots, h_n) = h_1 h_2 \cdots h_n \in G$. Observe that $f((h_1, \cdots, h_n) \cdot (h_1', h_2', \cdots, h_n')) = f(h_1h_1', h_2h_2', \cdots, h_nh_n') = h_1h_1'h_2h_2' \cdots h_nh_n' = h_1h_2 \cdots h_nh_1'h_2' \cdots h_n' = f(h_1, \cdots, h_n) \cdot f(h_1', \cdots, h_n')$. Therefore f is a homomorphism. However, recall from the remark that f is bijective, then f is a group isomorphism.

2. Take $H_i' \to H_i$ by $(e_1, \cdots, e_{i-1}, h_i, e_{i+1}, \cdots, e_n) \mapsto h_i$. This is clearly an isomorphism. Then there is $(h_1, \cdots, h_n) = (h_1, e, \cdots, e) \cdot (e, h_2, e, \cdots, e) \cdot \cdots \cdot (e, \cdots, e, h_n)$, which is an isomorphism between G and $H_1' \times H_2' \times \cdots H_n'$.

□

Remark 1.8.4. *For finite group* G *as the internal product, 2) in definition is equivalent to* $G = H_1H_2 \cdots H_n$ *and* $|G| = |H_1| \cdot |H_2| \cdot \cdots \cdot |H_n|$.

Indeed, ($\Rightarrow$) *note that there is a bijection* $H_1 \times \cdots \times H_n \to G$ *and* ($\Leftarrow$) *since* f *is surjective and equalite, then* f *is a bijection, hence 2) holds.*

Example 1.8.5. *Suppose* $\gcd(m,n) = 1$.

Note that $\mathbb{Z}/nm\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ *with* $[a]_{nm} \mapsto ([a]_n, [a]_m)$ *because of Chinese Remainder Theorem. In particular, we can write* $\mathbb{Z}/nm\mathbb{Z} = (m\mathbb{Z}/nm\mathbb{Z}) \times (n\mathbb{Z}/nm\mathbb{Z})$.

Proposition 1.8.6. *Let* G *be a finite group such that all Sylow subgroups of* G *are normal. Then* G *is the (internal) product of all Sylow subgroups.*

Proof. Denote $|G| = p_1^{k_1} p_2^{k_2} \cdots p_s^{k_s}$ where p_i are distinct primes. Define P_i as Sylow p_i-subgroup for $i = 1, \cdots, s$. Note that $|G| = |P_1||P_2| \cdots |P_s|$, and every Sylow subgroup is normal in G. By remark 1.8.4, it suffices to show that $G = P_1P_2 \cdots P_s$.

Take $g \in G$, define $q_i = \frac{|G|}{p_i^{k_2}}$, then $\gcd(q_1, q_2, \cdots, q_s) = 1$. Then by Bezout's Lemma, $\sum_{i=1}^{s} q_i m_i = 1$ for some $m_i \in \mathbb{Z}$.

Now, $g = g^1 = \prod_{i=1}^{s} (g^{q_i})^{m_i}$. Since $q_i \cdot p_i^{k_i} = |G|$, and $g^{|G|} = e$, we have $(g^{q_i})^{{p_i}^{k_i}} = e$. We know g^{q_i} generates a cyclic subgroup $H_i \subseteq G$ (p_i-subgroup) of order dividing $p_i^{k_i}$.

By the Second Sylow Theorem 1.7.13, $H_i \subseteq xP_ix^{-1} = P_i$ for some $x \in G$. Then $g^{q_i} \in H_i \subseteq P_i$, which means $(g^{q_i})^{m_i} \in P_i$. Therefore, $G = P_1P_2 \cdots P_s$. This concludes the proof. □

Corollary 1.8.7. *Let G be a group of order pq for prime p and q. Suppose $p > q$. If $p \not\equiv 1 \pmod q$, then G is cyclic.*

Proof. Let P_p and P_q be Sylow subgroups of order p and q, respectively.

Note that $[G : P_p] = \frac{|G|}{|P_p|} = \frac{pq}{p} = q$ is the smallest prime divisor of $|G| = pq$. Therefore, by **1.6.9**, $P_p \lhd G$.

Take $H = N_G(P_q)$, then $P_q \subseteq H \subseteq G$. Note that $|H| = q$ or pq, then $[G : H] = 1$ or p. However, by the Third Sylow Theorem **1.7.15**, the number of Sylow q-subgroups is congruent to 1 modulo q. Therefore, $[G : H] = 1$, so $G = H$, which means $P_q \lhd G$.

By proposition **1.8.6**, corollary **1.4.18**, theorem **1.3.4** and example **1.8.5**, $G = P_p \times P_q \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \cong \mathbb{Z}/pq\mathbb{Z}$, which means G is a cyclic group. This concludes the proof. □

Proposition 1.8.8. $|HK| = \frac{|H||K|}{|H \cap K|}$ *for H, K as subgroups of G.*

Proof. Consider the group homomorphism $f : H \times K \to HK$. This is clearly surjective. The equivalence class that sends elements in $H \times K$ to the same element in HK is exactly the set of elements $\{h_1k_1 = h_2k_2, h_i \in H, k_i \in K\}$. However, now $h_1^{-1}h_2 = k_1k_2^{-1} \in H \cap K$. Note that the number of pairs of (h_2, k_2) that makes the relation hold is exactly the number of elements in $h_1(H \cap K)$, which is just $|H \cap K|$. Therefore, $|HK| = \frac{|H||K|}{|H \cap K|}$. □

1.9 Nilpotent and Solvable Group

Definition 1.9.1 (Generated Subgroup). *For any group G and subset $S \subseteq G$, $\langle S \rangle$ is the smallest subgroup of G containing S. This is the subgroup generated by S.*

Proposition 1.9.2. $\langle S \rangle = \{x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}, x_i \in S, \varepsilon = \pm 1\}$.

Proof. Define $H = \{x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}, x_i \in S, \varepsilon = \pm 1\}$. Note that $H \subseteq G$ is a subgroup. [1]

Now for $x \in S$, $x = x^1 \in H$, so $S \subseteq H$, which means $\langle S \rangle \subseteq H$.

On the other hand, for $x_i \in S$, $x_i^{\varepsilon_i} \in S \subseteq \langle S \rangle$. Therefore, an arbitrary element $x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n} \in \langle S \rangle$. Therefore, $H \subseteq \langle S \rangle$. We conclude that $H = \langle S \rangle$. □

Remark 1.9.3. *If S satisfies $gSg^{-1} \subseteq S$ for all $g \in G$, then $\langle S \rangle \lhd G$.*

Example 1.9.4. *Let $S = \{g\}$, then $\langle S \rangle = \{g^k, k \in \mathbb{Z}\}$ is the cyclic group generated by g.*

Definition 1.9.5 (Commutator). *Let G be a group and $x, y \in G$. The commutator of x and y is $[x, y] = xyx^{-1}y^{-1}$.*

Property 1.9.6.

1. $[x, y] = e \iff xy = yx$
2. $g[x, y]g^{-1} = [gxg^{-1}, gyg^{-1}]$

[1] Note that even if S is empty, H still contains the empty product as an element, which is equivalent to the identity by definition. Therefore, H is not empty.

3. $[x, y]^{-1} = [y, x]$

Definition 1.9.7 (Commutator Subgroup/Derived Subgroup)**.** *The commutator subgroup (derived subgroup) is the subgroup generated by all commutators of G, denoted as $[G, G]$. In particular, an arbitrary element in $[G, G]$ has the form $[x_1, y_1] \cdot [x_2, y_2] \cdot \cdots \cdot [x_n, y_n]$ where $[x_i, y_i]$ is a generator of $x_i, y_i \in G$.*

Remark 1.9.8. *1. By **example 1.9.6**, $[G, G] \lhd G$.*

2. G is Abelian if and only if $[G, G] = \{e\}$.

Proposition 1.9.9. *Let $N \lhd G$. Then G/N is Abelian if and only if $[G, G] \subseteq N$.*

Proof.

$$\begin{aligned}
G/N \text{ Abelian} &\iff xN \cdot yN = yN \cdot xN \ \forall x, y \in G \\
&\iff xyN = yxN \ \forall x, y \in G \\
&\iff x^{-1}y^{-1}xyN = N \ \forall x, y \in G \\
&\iff [x^{-1}, y^{-1}] \in N \ \forall x, y \in G \\
&\iff [G, G] \subseteq N
\end{aligned}$$

□

Remark 1.9.10. *Observe that if $[G, G] \subseteq N \subseteq G$, then $N \lhd G$. Indeed, for arbitrary $g \in G$, $n \in N$, we have $gng^{-1}n^{-1} = h$ for some $h \in [G, G] \subseteq N$. Therefore, $gng^{-1} = hn \in N$. Hence, $N \lhd G$.*

*Therefore, a better interpretation of proposition **1.9.9** is the following: let $N \subseteq G$ be a subgroup. Then $[G, G] \subseteq N$ if and only if $N \lhd G$ and G/N is Abelian.*

Proposition 1.9.11. *If $f : G \to H$ is a homomorphism, H is Abelian, and N is a subgroup of G containing $\ker(f)$, then $N \lhd G$.*

Proof. By the first isomorphism theorem, we have $G/\ker(f) \cong \mathbf{im}(f) \subseteq H$. Since H is Abelian, we have $G/\ker(f)$ to be Abelian. By proposition, $[G, G] \subseteq \ker(f)$. Therefore, N has to contain $[G, G]$, which means $N \lhd G$. □

Remark 1.9.12 (Abelianization)**.** *Following remark **1.9.10**, take $N = [G, G] \lhd G$, it then follows that $G/[G, G]$ is Abelian. This group is called the Abelianization of group G.*

Proposition 1.9.13 (Universal Property of Abelian Groups)**.** *Let $f : G \to H$ be a group homomorphism where H is Abelian. Then the following diagram commutes:*

$$\begin{tikzcd}
G \arrow[rr, "f"] \arrow[dr, "\pi"'] & & H \\
& G/[G,G] \arrow[ur, dashed, "\exists! \bar{f}"'] &
\end{tikzcd}$$

Figure 1.3: Universal Property of Abelianization

Proof. By **remark 1.9.10**, $[G,G] \subseteq N$ if and only if $N \triangleleft G$ and G/N is Abelian. Take $N = \ker(f)$, then N is clearly normal, and $G/N \cong \mathrm{im}(f) \subseteq H$ must be Abelian, which means $[G,G] \subseteq N$. By the universal property **proposition 1.4.31**, we have the diagram as desired. □

Definition 1.9.14 (Solvable Group)**.** *Let G be a group. Define $G^{(0)} = G, G^{(1)} = [G,G], \cdots, G^{(i+1)} = [G^{(i)}, G^{(i)}]$. Therefore, $G = G^{(0)} \triangleright G^{(1)} \triangleright \cdots \triangleright G^{(n)} \triangleright \cdots$. Note that $G^{(i)}/G^{(i+1)}$ is Abelian. We say G is solvable if $G^{(n)} = \{e\}$ for some n.*

Property 1.9.15.

1. *G is solvable if and only if there is a sequence of subgroups $G = G_0 \supset G_1 \cdots$ such that $G_{i+1} \triangleleft G_i$ $\forall i$, G_i/G_{i+1} is Abelian, and $G_n = \{e\}$ for some n.*
2. *A subgroup of a solvable group is solvable.*
3. *If G is solvable and $N \triangleleft G$, then G/N is solvable.*
4. *Let $N \triangleleft G$, then G is solvable if and only if N and G/N are solvable.*

Proof.

1. Obviously if G is solvable, then $G_i = G^{(i)}$.

   Notice that G_i/G_{i+1} Abelian if and only if $[G_i, G_i] \subseteq G_{i+1}$. We show $G^{(i)} \subseteq G_i$ by induction on i. Suppose this is true, then $G^{(n)} \subseteq G_n = \{e\}$, which means $G^{(n)} = \{e\}$ is solvable.

   The case $i = 0$ is clear. Suppose the case is true at i, consider the case with $i+1$. By definition, $G^{(i+1)} = [G^{(i)}, G^{(i)}] \subseteq [G_i, G_i] \subseteq G_{i+1}$. This concludes the proof.

2. Let G be solvable. Then $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = \{e\}$, $G_{i+1} \supseteq [G_i, G_i]$.

   Let $H \subseteq G$ and define $H_i = H \cap G_i$.

   Since $G_{i+1} \triangleleft G_i$, then by the Second Isomorphism Theorem **1.5.6**, $H_{i+1} \triangleleft H_i$. Now, $H_i \cap G_{i+1} = H_{i+1} \subseteq H_i$, and so $[H_i, H_i] \subseteq [G_i, G_i] \cap H_i \subseteq G_{i+1} \cap H_i = H_{i+1}$.

   Hence, H_i/H_{i+1} is Abelian. Then by property 1, H is solvable.

3. Take $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = \{e\}$. Note that $G_{i+1} \supseteq [G_i, G_i]$.

   Now $G/N = GN/N = G_0/N \triangleright G_1N/N \triangleright \cdots \triangleright G_nN/N = N/N = \{e\}$ where $G_{i+1}N/N \supseteq [G_iN/N, G_iN/N]$. Indeed, for $g, g' \in G$ and $n, n' \in N$, we then have $[gnN, g'n'N] = [gN, g'N] = [g,g']N \in G_{i+1}N$. Therefore, G/N is solvable.

4. The $\Longrightarrow$ direction has been proven. We prove the $\Longleftarrow$ direction.

   Observe that $N = N_0 \triangleright N_1 \triangleright \cdots \triangleright N - n = \{e\}$ where N_i/N_{i+1} is Abelian.

   Now $G/N = F_0 \triangleright F_1 \triangleright \cdots \triangleright F_m = \{e\}$ is a sequence where F_i/F_{i+1} is Abelian. However, let $\pi : G \to G/N$ be the canonical homomorphism, and consider the preimage $G_i = \pi^{-1}(F_i)$, we have a corresponding sequence $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_m$. The nested subgroups are normal by the correspondence of preimage of the surjective homomorphism. Observe that $G_m = \pi^{-1}(F_m) = \ker(\pi) = N$.

Collecting the properties from above, we have $G = G_0 \rhd G_1 \rhd \cdots \rhd G_m = N = N_0 \rhd N_1 \rhd \cdots \rhd N_n = \{e\}$.

There is $\ker(G_i = \pi^{-1}(F_i) \twoheadrightarrow F_i) = N$, so by the First Isomorphism Theorem **1.5.3**, $F_i \cong G_i/N$. In particular, by the Third Isomorphism Theorem **1.5.7**, $F_i/F_{i+1} \cong (G_i/N)/(G_{i+1}/N) \cong G_i/G_{i+1}$. Therefore, G is solvable by property 1.

□

Example 1.9.16. *1. p-groups are solvable.*

This can be proven by induction on $|G|$. Since G is a p-group, then by proposition ***1.7.4****, $Z(G) \neq \{e\}$, and by definition $Z(G)$ is an Abelian group. Notice that the commutator subgroup of an Abelian group has to be trivial, then by definition $Z(G)$ is solvable. On the other hand, $G/Z(G)$ is another p-group, but by induction hypothesis it is also solvable. Therefore, by the previous properties, G is solvable.*

2. Let G be a finite group with $|G| = p \cdot q$ where p, q are prime. Then G is solvable.

If $p = q$, use the previous example. Suppose $p \neq q$, without loss of generality, assume $p > q$. Now let N be a Sylow p-subgroup, then $N \cong \mathbb{Z}/p\mathbb{Z}$ is Abelian. Then $[G : N] = q$, which is the smallest prime divisor of $|G|$. Therefore, $N \lhd G$. By the corollary, N is the only Sylow p-subgroup of G. In particular, N is Solvable by the previous remark.

On the other hand, G/N is a factor group of order q since $N \lhd G$. Therefore, G/N is also solvable. By the property above, G is solvable.

3. All groups of order less than 60 are solvable. A_5, with order 60, is not solvable.

The following text on nilpotent groups were not officially covered in lectures. The notes are collected through other sources and through homework problems.

Definition 1.9.17 (Nilpotent Group)**.** *A group G is called nilpotent if there is a sequence of subgroups*

$$G = G_0 \supset G_1 \supset \cdots \supset G_n = \{e\},$$

such that each G_i is normal in G and G_i/G_{i+1} is contained in the center of G/G_{i+1}.

Property 1.9.18. *1. If $H, K < G$ and $[H, K] < H$, then $K < N_G(H)$.*

2. If $H < G$, then $[G, H] = 1$ if and only if $H < Z(G)$.

3. If $H, K < G$ and $N \lhd G$ with $N < H, K$, then $[H/N, K/N] = [H, K]/(N \cap [H, K])$.

Proposition 1.9.19. *A sequence of subgroups $G = G_0 \supset G_1 \supset \cdots \supset G_n = \{e\}$ with $G_i \lhd G$ for each i satisfies $[G, G_i] < G_{i+1}$ for each i if and only if $G_i/G_{i+1} \subset Z(G/G_{i+1})$ for each i.*

Proof. By properties in **1.9.18**, then $[G, G_i] < G_{i+1}$ if and only if $[G/G_{i+1}, G_i/G_{i+1}] = \{e\}$, which happens if and only if $G_i/G_{i+1} < Z(G/G_i)$. ☐

Remark 1.9.20. *An equivalent definition of a nilpotent group G is that there exists a sequence of subgroups*

$$G = G_0 \supset G_1 \supset \cdots \supset G_n = \{e\},$$

such that such that each $G_i \lhd G$ and $[G, G_i] < G_{i+1}$ for each i.

Proposition 1.9.21.

1. *Finite products of nilpotent groups are nilpotent.*
2. *If $G/Z(G)$ is nilpotent, so is G.*
3. *Every abelian group is nilpotent.*
4. *Every p-group is nilpotent.*
5. *Every nilpotent group is solvable.*
6. *Let G be a nilpotent group and $H \subset G$ a subgroup different from G. Prove that $N_G(H) \neq H$.*
7. *Prove that a finite group is nilpotent if and only if it is isomorphic to the direct product of p-groups.*
8. *Any subgroup or quotient of a nilpotent group is nilpotent.*

Proof. See Homework 4. ☐

1.10 Symmetric and Alternating Group

Definition 1.10.1 (Symmetric Group, Cycle)**.** *Let $n \geq 1$, $X = \{1, 2, \cdots, n\}$. $S_n = \sum(X)$ is the Symmetric group of n symbols, with order $n!$.*

Recall that for a group G of order n, $G \hookrightarrow S_n$ is an embedding.

Take $\sigma \in S_n$, then $\sigma : X \xrightarrow{\cong} X$. Suppose there are distinct $a_1, \cdots, a_k \in X$ such that $\sigma(a_1) = a_2$, $\sigma(a_2) = a_3$, ..., $\sigma(a_k) = a_1$, and $\sigma(b) = b$ $\forall b \neq a_i$ $\forall i$.

We say $\sigma = (a_1\ a_2\ \cdots a_k)$ is a k-cycle. Note $\sigma^k = e$, and $ord(\sigma) = k$. Also note that $(a_1\ \cdots a_k) = (a_2\ a_3\ \cdots a_k\ a_1)$. The length of cycle is k, when $k = 0$, $\sigma = (\ ) = id$, k can be $0, 2, 3, \cdots, n$.

$\sigma = (i\ j)$ is called a transposition.

Example 1.10.2.

1. *$S_1 = \{e\}$.*
2. *$S_2 = \{e, (1\ 2)\}$.*

3. $S_3 = \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$.

   Let $\sigma = (1\ 2\ 3)$ *and* $\tau = (1\ 2)$, *note* $\sigma^3 = e$ *and* $\tau^2 = e$. *Then* $\sigma\tau = (1\ 2\ 3)(1\ 2) = (1\ 3)$, *and* $\tau\sigma = (1\ 2)(1\ 2\ 3) = (2\ 3)$.

   The subgroups of S_3 *are exactly the following:*

   $\langle\sigma\rangle = \{e, (1\ 2\ 3), (1\ 3\ 2)\} \lhd S_3$.

   $\langle\tau\rangle = \{e, (1\ 2)\}$, $\langle(1\ 3)\rangle = \{e, (1\ 3)\}$ *and* $\langle(2\ 3)\rangle = \{e, (2\ 3)\}$ *are not normal subgroups are* S_3.

4. *In* S_4, $(1\ 2)(3\ 4)$ *is a product of* 2*-cycles, but not a cycle itself. Observe that* $(1\ 2)(3\ 4) = (3\ 4)(1\ 2)$.

   In fact, an element in S_n *is always a product of these cycles.*

Theorem 1.10.3. *Every element in* S_n *is a product of disjoint cycles, i.e.* $\tau_1\tau_2\cdots\tau_s$. *Moreover,* τ_i*'s are unique (up to permutation).*

Proof. Take $\sigma \in S_n$, a bijection on X. Consider S_n acts on X with $H = \langle\sigma\rangle \subseteq S_n$, H acts on X. Now X is a disjoint union of H-orbits. i.e. $X = X_1 \coprod X_2 \coprod \cdots \coprod X_s$.

Consider $X_1 = \{a_1, \cdots, a_k\}$, and WLOG take $\sigma(a_1) = a_2$, $\sigma(a_2) = a_3$, ..., $\sigma(a_k) = a_1$. Then $\tau_1 = (a_1\ a_2\ \cdots a_k)$. In a similar fashion, $X_i \mapsto \tau_i$ cycle. Therefore, $\sigma = \tau_1\tau_2\cdots\tau_s$ as product of disjoint cycles.

On the other hand, if $\sigma = \tau_1\tau_2\cdots\tau_n$ is a product of disjoint cycles, then τ_i msut permute $X_i \subseteq X$, and $X = X_1 \coprod X_2 \coprod \cdots \coprod X_s$ as disjoint union. Therefore, σ acts transitively in each X_i, so X_i are the orbits of $H = \langle\sigma\rangle$, which shows that such τ_i is unique. □

Definition 1.10.4 (Length, Type)**.** *Consider* $\sigma = \tau_1\tau_2\cdots\tau_s$ *with corresponding* $X = X_1 \coprod X_2 \coprod \cdots \coprod X_s$, *then* $k_i = |X_i|$ *is defined as the length of* σ_i. *If we also count the* 1*-cycles (which we don't write down in the representations), then* $\sum\limits_{i=1}^{s} k_i = n$. *Therefore,* $(k_1, \cdots, k_s)$ *are uniquely determined up to permutation. We call this the type of* σ.

Example 1.10.5. *Suppose* $\sigma \in S_n$ *denoted as the cycle* $(a_1\ a_2\ \cdots\ a_k)$. *Let* $\tau \in S_n$. *What is* $\tau\sigma\tau^{-1}$*?*

If $b_i = \tau(a_i)$, *then* $(\tau\sigma\tau^{-1})(b_i) = \tau(\sigma(\tau^{-1}(b_i))) = \tau(\sigma(a_i)) = \tau(a_{i+1}) = b_{i+1}$*;For* $c \neq b_i\ \forall i$, $(\tau\sigma\tau^{-1})(c) = c$.

Therefore, $\tau\sigma\tau^{-1} = (b_1\ b_2\ \cdots\ b_k)$ *is a* k*-cycle as well.*

Remark 1.10.6. *If* $\sigma \in S_n$ *is a product of disjoint cycles, i.e.* $\sigma = \sigma_1\cdots\sigma_s$, *where* σ_i *is a* k_i*-cycle, then type*$(\sigma) = (k_1, \cdots, k_s)$.

Take $\tau \in S_n$, *then* $\tau\sigma\tau^{-1} = \tau\sigma_1\tau^{-1}\cdots\tau\sigma_s\tau^{-1}$, *then* $\tau\sigma_i\tau^{-1}$ *is a* k_i*-cycle, hence type*$(\tau\sigma\tau^{-1}) =$ *type*(σ).

If $\sigma, \sigma' \in S_n$, *type*$(\sigma) =$ *type*(σ'), *let* $\sigma = \sigma_1\cdots\sigma_s$ *and* $\sigma' = \sigma'_1\cdots\sigma'_s$ *where* σ_i *and* σ'_i *are* k_i*-cycles. Therefore we can write* $\sigma_i = (a_1\ \cdots\ a_n)$, $\sigma'_i = (a'_1\ \cdots\ a'_n)$. *If* $\tau \in S_n$ *is such that* $\tau(a_j) = a'_j$ *for all* $j = 1, \cdots, k_i$, *then* $\sigma'_i = \tau\sigma_i\tau^{-1}$.

In particular, there exists $\tau \in S_n$ *such that* $\sigma' = \tau\sigma\tau^{-1}$.

Proposition 1.10.7. *$\sigma, \sigma' \in S_n$ are conjugate if and only if type$(\sigma) =$ type(σ').*

Proof. See remark 1.10.6. $\square$

Remark 1.10.8. *Note that the number of conjugacy classes in S_n equals to the number of types and is equal to the number of partitions of n.*

Example 1.10.9.
1. *Consider S_3. Note that 3 can be represented by $1+1+1$, $1+2$ or 3 (up to permutation). Therefore, there are 3 conjugacy classes. They are identity e, transposition $(1\ 2)$ and 3-cycle $(1\ 2\ 3)$, respectively.*

2. *Consider S_4. Note that 4 can be represented by $1+1+1+1$, $1+3$, $1+1+2$, $2+2$ or 4 (up to permutation). Therefore, there are 5 conjugacy classes.*

Remark 1.10.10. *Suppose $\sigma = \sigma_1 \cdots \sigma_s$ as the product of disjoint cycles, where σ_i is a k_i-cycle, i.e. ord$(\sigma_i) = k_i$.*

Therefore, we know ord$(\sigma) =$ lcm$(k_1, \cdots, k_s)$.

Example 1.10.11. *Note that $N = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \subseteq S_4$ is a subgroup. Note that the subgroup is Abeliand and normal.*

We have $|S_4/N| = 6$ and is isomorphic to S_3.

In particular, S_4/N is solvable, but N is also solvable, which means S_4 is solvable.

Remark 1.10.12. *S_n is solvable for $n \leq 4$.*

Definition 1.10.13 (Monomial Matrix, Representation)**.** *Let $\sigma \in S_n$, $A^\sigma = (a^\sigma_{i,j})$ be an $n \times n$ matrix, where $a^\sigma_{i,j} = \begin{cases} 1 \text{ if } \sigma(j) = i \\ 0 \text{ otherwise} \end{cases}$. Then A^σ is called a monomial matrix.*

Note that $(A^\sigma A^\tau)_{i,j} = \sum_{k=1}^{n} (A^\sigma)_{i,k} \cdot (A^\tau)_{k,j} = a^\sigma_{i,j}\tau = (A^{\sigma\tau})_{i,j}$.

Observe that $A^\sigma \cdot A^\tau = A^{\sigma\tau}$, and $A^e = I_n$, and $A^\sigma \cdot A^{\sigma^{-1}} = I_n = A^{\sigma^{-1}} \cdot A^\sigma$. Therefore, the monomial matrices form a group in S_n.

In particular, $s : S_n \to GL_n(\mathbb{R})$ where $s(\sigma) = A^\sigma$ is a homomorphism, called the representation of σ.

Remark 1.10.14. *$GL_n(\mathbb{R})$ can be replaced by $GL_n(\mathbb{Z})$ or $GL_n(\mathbb{Q})$.*

Remark 1.10.15. *We have the composition $\varepsilon : S_n \xrightarrow{s} GL_n(\mathbb{Z}) \xrightarrow{\det} \mathbb{Z}^\times = \{\pm 1\}$.*

Note $\det(A^\sigma) = 1\ \forall \sigma \in S_n$.

Definition 1.10.16 (Even, Odd)**.** *$\sigma \in S_n$ is even if $\varepsilon(\sigma) = 1$, and $\sigma \in S_n$ is odd if $\varepsilon(\sigma) = -1$.*

Remark 1.10.17.
1. *Transpositions are always odd. This can be viewed from a matrix's perspective.*

2. ε *is surjective if* $n \geq 2$.

3. *The alternating group* $A_n = \ker(\varepsilon)$ *is the subgroup of all even elements in* S_n.

   In particular, $A_n \lhd S_n$, $S_n/A_n \cong \{\pm 1\}$, *therefore* $|A_n| = \frac{n!}{2}$ *for* $n \geq 2$.

Example 1.10.18. 1. $A_1 = S_1 = \{e\}$.

2. $S_2 \cong \mathbb{Z}/2\mathbb{Z}$, $A_2 = \{e\}$.

3. $A_3 \cong \mathbb{Z}/3\mathbb{Z}$.

4. $|A_4| = 12$ *is non-Abelian.*

5. $A_n \subseteq S_n$ *is solvable if* $n \leq 4$.

Remark 1.10.19. *For* $n \geq 3$, *we have* $Z(S_n) = \{e\}$; *for* $n \geq 4$, *we have* $Z(A_n) = \{e\}$.

Proposition 1.10.20. *Every element in* S_n *is a product of transpositions.*

Proof. We perform induction on n.

It is clear when $n = 1, 2$, since $S_2 = \{e, (1\ 2)\}$.

Suppose the case is true for $n-1$, we consider the case of n.

Take $\sigma \in S_n$, let $i = \sigma(n)$.

Case 1: $i = n$. Then $\sigma \in S_{n-1} = \{\tau \in S_n : \tau(n) = n\} \subseteq S_n$. By the induction hypothesis, σ is a product of transpositions.

Case 2: $i \neq n$. Consider $\tau = (i\ n)$, let $\sigma' = \tau \cdot \sigma$.

Then $\sigma'(n) = \tau(\sigma(n) = \tau(i) = n$. By case 1, $\sigma' = \tau_1 \cdots \tau_s$ is the product of transpositions. Then $\sigma = \tau^{-1} \cdot \sigma' = \tau_1 \cdots \tau_s$. ☐

Remark 1.10.21. *Consider* $\sigma \in S_n$, *then* $\sigma = \tau_1 \cdots \tau_s$ *where* τ_i *is a transposition.*

Note that σ *is even if* s *is even, and* σ *is odd if* s *is odd.*

Suppose $\sigma \in A_n$ *then* s *is even, so* $\sigma = (\tau_1\ \tau_2) \cdots (\tau_{s-1}\ \tau_s)$.

Corollary 1.10.22. A_n *is generated by products of two transpositions.*

Proof. See remark 1.10.21 above. ☐

Example 1.10.23.

$$\begin{aligned}
\sigma &= (a_1\ \cdots\ a_k) \\
&= (a_1\ a_k)(a_1\ \cdots\ a_{k-1}) \\
&= \cdots \\
&= (a_1\ a_k)(a_1\ a_{k-1}) \cdots (a_1\ a_2)
\end{aligned}$$

k-cycle is even when k is odd. k-cycle is odd when k is even.

Lemma 1.10.24. *A_n is generated by 3-cycles.*

Proof. It suffices to write $\tau_1\tau_2$ into a product of 3-cycles, where τ_i is a transposition. WLOG let $\tau_1 = (i\ j), \tau_2 = (k\ l)$.

Case 1: τ_1, τ_2 have two common symbols. Then $\tau_1 = \tau_2$, $\tau_1\tau_2 = e$.

Case 2: τ_1, τ_2 have one common symbol. i.e. $\tau_1 = (i\ j), \tau_2 = (j\ k)$. Then $\tau_1\tau_2 = (i\ j\ k)$, which is a 3-cycle.

Case 3: τ_1, τ_2 have no common symbols. Then $\tau_1\tau_2 = (i\ j)(k\ l) = (i\ j)(j\ k)(j\ k)(k\ l) = (i\ j\ k)(j\ k\ l)$. □

Lemma 1.10.25. *If $n \geq 5$, then every two 3-cycles in A_n are conjugate.*

Proof. Let $\sigma = (i\ j\ k)$ and $\tau = (l\ m\ n)$. Take $\rho \in S_n$ such that $\rho(i) = l$, $\rho(j) = m$ and $\rho(k) = n$. Therefore $\rho\sigma\rho^{-1} = \tau$.

If $\rho \in A_n$ we are done. Suppose not, then ρ has to be odd. Consider s, t different from i, j, k. Let $\varepsilon = (s, t)$, then $\sigma\varepsilon = \varepsilon\sigma$. In particular, $(\rho\varepsilon)\sigma(\rho\varepsilon)^{-1} = \rho(\varepsilon\sigma\varepsilon)\rho^{-1} = \rho\sigma\rho^{-1} = \tau$.

Therefore, $\rho\varepsilon \in A_n$. This concludes the proof. □

Definition 1.10.26 (Simple)**.** *A group $G \neq \{e\}$ is called simple if G has no non-trivial normal subgroup.*

Example 1.10.27.

1. *$\mathbb{Z}/p\mathbb{Z}$ for prime p is simple.*
2. *An Abelian group is simple if and only if it is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p.*

   Indeed, suppose a group G is Abelian and non-trivial. Let $e \neq a \in G$. If a generates an infinite cyclic group, then a^2 generates a proper subgroup and G cannot be simple. If $\langle a\rangle$ is finite, and G is simple, then $G = \langle a\rangle$. Let n be the order and suppose it is not prime. Then $n = rs$ for some $r, s \neq 1$, and $a^r \neq e$, so a^r generates a proper subgroup of G, contradiction, which means $G = \langle a\rangle$ must have order p. Hence, $G \cong \mathbb{Z}/p\mathbb{Z}$.
3. *Every non-Abelian simple group is not solvable.*

   Note that $\{e\} \neq [G, G] \lhd G$, then $[G, G] = G$. In particular, $G = G_1 = G_2 = \cdots = G_n = \cdots$ where G_i must all be $G = [G, G]$. Therefore, G is not solvable.
4. *S_n, A_n are solvable if $n \leq 4$. S_3, S_4, A_4 are not simple.*

   Indeed, note that $A_n \lhd S_n$, then S_n is not simple for $n \geq 3$.

Theorem 1.10.28. *A_n is simple for $n \geq 5$.*

Proof. Consider $\{e\} \neq N \lhd A_n$. We show that $N = A_n$.

It suffices to prove that N contains a 3-cycle σ. Suppose this is true, then $\forall\tau \in A_n$, $\tau\sigma\tau^{-1} \in N$ since $N \lhd A_n$. But from the previous lemma **1.10.25**, all 3-cycles in A_n are conjugates for $n \geq 5$.

Therefore, N contains all 3-cycles. However, recall from lemma **1.10.24** that A_n is generated by 3-cycles, so $A_n = N$.

Let $e \neq \sigma \in N$ be an element that fixes the largest number of symbols. We show that σ is a 3-cycle. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_s$ be the disjoint cycles.

Suppose all σ_i's are transpositions, then $\text{type}(\sigma) = (2, 2, \cdots, 2)$. Therefore, $\sigma \in N \subseteq A_n$ has to be even, which means s is even, so $s \geq 2$. Therefore, we can write $\sigma = (i\ j)(k\ l)\cdots$. Since $n \geq 5$, then there exists a symbol $r \neq i, j, k, l$. Take $\gamma = (k\ l\ r) \in A_n$. Let $\sigma' = [\gamma, \sigma] = (\gamma\sigma\gamma^{-1})\sigma^{-1} \in N$. Notice that $\gamma(i\ j)\gamma^{-1} = (\gamma(i)\ \gamma(j)) = (i\ j)$, and $\gamma(k\ l)\gamma^{-1} = (\gamma(k)\ \gamma(l)) = (l\ r)$

Claim 1.10.29. *$\sigma' \neq e$.*

Subproof. Observe that $\gamma\sigma\gamma^{-1} = (\gamma\sigma\gamma^{-1})(\gamma\sigma\gamma^{-1})\cdots = (i\ j)(l\ r)\cdots \neq (i\ j)(k\ l)\cdots = \sigma$. Therefore, $\gamma\sigma\gamma^{-1} \neq \sigma$, which means $\sigma' \neq \{e\}$. ■

Now $\sigma'(i) = \gamma\sigma\gamma^{-1}\sigma^{-1}(i) = \gamma\sigma\gamma^{-1}(j) = \gamma\sigma(j) = \gamma(i) = i$.

Therefore, σ' fixes i and j, but σ does not fix i and j.

Suppose σ fixes $p \neq r$, so $\sigma(p) = p$. Then $p \neq k, l$. Moreover, $\gamma(p) = p$, $\gamma^{-1}(p) = p$ and $\sigma^{-1}(p) = p$.

Hence, $\sigma' = \gamma\sigma\gamma^{-1}\sigma^{-1}$ fixes p.

Therefore, σ' fixes more symbols than σ, which is a contradiction.

Now suppose not all σ_i are transpositions. Without loss of generality, let σ_1 be length with at least 3. Therefore, we can write $\sigma_1 = (i\ j\ k\ \cdots)$. We want to show that $\sigma_1 = (i\ j\ k)$.

Claim 1.10.30. *There exists distinct symbols l and r such that $l, r \neq i, j, k$ and σ does not fix l, r.*

Subproof. Let $\sigma = \sigma_1 \cdots \sigma_s$. If $s \geq 2$, then $\sigma_2 = (l\ r\ \cdots)$, and we are done. If $s = 1$, $\sigma = \sigma_1 = (i\ j\ k\ \cdots)$, but $\sigma \neq (i\ j\ k\ l)$ is odd, then σ is at least a 5-cycle, i.e. $\sigma = (i\ j\ k\ l\ r\ \cdots)$. ■

Take $\gamma = (k\ l\ r)$ and $\sigma' = [\gamma, \sigma] = \gamma\sigma\gamma^{-1}\sigma^{-1} \in N$.

Claim 1.10.31. *$\sigma' \neq e$.*

Subproof. Note that $\gamma\sigma\gamma^{-1} = \gamma(i\ j\ k)\sigma_2\cdots\sigma_s\gamma^{-1} = (i\ j\ l\ \cdots)\cdots \neq (i\ j\ k\ \cdots)\cdots$. Therefore, $\gamma\sigma\gamma^{-1} \neq \gamma$, and so $\sigma' \neq e$. ■

Since $\sigma(j) = k$, then σ does not fix j. On the other hand, $\sigma'(j) = \gamma\sigma\gamma^{-1}\sigma^{-1}(j) = \gamma\sigma\gamma^{-1}(i) = \gamma\sigma(i) = \gamma(j) = j$. Therefore, σ' fixes j.

Let $\sigma(p) = p$, then $p \neq k, l, r$ since σ does not fix these elements. Then $\gamma(p) = p$, $\gamma^{-1}(p) = p$, $\sigma^{-1} = p$. In particular, $\sigma'(p) = p$. Again, σ' fixes more elements than σ, contradiction. This concludes the proof. □

Corollary 1.10.32. *A_n, S_n are not solvable if $n \geq 5$.*

Proof. A_n is simple but not Abelian, so not solvable.

S_n is not solvable because $A_n \lhd S_n$. □

Proposition 1.10.33. *A_n is the only non-trivial normal subgroup of S_n if $n \geq 5$.*

Proof. Consider $N \lhd S_n$. We want to show that N is either $\{e\}$, A_n or S_n.

Consider $f : A_n \hookrightarrow S_n \to S_n/N$. Then $\ker(f) = N \cap A_n \lhd A_n$. Therefore, $N \cap A_n = \{e\}$ or A_n. Suppose $N \cap A_n = A_n$, then $A_n \subseteq N \subseteq S_n$, which means $N = A_n$ or S_n. Suppose $N \cap A_n = \{e\}$, then f is injective, which means $A_n \hookrightarrow S_n/N$. In particular, $\frac{n!}{2} = |A_n| \leq |S_n/N| = \frac{n!}{|N|}$, so $|N| \leq 2$. Suppose $|N| = 2$, i.e. $N = \{e, \sigma\} \lhd S_n$. For all $\tau \in S_N$, $\tau N \tau^{-1} = N$, which means $\{e, \tau\sigma\tau^{-1}\} = \{e, \tau\}$, then $\tau\sigma\tau^{-1} = \sigma$ for all $\tau \in S_n$. In particular, $\sigma \in Z(S_n) = \{e\}$, contradiction. Therefore, $|N| = 1$, which means N is trivial. $\square$

1.11 Semidirect Product

Definition 1.11.1 (Internal Direct Product)**.** *Recall the definition: consider $K, H \lhd G$, then $G = H \times K$ is considered to be the internal direct product if*

1. *every $g \in G$ can be written uniquely as $g = hk$ for some $h \in H, k \in K$.*

   Equivalently,

2. *$H \times K \to G$ defined by $(h, k) \mapsto hk$ is a bijection.*

3. *$G = H \cdot K$ and $H \cap K = \{e\}$.*

4. *for finite G, $G = HK$ and $|G| = |H| \cdot |K|$.*

5. *for finite G, $H \cap K = \{e\}$ and $|G| = |H| \cdot |K|$.*

Analogously, we define internal semidirect products.

Definition 1.11.2 (Internal Semidirect Product)**.** *Consider $K, H \subseteq G$ where $H \lhd G$. Then G is the internal semidirect product of H and K, denoted by $G = H \rtimes K$, if all the equivalent conditions hold in the previous definition 1.11.1.*

Remark 1.11.3. *For $h_1, h_2 \in H$ and $k_1, k_2 \in K$, $(h_1k_1)(h_2k_2) = h_1(k_1h_2k_1^{-1})k_1k_2 = h_1(f_{k_1}(h_2))(k_1k_2) \in HK$, where $f_k : H \to H$ is defined as $f_k(h) = khk^{-1}$ for $k \in K, h \in H$.*

Note that $f_e = \mathbf{id}_H$, $f_k \circ f_{k'} = f_{kk'}$, and $(f_k)^{-1} = f_{k^{-1}}$. Therefore, this is a homomorphism.

Furthermore, $f_k \in Aut(H)$. Therefore, $f : K \to Aut(H)$ is a homomorphism where $f(k) = f_k$.

In particular, $f : K \to Aut(H) \hookrightarrow \sum(H)$, K acts on H by automorphisms.

Definition 1.11.4 (External Semidirect Product)**.** *Consider K, H as groups. $f : K \to Aut(H)$ is a homomorphism. Let $G = H \times K = \{hk : h \in H, k \in K\}$ be a set, with the product defined by $(h_1, k_1) \cdot (h_2, k_2) = (h_1 f(k_1) h_2, k_1k_2)$.*

G is a group based on this operation, called the external semidirect product of H and K with respect to f, denoted $G = H \rtimes_f K$.

Remark 1.11.5. *Let $G = H \rtimes_f K$ be the external semidirect product.*

Denote $H' = \{(h, e_K), h \in H\} \lhd G$, then $H' \cong H$.

Denote $K' = \{(e_H, k), k \in K\} \subseteq G$, then $K' \cong K$.

In particular, $(h, k) = (h, e_K) \cdot (e_H, K) \in H' \rtimes K'$. Therefore, $G = H' \rtimes K'$ as an internal semidirect product.

Remark 1.11.6. *Let $G = H \rtimes K$ be the internal semidirect product. Consider the bijection $H \times K \to G$ where $(h, k) \mapsto hk$. We can use f to define $H \rtimes K$ on the set $H \times K$. In particular, the map $H \rtimes_f K \xrightarrow{\cong} G$ is an isomorphism.*

Remark 1.11.7. *Both semidirect products are the usual product if and only if $f : K \to Aut(H)$ is trivial if and only if K acts on H trivially.*

Example 1.11.8.

1. *Consider $S_3 \supseteq H, K$ where $H = \langle(1\ 2\ 3)\rangle$ and $K = \langle(1\ 2)\rangle$. In particular, $H \lhd S_3$ and $H \cap K = \{e\}$, and $|H| \cdot |K| = 3 \cdot 2 = 6 = |S_3|$. Hence, S_3 is a semidirect product of $H \rtimes K$.*

2. *$S_4 \supseteq N = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$, and let $S_3 = K \subseteq S_4$. Note that $N \cap K = \{e\}$, and $|N| \cdot |K| = 4 \times 6 = 24 = |S_4|$. Therefore, $S_4 = N \rtimes K$. Observe that there is $f : S_3 \xrightarrow{\cong} Aut(N)$.*

3. *Consider $|G| = pq$ where $q < p$ are prime numbers. Moreover, let $p \equiv 1 \pmod q$. Note that $G_p \lhd G, G_q \subseteq G$, $G_p \cap G_q = \{e\}$, $|G_p| \times |G_q| = |G|$. Therefore, $G = G_p \rtimes G_q$.*

   Now consider $f : \mathbb{Z}/q\mathbb{Z} = G_q \to Aut(G_p) = Aut(\mathbb{Z}/q\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^\times$. Note that $(\mathbb{Z}/p\mathbb{Z})^\times$ is a cyclic group of order $p - 1$.

   Since $p \equiv 1 \pmod q$, so $q \mid p - 1$, then there is a nontrivial map $f([1]) = h \neq 1 \in H$, as $h^q = [1]$. Hence, $\mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z}$ is a non-Abelian group of order pq. Even though there are $q - 1$ maps, they are all isomorphic. Therefore, it is the unique non-trivial construction.

4. *Let C be a cyclic group of order n, where $\sigma \in C$ is a generator. Let $K = \{e, \tau\}$ be cyclic of order 2.*

   Consider $f : K \to Aut(C)$ where $f(e) = \mathbf{id}$ and $f(\tau) = (x \mapsto x^{-1})$.

   Now, D_{2n} is defined as the group $C \rtimes_f K$, the dihedral group, with generators σ, τ as $\sigma^n = e$ and $\tau^2 = e$.

   Note that $\tau\sigma\tau^{-1} = \sigma^{-1}$, which means $\tau\sigma\sigma^{-1}\tau$.

   In particular, with $f : K \to Aut(\mathbb{Z})$ defined by $\tau \mapsto (x \mapsto -x)$ and $e \mapsto e$, we have $\mathbb{Z} \rtimes_f K = D_\infty$.

Remark 1.11.9 (Classification of Small Order Groups)**.**

1. *Order 1: $\{e\}$*

2. *Order 2: $\mathbb{Z}/2\mathbb{Z}$*

3. *Order 3:* $\mathbb{Z}/3\mathbb{Z}$

4. *Order 4:* $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. *Since* $|G| = p^2$, *then* G *is Abelian. If there exists* $\sigma \in G$ *with order* p^2, *then* G *is cyclic, and is isomorphic to* $\mathbb{Z}/4\mathbb{Z}$. *If* $\forall \sigma \in G$ *we have* $\sigma^p = 1$, *then* $p \cdot \sigma = 0$, *which means* G *is a vector space over* $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. *In particular,* $G \cong \mathbb{F}_p \times \mathbb{F}_p$.

5. *Order 5:* $\mathbb{Z}/5\mathbb{Z}$

6. *Order 6: Note that* $6 = 2 \times 3$ *as product of two primes and* $3 \equiv 1 \pmod 2$. *In general, we can write* $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. *Then we have* $\mathbb{Z}/2\mathbb{Z} \to Aut(\mathbb{Z}/p\mathbb{Z}) = (\mathbb{Z}/p\mathbb{Z})^\times$. *Therefore for* $[x]^2 = [1]$, *we have* $x \equiv \pm 1 \pmod p$. *When* $x = [1]$, *we have* $G \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2p\mathbb{Z}$; *when* $x = [-1]$, *we have* $G = D_{2p}$.

   In particular, when $p = 3$, *we have two groups* $\mathbb{Z}/6\mathbb{Z}$ *and* D_6.

7. *Order 7:* $\mathbb{Z}/7\mathbb{Z}$

8. *Order 8:*

   a) *Suppose* $\exists x \in G$ *of order* 8, *then* $G \cong \mathbb{Z}/8\mathbb{Z}$.

   b) *Suppose* $\forall x \in G$, $x^2 = e$, *then* G *is a vector space over* $\mathbb{Z}/2\mathbb{Z}$. *In particular,* $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

   c) *Suppose* $\exists x \in G$ *with order* 4. *Denote* $H = \langle x \rangle$, *then* H *has order* 4 *and is a normal subgroup of* G.

      i. *Suppose* $\exists y \in G \backslash H$ *such that* $y^2 = e$, *then denote* $K = \langle y \rangle = \{e, y\}$. *Note that* $K \cap H. = \{e\}$, *and* $|K| \times |H| = |G|$, *then* $G = H \rtimes K$. *There is* $K = \mathbb{Z}/2\mathbb{Z} \xrightarrow{f} Aut(H) = (\mathbb{Z}/2\mathbb{Z})^\times$, *which is cyclic of order 2.*

         If f *is trivial, then* $G = H \times K = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. *If not, then* $G = D_8$.

      ii. *Suppose* $\forall y \in G \backslash H$, y *has order* 4. *Therefore,* G/H *is cyclic of order* 2. *Hence,* $y^2 H = (yH)^2 = H$ *with* $e \neq y^2 \in H$. *Then* $G = \{e, x, x^2, x^3, y, xy, x^2y, x^3y\}$.

         Observe that $y^2 \in H$ *has order* 2, *then* $y^2 = x^2$. *However,* $xy \neq yx$ *otherwise* $xyxy = x^2y^2 = x^4 = e$, *but* $xy \notin H$, *contradiction. Furthermore,* $H \ni yxy^{-1} \neq x$, *but the order of* yxy^{-1} *is the same as the order of* x, *which is* 4. *Therefore,* $yxy^{-1} = x^3$, *so* $yx = x^3y$. *In particular,* $G \cong Q_8$. *Note that* $c = x^2 = y^2$ *commutes with* x *and* y, *so* c *is in the center. Then* $Q_8 = \{e, x, y, xy, c, cx, cy, cxy\}$ *and* $cx = xc$, $cy = yc$, $x^4 = e = y^4$, $yx = cxy$.

         Finally, notice that $Q_8 = \mathbb{H} = \{\pm 1, \pm i, \pm j, \pm k\}$ *where* $i^2 = j^2 = -1$ *and* $k = ij = -ji$.

   Note that D_8 *has* 5 *elements of order* 2, *and* Q_8 *has 1 element of order* 2, *with other 3 groups are Abelian.*

9. *Note* $9 = 3^2$*, then the groups are* $\mathbb{Z}/9\mathbb{Z}$ *and* $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

10. *Since* $2 \times 5 = 10$*, then the possible groups are* $\mathbb{Z}/10\mathbb{Z}$ *and* D_{10}.

11. $\mathbb{Z}/11\mathbb{Z}$

12. *By Sylow's Theorem, there exists a subgroup* $H \subseteq G$ *of order* 4 *and a subgroup* $K \subseteq G$ *of order* 3*. We claim that at least one of* H *and* K *is normal in* G.

    Note that the number of Sylow 3-subgroups divides 4 and is equivalent to 1 modulo 3*. Suppose* K *is not normal in* G*, then* K *is not the unique Sylow 3-subgroup, which means there are four Sylow 3-subgroups. In particular, there are* $(3-1) * 4 = 8$ *non-identity elements. On the other hand, this means the Sylow 2-subgroup* H *has to be unique. Therefore,* $H \lhd K$*. Therefore, either* $G = H \rtimes K$ *or* $G = K \rtimes H$.

    Suppose $G \cong H \rtimes K$*. In particular, there is* $f : K \to Aut(H)$*. If* H *is cyclic, then* $Aut(H) = (\mathbb{Z}/4\mathbb{Z})^\times$ *of order* 2*, which means* f *is trivial, so* $G = H \times K = \mathbb{Z}/12\mathbb{Z}$*; If* $H = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$*, then* $Aut(H) = S_3$*. In particular,* $f : \mathbb{Z}/3\mathbb{Z} \to S_3$*. If* f *is trivial,* $G = H \times K = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$*. If* f *is not trivial,* $G = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes \mathbb{Z}/3\mathbb{Z} \cong A_4$*. (Note that* $S_4 \cong (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \rtimes S_3$*.)*

    Suppose $K \lhd G$*, then* $G = K \rtimes H$*. There is* $f : H \to Aut(K) = (\mathbb{Z}/3\mathbb{Z})^\times$ *cyclic of order 2. If* $H = \mathbb{Z}/4\mathbb{Z}$*,* f *is either trivial with* $G = K \times H = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} = \mathbb{Z}/12\mathbb{Z}$*, or* f *is the only non-trivial homomorphism* $f : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$*,* $[1]_4 \mapsto [1]_2$*,* $G = K \rtimes H = Dic_{12}$ *dicyclic group. If* $H = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$*,* f *is non-trivial,* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$*, with* $f' = f \circ g$ *for* $g \in Aut(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$*. Therefore,* $G = D_{12}$.

    We have two Abelian groups $\mathbb{Z}/12\mathbb{Z}$ *and* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ *(also known as the direct product of* $\mathbb{Z}/6\mathbb{Z}$ *and* $\mathbb{Z}/2\mathbb{Z}$*). The non-Abelian groups can be characterized by the following:*

	A_4	D_{12}	Dic_{12}
$H \lhd G$	✓		
$K \lhd G$		✓	✓
$H \cong \mathbb{Z}/4\mathbb{Z}$			✓
$H \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	✓	✓	

13. $\mathbb{Z}/13\mathbb{Z}$

14. *Since* $14 = 2 \times 7$*, we have* $\mathbb{Z}/14\mathbb{Z}$ *and* D_{14}.

15. *Recall from corollary* ***1.8.7*** *that this is cyclic, which means it is* $\mathbb{Z}/15\mathbb{Z}$.

16. *Groups start to be more complicated: there are* 14 *groups of order* 16.

Remark 1.11.10. *All groups above are either cyclic or semidirect product of two cyclic groups, except* Q_8.

Definition 1.11.11 (Short Exact Sequence)**.** *Consider a sequence $H \xrightarrow{s} G \xrightarrow{t} F$. Note that $t \circ s = 1$ if and only if $\ker(t) \supseteq im(s)$.*

We say this sequence is exact if $\ker(t) = im(s)$.

In particular, the sequence $1 \to G \xrightarrow{t} F$ is exact if and only if t is injective; the sequence $H \xrightarrow{s} G \to 1$ is exact if and only if s is surjective.

The sequence $\cdots \to G_1 \xrightarrow{s_1} G_2 \xrightarrow{s_2} G_3 \xrightarrow{s_3} G_4 \to \cdots$ is exact if every sequence $G_{i-1} \xrightarrow{s_{i-1}} G_i \xrightarrow{s_i} G_{i+1}$ is exact $\forall i$.

Note that $1 \to H \xrightarrow{s} G \xrightarrow{t} F \to 1$ is exact if and only if t is surjective and $im(s) = \ker(t)$. This is called a short exact sequence.

Example 1.11.12. *Suppose $H \lhd G$. Consider the short exact sequence*

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, hook, "s"] & G \arrow[r, two heads, "t"] & G/H \arrow[r] & 1 \end{tikzcd}$$

Figure 1.4: Standard Short Exact Sequence

We claim that every short exact sequence is isomorphic to this one.

Consider an arbitrary short exact sequence $1 \to H \xrightarrow{s} G \xrightarrow{t} F \to 1$. Note that $H = \ker(t) \lhd G$. Then by the First Isomorphism Theorem, $F = im(t) \cong G/\ker(t) = G/im(s) = G/H$. Therefore, we have

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, "s"] \arrow[d, equal] & G \arrow[r, "t"] \arrow[d, equal] & F \arrow[r] \arrow[d, "\cong"] & 1 \\ 1 \arrow[r] & H \arrow[r, hook, "s"] & G \arrow[r, two heads, "t"] & G/H \arrow[r] & 1 \end{tikzcd}$$

Figure 1.5: Isomorphism between the Sequences

Definition 1.11.13 (Split)**.** *A short exact sequence is split if $\exists K \subseteq G$ such that $t|_K : K \to F$ is an isomorphism.*

Equivalently, the short exact sequence is split if and only if there exists a group homomorphism $v : F \to G$ such that $t \circ v = id_F$. v is called a splitting of the short exact sequence. Note that the splitting may not be unique.

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, hook, "s"] & G \arrow[r, two heads, "t"] & G/H \arrow[r] \arrow[l, dashed, bend right, "v"'] & 1 \\ & & K \arrow[u, hook] \arrow[ur, "t|_K"', "\cong"] & & \end{tikzcd}$$

Figure 1.6: Split Short Exact Sequence

Remark 1.11.14. *Indeed, we can take $v = i \circ (t|_K)^{-1}$ where i is the inclusion map from K to G, for $x \in F$ we have $(t \circ v)(x) = t(v(x)) = x$, which means $t \circ v = id_F$.*

On the other hand, let $K = im(v) \subseteq G$. Then consider $id : F \xrightarrow{v} K \xrightarrow{t|_K} F$. Note that v has to be an isomorphism. Then since id is another isomorphism, then $t|_K$ has to be an isomorphism.

Example 1.11.15. 1. *Consider the following short exact sequence:*

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{s} \mathbb{Z}/4\mathbb{Z} \xrightarrow{t} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

where $s([1]_2) = s([1]_4)$ and $t([a]_4) = ([a]_2)$. However, this is not a split short exact sequence. Suppose such $K \subseteq \mathbb{Z}/4\mathbb{Z}$ exists, then $K \cong \mathbb{Z}/2\mathbb{Z}$, so $K = \ker(t)$, which means $t|_K : K \to \mathbb{Z}/2\mathbb{Z}$ has to be the zero map.

2. *Let $G = H \rtimes K$ where $H \lhd G$. Consider the following sequence:*

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, hook, "s"] & G \arrow[r, two heads, "t"] & K \arrow[r] & 1 \\ & & K \arrow[u, hook] & & \end{tikzcd}$$

Figure 1.7: Standard Split Short Exact Sequence

For arbitrary $g \in G$, there is unique $h \in H, k \in K$ such that $g = hk$. We now define $t : G \to K$ with $t(g) = k$ for any g, k defined above. In particular, the map has $\ker(t) = H$. Therefore, this is a short exact sequence. This sequence is also split. For $k = e \cdot k$, we have $t(k) = k$, so $t|_K = id_K$.

We claim that every split short exact sequence is isomorphic the sequence above.

Consider the arbitrary split short exact sequence below.

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, hook, "s"] & G \arrow[r, two heads, "t"] & G/H \arrow[r] & 1 \\ & & K \arrow[u, hook] \arrow[ur, "\cong", "t|_K"'] & & \end{tikzcd}$$

We claim that $G = H \rtimes K$. In particular, we show that $G = H \cdot K$ and $H \cap K = \{e\}$.

For $x \in G$, $y = t(x)$, there exists $k \in K$ such that $t(k) = y$. Therefore, $t(xk^{-1}) = t(x) \cdot t(k)^{-1} = y \cdot y^{-1} = e$. Therefore, let $h = xk^{-1}$, and we have $h = xk^{-1} \in \ker(t) = H$. Hence, $x = h \cdot k$.

Take $x \in H \cap K$, then $t(x) = e$ since $x \in H$, and $t|_K(x) = e$ since $x \in K$. However, $t|_K$ is an isomorphism, so $x = e$.

Hence, $G = H \rtimes K$ by definition.

Therefore, we have the following correspondence:

$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d, equal] & F \arrow[r] & 1 \\
1 \arrow[r] & H \arrow[r, hook] & H \rtimes K \arrow[r] & K \arrow[r] \arrow[u, "t|_K"', "\cong"] & 1
\end{tikzcd}$$

Figure 1.8: Isomorphism between the Sequences

Example 1.11.16. *Let $|G| = 8$ and take $H \subseteq G$ as a subgroup of order 4. Note that $H \lhd G$. Now, the short exact sequence*

$$1 \longrightarrow H \hookrightarrow G \twoheadrightarrow G/H \longrightarrow 1$$

is split if $G = D_8 = \langle \sigma, \tau | \sigma^4 = \tau^2 = 1 \rangle$ and $H = \langle \sigma \rangle$ and $K = \langle \tau \rangle$; it is not split if $G = D_8$.

1.12 Free Group

Definition 1.12.1 (Letter, Alphabet, Word)**.** *Let X be a set. Then $x \in X$ is called a letter x in the alphabet X. We call $x_1 x_2 \cdots x_n$ a word where $x_i \in X$ are letters.*

Remark 1.12.2. *Let S be the set of all words. For $v = x_1 \cdots x_n$ and $w = y_1 \cdots y_m$, define $v \cdot w = x_1 \cdots x_n y_1 \cdots y_m$. Note that S is still not a group.*

Consider $\bar{X}$ as "a copy" of X: using different notation for the exact same set. There is clearly a bijection between $\bar{x} \in \bar{X}$ and $x \in X$, with $\bar{\bar{x}} = x$.

For $X \cup \bar{X}$, let T be the set of all words in $X \cup \bar{X}$. We hope to let $\bar{X} = X^{-1}$ since we don't have inverses in the set yet.

We define the operation of a reduction $\mapsto$. Let $u = vx\bar{x}w$ where v, w are words and $x \in X \cup \bar{X}$. Then a reduction is $u = vx\bar{x}w \mapsto vw$.

We define an equivalence relation based on the reduction operation. We say two words $u, u' \in T$ are equivalent with $u \sim u'$ if $\exists u_0 = u,\ u_1, \cdots, u_n = u'$ in T such that for all i we have $u_i \mapsto u_{i+1}$ or $u_{i+1} \mapsto u_i$.

For example, we know $xy\bar{y}\bar{z}zt \mapsto x\bar{z}zt \mapsto xt$, and $xy\bar{y}\bar{z}zt \mapsto x\bar{z}zt \mapsto xt$, then $xy\bar{y}\bar{z}zt \sim x\bar{z}r\bar{r}zt$.

The set of equivalence classes $T/\sim$ should be a free group. Let $F(x)$ be the set of equivalence class of words (in $X \cup \bar{X}$). In particular, if v is a word, then $[v] \in F(X)$.

The equivalence class is well-defined as $[v] \cdot [w] = [vw]$, and if $v_1 \sim v_2$ and $w_1 \sim w_2$, by definition there is $[v_1 w_1] = [v_2 w_2]$ by reduction.

Claim 1.12.3. *$F(X)$ is a group on set X, namely the free group.*

Proof. 1. For arbitrary $[u], [v] \in F(X)$, there is $([u] \cdot [v]) \cdot [w] = [uv] \cdot [w] = [uvw] = [u] \cdot [vw] = [u] \cdot ([v] \cdot [w])$.

2. $e = [\_] = [\varnothing]$

3. For $u = x_1 \cdots x_n$ where $x_i \in X \cup \bar{X}$, define $v = \bar{x_n}\bar{x_{n-1}} \cdots \bar{x_1}$, then uv is reduced to $\varnothing$ and vu is also reduced to $\varnothing$. Therefore, we have found an inverse for u.

$\square$

For abbreviation, we write u for $[u]$, and u^{-1} for such v above. Every element in $F(X)$ can be written as $x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$ for some $x_i \in X$ and some $\varepsilon_i = \pm 1$.

Note that every equivalence class in $F(X)$ contains an irreducible word (a word of the minimal length i.e. cannot be further reduced). Indeed, this can be done by picking the word of smallest length, since a reduction operation at least reduces the length of 2.

Proposition 1.12.4. *Every equivalence class in $F(X)$ contains exactly one irreducible word.*

Proof. Suppose $u \sim v$ are irreducible words in the same equivalence class. Therefore, there is a sequence $w_1, w_2, \cdots, w_n$ where $w_1 = u$ and $w_n = v$, and $\forall i = 1, \cdots, n-1$, one of u_i, u_{i+1} is a reduction of the other, i.e. $w_i \mapsto w_{i+1}$ or $w_{i+1} \mapsto w_i$.

Let n be the length of the shortest possible sequence from u to v. We want to show that $n = 1$ and $u = w_1 = v$.

Assume $n \geq 2$, then $u = w_1 \leftarrow\!\shortmid w_2, \cdots, w_{n-1} \mapsto w_n = v$. Let w_i be the longest word among $w_2, \cdots, w_{n-1}$. Therefore, there must be a reduction $w_{i-1} \leftarrow\!\shortmid w_i \mapsto w_{i+1}$. Suppose the first reduction reduces $x\bar{x}$ and the second reduction reduces $y\bar{y}$. We split into cases.

Case 1: suppose $x\bar{x} = y\bar{y}$. Then we can write $w_{i-1} = ab$, $w_i = ax\bar{x}b$ and $w_{i+1} = ab$. Hence, $w_{i-1} = w_{i+1}$. However, that means we can delete w_{i-1} and w_i from the sequence, a contradiction since n is the smallest.

Case 2: suppose $x\bar{x}$ and $y\bar{y}$ overlaps, i,e, $y = \bar{x}$. Therefore, we have $w_{i-1} = axb$, $w_i = ax\bar{x}xb$ and $w_{i+1} = axb$. Similar to case 1, since $w_{i-1} = w_{i+1}$, we know this is a contradiction.

Case 3: suppose $x\bar{x}$ and $y\bar{y}$ don't overlap. Therefore, we have the following diagram:

$$\begin{tikzcd}
aby\bar{y}c \arrow[dr] & ax\bar{x}by\bar{y}c \arrow[l] \arrow[r] & ax\bar{x}bc \arrow[dl] \\
& abc &
\end{tikzcd}$$

Denote $w_i = abc$. Therefore, we can replace the sequence $w_1, \cdots, w_i, \cdots, w_n$ with $w_1, \cdots, w_i', \cdots, w_n$. However, the length of w_i' must be shorter than w_i, which is a contradiction.

Therefore, the irreducible word has to be unique in the equivalence class. $\square$

Remark 1.12.5. *For a set X, there is a bijection between $F(X)$ and the set of irreducible words.*

Example 1.12.6. *1. For $X = \varnothing$, $F(X) = \{e\}$.*

2. *For $X = \{x\}$, $F(X)$ consists of n-term words $xx\cdots x$ or $\bar{x}\bar{x}\cdots\bar{x}$. These words are irreducible. In particular, $F(X) \cong \mathbb{Z}$ is an infinite cyclic group generated by $x \in X$.*

3. *If $|X| \geq 2$, for $x \neq y \in X$, $xy \neq yx \in F(X)$. Therefore, $F(X)$ is not Abelian.*

Theorem 1.12.7 (Universal Property of Free Groups)**.** *Let X be a set and G be a group. Then every map $f : X \to G$ of sets extends uniquely to a group homomorphism $\tilde{f} : F(X) \to G$, i.e. $\tilde{f}(x) = f(x)\ \forall x \in X$.*

Proof. Take arbitrary $u \in F(X)$, then we can write $u = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$ for some $x_i \in X$.

Note that $\tilde{f}(x_i) = f(x_i)$ and $\tilde{f}(x_i^{-1}) = \tilde{f}(x_i)^{-1} = f(x_i)^{-1}$, and therefore $\tilde{f}(x_i^{\varepsilon_i}) = f(x_i^{\varepsilon_i})$. Therefore, $\tilde{f}(u) = f(x_1)^{\varepsilon_1} \cdots f(x_n)^{\varepsilon_n}$. Note that if such a homomorphism exists, then it must be unique given by f.

Note that a desired group homomorphism exists by using the definition above. The definition is well-defined. Suppose $v \mapsto u$ for $v = a \cdot x\bar{x} \cdot b$ and $u = ab$. Then $\tilde{f}(ax\bar{x}b) = \tilde{f}(a) \cdot f(x) \cdot f(x)^{-1} \cdot \tilde{f}(b) = \tilde{f}(a) \cdot \tilde{f}(b) = \tilde{f}(u)$. Hence, $\tilde{f}(uv) = \tilde{f}(u)\tilde{f}(v)$, hence f is a well-defined homomorphism indeed. This concludes the proof. $\square$

Remark 1.12.8. *Note that we don't have any relations between the generators at this point, which is why the group is called a "free group".*

Let H be a group and $R \subseteq H$ be a subset. We denote $\langle\langle R \rangle\rangle$ as the normal subgroup generated by R, which is the smallest subgroup containing R, and the intersections of all subgroups containing R.

Proposition 1.12.9. $\langle\langle R \rangle\rangle = \{(g_1\tau_1g_1^{-1})^{\varepsilon_1} \cdots (g_n\tau_ng_n^{-1})^{\varepsilon_n}, \tau_i \in R, g_i \in H, \varepsilon_i = \pm 1\} = \left\langle \left\langle \bigcup_{g \in G} gRg^{-1} \right\rangle \right\rangle.$

Proof. See Homework 6 problem 6. $\square$

Let X be a set and $F(X)$ be a free group. Suppose $R \subseteq F(X)$ is the subset of irreducible words, then let $G = F(X)/\langle\langle R \rangle\rangle$. Consider the map $X \hookrightarrow F(X) \xrightarrow{\pi} G$, which sends $x \in X$ to $x \in G$. Note that the set of $\{x \in G\}$ would generate G, then define the relation in G as $r = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} \in F(X)$, so $r \in \langle\langle R \rangle\rangle$ by definition. In particular, $r = e \in G$ by the mapping above, and we say R is the set of defining relations, with $r \in R$ is a relation in G.

We say that $G = F(X)/\langle\langle R \rangle\rangle = \langle X \mid R \rangle$ is the group defined by generators X and relations R.

Remark 1.12.10. *For a free group $F(X)$ over a set X, the group has no relations, i.e. $R = \varnothing$.*

Proposition 1.12.11. *Let $G = \langle X \mid R \rangle$ and H be a group. Define $f : X \to H$ as a map of sets such that $f(x_1)^{\varepsilon_1} \cdots f(x_n)^{\varepsilon_n} = e_H$ for all $x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} \in R$. Then there is a unique homomorphism $\tilde{f} : G \to H$ such that $\tilde{f}(x) = f(x)\ \forall x \in X$.*

Proof. Note that by extension of f, there is a homomorphism $\hat{f} : F(X) \to H$ such that $\hat{f}(x) = f(x)\ \forall x \in X$. Let $N = \ker(\hat{f}) \lhd F(X)$.

Consider $\tau = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} \in R$, $\hat{f}(\tau) = \hat{f}(x_1)^{\varepsilon_1} \cdots \hat{f}(x_n)^{\varepsilon_n} = f(x_1)^{\varepsilon_1} \cdots f(x_n)^{\varepsilon_n} = e_H$.
Since $R \subseteq \ker(\tilde{f}) = N$, then $\langle\langle R \rangle\rangle \subseteq N$.
Now $\bar{f}$ factors through the map as the following:

$$\begin{tikzcd}
F(X)/\langle\langle R \rangle\rangle \arrow[rr, "\tilde{f}"] & & H \\
& F(X) \arrow[ul, "\pi"] \arrow[ur, "\hat{f}"'] &
\end{tikzcd}$$

Figure 1.9: Factoring Property between Group and Corresponding Presentation

In particular, $xN \in G$ if and only if $x \in X$, then $\tilde{f}(x) = \tilde{f}(Nx) = \hat{f}(x) = f(x)$.

$\square$

We now consider the relationship backwards.

Let G be a group with a generating set $X \subseteq G$. The inclusion map $X \hookrightarrow G$ extends $\hat{f} : F(X) \twoheadrightarrow G$ by $x \mapsto x$. Let $N = \ker(\hat{f}) \lhd F(X)$. Let $R \subseteq N$ be a subset such that $\langle\langle R \rangle\rangle = N$. By the First Isomorphism Theorem, $G = \operatorname{im}(\hat{f}) \cong F(X)/N = F(X)/\langle\langle R \rangle\rangle = \langle X \mid R \rangle$. Therefore, $G \cong \langle X \mid R \rangle$.

Definition 1.12.12 (Presentation)**.** *$\langle X \mid R \rangle$ defined above is called the presentation of the group G.*

Proposition 1.12.13. *Every group G has a presentation.*

Proof. An explicit presentation is $G \cong \langle G \mid \{abc \mid a, b, c \in G \text{ with } abc = 1 \text{ in } G\} \rangle$. $\square$

Example 1.12.14.

1. *Consider $\mathbb{Z}/n\mathbb{Z}$. Note that $\mathbb{Z} = F(\{\sigma\}) = \langle \sigma \mid \varnothing \rangle$ for some arbitrary σ, with $\langle\langle \sigma^n \rangle\rangle = n\mathbb{Z}$. Therefore, $\mathbb{Z}/n\mathbb{Z} = \langle \sigma \mid \sigma^n \rangle$.*

2. *Consider the group D_{2n}. Note that the group is generated by σ and τ where $\sigma^n = e = \tau^2$ and $\tau\sigma\tau\sigma = e$.*

   We claim that $D_{2n} = \left\langle \sigma, \tau \mid \sigma^n, \tau^2, \tau\sigma\tau\sigma \right\rangle$, defined as a set named S. Now pick $\bar{\sigma}, \bar{\tau} \in S$. We have $\bar{\sigma}^n = e$, $\bar{\tau}^2 = e$, and therefore note that $\bar{\tau}^{-1} = \tau$. Notice that there is a defined surjective homomorphism from S to D_{2n} because S at least has $2n$ elements.

   For $v \in F(\sigma, \tau)$, we have $\bar{v} = \bar{\sigma}^{a_1}\bar{\tau}^{b_1}\bar{\sigma}^{a_2}\bar{\tau}^{b_2} \cdots = \bar{\sigma}^i\bar{\tau}^j$. In particular, $2n = |D_{2n}| \leq |S| \leq 2n$. Therefore, the two groups must have the same order, and we can then conclude that there is an isomorphism.

3. *$\left\langle \sigma, \tau \mid \sigma^2, \tau^2, (\sigma\tau)^2 \right\rangle = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

4. *$H = \left\langle x, y \mid x^2, y^2 \right\rangle \cong D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$.*

   Take $\sigma \in \mathbb{Z}$ and $\tau \in \mathbb{Z}/2\mathbb{Z}$. Note that $\tau\sigma\tau^{-1} = \sigma^{-1}$, and $\tau^2 = e$, so $(\tau\sigma)^2 = e$. In particular, take the map $x \mapsto \tau\sigma$ and $y \mapsto \tau$.

Furthermore, we can construct the inverse with $\tau \mapsto y$ and $\sigma \mapsto yx$. In particular, as $yx \in H$, we have $\langle yx \rangle \lhd H$ and $H = \langle yx \rangle \rtimes \langle \tau \rangle$.

5. *Consider $G \cong \left\langle x, y \mid x^2, y^3 \right\rangle$. Define the special linear group $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det = 1 \right\}$.*

   Then $G = SL_2(\mathbb{Z}) / \left\{ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$.

   Define $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. Note that $\sigma^2 = \tau^3 = e$. We may now obtain an isomorphism by mapping $x \mapsto \sigma$ and $y \mapsto \tau$.

Definition 1.12.15 (Free Product)**.** *Let $(G_i)_{i \in I}$ be a family of groups, then the free product is defined as $\coprod_{i \in I} G_i = F(\coprod_{i \in I} G_i) / \langle\langle R \rangle\rangle$, where $R = \coprod_{i \in I} (\{1_{G_i}\} \cup \{abc \mid a, b, c \in G_i, abc = 1_{G_i} \in G_i\}$.*

Remark 1.12.16. *The free product is the coproduct in the category of groups, sometimes denoted as $G * H$ instead of $G \coprod H$.*

Proposition 1.12.17. *Let $G_i = \langle A_i \mid R_i \rangle$, then $\coprod_{i \in I} G_i = \left\langle \coprod_{i \in I} A_i \mid \bigcup_{i \in I} R_i \right\rangle$. Moreover, $\coprod_{i \in I} G_i = \left\langle \bigcup_{i \in I} S_i \mid \bigcup_{i \in I} R_i \right\rangle$ where $G_i = \langle S_i \mid R_i \rangle$.*

Remark 1.12.18. *Analogously, if there are two disjoint sets S_1, S_2, then $\langle S_1 \mid R_1 \rangle * \langle S_2 \mid R_2 \rangle = \langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle$.*

Theorem 1.12.19 (Universal property of free products)**.** *Let $G = \coprod_{i \in I} G_i$ for groups $(G_i \mid i \in I)$. Then for any group H and homomorphisms $f_i : G_i \to H$, there is a unique $f : G \to H$ such that $f_i = f \circ \iota_i$ for each $i \in I$.*

$$\begin{tikzcd}
& H & \\
G_1 \arrow[ur, "f_1"] \arrow[r, "\iota_1"'] & G \arrow[u, dashed, "f"] & G_2 \arrow[ul, "f_2"'] \arrow[l, "\iota_2"]
\end{tikzcd}$$

Figure 1.10: Universal Property of Free Products

2 Elementary Category Theory

2.1 Introduction to Categories

Definition 2.1.1 (Category). *A category $\mathscr{C}$ consists of a class of objects ("dots") $\mathbf{Ob}(\mathscr{C})$ and a class of morphisms ("arrows") $\mathbf{Mor}(\mathscr{C})$ between the objects of $\mathscr{C}$.*

For objects $A, B \in \mathscr{C}$, a morphism $f : A \to B$ has A as the source and B as the target. For $f : A \to B$ and $g : B \to C$ as morphisms in $\mathscr{C}$, the composition $g \circ f$ is defined by $A \xrightarrow{g \circ f} C$, such that:

1. *Associativity holds: for $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.*
2. *$\forall A \in \mathbf{Ob}(\mathscr{C})$, there is a morphism $\mathbf{id}_A : A \to A$ such that $\forall f : X \to A$ morphism, $\mathbf{id}_A \circ f = f$, and $\forall g : A \to Y$ morphism, $g \circ \mathbf{id}_A = g$.*

Definition 2.1.2 (Small, Locally Small). *For objects $A, B \in \mathscr{C}$, $\mathbf{Mor}_{\mathscr{C}}(A, B)$ is the class of morphisms from A to B.*

A category $\mathscr{C}$ is locally small if $\mathbf{Mor}_{\mathscr{C}}(A, B)$ is a set for all objects $A, B \in \mathscr{C}$.

A category $\mathscr{C}$ is small if it is locally small and $\mathbf{Ob}(\mathscr{C})$ is a set.

Definition 2.1.3 (Isomorphism). *A morphism $f : A \to B$ in a category $\mathscr{C}$ is an isomorphism if $\exists g : B \to A$ such that $f \circ g = \mathbf{id}_B$ and $g \circ f = \mathbf{id}_A$. Such g is unique if exists, called the inverse of f, so $g = f^{-1}$, and $(f^{-1})^{-1} = f$.*

We denote $A \cong B$ if there exists an isomorphism $f : A \to B$.

Example 2.1.4.

1. *Denote $\mathbf{Set}$ as the category of sets. The objects of this category are sets and the morphisms are maps between sets.*

   Note $\mathbf{Mor}_{\mathbf{Set}}(X, Y) \subseteq X \times Y$ must be a set for $X, Y \in \mathbf{Set}$. Therefore, $\mathbf{Set}$ is locally small. Isomorphisms in $\mathbf{Set}$ are just bijections.

2. *Denote $\mathbf{Grp}$ as the category of groups. The objects of this category are groups and morphisms are the homomorphisms between groups.*

   Again, $\mathbf{Grp}$ is locally small, and isomorphisms between elements in $\mathbf{Grp}$ are just group isomorphisms.

3. *Denote $\mathbf{Ab}$ as the category of Abelian groups. This is a subcategory of $\mathbf{Grp}$.*

4. *Consider arbitrary set X, we can view the set as a category.*

   Here $\mathbf{Ob}(X) = X$ *and* $\mathbf{Mor}_X(x, x') = \begin{cases} \varnothing \text{ if } x \neq x' \\ \{(x, x')\} \text{ as identity if } x = x' \end{cases}$. *This is a small category, and the only isomorphism is the identity.*

5. *Let G be a group, then we can view the group as a category. Here* $\mathbf{Ob}(G) = *$ *and* $\mathbf{Mor}(*, *) = G$ *as a set, and the composition of morphisms is the group operation in G. Here, the identity morphisms is just the identity element of G.*

   This is a small category, and every morphisms in G is an isomorphism. Such G is called a groupoid.

6. *We can construct a group using the set $X = \{1, 2, \cdots, n\}$. Let the objects be the set X and let* $\mathbf{Mor}_X(i, j) = \begin{cases} \varnothing \text{ if } i > j \\ \{(i, j)\} \text{ if } i \leq j \end{cases}$. *The only isomorphisms in this category are the identities.*

   This can be generalized to a category **Pos** *of posets. One can also view the set of natural numbers $\mathbb{N}$ as a poset category.*

7. *Let $\mathscr{C}$ be a category, then we can define a category out of the morphisms of $\mathscr{C}$, denoted as* $\mathbf{Ar}(\mathscr{C})$.

   The objects of the category are morphisms $A \xrightarrow{f} B$ in $\mathscr{C}$, and the morphisms of any $A \xrightarrow{f} B$ and $A' \xrightarrow{f'} B'$ are a pair of morphisms $A \xrightarrow{g} A'$ and $B \xrightarrow{h} B'$ such that the diagram is commutative as follows:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "h"] \\ A' \arrow[r, "f'"] & B \end{tikzcd}$$

Figure 2.1: Morphisms in an Arrow Category

   i.e. $h \circ f = f' \circ g$.

8. *Let $\mathscr{C}$ be a category. The dual (opposite) category $\mathscr{C}^\circ$ has objects* $\mathbf{Ob}(\mathscr{C}^\circ) = \mathbf{Ob}(\mathscr{C})$ *(a copy $A^\circ \in \mathscr{C}^\circ$ for $A \in \mathscr{C}$) and morphisms* $\mathbf{Mor}_{\mathscr{C}^\circ}(A^\circ, B^\circ) = \mathbf{Mor}_{\mathscr{C}}(B, A)$, *i.e. a dual morphism $f^\circ : B \to A$ in $\mathscr{C}^\circ$ for $f : A \to B$ in $\mathscr{C}$.*

9. *Let $\mathscr{C}_1, \mathscr{C}_2$ be categories. Then we can define a product category $\mathscr{C}_1 \times \mathscr{C}_2$ from the two categories. The objects of $\mathscr{C}_1 \times \mathscr{C}_2$ are* $\mathbf{Ob}(\mathscr{C}_1 \times \mathscr{C}_2) = \{(A_1, A_2) : A_1 \in \mathbf{Ob}(\mathscr{C}_1), A_2 \in \mathbf{Ob}(\mathscr{C}_2)\}$, *and the morphisms for objects $(A_1, A_2), (B_1, B_2)$ in category $\mathscr{C}_1 \times \mathscr{C}_2$ are* $\mathbf{Mor}_{\mathscr{C}_1 \times \mathscr{C}_2}((A_1, A_2), (B_1, B_2)) = \mathbf{Mor}_{\mathscr{C}_1}(A_1, B_1) \times \mathbf{Mor}_{\mathscr{C}_2}(A_2, B_2)$.

10. *Let $\mathscr{C}$ be a category. Consider a subclass of objects $M \subseteq \mathbf{Ob}(\mathscr{C})$. We can derive a new category $\mathscr{C}'$ where the objects of the category are $\mathbf{Ob}(\mathscr{C}') = \mathbf{Ob}(\mathscr{C})\backslash M$ and the morphisms are $\mathbf{Mor}_{\mathscr{C}'}(A, B) = \mathbf{Mor}_{\mathscr{C}}(A, B)$ if $A, B \notin M$.*

Definition 2.1.5 (Initial/Final Object)**.** *Let $\mathscr{C}$ be a category. An object $A \in \mathscr{C}$ is initial if $\forall B \in \mathbf{Ob}(\mathscr{C})$, $\exists!$ morphism $A \to B$.*

An object $A \in \mathscr{C}$ is final (terminal) if $\forall B \in \mathbf{Ob}(\mathscr{C})$, $\exists!$ morphism $B \to A$.

Remark 2.1.6. *Note that the initial objects in $\mathscr{C}$ are exactly the final objects in $\mathscr{C}^\circ$.*

Proposition 2.1.7. *Every two initial objects are canonically isomorphic.*

Proof. Let A, A' be initial in category $\mathscr{C}$. Then there exists a unique morphism $f : A \to A'$ and a unique morphism $g : A' \to A$. In particular, $g \circ f : A \to A$ must be the identity morphism since A is an initial object, and $f \circ g : A' \to A'$ must also be the identity morphism.

Hence, f, g are isomorphic, then this induces a unique isomorphism. Hence, $A \cong A'$. □

Remark 2.1.8. *Similarly, two final objects are canonically isomorphic.*

Example 2.1.9. 1. *Consider* **Set**. *The initial object of this category is $\varnothing$, and the final objects of this category are the singleton sets.*

In particular, the set of maps from any set X to $\varnothing$ is $\varnothing$ if $X \neq \varnothing$ and is $\{\mathbf{id}_\varnothing\}$ if $X = \varnothing$.

2. *Consider* **Grp**. *The initial object and the final object of this category are both the identity group $\{e\}$.*

3. *For a group G defined as a category, recall that $\mathbf{Ob}(G) = \{*\}$ and $\mathbf{Mor}(*, *) = G$. Now, if $|G| > 1$, then there are no initial/final objects.*

4. *Consider the set $X = \{1, 2, \cdots, n\}$ as a category. Then 1 is the initial object and n is the final object.*

Notice that for a category $\mathscr{C}$ and $X, Y, X', Y' \in \mathbf{Ob}(\mathscr{C})$, with $f \in \mathbf{Mor}(Y, Y'), g \in \mathbf{Mor}(X, Y)$, then $f \circ g \in \mathbf{Mor}(X, Y')$. In particular, there is a map $f_* : \mathbf{Mor}(X, Y) \to \mathbf{Mor}(X, Y')$ such that $g \mapsto f \circ g$.

In a similar sense, consider $h \in \mathbf{Mor}(X, X')$, then $g \circ h \in \mathbf{Mor}(X, Y)$, and there is a map $h^* : \mathbf{Mor}(X', Y) \to \mathbf{Mor}(X, Y)$ with $g \mapsto g \circ h$.

Definition 2.1.10 (Product)**.** *For $X, Y \in \mathscr{C}$ as objects in a category, define the product of X and Y as $X \times Y \in \mathscr{C}$ with $p : X \times Y \to X$ and $q : X \times Y \to Y$ such that for all morphisms $f : Z \to X$ and $g : Z \to Y$, there is a unique $h : Z \to X \times Y$ with the property $p \circ h = f$ and $q \circ h = g$, i.e. the following diagram commutes:*

$$\begin{tikzcd}
Z \arrow[rrd, bend left, "f"] \arrow[rd, dashed, "\exists ! h"] \arrow[rdd, bend right, "g"'] & & \\
& X \times Y \arrow[r, "p"] \arrow[d, "q"] & X \\
& Y &
\end{tikzcd}$$

Figure 2.2: Universal Property of Product

In particular, this induces a bijection $\mathbf{Mor}(Z, X \times Y) \xrightarrow{p_*, q_*} \mathbf{Mor}(Z, X) \times \mathbf{Mor}(Z, Y)$.

Example 2.1.11. *1. Consider the category* **Set**. *The category has a usual product of sets.*

2. Consider the category **Grp**. *The category has a usual product of groups.*

3. Consider the category **Ab**. *The category has a usual product of Abelian groups.*

4. Consider the category from the set $X = \{1, 2, \cdots, n\}$ *where morphisms are the relations* $i \leq j$. *The product of this category is the minimal of* i *and* j.

For $f : X \to X'$ and $g : Y \to Y'$, they induce a morphism $f \times g : X \times Y \to X' \times Y'$ as follows:

$$\begin{tikzcd}
X \times Y \arrow[r] \arrow[d] \arrow[rd, dashed, "\exists ! f \times g"] & X \arrow[rd, "f"] & \\
Y \arrow[rd, "g"] & X' \times Y' \arrow[r] \arrow[d] & X' \\
& Y' &
\end{tikzcd}$$

Figure 2.3: Morphism Product

Moreover, the universal property can be induced by canonical isomorphism:

$$\begin{tikzcd}
Z \arrow[rd, dashed, "\exists !"'] \arrow[rrd, dashed, "\exists !"] \arrow[rrrd, bend left] \arrow[rrdd, bend right] & & & \\
& V \arrow[r, "\sim"] & X \times Y \arrow[r] \arrow[d] & X \\
& & Y &
\end{tikzcd}$$

Figure 2.4: Product Unique Up to Canonical Isomorphism

Proposition 2.1.12. *Let $X \times Y$ and $\widetilde{X \times Y}$ be two products of X and Y, then there is a unique isomorphism $X \times Y \xrightarrow[h]{\sim} \widetilde{X \times Y}$ such that the diagram*

$$\begin{tikzcd}
X \times Y \arrow[r] \arrow[d] \arrow[dr, "h", "\sim"'] & X \\
Y & \widetilde{X \times Y} \arrow[l] \arrow[u]
\end{tikzcd}$$

commutes.

Proof. Consider the category of pairs of morphisms with objects $(Z \xrightarrow{f} X, Z \xrightarrow{g} Y)$ and morphisms $(Z \xrightarrow{f} X, Z \xrightarrow{g} Y) \to (Z' \xrightarrow{f'} X', Z' \xrightarrow{g'} Y')$, which is a morphism from Z to Z' such that the diagram

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "f"] \arrow[dr, "h"] & Y \\
X & Z' \arrow[l, "f'"] \arrow[u, "g'"]
\end{tikzcd}$$

commutes.

In particular, $(X \times Y \xrightarrow{p} X, X \times Y \xrightarrow{q} Y)$ is a final object. $\square$

Remark 2.1.13. *We can define product of a family of objects in $\mathscr{C}$ by $\prod_{i \in I} X_i$ and morphisms $\prod_{i \in I} X_i \xrightarrow{p_j} X_j$.*

In particular, $\mathbf{Mor}(Z, \prod_{i \in I} X_i) \xrightarrow[\sim]{p_{i*}} \prod_{i \in I} \mathbf{Mor}(Z, X_i)$ is a bijection.

Definition 2.1.14 (Coproduct)**.** *Let $X, Y \in \mathscr{C}$. The coproduct $X * Y$ is an object together with two morphisms $X \to X * Y \leftarrow Y$ such that $\forall X \to Z, Y \to Z$, there exists a unique $X * Y \to Z$ such that the diagram*

$$\begin{tikzcd}
& Y \arrow[d] \arrow[ddr, bend left] & \\
X \arrow[r] \arrow[drr, bend right] & X * Y \arrow[dr, dashed, "\exists!"] & \\
& & Z
\end{tikzcd}$$

Figure 2.5: Universal Property of Coproduct

commutes.

Remark 2.1.15. *Note there is a bijection* $\mathbf{Mor}(X * Y, Z) \to \mathbf{Mor}(X, Z) \times \mathbf{Mor}(Y, Z)$*, sometimes also written as* $\mathbf{Mor}(\coprod_{i \in I} X_i, Z) \to \prod_{i \in I} \mathbf{Mor}(X_i, Z)$.

In particular, this tells us that $(X * Y)^\circ = X^\circ \times Y^\circ$*, so the coproduct is a dual notion of the product.*

Example 2.1.16.

1. *Consider the category* **Set**. *The coproduct is exact the disjoint union, i.e.* $X * Y = X \coprod Y$.

2. *Consider the category* **Grp**. *Consider the product* $G \times H$ *in the usual sense. However, one may notice that the product is not equivalent to the coproduct. Indeed, consider* $i : G \to G \times H$ *by* $g \mapsto (g, e_H)$ *and* $j : H \to G \times H$ *by* $h \mapsto (e_G, h)$. *We would have the following diagram:*

$$
\begin{tikzcd}
& H \arrow[d, "j"] \arrow[ddr, bend left, "t"] & \\
G \arrow[r, "i"] \arrow[drr, bend right, "s"'] & G \times H \arrow[dr, dashed, "\exists ! k"] & \\
& & Z
\end{tikzcd}
$$

Figure 2.6: Universal Property of Product/Coproduct in Abelian Groups

Here, $(g, h) = (g, e) \cdot (e, h) = i(g) \cdot j(h)$. *Furthermore, we have* $k(g, h) = k(i(g)) \cdot k(j(h)) = s(g) \cdot t(h)$. *Let this be the definition for our unique homomorphism* k. *Then we have* $k((g, h) \cdot (g', h')) = k(gg', hh') = s(gg') \cdot t(hh')$. *On the other hand, since* k *is a homomorphism, this is equivalent to* $k(g, h) \cdot k(g', h') = s(g) \cdot t(h) \cdot s(g') \cdot t(h')$. *This is true if and only if our choices of* G, H *are Abelian.*

3. *For the category of Abelian groups* **Ab**, *the coproduct is exactly the product, defined by the universal property in* ***Figure 2.6*** *above.*

4. *Reconsider the coproduct of groups. Let* $G = \langle X \mid R \rangle$ *and* $H = \langle Y \mid S \rangle$. *Define the coproduct by* $G * H = \langle X \coprod Y \mid R \cup S \rangle$*, where the unique homomorphism* $k : G * H \to Z$ *is generated by* $k(x) = s(x)$ *and* $k(y) = t(y)$ *for all* $x \in x \in G, y \in H$*, as shown in the figure below.*

$$
\begin{tikzcd}
& H \arrow[d] \arrow[ddr, bend left, "t"] & \\
G \arrow[r] \arrow[drr, bend right, "s"'] & G \times H \arrow[dr, dashed, "\exists! k"] & \\
& & Z
\end{tikzcd}
$$

Example 2.1.17. *1. Note that* $\mathbb{Z}/2\mathbb{Z} \cong \langle \sigma \mid \sigma^2 \rangle$ *as a presentation. Then* $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle \sigma, \tau \mid \sigma^2, \tau^2 \rangle = D_\infty$*, which is exactly the infinite dihedral group.*

2. $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z} = \langle \sigma, \tau \mid \sigma^2, \tau^3 \rangle = \mathbf{PSL}_2(\mathbb{Z})$*, the projective special linear group over* $\mathbb{Z}$*.*

Definition 2.1.18 (Subcategory, Full)**.** *Let* $\mathscr{C}$ *be a category, and let* $\mathscr{C}'$ *be a category such that*

- $\mathbf{Ob}(\mathscr{C}') \subseteq \mathbf{Ob}(\mathscr{C})$*, and*
- $\mathbf{Mor}_{\mathscr{C}'}(X, Y) \subseteq \mathbf{Mor}_{\mathscr{C}}(X, Y)$ *for all* $X, Y \in \mathbf{Ob}(\mathscr{C}')$*.*

Then $\mathscr{C}'$ *is a subcategory of* $\mathscr{C}$*.*

If $\mathbf{Mor}_{\mathscr{C}'}(X, Y) = \mathbf{Mor}_{\mathscr{C}}(X, Y)$ *for all* $X, Y \in \mathbf{Ob}(\mathscr{C}')$*, then we say* $\mathscr{C}'$ *is a full subcategory of* $\mathscr{C}$*.*

Example 2.1.19. *1.* **Ab** *is a full subcategory of* **Grp***.*

2. **Grp** *is a subcategory of* **Set***, but not full.*

3. Let $M \subseteq \mathbf{Ob}(\mathscr{C})$ *be a subclass of objects, then* $\mathscr{C} \backslash M$ *is a full subcategory of* $\mathscr{C}$*.*

2.2 Functor

Definition 2.2.1 (Functor)**.** *Let* $\mathscr{C}, \mathscr{D}$ *be categories. A (covariant) functor* $F : \mathscr{C} \to \mathscr{D}$ *assigns to every object* $C \in \mathscr{C}$ *an object* $FC \in \mathscr{D}$ *and to every morphism* $f : C \to D$ *in* $\mathscr{C}$ *a morphism* $Ff : FC \to FD$ *in* $\mathscr{D}$ *such that*

- $F(f \circ g) = Ff \circ Fg$*, and*
- $F(\mathbf{id}_A) = \mathbf{id}_{FA}$*.*

On the other hand, a (contravariant) functor would be $F : \mathscr{C}^\circ \to \mathscr{D}$ *that sends objects* $C^\circ \in \mathscr{C}^\circ$ *to* $FC \in \mathscr{D}$ *and morphisms* $f : C \to C'$ *in* $\mathcal{C}$ *to* $Ff : FC' \to FC$ *in* $\mathscr{D}$*.*

Remark 2.2.2. *A functor takes isomorphisms to isomorphisms. If there is* $f : X \to Y$ *and* $g : Y \to X$ *such that* $f \circ g = \mathbf{id}_Y$ *and* $g \circ f = \mathbf{id}_X$*, then correspondingly there is* $Ff : FX \to FY$ *and* $Fg : FY \to FX$ *such that* $Ff \circ Fg = \mathbf{id}_{FY}$ *and* $Fg \circ Ff = \mathbf{id}_{FX}$*.*

Example 2.2.3. *1. For arbitrary category $\mathscr{C}$, there is an identity functor $\mathbf{Id} : \mathscr{C} \to \mathscr{C}$.*

2. *For arbitrary categories $\mathscr{C}, \mathscr{D}$, there is a constant functor $F : \mathscr{C} \to \mathscr{D}$ such that for arbitrary $Y \in \mathbf{Ob}(D)$, there is $FX = Y$ for all $X \in \mathbf{Ob}(\mathscr{C})$ and $Ff = \mathbf{id}_Y$ for all $f \in \mathbf{Mor}_{\mathscr{C}}$.*

3. *For some categories $\mathscr{C}$ that are set-based (e.g. $\mathbf{Grp}, \mathbf{Ring}$, etc), there is a forgetful functor $F : \mathscr{C} \to \mathbf{Set}$ that "forgets" the structure and transform it back to a set.*

4. *Let $\mathscr{C}'$ be a subcategory of $\mathscr{C}$, then there is an inclusion functor $\mathbf{i} : \mathscr{C}' \to \mathscr{C}$.*

5. *Let I be a category of the form*

$$\begin{tikzcd} \cdot_1 \arrow[loop above] \arrow[r, "\alpha"] & \cdot_2 \arrow[loop above] \end{tikzcd}$$

Then a functor $F : I \to \mathscr{C}$ would map $\cdot_1$ to some object X in $\mathscr{C}$ and map $\cdot_2$ to some object Y in $\mathscr{C}$, and map α to some morphism $f : X \to Y$ in $\mathscr{C}$.

Therefore, a functor from I to $\mathscr{C}$ induces a morphism in $\mathscr{C}$.

Moreover, for a small category I, a functor $F : I \to \mathscr{C}$ is equivalent to a commutative diagram of shape I in shape I in $\mathscr{C}$.

For example, there is the following correspondence given by a functor from I to $\mathscr{C}$:

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr] & \cdot \arrow[d] & \Longrightarrow & A \arrow[r] \arrow[d] \arrow[dr] & B \arrow[d] \\
\cdot \arrow[r] & \cdot & \Longrightarrow & C \arrow[r] & D
\end{tikzcd}$$

6. *Let G be a group with an induced category $\underline{G}$. Then $\mathbf{Ob}(\underline{G}) = \{*\}$ and $\mathbf{Mor}(*, *) = G$.*

   A functor $F : \underline{G} \to \mathbf{Set}$ is just an G-action on a set.

Definition 2.2.4 (Functor Representation)**.** *Let $\mathscr{C}$ be a locally small category. Take some $X \in \mathbf{Ob}(\mathscr{C})$.*

Define a functor $R^X : \mathscr{C} \to \mathbf{Set}$ that sends objects Y to $\mathbf{Mor}_{\mathscr{C}}(X, Y)$ and morphisms $f : Y \to Y'$ to $f_* : \mathbf{Mor}(X, Y) \to \mathbf{Mor}(X, Y')$, i.e. sends $R^X(Y)$ to $R^X(Y')$.

If this is the case, then we say R^X is a functor represented by X.

Similarly, define $R_X : \mathscr{C}^\circ \to \mathbf{Set}$ by sending objects Y° to $\mathbf{Mor}_{\mathscr{C}}(Y, X)$ and morphisms $f^\circ : Y \to Y'$ to $f^* : \mathbf{Mor}(Y, X) \to \mathbf{Mor}(Y', X)$. If this is the case, then we say R_X is a functor corepresented by X.

Remark 2.2.5. *Observe that the functor R^X is actually just the covariant Hom functor $\mathbf{Hom}(X, -)$ and the functor R_X is just the contravariant Hom functor $\mathbf{Hom}(-, X)$. They are called "representation" because of their relation with the notion of representable functors we introduce later.*

Definition 2.2.6 (Full, Faithful)**.** *A functor $F : \mathscr{C} \to \mathscr{D}$ is faithful if $\mathbf{Mor}_{\mathscr{C}}(X,Y) \to \mathbf{Mor}_{\mathscr{D}}(FX,FY)$ is injective $\forall X, Y \in \mathscr{C}$.*

A functor $F : \mathscr{C} \to \mathscr{D}$ is full if $\mathbf{Mor}_{\mathscr{C}}(X,Y) \to \mathbf{Mor}_{\mathscr{D}}(FX,FY)$ is surjective $\forall X, Y \in \mathscr{C}$.

We say a functor is fully faithful if it is both faithful and full.

Example 2.2.7. *Let $\mathscr{C}' \subseteq \mathscr{C}$ be a subcategory, then $\mathscr{C}' \hookrightarrow \mathscr{C}$ is fully faithful if and only if $\mathscr{C}'$ is a full subcategory of $\mathscr{C}$.*

Definition 2.2.8 (Equivalence)**.** *A fully faithful functor $F : \mathscr{C} \to \mathscr{D}$ is called an equivalence if $\forall Y \in \mathbf{Ob}(\mathscr{D})$, there exists $X \in \mathbf{Ob}(\mathscr{C})$ such that $Y \cong FX$.*

In particular, one can say that there is a bijection between isomorphism classes in $\mathscr{C}$ and isomorphism classes in $\mathscr{D}$.

Example 2.2.9.

1. *Let $\mathscr{C}' \subseteq \mathscr{C}$ be a full subcategory. Therefore, $\forall X \in \mathbf{Ob}(\mathscr{C})$, there exists some $X' \in \mathbf{Ob}(\mathscr{C}')$ such that $X' \cong X$. Therefore, $\mathscr{C}' \hookrightarrow \mathscr{C}$ is an equivalence.*

   Notice that suppose $\mathscr{C}' = \mathscr{C} \backslash \{Z\}$ for some object Z. By the argument above, there is some $Y \in \mathbf{Ob}(\mathscr{C}')$ such that $Y \cong Z$. Moreover, for all objects $U \in \mathscr{C}'$, we have the following diagram:

$$\begin{tikzcd} Y \arrow[rr, bend left, "f"] \arrow[dr] & & Z \arrow[ll, bend left, "g"] \arrow[dl] \\ & U & \end{tikzcd}$$

   In particular, there is an isomorphism $\mathbf{Mor}(Y,U) \cong \mathbf{Mor}(Z,U)$.

   The idea is that one may delete extra copies of objects by using equivalences.

2. *Denote $\mathbf{Vect}(K)$ as the category of finite-dimensional vector spaces over K and linear maps between these vector spaces. This is equivalent to the category $\mathscr{C}_K$ with objects as non-negative numbers and morphisms $\mathbf{Mor}_{\mathscr{C}_K}(n,m)$ is the set of $m \times n$ matrices with K entries.*

   In particular, a functor $F : \mathscr{C}_K \to \mathbf{Vect}(K)$ would send objects n to K^n and send morphisms A ($m \times n$ matrices) to a linear transformation $A : K^n \to K^m$.

3. *Let $F : \mathscr{C} \to \mathscr{D}$ be a fully faithful functor. Let $D' \subseteq D$ be a subcategory consists of Y in D such that $Y \cong FX$ for some object $X \in \mathscr{C}$.*

   One can induce an equivalence $F'\mathscr{C} \xrightarrow{\sim} \mathscr{D}'$, then $\mathscr{C}$ is equivalent to the full subcategory $\mathscr{D}' \subseteq \mathscr{D}$.

Definition 2.2.10 (Functor Category, Natural Transformation, Natural Isomorphism)**.** *We define a category of functors (between categories $\mathscr{C}$ and $\mathscr{D}$), denoted by $\mathbf{Func}(\mathscr{C}, \mathscr{D})$, with functors $F : \mathscr{C} \to \mathscr{D}$ as objects. The morphisms of this category can be defined as follows.*

Let $F, G : \mathscr{C} \to \mathscr{D}$ by functors. A morphism $\alpha : F \to G$ in the functor category is the class of morphisms $FX \xrightarrow{\alpha_X} GX$ in $\mathscr{D}$ for every object X in $\mathscr{C}$ such that for every morphism $f : X \to Y$ in $\mathscr{C}$, the diagram

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\alpha_Y"] & GY \end{tikzcd}$$

Figure 2.7: Natural Transformation

commutes. Moreover, a morphism $\alpha : F \Rightarrow G$ in the functor category is called a natural transformation.

We say the natural transformation $\alpha : F \to G$ is an isomorphism (or just natural isomorphism) if $\alpha_X : FX \to GX$ is an isomorphism for all objects $X \in \mathscr{C}$.

Remark 2.2.11. *We also denote the functor category $\mathbf{Func}(\mathscr{C}, \mathscr{D})$ by $\mathscr{D}^{\mathscr{C}}$.*

Lemma 2.2.12 (Yoneda Lemma)**.** *Consider functor $F : \mathscr{C} \to \mathbf{Set}$ and some object $X \in \mathscr{C}$ for $\mathscr{C}$ locally small. Let $R^X : \mathscr{C} \to \mathbf{Set}$ maps object Y to $\mathbf{Mor}_{\mathscr{C}}(X, Y)$, and let $\alpha : R^X \to F$ be a natural transformation, where $\alpha_X : R^X X \to FX$. In particular, a map $\varphi : \mathbf{Mor}_{\mathbf{Func}}(R^X, F) \to FX$ is given by $\alpha_X(\mathbf{id}_X) = \varphi(\alpha) \in FX$. The lemma says that φ is a bijection. Moreover, the bijection is natural in both X and F.*

For clarity, we can write bijection $\varphi : \mathbf{Hom}(\mathbf{Hom}(X, -), F) \to FX$.

Proof. See *Category Theory in Context*, theorem 2.2.4. ☐

Remark 2.2.13. *Let $\alpha, \beta : R^X \to F$, if $\alpha_X(\mathbf{id}_X) = \beta_X(\mathbf{id}_X)$, then $\alpha = \beta$.*

For all $f : X \to Y$, there is $\alpha_Y(f) = Ff(\alpha_X(\mathbf{id}_X))$, but $\beta_Y(f) = Ff(\beta_X(\mathbf{id}_X))$, so $\alpha_Y = \beta_Y$ for all objects Y, then $\alpha = \beta$.

$$\begin{tikzcd} R^X X \arrow[r, "\alpha_X"] \arrow[d, "R^X f"'] & FX \arrow[d, "Ff"] \\ R^X Y \arrow[r, "\alpha_Y"] & FY \end{tikzcd}$$

Corollary 2.2.14 (Yoneda Embedding)**.**
- *The covariant version of the embedding states that $\mathbf{Mor}_{\mathbf{Func}}(R^X, R^{X'}) \cong R^{X'}(X) = \mathbf{Mor}_{\mathscr{C}}(X', X)$. In particular, $A \xrightarrow{\cong} B$ if and only if $R^B \cong R^A$.*
- *Considering the dual notion, the contravariant version of the embedding states that $\mathbf{Mor}_{\mathbf{Func}}(R_X, R_{X'}) \cong R_{X'}(X) = \mathbf{Mor}_{\mathscr{C}}(X, X')$. In particular, $A \xrightarrow{\cong} B$ if and only if $R_A \cong R_B$.*

Remark 2.2.15. *There is a functor $F : \mathscr{C}^\circ \to \mathbf{Func}(\mathscr{C}, \mathbf{Set})$ that takes $X \mapsto R^X$, where $R^X Y = \mathbf{Mor}(X, Y)$ and $(X' \to X) \mapsto (R^X \to R^{X'})$. Note that F is fully faithful. In particular, $\mathscr{C}^\circ$ is equivalent of a full subcategory of $\mathbf{Func}(\mathscr{C}, \mathbf{Set})$.*

We can also define a functor $G : \mathscr{C} \to \mathbf{Func}(\mathscr{C}^\circ, \mathbf{Set})$ in the dull notion by mapping X to R_X. The functor category $\mathbf{Func}(\mathscr{C}^\circ, \mathbf{Set})$ is called the presheaves on $\mathscr{C}$ in $\mathbf{Set}$.

Remark 2.2.16. *Every morphism between R^X, R^Y is of the form R^f for a unique $f : Y \to X$, defined as $R^f : R^X \to R^Y$ that maps $Z \mapsto R^f(Z) : R^X(Z) \to R^Y(Z)$ and $g \mapsto g \circ f$.*

Every isomorphism $R^X \xrightarrow{\sim} R^Y$ is given by a unique isomorphism $Y \xrightarrow{\sim} X$ up to canonical isomorphism.

Definition 2.2.17 (Representable)**.** *A functor $F : \mathscr{C} \to \mathbf{Set}$ is representable if $F \cong R^X$ for some $X \in \mathscr{C}$. X is uniquely determined by the functor F (if exists) up to isomorphism. (Equivalently, $R^X \xrightarrow{\sim} R^Y$ is given by unique $Y \xrightarrow{\sim} X$, and F is represented by X.)*

Analogously, $G : \mathscr{C}^\circ \to \mathbf{Set}$ is corepresentable if $G \cong R_X$ for some $X \in \mathscr{C}$.

Remark 2.2.18. *In particular, $F : \mathscr{C} \to \mathbf{Set}$ is represented (by $X \in \mathscr{C}$) if there exists an isomorphism $\alpha : F \xrightarrow{\sim} R^X$ such that*

$$\begin{tikzcd}
FY \arrow[r, "\alpha_Y", "\sim"'] \arrow[d, "Fg"] & R^X Y \arrow[r, "\cong"] \arrow[d, "R^X g"] & \mathbf{Mor}_{\mathscr{C}}(X, Y) \arrow[d, "g_*"] \\
FY' \arrow[r, "\alpha_Y", "\sim"'] & R^X Y' \arrow[r, "\cong"] & \mathbf{Mor}_{\mathscr{C}}(X, Y')
\end{tikzcd}$$

commutes.

Example 2.2.19.

1. *Consider a functor $F : \mathscr{C} \to \mathbf{Set}$ defined by $Y \mapsto \{*\}$ and $g \mapsto \mathbf{id}_*$ for all objects Y and morphisms g in $\mathscr{C}$.*

   We want to show that F is representable, so it suffices to find a representation object $X \in \mathscr{C}$. Therefore, for arbitrary object $Y \in \mathscr{C}$, there is an isomorphism $FY = \{*\} \cong \mathbf{Mor}_{\mathscr{C}}(X, Y)$. However, this means this set of morphism has to be a singleton for arbitrary object Y. In particular, X should be the initial object of $\mathscr{C}$ by definition.

2. *Let X be a set, and consider the group G along with the set of G-actions on X.*

   *Note that if there is a homomorphism $f : G \to G'$, then having G' acts on X would induce an action of G acting on X by the pullback action. (This is induced from **example 1.6.2**.)*

   Define a functor $F : \mathbf{Grp}^\circ \to \mathbf{Set}$ defined by mapping group G to the set of G-actions on X. We want to show that F is representable, so it suffices to check that there is some object O such that $F(G) = \{G-\text{actions on } X\} \xrightarrow{\cong} \mathbf{Hom}(G, O)$. This object O is exactly the symmetric group $\sum(X)$ by interpretation.

   We now check that the diagram commutes.

$$\begin{tikzcd}
F(G) = \{G - \text{actions on } X\} \arrow[r, "\sim"] & \mathbf{Hom}(G, \sum(X)) \\
F(G') = \{G' - \text{actions on } X\} \arrow[r, "\sim"] \arrow[u, "Ff"] & \mathbf{Hom}(G', \sum(X)) \arrow[u, "\text{pullback}"']
\end{tikzcd}$$

Here each action $g \cdot X$ in FG is defined as $f(g) \cdot x$ via the pullback action. For arbitrary $g \in G$, the morphism Ff maps the action $f(g) \cdot x$ to $g \cdot x$ defined as $f(g) \cdot x$. Therefore, there is $\varphi \in \mathbf{Hom}(G, \sum(X))$ given by $\varphi(g)(x) = g \cdot x = f(g) \cdot x$ as defined by the upper routine.

Taking the bottom routine, there is $\psi \in \mathbf{Hom}(G', \sum(X))$ defined by $\psi(f(g))(x) = f(g) \cdot x$. However, the pullback gives $(\psi \circ f)(g)(x) = \psi(f(g))(x) = f(g) \cdot x = g \cdot x = \varphi(g)(x)$. Therefore, the diagram above commutes by definition.

3. *Consider arbitrary $X, Y \in \mathbf{Ob}(\mathscr{C})$. Define a functor $F : \mathscr{C} \to \mathbf{Set}$ by $Z \mapsto R^X(Z) \times R^Y(Z)$.*

   One can check that this functor is represented by the coproduct object $X * Y \in \mathscr{C}$. In particular, $\mathbf{Mor}(X, Z) \times \mathbf{Mor}(Y, Z) \cong \mathbf{Mor}(X * Y, Z)$, so $R^X \times R^Y \cong R^{X*Y}$.

   Similarly, defining functor $G : \mathscr{C} \to \mathbf{Set}$ by the mapping $Z \mapsto R_X(Z) \times R_Y(Z)$, then the functor is represented by the product object $X \times Y \in \mathscr{C}$. In particular, $\mathbf{Mor}(Z, X) \times \mathbf{Mor}(Z, Y) \cong \mathbf{Mor}(Z, X \times Y)$, so $R_X \times R_Y \cong R_{X \times Y}$.

4. *Take $X \in \mathbf{Ob}(\mathscr{C})$. Define a functor $F : \mathbf{Set}^{\mathscr{C}} \to \mathbf{Set}$ that maps G to GX.*

   We want to show that F is a representable functor. Therefore, $F(G) = GX = \mathbf{Mor}_{\mathbf{Functors}}(O, G)$ for some functor $O : \mathscr{C} \to \mathbf{Set}$.

   Observe that by Yoneda Lemma, this is exactly the covariant hom functor R^X.

5. *Let X be a set. Define a functor $F : \mathbf{Grp} \to \mathbf{Set}$ by mapping G to the set of maps from X to G (as the underlying set).*

   We want to show that F is a representable functor, then it suffices to show that $\mathbf{Maps}(X, G) = \mathbf{Hom}(O, G)$ for some group O for all groups G. By the universal property of free groups, this group O is exactly the free group of X.

6. *Consider the forgetful functor $F : \mathbf{Grp} \to \mathbf{Set}$ by mapping each group G to its underlying set. We claim that this functor is representable. Indeed, take an object $\mathbb{Z}$, then for all groups G, there is $\underline{G} \cong \mathbf{Hom}(\mathbb{Z}, G)$ by corresponding $g \in G$ to a homomorphism generated by $1 \mapsto g$.*

7. *Take some integer $n > 0$. Define a functor $F : \mathbf{Grp} \to \mathbf{Set}$ by mapping a group G to a set $\{g \in G : g^n = e\}$. To show this functor is representable, it suffices to find an object O such that $\mathbf{Hom}(O, G) \cong \{g \in G : g^n = e\}$. One can take the object as $\mathbb{Z}/n\mathbb{Z}$, then there is a group homomorphism generated by $[1] \to g$ to element g.*

Definition 2.2.20 (Adjoint)**.** *Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors. Recall that the functor $\mathbf{Mor}(-, -) : \mathscr{C}^\circ \times \mathscr{C} \to \mathbf{Set}$ that maps $(X^\circ, Y) \mapsto \mathbf{Mor}(X, Y)$. We can construct two functors $\mathscr{C}^\circ \times \mathscr{D} \to \mathbf{Set}$:*

- $(X^\circ, Y) \mapsto \mathbf{Mor}_{\mathscr{D}}(FX, Y)$
- $(X^\circ, Y) \mapsto \mathbf{Mor}_{\mathscr{C}}(X, GY)$

We say F is a left adjoint of G (and G is a right adjoint of F) if the two functors above from $\mathscr{C}^\circ \times \mathscr{D} \to \mathbf{Set}$ are isomorphic.

In particular, that means $\mathbf{Mor}_{\mathscr{D}}(FX, Y) \cong \mathbf{Mor}_{\mathscr{C}}(X, GY)$. Moreover, this is natural in both X and Y.

Remark 2.2.21. *Note that if we fix $X \in \mathscr{C}$, then the functor $\mathscr{D} \to \mathbf{Set}$ defined as $Y \mapsto \mathbf{Mor}_{\mathscr{C}}(X, GY)$ is represented by FX.*

Remark 2.2.22. *Adjoint functors (for a given functor) are unique.*

Example 2.2.23.

1. *Consider the forgetful functor $K : \mathbf{Grp} \to \mathbf{Set}$ that takes a group G to the underlying set $\underline{G}$. Fix a set $X \in \mathbf{Set}$. We hope to construct a left adjoint functor $F : \mathbf{Set} \to \mathbf{Grp}$. By definition, it suffices to find F such that $\mathbf{Mor}_{\mathbf{Grp}}(FX, G) \cong \mathbf{Mor}_{\mathbf{Set}}(X, K(G) = \underline{G})$. Such F is exactly the free operation that takes a set X to a free group $F(X)$.*

2. *Consider the inclusion functor $K : \mathbf{Ab} \to \mathbf{Grp}$ with some $A \in \mathbf{Ab}$ and some $G \in \mathbf{Grp}$. We hope to find a left adjoint $F : \mathbf{Grp} \to \mathbf{Set}$, so that $\mathbf{Hom}(G, KA = A) = \mathbf{Hom}(FG, A)$. Note that the most convenient construction of functor F is the Abelianization that maps a group G to the Abelian group $G/[G, G]$.*

3. *Consider a functor $G : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ by mapping (X, Y) to $X \times Y$. We hope to construct a left functor $F : \mathscr{C} \to \mathscr{C} \times \mathscr{C}$. Note that we would have $\mathbf{Mor}_{\mathscr{C}}(Z, G(X, Y)) = \mathbf{Mor}_{\mathscr{C}\times\mathscr{C}}(FZ, X \times Y)$. Denote $FZ = (Z_1, Z_2)$, then $\mathbf{Mor}_{\mathscr{C}\times\mathscr{C}}(FZ, X \times Y) = \mathbf{Mor}_{\mathscr{C}}(Z_1, X) \times \mathbf{Mor}_{\mathscr{C}}(Z_2, Y)$, and note that $\mathbf{Mor}_{\mathscr{C}}(Z, G(X, Y)) = \mathbf{Mor}_{\mathscr{C}}(Z, X \times Y) = \mathbf{Mor}_{\mathscr{C}}(Z, X) \times \mathbf{Mor}_{\mathscr{C}}(Z, Y)$. However, this is the case if and only if $Z_1 = Z = Z_2$, which means $F(Z) = (Z_1, Z_2) = (Z, Z)$. This gives us a construction of diagonal functor F.*

Remark 2.2.24. *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor and $(X_i)_{i\in I}$ be a family of objects in $\mathscr{C}$. Let $p_j : \prod\limits_{i\in I} X_i \to x_j$ be the projections for all $j \in I$. Therefore, there is $Fp_j : F(\prod\limits_{i\in I} X_i) \to F(X_j)$. This induces a morphism $\alpha : F(\prod\limits_{i\in I}(X_i) \to \prod\limits_{i\in I} F(X_i)$.*

Definition 2.2.25 (Commutes with products)**.** *We say F commutes with products if α is an isomorphism for all families of objects $(X_i)_{i\in I}$.*

Proposition 2.2.26. *If $F : \mathscr{C} \to \mathscr{D}$ has a left adjoint, then F commutes with products.*

Proof. Take arbitrary $Z \in \mathscr{D}$, and let $G : \mathscr{D} \to \mathscr{C}$ be the left adjoint of F. Therefore,

$$\begin{aligned}\mathbf{Mor}_{\mathscr{D}}(Z, F(\prod_{i\in I} X_i)) &\cong \mathbf{Mor}_{\mathscr{C}}(GZ, \prod_{i\in I} X_i)\\ &\cong \prod_{i\in I}\mathbf{Mor}_{\mathscr{C}}(GZ, X_i)\\ &\cong \prod_{i\in I}\mathbf{Mor}_{\mathscr{D}}(Z, FX_i)\\ &\cong \mathbf{Mor}_{\mathscr{D}}(Z, \prod_{i\in I} FX_i)\end{aligned}$$

By the Yoneda Embedding, $F(\prod_{i\in I} X_i) \cong \prod_{i\in I} FX_i$. $\square$

Example 2.2.27. *This proposition gives us a loose criteria to check if a functor has adjoint or not.*

1. *Consider the forgetful functor $F : \mathbf{Grp} \to \mathbf{Set}$. Since this functor has a left adjoint, then F commutes with product, i.e. $F(\prod_{i\in I} G_i) \cong \prod_{i\in I} FG_i$, which means the set product structure is preserved from the group product structure.*

   However, F does not commute with coproduct, which means F does not have a right adjoint.

2. *Consider the inclusion functor $F : \mathbf{Ab} \hookrightarrow \mathbf{Grp}$. Since this functor has a left adjoint, then F commutes with the product. Again, the functor does not commute with coproduct, which means F has no right adjoint.*

Definition 2.2.28 (Inverse Limit)**.** *Consider a family of objects*

$$A_1 \xleftarrow{f_2} A_2 \xleftarrow{f_3} A_3 \xleftarrow{f_4} \cdots \xleftarrow{f_n} A_n \xleftarrow{f_{n+1}} \cdots$$

We define the inverse limit $\varprojlim (A_i)_{i\in I} = \{(a_i)_{i\in I} \mid f_i(a_i) = a_{i-1}\ \forall i\}$.

Remark 2.2.29. *Note that there is an I-shaped functor $F : I \to \mathscr{C}$ where I is a small category given by*

$$\cdot_1 \longleftarrow \cdot_2 \longleftarrow \cdot_3 \cdots$$

This functor corresponds the diagram above to the family of objects given above.

Since there are morphisms $\varprojlim (A_i)_{i\in I} \to A_j$ for all index j, for arbitrary object $X \in \mathscr{C}$, this induces a family of morphisms $\mathbf{Mor}_{\mathscr{C}}(X, \varprojlim (A_i)_{i\in I}) \to \mathbf{Mor}_{\mathscr{C}}(X, A_j)$ by applying the hom functor.

Therefore, there is a diagram

$$\begin{tikzcd} & & X \arrow[dll] \arrow[dl] \arrow[dr] & & \\ A_1 & A_2 \arrow[l] & \cdots \arrow[l] & A_n \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

This induces a bijection $\mathbf{Mor}_{\mathscr{C}}(X, \varprojlim(A_i)_{i\in I}) \xrightarrow{\sim} \mathbf{Mor}_{\mathbf{Func}}(\mathbf{id}_X, F)$ *from the diagram of* I *above. Furthermore, there is*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \prod_{i\in I} A_i \arrow[dl] \\ & \varprojlim(A_i)_{i\in I} & \end{tikzcd}$$

In particular, the inverse limit is an object in $\mathscr{C}$*.*

Definition 2.2.30 (Constant Functor, Constant Natural Transformation)**.** *For any object* $c \in \mathscr{C}$ *and any category* J*, the constant functor* $c : J \to \mathscr{C}$ *sends every object of* J *to* c *and every morphism in* J *to the identity morphism* $\mathbf{id}_c$*. The constant functors define an embedding* $\Delta : \mathscr{C} \to \mathbf{Func}(J, \mathscr{C})$ *that sends an object* c *to the constant functor at* c *and a morphism* $f : c \to c'$ *to the constant natural transformation, in which each component is defined to be the morphism* f*.*

Definition 2.2.31 (Cone)**.** *A cone over a diagram* $F : J \to \mathscr{C}$ *with summit* $c \in \mathscr{C}$ *is a natural transformation* $\lambda : c \to F$ *whose domain is the constant functor at* c*. The components* $(\lambda_j : c \to Fj)_{j\in J}$ *of the natural transformation are called the legs of the cone. Explicitly, the data of a cone over* $F : J \to \mathscr{C}$ *with summit* c *is a collection of morphisms* $\lambda_j : c \to Fj$*, indexed by the objects* $j \in J$*. A family of morphisms* $(\lambda_j : c \to Fj)_{j\in J}$ *defines a cone over* F *if and only if, for each morphism* $f : j \to k$ *in* J*, the following triangle commutes in* $\mathscr{C}$*:*

$$\begin{tikzcd} & c \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] & \\ Fj \arrow[rr, "Ff"'] & & Fk \end{tikzcd}$$

Definition 2.2.32 (Limit, Colimit)**.** *Let* I *be a small category and* $X \in \mathscr{C}$ *be an object. Let* $c_X : I \to \mathscr{C}$ *be the constant functor and* $F : I \to \mathscr{C}$ *be some other functor. A morphism* $X \to Y$ *induces a natural transformation* $c_X \to c_Y$ *, so we have a functor* $Cone(-, F) : \mathscr{C}^\circ \to \mathbf{Set}$ *given by* $X^\circ \to \mathbf{Mor}(c_X, F)$*, set of cones over* F *with summit* c*. The limit of* F *is an object* $\lim F$ *in* $\mathscr{C}$ *corepresenting this functor, if it exists.*

The colimit of F *is an object colim* F *representing the functor* $\mathscr{C} \to \mathbf{Set}$ *given by* $X \to \mathbf{Mor}(F, c_X)$*.*

Remark 2.2.33. $\mathbf{Mor}_{\mathscr{C}}(X, \lim F) \cong \mathbf{Mor}_{\mathbf{Func}}(c_X, F) = Cone(-, F)$.

$\mathbf{Mor}_{\mathscr{C}}(colim\ F, X) \cong \mathbf{Mor}_{\mathbf{Func}}(F, c_X) = Cone(F, -)$.

Remark 2.2.34 (Universal Property of Limit)**.** *Let* $(\lim F, \lambda : \lim F \to F)$ *be the limit over* $F : \mathcal{J} \to F$ *with object* $\lim F$ *and cone (natural transformation)* $\lambda : \mathbf{id}_{\lim F} \to F$*, such that for any other object* T *with cone* $\tau : \mathbf{id}_T \to F$*, there is a unique morphism* $u : T \to \lim F$ *such that the following diagram commutes for all* $j \in \mathcal{J}$*:*

$$\begin{tikzcd} T \arrow[rr, "\exists! u"] \arrow[dr, "\tau_j"'] & & \lim F \arrow[dl, "\lambda_j"] \\ & Fj & \end{tikzcd}$$

Figure 2.8: Universal Property of Limit

Proposition 2.2.35. *A limit is a terminal object in the category of cones over F.*

Example 2.2.36. *1. Let I be a set (as a category with no morphisms other than the identity morphisms). For the family of objects $(X_i)_{i\in I}$ in $\mathscr{C}$, the diagram F of shape I has $\lim F = \prod_{i\in I} X_i$ and colimit $\coprod_{i\in I} X_i$.*

2. Consider the following diagram F

$$\begin{tikzcd} & B \arrow[d] \\ A \arrow[r] & C \end{tikzcd}$$

The limit is $\lim F = \{(a,b) : f(b) = g(a)\}$, such that

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \lim F \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ & A \arrow[r, "g"'] & C \end{tikzcd}$$

Note that this is exactly the pullback, i.e. fiber product.

Moreover, the colimit of the diagram is C, which is the final object of the diagram.

3. Consider a group G as a category I, then $\mathbf{Ob}(I) = *$, $\mathbf{Mor}(*,*) = G$. Consider a functor $F : I \to \mathbf{Set}$. Note that for a set X, these morphisms $g \in G$ on X are equivalent to the G-actions on X.

Consider the diagram

$$\begin{tikzcd} Z \arrow[r, "\alpha"] \arrow[d, "\mathbf{id}"'] & X \arrow[d, "g\in G"] \\ Z \arrow[r, "\alpha"'] & X \end{tikzcd}$$

Then $g \circ \alpha = \alpha$ for all $g \in G$, which means $\alpha(z) \in X^G$. One can check that the limit is X^G with

$$\begin{tikzcd} Z \arrow[rr, "\alpha"] \arrow[dr, dashed, "\exists!"'] & & X \\ & \lim F = X^G \arrow[ur] & \end{tikzcd}$$

On the other hand, the colimit is the set of orbits $X/\sim$ where the equivalence $x \sim gx$ is given by the G-action.

Proposition 2.2.37. *If $F : \mathscr{C} \to \mathscr{D}$ has a left adjoint, then F commutes with limits.*

Proof. See Homework 9, problem 1. $\square$

Definition 2.2.38 (Equalizer)**.** *Let X, Y be sets with*

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

The equalizer $\mathbf{Eq}^{\mathbf{Set}}(f,g) = \{x \in X : f(x) = g(x)\} \subseteq X$ satisfies the universal property

$$\begin{tikzcd} Z \arrow[d, dashed, "\exists! k"'] \arrow[dr, "h"] & & \\ \mathbf{Eq}^{\mathbf{Set}}(f,g) \arrow[r, hook, "i"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

such that $fi = gi$, and for all sets Z and $h : Z \to X$ such that $fh = gh$, then there is a unique $k : Z \to \mathbf{Eq}^{\mathbf{Set}}(f,g)$ with $i \circ k = h$. In particular, this induces

$$\begin{tikzcd} \mathbf{Maps}(Z, \mathbf{Eq}^{\mathbf{Set}}(f,g)) \arrow[r, hook] & \mathbf{Maps}(Z, X) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathbf{Maps}(Z, Y) \end{tikzcd}$$

In general, for a category $\mathscr{C}$ and diagram $\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$, consider the functor $F : \mathscr{C}^\circ \to \mathbf{Set}$ that sends Z to $\mathbf{Eq}^{\mathbf{Set}}(f_*, g_*)$. The equalizer $\mathbf{Eq}(f,g)$ corepresents F.

There is the equalizer sequence

$$\begin{tikzcd} \mathbf{Mor}_{\mathscr{C}}(Z, \mathbf{Eq}(f,g)) \arrow[r, hook] & \mathbf{Mor}_{\mathscr{C}}(Z, X) \arrow[r, shift left, "f_*"] \arrow[r, shift right, "g_*"'] & \mathbf{Mor}(Z, Y) \end{tikzcd}$$

In particular, for $Z = \mathbf{Eq}(f,g)$, there is $\begin{tikzcd} \mathbf{Eq}(f,g) \arrow[r] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$.

Alternatively, the equalizer is the limit of the diagram of this shape.

In the dual argument, one can define the coequalizer by the following universal property:

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"] \arrow[dr, "k"'] & W \arrow[d, dashed, "\exists! a"] \\ & & Z \end{tikzcd}$$

Similarly, the coequalizer is the colimit of the morphism pair.

Remark 2.2.39. *The equalizer is always monic, and the coequalizer is always epic.*

2.3 Additive and Abelian Category

Definition 2.3.1 (Pre-additive Category)**.** *A category $\mathscr{C}$ is pre-additive if $\forall X, Y \in \mathscr{C}$, there is a given structure of Abelian group on $\mathbf{Mor}_{\mathscr{C}}(X, Y)$ such that the composition is bilinear:*

- $(f + f') \circ g = f \circ g + f' \circ g$ *for* $f, f' \in \mathbf{Mor}_{\mathscr{C}}(X, Y)$ *and* $g \in \mathbf{Mor}_{\mathscr{C}}(W, X)$.
- $f \circ (g + g') = f \circ g + f \circ g'$ *for* $f \in \mathbf{Mor}_{\mathscr{C}}(Y, Z)$ *and* $g, g' \in \mathbf{Mor}_{\mathscr{C}}(X, Y)$.

In particular, for $X, Y \in \mathbf{Ob}(\mathscr{C})$, $0 \in \mathbf{Mor}_{\mathscr{C}}(X, Y)$ is the zero morphism i.e. $f \circ 0 = 0$, $0 \circ g = 0$.

Proposition 2.3.2. *If $\mathscr{C}$ is pre-additive, then initial objects and final objects are the same.*

Proof. Let X be a final object with $0_X, 1_X \in \mathbf{Mor}_{\mathscr{C}}(X, X)$, then $0_X = 1_X$, Take $Y \in \mathbf{Ob}(\mathscr{C})$ with $f : X \to Y$, then $f = f \circ 1_X = f \circ 0_X = 0$. Therefore, f has to be unique, which means X is initial. We can use the same trick to show that an initial object is always final. In particular, a zero object is an object that is both initial and terminal. Therefore, object is zero if and only if it is final if and only if it is initial. ☐

Definition 2.3.3 (Biproduct)**.** *For $X, Y \in \mathscr{C}$, a biproduct of X and Y is (Z, i_1, i_2, p_1, p_2) denoted below*

$$\begin{tikzcd}
X \arrow[dr, "i_1"] & & X \\
& Z \arrow[ur, "p_1"] \arrow[dr, "p_2"] & \\
Y \arrow[ur, "i_2"] & & Y
\end{tikzcd}$$

Figure 2.9: Biproduct

such that $p_1 \circ i_1 = \mathbf{id}_X$, $p_2 \circ i_2 = \mathbf{id}_Y$, $p_1 \circ i_2 = 0$, $p_2 \circ i_1 = 0$, and $i_1 \circ p_1 + i_2 \circ p_2 = \mathbf{id}_Z$.

Proposition 2.3.4. *Let (Z, i_1, i_2, p_1, p_2) be a biproduct of X and Y. Then $Z = X \times Y$ with respect to p_1, p_2, and $Z = X * Y$ with respect to i_1, i_2.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
V \arrow[dr, dashed, "\exists ! h"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& Z \arrow[r, "p_1"] \arrow[d, "p_2"] & X \\
& Y &
\end{tikzcd}$$

Define $h = i_1 \circ f + i_2 \circ g : V \to Z$. The two triangles commute:

- $p_1 \circ h = p_1 \circ i_1 \circ f + p_1 \circ i_2 \circ g = \mathbf{id}_X \circ f + 0_X \circ g = f$
- $p_2 \circ h = p_2 \circ i_1 \circ f + p_2 \circ i_2 \circ g = 0_Y \circ f + \mathbf{id}_Y \circ g = g$

Furthermore, for $h, h' : V \to Z$, $p_1 \circ h = f = p_1 \circ h'$ and $p_2 \circ h = g = p_2 \circ h'$, therefore $h' = \mathbf{id}_Z \circ h' = (i_1 \circ p_1 + i_2 \circ p_2) \circ h' = i_1 \circ p_1 \circ h' + i_2 \circ p_2 \circ h' = i_1 \circ f + i_2 \circ g = h$. Therefore, h is unique. In particular, $Z = X \times Y$.

In a similar fashion we can prove that $Z = X * Y$. Therefore, Z as the biproduct $X \oplus Y$ is equivalent to both the product $X \times Y$ and the coproduct $X * Y$. □

Definition 2.3.5 (Additive Category). *A pre-additive category $\mathscr{C}$ is additive if $\mathscr{C}$ has zero object and finite products.*

Proposition 2.3.6. *In an additive category, every finite product is also a coproduct.*

Proof. Consider arbitrary objects $X, Y \in \mathscr{C}$, with the following diagram:

$$\begin{tikzcd}
 & Y \arrow[d, "i_2=(0,\mathbf{id}_Y)"] & \\
X \arrow[r, "i_1=(\mathbf{id}_X,0)"] & X \times Y \arrow[r, "p_1"] \arrow[d, "p_2"] & X \\
 & Y &
\end{tikzcd}$$

By definition, $p_1 \circ i_1 = \mathbf{id}_X$, $p_2 \circ i_2 = \mathbf{id}_Y$, and $p_2 \circ i_1 = 0$ and $p_1 \circ i_2 = 0$. Therefore, it suffices to check $i_1 \circ p_1 + i_2 \circ p_2 = \mathbf{id}_Z$.

Take $f = i_1 \circ p_1 + i_2 \circ p_2 : X \times Y \to X \times Y$. Then

$$\begin{aligned}
p_1 \circ f &= p_1 \circ (i_1 \circ p_1 + i_2 \circ p_2) \\
&= p_1 \circ i_1 \circ p_1 + p_1 \circ i_2 \circ p_2 \\
&= \mathbf{id}_X \circ p_1 + 0 \circ p_2 \\
&= p_1
\end{aligned}$$

Similarly, $p_2 \circ f = p_2$. Therefore, $(X \times Y, i_1, i_2, p_1, p_2)$ is a biproduct $X \oplus Y$, which is a coproduct. □

Example 2.3.7.

1. ***Ab** is additive.*
2. *A full subcategory of pre-additive category is pre-additive. A full subcategory of additive category that has finite products is additive. In particular, $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ is additive.*

3. *Consider the functor* $\mathscr{C} \hookrightarrow \mathbf{Functors}(\mathscr{C}^\circ, \mathbf{Ab})$, *then there is a correspondence of additive categories.*

Definition 2.3.8 (Additive Functor)**.** *Let A, B be additive categories. A functor $F : A \to B$ is called additive if $F(g+h) = F(g) + F(h)$ for all $g, h \in \mathbf{Mor}_A(X, Y)$ and $F(0) = 0$.*

Remark 2.3.9. *A key characteristic of an additive functor is that it preserves finite biproduct.*

Consider biproduct $(X \oplus Y, i_1, i_2, p_1, p_2)$ as a biproduct of X and Y. Then $(F(X \oplus Y), F(i_1), F(i_2), F(p_1), F(p_2))$ is a biproduct of $F(X)$ and $F(Y)$. Therefore, $F(X \oplus Y) \cong F(X) \oplus F(Y)$. In this sense, F commutes with products and coproducts as well.

Also, note that since $0_0 = 1_0$, then $F(0_0) = F(1_0)$, i.e. $0_{F(0)} = 1_{F(0)}$ indicates $F(0) = 0$.

Example 2.3.10.

1. *Take $Y \in \mathbf{Ob}(\mathcal{A})$. Consider the Hom functor $R^Y : A \to \mathbf{Ab}$ that sends $X \in \mathcal{A}$ to $\mathbf{Mor}_{\mathcal{A}}(Y, X)$ and morphism f to f_* where $f_*(g) = f \circ g$.*

   Observe that R^Y is an additive functor: $R^Y(g_1 + g_2)(f) = (g_1 + g_2)_*(f) = (g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f = R^Y(g_1)(f) + R^Y(g_2)(f)$ for all arbitrary morphism f.

2. *Notice that most functors are not additive. For example, consider the constant functor $F : \mathcal{A} \to \mathcal{B}$, where $B \in \mathbf{Ob}(\mathcal{B})$ is fixed, and $F(X) = B$ for all objects $X \in \mathcal{A}$ and $F(f) = \mathbf{id}_B$ for all morphisms f in $\mathcal{A}$.*

   Observe that $\mathbf{id}_B = F(f+g) \neq F(f) + F(g) = 2 \cdot \mathbf{id}_B$, which is true whenever $B \neq 0$.

Remark 2.3.11. *In the first example above, the hom functor is mapped into the category of Abelian groups. Here we are claiming that the set of morphisms, $\mathbf{Mor}_A(Y, X)$, has the "structure of a group", but not really an Abelian group. In fact, it is called a "group object", given by the properties of the data in the category. See the following definition. (Also, refer back to the definition of pre-additive category.)*

Definition 2.3.12 (Group Object)**.** *Let A be an additive category (or just a category with terminal object 1 as well as finite products). A group object G in A is an object together with morphisms*

- *$m : G \times G \to G$ (thought of as the "group multiplication")*
- *$e : 1 \to G$ (thought of as the "inclusion of the identity element")*
- *$\mathbf{inv} : G \to G$ (thought of as the "inversion operation")*

such that

- *m is associative, i.e. $m(m \times \mathbf{id}_G) = m(\mathbf{id}_G \times m)$ as morphisms $G \times G \times G \to G$, and where e.g. $m \times \mathbf{id}_G : G \times G \times G \to G \times G$; here we identify $G \times (G \times G)$ in a canonical manner with $(G \times G) \times G$.*

- *e is a two-sided unit of m, i.e. $m(\mathbf{id}_G \times e) = p_1$, where $p_1 : G \times 1 \to G$ is the canonical projection, and $m(e \times \mathbf{id}_G) = p_2$, where $p_2 : 1 \times G \to G$ is the canonical projection.*

- ***inv** is a two-sided inverse for m, i.e. if $d : G \to G \times G$ is the diagonal map, and $e_G : G \to G$ is the composition of the unique morphism $G \to 1$ (also called the counit) with e, then $m(\mathbf{id}_G \times \mathbf{inv})d = e_G$ and $m(\mathbf{inv} \times \mathbf{id}_G)d = e_G$.*

Example 2.3.13. *Consider the category* **Ab** *with morphism $f : A \to B$. There is the following sequence:*

$$A \twoheadrightarrow A/\ker(f) \xrightarrow{\cong} \mathbf{im}(f) \hookrightarrow B$$

In general, if $f : A \to B$ is a morphism in an additive category. Notice that here we define $\ker(f)$ *to be the equalizer for the morphism pair of f and zero morphism from A to B:*

$$\begin{tikzcd} \ker(f) \arrow[r, "i"] & A \arrow[r, "f"] & B \\ & X \arrow[ul, dashed, "\exists!"] \arrow[u] \arrow[ur, "0"'] & \end{tikzcd}$$

Then there is the following exact sequence:

$$0 \longrightarrow \mathbf{Mor}_A(X, \ker(f)) \xrightarrow{i_*} \mathbf{Mor}_A(X, A) \xrightarrow{f_*} \mathbf{Mor}_A(X, B)$$

Dually, we know the cokernel of f, i.e. $B/\mathbf{im}(f)$ is the coequalizer of f and zero morphism from A to B, described by the following diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "0"'] & B \arrow[r, "j"] \arrow[d] & \mathbf{coker}(f) \arrow[dl, dashed, "\exists!"] \\ & X & \end{tikzcd}$$

with the following exact sequence

$$0 \longrightarrow \mathbf{Mor}_A(\mathbf{coker}(f), X) \xrightarrow{j^*} \mathbf{Mor}_A(B, X) \xrightarrow{f^*} \mathbf{Mor}_A(A, X)$$

In particular, notice that $\ker(f_*)$ *is corepresented by* $\ker(f)$*, and* $\ker(f^*)$ *is represented by* $\mathbf{coker}(f)$*, in the following sense: for example, consider* $\ker(f_*)$ *as the set of morphisms* $\mathbf{Mor}_A(X, \ker(f))$*, then this is essentially a functor given by* $\mathbf{Mor}_A(-, \ker(f))$*.*

We sometimes write the coimage of f as $\mathbf{coim}(f) = A/\ker(f)$*.*

Definition 2.3.14 (Pre-Abelian Category)**.** *An additive category is pre-Abelian if all kernels and cokernels of morphisms exist.*

Example 2.3.15. *We want to find similar constructions as the one described in the previous example.*

Let $f : A \to B$ be a morphism. By definition, we have that $\mathbf{im}(f) = \ker(j : B \twoheadrightarrow \mathbf{coker}(f))$. Dually, $\mathbf{coim}(f) = \mathbf{coker}(i : \ker(f) \to A)$.

This induces the following universal property in terms of kernel:

$$
\begin{tikzcd}
\mathbf{im}(f) \arrow[r, "t"] \arrow[rr, bend left, "0"] & B \arrow[r, "j"] & \mathbf{coker}(f) \\
A \arrow[u, dashed, "\exists! k"] \arrow[ur, "f"] \arrow[urr, "0"] & &
\end{tikzcd}
$$

Moreover, if we add in the object $\ker(f)$, we have the following diagram:

$$
\begin{tikzcd}
\mathbf{im}(f) \arrow[r, "t"] \arrow[rr, bend left, "0"] & B \arrow[r, "j"] & \mathbf{coker}(f) \\
A \arrow[u, dashed, "\exists! k"] \arrow[ur, "f"] \arrow[urr, bend right, "0"] & & \\
\ker(f) \arrow[u, "i"] \arrow[uu, dashed, bend left] \arrow[uur, bend right, "0"] \arrow[uurr, bend right, "0"] & &
\end{tikzcd}
$$

By the universal property, the morphism $\ker(f) \to \mathbf{im}(f)$ is zero morphism.

On the other hand, we induce the following diagram from above:

$$
\begin{tikzcd}
\ker(f) \arrow[r, "i"] \arrow[dr, "0"] & A \arrow[r, "l"] \arrow[d, "k"] & \mathbf{coim}(f) \arrow[dl, dashed, "\exists! s"] \\
& \mathbf{im}(f) &
\end{tikzcd}
$$

This is true because recall that $\mathbf{coim}(f) = A/\ker(f)$, then by the universal property of the quotinet we have the diagram above.

Therefore, f is essentially a sequence:

$$
\begin{tikzcd}
A \arrow[r, "l"] \arrow[rr, bend left, "k"] & \mathbf{coim}(f) \arrow[r, "s"] & \mathbf{im}(f) \arrow[r, "t"] & B
\end{tikzcd}
$$

In this case, note that s is not necessary an isomorphism. i.e. The First Isomorphism Theorem may not hold in these cases.

Definition 2.3.16 (Abelian Category)**.** *A pre-Abelian category $\mathcal{A}$ is Abelian if $s : \mathbf{coim}(f) \to \mathbf{im}(f)$ is an isomorphism $\forall f : A \to B$ morphism in $\mathcal{A}$.*

Example 2.3.17. *1. If $\mathcal{A}$ is Abelian, then $\mathcal{A}^\circ$ is Abelian as well.*

2. *$\mathbf{Ab}$, R-mod (left module), Mod-R (right module) are Abelian categories.*

3. *The finite Abelian groups $\mathbf{FinAb} \subseteq \mathbf{Ab}$ is also an Abelian category.*

4. *Category of free Abelian groups (i.e. $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$) is a full subcategory of $\mathbf{Ab}$, but it is not even pre-Abelian.*

Example 2.3.18. *The construction of sequences provide us with other good constructions. Consider the following diagram:*

$$\begin{tikzcd}
\ker(f) \arrow[r] \arrow[d, "\exists!"] \arrow[rr, bend left, "0"] & A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
\ker(f') \arrow[r] & A' \arrow[r, "f'"] & B'
\end{tikzcd}$$

This induces the following diagram:

$$\begin{tikzcd}
\ker(f) \arrow[r] \arrow[d, "\exists!"'] \arrow[rr, bend left, "0"] \arrow[dr, dashed] \arrow[drr, dashed, "0"] & A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
\ker(f') \arrow[r] & A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

This induces a functor $\mathbf{Arr}(A) \to A$ by mapping f to $\ker(f)$.
In a dual argument, we have the following diagram with similar properties:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d] & \mathbf{coker}(f) \arrow[d] \\
A' \arrow[r, "f'"] & B' \arrow[r] & \mathbf{coker}(f')
\end{tikzcd}$$

Remark 2.3.19. *We also want to construct the notion of an exact sequence in these cases. Suppose we have the following diagram where $t : B \to \mathbf{coker}(s) \to C$ satisfies $t \circ s = 0$:*

$$\begin{tikzcd}
A \arrow[r, "s"] & B \arrow[r, "t"] \arrow[d] & C \\
& \mathbf{coker}(s) &
\end{tikzcd}$$

By rearranging it, we have the following diagram:

$$\begin{tikzcd}
A \arrow[r, "s"] \arrow[dr, "0"'] & B \arrow[r, "j"] \arrow[d, "t"] & \mathbf{coker}(s) \arrow[dl, dashed] \\
& C &
\end{tikzcd}$$

Note that the dashed morphism is induced by the universal property of cokernel (as a coequalizer).

This induces a morphism from $\ker(j)$ *to* $\ker(t)$. *Note that* $\ker(j) = \mathbf{im}(s)$, *then this induces a morphism from* $\mathbf{im}(s)$ *to* $\ker(t)$ *as well.*

Definition 2.3.20 (Exact Sequence)**.** *We say that the first diagram in the previous remark is exact if* $\mathbf{im}(s) \to \ker(t)$ *is an isomorphism (which implies* $t \circ s = 0$*).*

Definition 2.3.21 (Monomorphism, Epimorphism)**.** *Let* $\mathcal{A}$ *be an additive category, then* $f : A \to B$ *is a monomorphism if* $\forall g : X \to A$ *such that* $f \circ g = 0$, *we have* $g = 0$. *In particular, that is equivalent to having the morphism* $f_* : \mathbf{Mor}_{\mathcal{A}}(X, A) \to \mathbf{Mor}_{\mathcal{A}}(X, B)$ *by post-composing* f *as an injection.*

Let $\mathcal{A}$ *be an additive category, then* $f : A \to B$ *is an epimorphism if* $\forall g : B \to X$ *such that* $g \circ f = 0$, *we have* $g = 0$. *In paraticular, that is equivalent to having the morphism* $f^* : \mathbf{Mor}_{\mathcal{A}}(B, X) \to \mathbf{Mor}_{\mathcal{A}}(A, X)$ *by pre-composing* f *as a surjection.*

Proposition 2.3.22. *Let* $\mathcal{A}$ *be a pre-Abelian category, and let* $f : A \to B$ *be a morphism. The following are equivalent:*

1. *The sequence* $0 \to A \xrightarrow{f} B$ *is exact.*

2. $\ker(f) = 0$.

3. f *is a monomorphism.*

Proof. Observe that 1) and 2) are equivalent: $A \xrightarrow{f} B$ is exact if and only if $0 = \mathbf{im}(0) \xrightarrow{\sim} \ker(f)$.

We now show that 2) implies 3). Observe that $0 \to \mathbf{Mor}(X, \ker(f)) \to \mathbf{Mor}(X, A) \xrightarrow{f_*} \mathbf{Mor}(X, B)$ is an exact sequence of Abelian groups. Notice that f_* is an injection, which means f is a monomorphism.

Finally, we show that 3) implies 2). Since f is a monomorphism, then f_* is injective. Again, consider the sequence $0 \to \mathbf{Mor}(X, \ker(f)) \to \mathbf{Mor}(X, A) \xrightarrow{f_*} \mathbf{Mor}(X, B)$. In particular, $\mathbf{Mor}(X, \ker(f)) = 0$ for all $x \in \mathcal{A}$ because $f_* \circ i = f_* \circ 0$. Therefore, $\ker(f)$ is the final object in the category, which means $\ker(f) = 0$. $\square$

Lemma 2.3.23. *Let* $\mathcal{A}$ *be an Abelian category and* $f : A \to B$ *is a monomorphism. Then* $\mathbf{im}(f) = A$.

Proof. Note that $\mathbf{im}(f) \cong \mathbf{coim}(f) = \mathbf{coker}(\ker(f) \to A) = A$.

The first relation is by the definition of Abelian category. The second relation is a direct result from the definition. The last result is from the fact that $\ker(f) = 0$. $\square$

Remark 2.3.24. *In an Abelian category, if* $f : A \to B$ *is a monomorphism, then it is the kernel of* $g : B \to \mathbf{coker}(f)$ *(canonical surjective homomorphism) and zero morphism. Indeed,* $g \circ f = 0 \circ f = 0$ *by definition. Also, it satisfies the universal property because suppose there is some* $k : C \to B$

satisfies the same property $g \circ k = 0$. By definition, the image of k is contained in the image of f. By the lemma, there is $\mathbf{im}(f) \cong A$. In particular, there is some inverse $f' : B \to A$ of f. Therefore, the image of k is contained in A. Therefore, let $h : C \to A$ be defined as taking c to $f'(k(c)) \in A$. Therefore, we have $f \circ h = ff'k = k$ by definition. Note that since f is a monomorphism, so by left cancellation h is unique.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & B/\mathbf{im}(f) \\ C \arrow[u, dashed, "h"] \arrow[ur, "k"'] & & \end{tikzcd}$$

In a dual fashion, if $f : A \to B$ is an epimorphism, then it is the cokernel of $g : C \to A$.

Proposition 2.3.25. *In an Abelian category, the sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is exact if and only if $A = \ker(g)$.*

Proof. Recall that the sequence is exact if and only if f is a monomorphism and $\mathbf{im}(f) \xrightarrow{\sim} \ker(g)$.

By the previous lemma, $\mathbf{im}(f) = A$, so $A \cong \ker(g)$.

On the other hand, if $A = \ker(g)$, then there is the following exact sequence:

$$0 \longrightarrow \mathbf{Mor}(X, A) \xrightarrow{f_*} \mathbf{Mor}(X, B) \xrightarrow{g_*} \mathbf{Mor}(X, C)$$

This means that f is a monomorphism, so $\mathbf{im}(f) = A = \ker(g)$. □

From this point on, we work on Abelian categories unless specified otherwise.

Corollary 2.3.26. *Dually, the sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact if and only if $\mathbf{coker}(f) \cong C$.*

Proposition 2.3.27.

1. *A sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is exact in $\mathcal{A}$ if and only if for all objects X in $\mathcal{A}$, the sequence*

   $$0 \longrightarrow \mathbf{Mor}(X, A) \xrightarrow{f_*} \mathbf{Mor}(X, B) \xrightarrow{g_*} \mathbf{Mor}(X, C)$$

   is exact.

2. *A sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact in $\mathcal{A}$ if and only if for all objects X in $\mathcal{A}$, the sequence*

   $$0 \longrightarrow \mathbf{Mor}(C, X) \xrightarrow{g^*} \mathbf{Mor}(B, X) \xrightarrow{f^*} \mathbf{Mor}(A, X)$$

   is exact.

Proof. We prove the first statement. Note that the first sequence is exact if and only if $A = \ker(g)$ if and only if the second sequence is exact. □

Definition 2.3.28 (Exact)**.** *The sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact if and only if $A = \ker(g)$, $C = coker(f)$ *and* f *is a monomorphism and* g *is an epimorphism.*

Definition 2.3.29 (Left Exact, Right Exact, Exact)**.** *Let* $F : \mathcal{A} \to \mathcal{B}$ *be an additive functor between Abelian categories. We say that* F *is left exact if for every short exact sequence* $0 \to A \to B \to C \to 0$ *in* $\mathcal{A}$*, the sequence* $0 \to F(A) \to F(B) \to F(C)$ *is exact in* $\mathcal{B}$*.*

We say that F *is right exact if for every short exact sequence* $0 \to A \to B \to C \to 0$ *in* $\mathcal{A}$*, the sequence* $F(A) \to F(B) \to F(C) \to 0$ *is exact in* $\mathcal{B}$*.*

We say that F *is exact if it is both left exact and right exact.*

Example 2.3.30.

1. *Let* $X \in \mathcal{A}$*. Consider the covariant Hom functor* $R^X : \mathcal{A} \to \mathbf{Ab}$ *by mapping* $Y \mapsto \mathbf{Mor}_{\mathcal{A}}(X, Y)$*, then* R^X *is left exact.*

2. *Let* $X \in \mathcal{A}$*. Consider the contravariant Hom functor* $R_X : \mathcal{A} \to \mathbf{Ab}$ *by mapping* $Y \mapsto \mathbf{Mor}_{\mathcal{A}}(Y, X)$*, then* R_X *is left exact.*

Theorem 2.3.31 (Mitchell)**.** *Let* $\mathcal{A}$ *be a small Abelian category. Then there is a ring* R *and exact fully faithful functor* $F : \mathcal{A} \to R$*-modules, which is an Abelian category.*

Remark 2.3.32. *Consider the two parallel exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[r, "g"] \arrow[d, "v"] & C \arrow[r] \arrow[d, "w"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0
\end{tikzcd}$$

Then there is a corresponding diagram of exact sequences

$$\begin{tikzcd}
& 0 \arrow[r] & \ker(u) \arrow[r] & \ker(v) \arrow[r] & \ker(w) \arrow[dlll, "\delta"] & \\
& \mathbf{coker}(u) \arrow[r] & \mathbf{coker}(v) \arrow[r] & \mathbf{coker}(w) \arrow[r] & 0 &
\end{tikzcd}$$

Figure 2.10: Snake Lemma

where $\delta : \ker(w) \to coker(u)$ *is defined as the following:*

Take arbitrary $c \in \ker(w)$*, then* c *can be lifted back to* $b \in B$ *with* $g(b) = c$*. Then there is* $b' \in B'$ *correspondingly, and there is* $a' \in A'$ *as the lift for* $b' \in B'$*. Therefore, define* $c \mapsto \delta(c) = a' + \mathbf{im}(u) \in A'/\mathbf{im}(u) = \mathbf{coker}(u)$*.*

Lemma 2.3.33 (Snake Lemma)**.** *The sequence in* ***Figure 2.10*** *is exact.*

Proof. See Homework 9, problem 10. ☐

Proposition 2.3.34. *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence in an Abelian category. Then the following are equivalent:*

1. *$\exists h : C \to B$ such that $g \circ h = 1_C$.*
2. *$\exists k : B \to A$ such that $k \circ f = 1_A$.*
3. *There exists a biproduct (B, f, h, k, g).*
4. *The short exact sequence is isomorphic to*

$$0 \longrightarrow A \xrightarrow{(1,0)} A \oplus C \xrightarrow{(0,1)} C \longrightarrow 0$$

Definition 2.3.35 (Split)**.** *We say the short exact sequence is split if (1) - (4) above hold.*

Proof. We first show that (1) ⇒ (2).

Define $k' : 1_B - h \circ g : B \to B$. Then there is the following diagram:

$$\begin{tikzcd} & & B \arrow[dl, dashed, "k"'] \arrow[d, "k'"] \arrow[dr, dashed, "0"] & & \\ 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, bend left, "g"] & C \arrow[l, bend left, "h"] \arrow[r] & 0 \end{tikzcd}$$

Note that $g \circ k' = g \circ (1_B - h \circ g) = g - g \circ h \circ g = g - g = 0$.

Now $A = \ker(g)$, so there exists a unique $k : B \to A$ such that $f \circ k = k'$. In particular, $f \circ k \circ f = k' \circ f = (1_B - h \circ g) \circ f = f - h \circ g \circ f = f$. Thus, $f \circ (k \circ f - 1_A) = 0$, but f is a monomorphism, so $k \circ f = 1_A$.

Similarly, one can show that (2) ⇒ (1). So (1) and (2) are equivalent.

We now show that (2) implies (3). Note that we can use the fact that (1) and (2) are equivalent. Therefore, we have k and h: $g \circ h = 1_C$, $k \circ f = 1_A$. Then we have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, bend left, "f"] & B \arrow[l, bend left, "k"] \arrow[r, bend left, "g"] & C \arrow[l, bend left, "h"] \arrow[r] & 0 \end{tikzcd}$$

We know $g \circ f = 0$. Note $f \circ k \circ h = k' \circ h = (1_B - h \circ g) \circ h = h \circ h = 0$. But f is a monomorphism, so $k \circ h = 0$.

Finally, we check $f \circ k + h \circ g = 1_B$. This is obvious as $k' = f \circ k = 1_B - h \circ g$.

We then show that (3) implies (4). One can check that $(k, g) : B \to A \oplus C$ is an isomorphism such that the following diagram commutes.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "(k,g)"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r] & A \oplus C \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

Finally, we check that (4) ⇒ (1). This is obvious because we have $(0, 1) : C \to A \oplus C$ as an inverse:

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & A \oplus C \arrow[r, bend left] & C \arrow[l, bend left, "(0,1)"] \arrow[r] & 0
\end{tikzcd}
$$

□

Now let A be an Abelian category. Recall that for $X \in \mathbf{Ob}(\mathscr{C})$, $R^X : A \to \mathbf{Ab}$ that sends Y to $\mathbf{Mor}_A(X, Y)$ is left exact.

Definition 2.3.36 (Projective)**.** *We say X is projective if R^X is exact.*

Recall that if $0 \to A \to B \to C$ is a short exact sequence, then

$$
0 \longrightarrow \mathbf{Mor}(X, A) \xrightarrow{f_*} \mathbf{Mor}(X, B) \xrightarrow{g_*} \mathbf{Mor}(X, C)
$$

is exact as well. In particular, denote $B \twoheadrightarrow C$ as an epimorphism.
If X is projective, $\forall k : X \to C$, there exists $h : X \to B$ such that $g \circ h = k$.

Definition 2.3.37 (Lift)**.** *We say such morphism h is a lift:*

$$
\begin{tikzcd}
 & X \arrow[dl, dashed, "h"'] \arrow[d, "k"] \\
B \arrow[r, two heads] & C
\end{tikzcd}
$$

Figure 2.11: Lift

Remark 2.3.38. *Suppose $0 \to A \to B \to C \to 0$ is a short exact sequence where C is projective, then the short exact sequence splits because there is some $h : C \to B$ such that the following diagram commutes:*

$$
\begin{tikzcd}
 & C \arrow[dl, dashed] \arrow[d, "1_C"] \\
B \arrow[r] & C
\end{tikzcd}
$$

Dually, consider $R_X : A^\circ \to \mathbf{Ab}$.

Definition 2.3.39 (Injective)**.** *We say X is injective if R_X is exact.*

Thus, $\forall k : A \to X$, $\exists h : B \to X$ such that $h \circ f = k$. Here we denote $A \hookrightarrow B$ as a monomorphism. Then, we have:

$$
\begin{tikzcd}
A \arrow[r, hook, "f"] \arrow[d, "k"'] & B \arrow[dl, dashed, "h"] \\
X &
\end{tikzcd}
$$

Remark 2.3.40. *In particular, if $0 \to A \to B \to C \to 0$ is a short exact sequence where A is injective, then the short exact sequence splits.*

3 Ring Theory

3.1 Definition of Rings

Definition 3.1.1 (Ring). *A ring is a set R together with two binary operations $+, \cdot$, such that:*

1. *$(R, +)$ is Abelian group.*
2. *$\exists 1 \in R$ such that $1 \cdot x = x \cdot 1 = x$ for all $x \in R$.*
3. *$(xy)z = x(yz)$ for all $x, y, z \in R$.*
4. *$(x + y)z = xz + yz$, $z(x + y) = zx + zy$ for all $x, y, z \in R$.*

Finally, we say R is a commutative ring if $xy = yx$ for all $x, y \in R$.

Property 3.1.2.

1. *1 is unique.*
2. *$x \cdot 0 = 0 \cdot x = 0$ for all $x \in R$.*
3. *$(-x) \cdot y = -(xy) = x \cdot (-y)$ for all $x, y \in R$.*

Definition 3.1.3 (Invertible). *We say $x \in R$ is invertible if $\exists y \in R$ such that $xy = yx = 1$. We write $y = x^{-1}$, so $(x^{-1})^{-1} = x$ if it is well-defined. Moreover, $(x_1x_2)^{-1} = x_2^{-1}x_1^{-1}$.*

We denote $R^\times$ as the group of all invertible elements in R.

Remark 3.1.4. *We say $R = \{0\}$ is the zero ring, then $1 = 0$. Moreover, the converse is also true: if $1 = 0 \in R$, then R is the zero ring.*

Definition 3.1.5 (Division Ring). *A ring is called a division ring if $R \neq 0$ and every $x \neq 0$ is invertible, i.e. $R^\times = R \backslash \{0\}$.*

Remark 3.1.6. *A field is a commutative division ring.*

Definition 3.1.7 (Zero Divisor, Integral Domain). *If R is commutative, for $0 \neq x \in R$, x is called a zero divisor if $\exists 0 \neq y \in R$ such that $xy = 0$.*

R is called an integral domain if $R \neq 0$ is a commutative ring and has no zero divisors.

Remark 3.1.8. *Fields are integral domains.*

Example 3.1.9.

1. *$\mathbb{Z}$ is a ring, an integral domain, but not a field. In particular, $\mathbb{Z}^\times = \{\pm 1\}$.*

2. *$\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ are fields.*

3. *Let R be a ring with integer $n > 0$. $M_n(R)$ is the set of $n \times n$ matrices with entries in R. This is a ring as well. Note that $M_n(R)^\times = \mathbf{GL}_n(R)$, which is the group of all invertible $n \times n$ matrices with R-entries.*

4. *$\mathbb{Z}/n\mathbb{Z}$ is a commutative ring. It is an integral domain if and only if n is a prime integer, if and only if $\mathbb{Z}/n\mathbb{Z}$ is an integral domain. Moreover, $(\mathbb{Z}/n\mathbb{Z})^\times$ is a group of order $\varphi(n)$.*

5. *Let A be an Abelian group. Let R be the set of endomorphisms of A, i.e. the set of homomorphisms from A to itself. Then $R = \mathbf{End}_R(A) = \mathbf{Hom}(A, A)$ is a ring with usual addition and composition as multiplication, called the ring of endomorphisms of an Abelian group A. Note that $\mathbf{End}(A)^\times$ is the group of automorphisms of A, i.e. $\mathbf{Aut}(A)$.*

6. *Let $\mathbb{H}$ be a vector space over $\mathbb{R}$ with basis $\{1, i, j, k\}$. We can then figure out its multiplication table, which gives $k = ij = -ji$. Therefore, $\mathbb{H}$ is a ring, and is a non-commutative division ring in particular. We now look at the norm defined by $N(a + bi + cj + dk) = a^2 + b^2 + c^2 + d^2$ with $N(z_1 z_2) = N(z_1)N(z_2)$. In particular, $N(z) > 0$ if $z \neq 0$.*

   Denote $z = a+bi+cj+dk$, then let $\bar{z} = a-bi-cj-dk$, then $z\bar{z} = \bar{z}z = N(z)\cdot 1$. In particular, $z^{-1} = \frac{\bar{z}}{N(z)}$. This give the division ring structure.

   If we do the same thing over $\mathbb{C}$, then $\mathbb{H} \cong M_2(\mathbb{C})$, which is not a division ring.

7. *Let R be a ring. Let $R[t] = \{a_0 + a_1 t + \cdots + a_n t^n : a_i \in R\}$ be a set, then it is a ring in the usual sense. We call it the polynomial ring. Note that R is an integral domain if and only if $R[t]$ is an integral domain. However, $R[t]$ is never a field: t is not invertible.*

   One can add more variables into the polynomial ring: $R[s, t] = (R[s])[t]$. Moreover, for any set X of variables, we define $R[x] = \bigcup_{Y \subseteq X} R[Y]$ for finite sets Y.

   Moreover, let X be a set, then $R[X]$ is a polynomial ring with commuting variables in X. We also denote $R\langle X\rangle$ as the polynomial ring with non-commuting variables in X, i.e. $R\langle s, t\rangle \neq R[s, t]$. Alternatively, one can say that this is the set of R-linear combinations of monomials (a monomial is a word in X).

Definition 3.1.10 (Ring Homomorphism)**.** *Let R and S be rings. A map $f : R \to S$ is a ring homomorphism if*

- *$f(x + y) = f(x) + f(y)$*
- *$f(xy) = f(x)f(y)$*
- *$f(1_R) = 1_S$*

The collection of rings and the homomorphisms between them form a category of rings **Ring**.

Example 3.1.11. *1. $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ by taking $x \mapsto [x]_n$ is a ring homomorphism.*

2. *$\mathbb{Z} \hookrightarrow \mathbb{Q} \hookrightarrow \mathbb{R} \hookrightarrow \mathbb{C}$ are inclusion ring homomorphisms.*

3. *$0 : R \to S$ sends $1_R \mapsto 0_S$, which means this is not a ring homomorphism if $S \neq 0$.*

   e.g. $\mathbf{Mor}(\mathbb{Q}, \mathbb{Z}) = \varnothing$.

4. *In **Ring**, the initial object is $\mathbb{Z}$ and the terminal object is 0.*

5. *Consider the forgetful functor F : **Ring** $\to$ **Set**. There is a left adjoint, the free functor G : **Set** $\to$ **Ring** which takes a set X to the ring $\mathbb{Z}\langle X\rangle$. i.e. there is an isomorphism $\mathbf{Hom}_{\mathbf{Set}}(X, R) \cong \mathbf{Hom}_{\mathbf{Ring}}(GX = \mathbb{Z}\langle X\rangle, R)$, where $g(x_1x_2\cdots x_n) = f(x_1)f(x_2)\cdots f(x_n)$. This works because the mapping from $\mathbb{Z}\langle X\rangle$ to R is determined by sending x to τ_x. Note that this is analogous to the operations we have on free groups, so we call $\mathbb{Z}\langle X\rangle$ the free polynomial ring.*

   *Now consider the forgetful functor for the category of commutative rings (denoted as **CRing**) F : **CRing** $\to$ **Set**. It also has a left adjoint G : **Set** $\to$ **CRing** taking a set X to $\mathbb{Z}[X]$, i.e. $\mathbf{Hom}_{\mathbf{Set}}(X, R) \cong \mathbf{Hom}_{\mathbf{CRing}}(GX = \mathbb{Z}[X], R)$.*

6. *Consider a "semi-forgetful" functor F : **Ring** $\to$ **Grp** that sends $R \mapsto R^X$ and $(f : R \to S) \mapsto (Ff : R^\times \to S^\times)$. There is a left adjoint H : **Grp** $\to$ **Ring** that takes a group G to $\mathbb{Z}[G] = \{\sum_{g\in G} n_g \cdot g, n_g \in \mathbb{Z}$, where almost all $n_g = 0\}$. This is sometimes denoted at $\mathbb{Z}^{(G)}$, a set of maps between G and $\mathbb{Z}$ by sending $g \mapsto n_g$. The construction $\mathbb{Z}[G]$ is called the group ring of G.*

   In particular, denote a ring homomorphism $f : \mathbb{Z}[G] \to R$ by sending $G \subseteq \mathbb{Z}[G]^\times \mapsto F(R) = R^\times$. There is $\mathbf{Hom}_{\mathbf{Ring}}(\mathbb{Z}[G], R) \xrightarrow{\sim} \mathbf{Hom}(G, R^\times = F(R))$. One can define the inverse of f in the following way. Take $h : G \to R^\times$ a group homomorphism, then $f : \mathbb{Z}[G] \to R$ is defined by $f(\sum_{g\in G} n_g \cdot g) = \sum_{g\in G} n_g \cdot h(g) \in R$.

   Note that if G is an infinite cyclic group, then there is a generator t, now $\mathbb{Z}[G] = \mathbb{Z}[t, t^{-1}]$.

Definition 3.1.12 (Subring)**.** *Let S be a ring. A subset $R \subseteq S$ is a subring if $(R, +)$ is a subgroup of $(S, +)$, and for all $x, y \in R$, there is $xy \in R$, and we have $1_S \in R$.*

Note that this implies $1_R = 1_S$.

Example 3.1.13. *1. $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ are subrings.*

2. *Consider the subset $\left\{ \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \right\} \subseteq M_2(\mathbb{Q})$. Note that the subset is a ring with identity $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, but the identity of $M_2(\mathbb{Q})$ is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore, this is not a subring.*

3.2 Ideal

Definition 3.2.1 (Ideal). *Let R be a ring. A subset $I \subseteq R$ is called a left ideal if*

1. *$(I, +)$ is a subgroup of $(R, +)$.*
2. *For all $x \in R$, $y \in I$, we have $xy \in I$.*

Similarly, $I \subseteq R$ is a right adjoint if

1. *$(I, +)$ is a subgroup of $(R, +)$.*
2. *For all $x \in R$, $y \in I$, we have $yx \in I$.*

An ideal, or a two-sided ideal, is both a left ideal and a right ideal.

Example 3.2.2.

1. *There are two trivial ideals: the zero ideal $0 = \{0\} \subseteq R$ and the unit ideal $R \subseteq R$.*
2. *Let $a \in R$. $Ra = \{xa : x \in R\}$ is called the left-principal ideal (generated by a).*

   Similarly, $aR = \{ax : x \in R\}$ is the right principal ideal generated by a.
3. *Let $A \subseteq R$ be a subset. Denote $\langle A \rangle_l = \{\sum\limits_{a \in A} x_a \cdot a : x_a \in R,$ almost all x_a are zero$\}$ as the left ideal generated by A. Similarly, there is a right ideal generated by A.*

   Also note that Ra is the left ideal generated by the singleton set $\{a\}$.
4. *Let $I \subseteq R$ be a left ideal (respectively, right ideal, two-sided ideal) such that $I \cap R^\times \neq \varnothing$, then $I = R$ is the unit ideal.*

   Proof. Take $a \in I \cap R^\times$, then $1 = a^{-1} \cdot a \in I$. Therefore, for all $x \in R$, $x = x \cdot 1 \in I$, so $I = R$. ☐
5. *Let $I = \left\{ \begin{pmatrix} * & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \cdots & 0 \end{pmatrix} \right\} \subseteq M_n(R)$. Then I is a left ideal but not a right ideal.*
6. *Let $f : R \to S$ be a ring homomorphism, then $\ker(f) \subseteq R$ is a two-sided ideal.*

Definition 3.2.3 (Factor Ring). *Let $I \subseteq R$ be an ideal. Now R/I is a factor group. Define $(x+I)\cdot(y+I) = xy+I$. This is well-defined: if $x_1+I = x_2+I$ and $y_1+I = y_2+I$, then $x_1 - x_2 \in I$ and $y_1 - y_2 \in I$. Therefore, $x_1y_1 - x_2y_2 = (x_1y_1 - x_2y_1) + (x_2y_1 - x_2y_2) = (x_1 - x_2)y_1 + x_2(y_1 - y_2) \in I$. We now say R/I is a factor ring where $0_{R/I} = 0 + I = I$ and $1_{R/I} = 1 + I$.*

Remark 3.2.4. *Note that I has to be a two-sided ideal, i.e. the construction does not work on a left ideal or a right ideal.*

Consider the canonical ring homomorphism $\pi : R \twoheadrightarrow R/I$ that sends $a \mapsto a + I$. Then $\ker(\pi) = I$ has to be a two-sided.

The isomorphism theorems in groups also holds in rings, for example:

Theorem 3.2.5 (First Isomorphism Theorem of Rings)**.** *Let $f : R \to S$ be a ring homomorphism. Then* $\mathbf{im}(f)$ *is a subring of* S*. Moreover, the map* $\bar{f} : R/\ker(f) \to \mathbf{im}(f)$ *defined by* $\bar{f}(a+\ker(f)) = f(a)$ *is a ring isomorphism.*

Example 3.2.6. *Consider the surjective ring homomorphism* $f : \mathbb{R}[t] \twoheadrightarrow \mathbb{C}$ *that sends* $t \mapsto i$*,* $a + bt \mapsto a + bi$*,* $1 + t^2 \mapsto 1 + i^2 = 0$*, then* $\ker(f) = (1 + t^2) \cdot \mathbb{R}[t]$*.*

In particular, $\mathbb{C} \cong \mathbb{R}[t]/((1 + t^2) \cdot \mathbb{R}[t])$*. This is an algebraic definition of the set of complex numbers.*

Let R_i be rings for $i \in I$. Similar as in **Grp**, $\prod_{i\in I} R_i$ is the product in **Ring**.

Suppose R is the product of finitely many rings, i.e. $R = R_1 \times \cdots \times R_n$. Now let $e_i = (0, \cdots , 0, 1, 0, \cdots , 0) \in R$ for $i \in \{1, \cdots , n\}$ where the 1-entry is on the i-th slot. These elements satisfy the following properties:

1. Idempotent: $e_i^2 = e_i$.
2. Orthogonality: $e_ie_j = 0$ for all $i \neq j$.
3. Partition of Unity: $e_1 + \cdots + e_n = 1$.
4. $e_i \in Z(R)$: $e_ix = xe_i$ for all $x \in R$, for all i.

Note that $R_i = Re_i$, so $(xe_i)(ye_i) = xye_i \in R_i$. Therefore, R_i is a ring with identity e_i.

Consider the map $f : R_1 \times \cdots \times R_n \to R$ that sends $(x_1, \cdots , x_n) \mapsto x_1 + \cdots + x_n$. This is a ring homomorphism, where the multiplication comes from

$$\begin{aligned} f(x_1y_1, \cdots , x_ny_n) &= x_1y_1 + \cdots + x_ny_n \\ &= (x_1 + \cdots + x_n)(y_1 + \cdots + y_n) \end{aligned}$$

where $x_iy_j = 0$ for all $i \neq j$.

Moreover, for $x \in R$, $x = \sum_{i\in I} xe_i = f(xe_1, \cdots , xe_n)$. One can check that this is a ring isomorphism.

Example 3.2.7. *Let R be a ring. Take* $S = \left\{ \begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & * & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & * \end{pmatrix} \right\} \subseteq M_n(R)$*.*

Note that $e_i = e_{i,i}$*, i.e .having entry* 1 *on the* (i, i)*-th position and* 0 *elsewhere.*

In particular, $Se_i \cong R$*, so* $S \cong R \times \cdots \times R$*, with* n *copies.*

Theorem 3.2.8 (Chinese Remainder). *Let $I_1, \cdots, I_n$ be ideals in a ring R such that $I_k + I_l = R$ for all $k \neq l$. Let $a_1, \cdots, a_n \in R$. Then there is $a \in R$ such that $a \equiv a_i \pmod{I}_i$ for all $i = 1, \cdots, n$, i.e. $a - a_i \in I_i$.*

Proof. This can be done by induction on n.

Note $n = 1$ is obvious. Consider the case where $n = 2$, i.e. we have $a_1 = a_2 \in I = I_1 + I_2$, which means $a_1 - a_2 = x_1 + x_2$ for some $x_i \in I_i$.

Define $a = a_1 - x_1 = a_2 + x_2$, then such a satisfies $a - a_1 = -x_1 \in I_1$ and $a - a_2 = x_2 \in I_2$. Then we are done.

We use this idea in the inductive step, i.e. suppose case $n-1$ is true, show that the case is true at n.

By induction hypothesis, there exists $b \in R$ such that $b \equiv a_i \pmod{I}_i$ for all $i = 1, \cdots, n-1$.

We claim that $(\bigcap_{i \leq i \leq n-1} I_i) + I_n = R$.

By definition, $I_i + I_n = R$ for all $i = 1, \cdots, n-1$. Therefore, $x_i + y_i = 1$ for some $x_i \in I_i$ and $y_i \in I_n$ for $i = 1, \cdots, n-1$.

Now $\prod_{1 \leq i \leq n-1} (x_i + y_i) = 1$. By decomposing, $x_1 x_2 \cdots x_{n-1} \in \bigcap_{1 \leq i \leq n-1} I_i$, and the other terms in the product are monomials that contain at most one $y_i = 1$, which is in I_n.

Now, apply the $n = 2$ case to $\bigcap_{1 \leq i \leq n-1} I_i$ and I_n, and two elements b and a_n. In particular, there exists some $a \in R$ such that $a \equiv b \pmod{(}) \bigcap_{1 \leq i \leq n-1} I_i)$ and $a \equiv a_n \pmod{I}_n$.

This concludes the proof because $b \equiv a_i \pmod{I}_i$ for $i = 1, \cdots, n-1$ and so $a \equiv a_i \pmod{I}_i$ for $i = 1, \cdots, n-1$. □

Consider the map $f : R \to R/I_1 \times R/I_2 \times \cdots \times R/I_n$ that sends $a \mapsto (a + I_1, \cdots, a + I_n)$. The Chinese Remainder Theorem concludes that f is a surjective map. Furthermore, the kernel is $\bigcap_{1 \leq i \leq n} I_i$.

Therefore, $R / \bigcap_{1 \leq i \leq n} I_i \cong R/I_1 \times \cdots \times R/I_n$.

Example 3.2.9. *Consider $R = \mathbb{Z}$, and $I_i = \mathbb{Z} \cdot n_i$ where $i = 1, \cdots, m$ for $\gcd(n_i, n_j) = 1$ for all $i \neq j$, which is equivalent to saying $\mathbb{Z} \cdot n_i + \mathbb{Z} \cdot n_j = \mathbb{Z}$.*

Then $\bigcap_{1 \leq i \leq n} I_i = \mathbb{Z} \cdot (n_1 \cdots n_m)$.

Hence, $\mathbb{Z}/n_1 \cdots n_m \mathbb{Z} \cong \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_m\mathbb{Z}$.

We saw that the product in rings is the same as that in groups. However, the coproduct is different.

Consider a ring R with generating set $X \subseteq R$. Take $I = \ker(\mathbb{Z}\langle X \rangle \twoheadrightarrow R) \subseteq \mathbb{Z}\langle X \rangle$. Now $R \cong \mathbb{Z}\langle X \rangle / I$.

Suppose we have a family of rings $(R_i)_{i \in I}$ with $R_i \cong \mathbb{Z}\langle x_i \rangle / I_i$ where I_i is the kernel of $\mathbb{Z}\langle x_i \rangle \twoheadrightarrow R_i$, and $X_i \subseteq R_i$ is the generating subset of R_i.

Now $\coprod_{i \in I} R_i = \mathbb{Z}\left\langle \coprod_{i \in I} X_i \right\rangle / \langle \text{ideal generated by } I_i \rangle$. Note $I_j \subseteq \mathbb{Z}\langle X_j \rangle \subseteq \mathbb{Z}\left\langle \coprod_{i \in I} X_i \right\rangle$. This setting has the universal property as follows:

$$\begin{tikzcd} \mathbb{Z}\langle X_i \rangle \arrow[d] \arrow[dr, dashed, "f_i"] & \\ R_i \arrow[r] & S \end{tikzcd}$$

This induces $g : \mathbb{Z}\left\langle \coprod_{i\in I} X_i \right\rangle \to S$, which factors through ring homomorphism.

However, consider the category of commutative rings instead. Then $R_i \cong \mathbb{Z}[X_i]/I_i$ with the same setting as above.

In particular, $\coprod_{i\in I} R_i \cong \mathbb{Z}[\coprod_{i\in I} X_i]/\langle\text{ideal generated by } I_i\rangle$. Here, $R_1 \coprod R_2 = R_1 \otimes_{\mathbb{Z}} R_2$ is the tensor product.

Definition 3.2.10 (Prime Ideal). *Let R be a commutative ring and $P \subseteq R$ be an ideal. P is a prime ideal if $P \neq R$ and whenever $xy \in P$, either $x \in P$ or $y \in P$.*

This is equivalent to having $R/P \neq 0$ and R/P having no zero divisors, which is equivalent to having R/P as an integral domain.

Example 3.2.11. *Take $R = \mathbb{Z}$. Every ideal in $\mathbb{Z}$ is principal.*

Note that $\mathbb{Z}\cdot n$ is prime if and only if $n = 0$ or $n = \pm p$ for some prime p, i.e. prime p multiplied by a unit.

Definition 3.2.12 (Maximal Ideal). *Let R be a commutative ring and ideal $M \subseteq R$.*

We say M is maximal if $M \neq R$ and if $M \subseteq M' \subseteq R$ for some ideal M', then either $M' = M$ or $M' = R$.

Note that M is maximal if and only if ($R/M \neq 0$ and) R/M is a field.

Lemma 3.2.13. *A commutative ring R has exactly two ideals if and only if R is a field.*

Example 3.2.14.

1. *The zero ring has no prime or maximal ideals.*
2. *Let $R = \mathbb{Z}$ and $n \geq 0$. Then $n\mathbb{Z}$ is prime if and only if $n = 0$ or $n = p$ isprime. It is maximal if and only if $n = p$ is prime.*

Theorem 3.2.15 (Correspondence). *Let $I \subseteq R$ be an ideal in a ring. There is a bijective correspondence between ideals of R/I and ideals of R containing I, given by $J \mapsto \bar{J} = J/I$ and $\bar{J} \mapsto J = \pi^{-1}(J)$.*

Remark 3.2.16. *A maximal ideal is always a prime ideal. This is true because a field is always a ring.*

Note that zero rings have no maximal or prime ideals because for the quotient to be a field or domain, it has to be nonzero.

Theorem 3.2.17. *If $R \neq 0$, then there is a maximal ideal in R.*

Proof. The proof involves Zorn's Lemma.

Consider the set $A = \{I \subseteq R \text{ ideal} : I \neq R\}$. As $0 \neq R$, then $0 \in A$ and so $A \neq \varnothing$.

We say $I \leq J$ in $\mathcal{A}$ if $I \subseteq J$. This gives a partial order.

Let B be a chain of ideals included in $\mathcal{A}$. This means for all ideals $I, J \in B$, either $I \leq J$ or $J \leq I$.

Now let $K = \bigcup\limits_{I \in B} I$. Note that K is an ideal in R. Take arbitrary $x, y \in K$. By definition, $x \in I$ and $y \in J$ for some $I, J \in B$. Without loss of generality, $I \leq J$, so $x + y \in J \subseteq K$. Similarly, K is closed under scalar multiplication. Therefore, this verifies K is an ideal.

Note $1 \notin K$ so $K \neq R$. By definition, $K \supseteq I$ for all $I \in B$, i.e. $K \geq I$. Therefore, K is an upper bound of B, and is contained in A.

In particular, every chain in A has an upper bound in A. (Since A is not empty.)

By Zorn's Lemma, A has a maximal element M: if $M \subseteq I$, $I \in A$, then $M = I$.

Therefore, M is a maximal ideal in R. □

Corollary 3.2.18. *Every non-zero commutative ring has a prime ideal.*

Definition 3.2.19 (Principal Ideal Ring)**.** *Take $a \in R$, then aR is a principal ideal. We say R is a principal ideal ring if every ideal in R is principal.*

Example 3.2.20.
1. *Fields.*
2. $\mathbb{Z} \supseteq n\mathbb{Z}$.
3. $\mathbb{Z}/n\mathbb{Z}$ *is a principal ideal ring* $\forall n > 0$.

Note that the first two examples are also PID (principal ideal domain).

Definition 3.2.21 (Euclidean Ring)**.** *A Euclidean ring is a commutative ring R together with a function $\varphi : R\backslash\{0\} \to \mathbb{Z}^{\geq 0}$ such that for every $a, b \in R$, $a \neq 0$, there exists $q, r \in R$ such that $b = aq + r$, with either $r = 0$ or $\varphi(r) < \varphi(a)$.*

Theorem 3.2.22. *Every Euclidean ring is a principal ideal ring.*

Proof. Take ideal $I \subseteq R$ with $I \neq 0$. Now $\min\limits_{0 \neq a \in I} \varphi(a) = n \geq 0$.

Take $a \in I$ such that $\varphi(a) = n$. We claim that $I = aR$. Obviously $aR \subseteq I$.

Take $b \in I$. Then there exists q, r such that $b = aq + r$, where $r = 0$ or $\varphi(r) < \varphi(a) = n$.

If $\varphi(r) < \varphi(a)$, then $r = b - aq \in I$ as $b \in I$ and $aq \in I$, then $\varphi(r) < n$, contradiction. Hence, $b = aq$. It follows that $I = aR$, which concludes the proof. □

Example 3.2.23.
1. $R = \mathbb{Z}$ *with* $\varphi(a) = |a|$.
2. *Let F be a field, take $R = F[t]$ with* $\varphi(f) = \deg(f) \geq 0$.

   This setting is required for us to divide the highest coefficient, e.g. consider dividing $t + 1$ by $2t$ in $\mathbb{Q}[t]$, which is just $t + 1 = 2t \cdot \frac{1}{2} + 1$.

   Note that $R = \mathbb{Z}[t]$ is not a Euclidean ring, nor a PID: $2R + tR \subseteq R$ is not principal.

3. *Let* $R = \mathbb{Z}[i] = \{a + bi, a, b \in \mathbb{Z}\}$ *as the Gaussian integers, with* $\varphi(a+bi) = a^2 + b^2 = |a+bi|^2$, *i.e.* $\varphi(z) = |z|^2$.

   Why does this φ works?

   Consider $u, v \in R$ *with* $v \neq 0$. *We can write* $\frac{u}{v} = \alpha + \beta i \in \mathbb{C}$ *where* $\alpha, \beta \in \mathbb{R}$.

   We can find $a, b \in \mathbb{Z}$ *such that* $|\alpha - a| \leq \frac{1}{2}$, $|\beta - b| \leq \frac{1}{2}$. *i.e. give an approximation by integers.*

   Then $\frac{u}{v} = q + s$ *where* $q = a + bi$ *and* $s = (\alpha - a) + (\beta - b)i$, *one can see that* $|s|^2 < 1$.

   Now $u = vq + vs$, *but as* $u, vq \in R$, *we have* $r = vs \in R$. *This is the remainder.*

   In particular, $\varphi(r) = |r|^2 = |v|^2 \cdot |s|^2 < |v|^2 = \varphi(v)$.

   Therefore, the ring of Gaussian integers a Euclidean ring, and also a PID.

3.3 Factorization in Commutative Rings

Definition 3.3.1 (divisibility)**.** *Let* R *be a commutative ring. Let* $a, b \in R$ *with* $a \neq 0$.
We say b *is divisible by* a *if* $\exists c \in R$ *such that* $b = ac$.
Alternatively, we say a *divides* b, *i.e.* $a \mid b$, *which is true if and only if* $aR \supseteq bR$.

Remark 3.3.2. *Note that* $a \neq 0$ *if and only if* $aR \neq 0$, *and* $a \in R^\times$ *if and only if* $aR = R$.

Property 3.3.3.

1. *If* $a \mid b_1$ *and* $a \mid b_2$, *then* $a \mid b_1 + b_2$.
2. *If* $a \mid b$, *then* $a \mid bc$ *for all* c. *In particular,* $a \mid 0$.
3. *If* $a \mid b$ *and* $b \mid c$, *then* $a \mid c$.
4. *We say* $a \sim b$ *are associates if* $a \mid b$ *and* $b \mid a$, *i.e.* $aR = bR$.

   Let R *be an integral domain, then* $a \sim b$ *if and only if there exists* $u \in R^\times$ *such that* $b = au$.

   Indeed, if $a \mid b$ *and* $b \mid a$, *then* $b = ax = bxy$ *for some* y *such that* $a = by$. *In particular,* $1 = xy$ *for* $x \in R^\times$.

   Note that if $a \sim a'$ *and* $b \sim b'$, *then* $a \mid b$ *if and only if* $a' \mid b'$. *In particular,* $aR = a'R$ *and* $bR = b'R$.

Definition 3.3.4 (Prime)**.** *Let* R *be a domain, we say* $p \in R$ *is prime if*

1. $p \neq 0$,
2. $p \notin R^\times$,
3. *if* $p \mid ab$ *in* R, *then* $p \mid a$ *or* $p \mid b$.

Remark 3.3.5. *Note that* $p \in R$ *is prime if and only if* pR *is a prime ideal. (i.e.* $pR \neq 0, R$, *and* $ab \in pR$ *indicates* $a \in pR$ *or* $b \in pR$.*)*

Definition 3.3.6 (Irreducible)**.** *We say $c \in R$ is irreducible if*

1. $c \neq 0$,
2. $c \notin R^\times$,
3. *if* $c = ab$, *then either* $a \in R^\times$ *or* $b \in R^\times$.

Claim 3.3.7. *$c \in R$ is irreducible if and only if cR is maximal in the set of principal ideals $aR \neq R$.*

Proof. Suppose c is irreducible, then $cR \neq R$. Suppose $cR \subseteq aR$, then $c = ab$ for some b, then either $a \in R^\times$ or $b \in R^\times$.

If $a \in R^\times$, then $aR = R$. If $b \in R^\times$, then $cR = aR$.

Suppose cR is maximal in the set of principal ideals $aR \neq R$. Then $c = ab$ for $a \notin R^\times$. In particular, $cR \subseteq aR \neq R$, but cR is maximal, so $cR = aR$. In particular, $c = ab$ for some $b \in R^\times$. ☐

Example 3.3.8. $R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$.

Claim 3.3.9. *2 is irreducible but not prime in R.*

Proof. Note that $2 \mid 6 = (1+\sqrt{-5})(1-\sqrt{-5})$, but $2 \nmid 1 \pm \sqrt{-5}$: $\frac{1}{2} \pm \frac{1}{2}\sqrt{-5} \notin R$. Therefore, 2 is not prime.

Take $2 = xy$ for $x, y \in R$. Then $|x|^2, |y|^2 \in \mathbb{Z}$. Note $4 = |2|^2 = |x|^2|y|^2$. Without loss of generality, say $|x|^2 \leq 2$, then as $x = a + b\sqrt{-5}$ for $a, b \in \mathbb{Z}$, $|x|^2 = a^2 + 5b^2 \leq 2$.

Therefore, $b = 0$ and $|a| \leq 1$, which means $a = \pm 1$. In particular, $x = \pm 1 \in R^\times$. Therefore, 2 is irreducible by definition. ☐

Proposition 3.3.10. *Every prime element is irreducible.*

Proof. Let p be a prime. Suppose $p = ab$. Since $p \mid ab$, then $p \mid a$ or $p \mid b$. Suppose $a = pq$. Then $p = pqb$, which means $1 = qb$. Hence, $b \in R^\times$, which means p is irreducible. ☐

Proposition 3.3.11. *If R is a PID, then primes and irreducibles are the same.*

Proof. We only have to show that every irreducible element is prime.

Let $c \in R$ be irreducible. Then cR is maximal among principal ideals that are distinct from R. But every ideal in R is principal. Therefore, cR is a maximal ideal, which is a prime ideal, and so c is prime. ☐

Definition 3.3.12 (Unique Factorization)**.** *Let R be a domain. We say the factorization in R is unique if $c_1c_2\cdots c_n = d_1d_2\cdots d_m$ where c_i and d_j are irreducible, $n = m$, and there exists $\sigma \in S_n$ such that $d_i \sim c_{\sigma(i)}$ for all $i = 1, \cdots, n$.*

Definition 3.3.13 (Admit Factorization)**.** *We say R admits factorization if every $0 \neq x \in R$ with $x \notin R^\times$ can be written as $x = c_1c_2\cdots c_n$ for c_i irreducible.*

Definition 3.3.14 (Unique Factorization Domain)**.** *R is a unique factorization domain if R admits a unique factorization. We say R is a UFD.*

Theorem 3.3.15. *In a UFD, the primes and irreducibles are the same.*

Proof. Again, it suffices to show that every irreducible is a prime element. Take $c \in R$ to be irreducible. Consider $c \mid ab$. We can write $ab = cx$ for some $x \in R$.

Let $a = c_1 \cdots c_n$ and $b = d_1 \cdots d_m$ and $x = e_1 \cdots e_k$. Then $c_1 \cdots c_n d_1 \cdots d_m = ce_1 \cdots e_k$. Note that $c \sim c_i$ or $c \sim d_j$ for some i, j.

If $c \sim c_i \mid a$, then $c \mid a$. Similarly, if $c \sim d_j \mid b$, then $c \mid b$. Therefore, c is a prime. □

Theorem 3.3.16. *Let R admit factorization and suppose the primes and the irreducibles are the same. Then R is a UFD.*

Proof. Consider $c_1 \cdots c_n = d_1 \cdots d_m$ where c_i, d_j are irreducibles. Then $c_n \mid d_1 \cdots d_m$ where c_n is prime. In particular, $c_n \mid d_j$ for some j. We write $c_n x = d_j$ irreducible. But as c_n is irreducible, it is not a unit, then $x \in R^\times$, which means $d_j \sim c_n$. Without loss of generality, say $j = m$. Then $c_1 \cdots c_{n-1} = (xd_1)d_2 \cdots d_{m-1}$.

By performing induction on n, we conclude the proof. □

Proposition 3.3.17. *Let R be a commutative ring. The following are equivalent:*

1. *Every ideal of R is finitely generated.*
2. *For every chain of ideals $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$, there exists some $n > 0$ such that $I_n = I_{n+1} = \cdots$.*
3. *Every nonempty set of ideals contains a maximal ideal.*

Proof. We first show that 1) implies 2).

Take a chain of ideals $I_1 \subseteq I_2 \cdots \subseteq I_n \subseteq \cdots$. Take $\mathfrak{J} = \bigcup_{k \geq 1} I_k$, so $\mathfrak{J} = \sum_{i=1}^{n} a_i R$, where $a_i \in \mathfrak{J}$.

In particular, there exists $n > 0$ such that $a_1, \cdots, a_m \in I_n \subseteq \mathfrak{J}$. In particular, this indicates that $I_n = \mathfrak{J}$. However, $I_n \subseteq I_{n+i} \subseteq \mathfrak{J}$. Therefore, $I_{n+i} = \mathfrak{J}$ for all $i \geq 0$.

We now show that 2) implies 3).

Take a non-empty set of ideals. Take an ideal I_1 in the set. If it is maximal, we are done. If it is not maximal, it is contained in some ideal $I_2 \supsetneq I_1$. We perform this algorithm repeatedly. By property in 2), this algorithm has to stop at some point and we obtain a maximal element.

Finally, we show that 3) implies 1).

Let $I \subseteq R$ be an ideal. Consider the set $\{\mathfrak{J} \subseteq R \text{ ideals } : \mathfrak{J} \subseteq I, \mathfrak{J} \text{ is finitely generated}\}$. This set is not empty because it contains 0. In particular, it contains a maximal $\mathfrak{J}$. We claim that $I = \mathfrak{J}$. Suppose not, then $\mathfrak{J} \subsetneq I$, so there exists $a \in I \backslash \mathfrak{J}$. Then $\mathfrak{J} \subsetneq \mathfrak{J} + aR \subseteq I$, where $\mathfrak{J} + aR$ is still finitely generated because $\mathfrak{J}$ is finitely generated. But then $\mathfrak{J} + aR$ is in the set. This contradicts the fact that $\mathfrak{J}$ is maximal, contradiction. □

Definition 3.3.18. *If all the above properties hold, we say R is a Noetherian ring.*

Corollary 3.3.19. *Every PID is Noetherian.*

Theorem 3.3.20. *Noetherian domain admits factorization.*

Proof. Let $S = \{aR : a$ cannot be factored into product of irreducible elements$\}$. We want to show that $S = \varnothing$. Suppose not, then there is a maximal ideal $aR \in S$.

If a is irreducible, then it factors itself, so a is not irreducible, then $a = xy$ for some $x, y \notin R^\times$. In particular, $x \mid a$ and $y \mid a$. Therefore, $aR \subsetneq xR \notin S$ and $aR \subsetneq yR \notin S$. Therefore, x, y are products of irreducibles. Then so is a, contradiction. $\square$

Proposition 3.3.21. *Let R be a domain, then*

1. *If primes and irreducibles are the same in R, then R has unique factorization.*
2. *If R is Noetherian and primes and irreducibles are the same in R, then R is a UFD.*

Corollary 3.3.22. *Every PID is a UFD.*

Proof. It suffices to show that if R is a PID, then irreducibles in R are prime. Let $p \in R$ be irreducible and suppose that $p \mid ab$ but $p \nmid a$. Pick $d \in R$ so that $pR + aR = dR$. Then $d \mid p$ and $d \mid a$, but $p \nmid a$, so since p is irreducible, d is a unit, without loss of generality we can say $d = 1$. There exists $r, s \in R$ so that $pr + as = 1$. Then $prb + abs = b$, and the left hand side is divisible by p, so $p \mid b$ as desired. $\square$

Remark 3.3.23. *If $I, J \subseteq R$ are ideals, then $IJ = \{\sum_{i=1}^{n} x_i y_i : x_i \in I, y_i \in J\}$ is an ideal in R.*

In particular, if we multiply two principal ideals, we have $(aR)(bR) = abR$, which is still a principal ideal.

Similarly, if $a = c_1 \cdots c_n$, then $aR = (c_1R)\cdots(c_nR)$. This gives the existence of factorization of ideals. Also, if $(c_1R)\cdots(c_nR) = (d_1R)\cdots(d_mR)$ where c_i, d_j are irreducible, then the factorization is unique: $n = m$ and there exists $\sigma \in S_n$ such that $d_i \sim c_{\sigma(n)}$ for all $i = 1, \cdots, n$. Therefore, $d_iR = c_{\sigma(i)R}$.

Remark 3.3.24 (Greatest Common Divisor, Least Common Multiple)**.** *Let R be a UFD, and let $a_1, \cdots, a_n \in R$ be nonzero. Then there exists $c_1, \cdots, c_n$ distinct and irreducible, such that $a_i = u_i \prod_{j=1}^{m} c_j^{k_{ij}}$ where $k_{ij} \in \mathbb{Z}^{\geq 0}$ and $u_i \in R^\times$ (i.e. up to units).*

Correspondingly, $a_iR = \prod_{j=1}^{m} (c_jR)^{k_{ij}}$. This decomposition is unique up to permutation of terms.

One can define greatest common divisors as ideals: $\gcd(a_iR) = \prod_{j=1}^{m} (c_jR)^{s_j}$ where $s_j = \min_i(k_{ij})$. Similarly, we can define the least common multiples as ideals $\mathbf{lcm}(a_iR) = \prod_{j=1}^{m} (c_jR)^{s_j}$ where $s_j = \max_i(k_{ij})$.

We say ideals $a_1R, \cdots, a_nR$ *are relatively prime (or correspondingly,* $a_1, \cdots, a_n$ *are relatively prime) if* $\gcd(a_iR) = R$.

Note that greatest common divisors are up to units.

Proposition 3.3.25. *In a UFD, the greatest common divisor of a finite set of elements exists.*

Proof. Let $a_1, \cdots, a_n$ be elements in a UFD R, and let $p_1, \cdots, p_r$ be all of the primes appearing in the factorizations of $a_1, \cdots, a_n$ (up to units), so that for each i, $a_i = p_1^{e_{i,1}} \cdots p_r^{e_{i,r}}$ for $e_{ij} \geq 0$.

The greatest common divisor is then $\gcd(a_1, \cdots, a_n) = p_1^{\min(e_{1,1}, \cdots, e_{n,1})} \cdots p_r^{\min(e_{1,r}, \cdots, e_{n,r})}$. □

3.4 Factorization in Polynomial Rings

Let R be a commutative ring, then $R[x]$ is a polynomial ring. (Inductively, one can construct $R[x_1, \cdots, x_n]$.)

We aim to prove the following theorem in this section:

Theorem 3.4.1. *If R is a UFD, then so is $R[x]$.*

Note that if $R \to S$ is a ring homomorphism, then there is an induced homomorphism $R[x] \to S[x]$. Therefore, this is a functor from the category of rings to itself.

If R is a domain, then $\deg(fg) = \deg(f) + \deg(g)$, and $\deg(0) = -\infty$ by convention. Therefore, $\deg(f) \leq 0$ if and only if $f \in R$. Note that $R \subseteq R[x]$ is a subring.

Consider the invertible elements in this ring. Let $f \in R[x]^\times$, then if $fg = 1$, we have $\deg(f) + \deg(g) = 0$. Therefore, since the degrees are non-negative, we have $\deg(f) = 0$ and $\deg(g) = 0$, so $f, g \in R^\times$. Hence, $R[x]^\times = R^\times$.

We say that a polynomial $f \in R[x]$ is irreducible if f is an irreducible element of $R[x]$.

Definition 3.4.2 (Quotient Field)**.** *Let R be a domain, we define a field F containing R as the set of all pairs (a, b) where $a, b \in R$, $b \neq 0$. This is called the quotient field of R.*

We introduce the equivalence relation where $(a, b) \sim (a', b')$ if $ab' = a'b$.

We define $\frac{a}{b}$ is defined as the equivalence class of (a, b). Then F is the set of equivalence classes $\{\frac{a}{b} : a, b \in R, b \neq 0\}$. The operations defined on the set are

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} = \frac{a_1b_2 + a_2b_1}{b_1b_2}$$

$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = \frac{a_1a_2}{b_1b_2}$$

Note that F is a field because for $a, b \neq 0$, $(\frac{a}{b})^{-1} = \frac{b}{a}$.

In particular, there is an embedding $R \hookrightarrow F$ given by $a \mapsto \frac{a}{1}$. This homomorphism is unique.

Remark 3.4.3. *Define $F(x) = \{\frac{f}{g} : f, g \in R[x], g \neq 0\}$ to be a ring. This is called the quotient field of $R[x]$ (and of $F[x]$), also called the field of rational functions. Note that $F(x)$ contains both $R[x]$ and $F[x]$.*

Also note that $F[x]$ is not a field, but it is a PID (and a UFD).

Example 3.4.4. *$\mathbb{Z}[x]$ is not a PID: $\langle 2, x\rangle$ is not principal.*

Similarly, $F[x, y]$ is not a PID because $\langle x\rangle$ is not principal.

Remark 3.4.5. *Note that irreducible element are with respect to fields.*

1. *Consider $\mathbb{Z}[x] \subseteq \mathbb{Q}[x]$, where $2x$ is an element of both rings. However, $2x$ is not irreducible in $\mathbb{Z}[x]$, but it is irreducible in $\mathbb{Q}[x]$ (because 2 is a unit in $\mathbb{Q}[x]$.*

2. *Consider $\mathbb{R}[x] \subseteq \mathbb{C}[x]$, where $x^2 + 1$ is is an irreducible element of $\mathbb{R}[x]$, but not an irreducible element in $\mathbb{C}[x]$.*

Definition 3.4.6 (Content, Primitive)**.** *Let R be a UFD. Take $f = a_n x^n + \cdots + a_1 x + a_0$ for $a_i \in R$. Suppose $f \neq 0$.*

We say that $\gcd(a_0 R, \cdots, a_n R) = bR$ is the content of f, denoted $C(f)$.

We say that f is primitive if $C(f) = R$.

Remark 3.4.7. *If f is monic, then f is primitive.*

Also,

1. *$C(af) = aC(f)$ where $0 \neq a \in R$ and $0 \neq f \in R[x]$.*

2. *$C(f) = R$ for monic f.*

Lemma 3.4.8 (Gauss)**.** *If R is a UFD, and $f, g \in R[x]$ are primitive, then fg is primitive.*

Proof. Take $c \in R$ prime, then $cR \subsetneq R$ is prime.

Let $\bar{R} = R/cR$ be a domain. Then there is a surjection

$$\begin{aligned} R[x] &\to \bar{R}[x] \\ R &\mapsto \bar{R} \\ f &\mapsto \bar{f} \end{aligned}$$

Since f, g are primitive, then $\bar{f}, \bar{g} \neq 0$ in $\bar{R}[x]$ domain. Then $\bar{f}\bar{g} \neq 0$, which means $\bar{f}g \neq 0$. Therefore, not all coefficients of fg are divisible by c. In particular, fg is primitive by definition. □

Corollary 3.4.9. *$C(fg) = C(f) \cdot C(g)$.*

Proof. Let $f = a \cdot f'$ for f' primitive, then $C(f) = aR$.

Similarly, let $g = b \cdot g'$ for g' primitive, then $C(g) = bR$.

Then $fg = abf'g'$, where $f'g'$ is primitive by Gauss Lemma.

In particular, $C(fg) = abC(f'g') = abR = (aR) \cdot (bR) = C(f) \cdot C(g)$. □

Lemma 3.4.10. *Let f and g be non-zero polynomials in $R[x]$, and f is primitive. If $f \mid g$ in $F[x]$, then $f \mid g$ in $R[x]$.*

Remark 3.4.11. *Note that the primitive condition is necessary: note $2x \mid x^2 \in \mathbb{Q}[x]$, but $2x \nmid x^2 \in \mathbb{Z}[x]$.*

Proof. Let $g = fh$ where $h \in F[x]$, then there exists $0 \neq a \in R$ such that $a \cdot h \in R[x]$. Therefore, $ag = f \cdot (ah) \in R[x]$.

In particular, $aC(g) = C(ag) = C(f) \cdot C(ah) = C(ah)$. Note that all coefficients of ah are divisible by a.

Therefore, $h \in R[x]$. By definition, $f \mid g$ in $R[x]$. □

Lemma 3.4.12. *Let F be a UFD and let $f \in R[x]$ be irreducible, then f is primitive.*

Proof. Let dR to be the content of f for some $d \in R$. Then $f = d \cdot f'$ for some $f' \in R[x]$. Since f is irreducible, either d or f' has to be a unit. Obviously d has to be the unit. In particular, $C(f) = dR = R$. □

Lemma 3.4.13. *Let R be a UFD and let $f \in R[x]$ be a nonconstant polynomial. Then f is irreducible in $R[x]$ if and only if f is primitive and irreducible in $F[x]$.*

Proof. ($\Longrightarrow$): Since f is irreducible over UFD, then it is primitive. Suppose, towards contradiction that f is not irreducible in $F[x]$, then $f = gh$ for some non-constant polynomials $g, h \in F[x]$, i.e. $\deg(g), \deg(h) < \deg(f)$.

Note that g, h may have denominators in their coefficients. We multiply a certain constant a, then $ag \in R[x]$. We then divide the greatest common divisor b of the coefficients of ag, then we get a primitive polynomial $\frac{a}{b}g$. In particular, $g = \alpha \cdot g'$ and similarly $h = \beta \cdot h'$ for $\alpha, \beta \in F^\times$ and $g', h' \in R[x]$ are primitive.

Hence, $f = \alpha\beta g'h'$. So $g'h' \mid f$ in $F[x]$. Note that $g'h'$ is primitive by Gauss' Lemma, then by lemma, $gh \mid f$ in $R[x]$. In particular, $\alpha\beta \in R$.

We now write $f = (\alpha\beta g') \cdot h'$ in $R[x]$, which is a non-trivial factorization. This is a contradiction to the fact that f is irreducible in $R[x]$.

($\Longleftarrow$): We write $f = gh$ in $R[x]$. We need to show that g or h is an irreducible constant in R. Note that this is also a factorization in $F[x]$. Since f is irreducible in $F[x]$, then either g or h is a scalar in F. Since $F \cap R[x] = R$, we see that $g \in R$ or $h \in R$. Without loss of generality, say $g \in R$. Now $R = C(f) = g \cdot C(h)$, and so $g \in R^\times$. □

Theorem 3.4.14. *If R is a UFD, then so is $R[x]$.*

Proof. We prove by induction on the degree of polynomials that we can factor polynomial $f \in R[x]$.

When $\deg(f) = 0$, then $f \in R$ is a nonzero scalar. In particular, f factors as a product of irreducibles because R is a UFD. Note that irreducibles in R are still irreducible in $R[x]$.

Now assume that the case for $\deg(f) = n \geq 0$ is true. We want to prove the case for $\deg(f) = n+1 > 0$. Then $f = a \cdot f'$ for some $a \in R$ such that f' is primitive. Recall that $aR = C(f)$, then it is possible to assume f is primitive.

Assume $f = gh$ in $R[x]$ is a non-trivial factorization, i.e. g, h are not irreducible constants. Note that then g, h should not be constants, i.e. $g, h \notin R$: for example if $g \in R$, then $R = C(f) = g \cdot C(h)$, but that means $g \in R^\times$, contradiction.

Therefore, $\deg(g), \deg(h) < \deg(f)$. By induction, we can factor both g and h. Therefore, we can factor f.

This proves the existence of factorization. We now show its uniqueness. It suffices to show that every irreducible in $R[x]$ is a prime.

Take an irreducible polynomial f in $R[x]$. Suppose $f \mid gh$ where $g, h \in R[x] \subseteq F[x]$, where F is the quotient field of R. Therefore, $f \mid gh$ in $F[x]$ (which is a UFD and a PID). Now, since f is irreducible in $R[x]$, then that means f is irreducible in $F[x]$. Then f is prime in $F[x]$. Therefore, $f \mid g$ or $f \mid h$ in $F[x]$. Without loss of generality say $f \mid g$. Recall that f is primitive, then $f \mid g$ in $R[x]$ by the lemma. $\square$

Remark 3.4.15 (Factorization and Irreducible Elements in Polynomial Ring)**.** *Take $f \in R[x]$. If f is a constant, then $f \in R$ which is a UFD, so assume f is not a constant. Then we can factor f in $F[x]$. We write it as a product of irreducibles in $F[x]$: $f = g_1 g_2 \cdots g_k$. There exists $\alpha_i \in F^\times$ such that $g_i = \alpha_i \cdot h_i$, where $h_i \in R[x]$ is primitive. Observe that h_i is still irreducible, then by lemma, h_i is irreducible in $R[x]$. Now $f = (\alpha_1 \cdots \alpha_k) h_1 h_2 \cdots h_k$ is a factorization in $F[x]$, but since h_i are primitive, so $h_1 h_2 \cdots h_k$ is primitive, then $h_1 h_2 \cdots h_k \mid f$ in $R[x]$, and thus $\alpha_1 \cdots \alpha_k \in R$. Therefore, $f = (\alpha_1 \cdots \alpha_k) h_1 h_2 \cdots h_k$ is a factorization in $R[x]$.*

The irreducibles in $R[x]$ are:

1. *Irreducibles in R, i.e. constants.*
2. *Nonconstant primitive $h \in R[x]$ that are irreducible in $F[x]$.*

Theorem 3.4.16 (Eisenstein Criterion)**.** *Let R be a UFD with quotient field F. Let $f = a_n x^n + \cdots + a_1 x + a_0 \in R[x]$. Let $p \in R$ be an irreducible element such that*

1. *$p \nmid a_n$,*
2. *$p \mid a_i$ for $i = 1, 2, \cdots, n-1$,*
3. *$p^2 \nmid a_0$.*

Then f is irreducible in $F[x]$.

Proof. We first reduce the case to primitive polynomials. In general, we write $f = af'$ where $f' \in R[x]$ is primitive and $aR = C(f)$. Let $f = a_n x^n + \cdots + a_1 x + a_0$ for $a_i \in R$. We write $f' = b_n x^n + \cdots + b_1 x + b_0$ where $a_i = a \cdot b_i$. We claim that f' satisfies the same condition as f.

Note that

1. Since $p \nmid a_n$, then $p \nmid b_n$. Also, $p \nmid a$.
2. Since $p \mid a_i$ and $p \nmid a$, we have $p \mid b_i$.
3. Since $p^2 \nmid a_0$, then $p^2 \nmid b_0$.

Therefore, it suffices to prove the case for f': $f \sim f'$ in $F[x]$. Hence, assume that f is primitive from the start is reasonable.

Take $\bar{R} = R/pR$, then $\bar{R}$ is a domain because pR is prime. We have a homomorphism $R[x] \to \bar{R}[x]$ by sending $g \mapsto \bar{g}$. Then note that $\bar{f} = \bar{a_n} x^n$ where $\bar{a_n} \neq \bar{0}$. We need the primitive polynomial f to be irreducible in $F[x]$, which holds if and only if f is irreducible in $R[x]$.

Let $f = gh$ in $R[x]$. If we can show that $g \in R$, then $R = C(f) = gC(h)$ and so $g \in R^\times$.

Assume $\deg(g), \deg(h) < n$. Now $\bar{f} = \bar{g}\bar{h}$ in domain $R[x] \subseteq K[x]$, where $\bar{f} = \bar{a_n} x^n$ and K is the quotient field of $\bar{R}$. Therefore, we can write $\bar{g} = \alpha x^k$, $\bar{h} = \beta x^m$ for $\alpha, \beta \in K$, and $k, m > 0$.

In particular, as $\bar{g} \in \bar{R}[x]$, we know $\alpha \in \bar{R}$. Similarly, $\beta \in \bar{R}$. Note that $\bar{g}$ and $\bar{h}$ both have zero constant terms. Therefore, constant term of g and h are divisible by p. In particular, the constant term a_0 of $f = gh$ is divisible by p^2. However, $p^2 \nmid a_0$, contradiction. $\square$

Example 3.4.17. *Let $p \in \mathbb{Z}$ be prime. Consider the polynomial $f = x^{p-1} + x^{p-2} + \cdots + x + 1 \in \mathbb{Z}[x]$. We claim that f is irreducible in $\mathbb{Q}[x]$.*

Take $y = x - 1$, i.e. $x = y + 1$, then $f = \frac{x^p - 1}{x - 1} = \frac{(y+1)^p - 1}{y} = y^{p-1} + \binom{p}{1} y^{p-2} + \cdots + \binom{p}{p-2} y + \binom{p}{p-1}$. Note that the Eisenstein Criterion holds. Therefore, the claim is true indeed.

Remark 3.4.18 (Classification of Domains)**.** *The class of Euclidean Domains is contained in the class of Principal Ideal Domains (e.g. $\mathbb{Z}$), which is contained in the class of Unique Factorization Domains (e.g. $\mathbb{Z}[x], \mathbb{Z}[x_1, \cdots, x_n, \cdots]$). There is also a class of Noetherian domains, which also contains the class of Principal Ideal Domains. Note that $\mathbb{Z}[x]$ is both a UFD and a Noethereian Domain, $\mathbb{Z}[\sqrt{-5}]$ is a Noetherian domain but not a UFD, and $\mathbb{Z}[x_1, \cdots, x_n, \cdots]$ is UFD but not Noetherian.*

4 Module Theory

4.1 Definition

Definition 4.1.1 (Module)**.** *Let R be a ring (associative, with unit, but not necessarily commutative). A left R-module is an Abelian group M (written additively) together with an operation $R \times M \to M$ by sending $(a, m) \mapsto a \cdot m$ (scalar multiplication) such that*

1. $a(m_1 + m_2) = am_1 + am_2$,
2. $(a + b)m = am + bm$,
3. $(ab)m = a(bm)$,
4. $1 \cdot m = m$.

Similarly, one can define a right R-module as an Abelian group M (written additively) together with an operation $M \times R \to M$ by sending $(m, a) \mapsto m \cdot a$ (scalar multiplication) such that

1. $(m_1 + m_2)a = m_1a + m_2a$,
2. $m(a + b) = ma + mb$,
3. $m(ab) = (ma)n$,
4. $m \cdot 1 = m$.

Remark 4.1.2. *If R is commutative, then every left R-module can be viewed as a right R-module via $ma = am$.*

Without loss of generality, we work on the left R-modules from this point on.

Property 4.1.3.
1. $a \cdot 0 = 0$ *in* M.
2. $0 \cdot m = 0$.
3. $(-a)m = -(am) = a \cdot (-m)$.

Example 4.1.4.
1. *Let R be a field F, then R-modules are equivalent to vector spaces over F. Therefore, the notion of a module over ring is the generalization of the notion of a vector space over field.*

2. *Let $R = \mathbb{Z}$. We have the operation $\cdot$ given by $1 \cdot m = m$, $2 \cdot m = (1+1) \cdot m = m+m$, and so on. Therefore, the operation is uniquely determined. In this case, the R-modules are equivalent to Abelian groups.*

3. *Left (Right) ideals in R are left (right) R-modules. R is both a left and right R-module.*

4. *Let $f : R \to S$ be a ring homomorphism. Let M be a (left) S-module, then M has a structure of a (left) R-module via $a \cdot m = f(a) \cdot m)$ for $a \in R$ and $m \in M$. This is a pullback action with respect to f.*

5. *Let A be an Abelian group (written additively). Then $\mathbf{End}(A)$ is the ring of endomorphisms of A is given by the set of homomorphisms $\{f : A \to A\}$. Then A is a left $\mathbf{End}(A)$-module, with the operation defined by $f \cdot m = f(m)$ for $f \in \mathbf{End}(A)$ and $m \in A$.*

   There is more analogies with group theory. Let M be a left R-module. For $a \in R$, one can define left multiplication $l_a : M \to M$ by $l_a(m) = am$. We can then rewrite the module axioms:

   a) $l_a(m_1 + m_2) = l_a(m_1) + l_a(m_2)$, *which implies* $l_a \in \mathbf{End}(M)$.

   b) $l_{a+b}(m) = l_a(m) + l_b(m)$, *so the map* $\varphi : R \to \mathbf{End}(M)$ *where* $a \mapsto l_a$ *given by the previous axiom is additive.*

   c) $l_{ab}(m) = l_a(l_b(m)) = (l_a \circ l_b)(m)$. *This says that* φ *is also multiplicative.*

   d) $l_1(m) = m$, *i.e.* $l_1 = \mathbf{id}$. *This implies* φ *sends* 1 *to* 1.

   The properties above, shows that $\varphi : R \to \mathbf{End}(M)$ is a ring homomorphism. Therefore, every left module give raises to a ring homomorphism.

   We can reverse the construction as well. Suppose we have an Abelian group A (written additively), and $\varphi : R \to \mathbf{End}(M)$ is a ring homomorphism. Then we make A a left R-module by writing $a \cdot m = \varphi(a)(m)$.

   This induces a bijective correspondence between $\mathbf{Hom}_{\mathbf{Ring}}(R, \mathbf{End}(A))$ and left R-module structure on A.

   Note that for a ring homomorphism from R to $\mathbf{End}(A)$, a left R-module structure on A is given by the pullback of the canonical left $\mathbf{End}(A)$-module structure on A.

Definition 4.1.5 (Homomorphism)**.** *Let R be a ring and M, N be (left) R-modules. A map $g : M \to N$ is an R-module homomorphism if*

1. *g is a homomorphism of Abelian groups, and*

2. *$g(am) = a \cdot g(m)$ for all $a \in R$ and $m \in M$.*

The set of such morphisms is denoted as $\mathbf{Hom}_R(M, N)$, and is an Abelian group.

If we want to introduce categories, we consider R-**Mod** as a category of R-modules, with objects as left R-modules and morphisms as R-module homomorphisms.

Similarly, we can define a category of right R-modules, denoted **Mod**-R.

We will see that R-**Mod** (and similarly, **Mod**-R) is Abelian.

Property 4.1.6. *1. If R is commutative, then $\mathbf{R}-\mathbf{Mod} \cong \mathbf{Mod}-\mathbf{R}$ because left and right modules then coincide.*

2. If $f : R \to S$ is a ring homomorphism, then the pullback operation allows us to consider every S-module as R-module. We have a functor $f^(: \mathbf{S}-\mathbf{Mod} \to \mathbf{R}-\mathbf{Mod}$ given by $N \mapsto f^*N$, where the operation on f^*N is defined by $r \cdot_R n = f(r) \cdot_S n$.

Definition 4.1.7 (Submodule)**.** *If M is a left R-module, then a subgroup $N \subseteq M$ is called a submodule if $aN \subseteq N$ for all $a \in R$. Submodules are modules.*

Remark 4.1.8. *Let $\{N_i\}_{i\in I}$ be a family of submodules of M, then $\bigcap\limits_{i\in I} N_i \subseteq N$ is a submodule. However, the union of modules is generally not a module. Instead, we consider the sum of submodules, which is the smallest module containing the family: $\sum\limits_{i\in I} N_i = \{\sum\limits_{i\in I} n_i,\ \text{almost all } n_i = 0\} \subseteq M$.*

We can then define a factor module. If $N \subseteq M$ is a submodule, then $M/N = \{m+N, m \in M\}$ is a factor module defined by $a \cdot (m+N) = am + N$.

Let $g : M \to N$ be a R-module homomorphism. Then $\ker(g) \subseteq M$ and $\mathbf{im}(g) \subseteq N$ are submodules as well.

The three isomorphism theorems are also true in this setting, for example:

Theorem 4.1.9 (First Isomorphism Theorem)**.** *Let $g : M \to N$ be an R-module homomorphism. Then $> / \ker(g) \to \mathbf{im}(g)$ defined by $m + \ker(g) \mapsto g(m)$*

Remark 4.1.10. *The direct sums and products of this category is essentially the same as those in* **Ab***, because the forgetful functor (forgets the scalar product structure) $i : \mathbf{R}-\mathbf{Mod} \to \mathbf{Ab}$ has a left adjoint $A \mapsto R \otimes_{\mathbb{Z}} A$.*

Then let $(M_i)_{i\in I}$ be R-modules, we have $\prod\limits_{i\in I} M_i = \{(m_i)_{i\in I}, m_i \in M_i\}$ and $\coprod\limits_{i\in I} M_i = \{(m_i)_{i\in I}, m_i \in M_i,\ \text{almost all } m_i = 0\}$.

*Therefore, R-**Mod** (and similarly, **Mod**-R) should be Abelian.*

We can construct exact sequences and split exact sequences in this category.

Definition 4.1.11 (Finitely Generated)**.** *We say a (left) R-module is finitely generated if $\exists m_1, m_2, \cdots, m_n \in M$ such that every $M \in M$ can be written as a linear combination $m = \sum\limits_{1\le i\le n} a_i m_i$ for $a_i \in R$.*

4.2 Free Module

We first define the notion of a basis for modules.

Definition 4.2.1 (Basis, Free). *Let M be a (left) R-module. A subset $S \subseteq M$ is called a basis for M if every $m \in M$ can be written as $m = \sum_{s\in S} a_s \cdot s$ for unique $a_s \in R$ where almost all coefficients are zero.*

We say that M is free if M has a basis.

The "almost" condition is here to justify the summation operation.

Example 4.2.2. *For $R = \mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$ is not free, because $\tilde{1} = 3 \cdot \tilde{1}$.*

Every vector space is free (even if it is infinite-dimensional).

In cases of vector spaces, the cardinality of a basis is well-defined. However, for R-modules, different bases may have different cardinalities.

Remark 4.2.3 (Structure on a Free Module). *Let I be a set, then the coproduct $\coprod_{i\in I} R = R^{(I)} = \{(a_i)_{i\in I},$ almost all a_i are $0\}$. This module is free. For $i \in I$, let $e_i = (a_j)_{j\in I}$ where $a_j = 1$ if $j = i$ and $a)j = 0$ if $j \neq i$. Now, $\{e_i\}_{i\in I}$ forms a basis for $R^{(I)}$. Therefore, $R^{(I)}$ is free.*

In fact, if I has finitely many elements, we write $R^{(I)}$ as R^n, where n is the cardinality of I.

Suppose M is a free R-module, and we choose a basis $(m_i)_{i\in I}$ for M. We then have a well-defined homomorphism $R^{(I)} \to M$ by sending $(a_i)_{i\in I} \mapsto \sum_{i\in I} a_i m_i$. This is an isomorphism of modules because in definition, every element in M can be written uniquely as this sum.

As a conclusion, every free (left) R-module is isomorphic to $R^{(I)}$ for some set I.

Given by this setting, we can construct homomorphisms from free modules to other modules. If we have M be a (left) R-module, we can construct a set map $f : I \to M$, and then there is a R-module homomorphism given by $\bar{f} : R^{(I)} \to M$ such that $\bar{f}(ax) = \sum_{x\in I} a_x \cdot f(x)$.

Conversely, if we think of $I \subseteq R^{(I)}$, then an R-module homomorphism $g : R^{(I)} \to M$ can be restricted to a set map $f = g\mid_I : I \to M$.

*This induces an isomorphism $\mathbf{Mor}_{\mathbf{Set}}(I, M) \cong \mathbf{Hom}_R(R^{(I)}, M)$. This is essentially an adjunction between R-**Mod** and **Set**. The left adjoint is the forgetful functor that forgets the module structure, and the right adjoint takes a set X to the free module $R^{(X)}$. This is a typically forgetful-free adjunction.*

In particular, the hom functor from free module is exact, so if F is a free left R-module, then there is an isomorphism $F \cong R^{(X)}$ for some set X.

This gives an exact functor $\mathbf{R-Mod} \to \mathbf{Ab}$ that takes a module M to $\mathbf{Hom}_R(F, M)$. In general, the functor is left exact; the exactness comes from the free module. The right exactness comes from

$$\begin{tikzcd} & & & F = R^{(X)} \arrow[dl, dashed] \arrow[d] & \\ 0 \arrow[r] & P \arrow[r] & M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

The morphism from F to M is generated by the adjunction: it is the same as having

$$\begin{tikzcd} & X \arrow[dl, dashed, "h"'] \arrow[d, "f"] \\ M \arrow[r, two heads] & N \end{tikzcd}$$

where the map h is induced by the surjection: for $x \in X$, we have $f(x) \in N$, and there is $h(x) \in M$ that is a preimage of the map from M to N.

This shows that the functor is exact.

Another nice feature of free module is that every module is a factor module of a free module.

Let M be a (left) R-module, and pick a set of generators $X \subseteq M$. There is an embedding $X \hookrightarrow M$ gives an R-module homomorphism $g : R^{(X)} \twoheadrightarrow M$ by adjunction, which is a surjection because X is a generating set. Therefore, $M \cong R^{(X)}/\ker(g)$.

If M is finitely generated, then X can be chosen finite. Therefore, $M \cong R^n/(\cdots)$.

Finally, we can think about how to view morphisms between free modules. In general, if we have a collection of modules $(M_i)_{i\in I}$ and $(N_j)_{j\in J}$, then we can form a direct sum of M_i's and a direct product of N_j's, and we have

$$\mathbf{Hom}_R(\coprod_i M_i, \prod_j N_j) = \prod_{i,j} \mathbf{Hom}_R(M_i, N_j).$$

In particular, if I and J are finite, the product and the coproduct are the same. In that case, $\mathbf{Hom}_R(M_i, N_j)$ are just matrices formed by homomorphisms. Composition then corresponds to multiplication of matrices. In particular, if we take for all $M_i = R = N_j$ realized as a left module over itself, then we have $\mathbf{Hom}_R(R^n, R^m)$, which is just the set of $m \times n$ matrices. (Note that $\mathbf{Hom}_R(R, M) = M$.)

4.3 Projective and Injective Module

Since modules form an Abelian category, and we have defined projective and injective objects, then we don't actually have to define them again. Recall that

Definition 4.3.1 (Projective)**.** *A (left) R-module P is projective if the functor $\mathbf{Hom}_R(P, -)$ is exact.*

Remark 4.3.2. *Free modules are projective.*

Theorem 4.3.3. *A (left) R-module P is projective if and only if P is a direct summand of a free module, i.e. there exists a (left) R-module P' such that $P \otimes P'$ is free.*

Proof. Suppose P is projective, then the sequence

$$0 \longrightarrow N \longrightarrow F \longrightarrow P \longrightarrow 0$$

where F is free. This sequence is split because P is projective, and so $F \cong P \oplus N$.

Suppose $P \oplus N$ is free, then $\mathbf{Hom}_R(P, -)$ is corepresented by R_p. Then the represented functor $R_{P\oplus N} = R_P \oplus R_N$ and is exact because $P \oplus N$ is free. It is an easy exercise to see that R_P is exact, and so P is projective. $\square$

Example 4.3.4. *1. Take $R = R_1 \times R_2$ as a product of two rings. Take $P_1 = R_1 \times 0$ and $P_2 = 0 \times R_2$ as two ideals in R, and therefore are modules. In particular, we have $R \cong P_1 \oplus P_2$. Therefore, P_1 and P_2 are projectives.*

2. Let F be a field. Take $R = F[x, y, z]/(x^2 + y^2 + z^2 - 1) \cdot R[x, y, z]$. This is the ring of polynomial functions on the sphere S given by $x^2 + y^2 + z^2 = 1$.

Recall that the homomorphism between free modules is given by matrices. Therefore, consider the homomorphism $f : R^3 \to R$ given by $\begin{pmatrix} x & y & z \end{pmatrix}$, sending $\begin{pmatrix} f \\ g \\ h \end{pmatrix}$ to $xf + yg + zh$. This map is surjective, and is therefore split. We can define the retraction $R \to R^3$ given by the matrix $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Let P be the kernel of f. Then we have a split short exact sequence given by

$$0 \longrightarrow P \longrightarrow R^3 \xrightarrow{f} R \longrightarrow 0$$

Therefore, $R^3 \cong P \oplus R$. Hence, the kernel P is projective and stably free. In particular, $P = \{\begin{pmatrix} f \\ g \\ h \end{pmatrix} : xf + yg + zh = 0\}$. This is the R-module of tangent fields on a sphere.

Now, suppose P is free, i.e. $P \cong R^2$, then P has a basis given by $t, s \in P$. So for all $u \in S$, $\{t(u), s(u)\}$ forms a basis for the tangent plane at u. In particular, $t(u) \neq 0$ for all $u \in S$.

From the point of view of topology, if the base field $F = \mathbb{R}$, then there is no everywhere nonzero tangent vector field on the sphere.

Therefore, P is not free. (If the base field is $\mathbb{C}$, then it is free. Note that P is not free over any subfield of $\mathbb{R}$.)

Definition 4.3.5 (Injective)**.** *A (left) module Q is injective if $\mathbf{Hom}_R(-, Q)$ is exact. In particular, this means every exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & S \arrow[r] \arrow[dl, dashed] & T \arrow[r] & 0 \\ & Q & & & \end{tikzcd}$$

Remark 4.3.6. *One would expect a similar description of injective modules to exist, but there is none. The reason is that the dual category of the category of R-modules is not equivalent to a category of modules over some ring.*

Remark 4.3.7. *Now consider a special case of exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d] & R \arrow[r] \arrow[dl, dashed] & R/I \arrow[r] & 0 \\ & Q & & & \end{tikzcd}$$

where I is a left ideal in the ring R. Suppose Q is injective, then we have a natural extension as above. However, every homomorphism from R to Q is of the form that sends a to aq for a fixed element $q \in Q$.

Therefore, for every $f : I \to Q$, there exists $q \in Q$ such that $f(x) = xq$ for every $x \in I$.

This induces the following theorem, as a replacement of correspondence theorem of injective modules.

Theorem 4.3.8 (Baer)**.** *Let Q be a (left) R-module such that for every left ideal $I \subseteq R$ and every R-module homomorphism $f : I \to Q$, there is an element $q \in Q$ with $f(x) = xq$ for all $x \in I$, then Q is injective.*

Proof. Suppose we have a submodule $M \subseteq S$ for a module S, and we have a homomorphism g from S to Q. We use Zorn's Lemma and consider all possible extensions: the set of pairs $(\bar{M}, \bar{g})$, where $M \subseteq \bar{M} \subseteq S$ and $\bar{g} : \bar{M} \to Q$ is given by $\bar{g}\mid_M = g$. It is non-empty because we can take $\bar{M} = M$.

Observe the ordering on the set, given by $(M_1, g_1) \leq (M_2, g_2)$ when $M_1 \subseteq M_2$ and $g_1 = g_2 \mid_{M_1}$. By Zorn's Lemma, there exists a maximal pair (M', g').

The claim is that $M' = S$. If this is true, then g' is the extension we want, and we are done.

Suppose not, then there is $s \in S\backslash M'$. Define $M'' = M' + Rs \supsetneq M'$. We need to find $g'' : M'' \to Q$ extending g'.

Take $I = \{x \in R : xs \in M'\} \subseteq R$ to be a left ideal in R. There is now a map $f : I \to Q$ given by $x \mapsto g'(xs) \in Q$. This is well-defined because $xs \in M'$.

By assumption, there exists $q \in Q$ such that $f(x) = xq$. Then set$g''(m' + xs) = q'(m'') + xq$, and so $(M'', q'') \supsetneq (M', q')$. This gives a contradiction. $\square$

We now want to characterize the injective modules in principal ideal domains.

Definition 4.3.9 (Divisible)**.** *Let R be a PID, and let M be a R-module. We say M is divisible if $\forall m \in M$, $\forall 0 \neq a \in R$, there exists $m' \in M$ such that $m = a \cdot m'$.*

Proposition 4.3.10. *A module M over a PID R is injective if and only if M is divisible.*

Proof. Take arbitrary ideal I in R and take arbitrary homomorphism $f : I \to M$. Then M is injective if and only if the following extension exists.

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "f"] & R \arrow[dl, dashed] \\ & M & \end{tikzcd}$$

Since R is a PID, then $I = aR$ for some $a \in R$. Obviously we can assume $a \neq 0$. Now the map f is easy to understand becasue I is free with a basis given by $\{a\}$. Now, the mapping is determined by the single element a. Consider $f(a) = m \in M$ to be arbitrary, then we have $f(ax) = am$.

Now, this f can be extended: there exists $m' \in M$ such that $f(y) = ym'$. By substituting $y = a$, we have $m = f(a) = am'$. Therefore, $m = am'$, which implies divisibility.

Similarly we can see the other side of the proof. □

Example 4.3.11. *Consider $R = \mathbb{Z}$ (which is a PID), then $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Q}$ are divisible.*

In general, the factor module of divisible module is divisible, and so the factor module of injective module is injective.

Recall that every module is a factor module of a free module, and so it is a factor module of a projective module. The dual statement is that every module is a submodule of an injective module.

Proposition 4.3.12. *Consider $R = \mathbb{Z}$. Every group is a subgroup of a divisible group, so every group is a subgroup of an injective $\mathbb{Z}$-module.*

Proof. Take M to be an Abelian group. We want to embed M into a divisible group. We write M as a factor module of a free module, then $M = \mathbb{Z}^{(X)}/N$ for a set X, and $N \subseteq \mathbb{Z}^{(X)}$ is a submodule.

Therefore, we have $N \subseteq \mathbb{Z}^{(X)} \hookrightarrow \mathbb{Q}^{(X)}$, where $\mathbb{Q}^{(X)}$ is divisible. By factoring out the N, we have $M = \mathbb{Z}^{(X)}/N \hookrightarrow \mathbb{Q}^{(X)}/N$, where $\mathbb{Q}^{(X)}/N$ is divisible, so injective. □

4.4 Tensor Product

Let R be an arbitrary ring, let M be a right R-module and let N be a left R-module. We denote them M_R and $_RN$ respectively.

Definition 4.4.1 (Bilinear Form, Tensor Product)**.** *Let A be an Abelian group written additively. A bilinear form on $M \times N$ with values in A is a map $B : M \to N \to A$ such that*

1. $B(m_1 + m_2, n) = B(m_1, n) + B(m_2, n)$,
2. $B(m, n_1 + n_2) = B(m, n_1) + B(m, n_2)$,
3. $B(ma, n) = B(m, an)$ *for* $a \in R$.

Then Bil$(M, N; A)$ is the Abelian group of all bilinear forms $M \times N \to A$. For a homomorphism $A \to A$", this induces Bil$(M, N; A) \to$ Bil$(M, N; A')$.

When $(M_R, {}_RN)$ is fixed, there is a functor $F : \mathbf{Ab} \to \mathbf{Ab}$ that sends A to Bil$(M, N; A)$.

The tensor product $M \otimes_R N$ is an Abelian group representing this functor:

$$Bil(M, N; A) \xrightarrow{\sim} \mathbf{Hom}(M \otimes_R N, A)$$

which gives an isomorphism. This functor is natural in A.

A tensor product, if exists, is unique up to canonical isomorphism.

Example 4.4.2. *Consider $M = R$, i.e. the ring as a left and right module over itself. The bilinear form is $B : R \times N \to A$ given by $B(x, n) = B(1, xn)$. Moreover, if $f : N \to A$ takes $n \mapsto B(1, n)$, then it is a group homomorphism.*

Therefore, $f(xn) = B(1, xn) = B(x, n)$.

Hence, $Bil(R, N; A) = \mathbf{Hom}(N, A)$. In particular, $R \otimes_R N \cong N$ and $M \otimes_R R \cong M$.

We now show that a tensor product always exists.

Theorem 4.4.3. *$M \otimes_R N$ exists for every $(M_R, {}_R N)$.*

Proof. It suffices to find a construction: then all tensor products should be related by the canonical isomorphism.

Let $X = M \times N$ as the product of sets. Consider $C = \mathbb{Z}^{(X)}/G$, the factorization of free Abelian group of basis X and a subgroup G, where G is generated by elements of the form:

1. $(m_1 + m_2, n) - (m_1, n) - (m_2, n)$,

2. $(m, n_1 + n_2) - (m, n_1) - (m, n_2)$,

3. $(ma, n) - (m, an)$ for all $a \in R$.

To give a homomorphism $C \to A$ is just to give $B : \mathbb{Z}^{(X)} \to A$ such that

1. $B(m_1 + m_2, n) = B(m_1, n) + B(m_2, n)$.

2. $B(m, n_1 n_2) = B(m, n_1) + B(m, n_2)$,

3. $B(ma, n) = B(m, an)$.

This is to give a map $f : M \times N = X \to A$.

Therefore, we would have an isomorphism $\mathbf{Hom}(C, A) \cong \mathrm{Bil}(M, N; A)$.

Therefore, C is the representing object, and denoted $C = M \otimes_R N$.

□

Remark 4.4.4. *An element $m \otimes n$ in $M \otimes N$ is the coset of (m, n). Then $M \otimes_R N$ is generated by $m \otimes_R n$ for $m \in M$, $n \in N$.*

Remark 4.4.5. *Therefore, given by the isomorphism $\mathbf{Hom}(C, A) \cong Bil(M, N; A)$, we have $Bil(M, N; M \otimes_R N \cong \mathbf{Hom}(M \otimes_R N, M \otimes_R N$. The identity in the hom set is corresponding to a universal element B_{univ} in $Bil(M, N; M \otimes_R N)$, which gives $B_{univ} : M \times N \to M \otimes_R N$.*

Suppose we have some other bilinear form $B : M \times N \to A$, then B corresponds to some homomorphism f, with

$$\begin{tikzcd} M \times N \arrow[r, "B_{univ}"] \arrow[dr, "B"'] & M \otimes_R N \arrow[d, "f"] \\ & A \end{tikzcd}$$

Therefore, every bilinear form B is the composition of a homomorphism f and the universal bilinear form.

The universal bilinear form is now given by $B_{univ}(m,n) = m \otimes_R n = m \otimes n$ for $m \in M$ and $n \in N$. Therefore, the universal property can be rewritten as the following: for every bilinear form $B : M \times N \to A$, there exists a unique homomorphism $f : M \otimes_R N \to A$ such that $B(m,n) = f(m \otimes n)$.

The universal property itself may also define the tensor product.

Property 4.4.6. *1. $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$,*

2. $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$,

3. $ma \otimes n = m \otimes an$.

Remark 4.4.7. *Recall that $R \otimes_R N \cong N$. Indeed, $1 \otimes n$ corresponds to n and $a \otimes n$ corresponds to an.*

Remark 4.4.8. *We can also consider the functoriality. Suppose $f : M \to M'$ and $g : N \to N'$ are two R-module homomorphisms. Then we can look at the following composition B:*

$$M \times N \xrightarrow{f \times g} M' \times N' \longrightarrow M' \otimes_R N'$$

Then B is a bilinear form. Indeed, for example we have

$$\begin{aligned} B(m_1 + m_2, n) &= f(m_1 + m_2) \otimes g(n) \\ &= f(m_1) \otimes g(n) + f(m_2) \otimes g(n) \\ &= B(m_1, n) + B(m_2, n) \end{aligned}$$

Therefore, there exists a unique homomorphism $f \otimes g : M \otimes_R N \to M' \otimes_R N'$ such that $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$.

*This induces a functor Mod-$R \times R$-Mod $\to$ **Ab** given by $(M,N) \mapsto M \otimes_R N$ and $(f,g) \mapsto f \otimes g$.*

*If we fix ${}_RN$, then Mod-$R \to$ **Ab** is an additive functor that sends $M \mapsto M \otimes_R N$. In particular, we have the formula $(f_1 + f_2) \otimes g = f_1 \otimes g + f_2 \otimes g$.*

We now would like to know the properties of this additive functor.

Similarly, fix M_R, then we have

$$0 \longrightarrow N' \xrightarrow{h} N \xrightarrow{k} N'' \longrightarrow 0$$

exact in the category of R-modules.

By functoriality, we have an induced sequence

$$0 \longrightarrow \mathit{Bil}(M, N''; A) \longrightarrow \mathit{Bil}(M, N; A) \longrightarrow \mathit{Bil}(M, N'; A)$$

which is also exact:

$$\begin{tikzcd} M \times N'' \arrow[dr, "B''"] & \\ M \times N \arrow[u] \arrow[r, "B"] & A \\ M \times N' \arrow[u] \arrow[ur, "0"'] & \end{tikzcd}$$

Then $B''(m, n'') = B(m, n)$ by definition, where $n \in N$ is given by $k(n) = n''$. Therefore, this is independent of the choice on such maps.

When it comes to the definition of tensor product, equivalently, we have

$$0 \longrightarrow \boldsymbol{Hom}(M \otimes N'', A) \longrightarrow \boldsymbol{Hom}(M \otimes N, A) \longrightarrow \boldsymbol{Hom}(M \otimes N', A)$$

as exact sequence for arbitrary A.

This exactness on A is equivalent to the exactness of the following sequence on the right (because of contravariant properties):

$$M \otimes_R N' \xrightarrow{1_M \otimes h} M \otimes_R N \xrightarrow{1_M \otimes k} M \otimes_R N'' \longrightarrow 0$$

Therefore, $M \otimes_R -$ and $- \otimes_R N$ are both right exact.

Remark 4.4.9. *We now show that the tensor product is an additive functor by fixing one of the slots, i.e. commute with arbitrary direct sums.*

Let $(M_i)_{i \in I}$ be a family of right modules, and an arbitrary left R-module ${}_RN$. We want to show there is an canonical isomorphism $(\coprod_{i \in I} M_i) \otimes_R N \cong \coprod_{i \in I} M_i \otimes_R N$. The proof should be element-free.

The left-hand-side represents the functor of bilinear forms $\mathit{Bil}(\coprod_{i \in I} M_i, N; A)$. The right-hand-side represents the product $\prod_{i \in I} \mathit{Bil}(M_i, N; A)$. To see the two modules are isomorphic, it suffices to show that the two functors are isomorphic.

Consider the bilinear forms $B_i : M_i \times N \to A$. We can construct the bilinear form $B : \coprod_{i \in I} M_i \times N \to A$ by writing $B(\sum_{i \in I} m_i, n) = \sum_{i \in I} B_i(m_i, n)$. This induces an isomorphism of functors, with naturality in both slots.

Remark 4.4.10. *The tensor product is generated by the tensor product of elements. In particular, we have the following.*

Suppose $X \subseteq M_R$ and $Y \subseteq_R N$ are generating sets of modules.

We have homomorphisms $R^{(X)} \hookrightarrow M$ by sending $x \mapsto x$, and $R^{(Y)} \hookrightarrow N$ by sending $y \mapsto y$. ($R^{(X)}$ is viewed as a right R-module and $R^{(Y)}$ is viewed as a left R-module.)

Since the tensor product is left exact, we have surjections $R^{(X,Y)} = \coprod_{X \times Y} (R \otimes R)^{(X,Y)} = R^{(X)} \otimes R^{(Y)} \hookrightarrow M \otimes R^{(Y)} \hookrightarrow M \otimes N$. Therefore the map takes the generating element (x, y) to the tensor product $x \otimes y$.

Since this is a surjection, then $M \otimes_R N$ is generated by elements of the form $x \otimes y$ for $x \in X$ and $y \in Y$.

Example 4.4.11. *Suppose $I \subseteq R$ is a right ideal and let M be a left R-module. Then the short exact sequence*

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

is right exact when tensored with M:

$$I \otimes_R M \longrightarrow R \otimes_R M \longrightarrow R/I \otimes_R M \longrightarrow 0$$

Note that $R \otimes_R M$ is just canonically isomorphic to M. For $x \in I$, the first map α takes $x \otimes m \mapsto xm$, then the image of α is IM , which is an Abelian group generated by xm for $x \in I$ and $m \in M$, left submodule generated by these elements.

By exactness, we see that $R/I \otimes_R M$ is canonically factor to the group M/IM.

In particular, for integer n, the group A/nA is isomorphic to $(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} A$.

Remark 4.4.12. *Suppose we have a bimodule ${}_SM_R$ where R and S are rings. We assume the two module structures are related as follows: $(sm)t = s(mt)$.*

We can rewrite as follows: for $l_s : M \to M$ by $m \mapsto sm$ and $t_x : N \to N$ by $n \mapsto nx$. This says that $l_s \circ t_x = t_x \circ l_s$.

In particular, consider ${}_SM_R$ and ${}_RN$). We can form Abelian group $M \otimes_R N$. We now have left multiplication by s, which we can tensor along with the identity: $l_s \otimes 1_N : M \otimes_R N \to M \otimes_R N$. This is a group endomorphism, and it makes the tensor product $M \otimes_R N$ a left S -module. Therefore, we can write ${}_S(M \otimes_R N)$ where $s(m \otimes n) = sm \otimes n$.

Note that if R is commutative, then left and right R-modules coincide, i.e. ${}_RM_R$. Recall that we define $rm = mr$. This is now a bimodule over R. Therefore, if M, N are R-modules for commutative ring R, then so is the tensor product $M \otimes_R N$.

Remark 4.4.13. *Suppose we have M_S, ${}_SN_R$ and P_R. Then then tensor product $(M \otimes_S N)_R$ is a right R-module, and $(\mathbf{Hom}_R(N, P))_S$ is a group of homomorphisms of right R-modules, and has the right S-module structure.*

Having this in mind, we can write down the two Abelian groups through canonical isomorphisms, natural in all slots: $Bil(M, N; P) = \mathbf{Hom}_R(M \otimes_S N, P) \cong \mathbf{Hom}_S(M, \mathbf{Hom}_R(N, P))$. The first equation is not so precise: for the definition of tensor product, we view them as Abelian groups now.

For any bilinear form $M \times N \to P$ in this group, or more precisely, $M \otimes_S N \to P$, we can define the hom on the right by $m \mapsto (n \mapsto B(m,n))$. Conversely, we ca take a map in hom φ to the bilinear map $B(m,n) = \varphi(m)(n)$. One can check easily they are inverses to each other as isomorphisms.

There is a similar situation when we have left modules. Consider $({}_S M, {}_R N_S, {}_R P)$. The corresponding isomorphisms are given by $\mathbf{Hom}_R(N \otimes_S M, P) \cong \mathbf{Hom}_S(M, \mathbf{Hom}_R(N,P)$.

Remark 4.4.14 (Construction of Change of Ring). *Suppose we have $f : R \to S$ as a ring homomorphism, with ${}_R N, {}_S S_R$, this is by pulling elements back with $s \cdot t = s \cdot f(t)$.*

In this situation, we can form $S \otimes_R N$. But S is also a left module over itself, so it is a bimodule. Therefore, this is a left module, operates on the tensor of the form $s(x \otimes n) = sx \otimes n$.

In fact, we can get a functor of R-Mod $\to$ S-Mod by sending $M \mapsto S \otimes_R M$. Similarly, we can do the same on right modules.

Recall that we have a functor S-Mod $\to$ R-Mod given by the pullback construction with respect to f. It is not surprising that the two functors are adjoint to each other. In particular, for ${}_R N$ and ${}_S M$, we have $\mathbf{Hom}_S(S \otimes_R N, M) \cong \mathbf{Hom}_R(N, \mathbf{Hom}_S(S,M))$. Here $\mathbf{Hom}_S(S,M) \cong M$, but viewed as left R-module via the pullback. We get that the pullback functor $\mathbf{Hom}_S(S,-)$ is the right adjoint to the tensor product functor $S \otimes_R -$, i.e. extension of scalars.

We can now complete the proof that every module is a submodule of some injective module. We proved this for Abelian groups only. We now prove it for arbitrary modules.

Proposition 4.4.15. *Every module is a submodule of some injective module.*

Proof. Let M be an Abelian group. We use the only R-homomorphism $\mathbb{Z} \to R$ to view R as a $\mathbb{Z}$ module, and consider $\tilde{M} = \mathbf{Hom}_{\mathbb{Z}}(R, M)$, which is a left R-module.

Here we have ${}_{\mathbb{Z}} R_R$ and ${}_{\mathbb{Z}} M$.

Take any left R-module X. We can write the following formula: $\mathbf{Hom}_{\mathbb{Z}}(R \otimes_R X, M) \cong \mathbf{Hom}_R(X, \mathbf{Hom}_{\mathbb{Z}}(R, M))$. Note $R \otimes_R X$ is just X. We see that $\mathbf{Hom}_{\mathbb{Z}}(X, M) = \mathbf{Hom}_R(X, \tilde{M})$. Note that $\tilde{M}$ is the functor left adjoint to the pullback functor applied to X with respect to the homomorphism $\mathbb{Z} \to R$.

Suppose M is a divisible (therefore injective) Abelian group, then $\mathbf{Hom}_{\mathbb{Z}}(X, M)$ is an exact functor as a functor on X. Therefore, the functor $X \mapsto \mathbf{Hom}_R(X, \tilde{M})$ is also exact, now as functor R-Mod $\to$ $\mathbf{Ab}$.

Therefore, $\tilde{M}$ is an injective R-module.

So we have proven that the functor $\mathbf{Ab} \to R$-Mod that takes $M \mapsto \mathbf{Hom}_{\mathbb{Z}}(R, M) = \tilde{M}$ takes injectives to injectives.

Now we can prove that every left module can be embedded in some injective module.

Consider ${}_R M$, then $M \hookrightarrow Q$ is an embedding into a divisible (injective) Abelian group.

Therefore, applying the tilde construction to both, then since the hom functor is left exact, and the tilde is given by the hom functor, we still have an injection $\tilde{M} \hookrightarrow \tilde{Q}$. But now $\tilde{Q}$ is an injective R-module. We have $\tilde{M} = \mathbf{Hom}_{\mathbb{Z}}(R, M)$ is embedded in $\tilde{Q}$. So it suffices to embed $M \hookrightarrow \mathbf{Hom}_{\mathbb{Z}}(R, M)$ by $m \mapsto (\tau \mapsto \tau m)$. Therefore, in total we have an embedding $M \hookrightarrow \tilde{Q}$. $\square$

4.5 Modules over a Principal Ideal Domain

This is almost the simpliest situation to classify modules, only after the situation of field. Over PID, we can classify the finitely generated ones.

Definition 4.5.1 (Torsion)**.** *Let R be a domain and M be a R-module. An element $m \in M$ is called torsion if $\exists 0 \neq a \in R$ such that $am = 0$.*

All torsion modules form a submodule, called $M_{\text{tors}} \subseteq M$.

Definition 4.5.2 (Torsion, Torsion-free)**.** *M is a torsion module if all elements are torsions, $M_{tors} = M$.*

M is a torsion-free module if $M_{tors} = 0$.

Lemma 4.5.3. *M/M_{tors} is torsion free.*

Example 4.5.4. *R is torsion-free because it is a domain. Free modules are torsion-free as well.*

Note that for $R = \mathbb{Z}$,$\mathbb{Q}$ is torsion-free but not free. In particular, for $x, y \in \mathbb{Z}$, there exists $a, b \in \mathbb{Z}$ such that $ax + by = 0$. $\mathbb{Q}$ is an infinitely-generated Abelian group.

Remark 4.5.5. *Notice that a factor module of an injective module over a PID is injective, because injective means divisble over PID, and factor module of divisible module is still divisible.*

There is a dual module for projectives, every submodule of projective modules is projective.

Also, every submodule of a free module is free. We will only show this statement for finitely-generated modules, but this is true in general.

Definition 4.5.6 (Rank)**.** *Note that for $R^m \cong R^n$, we have $n = m$. We call this the rank of R^n.*

Remark 4.5.7. *For a free R-module F, we say F is finitely generated if and only if the rank is finite.*

Proposition 4.5.8. *Let M be a submodule of a free finitely-generated module F over a PID R. Then M is free and rank$(M) \leq$ rank(F).*

Remark 4.5.9. *Note that this holds only on PID. Consider $I \subseteq R$ be an ideal, then I is free if and only if I is principal, i.e. for $x, y \in I$ we have $y \cdot x + (-x) \cdot y = 0$.*

Proof. Let $x_1, \cdots, x_n$ be a basis for F, we prove by performing induction on n.

When $n = 1$, I is an ideal of R, so it is principal and so free.

Suppose the statement is true for $n - 1$, we now show the case for n. Consider the projection of free module $f : F \to R$ given by $f(\sum(a_i x_i) = a_n$. Then the kernel $\ker(f)$ is just the free module with basis $x_1, \cdots, x_{n-1}$.

We have $M \subseteq F$ as a submodule, and the image $f(M) = I \subseteq R$ is still an ideal.

Let us take the kernel of this particular (restricted) surjective map to be M', then we have the exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M \longrightarrow 0$$

Note $M' = M \cap \ker(f) \subseteq \ker(f)$. Therefore, by induction, M' is free of rank at most $n-1$.

Now I is free because it is principal, and so it is projective. In particular, the sequence above splits. Therefore, $M \cong M' \oplus I$. Both modules are free, where M' has rank at most $n-1$ and I has rank at most 1, so M has rank at most n. This concludes the proof. $\square$

Let R be a PID and M be a R-module. Recall that if we let $S = R\backslash\{0\}$, then the localization $S^{-1}R = F$ is the quotient field, or field of fractions. We know that R is a subring of F.

We can also localize the module, so $S^{-1}M$ is a vector space over F, which contains all fractions $\{\frac{m}{a}, a \neq 0\}$. Recall that $\frac{m}{a} = 0$ if and only if there exists $0 \neq b \in R$ such that $bm = 0$.

We also have the canonical map $M \to S^{-1}M$ that takes $m \in M$ to $\frac{m}{1}$. The kernel of this map is $\{m \in M : \exists 0 \neq b \in R : bm = 0\} = M_{\text{tors}}$.

Therefore, M is torsion-free if and only if $M \hookrightarrow S^{-1}M$ is an embedding.

Theorem 4.5.10. *A finitely generated torion-free module over a PID R is free.*

Proof. Since M is a torsion-free R-module, then $M \hookrightarrow S^{-1}M$, considered as a vector space over F of finite dimensions.

Let $x_1, \cdots, x_n$ be a basis for $S^{-1}M$. Let $N = Rx_1 + \cdots + Rx_n$. Then N is a free R-module with basis $x_1, \cdots, x_n$.

Now module M is finitely generated, by picking finitely many generators $m_1, \cdots, m_k$ where $m_i \in M \subseteq S^{-1}M$. Therefore, $m_i \in Fx_1 + \cdots + Fx_n$.

There exists $0 \neq a_i \in R$ such that $a_i m_i \in N$. If we take the product of all the a_i's, then let $a = a_1 \cdots a_k \neq 0$, so $am_i \in N$. Therefore, $a \cdot M \subseteq N$. But N is free, then aM is free.

However, there is an isomorphism $a \xrightarrow{\sim} aM$, so M is free. $\square$

Remark 4.5.11. *Suppose M is a finitely generated R-module over a PID R. There is a short exact sequence*

$$0 \longrightarrow M_{tors} \longrightarrow M \longrightarrow M/M_{tors} \longrightarrow 0$$

Note that M/M_{tors} is torsion-free, and is finitely generated, then it is free. In particular, it is projective, so the short exact sequence splits.

Therefore, $M \cong M_{tors} \oplus M/M_{tors} \cong M_{tors} \oplus R^n$ where n is the rank of M.

Now, $S^{-1}M \cong S^{-1}M_{tors} \oplus F^n$ where n is the rank of M. Note that $S^{-1}M_{tors} = 0$, killed by the localization. Therefore, this is nothing but $\dim_F(S^{-1}M)$.

The study of finitely generated modules can then be focused on torsion finitely generated modules.

Definition 4.5.12 (Primary)**.** *Let M be a torsion, finitely generated R-module. Take $0 \neq P \subseteq R$ as a non-zero prime ideal of R. Therefore, $P = p \cdot R = R \cdot p$ for some prime p.*

We say that $m \in M$ is P-primary if $P^n \cdot m = 0$, which is equivalent to $p^n m = 0$ for some $n > 0$.

We denote $M(P)$ as the set of all P-primary elements in M, also called the P-primary part of M.

Claim 4.5.13. *$M(P)$ is a submodule.*

Proof. A lot of things need to be checked. We only check that the sum is still in $M(P)$.

For $m_1, m_2 \in M(P)$, then $p^{k_1} \cdot m_1 = 0 = p^{k_2} m_2$ for some k_1, k_2. Let $k = \max(k_1, k_2)$, then $p^k \cdot m_i = 0$ for all $i = 1, 2$. Therefore, $p^k(m_1 + m_2) = 0$, which means $m_1 + m_2 \in< (P)$. □

Lemma 4.5.14. *Let $a_1, \cdots, a_n$ be relatively prime elements in a PID R. Then there exists $b_1, \cdots, b_n \in R$ such that $\sum_{i=1}^{n} b_i a_i = 1$.*

Proof. Take the ideal generated by relatively prime elements $I = Ra_1 + \cdots + Ra_n = cR$ is principal for some $0 \neq c \in R$.

Therefore, $c \mid a_i$ for all i, and so $c \in R^\times$ because elements are relatively prime. Therefore, $I = cR = R$. The ideal is just the unit ideal, so $1 \in I$. Therefore, one can find the desired linear combination. □

Remark 4.5.15. *The notion of relatively prime elements not only make sense in PID, but also in UFD. However, the statement is not true over UFD. For example, consider $R = F[x_1, x_2]$ where x_1, x_2 are relatively prime. Here we have $Rx_1 + Rx_2 \neq R$.*

Corollary 4.5.16. *Let M be a module over a PID R, and let $a_1, \cdots, a_n \in R$ be relatively prime and $m \in M$. If $a_i m = 0$ for all i, then $m = 0$.*

Proof. By lemma, we can find b_i's such that $\sum b_i a_i = 1$, then $m = 1 \cdot m = \sum b_i a_i m = 0$. □

Theorem 4.5.17. *Let M be a torsion, finitely generated module over a PID R. Then:*

1. *$M(P) = 0$ for almost all prime ideals $P \neq 0$.*
2. *$M = M(P_1) \oplus M(P_2) \oplus \cdots \oplus M(P_n)$ for some prime ideal P_i. In other words, M is the direct sum of finitely many primary submodules.*

Proof. Since M is finitely generated and torsion, then it can be killed by one element in the ring. In particular, $\exists 0 \neq a \in R$ such that $a \cdot M = 0$.

Claim 4.5.18. *If P is a prime ideal such that $a \notin P$, then $M(P) = 0$.*

Subproof. We write the ideal as $P = R \cdot p$. Since $a \notin P$, then $p \nmid a$. Then because every element $m \in M(P)$ is killed by a power of p, i.e. $p^n \cdot m = 0$, and killed by a, i.e. $a \cdot m = 0$. By corollary, this means $m = 0$, since a and p^n are relatively prime. Therefore, $M(P) = 0$. ■

If we factor a as a product of prime elements, i.e. $a = up_1^{t_1} \cdots p_s^{t_s}$ where p_i's are distinct primes, u is a unit, then $a \in P_i = Rp_i$ and $a \notin P \neq P_i$. (Note that the prime ideals P_i's are distinct.) This proves the first part.

Claim 4.5.19. *$M = \coprod M(P_i)$.*

Subproof. Take arbitrary $m \in M$ and write $a_i = \frac{a}{p_i^{t_i}}$ where $a_1, \cdots, a_s$ are relatively prime.

By lemma, $\exists b_i \in R$ such that $\sum\limits_{1 \le i \le s} b_i a_i = 1$, and so $m = \sum\limits_{1 \le i \le s} b_i a_i m$. In particular, $p^{t_i} b_i a_i m = b_i a m = 0$. Therefore, $am = 0$.

Now, $b_i a_i m \in M(P_i)$. ■

Therefore, $M = \sum M(P_i)$. We need to show that this is a direct sum, i.e. for $m_i \in M(P_i)$ such that $m_1 + \cdots + m_s = 0$. We need to show that all $m_i = 0$.

One can choose a power t such that $p_i^t \cdot m_i = 0$ for all i. Now we take integer k from $1, \cdots, s$, then it suffices to show that $m_k = 0$.

Now $q = \frac{p_1^t p_2^t \cdots p_s^t}{p_k^t}$. In particular, since p_i^t kills all m_i, then $qm_i = 0$ for all $i \neq k$. However, this means $q \cdot m_k = 0$ as well. On the other hand, $p_k^t \cdot m_k = 0$. However, q and p_k^t are relatively prime, so we have $q \cdot m_k = 0$ and $p_k^t \cdot m_k = 0$. By the corollary, we know $m_k = 0$.

□

This statement shows that every torsion-free finitely generated module is a direct sum of some primary modules, therefore this reduces our study to the study of primary modules.

Definition 4.5.20 (Cyclic)**.** *An R-module N is cyclic if N is generated by one element.*

Remark 4.5.21. *From homework, we know that every cyclic module N is isomorphic to the factor module R/I for some ideal $I \subseteq R$. Obviously R is generated by one element, and I is also generated by one element. Therefore, all cyclic modules are of this form.*

In particular, since $I = aR$ for some a, we should have $N \cong R/aR$, which is torsion-free if and only if $a \neq 0$.

Claim 4.5.22. *The module $N = R/aR$ is P-primary if and only if $aR = P^n$ for some n.*

Proof. The $\Leftarrow$ direction is clear. On the other hand, suppose N is P-primary, then write $P = Rp$, and we see that $p^n N = 0$ for some power n. Therefore, $p^n R \subseteq aR$. Therefore, $a \mid p^n$, but an element that divides p^n is also some power of p (up to units), so $a = up^m$. Therefore, $aR = p^m R = P^m$. □

Remark 4.5.23. *Therefore, cyclic P-primary modules all have the form R/P^n for some n.*

Definition 4.5.24 (Residual Field)**.** *Suppose $0 \neq P \subseteq R$ is a prime ideal, and consider P-primary R-modules. First of all, note that $K = R/P$ is a field, called the residue field of P.*

Remark 4.5.25. *Let R be a PID and $P = pR$ is a prime ideal in R. Let M be a P-primary finitely generated R-module over R. We have a sequence of submodules that form the filtration. More precisely, $M \supseteq P \cdot M \supseteq P^2 \cdot M \supseteq \cdots \supseteq P^n M = 0$. (Here we write $P = pR$ and $P \cdot M = pM$.) We can take the subsequence factor $P^i M / P^{i+1} M$. This is an R-module for sure. Now $P(P^i M / P^{i+1} M) = 0$, so the factor module is killed by P. In particular, $P^i M / P^{i+1} M$ is an R/P-module, but R/P is the* *residual K, so this is a vector space over K of finite dimension. Therefore, it makes sense to talk about the dimensions of each of these factor modules.*

Definition 4.5.26 (Length)**.** *We define the length of the module M to be $l(M) = \sum_{i=0}^{n-1} \dim_K(p^i M/p^{i+1}M) \geq 0$.*

Property 4.5.27. 1. *The length of a cyclic module $l(R/p^n R) = n$. This is because those factors $p^i M/p^{i+1}M \cong p^i R/p^{i+1}R \cong R/P = K$. This is a one-dimensional vector space, so when summing the dimension up for $i = 0, \cdots, n-1$, we have n. Why does the last isomorphism hold? Observe that*

$$P \longrightarrow R \xrightarrow{\cdot p^i} p^i R/p^{i+1R} \longrightarrow 0$$

is an exact sequence as P is a kernel of $\cdot p^i$. The result follows from the first isomorphism theorem.

2. *If M, N are P-primary finitely generated modules, then $l(M \oplus N) = l(M) + l(N)$.*

3. *If $0 \neq N \subseteq M$ is a submodule, and both are P-primary finitely generated modules, then $l(M/N) < l(M)$. We can denote $M' = M/N$. Then there is a natural surjection $\varphi_i : p^i M/p^{i+1}M \twoheadrightarrow p^i M'/p^{i+1}M'$, given by the surjection $M \twoheadrightarrow M'$. We need strict equality.*

   Note that there exists some unique i such that $N \subseteq p^i M$ but $N \not\subseteq P^{i+1}M$. The image of N in $p^i M/p^{i+1}M$ is nonzero (because $N \not\subseteq P^{i+1}M$), but the image of N in $p^i M'/p^{i+1}M'$ is zero, so the kernel of φ_i is non-zero.

We now write ${}_pM = \{m \in M : pm = 0\} = \ker(M \xrightarrow{p} M)$. Note that this is a submodule, and the ideal P acts on ${}_pM$ trivially, i.e. $P \cdot_p M = 0$. Therefore, ${}_pM$ is a vector space over $K = R/P$.

Lemma 4.5.28. *Given by the setting above, assume that $p^n M = 0$ but $p^{n-1}M \neq 0$. If $\dim_P(M) = 1$, then $M = R/P^n = R/p^n R$.*

Proof. By assumption, there exists an element $x \in M$ such that $p^{n-1}x \neq 0$.

Claim 4.5.29. *If $ax = 0$ for some $a \in R$, then $p^n \mid a$.*

Subproof. We write $a = p^m \cdot b$ for $\gcd(p, b) = 1$. We need to show that $m \geq n$. Suppose, towards contradiction, that $m < n$. Therefore, $b(p^m x) = ax = 0$, and $p^{n-m}p^m x = p^n x = 0$. However, p^{n-m} and b are relatively prime, so by lemma we conclude $p^m x = 0$. However, $p^{n-1}x \neq 0$, contradiction. ■

Consider the homomorphism $R \to M$ given by $a \mapsto ax$. Since $P^n \cdot x = 0$, then P^n is contained in the kernel.

Consider $f : R/P^n = R/p^n \cdot R \to M$ given by $a + P^n \mapsto ax$.

Suppose $f(a + P^n) = ax = 0$. Then by claim $a \in P^n$. Therefore, f is injective. Also, we can show that every $y \in M$ is contained in Rx. We can pick smallest k such that $p^k y = 0$. We now do induction on k.

If $k = 1$, $py = 0$, with $y \in_p M \ni p^{n-1}x \neq 0$. Since $\dim_P(M) = 1$, there exists $b \in R$ such that $y = b \cdot p^{n-1}x \in Rx$.

Suppose the case is true for $k - 1$, we prove the case for k. Take $p^{k-1}(py) = 0$, by induction, $py \in Rx$, $py = ax$ for some $a \in R$. Now $0 = p^n y = p^{n-1}ax$. By the claim, $p^n \mid p^{n-1}a$, so $p \mid a$, and we can write $a = pb$ for some $b \in R$. Therefore, $py = pbx$, so $p(y - bx) = 0$. Therefore, $y - bx \in_P M \subseteq Rx$. So $y \in Rx$. Therefore, the map is surjective. In particular, we have an isomorphism as desired. ☐

Proposition 4.5.30. *Let $p^n M = 0$ but $p^{n-1}M \neq 0$. Then there is a surjective R-module homomorphism $M \twoheadrightarrow R/p^n R$.*

Proof. We perform induction on $l(M)$. We pick $x \in M$ such that $p^{n-1}x \neq 0$. However, $p^n x = 0$ would indicate $0 \neq p^{n-1}x \in_P M$, and so the dimension $\dim_K({}_PM) > 0$.

If $\dim({}_PM) = 1$, by lemma, $M \cong R/p^n R$.

If $\dim({}_PM) > 1$, then there exists a nonzero subspace (submodule) $N \subseteq_p M$ such that $Rp^{n-1}x \not\subseteq N$. Consider the factor module $M' = M/N$, then $l(M') < l(M)$. Then $p^{n-1}(x+N)$ as an element of M' is not equal to N. Therefore, $p^{n-1}M' \neq 0$, as it contains some $x + N \in p^{n-1}M'$, but $p^n M' = 0$. By the induction step, there exists a surjective homomorphism composed by $M \twoheadrightarrow M' \twoheadrightarrow R/p^n R$. ☐

Theorem 4.5.31. *Every finitely generated P-primary R-module M is isomorphic to a direct sum of cyclic modules R/P^k.*

Proof. We prove by performing induction on $l(M)$. As usual, we choose n such that $p^n M = 0$ but $p^{n-1}M \neq 0$. In particular, M is a R/P^n-module because P^n kills the module. By proposition, there is a surjective R-module homomorphism $M \twoheadrightarrow R/P^n$, and can be embedded in the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow\!\!\!\!\!\rightarrow R/P^n \longrightarrow 0$$

Note that this is a short exact sequence of R/P^n modules. The last module R/P^n is free and so projective, then the sequence splits. We consider this as a splitting on R-modules. So $M \cong N \oplus (R/P^n)$ as R-modules.

In particular, $l(N) = l(M) - n < l(M)$. By induction, N is a direct sum of cyclic modules. ☐

Therefore, collecting the results we saw above, if we let M be a finitely generated R-module over a PID R, then R is a direct sum of modules of the form R and R/P^n for prime ideal P's. Note that every module here is cyclic, so every finitely generated R-module over PID is a direct sum of cyclic modules.

We just saw such decomposition holds, but we still need to check uniqueness.

Remark 4.5.32. *Recall that $M = R^n \oplus M_{tors}$, where $n = \dim_F(S^{-1}M)$ and is the rank of M. Here $S = R\backslash\{0\}$. It sufficient to prove that the torsion part has unique decomposition, up to permutation of terms.*

Therefore, consider $M = M_{tors}$, so M is a finite direct sum of $M(P)$'s. To prove uniqueness, we consider $M = M(P)$ as some P-primary ideal. We write $M = \coprod_{i=1}^{\infty}(R/P^iR)^{\oplus s_i}$. It suffices to express the integer s_i in terms of the module M in a unique way.

We use the following computations: suppose $N = R/p^nR$ is a cyclic module. Then $p^{k-1}N = p^{k-1}R/p^nR$ and $p^kN = p^kR/p^nR$, and $p^{k-1}N/p^kN \cong R/pR$ for $k \leq n$. However, if $k > n$, $p^{k-1}N = 0$ because p^{k-1} kills the module.

To remember, $\dim_K(p^{k-1}N/p^kN) = \begin{cases} 1, & \text{if } k = 1, \cdots, n \\ 0, 7 \text{ if } k > n & \end{cases}$.

Now $l_K = \dim_K(p^{k-1}M/p^kM) = s_k + s_{k+1} + \cdots$. Therefore, $s_k = l_k - l_{k+1}$.

Let M be a torsion module over a PID R. Then there exists distinct prime ideals $P_1, \cdots, P_k$ such that $M \cong R/P_1^{\alpha_{11}} \oplus R/P_1^{\alpha_{12}} \oplus \cdots \oplus R/P_1^{\alpha_{1t_1}} \oplus R/P_2^{\alpha_{21}} \oplus \cdots \oplus R/P_2^{\alpha_{2t_2}} \oplus \cdots \oplus R/P_k^{\alpha_{k1}} \oplus \cdots \oplus R/P_k^{\alpha_{kt_k}}$, and without loss of generality we have $\alpha_{11} \geq \alpha_{12} \geq \cdots$, $\alpha_{21} \geq \alpha_{22} \geq \cdots$, $\cdots$, $\alpha_{k1} \geq \alpha_{k2} \geq \cdots$.

The family $\{P_i^{\alpha_{ij}}\}$ is called the set of elementary divisors of M, also known as $ED(M)$. This family of elementary divisors is unique up to permutation of terms.

In particular, if M is a finitely generated module, then $M = R^n \oplus M_{\text{tors}}$. If N is finitely generated as well, then $M \cong N$ if and only if they have the same rank and the same elementary divisor, i.e. $\text{rank}(M) = \text{rank}(N)$, $ED(M) = ED(N)$.

Theorem 4.5.33 (Elementary Divisor Form)**.** *Two finitely generated R-modules over a PID are isomorphic if and only if they have the same rank and the same families of elementary divisors.*

Given by the structure above, by applying the Chinese Remainder Theorem, we have $R/P_1^{\alpha_{1j}} \oplus R/P_2^{\alpha_{2j}} \oplus \cdots \oplus R/P_k^{\alpha_{kj}} = R/I_j$ where $I_j = \prod_{i=1}^{k} P_{i,j}^{\alpha_{i,j}}$. Now $M \cong R/I_1 \oplus R/I_2 \oplus \cdots \oplus R/I_s$ for some $s = \max_{1 \leq i \leq k}(t_i)$. In particular, $I_1 \subset I_2 \subset \cdots I_s$. We can write every ideal here as a principal ideal, e.g. $I_j = a_jR$ for some $a_j \in R$. Equivalently, we have $a_s \mid a_{s-1} \mid \cdots \mid a_2 \mid a_1$.

Conversely, if we know the ideals, we can write down the matrices, by factoring the ideals into the powers of prime ideals. The family of those ideals $\{I_s, I_{s-1}, \cdots, I_1\}$ is called the family of invariant factors of M, denoted $IF(M)$, and are determined uniquely. Sometimes we just write it in terms of $\{a_s, a_{s-1}, \cdots, a_1\}$ and call them the invariant factors (but those are not uniquely determined, since there can be multiple generators for an ideal).

In particular, the two forms are equivalent, and so we have the following theorem:

Theorem 4.5.34 (Invariant Factor Form)**.** *Two finitely generated R-modules are isomorphic if and only if they have the same rank and the same invariant factors.*

Remark 4.5.35 (How to compute the two forms?)**.** *Take M to be a finitely generated R-module, then it is a factor module of a finitely generated free R-module F, and there is a submodule $N \subseteq F$ such that $M \cong F/N$. Moreover, N is free because it is a submodule of the free module.*

We get to choose a basis $\{x_1, x_2, \cdots, x_n\}$ for F, and let $\{y_1, y_2, \cdots, y_n\}$ be a set that generates N, where $m \leq n$. Because $N \subseteq F$, then $y_1 = a_{11}x_1 + a_{21}x_2 + \cdots + a_{n1}x_n$, $y_2 = a_{12}x_2 + \cdots + a_{n2}x_n$, up until $y_m = a_{1m}x_1 + a_{2m}x_2 + \cdots + a_{nm}x_n$. We then construct a matrix $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$, as the transpose of the system of equations above.

Suppose A is of the form $\begin{pmatrix} t_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & t_2 & 0 & \cdots & \cdots & \cdots & \vdots \\ 0 & \cdots & \ddots & \cdots & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & t_k & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$, such that $t_i \neq 0$ and $t_1 \mid t_2 \mid \cdots \mid t_k$, then we have $y_i = \tau_i x_i$ for $i \leq k$, and $y_i = 0$ for $i > k$. Now, $M = R/t_1R \oplus R/t_2R \oplus \cdots \oplus R/t_kR \oplus R \oplus R \cdots \oplus R$, where there are $m - k$ terms of R-summands.

Recall that $t_1 \mid t_2 \mid \cdots \mid t_k$. Therefore, the invariant factors of M are just the invariant factors of M_{tors}, which is $(t_1R, \cdots, t_kR)$.

Although the matrix A we considered is very preliminary, we can introduce the following operations so that we get to consider an arbitrary matrix:

1. *Transposition of two rows/columns. Such operations don't change M, N or F.*

2. *Subtraction from a row (respectively, column) a multiple of another row (respectively, column). This operation changes the basis elements, but doesn't change the modules M, N or F.*

3. *Multiplication of a row/column by a unit of R. Again, this does not change the modules.*

Note that by applying the three operations, we can get to a simplified form as denoted above.

Example 4.5.36. *Consider $R = \mathbb{Z}$, so the R-modules are the Abelian groups. Consider $M = \mathbb{Z}^2 / \left\langle \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\rangle$. We take the standard basis of $\mathbb{Z}^2$, i.e. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, with $y_1 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$. Therefore, $A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$. We then have*

$$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \to \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} \to \begin{pmatrix} 2 & 4 \\ 0 & -6 \end{pmatrix} \to \begin{pmatrix} 2 & 0 \\ 0 & -6 \end{pmatrix} \to \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$$

Therefore, the invariant factor form of M is $\{2,6\} = \{2\mathbb{Z}, 6\mathbb{Z}\}$. Therefore, $M = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$. Now, $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, so $M \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. Hence, the elementary divisor form of M is given by $\{2,2,3\}$.

We now want to apply our results to PIDs. In particular, for the ring $R = \mathbb{Z}$, the R-modules are exactly the Abelian groups. This would help us classify the finitely-generated Abelian groups.

4.6 Finitely-generated Abelian Groups

Let $R = \mathbb{Z}$. Corresponding to the results above, we have two forms of the main theorem:

Theorem 4.6.1 (Elementary Divisor Form). *Every finitely generated Abelian group is isomorphic to a direct sum of cyclic groups, i.e. $\mathbb{Z}$ or $\mathbb{Z}/p^n\mathbb{Z}$ for some prime p. Two groups are isomorphic if and only if they have the same rank and the same elementary divisors.*

Theorem 4.6.2 (Invariant Factor Form). *Every finitely generated Abelian group is isomorphic to a direct sum of the form $\mathbb{Z}^m \oplus \mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_s\mathbb{Z}$ with $a_1 \mid a_2 \mid \cdots \mid a_s$. The ideals $a_1\mathbb{Z}, \cdots a_s\mathbb{Z}$ are uniquely determined.*

Moreover, if we assume the integers are positive, then the integers are uniquely determined. Two groups are isomorphic if and only if they have the same rank and the same invariant factors.

Although it is very obvious in this case, the result is not very obvious in general.

4.7 Canonical Form of a Linear Operator

Let F be a field, and V is a vector space of finite dimension over F. Let $S : V \to V$ be a linear operator. Of course, V can be viewed as a module over the field, and then S is just an endomorphism over the module V. We try to classify these linear operators.

Let $R = F[x]$, then it is a Euclidean domain and then a PID. We now get to define an R-module structure on V: let $a_i \in F$, then the scalar multiplication is defined by $(a_n x^n_+ \cdots a_1 x + a_0) \cdot v = a_n S^n(v) + \cdots + a_1 S(v) + a_0 v$. Conversely, suppose M be a R-module, then because F is a subring of R, then M becomes a F-module, and therefore is a vector space over F. Define $T : M \to M$ by $T(m) = x \cdot m$. Then T is a linear operator over the vector space M.

Moreover, if M is a finitely generated module (not necessarily of finite dimension), then $M \cong R^k \oplus M_{\text{tors}}$ for some k. Note that R^k is infinite-dimensional if k is positive. We will see later that M_{tors} always has a finite dimension. Therefore, $\dim(M) < \infty$ if and only if M is torsion as an R-module.

We now can translate between the language of R-modules (where R is a polynomial ring), Linear Operators and Matrices.

(Torsion Finitely-generated) R-modules	Linear Operators	Matrices
Module V	$S : V \to V$, $S(v) = x \cdot v$	$[S]_{\mathcal{B}}$ as $n \times n$ matrix
Direct sum operation $V_1 \oplus V_2$	$S_1 \oplus S_2 : V_1 \oplus V_2 \to V_1 \oplus V_2$ for $(S_1 \oplus S_2) * v_1, v_2) = (S_1(v_1), S_2(v_2))$	$[S_1 \oplus S_2]_{\mathcal{B}_1 \cup \mathcal{B}_2} = \begin{pmatrix} [S_1]_{\mathcal{B}_1} & 0 \\ 0 & [S_2]_{\mathcal{B}_2} \end{pmatrix}$
Isomorphism $\alpha : V_1 \xrightarrow{\cong} V_2$ for $\alpha(f \cdot v) = f \cdot \alpha(v)$, $f \in R, v \in V_1$	For $S_i : V_i \to V_i$, $S_i(v) = xv$, $S_2 \circ \alpha = \alpha \circ S_1$ commutes: $S_1 \cong S_2$ iff $\exists \alpha : V_1 \to V_2 : \alpha \circ S_1 = S_2 \circ \alpha$	$[S_1]_{\mathcal{B}_1}$ and $[S_2]_{\mathcal{B}_2}$ are similar: $[S_2]_{\mathcal{B}_2} = A \cdot [S_1]_{\mathcal{B}_1} \cdot A^{-1}$ where A is the matrix of α
Cyclic R-module R/fR	$S : V \to V$ is cyclic	Companion Matrix $C(f)$

Figure 4.1: Relationship between (Torsion Finitely-generated) R-modules, Linear Operators and Matrices

Remark 4.7.1 (Cyclic Correspondence). *Without loss of generality, we can write f as a monic polynomial $f = x^n + a_{n-1}x^{n-1} + \cdots + a_1 + a_0 \in F[x]$. There is a canonical map $R = F[x] \to M = R/fR$ by sending $g \mapsto g\bar{g}$.*

We claim that $\{\bar{1}, \bar{x}, \bar{x}^2, \cdots, \bar{x}^{n-1}\}$ is a basis for M. In particular, $\dim_F(M) = n = \deg(f) < \infty$.

For $\bar{g} \in M$, $g = f \cdot q + t$ where $\deg(t) < n$. So $t = b_0 + b_1x + \cdots + b_{n-1}x^{n-1}$, hence $\bar{g} = \bar{f}\bar{q} + \bar{t} = \bar{t} = b_0 \cdot \bar{1} + b_1\bar{x} + \cdots + b_{n-1}\bar{x}^{n-1}$. Moreover, suppose $c_0 \cdot \bar{1} + c_1 \cdot \bar{x} + \cdots + c_{n-1}\bar{x}^{n-1} = 0$. We want to show that $c_i = 0$ for all i. Let $h = c_0 + c_1x + \cdots + c_{n-1}x^{n-1} \in fR$, then $f \mid h$, so $\deg(f) = n > \deg(h)$. Hence, $h = 0$, and so $c_i = 0$ for all i.

Therefore, $\{\bar{1}, \bar{x}, \bar{x}^2, \cdots, \bar{x}^{n-1}\}$ is a basis for $M = R/fR$. Let $S : M \to M$ be the operator $S(\bar{g}) = x\bar{g}$. Therefore, $S(\bar{x}^i = x \cdot \bar{x}^i = \bar{x}^{i+1}$ for $i < n-1$, and $S(\bar{x}^{n-1}) = \bar{x}^n = -a_0 \cdot \bar{1} - a_1 \cdot \bar{x} - \cdots - a_{n-1}\bar{x}^{n-1}$. Moreover, we get the matrix $[S]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$. *This is exactly the companion matrix of f, denoted $C(f)$.*

Theorem 4.7.2 (Invariant Factors and Elementary Divisors for Operators). *Let V be a finite-dimensional vector space over F, and $S : V \to V$ is a linear operator. Then*

1. *(Invariant Factor Form) there exists unique monic polynomials $f_1 \mid f_2 \mid \cdots \mid f_r$ such that the matrix of S in some basis is the block diagonal matrix of the form diag$(C(f_1), C(f_2), \cdots, C(f_r))$. This matrix is then unique. This is called the canonical form of S.*

2. *(Elementary Divisor Form) there exists polynomials $p_1^{k_1}, p_2^{k_2}, \cdots, p_s^{k_s}$ (unique up to permutation) where p_i's are monic irreducible polynomials, such that the matrix of S in some basis is of the form diag$(C(p_1^{k_1}), C(p_2^{k_2}), \cdots, C(p_s^{k_s}))$.*

Theorem 4.7.3. *Let A be an $n \times n$ matrix over a field F. Then*

1. *(Invariant Factor Form) there are unique monic polynomials $f_1 \mid f_2 \mid \cdots \mid f_r$ such that A is similar to diag$(C(f_1), C(f_2), \cdots, C(f_r))$, which is called the canonical form of A.*
2. *(Elementary Divisor Form) there are $p_1^{k_1}, p_2^{k_2}, \cdots, p_s^{k_s}$ unique up to permutation such that AA is similar to the block diagonal matrix diag$(C(p_1^{k_1}), C(p_2^{k_2}), \cdots, C(p_s^{k_s}))$.*

Remark 4.7.4. *Let A be an $n \times n$ matrix over the field. How to find its canonical form?*

Correspondingly, there is a matrix $x \cdot I_n - A = \begin{pmatrix} x - a_{11} & a_{12} & \cdots & -a_{1n} \\ -a_{21} & x - a_{22} & \cdots & -a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & x - a_{nn} \end{pmatrix}$ *over* $R = F[x]$.

The determinant $\det(xI_n - A) = p_A(x)$ *is called the characteristic polynomial of A, and is monic of degree n. It is also equivalent to the product of all invariant factors.*

Consider the submodule $N \subseteq R^n$, generated by the columns of $xI_n - A$.

Lemma 4.7.5. $\dim_F(R^n/N) = n$.

Proof. Denote $F = R^n \supseteq N$. Let y_i be the i-th column of $xI_n - A$. We want to find the invariant factors of the factor module R^n/N.

By elementary transformations, we can transform $xI_n - A$ to $\mathrm{diag}(f_1, f_2, \cdots, f_n)$. (Indeed, elementary transformations only change the determinant by a scalar.) Then $p_A(x) = f_1 f_2 \cdots f_n$ and $n = \deg(p_A) = \sum \deg(f_i) = \sum \dim(R/f_iR) = \dim_F(R^n/N)$ since $R^n/N \cong R/f_1R \oplus R/f_2R \oplus \cdots \oplus R/f_nR$. Thus, the invariant factors of R^n/N are exactly $\{f_1, f_2, \cdots, f_n\}$, where $f_1 \mid f_2 \mid \cdots \mid f_n$. □

Now, suppose $S : V \to V$ is a linear operator on vector space V, and choose a basis $\{v_1, \cdots, v_n\}$ for V. Let $A = [S]_{\mathcal{B}}$. We define $g : R^n \to V$ such that $g(f_1, f_2, \cdots, f_n) = f_1(S)(v_1) + f_2(S)(v_2) + \cdots + f_n(S)(v_n)$. This is a R-module homomorphism.

Now, if we apply the first column of $xI_n - A$, we get $g(x - a_{11}, -a_{21}m \cdots, -a_{n1}) = S(v_1) - a_{11} - a_{21}v_1 - \cdots - a_{n1}v_n = 0$. This is true for any column of $xI_n - A$. Therefore, $g(N) = 0$ where N is the submodule generated by the columns. Hence, $N \subseteq \ker(g)$, and so g factors as $g : R^n \to R^n/N \twoheadrightarrow V$. By lemma, R^n/N is n-dimensional, and V is also n-dimensional. Therefore, $h : R^n/N \to V$ is an isomorphism between R-modules.

The goal now is to find the invariant factors of this module V, which is the same as looking for the invariant factors of R^n/N. We can do some by performing elementary transformations on $xI_n - A$, and get a diagonal matrix of the form diag$(f_1, f_2, \cdots, f_n)$ where $f_1 \mid f_2 \mid \cdots \mid f_n$ are monic polynomials, and $V \cong R/f_1R \oplus R/f_2R \oplus \cdots \oplus R/f_nR$. However, note that some of the f_i's are units. WLOG say $f_1 = f_2 = \cdots = f_k = 1$ and $\deg(f_m) > 0$ for all $m > k$. Therefore, the invariant factors of S (or invariant factors of A, or invariant factors of V) are just $\{f_{k+1}, \cdots, f_n\}$.

Example 4.7.6. *1. Let* $A = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}$. *Then* $xI_2 - A = \begin{pmatrix} x & 2 \\ 1 & x-3 \end{pmatrix}$. *By elementary operations, this matrix can be transformed into the form* $\begin{pmatrix} 1 & 0 \\ 0 & x^2-3x+2 \end{pmatrix}$. *Therefore, the invariant factors of* A *is* $\{x^2 - 3x + 2\}$ *as* 1 *is a unit.*

The canonical form of the matrix is just the companion matrix $C(x^2 - 3x + 2) = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$. *Note that this is similar to matrix* A.

2. Find representatives of conjugacy classes in $G = GL_2(\mathbb{Z}/p\mathbb{Z})$ *where* p *is a prime. Let* $F = \mathbb{Z}/p\mathbb{Z}$, *and note that* $G = (p^2 - 1)(p^2 - p)$. *Take a* 2×2 *matrix* $A \in G$, *then the matrix is invertible with determinant nonzero. We take the invariant factors of* A. *We know that up to conjugacy, matrix* A *is uniquely determined by the factors* $f_1, f_2, \cdots, f_s$ *(non-constant monic polynomials such that* $f_1 \mid f_2 \mid \cdots \mid f_s$.

Recall that $p_A(x)$ *is the product of invariant factors, so* n *is the sum of degrees of the invariant factors. Therefore, the sum of degrees* $f_1, \cdots, f_s$ *is* 2. *Also, given that* $\det(A) \neq 0$ *and* $\det(A) = \pm p_A(0)$, *so* $p_A(0) \neq 0$, *i.e.* $f_i(0) \neq 0$ *for all* i. *There are two cases. Either 1) there is only one invariant factor* $f_1 = x^2 + ax + b$ *for* $b \neq 0$, *or 2) there are two invariant factors* f_1, f_2, *so* $f_1 = f_2 = x + c$ *for* $c \neq 0$. *The first case has* $p(p-1)$ *classes and the second case has* $p - 1$ *classes. Therefore, there are* $p^2 - 1$ *conjugacy classes in* G.

In the first case, the representation is given by $\begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix}$, *where* $a \in F$ *and* $0 \neq b \in F$. *In the second case, the representation is given by* $\begin{pmatrix} -c & 0 \\ 0 & -c \end{pmatrix}$ *where* $0 \neq c \in F$.

Remark 4.7.7. *Let* V *be a vector space as an* R*-module with operator* $S : V \to V$. *Consider* $\{f \in R : f \cdot V = 0\}$, *i.e. having* $f(S)(V) = 0$. *This set is called the annihilators of* V, *denoted* $\boldsymbol{Ann}(V) \subseteq R$ *as an ideal. Hence, it can be generated by one element* $0 \neq f_{min} \cdot R$ *which is monic. This is called the minimal polynomial.*

Note that $f_{min} \cdot V = 0$, *and if* $g \cdot V = 0$ *is annihilator, then* $f_{min} \mid g$.

Now, the invariant factors of V *are* $f_1, f_2, \cdots, f_s$ *and* $V = \coprod_{i=1}^{s} R/f_iR$ *and* $Ann(R/f_iR) = f_iR$, *where* $f_1 \mid f_2 \mid \cdots \mid f_s$. *In particular,* $f_{min} = f_s$.

Example 4.7.8. *Classify* 4×4 *matrices over* $\mathbb{R}$ *such that* $(A - 3I)^2 = 0$.

The invariant factors of V *should look like* $f_1, \cdots, f_s$. *Now,* $f_s = f_{min} \mid (x-3)^2$. *Moreover, the sum of degrees of invariant factors are just* 4.

If $f_s = (x-3)^2$, *then the collection can be* $\{(x-3)^2, (x-3)^2\}$ *or* $\{x-3, x-3, (x-3)^2\}$. *If* $f_s = x - 3$, *then the collection should be* $\{x-3, x-3, x-3, x-3\}$.

For $\{(x-3)^2, (x-3)^2\}$*, the corresponding matrix is given by* $\begin{pmatrix} 0 & -9 & 0 & 0 \\ 1 & 6 & 0 & 0 \\ 0 & 0 & 0 & -9 \\ 0 & 0 & 1 & 6 \end{pmatrix}$. *For* $\{x-3, x-3, (x-3)^2\}$*, the corresponding matrix is given by* $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -9 \\ 0 & 0 & 1 & 6 \end{pmatrix}$. *For* $\{x-3, x-3, x-3, x-3\}$*, the corresponding matrix is given by* $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

Therefore, every matrix satisfying the conditions in the example is similar (conjugate) to one of these three matrices.

Remark 4.7.9. *Suppose* A *has invariant factors* $f_1, f_2, \cdots, f_s$*. Then*

1. $f_1 \mid f_2 \mid \cdots \mid f_s$.
2. $\prod f_i = p_A$.
3. $f_s = f_{min}$.
4. p_A *and* f_s *has the same irreducible factors. It follows that* $f_{min} \mid p_A$.
5. *The invariant factors of* A *does not depend on the base field. In particular, if* $L \supseteq F$ *are fields, then the invariant factors of* A *over* F *should be the same as the invariant factors of* A *over* L.

Example 4.7.10. *Let* A *and* B *be matrices of* F*, with* $L \supseteq F$*. Then* $A \sim B$ *over* F *if and only if* $A \sim B$ *over* L.

4.8 Jordan Canonical Form

Even though we haven't really talked about elementary divisors, they are particularly useful in Jordan canonical form.

Recall that for $A : V \to V$, $\lambda \in F$ is an eigenvalue of A if $Av = \lambda v$ for some $0 \neq v \in V$. (They are exactly the roots of the characteristic polynomial p_A.) Every $v \in V$ such that $Av = \lambda v$ is called an eigenvector of A for the eigenvalue λ. We then have $E_\lambda = \{\text{eigenvalues of } A\} \subseteq V$ called the eigenspace of A with respect to λ.

Proposition 4.8.1. *The following are equivalent:*

1. *A is diagonalizable.*

2. *There exists a basis of eigenvectors.*

3. *V is a direct sum of all eigenspaces, i.e. $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$.*

4. *All elementary divisors of A are linear.*

5. *All invariant factors of A are products of distinct linear polynomials;*

6. *f_{min} is a product of distinct linear polynomials.*

In this case, the characteristic polynomial is split, i.e. it is a product of linear factors.

Proof. Linear Algebra. □

Example 4.8.2. *The following are equivalent:*

1. *V is cyclic.*

2. *The set of invariant factors is a singleton $\{f\}$. In particular, $p_A = f$.*

3. *$f_{min} = p_A$.*

4. *All elementary divisors are pairwise relatively prime.*

Let $S : V \to V$ be a linear operator. Assume that p_S is split, so $p_S(x) = \prod_{i=1}^{n}(x - \lambda_i)$ where $n = \dim(V)$. As the product of elementary divisors is p_S, then every elementary divisor is of the form $(x-\lambda)^k$, where $\lambda = \lambda_i$ for some i. Then we examine the cyclic summand $M = R/(x-\lambda)^k R$ where R is the polynomial ring. We would like to find a basis of the vector space. An obvious basis is $\bar{1}, \bar{x}, \bar{x}^2, \cdots, \bar{x}^{k-1}$ for M. Another basis is $\bar{1}, \overline{x-\lambda}, \cdots, \overline{(x-\lambda)^{k-1}}$, where we consider $y = x - \lambda$. In particular, $x \cdot \overline{(x-\lambda)^i} = (x-\lambda)\overline{(x-\lambda)^i} + \lambda\overline{(x-\lambda)^i} = \overline{(x-\lambda)^{i+1}} + \lambda(x \bar{-} \lambda)^i$, and $\overline{(x-\lambda)^k} = 0$, so $x\overline{(x-\lambda)^{k-1}} = \lambda \cdot \overline{(x-\lambda)^{k-1}}$. Now the matrix S in the new basis is given by $\begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \lambda \end{pmatrix}$. This matrix is denoted $J(\lambda, k)$, which is a $j \times j$ with respect to eigenvalue λ, and is called a Jordan block.

Theorem 4.8.3 (Jordan Canonical Form)**.** *Let $S : V \to V$ be a linear operator in a finite-dimensional vector space V. Assume that the characteristic polynomial p_S is split. Then there is a basis $\mathcal{C}$ for V such that $[S]_{\mathcal{C}} = diag(J(\lambda_1, k_1), J(\lambda_2, k_2), \cdots, J(\lambda_s, k_s))$. The Jordan blocks $J(\lambda_i, k_i)$ are uniquely determined up to permutation. The matrix is called the Jordan canonical form of S.*

5 Field Theory

5.1 Field Extensions

Proposition 5.1.1. *Every field homomorphism is injective.*

Proof. Suppose $\alpha : F \to K$ is a field homomorphism, then $\ker(\alpha) \subset F$ is a non-trivial ideal. Therefore, α is injective. $\square$

Remark 5.1.2. *In particular, F is isomorphic to the subfield $\alpha(F) \subseteq K$.*

Definition 5.1.3 (Field Extension)**.** *Let $F \subseteq K$ be a subfield. We say that K is an extension of F and write K/F.*

If K/F is a field extension, then $F \hookrightarrow K$ is an embedding, i.e. an injective field homomorphism. Conversely, if $\alpha : F \to K$ is a field homomorphism, then we can identify F as a subfield of K. Specifically, we have $F \cong \alpha(F) \subseteq K$, and $K/\alpha(F)$ is a field extension, i.e. K/F is a field extension.

There is an obvious category of fields, which is a subcategory of the category of rings. However, we can get a different taste of a category on fields.

Definition 5.1.4 (Category of Field Extensions)**.** *Let F be a field. The category of field extensions of F has objects as field extensions K/F and morphisms from K/F to L/F is a field homomorphism $\alpha : K \to L$ that is the identity homomorphism on subfield F, i.e. $\alpha(x) = x$ for all $x \in F$.*

Equivalently, the objects are field homomorphisms $F \to K$ for fixed F, and morphisms between two field homomorphisms $F \to K$ and $F \to L$ are field homomorphisms $K \to L$ such that the related diagrams commute:

$$\begin{tikzcd} & F \arrow[dl] \arrow[dr] & \\ K \arrow[rr, "\alpha"'] & & L \end{tikzcd}$$

We denote this category as Fields$/F$.

Suppose K/F is a field extension, then K is a module over itself, and is then a module over F (as a vector space). We denote $[K : F] = \dim_F(K)$ as the degree of K over F.

Example 5.1.5. *1. $[K : F] = 1$ if and only if $K = F$, and we call F/F as the trivial extension.*

2. $\mathbb{C}/\mathbb{R}$ has a basis $\{1, i\}$ for $\mathbb{C}$ over $\mathbb{R}$, so $[\mathbb{C} : \mathbb{R}] = 2$.

3. *Note* $[\mathbb{R}:\mathbb{Q}]=\infty$ *because the extension does not have a finite basis.*

Proposition 5.1.6. *Let $L/K/F$ be field extensions. Then $L:K]=[L:K]\cdot[K:F]$. We can read this even if some terms are ∞. In particular, the extension L/F is finite if and only if L/K and K/F are finite.*

Proof. Let us choose a basis $\{x_i\}_{i\in I}$ for K/F, so $x_i\in K$, and another basis $\{y_j\}_{j\in J}$ for L/K, so $x_j\in L$.

Claim 5.1.7. $\{x_iy_j\}_{i\in I,j\in J}$ *is a basis for* L/F.

Subproof. Suppose $\sum\limits_{x\in I,j\in J} a_{ij}x_iy_j=0$ for $a_{ij}\in F$. Now $\sum\limits_{y\in J}(\sum\limits_{i\in I}a_{ij}x_i)y_j=0$ where $\sum\limits_{i\in I}a_{ij}x_i\in K$. However, since y_j's are linearly independent over K, then $\sum\limits_{i\in I}a_{ij}x_i=0$ for all j, and since x_i's are linearly independent over F, then $a_{ij}=0$ for all i,j. Hence, they are linearly independent. We now have to show that they generate the whole space.

Let $v\in L$, because y_j's generate L over K, then $v=\sum\limits_{j\in J}u_jy_j$ for some $u_j\in K$. Now since x_i's generate K over F, then every $u_j=\sum\limits_{x\in I}a_{ij}x_i$ for some $a_{ij}\in F$. Now $v=\sum\limits_{i,j}a_{ij}x_iy_j$. This concludes the proof of the claim. ■

The statement automatically follows from the claim.

□

Corollary 5.1.8. *If $L/K/F$ are finite, then $[K:F]\mid[L:F]$ and $[L:K]\mid[L:F]$.*

Example 5.1.9. *For $L/K/F$, suppose $[L:F]=p$ is prime, so either $[K:F]=1$ or $[L:K]=1$, so either $K=F$ or $L=K$.*

Corollary 5.1.10. *If $F_1\subseteq F_2\subseteq\cdots\subseteq F_n$ is a tower of field extensions, then $[F_n:F_1]=\prod\limits_{i=1}^{n-1}[F_{i+1}:F_i]$.*

Lemma 5.1.11. *Let K be a field and $S\subseteq K$ is a subset. Then there is a unique smallest subfield of K containing S.*

Proof. Take the intersection of all subfields of K containing S. Note that this is still a field. □

Definition 5.1.12. *Let K/F be a field extension and $T\subseteq K$ is a subset. Denote set $S=T\cup F$. We can denote $F(T)$ as the smallest subfield of K containing S. Note that $F\subseteq F(T)\subseteq K$, then $F(T)$ is the smallest subfield of K containing F and T, and called the field generated by T over K.*

Suppose T is finite, i.e. $T=\{\alpha_1,\cdots,\alpha_n\}$. Then we can write $F(T)=F(\alpha_1,\cdots,\alpha_n)$.

Lemma 5.1.13. *Let K/F be a field extension, and let $\alpha_1,\cdots,\alpha_n\in K$. Then*

$$F(\alpha_1,\cdots,\alpha_n)=\{\frac{f(\alpha_1,\cdots,\alpha_n)}{g(\alpha_1,\cdots,\alpha_n)}:f(\alpha_1,\cdots,\alpha_n),g(\alpha_1,\cdots,\alpha_n)\in F[x_1,\cdots,x_n],g(\alpha_1,\cdots,\alpha_n)\neq 0\}.$$

Proof. Let L denote the set on the right hand side. Note that L is a field containing F. By definition, $F(\alpha_1, \cdots, \alpha_n) \subseteq L$.

On the other hand, note $\alpha_i = \frac{\alpha_i}{1} \in F(\alpha_1, \cdots, \alpha_n)$, then $\frac{f(\alpha_1,\cdots,\alpha_n)}{g(\alpha_1,\cdots,\alpha_n)} \in F(\alpha_1, \cdots, \alpha_n)$, and so $L \subseteq F(\alpha_1, \cdots, \alpha_n)$. □

We can now define a similar structure.

Definition 5.1.14. *For K/F field extension, let $\alpha_1, \cdots, \alpha_n \in K$, then $F[\alpha_1, \cdots, \alpha_n] = \{f(\alpha_1, \cdots, \alpha_n : f(x_1, \cdots, x_n) \in F[x_1, \cdots, x_n]\}$ is a ring (and may not be a field).*

Note that $F \subseteq F[\alpha_1, \cdots, \alpha_n] \subseteq F(\alpha_1, \cdots, \alpha_n)$.

Remark 5.1.15. *$F[\alpha_1, \cdots, \alpha_n] = F(\alpha_1, \cdots, \alpha_n)$ if and only if $F[\alpha_1, \cdots, \alpha_n]$ is a field.*

Example 5.1.16.

1. *Let x be a variable over F. So $F \subseteq F[x] \subseteq F(X) = K$, so $F[x]$ is the polynomial ring (but not a field), and $K = F(x)$ is the ring of rational functions, which is a field. Here $T = \{x\}$.*
2. *Consider $\mathbb{C}/\mathbb{R}$. Take $T = \{i\}$. Then $\mathbb{R}[i] = \{a + bi : a, b \in \mathbb{R}\} = \mathbb{C}$ is a field, and so $\mathbb{R}[i] = \mathbb{C} = \mathbb{R}(i)$.*

Definition 5.1.17 (Algebraic, Transcendental)**.** *Suppose K/F is a field extension, then $\alpha \in K$ is called algebraic over F if there exists a nonzero polynomial $f \in F[x]$ such that $f(\alpha) = 0$.*

If α is not algebraic, then α is called transcendental over F.

A field extension K/F is algebraic if every element $\alpha \in K$ is algebraic over F.

Example 5.1.18.

1. *$\alpha \in F$ is algebraic over F, because $x - \alpha \in F[x]$.*
2. *Suppose $\alpha \in L/K/F$. If α is algebraic over F, then α is algebraic over K: $f \in F[x] \subseteq K[x]$.*
3. *If $\alpha \in K$ is transcendental over F, then $F[\alpha] \cong F[x]$. More precisely, $F[x] \to F[\alpha]$ sending $g \mapsto g(\alpha)$ is an isomorphism. Moreover, $F(x) \cong F(\alpha)$, and α plays the role of a variable.*
4. *$x \in F(x)$ is transcendental over F.*

Theorem 5.1.19. *Let $\alpha \in K/F$ be algebraic over F. Then*

1. *There is a unique monic irreducible polynomial $m_\alpha \in F[x]$ such that $m_\alpha(\alpha) = 0$.*
2. *If $f(\alpha) = 0$ for $f \in F[x]$, then $m_\alpha \mid f$.*
3. *The elements $1, \alpha, \alpha_2, \cdots, \alpha^{n-1}$, where $n = \deg(m_\alpha)$ form a basis for the extension $F(\alpha)$ over F. In particular, $[F(\alpha) : F] = \deg(m_\alpha)$.*
4. *$F(\alpha) = F[\alpha]$. In particular, this holds if and only if α is algebraic.*

Proof. Consider $\varphi : F[x] \to K$ given by $\varphi(g) = g(\alpha)$. Then $\text{im}(\varphi) = F[\alpha]$. Now $\ker(\varphi) \subseteq F[x]$ is a nonzero ideal, and is generated by one element, i.e. $\ker(\varphi) = m_\alpha \cdot F[x]$, where m_α is monic. Now every $f \in F[x]$ such that $f(\alpha) = 0$ is contained in $f \in \ker(\varphi)$, and so $m_\alpha \mid f$. This proves 2). Now, the factor ring $F[x]/m_\alpha \cdot F[x] \cong \text{im}(\varphi) \subseteq K$ as a subring. Since K is a field, then it is a domain, and so $\text{im}(\varphi)$ is a domain, so the factor ring is a domain, and so the ideal is prime, hence m_α is irreducible. This proves 1). For the map $F[x]/m_\alpha \cdot F[x] \to \text{im}(\varphi) \subseteq K$, we have that $\bar{x} \mapsto \alpha$. We know that $\bar{1}, \bar{x}, \cdots, \bar{x}^{n-1}$ is a basis of the factor ring, and so it is a basis for the image of φ. In particular, $1, \alpha, \alpha^2, \cdots, \alpha^{n-1}$ forms a basis for $F[\alpha]$ over F. Because $m_\alpha \cdot F[x]$ is a nonzero prime ideal, so it is maximal. Hence, the factor ring is a field, and so the image $F(\alpha) = \text{im}(\varphi)$ is a field. Therefore, $F[\alpha] = F(\alpha)$. $\square$

Remark 5.1.20. *This unique monic irreducible polynomial m_α is called the minimal polynomial of α over F. The degree of the extension is then determined by the degree of the minimal polynomial. The degree of this element α is just the degree of the polynomial, i.e.* $\deg(\alpha) = \deg(m_\alpha)$.

Remark 5.1.21. *Given $\alpha \in K/F$, we want to know how to find the minimal polynomial. In particular, we want to find some polynomial $f \in F[x]$ such that $f(\alpha) = 0$. Moreover, if f is not irreducible, i.e. $f = gh$ as monic non-constant polynomials, then either $g(\alpha) = 0$ or $h(\alpha) = 0$, and by continuing the factorization, we can find the minimal polynomial.*

We saw before that $F(\alpha) \cong F[x]/m_\alpha \cdot F[x]$. We can reverse the procedure as follows: suppose $m \in F[x]$ is a monic irreducible polynomial, then it generates prime (and therefore maximal) ideal. Therefore, the residual ring $F[x]/m \cdot F[x]$ is a field because the ideal is maximal. Moreover, consider $F \hookrightarrow F[x]/m \cdot F[x]$ which is an embedding. If we denote $K = F[x]/m \cdot F[x]$, then K/F is a field extension. Take $\alpha = \bar{x} \in K$. Then $m(\alpha) = 0$, and m is monic irreducible, therefore $m = m_\alpha$ is the minimal polynomial of α. Moreover, α generates the field: $F[\alpha] = K = F(\alpha)$. The extension degree is thus given by $[K : F] = \deg(m)$.

Example 5.1.22.

1. *$\mathbb{C} = \mathbb{R}[i] = \mathbb{R}(i)$ where $i^2 + 1 = 0$. The polynomial $x^2 + 1$ is irreducible in $\mathbb{R}[x]$, therefore, this is isomorphic to the factor ring $\mathbb{R}[x]/(x^2+1)\mathbb{R}[x]$. The degree of i is the degree of the polynomial, which is 2.*

2. *What is $\mathbb{Q}(\sqrt{3})/\mathbb{Q}$? Note that $\sqrt{3}$ is a root of $x^2 - 3$ over $\mathbb{Q}$, which is a irreducible polynomial, so the degree of extension is 2, with $\deg(\sqrt{3}) = 2$. Degree 2 extensions are also called quadratic extensions.*

3. *Let p be a prime integer. Denote $\xi_p = \cos(\frac{2\pi}{p}) + i \cdot \sin(\frac{2\pi}{p})$ where $(\xi_p)^p = 1$ and $\xi_p \neq 1$. In particular, ξ_p is a root of $x^p - 1 = (x-1)(x^{p-1} + \cdots + x + 1)$, and therefore it is a root of $x^{p-1} + \cdots + x + 1$. By Eisenstein's criterion, this polynomial is irreducible over $\mathbb{Q}$, so it is the minimal polynomial of ξ_p, i.e. $m_{\xi_p} = x^{p-1} + \cdots + x + 1$, and $[\mathbb{Q}(\xi_p) : \mathbb{Q}] = p - 1$.*

Corollary 5.1.23. *Let $\alpha \in K/F$. Then α is algebraic over F if and only if $[F(\alpha) : F]$ is finite.*

Proof. ($\Rightarrow$) is true by the theorem.

($\Leftarrow$): consider the elements $1, \alpha, \alpha^2, \cdots, \alpha^n$ which are linearly dependent for large enough n, i.e. $n \geq [F(\alpha) : F]$. Therefore, $\sum_{i=0}^{n} a_i \alpha^i = 0$ for some nontrivial combination $a_i \in F$. Therefore, α is algebraic over F. □

Corollary 5.1.24. *A finite field extension is algebraic, i.e. all elements in this extension are algebraic over the base field.*

Proof. Take $\alpha \in K/F$. The extension generated is $F(\alpha) \subseteq K$ and with $[K : F] < \infty$. Therefore, $[F(\alpha) : F] < \infty$. By the previous corollary, α is algebraic over F. □

Corollary 5.1.25. *Let $\alpha_1, \cdots, \alpha_n \in K/F$ be algebraic over F. Then $F(\alpha_1, \cdots, \alpha_n) = F[\alpha_1, \cdots, \alpha_n]$, and this is a finite field extension of F. In particular, $F(\alpha_1, \cdots, \alpha_n)/F$ is algebraic.*

Proof. The last statement simply follows the first two statements. We now prove by induction on n.

Case $n = 1$: this is true by the theorem.

Suppose this is true for case $n - 1$, we now show the case at n. Now α_n is algebraic over F, and so it is algebraic over $F(\alpha_1, \cdots, \alpha_{n-1})$, which is equivalent to $F[\alpha_1, \cdots, \alpha_{n-1}]$ by induction hypothesis. Therefore, we know that $F(\alpha_1, \cdots, \alpha_{n-1})(\alpha_n) = F(\alpha_1, \cdots, \alpha_n) = F(\alpha_1, \cdots, \alpha_{n-1})[\alpha_n] = F[\alpha_1, \cdots, \alpha_{n-1}][\alpha_n] = F[\alpha_1, \cdots, \alpha_n]$ by induction.

Therefore, we obtain the extension $F(\alpha_1, \cdots, \alpha_n)/F(\alpha_1, \cdots, \alpha_{n-1})/F$, which are both finite, and so the tower of finite extension is finite. □

Theorem 5.1.26. *Let K/F be a field extension. Then the set $E \subseteq K$ of all algebraic over F elements is a subfield of K containing F.*

Proof. Suppose $\alpha, \beta \in E$, then $F(\alpha, \beta)/F$ is an algebraic extension. Note that $\alpha + \beta, \alpha\beta, \alpha^{-1} \in F(\alpha, \beta)/F$, and so E is generated as a field. □

Theorem 5.1.27. *Let L/K and K/F be field extensions. Then L/F is algebraic if and only if L/K and K/F are algebraic.*

Proof. ($\Rightarrow$): since $K \subseteq L$, then K/F is algebraic. Take $\alpha \in L/F$, then it is algebraic, so $\alpha \in L/K$ is also algebraic.

($\Leftarrow$): take $\alpha \in L$. By assumption, it is algebraic over K. Now there exists nonzero polynomial $f = \sum_{i=0}^{n} \beta_i x^i \in K[x]$ such that $f(\alpha) = 0$. Take $E = F(\beta_1, \cdots, \beta_n)$, which is generated by finitely many algebraic elements over F, so it is algebraic over F. In particular, $[E : F] < \infty$. Note that $\alpha \in L$ is algebraic over E since $f \in E[x]$, and so $[E(\alpha) : E] < \infty$. Therefore, $[F(\alpha) : F] \leq [E(\alpha) : F] = [E(\alpha) : E] \times [E : F]$, but both field extension degrees are finite, so α is algebraic over F. □

Property 5.1.28. *A property $\mathcal{P}$ of field extensions is "good" if for field extensions $L/K/F$, $\mathcal{P}(L/F)$ holds if and only if $\mathcal{P}(L/K)$ and $\mathcal{P}(K/F)$ hold.*

In particular, the algebraic property $\mathcal{P}$ = algebraic is good.

Theorem 5.1.29. *Let $f \in F[x]$ be a non-constant polynomial. Then there exists a field extension K/F such that $[K : F] \leq \deg(f)$ and f has a root in K.*

Proof. We proved the case when f is irreducible. For general f, there exists a irreducible polynomial $g \mid f$. Then take $K = F[x]/gF[x]$, then g has a root in K, and hence f has a root in K and the degree of extension $[K : F] = \deg(g) \leq \deg(f)$. □

Corollary 5.1.30. *Let $f \in F[x]$ be a non-constant polynomial. Then there is a field extension K/F such that $[K : F] \leq \deg(f)!$ and f is split over K.*

Proof. This can be done by induction on the degree of f. It is trivial if the degree is 1: take $K = F$. For the induction step, by the theorem, we find a field extension L/F such that $[L : F] \deg(f)$ and f has a root $\alpha \in L$. Then we write $f = (x - \alpha) \cdot g$, so $g \in L[x]$ has degree $\deg(g) = \deg(f) - 1$. By induction, there exists a field extension K/L such that g is split over K and $[K : L] \leq \deg(g)!$. Therefore, f is split over K and $[K : F] = [K : L] \times [L : F] \leq \deg(g)! \times \deg(f) \leq \deg(f)!$. □

Definition 5.1.31 (Splitting Field)**.** *Let $f \in F[x]$ be a non-constant polynomial. A field extension K/F is called a splitting field of f (over F) if*

1. *f is split over K, i.e. $f = a \cdot (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$ where $a \in F$ and $\alpha_i \in K$ are all roots of f in K.*
2. *$K = F(\alpha_1, \cdots, \alpha_n)$.*

Example 5.1.32.

1. *If f is split over F, then F/F is a splitting field.*
2. *Suppose $f = x^3 - 1$ over $F = \mathbb{Q}$. Note $x^3 = (x-1)(x^2+x+1)$, where x^2+x+1 is irreducible over F. The roots are exactly $\frac{-1\pm\sqrt{-3}}{2}$. Therefore, $\mathbb{Q}(\sqrt{-3})/\mathbb{Q}$ is the splitting field for f.*

Proposition 5.1.33. *A non-constant polynomial $f \in F[x]$ has a splitting field of degree at most $\deg(f)!$.*

Proof. Similar as above, we find a field extension K/F of degree at most $\deg(f)!$ such that f is split over K. Let $\alpha_1, \cdots, \alpha_n$ are roots of f in K. Then $L = F(\alpha_1, \cdots, \alpha_n)$ is a splitting field and $L \subseteq K$. Therefore, $[L : K] \leq \deg(f)!$. □

Remark 5.1.34. *If K/F is a field extension such that f is split over K, then K contains a unique splitting field of F. Indeed, the field is the only splitting field inside K.*

Remark 5.1.35 (Irreducibility of polynomials of small degree)**.** *If $\deg(f) = 2$ and α is a root of f, then $f = (x - \alpha)(ax + b)$, so f is split. Therefore, a degree 2 polynomial f is split, if and only if f has a root, if and only if f is not irreducible. Similar results hold for polynomials of degree 3.*

Definition 5.1.36. *Suppose K/F and K_1/F_1 are two field extensions. Suppose we have a field homomorphism $\varphi : F \to F_1$. A field homomorphism $\psi : K \to K_1$ is called an extension of φ if $\psi(a) = \varphi(a)$ for all $a \in F$.*

Suppose further that $f = a_n x^n + \cdots + a_1 x + a_0 \in F[x]$. We can denote $\varphi(f) = \varphi(a_n) \cdot x^n + \cdots + \varphi(a_1) \cdot x + \varphi(a_0) \in F_1[x]$.

Proposition 5.1.37. *Let $K = F(\alpha)/F$ be a finite field extension. Let $f = m_\alpha \in F[x]$ be the minimal polynomial of α. Suppose $\varphi : F \to F_1$ is a field homomorphism and K_1/F_1 is a field extension as above. Then*

1. *if $\psi : K \to K_1$ is an extension of φ, then $\psi(\alpha)$ is a root of the polynomial $\varphi(f) \in F_1[x]$.*

2. *For any root α_1 of $\varphi(f)$ in K_1, there exists a unique extension $\psi : K \to K_1$ of φ such that the image $\psi(\alpha) = \alpha_1$.*

Proof.

1. Since $f(\alpha) = 0$, we denote $f = a_n x^n + \cdots + a_1 x + a_0$ for $a_i \in F$ and apply ψ and get $\varphi(f)(\psi(f)) = 0$, therefore $\psi(\alpha)$ is a root of $\varphi(f)$.

2. Let $\psi' : F[x] \to K_1$ be the evaluation of polynomial at α_1, i.e. $\psi'(g) = \varphi(g)(\alpha_1)$, then $\psi'(f) = \varphi(f)(\alpha_1) = 0$. Therefore, $f \in \ker(\psi')$. Hence, ψ' factors $\psi : F[x]/f \cdot F[x] \to K_1$. Note that $F[x]/f \cdot F(x) \cong K = F(\alpha)$, so $\psi(\alpha) = \psi'(x) = \alpha_1$.

□

Corollary 5.1.38. *Given the setting in the proposition above, the number of extensions of φ is at most* $\deg(f) = \deg(\alpha) = [K : F]$.

Theorem 5.1.39. *Let K/F be a splitting field of a nonconstant polynomial $f \in F[x]$ and $\varphi : F \to F_1$ is a field isomorphism. Let K_1/F_1 be a splitting field of $\varphi(f) \in F_1[x]$. Then there exists a field isomorphism $\psi : K \to K_1$ that extends φ.*

Proof. We prove by induction on $n = \deg(f)$.

If $n = 1$, then the polynomial is linear and thus split, so $K = F$ and $K_1 = F_1 <$ then $\psi = \varphi$.

Suppose the theorem is true for case $n - 1$, we want to show the case for n. Let $\alpha \in K$ be a root of f. Therefore, $f = (x - \alpha) \cdot g$ for some $g \in F(\alpha)[x]$. Let m_α be the minimal polynomial of α over F. In particular, $m_\alpha \mid f$. Therefore, $\varphi(n_\alpha) \mid \varphi(f)$. Since $\varphi(f)$ is split over K_1 by assumption, then $\varphi(m_\alpha)$ is also split over K_1. Take a root α_1 of $\varphi(m_\alpha)$ in K_1. By the proposition above, there exists a field homomorphism $\varphi' : F(\alpha) \to F_1(\alpha_1)$ extending φ such that $\varphi(\alpha) = \alpha_1$. The map φ' is clearly surjective because α is mapped to α_1. It is also injective since it is a field homomorphism. Therefore, φ' is a field isomorphism. Now $\varphi(f) = \varphi'((x - \alpha) \cdot g) = \varphi'(g)$, so $\varphi'(g) \in F_1(\alpha_1)[x]$. Observe that $g \mid f$ is split over K because f is split over K. Moreover, the roots of f in K are the same as the roots of $g \cup \{\alpha\}$. Therefore, the field K is generated by all roots of g in K over $F(\alpha)$, since K is generated over F by all roots of f. Therefore, $K/F(\alpha)$ is a splitting field of g. Similarly,

$K_1/F_1(\alpha_1)$ is a splitting field of $\psi'(g)$ for $\psi' : F(\alpha \xrightarrow{\cong} F_1(\alpha_1)$. By applying the inductive hypothesis over $\psi' : F(\alpha) \to F_1(\alpha_1)$ with $g \in F(\alpha)[x]$, we conclude that ψ' extends to an isomorphism of splitting fields $\psi : K \xrightarrow{\cong} K_1$. Since ψ extends to ψ' and ψ' extends to φ, then ψ extends to φ. $\square$

Remark 5.1.40. *We can restate the theorem as the following. For a base field F, there is a category of field extensions over F. Two elements of this category are K/F and K_1/F. Then K/F and K_1/F are isomorphic if there exists $\psi : K \to K_1$ such that $\psi(a) = a$ for all $a \in F$. Equivalently, we say that ψ extends the identity isomorphism from F to itself.*

Theorem 5.1.41. *Let $f \in F[x]$ be a non-constant polynomial and K/F and K_1/F are two splitting fields of the polynomial. Then K/F and K_1/F are isomorphic over F.*

Proof. Apply the previous theorem to the case where $\varphi = \mathbf{id}_F$ and $F_1 = F$. $\square$

5.2 Finite Fields

Definition 5.2.1 (characteristic)**.** *The characteristic of a field F is the smallest positive integer n such that the n-term summation of 1_F is 0_F. If this smallest positive integer exists, then the field has characteristic n; if not, then we say the field has characteristic 0.*

Remark 5.2.2. *Let F be an arbitrary field. Note that $\mathbb{Z}$ is an initial object in the category of rings, then there exists a unique morphism $f : \mathbb{Z} \to F$ that maps $1_{\mathbb{Z}} \to 1_F$. We also have $\mathbb{Z}/\ker(f) \cong$ im$(f) \subseteq F$. Note that the image of f is a domain, so $\ker(f)$ is a prime ideal in $\mathbb{Z}$. Therefore, either*

1. *$\ker(f) = 0$, so characteristic of F is char$(F) = 0$, with $\mathbb{Z} \subseteq F$. This means that the summation of n terms of 1_F is always nonzero for all positive integer n.*

   Note that if we take $0 \neq n \in \mathbb{Z}$, then $n^{-1} \in F$, so we can consider fractions and then extends $\mathbb{Z}$ to $\mathbb{Q}$ as a field. Hence, $\mathbb{Q}$ is the smallest subfield (also called the prime subfield) of any field F.

2. *$\ker(f) = p\mathbb{Z}$ where p is prime. Then we say the characteristic of F is char$(F) = p$ (with similar reasoning as above), and therefore $\mathbb{Z}/p\mathbb{Z} \hookrightarrow F$ is the smallest subfield (also called the prime subfield) of any field F.*

Therefore, if a field F has characteristic 0, then it contains $\mathbb{Q}$ as the smallest subfield; if a field F has characteristic p, then it contains $\mathbb{Z}/p\mathbb{Z}$ as the smallest subfield.

In particular, $\mathbb{Z}/p\mathbb{Z}$ has characteristic $\mathbb{Z}$. Therefore, it does not contain a non-trivial subfield.

Moreover, notice that a field either has characteristic p for some prime p, or has characteristic 0.

Remark 5.2.3 (Freshman's Dream, Frobenius Homomorphism)**.** *When a field has characteristic p, then $(a + b)^p = a^p + b^p$ for all $a, b \in F$.*

Therefore, the map $f : F \to F$ given by $f(x) = x^p$ in such field F is an injective field homomorphism: $f(a+b) = (a+b)^p = a^p + b^p = f(a) + f(b)$ and $(ab)^p = a^p b^p$. This is called the Frobenius homomorphism.

Also note that $(a+b)^{p^k} = a^{p^k} + b^{p^k}$.

Definition 5.2.4 (Multiplicity, Simple Root, Derivative)**.** *Let $f \in F[x]$ be a polynomial where F is a field of positive characteristic. Suppose $\alpha \in F$ is a root of f, then $f(\alpha) = 0$. Therefore, $f = (x-\alpha)^k \cdot h$ for some $h \in F[x]$ and some positive integer k such that $h(\alpha) \neq 0$. This number k is called the multiplicity of α. In particular, if $k = 1$, then α is called a simple root of f.*

Suppose we denote $f = a_n x^n + \cdots + a_1 x + a_0$ for some $a_i \in F$. Then the derivative of f is denoted $f' = a_n \cdot n x^{n-1} + \cdots + a_1$. In particular, note that $(f+g)' = f' + g'$ and $(fg)' = f'g + fg'$.

Lemma 5.2.5. *Let $f \in F[x]$ be a polynomial over F and $\alpha \in F$ is a root of f. Then α is a simple root of f if and only if $f'(\alpha) \neq 0$.*

Proof. We write $f = (x-\alpha) \cdot g$ and compute the derivative. Note that $f' = g + (x-\alpha) \cdot g'$, then $f'(\alpha) = g(\alpha)$. This is nonzero if and only if α is simple. □

Definition 5.2.6 (Greatest Common Divisor)**.** *The greatest common divisor of two polynomials f and g is a monic polynomial h of the highest possible degree such that $h \mid f$ and $h \mid g$. We denote it $\gcd(f,g) = h$. In particular, when considering constant polynomials, this notion is exactly the same one as the conventional definition.*

Corollary 5.2.7. *If $\gcd(f, f') = 1$, then every root of f is simple.*

Proof. If α is a root of f, then $x - \alpha \mid f$, so $x - \alpha \nmid f'$, hence $f'(\alpha) \neq 0$, so α is simple. □

Remark 5.2.8. *If $\gcd(f, f') = 1$, and K/F is a field extension, then f and f' are still relatively prime over K, hence all roots of f over K are simple over a splitting field: $f = a(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$ where all α_i are distinct.*

Definition 5.2.9. *A finite field F is a field of finitely many elements.*

Remark 5.2.10. *The characteristic of a finite field F is a positive prime $p > 0$. Then there is a prime subfield $F_0 \subseteq F$, which is $F_0 \cong \mathbb{Z}/p\mathbb{Z}$. Moreover, if we denote $[F : F_0] = n$, then $x_1, x_2, \cdots, x_n$ can form a basis for F/F_0. Therefore, $F = \{\sum_{i=1}^{n} a_i x_i, a_i \in F_0\}$. Hence, $|F| = p^n$. Therefore, a finite field must have order p^n for some p and some n.*

Theorem 5.2.11. *For any prime integer p and integer $n > 0$, there exists a finite field F with exactly p^n elements. Moreover, every two such finite fields are isomorphic.*

Proof. We write $q = p^n$. We first show the existence. Consider the polynomial $f = x^q - x$ over $\mathbb{Z}/p\mathbb{Z}$. Let F be a splitting field of f over $\mathbb{Z}/p\mathbb{Z}$. Let S be the set of all roots of f in F, so $S \subseteq F$. Note $f = qx^{q-1} - 1 = -1$ since $q = 0$ in $\mathbb{Z}/p\mathbb{Z}$, so $\gcd(f, f') = 1$. Therefore, all the roots of f in F

are simple. Hence, $|S| = q$. Suppose α, β are roots of f, i.e. $\alpha^q = \alpha$ and $\beta^q = \beta$. Hence, $\alpha + \beta$ and $\alpha\beta$ are also roots. Moreover, for $\alpha \neq 0$, α^{-1} is also a root. Hence, the set of roots $S \subseteq F$ is a subfield of F, consisting of all the roots. Since F is generated by all the roots, then $S = F$. But that means $|F| = q$. Therefore, we always have a field of p^n elements. In fact, $x^q - x = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_q)$ over F, so $F = \{\alpha_1, \alpha_2, \cdots, \alpha_q\}$.

We now show its uniqueness. Denote $|F| = q = p^n$, so $|F^\times| = q - 1$, so $x^{q-1} = 1$ for all $x \in F^\times$. Therefore, $x^q = x$ for all $x \in F$. Hence, all elements of F are roots of $f = x^q - x$. This means f is split over F. Moreover, F is generated by all the roots, then this means $F/(\mathbb{Z}/p\mathbb{Z})$ is a splitting field of f. However, the splitting field is unique up to isomorphism. Therefore, every two fields of order q are isomorphic. $\square$

Since finite fields of a certain order are uniquely determined, we denote $\mathbb{F}_q$ to be a field of q elements for $q = p^n$, uniquely up to isomorphism.

Example 5.2.12. *1.* $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

2. $\mathbb{F}_{p^2} \neq \mathbb{Z}/p^2\mathbb{Z}$. *For example,* $\mathbb{F}_4 = \mathbb{F}_2[x]/(x^2 + x + 1)\mathbb{F}_2[x]$, *where* $x^2 + x + 1$ *is the only irreducible polynomial of degree* 2 *over* $\mathbb{F}_2$. *Let* $\alpha = \bar{x}$, *then* $\mathbb{F}_4 = \{0, 1, \alpha, \alpha + 1\}$. *Notice that* $\alpha(\alpha + 1) = \bar{x}(\bar{x} + 1) = \bar{x}^2 + \bar{x} = 1$ *because* $\bar{x}^2 + \bar{x} + 1 = 0$. *Moreover,* $\alpha^2 = \bar{x}^2 = \bar{x} + 1 = \alpha + 1$. *Therefore,* $\mathbb{F}_4 \neq \mathbb{Z}/4\mathbb{Z}$.

Theorem 5.2.13. *Let F be a field and $A \subseteq F^\times$ is a finite subgroup. Then A is cyclic.*

Proof. Note that A is a product of primary components, i.e. $A = \prod_{p \text{ prime}} A[p]$, where $A[p]$ is the product of cyclic groups of the form $\mathbb{Z}/p^k\mathbb{Z}$.

Let us take the set $\{x \in A[p] : x^p = 1\}$. Note that the set has p^a elements, where a is the number of cyclic groups. Note that the elements in this set are roots of $x^p - 1$, so $p^a \leq p$, which means the number of cyclic groups is at most 1, so $A[p]$ is cyclic. By Chinese Remainder Theorem, the product of cyclic groups that are pairwise relatively prime is also cyclic. Hence, A is cyclic. $\square$

Corollary 5.2.14. *$\mathbb{F}_q^\times$ is cyclic. In particular, $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic.*

Definition 5.2.15 (Simple Field Extension)**.** *A field extension K/F is simple if $\exists \alpha \in K$ such that $K = F(\alpha)$.*

Corollary 5.2.16. *Every finite extension of a finite field is simple.*

Proof. Suppose K/F is an extension such that F is a finite field and K/F is a finite extension. Therefore, K is a finite field. Then $K^\times$ is cyclic, so it is generated by $\alpha \in K^\times$. This implies that $K = F(\alpha) = F[\alpha]$. $\square$

Remark 5.2.17. *For $q = p^n$ and $s = p^m$, then $\mathbb{F}_q/\mathbb{F}_s$ is a field extension if and only if $m \mid n$.*

5.3 Normal Extensions

Lemma 5.3.1. *Let E/F be a finite field extension, and $\sigma : F \to L$ is a field homomorphism. Then there is a finite field extension M/L and an extension $\tau : E \to M$ over σ.*

Proof. Note that $E = F(\alpha_1, \cdots, \alpha_n)$ for $\alpha_i \in E$. We prove the statement by induction on n.

Suppose $n = 1$. Then $E = F(\alpha)$, and we take the minimal polynomial $m_\alpha \in F[x]$. Let M/L be a splitting field of $\sigma(m_\alpha) \in L[x]$. Therefore, this is a finite field extension as well. Now $\sigma(\alpha) = \beta \in M$ is a root of $\sigma(m_\alpha)$. Therefore, there exists a unique extension $\tau : E \to M$ such that $\tau(\alpha) = \sigma(\alpha) = \beta$.

$$\begin{tikzcd} F(\alpha) \arrow[r, dashed, "\exists!\tau"] & M \\ F \arrow[u, dash] \arrow[r, "\sigma"'] & L \arrow[u, dash] \end{tikzcd}$$

Now, suppose we have proven the case for $n-1$, we now prove the case for n. In a similar fashion, we have the diagram

$$\begin{tikzcd} E = E'(\alpha_1, \cdots, \alpha_{n-1}) \arrow[r, "\tau"] & M \\ E' = F(\alpha_n) \arrow[u, dash] \arrow[r, "\tau'"] & M' \arrow[u, dash] \\ F \arrow[u, dash] \arrow[r, "\sigma"'] & L \arrow[u, dash] \end{tikzcd}$$

where τ extends σ and the extension M/L is finite. $\square$

Proposition 5.3.2. *Let E/F be a finite field extension. The following are equivalent:*

1. *E is the splitting field of some polynomial f over F.*
2. *For every finite extension M/E and every field homomorphism $\sigma : E \to M$ over F, we have $\sigma(E) = E$.*
3. *Every irreducible polynomial $f \in F[x]$ that has a root in E is split over E.*

Definition 5.3.3 (Normal Extension)**.** *We say an extension is normal if it satisfies all of the above.*

Proof. We first prove that (1) $\Rightarrow$ (2). Since E is a splitting field of $f \in F[x]$, $E = F(\alpha_1, \cdots, \alpha_n)$ where α_i are all roots of f over E and f is split over E. Now $\sigma(\alpha_i)$ is a root of $\sigma(f) = f$. Therefore, $\sigma(\alpha_i) = \alpha_j$ for some j, so $\alpha_j \in E$. Hence, $\sigma(E) \subseteq E$. Consider $\sigma : E \hookrightarrow E$ as a linear map over F with E/F finite, then σ is an isomorphism. Then $\sigma(E) = E$.

We now prove that (2) $\Rightarrow$ (3). Let α be a root of f in E, let L be a splitting field of f over E, and let β be a root of f in L. Then there exists a unique F-homomorphism $\sigma : F(\alpha) \to L$ with $\sigma(\alpha) = \beta$. By the lemma, there exists a finite extension M/L and $\tau : E \to M$ extending σ:

$$
\begin{tikzcd}
 & M \arrow[d, dash] \\
 & L \arrow[d, dash] \\
E \arrow[uur, dashed, "\tau"] \arrow[d, dash] & E \arrow[dd, dash] \\
F(\alpha) \arrow[uur, "\sigma"] \arrow[d, dash] & \\
F \arrow[r, equal] & F
\end{tikzcd}
$$

Since $\tau(E) = E$, we have $\beta = \tau(\alpha) \in E$. Therefore, all the roots of f are in E, so f is split over E.

Finally, we prove that (3) $\Rightarrow$ (1). Let $E = F(\alpha_1, \cdots, \alpha_n)$. Let $f_i = m_{\alpha_i}$ for $i = 1, \cdots, n$, so are irreducible. As α_i is a root of f_i in E, then every f_i splits over E. Now $f = f_1 \cdots f_n \in F[x]$ is split over E. Then E is generated by all roots of f over F. Therefore, E is a splitting field of f. $\square$

Remark 5.3.4 (Normality Test)**.** *If $E = F(\alpha_1, \cdots, \alpha_n)$, then E/F is normal if and only if m_{α_i} splits over E for all i.*

Example 5.3.5.

1. *Extension of degree* 1 *and* 2 *are normal. Therefore, F/F is normal. Suppose E/F such that $[E : F] = 2$, then we have $\alpha \in E\backslash F$, so $E = F(\alpha)$. Let $f = m_\alpha$, then* $\deg(f) = 2$, *so $f = (x-\alpha)(x-\beta)$ is split over E with $\beta \in E$.*

2. *Consider the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ where $\alpha = \sqrt[3]{2}$, then $m_\alpha = x^3 - 2$, which is irreducible by Eisenstein's criterion. Now $m_\alpha = (x - \sqrt[3]{2})(x^2 + \sqrt[3]{2}x + \sqrt[3]{4})$, so it is not split because $x^2 + \sqrt[3]{2}x + \sqrt[3]{4}$ has no roots in $\mathbb{Q}(\sqrt[3]{2}) \subseteq \mathbb{R}$. Therefore, the extension is not normal.*

Corollary 5.3.6. *If $L/E/F$ is a tower of field extensions and L/F is normal, then so is L/E.*

Proof. If L is a splitting field of $f \in F[x] \subseteq E[x]$, then L is a splitting field of f over E. Therefore, L/E is normal. $\square$

Example 5.3.7.

1. *Let $F = \mathbb{Q}$ and note that $x^3 - 2 = (x - \sqrt[3]{2})(x - \xi\sqrt[3]{2})(x - \xi^2\sqrt[3]{2})$ where $\xi^3 = 1$ but $\xi \neq 1$ is a root of unity. Therefore $\mathbb{Q}(\sqrt[3]{2}, \xi)$ is the splitting field of $x^3 - 2$ over $\mathbb{Q}$. Therefore, $\mathbb{Q}(\sqrt[3]{2}, \xi)/\mathbb{Q}$ is a normal extension, but note that $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not a normal extension:*

$$
\begin{tikzcd}
\mathbb{Q}(\sqrt[3]{2},\xi)/\mathbb{Q} \arrow[d, dash] \arrow[dd, dash, bend right=60, "normal"'] \\
\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q} \arrow[d, dash, "not\ normal"] \\
\mathbb{Q}
\end{tikzcd}
$$

Note that $\mathbb{Q}(\sqrt[3]{2}, \xi)/\mathbb{Q}(\sqrt[3]{2})$ *is quadratic, hence normal, but* $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ *is not normal because the minimal polynomial* $x^3 - 2$ *of* $\sqrt[3]{2}$ *does not split in* $\mathbb{Q}(\sqrt[3]{2})[x]$*. More generally,* $\mathbb{Q}(\sqrt[n]{2})/\mathbb{Q}$ *is not normal for* $n \geq 3$*.*

2. *Note that the extensions* $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$ *and* $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ *are normal, as both are quadratic, but* $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$ *is not normal because* $x^4 - 2$ *does not split over* $\mathbb{Q}(\sqrt[4]{2}) \subseteq \mathbb{R}$*.*

$$\begin{tikzcd}
\mathbb{Q}(\sqrt[4]{2}) \arrow[d, dash, "normal"] \\
\mathbb{Q}(\sqrt{2}) \arrow[d, dash, "normal"] \\
\mathbb{Q} \arrow[uu, dash, bend left, "not\ normal"]
\end{tikzcd}$$

Remark 5.3.8. *Therefore, normality is not a good property.*

Definition 5.3.9 (Normal Closure)**.** *Let* K/F *be a finite field extension. A normal closure of* K/F *is a tower* $E/K/F$ *such that* E/F *is normal, and if* E' *is a field such that* $K \subseteq E' \subseteq E$ *and* E'/F *is normal, then* $E' = E$*.*

Theorem 5.3.10. *Let* K/F *be a finite field extension. Then a normal closure exists and it is unique up to isomorphism over* K*. (Similarly, over* F*.)*

Proof. Let $K = F(\alpha_1, \cdots, \alpha_n)$ and $f_i = m_{\alpha_i}$. Now let $f = f_1 \cdots, f_n$. Denote E as the splitting field of f over K. Therefore, E/F is generated by all roots of f as well. Therefore, E/F is a splitting field of f, and thus is normal.

Suppose we have

$$\begin{tikzcd}
 & E \arrow[d, dash] \arrow[ddd, dash, bend left, "normal"] \\
E' \arrow[ur, dash] \arrow[dr, dash] \arrow[ddr, dash, "normal"'] & \\
 & K \arrow[d, dash] \\
 & F
\end{tikzcd}$$

Since f_i is irreducible and has roots α_i in K (so also in E'), then by definition, f_i is split over E'. Therefore, f splits over E', which means all roots of f in E are already in E'. But E is generated by all the roots, so $E = E'$.

We now show that the normal closure is unique up to isomorphism over K. To prove this, we prove the following claim. This is sufficient because the splitting field of a polynomial is unique up to isomorphism over the ground field.

Claim 5.3.11. *Let $K = F(\alpha_1, \cdots, \alpha_k)$, $f_i = m_{\alpha_i}$ and $f = f_1 f_2 \cdots f_k$. E is then a splitting field of f over K.*

Subproof. We see that f_i is irreducible over F and has root α_i in $K \subseteq E$, so E/F is normal. Therefore, f_i is split over E, and so f is split over E.

Let K' be the field extended from K by the roots of f, so $K \subseteq K' \subseteq E$. Note that f is split over K'. Now K' is also the field extended from F by the roots of f, since K can be extended from F by $\alpha_1, \cdots, \alpha_n$, which are some roots of f. Therefore, K'/F is normal, ans so $K' = E$. Therefore, E is generated by all roots of f. Hence, E/K is a splitting field over f. ■

The statement then follows from the claim. □

Remark 5.3.12. 1. *Suppose K/F to be a finite field extension, and $f = f_1 \cdots f_n$. Take any field extension L/K such that f is split over L. Consider E to be the field extended from K by all roots of f in L. Then E is a normal closure of K/F, with $E \subseteq L$.*

2. *Following the notation above, the normal closure of K/F inside L is unique.*

5.4 Separable Extensions

Lemma 5.4.1. *Let $f \in F[x]$ be a non-constant polynomial. Then the following are equivalent:*

1. *f and f' are relatively prime.*
2. *Over any field extension K/F, f has no multiple roots.*
3. *There is a field extension K/F such that f is split over K and has no multiple roots.*

Proof. We first prove that (1) ⇒ (2). Since $\gcd(f, f') = 1$ over K, then f has no multiple roots over K.

We now prove (2) ⇒ (3). Take any splitting field K/F of f.

Finally, we prove that (3) ⇒ (1). For all roots α of f in K, we have $f'(\alpha) \neq 0$, but $f(\alpha) = 0$. Therefore, $x - \alpha$ does not divide f' for all root α. Hence, $\gcd(f, f') = 1$. □

Definition 5.4.2 (Separable Polynomial)**.** *A non-constant polynomial $f \in F[x]$ is separable if f satisfies all of the above.*

Corollary 5.4.3. 1. *If $f \in F[x]$ is separable, then for all field extensions K/F, $f \in K[x]$ is also separable over K.*

2. *If f is separable and $g \mid f$ is a non-constant divisor, then g is separable.*

Proof. 1. The notion of relatively prime is independent on the fields.

2. Take K/F as in (3) as the above lemma, so f is split over K and has no multiple roots, then so it g. Hence, g is separable.

□

Proposition 5.4.4. *An irreducible polynomial $f \in F[x]$ is separable if and only if $f' \neq 0$.*

Proof. If f is separable, then $\gcd(f, f') = 1$ if and only if $f' \neq 0$. □

Example 5.4.5. *Consider a field F of characteristic $p > 0$, and let $a \in F^\times$. Take the polynomial $f = x^p - a$. The derivative of f is $f' = px^{p-1} = 0$ since $p' = 0$ in F. Therefore, f is not separable.*

In fact, if $a \notin (F^\times)^p$, then f is irreducible. Therefore, there are irreducible polynomials that are not separable.

Let K/F be a splitting field, β be a root of f, so $\beta^p = a$. Then $f = (x - \beta)^p$ over K has multiple roots, i.e. not irreducible.

Definition 5.4.6 (Perfect Field)**.** *A field F is perfect if either it has characteristic 0 or having characteristic $p > 0$ but $F^\times = (F^\times)^p$.*

Proposition 5.4.7. *Every irreducible polynomial over a perfect field is separable.*

Proof. Take $f \in F[x]$ irreducible over perfect field F. If suffices to show that $f' \neq 0$. Notice that having $(ax^n)' = anx^{n-1}$, this equals to 0 if and only if $p \mid n$. This is fine when F has characteristic 0. If the characteristic of F is $p > 0$, then suppose $f' = 0$. Then $f = a_0 + a_1x^p + a_2x^{2p} + \cdots + a_mx^{mp}$. Since F is perfect, then $a_i = b_i^p$ for some $b_i \in F$. Therefore, $f = (b_0 + b_1x + \cdots + b_mx^m)^p$. This is not irreducible, contradiction. □

Example 5.4.8.

1. *$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are perfect.*
2. *Finite fields are perfect. Note that we have the Frobinius map $f : F \to F$ as $x \mapsto x^p$ as a field homomorphism. Therefore, this must be injective. Since F is finite, we have a bijection.*
3. *Consider a field F of characteristic p. Then $F(x)$ is not perfect, since x is not a p-th power of rational functions.*

Definition 5.4.9 (Separable Element)**.** *Let K/F be a field extension and $\alpha \in K$ is algebraic over F (so the extension is finite). Then α is separable over F if the minimal polynomial m_α is separable.*

Note that if F is perfect, then every algebraic element α is separable.

Lemma 5.4.10. *Let $L/K/F$ be a tower of field extension, and $\alpha \in L$ is separable over F. Then α is separable over K.*

Proof. Let m_α be the minimal polynomial of α over F, then it is separable. If g is the minimal polynomial of α over K, then $g \mid m_\alpha$. But every divisor of separable polynomial is separable, so α is separable over K. □

Lemma 5.4.11. *Let K/F be a finite field extension, $\sigma : K \to L$ be a field homomorphism. Then there are at most $[K : F]$ extensions $K \to L$ of σ.*

Proof. As usual, write $K = F(\alpha_1, \cdots, \alpha_n$. We prove by induction on n.

When $n = 1$, then $K = F(\alpha)$, so $[K : F] = \deg(m_\alpha)$. Suppose

$$\begin{tikzcd} K \arrow[dd, dash] \arrow[dr, "\tau"] & \\ & L \\ F \arrow[ur, "\sigma"] & \end{tikzcd}$$

Then $\tau(\alpha)$ is a root of $\sigma(m_\alpha)$ in L. Therefore, there is a correspondence between extensions of τ and the roots of $\sigma(m_\alpha)$ in L.

Therefore, the number of roots is less than or equal to $\deg(m_\alpha) = [K : F]$.

In general, consider

$$\begin{tikzcd} K = F'(\alpha_1, \cdots, \alpha_{n-1}) \arrow[d, dash] \arrow[dr, dashed, "\rho"] & \\ F' = F(\alpha_n) \arrow[d, dash] \arrow[r, dashed, "\tau"] & L \\ F \arrow[ur, "\sigma"] & \end{tikzcd}$$

Then ρ extends τ and τ extends σ. Now the number of choices of τ is at most $[F' : F]$. But now for every τ, the number of extensions ρ of τ is less than or equal to $[K : F']$. Therefore, the number of extensions of σ is at most $[F' : F] \cdot [K : F'] = [K : F]$. $\square$

Definition 5.4.12 (Separable Extension)**.** *A finite field extension K/F is separable if there is a field homomorphism $\sigma : F \to L$ that has exactly $[K : F]$ extensions $K \to L$.*

Proposition 5.4.13. *A finite field extension $F(\alpha)/F$ is separable if and only if α is separable over F.*

Proof. Denote $K = F(\alpha)$.

Suppose K/F is separable, then $\sigma : F \to L$ has exactly $[K : F]$ extensions from K to L. Take $f = m_\alpha$. Now $\sigma(f)$ has exactly $[K : F] = \deg(f)$ roots in l. Then f is split over L and has no multiple roots in L. By definition, f is separable, so α is separable.

Suppose α is separable over F, and let L be the splitting field of f over F. Then f has exactly $[K : F] = \deg(f)$ roots in L. There are exactly $[K : F]$ extensions from K to L. Hence, K/F is separable by definition. $\square$

Lemma 5.4.14. *Let F be an infinite field, and L/F is a field extension, and $g \in L[x_1, x_2, \cdots, x_n]$ is a nonzero polynomial. Then there exists $a_1, a_2, \cdots, a_n$ such that $g(a_1, \cdots, a_n) \neq 0$.*

Proof. We can do induction on n. When $n = 1$, since every polynomial has finitely many roots, but F is infinite, then there is an element of F that is not a root.

Suppose we prove the case for $n-1$, we show the case at n. Let $g = g_0 + g_1 x_n + \cdots + g_m x^m$ where $g_i \in K[x_1, \cdots, x_{n-1}]$. Since g is nonzero, then there exists some i such that $g_i \neq 0$. By induction, there exists $a_1, a_2, \cdots, a_{n-1} \in F$ such that $g_i(a_1, \cdots, a_{n-1}) \neq 0$, so $g(a_1, \cdots, a_{n-1}, x_n)$ is a nonzero polynomial in $L[x_n]$. By case $n = 1$, there exists $a_n \in F$ such that $g(a_1, \cdots, a_n) \neq 0$. $\square$

Remark 5.4.15. *The statement is false when F is finite. Take $F = \mathbb{F}_q$, then every element in F is a root of the polynomial $f = x^q - x$, so $f(a) = 0$ for all $a \in F$ but $f \not\equiv 0$.*

Corollary 5.4.16. *Let $g_1, g_2, \cdots, g)m \in L[x_1, x_2, \cdots, x_n]$ be distinct polynomials. Then there exists $a_1, a_2, \cdots, a_n \in F$ such that $g_i(a_1, \cdots, a_n)$ are distinct.*

Proof. Apply the lemma to the product $\prod\limits_{i<j}(g_i - g_j)$. $\square$

Theorem 5.4.17 (Primitive Element Theorem)**.** *Let K/F be a finite separable extension. Then $K = F(\alpha)$ for some $\alpha \in K$.*

Proof. If F is finite, then so it K. We know that K/F is simple. Therefore, we may assume that F is infinite. Since the extension is separable, then there is a field homomorphism $\sigma : F \to L$ that has $m = [K : F]$ extensions $\tau_1, \cdots, \tau_m : K \to L$. By writing $K = F(\alpha_1, \cdots, \alpha_n)$, we consider $f = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \in K[x_1, x_2, \cdots, x_n]$.

Now $\tau_i(f) = \tau_i(\alpha_1)x_1 + \cdots + \tau_i(\alpha_n)x_n \in L[x_1, \cdots, x_n]$ for $i = 1, \cdots, m$. Since α_i generates the field K/F and all τ_i's are distinct, so $\forall i \neq j$, there exists α_k such that $\tau_i(\alpha_k) \neq \tau_j(\alpha_k)$, so $g_i \neq g_j$. Therefore, the polynomials are distinct.

By the corollary, there exists $a_1, a_2, \cdots, a_n \in F$ such that the elements $\beta_i = \tau_i(f)(a_1, \cdots, a_n) = \tau_i(\alpha_1)a_1 + \cdots + \tau_i(\alpha_n)a_n \in L$ are distinct.

Let $\beta = \alpha_1 a_1 + \cdots + \alpha_n a_n \in K$. Then $\beta_i = \tau_i(\beta) \in L$ for $i = 1, \cdots, m$ and are pairwise distinct.

Set $K' = F(\beta)$, so it is a subfield of K. Then we have $K/K'/F$. Note that by restricting to $\tau_i \mid_{K'}: K' \to L$, becuase $\beta \in K'$ and $\tau_i(\beta) = \beta_i$ are distinct, then $\tau_i \mid_{K'}$ are distinct. Hence, $\tau_i \mid_{K'}: K' \to L$ are extensions of $\sigma : F \to L$.

$$\begin{tikzcd}
K \arrow[d, dash] \arrow[dr, "\tau_i"] & \\
K' \arrow[r] \arrow[d, dash] & L \\
F \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

Now note that m is bounded above by the number of extensions $K' \to L$ of σ, which is bounded by $[K' : F] \leq [K : F] = m$. Therefore, $K' = K$. Hence, $K = F(\beta)$. $\square$

Example 5.4.18. $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(a\sqrt{2} + b\sqrt{3})$.

Example 5.4.19. *Suppose F is a field of characteristic $p > 0$. Then $F(x)/F(x^p)$ is a degree-p extension because $m_x = t^p - x^p$.*

Moreover, $F(x,y)/F(x^p,y^p)$ is an extension of degree p^2 because both $F(x,y)/F(x,y^p)$ and $F(x,y^p/F(x^p,y^p)$ has degree p.

Take a rational function $h \in F(x,y)$, then $h^p \in F(x^p,x^p)$, then $F(x^p,y^p)(h)/F(x^p,y^p)$ has degree at most p. Therefore, $F(x^p,y^p)(h) \neq F(x,y)$. So $F(x,y)/F(x^p,y^p)$ is not simple (and not separable).

Proposition 5.4.20. *Let $E/K/F$ be finite field extensions. Then E/F is finite if and only if E/K and K/F are separable.*

Remark 5.4.21. *Separability is a good property.*

Proof. Suppose E/F is separable. Then there exists $\sigma : F \to L$ having exactly $[E:F]$ extensions $E \to L$:

$$\begin{tikzcd} E \arrow[d, dash] \arrow[dr, "\tau"] & \\ K \arrow[r, "\rho"] \arrow[d, dash] & L \\ F \arrow[ur, "\sigma"'] & \end{tikzcd}$$

So σ has at most $[K:F]$ extensions ρ, and each ρ has at most $[E:K]$ extensions τ. Moreover, the total degree $[E:F] = [K:F] \times [E:K]$, so σ has exactly $[K:F]$ extensions, so K/F is separable. Also, every ρ has exactly $[E:K]$ extensions τ, so E/K is separable.

Conversely, suppose both E/K and K/F are separable. We choose $\alpha \in E$ such that $E = K(\alpha)$. Let $\sigma : F \to L$ has exactly $m = [K:F]$ extensions, so $\tau_i : K \to L$:

$$\begin{tikzcd} E = K(\alpha) \arrow[d, dash] & \\ K \arrow[r, "\tau_i"] \arrow[d, dash] & L \\ F \arrow[ur, "\sigma"'] & \end{tikzcd}$$

Consider $f = m_\alpha$ over K, with E/K separable, we know α is separable over K, so f is separable over K, with $\deg(f) = [E:K]$. Consider $f_i = \tau_i(f) \in L[x]$. Then $g = f_1 f_2 \cdots f_m \in L[x]$.

Let M be a splitting field of g over L, then all f_i's are split over M. Note that every polynomial f_i is separable. So f_i has exactly $\deg(f_i)$ roots in M. Therefore, for every i, the extension from E to M of τ_i are in one-to-one correspondence with roots of f_i in M. So the number of extensions from E to M of τ_i is equal to $\deg(f_i)$. We then get $\sum \deg(f_i) = \deg(g) = m \times \deg(f) = [E:K] \times [K:F] = [E:F]$ extensions ρ of σ. By definition, this means that E/F is separable. $\square$

Corollary 5.4.22. *Let K/F be a finite field extension. The following are equivalent:*

1. *K/F is separable.*
2. *Every $\alpha \in K$ is separable over F.*
3. *$K = F(\alpha_1, \cdots, \alpha_n)$, where α_i is separable over F.*
4. *$K = F(\alpha)$ where α is separable over F.*

Proof. (1) $\Rightarrow$ (2): Note that $F \subseteq F(\alpha) \subseteq K$, so $F(\alpha)/K$ is separable, so α is separable.

(2) $\Rightarrow$ (3): Take any generators $\alpha_1, \alpha_2, \cdots, \alpha_n$.

(3) $\Rightarrow$ (1): We do mathematical induction on n. The base case is easy. As for the inductive step, consider the extension $K/K'/F$ with $K' = F(\alpha_1, \cdots, \alpha_{n-1})$ and $K = K'(\alpha_n)$. Note that both K/K' and K'/F are separable, so K/F is separable.

(1) $\Rightarrow$ (4): Theorem.

(4) $\Rightarrow$ (1): Trivial. □

Corollary 5.4.23. *Every finite field extension over a perfect field is separable.*

Proof. Suppose K/F is a finite field extension over perfect field F. Take an arbitrary element $\alpha \in K$, then m_α is irreducible, and so it is separable. Therefore, α is separable, so K/F is separable. □

5.5 Galois Field Extensions

Definition 5.5.1 (Galois Group). *Let E/F be a finite field extension. Consider the set $\{\alpha : E \to E \text{field isomorphisms over } F\}$. This set forms a group $Gal(E/F)$, called the Galois group of E/F.*

Remark 5.5.2. *If $\sigma \in Gal(E/F)$, then $\sigma(a) = a$ for all $a \in F$. Moreover, for $a \in F$ and $x \in E$, then $\sigma(ax) = \sigma(a)\sigma(x) = a\sigma(x)$. Therefore, σ is F-linear.*

Proposition 5.5.3. *Suppose E/F is a finite field extension. Then*

1. *$|Gal(E/F)| \leq [E : F]$.*
2. *$|Gal(E/F)| = [E : F]$ if and only if E/F is normal and separable.*

Proof.

1. Note that every $\sigma \in \mathrm{Gal}(E/F)$ is an extension of the inclusion of the inclusion $F \hookrightarrow E$ to a field homomorphism $\sigma : E \to E$. Therefore, the order of the Galois group $\mathrm{Gal}(E/F)$ is bounded above by the number of extensions, which is bounded above by the degree $[E : F]$.
2. Suppose $|\mathrm{Gal}(E/F)| = [E : F]$, then the inclusion $F \hookrightarrow E$ has at least $|\mathrm{Gal}(E/F)| = [E : F]$ extensions from E to E. Therefore, E/F is separable. Take an arbitrary field extension M over E and let $\sigma : E \to M$ be an extension over the identity of F. Then

$$\begin{tikzcd} & M \arrow[d, dash] \\ E \arrow[ur] \arrow[d, dash] & E \arrow[d, dash] \\ F \arrow[r, equal] & F \end{tikzcd}$$

We need to show that $\sigma(E) = E$. Notice that the number of such σ is bounded above by $[E : F]$. Also, for every $\tau \in \text{Gal}(E/F)$, it satisfies $\tau : E \xrightarrow{\cong} E$, so $E \xrightarrow{\tau} E \hookrightarrow M$. We have $|\text{Gal}(E/F)| = [E : F]$ such compositions $E \hookrightarrow M$, so σ is of this form. Therefore, $\sigma(E) = \tau(E) = E$. hence, we also have normality.

Conversely, suppose E/F is normal and separable. Since the extension is separable, so we get to write $E = F(\alpha)$ for some $\alpha \in E$. Denote $f = m_\alpha \in F[x]$, which is irreducible. Also, $f(\alpha) = 0$. Since the extension E/F is normal, so f is split over E. Since E/F is separable, then f is separable, which means it has no multiple root. Therefore, f has exactly $[E : F]$ roots in E. For every root of β of f in E, there is a unique field homomorphism $\sigma : E \to E$ such that $\sigma(\alpha) = \beta$. This is now an injective linear map of finite-dimensional vector spaces. Therefore, σ is an isomorphism.

Therefore, we have found $[E : F]$ extensions $E \xrightarrow{\cong} E$ over F. Every such extension is an element in the Galois group, so the size of the Galois group is at least $[E : F]$. But the Galois group also has size of at most $[E : F]$, so it has exactly $[E : F]$ elements.

□

Definition 5.5.4 (Galois Extension). *A finite field extension E/F is called Galois if $|Gal(E/F)| = [E : F]$, or equivalently, E/F is normal and separable.*

Example 5.5.5. *1. $Gal(\mathbb{C}/\mathbb{R}) = \{e, conjugation\}$.*

2. *Consider a field F with characteristic not 2. Take $a \in F^\times$ that is not a square. In this case, $f = x^2 - a$ is irreducible and separable because the derivative is nonzero. Hence, the splitting field E of the polynomial f is separable. Note that $f = (x - \sqrt{a})(x + \sqrt{a})$ so $E = F(\sqrt{a})$, with $[E : F] = 2$. Therefore, E/F is normal and so Galois. The Galois has two elements, one is the identity, the other is σ, with $\sigma(\sqrt{a}) = -\sqrt{a}$. Denote $\alpha = x + y\sqrt{a}$ to be an arbitrary element with $x, y \in F$, then $\sigma(\alpha) = x - y\sqrt{a}$.*

3. *Consider a field F with characteristic 2. Consider $a \in F$ with $f = x^2 + x + a \in F[a]$. Then $f' = 2x + 1 = 1$, so f is separable. Assume f has no root in F, then f is irreducible. Again, take E to be the splitting field of the polynomial f over F. Let $\alpha, \beta \in E$ be a root of f, so $\alpha + \beta = 1$, so $\beta = 1 - \alpha = 1 + \alpha$. Therefore, $f = (x - \alpha)(x - 1 - \alpha)$ over E. We see that $[E : F] = 2$ and the extension is separable, so it is Galois. The Galois group then has two elements, one is the identity, the other element is σ with $\sigma(\alpha) = 1 + \alpha$. Hence, $x + y\alpha \in E$ is sent to $x + y(1 + \alpha)$.*

4. *Let q be a power of a prime. Consider the extension $\mathbb{F}_{q^n}/\mathbb{F}_q$ of degree n. Note that the Frobenius homomorphism $\sigma : \mathbb{F}_{q^n} \to \mathbb{F}_{q^n}$ defined by $\sigma(x) = x^q$ satisfies $\sigma(x) = x^q = x$ for $x \in \mathbb{F}_q$. Therefore, $\sigma \in Gal(\mathbb{F}_{q^n}/\mathbb{F}_q)$. Consider σ^i such that the map becomes the identity, then $\sigma^i(x) = x^{q^i} = x$ should hold for all x. However, the multiplication group of $\mathbb{F}_{q^n}$ should be cyclic, so $x^{q^i-1} = 1$, hence the order of the multiplication group $q^n - 1$ divides $q^i - 1$. In particular, $n \leq i$. In fact, the smallest i is just n. Therefore, n is the order of σ in the Galois group, so the Galois group has at least n elements, but n is also the upper bound because it is the degree of extension. Hence, the Galois group has order n, and is exactly the cyclic group generated by the Frobenius map.*

5. *Suppose E/F is Galois, then let $G = Gal(E/F)$. Since it is separable, then $E = F(\alpha)$ for some $\alpha \in E$. Take $f = m_\alpha$, then it is irreducible over F and has $f(\alpha) = 0$. Since E/F is normal, then f is split over E. Because f is separable, f has exactly $[E : F]$ roots in E. Say the roots of f are $X = \{\alpha_1 = \alpha, \alpha_2, \cdots, \alpha_n\} \subseteq E$. Pick $\sigma \in G$, then it takes a root to another root, so $\sigma(\alpha_i) = \alpha_j$ for some j.*

   Consider G acting on the set X of all roots of f in E.

   Claim 5.5.6. *G acts simply transitively.*

   Proof. Take any $\beta \in X$, then there exists $\sigma \in G$ such that $\sigma(\alpha) = \beta$. Then G acts transitively. Moreover, this choice is unique, so G acts simply transitively. $\square$

   Note that every set's group action is simply transitive if it is isomorphic to the group acting on itself by left translation, so $X \cong G$ as finite G-sets.

   Consider $E = \mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}$. This is an extension of degree 4: suppose $\alpha = \sqrt{2+\sqrt{2}}$, then the minimal polynomial is $f = (x-\alpha)(x+\alpha)(x-\sqrt{2-\sqrt{2}})(x+\sqrt{2-\sqrt{2}}) = x^4 - 4x^2 + 2$, which is irreducible by Eisenstein criterion. Since $f(\alpha) = 0$ and f is irreducible, then $f = m_\alpha$, and so $[E : \mathbb{Q}] = 4$.

   We see that $\beta = \sqrt{2-\sqrt{2}}$ is another root of f, and $\alpha^2 = 2+\sqrt{2}$, then $\sqrt{2} = \alpha^2 - 2 \in E$. Moreover, $\alpha \cdot \sqrt{2-\sqrt{2}} = \sqrt{2} \in E$. Therefore, $\beta \in E$. Therefore, there exists $\sigma : E \to E$ over $\mathbb{Q}$ such that $\sigma(\alpha) = \beta$. hence, $\sigma(\sqrt{2+\sqrt{2}} = \sqrt{2-\sqrt{2}}$. Therefore, $\sigma \in G = Gal(E/\mathbb{Q})$.

   Note that $\sigma^2(\alpha) = \sigma(\sigma(\alpha)) = \sigma(\beta) = \sigma(\sqrt{2-\sqrt{2}} = \sigma(\frac{\sqrt{2}}{\sqrt{2+\sqrt{2}}}) = \frac{\sigma(\sqrt{2}}{\sigma(\alpha)}$. Note $\sqrt{2} = \alpha^2 - 2$, then $\sigma(\sqrt{2}) = \sigma(\alpha)^2 - 2 = \beta^2 - 2 = (2-\sqrt{2}) - 2 = -\sqrt{2}$. Therefore, $\sigma^2(\alpha) = -\alpha$, so it is not the identity. Moreover, $\sigma^3(\alpha) = \sigma(\sigma^2(\alpha) = \sigma(-\alpha) = -\beta$, so σ^2 is not identity as well. Hence, the Galois group has to be $G = \{e, \sigma, \sigma^2, \sigma^3\}$, which is a cyclic group of order 4. Therefore, $E/\mathbb{Q}$ is Galois.

6. *Consider the extension $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$. Note that both $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ are extensions of degree 2, so $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ has degree 4, and is the splitting field of $(x^2-2)(x^2-3)$. Therefore, this is a Galois extension of degree 4, with Galois group G of order 4. For $\sigma \in G$,*

$\sigma(\sqrt{2}) = \pm\sqrt{2}$ *and* $\sigma(\sqrt{3}) = \pm\sqrt{3}$, *therefore we have at most four possibilities (because the automorphism must send a root to another root of its minimal polynomial, by the example above). Consider* σ *that takes* $\sqrt{2} \mapsto -\sqrt{2}$ *and* $\sqrt{3} \mapsto \sqrt{3}$, *and* τ *that takes* $\sqrt{2} \mapsto \sqrt{2}$ *and* $\sqrt{3} \mapsto -\sqrt{3}$. *Therefore,* $G = \{e, \sigma, \tau, \sigma\tau\}$, *so* $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

7. *Consider the field extension* $\mathbb{Q}(\sqrt[3]{2}, \xi)/\mathbb{Q}$ *where* $\xi^3 = 1$ *but* $\xi \neq 1$. *We can just say* $\xi = \frac{-1+\sqrt{-3}}{2}$. *Note that the extension has degree* 6, *because* $\mathbb{Q}(\sqrt[3]{2}, \xi)/\mathbb{Q}(\sqrt[3]{2})$ *has degree* 2 *and* $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ *has degree* 3. *The extension is separable and normal because it is the splitting field of* $x^3 - 2$. *Therefore, the extension is Galois, and the Galois group* G *has order* 6. *For* $\sigma \in G$, *it should send* $\sigma(\sqrt[3]{2}) = \xi^i \cdot \sqrt[3]{2}$ *where* $i = 0, 1$ *or* 2, *and* $\sigma(\xi) = \xi^j$ *for* $j = 1, 2$. *Hence, this is the six choices we want. Consider* σ *that sends* $\sqrt[3]{2}$ *to* $\xi \cdot \sqrt[3]{2}$ *and sends* ξ *to* ξ. *Therefore,* $\sigma^3 = e$. *Also consider* τ *that sends* $\sqrt[3]{2})$ *to* $\sqrt[3]{2}$ *and sends* ξ *to* ξ^2. *So* $\xi^2 = e$. *Therefore,* $\tau\sigma\tau = \sigma^2$. *The group* G *is essentially* S_3.

Theorem 5.5.7 (Artin)**.** *Let* E *be any field and* G *is a finite subgroup of* $Aut(E)$. *Set* $F = E^G := \{x \in E : \sigma(x) = x \ \forall \sigma \in G\} \subseteq E$, *then* E/F *is a field extension. We claim that* E/F *is a Galois field extension with Galois group* $Gal(E/F) = G$.

Proof.

Claim 5.5.8. *Every* $\alpha \in E$ *is algebraic and separable over* F *and* $\deg(\alpha) = [F(\alpha) : F] \leq |G|$.

Subproof. Denote $S = \{\sigma(\alpha), \sigma \in G\} \subseteq E$. For $\tau \in G$, then $\tau S = S$. Note $|S| \leq |G|$. Consider the polynomial $f = \prod_{\sigma \in S} (x - s) \in E[x]$, with $\deg(f) = |S|$. However, for $\tau \in G$, $\tau f = \prod_{s \in S} (x - \tau s) = f$, so $f \in F[x]$. Note that f is separable and α is a root of f, so α is separable, and is algebraic over F. The degree is then $[F(\alpha) : F] = \deg(m_\alpha) \leq \deg(f) = |S| \leq |G|$. ■

Claim 5.5.9. $[E : F] \leq |G|$.

Subproof. Suppose not, then $[E : F] > |G|$, then there are linearly independent elements $\alpha_1, \cdots, \alpha_n \in E$ over F, and $n > |G|$.

Note that $F(\alpha_1, \cdots, \alpha_n)/F$ is an extension of degree at least $n > |G|$. Note that this is separable over F, so it is generated by one element, i.e. $F(\alpha_1, \cdots, \alpha_n) = F(\alpha)$ for some $\alpha \in E$, then $[F(\alpha) : F] > |G|$, contradiction. ■

Recall that for a finite field extension, we should have $[E : F] \geq |G|$. By the second claim, $[E : F] \leq G$. Therefore, $[E : F] = |G|$. Moreover, we get to write $|G| \geq [E : F] \geq |\text{Gal}(E/F)| \geq |G|$ since $G \subseteq \text{Gal}(E/F)$. Therefore, $[E : F] = |\text{Gal}(E/F)|$, so E/F is Galois. Because G is a subgroup of the Galois group, then $\text{Gal}(E/F) = G$. □

Example 5.5.10. 1. *Let* K *be a field, take* $E = K(x_1, x_2, \cdots, x_n)$. *We claim that this is the quotient field of* $K[x_1, x_2, \cdots, x_n]$. *Take* $S_n \subseteq Aut(E)$, *so it permutes the* x_i*'s. Then* $E^{S_n} \subseteq E$ *as a subfield of symmetric functions in* E. *Note that* $E^{S_n} = F(s_1, s_2, \cdots, s_n)$ *where* s_i *is the*

i-th standard symmetric function $\sum x_{j_1}x_{j_2}\cdots x_{j_i}$. From Artin's Theorem, E/E^{S_n} is Galois, and Gal$(E/E^{S_n}) = S_n$.

2. *Let G be a finite group, and we know we get to embed G into some S_n. Note $G \subseteq S_n \subseteq Aut(E)$. Applying Artin's theorem to G, we see that Gal$(E/E^G) = G$. Therefore, every finite group is the Galois group of some field extension.*

3. *Consider the smallest field of characteristic 0, which is $\mathbb{Q}$. The inverse Galois problem asks whether there is a Galois extension $E/\mathbb{Q}$ with Gal$(E/\mathbb{Q}) \cong G$ for some finite group G. This remains an open question, but it is known that every finite Abelian group and every symmetric group can be realized in such form.*

Remark 5.5.11. *There are two maps that give a correspondence: let E/F be a Galois extension and $G =$ Gal(E/F). Given a field L with $F \subseteq L \subseteq E$, we obtain a subgroup of G given by $\{\sigma \in G \mid \sigma(x) = x \ \forall x \in L\} =$ Gal(E/L). Conversely, given $H \subseteq G$, we obtain a subfield L with $F \subseteq L \subseteq E$ by setting $L = E^H$. More precisely, the mappings are given by $K \mapsto$ Gal$(E/K) \subseteq$ Gal$(E/F) = G$ (from intermediate field K to a subgroup of G) and $H \mapsto E^H$ for $H \subseteq G$ and $F \subseteq E^H \subseteq E$ (from a subgroup H of G to an intermediate field E^H of E/F), respectively.*

Theorem 5.5.12. *The two maps are inverses to each other. (In particular, they are bijections.)*

Proof. Take an intermediate field $F \subseteq K \subseteq E$ of E/F. By the first mapping, we get $H = \mathrm{Gal}(E/K)$; by the second mapping, we get E^H. To show that that this is a bijection, we need to show that $E^H = K$.

Note that H is identity on K, so $K \subseteq E^H$. Since E/K is normal and separable, then it is Galois, and so $H = \mathrm{Gal}(E/K)$. In particular, the order of the extension is $[E : K] = |H|$. By Artin's theorem, E/E^H is Galois, and $\mathrm{Gal}(E/E^H) = H$. In particular, the degree $[E : E^H] = |H|$. Therefore, $E^H = K$ because $K \subseteq E^H$.

For the other composition, let H be a subgroup of G, and we get $K = E^H$, then get $\mathrm{Gal}(E/K$. We need to show that the Galois group is just H. By Artin's theorem, we have that $\mathrm{Gal}(E/K) = \mathrm{Gal}(E/E^H) = H$. ☐

Property 5.5.13.

1. *Suppose we have the extension $E/K_2/K_1/F$, then Gal$(E/K_1) \supseteq$ Gal(E/K_2). Similarly, if we have $H_1 \subseteq H_2 \subseteq G$, then $E^{H_1} \supseteq E^{H_2}$. In particular, the largest subgroup (E itself) should correspond to the trivial subgroup, and the smallest subgroup (F itself) should correspond to G, so $F = E^G$. This gives a correspondence between subgroup H and E^H.*

2. *Suppose $H \subseteq G$ is a subgroup and $K = E^H$, then Gal$(E/K) = K$, and $[E : K] = |H|$.*

3. *Take $\sigma \in G$ Galois group and $H \subseteq G$ is a subgroup, then we have the conjugation subgroup $\sigma H \sigma^{-1} \subseteq G$. Note that $E^{\sigma H \sigma^{-1}} = \sigma(E^H)$.*

   Proof. Note that $x \in E^{\sigma H \sigma^{-1}}$ if and only if $\sigma\tau\sigma^{-1}(x) = x$ for all $\tau \in H$ if and only if $\tau\sigma^{-1}(x) = \sigma^{-1}(x)$ for all $\tau \in H$ if and only if $\sigma^{-1}(x) \in E^H$ if and only if $x \in \sigma(E^H)$. ☐

4. *Let E/F be a Galois field extension with $G = Gal(E/F)$ and $H \subseteq G$ is a subgroup. Then E^H/F is normal if and only if $H \lhd G$.*

   In this case, $Gal(E^H/F) \cong G/H$.

   Proof. Suppose E^H/F is normal. Take $\sigma \in G$ such that

$$\begin{tikzcd} E \arrow[r, "\sigma", "\sim"'] & E \\ E^H \arrow[u, dash] \arrow[r, dashed, "\cong"] \arrow[ur] & E^H \arrow[u, dash] \\ F \arrow[u, dash] \arrow[r, equal] & F \arrow[u, dash] \end{tikzcd}$$

   Now $\sigma(E^H) = E^H$. We have the restriction res : $G \to \mathrm{Gal}(E^H/F)$ by sending $\sigma \mapsto \sigma\,|_{E^H}$: $E^H \to E^H$, then $\ker(\mathrm{res}) = \mathrm{Gal}(E/E^H) = H \lhd G$.

   Conversely, suppose $H \lhd G$. Take $\sigma \in G$, then $\sigma H \sigma^{-1} = H$, $E^{\sigma H \sigma^{-1}} = \sigma(E^H)$, so $\sigma(E^H) = E^H$. Then there is a restriction map res : $G \to \mathrm{Gal}(E^H/F)$ given by

$$\begin{tikzcd} G \arrow[rr] \arrow[dr] & & \mathrm{Gal}(E^H/F) \\ & G/H \arrow[ur, hook, "\sim"'] & \end{tikzcd}$$

   where $H = \ker(\mathrm{res})$.

   So $|\mathrm{Gal}|(E^H/F) \geq |G/H| = [G : H] = [E^H : F] \geq |\mathrm{Gal}(E^H/F)|$. Note that the first inequality is equal if and only if we have an isomorphism, and the second inequality is equal if and only if we have E^H/F Galois and normal. Hence, we have an isomorphism $G/H \to \mathrm{Gal}(E^H/F)$ by sending σH to $\sigma\,|_{E^H}$. $\square$

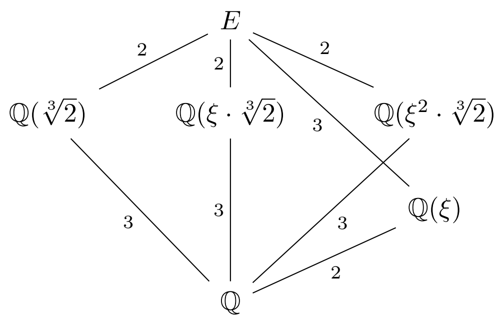
Example 5.5.14. 1. *Consider $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$, $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

$$\begin{array}{ccccc} & & E & & \\ & \diagup & | & \diagdown & \\ \mathbb{Q}(\sqrt{2}) & & \mathbb{Q}(\sqrt{6}) & & \mathbb{Q}(\sqrt{3}) \\ & \diagdown & | & \diagup & \\ & & \mathbb{Q} & & \end{array}$$

2. *$E = \mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q}$, and G is the cyclic group of order 4.*

$$\begin{array}{c} E \\ | \\ \mathbb{Q}(\sqrt{2}) \\ | \\ \mathbb{Q} \end{array}$$

3. $E = \mathbb{Q}(\xi, \sqrt[3]{2})$, *where* $\xi^3 = 1$ *but* $\xi \neq 1$, *and* $G = S_3$. *There are* 3 *subgroups of order* 2, 1 *subgroup (normal) of normal* 3.



Proposition 5.5.15. *Let* M/F *be a field extension,* $K \subseteq K \subseteq M$, $F \subseteq L \subseteq M$. KL *is the smallest subfield of* M *containing both* K *and* L.

Proof. Denote $K = F(\alpha_1, \cdots, \alpha_n)$, then $KL = L(\alpha_1, \cdots, \alpha_n)$.

$$\begin{tikzcd} & KL \arrow[dl, dash] \arrow[dr, dash] & \\ K \arrow[dr, dash] & & L \arrow[dl, dash] \\ & F & \end{tikzcd}$$

□

Theorem 5.5.16. *Assume that* K/F *is Galois. Then* KL/L *is also Galois. The restriction map* $res : Gal(KL/L) \to Gal(K/F)$ *is well-defined and it yields an isomorphism*

$$Gal(KL/L) \xrightarrow{\cong} Gal(K/K \cap L).$$

Proof. Note that we can write $K = F(\alpha_1, \cdots, \alpha_n)$. Since K/F is separable, then α_i are separable over F, then they are separable over L, so KL/L is separable. Note that K is the splitting field of $f \in F[x]$ where $f = \prod m_{\alpha_i}$. Then KL/L is a splitting field of $f \in L[x]$. Therefore, KL/L is Galois.

Take $\sigma \in \mathrm{Gal}(KL/L)$. Then $\sigma |_K (K) = K$.

$$\begin{tikzcd} KL \arrow[r, "\sigma", "\sim"'] \arrow[d, dash] & KL \arrow[d, dash] \\ K \arrow[ur, "\sigma|_K"'] \arrow[r, dashed] \arrow[d, dash] & K \arrow[d, dash] \\ F \arrow[r, equal] & F \end{tikzcd}$$

Therefore, the restriction is well-defined: $\mathrm{Gal}(KL/L) \to \mathrm{Gal}(K/F)$, and gives $\sigma \in \mathrm{Gal}(K/F)$. Suppose σ acts as the identity on K, i.e. $\sigma |_K = \mathbf{id}_K$, then $\sigma(\alpha_i) = \alpha_i$ for each α_i, so σ acts as the identity on L. However, now $KL = L(\alpha_1, \cdots, \alpha_n)$, so $\sigma = \mathbf{id}_{KL}$. Therefore, the restriction is injective.

Now $\sigma \in \mathrm{Gal}(KL/L)$, $\sigma\mid_L = \mathbf{id}_L$, so $\sigma\mid_{K\cap L} = \mathbf{id}_{K\cap L}$. Therefore, $\mathrm{res}(\sigma) = \sigma\mid_K \in \mathrm{Gal}(K/K\cap L)$, so $\mathrm{im}(\mathrm{res}) \subseteq \mathrm{Gal}(K/K\cap L)$.

Denote $H = \mathrm{im}(\mathrm{res})$. Now $K^H = \{x \in K : \sigma\mid_K (x) = \sigma(x) = x \ \forall \sigma \in \mathrm{Gal}(KL/L)\} \subseteq L$. Now $K^H \subseteq K \cap L$, so $\mathrm{Gal}(K/K^H) = \mathrm{Gal}(K/K\cap L)$. Therefore, $\mathrm{im}(\mathrm{res}) = \mathrm{Gal}(K/K\cap L)$. $\square$

Corollary 5.5.17. *If $K \cap L = F$, i.e. K and L are linearly disjoint over F, then Gal$(KL/L) \cong$ Gal(K/F).*

Theorem 5.5.18. *Assume that both K/F and L/F are Galois. Then KL/F is Galois and the restriction map res : Gal$(KL/F) \to$ Gal$(K/F) \times$ Gal(L/F) is injective. If $K \cap L = F$, then res is an isomorphism. Moreover, Gal$(KL/F) =$ Gal$(KL/L) \times$ Gal(KL/K) as an internal direct product.*

Proof. If K is a splitting field of $f \in F[x]$, L is a splitting field of $g \in F[x]$, then KL is a splitting field of fg. Therefore, KL/F is normal.

Moreover, since $K = F(\alpha_1, \cdots, \alpha_n)$ is separable, then $KL = L(\alpha_1, \cdots, \alpha_n)$ is also separable. Hence, KL/F is separable, so it is Galois.

Take $\sigma \in \mathrm{Gal}(KL/F)$, then $\sigma\mid_K = \mathbf{id}_K$, $\sigma\mid_L = \mathbf{id}_L$, so it is in $\mathrm{Gal}(K/L)$, then $\sigma = \mathbf{id}$. Hence, res is injective.

Moreover, suppose K and L are linearly disjoint over F, then for $\tau \in \mathrm{Gal}(K/F)$ and $\rho \in \mathrm{Gal}(L/F)$, note that there is $\tau' \in \mathrm{Gal}(KL/L)$ and $\rho' \in \mathrm{Gal}(KL/K)$ that can be restricted to the two maps. Hence, $\tau'\rho'\mid_K = \tau$, $\tau'\rho'\mid_L = \rho$, so $\tau(\tau'\rho') = (\tau, \rho)$, so res is an isomorphism.

In fact, $\mathrm{Gal}(K/F) \cong \mathrm{Gal}(KL/L)$ and $\mathrm{Gal}(L/F) \cong \mathrm{Gal}(KL/K)$, both of which are subgroups of $\mathrm{Gal}(KL/F)$, satisfy $\mathrm{Gal}(KL/F) = \mathrm{Gal}(KL/L) \times \mathrm{Gal}(KL/K)$ as an internal direct product. $\square$

5.6 Cyclotomic Field Extensions

Example 5.6.1. *Take a field F of characteristic $p > 0$, let $x \in F$ that is a root of unity of degree p, i.e. $x^p = 1$. Then $0 = x^p - 1 = (x-1)^p$, so $x - 1 = 0$, then $x = 1$.*

Let F be a field and n is an integer that is prime to char(F) (if the characteristic is 0, then the restriction is empty). The polynomial $f = x^n - 1$ has derivative $f' = nx^{n-1} \neq 0$, then $\gcd(f, f') = 1$, so f is separable. Consider F_n/F as the splitting field of polynomial f. This is a separable field extension and is unique up to isomorphism. Moreover, it is normal, so F_n/F is Galois.

Definition 5.6.2 (Cyclotomic Field Extension)**.** *The extension structure above F_n/F is called the n-cyclotomic field extension of F.*

Remark 5.6.3. *We want to determine the structure of the Galois group of F_n/F. Recall that if we denote $\mu_n = \{x \in F_n : x^n = 1\} \subseteq F_n^\times$ as the field of root of unity, then it is also cyclic of order n. We know that the group is generated by $\varphi(n)$ elements, where φ is the Euler function. Suppose* *we choose a generator $\xi_n \in \mu_n$ (a primitive n-th root of unity), then $\forall \xi \in \mu_n$, $\xi = (\xi_n)^i$ for some i, where i is unique modulo n. Hence, $i + n\mathbb{Z} \in \mathbb{Z}/n\mathbb{Z}$ is well-defined. We can also conclude that the field F_n is generated by a primitive root, u.e. $F_n = F(\xi_n)$.*

Take $\sigma \in Gal(F_n/F)$, then it sends a root of unity to another root of unity, i.e. $\sigma(\xi_n) = (\xi_n)^i$ for some i, where $\gcd(i, n) = 1$. Now suppose the map $\chi : Gal(F_n/F) \to (\mathbb{Z}/n\mathbb{Z})^\times$ sends $\sigma \mapsto [i]_n$, then χ is a homomorphism: suppose in addition that $\tau(\xi_n) = (\xi_n)^j$, then $\sigma\tau(\xi_n) = (\xi_n)^{ij}$. Moreover, if we take a root of unity $\xi \in \mu_n$, then $\xi = (\xi_n)^k$, and $\sigma(\xi) = \sigma(\xi_n^k) = \sigma(\xi_n)^k = (\xi_n)^{ij} = \xi^i$. Therefore, the formula $\sigma(\xi_n) = (\xi_n)^i$ should hold for any root. Hence, we see that χ is independent on the choice of ξ_n.

Claim 5.6.4. *χ is injective.*

Proof. Take $\sigma \in \ker(\chi)$, then $\sigma(\xi_n) = \xi_n = (\xi_n)^i$, so $i \equiv 1 \pmod{n}$. Therefore, $[i]_n = [1]_n$. □

Hence, we can identify canonically the Galois group of an cyclotomic field extension with the group $(\mathbb{Z}/n\mathbb{Z})^\times$, i.e. $Gal(F_n/F) \hookrightarrow (\mathbb{Z}/n\mathbb{Z})^\times$. In particular, $Gal(F_n/F)$ is Abelian.

Remark 5.6.5. *Suppose $F = \mathbb{Q}$. Let Φ_n be the minimal polynomial of ξ_n of degree n. Therefore, $\Phi_n \in \mathbb{Q}[x]$ is monic. Note that $\exists \alpha \in \mathbb{Q}$ such that $\tilde{\Phi}_n := \alpha\Phi_n \in \mathbb{Z}[x]$ is primitive. We know that every primitive root of unity ($\xi_n^n - 1 = 0$, so ξ_n is a root of $f = x^n - 1$, then the minimal polynomial $\Phi_n \mid (x^n - 1)$ and $\tilde{\Phi}_n \mid (x^n - 1)$ in $\mathbb{Q}$. Note that both polynomials are primitive. By Gauss' Lemma, then they are also divisible in $\mathbb{Z}[x]$. Hence, $x^n - 1 = \tilde{\Phi}_n \cdot g$ for some $g \in \mathbb{Z}[x]$. Hence, the leading coefficient of $\tilde{\Phi}_n$ must be ± 1. However, Φ_n is monic, so $\alpha = \pm 1$. We deduce that $\Phi_n \in \mathbb{Z}[x]$. Therefore, the minimal polynomial has integer coefficients. The polynomial Φ_n is called the n-th cyclotomic polynomial (over $\mathbb{Q}$).*

Lemma 5.6.6. *Let p be a prime integer such that $p \nmid n$. Then $(\xi_n)^p$ is a root of Φ_n.*

Proof. We write $x^n - 1 = \Phi_n \cdot g$ where $g \in \mathbb{Z}[x]$. Suppose, towards contradiction, that $(\xi_n)^p$ is not a root of Φ_n, then $(\xi_n)^p$ is a root of g. Therefore, $g((\xi_n)^p) = 0$.

Observe that for $g_1(x) = g(x^p)$, then $g_1(\xi_n) = g((\xi_n)^p) = 0$, so ξ_n is a root of g_1. Therefore, the minimal polynomial $\Phi_n \mid g_1$ in $\mathbb{Z}[x]$. Consider the canonical homomorphism $\mathbb{Z} \to \mathbb{F}_p = \mathbb{Z}/pmathbbZ$ and correspondingly $\mathbb{Z}[x] \to \mathbb{F}_p[x]$ that sends h to $\bar{h}$.

Note that $\bar{g}(x) = \bar{g}(x^p)$, but for any $a \in \mathbb{F}_p$, then $a^p = a$, so $\bar{g}(x) = \bar{g}(x^p) = \sum a_i x^{pi} = \sum a_i^p x^{pi} = \sum (a_i x^i)^p = \bar{g}^p$. Therefore, $\bar{g}_1 = (\bar{g})^p$ in $\mathbb{F}_p[x]$.

Recall that $\Phi_n \mid g_1$, then $\bar{\Phi}_n \mid \bar{g}_1 = (\bar{g})^p$ over $\mathbb{F}_p$. Let l be an irreducible divisor of $\bar{\Phi}_n$ in $\mathbb{F}_p[x]$. Then $l \mid \bar{\Phi}_n \mid (\bar{g})^p$, so $l \mid \bar{g}$. Recall that $x^n - 1 = \Phi_n \cdot g$, so $x^n - \bar{1} = \bar{\Phi}_n \cdot \bar{g}$, so $l \mid \bar{\Phi}_n$ and $l \mid \bar{g}$, hence $l^2 \mid x^n - 1$ in $\mathbb{F}_p[x]$. This is a contradiction because $x^n - \bar{1}$ is separable polynomial (as $\bar{f}' = nx^{n-1} \neq 0$), so it cannot be divided by a square of a irreducible, contradiction. □

Corollary 5.6.7. *All primitive roots of* 1 *of degree n are the roots of Φ_n. In particular,* $\deg(\Phi_n) \geq \varphi(n)$.

Proof. Let ξ be a primitive root of degree n of 1. Then $\xi = (\xi_n)^i$, so $\gcd(i, n) = 1$. We write $i = p_1 p_2 \cdots p_k$ as a product of primes, but that means the primes do not divide n. We apply the lemma k times, then $(\xi_n)^{p_1}, (\xi_n)^{p_1 p_2}, \cdots, (\xi_n)^{p_1 \cdots p_k}$ are the roots of Φ_n. $\square$

Theorem 5.6.8. *$Gal(\mathbb{Q}_n/\mathbb{Q}) = (\mathbb{Z}/n\mathbb{Z})^\times$. Moreover, $[\mathbb{Q}_n : \mathbb{Q}] = \varphi(n)$, and $\Phi_n(x) = \prod_{\xi \text{ primitive } n\text{th root of } 1} (x - \xi)$, and should be independent on ξ_n.*

Proof. We know that $\varphi(n) \leq \deg(\Phi_n) = [\mathbb{Q}_n : \mathbb{Q}] = |\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})| \leq |(\mathbb{Z}/n\mathbb{Z})^\times| = \varphi(n)$. Therefore, $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) = (\mathbb{Z}/n\mathbb{Z})^\times$ and $[\mathbb{Q}_n : \mathbb{Q}] = \varphi(n)$, and $\Phi_n(x) = \prod_{\xi \text{ primitive } n\text{th root of } 1} (x - \xi)$. $\square$

Remark 5.6.9. *$x^n - 1 = \prod_{d|n} \Phi_d$. Indeed, note that $x^n - 1 = \prod_{\xi \text{ primitive } n\text{th root of } 1} (x - \xi)$, but every root of unity is primitive for exactly one integer. Therefore by taking $\xi \in \mu_n$, if d is the order of ξ in μ_n, then $d \mid n$ and ξ is a primitive dth root of unity. Hence, Φ_d should be a linear term of the form $\Phi_d = x - \xi$.*

When $d = n$, we have $\Phi_n = \frac{x^n - 1}{\prod_{d|n, d \neq n} \Phi_d}$.

Example 5.6.10.

1. $\Phi_1 = x - 1$.
2. $\Phi_2 = x + 1$.
3. $\Phi_p = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + x + 1$ *for p prime, as $\varphi(p) = p - 1$.*
4. $\Phi_3 = x^2 + x + 1$.
5. $\Phi_4 = \frac{x^4 - 1}{(x-1)(x+1)} = x^2 + 1 = (x + i)(x - i)$.
6. $\Phi_5 = x^4 + x^3 + x^2 + x + 1$.
7. $\Phi_6 = x^2 - x + 1$, *as $\varphi(6) = 2$.*
8. *All cyclotomic polynomials are irreducible polynomials over $\mathbb{Q}$, because they are minimal polynomials.*
9. *$\forall n < 105$, all coefficients of Φ_n are 0 or ± 1. $\Phi_{105} = x^{48} + x^{47} + x^{46} - x^{43} - x^{42} - 2x^{41} + \cdots - 2x^7 + \cdots$. Note that $105 = 3 \times 5 \times 7$ is the product of first three odd primes.*

5.7 Galois Group of a Polynomial

Definition 5.7.1 (Galois Group of a Polynomial)**.** *Let $f \in F[x]$ be a separable polynomial (and so it is non-constant) over a field F of characteristic 0. Take E/F as the splitting field of f. We know that E exists and is unique up to isomorphism. Therefore, E/F is normal and separable, so it is Galois.*

Now $Gal(E/F)$ is called the Galois group of f, also denoted $Gal(f)$.

Example 5.7.2. *1. Gal($x^n - 1$) $= (\mathbb{Z}/n\mathbb{Z})^\times$ over $\mathbb{Q}$.*

2. Let K be a field and $E = K(x_1, \cdots, x_n)$ be a field. Note that S_n acts on E by permutation of variables, and $E^{S_n} = K(s_1, s_2, \cdots, s_n)$ is generated by standard symmetric functions over K. We denote $F = E^{S_n}$. Consider $f = \prod_{i=1}^{n}(x - x_i) = x^n - s_1x^{n-1} + s_2x^{n-2} - \cdots + (-1)^n s_n \in F[x]$. We call this the generic polynomial. The coefficients $s_1, \cdots, s_n$ are algebraically independent. We can conclude that E is the splitting field of F over F because the polynomial f splits in E, and E is generated by the roots. The Galois group of f is given by Gal(f) = Gal(E/F) $= S_n$.

Proposition 5.7.3. *Let E/F be a Galois field extension, and $\alpha \in F$. Let $S = \{\sigma(\alpha) : \sigma \in G =$ Gal$(E/F)\}$. Then $\deg(\alpha) = |S|$ and the minimal polynomial $m_\alpha = \prod_{\beta \in S}(x - \beta)$.*

Proof. Consider the extension $E/F(\alpha)/F$, where we denote $H = \mathrm{Gal}(E/F(\alpha)) \subseteq G$. Here G acts on S transitively such that $H = \mathrm{stab}(\alpha)$ because the action is trivial. By definition, $\deg(\alpha) = [F(\alpha) : F] = [G : H] = |S|$. Also note that $f = \prod_{\beta \in S}(x - \beta)$ is $G-$ stable: $\sigma f = f$ for all $\sigma \in G$. Therefore, $f \in F[x]$. Since $\alpha \in S$, then $f(\alpha) = 0$. Therefore, $m_\alpha \mid f$, but $\deg(m_\alpha) = |S| = \deg(f)$, then since both polynomials are monic, we conclude that $m_\alpha = f$. □

Example 5.7.4. *Let $\alpha = \sqrt{2} + \sqrt[3]{5}$ over $\mathbb{Q}$. For $\xi^3 = 1$ such that $\xi \neq 1$, we know*

$$\begin{tikzcd}
& \alpha \in \mathbb{Q}(\sqrt{2}, \sqrt[3]{5}, \xi) \arrow[dl, dash] \arrow[dr, dash] \arrow[dd, dash, "G"] & \\
\mathbb{Q}(\sqrt{2}) \arrow[dr, dash, "2"'] & & \mathbb{Q}(\sqrt[3]{5}, \xi) \arrow[dl, dash, "6"] \\
& \mathbb{Q} &
\end{tikzcd}$$

Note that the intermediate fields are linear disjoint, then $G \cong \mathbb{Z}/2\mathbb{Z} \times S_3$. Let ρ be the element from $\mathbb{Z}/2\mathbb{Z}$, and σ and τ are elements of S_3 as the 3-cycle and the 2-cycle, respectively. We then can denote $\rho(\sqrt{2}) = -\sqrt{2}$, $\rho(\sqrt[3]{5}) = \sqrt[3]{5}$ and $\rho(\xi) = \xi$; $\sigma(\sqrt{2}) = \sqrt{2})$, $\sigma(\sqrt[3]{5}) = \xi \cdot \sqrt[3]{5}$ and $\sigma(\xi) = \xi$; $\tau(\sqrt{2}) = \sqrt{2}$, $\tau(\sqrt[3]{5}) = \sqrt[3]{5}$, and $\tau(\xi) = \xi^{-1} = \xi^2$.

In particular, $\sigma(\alpha) = \pm\sqrt{2} + \xi^i \cdot \sqrt[3]{5}$, where $i = 0, 1, 2$, so there are 6 possibilities, so $|S| = 6 =$ $\deg(\alpha)$. We can write $\alpha - \sqrt{2} = \sqrt[3]{5}$, then $(\alpha - \sqrt{2})^3 = 5$, so $m_\alpha = (x^3 + 6x - 5)^2 - 2(3x^2 + 2)$.

5.8 Algebraically Closed Field

Proposition 5.8.1. *Let F be a field. The following are equivalent:*

1. *F has no non-trivial finite field extensions.*

2. *Every irreducible polynomial in $F[x]$ is linear.*

3. *Every non-constant polynomial in $F[x]$ has a root in F.*

4. *Every non-constant polynomial in $F[x]$ is split.*

Proof. (1) $\Rightarrow$ (2): Let f be irreducible, then $F[x]/f \cdot F[x]$ is a field extension over F of degree $\deg(f)$. However, F has no non-trivial field extensions, so f is linear.

(2) $\Rightarrow$ (3): Take any non-constant polynomials, we write it as the product of linear terms, then there has to be a root in F.

(3) $\Rightarrow$ (4): Since $f(\alpha) = 0$, then $f = (x - \alpha) \cdot g$.

(4) $\Rightarrow$ (2): if every polynomial is split, then every irreducible is linear.

(2) $\Rightarrow$ (1): Suppose K/F is a finite field extension. Consider $\alpha \in K$. We know that $\deg(m_\alpha) = [F(\alpha) : F]$. But m_α is irreducible, so it is linear, then $[F(\alpha) : F] = 1$, which means $\alpha \in F$. Therefore, $K = F$. □

Definition 5.8.2 (Algebraically Closed)**.** *If F is a field satisfying the four conditions above, then F is called algebraically closed.*

Theorem 5.8.3. *$\mathbb{C}$ is algebraically closed.*

Proof.

Claim 5.8.4. *$\mathbb{R}$ has no non-trivial odd-degree extension.*

Subproof. Every finite extension is generated by one element since it is separable. Suppose $[\mathbb{R}(\alpha) : \mathbb{R}] = \deg(m_\alpha)$ is odd, then m_α is irreducible of odd degree, but that means m_α has to have a real root, then $\deg(m_\alpha) = 1$. ■

Claim 5.8.5. *$\mathbb{C}$ has no quadratic extensions.*

Subproof. If $z = \tau \cdot (\cos(\varphi) + i \sin(\varphi))$ is a complex number, then $t = \sqrt{\tau}(\cos(\frac{\varphi}{2}) + i \sin(\frac{\varphi}{2})$ satisfies $t^2 = z$. Therefore, every complex number is a square, so every quadratic polynomial has a root. ■

Let $K/\mathbb{C}$ be a finite extension. We want to show that $K = \mathbb{C}$. We have a tower $K/\mathbb{C}/\mathbb{R}$, then it is a finite extension. Replacing K by a normal closure of K over $\mathbb{R}$ (up to isomorphism), we may assume that $K/\mathbb{R}$ is Galois. Let $G = \mathrm{Gal}(K/\mathbb{R})$, let $P \subseteq G$ be a Sylow 2-subgroup. Therefore, note that $[K^P : \mathbb{R}] = [G : P]$ is odd. Therefore,$K^P = \mathbb{R} = K^G$, then $G = P$, so G is a 2-group.

Let $H = \mathrm{Gal}(K/\mathbb{C}) \subseteq G$ of index 2. We need to show that $H = \{e\}$. Suppose not, then there exists a subgroup $I \subseteq H$ of index 2. Therefore, we have $\mathbb{C} \subseteq K^I \subseteq K$, but $[K^I : \mathbb{C}] = [H : T] = 2$, contradiction. Therefore, H is trivial and $K = \mathbb{C}$. □

Definition 5.8.6 (Algebraic Closure)**.** *Let F be a field. A field extension F_{alg}/F is called an algebraic closure of F if*

1. *F_{alg} is algebraically closed.*

2. F_{alg}/F is algebraic.

Example 5.8.7. *$\mathbb{Q}_{alg}$ is the field of algebraic elements in $\mathbb{C}$.*

Theorem 5.8.8. *F_{alg} exists for every field F.*

Proof. Let S be the set of all non-constant polynomials in $F[x]$. For all $f \in S$ we take a variable x_f. Denote $R = F[x_f]_{f\in S}$. Let $I \subseteq R$ be the ideal that is generated by $f(x_f)$ for all $f \in S$.

Claim 5.8.9. *$I \neq R$.*

Subproof. Suppose $I = R$, then $1 = \sum\limits_{f\in T} f(x_f) \cdot g_f$ where $g_f \in R$ and $T \subseteq S$ is a finite subset. Take $h(t) = \prod\limits_{f\in T} f(t) \in F[t]$ to be a non-constant polynomial. Let L/F be a splitting field of h. IN particular, all $f \in T$ are split over L, so $f(a_f) = 0$ for some $a_f \in L$. Take $x_f = a_f$, then $1 = \sum\limits_{f\in T} f(a_f) \cdot g_f(\cdots) = 0$, contradiction. ■

Since $I \neq R$, there exists a maximal ideal M such that $I \subseteq M \subsetneq R$. Let $F_1 = R/M$ to be a field extension over F. We have $I = f(x_f) + I \in R/I \twoheadrightarrow R/M = F_1$. Therefore, if $\bar{x}_f$ is the image of x_f in the field F_1, then $f(\bar{x}_f) = 0$ in F_1. In particular, every $f \in S$ has a root in F_1.

Denote $F = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq$, where we repeat the procedure as above. Then $F_{\text{alg}} = \bigcup\limits_{i=0}^{\infty} F_i$.

Claim 5.8.10. *F_{alg} is an algebraic closure of F.*

Subproof. Let $f \in F_{\text{alg}}[x]$ be a non-constant polynomial. Then $f \in F_i[x]$ for some i. By construction, f has a root in $F_{i+1} \subseteq F_{\text{alg}}$. Therefore, F_{alg} is algebraically closed. ■

Claim 5.8.11. *F_{i+1}/F_i is algebraic.*

Subproof. It suffices to show that F_1/F_0 is algebraic, and the rest are similar.

Note that $F_1 = R/M$ is generated by the images of generators x_f of R, where R is a polynomial ring. Note that $f(\bar{x}_f) = 0$, so $\bar{x}_f$ is algebraic over F, so F_1/F is algebraic. ■

□

Remark 5.8.12. *Suppose we have $F \hookrightarrow F_{alg}$ and a finite field extension E/F. How do we embed E into F_{alg}?*

Note that there exists an embedding $E \hookrightarrow M$, such that M/F_{alg} is finite. However, $M = F_{alg}$ because F_{alg} is algebraically closed, so we get the desired embedding. Note that this embedding is not unique.

5.9 Radical Field Extensions

Definition 5.9.1 (n-Radical). *Let F be a field of characteristic 0. A field extension $K = F(\alpha)$ over F is n-radical if $\alpha^n \in F$.*

Proposition 5.9.2. *Let K/F be a n-radical field extension. If $\xi_n \in F$, then K/F is a cyclic field extension of degree dividing n.*

Proof. Let $K = F(\alpha)$, and denote $a = \alpha^n \in F$. Then K/F is a splitting field of $x^n - a = \prod_{i=0}^{n-1}(x - \xi_n^i \cdot \alpha) \in K[x]$.

Let $G = \mathrm{Gal}(K/F)$ and take $\sigma \in G$, then $\sigma(\alpha)^n = \sigma(\alpha^n) = \sigma(a) = a$, so $\sigma(\alpha)$ is a root of $x^n - a$. Therefore, denote $\sigma(\alpha) = \xi^i \cdot \alpha$ for some i. We have a well-defined map $f : G \to \mathbb{Z}/n\mathbb{Z}$ where $f(\sigma) = i + n\mathbb{Z}$. In particular, if we also take $\tau \in G$ such that $\tau(\alpha) = \xi^j \cdot \alpha$, then $(\sigma\tau)(\alpha) = \sigma(\xi^j \cdot \alpha) = \sigma(\xi^i)\sigma(\alpha) = \xi^j \cdot \xi^i \cdot \alpha = \xi^{i+j}\alpha$. Therefore, $f(\sigma\tau) = i + j + n\mathbb{Z} = f(\sigma) + f(\tau)$. Therefore, f is a homomorphism. Moreover, note that if $f(\sigma) = 0 + n\mathbb{Z}$, then $\sigma(\alpha) = \xi^0 \cdot \alpha = \alpha$, so $\sigma = \mathbf{id}$. Therefore, f is injective. In particular, we have an embedding $G \hookrightarrow \mathbb{Z}/n\mathbb{Z}$, so G is cyclic of order dividing n. $\square$

Remark 5.9.3. *Let L/F be Galois with $G = Gal(L/F)$. The vector space $\mathbf{End}_F(L)$ is also a vector space over L over, or just a L-module. In particular, every element of G is an endomorphism of L over F, so G is a subset of $\mathbf{End}_F(L)$.*

Lemma 5.9.4. *G is a linearly independent subset of $\mathbf{End}_F(L)$ over L.*

Proof. Suppose we have $\sum_{i=1}^{n} x_i\sigma_i = 0$ for $x_i \in L, \sigma_i \in G$, where not all $x_i = 0$. In particular, notice that the smallest number of non-zero terms is 2, then we have $x_1 \neq 0 \neq x_2$ without loss of generality, and assume that the number of non-zero coefficients is at its minimum. For all $y \in E$, we have $\sum x_i\sigma_i(y) = 0$, so $\sum x_i\sigma_i(yz) = (\sum_i x_i\sigma_i(y)\sigma_i)(z) = 0$ for any z. Multiplying the initial linear dependence by $\sigma_1(y)$, and choosing $y \in L$ so that $\sigma_1(y) \neq \sigma_y(y)$, we get by subtracting that $\sum_{i=1}^{n} x_i(\sigma_i(y) - \sigma_1(y))\sigma_i = 0$. The number of non-zero coefficients is smaller, but not zero since $x_2(\sigma_2(y) - \sigma_1(y)) \neq 0$, so we have the required contradiction. $\square$

Proposition 5.9.5 (Hilbert Theorem 90). *Let L/F be a cyclic field extension of degree n. If $\xi_n \in F$, then L/F is n-radical.*

Proof. Let σ be the generator of $\mathrm{Gal}(L/F) = \{\mathbf{id}, \sigma, \cdots, \sigma^{n-1}\}$, then consider $\sum_{k=0}^{n-1} \xi_n^{-k}\sigma^k \neq 0$. There exists $y \in L$ such that $\alpha = \sum_{k=0}^{n-1} \xi_n^{-k}\sigma^k(y) \neq 0$, and we claim that $L = F(\alpha)$. To see this, note that $\sigma(\alpha) = \sum_{i=0}^{n-1} \xi^{-i} \cdot \sigma^{i+1}(x) = \xi \cdot \sum_{i=0}^{n-1} \xi^{-(i+1)} \cdot \sigma^{i+1}(x) = \xi\alpha$. Therefore, $\sigma(\alpha^n) = \sigma(\alpha)^n = \alpha^n$. Here $\alpha^n \in F$, so $F(\alpha)/F$ is n-radical. Moreover, the values $\sigma^i(\alpha) = \xi_n^i \cdot \alpha$ are distinct, so $\deg(\alpha) = n$. Since $[L : F] = [F(\alpha) : F] = n$, then we have $L = F(\alpha)$. $\square$

Definition 5.9.6 (Radical Extension)**.** *A field extension L/F is radical if there is a tower of field extensions $F = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_m = L$ such that F_{i+1}/F_i is n_i -radical for some n_i, $i = 0, 1, \cdots, m-1$.*

Property 5.9.7. *1. If K/F and L/K are radical, then so is L/F.*

2. Suppose $L = F(\alpha_1, \cdots, \alpha_m)$ with $\alpha_i^{n_i} \in F$, then L/F is radical.

3. If K/F is radical and L/F is any field extension, then KL/L is radical.

Lemma 5.9.8. *Every radical field extension L/F is contained in a normal radical field extension E/F.*

Proof. Let $L = F(\alpha_1, \cdots, \alpha_m)$ with $\alpha_i^{n_i} \in E_{i-1} = F(\alpha_1, \cdots, \alpha_{i-1})$ for each i, and let $L = E_m$. We induct on m. When $m = 0$, we take $E = L = F$. Suppose the result holds for $m-1$. Then we can embed E_{m-1}/F into a normal radical extension K_{m-1}/F.

$$\begin{tikzcd}
 & K_m \arrow[dd, no head] \\
E_m \arrow[ur, dashed, "\psi"] \arrow[dd, no head] & \\
 & K_{m-1} \arrow[dl, no head] \arrow[ddl, no head] \\
E_{m-1} \arrow[d, no head] & \\
F &
\end{tikzcd}$$

Let K_{m-1} be the splitting field of $g \in F[x]$ over F, so K_{m-1}/F is Galois with $G = \mathrm{Gal}(K_{m-1}/F)$. Let $L = E_{m-1}(\alpha)$ with $\alpha^n = a \in E_{m-1}$ for some n and a. Let $H = \prod_\sigma \sigma(m_\alpha) \in F$. Define K_m to be the splitting field of H over K_{m-1}. Then gh splits in $K_m[x]$ and $gh \in F[x]$, and K_m is generated over F by all roots of gh, as the roots of g generate K_{m-1} over F and the roots of h generate K_m over K_{m-1}. Hence K_m/F is normal, so it remains to find an embedding of E_m into K_m. Since $f = m_\alpha \mid h$ and h is split over K_m, in particular f has a root in K_m. Using this root, we embed E_m into $K_m>$ To see that K_m/F is radical, we have that K_m is generated over K_{m-1} by the roots of h. If β is a root of h, then $h(\beta) = 0$, so $(\psi f)(\beta) = 0$, hence $f(\psi^{-1}\beta) = 0$. Since $f \mid x^n - a$, we have $\psi^{-1}(\beta)^n = a$, so $\beta^n = \psi(a) \in K_{n-1}$. Thus, $K_m = K_{m-1}(\beta)$ is n-radical. ☐

Definition 5.9.9 (Solvable)**.** *Let $f \in F[x]$ be a polynomial over a field (of characteristic 0). We say that the equation $f(x) = 0$ is solvable by radicals if f is split in a radical extension of F.*

Theorem 5.9.10. *A non-constant polynomial $f \in F[x]$ in a field of characteristic 0 is solvable by radicals if and only if Gal(f) is solvable.*

Proof. ($\Rightarrow$): Let L/F be a radical field extension such that f is split over L. By the lemma last time, we may assume that L/F is normal, and therefore Galois. Since L/F is radical, then there exists $F_0 = F \subseteq F_1 \subseteq \cdots \subseteq F_m = L$ such that $F_{i+1} = F_i(\alpha_i)$ where $\alpha_i^{n_i} \in F_i$.

Let n be the least common multiple of n_i's. Let $F' = F(\xi_n)$ be the cyclotomic extension of F over nth roots of unity, and similarly $L' = L(\xi_n)$. Note that $L/E/F$ is a field extension where E is the splitting field of f over F.

We now construct $F_0' = F' \subseteq F_1' \subseteq \cdots \subseteq F_m' = L'$ where $F_{i+1}' = F_i'(\alpha_i)$.

Note that L/F is Galois, then L'/F' is also Galois. Let $G = \mathrm{Gal}(L'/F')$ and $H_i = \mathrm{Gal}(L'/F_i')$. Then $G = H_0 \supseteq H_1 \supseteq \cdots \supseteq H_m = \{e\}$. Since $\xi_n \in F'$, each F_{i+1}'/F_i' is Galois cyclic, so $H_{i+1} \lhd H_i$ with $\mathrm{Gal}(F_{i+1}'/F_i') = H_i/H_{i+1}$ cyclic. Hence G is solvable. The extension F'/F is cyclotomic, hence Abelian, so L'/F is solvable as well. Therefore, E/F is solvable because it is a factor group of $\mathrm{Gal}(L'/F)$.

($\Leftarrow$): Suppose E/F is the splitting field of $f \in F$, take $G = \mathrm{Gal}(f) = \mathrm{Gal}(E/F)$ to be solvable. Let $n = |G|$, then we write $F' = F(\xi_n)$ and $E' = E(\xi_n)$. Since $\mathrm{Gal}(E/F)$ is solvable, $\mathrm{Gal}(E'/F') \hookrightarrow \mathrm{Gal}(E/F)$ is solvable. Take a descending sequence of subgroups $\mathrm{Gal}(E'/F') = H_0 \lhd H_1 \lhd \cdots \lhd H_m = \{e\}$ with H_i/H_{i+1} cyclic. Setting $F_i' = (E')^{H_i}$, we obtain a tower of cyclic extensions $F_0' = F' \subseteq F_1' \subseteq \cdots \subseteq F_m' = E'$, and $\mathrm{Gal}(F_{i+1}'/F_i') \cong H_i/H_{i+1}$ is cyclic, so F_{i+1}'/F_i' is n_i-radical, where $n_i = [F_{i+1}'" F_i]$. Therefore, E'/F' is radical. Since $F' = F(\xi_n)$ is radical, then E'/F is radical, but $E \subseteq E'$, then E/F is radical. Hence, f is solvable by radicals. □

Example 5.9.11. *Denote $f = F[x]$ to be a non-constant polynomial. Let E/F be the splitting field so that G = Gal(E/F) = Gal(f). Consider the set of roots of f in E given by $X = \{\alpha_1, \cdots, \alpha_n\}$ where $n \leq \deg(f)$. Take $\sigma \in G$, then $\sigma(\alpha_i) = \alpha_j$ for some j. Consider G acting on the set X, then there is an injective map $G \to S(X) = S_n$ since E is generated by the roots. Therefore, we can consider $G \hookrightarrow S_n$ as a subgroup.*

For example, consider $f = x^n - 1$ over $\mathbb{Q}$, then $G = (\mathbb{Z}/n\mathbb{Z})^\times \hookrightarrow S_n$. Or suppose f is generic, then $G = S_n$. In particular, if $n \leq 4$, S_n is solvable, so G is solvable, then f is solvable by radicals. If $n \geq 5$, then S_n is not solvable, so the generic f of degree n is not solvable by radicals. Therefore, in this case we cannot write down the roots in radicals.

Proposition 5.9.12. *Let $f \in \mathbb{Q}[x]$ be irreducible and $\deg(f) = p$ prime. Assume that f has exactly two non-real roots, then G = Gal(f) = S_p.*

Proof. By action on the group, we have $G \hookrightarrow S_p$. Because f is irreducible, G acts transitively on the set of p roots of f over the splitting field. Let $H \subseteq G$ be the stabilizer of some root with $[G : H] = |$ orbit of the root $| = p$. Therefore, $p \mid |G|$. By Cauchy Theorem, there exists $\sigma \in G \subseteq S_p$ such that the order is p. Therefore, σ is a p-cycle.

Moreover, note that complex conjugation τ is also in G and in S_p, it is a transposition since f has exactly two non-real complex roots, which are conjugate. However, we know that S_p is generated by a p-cycle and a transposition, so $G = S_p$. $\square$

Example 5.9.13. *$f = x^5 - 4x + 2 \in \mathbb{Q}[x]$ is irreducible with two non-real roots, so it is not solvable by radicals over $\mathbb{Q}$.*

Lemma 5.9.14. *For every finite Abelian group G, there exists n such that there is a surjective homomorphism $(\mathbb{Z}/n\mathbb{Z})^\times \twoheadrightarrow G$.*

Proof. Write $G = \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_s\mathbb{Z}$. Find distinct primes $p_1, \cdots, p_s$ such that $p_i \equiv 1 \pmod{m_i}$. Take $n = p_1 \cdots p_s$. $\square$

Corollary 5.9.15. *For every finite Abelian group G, there is an extension $E/\mathbb{Q}$ with Galois group G.*

5.10 Kummer Theory

Definition 5.10.1 (Kummer Extension)**.** *Let F be a field and $n > 0$ is an integer, and the characteristic of F does not divide n. Also assume that $\xi_n \in F$. Pick $a_1, a_2, \cdots, a_m \in F^\times$ and let L/F be a splitting field of the polynomial $(x^n - a_1)(x^n - a_2)\cdots(x^n - a_m)$, which is separable. Therefore, the extension is Galois. We want to study the Galois group $G = Gal(L/F)$. In particular, denote $L = F(\sqrt[n]{a_1}, \cdots, \sqrt[n]{a_m})$.*

In particular, L/F is called a Kummer extension.

Example 5.10.2. *$\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6})$ is a Kummer extension over $\mathbb{Q}$.*

Remark 5.10.3. *Let $A \subseteq F^\times$ be a subgroup generated by $(F^\times)^n$ and $a_1, a_2, \cdots, a_m$. Therefore $F^{\times n} \subseteq A \subseteq F^\times$, then $A/F^{\times n} \subseteq F^\times/F^{\times n}$, where the latter is a $\mathbb{Z}/n\mathbb{Z}$-module.*

By taking $a \in A$, we have $a = x^n$ for some $x \in L^\times$. Therefore note that $a_i = (\sqrt[n]{a_i})^n$ in L.

Observe that for any $\sigma \in G$, we have $(\frac{\sigma x}{x})^n = \frac{\sigma(x^n)}{x^n} = \frac{\sigma(a)}{a} = \frac{a}{a} = 1$, so $frac\sigma xx \in \mu_n \subseteq F^\times$ is a root of unity.

Suppose $x^n = a = y^n$, then $y = \xi \cdot x$ for some $\xi \in \mu_n \subseteq F^\times$, therefore $\frac{\sigma(y)}{y} = \frac{\sigma(\xi)\cdot\sigma(x)}{\xi\cdot x} = \frac{\sigma(x)}{x}$, which means the $\sigma(x)$ does not depend on choice of x. Therefore, we have $A \to \mu_n$ where $a \mapsto \frac{\sigma x}{x}$ where $x^n = a$ and $x \in L^\times$. Suppose we have $b \mapsto \frac{\sigma y}{y}$ where $y^n = b$ so $(xy)^n = ab$. Then $ab \mapsto \frac{\sigma(xy)}{xy} = \frac{\sigma x}{x} \cdot \frac{\sigma y}{y}$. Hence, the map is a homomorphism.

Also note that for $a \in A$ we have $a^n \mapsto \frac{\sigma(a)}{a} = 1$ and take $x = a \in F^\times$, so A^n is contained in the kernel of the map. Therefore, $A/A^n \to \mu_n$ is a well-defined homomorphism for all $\sigma \in G$, by sending $aA^n \mapsto \frac{\sigma x}{x}$ with $x^n = a$.

We now have a canonical map $G \times (A/A^n) \to \mu_n$ by sending $(\sigma, aA^n) \mapsto \frac{\sigma x}{x}$ where $x^n = a$. This is a homomorphism if we fix the first argument, and is linear if we fix the second argument (so bilinear).

Indeed, for $\sigma, \tau \in G$ and $a \in A$, we have $(\sigma\tau, a) \mapsto \frac{\sigma\tau(x)}{x} = \frac{\sigma\tau(x)}{\sigma(x)} \cdot \frac{\sigma(x)}{x} = \sigma(\frac{\tau(x)}{x}) \cdot \frac{\sigma(x)}{x} = \frac{\sigma(x)}{x} \cdot \frac{\tau(x)}{x}$ because $\frac{\tau(x)}{x} \in \mu_n \subseteq F^\times$. This map structure is called a pairing.

We can then construct a homomorphism $\varphi : G \to \mathbf{Hom}(A/(F^\times)^n, \mu_n)$ that sends $\sigma \mapsto (\bar{a} \mapsto \frac{\sigma x}{x})$.

We want to understand $B^* = \mathbf{Hom}(B, \mu_n)$ where $n \cdot B = 0$. This is called the characteristic group of B. If $B = \mathbb{Z}/k\mathbb{Z}$ where $k \mid n$, then $\mathbf{Hom}(\mathbb{Z}/k\mathbb{Z}, \mu_n) = \mu_k$. In particular, B^* is a cyclic group of order k. Therefore, $B^* \cong B$, but not canonically.

In general, if B is finite and Abelian, then $B = \coprod C_i$ where C_i are cyclic groups such that $n \cdot C_i = 0$. Then $B^* \cong \coprod C_i^* \cong \coprod c_i \cong B$ in a non-canonically way.

In particular, observe that $\mathbf{Hom}(A/(F^\times)^n \cong A/(F^{\times n})$.

Claim 5.10.4. *φ is injective.*

Proof. Take $\sigma \in \ker(\varphi)$, with $x_i = \sqrt[n]{a_i} \in K$. Then $\sigma(\bar{a}_i) = \frac{\sigma x_i}{x_i} = 1$ so $\sigma(x_i) = x_i$ for all i. Since $K = F(x_1, \cdots, x_m)$, then $\sigma = \beth$. □

In particular, $G \hookrightarrow A/(F^{\times n})$, and since G is Abelian, then $G^n = e$ and $|G| \leq |A/(F^{\times n})|$.

Denote $\psi : A/(F^\times)^n \to \mathbf{Hom}(G, \mu_m)$.

Claim 5.10.5. *ψ is injective.*

Proof. Let $\bar{a} \in A/(F^\times)^n$ such that $\psi(\bar{a}) = e$.

$\psi(\bar{a})$ takes $\frac{\sigma x}{x}$ where $x^n = a$. But since $\psi(\bar{a}) = e$, then $\frac{\sigma x}{x} = 1$ for all σ, hence $x \in K^G = F$, so $\alpha = x^n \in (F^\times)^n$, so $\bar{a} = e$. □

Note $\mathbf{Hom}(G, \mu_m) = G^* \cong G$. Then $|A/(F^\times)^n| \leq |G|$, so $|A/(F^\times)^n| = |G|$. Therefore, φ and ψ are isomorphisms.

Theorem 5.10.6 (Kummer)**.** *Let F be a field and $n > 0$ is an integer, with char$(F) \nmid n$. Let $a_1, \cdots, a_n \in F^\times$, and K is the splitting field of $(x^n - a_1)(x^n - a_2) \cdots (x^n - a_m)$. Then K/F is Galois and the map $\varphi : G \to \mathbf{Hom}(A/(F^\times)^n, \mu_n)$ is an isomorphism, where $G =$ Gal(K/F). ($A \subseteq F^\times$ is a subgroup generated by $(F^\times)^n$ and a_i.)*

Example 5.10.7. *Consider Gal$(\mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{6}, \sqrt{10}, \sqrt{15})/\mathbb{Q}$. We just need to look at $\mathbb{Q}^\times/\mathbb{Q}^{\times 2} \ni \langle \bar{2}, \bar{5}, \bar{6}, \bar{10}, \bar{15} \rangle$, which is a vector space over $\mathbb{F}_2$.*

So we can write $\{\bar{-1}, \bar{2}, \bar{3}, \bar{5}, \cdots\}$ as a basis of $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$, so $\bar{2} = \bar{2}$, $\bar{5} = \bar{5}$, $\bar{6} = \bar{2} \cdot \bar{3}$, $\bar{10} = \bar{2} \cdot \bar{5}$ and $\bar{15} = \bar{3} \cdot \bar{5}$, then we can express these elements by basis elements $\{2, 3, 5\}$.

Therefore, Gal$(\mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{6}, \sqrt{10}, \sqrt{15})/\mathbb{Q} = (\mathbb{Z}/2\mathbb{Z})^3$.

Remark 5.10.8. *Suppose $\sigma \in G$, then $\varphi(\sigma) = f : A/(F^\times)^n \to \mu_n$. By taking $a_i \in A$, then $x_i = \sqrt[n]{a_i}$ satisfies $x_i^n = a_i$. Now $\sigma(x_i) = \sigma(\sqrt[n]{a_i}) = f(\bar{a}_i)$. Then $\sigma(\sqrt[n]{a_i}) = f(\bar{a}_i) \cdot \sqrt[n]{a_i}$ where $f(\bar{a}_i) \in \mu_n$.*

5.11 Infinite Galois Field Extensions

Definition 5.11.1. *Consider L/F to be an algebraic extension, but should be an infinite extension. We say L/F is separable if every element of L is separable over F. L/F is normal if the following equivalent conditions holds:*

1. *L is a splitting field of a set of polynomials over F.*
2. *Every irreducible polynomial in $F[x]$ that has a root in L is split over L.*
3. *L is the union of all subfields K such that $F \subseteq K \subseteq L$ such that K/F is finite and normal.*

We say L/F is Galois if L/F is separable and normal. Denote $G = Gal(L/F)$ to be the group of automorphisms $\sigma : L \to L$ that is identity over F.

Remark 5.11.2. *Let L/F be a Galois field extension with $G = Gal(L/F)$. We can write $L = \bigcup_{i \in I} L_i$ where L_i/F is finite and Galois. Take $\sigma \in G$, then $\sigma(L_i) = L_i$, so we can restrict σ to L_i and get an automorphism $\sigma\mid_{L_i}: L_i \to L_i$ over F. Therefore, we have $G \to Gal(L_i/F)$ that sends $\sigma \mapsto \sigma\mid_{L_i}$.*

Suppose I is ordered, with $i < j$ if $L_i \subseteq L_j$. We can view I as a small preorder category, with $L_j \to L_i$ a morphism in the category if $L_i \subseteq L_j$. Then we have $G \to \prod_{i \in I} Gal(L_i/F)$, with $(\sigma\mid_{L_j}\mid_{L_i} = \sigma\mid_{L_i}$, then we have the map $G \to \lim_{i \in I} Gal(L_i/F)$ and can be expressed explicitly as $\{(\sigma_i)_{i \in I} : \sigma_j\mid_{L_i} = \sigma_i \text{ when } i < j\}$, expressed as an inverse limit of finite groups. We can show that this map is an isomorphism. Then G is a profinite group.

Moreover, this is a topological group. We observe that the finite group has discrete topology, then the product of those groups gives a product topology. The product of compact spaces is still quasi-compact. The limit is a closed subset in the product, so it is also quasi-compact. Therefore, G is quasi-compact. (Profinite groups are quasi-compact.) For example, $\mathbb{Z}$ cannot be a Galois group because we cannot introduce any non-trivial topology on $\mathbb{Z}$ so that it becomes quasi-compact.

Also, G is Hausdorff.

Definition 5.11.3 (Profinite Group)**.** *A group that is isomorphic to a limit of finite groups is called a profinite group.*

Remark 5.11.4. *Suppose L/F is a Galois field extension, and $G = Gal(L/F)$, and let $L/K/F$ be an intermediate extension. Then L/K is also Galois and $H = Gal(L/K) \subseteq G$ is a subgroup. Therefore, $H = \{\sigma \in G : \sigma\mid_K = \mathbf{id}_K\}$. But K is a union of finite field extensions, so $K = \bigcup_i K_i$, where K_i/F is finite, then $H = \{\sigma \in G : \sigma\mid_K = \mathbf{id}_K\} = \bigcap_i \{\sigma \in G : \sigma\mid_{K_i} = \mathbf{id}_{K_i}\} = \bigcap_i Gal(L/K_i)$. Moreover, note that $H_i = Gal(L/K_i) \subseteq G$ is open in the topology, and $G = \bigcup_i gH_i$ where each coset is open, and there are finitely many of them, so the coset gH_i is closed. Therefore, $H = Gal(L/K)$ is closed in G.*

Theorem 5.11.5. *Suppose $G = Gal(L/F)$, then the set of intermediate fields in L/F and the set of closed subgroups in G are isomorphic: on one hand, we send an intermediate field K to $Gal(L/K)$, and on the other hand, we send a closed subgroup H to L^H. The two maps are bijection inverses of each other.*

Example 5.11.6.

1. *For a field F embedded into the algebraic closure $F \hookrightarrow F_{alg}$, this embedding is not a Galois extension because it is not necessary separable. Instead, we take $F_{sep} \subseteq F_{alg}$ of all separable elements. Then F_{sep}/F is Galois, and F_{sep} is called the separable closure of F. Therefore, $\Gamma_F := Gal(F_{sep}/F)$ is called the absolute Galois group of field F.*

   In particular, $Gal(K/F) = \Gamma_F/\Gamma_K$, with $F_{sep} = K_{sep}$. We don't really understand the structure, even for $\Gamma_{\mathbb{Q}}$ at this point.

2. $\Gamma_{\mathbb{R}} = \mathbb{Z}/2\mathbb{Z}$.

3. *Suppose F is a finite field $\mathbb{F}_q$, then there exists exactly one extension of this field of degree n, namely $\mathbb{F}_{q^n}/\mathbb{F}$. The Galois group of this field extension is canonically isomorphic to $\mathbb{Z}/n\mathbb{Z}$. Then $\Gamma_F = \lim \mathbb{Z}/n\mathbb{Z} = \{(a_n \in \mathbb{Z}/n\mathbb{Z}) : \forall k \mid n, a_k \equiv a_n \pmod{k}\mathbb{Z}\}$. This group is known as the completion of $\mathbb{Z}$, namely the group of profinite integers $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, as the product of all p-adic integers with prime p. This group has the cardinality continuum. We then have $\mathbb{Z} \hookrightarrow \hat{\mathbb{Z}}$ as a dense embedding. Note $\mathbb{Z}$ is not a Galois group but $\hat{\mathbb{Z}}$ is an absolute Galois group.*

6 Hilbert's Nullstellensatz

6.1 Hilbert Basis Theorem

Definition 6.1.1 (ACC,DCC)**.** *Let R be a ring and M is a (left) R-module.*

The ascending chain condition (ACC) is that every sequence $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq \cdots$ of submodules of M is stable, i.e. there exists some $m \in \mathbb{N}$ such that $M_k = M_m$ for all $k \geq m$.

The descending chain condition (DCC) is that every sequence $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ of submodules of M is stable, i.e. there exists some $m \in \mathbb{N}$ such that $M_k = M_m$ for all $k \geq m$.

Proposition 6.1.2. *Let R be a ring and M is a (left) R-module. The following are equivalent:*

1. *ACC (respectively, DCC) condition.*
2. *Every non-empty set of submodules of M has a maximal (respectively, minimal) element.*

Definition 6.1.3. *Let R be a ring. Let M be a (left) R-module. We say M is Noetherian (respectively, Artinian) if M satisfies ACC (respectively, DCC).*

R is a (left) Noetherian (respectively, Artinian) if R as a (left) module over R is Noetherian (respectively, Artinian).

Example 6.1.4.

1. *Fields are Noetherian and Artinian.*
2. *$\mathbb{Z}$ is Noetherian but not Artinian.*

Proposition 6.1.5. *Let $0 \to N \to M \xrightarrow{f} P \to 0$ be a short exact sequence of (left) R-modules. Then M is Noetherian (respectively, Artinian) if and only if both N and P are.*

Proof. We only prove the case for Noetherian. The case for Artinian is analogous.

($\Rightarrow$): consider $N_1 \subseteq N_2 \subseteq \cdots \subseteq N \hookrightarrow M$ and $P_1 \subseteq P_2 \subseteq \cdots \subseteq P$, so the sequence is stable and N is Noetherian. Moreover, consider $f^{-1}(P_1) \subseteq f^{-1}(P_2) \subseteq \cdots \subseteq f^{-1}(P) = M$, then it is stable, and so $\{P_i\}_{i\geq 1}$ is stable.

($\Leftarrow$): Take $M_1 \subseteq M_2 \subseteq \cdots \subseteq M$. Then $f(M_1) \subseteq f(M_2) \subseteq \cdots \subseteq P$ is stable. Hence, $M_1 \cap N \subseteq M_2 \cap N \subseteq \cdots \subseteq N$ is stable. Therefore, there exists some n such that $f(M_k) = f(M_n)$ and $M_k \cap N = M_n \cap N$ for all $k \geq n$. Therefore, $M_k = M_n$ for all $k \geq n$, so $\{M_n\}_{n\geq 1}$ is stable. $\square$

Corollary 6.1.6. *If $M_1, \cdots, M_n$ are Noetherian (respectively, Artinian), then so is $M_1 \oplus M_2 \oplus \cdots \oplus M_n$.*

Proposition 6.1.7. *Suppose $f : R \to S$ is a surjective ring homomorphism. Let M be a (left) S-module. Then M is a Noetherian (respectively, Artinian) S-module if and only if M is a Noetherian (respectively, Artinian) R-module.*

Proof. Again, we only prove the case for Noetherian. A similar proof works for the Artinian case.

($\Rightarrow$): Consider $N_1 \subseteq N_2 \subseteq \cdots \subseteq M$ as a chain of R-submodules. These are S-submodules by surjectivity of f. Therefore, we conclude stability.

($\Leftarrow$): Consider $M_1 \subseteq M_2 \subseteq \cdots \subseteq M$ as S-submodules. Then they are also R-submodules, so they are stable as well. $\square$

Corollary 6.1.8. *Suppose $f : R \to S$ is a surjective ring homomorphism. If R is (left) Noetherian (respectively, Artinian), so is S.*

Proof. Because R is Noetherian (respectively, Artinian), then S is also Noetherian (respectively, Artinian) as R-module, then S is Noetherian (respectively, Artinian) as a S-module by proposition. $\square$

Proposition 6.1.9. *Suppose R is a (left) Noetherian (respectively, Artinian). Then every finitely generated (left) R-module is Noetherian (respectively, Artinian).*

Proof. If R is Noetherian (respectively, Artinian), then R^n is also Noetherian (respectively, Artinian). Therefore, the factor module $M = R^n/N$ is Noetherian (respectively, Artinian). $\square$

Proposition 6.1.10. *Every Noetherian R-module is finitely generated.*

Remark 6.1.11. *This proposition acts as the converse of the previous proposition, and it only holds for Noetherian modules.*

Proof. Suppose M is a Noetherian R-module that is not finitely generated, then we consider the following chain: we first take $N_1 = Rm_1$ for $m_1 \in M$. Then $N_1 \neq M$. Take $N_2 = Rm_1 + Rm_2$ for $m_2 \in M \backslash N_1$. Then $N_2 \neq M$. We proceed inductively, and we get a chain of modules $N_1 \subseteq N_2 \subseteq \cdots$ that is not stable and each module is finitely generated. $\square$

Proposition 6.1.12. *Let R be a (left) Noetherian ring. Every submodule of a finitely-generated (left) R-module is finitely generated.*

Proof. Suppose we have a submodule $N \subseteq M$ where M is a finitely-generated module. Then M is Noetherian, so N is Noetherian, then N is finitely generated. $\square$

Proposition 6.1.13. *Let R be a ring. It is a (left) Noetherian ring if and only if every ideal of R is finitely generated.*

Proof. ($\Rightarrow$): Take $I \subseteq R$ as a (left) ideal, then it is a (left) R-module, so it is finitely generated.

($\Leftarrow$): Consider a chain $I_1 \subseteq I_2 \subseteq \cdots \subseteq R$, then $I = \bigcup_{k=1}^{\infty} I_k$ is a (left) ideal, and is finitely generated. Let $I = (x_1, \cdots, x_n)$. Then $\{x_1, \cdots, x_n\} \subseteq I_N$ for some N. Therefore, $I = I_N = I_{N+1} = \cdots$, so the sequence is stable. □

Theorem 6.1.14 (Hilbert Basis Theorem)**.** *Let R be a (left) Noetherian ring. Then so is $R[x_1, \cdots, x_n]$.*

Proof. It suffices to show that $R[x]$ is a Noetherian ring, the rest just follows from induction. We show that every left ideal is finitely generated. Let $I \subseteq R[x]$ be a left ideal. Now for all $f \in I$, we write $f = a_n x^n + \cdots + a_0$. Look at $J = \{$ highest coefficients $a_n \in I\} \subseteq R$. Then J is a (left) ideal, so it is finitely generated by $\{a_1, \cdots, a_n\}$. Let $f_i \in I$ have highest coefficient a_i and degree n_i. Let $n = \max_i n_i$.

Consider $M = R \oplus Rx \oplus \cdots Rx^{n-1}$. It is a free finitely generated R-submodule of $R[x]$ because it is Noetherian.

Therefore, $I \cap M \subseteq M$ is finitely generated by $g_1, \cdots g_s$ as a R-module.

Claim 6.1.15. *I is generated by $f_1, \cdots, f_m, g_1, \cdots, g_s$ as a $R[x]$-module. Take $h \in I$. We induct on* $\deg(h)$.

Subproof. Suppose $\deg(h) < n$, then $h \in I \cap M$ and is generated by $g_1, \cdots, g_s$.

Suppose $\deg(h) \geq n$, then we use induction. Let $h = ax^r + \cdot$, then a is generated by $a_1, \cdots, a_n$. Then there is some linear combination $\sum_i b_i x^{r_i} f_i$ with highest term ax^r. Therefore, $h - \sum_i b_i x^{r_i} f_i$ has degree one less. By induction, it is generated by f_i's and g_j's. Therefore, h is generated by f_i's and g_j's. ■

□

6.2 Hilbert's Nullstellensatz

Definition 6.2.1 (Finite, Finite Type)**.** *Suppose there is a commutative ring S with a commutative subring R. We say S is finite overR is S is a finitely generated R-module. So there exists $s_1, \cdots, s_n$ such that for all $s \in S$, $s = \sum_i \tau_i s_i$ where $\tau_i \in R$.*

We say S is of finite type over R if S is finitely generated as a ring over R. Therefore, there exists $s_1, \cdots, s_n \in S$ such that for all $s \in S$, $s = f(s_1, \cdots, s_n)$ for some $f \in R[x_1, \cdots, x_n]$.

Remark 6.2.2. *Finite implies finite type, but not the other way around.*

Hilbert's Nullstellensatz shows when does the two notions become the same.

Corollary 6.2.3 (From Hilbert's Basis Theorem)**.** *Let $R \subseteq S$ be commutative rings, S of finite type over R. If R is Noetherian, then so is S.*

Proof. Let $s_1, \cdots, s_n \in S$ be generators. Then there is a surjective homomorphism given by $R[x_1, \cdots, x_n] \twoheadrightarrow S$ given by $x_i \mapsto s_i$. Note that by Hilbert's Basis theorem, we know $R[x_1, \cdots, x_n]$ is a Noetherian ring. Therefore, S is Noetherian. $\square$

Lemma 6.2.4. *Let $R \subseteq S \subseteq T$ be commutative rings such that*

1. *R is Noetherian.*
2. *T is of finite type over R.*
3. *T is finite over S.*

Then S is of finite type over R.

Proof. Let $T = R[x_1, \cdots, x_n]$ where x_i's are generators of T over R. Moreover, $T = \sum_{j=1}^{m} S \cdot y_j$ for $y_j \in T$. Then $x_i = \sum_j a_{ij} y_j$ where $a_{ij} \in S$ and $y_i y_j = \sum_k b_{ijk} y_k$ where $b_{ijk} \in S$.

Let $S_0 = R[a_{ij}, b_{ijk}]$ be generated by the two sets of coefficients. Then $R \subseteq S_0 \subseteq S \subseteq T$. By the corollary, S_0 is Noetherian.

Claim 6.2.5. *T is finite over S_0. We can show that $T = \sum_j S_0 \cdot y_j$.*

Subproof. Recall that $\sum S_0 y_j$ and $y_i y_j \in \sum S_0 \cdot y_j$. Then $y_i y_j y_k \in \sum_i S_0 y_j y_k \in \sum S_0 y_j$. ■

Now $S \subseteq T$ is a S_0-submodule, and T is a finitely generated S_0-module.

Since S_0 is Noetherian, S is a finitely-generated S_0-module. Therefore, S is finite over S_0, and so it is of finite type over S_0. However, S_0 is finite type over R, so S is of finite type over R. $\square$

Proposition 6.2.6. *Let E/F be a field extension such that E is of finite type (as a ring) over F. Then E is finite over F, i.e. $[E : F] < \infty$.*

Proof. We first prove a special case, that is suppose $E = F(x_1, \cdots, x_n)$ where x_i's are algebraically independent.

Claim 6.2.7. *$E = F$.*

Proof. Since E is of finite type over F, then $E = F[f_1, \cdots, f_m]$ where $f_i = \frac{g_i}{h}$ for $g_i, h \in F[x_1, \cdots, x_n]$. Note that every element in E is of the form $\frac{g}{h^k}$, for $g \in F[x_1, \cdots, x_n]$. Suppose $n > 0$, then there exists an irreducible polynomial $p \in F[x_1, \cdots, x_n]$ such that $p \nmid h$. Therefore, $\frac{1}{p}$ is not of the form $\frac{g}{h^k}$ in E, contradiction. Therefore, $n = 0$. Hence, $E = F$. ■

We now prove the general case, with $E = F[f_1, \cdots, f_m] = F(f_1, \cdots, f_m)$. Choose a maximal algebraically independent subset in $\{f_1, \cdots, f_m\}$, denoted $\{f_1, \cdots, f_k\}$ without loss of generality.

Let $K = F(f_1, \cdots, f_k)$, then $K \cong F(x_1, \cdots, x_k)$. On the other hand, if we add f_i to the set where $i > k$, then the set is algebraically dependent. Therefore, f_i is algebraic over $\{f_1, \cdots, f_k\}$. Hence,

f_i is algebraic over K for all $i \in \{1, \cdots, m\}$. Therefore, E/K is algebraic and is finitely-generated. Therefore, E/K is finite.

By lemma, K is of finite type over F. But $K = F(x_1, \cdots, x_k)$ is also purely transcendental. By the special case, $K = F$, so $[E : F] < \infty$. $\square$

Theorem 6.2.8 (Hilbert's Nullstellensatz, Weak Form). *Let F be an algebraically closed field. Let $f_1, \cdots, f_m \in F[x_1, \cdots, x_n] = R$. The following are equivalent:*

1. *There is no $a = (a_1, \cdots, a_n) \in F^n$ such that $f_i(a) = 0$ for all i.*
2. *f_i generates the unit ideal in $R = F[x_1, \cdots, x_n]$.*

Proof. (2) $\Rightarrow$ (1): If we have a linear combination $\sum\limits_i f_i g_i = 1$, then $\sum\limits_i f_i(a) g_i(a) = 1$. Then there exists i such that $f_i(a) \neq 0$.

(1) $\Rightarrow$ (2): Suppose $R \neq \sum\limits_i f_i R \subseteq M$ for some maximal ideal M. Now $F \hookrightarrow R \twoheadrightarrow R/M$ into the field gives a field extension of F and is of finite type over F. By proposition, it is a finite field extension. But F is algebraically closed. Then it is the trivial extension. Hence, $F \xrightarrow{\cong} R/M$ that sends $a_i \mapsto \bar{x}_i$, $f_j \mapsto \overline{f_i(x_1, \cdots, x_n)} = 0$ since f_j's are in M.

Then $f_i(a) = 0$ for all j, contradiction. $\square$

Remark 6.2.9. *If F is not algebraically closed, e.g. $\mathbb{R}$, we have $x^2 + 1$ with no roots, but it does not generate the unit ideal.*

Theorem 6.2.10 (Hilbert's Nullstellensatz, Alternate Weak Form). *Let K be a field and L is a K-algebra such that L is finitely-generated as a K-algebra and is a field, then L is algebraic over K, and L/K is a finite field extension.*

Corollary 6.2.11. *Suppose, in addition to the above alternate form, that K is algebraically closed, then every maximal ideal of $A = K[X_1, \cdots, X_n]$ is of the form*

$$\mathfrak{m} = (X_1 - a_1, \cdots, X_n - a_n)$$

for some $a_1, \cdots, a_n \in K$; the map $K[X_1, \cdots, X_n] \to K[X_1, \cdots, X_n]/\mathfrak{m} = K$ is given by the natural evaluation map. Hence, there is a natural one-to-one correspondence between K^n and ideals A in Spec$(\mathfrak{m})$ given by $(a_1, \cdots, a_n) \leftrightarrow (X_1 - a_1, \cdots, X_n - a_n)$.

Definition 6.2.12 (Variety). *Let K be a field. A variety $V \subseteq K^n$ is a subset of the form*

$$V = V(J) = \{P = (a_1, \cdots, a_n) \in K^n \mid f(P) = 0 \ \forall f \in J\},$$

where $J \subseteq K[X_1, \cdots, X_n]$ is an ideal. Note that $J = (f_1, \cdots, f_m)$ is finitely generated, so that a variety V is defined by

$$f_1(P) = \cdots = f_m(P) = 0,$$

that is, it is a subset $V \subseteq K^n$ defined as the simultaneous solutions of a number of polynomial equations.

Proposition 6.2.13. *Suppose K is an algebraically closed field and that $A = K[x_1, \cdots, x_n]$ is a finitely-generated K-algebra of the form $A = K[X_1, \cdots, X_n]/J$ where J is an ideal of $K[X_1, \cdots, X_n]$, then every maximal ideal of A is of the form $(x_1 - a_1, \cdots, x_n - a_n)$ for some point $(a_1, \cdots, a_n) \in V(J)$. Therefore, there is a one-to-one correspondence between $V(J)$ and maximal ideals of A given by $(a_1, \cdots, a_n) \leftrightarrow (X_1 - a_1, \cdots, X_n - a_n)$.*

Proof. The ideals of A are given by ideals of $K[X_1, \cdots, X_n]$ containing J, so every maximal ideal of A is of the form $(x_1 - a_1, \cdots, x_n - a_n)$ for some $a_1, \cdots, a_n$ such that $J \subseteq (X_1 - a_1, \cdots, X_n - a_n)$. However, since $(X_1 - a_1, \cdots, X_n - a_n)$ is just the kernel of the evaluation map on f, it then follows that $J \subseteq (X_1 - a_1, \cdots, X_n - a_n)$ if and only if $f(a_1, \cdots, a_n) = 0$ for all $f \in J$, i.e. $(a_1, \cdots, a_n) \in V(J)$. $\square$

More formally, we have the following correspondence.

Remark 6.2.14. *A variety $X \subseteq K^n$ is by definition equal to $X = V(J)$ for some ideal J of $K[X_1, \cdots, X_n]$, so V gives a map from the set of ideals of $K[X_1, \cdots, X_n]$ to the subsets of K^n. Conversely, there is a map I from subsets of K^n to ideals of $K[X_1, \cdots, X_n]$, defined by taking a subset $X \subseteq K^n$ into the ideal*

$$I(X) = \{f \in K[X_1, \cdots, X_n] \mid f(P) = 0 \ \forall P \in X\}.$$

One important property is that: if $J \subseteq J'$, then $V(J) \supseteq V(J')$; if $X \subseteq Y$, then $I(X) \supseteq I(Y)$. Moreover, $X \subseteq V(I(X))$ for any subset X, and $X = V(I(X))$ if and only if X is a variety. Conversely, $J \subseteq I(V(J))$ for any ideal J.

Theorem 6.2.15 (Hilbert's Nullstellensatz, Strong Form)**.** *Let F be an algebraically closed field. Let $f_1, \cdots, f_m, f \in F[x_1, \cdots, x_n] = R$. The following are equivalent:*

1. *If $a \in F^n$ is such that $f_i(a) = 0$ for all i, then $f(a) = 0$.*
2. *There exists $k > 0$ such that $f^k \in \sum R \cdot f_i$, so f is in the radical $\sqrt{\sum R \cdot f_i}$.*

Proof. (2) $\Rightarrow$ (1): For the k as specified, we have $f^k = \sum\limits_i f_i g_i$, so $f(a)^k = 0$, then $f(a) = 0$.

(1) $\Rightarrow$ (2): Consider $R[t] = F[x_1, \cdots, x_n, t]$. Now let $f_{m+1} = 1 - t \cdot f \in S \ni f_1, \cdots, f_m$. Note $f_1, \cdots, f_{m+1}$ have no common zero: if $f_1(a) = \cdots = f_{m+1}(a) = 0$, then $f(a) = 0$, so $f_{m+1}(a) = 1$. By the weak form of the Nullstellensatz, $f_1, \cdots, f_{m+1}$ generate the unit ideal in S, so $1 = \sum\limits_{i=1}^{m} f_i g_i + f_{m+1} g_{m+1}$ for $g_1, \cdots, g_{m+1} \in S$.

Let $t = \frac{1}{f}$ in $F(x_1, \cdots, x_n)[t]$, then f_{m+1} vanishes: $1 = \sum_{i=1}^{m} f_i \cdot \tilde{g}_i$ where $\tilde{g}_i = \frac{h_i}{f^k}$ where $h_i \in R$. Therefore, $f^k = \sum_{i=1}^{m} f_i h_i \in \sum R \cdot f_i$. □

Theorem 6.2.16 (Hilbert's Nullstellensatz, Alternate Strong Form)**.** *Let K be an algebraically closed field. Then:*

1. *If $J \subsetneq K[X_1, \cdots, X_n]$, then $V(J) \neq \varnothing$.*
2. *$I(V(J))$ is the radical of J. Therefore, for $f \in K[X_1, \cdots, X_n]$, $f(P) = 0$ for all $P \in V$ if and only if $f^n \in J$ for some n.*

Proposition 6.2.17. *Let F be an algebraically closed field and set $R = F[x_1, \cdots, x_m]$, with $a = (a_1, \cdots, a_n) \in F^n$. We define $M_a = \{f \in R : f(a) = 0\}$ to be an ideal in R. Then*

1. *M_a is a maximal ideal in R.*
2. *Every maximal ideal of R is $M - a$ for $a \in F^n$.*

Proof.
1. Denote $\alpha_a : R \twoheadrightarrow F$ that sends $f \mapsto f(a)$ to be a surjective ring homomorphism. By the first isomorphism theorem, $M_a = \ker(\alpha_a)$, so $R/M_a \cong F$ is a field, then M_a is maximal.
2. Take $M \subseteq R$ as a maximal ideal, so $M = \sum_{i=1}^{m} f_i R_i$ for some $f_i \in R$. By Hilbert's Nullstellsatz, there exists $a \in F^n$ such that $f_i(a) = 0$ for all i. Then for all $g \in M$, $g(a) = 0$, so $M \subseteq M_a$. But M is maximal, so $M = M_a$.

□

Definition 6.2.18 (Irreducible Variety)**.** *A variety $X \subseteq K^n$ is irreducible if it is non-empty and not the union of two proper subvarieties, i.e. $X = X_1 \cup X_2$ as varieties if and only if $X = X_1$ or $X = X_2$.*

Proposition 6.2.19. *A variety X is irreducible if and only if $I(X)$ is prime.*

Proof. Set $I = I(X)$; if I is not prime, take $f, g \in A \backslash I$ such that $fg \in I$; now define ideals $J_1 = (I, f)$ and $J_2 = (I, g)$. Since $f \notin I(X)$, then $V(J_1) \subsetneq X$, and similarly $V(J_2) \subsetneq X$, and so $X = V(J_1) \cup V(J_2)$ must be reducible. We can prove the converse in a similar manner. □

Corollary 6.2.20. *Let K be a algebraically closed field. Then there is a one-to-one correspondence between V and I:*

1. *Radical ideals J of $K[X_1, \cdots, X_n]$ corresponds to varieties $X \subseteq K^n$.*
2. *Considering the subsets of the two structure, we have a second correspondence: prime ideals P of $K[X_1, \cdots, X_n]$ corresponds to the irreducible varieties $X \subseteq K^n$.*

Proposition 6.2.21. *Let $A = K[x_1, \cdots, x_n]$ be a finitely generated K-algebra where K is an algebraically closed field; write J for the ideal of relations holding between $x_1, \cdots, x_n$, so that $A = K[X_1, \cdots, X_n]/J$. Then there is the one-to-one correspondence between the prime ideals of A and irreducible subvarieties $X \subseteq V(J)$.*

Proof. We know that maximal ideals correspond one-to-one with points of $V(J)$. Moreover, because prime ideals of A correspond to prime ideals of $K[X_1, \cdots, X_n]$ containing J, then by the above corollary, every prime ideal P of A is of the form $P = I(X)$ modulo J for an irreducible variety $X \subseteq K^n$ with $J \subseteq P = I(X)$. This condition is equivalent to $V(J) \supseteq V(P) = V(I(X)) = X$. $\square$

The concept of variety is deeply connected with Zariski topology.

7 Dedekind Domain

7.1 Definitions

Definition 7.1.1 (Product Ideal)**.** *Let R be a domain, and $I, J \subseteq R$ are ideals, then the product ideal IJ is the ideal generated by xy for $x \in I$ and $y \in J$. Note $xR \cdot yR = xyR$.*

Example 7.1.2. *Consider $R = \mathbb{Z}[\sqrt{-5}]$, but $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$, so R does not have unique factorization. Indeed, note $2R \cdot 3R = (1+\sqrt{-5})R \cdot (1-\sqrt{-5})R$.*

Consider $P_1 = 2R + (1+\sqrt{-5})R = \langle 2, 1+\sqrt{-5}\rangle$. Then

$$2 \in P_1^2 = \left\langle 4, 2+2\sqrt{-5}, (1+\sqrt{-5})^2 = -4+2\sqrt{-5}\right\rangle = 2R.$$

Therefore, $P_1 = 2R$ is not a principal ideal domain.

Consider $P_2 = \langle 3, 1+\sqrt{-5}\rangle$ and $P_3 = \langle 3, 1-\sqrt{-5}\rangle$. Now

$$3 \in P_2 \cdot P_3 = \left\langle 9, 3-3\sqrt{-5}, 3+3\sqrt{-5}, 6\right\rangle = 3R.$$

Also note that $P_1 \cdot P_2 = \left\langle 6, 2+2\sqrt{-5}, 3+3\sqrt{-5}, (1+\sqrt{-5})^2\right\rangle = (1+\sqrt{-5})R$ and $P_1 \cdot P_3 = (1-\sqrt{-5})R$ by similar calculations.

In particular, we have $P_1^2 \cdot P_2P_3 = P_1P_2 \cdot P_2P_3$. Notice that we have uniqueness in this case. The ideal is factored into unique prime ideals, which is the point of Dedekind domains.

Definition 7.1.3 (Divisible)**.** *Let $A, B \subseteq R$ be ideals where $B \neq 0$. We say A is divisible by B if there exists an ideal $C \subseteq R$ such that $A = BC \subseteq B$. We denote $B \mid A$. In particular, $A \subseteq B$.*

Remark 7.1.4. *Notice that $bR \mid aR$ if and only if $aR \subseteq bR$ if and only if $b \mid a$, which holds for principal ideals. However, in general, this is false.*

Example 7.1.5. *Consider $R = F[x,y]$, where $A = xR$, $B = xR + yR$, then $A \subseteq B$ but $B \nmid A$.*

Definition 7.1.6 (Dedekind Ring)**.** *A domain R is a Dedekind ring if for every two ideals $A \subseteq B \subseteq R$, there is an ideal $C \subseteq R$ such that $A = BC$. We say C is the quotient in this case.*

Example 7.1.7. *Every PID is Dedekind.*

Property 7.1.8 (Cancellation Law)**.** *Suppose $A, A', B \subseteq R$ are non-zero ideals for Dedekind ring R. If $AB = A'B$, then $A = A'$.*

Proof. Take $0 \neq b \in B$, then $bR \subseteq B$. Therefore, there exists an ideal $C \subseteq R$ such that $bR = BC$. Then $ABC = A'BC$, so $Ab = A'b$, which means $A = A'$. □

Proposition 7.1.9. *Every ideal of a Dedekind ring R is a finitely generated projective R-module.*

Proof. Take $0 \neq A \subseteq R$ as an ideal. Then $\exists 0 \neq a \in A$, so $aR \subseteq A$. Hence, there exists ideal $B \subseteq R$ such that $aR = AB$. In particular, $a = \sum_{i=1}^{n} x_{-}y_i$ where $x_i \in A$ and $y_i \in B$. Define

$$\begin{aligned} f : R^n &\to A \\ (\tau_1, \cdots, \tau_n) &\mapsto \sum_{i=1}^{n} \tau_i x_i \in A \\ g : A &\to R^n \\ x &\mapsto (\frac{xy_1}{a}, \cdots, \frac{xy_n}{a}) \in R^n \end{aligned}$$

Note $x \in A, y \in B$, then $xy \in AB = aR$, so $\frac{xy}{a} \in R$. Then $(f \circ g)(x) = f(g(x)) = \sum_{i=1}^{n} \frac{xy_i}{a} x_i = x$. Therefore, $f \circ g = 1_A$. We then have

$$\begin{tikzcd} 0 \arrow[r] & \ker(f) \arrow[r] & R^n \arrow[r, "f"] & A \arrow[r] \arrow[l, bend left, "g"] & 0 \end{tikzcd}$$

splits.

Therefore, $R^n = \ker(f) \oplus A$, so A is a finitely-generated projective module. □

Corollary 7.1.10. *Every Dedekind ring is Noetherian.*

Definition 7.1.11 (Krull Dimension)**.** *If R is a commutative ring, consider a chain of $n+1$ prime ideals*

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$$

where we call this chain of length n. The dimension $\dim(R)$ is the maximal length of chain of prime ideals in R. This is the Krull dimension.

Example 7.1.12. *1. For a field F, $\dim(F) = 0$.*

2. For a PID R, $\dim(R) = 1$; $\dim(\mathbb{Z}) = 1$.

3. Suppose R is a domain. Note that 0 is prime. Therefore, $\dim(R) \leq 1$ if and only if every non-zero prime ideal is maximal.

4. $\dim(F[x_1, \cdots, x_n]) = n$.

Proposition 7.1.13. *If R is a Dedekind ring, $\dim(R) \leq 1$.*

Proof. Let $P \subseteq R$ be a non-zero prime ideal. Suppose, towards contradiction, that P is not maximal. Suppose $Q \supseteq P$ a prime ideal, then there exists an ideal $A \subseteq R$ such that $P = Q \cdot A$. Hence, either $Q \subseteq P$ or $A \subseteq P$. If not, then there exists $xinQ \backslash P$ with $y \in A \backslash P$ and $xy \notin P$, contradiction.

Suppose $A \subseteq P$, then $QA \subseteq QP$, but then we know $P = QA \subseteq QP \subseteq P$. Therefore, $QA = QP$, so $A = P$, but $P = QP$, then $RP = P = QP$, which means $R = Q$, contradiction. Therefore, $P = Q$, another contradiction. $\square$

Theorem 7.1.14. *Let R be a Dedekind domain. Then every non-zero ideal $I \subseteq R$ is a product of primes: $I = P_1 P_2 \cdots P_n$. The prime ideals $P_1, \cdots, P_n$ are unique up to permutation.*

Proof. Clearly we know R is Noetherian.

Let $A = \{I \subseteq R \text{ ideal } : I \neq 0, I \neq R, I \text{ is not such product }\}$. Suppose $A \neq \varnothing$, then it has a maximal element I. In particular, $I \neq R$, and there exists a maximal ideal $M \subseteq R$ such that $I \subseteq M \notin A$.

There exists an ideal $Y \subseteq R$ such that $I = M \cdot Y \subsetneq Y = R \cdot Y$. Clearly $Y \neq R$, otherwise $I = M$. Therefore, $y \notin A$, so $Y = P_1 \cdots P_n$ for P_i prime. SO $I = M \cdot P_1 \cdots P_n$, so $I \notin A$, contradiction.

Note $P_1 \cdots P_n \subseteq P_1$. Since P_1 is prime, then $Q_i \subseteq P_1$ for some i. Recall that the dimension of a Dedekind domain is 1, so every non-zero prime ideal is maximal. Therefore, Q_i and P_1 are maximal. Hence, $Q_i = P_1$. Without loss of generality, say $Q_1 = P_1$, then $Q_2 \cdots Q_m = P_2 \cdots P_n$. We proceed by induction. $\square$

7.2 Integral Elements

Definition 7.2.1 (Integral Element)**.** *Let $R \subseteq S$ be commutative rings. An element $x \in S$ is called integral over R if there exists a polynomial $f \in R[x]$ such that $f(s) = 0$.*

Example 7.2.2.

1. *If R and S are fields, then integral elements are equivalent to algebraic elements.*
2. *Every element $r \in R \subseteq S$ is integral over R: take $f = x - t$.*

Definition 7.2.3 (Faithful Module)**.** *Let R be a commutative ring and M is a R-module. We say M is faithful if $\forall 0 \neq r \in R$, $r \cdot M \neq 0$. Equivalently, R as an Abelian group generates the injective homomorphism $R \to \mathbf{End}(M)$.*

Example 7.2.4. *Rings $R \subseteq S$ indicates S is a faithful R-module.*

Proposition 7.2.5. *Let $R \subseteq S$ be rings and $s \in S$. The following are equivalent:*

1. *s is integral over R.*
2. *$R[s]$ is finitely generated as a R-module.*

3. *There is a faithful $R[s]$-module that is finitely generated as a R-module.*

Proof. (1) $\Rightarrow$ (2): Let $f \in R[x]$ be monic, $f(s) = 0$. Let $n = \deg(f)$. By observing $s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_0 = 0$ where $a_i \in R$, we have $R[s] = \sum\limits_{i=0}^{n-1} Rs^i$ is finitely generated.

(2) $\Rightarrow$ (3): $R[s]$ is a faithful (as $R[s]$-module) finitely-generated R-module.

(3) $\Rightarrow$ (1): Suppose M is a faithful $R[s]$ -module, finitely generated as a R-module Let M be generated by $m_1, \cdots, m_n$. We write $sm_i = \sum\limits_{j=1}^{n} a_{ij}m_j$ with $a_{ij} \in R$. They form an $n \times n$ matrix A over R.

Let $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ m_n \end{pmatrix}$, then $s \cdot X = A \cdot X$, so $(s \cdot I - A)X = 0>$ This gives a matrix over $R[s]$. Note that for an $n \times n$ matrix B, there exists an adjoint (cofactor) matrix B' such that $B' \cdot B = \deg(B) \cdot I$ over commutative rings. Then we have $(sI \tilde{-} A)(sI - A)X = 0$, so $\deg(s \cdot I - A)X = 0$. Hence, $\deg(S \cdot I - A) \cdot m_i = 0$ for all i. Therefore, $\deg(s \cdot I - A) \cdot M = 0$.

Since M is faithful as a $R[s]$-module, then $\deg(s \cdot I - A) = 0$. Consider $f(x) = \deg(xI - A) \in R[x]$, which is monic of degree n. Then $f(s) = 0$, so s is integral over R. ☐

Corollary 7.2.6. *Let $R \subseteq S$ be rings and $s_1, \cdots, s_n \in S$ are integral over R. Then $R[s_1, \cdots, s_n]$ is finitely generated as an R-module.*

Proof. We proceed by induction.

The base case is trivial. Suppose we know $R' = R[s_1, \cdots, s_{n-1}]$ is finitely generated as R-module, then $R[s_1, \cdots, s_n] = R'[s_n]$ which is a finitely generated R'-module. But since s_n is integral over R, then s_n is integral over R'. We then can conclude the proof easily. ☐

Proposition 7.2.7. *Let $R \subseteq S$ be rings. Then the set S' of all integral elements in S over R is a subring over S: $R \subseteq S' \subseteq S$.*

Proof. Obviously $R \subseteq S' \subseteq S$. We show that S' is a ring. For $x, y \in S$, we show $x + y, xy \in S$.

Let $z = x + y$. Note that the proof still works if we set $z = xy$. Then we have rings $R[z] \subseteq R[x, y]$. But $R[x, y]$ is faithful as a $R[z]$-module, so by corollary it is integral as a R-module.

By proposition, z is integral over R, so $z \in S'$. ☐

Definition 7.2.8 (Integral Closure, Integral, Integrally Closed, Normal)**.** *S' is called the integral closure of R in S.*

If $S' = S$, we say that S is integral over R. If $S' = R$, we say that R is integrally closed in S.

Let R be a domain (or commutative ring), we embed $R \subseteq F$, which is the quotient field of R. We say R is normal if R is integrally closed in F.

Proposition 7.2.9. *Let $R \subseteq T \subseteq S$ be rings such that T is integral over R and $s \in S$ is integral over T. Then s is integral over R.*

Proof. Consider $s^n + t_1 s^{n-1} + \cdots + t_{n-1}s + t_n = 0$ for $t_i \in T$. Then $R \subseteq T' = R[t_1, \cdots, t_n] \subseteq T$, and t_i's are integral over R. Hence, T' is finitely generated as a R-module.

Now $s \in S$ is integral over T', then $T'[s]$ is finitely generated as T'-module. By transitivity, $T'[s]$ is finitely generated as an R-mod. But $T'[s]$ is faithful as $R[s]$ module. Therefore, by proposition, s is integral over R. □

Corollary 7.2.10. *Let $R \subseteq S$ be rings. Then the integral closure of R in S is integrally closed in S.*

Example 7.2.11. *Suppose we have a subring R in a field K, and L/K is an algebraic field extension. Then the integral closure of R in L is normal.*

Suppose we have a domain R with a quotient field F, and L/K is an algebraic field extension, with $S \subseteq L$ is the integral closure of R in L. Then S is normal and the quotient field of S is L.

Proof. Let $x \in L$ be algebraic over F. Note that there exists $a_i \in F$ such that

$$x^n + \alpha_1 x^{n-1} + \cdots + \alpha_{n-1}x + \alpha_n = 0.$$

We can then set $ax^n + a_1x^{n-1} + \cdots + a_n = 0$ for $a, a_i \in R$. Then we have $(ax)^n + a_1(ax)^{n-1} + \cdots + a^{n-1}a_n = 0$.

Note $y = ax$ is integral over R, then $y \in S$, and we have $x = \frac{y}{a}$, so $y \in S$, $a \in R \subseteq S$.

Now S is integrally closed in L, which is the quotient field of S then, so S is normal. □

Theorem 7.2.12. *Every Dedekind ring is normal.*

Proof. Let F be the quotient field of R, and take $x \in F$ integral over R. Note that $F \supseteq R[x]$ is finitely generated as a R-module, so $\exists 0 \neq y \in R$, and we get to define $A := y \cdot R[x] \subseteq R$ as a non-zero ideal. Then $x \cdot R[x] \subseteq R[x]$, hence $xA \subseteq A$.

Denote $x = \frac{a}{b}$ where $a, b \in R$, then $\frac{a}{b} \cdot A \subseteq A$, so $aA \subseteq bA$.

Since R is Dedekind, then there exists an ideal $B \subseteq R$ such that $(aR)A = aA = bAB = (bB)A$, so $aR \subseteq bB \subseteq bR$, so $x = \frac{a}{b} \in R$. □

Lemma 7.2.13. *Let R be a Noetherian normal domain with F as its quotient field. Let $x \in A$ and $A \subseteq R$ be a non-zero ideal such that $xA \subseteq A$, then $x \in R$.*

Proof. Note that A is a faithful $R[x]$-module and is finitely generated as an R-module, then x is integral over R. By normality, $x \in R$. □

Theorem 7.2.14. *A domain R is Dedekind if and only if R is Noetherian, normal and* $\dim(R) \leq 1$.

Proof. ($\Rightarrow$): this can be easily concluded from the knowledge we have.

($\Leftarrow$):

Claim 7.2.15. *Every non-zero ideal of R contains the product of finitely many prime ideals.*

Proof. We apply the Noetherian induction.

Let $0 \neq A \subseteq R$ be an ideal with $A \neq R$. (If $A = R$, we take the empty product.) Suppose A is not prime, so $\exists x, y \in R$ such that $xy \in A$ but $x, y \notin A$. Now $(A + xR)(A + yR) \subseteq A$ contains the product of primes, where $A + xR \neq A$ and $A + yR \neq A$. $\square$

Suppose $0 \neq A \subseteq B \subseteq R$ are ideals, then there exists an ideal C such that $A = BC$, which can be proven by Noetherian induction on B.

Claim 7.2.16. *$\exists x \in F \backslash R$ such that $xB \subseteq R$.*

Proof. Take $0 \neq b \in B$. By the previous claim, there exists

$$P_1 \cdots P_n \subseteq bR \subseteq B \subseteq P,$$

where P_i's are primes (also maximals) and P_n is the smallest, and P is also prime (also maximal). Then $P_1 \subseteq P_n \subseteq P$, so there exists some i such that $P_i \subseteq P$. But P and P_i are maximal ideals, then $P_i = P$. Without loss of generality, say $i = 1$, then $P = P_1$. Now $P_2 \cdots P_n \not\subseteq bR$, so there exists $c \in P_2 \cdots P_n$ such that $c \notin bR$ and $x = \frac{c}{b} \notin R$. So $cP_1 \subseteq P_1 \cdots P_n \subseteq bR$, then $\frac{c}{b} P_1 \subseteq R$, then $\frac{c}{b} B \subseteq R$. $\square$

Claim 7.2.17. *$xB \not\subseteq B$.*

Proof. Indeed, otherwise $xB \subseteq B$, then by lemma $x \in R$, contradiction. $\square$

Let $B' = B + xB \subseteq R$, then $B \not\subseteq B'$, and $A \subseteq B \subseteq B'$. By induction, $\exists C' \subseteq R$ such that $A = B'C' = B \cdot (R + xR) \cdot C'$. Take $C = (R + xR) \cdot C'$. It suffices to show that $C \subseteq R$ is an ideal. Indeed, for $c \in C$, $cB \subseteq A \subseteq B$, then by lemma we know $c \in R$, so $C \subseteq R$. $\square$

Definition 7.2.18 (Trace)**.** *Let L/K be a finite field extension, so we can view L as a vector space over K. Take $\alpha \in L$, then there exists a map, namely the left multiplication $m_\alpha : L \to L$ that takes $x \mapsto \alpha x$, which makes it a K-linear transformation. The trace of α, denoted $Tr_{L/K}(\alpha)$, can then be defined as the trace of this linear transformation in the linear algebra sense.*

Alternatively, if L/K is a separable extension, then we can define the trace $Tr_{L/K}(\alpha)$ as $Tr(x) = \sum_{\tau \in Gal(E/L)} \tau(x) \in E$, where E is the normal closure over L, i.e. E/K is Galois.

Theorem 7.2.19. *Let R be a Dedekind ring with quotient field F. Let L/F be a finite field extension and S is the integral closure of R in L. Then S is also Dedekind with quotient field L.*

Proof. For this proof, we assume that L/F is separable, which is reasonable. Denote $G = \mathrm{Gal}(E/F)$ where E is the normal closure over L/F.

Consider the homomorphisms from L to E, then for any $x \in L$, we can consider the trace as the sum of all the Galois conjugates of x, i.e. $\mathrm{Tr}(x) = \sum_{\tau \in \mathrm{Gal}(E/L)} \tau(x) \in E$.

Note that take any $\sigma \in G$, we then have $\sigma\tau : L \xrightarrow{\tau} E \xrightarrow{\sigma} E$. Moreover, $\sigma\mathrm{Tr}(x) = \sum\limits_{\tau} \sigma\tau(x) = \mathrm{Tr}(x)$, so $\mathrm{Tr}(x) \in E^G = F$.

In particular, we can the trace map $\mathrm{Tr}(L \to F)$ is linear, with $\mathrm{Tr}(x+y) = \mathrm{Tr}(x) + \mathrm{Tr}(y)$.

Claim 7.2.20. *Tr* $\neq 0$.

Subproof. $L = F(\alpha)$, then $1, \alpha, \cdots, \alpha^{n-1}$ is a basis for L/F, where $n = [L : F]$.

We have distinct homomorphisms $\tau_1, \cdots, \tau_n : L \to E$, then $\tau_i(\alpha^j) = (\tau_i(\alpha))^j$. This gives an $n \times n$ matrix with non-zero determinant. Then $\mathrm{Tr}(\alpha^i) = \sum\limits_{i} \tau_i(\alpha^j) \neq 0$ for some j. Hence, $\mathrm{Tr} \neq 0$. ■

Claim 7.2.21. *Tr*$(S) \subseteq R$.

Subproof. Recall that $\mathrm{Tr}(x) = \sum\limits_{\tau : L \to E} \tau x$. For $x \in S$, it is always integral over R, with $f(x) = 0$ for some $f \in R[x]$ monic. Then $f(\tau x) = 0$, so τx is integral over R for all τ. Therefore, $\mathrm{Tr}(x)$ is integral over R. But $\mathrm{Tr}(x) \in F$, and since R is normal, then $\mathrm{Tr}(x) \in R$. ■

We know that L is the quotient field of S, now $\forall x \in L$, $x = \frac{s}{a}$ for $s \in S$, $a \in R$. Let $\{x_1, \cdots, x_n\}$ be a basis for L/F, so every x_i is of the form $\frac{s_i}{a_i}$. Then we may assume that $x_i \in S$ since they are invertible. We define the map

$$\begin{aligned} f : L &\to F \times F \times \cdots \times F = F^n \\ y &\mapsto (\mathrm{Tr}(x_1 y), \cdots, \mathrm{Tr}(x_n y). \end{aligned}$$

Claim 7.2.22. $\ker(f) = 0$.

Subproof. Let $y \in \ker(f)$. Then $\mathrm{Tr}(x_i y) = 0$ for all i. Recall $\mathrm{Tr}(ay) = a \cdot \mathrm{Tr}(y)$ for $a \in F$, $y \in L$. Then $\mathrm{Tr}(zy) = 0$ for all $z \in L$. But $\mathrm{Tr} \neq 0$, then $y = 0$. ■

Therefore, we can easily see that f is an isomorphism.

Consider $f|_S : S \to R \times \cdots \times R$ which is R-linear and injective: note that $y \in S$ indicates $x_i y \in S$, so $\mathrm{Tr}(x_i y) \in R$.

Therefore, we have an embedding $S \hookrightarrow R^n$ as an R-submodule.

Since R is Noetheriand and Dedekind, then S is finitely generated as an R-module. Therefore, S is of finite type over R, so S is Noetherian.

Since S is integrally closed in L, we can easily conclude that S is normal.

Finally, we show that $\dim(S) \leq 1$. Let $0 \neq P \subseteq S$ be a prime ideal. It suffices to show that P is maximal. Define $Q := P \cap R$, then it is prime in R.

Claim 7.2.23. $Q \neq 0$.

Subproof. Take $0 \neq x \in P$, then it is integral over R. Then we have $x_n + a_1 x^{n-1} + \cdots + a_n = 0$ for some $a_i \in R$. Note $a_n \neq 0$, then $a_n \in Sx \subseteq P$. Since $a_n \in R$, so $0 \neq a_n \in Q$. ■

Therefore, Q is maximal, then R/Q is a field.

Consider the map $R \subseteq S \twoheadrightarrow S/P$, we then have $R/Q \hookrightarrow S/P$, where R/Q is a field and S/P is a ring as a finitely-generated R/Q-vector space, so it is essentially a domain.

Consider $l_u : S/P \to S/P$ as left-multiplication. This map is injective and so an isomorphism. Therefore, $u \in (S/P)^\times>$ Hence, S/P is a field, and so P is maximal. □

Example 7.2.24. *1. Suppose $L/\mathbb{Q}$ is a finite field extension with $R = \mathbb{Z}$. Then S is always Dedekind.*

In particular, suppose L is a quadratic extension over $\mathbb{Q}$, i.e. $L = \mathbb{Q}(\sqrt{d})$ where $d \neq 0$ and is square-free. Now

$$S = \begin{cases} \mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} : a, b \in \mathbb{Z}\}, & \text{if } d \not\equiv 1 \pmod 4 \\ \mathbb{Z}[\frac{1+\sqrt{d}}{2}], & \text{if } d \equiv 1 \pmod 4 \end{cases}.$$

In particular, $\mathbb{Z}[\sqrt{-5}]$ is a Dedekind ring, but $\mathbb{Z}[\sqrt{5}]$ is not because it is not integrally closed. Instead, $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ is.

2. Let $R = F[x]$ where F is a field. Then we have

$$\begin{tikzcd} S \arrow[r, hook, "subset"] \arrow[d, dash] & L \arrow[d, dash, "finite"] \\ F[x] \arrow[r, hook, "subset"'] & F(x) \end{tikzcd}$$

with S Dedekind. Note that S is the ring of regular polynomial functions on a regular affine algebraic curve over F. For example, let $y = \sqrt{1-x^2}$, then $L = F(x)(y)$ satisfies a circle $x^2 + y^2 = 1$. Then $S = F[x, y] = F[X, Y]/(X^2 + Y^2 - 1)$.

Remark 7.2.25. *The intersection of Dedekind domain and UFD is exactly the PIDs.*

It is obvious that PIDs are in the intersection. Then if suffices to show that every prime ideal is principal.

Take $0 \neq P \subseteq R$ prime, then take $0 \neq x \in P$ with $x = p_1 \cdots p_n$ primes in P. Then $P_i \in P$ for some i. Then $P_i R \subseteq R$. But we know that $p_i R$ is prime, then $P = p_i R$ is a maximal ideal.

7.3 Discrete Valuation Ring (DVR)

Definition 7.3.1 (Discrete Valuation)**.** *Let F be a field. $v : F^\times \to \mathbb{Z}$ is called a discrete valuation if*

1. $v(xy) = v(x) + v(y)$,

2. $v(x + y) \geq \min(v(x), v(y))$.

We also define $v(0) = \infty$.

Example 7.3.2. 1. *Let R be a Dedekind ring. Let F be the quotient field of R, and $0 \neq P \subseteq R$ be prime. We define a discrete valuation $v_P : F^\times \to R$ as for $0 \neq x \in R$, $xR = P^i \cdot$ (product of other ideals) for $i \geq 0$. Then $v_P(x) = i$. For $x \in R^\times$, we write $x = \frac{y}{z}$ for $y, z \in R \backslash \{0\}$. Then $v_P(x) = v_P(y) - v_P(z)$. Here v_P is called the discrete valuation with P, or just p-adic discrete valuation on F.*

2. *Suppose $R = \mathbb{Z}$, $P = p\mathbb{Q}$ and $F = \mathbb{Q}$. Then $v_p(x) = i$ where $x = p^i \frac{a}{b}$ for $p \nmid a, \nmid b$.*

Proposition 7.3.3 (Ostrowsky)**.** *There are the only valuations of $\mathbb{Q}$, i.e. the non=trivial absolute value on $\mathbb{Q}$ is equivalent to either the usual real absolute value or a p-adic absolute value.*

Example 7.3.4. *Let K be a field, $F = K(x)$, $P = pK[x]$ for p monic irreducible. Then $v_p(f) = i$ for $f = p^i \frac{a}{b}$, $p \nmid a, p \nmid b$, and $v_\infty(\frac{g}{h}) = \deg(h) - \deg(g)$.*

Definition 7.3.5 (Valuation Ring, Discrete Valuation Ring)**.** *Let F be a field and $v : F^\times \to \mathbb{Z}$ is a discrete valuation. Set $v(0) = \infty$. $Rv = \{x \in F : v(x) \geq 0\}$ is a subring of F, called the valuation ring.*

A domain R is a discrete valuation ring (DVR) if $R = R_v$ for some valuation on the quotient field F.

Example 7.3.6. 1. *Let F be a field and let $v = 0$ on F, then $Rv = F$, so F is a DVR.*

2. *Let $F = \mathbb{Q}$ and $p \in \mathbb{Z}$ be a prime. Then $R_{v_p} = \{\frac{a}{b} : p \nmid b\} = \mathbb{Z}_p$, namely the localization at $p\mathbb{Z}$.*

Definition 7.3.7 (Local Ring)**.** *We say that a ring R is a local ring if any of the following properties hold:*

1. *R has a unique (left/right) maximal ideal.*

2. *It is a non-trivial ring and the sum of any two non-units in R is a non-unit.*

3. *It is a non-trivial ring such that if x is any element of R, then x or $1 - x$ is a unit.*

Proposition 7.3.8. *If R is a Dedekind ring and $P \subseteq R$ is a nonzero prime ideal, then $R \subseteq R_{v_p} = R_p \subseteq F$, where F is the quotient field of R.*

If R is any commutative ring and $P \subseteq R$ is a prime ideal, then R_p is a local ring with unique prime/maximal ideal P_p.

In general, if $v : F^\times \to \mathbb{Z}$ is a discrete valuation, then R_v is local with unique maximal ideal $M = \{x \in F : v(x) > 0\}$. Note that every $x \in R_v \backslash M$ is invertible.

Theorem 7.3.9. *The following are equivalent:*

1. *DVR.*
2. *Local PID.*
3. *Local Dedekind ring.*

Proof. (1) $\Rightarrow$ (2): Let R be a DVR. Then R is local with maximal ideal M.

Claim 7.3.10. *Every non-zero ideal $I \subseteq R$ is of the form $\pi^i R$ for $i \geq 0$ and $v(\pi) = 1$. In particular, R is a PID.*

Subproof. Assume $v : F^\times \to \mathbb{Z}$ is nonzero, otherwise $R = F$ is a PID. Now $\text{im}(v) \subseteq \mathbb{Z}$ is an ideal, so $\text{im}(v) = n\mathbb{Z}$. We can divide by n to let $\text{im}(v) = \mathbb{Z}$. Let $i = \min_{x \in I} v(x)$ and fix $\pi \in F$ with $v(\pi) = 1$. Then $v(\frac{x}{\pi^i}) = v(x) - v(\pi^i) = v(x) - iv(\pi) \geq i - i = 0$, so $\frac{x}{\pi^i} \in R$, so $I \subseteq \pi^i R$. Conversely, for all $x \in I$ with $v(x) = i$, $v(\frac{x}{\pi^i} = 0$, then $\frac{x}{\pi} \in R^\times$, so $\pi^i = \frac{\pi^i}{x} \cdot x \in I$, then $I = \pi^i R$. ■

(2) $\Rightarrow$ (3): trivial.

(3) $\Rightarrow$ (1): If R is a field then R is clearly a DVR. Let R be a local Dedekind ring with unique maximal ideal $M \neq 0$, then M is the only non-zero prime ideal in R. By factorization, every ideal is of the form M^i, then take $v = v_M : F^\times \to \mathbb{Z}$, which is the only valuation of R.

Claim 7.3.11. *$R = R_v$, then R is a DVR.*

Subproof. Note that every $x \in R \backslash \{0\}$ satisfies $v(x) \geq 0$, so $x \in Rv$, then $R \subseteq R_v$. (By factoring xR into powers of M, we see that $v(x) \geq 0$.) Now for every element $x = \frac{a}{b} \in R_v \backslash \{0\}$, we let $aR = M^i$ and $bR = M^j$. Then $v(x) = i - j \geq 0$, so $M^i \subseteq M_j$, which means $aR \subseteq bR$, then $x \in R$, so $R = R_v$. ■

□

Remark 7.3.12. *If R is a DVR and M is the unique maximal ideal, then all non-zero ideals form a chain $M \supseteq M^2 \supseteq \cdots \supseteq M^i \supseteq \cdots$.*

If $a \in R$ non-zero, we write $aR = M^i$ and then $i = v(a)$, so $aR = M^{v(a)}$.

Remark 7.3.13. *In general, let R be a Dedekind ring and $P \subseteq R$ a non-zero prime. Pick any $x \in R \backslash \{0\}$. Then R_p is a local Dedekind ring, so it is a DVR with discrete valuation $v = v_p : F^\times \to \mathbb{Z}$. By factorization, we have $xR = P^i \cdot P_1 \cdots P_n$, so $(xR)_p = P_p^i$ since $(P_j)_p = R_p$ when $P_j \not\subseteq P$, also $i = v_p(x)$. Therefore, $xR = \prod_{p \text{ non-zero prime}} P^{v_p(x)}$, which is always finite.*

Definition 7.3.14 (Fractional Ideal)**.** *Let R be a Noetherian domain and F is the quotient field. A fractional ideal of R is a finitely-generated R-submodule of F.*

Example 7.3.15. *1. All ideals are fractional ideals.*

2. *If $I \subseteq F$ is any fractional ideal and $x \in F^\times$, then $xI \cong I$ is also a fractional ideal.*

Remark 7.3.16. *Let R be a Dedekind ring with F as its quotient field. A fractional ideal A is a finitely-generated R-module by definition. There exists $0 \neq a \in R$ such that $aA \subseteq R$ as an ideal. Conversely, every ideal is a fractional ideal.*

Let $A, B \in \mathit{Frac}(R)$, which is the set of fractional ideals. Then $A = \sum a_i R$, $B = \sum b_j R$. $AB = \sum a_i b_j R$, so $AB \in \mathit{Frac}(R)$.

Proposition 7.3.17. *$\mathit{Frac}(R)$ is an Abelian group.*

Proof. The operation comes for free and the identity is R itself. Take $A \in \mathrm{Frac}(R)$, then there exists $0 \neq a \in R$ such that $I = aR \subseteq R$ is an ideal. Pick any $0 \neq x \in I$, so that $xR \subseteq I$, there exists an ideal $J \subseteq I$ such that $xR = IJ = aA \cdot J$. Then $R = A \cdot \frac{a}{x} J$, so $A^{-1} = \frac{a}{x} \cdot J \in \mathrm{Frac}(x)$. $\square$

Remark 7.3.18. *Let $A \in \mathit{Frac}(R)$. Find $a \in R$ such that $aA = I \subseteq R$ is an ideal. Note $A = P_1 \cdots P_s$ as a product of prime ideals. Also, $aR = Q_1 \cdots, Q_t$ is also a product of prime ideals. Now $A = (a^{-1}R) \cdot I = Q_1^{-1} \cdots Q_t^{-1} P_1 \cdots P_s$. So every fractional ideal is a product of primes and their inverses.*

Therefore, denote $A = S_1^{\alpha_1} \cdots S_r^{\alpha_r}$ where $S_i \subseteq R$ are primes with $\alpha_i \in \mathbb{Z}$. Note that such decomposition is unique. Then $\mathit{Frac}(R)$ is a free Abelian group with a canonical basis of prime ideals.

Note that if we start with ideals only, we only get a monoid or semi-group.

Remark 7.3.19. *A fractional ideal A is principal if $A = xR$ for $x \in F^\times$. Note that $(xR)(yR) = xyR$ is also principal. Therefore, principal fractional ideals form a subgroup $\mathit{PFrac}(R) \subseteq \mathit{Frac}(R)$. There is a surjective homomorphism with kernel as $R^\times \subseteq F^\times$:*

$$F^\times \twoheadrightarrow \mathit{PFrac}(R) x \mapsto xR$$

By the First Isomorphism Theorem, $\mathit{PFrac}(R) \cong F^\times / R^\times$. We also denote $\mathit{Cl}(R) = \mathit{Frac}(R)/\mathit{PFrac}(R)$ to be the class group of R. We then have an exact sequence of Abelian groups

$$1 \longrightarrow R^\times \longrightarrow F^\times \longrightarrow \mathit{Frac}(R) \longrightarrow \mathit{Cl}(R) \longrightarrow 1$$

where $A \in \mathit{Frac}(R)$ is sent to $[A] \in \mathit{Cl}(R)$.

Note that $\mathit{Cl}(R) = 1$ if and only if every fractional ideal is principal if and only if every ideal is principal if and only if R is a PID.

Example 7.3.20. 1. *Let R be the ring of algebraic integers, then the class group is finite.*

In particular, if $R = \mathbb{Z}[\sqrt{-5}]$, then $\mathit{Cl}(R) = \{[R], [I]\}$, which is a cyclic group generated by $[I]$. Indeed, $2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$, with $I = 2R + (1 + \sqrt{-5})R$, then $I^2 = 2R$.

2. *Let K be a field and we have $F/K(x)$ as a finite field extension. Let R be the integral closure of $K[x]$ in F. If K is finite, then $\mathit{Cl}(R)$ is also finite.*

In particular, let K be a field of char$(K) \neq 2$. Consider $K(x)(\sqrt{1-x^2})/K(x)$, then $R = K[x,y]$ where $y = \sqrt{1-x^2}$. Hence, $R = K[x,y] = K[X,Y]/(X^2+Y^2-1)$.

Observe that $x^2+y^2=1$, so $x^2 = 1-y^2 = (1+y)(1-y)$. Let $K = \mathbb{Q}$ or $\mathbb{R}$, then we have the factorization with $I = xR + (1-y)R$ and $J = xR + (1+y)R$, which are both prime but not principal. Moreover, $IJ = xR$ and $I^2 = (1-y)R$, $J^2 = (1+y)R$ with $(IJ)^2 = I^2J^2$. Therefore, $Cl(R) = \{[R], [I] = [J]\}$.

However, if $K = \mathbb{C}$, then $Cl(R) = 1$, so R is a PID in this case.

7.4 Modules over Dedekind Rings

Recall that the PIDs are exactly the intersection of Dedekind rings and the UFDs. We want to find a similar classification of modules over PIDs as the usual modules.

Let R be a Dedekind ring and M is a R-module. Recall that $M_{\text{tors}} = \{m \in M : \exists 0 \neq a \in R$ such that $a \cdot m = 0\}$. We say M is torsion if $M = M_{\text{tors}}$ and M is torsion-free if $M_{\text{tors}} = 0$.

Let M be a torsion finitely generated R-module, and let $0 \neq P \subseteq R$ be a prime ideal. We say M is P-primary if $P^n \cdot M = 0$ for some $n > 0$.

We have $M[P] = \{m \in M : P^n \cdot m = 0$ for some $n > 0\}$ as a finitely-generated submodule of M , called the P-primary component of M.

Recall $M = \coprod_{0 \neq P \subseteq R \text{ prime}} M[P]$. Note that the same proof works: every two distinct non-zero prime ideals are coprime, i.e. $P + Q = R$.

Now let M be a P-primary finitely-generated torsion R-module. Take $r \in R \backslash P$. Then $l_r : M \to M$ that sends $m \mapsto rm$ is an automorphism on M. Indeed, $rR + P = R$.

Suppose $S \subseteq R$ is a multiplicative subset and M is a R-module. Then for all $s \in S$, $l_s : M \to M$ is an isomorphism, then M has the structure of a module over $S^{-1}R$: note that $\frac{r}{s} \cdot m = l_s^{-1}(rm)$.

Therefore, we can localize by $S = R \backslash P$. Now $R_p = S^{-1}R$, then M is a finitely-generated R_p-module. Then R_p is a local Dedekind ring, and so it is a PID. Therefore, N is a direct sum of cyclic modules $M = R_p/P_p^n$.

Note that $M = R_p/P_p^n \cong R/P^n$, because $R_p/P_p^n = S^{-1}(R/P^n)$, and the localization acts as an isomorphism towards R/P^n since multiplication by any s is an isomorphism.

Theorem 7.4.1 (Invariant Form)**.** *Let M be a torsion finitely-generated module over a Dedekind ring R, then there are ideals*

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_r$$

of R such that $M \cong R/A_1 \oplus \cdots R/A_r$. The ideals A_i are unique and are called the invariant form of M.

Theorem 7.4.2 (Elementary Divisor)**.** *Let M be a torsion finitely-generated module over a Dedekind R, then there are non-zero prime ideals $P_1, \cdots, P_s$ of R and positive integers $k_1, \cdots, k_s$ such that*

$$M \cong R/P_1^{k_1} \oplus \cdots \oplus R/P_s^{k_s}.$$

The ideals P_i and integers k_i are unique up to permutation. The family $\{P_i^{k_i}\}_{i\geq 1}$ is called the elementary divisors of M.

Lemma 7.4.3. *Let M be a torsion-free finitely-generated module over a Dedekind ring R. Then M is isomorphic to a submodule of R^n for some n.*

Proof. We localize with $S = R\backslash\{0\}$, so $S^{-1}R = F$, where F is the quotient field of R. Then $S^{-1}M$ is a vector space over F, so $S^{-1}M \cong F^n$. Here we call this n to be the rank of M.

Consider the map $M \to S^{-1}M$ by sending $m \mapsto \frac{m}{1}$, then the kernel of the map $\ker(M \to S^{-1}M) = M_{\text{tors}} = 0$. Hence, we know $M \hookrightarrow S^{-1}M \cong F$. In particular, there exists $0 \neq a \in R$ such that

$$M \underset{\sim}{\xrightarrow{a}} aM \hookrightarrow R^n$$

because M is torsion-free. □

Corollary 7.4.4. *$M \cong A_1 \oplus \cdots \oplus A_n$ where A_i are ideals in R. In particular, M is projective.*

Proof. We prove by induction on $n = \text{rank}(M)$. The base case is trivial. We now suppose the case is true at $n-1$, we now show the case for n.

By lemma, consider $M \hookrightarrow R^n$. Then

$$\ker(M \subseteq R^n \to R) = N \subseteq R^{n-1}.$$

Now $A = \text{Im}(M \subseteq R^n \to R) \subseteq R$ is an ideal. Then we have

$$0 \longrightarrow N \hookrightarrow M \longrightarrow A \longrightarrow 0$$

to be a split short exact sequence since A is projective. Then $M \cong N \oplus A$. We use induction to conclude the proof. □

Let M be a finitely-generated R-module, then

$$0 \longrightarrow M_{\text{tors}} \longrightarrow M \longrightarrow M/M_{\text{tors}} \longrightarrow 0$$

Now torsion-free implies projective, so the short exact sequence splits. Then

$$M \cong M_{\text{tors}} \oplus M/M_{\text{tors}}.$$

Consider fractional ideals A, B of R (which implies they are non-zero, and take $x \in B \cdot A^{-1}$ from the fractional ideal. Consider

$$\begin{aligned} f_x : A &\to B \\ a &\mapsto xa \end{aligned}$$

which is well-defined since $xa \in (BA^{-1}) \cdot A = B$. Then f_x is a R-module homomorphism. Now there is

$$\begin{aligned} BA^{-1} &\to \mathbf{Hom}_R(A, B) \\ x &\mapsto f_x \end{aligned}$$

Lemma 7.4.5. *This is an isomorphism of R-modules.*

Proof. Suppose $f_x = f_y$, then take $0 \neq a \in A$, we have

$$xa = f_x(a) = f_y(a) = ya,$$

so $x = y$. Consider $f : A \to B$. For $0 \neq a_0 \in A$ and $a \in A$,

$$a_0 \cdot f(a) = f(a_0 a) = a \cdot f(a_0).$$

Therefore, $f(a) = xa$, where $x = \frac{f(a_0)}{a_0}$. Now $xa \in B$ for all $a \in A$, so $xA \subseteq B$. Then $x \in xR = xAA^{-1} \subseteq BA^{-1}$, so $f = f_x$. $\square$

Consider the map

$$\begin{aligned} \mathbf{Hom}_R(B, C) \times \mathbf{Hom}_R(A, B) &\to \mathbf{Hom}_R(A, C) \\ (f, g) &\mapsto f \circ g \\ CB^{-1} \times BA^{-1} &\mapsto CA^{-1} \end{aligned}$$

Therefore we can consider the map in two ways, as a composition and as a product operation. Note that $A_1 \oplus \cdots \oplus A_n \xrightarrow{f} B_1 \oplus \cdots \oplus B_m$ is given by a matrix with entries in $B_j A_i^{-1}$.

Remark 7.4.6. *Let C be a fractional ideal. Observe that $(B_j C)(A_i C)^{-1} = B_j A_i^{-1}$. In other words, f gives a canonical homomorphism*

$$g : A_1 C \oplus \cdots \oplus A_n C \to B_1 C \oplus \cdots \oplus B_m C,$$

and so if f is an isomorphism, then so is g.

Suppose for fractional ideals A_i, B_j we have

$$M = A_1 \oplus \cdots \oplus A_n \cong B_1 \oplus \cdots \oplus B_m,$$

then $n = m$. Indeed, let $S = R \backslash \{0\}$ and $S^{-1}A_i \cong F \cong S^{-1}B_j$, then $S^{-1}M \cong F^n \cong F^m$, so $n = m$.

Moreover, the isomorphism is given by an $n \times n$ matrix C with entries in $B_i A_j^{-1} \in A$, then C is invertible.

Claim 7.4.7. *Take $a_i \in A_i$ for all i, then*

$$\det(C) \cdot a_1 a_2 \cdots a_n \in B_1 \cdots B_m.$$

Proof. Let $D = C \cdot \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$, then $d_{ij} = c_{ij} \cdot a_j \in B_i A_j^{-1} A_j = B_i$.

Now

$$\deg(C) \cdot a_1 \cdots a_n = \deg(D) \in B_1 \cdots B_n.$$

□

It then follows that

$$\deg(C) \cdot A_1 \cdots A_n \subseteq B_1 \cdots B_m.$$

The same argument works on the inverse of the isomorphism, so

$$\deg(C^{-1}) \cdot B_1 \cdots B_m \subseteq A_1 \cdots A_n.$$

Therefore,

$$\begin{cases} \deg(C) \cdot A_1 \cdots A_n = B_1 \cdots B_m \\ \deg(C^{-1}) \cdot B_1 \cdots B_m = A_1 \cdots A_n \end{cases}.$$

Therefore, $[A_1 \cdots A_n] = [B_1 \cdots B_m]$ in the class group $\mathrm{Cl}(R)$. We define it to be the determinant $\det(M)$, the determinant of M in the class group.

Lemma 7.4.8. *Let A and B be fractional ideals $P \subseteq R$ is a non-zero prime ideal, then $A \oplus BP \cong AP \oplus B$.*

Proof. We first assume that $B = R$. Then it suffices to show that $A \oplus P \cong AP \oplus R$.

Note $A^{-1} \cdot A = R$, so $\exists x_i \in A^{-1}$, $a_i \in A$ such that $\sum x_i a_i = 1$. Therefore, there exists some i such that $x_i a_i \notin P$. Consider

$$\begin{aligned} f : A &\to R \\ a &\mapsto x_i a \in R. \end{aligned}$$

Then $\mathrm{im}(f) \ni x_i a_i \notin P$. Then consider

$$h : A \oplus P \xrightarrow{(f,g)} R$$

for $g : P \hookrightarrow R$, so $\mathrm{im}(h) = \mathrm{im}(f) + \mathrm{im}(g)$. But $\mathrm{im}(f) \not\subseteq P$ and $\mathrm{im}(g) = P$, which is a maximal ideal, then $\mathrm{im}(h) = R$.

Now let $N = \ker(h)$, then

$$0 \longrightarrow N \hookrightarrow A \oplus P \xrightarrow{h} R \longrightarrow 0$$

is a split short exact sequence. Then

$$A \oplus P \cong N \oplus R.$$

Therefore, if we denote F as the quotient field of R, then $F \oplus F \cong S^{-1}N \oplus F$, therefore $N \hookrightarrow S^{-1}N \cong F$ where N is a finitely-generated R-submodule of F.

Then N is a fractional ideal, so

$$[A \cdot P] = \deg(A \oplus P) = \det(N \oplus R) = [N \cdot R] = [N]$$

in $\mathrm{Cl}(R)$. Then $A \cdot P \cong xN \cong N$ with $x \in F^{\times}$.

This proves the special case. In general, we know $AB^{-1} \oplus P \cong AB^{-1}P \oplus R$ by the special case. Therefore, $A \oplus BP \cong AP \oplus B$. $\square$

Theorem 7.4.9. *Let R be a Dedekind domain. Then*

1. *Every torsion-free finitely-generated R-module M is isomorphic to $I \oplus R^{n-1}$ where $n = rank(M)$, and $I \subseteq R$ is an ideal such that $[I] = \det(M)$.*

2. *Two torsion-free finitely-generated R-module M and N are isomorphic if and only if $rank(M) = rank(N)$ and $\det(M) = \det(N)$ in $Cl(R)$.*

Proof. 1. Note $A \oplus BP \cong AP \oplus B$.

Claim 7.4.10. *For every two ideals I and J in R, $I \oplus J \cong IJ \oplus R$.*

Subproof. Let $J = P_1 \cdots P_s$ where P_i are primes, then

$$\begin{aligned} I \oplus J &= I \oplus P_1 \cdots P_s \\ &= IP_1 \oplus RP_2 \cdots P_s \\ &= IP_1 \cdots P_s \oplus R \\ &= IJ \oplus R. \end{aligned}$$

■

Now $M \cong I_1 \oplus \cdots I_n \cong I_1 I_2 \oplus R \oplus I_3 \oplus \cdots \oplus I_n = I_1 I_2 I_3 \oplus R \oplus I_4 \oplus \cdots \oplus I_n$, so $M \cong I \oplus R^{n-1}$, where n is the rank of M, and $I_j \subseteq R$ are ideals such that $I = I_1 \cdots I_n$. Therefore, $[I] = [I_1] \cdots [I_n] = \det(M)$.

2. Denote n as the rank of M and N, then $M \cong I \oplus R^{n-1}$ and $N \cong J \oplus R^{n-1}$, so $[I] = \det(M) = \det(N) = [J]$ in $\mathrm{Cl}(R)$, so there exists $x \in F^{\times}$ such that $J = xI$, which means $M \cong N$.

□

7.5 Picard Group

Let R be a commutative ring.

Definition 7.5.1 (Spectrum, Jacobson Radical)**.** *The (prime) spectrum of R is the set of all prime ideals in R, denoted $Sepc(R)$. The maximal spectrum of R is the set of maximal ideals of R, denoted $Specm(R)$.*

The Jacobson radical is defined as the intersection of all maximal ideals for commutative rings, i.e. $J(R) = \bigcap\limits_{M \in Specm(R)} M$.

Proposition 7.5.2 (Nakayama Lemma, Statement 1)**.** *Let I be an ideal in R, and M a finitely-generated module over R. If $IM = M$, then there exists $r \in R$ such that $r \equiv 1 \pmod I$ such that $rM = 0$.*

Corollary 7.5.3 (Nakayama Lemma, Statement 2)**.** *If $J(R)M = M$, then $M = 0$.*

Lemma 7.5.4. *If R is local, then every finitely-generated projective R-module is free.*

Proof. Suppose $M \subseteq R$ is the maximal ideal, with $K = R/M$. Let P be a finitely-generated projective R-module. Now $P/MP (= P \otimes_R K)$ as a K-space of finite dimension.

Let $p_1, \cdots, p_n \in P$, then $\{\bar{p}_1, \cdots, \bar{p}_n\}$ is a basis for P/MP over K. Note $\bar{p}_i = p_i + MP \in P/MP$.

Claim 7.5.5. *$\{P_1, \cdots, P_n\}$ is a basis for P over R.*

Subproof. Take $Q = \sum R \cdot p_i \subseteq P$. Now $N = P/Q$, then $N/M \cdot N = (P/MP)/((Q+MP)/MP) = 0$, where $\bar{p}_i$ in $Q + MP$ generates P.

By Nakayama Lemma, $N = 0$. Therefore, $P = Q$, so $p_i's$ generate P.

Consider the short exact sequence

$$0 \longrightarrow S \longrightarrow R^n \xrightarrow{\varphi} P \longrightarrow 0$$

where $\varphi(e_i) = p_i$. Note that φ is a surjection. To show it is an isomorphism, it suffices to show that $\ker(\varphi) = S = 0$. Also note that the sequence

$$S/MS \longrightarrow K^n \xrightarrow{\sim} P/MP \longrightarrow 0$$

has an isomorphsim where we send $e_i \mapsto \bar{p}_i$. Therefore, both sequence split by projective. Therefore, $S/MS = 0$, then by Nakayama Lemma, $S = 0$, so φ is an isomorphism. ■

□

Remark 7.5.6. *Let $P \subseteq R$ be a prime ideal. The local ring $R_P = S^{-1}R$, where $S = R \backslash P$.*

Denote M as an R-module, then $M_P = S^{-1}M$ is an R_P-module.

If $M_P = 0$ for all P, then $M = 0$.

If M is a finitely-generated projective module, then by lemma, M_P is a finitely-generated free R_P-module.

Considering $rank : Spec(R) \to \mathbb{Z}^{\geq 0}$ as a map that sends $P \mapsto rank(M_P)$, we have $rank(M_P) \in \mathbb{Z}^{\geq 0}$. Therefore, $M = 0$ if and only if $rank = 0$.

If M and N are finitely-generated projective R-module, then $M \otimes_R N$ is a finitely-generated projective R-module. Therefore, $rank_{M \otimes N} = rank(M) \cdot rank(N)$. If $rank_M = rank_N = 1$, then $rank_{M \otimes_R N} = 1$ (with $M_P \cong R_P \cong N_P$). Therefore, we have a monoid structure on ranks. Moreover, this is a group.

Let M be a finitely-generated rank-1 projective R-module. Then $M^* = Hom_R(M, R)$ is the dual R-module.

Claim 7.5.7. *M^* is a finitely-generated rank-1 projective R-module.*

Proof. $M \oplus N \cong R^n$, so $M^* \oplus N^* \cong (R^*)^n = R^n$. Hence, M^* is a finitely-generated projective R-module. Although localization doesn't usually commute with hom functors, we have

$$(M^*)_P \cong \mathbf{Hom}_{R_P}(R_P \cong M_P, R_P) \cong R_P,$$

so $\operatorname{rank}(M^*) = 1$. □

Let M be a finitely-generated rank-1 projective R-module with map $f : M \to R$, then we have a map

$$
\begin{aligned}
M^* \otimes_R M &\to R \\
f \otimes m &\mapsto f(m)
\end{aligned}
$$

Claim 7.5.8. *This map is an isomorphism.*

Proof. It suffices to check this is an isomorphism after localization. (We want to check that the kernel and cokernel are both 0m, but they commute with the localization functor as well.)

Consider $(M_P)^* \otimes_{R_P} M_P \to R_P$. As $M_P \cong R_P$, pick $x \in M_P$, then $\{x\}$ becomes a basis. Hence, $(M_P)^* \cong R_P$. We can pick some $f \in (M_P)^*$, so that $f(x) = 1$, then $\{f\}$ is a basis of $(M_P)^*$.

Therefore, $f \otimes x \mapsto f(x) = 1$ by the mapping, and observe that $\{1\}$ is a basis of R_P. We then have an isomorphism. $\square$

Therefore, $M^* \otimes_R M \cong R$. We can now define the Picard group.

Definition 7.5.9 (Picard Group)**.** *For a commutative ring R, the Picard group $Pic(R)$ is the set of isomorphism classes of finitely-generated rank-1 R-modules, with operation $\otimes$ and unit R.*

Remark 7.5.10. *Let R be a Dedekind ring and I is a fractional ideal, then I is a finitely-generated R-module. Let $P \subseteq R$ be prime, then I_P is a fractional ideal of R_P, which is a PID, so $I_P = xR_P \cong R_P$. Then $rank_I = 1$. Hence, I is a finitely-generated rank-1 projective R-module.*

Now consider the map

$$
\begin{aligned}
Frac(R) &\to Pic(R) \\
I \cdot J &\mapsto I \cdot J \cong I \otimes J
\end{aligned}
$$

If $M \in Pic(R)$, then $M \cong I$ is an ideal. We observe that the map is surjective. For any I in the kernel of the map, we have $I \cong R$, so I is a principal ideal of R. Therefore, the kernel is exactly the principal ideals of R.

In particular, we have $Cl(R) \xrightarrow{\cong} Pic(R)$.

8 Representation Theory

8.1 Simple and Semisimple Modules

Remark 8.1.1. *To measure complexity of a ring, we can measure the complexity of category of modules over it.*

Definition 8.1.2 (Simple Module)**.** *Let R be a ring. A non-zero (left) R-module M is simple if M has no proper submodules besides M and 0.*

Lemma 8.1.3. *Let R be a ring and M is a (left) R-module. Then M is simple if and only if $M \cong R/I$ for (left) maximal ideal I.*

Proof. ($\Rightarrow$): Let M be a simple and $m \in M$ be non-zero. Then define $f : R \to M$ by $r \mapsto rm$. Then $\mathrm{im}(f) \neq 0$, so $\mathrm{im}(f) = M$ by simpleness. Therefore, $I = \ker(f) < R$ is a left ideal and $M \cong R/I$. We have the correspondence between submodules of M/I and ideals in R containing I, so I is maximal.

($\Leftarrow$): Use the same correspondence. □

Corollary 8.1.4. *Every non-zero ring has a simple (left) module.*

Proposition 8.1.5 (Simplicity Test)**.** *Let R be a ring and A is a (left) R-module. Then M is simple if and only if $M \neq 0$ and $M = Rm$ for any non-zero $m \in Rm$.*

Proof. ($\Rightarrow$): Let $N < M$ be a non-zero submodule, then for any non-zero $n \in N$, $Rn < N < M$, so $M = N$.

($\Leftarrow$): $Rm < m$ is a non-zero submodule, so $Rm = M$. □

Example 8.1.6.

1. *Suppose F is a field or a division ring, then every (left) module is free according to Zorn's lemma. Therefore, the only simple module is F.*

2. *Let D be a division ring. Take $R = M_n(D)$, then $M = D^n$ (viewed as column vectors) is a left R-module. Hence, for every non-zero $m_1, m_2 \in D$, there exists $r \in End(M) = R$ such that $rm_1 = m_2$, which is similar to the case in vector spaces. Hence, $Rm = M$ for every non-zero $m \in M$, then M is simple.*

3. *Let $R = \mathbb{Z}$. The maximal ideals are $p\mathbb{Z}$ for p prime, so all simple $\mathbb{Z}$-modules are of the form $\mathbb{Z}/p\mathbb{Z}$.*

Theorem 8.1.7 (Schur Lemma)**.** *Let R be a ring, and M, N are simple (left) R-modules. Suppose $f : M \to N$ is an R-linear map, then $f = 0$ or f is an isomorphism.*

Proof. Suppose $f \not\equiv 0$, then $\operatorname{im}(f) \neq 0$ and $\ker(f) \neq M$, so $\operatorname{im}(f) = N$, and $\ker(f) = 0$ by simpleness. □

Corollary 8.1.8. *If M is a simple (left) R-module, then $End(M)$ is a division ring.*

Definition 8.1.9 (Semisimple Module)**.** *A (left) R-module M is semisimple if $M \cong \coprod_{i\in I} M_i$ where all M_i are simple.*

Remark 8.1.10. *Semisimpleness implies simple. Note that 0 is semisimple but not simple.*

Definition 8.1.11 (Semisimple Ring)**.** *A ring R is (left) semisimple if R is semisimple as a (left) R-module, i.e. $R \cong \coprod_{i\in I} L_i$ as an internal product for non-zero minimal (left) ideals.*

Example 8.1.12.

1. *Let D be a division ring and $R = M_n(D)$, then $M = D^n$ viewed as column vectors. For all $1 \le i \le n$, let $L_i \subseteq R$ be a left ideal whose only non-zero column is the i-th one. Then $L_i \cong M$ is simple, and $R \cong \coprod_{i=1}^{n} L_i$ is semisimple.*
2. *If $R_1, \cdots, R_n$ are semisimple, so is $R_1 \times R_2 \times \cdots \times R_n$. Therefore, $R = M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ as above is semisimple. Actually,every semisimple ring is of this form.*

Remark 8.1.13. *Suppose $R = \coprod_{i\in I} L_i$ where $L_i < R$ are left ideals. Write $1 = \sum_{i\in I} e_i$ for $e_i \in L_i$, where almost all e_i are 0. Let $J = \{i \in I : e_i \neq 0\}$, then for all $a \in R$, $a = \sum_{i\in I} ae_i = \sum_{i\in J} ae_i$. Then $R = \coprod_{i\in J} L_i$ is a finite sum of ideals.*

Also, $e_j = \sum_{i\in J} e_i e_j \in L_i$, so $e_i e_j = e_j$ if $i = j$ and $e_i e_j = 0$ if $i \neq j$.

Consider $(*)$ condition: $\{e_i\}_{i\in J}$ are orthogonal idempotent elements and they partition 1. Note that this condition does not need to distinguish between left and right ones.

Conversely, if $\{e_i\}_{i\in J}$ satisfies $(*)$ then $L_i = Re_i$ are left ideals and $R = \coprod_{i\in J} L_i$.

Proposition 8.1.14. *Left semisimpleness and right semisimpleness are equivalent.*

Proof. Let R be a left semisimple ring so that $R = \coprod_{i\in I} Re_i$ and Re_i are minimal. Then $R = \coprod_{i\in I} e_i R$. For arbitrary i, we show that $e_i R$ is simple by simplicity test.

Let $e = e_i$. Take $0 \neq a \in eR$, then $ea = a$. Therefore, $\sum e_i = 1$, and so $\sum ae_i = a \neq 0$, so there exists j such that $ae_j \neq 0$. Now $0 \neq Rae_j \subseteq Re_j$ is simple, then $Rae_j = Re_j$, so $\exists b \in R$ such that $bae_j = e_j$.

Take $f : Re \to Re_j$ by sending $x \mapsto xae_j$, this is a homomorphism of left R-modules. Now $f(e) = eae_j = ae_j \neq 0$, so $f \not\equiv 0$. By Schur's lemma, f is an isomorphism. Now $f(abe) = abeae_j = abae_j = ae_j = f(e)$, so $abe = e \in aR$. □

Definition 8.1.15 (Minimal Ideal). *The definition of a minimal ideal of ring R is equivalent to the following conditions:*

- *N is nonzero and if K is an ideal of R with $K \subseteq N$, then either K is trivial or $K = N$.*
- *N is a simple R-module.*

Lemma 8.1.16. *A (left) Rmodule M is semi-simple if and only if M is a sum of simple submodules.*

Proof.

- If M is semisimple, then $M = \bigoplus_{j\in J} S_j$, where every S_j is simple. Given a subset $I \subseteq J$, define $S_I = \bigoplus_{j\in I} S_j$. If N is a submodule of M, we see, using Zorn's Lemma, that there exists a subset I of J maximal with $S_I \cap N = \{0\}$. We claim that $M = N \oplus S_I$, which will follow if we prove that $S_j \subseteq N + S_I$ for all $j \in J$. This inclusion holds, obviously, if $j \in I$. If $j \notin I$, then the maximality of I gives $(S_j + S_I) \cap N \neq \{0\}$. Thus, $s_j + s_I = n \neq 0$ for some $s_j \in S_j$, $s_I \in S_I$, and $n \in N$, so that $s_j = n - s_I \in (N + S_I) \cap S_j$. Now $s_j = 0$, lest $s_I \in S_I \cap N = \{0\}$. Since S_j is simple, we have $(N + S_I) \cap S_j = S_j$; that is, $S_j \subseteq N + S_I$.

- Suppose, conversely, that every submodule of M is a direct summand. We begin by showing that each nonzero submodule N contains a simple submodule. Let $x \in N$ be nonzero; by Zorn's Lemma, there is a submodule $Z \subseteq N$ maximal with $x \notin Z$. Now Z is a direct summand of M, by hypothesis, and so Z is a direct summand of N, i.e. $N = Z \oplus Y$. We claim that Y is simple. If Y is a proper nonzero submodule of Y , then $Y = Y \oplus Y$ and $N = Z \oplus Y = Z \oplus Y \oplus Y$. Either $Z \oplus Y$ or $Z \oplus Y$ does not contain x [lest $x \in (Z \oplus Y) \cap (Z \oplus Y) = Z$], contradicting the maximality of Z. Next, we show that M is semisimple. By Zorn's Lemma, there is a family $(S_k)_{k\in K}$ of simple submodules of M maximal with the property that they generate their direct sum $D = \bigoplus_{k\in K} S_k$. By hypothesis, $M = D \oplus E$ for some submodule E. If $E = \{0\}$, we are done. Otherwise, $E = S \oplus E$ for some simple submodule S, by the first part of our argument. But now the family $\{S\} \cup (S_k)_{k\in K}$ violates the maximality of $(S_k)_{k\in K}$, a contradiction.

$\square$

Lemma 8.1.17. *Let R be a semi-simple ring, $R = \coprod L_i$ where L_i's are minimal (left) ideals. Then every simple (left) R-module is isomorphic to L_i for some i.*

Proof. Note that $\mathbf{Hom}_R(R, M) = M$. Apply this to M as simple modules. Therefore, there exists a non-zero $R = \coprod L_i \to M$. Therefore, there exists a non-zero map $L_i \to M$ for some I. By Schur's lemma, $L_i \cong M$. $\square$

Theorem 8.1.18. *Let R be a ring. The following are equivalent.*

1. *R is semi-simple.*
2. *Every (left) R-module is semi-simple.*
3. *Every (left) R-module is projective.*

4. *Every (left) R-module is injective.*

5. *All short exact sequences split.*

Proof. (1) $\Rightarrow$ (2): Denote $R = \coprod L_i$. Let M be any left R-module. Take $m \in M$ and $L_i m \subseteq M$, then $L_i m \subseteq L_i$ is simple, so $L_i m = \begin{cases} 0 & , \\ L_i & \text{, simple} \end{cases}$. Then $M = R \cdot M = \sum_{m \in M, i} L_i M$ is the sum of simple submodules. By lemma, M is semi-simple.

(2) $\Rightarrow$ (3): Denote $M = \coprod M_j$ as a direct sum of simple modules. Since (2) $\Rightarrow$ (1) trivially, R is semi-simple, then $R = \coprod L_i$ simple modules and every M_j is isomorphic to L_i for some i. Then L_i projective implies M_i is projective. Therefore, M is projective.

(3) $\Rightarrow$ (5): Note that the sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

is split since P is projective.

(5) $\Rightarrow$ (4): M is injective if for every module Y and a submodule $X \subseteq Y$, every homomorphism $X \to M$ extends to $Y \to M$. Therefore we have

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[r] \arrow[l, bend right, "h"'] \arrow[dl, dashed] & Z \arrow[r] & 0 \\ & M & & & \end{tikzcd}$$

and induces the map from $Y \to M$ by $f \circ h$, since $(f \circ h) \circ g = f \circ (h \circ g) = f$.

(4) $\Rightarrow$ (1): Prove that R is a sum of minimal left ideals. Let I be the sum of all minimal left ideals. We show that $I = R$. Suppose that $I \neq R$, so $I < R$, then there exists a maximal left ideal $I \subseteq M \subseteq R$. Consider $0 \to M \to R \to R/M \to 0$. This splits because M is injective. Hence, there exists a submodule $N \subseteq R$ such that $N \cong R/M$. Now N is simple because R/M is simple. Because $M \cap N = 0$, we have $I \cap N = 0$. Hence, $I + N \supsetneq I$. But $I + N$ is a sum of simple modules, contradiction. $\square$

Recall that if $D_1, \cdots, D_s$ are division rings, then $R = M_{n_1}(D_1) \times \cdots \times M_{n_s}(D_s)$ is semi-simple. We also have the converse result.

Theorem 8.1.19. *Every semi-simple ring is of the form as above.*

Proof. Let R be the direct sum of minimal let ideals. Then $R \cong \coprod_{i=1}^{s} L_i^{\oplus n_i}$, where $L_1, \cdots, L_s$ are all non-isomorphic minimal left ideals. Now we can write $R = \mathbf{Hom}_R(R, R) = \mathbf{Hom}_R(\coprod L_i^{\oplus n_i}, \coprod L_j^{\oplus n_j}) = \mathbf{Hom}_R(\coprod N_i, \coprod N_j)$ as we denote $N_i = L_i^{\oplus n_i}$. This is just the set of matrices $\left\{ \begin{pmatrix} s_{11} & \cdots s_{1s} \\ \vdots & \ddots & \vdots \\ s_{s1} & \cdots & s_{ss} \end{pmatrix} : \right.$

$s_{ij} = \mathbf{Hom}_R(N_j, N_i)\Big\}$. If $i \neq j$, $\mathbf{Hom}_R(N_j, N_i) = 0$ since $\mathbf{Hom}_R(L_j, L_i) = 0$. Now for $i = j$, $\mathbf{Hom}_R(N_i, N_i) = \mathbf{Hom}_R(L_i^{\oplus n_i}, L_i^{\oplus n_i}) = M_{n_i}(D_i)$, where $D_i = \mathbf{Hom}_R(L_i, L_i) = \mathbf{End}_R(L_i)$, which is a division ring. Note that all matrices in the set above is diagonal, so we can write

$$R = \left\{ \begin{pmatrix} M_{n_1}(D_1) & 0 & \cdots & 0 \\ 0 & M_{n_2}(D_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{n_s}(D_s) \end{pmatrix} \right\} = M_{n_1}(D_1) \times \cdots \times M_{n_s}(D_s).$$

□

Remark 8.1.20. *R has exactly s simple modules $N_1, \cdots, N_s$, up to isomorphism. We have $D_i = End_R(N_i)$ and $\dim_{D_i}(N_i) = n_i$. Therefore, s does not depend on the decomposition.*

From the proof, we have

$$R = \mathbf{Hom}_R(R, R) = End_R(R)$$
$$x \mapsto l_x$$

should be viewed as a right module, with $l_x \circ l_y = l_{xy}$. Note $D_i = End_R(L_i)$ for L_i minimal right ideal.

Suppose $R = M_n(D) \cong L^{\oplus n}$ where L is a right module. Consider

$$D \to End_R(L)$$
$$x \mapsto l_x$$

and this is an isomorphism. Indeed, by writing $D \xrightarrow{f} \mathbf{Hom}_R(L, L)$, we have

$$D^n \xrightarrow{f^{\oplus n}} \mathbf{Hom}_R(R = L^{\oplus n}, L) = L \cong D^n$$

so $f^{\oplus n}$ is the identity map, then f is an isomorphism. As $R \cong L^{\oplus n} = L \oplus \cdots \oplus L$, but M is a right simple R-module, so $M \cong L$. Therefore, $D \cong End_R(M)$, $n = \dim_D(M)$, which are both unique for a ring R.

In general, when $R = M_{n_1}(D_1) \times \cdots \times M_{n_s}(D_s)$, let $K_i = (0, \cdots, 0, L_i, 0, \cdots, 0)$, so L_i's are all simple right R-modules up to iomsophism. s is the number of all such modules. Now $D_i = End_R(L_i)$, $n_i = \dim_{D_i}(L_i)$ are both unique.

If we use left module structure instead, we will recover a simple left R-module structure with L_i's as minimal left ideals, then $D_i^{op} \cong End_R(L_i')$. For M a (left) R-module such that $M \cong L^{oplusa_i} \oplus \cdots \oplus L_s^{\oplus a_s}$ the information is essentially the tuple of dimensions, and

$$R\text{-}Mod \cong D_1\text{-}Mod \times \cdots \times D_s\text{-}Mod M_n(D)\text{-}Mod \cong D\text{-}Mod$$

Note that the second isomorphism works for all rings D. This is called the Monta equivalence. Let D be a ring and $R = M_n(D)$, then ${}_R D^n_D$ acts as a bimodule, and the operations commute:

$$\begin{aligned} D\text{-}Mod &\leftrightarrow M\text{-}Mod \\ N &\mapsto D^n \otimes_D N \\ \mathbf{Hom}_R(D^n, M) &\leftarrow M \end{aligned}$$

8.2 Jacobson Radical

Definition 8.2.1 (Radical)**.** *Let R be a ring and M is a (left) R-module. Recall that the radical of M is the intersection of all submodules of M, denoted Rad(M). If the intersection is empty, then we say Rad$(M) = M$.*

Remark 8.2.2. *Some modules don't have maximal submodules. The proof for maximal ideals on Zorn's lemma does not work here, because the union of the modules is the entire module M: the union contains identity element, unlike the union of ideals, which doesn't contain the identity element.*

A submodule $N \subseteq M$ is maximal if and only if M/N is simple.

Example 8.2.3.

1. $Rad_{\mathbb{Z}}(\mathbb{Z}) = \bigcap\limits_{p\ prime} p\mathbb{Z} = 0.$

2. *$Rad_{\mathbb{Z}}(\mathbb{Q}) = \mathbb{Q}$ because it has no maximal submodule. If N is maximal, then $\mathbb{Q}/N$ is simple, so $\mathbb{Q}/N \cong \mathbb{Z}/p\mathbb{Z}$. We then have $Q \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$, but $\mathbb{Q}$ is divisible and $\mathbb{Z}/p\mathbb{Z}$ is not.*

3. $Rad_R(M/Rad_R(M)) = 0.$

Proposition 8.2.4. *Let M be a (left) R-module.*

1. *Let M be semisimple, then Rad$(M) = 0$.*

2. *If M is Artinian, and Rad$(M) = 0$, then M is semisimple.*

Proof.

1. Since M is semisimple, we have $M = \coprod\limits_{i\in I} M_i$ of simple modules. Write $N_j = \coprod\limits_{i\neq j} M_j \subseteq M$, then $M/N_j = M_j$ simple, so N_j is maximal. hence, $\bigcap\limits_j N_j = 0$, so Rad$(M) = 0$.

2. Let N be the sum of all simple submodules of M. Assume $N \neq M$, then there exists a minimal submodule $N' \subseteq N$ such that $N + N' = M$.

   Claim 8.2.5. $N \cap N' = 0.$

Subproof. Note that $N' \neq 0$ since $M \neq N$. Assume that $N \cap N' \neq 0$, since $\mathrm{Rad}(M) = 0$, then there exists a maximal submodule $M' \subseteq M$ such that $N \cap N' \nsubseteq M'$. Now $M' \subsetneq (N \cap N') + M' = M$.

Claim 8.2.6. $N + (M' \cap N') = M$.

Subproof. Take $m \in M$, then $m = x + y$ for $x \in N$ and $y \in N'$. Now $y = z + m'$ where $z \in N \cap N'$ and $m' \in M'$. Then $m' = y - z \in N'$. Hence, $m' \in M' \cap N'$. Now $m' = (x + z) + m'$, where $x + z \in N$ and $m' \in M' \cap N'$. ■

Because $N + N' = M$ and $N + (M' \cap N') = M$, then by minimality of N', $M' \cap N' = N'$, so $N' \subseteq M'$. But $N \cap N' \nsubseteq M'$, contradiction. ■

Now $M = N \oplus N'$ with $N' \neq 0$. N' contains a simple submodule P because M is Artinian. But $P \nsubseteq N$ by definition of N, contradiction.

□

Lemma 8.2.7. *Consider a ring R as its own left module. Now $Rad_R(R) = \{a \in R : 1 - xa \text{ has left inverse } \forall x \in R\}$.*

Proof. $\subseteq$: Take $a \in \mathrm{Rad}_R(R)$. Suppose $1 - xa$ has no left inverse, then $R(1 - xa) \neq R$, with $R(1 - xa) \subseteq M \subseteq R$ where M is a maximal left ideal. For $a \in \mathrm{Rad}(R)$, $xa \in \mathrm{Rad}(R)$, then $1 = (1 - xa) + xa \in M$ because $1 - xa \in M$ and $xa \in \mathrm{Rad}(R) \subseteq M$, contradiction.

$\supseteq$: Let $M \subseteq R$ be a maximal left ideal, $a \in R>$ Suppose $1 - xa$ has a left inverse for all $x \in R$. Suppose, towards contradiction, that $a \notin M$, then $a \nsubseteq M$, so $M \not\supseteq Ra + M = R$. Then $1 = xa + y$ where $xa \in R$ and $y \in M$, so $y = 1 - xa \in M$ has a left inverse, then $zy = 1$ is in M. However, $1 \notin M$ because it is a maximal ideal, so we reach a contradiction. □

Lemma 8.2.8. *If $1 - ab$ is left invertible, so is $1 - ba$.*

Proof. Suppose $c(1 - ab) = 1$, then $(1 + bca)(1 - ba) = 1$. □

Proposition 8.2.9. *$Rad_R(R) = \{a \in R : 1 - xay \in R^\times, \forall x, y \in R\}$.*

Proof. We know $\mathrm{Rad}_R(R) = \{a \in R : 1 - xy \text{ is left invertible } \forall x \in R\}$, so $\Leftarrow$ direction is clear.

($\Rightarrow$): $a \in \mathrm{Rad}_R(R)$, $x, y \in R$, we have $1 - yxa$ is left invertible. By lemma, $1 - xay$ is left invertible, let $b(1 - xay) = 1$. Then $1 + ybxa$ is left invertible, so by lemma $1 + bxay$ is left invertible. But $1 + bxay = b$, so b (and $1 - xay = b^{-1}$) is invertible. □

Remark 8.2.10. *This characterization is symmetric in left and right. $Rad_R(R)$ is a two-sided ideal in R, called the Jacobina radical of R, or $J(R)$. $Rad_R(R)$ is also the intersection of all maximal right ideals. If $R \neq 0$, then $J(R) \neq R$ (maximal ideal exists).*

Theorem 8.2.11. *Let R be a ring. Then R is semisimple if and only if R is Artinian and $J(R) = 0$.*

Remark 8.2.12. *Sometimes R doesn't have any non-trivial biideals.*

Proof. ($\Rightarrow$): Suppose $R = M_{n_1}(D_1) \times \cdots \times M_{n_s}(D_s)$ for $D_1, \cdots, D_s$ division rings, so $M_{n_i}(D_i)$'s are simple components of R, unique up to isomorphism. When $R = M_n(D)$, $D \hookrightarrow R$ by $d \mapsto \mathrm{diag}(d, \cdots, d)$, so left R-modules are left D-modules. For all $R \supseteq I_1 \supseteq I_2 \supseteq \cdots$, we have $\infty > \dim_D(R) \geq \dim_D(I_1) \geq \cdots \geq 0$, so the sequence stabilizes.

Claim 8.2.13. *$M_n(D)$ has no non-trivial two-sided ideals.*

Subproof. Suppose $I \subseteq M_n(D)$ is and $I \neq 0$, then let $x = \sum_{i,j} d_{ij} e_{ij} \in I$ be non-zero. Suppose $d_{kl} \neq 0$, then for all s, $e_{sk} \times e_{ls} = d_{kl} e_{ss} \in I$, so $e_{ss} \in I$ for all s. Hence, $1 = \sum_s e_{ss} \in I$. ■

Claim 8.2.14. *$J(R) = 0$.*

Subproof. Note that the set

$$\{\begin{pmatrix} 0 & \cdots & 0 & * & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & * & 0 & \cdots & 0 \end{pmatrix} \in M_n(D)\}$$

is a maximal left ideal, so $J(M_n(D)) = 0$. Therefore, $J(R) = \prod_i J(M_{n_i}(D_i)) = 0$. ■

□

Definition 8.2.15 (Simple)**.** *A ring R is simple if $R \neq 0$ and R has no non-trivial two-sided ideals.*

Example 8.2.16. *$M_n(D)$ is simple for D division ring.*

Theorem 8.2.17. *Every simple Artinian ring is isomorphic to $M_n(D)$ for some D division ring.*

Proof. Let $R \neq 0$ be a simple Artinian ring. $J(R) \neq R$ is a two-sided ideal, so $J(R) = 0$. Therefore, R is semisimple. Write $R = M_{n_1}(D_1) \times \cdots \times M_{n_s}(D_s)$. If $s \geq 2$, then $M_{n_1}(D_1) \times 0 \times \cdots \times 0 \subseteq R$ is a non-trivial two-sided ideal. □

8.3 Algebra

Definition 8.3.1 (R-Algebra)**.** *Let R be a commutative ring and S be a ring. S is an R-algebra if S has a structure of R-module such that*

1. *Two addition structures are the same.*

2. *$\forall a \in R$, $x, y \in S$, there is $a(xy) = (ax)y = x(ay)$.*

Remark 8.3.2. *For all $a \in R$ and $x \in S$, there is $ax = a(1_s \cdot x) = (a1_s) \cdot x$. Therefore, scalar products corresponds with products. Consider $R \to S$ by sending $a \mapsto a1_s$, then this is a ring homomorphism. Also, $\forall a \in R, x \in S$, we have $f(a)x = xf(a)$: $f(a)x = (a1_s) \cdot x) = ax$ and $xf(a) = x(a1_s) = a(x \cdot 1_s) = ax$. Then $im(f) \subseteq Z(S)$.*

Claim 8.3.3. *Conversely, suppose $f : R \to S$ is a ring homomorphism where R is commutative and $im(f) \subseteq Z(S)$, then S can be given an R-algebra structure.*

Proof. Note $ax = f(a) \cdot x$. Check the necessary conditions. □

Definition 8.3.4 (Category, Homomorphism)**.** *Let R be a commutative ring , then $\mathbf{Alg}(R)$ is the category of R-algebras. The morphisms in $\mathbf{Alg}(R)$ are R-algebra homomorphisms that*

1. *respect all structures, or*

2. *by claim, the following diagram commutes by the homomorphism from $S \to T$:*

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ S \arrow[rr] & & T \end{tikzcd}$$

Remark 8.3.5. *In particular, $\mathbf{Alg}(\mathbb{Z})$ is the category of rings. In $\mathbf{Alg}(R)$, the initial object is R, the final object is 0. Products are the same as products in the category of rings, but the coproducts are complicated. However, in $\mathbf{CAlg}(R)$, category of commutative rings over R, the coproduct of S and T is $S \otimes_R T$ by $(x \otimes y)(x' \otimes y') = xx' \otimes yy'$.*

So for all $f : S \to V$ and $g : T \to V$, we have $S \otimes_R T \to V$ by $x \otimes y \mapsto f(x)g(y)$. Note that V needs to be commutative.

8.4 Representation of Finite Groups

We can use three different languages to describe the groups.

Definition 8.4.1 (First Language: G-space)**.** *Let G be a group and F be a field. A G-space is a vector space over F, together with an G-action by linear operators:*

1. $g(v_1 + v_2) = gv_1 + gv_2)$,

2. $g(\lambda v) = \lambda(gv)$ *for all* $\lambda \in F$,

3. $(gh)v = g(hv)$,

4. $ev = v$.

The first two properties describe linearity, and the last two properties describe the G-action.

Definition 8.4.2 (Second Language: Representation)**.** *A representation of G over F is a group homomorphism $\rho : G \to GL(V)$ for some vector V over F.*

Remark 8.4.3. *G-spaces corresponds with representations by $gv = \rho(g)(v)$.*

Definition 8.4.4 (Third Language: Group Action)**.** *Let G be a group and F be a field. The group algebra is the vector space spanned by G:*

$$F[G] = \{\sum_{g\in G} G_g g : a_g \in F, \text{ almost all } 0\} = \{f : G \to F : f(g) = 0 \text{ for almost all } g\}.$$

There is a multiplication operation on $F[G]$ following the multiplication on G, given by

$$\left(\sum_{g\in G} a_g g\right)\left(\sum_{h\in G} b_h h\right) = \sum_{g\in G}\sum_{h\in G} a_g b_h gh = \sum_{l\in G}\left(\sum_{gh=l} a_g b_h\right) l.$$

We can view $G \hookrightarrow F[G]$ with only one non-zero coefficient. Then G is a basis of $F[G]$ and $\dim(F[G]) = |G|$.

Remark 8.4.5. *$F \hookrightarrow F[G]$ is given by $\lambda \mapsto \lambda \cdot e$, then F is a subring of $F[G]$.*

Remark 8.4.6 (Naturality)**.** *If we have a group homomorphism $H \to G$, then there is a F-algebra homomorphism $F[H] \to F[G]$. Therefore, we have a functor* **Grp** $\to F-$**Algebra** *by $G \mapsto F[G]$. This functor also has a right adjoint $S \mapsto S^\times$ such that*

$$\begin{tikzcd}
\left(f : G \to S^\times\right) \arrow[r, bend left, "extend"] & \left(h : F[G] \to S\right) \arrow[l, bend left, "restrict"]
\end{tikzcd}$$

where G is invertible.

Remark 8.4.7. *Suppose V is a G-space, then it is a left $F[G]$-module with the structure* $\left(\sum_{g\in G} a_g g\right) \cdot v = \sum_{g\in G} a_g(gv)$. *Moreover, the left $F[G]$-module structure then gives the structure of a G-space by restricting to $G \subseteq F[G]$. Therefore, we have*

	G-spaces	*Representations*	*Left $F[G]$-modules*
Basic Info	V 0 V *with* $\dim(V)=1$ $V=F$, *trivial action*	$\rho: G \to GL(V)$ $\rho: G \to \{e\}$ *zero representation* $\rho: G \to F^{\times}$ *as character* $\rho: G \to F^{\times}$, $\rho(g)=1$ *as trivial representation*	V 0 V *with* $\dim(V)=1$ $V=F$, $(\sum_{g\in G} a_g g)v = \sum_{g\in G} a_g v$
Category (all are Abelian)	*Vector spaces as Objects, Linear mappings $V \to W$ that preserves G-actions as Morphisms*	*Representations as Objects, Linear maps $f: V \to W$ such that $f(\rho(g)(v)) = \rho(g)f(v)$ as Morphisms*	*Category of $F[G]$-modules*
Direct Sum	$V \otimes W$: $g(v+w)=gv+gw$	$\rho \oplus \mu : G \to GL(V \oplus W)$	$V \oplus W$ *as $F[G]$-modules*
Isomorphism	*G-equivalent isomorphisms of vector Spaces*	*Isomorphisms of vector spaces $f: V \to W$ such that* $\begin{array}{ccc} G & \xrightarrow{\rho} & GL(V) \\ & {}_{\mu}\searrow & \sim\downarrow \text{conjugate by } f \\ & & GL(W) \end{array}$	*Module Isomorphisms*

Figure 8.1: Relationship between G-spaces, Representations and $F[G]$-Modules

When V is finite-dimensional, $GL(V) = GL_n(F)$, so a representation represents G as matrices $\rho: G \to GL_n(F)$, $\mu: G \to GL_m(F)$. This is (almost) another language: for $\rho \oplus \mu : G \to GL_{m+n}(F)$ that sends $g \mapsto \begin{pmatrix} \rho(g) & 0 \\ 0 & \mu(g) \end{pmatrix}$, where $\rho \cong \mu$ if and only if $\exists A \in GL_n(F)$ such that for all $g \in G$, $\mu(g) = A\rho(g)A^{-1}$.

Example 8.4.8. *Let G be a finite group of order n, then $F[G] = F[t]/(t^n - 1)$. When $F = \mathbb{Q}$, $\mathbb{Q}[G] = \prod_{d|n} \mathbb{Q}[t]/\varphi(d) = \prod_{d|n} \mathbb{Q}(\xi_d)$. Moreover, if G is commutative, then the group algebra is also commutative.*

Theorem 8.4.9. *Let G be a finite group and F be a field. Then $F[G]$ is semisimple as a ring if and only if $char(F) \nmid |G|$. In particular, if $char(F) = 0$, then every $F[G]$ structure is semisimple.*

Proof. ($\Rightarrow$): For $\varepsilon(g) = 1$, we have $F[G] \xrightarrow{\varepsilon} F \to 0$ as a short exact sequence. Then it is a surjective $F[G]$-module homomorphism (F is the $F[G]$-module of trivial representation). Note that the sequence splits, so there exists a section $f : F \to F[G]$. Then for all $g \in G$, $g \cdot f(1) = f(g \cdot 1) = f(1)$, so $f(1) = F[G]^G = F \cdot N$ where $N = \sum_{g\in G} g$. Note that $f(1) = \lambda N$, then $1 = \varepsilon(f(1)) = \varepsilon(\lambda N) = \lambda|G|$, so $|G| \neq 0$, and so $\mathrm{char}(F) \neq |G|$.

($\Leftarrow$): Consider an arbitrary short exact sequence of $F[G]$-modules:

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} P \longrightarrow 0$$

then there exists $h : M \to N$ such that $hf = 1$. Note that only linear $\mathbf{Hom}_F$ needs $\mathbf{Hom}_{F[G]}$. We set $\bar{h} = \frac{1}{|G|} \sum_{g \in G} g^{-1}h(gm)$. We only need to check that $\bar{h}f = 1$, with $\bar{h}$ an $F[G]$-linear map. Therefore, every short exact sequence splits, so $F[G]$ is semisimple. □

Remark 8.4.10. *char*$(F) \nmid |G|$ *if and only if* $F[G]$ *is semisimple. Then* $F[G] = M_{d_1}(D_1) \times \cdots \times M_{d_r}(D_r)$*. All simple* $F[G]$*-modules are* $L_i = 0 \times \cdots \times 0 \times D_i^{d_i} \times 0 \times \cdots \times 0$*. Now* $|G| = \sum_{i=1}^{r} d_i^2 \dim(D_i) < \infty$*, where* $D_i = End_{F[G]} L_i$ *is also an* F*-algebra. In particular,* $\dim_F L_i = d_i \dim(D_i) < \infty$*.*

Claim 8.4.11. *Let* M *be an* $F[G]$*-module with* $\dim_F(M) < \infty$*, then* $M = L_1^{\oplus a_1} \oplus \cdots \oplus L_r^{\oplus a_r}$ *with integers* $a_1, \cdots, a_r$ *uniquely determined by* M*. Note that this also works for* $\dim_F(M) = \infty$*.*

Remark 8.4.12 (Translation)**.** *M finite dimensional G-space, L_i simple G-spaces.*

Remark 8.4.13 (Translation)**.** *Let* $\rho : G \to GL(V)$ *be a representation with* $\dim_F(V) < \infty$*. We have* $\rho_i : G \to GL(L_i)$ *for* $1 \leq i \leq r$ *are irreducible representations of* G*. For all* ρ*, there is* $\rho = \rho_1^{\oplus a_1} \oplus \cdots \oplus \rho_r^{\oplus a_r}$*, so there exists a basis of* V *such that* ρ *is given by block matrices* $\rho_1, \cdots, \rho_r$*.*

From now on, we can assume F is algebraically closed with characteristic 0.

Lemma 8.4.14. *Let* D *be a finitely-dimensional division* F*-algebra over an algebraically closed field* F*, then* $D = F$*.*

Proof. Note we have $F \hookrightarrow D$ by $a \mapsto a \cdot 1$. For all $d \in D$, we have the set $\{1, d, d^2, \cdots\}$ that is linearly dependent over F. Then d is a root of $f(x) = \sum_{i=0}^{n} a_i x^i \in F[x]$. Now because F is algebraically closed, then $f(x) = a_n(x - b_1) \cdots (x - b_n)$. Note $f(d) = 0$ and D is a division ring, so $d - b_i = 0$ for some i. Therefore, $d = b_i \in F$, so $D = F$. □

Remark 8.4.15. *Now* $F[G] = M_{d_1}(F) \times \cdots \times M_{d_r}(F)$*, so* $|G| = d_1^2 + \cdots + d_r^2$*. Note simple modules are* $L_i = 0 \times \cdots \times 0 \times F^{d_i} \times 0 \times \cdots \times 0$*, with* $\dim_F(L_i) = d_i$*. Therefore,* $|G| = \sum_{i=1}^{r} d_i^2 = \sum_{i=1}^{r} (\dim_F(\rho_i))^2$*.*

Example 8.4.16. *$F[G]$ as a left $F[G]$-module is called a regular $F[G]$-module, regular G-space or regular representation.*

We have $F[G] = M_{d_1}(F) \times \cdots \times M_{d_r}(F) \cong L_1^{\oplus d_1} \oplus \cdots \oplus L_r^{\oplus d_r}$*, which are also corresponding to sum of columns of matrices. Therefore,* $\rho_{reg} = \rho_1^{\oplus d_1} \oplus \cdots \oplus \rho_r^{\oplus d_r}$ *for* $d_i = \dim_F(\rho_i)$*.*

This creates a new question: how to find irreducible representations?

Lemma 8.4.17. $Z(M_d(F)) = F$*.*

Proof. $a = (a_{ij})$ is in the center, compute $al_{kl} = l_{kl}a$, then $a_{ij} = 0$ for $i \neq j$, all a_{ii} are equal. □

Remark 8.4.18. *Now $F[G] = M_{d_1}(F) \times \cdots M_{d_r}(F)$, so $Z(F[G]) \cong \prod_{i=1}^{r} M_{d_i}(F) \cong F^r$, so the number of irreducible representations is just $\dim_F(Z(F[G]))$. Now $u = \sum_{g \in G} a_g g \in Z(F[G])$ if and only if $ux = xu$ for all $x \in G$. Therefore, we have*

$$\sum_{g \in G} a_g gx = \sum_{g \in G} a_g xg = \sum_{g \in G} a_g (xgx^{-1})x = \sum_{g' \in G} a_{x^{-1}g'x} g' x,$$

so $a_g = a_{x^{-1}gx}$ for all $g, x \in G$.

Let $G = C_1 \sqcup \cdots \sqcup C_s$ be the disjoint union of conjugacy classes, and let $v_i = \sum_{g \in C_i} g$, then $\{v_i\}_{i \in I}$ forms a basis for $Z(F[G])$. Therefore, we conclude the following.

Theorem 8.4.19. *The number of conjugacy classes is the same as the number of irreducible representations.*

Remark 8.4.20. *Although they are equal, these two sets do not have a "good" bijection.*

Consider $G \to \mathrm{GL}_1(F) = F^\times$.

Proposition 8.4.21. *Let G be a finite group. The following are equivalent:*

1. *G is Abelian.*
2. *Every irreducible representation has dimension 1.*
3. *The number of irreducible representations is $|G|$.*

Proof. Recall $F[G] = M_{d_1}(F) \times \cdots \times M_{d_r}(F)$ where r is the number of irreducible representations $\rho_1, \cdots, \rho_r$, and $d_i = \dim(\rho_i)$. Then G is Abelian if and only if $F[G]$ is commutative if and only if $d_1 = d_2 = \cdots = d_r = 1$. Therefore, 1) $\iff$ 2). Also, because $|G| = \sum_{i=1}^{r} d_i^2 \geq \sum_{i=1}^{r} 1^2 = r$, so $r = |G|$ if and only if all d_i's are 1. Therefore, 2) $\iff$ 3). $\square$

Therefore, for Abelian group G we have $|\mathbf{Hom}(G, F^\times)| = |G|$. Now, let G be an arbitrary finite group with homomorphism $\rho : G \to F^\times$. Note that there is the canonical decomposition into the Abelianization $G^{ab} = G/[G, G]$. Then $\mathbf{Hom}(G, F^\times) = \mathbf{Hom}(G^{ab}, F^\times)$. Hence, G has exactly $|G^{ab}| = [G : [G, G]]$ 1-dimensional representations. Note that 1-dimensional representations are irreducible.

Example 8.4.22. *1. Suppose $G = S_n$. Note that the number of irreducible representations is the number of conjugacy classes.*

Note F^n is a S_n-space, called the standard S_n-space. Note that

$$0 \longrightarrow M \longrightarrow F^n \twoheadrightarrow F \longrightarrow 0$$

where the map $F^n \to F$ is a surjective homomorphism of modules given by $(a_i) \mapsto \sum a_i$. Moreover, there is a section $F \to F^n$ given by $1 \mapsto \frac{1}{n}\sum g_i$. Now consider the kernel M, then $F^n \cong M \oplus F$. So M has $n-1$-dimensions.

We have $\rho_{st} = \rho'_{st} \oplus \mathbb{1}$, where $\mathbb{1}$ is the trivial action that sends every element to identity, and ρ'_{st} is irreducible. Then ρ'_{st} has dimension $n-1$.

For arbitrary n, consider $S_n \to F^\times$, but $[S_n, S_n] = A_n$, so $S_n/[S_n, S_n]$ is cyclic of order 2, then there are two representations $\sigma \mapsto Sgn(\sigma) = \pm 1$. In particular, for $G = S_3$, we have $d_n = 1, d_2 = 1, d_3 = 1$, with $\sum d_i^2 = 6 = |S_3|$.

For $G = S_4$, we have 5 representations by checking decomposition of 4, with two dimension-1 representations, then by decomposition we know $d_1 = d_2 = 1$, $d_3 = 2$, $d_4 = d_5 = 3$.

2. *For $G = D_8$ or Q_8, note $|G^{ab}| = 4$, $r = 3$, then $d_1 = 1, d_2 = 1, d_3 = 1, d_4 = 1, d_5 = 2$. In particular, $F[G] = F \times F \times F \times F \times M_2(F)$ for algebraically closed field, e.g. $\mathbb{C}$.*

   If $F = \mathbb{Q}$, the formula still holds for $G = D_8$, but $\mathbb{Q}[Q_8] = F \times F \times F \times F \times \mathbb{H}$, where $\mathbb{H} = M_1(\mathbb{H})$.

Remark 8.4.23 (Open Problem)**.** *Suppose G, H are groups such that $\mathbb{Z}[G] \cong \mathbb{Z}[H]$, does $G \cong H$ hold?*

8.5 Characters

Definition 8.5.1 (Character)**.** *Suppose we have a representation $\rho : G \to GL(V)$ for finite group G and $\dim(V) < \infty$, take $g \in G$, then the trace is $Tr(\rho(g)) = \chi_\rho(g)$. Here $\chi_\rho : G \to F$ is the character of ρ.*

Property 8.5.2.

1. $\rho \cong \rho' \Rightarrow \chi_\rho = \chi_{\rho'}$.
2. $\chi_{\rho\oplus\rho'} = \chi_\rho + \chi_{\rho'}$.
3. $\chi_\rho(hgh^{-1}) = \chi_\rho(g)$.
4. $\chi_\rho(e) = \dim(\rho)$.
5. *For 1-dimensional $\rho : G \to F^\times$, we have $\chi_\rho = \rho$.*

Example 8.5.3. *For $\rho_{reg} : G \to GL(F[G])$, $\chi_{reg} := \chi_{\rho_{reg}}$. Then G creates a basis for $F[G]$, so for any $g \in G$, $\rho_{reg}(g)(h) = gh$. Note $\rho_{reg}(g)$ is monomimal. Moreover, $gh \neq h$ for $g \neq e$, so* $\chi_{reg}(g) = \begin{cases} 0, & \text{if } g \neq e \\ |G|, & \text{if } g = e \end{cases}$.

Let $\rho_1, \cdots, \rho_r$ be characters of irreducible representations (irreducible characters). Then $\rho_{reg} = \rho_1^{d_1} \oplus \cdots \oplus \rho_r^{d_r}$ for $d_i = \dim(\rho_i)$. Hence, $\chi_{reg} = \sum_{i=1}^r d_i\chi_i$, where $\chi_{reg}(g) = \begin{cases} 0, & \text{if } g \neq e \\ |G|, & \text{if } g = e \end{cases}$.

Remark 8.5.4. *χ extends to $F[G]$ in the natural sense with $\chi(g) = tr(l_g)$.*

Remark 8.5.5. *Let $F[G] = M_{d_1}(F) \times \cdots \times M_{d_r}(F)$ where $e_1, \cdots, e_r$ are orthogonal idempotents that partition 1. Let M_j be the corresponding simple modules with $\dim(M_j) = d_j$, then $M_j = 0 \times \cdots \times L_j \times \cdots \times 0$, where L_j is the minimal j-th component, $L_j = F[G]e_j$. Let $m \in M_j$, then $\chi_j(e_i m) = \begin{cases} \chi_j(m), & i = j \\ 0, & j \neq j \end{cases}$, with $\chi_j(e_i m) = \chi_j(e_i^2 m e_i^{-1}) = \chi_j(m)$.*

Let us write $e_i = \sum\limits_{g \in G} a_{ig} g$ for $a_{ig} \in F$. Then $\chi_{reg}(e_i g^{-1}) = \chi_{reg}(\sum\limits_{h \in G} a_{ih} h g^{-1}) = \sum\limits_{h \in G} a_{ih} \chi_{reg}(h g^{-1}) = |G| a_{ig}$, but $\chi_{reg}(e_i g^{-1}) = \sum\limits_{j=1}^{r} d_j \chi_j(e_i g^{-1}) = d_i \chi_i(g^{-1})$. Hence, $e_i = \frac{d_i}{|G|} \sum\limits_{g \in G} \chi_i(g^{-1}) g$.

Remark 8.5.6. *Let $Ch(G) = \{f : G \to F : f(ghg^{-1}) = f(h), \forall g, h \in G\}$, then $\dim(Ch(G))$ is just the number of conjugacy classes in G. Moreover, $Ch(G)$ has a bilinear form $\langle \chi, \eta \rangle = \frac{1}{|G|} \sum\limits_{g \in G} \chi(g^{-1}) \eta(g) \in F$.*

Proposition 8.5.7. *The irreducible representations $\chi_1, \cdots, \chi_r$ form an orthonormal basis of $Ch(G)$.*

Proof. Note $\chi_j(e_i) = d_i \delta_i$. Also,

$$\chi_j(e_i) = \chi_j(\frac{d_i}{|G|} \sum_{g \in G} \chi_i(g^{-1}) g) = \frac{d_i}{|G|} \sum_{g \in G} \chi_i(g^{-1}) \chi_j(g) = d_i \langle \chi_i, \chi_j \rangle .$$

From orthonormality, we conclude that we have a basis. □

Theorem 8.5.8. *Suppose F is an algebraically closed field with $char(F) \neq |G|$. Let G be a finite group with $\rho_1, \cdots, \rho_r$ as irreducible representations of G, with irreducible characters $\chi_1, \cdots, \chi_r$ correspondingly. Then*

1. *Every representation ρ of G is isomorphic to $\rho_1^{\oplus n_1} \oplus \cdots \oplus \rho_r^{\oplus n_r}$ where $n_i = \langle \chi_\rho, \chi_i \rangle \in \mathbb{Z}$ and $\chi_\rho = \sum\limits_{i=1}^{r} n_i \chi_i$.*

2. *Two representations ρ and μ are isomorphic as G-spaces if and only if $\chi_\rho = \chi_\mu$.*

3. *A representation ρ is irreducible if and only if $\langle \chi_\rho, \chi_\rho \rangle = 1$.*

Proof.

1. By induction.

2. From $\chi_\rho = \chi_\mu$ we get a decomposition of ρ and of μ, then apply the first property.

3. We have $\rho = \rho_1^{\oplus n_1} \oplus \cdots \oplus \rho_r^{\oplus n_r}$, then $\langle \chi_\rho, \chi_\rho \rangle = \sum\limits_{i=1}^{r} n_i^2$.

□

Example 8.5.9. *1. Consider $G = S_n$. We have the standard representation ρ_{st} acts on F^n, then $\chi_{st}(\sigma)$ is the number of fixed entries of an entry $\sigma \in S_n$. In particular, $\langle \mathbb{1}, \rho_{st} \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} Fix(\sigma) = 1$. Also, $\rho_{st} = \mathbb{1} \oplus \rho'_{st}$ where both $\mathbb{1}$ and ρ'_{st} are irreducible, then we have*

$$\frac{1}{n!} \sum_{\sigma \in S_n} Fix(\sigma)^2 = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_{st}(\sigma)\chi_{st}(\sigma^{-1}) = \langle \chi_{st}, \chi_{st} \rangle = 2.$$

2. Suppose $G = S_3$. Then the three characters χ_1, χ_2, χ_3 are 1, 1 and 2, respectively with $\rho_i : G \to F^\times$. In particular, G is generated by σ and τ where $\sigma^3 = 1$, $\tau^2 = 1$ and $\tau\sigma\tau = \sigma^{-1}$. Therefore we have

	1	σ	σ^2	τ	$\sigma\tau$	$\sigma^2\tau$
$\chi_1 = \mathbb{1}$	1	1	1	1	1	1
χ_2	1	1	1	-1	-1	-1
χ_3	2	-1	-1	0	0	0
χ_{st}	3	0	0	1	1	1

Figure 8.2: Character Table of S_3

Note $\chi_{reg} = \sum_i d_i \chi_i$ with $\sum_i d_i \chi(\varepsilon) = \begin{cases} |G|, & \varepsilon = 1 \\ 0, & \varepsilon \neq 1 \end{cases}$. Because $\chi_{st}(\varepsilon) = Fix(\varepsilon)$, we have $\chi_{st} = \chi_1 + \chi_3$, hence $\rho_{st} = \mathbb{1} \oplus \rho'_{st}$.

3. Suppose $G = Q_8$, then the five characters $\chi_1, \chi_2, \chi_3, \chi_4, \chi_5$ are 1, 1, 1, 1 and 2, respectively with $\rho_i : G \to F^\times$. Note that Q_8 is generated with i, j such that $i^2 = \varepsilon = j^2$, $ji = \varepsilon ij$, $\varepsilon i = i\varepsilon$ and $\varepsilon j = j\varepsilon$. We have the following character table:

	1	i	j	ij	ε	εi	εj	εij
χ_1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	-1	1	-1
χ_3	1	1	-1	-1	1	1	-1	-1
χ_4	1	-1	-1	1	1	-1	-1	1
χ_5	2	0	0	0	-2	0	0	0

Figure 8.3: Character Table of Q_8

Note that there is the canonical decomposition

$$Q_8 \longrightarrow Q_8 / \langle \varepsilon \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow F^\times$$

Therefore we have the map from $G \to GL_2(F)$ given by $i \mapsto \begin{pmatrix} \sqrt[4]{-1} & 0 \\ 0 & -\sqrt[4]{1} \end{pmatrix}$, $j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

and $\varepsilon \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. *Recall* $\mathbb{Q}[Q_8] = F \times F \times F \times F \times \mathbb{H}$ *with dimensions* 1, 1, 1, 1 *and* 4, *respectively.*

8.6 Hurwitz Theorem

Recall in $\mathbb{C}$ we have the norm as a function $N(x+yi) = (x+yi)(x-yi) = x^2+y^2$, and it is multiplicative that $N(z_1z_2) = N(z_1)N(z_2)$. Similarly, in $\mathbb{H}$, for $q = x_1 + x_2i + x_3j + x_4ij$, we have $N(q) = x_1^2 + x_2^2 + x_3^2 + x_4^2 = q \cdot \bar{q}$, where $\bar{q} = x_1 - x_2i - x_3j - x_4ij$.

Similarly, $(x_1^2 + x_2^2 + x_3^2 + x_4)^2(y_1^2 + y_2^2 + y_3^2 + y_4^2) = f_1^2 + f_2^2 + f_3^2 + f_4$ where $f_i \in \mathbb{Z}[x, y]$.

A more common form of such case is the Cayley Algebra (Octonion Algebra), which is non-associative with N mapping an element to $\sum_{i=1}^{8} x_i^2$. Informally, Hurwitz theorem states that such formula only works when $n = 1, 2, 4, 8$.

Theorem 8.6.1 (Hurwitz)**.** *If there are* $f_1, \cdots, f_n \in \mathbb{C}[x_1, \cdots, x_n, y_1, \cdots, y_n]$ *such that*

$$(\sum_{i=1}^{n} x_i) \cdot (\sum_{i=1}^{n} y_i) = \sum_{i=1}^{n} f_i^2,$$

then $n = 1, 2, 4$ *or* 8.

In other words, the only Euclidean Hurwitz algebras are the real numbers, the complex numbers, the quaterions and the octonians.

Proof. Denote $f_i = \sum_j a_{ij}(x) \cdot y_j$, where $a_{ij}(x)$ are linear homogeneous in x. Then

$$\begin{aligned} \sum_i f_i^2 &= \sum_i \sum_j a_{ij}(x)^2 y_j^2 + 2 \sum_i \sum_{j<k} a_{ij}(x) \cdot a_{ik}(x) \cdot y_j y_k \\ &= (\sum_i x_i^2)(\sum_i y_i^2). \end{aligned}$$

For all j, $\sum_i a_{ij}(x)^2 = \sum_i x_i^2$, and $\sum_i a_{ij}(x) \cdot a_{ik}(x) = 0$ for all $j \neq k$.

Take $A = A(x) = (a_{ij}(x))$, an $n \times n$ matrix of homogeneous linear polynomials. Then

$$A_t \cdot A = (\sum_i x_i^2) \cdot I_n,$$

equivalent to the original matrix.

We now write $A = \sum_i A_i x_i$, where A_i is an $n \times n$ matrix over $\mathbb{C}$:

$$(\sum_i A_i^t \cdot x_i)(\sum_j A_j x_j) = (\sum_i x_i^2) \cdot I_n.$$

Note $A_i^t \cdot A_i = I_n$, with $A_i^t \cdot A_j + A_j^t \cdot A_i = 0$ for all $i \neq j$. Denote $B_i = A_n^t \cdot A_i$ for $i = 1, \cdots, n-1$, then $B_i^t = A_n^t \cdot A_n = -A_n^t \cdot A_i = -B_i$, which shows a skew-symmetry property. Moreover, $B_i^2 = A_n^t A_i A_n^t A_i = -A_i^t A_n A_n^t A_i = A_i^t A_i = I$. For $i \neq j$, we have

$$\begin{aligned} B_i B_j + B_j B_i &= A_n^t A_i A_n^t A_j + A_n^t A_j A_n A_i \\ &= -A_i^t A_j - A_j^t A_i \\ &= 0. \end{aligned}$$

Overall, the $n \times n$ matrices B_i over $\mathbb{C}$ for $i = 1, \cdots, n-1$ satisfies

- $B_i^2 = -1$.
- $B_i B_j = -B_j B_i$, $i \neq j$.

Therefore, this now looks more like a representation of a group.

Let G be a group generated by $a_1, \cdots, a_{n-1}, \varepsilon$ with relations:

- $a_i^2 = \varepsilon$.
- $a_j a_i = \varepsilon a_i a_j$ for all $i \neq j$.
- $\varepsilon^2 = 1$.
- $\varepsilon a_i = a_i \varepsilon$.

This is called the generalized quaternion group. Note that every $g \in G$ has the form $g = \varepsilon^s \cdot a_1^{t_1} \cdots a_{n-1}^{t_{n-1}}$, where $s, t_1, t_2, \cdots, t_{n-1}$ are 0 and 1. Then $|G| = 2^n$, and $[G, G] = \langle \varepsilon \rangle$.

In particular, we have $a_i \mapsto B_i$ with $\varepsilon \mapsto -I$, which is an n-dimensional representation of G. $\square$

Now observe that $\rho : G \to \mathrm{GL}(V)$ irreducible, with $F[G] \to \mathbf{End}(V)$. Note that there is a generated map $Z(F[G]) \to \mathbf{End}_{F[G]}(V) = F$, by having the center acting by scalar multiplication.

Proposition 8.6.2. *Let $C(g)$ be the conjugacy classes of $g \in G$, and let ρ be an irreducible representation of G of dimension d. Denote $\chi = \chi_\rho$. Then $\frac{1}{d}|C(g)|\chi(g)$ is an algebraic integer (we can assume $F = \mathbb{C}$).*

Proof. Take $g \in G$. Let $x = \sum_{h \in C(g)} h \in Z(F[G])$. Let $f : Z(F[G]) \to F$ and $\rho : F[G] \to \mathbf{End}(V)$ as above. Let $\alpha = f(x)$, then $\rho(x)$ is a diagonal matrix where every entry is α. Then we denote $d\alpha = \mathrm{Tr}(\rho(x)) = \chi(x) = \sum_{h \in C(g)} \chi(h) = |C(g)|\chi(g)$. Therefore, character is invariant under conjugation. Hence, we have

$$\frac{|C(g)|\chi(g)}{d} = \alpha.$$

Note x has integral coefficients, so $x \in Z(\mathbb{Z}[G])$. Therefore is now an induced map $\bar{f} : Z(\mathbb{Z}[G]) \to \mathbb{C}$, and $\alpha \in \mathrm{im}(\bar{f}) \subseteq \mathbb{C}$, and $\mathrm{im}(\bar{f})$ is a finitely generated subring.

Therefore, $\mathbb{Z}[\alpha]$ is a faithful $\mathbb{Z}[\alpha]$-module, finitely generated subring of $\mathbb{C}$. Therefore, α is an algebraic integer. □

Theorem 8.6.3. *Let d be the dimension of an irreducible representation of G over $\mathbb{C}$. Then $d \mid |G|$.*

Proof. Let $n = |G|$ and let χ be the character. Then we know

$$1 = \langle \chi, \chi \rangle = \frac{1}{n} \sum_{g \in G} \chi(g^{-1})\chi(g).$$

Let $G = C(g_1) \sqcup C(g_2) \sqcup \cdots \sqcup C(g_r)$ be the conjugacy classes. Then we have

$$1 = \frac{1}{n} \sum_{i=1}^{r} |C(g_i)| \chi(g_i^{-1})\chi(g_i).$$

Hence, $\frac{n}{d} = \sum\limits_{i=1}^{r} \frac{|C(g_i)|}{d} \chi(g_i)\chi(g_i^{-1})$, where g_i and g_i^{-1} are both algebraic integers.

Therefore, $\frac{n}{d}$ is also an algebraic integer, so $d \mid n$. □

Now suppose $F \subseteq K$ and both are algebraically closed with characteristic 0. Now for every representation $\rho L : G \to \mathrm{GL}_n(F)$, we can compose $G \to \mathrm{GL}_n(F) \hookrightarrow \mathrm{GL}_n(K)$. As a functor, we know $\mathrm{Rep}_F(G) \to \mathrm{Rep}_K(G)$ by $M \mapsto K \otimes_F M$. We then have $K[G] \cong K \otimes_F F[G]$.

Claim 8.6.4. *ρ is irreducible if and only if ρ_K is irreducible.*

Proof. Note $\chi_\rho = \chi_{\rho_K}$ and irreducible if and only if $\langle \chi, \chi \rangle = 1$. This is true in algebraically closed fields with characteristic 0. □

Also note that $\dim(\rho) = \dim(\rho_K)$. Therefore, irreducible representations over K are exactly those obtained from irreducible representations over F.

For F algebraically closed and characteristic 0, we have a one-to-one correspondence between irreducible representations, so it suffices to prove for $\mathbb{C}$ only.

$$\begin{tikzcd}
F \arrow[rr, dash] \arrow[dr, dash] & & \mathbb{C} \arrow[dl, dash] \\
& \mathbb{Q}_{alg} &
\end{tikzcd}$$

8.7 Tensor Product of Representations

Definition 8.7.1 (Tensor Product of Representation)**.** *Suppose $\rho_1 : G_1 \to GL(V)$ and $\rho_2 : G_2 \to GL(W)$, then $V \otimes W$ is a $G_1 \times G_2$-space by $(g_1, g_2)(v \otimes w) = g_1 v \otimes g_2 w$. This is well-defined because the left hand side is a bilinear map. This is the tensor product of representations $\rho_1 \otimes \rho_2$. Furthermore,* *we have* $\dim(\rho_1 \otimes \rho_2) = \dim(\rho_1) \cdot \dim(\rho_2)$. *If* $\{x_1, \cdots, x_m\}$ *is a basis for* V *and* $\{y_1, \cdots, y_n\}$ *is a basis for* W*, then* $\{x_i \otimes y_i\}_{i,j}$ *is a basis for* $V \otimes W$.

Let $g_1 \in G$ and $g_2 \in G_2$, and let $\rho_1(g_1)(x_i) = \cdots + a_i x_i + \cdots$ and $\rho_2(g_2)(y_j) = \cdots + b_j y_j + \cdots$, so $(\rho_1 \otimes \rho_2)(g_1, g_2)(x_i \otimes y_j) = \cdots + a_i b_j (x_i \otimes y_j) + \cdots$.

In particular, we know

$$\chi_{\rho_1 \otimes \rho_2} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j = (\sum_{i=1}^{m} a_i)(\sum_{j=1}^{n} b_j) = \chi_{\rho_1} \cdot \chi_{\rho_2}.$$

Let $\rho_i, \mu_i u$ be the representations of G_i for $i = 1, 2$. Therefore, we have

$$\begin{aligned} \langle \rho_1 \otimes \rho_2, \mu_1 \otimes \mu_2 \rangle &= \frac{1}{|G_1||G_2|} \sum_{g_1 \in G_1} \sum_{g_2 \in G_2} \chi_{\rho_1 \otimes \rho_2}(g_1 g_2) \chi_{\mu_1 \otimes \mu_2}(g_1^{-1} g_2^{-1}) \\ &= \frac{1}{|G_1||G_2|} \sum_{g_1 \in G_1} \sum_{g_2 \in G_2} \chi_{\rho_1}(g_1) \chi_{\rho_2}(g_2) \chi_{\mu_1}(g_1^{-1}) \chi_{\mu_2}(g_2^{-1}) \\ &= \langle \rho_1, \mu_1 \rangle \langle \rho_2, \mu_2 \rangle . \end{aligned}$$

Hence, if ρ_1, ρ_2 are irreducible, so is $\rho_1 \otimes \rho_2$.

Claim 8.7.2. *Let* $\rho_i^{(1)}, \cdots, \rho_i^{(r_i)}$ *be all the irreducible representations of* G_i *for* $i = 1, 2$. *Then* $\{\rho_1^{(i)} \otimes \rho_2^{(j)}\}_{i,j}$ *are all the irreducible representations of* $G_1 \times G_2$.

Proof. Look at the number of conjugacy classes or sum of square of dimensions. $\square$

Note that this only works for algebraically closed field with characteristic 0.

Now suppose we have the map $H \xrightarrow{f} G \to \mathrm{GL}(V)$, then f gives a functor $\mathrm{Rep}(G) \xrightarrow{f^*} \mathrm{Rep}(H)$. In particular, if $H < G$, then f is the restriction functor. However, the restriction functor does not preserve irreducibility.

If $\rho_1 : G \to \mathrm{GL}(V_1)$ and $\rho_2 : G \to \mathrm{GL}(V_2)$, then we have a map $\rho_1 \otimes \rho_2 : G \times G \to \mathrm{GL}(V_1 \otimes V_2)$. We can now restrict to the diagonal functor

$$G \xhookrightarrow{\Delta} G_1 \times G_2.$$

This is also called tensor product of $\rho_1 \otimes \rho_2$. Note that this "tensor product" may not preserve irreducibility as well.

Now $\oplus$ and $\otimes$ are operations that make $\mathrm{Rep}(G)$ a tensor category. The set of isomorphisms of $\mathrm{Rep}(G)$ is a ring with the two operations. This gives a free Abelian group with basis irreducible representations.

Definition 8.7.3 (Representation Ring)**.** *Let* G *be a finite group and let* $\rho_1, \cdots, \rho_r$ *irreducible representations. We now define*

$$R(G) = \{\sum_{i=1}^{r} a_i p_i, a_i \in \mathbb{Z}\}$$

as the free Abelian group generated by $[\rho_1], \cdots, \rho_r]$. Note that $R(G)$ is a ring: note $\rho_i \otimes \rho_j = \coprod_{k=1}^{r} \rho_k^{\oplus b_k}$ and set $[\rho_i][\rho_j] = \sum_{k=1}^{r} b_k \rho_k = [\rho_i \otimes \rho_j]$.

We can check $[\varepsilon] \cdot [\mu] = [\varepsilon \otimes \mu]$ for all representations. Therefore, the multiplication operation is associative. The identity is given by $[\mathbb{1}]$, and $R(G)$ is called the representation ring.

Without using irreducible representations, another way to define $R(G)$ is using generators and relations. The generators are given by isomorphism classes of all (finite-dimensional) representations. The relations are given by the generators commuting, with $[\varepsilon \oplus \mu] = [\varepsilon] + [\mu]$. Then $[\varepsilon] = [\coprod_{i=1}^{r} b_i \rho_i] = \sum_{i=1}^{r} b_i [\rho_i]$. This agrees with $R(G)$ above. One needs to show that $[\rho_1], \cdots, [\rho_i]$ are linearly independent.

Definition 8.7.4 (Grothendieck Group/Ring)**.** *In general, $R(G)$ can be defined for any category with direct sum/tensor product. This is called the Grothendieck group/ring.*

Now let A be the set of isomorphism classes of representations, then A is actually a monoid with respect to $\oplus$. Consider $A^+ = A \times A/\sim$ where $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 \oplus y_2) = y_1 \oplus x_2$. Then this is a group with component-wise addition. In particular, we have $(x, y)^{-1} = (y, x)$ since $(x, y) + (y, x) = (x \oplus y, y \oplus x) \sim (0, 0)$. In general, this is a functor that is the left adjoint of the forgetful functor:

Consider A in **CMon** and G in **Ab**. Now any map $f : A \to G$ is corresponding to the map $A^+ \to G$ by setting $(x, y) \mapsto f(x) - f(y)$, then we have $\mathbf{Hom}_{\mathbf{CMon}}(A, G) = \mathbf{Hom}_{\mathbf{Ab}}(A^+, G)$. We can then define $R(G) = A^+$.

Recall that $\mathrm{Ch}(G) = \{f : G \to F, f(ghg^{-1}) = f(h)\}$ is a vector space. Now $R(G)$ is the subgroup of $\mathrm{Ch}(G)$ generated by χ_ρ for all representations ρ, which is essentially the same as the free Abelian subgroup generated by all irreducible characters $\chi_1, \cdots, \chi_\rho$. The product in $R(G)$ is the usual product in F, given by $\chi_{\rho \oplus \mu} = \chi_\rho \cdot \chi_\mu$. This is convenient for computation.

Example 8.7.5. *Suppose G is a finite Abelian group. Then $G^* = \mathbf{Hom}(G, F^\times)$ are all irreducible characters/representations, with $R(G) = \mathbb{Z}[G^*]$. Then G^* is called the character group, with $G^* \cong G$ as a non-canonical isomorphism.*

Example 8.7.6. *Recall $G = S_3 = \langle \sigma, \tau : \sigma^3 = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$. We also had the character table*

	1	σ	σ^2	τ	$\sigma\tau$	$\sigma^2\tau$
$\chi_1 = \mathbb{1}$	1	1	1	1	1	1
χ_2	1	1	1	-1	-1	-1
χ_3	2	-1	-1	0	0	0

As a group, $R[G] = \mathbb{Z}\cdot\mathbb{1}\oplus\mathbb{Z}\cdot x\oplus\mathbb{Z}\cdot y$. As a ring, we know $\mathbb{1}$ is the identity, and $x^2=\mathbb{1}$, $xy = yx = y$, with $y^2 = \mathbb{1}+x+y$. Therefore,

$$R(G)\cong\mathbb{Z}[X,Y]/(X^2-1, XY-Y, Y^2-Y-X-1).$$

Theorem 8.7.7. *Let G be a finite group and ρ is a irreducible representation, set $d=\dim(\rho)$, then $d \mid [G:Z(G)]$. Recall we have shown that $d\mid |G|$.*

Proof. Denote $\rho: G\to \mathrm{GL}(V)$ with $\dim(V)=d$. Consider $\rho^{\otimes m}: G^m\to\mathrm{GL}(V^{\otimes m})$, then $\dim(\rho^{\otimes m}) = d^m$. Let $S\subseteq G^m$ by $S=\{(g_1,\cdots,g_m)\in Z(G), g_1\cdots g_m = 1\}$. Then S is a normal subgroup, with $|S| = |Z(G)|^{m-1}$. Now ρ is irreducible, so for all $g\in Z(G)$, there exists $\alpha\in F$ such that $gv=\alpha v$ for all $v\in V$.

Consider $(g_1,\cdots,g_m)(v_1\otimes\cdots\otimes v_m) = (\alpha_1 v_1)\otimes\cdots\otimes(\alpha_m v_m) = (\alpha_1\cdots\alpha_m)(v_1\otimes\cdots\otimes v_m)$. Note that if $gv=\alpha v$ and $hv=\beta v$, then $(gh)v=\alpha\beta v$.

We have $\alpha_1\alpha_2\cdots\alpha_n = 1$ since $g_1\cdots g_m=1$. Hence, S acts on $V^{\otimes m}$ by identity, so $\rho^{\otimes m}(s)=I$. Therefore, $S\subseteq\ker(\rho)$. Then $\rho: G^m/S\to \mathrm{GL}(V^{\otimes m}$. Note $\rho^{\otimes m}: G^m\to \mathrm{GL}$ is still irreducible. Then $d^m\mid [G^m:S] = |G|^m/|Z(G)|^{m-1}$ for all m. Therefore, we have $d\mid |G|/|Z(G)|$. □

Theorem 8.7.8 (Burnside's pq-Theorem)**.** *Let p and q be prime integers. Every group of order $p^a q^b$ is solvable for all $a,b\in\mathbb{Z}_{\geq 0}$.*

We would develop the proof for the theorem gradually.

Lemma 8.7.9. *Let G be a finite group and ρ is an irreducible representation over $\mathbb{C}$ of dimension d. Denote $\chi=\chi_\rho$. Let $C\subseteq G$ be a conjugacy class such that $\gcd(|C|,d)=1$. Then every element $g\in C$ either satisfies $\chi(g)=0$ or $\rho(g)$ is a scalar matrix.*

Proof. Suppose $a|c|+bd=1$ with $a,b\in\mathbb{Z}$, then

$$a\frac{|c|\chi(g)}{d}+b\chi(g) = \frac{\chi(g)}{d}$$

where $\chi(g)$ and $\frac{|c|\chi(g)}{d}$ are algebraic integers. Therefore, $\frac{\chi(g)}{d}$ is an algebraic integer. Also, $|\chi(g)|\leq d$ and if $|\chi(g)|=d$ then $\rho(g)$ is a scalar matrix.

Suppose $\alpha=\frac{\chi(g)}{d}$ with $\alpha<1$. Let $n=|G|$. Let $\Gamma=\mathrm{Gal}(\mathbb{Q}(\xi_n)/\mathbb{Q})$. Write $\chi(g)=\chi_1+\cdots+\chi_d$ for $\chi_1,\cdots,\chi_d\in\mu_n$, the set of primitive n-th root of unity. For all $\gamma\in\Gamma$, we have $\gamma\chi(g)=\gamma\chi_1+\cdots+\gamma\chi_d$. Hence, we know $|\gamma\chi(g)|\leq|\gamma\chi_1|+\cdots+|\gamma\chi_d| = d$. Therefore,

$$|\gamma\alpha| = |\frac{\gamma\chi(g)}{d}|\leq 1.$$

Now $c=\prod_{\gamma\in\Gamma}\gamma\alpha\in\mathbb{Q}(\xi_n)^\tau=\mathbb{Q}$, where $\mathbb{Q}(\xi_n)^\tau$ is the set of fixed points of the Galois group. Moreover, we know $|c|=\prod_{\gamma\in\Gamma}|\gamma\alpha|<1$, where $|\gamma\alpha|<1$ when $\gamma=\mathbf{id}$. But c is also an algebraic integer, so $c=0$. Therefore, $\alpha=0$, and so $\chi(g)=0$. □

Proposition 8.7.10. *Suppose $C \subseteq G$ is a conjugacy class, and $|C| = p^a > 1$ for prime p. Then G is not simple.*

Proof. Suppose G is simple. Let $\rho_1, \cdots, \rho_r$ be irreducible representations of G. Let $\chi_1, \cdots, \chi_r$ be their characters. Let $d_1, \cdots, d_r$ be their dimensions. Also set $\rho_1 = \mathbb{1}$.

Claim 8.7.11. *If $p \nmid d_i$ for some $i > 1$, then $\chi_i(g) = 0$ for all $g \in C$.*

Subproof. Set $H = \{g \in G : \rho_i(g) \text{ is a scalar matrix}\}$, then $H \lhd G$. Note $\ker(\rho_i) \lhd G$ and $\ker(\rho_i) \neq G$ since $\rho_i \neq \mathbb{1}$ for G simple. Therefore, ρ_i is injective. ■

If $G = H$, then $G \cong \mathrm{im}(\rho_i)$ is Abelian, contradiction. Therefore, $H = \{e\}$. Also $e \notin C$ since $|C| > 1$. Then $\chi(g) = 0$ for all $g \in C$ by lemma.

Now note that $\chi_{reg} = \sum_{i=1}^{r} d_i\chi_i$ for all $g \in C$, and $0 = \chi_{reg}(g) = 1 + \sum_{i=2}^{r} d_i\chi_i(g)$ because $g \neq e$. Hence,

$$-\frac{1}{p} = \sum_{i=2}^{r} \frac{d_i\chi_i(g)}{p}.$$

If $p \mid d_i$, then $\frac{d_i\chi_i(g)}{p}$ is an algebraic integer. If $p \nmid d_i$, then $\chi_i(g) = 0$. Therefore, $-\frac{1}{p}$ is an algebraic integer, contradiction. □

We now prove the Burnside Theorem above.

Proof. Assume $p \neq q$ and $a, b > 0$. Otherwise the case is known. Let $|G| = p^a q^b$. It suffices to show G is not simple by induction.

Let $Q < G$ be a Sylow q-subgroup. Then $[G : Q] = p^a$ with $Q \neq 1$. Let $g \in Z(Q)$ be non-trivial, and note non-trivial q-groups have non-trivial center. Then let H be the centralizer of g in G, then $Q < H < G$, so $[G : H] = p^r$ for some $r \geq 0$. Let $C \subseteq G$ be the conjugacy class of g in G. Then $|C| = \frac{|G|}{|H|} = p^r$. Note that G acts on C by conjugation by the orbit-stabilizer theorem.

If $|C| = 1$, then $g \in Z(G)$, so $\langle g \rangle \lhd G$. Then either G has a proper non-trivial subgroup $\langle g \rangle$ or G is cyclic, therefore G is solvable. (Actually for $g \in Q$ we know the order of g is q^s, then $\langle g \rangle \neq G$. If $|C| > 1$, then G is not simple by proposition. □

8.8 Simple Algebra

Fix a field F and let A be an F-algebra. Denote A as a ring and a vector space over F with compatible operations. There is a ring homomorphism $F \to Z(A)$ if $A \neq 0$ by sending $x \mapsto x \cdot 1$. This map is injective since F is a field. Then we can identify F as a subfield of $Z(A)$.

Let A, B be F-algebras, then $A \otimes_F B$ is also an F-algebra by $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1b_1 \otimes a_2b_2$.

Property 8.8.1. *1.* $\dim_F(A \otimes_F B) = \dim_F(A)\dim_F(B)$.

2. *Let $(a_i)_{i\in I}$ be a basis of A. Then every element of $A \otimes_F B$ can be uniquely written as $\sum_i a_i \otimes b_i$ for $b_i \in B$. This also works for B because of symmetry argument.*

3. *$F \otimes_F A \cong A \cong A \otimes_F F$ canonically.*

4. *$A \otimes_F B \cong B \otimes_F A$ canonically.*

5. *$(A \otimes_F B) \otimes_F C \cong A \otimes_F (B \otimes_F C)$ canonically.*

6. *The set of all $n \times n$ matrices $M_n(F)$ is an F-algebra. For all F-algebra A, we have $M_n(F) \otimes_F A \cong M_n(A)$.*

7. *$M_n(F) \otimes_F M_m(F) \cong M_{mn}(F)$. This is true by viewing it as $\mathbf{End}(V) \otimes F\mathbf{End}(W) \xrightarrow{\sim} \mathbf{End}(V \otimes_F W)$ by "$\alpha \otimes \beta \mapsto \alpha \otimes \beta$". Note $\dim(\mathbf{End}(V)) = m$, $\dim(\mathbf{End}(W)) = n$ and $\dim(\mathbf{End}(V \otimes_F W)) = mn$.*

We now focus on simple algebra of finite dimensions. Recall the following proposition:

Proposition 8.8.2. *Let A be an F-algebra and $\dim_F(A) < \infty$. The following are then equivalent:*

1. *A is simple.*

2. *$A \neq 0$, A is semisimple and has only one simple A-algebra.*

3. *$A \cong M_n(D)$ for D a division F-algebra.*

Note that here $D = \mathbf{End}_A(M)$ where M is a (unique) simple left A-module.

Remark 8.8.3. *Note that there is the map*

$$\begin{aligned} F &\subseteq Z(A) \\ x &\mapsto x \cdot 1. \end{aligned}$$

One can prove that $Z(A)$ is a field, and we can view A as a $Z(A)$-algebra.

Definition 8.8.4 (Simple Algebra)**.** *An F-algebra A is simple if $Z(A) = F$.*

Remark 8.8.5. *Every F-algebra is simple over $Z(A)$.*

Definition 8.8.6 (Central Algebra)**.** *An F-algebra A is called a central simple algebra over F if A is simple and $Z(A) = F$.*

Example 8.8.7. *Note $M_n(F) \supseteq F$ is central.*

Definition 8.8.8 (Centralizer)**.** *Suppose $S \subseteq A$ is a subalgebra, the centralizer is $C_A(S) = \{x \in A : xs = sx \forall s \in S\} \subseteq A$.*

Remark 8.8.9. *$C_A(F) = A$ and $C_A(A) = Z(A)$.*

Remark 8.8.10. *We can view A and B as subalgebras of $A \otimes_F B$ by sending $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$.*

Denote $a = \sum_i \alpha_i a_i$ where $(a_i)_i$ forms a basis for a. We can also write $(b_j)_j$ as a basis for B with $b_1 = 1$ without loss of generality. Therefore, there is the mapping $a_i \mapsto a_i \otimes b_1$. Note that $(a_i \otimes b_j)$ forms a basis for $A \otimes_F B$.

Therefore, there is the relation

$$(a \otimes 1)(1 \otimes b) = a \otimes b = (1 \otimes b)(a \otimes 1).$$

Consider $S \subseteq A$ and $T \subseteq B$ as subalgebras. Then we have $C_A(S) \subseteq A$ and $C_B(T) \subseteq B$, which means $C_A(S) \otimes C_B(T) \subseteq A \otimes B$. We also know that $S \otimes T \subseteq A \otimes B$ and therefore $C_{A \otimes B}(S \otimes T) \subseteq A \otimes B$. A obvious question is the relation between these two structures.

Proposition 8.8.11. *$C_{A \otimes B}(S \otimes T) = C_A(S) \otimes C_B(T)$.*

Proof. The $\supseteq$ direction is obvious. We prove the other one.

Let $(a_i)_i$ be a basis of A, then $C_{A \otimes B}(S \otimes T) \supseteq \sum a_i \otimes b_i = u$ for $b_i \in B$. For $t \in T$, because $1 \otimes t \subseteq S \otimes T$, then we have $(\sum_i a_i \otimes b_i)(1 \otimes t) = (1 \otimes t) \sum_i (a_i \otimes b_i)$, and so we get $\sum_i a_i \otimes b_i t = \sum_i a_i t b_i$.

Hence, $\sum a_i \otimes (b_i t - t b_i) = 0$. Therefore $b_i t - t b_i = 0$ for all t and all i, then $b_i \in Z_B(T)$ must be true.

Now, there is a basis (b_i) of $C_B(T)$ such that $u = \sum a_i \otimes b_i$ for some $a_i \in A$. Take $s \in S$, then

$$(\sum_i a_i \otimes b_i)(s \otimes 1) = (s \otimes 1)(\sum a_i \otimes b_i).$$

Therefore, $\sum a_i s \otimes b_i = \sum s a_i \otimes b_i$, and so $\sum (a_i s - s a_i) \otimes b_i = 0$. Therefore, $a_i s - s a_i = 0$ for all $s \in S$ and all i. Hence, $a_i \in C_A(S)$ and so $u \in C_A(S) \otimes C_B(T)$. $\square$

Corollary 8.8.12. *$Z(A \otimes_F B) = Z(A) \otimes_F Z(B)$.*

Corollary 8.8.13. *If A and B are central algebras, so is $A \otimes_F B$.*

Example 8.8.14. *Let L/F be a finite field extension. Then L is a simple F-algebra. There is a map $f : L \otimes L \to L$ that sends $x \otimes y \mapsto xy$ and this is a homomorphism with $xx' \otimes yy' \mapsto xx'yy'$ by taking $x \otimes y \mapsto xy$ and $x' \otimes y' \mapsto x'y'$. Moreover, this is a surjective algebra homomorphism. Let $I = \ker(f)$, then $\dim(I) = n^2 - n > 0$ for $n = [L : F] > 1$. Therefore, I is a proper two-sided ideal in $L \otimes_F L$, and so $L \otimes_F L$ is not simple.*

Proposition 8.8.15. *Let A and B be simple F-algebras and A is central. Then $A \otimes_F B$ is simple.*

Proof. Let $0 \neq I \subseteq A \otimes_F B$ be a two-sided ideal. Take $0 \neq u \in I$. Then $u = \sum_{i=1}^{n} a_i \otimes b_i$ for b_i linearly independent in B and n is the smallest possible.

For $a_1 \neq 0$, $Aa_1A \subseteq A$ is a two-sided ideal. Because A is simple, then $Aa_1A = A$. Therefore, there is $1 = \sum\limits_j x_j a_i y_j$ for $x_j, y_j \in A$. Then $I \ni \sum\limits_j (x_j \otimes 1)u(y_j \otimes 1) = \sum\limits_{i,j} x_j a_i y_j \otimes b_i = \sum\limits_i (\sum\limits_j x_j a_i y_j) \otimes b_i = 1 \otimes b_1 + a'_2 \otimes b_2 + \cdots + a'_n \otimes b_n$ because $\sum\limits_j x_j a_i y_j = 1$ if $i = 1$. We now set $v = 1 \otimes b_1 + a'_2 \otimes b_2 + \cdots + a'_n \otimes b_n$.

For $a \in A$, $I \ni (a \otimes 1)v - v(a \otimes 1) = \sum\limits_{i=2}^{n} (aa'_i - a'_i a) \otimes b_i = 0$, and so $aa'_i = a'_i a$ for all $a \in A$ and all $i > 1$. Then $a'_i \in Z(A) = F$.

We now have $0 \neq v = 1 \otimes b_1 + a'_2 \otimes b_2 + \cdots + a'_n \otimes b_n = 1 \otimes b$ by linear independence. Then $b \neq 0$, and so $BbB = B$ since B is simple. Hence, $1 = \sum\limits_j s_j t b_j$ for $s_j, t_j \in B$. Therefore, $I \ni \sum\limits_j (1 \otimes s_j) v (1 \otimes t_j) = 1 \otimes \sum\limits_j s_j b t_j = 1 \otimes 1 = 1_{A \otimes B}$. □

Corollary 8.8.16. *If A and B are central simple algebras, then so is $A \otimes B$.*

Note $F \otimes_F A = A$. This gives a monoidal structure of algebra. If we factor out central simple algebra by some equivalence relation, we get a group, namely the Brauer group.

8.9 Brauer Group

Consider a central simple (finite-dimensional) F-algebra for a fixed field F. We define the equivalence relation $A \sim B$ to be that $M_n(A) \cong M_m(B)$ for some m, n. This relation is indeed an equivalence relation, with reflexivity and symmetry clear. The transitivity follows from that if $M_n(A) \cong M_m(B)$ and $M_k(B) \cong M_s(C)$, then by tensoring the equations on the right with $M_k(F)$ and $M_m(F)$ respectively, we have $M_{nk}(A) \cong M_{mk}(B) \cong M_{ms}(C)$. Therefore, this is an equivalence relation indeed.

Proposition 8.9.1. *Let $A_1 = M_{n_1}(D_1)$ and $A_2 = M_{n_2}(D_2)$ be two central simple F-algebras with D_1, D_2 division F-algebras. Then $A_1 \sim A_2$ if and only if $D_1 \cong D_2$.*

Proof. If $A_1 \sim A_2$, then $M_{s_1}(A_1) \cong M_{s_2}(A_2)$, so $M_{s_1 n_1}(D_1) \cong M_{s_2 n_2}(D_2)$, hence $D_1 \cong D_2$. Conversely, $M_{n_2}(A_1) \cong M_{n_1 n_2}(D_1) \cong M_{n_1 n_2}(D_2) \cong M_{n_1}(A_2)$, hence $A_1 \sim A_2$. □

Therefore, the class $[A]$ of $A = M_n(D)$ is $\{M_i(D)\}$ for $i \geq 1$. In particular, $D \in [A]$, so we have a correspondence between equivalence classes and central division F-algebras.

Write $\mathrm{Br}(F)$ for the set of equivalence classes with operation $[A][B] = [A \otimes_F B]$. The operation is well-defined: if $A_1 \sim A_2$, i.e. $M_{s_1}(A_1) \cong M_{s_2}(A_2)$ and $B_1 \sim B_2$, i.e. $M_{t_1}(B_1) \cong M_{t_2}(B_2)$, then

$$M_{s_1 t_1}(A_1 \otimes_F B_1) \cong M_{s_1}(A_1) \otimes_F M_{t_1}(B_1) \cong M_{s_2}(A_2) \otimes_F M_{t_2}(B_2) \cong M_{s_2 t_2}(A_2 \otimes_F B_2),$$

i.e. $A_1 \otimes_F B_1 \sim A_2 \otimes_F B_2$.

Theorem 8.9.2. *The set $Br(F)$ is an Abelian group.*

Proof. The operation is obviously commutative and associative. The class $[F]$ is the identity. Let A be a central simple algebra of finite dimension over F. We show that $[A]^{-1} = [A^{\mathrm{op}}]$. Consider a map

$$\begin{aligned} f : A \otimes_F A^{\mathrm{op}} &\to \mathrm{End}_F(A) \\ f(x \otimes y^{\mathrm{op}})(a) &= xay. \end{aligned}$$

This is a homomorphism of simple F-algebras of the same dimension, hence f is an isomorphism. It follows that $[A][A^{\mathrm{op}}] = [\mathrm{End}_F(A)] = [F] = 1$. □

Definition 8.9.3 (Brauer Group)**.** *The Abelian group $Br(F)$ is the Brauer group of F.*

Remark 8.9.4. *Every class $[A]$ in $Br(F)$ contains a central division algebra that is unique up to isomorphism. Thus, we have a bijection between the set $Br(F)$ and the set of isomorphism classes of central division F-algebras of finite dimension.*

Note that $Br(F) = 1$ if and only if every central division F-algebra of finite dimension is F.

Example 8.9.5. *If F is algebraically closed, then $Br(F) = 1$.*

Theorem 8.9.6. *If F is a finite field, then $Br(F) = 1$.*

Proof. Let $F = \mathbb{F}_q$ and let A be a central division F-algebra of finite dimension. We show that $A = F$.

Suppose $\dim_F(A) = n$, so $|A| = q^n$. Hence $|A^\times| = q^n - 1$. For any $a \in A$ non-zero, the centralizer $C_A(a) \subseteq A$ is a subspace, so $|C_A(a)| = q^k$ for some k, hence $|C_{A^\times}(a)| = q^k - 1$. Note that k divide n as $\frac{n}{k}$ is the rank of A as a module over the division algebra $C_A(a)$. Therefore, the conjugacy class of a in $A^\times$ has $\frac{q^n-1}{q^k-1}$ elements. The elements of $Z(A)^\times = F^\times$ have conjugacy classes of size 1, so there are exactly $q - 1$ of them. As $A^\times$ is the disjoint union of conjugacy classes, we have

$$q^n - 1 = \sum_{k<n} \frac{q^n - 1}{q^k - 1} + (q - 1).$$

If k divides n and $k < n$, the polynomial $\frac{x^n-1}{x^k-1}$ is divisible by the cyclotomic polynomial $\Phi_N(x)$, hence $\Phi_n(q)$ divides $\frac{q^n-1}{q^k-1}$. It follows that $\Phi_n(q)$ divides $q - 1$, hence $|\Phi_n(q)| \leq q - 1$. On the other hand, $\Phi_n(x) = \prod(x - \xi)$, where the product is taken over all primitive n-th roots of unity ξ, hence $\Phi_n(q) = \prod(q - \xi)$. As $|q - \xi| \geq q - 1 \geq 1$, we must have $n = 1$. □

Example 8.9.7. *The quaternion algebra $\mathbb{H}$ is a central $\mathbb{R}$-algebra of dimension 4, so $Br(\mathbb{R}) \neq 1$. If F is a field of characteristic not 2 and $a, b \in F^\times$. The F-algebra $(a, b)_F$ with basis $\{1, i, j, k\}$ and multiplication table $i^2 = a$, $j^2 = b$ and $j = ij = -ji$ is called the (generalized) quaternion algebra. We will see that $(a, b)_F$ is a central simple algebra over F.*

Example 8.9.8. *An anti-automorphism of an F-algebra A is a linear automorphism $\sigma : A \to A$ such that $\sigma(x + y) = \sigma(x) + \sigma(y)$ and $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in A$. An anti-automorphism σ can be viewed as an isomorphism betweedn A and A^{op}. If an anti-automorphism $\sigma \circ \sigma = \mathbf{id}_A$, we say that σ is an involution.*

If A is a central simple F-algebra that admits an anti-automorphism, then $A \cong A^{op}$ and hence $[A]^{-1} = [A]$ in $Br(F)$.

Theorem 8.9.9 (Noether-Skolem)**.** *Let A be a finite-dimensional central simple algebra over F, and let $S, T \subseteq A$ be simple subalgebras. Let $f : S \to T$ be an F-algebra isomorphism. Then there exists $a \in A^\times$ such that $f(s) = asa^{-1}$ for all $s \in S$.*

Proof. Regard A as a right $(A^{\mathrm{op}} \otimes_F S)$-module in two ways. First, we define

$$a \cdot (b^{\mathrm{op}} \otimes s) = bas.$$

Second, we define

$$a * (b^{\mathrm{op}} \otimes s) = baf(s).$$

Since S is simple and A^{op} is central simple, $A^{\mathrm{op}} \otimes_F S$ is simple. Over a simple algebra every two right modules of the same dimension are isomorphic. Therefore, the two module structures are isomorphic. Let $g : A \to A$ be an isomophism, so that

$$g(bas) = bg(a)f(s)$$

for all $a, b \in A$ and $s \in S$. For $a = s = 1$, we get $g(b) = bg(1)$. As g is an isomorphism, this implies $g(1)$ left invertible, hence right invertible since A has finite dimension over F. For $a = b = 1$, we get $sg(1) = g(s) = g(1)f(s)$, so $f(s) = g(1)^{-1}sg(1)$ as desired. ☐

Remark 8.9.10. *The condition that A is central cannot be dropped. Otherwise, take $S = T = A$ to be a (non-trivial) Galois field extension of F. For $S = T = A$, we get $Aut_{\mathbf{F}-alg}(A) \cong A^\times/F^\times$ for the F-algebra automorphism group, with the action by conjugation. If $A = M_n(F)$, then $A^\times = GL_n(F)$ and $Aut_{\mathbf{F}-alg}(M_n(F)) = GL_n(F)/F^\times = PGL_n(F)$.*

Example 8.9.11. *Let S be an F-algebra and $B = End_F(S)$. Then $S \subseteq B$ by left multiplication and $S^{op} \subseteq B$ by right multiplication. In fact, $S^{op} = C_B(S)$ and $S = C_B(S^{op})$. Indeed, $f \in C_B(S)$ if and only if $f(ax) = af(x)$ for all $a, x \in A$. Plugging $x = 1$, we get $f(a) = af(1)$, i.e. f is right multiplication by $f(1)$. Conversely, if $f(a) = ab$ for some $b \in A$, then $f(ax) = (ax)b = a(xb) = af(x)$, i.e. $f \in C_B(S)$.*

Theorem 8.9.12 (Double Centralizer Theorem)**.** *Let A be a central simple algebra over F and let $S \subseteq A$ be simple subalgebra. Then*

1. *$C_A(S)$ is simple with $Z(C_A(S)) = S \cap C_A(S) = Z(S)$.*

2. $(\dim S) = (\dim C_A(S)) = \dim(A)$.

3. $C_A(C_A(S)) = S$.

Proof. 1. Let $S \subseteq B = \mathrm{End}_F(S)$. Then $C_B(S) = S^{\mathrm{op}}$. We have

$$S = S \otimes F \subseteq A \otimes_F B$$

and

$$S = F \otimes_F S \subseteq A \otimes_F B.$$

The first inclusion has

$$C_{A \otimes B}(S \otimes F) = C_A(S) \otimes C_B(F) = C_A(S) \otimes B,$$

while the second inclusion has

$$C_{A \otimes B}(F \otimes S) = C_A(F) \otimes C_B(S) = A \otimes S^{\mathrm{op}},$$

which is simple. By Noether-Skolem, $S \otimes F$ and $F \otimes S$ are conjugate. Hence their centralizers $C_A(S) \otimes B$ and $A \otimes S^{\mathrm{op}}$ are conjugate, hence isomorphic. As $A \otimes S^{\mathrm{op}}$ is simple, so is $C_A(S) \otimes B$ and hence $C_A(S)$ is simple.

For the equalities, that $Z(S) = S \cap C_A(S)$ is clear. By the third result, $Z(C_A(S)) = C_A(S) \cap C_A(C_A(S)) = C_A(S) \cap S$.

2. We have $(\dim C_A(S))(\dim B) = (\dim A)(\dim S^{\mathrm{op}})$, and the result follows from $\dim B = (\dim S)^2$.

3. By the second result, $\dim C_A(C_A(S)) = \dim S$ and $S \subseteq C_A(C_A(S))$, so $C_A(C_A(S)) = S$.

$\square$

Corollary 8.9.13. *Let S be a central simple subalgebra of a central simple algebra A. Then $A = S \otimes_F C_A(S)$.*

Proof. Consider the F-algebra homomorphism $f : S \otimes_F C_A(S) \to A$ given by $f(x \otimes y) = xy$. By the theorem, $S \otimes_F C_A(S)$ is a simple F-algebra of the same dimension as A. Hence, f is an isomorphism. $\square$

Remark 8.9.14. *Let A be a central simple algebra over F and let L/F be a field extension. Then $A_L = A \otimes_F L$ is a central simple L-algebra, as it is simple and $Z(A \otimes_F L) = Z(A) \otimes_F Z(L) = F \otimes_F L = L$. Moreover, $\dim_L A_L = \dim_F A$.*

Suppose $A \sim B$ over F. Then $M_n(A) \cong M_n(B)$ for some n and m, so $M_n(A_L) \cong M_m(B_L)$. Therefore, $M_n(A_L) \cong M_m(B_L)$, so $A_L \cong B_L$ over L. Thus, we have a group homomorphism $Br(F) \to Br(F)$ given by extension of scalars $[A] \mapsto [A_L]$.

Proposition 8.9.15. *If A is a central simple algebra over F, then* $\dim_F(A) = n^2$ *for some n.*

Proof. Let L be the algebraic closure of F. Then A_L is a central simple algebra over L, so $A_L \cong M_n(L)$ for some n. Then $\dim_F(A) = \dim_L(A_L) = n^2$. □

The value n is called the degree of A. Then $\deg(M_k(A)) = k\deg(A)$. Let A be a central simple algebra over F with $A \cong M_k(D)$ for some central division F-algebra D. If $m = \deg(D)$ and $n = \deg(A)$, then $n = km$. The value m is the index of A, denoted $\mathrm{ind}(A)$. From the definition, $\mathrm{ind}(A) \mid \deg(A)$, with equality if and only if A is a division algebra.

8.10 Maximal Subfield

If A is a central simple algebra over F, then $(\deg A)^2 = \dim_F(A)$. Writing $A = M_s(D)$ for a central division F-algebra, the index of A is $\mathrm{ind}(A) = \deg(D)$, so $\deg(A) = s\mathrm{ind}(A)$ and $\deg(D) = \mathrm{ind}(D)$.

Let D be a central division algebra over F and let $L \subseteq D$ be a subalgebra. Then L is a division subalgebra and L is a field extension of F if L is commutative. In the latter case, we will simply say that L is a subfield, with the containment of F understood.

Proposition 8.10.1. *If $L \subseteq D$ is a subfield, then L is maximal if and only if $C_D(L) = L$.*

Proof. ($\Rightarrow$): Suppose $\alpha \in C_D(L)$, then $L \subseteq L[\alpha] \subseteq D$ and $L[\alpha]$ is a subfield of D, so $L[\alpha] = L$.

($\Leftarrow$): Let $L' \subseteq D$ be a subfield containing L. Then $L' \subseteq C_D(L) = L$, so $L' = L$. □

Corollary 8.10.2. *Let L be a maximal subfield of a central division F-algebra D. Then $[L : F] = \deg(D)$.*

Proof. The double centralizer theorem gives $(\dim L)^2 = (\dim L)(\dim C_D(L)) = \dim D = (\deg D)^2$. □

Corollary 8.10.3. *Let L be a subfield of D. Then $[L : F]$ divides* $\deg D$.

Proof. There is a maximal subfield L' of D containing L. Hence $[L : F]$ divides $[L' : F] = \deg D$. □

Example 8.10.4. *Let D be a finite division ring. Then $F = Z(D)$ is a finite field and D is central as an F-algebra. Let L be a maximal subfield of D. Let $\alpha \in D^\times$ and L' a maximal subfield of D containing α. Then $[L : F] = \deg(D) = [L : F]$. As F is a finite field, the fields L and L' are isomorphic over F, hence conjugate by Noether-Skkolem theorem. It follows that $\alpha \in \beta L^\times \beta^{-1}$ for some $\beta \in D^\times$.*

We have proved that $D^\times = \bigcup_{\beta \in D^\times} \beta L^\times \beta^{-1}$, so since the groups are finite, $L\times = D^\times$. Hence $L = D$. Computing dimensions, it follows that $\deg D = 1$.

Let A be a central simple algebra over F and let K/F be a field extension. Then $A_K = A \otimes_F K$ is a central simple algebra over K and $\deg_F A = \deg_K A_K$.

Definition 8.10.5 (Splitting Field)**.** *A central simple F-algebra A is split over F if $A \cong M_n(F)$ for $n = \deg A$. Let A be a central simple F-algebra and K/F a field extension. We say that K is a splitting field of A (or A is split over K) if A_K is split over K.*

Equivalently, A is split over K if $[A] \in \ker(\mathrm{Br}(F) \to \mathrm{Br}(K))$. If K is an algebraic closure of F, then $\mathrm{Br}(K)$ is trivial, so every central simple algebra is split over the algebraic closure.

Remark 8.10.6. *If A is an F-algebra such that $A_K = A \otimes_F K \cong M_n(K)$ for some n, then A is a central simple algebra over F of degree n. In fact, the central simple algebras over F are of this form for some K. These are referred to as twisted forms of $M_n(F)$, since $A \otimes_F K \cong M_n(K) = M_n(F) \otimes_F K$.*

Proof. Computing dimensions, $\dim_F A = \dim_K A_K$. We have

$$Z(A) \otimes_F K = Z(A \otimes_F K) = K = F \otimes_F K$$

and $F \subseteq Z(A)$, so computing dimensions, $Z(A) = F$. Hence A is central. To see that A is simple, if $I \subseteq A$ is a two-sided ideal, then $I \otimes_F K \subseteq A \otimes_F K = M_n(K)$ is a two-sided ideal, so $I \otimes_F K$ is 0 or $A|otimes_F K$. Hence I is either 0 or A. ☐

Theorem 8.10.7. *Let A be a central simple algebra over F with $\deg(A) = n$. Let $L \subseteq A$ be a subfield with $[L : F] = n$. Then L is a splitting field of A.*

Proof. Since $A \otimes_F L$ and $M_n(L)$ are central simple algebras of the same dimension, it suffices to find any homomorphism. Define $f : A \otimes_F L \to \mathrm{End}_L(A) \cong M_n(L)$ with A viewed as a right L-module by $f(a \otimes l)(m) = aml$. ☐

Corollary 8.10.8. *Every maximal subfield of a central division algebra D is a splitting field of D.*

Corollary 8.10.9. *Every central simple algebra A over F has a splitting field L such that $[L : F] =$ ind(A).*

Proof. Write $A = M_s(D)$ for a central division algebra D of degree $n = \mathrm{ind}(A)$. Then a maximal subfield L of D is a splitting field for D, hence for A. ☐

Let D be a central division F-algebra and $\alpha \in D$. Then $F[\alpha] \subseteq D$ is a subfield and $[F[\alpha] : F] < \infty$, so α is algebraic over F.

Lemma 8.10.10. *Let D be a central division F-algebra with $D \neq F$. Then there exists $\alpha \in D \backslash F$ which is separable over F.*

Proof. If $\mathrm{char}(F) = 0$, then we are done. Otherwise, let $p = \mathrm{char}(F) > 0$. Suppose all $\alpha \in D \backslash F$ are not separable. Pick $\alpha \in D \backslash F$. Then the maximal separable extension of F contained in $F(\alpha)$ is F, so $F(\alpha)/F$ is purely inseparable. Therefore, $\alpha^{p^n} \in F$ for some n. Choose n as small as possible and let $\beta = \alpha^{p^{n-1}}$, so $\beta^p \in F$.

Define $f : D \to D$ by $f(a) = \beta a - a\beta$. Then $f \neq 0$, since D is central and $D \neq F$, while $f^p(a) = \beta^p a - a\beta^p = 0$. Thus f is nilpotent, so we can choose the smallest $k > 1$ with $f^k = 0$.

Let $\gamma = f^{k-1}(\delta) \neq 0$ for some $\delta \in D$, so then $f(\gamma) = 0$. If $\varepsilon = f^{k-2}(\delta)$, then $\gamma = f(\varepsilon) = \beta\varepsilon - \varepsilon\beta$ and $\beta\gamma - \gamma\beta = 0$, i.e. β and γ commute. Since D is a division algebra, we can write $\gamma = \beta\zeta$ for some $\zeta \in D$. Note that β, γ and ζ commute. Then $\beta\zeta = \zeta\beta$, so

$$\beta = \gamma\zeta^{-1} = (\beta\varepsilon - \varepsilon\beta)\zeta^{-1} = \beta\varepsilon\zeta^{-1} - \varepsilon\beta\zeta^{-1} = \beta\varepsilon\zeta^{-1} - \varepsilon\zeta^{-1}\beta = \beta\theta - \theta\beta$$

for $\theta = \varepsilon\zeta^{-1}$. Thus $1 = \theta - \beta^{-1}\theta\beta$, hence $\theta = 1 + \beta^{-1}\theta\beta$, so

$$\theta^{p^m} = (1 + \beta^{-1}\theta\beta)^{p^m} = 1 + \beta^{-1}\theta^{p^m}\beta = 1 + \theta^{p^m},$$

for large m since $\theta^{p^m} \in F$, a contradiction. □

Corollary 8.10.11. *Every central division F-algebra admits a maximal subfield which is separable over F.*

Proof. Let $L \subseteq D$ be the maximal separable subfield extending F. Then $L \subseteq C_D(L)$, with equality if and only if L is a maximal subfield of D. If $L \neq C_D(L)$, since $C_D(L)$ is central division L-algebra, by the lemma, there exists $\alpha \in C_D(L) \backslash L$ such that $L(\alpha)/L$ is non-trivial and separable, but then $L(\alpha)/F$ is separable, contradicting maximality of L as a separable extension. □

Corollary 8.10.12. *Every central simple F-algebra is split by a (finite) separable extension of F.*

Proof. Let A be a central simple F-algebra and write $A = M_s(D)$ for D a central division F-algebra. Let $L \subseteq D$ be a maximal subfield which is separable over F. Then L is a splitting field for D, so also for A. □

Example 8.10.13. *If F is separably closed, i.e. it has no non-trivial separable extensions, then $Br(F) = 1$. One can construct the separable closure of a field by taking all separable elements in an algebraic closure.*

Theorem 8.10.14. *Let A be a central simple F-algebra and K/F be a field extension.*

1. *$ind(A_K) \mid ind(A)$;*

2. *If K/F is a finite field extension, then $ind(A) \mid [K : F] \cdot ind(A_K)$. Moreover, if $A_K = M_s(D)$ for a central division K-algebra D, then $D \hookrightarrow M_p(A)$ for $p = [K : F] ind(A_K)/ind(A)$.*

Proof. 1. Let $A = M_n(E)$ for a division algebra E, then $\text{ind}(A) = \deg(E)$. We have $A_K = M_n(E_K)$, so $\text{ind}(A_K) = \text{ind}(E_K) \mid \deg(E_K) = \deg(E) = \text{ind}(A)$.

2. First suppose A is a division algebra. Let $r = [K : F]$ and consider the embedding $K \hookrightarrow \mathrm{End}_F(K) = M_r(F)$ via left multiplications. Therefore,

$$M_s(F) \subseteq M_s(D) \cong A_K = A \otimes K \hookrightarrow A \otimes M_r(F) = M_r(A).$$

Let $C = C_{M_r(A)}(M_s(F))$. Since $M_s(F)$ and $M_r(A)$ are central simple algebras, C is also central simple and we have $M_s(C) \cong M_s(F) \otimes C \cong M_r(A)$. As A is division algebra, we have $C \cong M_p(A)$, where $p = \frac{r}{s}$. We have $s = \frac{\deg(A_K)}{\deg(D)} = \frac{\mathrm{ind}(A)}{\mathrm{ind}(A_K)}$, hence $p = [K : F] \cdot \frac{\mathrm{ind}(A_K)}{\mathrm{ind}(A)}$, i.e. $\mathrm{ind}(A)$ divides $[K : F]\mathrm{ind}(A_K)$. Note that $D \subseteq C \cong M_p(A)$.

In the general case, we write $A = M_n(E)$ for a division algebra E. We have $\mathrm{ind}(E) = \mathrm{ind}(A)$ and $\mathrm{ind}(E_K) = \mathrm{ind}(A_K)$. Also, by the above, $D \hookrightarrow M_p(E) \subseteq M_p(A)$.

□

Corollary 8.10.15. *If a finite extension K/F splits a central simple F-algebra A, then ind$(A) \mid [K : F]$.*

Corollary 8.10.16. *If A is a central simple F-algebra and K/F is a splitting field for A of degree $r$ind(A), then $K \hookrightarrow M_r(A)$. If A is a division algebra and $[K : F] =$ ind(D), then K is isomorphic to a maximal subfield of A.*

Proposition 8.10.17. *Let D be a division algebra, then the intersection of the subfields of D and the splitting fields of D is exactly the maximal subfields of D.*

8.11 Cyclic Algebra

Definition 8.11.1 (Cyclic Algebra)**.** *Let L/F be a cyclic field extension with Galois group $G =$ Gal(L/F) generated by σ. Let $n = [L : F]$ and $a \in F^\times$. The cyclic algebra $(L/F, \sigma, a)$ is the F-algebra given by*

$$A = (L/F, \sigma, a) = \bigoplus_{i=0}^{n-1} L \cdot u = (L \cdot 1) \oplus (L \cdot u) \oplus \cdots \oplus (L \cdot u^{n-1}),$$

where $1, u, \cdots, u^{n-1}$ is a basis for L/F. The multiplication is defined by $u^n = a \cdot 1$ and extending the relations $(xu^i)(yu^j) = x\sigma^i(y)u^{i+j}$ for $x, y \in L$. In particular, $uyu^{-1} = \sigma(y)$.

Example 8.11.2. *1. Suppose char$(F) \neq 2$. Let $L = F(\sqrt{b}) = F[j]/(j^2 - b)$ for $b \in F$ not a square. Then for $a \in F^\times$, we have*

$$(L/F, \sigma, a) = (L \cdot 1) \oplus (L \cdot i) = (F \cdot 1) \oplus (F \cdot i) \oplus (F \cdot j) \oplus (F \cdot ji)$$

with $i^2 = a$, $j^2 = b$, $ij = -ji$. Hence $(L/F, \sigma, a) = (a, b)_F$ is the generalied quaternion algebra. The usual quaternions are $\mathbb{H} = (\mathbb{C}/\mathbb{R},$ conjugation $, -1)$.

2. *If char$(F) = 2$, then polynomials $x^2 + x + a$ for $a \in F$ are separable. Let $L = F(\theta)$ for θ a root of $x^2 + x + a$ (assumed irreducible). Then $\sigma(\theta) = \theta + 1$, so $(L/F, \sigma, a)$ has basis $\{1, \theta, u, \theta u\}$ with relations $\theta^2 + \theta + a = 0$, $u^2 = a$, $u\theta = (\theta + 1)u$.*

Proposition 8.11.3. *$A = (L/F, \sigma, a)$ is a central simple algebra.*

Proof. Suppose $s = \sum_i \alpha_i u^i \in Z(A)$ where $\alpha \in L$ and let $\beta \in L$. Then

$$0 = \beta s - s\beta = \sum_i (\alpha_i \beta - \alpha_i \sigma^i(\beta))u^i,$$

hence $\alpha_i(\beta - \sigma^i(\beta)) = 0$ for all i. If $i \neq 0$, then we can choose β so that $\sigma^i(\beta) \neq \beta$, so then $\alpha_i = 0$. Hence $s = \alpha_0 \cdot 1$, so $C_A(L) = L$. From $us = su$, we get $\sigma(\alpha_0) = \alpha_0$. This shows that $\alpha_0 \in F$, so $Z(A) = F$.

Let $0 \neq I \subseteq A$ be an ideal. We must show that $1 \in I$. Let $s = \sum_i \alpha_i u^i \in I \neq 0$ have the smallest number of non-zero terms. By replacing s with su^k for some k, we can suppose $\alpha_0 \neq 0$. For $\beta \in L$, we have $\beta s - s\beta = \sum_i \alpha_i(\beta - \sigma^i(\beta))u^i \in I$. For $i = 0$, we get 0, so $\beta s - s\beta = 0$. Therefore, $\alpha_i = 0$ for $i \neq 0$, so $s = \alpha_0 \cdot 1$ for $\alpha_0 \in L$ non-zero. Hence, $\alpha_0^{-1} s = 1 \in I$. □

Therefore, A is a central simple algebra of dimension n^2 containing L as a subfield of dimension n over F. In particular, L/F is a splitting field for A, so

$$[A] = \ker(\mathrm{Br}(F) \to \mathrm{Br}(L)) =: \mathrm{Br}(L/F)$$

(the relative Brauer group). If A is a division algebra, then L is also a maximal subfield of A.

It can also be shown that $C(L/F, \sigma, a)$ and $C(L/F, \sigma^i, a^i)$ are isomorphic for i coprime to n.

Lemma 8.11.4. *Let L/F be a cyclic field extension of degree n and let A be a central simple algebra of degree n over F. If $L \hookrightarrow A$, then $A \cong C(L/F, \sigma, a)$ for some σ generating $G = Gal(L/F)$, and $a \in F^\times$.*

Proof. By Noether-Skolem theorem, $\sigma : L \to L$ extends to an inner automorphism $\sigma(\alpha) = \beta\alpha\beta^{-1}$ for some $\beta \in A^\times$ and all $\alpha \in L$. Then $\alpha = \sigma^n(\alpha)$ shows that $\beta^n \in C_A(L) = L$. Since $\beta^n = \sigma(\beta^n)$, in fact $\beta^n \in F$. Take $a = \beta^n$, then define a map

$$\begin{aligned} C(L/F, \sigma, a) &\to A \\ \alpha \in L &\mapsto \alpha \in L \subseteq A \end{aligned}$$

and $u \mapsto \beta$. It is easily checked that this is well-defined and a map of central simple algebras of the same dimension, hence an isomorphism. □

Proposition 8.11.5. *Let L/F be a cyclic extension. Then*

$$Br(L/F) = \{[C(L/F, \sigma, a)] \mid a \in F^\times\}.$$

Proof. Let $[A] \in \mathrm{Br}(L/F)$ for A a division algebra. Then $\deg(A) = \mathrm{ind}(A) = m$. We know that $n = [L : F]$ is divisible by m, so $n = mk$ for some k and $L \hookrightarrow M_k(A)$. The degree of $M_k(A)$ is $km = n$, so there is a cyclic algebra $C(L/F, \sigma, a)$ isomorphic to $M_k(A)$, hence $[A] = [C(L/F, \sigma, a)]$. ☐

Lemma 8.11.6. $C(L/F, \sigma, 1) \cong M_n(F)$ *for* $n = [L : F]$.

Proof. Define an F-algebra isomorphism $C(L/F, \sigma, 1) \to \mathrm{End}_F(L) = M_n(F)$ by $\alpha \in L \mapsto l_\alpha \in \mathrm{End}_F(L)$ and $1 \mapsto \sigma$. ☐

Lemma 8.11.7. *Let L/F be cyclic extension of degree n, $\sigma \in Gal(L/F)$ be a generator, and $a, b \in F^\times$. Then $C(L/F, \sigma, a) \cong C(L/F, \sigma, b)$ if and only if $b/a \in N_{L/F}(L^\times)$.*

Proof. ($\Rightarrow$): Let $f : C(L/F, \sigma, a) \to C(L/F, \sigma, b)$ be an isomorphism. Then $f(L)$ and L are isomorphic subfields of $C(L/F, \sigma, b)$, so by Noether-Skolem theorem, we can modify f by conjugation to suppose f fixes L. If u gneerates $C(L/F, \sigma, a)$ and v generates $C(L/F, \sigma, b)$, then $f(u)$ and v act by conjugation in the same way on $L \subseteq C(L/F, \sigma, b)$. Hence, $f(u)v^{-1}$ is in the centralizer of L, which is L itself, so $f(u) = \alpha^{-1}v$ for some $\alpha \in L^\times$. It follows by computation that $b = aN_{L/F}(\alpha)$.

($\Leftarrow$): Suppose $b = aN_{L/F}(\alpha)$ for some $\alpha \in L^\times$. Let u be a generator of $C(L/F, \sigma, a)$ and v be a generator of $C(L/F, \sigma, b)$. We can then define a homomorphism $C(L/F, \sigma, a) \to C(L/F, \sigma, b)$ by fixing $L^\times$ and mapping $u \mapsto \alpha^{-1}v$. Since the two algebras are central simple algebras, the homomorphism is automatically an isomorphism. ☐

Corollary 8.11.8. $[C(L/F, \sigma, a)] = 1$ *if and only if* $a \in N_{L/F}(L^\times)$.

Example 8.11.9. *Let $F = \mathbb{F}_q$ be a finite field. We have $Br(F) = \bigcup_{L/F} Br(L/F)$ with L/F ranging over all finite extensions. Since F is finite, L/F is cyclic and $N_{L/F} : L^\times \to F^\times$ is surjective, so $Br(L/F) = 1$.*

Let L/F be cyclic and $\sigma \in Gal(L/F)$ be a generator. Define $f : F^\times \to Br(L/F)$ given by $a \mapsto [C(L/F, \sigma, a)]$.

Theorem 8.11.10. *If L/F is a cyclic field extension, f is a surjective homomorphism and* $\ker(f) = N_{L/F}(L^\times)$. *In particular,*

$$Br(L/F) \cong F^\times / N_{L/F}(L^\times).$$

Consider $p : L \otimes_F L \to L^n$ by $p(x \otimes y) = (xy, x\sigma(y), \cdots, x\sigma^{n-1}(y))$.

Proposition 8.11.11. *p is an F-algebra isomorphism.*

Proof. Write $L = F(\alpha) = F[t]/(f)$ with $f(t) = (t-\alpha)\cdots(t-\sigma^{n-1}(\alpha)) \in L[t]$. Then $L \otimes_F L = L[t]/(f)$ and the map p takes $g \in L[t]/(f)$ to $(g(\alpha), \cdots, g(\sigma^{n-1}(\alpha)))$. This is an isomorphism by the Chinese Remainder Theorem. $\square$

If $G = \mathrm{Gal}(L/F)$, then G acts on $L \otimes_F L$ by $\sigma(x \otimes y) = \sigma(x) \otimes \sigma(y)$. If G acts on L^n component-wise, then p respects the action of G, so $(L \otimes_F L)^G \cong F^n$.

Lemma 8.11.12. *Let A be a central simple algebra of degree n over F. If $F^n \hookrightarrow A$ as a subalgebra, then $A \cong M_n(F)$.*

Proof. We have $A \cong \mathrm{End}_D(V) \cong M_k(D)$ for some central division F-algebra D and V a D-module of rank k. Let $e_1, \cdots, e_n \in F^n$ be orthogonal idempotents. Then $V = e_1(V) \oplus \cdots \oplus e_n(V)$ gives $\mathrm{rank}_D(V) \geq n$. On the other hand, if $\deg(D) = m$, then $n = km$, so $\mathrm{rank}_D(V) = k = \frac{n}{m} \geq n$, so $m = 1$ and $k = n$, so $D = F$. $\square$

Proposition 8.11.13. $[C(L/F, \sigma, a)] \cdot [C(L/F, \sigma, b)] = [C(L/F, \sigma, ab)] \in Br(L/F)$.

Proof. It suffices to show that

$$C(L/F, \sigma, a) \otimes_F C(L/F, \sigma, b) \cong M_n(C(L/F, \sigma, ab)).$$

To do this, we find an embedding of $C(L/F, \sigma, ab)$ into the tensor product with centralizer $M_b(F)$. Let

$$A = C(L/F, \sigma, a) = \bigoplus_i Lu^i$$

and

$$B = C(L/F, \sigma, b) = \bigoplus_i Lv^i.$$

Then $A \otimes_F B = \bigoplus (L \otimes_F L)(u^i \otimes v^j)$. If $D = C(L/F, \sigma, ab) = \bigoplus Lw^i$, then

$$\bigoplus (L \otimes_F F)(u^i \otimes v^i) \cong D$$

by $u \otimes v \mapsto w$, which embeds in $A \otimes_F B$. Note that the diagonal G-action on $L \otimes_F L = L^n$ coincides with the component-wise G-action. Hence, the centralizer of D contains $(L \otimes_F L)^G = F^n$, so the centralizer of D is $M_n(F)$ by the lemma. $\square$

Algebra Qualification Exam Problems

This section consists of knowledge and algebra qualification exam problems that were discussed during the discussion. Solutions were originally provided by Matthew Gherman, a PhD student at UCLA.

1 MATH 210A Discussion 1, October 5, 2021

A group G acts on a set X if it satisfies:

(a) $e \cdot x = x$.

(b) $g \cdot (h \cdot x) = (gh) \cdot x$.

For example, S_n acts on $X = \{1, 2, \cdots, n\}$ as permutation.

Cayley's Theorem: there is an injection $G \xhookrightarrow{\varphi} S_n$ where $|G| = n$.

Suppose G acts on X, then define $\text{orb}(X) = \{y \in X : g \cdot x = y \text{ for some } g \in G\}$, define $\text{Stab}(x) = \{g \in G : g \cdot x = x\}$. Note that the stabilizer is a subgroup of G, but not necessarily normal.

Orbit-Stabilizer Theorem: take $x \in X$, then $|\text{Orb}(x)| = [G : \text{Stab}(x)]$.

Problem 1.1 (Fall 2014 - Problem 6)**.** *Let G be a finite group and let p be the smallest prime number dividing the order of G. Assume G has a normal subgroup H of order p. Show that H is contained in the center of G.*

Proof. Recall that the center of G is defined by $\{g \in G : gh = hg \ \forall h \in G\}$.

Consider an group action from G to the normal subgroup H defined by conjugation: $g \cdot h = ghg^{-1}$. Note that the fixed points in the action are exactly the elements of H that are in $Z(G)$. Since the size of H is the sum of sizes of different orbits, we have $|H| = |Z(G) \cap H| + \sum\limits_{h_i \notin Z(G)} |\text{Orb}(h_i)|$. Observe that $e \in Z(G) \cap H$, then $|Z(G) \cap H| \geq 1$. On the other hand, note that $|\text{Orb}(h_i)| < p$ for all $h_i \notin Z(G)$. By the Orbit-Stabilizer Theorem, $|\text{Orb}(h_i)| = \frac{|G|}{|\text{Stab}(h_i)|}$, and so $|\text{Orb}(h_i)| \mid |G|$ for any i. Therefore, since p is the smallest prime that divides $|G|$, and $|H| = p$, then if there are orbits not in $Z(G)$, the orbits must have size 1, contradiction. Therefore, there are no elements in H that are not in $Z(G)$. Hence, $H \subseteq Z(G)$. $\square$

Problem 1.2 (Spring 2016 - Problem 9)**.** *Show that if G is a finite group acting transitively on a set X with at least two elements, then there exists $g \in G$ which fixes no point of X.*

Proof. **Method 1:** By Burnside's Lemma **1.6.13**, the number of orbits is equivalent to $\frac{1}{|G|}\sum_{g\in G}|X^g|$. Therefore, $|G| = \sum_{g\in G}|X^g|$. In particular, the element e fixes all elements in X, which means e fixes at least two elements. By pigeonhole principle, there exists some element in G that fixes no point in X.

Method 2: Take $|G| = n$ and $|X| = k \geq 2$. Define $\text{Fix}(g) = \{x \in X : g\cdot x = x\}$. For each element $g \in \text{Stab}(x)$, we have $x \in \text{Fix}(g)$ and vice versa. In particular, $\sum_{x\in X}|\text{Stab}(x)| = \sum_{g\in G}|\text{Fix}(g)|$. By Orbit-Stabilizer Theorem, since G is finite, $|\text{Stab}(x)| = \frac{|G|}{|\text{Orb}(x)|}$ for all $x \in X$. Note that $|\text{Orb}(x)| = |X| = k$ since G acts transitively, and so $|\text{Stab}(x)| = \frac{n}{k}$. In particular, $\sum_{g\in G}|\text{Fix}(g)| = \frac{n}{k}\cdot k = n$. However, note that $|\text{Fix}(e)| \geq 2$ since e fixes all elements in X. Therefore, by pigeonhole principle, there exists some element in G that fixes no point in X. □

Problem 1.3. *[Fall 2018 - Problem 1] Let $Q_8 = \{\pm1, \pm i, \pm j, \pm k\}$ be the quaternion group of order* 8*.*

(a) Show that every non-trivial subgroup of Q_8 contains -1.

(b) Show that Q_8 does not embed in the symmetric group S_7 (as a subgroup).

Proof. (a) Take arbitrary non-trivial subgroup $G \subseteq Q_8$. If $-1 \in G$, then we are done. Suppose $-1 \notin G$, then since G is non-trivial, one of $\pm i, \pm j, \pm k$ should be in Q_8. However, observe that $(\pm i)^2 = (\pm j)^2 = (\pm k)^2 = -1 \in G$, contradiction. Then -1 has to be contained in every non-trivial subgroup of Q_8.

(b) Let us take $\varphi : Q_8 \to S_7$ as an injective group homomorphism, which is defined as a group action of Q_8 on the set $X = \{x_1, \cdots, x_7\}$ via $g \cdot x_i = x_{\varphi(g)(i)}$ for arbitrary $g \in Q_8$. The orbits of the action partition X so $|X| = \sum_{x\in X}|\text{Orb}(x)|$. By Orbit-Stabilizer Theorem, $|\text{Orb}(x)| = \frac{|Q_8|}{|\text{Stab}(x)|}$. For arbitrary $x \in X$, $\text{Stab}(x) \subseteq Q_8$ has to be non-trivial, otherwise $|\text{Orb}(x)| = \frac{8}{1} = 8 > 7$, contradiction. Recall that -1 is contained in every non-trivial subgroup of Q_8, then $-1 \in \text{Stab}(x)$ for all $x \in X$. In particular, -1 fixes all elements of X, which means $\varphi(1) = \varphi(-1) = e$. In particular, φ is not injective, contradiction.

□

Problem 1.4 (Spring 2019 - Problem 8)**.** *Prove that every finite group of order n is isomorphic to a subgroup of $GL_{n-1}(\mathbb{C})$.*

Proof. By Cayley's Theorem, there is an injective homomorphism from G to S_n. There is an injective homomorphism from S_n to $\text{GL}_n(\mathbb{C})$ given by permuting the elements of $\mathbb{C}^n$ once a basis has been chosen. Take $(1, 1, \cdots, 1) = v \in \mathbb{C}^n$, which is an eigenvector for each permutation matrix. Each permutation matrix in the basis $\beta = \{v, e_2, \cdots, e_n\}$ for $\mathbb{C}^n$ will be a block matrix of $\begin{pmatrix}1\end{pmatrix}$ and a permutation matrix in $\text{GL}_{n-1}(C)$. Thus there is an injective homomorphism of S_n to $\text{GL}_{n-1}(C)$. Compose this with the injection from Cayley's Theorem to prove the claim. □

Problem 1.5 (Spring 2020 - Problem 7)**.** *Let G be a finite p-group and $1 \neq N \trianglelefteq G$ be a non-trivial normal subgroup.*

(a) Show that N contains a non-trivial element of the center $Z(G)$ of G.

(b) Give an example where $Z(N) \not\subseteq Z(G)$.

Proof. (a) Conjugating elements of N by G is a group action since N is a normal subgroup. The fixed points of the action are exactly the elements of N in $Z(G)$. Thus $|N| = |Z(G) \cap N| + \sum_{h \in N, h \notin Z(G)} |\text{Orb}(h)|$. The identity is contained in N and $Z(G)$ which implies $|N \cap Z(G)| \geq 1$. By Orbit-Stabilizer Theorem, $|\text{Orb}(h)| = \frac{|G|}{|Stab(h)|}$, which is divisible by p. Then $|N| - \sum_{h \in N, h \in Z(G)} |\text{Orb}(h)| = |Z(G) \cap N|$ is divisible by p, and there is some non-trivial element of $Z(G) \cap N$.

(b) Take $G = D_4$, the dihedral group of order 8. Let $N = \langle r \rangle$ be the cyclic subgroup of G generated by rotation by $\frac{\pi}{2}$ counter-clockwise. Then $Z(N) = N$ but $Z(G) = \langle r \rangle$.

We will cover the non-finite case when we talk about Sylow p-subgroups. ☐

2 MATH 210A Discussion 2, October 12, 2021

A free group F is a group of all words on a set of letters S. In particular, there is the universal property illustrated by the following diagram:

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, hook] & G \\ F \arrow[ur, "\varphi"] & \end{tikzcd}$$

Figure 8.1: Universal Property of Free Group

In other words, given a function f from S to G, there exists a unique group homomorphism $\varphi : F \to G$ such that φ restricts to f on S.

The commutator subgroup of a group G is defined by $[G : G] = \langle ghg^{-1}h^{-1}, g, h \in G \rangle$. Note that $[G, G] \triangleleft G$, and $G/[G, G]$ is an Abelian Group. Moreover, if G/N is Abelian, then $[G, G] \subseteq N$.

Problem 2.1 (Spring 2017 - Problem 3)**.** *Find the number of subgroups of index 3 in the free group $F_2 = \langle u, v \rangle$ on two generators.*

Proof. Denote $X = \{1, 2, 3\}$ as a set of order 3. To proceed, we first show that there is a bijection between subgroups of F_2 with index 3 and stabilizers of transitive group actions on a set of three elements.

Suppose there is a transitive group action of F_2 on X. Then notice that $\text{Stab}(1) \subseteq G$ is a subgroup and $[G : \text{Stab}(1)] = |\text{Orb}(1)| = 3$ according to the Orbit-Stabilizer Theorem. On the other hand, suppose we have a subgroup $H \subseteq F_2$ of index 3. By definition, the coset F_2/H has order 3. We then have a transitive group action of F_2 on F_2/H given by left multiplication. Take $g \in F_2$. We have $g \cdot H = H$ if and only if $g \in H$. As a result, $\text{Stab}(H) = H$. This gives the desired correspondence relation.

Therefore, it suffices to find the number of transitive group actions of F_2 on the set $X = \{1, 2, 3\}$ with $H = \text{Stab}(1)$ without loss of generality. Since $|X| = 3$, this is equivalent to finding a homomorphism $\varphi : F_2 \to S_3$ whose image contains a 3-cycle. The image of u and v under φ uniquely determines φ by the universal property of free groups.

We will break into cases. Note that the element 2 and 3 in X can are interchangeable so $\varphi(u) = (1\ 3)$ cases produce the same stabilizers of 1 as the $\varphi(u) = (1\ 2)$ cases. Similarly, we do not have to consider the cases where $\varphi(u) = (1\ 3\ 2)$. We then have the following list:

$$\begin{gathered}
\varphi(u) = e \text{ implies } \varphi(v) \in \{(1\ 2\ 3), (1\ 3\ 2)\} \\
\varphi(u) = (1\ 2) \text{ implies } \varphi(v) \in \{(1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\} \\
\varphi(u) = (2\ 3) \text{ implies } \varphi(v) \in \{(1\ 2), (1\ 3), (1\ 2\ 3), (1\ 3\ 2)\} \\
\varphi(u) = (1\ 2\ 3) \text{ implies } \varphi(v) \in \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}
\end{gathered}$$

The symmetry of element 2 and 3 also allows us to remove the cases $\{\varphi(u) = e, \varphi(v) = (1\ 3\ 2)\}$, $\{\varphi(u) = (2\ 3), \varphi(v) = (1\ 3)\}$ and $\{\varphi(u) = (2\ 3), \varphi(v) = (1\ 3\ 2)\}$. Therefore, we are left with 13 suitable group homomorphisms $\varphi : F_2 \to S_3$ for which $\text{Stab}(1)$ determines all distinct subgroups of F_2 of index 3. This concludes the proof. $\square$

Problem 2.2 (Fall 2017 - Problem 2). *Let G be a finite group of order a power of a prime number p. Let $\Phi(G)$ be the subgroup of G generated by elements of the form g^p for $g \in G$ and $ghg^{-1}h^{-1}$ for $g, h \in G$. Show that $\Phi(G)$ is the intersection of the maximal proper subgroups of G.*

Proof. Let G be a p-group that acts on a finite set X. We will first show that $|X^G| \equiv |X| \pmod p$ where $X^G = \{x \in X : |\text{Orb}(x)| = 1\}$. THe orbits partition X so $|X| = |X^G| + \sum_{x \in X, x \notin X^G} |\text{Orb}(x)|$. By the Orbit-Stabilizer Theorem, $|\text{Orb}(x)| = [G : \text{Stab}(x)] = \frac{|G|}{|\text{Stab}(x)|}$ with $|G|$ finite. For each $x \notin X^G$, we have $|\text{Orb}(x)| = \frac{|G|}{|\text{Stab}(x)|} > 1$, so p will divide $\frac{|G|}{|\text{Stab}(x)|} = |\text{Orb}(x)|$. Therefore, $|X| \equiv |X^G| \pmod p$.

Now take $|G| = p^k$ for some k. Let $H \subset G$ be a maximal proper subgroup of G so $|H| = p^{k-1}$. Let H act on the left cosets of H in G by left multiplication. If $aH \in X^H$, then $b(aH) = aH$ for all $b \in H$. Thus, $aba^{-1} \in H$ and $a \in N_G(H)$. Similarly, taking some $a \in N_G(H) gives aH \in X^H$. Therefore, $X^H = [N_G(H) : H]$ and the above result implies $[N_G(H) : H] \equiv [G : H] \equiv 0 \pmod p$. Then index $[N_G(H) : H]$ divides $[G : H]$, so it is either 1 or p. We conclude that $[N_G(H) : H] = p$ and $N_G(H) = G$ since $|H| = p^{k-1}$. Thus, $H \lhd G$, and the set G/H is a factor group of order p. The only possible group is the cyclic group $\mathbb{Z}/p\mathbb{Z}$, so $G/H \cong \mathbb{Z}/p\mathbb{Z}$.

If $g \notin H$ for some $g \in G$, then gH is a generator of G/H so $(gH)^p = g^pH = H$. H contains elements of the form g^p for $g \in G$. Further, G/H is Abelian so the canonical projection $p : G \to G/H$

factors through $\pi : G/[G,G]$ for the commutator subgroup $[G,G]$. Thus, $\ker(\pi) = [G,G] \subseteq \ker(p) = H$, and H contains all elements of the form $ghg^{-1}h^{-1}$ for $g, h \in H$. Therefore, $\Phi(G)$ is contained in the intersection of the maximal proper subgroups of G.

For each $g \in G$, we want to show that there is a maximal proper subgroup $M \subset G$ that does not contain g. The commutator subgroup of G is normal. Let $g, h \in G$. Then $hg^p h^{-1} = (hgh^{-1})^p \in \Phi(G)$, so $\Phi(G)$ is a normal subgroup of G. Every element $g \in G \backslash \Phi(G)$ corresponds to a coset $\bar{g} = g\Phi(g) \in G/\Phi(G)$. By $g^p \in \Phi(G)$ for all $g \in G$, $G/\Phi(G)$ is a group where each element divides order p. Since the commutator subgroup is contained in $\Phi(G)$, then $G/\Phi(G)$ is a finite Abelian group with only elements of order dividing p. We can view $G/\Phi(G)$ as an F_p-vector space so take an F_p-basis $\{\bar{g}, \bar{x_1}, \cdots, \bar{x_k}\}$ for $G/\Phi(G)$. Let x_i be a lift of $\bar{x_i}$ in G. Define the subgroup M to be the one generated by $\Phi(G) \cup \{x_1, \cdots, x_k\}$. Since $g \notin M$ by construction, M is a proper subgroup of G. Further, $\langle M \cup \{g\} \rangle = G$ so G is a maximal proper subgroup of G that does not contain g. We conclude that the intersection of the maximal proper subgroups of G is contained in $\Phi(G)$. ☐

Problem 2.3 (Fall 2018 - Problem 2). *Let G be a finitely generated group (assumed to be infinite) having a subgroup of finite index $n > 1$. Show that G has finitely many subgroups of index n and has a proper characteristic subgroup (i.e. preserved by all automorphisms) of finite index.*

Proof. Note that there are finite groups for which the statement does not hold. Conjugation by an element of a group is an automorphism of the group (called an inner automorphism). Thus every characteristic subgroup of a group is normal. The finite group A_5 is simply and thus contains no non-trivial characteristic subgroups. Assume G is infinite.

Let $H \subseteq G$ bea. subgroup of index n. Then G acts on the set of left cosets $G/H = \{g_1H, \cdots, g_nH\}$ via left multiplication. This defines a group homomorphism $\varphi : G \to S_n$ such that $g \cdot g_i H = g_{\varphi(g)(i)}H$. Note that $g \cdot H = H$ if and only if $g \in H$. Thus, $\mathrm{Stab}(H) = H$ implies a one-to-one correspondence between the index n subgroups of G and homomorphisms $\varphi : G \to S_n$. Let G be generated by $\{x_1, \cdots, x_k\}$. Then the image of each x_i in S_n determine uniquely each homommorphism $\varphi : G \to S_n$. There are $n!$ choices for the image of each x_i so there are finitely many homomorphisms $\varphi : G \to S_n$. We conclude there are finitely many index n subgroups of G.

Let $\sigma \in \mathrm{Aut}(G)$ and $H \subseteq G$ be the index n subgroup in the problem statement. Now $\sigma(H)$ is a subgroup of H since σ is an automorphism. Note that the cosets are $\sigma(G)/\sigma(H) = G/\sigma(H) = \{\sigma(g_1)\sigma(H), \cdots, \sigma(g_n)\sigma(H)\}$ so $\sigma(H)$ is an index n subgroup of G. Define $N = \bigcap_{\sigma \in \mathrm{Aut}(G)} \sigma(H)$. There are finitely many index n subgroups of G so $N = \bigcap_{i=1}^{m} H_i$ for some index n subgroups $H_i \subseteq G$. We want to show that N is a proper characteristic subgroup of finite index in G. It is clear that N is a subgroup that is fixed under all automorphisms of G. We can define a group action of G on $\prod_{i=1}^{m} G/H_i$, by component-wise left multiplication. Then $\mathrm{Stab}(H_1, H_2, \cdots, H_m) = \bigcap_{i=1}^{m} H_i = N$ since $gH_i = H_i$ if and only if $g \in H_i$. By Orbit-Stabilizer Theorem, $[G : N] = [G : \mathrm{Stab}(H_1, \cdots, H_m) = |\mathrm{Orb}(H_1, \cdots, H_m| \leq |\mathrm{Orb}(H_1)| \cdots |\mathrm{Orb}(H_n)| = [G : H_1] \cdots [G : H_m]$.

Since each H_i is of finite index, $[G : N]$ is finite. Therefore, N is a characteristic subgroup of G of finite index. Note that N cannot be all of G since it is a subgroup of H and N is not trivial since it is a finite index subgroup of an infinite group. $\square$

Problem 2.4 (Fall 2015 - Problem 8). *Let F be a field. Show that the group $SL(2, F)$ is generated by the matrices* $\begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix}$ *for element $e \in F$.*

Proof. The group $\mathrm{SL}(2, F)$ is all 2×2 contains matrices with determinant 1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a general matrix in $\mathrm{SL}(2, F)$. Case 1: If $a = 0$ or $d = 0$, then $c = -b^{-1}$.

$$\begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -e^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 0 & e \\ -e^{-1} & 1 \end{pmatrix}$$
$$\begin{pmatrix} 0 & e \\ -e^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & e(1-a) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & e \\ -e^{-1} & a \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 \\ -e^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & e \\ -e^{-1} & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & e(1-a) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & e \\ -e^{-1} & 0 \end{pmatrix} = \begin{pmatrix} a & e \\ -e^{-1} & 0 \end{pmatrix}$$

Case 2: If $b = 0$ or $c = 0$, then $d = a^{-1}$.

$$\begin{pmatrix} 0 & -be \\ e^{-1}b^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & e \\ -e^{-1} & -e^{-1}b^{-1}a \end{pmatrix} = \begin{pmatrix} b & a \\ 0 & b^{-1} \end{pmatrix}$$
$$\begin{pmatrix} 0 & -e^{-1} \\ e & -e^{-1}b^{-1}a \end{pmatrix} \begin{pmatrix} 0 & e^{-1}b^{-1} \\ -be & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ a & b^{-1} \end{pmatrix}$$

Case 3: Assuming nonzero $a, b, c, d \in F$, then $A = \begin{pmatrix} d^{-1}(1+bc) & b \\ c & d \end{pmatrix}$.

$$\begin{pmatrix} b & a \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} c & 0 \\ d & c^{-1} \end{pmatrix} = \begin{pmatrix} bc + ad & ac^{-1} \\ b^{-1}d & b^{-1}c^{-1} \end{pmatrix}$$

Then $(b^{-1}c^{-1})^{-1}(1 + (ac^{-1})(b^{-1}d)) = bc(1 + ab^{-1}c^{-1}d)bc + ad$, the first row, first column entry above. We conclude $\mathrm{SL}(2,F)$ is generated by the matrices $\begin{pmatrix}1 & e\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\ e & 1\end{pmatrix}$ for element $e \in F$. $\square$

Problem 2.5 (Fall 2015 - Problem 10)**.** *Let p be a prime number. For each abelian group K of order p^2, how many subgroups H of $\mathbb{Z}^3$ are there with $\mathbb{Z}^3/H$ isomorphic to K?*

Proof. Note that $\mathbb{Z}^3$ is Abelian so each subgroup $H \subseteq \mathbb{Z}^3$ is normal. Let S be the set of surjective group homomorphisms $f : \mathbb{Z}^3 \to K$ and T be the set of all subgroups H of $\mathbb{Z}^3$ for which $\mathbb{Z}^3 \cong K$. Then define a set map $\Phi : S \to T$ by $\Phi(f) = \ker(f)$. Note that $\mathrm{Aut}(K)$ acts on S by post-composition. Denote by $S/\mathrm{Aut}(K)$ the set of orbits of S under the action by $\mathrm{Aut}(K)$. Let $\sigma \in \mathrm{Aut}(K)$, then $\ker(\sigma \circ f) = \ker(f)$ since σ is injective. As a result, $\bar{\Phi} : S/\mathrm{Aut}(K) \to T$ is a well-defined set map. Surjectivity of $\bar{\Phi}$ follows from the fact that each subgroup H for which $\mathbb{Z}^3/H \cong K$ defines a surjective group homomorphism $f : \mathbb{Z}^3 \to \mathbb{Z}^3/H \cong K$.

We want to show that $\bar{\Phi}$ is injective. Let $f, g \in S$ such that $\ker(f) = \ker(g)$. By the universal property of quotients, f factors through $\mathbb{Z}^3/\ker(f)$, and there is some isomorphism $\alpha : \mathbb{Z}^3/\ker(f) \to K$ such that $\alpha \circ \pi = f$ for $\pi : \mathbb{Z}^3 \to \mathbb{Z}^3/H$ the canonical quotient homomorphism. Similarly, $\beta \circ \pi = g$ for an isomorphism $\beta : \mathbb{Z}^3/\ker(g) \to K$. Then $f = (\alpha \circ \beta^{-1}) \circ g$ where $(\alpha \circ \beta^{-1} \in \mathrm{Aut}(K)$, and f and g are in the same $\mathrm{Aut}(K)$-orbit of S. We conclude that $\bar{\Phi}$ is a bijection.

It is sufficient to find the number of surjective group homomorphisms $f : \mathbb{Z}^3 \to K$ for each K. There are only two Abelian groups of order p^2: $\mathbb{Z}/p^2\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Case 1: Let $K = \mathbb{Z}/p^2\mathbb{Z}$. We need only find images for the 3 generators of the free Abelian group $\mathbb{Z}^3$. Let $x, yin\mathbb{Z}/p^2\mathbb{Z}$ be non-generating elements. They are classes represented by integers divisible by p. Then representatives of $x+y$ are divisible by p and $x+y$ does not generate $\mathbb{Z}/p^2\mathbb{Z}$. Thus at least one of the generators of $\mathbb{Z}^3$ must map to a generator of $\mathbb{Z}/p^2\mathbb{Z}$ in order for the homomorphism to be surjective. There are $\varphi(p^2) = p^2 - p$ generators of $\mathbb{Z}/p^2\mathbb{Z}$ for Euler's totient function $\varphi/$. There are p^6 total homomorphisms and p^3 homomorphisms that are not surjective. Since $|\mathrm{Aut}(\mathbb{Z}/p^2\mathbb{Z})| = \varphi(\mathbb{Z}/p^2\mathbb{Z} = p^2 - p$, there are $\frac{p^6-p^3}{p^2-p} = p^4 + p^3 + p^2$ total subgroups H of $\mathbb{Z}^3$ for which $\mathbb{Z}^3/H \cong \mathbb{Z}/p^2\mathbb{Z}$.

Case 2: Let $K = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Once again, we need only find images for the 3 generators of the free Abelian group $\mathbb{Z}^3$. Note that K is no longer generated by just one element. For the homomorphism to be surjective, we need the image of at least two of the generators of $\mathbb{Z}^3$ to map to generators of K. This equates to sending one generator to a non-trivial element $a \in K$ and sending a second to an element outside the subgroup generated by a in K. The subgroup generated by a will have order p. We have three scenarios. If the first generator is sent to a nonzero $a \in K$, we have $(p^2-1)(p^2-p)p^2 + (p^2-1)p(p^2-p)$ options depending on the image of the second generator. If the first generator is sent to zero, we have $(p^2-1)(p^2-p)$ options. In total, we have $p^6 - p^4 - p^3 + p$ surjective homomorphisms. There are $(p^2-1)(p^2-p) = p^4 - p^3 - p^2 - p$ automorphisms of K which implies $\frac{p^6-p^4-p^3+p}{p^4-p^3-p^2+p} = p^2 + p + 1$ subgroups H of $\mathbb{Z}^3$ such that $\mathbb{Z}^3/H \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. $\square$

Problem 2.6 (Spring 2017 - Problem 1 / Fall 2019 - Problem 6). *Classify all finite subgroups of $GL(2,\mathbb{R})$ up to conjugacy.*

Proof. Each conjugacy class of matrices in $\mathrm{GL}(2,\mathbb{R})$ has a unique representative in rational canonical form. For 2×2 matrices, the invariant factors of $A \in \mathrm{GL}(2,\mathbb{R})$ could be $\{f\}$ for $f = x^2 - ax - b \in \mathbb{R}[x]$ or $\{g,h\}$ where $g \mid h$. Since the sum of the degrees of g and h is 2, we see that $\deg(g) = \deg(h) = 1$. We can take g and h to be monic so $g = h = x - c$ for some $c \in \mathbb{R}$. Thus, the possible canonical forms for a matrix in $\mathrm{GL}(2,\mathbb{R})$ are $\begin{pmatrix} 0 & b \\ 1 & a \end{pmatrix}$ or $\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$ for $a,b,c \in \mathbb{R}$. Each conjugacy class of $\mathrm{GL}(2,\mathbb{R})$ has a representative of the form above. ☐

3 MATH 210A Discussion 3, October 19, 2021

Let H, K be groups, then $H \lhd_\varphi K = H \times K$ as a set where $\varphi : K \to \mathrm{Aut}(H)$ and $(h_1,k_1)\cdot(h_2,k_2) = (h_1 \cdot \varphi(k_1)(h_2), k_1k_2$.

Example: If φ is trivial, then $H \lhd_\varphi K = H \times K$. $D_4 = \langle r\rangle \lhd_\varphi \langle s\rangle$ where $\varphi : \langle s\rangle \to \mathrm{Aut}(\langle r\rangle)$ mapping $s \mapsto (r \mapsto r^3)$.

Note that $H \lhd (H \lhd_\varphi K)$.

If H, K subgroups of G and $H \lhd G$ and $HK = G$ and $H \cap K = \{e\}$, then $G \cong H \lhd_\varphi K$.

Problem 3.1 (Spring 2015 - Problem 8). *Let G be a finite group of order pq, where p and q are distinct primes. Show that:*

(a) G has a normal subgroup distinct from 1 and G.

(b) if $p \not\equiv 1 \pmod q$ and $q \not\equiv 1 \pmod p$, then G is Abelian.

Proof. (a) Without loss of generality, assume $p > q$. Let m_p denote the number of Sylow p-subgroups of G. By Sylow's Third Theorem, $m_p \equiv 1 \pmod p$ and $m_p | q$. Since q is prime, m_p is either 1 or q. But $q \not\equiv 1 \pmod p$ since $p > q$. Thus, $m_p = 1$. Conjugation of a subgroup $H \subset G$ by $g \in G$ is again a subgroup of G of order $|H|$. Thus we will obtain a Sylow p-subgroup of G when we conjugate a Sylow p-subgroup by any element $g \in G$. Since we have a unique Sylow p-subgroup $P \subset G$, $gPg^{-1} = P$ and P is normal in G.

(b) Without loss of generality, assume $p > q$. Note that this is now a direct result of corollary 1.8.7. The following is a similar argument.

By part (a), the Sylow p-subgroup $P \subset G$ is a normal subgroup of G. Sylow's Theorems imply the existence of some Sylow q-subgroup $Q \subset G$. The subgroup $P \cap Q$ is a subgroup of both P and Q. Then $|P \cap Q| = 1$ since $|P|$ and $|Q|$ are relatively prime. All of this implies $G = P \rtimes_\varphi Q$ for some group homomorphism $\varphi : Q \to \mathrm{Aut}(P)$. We have $\mathrm{Aut}(P) \cong \mathbb{Z}/(p-1)\mathbb{Z}$. The generator $a \in Q$ has order q so it needs to map to an element of order dividing q, leaving

1 or q. By assumption, $p \not\equiv 1 \pmod{q}$ so $\varphi(a)$ is the identity automorphism. Thus $G \cong P \times Q$ for P, Q cyclic (which implies Abelian). We conclude that G is Abelian.

□

Problem 3.2 (Fall 2015 - Problem 5). *(a) Let G be a group of order $p^e v$ with v and e positive integers, p prime and $p > v$, and v is not a multiple of p. Show that G has a normal Sylow p-subgroup.*

(b) Show that a non-trivial finite p-group has non-trivial center.

Proof. (a) By Sylow's Third Theorem, the number of Sylow p-subgroups m_p satisfies $m_p \equiv 1 \pmod{p}$ and m_p divides v. Thus $m_p = kp + 1$ for $k \geq 0$. However, $p > v$ and $m_p | v$ implies $k = 0$. We conclude $m_p = 1$. Let P be the unique Sylow p-subgroup of G. Following the argument in the previous problem (or by corollary **1.8.7**), we know P is a normal Sylow p-subgroup.

(b) Note that this is a direct result of **1.7.4**. We can also prove it by the following argument.

Let H be a nontrivial finite p-group. Thus $|H| = p^k$ for $k > 0$. Act on the set H by H via conjugation. An element is fixed by conjugation if and only if the element is in the center of H. The class equation implies $|H| = |Z(H)| + \sum_{h \in H, h \notin Z(H)} |\mathrm{Orb}(h)|$. We have $p||H|$ and $|\mathrm{Orb}(h)| = [G : \mathrm{Stab}(h)]$ by Orbit-Stabilizer Theorem. Thus $p||\mathrm{Orb}(h)|$ for each $h \notin Z(H)$. We conclude that p divides $|Z(G)| = |H| - \sum_{h \in H, \notin Z(H)} |\mathrm{Orb}(h)|$. Note $|Z(H)| > 1$ since the identity of H is contained in the center. Thus $|Z(H)| \geq p$ so H has a nontrivial center.

□

Problem 3.3 (Fall 2017 - Problem 1). *Let G be a finite group, p a prime number, and S a Sylow p-subgroup of G. Let $N = \{g \in G | gSg^{-1} = S\}$. Let X and Y be two subsets of $Z(S)$ such that there is $g \in G$ with $gXg^{-1} = Y$. Show that there exists $n \in N$ such that $nxn^{-1} = gxg^{-1}$ for all $x \in X$.*

Proof. Let G act on a set X with $g \cdot x = y$ for $g \in G$ and $x, y \in X$. We want to show that $\mathrm{Stab}(Y) = g\mathrm{Stab}(x)g^{-1} \subset G$. Let $h \in \mathrm{Stab}(y)$. Then $g^{-1}hg \cdot x = g^{-1}h \cdot y = g^{-1} \cdot y = x$ so $g^{-1}hg \in \mathrm{Stab}(x)$. We have $g^{-1}\mathrm{Stab}(y)g \subset \mathrm{Stab}(x)$. Next let $k \in \mathrm{Stab}(x)$. Then $gkg^{-1} \cdot y = gk \cdot x = g \cdot x = y$ and $g\mathrm{Stab}(x)g^{-1} \subseteq \mathrm{stab}(y)$. Since conjugation by an element of a group is an invertible operation, $\mathrm{Stab}(y) = g\mathrm{Stab}(x)g^{-1}$.

We can define an N-action on S via conjugation. Define $\mathrm{Stab}(X) = \bigcap_{x \in X} \mathrm{Stab}(x) \subset G$. Since $X, Y \subseteq Z(S)$, we have $S \subset \mathrm{Stab}(X)$ and $S \subset \mathrm{Stab}(Y)$. Note that S is a Sylow p-subgroup of $\mathrm{Stab}(X)$ and $\mathrm{Stab}(Y)$. By the result above applied to each $y \in Y$, we have $\mathrm{Stab}(Y) = g\mathrm{Stab}(X)g^{-1}$. Conjugation preserves the order of subgroups so $gSg^{-1} \subset \mathrm{Stab}(Y)$ is a Sylow p-subgroup of $\mathrm{Stab}(Y)$. By Sylow's Second Theorem, the two Sylow p-subgroups S and gSg^{-1} are conjugate in $\mathrm{Stab}(Y)$. Thus there exists an $h \in \mathrm{Stab}(Y)$ such that $h(gSg^{-1})h^{-1} = S$. We note that $hg \in N$. Additionally,

$(hg) \cdot h = \cdot(gxg^{-1}) = gxg^{-1}$ since $h \in \text{Stab}(Y)$. Let $n = hg \in N$ and $nxn^{-1} = gxg^{-1}$ for all $x \in X$. □

Problem 3.4 (Spring 2018 - Problem 9)**.** *Show that there is no simple group of order* 616*.*

Proof. Again, following the argument in the previous problem (or by corollary **1.8.7**), it suffices to find a unique Sylow p-subgroup for some prime p, then that subgroup would be normal and we know the group is not simple by definition.

Let G be a group with order $616 = 2^3 \cdot 7 \cdot 11$. By Sylow's Third Theorem, the number of Sylow 11-subgroups m_{11} divides 56 and is congruent to 1 modulo 11. Thus we could have $m_{11} = 1$ or $m_{11} = 56$. Suppose, towards contradiction, that $m_{11} = 56$. Next, the number of Sylow 7-subgroups m_7 divides 88 and is congruent to 1 modulo 7. We could have $m_7 = 1, 8, 22, 88$. The argument will work for larger choices for m_7 so assume $m_7 = 8$. The intersection of a Sylow 7-subgroup and Sylow 11-subgroup must be trivial by an order consideration. Thus the Sylow subgroups chosen account for $(11 + 55(10)) + 8(6) = 609$ elements. A Sylow 2-subgroup of G will have order 8. As a result, there can be at most one Sylow 2-subgroup. Sylow's Theorems imply the existence of a Sylow 2-subgroup so $m_j = 1$ for some $j \in \{2, 7, 11\}$. By the above argument, we conclude that G has a normal subgroup and G is not simple. □

Problem 3.5 (Fall 2020 - Problem 1)**.** *Let $p < q < r$ be primes and G a group of order pqr. Prove that G is not simple and, in fact, has a normal Sylow r-group.*

Proof. We will first prove that G is not simple. Let n_p be the number of distinct Sylow p-subgroups, n_q be the number of distinct Sylow q-subgroups, and n_r be the number of distinct Sylow r-subgroups. By Sylow's Third Theorem, we know the following

$$\begin{aligned} n_p &\equiv 1 \pmod p, n_p \mid qr \\ n_q &\equiv 1 \pmod q, n_q \mid pr \\ n_r &\equiv 1 \pmod r, n_r \mid pq \end{aligned}$$

We conclude that $n_r = 1, p, q, pq$.Since $r > p$ and $r > q$, p and q can't be congruent to 1 modulo r. Thus $n_r = 1$ or $n_r = pq$. If $n_r = 1$, we're done so assume $n_r = pq$. Every Sylow r-subgroup contains the identity and $r - 1$ order r elements of G. Thus there are $pq(r-1) = pqr - pq$ order r elements of G. Similarly, $n_q = 1, p, r, pr$. Since $q > p$, p can't be congruent to 1 modulo q. If $n_q = 1$, we're done so assume that $n_q = r$, the smallest other possibility. As above, there are $r(q-1) = rq - r$ elements of order q in G. We have $n_p = 1, q, r, qr$ so assume that $n_p = q$. Then there are $q(p-1) = pq - q$ elements of order p in G. In total this accounts for

$$(pqr - pq) + (rq - r) + (p - q) + 1 = pqr + rq - r - q + 1$$

elements of G. Since r and q are greater than 1, $rq \geq r + q$ and this exceeds the order of G. Thus there is some normal Sylow subgroup and G is not simple.

Let N be a normal Sylow subgroup of G. If $|N| = r$, we are done so assume $|N| = q$ without loss of generality. Then G/N is a group of order pr, which implies that G/N has a normal subgroup of order r. By the subgroup correspondence, there is a normal subgroup H of G containing N for which H/N is order r. Thus $|H| = qr$ and H contains a normal subgroup of order r denoted P_r. We want to prove that P_r is normal in G. Let $g \in G$. Then $|gP_rg^{-1}| = r$ and $gPrg^{-1} \subset H$ since H is normal in G. Since P_r is a normal Sylow r-subgroup of H, P_r is the unique Sylow r-subgroup of H. We conclude that $gPrg^{-1} = P_r$ and P_r is normal in G. □

Problem 3.6 (Fall 2020 - Problem 2). *Show that groups of order $231 = 3 \times 7 \times 11$ are semi-direct products and show that there are exactly two such groups up to isomorphism.*

Proof. Let G be a group of order 231 with P_3 a Sylow 3-subgroup, P_7 a Sylow 7-subgroup, and P_{11} a Sylow 11-subgroup. Since $|P_i \cap P_j| = 1$ for distinct i and j in $\{3, 7, 11\}$, we conclude that $|G| = |P_3||P_7||P_{11}|$ and $G = P_3P_7P_{11}$. By the previous problem P_{11} is normal in G. Let n_7 be the number of distinct Sylow 7-subgroups in G. Sylow's Third Theorem proves that $n_7 \equiv 1 \pmod 7$ and $n_7 \mid 33$. The only option is $n_7 = 1$ and P_7 is normal in G. Thus the cyclic subgroup P_7P_{11} of order 77 is normal in G and $G \cong P_7P_{11} \rtimes_\varphi P_3$. We have $\mathrm{Aut}(P_7P_{11}) \cong \mathbb{Z}/6Z \times \mathbb{Z}/10\mathbb{Z}$ and P_3 cyclic of order 3. Therefore, $\varphi : P_3 \to \mathrm{Aut}(P_7P_{11})$ is either trivial or sends a generator of P_3 to an order 3 element of $\mathbb{Z}/6\mathbb{Z}$. The cases of the latter produce isomorphic semi-direct products so there are only two groups of order 231 up to isomorphism. □

4 MATH 210A Discussion 4, October 26, 2021

Definition 4.1 ((Linear) Representation). *A (linear) representation is a homomorphism $\rho : G \to GL(V)$ for some vector space V over a field F. The dimension of ρ is the dimension of V over field F.*

Definition 4.2 (Invariant, Irreducible). *Let $\rho :: G \to GL(V)$ be a representation of a group G. A subspace W of V is G-invariant if $\rho(G) \cdot W \subseteq W$.*

An irreducible representation $\rho : G \to GL(V)$ is one in which there is no non-trivial, proper G-invariant subspace of V (up to change of basis).

Definition 4.3 (Group Algebra). *A group algebra $F[G]$ is a free vector space over F generated by G. Group multiplication on the ring $F[G]$ is defined by $h \cdot \sum a_g g = \sum a_g \cdot (hg)$ for $a_g \in F$, $g \in G$.*

Example 4.4. *Let G be cyclic of order n generated by $g \in G$. Then every element of $F[G]$ can be written as $\sum\limits_{i=0}^{n-1} a_i g^i$ where $a_i \in F$. The multiplication works as $g \cdot (\sum\limits_{i=0}^{n-1} a_i g^i) = a_{j-1}e + \sum\limits_{i=1}^{n-1} a_{i-1}^i$ where e is the identity element of G.*

A representation $\rho : G \to \mathrm{GL}(V)$ gives V a $F[G]$-module structure. An $F[G]$-module V defines a representation $\rho : G \to \mathrm{GL}(V)$ based on the action of each $g \in G$ on V. e.g. Take a basis $\{v_1, \cdots, v_n\}$ for V, then $g \cdot v_i = \sum\limits_{j=1}^{n} a_j v_j$ will map it to $\rho(g) \in \mathrm{GL}(V)$.

Further, isomorphic $F[G]$-modules correspond to isomorphic representations, or representations that differ by a base change. i.e. two representations ρ, μ are isomorphic if $\mu(g) = P\rho(g)P^{-1}$ for all $g \in G$. Thus the two languages are equivalent, and we will use the $F[G]$-module interpretation to find some nice properties of representations. For the qualifying exam, we will almost always take $F = \mathbb{C}$ and G finite.

Since $\mathbb{C}$ is characteristic 0 and algebraically closed, by Artin-Wedderburn Theorem, $\mathbb{C}[G]$ is a semi-simple F-algebra, so in particular $\mathbb{C}[G] \cong \prod\limits_{i=1}^{k} M_{d_i}(\mathbb{C})$ where M_{d_i} is some $d_i \times d_i$ matrix over $\mathbb{C}$ and vice versa. Each component $M_{d_i}(\mathbb{C})$ defines an irreducible representation ρ_i of dimension d_i over $\mathbb{C}$. We have $\dim_{\mathbb{C}}(\mathbb{C}[G]) = |G|$ by counting the basis elements and $\dim_{\mathbb{C}}(M_{d_i}(G)) = d_i^2$ by the theorem, so $\sum\limits_{i=1}^{k} d_i^2 = |G|$. In particular, the sum of squares of irreducible representations is $|G|$.

The center of the group algebra, $Z(\mathbb{C}[G])$, is the set of all elements $\alpha \in \mathbb{C}[G]$ that commute with each basis element $g \in \mathbb{C}[G]$. If $\alpha = \sum\limits_{g\in G} a_g g$ for $a_g \in F$ and $g \in G$, we can show that $\alpha \in Z(\mathbb{C}[G])$ if and only if $a_g = a_{g'}$ whenever g, g' in the same conjugacy class. Let $G = \bigcup\limits_{i=1}^{l} C_i$ by the union of distinct disjoint conjugacy classes, then a basis for $Z(\mathbb{C}[G])$ is just $\{k_1, \cdots, k_l\}$ for $k_i = \sum\limits_{g\in C_i} g$. The dimension of $Z(\mathbb{C}[G])$ is (magically, with no connection) the number of conjugacy classes. On the other hand, $Z(M_{d_i}(\mathbb{C})) = \mathbb{C} \cdot I_{d_i}$ is one-dimensional, so the number of irreducible representations of G over $\mathbb{C}$ is equal to the number of conjugacy classes of G.

Example 4.5. *One-dimensional representations are easy to consider. By definition, two representations $\rho, \mu : G \to GL(\mathbb{C})$ are isomorphic if and only if $\rho = \mu$.*

Suppose $\rho : G \to GL_1(\mathbb{C})$ is a one-dimensional irreducible representation. Then $GL_1(\mathbb{C}) \cong \mathbb{C}^\times$, so ρ factors through $G/[G,G]$. We conclude that the number of one-dimensional irreducible representations of G is equal to the order of $G/[G,G]$.

$$\begin{tikzcd}
G \arrow[rr, "\rho"] \arrow[dr] & & GL_1(\mathbb{C}) \arrow[r, "\cong"] & \mathbb{C}^\times \\
& G/[G,G] \arrow[ur] & &
\end{tikzcd}$$

Figure 8.2: One-dimensional Irreducible Representation

Example 4.6. *For any Abelian group A, $|A|$ is the number of distinct conjugacy classes, which implies the number of non-isomorphic irreducible representations. Therefore, $\sum d_i^2 = |A|$, so $d_i = 1$ for all i.*

Definition 4.7 (Character)**.** *Let $\rho : G \to GL(V)$ be a representation. The character of ρ is defined as $\mathcal{X}_\rho(g) = tr(\rho(g))$.*

Example 4.8. *Let $\rho : G \to GL(V)$ and $\mu : G \to GL(W)$ be representations of a group G.*

1. *If* $\dim(\rho) = 1$*, then* $\mathcal{X}_\rho(g) = \rho(g)$*.*

2. *Suppose two representations* ρ, μ *are isomorphic, then* $\mathcal{X}_\rho(g) = \mathcal{X}_\mu(g)$ *for all* $g \in G$*.*

3. $\mathcal{X}_{\rho\oplus\mu}(g) = \mathcal{X}_\rho(g) + \mathcal{X}_\mu(g)$ *for all* $g \in G$*.*

4. $\mathcal{X}_\rho(hgh^{-1}) = \mathcal{X}_\rho(g)$ *for all* $g, h \in G$*.*

5. $\mathcal{X}_\rho(e) = \dim(\rho)$ *for identity* $e \in G$*.*

Definition 4.9. *Let Ch*(G) *be the vector space of functions* $G \to F$ *which are constant on conjugacy classes. Note that the characters of a group* G *are elements of this vector space. We can define a bilinear form* B : *Ch*$(G) \times$ *Ch*$(G) \to F$ *on Ch*(G) *as* $B(\mathcal{X}_\rho, \mathcal{X}_\mu) = \frac{1}{|G|}\sum \mathcal{X}_\rho(g^{-1}\mathcal{X}_\mu(g)$*. Then the characters of irreducible representations* ρ_i *form an orthonormal basis for Ch*(G) *with respect to* B*. Indeed,* $\dim$ *Ch*$(G) = k$ *is the number of conjugacy classes, or equivalently the number of irreducible representations.*

Theorem 4.10. *Let* $\rho_1, \cdots, \rho_k$ *be the irreducible representations of a finite group* G *over* $\mathbb{C}$ *with corresponding characters* $\mathcal{X}_1, \cdots, \mathcal{X}_k$*.*

1. *Every finite-dimensional representation* ρ_i *is isomorphic to* $\bigoplus_{i=1}^{k} \rho_i^{B(\mathcal{X}_\rho, \mathcal{X}_i)}$*.*

2. *Two representations* ρ *and* μ *are isomorphic if and only if* $\mathcal{X}_\rho = \mathcal{X}_\mu$*.*

3. *A representation* ρ *is irreducible if and only if* $B(\mathcal{X}_\rho, \mathcal{X}_\rho) = 1$*.*

Example 4.11. *Let* $\rho_1, \cdots, \rho_k$ *be the irreducible representations of a finite group* G *with corresponding characters* $\mathcal{X}_1, \cdots, \mathcal{X}_k$*. Let* $C_1, \cdots, C_k$ *be the conjugacy classes of* G*. Then* $\sum_{i=1}^{k} \overline{\mathcal{X}_i(g_{j_1})}\mathcal{X}_i(g_{j_2}) = 0$ *where* $g_{j_1} \in C_1$ *and* $g_{j_2} \in C_2$ *and* $j_1 \neq j_2$*. (Similar to an inner product.)*

Example 4.12. *The regular representation of* G *is given by acting on the vector space* $F[G]$ *by left multiplication. The representation will be* $\rho : G \to GL_n(F)$ *where* $n = |G|$*. Each element* $\rho(g)$ *is a permutation matrix. Let* $\{g_1, \cdots, g_n\}$ *be the group elements of* G *which form a basis for* $F[G]$ *as an* F*-vector space. If* g *is not the identity, then it fixes no elements of the basis and* $\mathcal{X}_\rho(g) = tr(\rho(g)) = 0$*. Let the irreducible representations of* G *be* $\rho_1, \cdots, \rho_k$ *with characters* $\mathcal{X}_1, \cdots, \mathcal{X}_k$*. We find that the regular representation breaks down as* $\rho = \bigoplus_{i=1}^{k} rho_i^{d_i}$ *and* $\mathcal{X}_\rho = \sum_{i=1}^{k} d_i\mathcal{X}_i$*. The regular representation can be helpful in coming up with higher dimensional representations for a group since each irreducible representation is a direct summand.*

Most of the qualification exam problems on representation theory will ask us to find the character table of a given group G. The character table for a group G is constructed as follows. Each row will represent the character of an irreducible representation, which we will denote $\mathcal{X}_1, \cdots, \mathcal{X}_k$. Each column will represent a conjugacy class of G, which we will denote $C_1, \cdots, C_k$. By above, the table will have the same number of rows and columns. The ith row, jth column entry of the table will be $\mathcal{X}_i(g_j)$ for $g_j \in C_j$.

A typical approach to one of these problems includes the following steps.

1. Find the conjugacy classes of G. The number of conjugacy classes is the number of irreducible representations.

2. The order of $G/[G,G]$ is the number of one-dimensional irreducible representations.

3. Let d_i denote the dimension of the irreducible representation ρ_i with character $\mathcal{X}_i$. The equation $|G| = \sum_{i=1}^{k} d_i^2$ along with the number of one-dimensional irreducible representations can sometimes help us determine the dimensions of other irreducible representations.

4. The column corresponding to the conjugacy class of the identity will be populated with the dimensions of each irreducible representation.

5. The trivial one-dimensional representation $\rho(g) = 1$ will provide a row of all 1's.

6. The rows satisfy an orthogonality condition $\sum_{i=1}^{k} |C_i|\overline{\mathcal{X}_{j_1}(g_i)}\mathcal{X}_{j_2}(g_i) = 0$ for $j_1 \neq j_2$ and some representative $g_i \in C_i$. Further, $\sum_{i=1}^{k} |C_i||\mathcal{X}_j(g_i)|^2 = |G|$.

7. The columns satisfy an orthogonality condition $\sum_{i=1}^{k} \overline{\mathcal{X}_i(g_{j_1})}\mathcal{X}_i(g_{j_2}) = 0$ where $g_{j_1} \in C_{j_1}$ and $g_{j_2} \in C_{j_2}$ and $j_1 \neq j_2$.

5 MATH 210A Discussion 5, November 2, 2021

Problem 5.1 (Fall 2015 - Problem 7 / Spring 2016 - Problem 8)**.** *Show that the symmetric group S_4 has exactly two isomorphism classes of irreducible complex representations of dimension* 3*. Compute the characters of these two representations.*

Proof. We first show that $|S_4/[S_4,S_4]| = 2$. Observe that every element in S_4 has to be an even permutation, so $[S_4,S_4] \subseteq A_4$. Furthermore, observe that the non-identity elements of A_4 has to be of the form $(i\ j)(k\ l)$ or $(i\ j\ k)$. One can show that they are all elements in $[S_4,S_4]$, then $[S_4,S_4] = A_4$. In particular, $|S_4/[S_4,S_4]| = 2$, therefore there are two 1-dimensional irreducible representations.

Now notice that each one-dimensional representation of S_4 is a group homomorphism $\rho : S_4 \to \mathbb{C}^\times$, but since $\mathbb{C}^\times$ is Abelian, then ρ factors uniquely through the Abelian group $S_4/[S_4,S_4]$. If two one-dimensional representations are equal on $S_4/[S_4,S_4]$, then they are equal as homomorphisms from S_4. Thus, the number of one-dimensional representations of $S_4/[S_4,S_4]$ is equal to the number of one-dimensional representations of S_4. By above, $S_4/[S_4,S_4]$ has two conjugacy classes so it has two one-dimensional irreducible representations. We conclude that S_4 should have two one-dimensional representations. (This works for one-dimensional irreducible representations of any group.)

The two 1-dimensional irreducible representations are obvious: the trivial representation and the sign representation (mapping permutations to their signs). Note S_n breaks into cycle type conjugacy classes. S_4 has 5 conjugacy classes, then there are 5 irreducible representations, so $S_4 = \sum_{i=1}^{5} d_i^2$, where $24 = 1 + 1 + a^2 + b^2 + c^2$ for some $2 \leq a, b, c \leq 3$. By solving it we have $a = 2, b = 3, c = 3$.

Note that there are five conjugacy classes in S_4, which are e, $(1\ 2)$, $(1\ 2\ 3)$, $(1\ 2)(3\ 4)$ and $(1\ 2\ 3\ 4)$.

Obviously, the trivial representation maps all elements to 1 so the row has all 1's as trace. As for the sign representation, it maps e to 1, $(1\ 2)$ to -1, $(1\ 2\ 3)$ to 1, $(1\ 2)(3\ 4)$ to 1, and $(1\ 2\ 3\ 4)$ to -1.

We now find the two irreducible representations of dimension 3. Define the vector space $V = \{(v_i) \in \mathbb{R}^4 : \sum_{i=1}^{4} v_i = 0\}$, then V has a left S_4 action via $\sigma(v_i) = v_{\sigma(i)}$ for $\sigma \in S_4$ and $\{(-1,1,0,0), (-1,0,1,0), (-1,0,0,1)\}$ as a basis for V. The action described gives an irreducible representation for S_4 since $(2\ 3)(-1,1,0,0) = (-1,0,1,0)$ and $(2\ 4)(-1,1,0,0) = (-1,0,0,1)$. In other words, there is no S_4-invariant subspace of V. Let $\rho : S_4 \to M_3(\mathbb{C})$ denote this 3-dimensional irreducible representation.

We find the characteristics of ρ with respect to each conjugacy class. Obviously ρ has characteristics 3 on the identity class since there is no permutation, i.e. identity matrix. For $(1\ 2)$, the permutation maps the basis to $\{(1,-1,0,0), (0,-1,1,0), (0,-1,0,1)\}$, which gives the matrix $\begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and has trace 1. Therefore, the characteristic is 1. For $(1\ 2\ 3)$, the basis is mapped to $\{(0,-1,1,0), (1,-1,0,0), (0,-1,0,1)\}$, which forms a matrix $\begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which has trace 0. Therefore, the characteristic is 0. Similarly, $(1\ 2)(3\ 4)$ maps the basis to $\{(1,-1,0,0), (0,-1,0,1), (0,-1,1,0)\}$, which is presented by the matrix $\begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and has trace -1. Therefore, the characteristic is -1. Finally, we can find the characteristic of ρ for conjugacy class $(1\ 2\ 3\ 4)$ is -1.

From the above, there is an irreducible representation $\rho \otimes \text{sign}$ of $S_4 \times S_4$. By including S_4 along the diagonal of $S_4 \times S_4$, we can make a representation of S_4 denoted $\rho \otimes \text{sign}$. In particular, $\mathcal{X}_{\rho\otimes\text{sign}}(g) = \mathcal{X}_\rho(g)\mathcal{X}_{\text{sign}}(g)$. We have an inner product on the space of class functions such as $\langle \mathcal{X}_\mu, \mathcal{X}_\upsilon \rangle = \frac{1}{|G|} \sum_{g \in G} \mathcal{X}_\mu(g)\mathcal{X}_\upsilon(g^{-1})$. We know that $\langle \mathcal{X}_{\rho\otimes\text{sign}}, \mathcal{X}_{\rho\otimes\text{sign}} \rangle = 1$ if and only if $\rho \otimes \text{sign}$ is an irreducible representation. We note that the number of elements in each conjugacy class are $1, 6, 8, 3, 6$ respectively. Since g^{-1} and g are in the same conjugacy class for all $g \in S_4$, $\langle \mathcal{X}_{\rho\otimes\text{sign}}, \mathcal{X}_{\rho\otimes\text{sign}} \rangle = \frac{1}{24}(1\times 9 + 6\times 1 + 8\times 0 + 3\times 1 + 6\times 1) = 1$. Thus, $\rho\otimes\text{sign}$ is the other irreducible representation of S_4.

We can draw the following table:

#elements	1	6	8	3	6
	e	(1 2)	(1 2 3)	(1 2)(3 4)	(1 2 3 4)
$\mathcal{X}_{\text{trivial}}$	1	1	1	1	1
$\mathcal{X}_{\text{sign}}$	1	-1	1	1	-1
$\mathcal{X}_{\mu}$	2				
$\mathcal{X}_{\rho}$	3	1	0	-1	-1
$\mathcal{X}_{\rho\otimes\text{sign}}$	3	-1	0	-1	1

where μ is the unknown irreducible representation of dimension 2.

By using restrictions mentioned in last discussion, we may fill out the rest of the table.

#elements	1	6	8	3	6
	e	(1 2)	(1 2 3)	(1 2)(3 4)	(1 2 3 4)
$\mathcal{X}_{\text{trivial}}$	1	1	1	1	1
$\mathcal{X}_{\text{sign}}$	1	-1	1	1	-1
$\mathcal{X}_{\mu}$	2	0	-1	2	0
$\mathcal{X}_{\rho}$	3	1	0	-1	-1
$\mathcal{X}_{\rho\otimes\text{sign}}$	3	-1	0	-1	1

Figure 8.3: Character Table for S_4

$\square$

Problem 5.2 (Fall 2016 - Problem 4)**.** *Let D be a Dihedral group of order $2p$ with normal cyclic subgroup C of order p for an odd prime p. Find the number of n-dimensional irreducible representations of D (up to isomorphisms) over C for each n, and justify your answer.*

Proof. Note that a dihedral group of order $2p$ is represented by $D = \langle r, s : r^p = s^2 = e, rs = sr^{-1}\rangle$. We first find the commutator subgroup $[D,D] \subseteq D$. Any element of the commutator subgroup is of the form $(r^i s)(r^j s)(r^i s)^{-1}(r^j s)^{-1}$ for some $0 \le i, j \le p-1$. Reducing this, we end up with r^{2i-2j}. Further, $r^{\frac{p+1}{2}} s r^{p-\frac{p+1}{2}} s^{-1} = r^{\frac{p+1}{2}} r^{\frac{p+1}{2}} ss = r^{\frac{2p+2}{2}} = r \in [D,D]$. Thus, $[D,D]$ is the subgroup of D generated by r and $|D/[D,D]| = 2$. There are two non-isomorphic classes of one-dimensional representations of D.

We now classify the conjugacy classes of D. Note that it is sufficient to conjugate each element only by the generators r and s. The identity makes up one conjugacy class. When we conjugate s we notice $r^i s r^{p-i} = r^{2i} s$. Since p is odd, we can continue this process to obtain the conjugacy class $\{s, rs, \cdots, r^{p-1}s\}$. When we conjugate r^i we have $sr^i s^{-1} = sr^i s = r^{p-i}$ for $1 \le i \le p-1$. Conjugating by s again yields $sr^{p-i}s^{-1} = sr^{p-i}s = r^i$. Thus, we have the conjugacy classes r^i, r^{p-i} for $1 \le i \le \frac{p-1}{2}$. In total, this gives $\frac{p+3}{2}$ conjugacy classes.

Using the intuition of D as permutations of vertices of a regular p-gon, we can construct the classes of 2-dimensional irreducible representations. Define the rotation by $\frac{2\pi k}{p}$ counterclockwise in plane,

$\varphi_k(r) = \begin{pmatrix} \cos(\frac{2\pi k}{p}) & -\sin(\frac{2\pi k}{p}) \\ \sin(\frac{2\pi k}{p}) & \cos(\frac{2\pi k}{p}) \end{pmatrix}$ and the reflection about the x-axis in the plane, $\varphi_k(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for $1 \leq k \leq \frac{p-1}{2}$. Each φ_k is an irreducible representation of D since there are no subspaces of $\mathbb{C}^2$ invariant under these transformations. Further, these are non-isomorphic irreducible representations since the characters $\mathcal{X}_{\varphi_k}(r) = 2\cos(\frac{2\pi k}{p})$ differ for each k.

The sum of the squares of the dimensions of these representations is $1+1+(\frac{p-1}{2})2^2 = 2+(2p-2) = 2p$, the order of the group. Thus these are all isomorphism classes of irreducible representations of D over $\mathbb{C}$. We conclude that there are two one-dimensional and $\frac{p-1}{2}$ two-dimensional isomorphism classes of irreducible complex representations of D.

□

Problem 5.3 (Spring 2017 - Problem 2). *Let G be the group with presentation*

$$\langle x, y : x^4 = 1, y^5 = 1, xyx^{-1} = y^2 \rangle,$$

which has order 20. *Find the character table of G.*

Proof. We first find the conjugacy classes of G. Note that we only need to check conjugation by the generators x and y. Since $xy = y^2x$, we can write each element of G as y^ix^j for some $0 \leq i < 5$ and $0 \leq j < 4$. Additionally,

$$(y^ix^j)(y^kx^l)(y^ix^j)^{-1} = y^{i+2^jk}x^{j+l}x^{-j}y^{-i} = y^{i+2^jk}x^l$$

so the exponent of x remains unchanged by conjugation. By the formula above, conjugating y^kx^l by y will result in $y^{k-1}x^l$. Thus the conjugacy classes are

$$\{1\}, \{y, y^2, y^3, y^4\}, \{x, yx, y^2x, y^3x, y^4x\}, \{x^2, yx^2, y^2x^2, y^3x^2, y^4x^2\}, \{x^3, yx^3, y^2x^3, y^3x^3, y^4x^3\},$$

which implies 5 isomorphism classes of irreducible representations. We will now find the commutator subgroup $[G, G]$. The generators of $[G, G]$ have the form $(y^ix^j)(y^kx^l)(y^ix^j)^{-1} = (y^{-i+2^jk}x^l)x^{-l}y^{-k} = y^{-i+(2^j-1)k}$. We can pick $i = 4, j = 0, k = 0$ and $l = 1$, which implies $[G, G]$ is the cyclic subgroup of G generated by y. Then the number of isomorphism classes of one-dimensional representations is $|G/[G, G]| = 4$ by the argument in problem 7 in Fall 2015. There are 4 one-dimensional representations and 5 conjugacy classes. Since the order of G is the sum of the squares of the irreducible representations, $20 = 1^2 + 1^2 + 1^2 + 1^2 + k^2$ so $k = 4$.

We will now determine the 4 one-dimensional representations. Since x is order 4, it must map to ± 1, $\pm i$ in $\mathbb{C}^\times$. Similarly, y is order 5 so y must map to a fifth root of unity in $\mathbb{C}^\times$. The character is equal to the representation in the one-dimensional case so the representation is the same on each conjugacy class. Let $\rho_i : G \to \mathbb{C}^\times$ be one-dimensional representations for $1 \leq i \leq 3$ and $\mu : G \to \mathbf{GL}_4(\mathbb{C})$ be the 4-dimensional irreducible representation. For $\rho_i : G \to \mathbb{C}^\times$, $\rho_i(y) = \rho_i(y^2) = \rho_i(y)^2$ so $\rho_i(y) = 1$. We can fill in the character table below based on the image of x. The last row of the table is found by column orthogonality.

	e	y	x	x^2	x^3
$\mathcal{X}_{\text{trivial}}$	1	1	1	1	1
$\mathcal{X}_{\rho_1}$	1	1	i	-1	$-i$
$\mathcal{X}_{\rho_2}$	1	1	-1	1	-1
$\mathcal{X}_{\rho_3}$	1	1	$-i$	-1	i
$\mathcal{X}_{\mu}$	4	-1	0	0	0

Figure 8.4: Character Table for $\langle x, y : x^4 = 1, y^5 = 1, xyx^{-1} = y^2 \rangle$

□

Problem 5.4 (Spring 2018 - Problem 6)**.** *Let G be a group with a normal subgroup $N = \langle y, z \rangle$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Suppose that G has a subgroup $Q = \langle x \rangle$ isomorphic to the cyclic group $\mathbb{Z}/3\mathbb{Z}$ such that the composition $Q \subset G \to G/N$ is an isomorphism. Finally, suppose that $xyx^{-1} = z$ and $xzx^{-1} = yz$. Compute the character table of G.*

Proof. We will find the conjugacy classes of G. Since $xy = zx$ and $xz = yzx$, we can write every element of G as $y^i z^j x^k$ for $0 \leq i, j \leq 1$ and $0 \leq i \leq 2$. The relations allow reduction to the form $y^i z^j x^k$ without changing the x component. As a result, conjugation by any element will preserve the x component of any element. We will show that the conjugacy classes are based on the exponent of x. The relations of G produce the conjugacy class $\{y, z, yz\}$. In the equations below, we start with x.

$$yxy^{-1} = yxy = zx$$
$$y(zx)y^{-1} = yz^2x = yz$$
$$z(zx)z^{-1} = xz = yzx$$

A similar argument starting with x^2 gives the conjugacy class breakdown below.

$$\{e\}, \{y, z, yz\}, \{x, yx, zx, yzx\}, \{x^2, yx^2, zx^2, yzx^2\}.$$

Note that $|G| = 12$. Thus, the sum of 1 and three squares needs to be $|G| = 12$. We cannot have an irreducible representations of dimension higher than three. The only option is $12 = 1^2 + 1^2 + 1^2 + 3^2$ so there should be three isomorphism classes of one-dimensional representations and on isomorphism class of 3-dimensional irreducible representations.

We will first classify the characters of the one-dimensional irreducible representations. Let $\rho_i : G \to \mathbb{C}^\times$ for $1 \leq i \leq 3$ be the one-dimensional representations. Since y and z are order 2 elements of G, they must map to ± 1 in $\mathbb{C}^\times$. Similarly, x will be sent to a third root of unity. The group $\mathbb{C}^\times$ is Abelian so $\rho(z) = \rho(xyx^{-1}) = \rho(x)\rho(y)\rho(x)^{-1} = \rho(y)$ and $\rho(yz) = \rho(xzx^{-1}) = \rho(x)\rho(z)\rho(x)^{-1} = \rho(z)$. Let ξ be a primitive third root of unity. We find the final row of the character table by column orthogonality and the identity $\sum_{i=1}^{3} \xi^i = 0$.

	e	y	x	x^2
$\mathcal{X}_{\text{trivial}}$	1	1	1	1
$\mathcal{X}_{\rho_1}$	1	1	ξ	ξ^2
$\mathcal{X}_{\rho_2}$	1	1	ξ^2	ξ
$\mathcal{X}_{\mu}$	3	-1	0	0

Figure 8.5: Character Table for group G

□

Problem 5.5 (Fall 2018 - Problem 11). *Let G be a finite group, ω be a primitive third root of* 1 *in* $\mathbb{C}$ *and suppose that the complex character table of G contains the row*

$$1 \quad \omega \quad \omega^2 \quad 1$$

Determine the whole complex character table of G, the order of the group and the order of its conjugacy classes.

Proof. Note that the number of columns, four, determines the number of conjugacy classes of G and the number of isomorphism classes of irreducible representations. The first row of character table corresponds to the trivial representation. Let $\rho : G \to \mathbb{C}$ be the one-dimensional representation desfcribed in the row given. Then we can construct a one-dimensional representation $\rho \otimes \rho : G \times G \to \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$. By including G in $G \times G$ via the diagonal homomorphism, we find $\rho \otimes \rho$ describes a one-dimensional representation with $\mathcal{X}_{\rho\otimes\rho}(g) = \mathcal{X}_\rho(g)^2$. Since the characters $\mathcal{X}_{\rho\otimes\rho}$ differ from the current rows, $\rho \otimes \rho$ describes a distinct isomorphism class of one-dimensional representations.

By orthogonality of the second/third column and the first column, we find the zeros in the fourth row. Let $a = \mathcal{X}_\mu(e)$ and $b = \mathcal{X}_\mu(g)$ for $g \in C_4$. Then $ab = -3$ by the orthogonality of columns one and four. Since a represents the dimension of the irreducible representation $\mu : G \to M_a(\mathbb{C})$, $a > 0$ is an integer so $b \in \mathbb{Q}$. With $|G|$ finite, the trace of $\mu(g)$ is the sum of eigenvalues that are all roots of unity. Thus $b \in \mathbb{Q}$ is an algebraic integer so $b \in \mathbb{Z}$. We conclude that $a = 1$ and $b = -3$ or $a = 3$ and $b = -1$. If $a = 1$, then $|G| = 4$. The order of some $g \in C_2$ must be divisible by 3 since $\rho(g^3) = \rho(g)^3 = 1$. This contradicts the order of G so $a \neq 1$. Thus, $a = 3$ and $b = -1$.

As a result, $|G| = 1^2 + 1^2 + 1^2 + 3^2 = 12$. The rows are orthonormal under the inner product $\langle v, w\rangle = \frac{1}{|G|}\sum_{i=1}^{4} |C_i| v_i \overline{w_i}$. Row three implies $1 = \frac{9+|C_4|}{12}$ and $|C_4| = 3$. The inner product of rows two and one gives $0 = \frac{1+|C_2|\omega^2+|C_3|\omega+3}{12}$. Thus, $|C_2| = |C_3|$ with 8 elements between the two conjugacy classes. We conclude $|C_2| = |C_3| = 4$.

	$C_1 = \{e\}$	C_2	C_3	C_4
$\mathcal{X}_{\text{trivial}}$	1	1	1	1
$\mathcal{X}_{\rho}$	1	ω	ω^2	1
$\mathcal{X}_{\rho\otimes\rho}$	1	ω^2	ω	1
$\mathcal{X}_{\mu}$	3	0	0	-1

Figure 8.6: Character Table for group G

□

Problem 5.6 (Fall 2019 - Problem 7). *Let G be the group of order 12 with presentation*

$$G = \langle g, h \mid g^4 = 1, h^3 = 1, ghg^{-1} = h^2 \rangle.$$

Find the conjugacy classes of G and the values of the characters of the irreducible complex representations of G of dimension greater than 1 on representatives of these classes.

Proof. The final relation of G implies that $gh = h^2g$ and $gh^2 = hg$. We can use these relations to write every element of G as g^ih^j for $0 \leq i \leq 3$ and $0 \leq j \leq 2$. Further, we have the relations $h^2g^3 = g^3h$ and $hg^3 = g^3h^2$ by inverting the above relations. Clearly, $C_1 = \{e\}$ is a conjugacy class. The relations

$$\begin{aligned} ghg^{-1} &= ghg^3 = h^2 \\ gh^2g^{-1} &= gh^2g^3 = h \end{aligned}$$

show that $C_2 = \{h, h^2\}$ is a conjugacy class. We find

$$\begin{aligned} hgh^{-1} &= hgh^2 = gh \\ h(gh)h^{-1} &= gh^2 \\ g(gh)g^{-1} &= g^2hg^3 = gh^2 \\ h(gh^2)h^{-1} &= hgh = g \\ g(gh^2)g^{-1} &= g^2h^2g^3 = gh \end{aligned}$$

so $C_3 = \{g, gh, gh^2\}$ is a conjugacy class. By similar computation, we have conjugacy class $C_4 = \{g^3, g^3h, g^3h^2\}$. The equations

$$\begin{aligned}
hg^2h^{-1} &= hg^2h^2 = gh^2gh^2 = g^2 \\
h(g^2h)h^{-1} &= hg^2 = gh^2g = g^2h \\
g(g^2h)g^{-1} &= g^3hg^3 = g^2h^2 \\
h(g^2h^2)h^{-1} &= hg^2h = gh^2gh = g^2h^2 \\
g(g^2h^2)g^{-1} &= g^3h^2g^3 = g^2h
\end{aligned}$$

prove that $C_5 = \{g^2\}$ and $C_6 = \{g^2h, g^2h^2\}$ are conjugacy classes. All elements of G have been placed in conjugacy classes.

The commutator $[G,G]$ has elements of the form $ghg^{-1}h^{-1} = ghg^3h^2 = h$. Thus, $\langle h \rangle \subseteq [G,G]$. We see that $G/\langle h \rangle$ is cyclic of order 4 and, thus, Abelian. We conclude $[G,G] = \langle h \rangle$ and there are $|G/[G,G]| = 4$ one-dimensional non-isomorphic irreducible representations of G. Each one-dimensional $\rho_i : G \to \mathbb{C}^\times$ sends h to 1. The image of g must be a fourth root of unity. Further, $12 = 4 + a^2 + b^2$ for a and b the dimensions of the other irreducible representations of G. We see that $a < 3$ and $b < 3$ so $a = b = 2$ so we obtain the following character table:

	e	h	g	g^2	g^3	g^2h
$\mathcal{X}_1$	1	1	1	1	1	1
$\mathcal{X}_2$	1	1	i	-1	$-i$	-1
$\mathcal{X}_3$	1	1	-1	1	-1	1
$\mathcal{X}_4$	1	1	$-i$	-1	i	-1
$\mathcal{X}_5$	2					
$\mathcal{X}_6$	2					

We will construct a two-dimensional irreducible representation of G over $\mathbb{C}$. Define a set map μ on the generators

$$\mu(g) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\mu(h) = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{\frac{4\pi i}{3}} \end{pmatrix}$$

Then the image of g has order 4 in $\mathbf{GL}_2(\mathbb{C})$ and the image of h has order 3 in $\mathbf{GL}_2(\mathbb{C})$. Further,

$$\begin{aligned}\mu(ghg^{-1}) &= \mu(g)\mu(h)\mu(g)^{-1}\\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{\frac{4\pi i}{3}} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\\ &= \begin{pmatrix} e^{\frac{4\pi i}{3}} & 0 \\ 0 & e^{\frac{2\pi i}{3}} \end{pmatrix}\\ &= \mu(h)^{-1}\end{aligned}$$

so $\mu : G \to \mathbf{GL}_2(G)$ is a group homomorphism as desired. There is no non-trivial, proper G-invariant subspace of $\mathbb{C}^2$ which proves μ is irreducible. Compute the characters $\mathcal{X}_5$ by taking the traces of the relevant matrices. We can complete the final row of the character table by column orthogonality of column j with column 1.

	e	h	g	g^2	g^3	g^2h
$\mathcal{X}_1$	1	1	1	1	1	1
$\mathcal{X}_2$	1	1	i	-1	$-i$	-1
$\mathcal{X}_3$	1	1	-1	1	-1	1
$\mathcal{X}_4$	1	1	$-i$	-1	i	-1
$\mathcal{X}_5$	2	-1	0	-2	0	1
$\mathcal{X}_6$	2	-1	0	2	0	-1

Figure 8.7: Character Table of $\langle g, h \mid g^4 = 1, h^3 = 1, ghg^{-1} = h^2 \rangle$

$\square$

6 MATH 210A Discussion 6, November 9, 2021

A small category is one in which the objects and morphisms form a set. A locally small category $\mathscr{C}$ is one in which $\mathbf{Hom}_{\mathscr{C}}(X, Y)$ is a set for all $X, Y \in \mathbf{Ob}(\mathscr{C})$.

Example 6.1. *Lists of categories that would be on the qualification exam:*

1. *Category of sets.*

2. *Category of groups.*

3. *Category of Abelian groups ($\mathbb{Z}$-modules). This is a full subcategory of the category of groups (objects in the category of Abelian groups would keep all morphisms between these objects in the category of groups).*

4. *Category of rings.*

5. *Category of commutative rings.*

6. *Category of R-modules.*

Example 6.2. 1. *In the* **Set** *category, product is Cartesian product, coproduct is disjoint union.*

2. *In the* **Grp** *category, product is direct product, coproduct is free product.*

3. *In the* **Ab** *category, product is direct product, coproduct is direct sum. (They are the same on finite families.)*

4. *In the* **Ring** *category, product is direct product, coproduct is similar to free product of groups.*

5. *In the* **CRing** *category, product is the direct product, coproduct is the tensor product over $\mathbb{Z}$.*

6. *In the* **R-mod** *category, finite product and coproduct are the same, called the direct sum.*

For each term we define, we want to know the technical construction in each of the above categories. Once we are familiar with each category, we will hopefully be able to come up with simple counterexamples to qualification exam problems.

Note that by the universal properties of products and coproducts, we can construct the following bijections

$$\mathbf{Hom}_{\mathscr{C}}(X, \prod_i Y_i) \cong \prod_i \mathbf{Hom}_{\mathscr{C}}(X, Y_i).$$
$$\mathbf{Hom}_{\mathscr{C}}(X, \coprod_i Y_i) \cong \coprod_i \mathbf{Hom}_{\mathscr{C}}(X, Y_i).$$

Problem 6.3 (Spring 2015 - Problem 1). *What are the coproducts in the category of groups?*

Proof. We will define the free product of a family of groups G_i for $i \in I$. As a set $\coprod_{i\in I} G_i$ is all words on the letters $\bigcup_{i\in I} G_i$. We reduce letters from the same group via the group multiplication. Define the group operation as concatenation. The identity element is the empty word, concatenation is associative, and the inverse of a reduced word $g_1 \cdots g_n$ is $g_n^{-1} \cdots g_1^{-1}$. Thus, the free product of a family of groups is a group.

Define the inclusion homomorphisms $i_k : G_j \to \coprod_{k\in I} G_k$ as $i_j(g) = g$. We want to show that $\coprod_{i\in I} G_i$ satisfies the universal property of the coproduct. Let $f_i : G_i \to A$ be a family of group homomorphisms. For the diagram below to commute, $h : \coprod_{k\in I} G_k \to A$ must be defined as $h(g) = f_j(g)$ for $g \in G_j$. Then we extend h to a group homomorphism. For a reduced word $g_1 \cdots g_n \in \coprod_{k\in I} G_k$, we have $h(g_1 \cdots g_n) = h(g_1) \cdots h(g_n) = f_{j_1}(g_1) \cdots f_{j_n}(g_n)$ for $g_i \in G_{j_i}$. Since h is uniquely determined by the collection of homomorphisms $\{f_j\}_{j\in I}$, the free product is the coproduct in the category of groups.

$$
\begin{tikzcd}
 & G_{j_2} \arrow[d, "i_{j_2}"] \arrow[ddr, bend left, "f_{j_2}"] & \\
G_{j_1} \arrow[r, "i_{j_1}"] \arrow[drr, bend right, "f_{j_1}"'] & \coprod_{k\in I} G_k \arrow[dr, dashed, "h"] & \\
 & & A
\end{tikzcd}
$$

□

Problem 6.4 (Fall 2018 - Problem 8)**.** *Give an example of a diagram of commutative rings whose colimit in the category of commutative rings is different from its colimit in the larger category of rings (and ring homomorphisms).*

Proof. We will show that the coproduct of two commutative rings is the tensor product over $\mathbb{Z}$. Let A, B, C be commutative rings with ring homomorphisms $f : A \to C$ and $g : B \to C$. We need $h(i_A(a)) = h(a \otimes 1) = f(a)$ and $h(i_B(b)) = h(1 \otimes b) = g(b)$ for $a \in A$ and $b \in B$. Extend h to a commutative ring morphism so $h(a \otimes b) = f(a)g(b)$ for $a \otimes b \in A \otimes_{\mathbb{Z}} B$. Thus h is the unique commutative ring homomorphism that causes the diagram to commute.

$$
\begin{tikzcd}
 & B \arrow[d, "i_B"] \arrow[ddr, bend left, "g"] & \\
A \arrow[r, "i_A"] \arrow[drr, bend right, "f"'] & A \otimes_{\mathbb{Z}} B \arrow[dr, dashed, "h"] & \\
 & & C
\end{tikzcd}
$$

We will now show that the tensor product over $\mathbb{Z}$ is not the coproduct in the category of rings. Let $A = B = C = M_2(\mathbb{Q})$ and take $f = g = \mathbf{id}_{M_2(\mathbb{Q})}$. Then $h : M_2(\mathbb{Q}) \otimes_{\mathbb{Z}} M_2(\mathbb{Q}) \to M_2(\mathbb{Q})$ can be defined as $h(a \otimes b) = ab$ or $h(a \otimes b) = ba$. These two ring homomorphisms are not equal since $M_2(\mathbb{Q})$ is not commutative. Thus $M_2(\mathbb{Q}) \otimes_{\mathbb{Z}} M_2(\mathbb{Q})$ does not satisfy the universal property of the coproduct.

□

7 MATH 210A Discussion 7, November 16, 2021

A covariant functor $F : \mathscr{C} \to \mathscr{D}$ satisfies:

- $F(\mathbf{id}_X) = \mathbf{id}_{FX}$
- $F(g \circ f) = F(g) \circ F(f)$

A natural transformation $\eta : F \to G$ is a collection of morphisms $\eta_X : FX \to GX$ for each object $X \in \mathbf{Ob}(\mathscr{C})$, such that

$$\begin{array}{ccc} FX & \xrightarrow{\eta_X} & GX \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FY & \xrightarrow[\eta_Y]{} & GY \end{array}$$

A functor F is represented by X if $F \cong \mathbf{Hom}(X, -) = R^X$ is naturally isomorphic.

Lemma 7.1 (Yoneda Lemma)**.** *Let $\mathscr{C}$ be locally small, and let $X \in \mathbf{Ob}(\mathscr{C})$. Take a functor $F : \mathscr{C} \to \mathbf{Set}$. The lemma claims that $\varphi : \mathbf{Nat}(R^X, F) \to FX$ is a bijection, which takes η to $\eta_X(\mathbf{id}_X)$.*

Corollary 7.2 (Yoneda Embedding)**.** $\mathbf{Nat}(R^X, R^Y) \cong \mathbf{Hom}_{\mathscr{C}}(Y, X)$.

In particular, if $\mathbf{Hom}_{\mathscr{C}}(A, B) \cong \mathbf{Hom}_{\mathscr{C}}(C, B)$ for all objects B, then $A \cong C$ by Yoneda Lemma.

For functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$, F is a left adjoint of G if $\mathbf{Hom}_{\mathscr{D}}(FX, Y) \cong \mathbf{Hom}_{\mathscr{C}}(X, FY)$, which is natural in both X and Y.

Definition 7.3 (Adjunction Pair)**.** *Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be two functors. We say that F and G form an adjunction pair $F \dashv G$ with F as a left adjoint of G and G as a right adjoint of F if $\mathbf{Hom}_{\mathscr{D}}(FX, Y) \cong \mathbf{Hom}_{\mathscr{C}}(X, GY)$ for all $X \in \mathbf{Mor}(\mathscr{C})$ and $Y \in \mathbf{Mor}(\mathscr{D})$ such that the family of bijections is natural in X and Y.*

Alternatively, F and G form an adjunction pair with F a left adjoint to G and G a right adjoint to F if there are natural transformations $\varepsilon : FG \to 1_{\mathscr{C}}$ and $\eta : 1_{\mathscr{D}} \to GF$ such that

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F$$

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G$$

are the identity transformations on F and G respectively. We can ε the counit and η is unit.

Problem 7.4 (Spring 2015 - Problem 2)**.** *Let $\mathscr{C}$ be the category of groups and $\mathscr{C}'$ be its full subcategory with objects the Abelian groups. Let $F : \mathscr{C}' \to \mathscr{C}$ be the inclusion functor. Determine the left adjoint of F and show that F has no right adjoint.*

Proof. Let $f : G \to H$ be a group homomorphism where H is Abelian. The commutator subgroup $[G, G]$ is a subgroup generated by $\{g_1 g_2 g_1^{-1} g_2^{-1} \in G : g_1, g_2 \in G\}$. For $g_1, g_2 \in G$, we have $(g_1[G, G])(g_2[G, G]) = g_1 g_2[G, G] = g_1 g_2(g_2^{-1} g_1^{-1} g_2 g_1)[G, G] = g_2 g_1[G, G] = (g_2[G, G])(g_1[G, G])$. Thus, $G/[G, G]$ is an Abelian group. Note $f(g_1 g_2) = f(g_1)f(g_2) = f(g_2)f(g_1) = f(g_2 g_1)$ and $f([G, G]) = 0$. Since $[G, G \subseteq \ker(f)$, there is a unique Abelian group homomorphism $h : G/[G, G] \to H$ such that $ph = f$ for projection $p : G \to G/[G, G]$.

We will define the functor $L : \mathscr{C} \to \mathscr{C}'$ as $L(G) = G/[G, G]$. Note that a morphism of groups $f : G \to H$ gives a unique morphism $\bar{f} : G \to H/[H, H]$ by composing with the projection. Since

$H/[H,H]$ is an Abelian group, the above argument implies $\bar{f}$ factors uniquely through $G/[G,G]$ as $\bar{f} = pg$ for $p : G \to [G,G]$ the projection. Note that $g(a[G,G]) = f(a)[H,H]$ for $a \in G$. Define $L(f) = g$. Let $1_G : G \to G$ be the identity group homomorphism. Then $\overline{1_G} : G \to G/[G,G]$ factors uniquely through $G/[G,G]$ as the identity on $G/[G,G]$. We have $L(1_G) = 1_{L(G)}$. Now let $f : G \to H$ and $g : H \to I$ be two group homomorphisms. Then $gf : G \to I$ gives $L(gf) = h$ for $h : G/[G,G] \to I/[I,I]$ an Abelian group homomorphism defined as $h(a[G,G]) = (g(f(a))[I,I]$. Now $L(f) : G/[G,G] \to H/[H,H]$ gives $L(f)(a[G,G]) = f(a)[H,H]$ and $L(g) : H/[H,H] \to I/[I,I]$ gives $L(g)(f(a)[H,H]) = g(f(a))[I,I]$. Thus $L(gf) = L(g)L(f)$ and L is a covariant functor.

We want to show that $\mathbf{Hom}_{\mathscr{C}}(A, F(B))$ and $\mathbf{Hom}_{\mathscr{C}'}(L(A), B)$ are in bijective correspondence for $A \in \mathbf{Ob}(\mathscr{C})$ and $B \in \mathbf{Ob}(\mathscr{C}')$ and the bijection is functorial in A and B. As we have seen, some $\in \mathbf{Hom}_{\mathscr{C}}(A, F(B))$ factors uniquely through $L(A) = A/[A,A]$ since B is an Abelian group. Define the natural isomorphism Φ whereby $\Phi_{A,B}(f)$ is this unique morphism. Thus $\mathbf{Hom}_{\mathscr{C}}(A, F(B)) \cong \mathbf{Hom}_{\mathscr{C}'}(L(A), B)$ via $\Phi_{A,B}$. Let $g : A' \to A$ be a morphism of groups. Then we want to show the diagram below commutes. Note that $g([A,A']) \subseteq [A,A] = \ker(A \to A/[A,A])$ so $L(g)$ factors uniquely through $A'/[A',A']$. We note that $L(g) : A'/[A',A'] \to [A,A]$ is this unique morphism. Then $\Phi_{A,B}(f) \circ L(g) : A'/[A',A'] \to B$ descends from $f \circ g : A' \to A \to B$. By construction, $\Phi_{A',B}(f \circ g)$ descends from $f \circ g$. The uniqueness of these morphisms implies $\Phi_{A,B}(f) \circ L(g) = \Phi_{A',B}(f \circ g)$ and we are functorial in A. A similar argument shows the bijection is functorial in B. We conclude that L is a left adjoint to F.

$$\begin{array}{ccc}
\mathbf{Hom}_{\mathscr{C}}(A, F(B)) & \xrightarrow{\Phi_{A,B}} & \mathbf{Hom}_{\mathscr{C}'}(L(A), B) \\
{\scriptstyle -\circ g}\downarrow & & \downarrow{\scriptstyle -\circ L(g)} \\
\mathbf{Hom}_{\mathscr{C}}(A', F(B)) & \xrightarrow[\Phi_{A',B}]{} & \mathbf{Hom}_{\mathscr{C}'}(L(A'), B)
\end{array}$$

We will show that F does not have a right adjoint. We will first prove that a left adjoint functor F preserves coproducts. Let G be the right adjoint. Let A_i be the objects of $\mathscr{C}$ and B an object of $\mathscr{D}$. Then

$$\begin{aligned}
\mathbf{Hom}_{\mathscr{C}}(F(\coprod_i A_i), B) &\cong \mathbf{Hom}_{\mathscr{D}}(\coprod_i A_i, B) \\
&\cong \prod_i \mathbf{Hom}_{\mathscr{D}}(A_i, G(B)) \\
&\cong \prod_i \mathbf{Hom}_{\mathscr{C}}(F(A_i), B) \\
&\cong \mathbf{Hom}_{\mathscr{C}}(\coprod_i F(A_i), B)
\end{aligned}$$

By Yoneda Lemma, $F(\coprod_i A_i) \cong \coprod_i F(A_i)$. The coproduct in the category of groups is the free product while the coproduct in the category of Abelian groups is the direct sum. The free product

$\mathbb{Z} * \mathbb{Z}$ is not isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ so F does not have a right adjoint. $\square$

Problem 7.5 (Fall 2017 - Problem 10)**.** *Let $\mathscr{C}$ be a category with finite products, and let $\mathscr{C}^2$ be the category of pairs of objects of $\mathscr{C}$ together with morphisms $(A, A' \to (B, B')$ of pairs consisting of pairs $(A \to B, A' \to B')$ of morphisms in $\mathscr{C}$. Let $F : \mathscr{C}^2 \to \mathscr{C}$ be the direct product functor (that takes pairs of objects and morphisms to their products).*

1. *Find a left adjoint to F.*
2. *For $\mathscr{C}$ the category of Abelian groups, determine whether or not F has a right adjoint.*

Proof. 1. Let $C, D \in \mathbf{Ob}(\mathscr{C})$ and $f \in \mathbf{Hom}_{\mathscr{C}}(C, D)$. Define $L : \mathscr{C} \to \mathscr{C}^2$ as $L(C) = (C, C)$ and $L(f) : L(C) \to L(D)$ as (f, f). Then $L(1_C) = (1_C, 1_C) = 1_{L(C)}$. Additionally, $L(gf) = (gf, gf) = (g, g) \circ (f, f) = L(g)L(f)$ for a morphism $g \in \mathbf{Hom}_{\mathscr{C}}(D, E)$ and $E \in \mathbf{Ob}(\mathscr{C})$. Thus, L is a functor.

By the universal property of the direct product, there is a unique morphism $h : C \to A \prod B$ for each pair of morphisms $(f, g) : (C, C) \to (A, B)$ such that $\pi_A \circ h = f$ and $\pi_B \circ h = g$. Define a natural transformation $\Phi : \mathbf{Hom}_{\mathscr{C}^2}(L(-), -) \to \mathbf{Hom}_{\mathscr{C}^2}(-, F(-))$ so that $\Phi_{C,(A,B)} : \mathbf{Hom}_{\mathscr{C}}(L(C), (A, B)) \to \mathbf{Hom}^2_{\mathscr{C}}(C, F(A, B))$ gives $\Phi_{C,(A,B)}(f, g)$ defined as h. Let $K \in \mathbf{Hom}_{\mathscr{C}}(C', C)$ for $C' \in \mathbf{Ob}(\mathscr{C})$. We want to show the following diagrma commutes.

$$\begin{tikzcd}
\mathbf{Hom}_{\mathscr{C}}(L(C), (A, B)) \arrow[r, "\Phi_{C,(A,B)}"] \arrow[d, "{(-\circ k, -\circ k)}"'] & \mathbf{Hom}_{\mathscr{C}^2}(C, F(A, B)) \arrow[d, "-\circ k"] \\
\mathbf{Hom}_{\mathscr{C}}(L(C'), (A, B)) \arrow[r, "\Phi_{C',(A,B)}"'] & \mathbf{Hom}_{\mathscr{C}^2}(C', F(A, B))
\end{tikzcd}$$

Let $(f, g) \in \mathbf{Hom}_{\mathscr{C}}(L(C), (A, B)) = \mathbf{Hom}_{\mathscr{C}}((C, C), (A, B))$. We have $\Phi_{C,(A,B)}(f, g) \circ k$ is a morphism from C' to $A \prod B$ for which $\pi_A \circ (\Phi_{C,(A,B)}(f, g) \circ k) = f \circ k$ and $\pi_B \circ (\Phi_{C,(A,B)}(f, g) \circ k) = g \circ k$. Further, $h = \Phi_{C',(A,B)}(f \circ k, g \circ k)$ is the unique morphism $C' \to A \prod B$that commutes with $f \circ k$ and $g \circ l$ under projection morphisms.

$$\begin{tikzcd}
C' \arrow[dr, dashed, "h"] \arrow[drr, bend left, "g\circ k"] \arrow[ddr, bend right, "f\circ k"'] & & \\
& A \prod B \arrow[r, "\pi_B"] \arrow[d, "\pi_A"] & B \\
& A &
\end{tikzcd}$$

Thus the universal property of the direct product implies $\Phi_{C,(A,B)}(f, g) \circ k = \Phi_{C',(A,B)}(f \circ k, g \circ k)$ and the desired diagram commutes. By a similar argument, we obtain naturality in (A, B). We conclude that L is a left adjoint to F.

2. Since Abelian groups is an Abelian category, finite products and coproducts are isomorphic. Define $R : \mathscr{C} \to \mathscr{C}^2$ as $R(C) = (C, C)$ and $R(f) = (f, f)$ for $f \in \mathbf{Hom}_{\mathscr{C}}(C, D)$. Then $R(1_C) = (1_C, 1_C) = 1_{R(C)}$. Additionally, $R(gf) = (gf, gf) = (g, g) \circ (f, f) = R(g)R(f)$ for a morphism $g \in \mathbf{Hom}_{\mathscr{C}}(D, E)$ and $E \in \mathbf{Ob}(\mathscr{C})$. Thus, R is a functor.

   By the universal property of the coproduct, there is a unique morphism $h : A \coprod B \to C$ for each pair $(f, g) : (A, B) \to (C, C)$ such that $h \circ i_A = f$ and $h \circ i_B = g$. Define the natural transformation $\Phi : \mathbf{Hom}_{\mathscr{C}^2}(-, R(-)) \to \mathbf{Hom}_{\mathscr{C}}(F(-), -)$ as $\Phi_{(A,B),C}(f, g)$ defined as h. As in (a), the universal property of the coproduct implies naturality in (A, B) and C. We conclude that R is a right adjoint to F.

□

Problem 7.6 (Fall 2016 - Problem 8)**.** *Prove that if a functor $F : \mathscr{C} \to$ **Set** has a left adjoint functor, then F is representable.*

Proof. Let $L : \mathbf{Set} \to \mathscr{C}$ be the left adjoint to F. Then we know that $\Phi_{A,B} : \mathbf{Hom}_{\mathscr{C}}(L(A), B) \cong \mathbf{Hom}_{\mathbf{Set}}(A, F(B))$ for some natural isomorphism Φ and $A \in \mathbf{Ob}(\mathbf{Set})$ and $B \in \mathbf{Ob}(\mathscr{C})$. Let $A = \{*\}$ be a set with one element. Then $\mathbf{Hom}_{\mathbf{Set}}(A, F(B)) \cong F(B)$ as sets via the morphism $h_B : \mathbf{Hom}_{\mathbf{Set}}(A, F(B)) \to F(B)$ with $h_B(\alpha)$ defined as $\alpha(*)$. Thus, $\mathbf{Hom}_{\mathscr{C}}(L(A), B) \cong \mathbf{Hom}_{\mathbf{Set}}(A, F(B)) \cong F(B)$ for all $B \in \mathbf{Ob}(\mathscr{C})$.

Define a natural transformation $\eta_B : \mathbf{Hom}_{\mathscr{C}}(L(A), B) \to F(B)$ by $\eta_B(f) = \Phi_{A,B}(*)$. Since $\Phi_{A,B}$ is an isomorphism and $\mathbf{Hom}(A, F(B)) \cong F(B)$ by choosing the image of $* \in A$, we conclude that η_B is an isomorphism for each $B \in \mathbf{Ob}(\mathscr{C})$. Let $f \in \mathbf{Hom}_{\mathscr{C}}(L(A), B)$, and let $g : B \to C$ be a morphism in $\mathscr{C}$ for $C \in \mathbf{Ob}(\mathscr{C})$. We want to show the diagram below commutes. Since Φ is a natural transformation, the square on the left commutes. The square on the right commutes since $F(g)(h_B(\alpha)) = F(g)(\alpha(*))$ and $h_C(F(g) \circ \alpha) = (F(g) \circ \alpha)(*)$ for $\alpha \in \mathbf{Hom}_{\mathbf{Set}}(A, F(B))$. Therefore, the diagram commutes. We conclude that F is represented by $L(A) \in \mathbf{Ob}(\mathscr{C})$.

$$\begin{tikzcd}
\mathbf{Hom}_{\mathscr{C}}(L(A), B) \arrow[r, "\Phi_{A,B}"] \arrow[d, "g\circ-"'] \arrow[rr, bend left, "\eta_B"] & \mathbf{Hom}_{\mathbf{Set}}(A, F(B)) \arrow[r, "h_B"] \arrow[d, "F(g)\circ-"] & F(B) \arrow[d, "F(g)"] \\
\mathbf{Hom}_{\mathscr{C}}(L(A), C) \arrow[r, "\Phi_{A,C}"'] \arrow[rr, bend right, "\eta_C"'] & \mathbf{Hom}_{\mathbf{Set}}(A, F(C)) \arrow[r, "h_C"'] & F(C)
\end{tikzcd}$$

□

8 MATH 210A Discussion 8, November 23, 2021

Definition 8.1 (Initial/Terminal Object). *An initial object of a category $\mathscr{C}$ is an object I such that, for every object X of $\mathscr{C}$, there exists one and only one morphism $I \to X$. A terminal object of a category $\mathscr{C}$ is an object T such that, for every object X of $\mathscr{C}$, there is one and only one morphism $X \to T$.*

Problem 8.2 (Spring 2016 - Problem 2). *Consider the functor F from commutative rings to Abelian groups that takes a commutative ring R to the group R^* of invertible elements. Does F have a left adjoint? Does F have a right adjoint?*

Proof. We will show that F has a left adjoint. Define the functor $L : \mathbf{Ab} \to \mathbf{CRing}$ as $L(A) = \mathbb{Z}[A]$, the group ring over $\mathbb{Z}$. For an Abelian group morphism $f : X \to Y$, we define $L(f) : \mathbb{Z}[X] \to \mathbb{Z}[Y]$ as $L(f)(x) = f(x)$ and extend $\mathbb{Z}$-linearly. Note that $L(f)$ is well-defined since $x \in X$ is a unit in $\mathbb{Z}[X]$ and it maps to a unit in $\mathbb{Z}[Y]$. Additionally, $L(f)$ is a unique commutative ring morphism that agrees with f on X since $\mathbb{Z}$ is initial in **CRing**. Let $1_X : X \to X$ be the identity morphism. Then $L(1_X)(\sum\limits_{x\in X} a_x x) = \sum\limits_{x\in X} a_x x$ and $L(1_X) = 1_{L(X)}$ for $a_x \in \mathbb{Z}$. Let $f : X \to Y$ and $g : Y \to Z$ be two Abelian group morphisms. Then $L(gf)(\sum\limits_{x\in X} a_x x) = \sum\limits_{x\in X} a_x g(f(x)) = L(g)(\sum\limits_{x\in X} a_x f(x)) = L(g)(L(f)(\sum\limits_{x\in X} a_x x))$ for $a_x \in \mathbb{Z}$. Thus, $L(gf) = L(g)L(f)$ and L is a functor indeed.

We want to show that L is a left adjoint to F. Let $f : A \to F(B)$ be an Abelian group morphism for $A \in \mathbf{Ob}(\mathbf{Ab})$ and $B \in \mathbf{Ob}(\mathbf{CRing})$. Define a natural transformation $\Phi_{A,B} : \mathbf{Hom}_{\mathbf{Ab}}(A, F(B)) \to \mathbf{Hom}_{\mathbf{CRing}}(L(A), B)$ by $\Phi_{A,B}(f)(x) = f(x)$ and extend $\mathbb{Z}$-linearly. By above, this is well-defined and the unique commutative ring morphism that agrees with f on X. Since units must map to units in a commutative ring morphism, every $h \in \mathbf{Hom}_{\mathbf{CRing}}(L(A), B)$ restricts to a morphism in $\mathbf{Hom}_{\mathbf{Ab}}(A, F(B))$. Thus, $\Phi_{A,B}$ is a bijection. We want to show that the bijection is functorial in A and B. Let $g : A' \to A$ be a morphism of Abelian groups. We want the diagram below to commute. Let $f \in \mathbf{Hom}_{\mathbf{Ab}}(A, F(B))$ as before. Then $\Phi_{A,B}(f) \circ L(g) : L(A') \to B$ extends the morphism $f \circ g : A' \to F(B)$. By definition, $\Phi_{A',B}(f \circ g)$ is also a morphism that extends $f \circ g$. The uniqueness in our choices of this morphism implies $\Phi_{A,B}(f) \circ L(g) = \Phi_{A',B}(f \circ g)$ and the diagram commutes. The argument for B is similar so the bijection is functorial in A and B. Therefore, L is a left adjoint to F.

$$\begin{tikzcd}
\mathbf{Hom}_{\mathbf{Ab}}(A, F(B)) \arrow[r, "\Phi_{A,B}"] \arrow[d, "-\circ g"'] & \mathbf{Hom}_{\mathbf{CRing}}(L(A), B) \arrow[d, "-\circ L(g)"] \\
\mathbf{Hom}_{\mathbf{Ab}}(A', F(B)) \arrow[r, "\Phi_{A',B}"'] & \mathbf{Hom}_{\mathbf{CRing}}(L(A'), B)
\end{tikzcd}$$

We will now show that left adjoints preserve initial objects. Let $L : \mathscr{C} \to \mathscr{D}$ and $R : \mathscr{D} \to \mathscr{C}$ be an adjoint pair. Let $A \in \mathbf{Ob}(\mathscr{C})$ be an initial object. Then $\mathbf{Hom}_{\mathscr{D}}(L(A), B) \cong \mathbf{Hom}_{\mathscr{C}}(A, R(B))$ for any $B \in \mathbf{Ob}(\mathscr{D})$. But A being an initial object in $\mathscr{C}$ implies $\mathbf{Hom}_{\mathscr{C}}(A, R(B))$ has only one element. We conclude that $\mathbf{Hom}_{\mathscr{D}}(L(A), B)$ has only one element and $L(A)$ is initial in $\mathscr{D}$. We want to show that F does not have a right adjoint. We note that $\mathbb{Z}$ is initial in **CRing**, but $F(\mathbb{Z}) \cong \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$ since ± 1 are the only units in $\mathbb{Z}$. The Abelian group $\mathbb{Z}/2\mathbb{Z}$ is not initial since $\mathbf{Hom}_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ has two elements, the trivial morphism and an isomorphism. Thus, F cannot have a right adjoint. $\square$

Definition 8.3 (Adjunction Pair in Units)**.** *Alternatively, F and G form an adjunction pair with F a left adjoint to G and G a right adjoint to F if there are natural transformations $\varepsilon : FG \to 1_{\mathscr{C}}$ and $\eta : 1_{\mathscr{D}} \to GF$ such that*

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F$$

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G$$

are the identity transformations on F and G respectively. We call η the unit and ε the counit.

To derive the unit and counit fro our earlier definition, note that an adjoint pair $F : \mathscr{C} \to \mathscr{D}$ and $G; \mathscr{D} \to \mathscr{C}$ defines a natural transformation of functors $\mathbf{Hom}_{\mathscr{D}}(F(-), -) \to \mathbf{Hom}_{\mathscr{C}}(-, G(-))$. For an object X in $\mathscr{C}$, we obtain a morphism $\mathbf{Hom}_{\mathscr{D}}(F(X), F(X)) \to \mathbf{Hom}_{\mathscr{C}}(X, G(F(X)))$ that sends $\mathbf{id}_{F(X)}$ to a morphism $X \to G(F(X))$. There is a similar setup for an object $Y \in \mathscr{D}$. These adjunction maps are functorial in X and Y so we obtain the unit and counit described above.

Definition 8.4 (Faithful, Full)**.** *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor of locally small categories. Define the set map $\varphi_{X,Y} : \mathbf{Hom}_{\mathscr{C}}(X, Y) \to \mathbf{Hom}_{\mathscr{D}}(F(X), F(Y))$ to be $\varphi_{X,Y}(f) = F(f)$ for any two objects X and Y of $\mathscr{C}$. The functor F is faithful if $\varphi_{X,Y}$ is injective for each pair of objects. The functor F is full if $\varphi_{X,Y}$ is surjective for each pair of objects.*

Problem 8.5 (Fall 2018 - Problem 7)**.** *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor with a right adjoint G. Show that F is fully faithful if and only if the unit of the adjunction $\eta : \mathbf{id}_{\mathscr{C}} \to GF$ is an isomorphism.*

Proof. Let $\varepsilon : FG \to 1_{\mathscr{D}}$ be the counit of the adjunction.

($\Rightarrow$)

Assume F is fully faithful. We will show that $\eta_Y : Y \to GF(Y)$ is an isomorphism. Let $f, g : X \to Y$ be morphisms in $\mathscr{C}$ such that $\eta_Y \circ f = \eta_Y \circ g$. By the adjunction, $\mathbf{Hom}_{\mathscr{C}}(X, GF(Y)) \cong \mathbf{Hom}_{\mathscr{D}}(F(X), F(Y))$ so $\eta_Y \circ f$ and $\eta_Y \circ g$ maps to the same morphism $h : F(X) \to F(Y)$. Since F is fully faithful, $F_{X,Y} : \mathbf{Hom}_{\mathscr{C}}(X, Y) \to \mathbf{Hom})_{\mathscr{D}}(F(X), F(Y))$. Thus, $f = g$, and η_Y is left-cancellative. Since F is full, we have $h : GF(X) \to X$ such that $F(h) = \varepsilon_{F(X)}$ for each $X \in \mathbf{Ob}(\mathscr{C})$. Then

$$\varepsilon_{F(X)} \circ F(\eta_X \circ h) = (\varepsilon_{F(X)} \circ F(\eta_X)) \circ F(h) = F(h) = \varepsilon_{F(X)} = \varepsilon_{F(X)} \circ F(1_X)$$

for all $X \in \mathbf{Ob}(\mathscr{C})$. Note that F is faithful so $\eta_X \circ h = 1_X$ and η_X is right cancellative. We conclude that η is an isomorphism.

($\Leftarrow$)

Assume η is an isomorphism. Let $f \in \mathbf{Hom}_{\mathscr{C}}(X, Y)$. Since η_Y is an isomorphism, $\eta_Y \circ -$ is a natural isomorphism $\mathbf{Hom}_{\mathscr{C}}(X, Y) \cong \mathbf{Hom}_{\mathscr{C}}(X, GF(Y))$. Via the adjunction, $\varepsilon_{F(X)} \circ F(\eta_Y \circ f) = \varepsilon_{F(Y)} \circ F(\eta_Y) \circ F(f) = F(f)$. As a result, $\mathbf{Hom}_{\mathscr{C}}(X, Y) \cong \mathbf{Hom}_{\mathscr{C}}(X, GF(Y)) \cong \mathbf{Hom}_{\mathscr{D}}(F(X), F(Y))$ via $F_{X,Y}$ and F is fully faithful.

$$\begin{tikzcd}
X \arrow[d, "f"'] \arrow[dr, "\eta_Y \circ f"] & \\
Y \arrow[r, "\eta_Y"'] & GF(Y)
\end{tikzcd}$$

$$\begin{tikzcd}
& F(X) \arrow[d, "F(\eta_Y \circ f)"'] \arrow[dr, bend left, "F(f)"] & \\
F(Y) \arrow[r, "F(\eta_Y)"] & FGF(Y) \arrow[r, "\varepsilon_{F(Y)}"] & F(Y)
\end{tikzcd}$$

$\square$

9 MATH 210A Discussion 9, November 30, 2021

Definition 9.1 (Monomorphism, Epimorphism)**.** *A monomorphism is a left-cancellative morphism. In other words, f is a monomorphism if, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$ for all suitable morphisms g_1 and g_2.*

An epimorphism is a right-cancellative morphism. In this case, g is an epimorphism if, $f_1 \circ g = f_2 \circ g$ implies $f_1 = f_2$ for all suitable morphisms f_1 and f_2.

Monomorphisms are the categorical analog of injective functions while epimorphisms are the categorical analog of surjective functions.

Problem 9.2 (Fall 2015 - Problem 1)**.** *Show that the inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism in the category of rings with multiplicative identity.*

Proof. We want to show that $f : \mathbb{Z} \to \mathbb{Q}$ is right cancellative. Let $g, h : \mathbb{Q} \to R$ be ring homomorphisms such that $gf = hf$ for R a ring with identity. For $a, b \in \mathbb{Z}$ we have $g(\frac{a}{b}) = g(a)g(b^{-1}) = g(a) = g(b)^{-1} = h(a)h(b)^{-1} = h(\frac{a}{b})$ since $g(a) = g(f(a)) = h(f(a)) = h(a)$ for all $a \in \mathbb{Z}$. We conclude $g = h$ and f is an epimorphism. $\square$

Problem 9.3 (Fall 2015 - Problem 1)**.** *Let $\mathscr{C}$ be a category. A morphism $f : A \to B$ in $\mathscr{C}$ is called an epimorphism if for any two morphisms $g, h : B \to X$ in $\mathscr{C}$, $g \circ f = h \circ f$ implies $g = h$. Let* **ALG** *be the category of $\mathbb{Z}$-algebras, and let* **MOD** *be the category of $\mathbb{Z}$-modules.*

(a) *Prove that in* **MOD***, $f : M \to N$ is an epimorphism if and only if f is a surjection.*

(b) *In* **ALG***, does the equivalence of epimorphism and surjection hold? If yes, prove the equivalence, and if no, give a counterexample (as simple as possible).*

Proof. (a) ($\Rightarrow$)

We will prove the contrapositive. Assume that $f : M \to N$ is not surjective. Then $\mathbf{im}(f) \subseteq N$ is a proper $\mathbb{Z}$-submodule. We define $\pi : N \to N/\sqsupset\gtrdot(f)$ the canonical projection and $g : N \to N/\mathbf{im}(f)$ the zero $\mathbb{Z}$-module homomorphism. Then gf and πf are zero maps so $gf = \pi f$. Let $n \in N$ such that $n \notin \mathbf{im}(f)$. Then $g \neq \pi$ since $g(n) = 0 + \mathbf{im}(f)$ while $g(n) = n + \mathbf{im}(f) \neq 0 + \mathbf{im}(f)$. We conclude that f is not an epimorphism.

($\Leftarrow$) Let $f : M \to N$ be surjective. Let $g, h : N \to P$ be $\mathbb{Z}$-module homomorphisms such that $gf = hf$. Let $n \in N$, then $n = f(m)$ for some $m \in M$. As a result, $g(n) = g(f(m)) = h(f(m)) = h(n)$ so $g = h$. We conclude that f is right-cancellative and f is an epimorphism.

(b) Let $i : \mathbb{Z} \to \mathbb{Q}$ be the canonical inclusion morphism of $\mathbb{Z}$-algebras. By problem 1 in Fall 2015, the morphism is a non-surjective epimorphism.

□

In more recent years, there have been problems about Abelian categories. The prototypical example of an Abelian category is R-Mod for a ring R. If the finite product and finite coproduct constructions are isomorphic in a category, we refer to the operation as a direct sum. As an example, see the direct sum in **Ab**.

Definition 9.4 (Zero Object)**.** *An object that is both initial and terminal is a zero object.*

Definition 9.5 (Additive Category)**.** *An additive category is a category admitting a zero object, any two pairs of objects admits a direct sum, and every Hom set has an Abelian group structure.*

Definition 9.6 (Kernel, Cokernel)**.** *The kernel of a morphism $f : X \to Y$ is an object* $\ker(f)$ *together with a morphism $i : \ker(f) \to X$ that satisfies the following universal property. If there is a morphism $g_X : Z \to X$ such that $f \circ g = 0_{Z,Y}$, then there is a unique morphism $u : Z \to \ker(f)$ such that $g = i \circ u$.*

$$\begin{tikzcd}
 & X \arrow[dr, "f"] & \\
 & \ker(f) \arrow[u, "i"] \arrow[r, "0"] & Y \\
Z \arrow[uur, bend left, "g"] \arrow[ur, dashed, "u"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

The dual notion (where we flip the arrows) is a cokernel of $f : X \to Y$ denoted $\mathbf{coker}(f)$*.*

In a category like R-mod, the categorical kernel of a morphism $f : X \to Y$ is an equivalent notion to that of our standard element-wise kernel. Further, the cokernel can be thought of as $Y/\mathbf{im}(f)$.

Definition 9.7 (Abelian Category)**.** *An Abelian category is an additive category in which each morphism has a kernel and cokernel and, for each $f : X \to Y$, the canonical morphism* $\mathbf{coker}(\ker(f)) \to \ker(\mathbf{coker}(f))$ *is an isomorphism.*

The purpose of Abelian categories is that it is the most general setting in which we can discuss exact sequences. Homological algebra is the study of Abelian categories.

Definition 9.8 (Projective)**.** *An object P is projective if for any epimorphism $e : E \to X$ and morphism $f : P \to X$, there is a morphism $g : P \to E$ such that $e \circ g = f$.*

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dashed, "g"'] \\ E \arrow[r, two heads, "e"] & X \end{tikzcd}$$

Definition 9.9 (Injective)**.** *An object P is injective if for any monomorphism $m : X \to Y$ and morphism $g : X \to Q$, there is a morphism $h : Y \to Q$ such that $h \circ m = g$.*

$$\begin{tikzcd} X \arrow[r, hook, "m"] \arrow[d, "g"'] & Y \arrow[dl, dashed, "h"] \\ Q & \end{tikzcd}$$

Example 9.10. *In the category of Abelian groups, the projective objects are free Abelian groups. The injective objects in the category of Abelian groups are necessarily divisible. Assuming the axiom of choice, every divisible group is injective.*

Problem 9.11 (Spring 2019 - Problem 10)**.** *Let $\mathscr{C}$ be an Abelian category. Prove the following are equivalent:*

1. *Every object of $\mathscr{C}$ is projective.*
2. *Every object of $\mathscr{C}$ is injective.*

Proof. 1. (1) $\Rightarrow$ (2):

Assume that every object is projective. Let $m : X \to Y$ be a monomorphism for which there is a morphism $g : X \to Q$. We can build the short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{q} C \longrightarrow 0$$

where $C = \mathbf{coker}(m)$. We have the diagram

$$\begin{tikzcd} & C \arrow[d, "1_C"] \arrow[dl, dashed] \\ Y \arrow[r, two heads, "q"'] & C \end{tikzcd}$$

The morphism $C \to Y$ guaranteed by C projective is a splitting. In an Abelian category left and right split are equivalent so there is a morphism $s : Y \to X$ such that $s \circ m = 1_X$. Define $h = g \circ s$ and

$$h \circ m = (g \circ s) \circ m = g \circ (s \circ m) = g$$

Thus Q is injective.

2. (2) $\Rightarrow$ (1):

   Make a similar argument since a short exact sequence will split with an injective first entry.

$\square$

10 MATH 210B Discussion 1, January 6, 2022

11 MATH 210B Discussion 2, January 13, 2022

12 MATH 210B Discussion 3, January 20, 2022

13 MATH 210B Discussion 4, January 27, 2022

14 MATH 210B Discussion 5, February 3, 2022

15 MATH 210B Discussion 6, February 10, 2022

16 MATH 210B Discussion 7, February 17, 2022

17 MATH 210B Discussion 8, February 24, 2022

18 MATH 210B Discussion 9, March 3, 2022

19 MATH 210B Discussion 10, March 10, 2022

Homework Problems

1 MATH 210A Homework 1

Problem 1.1. *Let $a_1, a_2, \cdots, a_n$ be elements of a group G. Define the product of the a_i's by induction: $a_1 a_2 \cdots a_n = (a_1 a_2 \cdots a_{n-1}) a_n$.*

(a) Prove that $a_1 a_2 \cdots a_n b_1 b_2 \cdots b_m = (a_1 a_2 \cdots a_n)(b_1 b_2 \cdots b_m)$.

(b) Prove that $a_1 a_2 \cdots a_n$ is equal to the product of the a_i's with the parentheses inserted arbitrarily.

(c) Prove that the number of different parenthesizations of $a_1 a_2 \cdots a_n$ is equal to the Catalan number $C_{n-1} = \frac{1}{n} \times \binom{2n-2}{n-1}$.

Proof. (a) Consider arbitrary elements $a_1, a_2, \cdots, a_n, b_1, \cdots, b_m \in G$. We prove the statement by mathematical induction on m.

The base case is when $m = 1$, then we have $a_1 a_2 \cdots a_n b_1 = (a_1 a_2 \cdots a_n)(b_1)$ by the definition of products stated in the problem.

Suppose that the statement is true when $m = k$ for some $k \geq 1$. Then by the inductive hypothesis, $a_1 a_2 \cdots a_n b_1 b_2 \cdots b_k = (a_1 a_2 \cdots a_n)(b_1 b_2 \cdots b_k)$. Now we consider the case where $m = k + 1$. Note that we have

$$\begin{aligned} a_1 a_2 \cdots a_n b_1 b_2 \cdots b_{k+1} &= (a_1 a_2 \cdots a_n b_1 b_2 \cdots b_k) b_{k+1} \text{ by definition of product} \\ &= ((a_1 a_2 \cdots a_n)(b_1 b_2 \cdots b_k)) b_{k+1} \text{ by induction assumption} \\ &= (a_1 a_2 \cdots a_n)((b_1 b_2 \cdots b_k) b_{k+1}) \text{ by associativity in } G \\ &= (a_1 a_2 \cdots a_n)(b_1 b_2 \cdots b_k b_{k+1}) \text{ by definition of product} \end{aligned}$$

Therefore, the induction holds, and $a_1 \cdots a_n b_1 \cdots b_m = (a_1 \cdots a_n)(b_1 \cdots b_m)$ for arbitrary $n, m \geq 1$.

(b) Take arbitrary $n \geq 1$.

Notice that regardless how parentheses are inserted on $a_1a_2\cdots a_n$, the result would always be equivalent to $(a_1\cdots a_{k_1})(a_{k_1+1}\cdots a_{k_2+1})\cdots(a_{k_m+1}\cdots a_n)$ for arbitrary $m\geq 1$. [1]

It suffices to show that $a_1a_2\cdots a_n = (a_1\cdots a_{k_1})(a_{k_1+1}\cdots a_{k_2+1})\cdots(a_{k_m+1}\cdots a_n)$. We proceed by induction on m.

When $m=1$, then we have $a_1a_2\cdots a_n = (a_1\cdots a_{k_1})(a_{k_1+1}\cdots a_n)$, which is true according to part (a).

Suppose the statement is true for some $m=b\geq 1$, then we want to prove the case for $m=b+1$. Consider $(a_1\cdots a_{k_1})(a_{k_1+1}\cdots a_{k_2+1})\cdots(a_{k_{b+1}+1}\cdots a_n)$. Since the inductive assumption holds for arbitrary n, then by part (a) we have

$$\begin{aligned}(a_1\cdots a_{k_1})(a_{k_1+1}\cdots a_{k_2+1})\cdots(a_{k_{b+1}+1}\cdots a_n) &= (a_1\cdots a_{k_{b+1}})(a_{k_{b+1}+1}\cdots a_n)\\ &= a_1\cdots a_n\end{aligned}$$

Therefore, the induction process holds, and that concludes the proof.

(c) We first show that the number we want is exactly $\binom{2n-2}{n-1}-\binom{2n-2}{n}$.

Notice that with the definition of product given in the problem, a multiplication with n terms can be considered as an operation with $n-1$ pair of parentheses. [2]

Therefore, the problem can be simplified as finding the number of possible valid $n-1$ parentheses combination. To list $n-1$ pairs of parentheses, there are $\binom{2n-2}{n-1}$ possible combinations. It suffices to show that among them there are $\binom{2n-2}{n}$ invalid combinations. Observe that an invalid representation is at some point having more right parentheses compared to left parentheses. Name the position that this phenomenon first happens as position k. That means before that point, there should be equally many left parentheses and right parentheses.

Let us interchange the left and right parentheses for the first k positions, then the list have $n-2$ right parenthesis and n left parenthesis. Moreover, for a list that consists of $n-2$ right parenthesis and n left parenthesis, there must be at some point that the number of left parenthesis first exceeds the number of right parenthesis. Notice that this is a correspondence (bijection) with the case discussed above, because if we denote i as the first position where that phenomenon happens, then interchanging the left and right parentheses would make the list consists of $n-1$ left and right parenthesis respectively. Therefore, the number of illegal representations is the number of representations with $n-2$ right parenthesis and n left parenthesis, which is $\binom{2n-2}{n}$.

[1] Note that If there are parentheses over these parentheses, then by group associativity, they can be transformed into this form as well.

[2] The problem requires the following simplification: an k-term multiplication should always be considered as a nested set of $k-1$ parenthesis. This is also evident from the definition of product given above. From mathematical induction, we would know there is always $n-1$ times multiplication for an n-term product. Hence, there are $n-1$ pair of parentheses.

Therefore, the number of valid representations is $\binom{2n-2}{n-1} - \binom{2n-2}{n}$.

We now show that $\binom{2n-2}{n-1} - \binom{2n-2}{n}$ is equivalent to the Catalan number $C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$. Notice that

$$\begin{aligned}\binom{2n-2}{n} &= \frac{(2n-2)!}{(n-2)!n!} \\ &= \frac{n-1}{n} \times \frac{(2n-2)!}{(n-1)!(n-1)!} \\ &= \frac{n-1}{n}\binom{2n-2}{n-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}\binom{2n-2}{n-1} - \binom{2n-2}{n} &= \binom{2n-2}{n-1} - \frac{n-1}{n}\binom{2n-2}{n-1} \\ &= \frac{1}{n}\binom{2n-2}{n-1}.\end{aligned}$$

This concludes the proof.

□

Problem 1.2. *(a) Prove that for every integer $n > 0$, the set of all complex n-th roots of unity is a group with respect to complex multiplication. Show that this group is cyclic.*

(b) Prove that if G is a cyclic group of order n and k divides n, then G contains exactly one subgroup of order k.

Proof. (a) Take arbitrary integer $n > 0$. The set of complex root of unity is exactly $S = \{e^{\frac{2\pi k i}{n}} \mid k = 0, \cdots, n-1\}$. We first show that S is a group with respect to complex multiplication.

- The operation is closed in S. Indeed, $e^{\frac{2\pi a i}{n}} \cdot e^{\frac{2\pi b i}{n}} = e^{\frac{2\pi(a+b)i}{n}} \in S$ since $e^{2\pi i} = 1$ and Euclidean algorithm holds.
- The operation is associative. For arbitrary elements $x = e^{\frac{2\pi a i}{n}}$, $y = e^{\frac{2\pi b i}{n}}$, $z = e^{\frac{2\pi c i}{n}}$ in S, we have

$$\begin{aligned}(xy)z &= (e^{\frac{2\pi ai}{n}} \cdot e^{\frac{2\pi bi}{n}}) \cdot e^{\frac{2\pi ci}{n}} \\ &= e^{\frac{2\pi(a+b)i}{n}} \cdot e^{\frac{2\pi ci}{n}} \\ &= e^{\frac{2\pi(a+b+c)i}{n}} \\ &= e^{\frac{2\pi ai}{n}} \cdot e^{\frac{2\pi(b+c)i}{n}} \\ &= e^{\frac{2\pi ai}{n}} \cdot (e^{\frac{2\pi bi}{n}} \cdot e^{\frac{2\pi ci}{n}}) \\ &= x(yz)\end{aligned}$$

- There exists a unit $e = 1 = e^{\frac{2\pi\cdot 0\cdot i}{n}} \in S$. Indeed, $1 \cdot e^{\frac{2\pi ki}{n}} = e^{\frac{2\pi ki}{n}} \cdot 1 = e^{\frac{2\pi ki}{n}}$ for every element in S.
- For arbitrary element $e^{\frac{2\pi ki}{n}} \in S$, there exists an element $e^{\frac{2\pi(n-k)i}{n}} \in S$ such that $e^{\frac{2\pi ki}{n}} \cdot e^{\frac{2\pi(n-k)i}{n}} = 1 \in S$. Hence, an inverse always exists.

Collecting properties above, we know S is a group over complex multiplication indeed. Finally, $e^{\frac{2\pi i}{n}} \in S$ is clearly a generator. Therefore S is cyclic by definition.

(b) Note that when saying $k \mid n$ we also assume $k > 0$. We first show that such subgroup exists.

Denote $G = \langle g \rangle$ for some $g \in G$. In particular, g has order n. i.e. $g^n = e$. Since $k \mid n$, then $m = \frac{n}{k} \in \mathbb{N}$ and is nonzero.

I claim that $\langle g^m \rangle$ is a desired subgroup of G. First of all, notice that $\langle g^m \rangle \subseteq G$. We can also show that $\langle g^m \rangle$ contains k elements, and the elements are exactly $g^m, (g^m)^2, \cdots, (g^m)^k = g^n = e$. Indeed, any higher power can be reduced to these elements since $g^n = e$. Then it suffices to show that any two elements in this list are distinct. Suppose, that $(g^m)^a = (g^m)^b$ for some $1 \le a, b \le k$. Then $(g^m)^a \cdot (g^m)^{k-a} = (g^m)^b \cdot (g^m)^{k-a}$, thus $e = (g^m)^k = (g^m)^{k+b-a}$. Therefore, $n \mid m(k+b-a)$ and so $n \mid m(b-a)$. Since $n = km$, then $k \mid (b-a)$. But since $1 \le a, b \le k$, then we must have $b = a$. Therefore, the set $\langle g^m \rangle$ contains k elements indeed.

All there is left to show is that $\langle g^m \rangle$ is a group, with operation carried over from G.

- Note that the operation $\cdot$ on $\langle g^m \rangle$ is closed: we always have $(g^m)^a \cdot (g^m)^b = (g^m)^{a+b} \in \langle g^m \rangle$.
- Also, $\langle g^m \rangle$ is associative. For any elements $x = (g^m)^a, y = (g^m)^b, z = (g^m)^c \in \langle g^m \rangle$, in $\langle g^m \rangle$ we have

$$
\begin{aligned}
(xy)z &= ((g^m)^a \cdot (g^m)^b) \cdot (g^m)^c \\
&= (g^m)^{a+b} \cdot (g^m)^c \\
&= (g^m)^{a+b+c} \\
&= (g^m)^a \cdot (g^m)^{b+c} \\
&= (g^m)^a \cdot ((g^m)^b \cdot (g^m)^c) \\
&= x(yz)
\end{aligned}
$$

- Obviously $e = e_G \in \langle g^m \rangle$ since $g^n = (g^m)^k = e$ and $ex = xe = e$ for all elements.
- For arbitrary element $x = (g^m)^a \in \langle g^m \rangle$, we have an inverse $y = (g^m)^{k-a}$ obviously with $xy = (g^m)^k = g^n = e$ and $yx = e$ as well.

Therefore, we have verified that $\langle g^m \rangle$ is a group, and since $\langle g^m \rangle \subseteq G$ and has exactly k elements, then there is a subgroup of G of order k.

We now show that the subgroup of order k is unique. Consider an arbitrary subgroup $H \subseteq G$ with order k. We claim that H must be cyclic. If $k = 1$, then $H = \{e\}$ is obviously cyclic. Consider $k > 1$. Observe that elements of H must be of form g^i for some i between 1 and n. Since H has more than one element, and by well-ordering principle, then there exists some minimal $0 < a < n$ such that $g^a \in H$. Obviously $\langle g^a \rangle \subseteq H$. Moreover, consider arbitrary $g^i \in H$. Then we clearly have $i \geq a$. By Euclidean Algorithm, there is $i = aq+r$ for $0 \leq r < a$. Then $g^i = g^{aq} \cdot g^r$, and so $g^r = g^i \cdot (g^a)^{-q}$. Since $g^i \in H$ and $g^a \in H$, then $g^r \in H$. However, since $r < a$ and a is the minimal positive number such that $g^a \in H$, then $r = 0$. Hence, $g^i = (g^a)^q$. In particular, $g^i \in \langle g^a \rangle$, which means $H \subseteq \langle g^a \rangle$. Hence, $H = \langle g^a \rangle$ for some a.

Suppose there is a subgroup $H \subseteq G$ with order k. Followed from the argument above, there is some integer $0 < a \leq n$ such that $H = \langle g^a \rangle$. If $a = m$ then obviously H is identical to $\langle g^m \rangle$. Suppose $a < m$, then $(g^a)^k \neq e$ since $n = mk > ak$ is the smallest positive integer such that $g^n = e$. This is a contradiction since H must have order k. Suppose $a > m$ with $(g^a)^k = e$, then $mk \mid ak$, and so $m \mid a$. In particular, $H \subseteq \langle g^m \rangle$. However, since the two groups have the same size, then $H = \langle g^m \rangle$. Hence, the group $\langle g^m \rangle$ is the unique subgroup of G with order k. □

Problem 1.3. *(a) Show that if K and N are two finite subgroups of a group G of relatively prime orders, then $K \cap N = \{e\}$.*

(b) Show that if a group G has only finite number of subgroups, then G is finite.

Proof. (a) Suppose $|K| = p$ and $|N| = q$ such that $\gcd(p, q) = 1$. Observe that $e_K = e_G$ since the property is shared. Similarly $e_N = e_G = e_K$.

Obviously $\{e\} \subseteq K \cap N$. It suffices to show that $K \cap N \subseteq \{e\}$. Take arbitrary element $x \in K \cap N$. In particular, $x \in K$ and $x \in N$. Recall from a corollary of Lagrange Theorem that $x^p = e_K = e_G$ and $x^q = e_N = e_G$. By Bezout's Lemma, there exists integers a, b such that $ap + bq = 1$, hence we have $e = (x^p)^a = x^{ap} = x^{1-bq} = x \cdot (x^q)^{-b} = x \cdot e = x$. Therefore, $K \cap N \subseteq \{e\}$, and it follows that $K \cap N = \{e\}$.

(b) Consider a group G with finite number of subgroups. In particular, G must have finitely many cyclic subgroups, which are the groups of the form $\langle a \rangle$ where $a \in G$.

I claim that G is the union of all the cyclic subgroups. Obviously the union of all cyclic subgroups is a subset of G by definiton. On the other hand, an arbitrary element $a \in G$ is in the cyclic subgroup $\langle a \rangle \subseteq G$. Therefore, G is a subset of the union. Hence, G is the union of all the cyclic subgroups.

Consider arbitrary cyclic subgroup $\langle a \rangle \subseteq G$ for $a \in G$. I claim that $\langle a \rangle$ has finite order. Suppose that $\langle a \rangle$ has infinite order instead, then $a^n \neq a^m$ for any distinct $n, m > 0$, otherwise we have $a^{n-m} = e$ or $a^{m-n} = e$, a contradiction to the fact that $\langle a \rangle$ has infinite order. In particular, $\langle a \rangle$ has infinitely many subgroups, because for arbitrary $n > m > 0$, $a^m \notin \langle a^n \rangle$ and so $\langle a^n \rangle \neq \langle a^m \rangle$ but that would mean G also has infinitely many subgroups, contradiction. Therefore, $\langle a \rangle$ must have finite order.

Now, by using arguing through cardinality, we know the union of finitely many finite sets is finite. Therefore, G has to be finite.

□

Problem 1.4. *(a) Let G be a group of order n. Show that $a^n = e$ for all $a \in G$.*

(b) Prove that a group G is cyclic if and only if there is an element $a \in G$ with ord$(a) = |G|$.

(c) Show that every group of prime order is cyclic.

Proof. (a) This is exactly corollary **1.4.16**. For any finite group G with order n and subgroup H, we have $|H| \mid |G|$ by Lagrange Theorem. Furthermore, for any element $a \in G$, the order of a is the same as the order of $\langle a \rangle$ by remark **1.3.5**, and so the order of a divides the order of G. Hence, $|G| = \text{ord}(a) \cdot k$ for some $k \in \mathbb{Z}$. Hence, $a^n = (a^{\text{ord}(a)})^k = e^k = e \in G$.

(b) Suppose G is a cyclic group, then we have $G = \langle a \rangle$ for some $a \in G$. Recall that the order of element $a \in G$ is just the order of group $\langle a \rangle = G$. Therefore, this is always true.

Suppose there exists an element $a \in G$ such that the order of a is the order of G. Then $\langle a \rangle$ and G has the same order. However, $\langle a \rangle \subseteq G$ obviously, then $G = \langle a \rangle$. Hence, G is cyclic. [3]

[3] This direction only makes sense when G is finite. Consider the general linear group over $\mathbb{R}$ for $n > 1$, i.e. $\mathrm{GL}_n(\mathbb{R})$. Notice that this group has infinite order, and the identity matrix as an element also have infinite order. However, this group is not cyclic.

(c) This is exactly corollary 1.4.18. Consider arbitrary group G with order p prime. In particular, there exists some $g \in G$ such that $g \neq e$. By Lagrange's Theorem, the order of g divides the order of G. In particular, since G has prime order, then g either has order 1 or has order p. However, $g \neq e$, which means g cannot have order 1, and so g has order p. Moreover, observe that the order of $g \in G$ is the order of $\langle g \rangle$ as a subgroup of G. (The notation makes sense because G is a finite group.) Then $\langle g \rangle$ has G has the same (finite) order, but $\langle g \rangle$ is obviously a subgroup of G, then $G = \langle g \rangle$. In particular, G is cyclic.

□

Problem 1.5. *(a) Show that if $a^2 = e$ for all elements a of a group G, then G is Abelian.*

(b) Prove that if G is a finite group of even order, then G contains an element a such that $a^2 = e$ and $a \neq e$.

(c) Show that every subgroup of index 2 is normal.

Proof. (a) Take arbitrary element $a, b \in G$. It suffices to show that $ab = ba$.

Note that $abba = a(bb)a = aa = e$. Furthermore, $abab = (ab)^2 = e$. Therefore, $abba = abab$, and so $(ab)(ba) = (ab)(ab)$. By left cancellation, we have $ab = ba$. It is then obvious that G is Abelian.

(b) Let G be a finite group of even order. Suppose that there is no element $a \in G$ such that $a^2 = e$ and $a = e$. Then every element $a \neq e$ must have an inverse $b \neq a$. In particular, that means G must have odd order because we can group elements in pairs, i.e. an element and its inverse, except for the identity e, which is its own inverse. (The cardinality argument works because G is finite.) This is a contradiction to the fact that G has even order. Therefore, there must be some element $a \neq e$ such that $a^2 = e$.

(c) Consider a group G with subgroup H of index 2. It suffices to show that $gHg^{-1} = H$ for every $g \in G$.

Suppose $g \in H$. Obviously $gHg^{-1} \subseteq H$. On the other hand, for arbitrary $h \in H$, notice that $h = g(g^{-1}hg)g^{-1} \in gHg^{-1}$. Therefore we know $H = gHg^{-1}$.

Suppose $g \notin H$. Consider the left cosets of H in G. Since H is of index 2, then the left cosets are exactly H and gH, then since G is the union of all left cosets, $gH = G \backslash H$. In a similar fashion, we know $Hg = G \backslash H$ as well. Hence, $gH = Hg$, i.e. $H = gHg^{-1}$.

Therefore, $H = gHg^{-1}$ for all $g \in G$. Hence, H is normal.

□

Problem 1.6. *Find all groups (up to isomorphism) of order ≤ 5. What is the smallest order of a non-cyclic group?*

Proof. Obviously all groups of order 1 are isomorphic to the trivial group $\{e\}$.

By problem 4(c), groups of order 2, 3 and 5 are cyclic. Therefore, they are always isomorphic to $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/5\mathbb{Z}$ respectively (or just the cyclic groups of these orders).

By filling out multiplication table, we will see that groups of order 4 are either isomorphic to $\mathbb{Z}/4\mathbb{Z}$ or V_4, the Klein 4-group. We first have

	e	a	b	c
e	e	a	b	c
a	a			
b	b			
c	c			

Note that $a^2 = e$ or $a^2 = b$. ($a^2 = c$ is just isomorphic to the case where $a^2 = b$.) In the first case, we have

	e	a	b	c
e	e	a	b	c
a	a	e		
b	b			
c	c			

Note that $ab \neq b$ because $a \neq e$, then $ab = c$. Similarly, $ba = c$. Then we know $ac = ca = b$. Finally, we have either

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

or

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

Note that the first group is isomorphic to V_4, the Klein 4-group, and the second group is isomorphic to $\mathbb{Z}/4\mathbb{Z}$.

In the second case, we have

	e	a	b	c
e	e	a	b	c
a	a	b		
b	b			
c	c			

Then we know $ba = c$ and $ca = e$. Similarly $ab = c$ and $ac = e$. (Otherwise cancellation law gives us a contradiction.) Then we can fill out the following table.

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

This is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ as well.

Therefore, all groups of order 4 are isomorphic to either V_4 or $\mathbb{Z}/4\mathbb{Z}$.

Therefore, the following are all the groups with order not exceeding 5.

$\{e\}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, V_4 (Klein 4), $\mathbb{Z}/5\mathbb{Z}$.

In particular, only one of them is non-cyclic, which is V_4 (observed from above). Therefore, the smallest order of a non-cyclic group if 4. $\square$

Problem 1.7. *Find all subgroups of the symmetric group S_3. Determine normal subgroups.*

Proof. Note that S_3 is essentially the group of all permutations of a three-element set. Therefore, we can use permutation notations to denote all the subgroups. Also recall that the order of subgroup divides the order of S_3. Therefore, the order of subgroup must be $1, 2, 3, 6$.

Obviously S_3 and $\{e\}$ are subgroups as well.

Subgroups of order 2 must contain identity, then we can only have: $\{e, (1\ 2)\}$, $\{e, (1\ 3)\}$, $\{e, (2\ 3)\}$.

Subgroups of order 3 must involve actions on all elements because an action on two elements must have order 2 (i.e. $(1\ 2)(1\ 2) = e$), which is impossible as we have shown. Moreover, the composition of two non-identity elements must be the identity, so we have: $\{e, (1\ 2\ 3), (1\ 3\ 2)\}$.

Therefore, these are all subgroups of S_3.

Recall from problem 5(c) that every subgroup of index 2 is normal, therefore the subgroup $\{e, (1\ 2\ 3), (1\ 3\ 2)\}$ must be normal. Moreover, $\{e\}$ and S_3 are clearly normal by checking the definition.

We can check that $\{e, (1\ 2)\}$ is not a normal subgroup. Indeed, $(1\ 2\ 3)(1\ 2)(1\ 2\ 3)^{-1} = (1\ 2\ 3)(1\ 2)(1\ 3\ 2) = (3\ 2\ 1) \notin \{e, (1\ 2)\}$. With the same argument, we would know that $\{e, (1\ 3)\}$, $\{e, (2\ 3)\}$ are not normal as well. Therefore, the normal subgroups are exactly $S_3, \{e\}$ and $\{e, (1\ 2\ 3), (1\ 3\ 2)\}$. $\square$

Problem 1.8. *Let n be a natural number. Show that the map*

$$f : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}, f(a + \mathbb{Z}) = na + \mathbb{Z}$$

is a well defined homomorphism. Find $\ker(f)$ *and* $im(f)$*.*

Proof. We first show that f is well-defined. Take arbitrary $a \in \mathbb{Q}$. Take arbitrary $b \in \mathbb{Z}$. Note that $f((a+b)+\mathbb{Z}) = n(a+b)+\mathbb{Z} = na+nb+\mathbb{Z} = na+\mathbb{Z} = f(a+\mathbb{Z})$. Therefore, f is well-defined.

We now show that f is a homomorphism. Take arbitrary $x + \mathbb{Z}, y + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$. Then

$$\begin{aligned} f((x+\mathbb{Z}) + (y+\mathbb{Z})) &= f((x+y)+\mathbb{Z}) \\ &= n(x+y)+\mathbb{Z} \\ &= (nx+\mathbb{Z}) + (ny+\mathbb{Z}) \\ &= f(x) + f(y) \end{aligned}$$

Therefore, f is a homomorphism.

Suppose we have some $a \in \mathbb{Q}$ such that $f(a+\mathbb{Z}) = na+\mathbb{Z} = 0$. Then $na \in \mathbb{Z}$. In particular, since n is fixed, then a is the set of rational numbers such that $na \in \mathbb{Z}$. Hence, $\ker(f) = \{a + \mathbb{Z} : a \in \mathbb{Q} \text{ such that } na \in \mathbb{Z}\}$.

We claim that $\mathrm{im}(f) = \mathbb{Q}/\mathbb{Z}$. In particular, we show that f is surjective. Take arbitrary $a + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$. Then we have $f(\frac{a}{n} + \mathbb{Z}) = a + \mathbb{Z}$ with $\frac{a}{n} \in \mathbb{Q}$. Therefore, f is surjective, which means $\mathrm{im}(f) = \mathbb{Q}/\mathbb{Z}$.[4] □

Problem 1.9. *(a) Show that $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.*

(b) Prove that $n\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z}$.

Proof. (a) Let us define $f : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ by $[a]_6 \mapsto [a]_2 \times [a]_3$. Note that f is well-defined because $f(a+6b) = ([a+6b]_2, [a+6b]_3) = ([a]_2, [a]_3) = f(a)$.

We show that f is a homomorphism. Consider arbitrary $[x]_6, [y]_6 \in \mathbb{Z}/6\mathbb{Z}$, then

$$\begin{aligned} f([x]_6 + [y]_6) &= f([x+y]_6) \\ &= ([x+y]_2, [x+y]_3) \\ &= ([x]_2, [x]_3) + ([y]_2, [y]_3) \\ &= f([x]_6) + f([y]_6) \end{aligned}$$

[4] The answer actually depends on what we consider as "natural number". The answer above considers $0 \notin \mathbb{N}$. Suppose $0 \in \mathbb{N}$, then when we pick $n = 0$, then the map is just $f(a+\mathbb{Z}) = 0 + \mathbb{Z}$ for all $a \in \mathbb{Q}$. Hence, f is still a homomorphism, $\ker(f) = \{a + \mathbb{Z} \mid a \in \mathbb{Q}\}$, and $\mathrm{im}(f) = \{[0] + \mathbb{Z}\}$.

Hence, f is a homomorphism. We now show that f is a bijection.

We first show that f is surjective using brute-force. Observe that $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ has 6 elements: $([0]_2, [0]_3)$, $([0]_2, [1]_3)$, $([0]_2, [2]_3)$, $([1]_2, [0]_3)$, $([1]_2, [1]_3)$, $([1]_2, [2]_3)$. We have the following list:

$$f([0]_6) = ([0]_2, [0]_3)$$
$$f([1]_6) = ([1]_2, [1]_3)$$
$$f([2]_6) = ([0]_2, [2]_3)$$
$$f([3]_6) = ([1]_2, [0]_3)$$
$$f([4]_6) = ([0]_2, [1]_3)$$
$$f([5]_6) = ([1]_2, [2]_3)$$

Therefore, f is surjective indeed. We now show that f is injective. Consider $f([x]_6) = f([y]_6)$ for some x, y. Then $x \equiv y \pmod 2$ and $x \equiv y \pmod 3$. Therefore, $x - y$ is divisible by 2 and 3 respectively, which means $x - y$ is divisible by 6 since 2 and 3 are coprime. Hence, $x \equiv y \pmod 6$. Therefore, $[x]_6 = [y]_6$. Therefore, f is injective as well, so f is bijective. Therefore, since f is a bijective homomorphism, then f is an isomorphism. By definition, $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

(b) Define $f : n\mathbb{Z}/mn\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ by $[nx]_{mn} \mapsto [x]_m$ for arbitrary $x \in \mathbb{Z}$. We claim that this is a well-defined homomorphism.

We first show that f is well-defined. For arbitrary $x, y \in \mathbb{Z}$ with $[nx]_{mn} = [ny]_{mn}$, we have $f([nx]_{mn}) = [x]_m$ and $f([ny]_{mn}) = [y]_m$. Note that $(nx - ny) \equiv 0 \pmod{mn}$, and so $nx - ny = amn$ for some $a \in \mathbb{Z}$. By cancellation (since $n > 0$), $x - y = am$, hence $x \equiv y \pmod m$. Therefore, $[x]_m = [y]_m$. In particular, $f([nx]_{mn}) = f([ny]_{mn})$. Therefore f is well-defined.

Consider arbitrary $x, y \in \mathbb{Z}$. Then

$$\begin{aligned} f([nx]_{mn} + [ny]_{mn}) &= f([n(x+y)]_{mn}) \\ &= [x+y]_m \\ &= [x]_m + [y]_m \\ &= f([nx]_{mn}) + f([ny]_{mn}) \end{aligned}$$

Therefore f is a homomorphism.

We now show that f is a bijection. Take arbitrary $x \in \mathbb{Z}$ and corresponding element $[x]_m \in \mathbb{Z}/m\mathbb{Z}$. Note that $f([nx]_{mn}) = [x]_m$. Therefore, f is surjective. Suppose for some $x, y \in \mathbb{Z}$ we have $f([nx]_{mn}) = f([ny]_{mn})$, then $[x]_m = [y]_m$, which means $x - y \equiv 0 \pmod m$. Hence, $nx - ny \equiv 0 \pmod{mn}$. Therefore, $[nx]_{mn} = [ny]_{mn}$. Thus, f is injective as well. Therefore, f is a bijection.

Collecting properties from above, f is an isomorphism. Therefore $n\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z}$ by definition.

$\square$

Problem 1.10. *Let $f : G \to H$ be a surjective group homomorphism and let H' be a subgroup in H.*

(a) Show that $f^{-1}(H')$ is a subgroup of G.

(b) Prove that the assignment $H' \mapsto f^{-1}(H')$ yields a bijection between the set of all subgroups of H and the set of all subgroups of G that contain $\ker(f)$.

(c) Prove that a subgroup H' is normal in H if and only if $f^{-1}(H')$ is normal in G. Show that $G/f^{-1}(H') \cong H/H'$.

Proof. (a) By definition $f^{-1}(H')$ is the preimage of $H' \subseteq H$ in G, so $f^{-1}(H') \subseteq G$. It suffices to show that $f^{-1}(H')$ is a group with operation $\cdot_G$. Note that $f^{-1}(H')$ is associative since it is a subset of group G. Note that $e_G \in f^{-1}(H')$ because $f(e_G) = e_H \in H'$ according to the definition of a group homomorphism. Finally, for arbitrary $g \in f^{-1}(H')$, we know $f(g^{-1}) = f(g)^{-1}$ by group homomorphism. However, since $g \in f^{-1}(H')$, then $f(g) \in H'$, and so $f(g)^{-1} \in H'$. Hence, $g^{-1} \in f^{-1}(H')$. Thus, $f^{-1}(H') \subseteq G$ is a subgroup according to the properties above.

(b) For arbitrary subgroup H' of H, we prove that $f^{-1}(H')$ contains $\ker(f)$. Indeed, take arbitrary $g \in \ker(f)$, then $f(g) = e_H \in H'$. Hence, $f^{-1}(H')$ must contain $\ker(f)$.

We claim that the assignment $\varphi : H' \mapsto f^{-1}(H')$ has an inverse assignment $\theta : K \mapsto f(K)$ for subgroups K of G that contain $\ker(f)$.

We first show that $\theta \circ \varphi$ is the identity map for subgroup H' of H. Clearly $\theta\varphi(H') \subseteq H'$. On the other hand, for arbitrary element $h \in H'$, since f is surjective, then there is $f(h') = h$. Furthermore, since $f(h') = h$, then $h' \in f^{-1}(H')$. Therefore, $h \in \theta\varphi(H')$, and therefore $\theta\varphi(H') = H'$.

On the other hand, $\varphi \circ \theta$ is the identity map for subgroup G' of G that contains $\ker(f)$. Indeed, note that $G \subseteq \varphi\theta(G')$. On the other hand, pick arbitrary $g \in \varphi\theta(G')$. Then we have $f(g) = f(g')$ for some $g' \in G'$. However, that means $f(gg'^{-1}) = e$. In particular, $gg'^{-1} \in \ker(f)$. However, since $g' \in G'$ and $gg'^{-1} \in \ker(f) \subseteq G'$, then $g \in G'$ as well. Hence, $\varphi \circ \theta$ is the identity map for such subgroups indeed.

Therefore, the assignment $H' \mapsto f^{-1}(H')$ between subgroups of H and subgroups of G that contain $\ker(f)$ is invertible with an inverse as specified above. Therefore, such assignment must be a bijection.

(c) Let $H' \lhd H$ be a normal subgroup. Recall that $f^{-1}(H')$ is a subgroup of G. Now take arbitrary $g \in G$. Consider arbitrary $h \in f^{-1}(H')$. Then $f(ghg^{-1}) = f(g)f(h)f(g)^{-1}$. Since H' is a normal subgroup of H, then $f(ghg^{-1}) \in H'$. Hence, $ghg^{-1} \in f^{-1}(H')$, so $gf^{-1}(H')g^{-1} \subseteq f^{-1}(H')$. On the other hand, for arbitrary $h \in f^{-1}(H')$, we have $h = g(g^{-1}hg)g^{-1}$ where $g^{-1}hg \in f^{-1}(H')$ for the exact reason above, then $h \in gf^{-1}(H')g^{-1}$, hence $f^{-1}(H') = gf^{-1}(H')g^{-1}$, so $f^{-1}(H')$ is a normal subgroup of G by definition.

Let $f^{-1}(H') \lhd G$ for some subgroup $H' \subseteq H$. We show that $H' \lhd H$. Take arbitrary $h \in H$. We show that $hH'h^{-1} = H'$. Take arbitrary $h' \in H'$. We show that $hh'h^{-1} \in H'$. Since f is surjective, then there exists $f(g) = h$ and $f(g') = h'$ for $g \in G$ and $g' \in f^{-1}(H')$. Then since $f^{-1}(H') \lhd G$, then $gg'g^{-1} \in f^{-1}(H')$. Hence, $hh'h^{-1} \in H'$. In particular, we have $hH'h^{-1} \subseteq H'$. On the other hand, take arbitrary $h' \in H'$. We have $h' = h(h^{-1}h'h)h^{-1}$, and $h^{-1}h'h \in H'$ for the exact reason as above. Therefore $h' \in hH'h^{-1}$. Hence, $H' = hH'h^{-1}$, and so $H' \lhd H$.

Finally, we show that $G/f^{-1}(H') \cong H/H'$ as groups (which means we assume $f^{-1}(H') \lhd G$ and $H' \lhd H$, otherwise if we talk about set isomorphisms then it is just an obvious bijection by $gf^{-1}(H') \mapsto f(g)H'$ since f is surjective). Define $\pi : H \to H/H'$ as the canonical homomorphism which is surjective. Since the composition of two surjective homomorphism is a surjective homomorphism, we know $\pi \circ f$ is a surjective homomorphism. In particular, we claim that the kernel of $\pi \circ f$ is $f^{-1}(H')$. Indeed, for arbitrary $x \in f^{-1}(H')$, we have $f(x) \in H'$ and so $\pi \circ f(x) \in H'$. On the other hand, for arbitrary element x in $\ker(\pi \circ f)$, we have $\pi \circ f(x) \in H'$, which means $f(x) \in H'$ by the definition of canonical homomorphism. Therefore, $\ker(\pi \circ f) = f^{-1}(H')$ as a subgroup. Therefore, by the first isomorphism theorem, we have $G/f^{-1}(H') \cong H/H'$ as desired.

□

2 MATH 210A Homework 2

Problem 2.1. *Let G be a group and $a, b \in G$.*

(a) Prove that $a^n \cdot a^m = a^{n+m}$ and $(a^n)^m = a^{nm}$ for all $n, m \in \mathbb{Z}$.

(b) Prove that $ord(a^n) = \frac{ord(a)}{\gcd(n, ord(a))}$ if $ord(a) < \infty$.

(c) Prove that $ord(ab) = ord(a) \cdot ord(b)$ if a and b commute and $\gcd(ord(a), ord(b)) = 1$.

Proof. (a) We first prove that $a^n \cdot a^m = a^{n+m}$.

Suppose $n = 0$. Then $a^n \cdot a^m = e \cdot a^m = a^m = a^{n+m}$ as required. Similar result holds when $m = 0$. Suppose $n > 0, m > 0$, then $a^n \cdot a^m = (a \cdot a \cdot \cdots \cdot a) \cdot (a \cdot a \cdot \cdots \cdot a)$ where the first pair of parentheses contains n terms and the second pair contains m terms. Recall from homework 1 that this is equivalent to $(n+m)$-term operation a^{n+m}, since the parentheses can be inserted arbitrarily. Suppose $n < 0, m < 0$, then $a^n = (a^{-1})^{-n}$ and $a^m = (a^{-1})^{-m}$. This is exactly the same as the previous case with a substituted by a^{-1}, which makes the result $a^n \cdot a^m = (a^{-1})^{-n} \cdot (a^{-1})^{-m} = (a^{-1})^{-(n+m)} = a^{n+m}$. Therefore, the operation also holds. Suppose $n > 0$ but $m < 0$. Therefore, $a^n \cdot a^m = a^n \cdot (a^{-1})^{-m} = (a \cdot a \cdot \cdots \cdot a) \cdot (a^{-1} \cdot a^{-1} \cdot \cdots \cdot a^{-1})$ which is a product of n-term and $(-m)$-term operation. By associativity, $\min(n, -m)$ terms get cancelled. What is left is either $a^{n-(-m)} = a^{n+m}$ or $(a^{-1})^{-m-n} = a^{m+n}$. Therefore, the operation is well-defined and $a^n \cdot a^m = a^{m+n}$. In exactly the same manner, we know $a^n \cdot a^m = a^{m+n}$ when $n < 0$ and $m > 0$. That concludes the proof.

We now show that $(a^n)^m = a^{nm}$. Suppose $n = 0$, then $(a^n)^m = e^m = e = a^0$. Suppose $m = 0$, then $(a^n)^m = e = a^0$. If $n > 0$ and $m > 0$, then $(a^n)^m$ is a m-term operation on a^n, which has n terms of a in it. Therefore, there are eventually $n \times m$ terms multiplying, which means the result is a^{nm}. If $n < 0$ and $m < 0$, then

$$\begin{aligned}
(a^n)^m &= (((a^{-1})^{-n})^{-1})^{-m} \\
&= ((a^{-1}a^{-1} \cdots a^{-1})^{-1})^{-m}
\end{aligned}$$

Since parentheses can be arbitrarily asserted, we write the term $(a^{-1}a^{-1} \cdots a^{-1})^{-1}$ as $((a^{-1} \cdots a^{-1})(a^{-1}))^{-1}$, then by definition this is equivalent to $(a^{-1})^{-1}(a^{-1} \cdots a^{-1})^{-1} = a(a^{-1} \cdots a^{-1})^{-1}$, then proceeding by induction we have an n-term product of a‘s, i.e. $(a^{-1}a^{-1} \cdots a^{-1})^{-1} = aa \cdots a$. Therefore, $(a^n)^m$ is the $(-m)$-term product of $(-n)$-term products of a's, which is just (mn)-term product of a's, which means $(a^n)^m = a^{nm}$.

If $n > 0$ and $m < 0$, then $(a^n)^m = ((a^n)^{-1})^{-m} = (a^{-1} \cdots a^{-1})^{-m}$ by similar induction, which means $(a^n)^m$ is an $(-m)$-term product of n-term products of a^{-1}, which is just a $(-mn)$-term product of a^{-1}. In particular, $(a^n)^m = (a^{-1})^{-nm}$. Note that by definition this is equivalent to $a^{-(-nm)} = a^{nm}$. Therefore, the operation also holds. The case where $n < 0$ and $m > 0$ follows in a similar manner.

(b) Take arbitrary $a \in G$ such that $\text{ord}(a) < \infty$. In particular, there exists a smallest positive integer k such that $a^k = e \in G$. Take $d = \gcd(n, k) > 0$. Then $n = dn_0$ and $k = dk_0$ where $k_0 > 0$ with $\gcd(n_0, k_0) = 1$. It suffices to show that $\text{ord}(a^n) = k_0$.

Note that $(a^n)^{k_0} = a^{dn_0k_0} = (a^k)^{n_0} = e^{n_0} = e$. Furthermore, Consider a positive integer k_1 such that $(a^n)^{k_1} = e$. Note that $k_0 \mid k_1$, otherwise Euclidean Algorithm would conclude that $(a^n)^{k_1} \neq e$ following the factorization above. In particular, $0 < k_0 < k_1$. Hence, k_0 is the smallest positive integer b such that $(a^n)^b = e$. Hence, $\text{ord}(a^n) = k_0 = \frac{k}{d} = \frac{\text{ord}(a)}{\gcd(n, \text{ord}(a))}$.

(c) Take $a, b \in G$ such that $ab = ba$ and $\gcd(\text{ord}(a), \text{ord}(b)) = 1$. We explicitly write $\text{ord}(a) =$

$n > 0$ and $\text{ord}(b) = m > 0$. Denote $\text{ord}(ab) = k$. It suffices to prove that $k = nm$. Note that $\text{ord}(ab) > 0$ by definition.

Since $\gcd(n, m) = 1$, then by Homework 1 Problem 3 we have $\langle a\rangle \cap \langle b\rangle = e$. Therefore, consider integer i such that $b^i \neq e$, then there is no integer j such that $a^j = (b^i)^{-1}$, otherwise the inverse $b^i \in \langle a\rangle$, contradiction. Similarly, consider integer i such that $a^i \neq e$, then there is no integer j such that $b^j = (a^i)^{-1}$ for the same reason. Therefore, since $(ab)^k = a^k b^k = e$, then $a^k = e$ and $b^k = e$. In particular, $\text{lcm}(n, m) \mid k$. Observe that $(ab)^{nm} = (a^n)^m \cdot (b^m)^n = e^m \cdot e^n = e$. Therefore $k \mid nm$. Since $\gcd(n, m) = 1$, then $\text{lcm}(n, m) = \frac{nm}{\gcd(n,m)} = nm$. Hence, $k = nm$ as desired.

$\square$

Problem 2.2. *Let $H \subseteq G$ be a subgroup. Show that the correspondence $Ha \mapsto (Ha)^{-1} = a^{-1}H$ is a bijection between the sets of right and left cosets of H in G.*

Proof. We first show that the map f that maps right cosets to their inverses is well-defined. For $Ha = Hb$, by definition $ab^{-1} \in H$. Therefore, $ba^{-1} \in H$ as well. In particular, $ba^{-1}H = H$, hence $a^{-1}H = b^{-1}H$. Therefore, $f(Ha) = (Ha)^{-1} = a^{-1}H = b^{-1}H = f(Hb)$. Hence, the map is well-defined.

Furthermore, f is invertible. Indeed, the inverse g maps aH to $(aH)^{-1} = Ha^{-1}$. Note that $gf(Ha) = g(a^{-1}H) = Ha$, and similarly $fg(aH) = f(Ha^{-1}) = aH$. Therefore, this is a well-defined bijective correspondence indeed. $\square$

Problem 2.3. *Let $H \subseteq G$ be a subgroup. Suppose that for any $a \in G$ there exists $b \in G$ such that $aH = Hb$. Show that H is normal in G.*

Proof. Take arbitrary $g \in G$. Note that there exists some $g' \in G$ such that $gH = Hg'$, so $gHg'^{-1} = H$. However, since $gH = Hg'$, then $g' = eg' = gh$ for some $h \in H$. Hence, $gH(gh)^{-1} = H$. In particular, $gHg^{-1} = gHh^{-1}g^{-1} = H$, which means $H \lhd G$. $\square$

Problem 2.4. *Let G be the smallest subgroup of $GL_2(\mathbb{C})$ containing $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.*

(a) Prove that G is a non-Abelian group of order 8.

(b) Determine all subgroups of G and prove that they are all normal in G.

Proof. (a) We claim that $G = \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$. Observe that $G \subseteq \text{GL}_2(\mathbb{C})$, then the cyclic group must be subgroup of this general linear group.

By the properties of a group, any subgroup of the general linear group that contains the two elements $a = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, must also contain $a^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = b^2$, $a^3 =$

$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = a^{-1}$, $a^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = e_{\mathrm{GL}} = b^4$, $b^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = b^{-1}$, $ab = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $ba = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = (ab)^{-1}$. Therefore, any subgroup must have order of at least 8. Note that one can check that a set with the eight elements above closed under multiplication, with $ab = (ba)^{-1}$. Since a subgroup must contain these eight elements, then G is exactly the group above, with order 8.

Finally, we show that G is non-Abelian. Indeed, note that $ab \neq ba$ from above.

(b) By Lagrange Theorem, a subgroup of G must have either $1, 2, 4$ or 8 elements.

First, there are two trivial subgroups, $\{e\}$ and G itself. These are all the subgroups with 1 or 8 elements.

There is only one subgroup with two elements since such subgroups must contain the identity, then the other element must be of order 2. The only element that fulfills such requirement is $a^2 = b^2$. Hence, the subgroup with two elements is $H = \{a^2 = b^2, e\}$.

There are also obviously two cyclic subgroups $\langle a \rangle$ and $\langle b \rangle$ of order 4. Finally, there is another subgroup of order 4, which is $\{e, a^2 = b^2, ab, ba\}$ since $(ab)^2 = (ba)^2 = a^2 = b^2$.

We now show that all subgroups above are normal. Clearly the two trivial subgroups are normal. Recall that a subgroup of index 2 must be normal, so the subgroups of order 4 must be normal. Finally, the subgroup of order 2 is normal: we show that $gH = Hg$ with H defined above. Take $h = a^2 = b^2$, since the only other element is identity and the proof is trivial. If pick $g = ab$, then $(ab)h = ab^3 = a^3b = h(ab)$, similar property holds if we pick $g = ba$. If we pick $g = a^k$ or b^k for some k, then $gh = hg = a^{k+2}$ or b^{k+2}. Therefore, $ghg^{-1} \in H$. Hence, $gH = Hg$ indeed. Hence, every subgroup of G is normal.

□

Problem 2.5. *Let N be a subgroup in the center $Z(G)$ of G. Show that N is normal in G. Prove that if the factor group G/N is cyclic, then G is Abelian.*

Proof. We first show that $N \lhd G$.

Note that for any element $g \in N$, we have $\forall h \in H$, $gh = hg$. Then $hgh^{-1} = g$ for all $h \in H$. Hence, $hNh^{-1} \subseteq N$ by definition, and by proposition **1.4.22** we have $N \lhd G$.

Therefore, the factor group G/N is well-defined. Suppose G/N is cyclic, we want to show that G is Abelian. Since G/N is cyclic, then for some $a \in G$, we have $G/N = \langle aN \rangle$. In particular, for arbitrary $g \in G$, we have $g = a^m \cdot n$ for some $m > 0$ and $n \in N$. Consider arbitrary elements $g, g' \in G$. Following the argument above, we can write $g = a^{m_1} \cdot n_1$ and $g' = a^{m_2} \cdot n_2$ for some $m_i > 0$ and $n_i \in N$. Therefore,

$$\begin{aligned} gg' &= a^{m_1} \cdot n_1 \cdot a^{m_2} \cdot n_2 \\ &= a^{m_1}(n_1 a^{m_2})n_2 \\ &= a^{m_1} a^{m_2} n_1 n_2 \\ &= a^{m_1} a^{m_2} n_2 n_1 \\ &= a^{m_2} a^{m_1} n_2 n_1 \\ &= a^{m_2}(n_2 a^{m_1})n_2 \\ &= (a^{m_2} n_2)(a^{m_1} n_2) \\ &= g'g \end{aligned}$$

since $n_i \in N$ which is a normal subgroup. Therefore, G is Abelian indeed. □

Problem 2.6. *Prove that if a group G contains a subgroup H of finite index n, then G contains a normal subgroup N of finite index dividing $n!$ such that $N \subseteq H$. Show that $n!$ cannot be replaced by a smaller integer. (Hint: Consider the homomorphism of G to the symmetric group of all left cosets of H in G taking any $x \in G$ to f_x defined by $f_x(aH) = xaH$.)*

Proof. We define a group homomorphism $f : G \to S_{G/H}$ defined by $x \mapsto f_x : G/H \to G/H$ where $f_x(aH) = xaH$. Recall that $\ker(f) \lhd G$. We show that $\ker(f) \subseteq H$. Indeed, take arbitrary $g \in \ker(f)$, then $f(g) = f_g = \mathrm{id}_{G/H}$, and in particular $gH = H$. Therefore, $g \in H$. Hence, $\ker(f) \subseteq H$. By the first isomorphism theorem, $G/\ker(f) \cong \mathrm{im}(f) \subseteq S_{G/H}$. However, note that H in G has finite index n, which means $\mathrm{im}(f)$ has to be a finite group in $S_{G/H}$. Therefore, $G/\ker(f)$ is finite and by definition the index of $\ker(f)$ in G is finite. Finally, recall that since H in G has index n, then G/H has cardinality n, which means $S_{G/H}$ has cardinality $n!$. The image of f, as a subgroup of $S_{G/H}$, must have $|\mathrm{im}(f)| \mid S_{G/H} = n!$. Therefore, $|G/\ker(f)| \mid n!$, which means the index of $\ker(f)$ in G divides $n!$.

To show the second claim, it suffices to show that $\ker(f)$ is the largest normal subgroup of G contained in H. Note that by definition the kernel of this action is exactly the intersection of all stabilizers when indexed on $xH \in G/H$, which means:

$$
\begin{aligned}
\ker(f) &= \bigcap_{xH \in G/H} \mathbf{stab}(xH) \\
&= \{g \in G : gxH = xH \ \forall x \in G\} \\
&= \{g \in G : x^{-1}gxH = H \ \forall x \in G\} \\
&= \{g \in G : x^{-1}gx \in H \ \forall x \in G\} \\
&= \{g \in G : g \in xHx^{-1} \ \forall x \in G\} \\
&= \bigcap_{x \in G} xHx^{-1}
\end{aligned}
$$

Suppose K is a normal subgroup of G contained in H, then for all $x \in G$, $K = xKx^{-1} \subseteq xHx^{-1}$. In particular, $K \subseteq \ker(f)$. Moreover, one can see that there are cases where index of $\ker(f)$ would be $n!$, which means this is the smallest index possible. This concludes the proof. □

Problem 2.7. *Let H be a subgroup of a group G. For any $x, y \in G$ write $x \sim y$ if $x^{-1}y \in H$. Prove that $\sim$ is an equivalence relation on G with the equivalence classes all left cosets of H in G.*

Proof. Take arbitrary $x \in G$. Then $x \sim x$ since $x^{-1}x = e_G \in H$ by definition.

Take arbitrary $x, y \in G$ such that $x \sim y$. Therefore $x^{-1}y \in H$. In particular, we have $(x^{-1}y)^{-1} = y^{-1}x \in H$. By definition, $y \sim x$.

Take arbitrary $x, y, z \in G$ such that $x \sim y$ and $y \sim z$. Therefore $x^{-1}y, y^{-1}z \in H$. Therefore, $(x^{-1}y)(y^{-1}z) = x^{-1}z \in H$. By definition, $x \sim z$.

Collecting the properties above, we know $\sim$ is an equivalence relation. Note that an arbitrary equivalence class with respect to $\sim$ is $[x] = \{y \in G \mid y \sim x\}$. It suffices to show that $[x] = xH$. Indeed, for arbitrary $y \in [x]$, we have $x^{-1}y \in H$, which means $xh = y$ for some $h \in H$, hence $y \in xH$. Therefore, $[x] \subseteq xH$. On the other hand, for arbitrary $y \in xH$, $y = xh$ for some $h \in H$, so $x^{-1}y \in H$. Hence $xH \subseteq [x]$, and $xH = [x]$. This concludes the proof. □

Problem 2.8. *Let N be a normal subgroup of a group G. Prove that the assignment $H \mapsto H/N$ establishes*

(i) A bijection between the set of all subgroup of G containing N and all subgroups of G/N;

(ii) A bijection between the set of all normal subgroup of G containing N and all normal subgroups of G/N.

Construct the inverse bijections using the canonical homomorphism $\pi : G \to G/N$.

Proof. Observe that G/N is a well-defined group.

Take H as a subgroup of G containing N. We first show that H/N is indeed a subgroup of G/N. Note that since $N \triangleleft G$, then $N \triangleleft H$ by definition, and so H/N has to be a factor group. In particular, since $H/N \subseteq G/N$ by definition, then H/N is a subgroup of G/N indeed.

We define the inverse assignment by $G' \mapsto \pi^{-1}(G')$ for subgroup G' of G/N where π is the canonical (surjective) homomorphism $\pi : G \to G/N$. Note that the map is well-defined because $\pi^{-1}(G')$ has to be a subgroup of G according to homework 1. Take an arbitrary subgroup $G' \subseteq G/N$, then G' contains some elements of form gN for some $g \in G$. By definition, $\pi^{-1}(G') = \{g \in G : gN \in G'\}$ is a subgroup of G. Furthermore, $\pi^{-1}(G')$ contains N because for arbitrary $g \in N$, $gN = N \in G'$ by definition. Notice that we have also proven that every subgroup of G/N takes the form H/N for some subgroup $H = \pi^{-1}(G)$ of G. This gives us the desired correspondence. In particular, we have $H \mapsto H/N \mapsto \pi^{-1}(H/N) = H$, and $G' \mapsto H = \pi^{-1}(G') \mapsto H/N = G'$.

Take normal subgroup H of G that contains N, then N is also a normal subgroup of H, i.e. $N \lhd H \lhd G$. In particular, H/N is a well-defined factor group. Take arbitrary element $gN \in G/N$, then take arbitrary $hN \in H/N$. Then

$$\begin{aligned}
(gN) \cdot (hN) \cdot (gN)^{-1} &= (gN) \cdot (hN) \cdot (g^{-1}N) \\
&= (gh)N \cdot g^{-1}N \\
&= ghg^{-1}N \text{ where } ghg^{-1} \in H \\
&\in H/N
\end{aligned}$$

Therefore, $gN \cdot H/N \cdot (gN)^{-1} \in H/N$ for arbitrary $gN \in G/N$. Hence, by definition $H/N \lhd G/N$.

Recall we just proved that a subgroup of G/N takes the form H/N for some subgroup H of G. Suppose H/N is a normal subgroup of G/N, then we want to prove that $N \subseteq H \lhd G$. Recall that $N \subseteq H$ as we proved in the previous part. It suffices to show that $H \lhd G$. Take arbitrary $g \in G$ and arbitrary $h \in H$, then $(gN) \cdot (hN) \cdot (gN)^{-1} \in H/N \lhd G/N$ by definition. In particular, by the operation on factor groups, we have $(ghg^{-1})N \in H/N$, then by definition $ghg^{-1} \in H$, hence $H \lhd G$.

Recall from part (i) that the two maps form a correspondence between subgroups, then by restricting domains onto normal subgroups, we have another correspondence between normal subgroups. In particular, $H \lhd G$ is mapped to $H/N \lhd G/N$, then mapped back to $H \lhd G$ according to the previous map. On the other hand, $G' = H/N \lhd G/N$ is mapped to $\pi^{-1}(G') = H \lhd G$ then mapped back to $H/N \lhd G$ according to the previous map. (This inverse bijection is constructed using the canonical homomorphism π from the previous part as required.) $\square$

Problem 2.9. *(a) Show that the group Inn(G) of all inner automorphisms of a group G (given by $a \mapsto gag^{-1}$ for some $g \in G$) is a normal subgroup in the group Aut(G) of all automorphisms of G.*

(b) Find all automorphisms of all (finite and infinite) cyclic groups.

Proof. (a) First recall that $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$.

Take an arbitrary automorphism $f : G \to G$ in $\mathrm{Aut}(G)$. It suffices to show that $f\mathrm{Inn}(G)f^{-1} \subseteq \mathrm{Inn}(G)$. Take an arbitrary inner automorphism $g : G \to G$ defined by $a \mapsto gag^{-1}$ for some

$g \in G$. Suppose $f(g_1) = g_2$. Note that fgf^{-1} is a composition of isomorphisms, therefore it is an isomorphism itself. Then

$$\begin{aligned} fgf^{-1}(g_2) &= fg(g_1) \\ &= f(gg_1g^{-1}) \\ &= f(g)f(g_1)f(g^{-1}) \\ &= f(g)g_2f(g^{-1}) \\ &= f(g)g_2f(g)^{-1} \end{aligned}$$

therefore fgf^{-1} maps arbitrary g_2 to $f(g)g_2f(g)^{-1}$ for some $f(g) \in G$. Therefore, fgf^{-1} is also an inner automorphism. Hence, $\mathrm{Inn}(G) \subseteq \mathrm{Aug}(G)$ is a normal subgroup indeed.

(b) Take arbitrary $G = \langle x \rangle$. Recall that every cyclic group is either isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ for some $n > 0$.

Suppose $G \cong \mathbb{Z}$. Then G is an infinite group. We claim that the generators of G are either $\langle x \rangle$ or $\langle -x \rangle$ (notation due to the fact that cyclic groups must be Abelian). Obviously they are generators indeed. Suppose otherwise, then there exists some other generator nx for $n \in \mathbb{Z}$. Note that here nx is defined as the n-term sum of x if $n \geq 0$, and defined as the $(-n)$-term sum of $-x$ if $n < 0$. Obviously $n \neq 0$. WLOG say $n > 0$, then $n \geq 2$. We claim that nx does not generate x. Indeed, suppose otherwise, then $m(nx) = x$ for some $m \in \mathbb{Z}$. However, that means $(mn-1)x = e$. In particular, that means G is finite, contradiction. Therefore, $\langle x \rangle$ or $\langle -x \rangle$ are the only generators.

Observe that an automorphism $f : G \to G$ must map a generator to another generator, otherwise if a generator is mapped to a non-generator, then the map can not surjective, contradiction. Therefore, f either map $x \mapsto x, -x \mapsto -x$ or $x \mapsto -x, -x \mapsto x$. Therefore, $\mathbf{id}, -\mathbf{id}$ are the only automorphisms for infinite cyclic group.

Suppose $G \cong \mathbb{Z}/n\mathbb{Z}$ for some $n > 0$. Recall from remark in example **1.6.4** that an automorphism must map a generator to another. In particular, an automorphism from $\mathbb{Z}/n\mathbb{Z}$ to itself is uniquely defined by the map for $[1]_n$ because of the operation. Therefore, $[1]_n$ can be mapped to any $[a]_n$ where $\gcd(a, n) = 1$, which is a requirement for being the generator. Therefore, there are exactly $\varphi(n)$ automorphisms, where φ is the Euler function.

Therefore, this classifies all automorphisms on cyclic groups.

□

Problem 2.10. *Let x and x' be two elements in the same orbit under some action of a group G on a set. Show that the stabilizers $Stab(x)$ and $Stab(x')$ are conjugate in G.*

Proof. By definition, $x' = g \cdot x$ for some $g \in G$ and some action $\cdot$. Let us define the stabilizers $\text{Stab}(x) = \{h \in G : h \cdot x = x\}$ and $\text{Stab}(x') = \{h \in G : h \cdot x' = x'\}$. It suffices to show that $\text{Stab}(x') = g\text{Stab}(x)g^{-1}$. Take $g_1 \in \text{Stab}(x')$. Then $g_1 \cdot x' = x'$. In particular, $g_1 g \cdot x = g \cdot x$. In particular, $g^{-1}g_1 g x = x$, then $g^{-1}g_1 g \in \text{Stab}(x)$. Hence, $g^{-1}\text{Stab}(x')g \subseteq \text{Stab}(x)$, so $\text{Stab}(x') \subseteq g\text{Stab}(x)g^{-1}$.

On the other hand, consider $g_2 \in \text{Stab}(x)$. Then $gg_2g^{-1}x' = gg_2x = gx = x'$. In particular, $gg_2g^{-1} \in \text{Stab}(x')$. Therefore, $g\text{Stab}(x)g^{-1} \subseteq \text{Stab}(x')$. We then conclude that $\text{Stab}(x') = g\text{Stab}(x)g^{-1}$, which means the two stabilizers are conjugate in G. □

3 MATH 210A Homework 3

Problem 3.1. *Let a group G act on two sets X and Y. We say that X and Y are G-isomorphic if there is a bijection $f : X \to Y$ such that $f(gx) = g(f(x))$ for every $x \in X$ and $g \in G$. Prove that if G acts on X transitively, then X is G-isomorphic to the set of left cosets G/H for some subgroup $H \subset G$ (with the action of G on G/H by left translations).*

Proof. Fix arbitrary $a \in X$ and consider $H = \text{Stab}(a)$ as a subgroup of G, the stabilizer of $a \in X$ under the group action of G. Since G acts on X transitively, then for any $x \in X$, there exists $g_x \in G$ such that $x = g_x \cdot a$. Therefore, $g_x = xa^{-1}$.

We define $f : X \to G/H$ as $f(x) = (xa^{-1})H$. Notice that for any $g \in G$ and any $x \in X$, we have $f(gx) = (gxa^{-1})H = g(xa^{-1})H = g(f(x))$.

We first show that f is well-defined. Suppose $x = g_1 a = g_2 a$, which means $g_1^{-1}g_2$ fixes a, and so $g_1^{-1}g_2 \in H$, hence $f(g_1 a) = g_1 f(a) = g_1 H = g_2 H = f(g_2 a)$.

In order to show that f is bijective, we define an inverse $f'(gH) = g \cdot a$. Note that f' is well-defined because suppose $g_1 H = g_2 H$, then $f'(g_1 H) = g_1 \cdot a = g_2 \cdot a = f'(g_2 H)$ because $g_2^{-1}g_1 \in H$ then $(g_2^{-1}g_1) \cdot a = a$. Now define $b = ga$. Then $f'f(b) = f'(gH) = ga = b$, and $ff'(gH) = f(g \cdot a) = f(b) = ba^{-1} = g$. Therefore, f has an inverse and is therefore bijective. By definition, X is G-isomorphic to the set of left cosets G/H where H is the stabilizer of $a \in X$. □

Problem 3.2. *A transitive G-action on X is called simply transitive if the stabilizers $G(x)$ are trivial for all $x \in X$.*

(a) *Prove that if G acts simply transitively on X, then X is G-isomorphic to G with the G-action by left translations.*

(b) *Let G act on a non-empty set X. Consider the map $f : G \times X \to X \times X$ defined by $f(g, x) = (gx, x)$. Prove that the action is simply transitive if and only if f is a bijection.*

Proof. (a) Fix $a \in X$. Since G acts transitively on X, then according to problem 1, X is G-isomorphic to the set of left cosets G/H for $H = \text{Stab}(a)$ (with the left translation from G

onto the cosets, as described in problem 1). However, by definition $\mathrm{Stab}(a) = \{e\}$ and so $G/H = G$. Therefore, X is G-isomorphic to G acting on itself by left translation.

(b) $\Rightarrow$:

We first show that f is injective. Suppose $f(g_1, x_1) = f(g_2, x_2)$, then $(g_1x_1, x_1) = (g_2x_2, x_2)$. Clearly $x_1 = x_2$ and so $g_2^{-1}g_1x_1 = x_1$, which means $g_2^{-1}g_1 \in \mathrm{Stab}(x_1) = \{e\}$. Therefore, $g_1 = g_2$. Thus, f is injective.

We show that f is surjective. Take arbitrary elements $(x_1, x_2) \in X \times X$. Since the group action is transitive, then there exists some $g \in G$ such that $g \cdot x_2 = x_1$. In particular, $f(g, x_2) = (gx_2, x_2) = (x_1, x_2)$. Therefore, f is surjective and therefore bijective.

$\Leftarrow$:

Suppose f is bijective. We want to show that the group action is simply transitive. We first show that the action is transitive. Take arbitrary $x_1, x_2 \in X$. Since f is a bijection, then there is some $g \in G, x \in X$ such that $f(g, x) = (x_1, x_2)$. In particular, $x = x_2$ and $g \cdot x = g \cdot x_2 = x_1$. Hence, by definition the group action is transitive.

Take arbitrary $x \in X$. Take arbitrary $g \in \mathrm{Stab}(x)$. Observe that for element $(x, x) \in X \times X$, we have $f(g, x) = (gx, x) = (x, x) = (ex, x) = f(e, x)$. However, since f is injective, we then have $g = e$. Therefore, $\mathrm{Stab}(x) = \{e\}$, and it then follows that the group action is simply transitive.

$\square$

Problem 3.3. *Let G and H be two isomorphic groups. Show that there are two simply transitive actions of the groups $Aut(G)$ and $Aut(H)$ on the set X of all group isomorphisms between G and H.*

Proof. Let X be the set of all the group isomorphisms between G and H. (I assume this fixes the direction $G \mapsto H$, otherwise the operation does not really look well) We first show that there is a transitive action of the group $\mathrm{Aut}(G)$ on X. Take arbitrary $g \in \mathrm{Aut}(G)$ and arbitrary $x \in X$. We define $g \cdot x = x \circ g^{-1}$, where g^{-1} is the inverse of g, and therefore $g \cdot x \in X$.

We first show that the defined operation is a group action. Observe that $\mathrm{id}_G \cdot x = x \circ \mathrm{id}_G = x$, and for $g_1, g_2 \in \mathrm{Aut}(G)$, we have $g_1 \cdot (g_2 \cdot x) = g_1 \cdot (x \circ g_2^{-1}) = x \circ g_2^{-1} \circ g_1^{-1} = x \circ (g_1 \circ g_2)^{-1} = (g_1 \circ g_2) \cdot x$ since g_1, g_2 are isomorphisms. We now show that the group action is transitive. Take arbitrary isomorphisms $x_1, x_2 : G \to H$. Define $g \in \mathrm{Aut}(G)$ as $g = x_1^{-1} \circ x_2 : G \to G$. In particular, since g is a composition of isomorphisms, and it is an automorphism. Note that $g \cdot x_1 = x_1 \circ g = x_2$. By definition, the action $\cdot$ is transitive. Finally, we show that $\cdot$ is simply transitive. Consider arbitrary $x \in X$. Consider a stabilizer $g \in \mathrm{Aut}(G)$ of x, i.e. $g \cdot x = x$. By defintion, $x \circ g^{-1} = x$. By cancellation (since x is isomorphism), then $g^{-1} = \mathrm{id}_G$. In particular, $g = \mathrm{id}_G$. Therefore, the only stabilizer of x has to be the identity, which shows that $\cdot$ is simply transitive.

Now consider arbitrary $h \in \mathrm{Aut}(H)$ and arbitrary $x \in X$. We define the operation $h \cdot x = h \circ x \in X$.

We first show that the defined operation is a group action. Observe that $\mathrm{id}_H \cdot x = \mathrm{id}_H \circ x = x$, and for $h_1, h_2 \in \mathrm{Aut}(H)$, we have $h_1 \cdot (h_2 \cdot x) = h_1 \cdot (h_2 \circ x) = (h_1 \circ h_2) \circ x = (h_1 \circ h_2) \cdot x$ since h_1, h_2 are isomorphisms. We now show that the group action is transitive. Take arbitrary isomorphisms $x_1, x_2 : G \to H$. Define $h \in \mathrm{Aut}(H)$ as $h = x_2 \circ x_1^{-1} : H \to H$. In particular, since h is a compositin of isomorphisms, then it is an automorphism. Note that $h \cdot x_1 = h \circ x_1 = x_2$. By definition, the action $\cdot$ is transitive. Finally, we show that $\cdot$ is simply transitive. Consider arbitrary $x \in X$. Consider a stabilizer $h \in \mathrm{Aut}(H)$ of x, then $h \cdot x = x$. By definition, $h \circ x = x$. By cancellation (since x is isomorphism), then $h = \mathrm{id}_H$. Therefore, the only stabilizer of x has to be the identity, which shows that $\cdot$ is simply transitive. □

Problem 3.4. *Let H be a p-subgroup of a finite group G. Show that if H is not a Sylow p-subgroup, then $N_G(H) \neq H$.*

Proof. Since G is finite, take $|G| = p^n \cdot m$ for some $\gcd(p, m) = 1$. Since H is not a Sylow p-subgroup, then $|H| = p^k$ for some $k < n$. Recall from lemma **1.7.6**, $[N_G(H) : H] \equiv [G : H] \pmod p$, but since G is finite, then $\frac{|N_G(H)|}{|H|} \equiv \frac{|G|}{|H|} \equiv 0 \mod p$. Therefore $\frac{|N_G(H)|}{|H|} \equiv 0 \pmod p$. However, obviously $|N_G(H)| \geq |H| \geq 1$ since H is a p-subgroup, then $|N_G(H)| \neq 0$. Hence, we must have $|N_G(H)| \geq p|H|$, so $N_G(H) \neq H$. □

Problem 3.5. *Let G be a p-group and let k be a divisor of $|G|$.*

(a) Prove that G contains a normal subgroup of order k.

(b) Prove that every group of order p^2 (for a prime p) is Abelian.

Proof. (a) Since G is a p-group, we have $|G| = p^n$ for some positive integer n. Therefore $k = p^m$ for some $0 \leq m \leq n$. We prove by induction on m. Clearly the statement is true when $m = 0$ as $\{e\} \lhd G$. Suppose the statement is true for $m = l$ for some $0 \leq l < m$. We want to show that the statement is true for $m = l + 1$.

By the inductive hypothesis, there is a group $H \lhd G$ of order p^l. In particular, there is a factor group G/H, which is also a p-group since $l < n$. Now, the center of G/H, denoted as $Z(G/H)$, is a normal subgroup of G/H, and in particular $Z(G/H)$ is not trivial since G/H is a p-group. Therefore, $Z(G/H)$ is also a p-group. By Cauchy's theorem, there is an element $gH \in Z(G/H)$ by order p. In particular, $\langle gH \rangle \subseteq Z(G/H)$. Recall that $Z(G/H) \lhd G/H$, then for arbitrary $g_1H \in G/H$ and arbitrary $g_2H \in \langle gH \rangle$, since g_2H is in the center of G/H, then $g_1H \cdot g_2H \cdot (g_1H)^{-1} = g_2H$, which means $\langle gH \rangle \lhd G/H$. Recall from Homework 2 Problem 8, the preimage of $\langle gH \rangle$ in the bijective correpsondence between subgroups containing H and subgroups of G/H must be a normal subgroup as well. Furthermore, the preimage of $\langle gH \rangle$ in G is exactly the elements in the union of the cosets H, gH, g^2H, $\cdots$, $g^{p-1}H$. In particular, since G is a finite group, then the preimage is a group that contains $p \times |H| = p \times p^l = p^{l+1}$ elements. This concludes the proof.

(b) Take arbitrary group G of order p^2. Then since G is a p-group, then $Z(G) \neq \{e\}$. In particular, $Z(G)$ is a normal subgroup of G, then $Z(G)$ either of order p or of order p^2.

Suppose $Z(G)$ has order p^2, then $Z(G) = G$, then by definition G is Abelian automatically.

Suppose $Z(G)$ has order p. Then since $Z(G) \lhd G$, we have a factor group $G/Z(G)$ of order p, which must by cyclic. Recall from Homework 2 Problem 5, G is Abelian.

□

Problem 3.6. *Let $H \subseteq G$ be a subgroup and let $P \subseteq H$ be a Sylow p-subgroup. Prove that there is a Sylow p-subgroup $Q \subseteq G$ such that $Q \cap H = P$.*

Proof. Let us write $G = p^n \cdot m$ for some $\gcd(m, p) = 1$. Note that $p \mid |H|$ since otherwise there cannot be a Sylow p-subgroup in H. Therefore we have $H = p^{n_1} \cdot m_1$ for some $n_1 < n$ and $m_1 \mid m$. In particular, $|P| = p^{n_1}$.

Since $H \subseteq G$ is a subgroup, then the Sylow p-subgroup P of H must be a p-subgroup of G. By the First Sylow Theorem, there exists a Sylow p-subgroup of G, namely $Q \subseteq G$ that contains P. Therefore, $P \subseteq Q \cap H$. However, recall that the intersection of two subgroups must be a subgroup as well, so $Q \cap H$ is a subgroup. Now by definition $|Q| = p^n$, then by Lagrange Theorem, $|Q \cap H| = p^{n_2}$ for some $n_2 \leq n_1$. However, recall that $|P| = p^{n_1}$, so $P = Q \cap H$ as $n_1 = n_2$. □

Problem 3.7. *(a) A subgroup H of G is called characteristic, if $f(H) = H$ for every automorphism f of G. Show that a characteristic subgroup H is normal in G.*

(b) Prove that if K is a characteristic subgroup of H and H is a characteristic subgroup of G, then K is characteristic in G.

Proof. (a) Recall that a conjugation action from G to itself is an inner automorphism. Recall that an inner automorphism is also an automorphism. Therefore, since H is a characteristic subgroup, then a G-conjugation f maps $f(H) = H$, and in particular $gHg^{-1} = H$ for some $g \in G$. Therefore, H is obviously normal.

(b) Consider arbitrary G-automorphism $f : G \to G$. Since H is a characteristic subgroup of G, then $f(H) = H$. In particular, the restriction map $f\mid_H : H \to H$ is an automorphism on H by definition. However, since K is characteristic in H, then $f\mid_H (K) = K$. Therefore, $f(K) = K$. Since f is an arbitrary G-automorphsim, then K is characteristic in G.

□

Problem 3.8. *(a) For any two subgroups K and H of a group G denote by $[K, H]$ the subgroup in G generated by the commutators $[k, h] = khk^{-1}h^{-1}$ for all $k \in K$ and $h \in H$. Show that if K and H are normal in G, then so is $[K, H]$.*

(b) Prove that $[G, H]$ is normal in G for every subgroup $H \subseteq G$.

Proof. (a) Suppose $K \lhd G$ and $H \lhd G$. Take arbitrary $g \in G$, we first show that for arbitrary commutator $[k, h]$, $g[k, h]g^{-1} \in [K, H]$. Indeed, $g[k, h]g^{-1} = gkhk^{-1}h^{-1}g^{-1} = (gkg^{-1})(ghg^{-1})(gk^{-1}g^{-1})(gh^{-1}g^{-1}) = (gkg^{-1})(ghg^{-1})(gkg^{-1})^{-1}(ghg^{-1})^{-1}$. However, since $K \lhd G$, then $gkg^{-1} \in K$, similarly $ghg^{-1} \in H$, so $g[k, h]g^{-1} \in [K, H]$. Now take arbitrary $g \in G$ and arbitrary element $a = [k_1, h_1] \cdot [k_2, h_2] \cdot \cdots \cdot [k_n, h_n] \in [K, H]$. It suffices to show that $gag^{-1} \in [K, H]$.

Unpacking the definition, we have

$$\begin{aligned} gag^{-1} &= g[k_1, h_1] \cdots [k_n, h_n]g^{-1} \\ &= g[k_1, h_1]g^{-1} \cdot g[k_2, h_2]g^{-1} \cdot \cdots \cdot g[k_n, h_n]g^{-1} \\ &\in [K, H] \end{aligned}$$

This concludes the proof.

(b) Consider arbitrary $H \subseteq G$. We show that $[G, H] \lhd G$. Take arbitrary $g \in G$. Consider arbitrary commutator $[g', h] \in [G, H]$, then $g[g', h]g^{-1} = gg'hg'^{-1}h^{-1}g^{-1} = (gg')h(gg')^{-1}h^{-1}hgh^{-1}g^{-1} = [gg', h] \cdot [h, g] = [gg', h] \cdot [g, h]^{-1} \in [G, H]$. Now, for arbitrary element $a = [g_1, h_1] \cdot [g_2, h_2] \cdot \cdots \cdot [g_n, h_n] \in [G, H]$, we have $gag^{-1} = (g[g_1, h_1]g^{-1})(g[g_2, h_2]g^{-1}) \cdots (g[g_n, h_n]g^{-1}) \in [G, H]$. Therefore, $[G, H] \lhd G$.

□

Problem 3.9. *Assume that a subset $S \subset G$ of a group G satisfies $gSg^{-1} \subset S$ for all $g \in G$. Prove that the subgroup generated by S is normal in G.*

Proof. Note that this is exactly remark **1.9.3**.

Recall from lemma **1.9.2** that $\langle S \rangle = \{x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}, x_i \in S, \varepsilon_i = \pm 1\}$. Therefore, take arbitrary $g \in G$ and arbitrary $x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n} \in \langle S \rangle$ for $x_i \in S$ and $\varepsilon_i = \pm 1$. Now, $gx_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n} g^{-1} = (gx_1^{\varepsilon_1} g^{-1})(gx_2^{\varepsilon_2} g^{-1}) \cdots (gx_n^{\varepsilon_n} g^{-1})$. Suppose $\varepsilon_i = -1$ for some i, then note that we have $gx_i^{\varepsilon_i} g^{-1} = (gx_i g^{-1})^{-1} \in S$. Therefore, since the operation is closed, then $gx_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n} g^{-1} \in S$. It then follows that $\langle S \rangle \lhd G$. □

Problem 3.10. *Let N be an Abelian normal subgroup in a finite group G. Assume that the orders $|G/N|$ and $|Aut(N)|$ are relatively prime. Prove that N is contained in the center of G.*

Proof. Since $N \lhd G$, we define $f : G \to \mathrm{Aut}(N)$ by $f(g) = f_g : n \mapsto gng^{-1} \in N$. This is a homomorphism because for $g_1, g_2 \in G$, we have $f(g_1 g_2) = f_{g_1 g_2} = f_{g_1} \circ f_{g_2} = f(g_1)f(g_2)$. Notice that the kernel of the map is exactly the set of elements $g \in G$ such that $gng^{-1} = n \ \forall n \in N$. In particular, since N is an Abelian subgroup of G, we know $N \subseteq \ker(f)$. By the First Isomorphism Theorem, we have $G/\ker(f) \cong \mathrm{im}(f) \subseteq \mathrm{Aut}(N)$. However, by the Third Isomorphism Theorem, since $N \subseteq \ker(f)$ is a normal subgroup of G, then $(G/N)/(\ker(f)/N) \cong G/\ker(f) \cong \mathrm{im}(f)$. In particular, $\frac{|G|}{|\ker(f)|} \mid |\mathrm{Aut}(N)|$ and $\frac{|G|}{|\ker(f)|} \mid \frac{|G|}{|N|}$. However, recall that the orders $|G/N|$ and $|\mathrm{Aut(N)}|$ are relatively prime, then $|G| = |\ker(f)|$. Therefore, $\forall g \in G$, we have $gng^{-1} = n \ \forall n \in N$. Rewriting the definition, we have $\forall n \in N$, $gn = ng \ \forall g \in G$. Therefore, $N \subseteq Z(G)$. $\square$

4 MATH 210A Homework 4

Problem 4.1. *Let P be a Sylow subgroup of a finite group G and $H = N_G(P)$. Prove that $N_G(H) = H$.*

Proof. Without loss of generality, say P is a Sylow p-subgroup.

It suffices to show that $N_G(H) \subseteq H$. Take arbitrary $g \in N_G(H)$, then $gPg^{-1} \in H$. Recall that $P \cong gPg^{-1}$ is a Sylow subgroup, then H contains the Sylow p-subgroup gPg^{-1}. By the second Sylow Theorem, two Sylow p-subgroups are conjugate, so $H \supseteq gPg^{-1} = hPh^{-1}$ for some $h \in H$. In particular, $g^{-1}hPh^{-1}g = P$, and so $(g^{-1}h)P(g^{-1}h)^{-1} = P$. Hence, $g^{-1}h \in N_G(P) = H$, and so $g \in H$ by definition. Hence, $N_G(H) \subseteq H$ and it then follows that $N_G(H) = H$. $\square$

Problem 4.2. *Let G be a group and K, H two normal subgroups of G. Prove that $G = H \times K$ (internal product) if and only if $G = HK$ and $H \cap K = \{e\}$.*

Proof. Suppose $G = H \times K$ as an internal product. By definition, every $g \in G$ can be uniquely written as $g = hk$ for $h \in H, k \in K$. Therefore, there exists an isomorphism $G \to HK$ such that $g \mapsto hk$ where $g = hk$. Therefore, $G = HK$. In particular, $H \cap K = \{e\}$, otherwise there is some $e \neq g \in H \cap K$, which means $g = eg = ge \in HK$, then g cannot be uniquely written as a product of elements in H and K, contradiction. This concludes one direction of the proof.

Suppose $G = HK$ and $H \cap K = \{e\}$. By definition, it suffices to show that every $g \in G$ can be uniquely written as $g = hk$ for some $h \in H, k \in K$. Suppose we have $g = h_1k_1 = h_2k_2$ for some $h_1, h_2 \in H$ and $k_1, k_2 \in K$. We then have $H \ni h_2^{-1}h_1 = k_2k_1^{-1} \in K$, which means $h_2^{-1}h_1 = k_2k_1^{-1} \in H \cap K = \{e\}$. Hence, $h_1 = h_2, k_1 = k_2$. This concludes the proof. $\square$

Problem 4.3. *A group G is called nilpotent if there is a sequence of subgroups*

$$G = G_0 \supset G_1 \supset \cdots \supset G_n = \{e\},$$

such that each G_i is normal in G and G_i/G_{i+1} is contained in the center of G/G_{i+1}.

(a) *Prove that if $G/Z(G)$ is nilpotent, so is G.*

(b) *Prove that every Abelian group is nilpotent.*

(c) *Prove that every p-group is nilpotent.*

(d) *Prove that every nilpotent group is solvable.*

Proof. (a) Suppose $G/Z(G)$ is nilpotent, then by definition there is a sequence of subgroups $G/Z(G) = G_0 \supset G_1 \supset \cdots \supset G_n = \{e\}$. Recall from a conclusion we proved in Homework 2 problem 8, a subgroup G_1 of $G/Z(G)$ is essentially $G_1 = H_1/Z(G)$ for some H_1 as a subgroup of G. Therefore, we have $G/Z(G) \supset H_1/Z(G) \supset \cdots \supset H_n/Z(G) = \{e\}$ where each $H_i/Z(G) \lhd G/Z(G)$ and $(H_i/Z(G))/(H_{i+1}/Z(G)) \subseteq Z((G/Z(G))/(H_{i+1}/Z(G)))$. By the correspondence result of Homework 2 problem 8, we have $G = G_0 \supset H_1 \supset \cdots \supset H_n = Z(G)$, and can extend it with $Z(G) \supset H_{n+1} = \{e\}$. Indeed, note that By the third isomorphism theorem, $H_i/H_{i+1} \subseteq Z(G/H_{i+1})$ for all i, and each $H_i \lhd G$. By definition, G is nilpotent.

(b) Take arbitrary Abelian group G. Observe that every subgroup of G must be normal. Furthermore, $Z(G) = G$. Therefore, we have a sequence $G \supset \{e\}$. In particular, $G/\{e\} = G \subseteq Z(G)$. Therefore, G is nilpotent, and every Abelian group has to be nilpotent.

(c) Denote $Z_1(G) = Z(G)$ and for any $Z_i(G)$, define $Z_{i+1}(G)$ as the subgroup containing $Z_i(G)$ such that $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$. This is well-defined: the center of a group is essentially a subgroup, then $Z(G/Z_i(G))$ is a subgroup $H/Z_i(G)$ of $G/Z_i(G)$ for $H \subseteq G$ by the correspondence we proved in Homework 2 Problem 8. Note that any subgroup of G has to be a p-group as well, then the center cannot be trivial. Therefore, $Z_{i+1}(G) \supset Z_i(G)$ always hold unless $Z_i(G) = G$.

Observe that G is finite, then G must satisfy the ascending chain condition, which means we have an ascending chain $\{e\} \subset Z(G) = Z_1(G) \subset \cdots \subset Z_i(G) \subset Z_{i+1}(G) = G$ for some i. By reversing the order of the chains, we have a sequence of subgroups $G = G_0 \supset G_1 \supset \cdots \supset G_n = Z(G) \supset \{e\}$. Observe that G_i/G_{i+1} is the center of G/G_{i+1}, and for arbitrary G_i, and arbitrary $g \in G$, we want to show that $(gG_ig^{-1})/G_{i+1}$ is contained in $Z(G/G_{i+1})$. Indeed,

$$\begin{aligned}
(gG_ig^{-1})/G_{i+1} &= gG_{i+1} \cdot G_i/G_{i+1} \cdot g^{-1}G_{i+1} \\
&= gG_{i+1} \cdot Z(G/G_{i+1}) \cdot g^{-1}G_{i+1} \\
&= Z(G/G_{i+1}) \cdot gG_{i+1} \cdot g^{-1}G_{i+1} \\
&= Z(G/G_{i+1})
\end{aligned}$$

Collecting the properties above, we know G is nilpotent. Hence, every p-group is nilpotent.

(d) Let G be nilpotent, then by definition there is $G = G_0 \supset G_1 \supset \cdots \supset G_n = \{e\}$, where $G_i \lhd G$ and $G_i/G_{i+1} \subseteq Z(G/G_{i+1})$. By definition, $G_{i+1} \lhd G_i$ for any i. Furthermore, since a center must be Abelian, then G_i/G_{i+1} is contained in an Abelian group, which means G_i/G_{i+1} is also Abelian. Therefore, G is solvable by a property we proved.

□

Problem 4.4. *(a) Let G be a nilpotent group and $H \subset G$ a subgroup different from G. Prove that $N_G(H) \neq H$.*

(b) Prove that a finite group is nilpotent if and only if it is isomorphic to the direct product of p-groups.

Proof. (a) Since G is nilpotent, then consider a sequence of subgroups $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = \{e\}$ with $G_i \triangleleft G$ and $G_i/G_{i+1} \subset Z(G/G_{i+1})$ for each i.

Note that for arbitrary $K \subseteq G$, $[G, K] = 1$ if and only if $gh = hg$ for all $g \in G, h \in K$ if and only if $K \subseteq Z(G)$. Therefore, $G_i/G_{i+1} \subset Z(G/G_{i+1})$ if and only if $[G/G_{i+1}, G_i/G_{i+1}] = 1$.

If $[G/G_{i+1}, G_i/G_{i+1}] = 1$, then for any $gG_{i+1} \in G/G_{i+1}$ and any $g'G_{i+1} \in G_i/G_{i+1}$, we have $(gg'g^{-1}g'^{-1})G_{i+1} = G_{i+1}$. In particular, $gg'g^{-1}g'^{-1} \in G_{i+1}$. Hence, $[G, G_i] \subset G_{i+1}$. On the other hand, if $[G, G_i] \subseteq G_{i+1}$, then $gG_{i+1} \cdot g'G_{i+1} \cdot g^{-1}G_{i+1} \cdot g'^{-1}G_{i+1} = (gg'g^{-1}g'^{-1})G_{i+1} \in G_{i+1}$, which means $[G/G_{i+1}, G_i/G_{i+1}] \subseteq \{e\}$, and so $[G/G_{i+1}, G_i/G_{i+1}] = \{e\}$. Therefore, collecting the properties above, $G_i/G_{i+1} \subset Z(G/G_{i+1})$ if and only if $[G, G_i] \subset G_{i+1}$.

Now we prove the statement. Note that for the proper subgroup $H \subset G$, there exists some unique i such that $G_i \subseteq H$ and there exists some $a \in G_{i-1} \backslash H$. Therefore, we have $[H, G_{i-1}] \subset [G, G_{i-1}] \subset G_i \subset H$.

In particular, for arbitrary $h \in H, g \in G_{i-1}$, we have $hgh^{-1}g^{-1} \in H$, so $gh^{-1}g^{-1} \in H$, which means $g \in N_G(H)$. Hence, $G_{i-1} \subseteq N_G(H)$. In particular, $a \in N_G(H)$ but $a \notin H$, which means $N_G(H) \neq H$.

(b) $\Leftarrow$:

It suffices to show the case for $G \cong H \times K$. The case for group product with more multiplication terms would follow from a more general argument.

Let H, K be nilpotent and let $G = H \times K$ be the external direct product. Note that $[(h_1, k_1), (h_2, k_2)] = (h_1, k_1)(h_2, k_2)(h_1, k_1)^{-1}(h_2, k_2)^{-1} = ([h_1, h_2], [k_1, k_2])$ for arbitrary $h_i \in H, k_i \in K$. Therefore, $[H \times K, H \times K] = [H, H] \times [K, K]$. Let $H_0 = H$ and $K_0 = K$, and $H_i = [H, H_{i-1}]$ and $K_i = [K, K_{i-1}]$. Since H and K are nilpotent, then notice that we have the following two sequences of subgroups: $H = H_0 \triangleright H_1 \triangleright \cdots \triangleright H_m = \{e\}$ and $K = K_0 \triangleright K_1 \triangleright \cdots \triangleright K_n = \{e\}$. This is given by the property we proved in part (a). Define $G_i = H_i \times K_i$, therefore, for some i we have $G_i = H_i \times K_i = \{e\}$. In particular, G is nilpotent.

$\Rightarrow$:

Let G be a finite group. By theorem in class, it suffices to show that all Sylow subgroups of G are normal. Take arbitrary Sylow p-subgroup P. Define $H = N_G(P)$. By problem 1, we know $N_G(H) = H$. Then by part (a) of the question, H is not a proper subgroup of G, which means $N_G(P) = H = G$. In particular, $P \triangleleft G$.

$\square$

Problem 4.5. *Show that A_4 has no subgroups of order* 6.

Proof. Suppose, towards contradiction, that A_4 has a subgroup H of order 6. Recall that A_4 has order 12, then $H \lhd A_4$. In particular, A_4/H is a factor group. Recall from lemma in class that A_4 is generated by 3-cycles, then let x be a 3-cycle not in H. Since a 3-cycle has order 3, then A_4 is partitioned by H, xH, x^2H. However, A_4/H is of order 2, and since $H \neq xH$ since $x \notin H$, we must have $x^2H = H$ or $x^2H = xH$. However, in the first case, we have $x^3H = H = xH$ so $x \in H$, a contradiction. In the second case, we have $x \in H$ directly, another contradiction. Therefore, there is no such H of order 6. □

Problem 4.6. *(a) Prove that S_n is generated by* $(1\ 2), (1\ 3), \cdots (1\ n)$.

(b) Prove that S_n is generated by two cycles $(1\ 2)$ *and* $(1\ 2\ \cdots\ n)$.

Proof. (a) Recall that S_n is generated by transpositions. Therefore, it suffices to show that every transposition is generated by $(1\ 2), (1\ 3), \cdots, (1\ n)$. Observe that $(i\ j) = (1\ i)(1\ j)(1\ i)$. This concludes the proof.

(b) Observe that $(i\ i+1) = (1\ 2\ \cdots\ n)(i-1\ i)(1\ 2\ \cdots\ n)^{-1}$ for $i \geq 2$. Therefore, $(1\ 2)$ and $(1\ 2\ \cdots\ n)$ generates all $(i\ i+1)$ for $1 \leq i \leq n-1$. Futhermore, we have $(1\ i+1) = (i\ i+1)(1\ i)(i\ i+1)$. By proceeding inductively, we know $(1\ 2)$ and $(1\ 2\ \cdots\ n)$ generates all $(1\ i+1)$ for $1 \leq i \leq n-1$. Recall that these are exactly the elements that generate S_n, which means $(1\ 2)$ and $(1\ 2\ \cdots\ n)$ generates S_n.

□

Problem 4.7. *(a) Show that the centralizer of A_n in S_n (the subgroup in S_n consisting of all elements, which commute with all elements in A_n) is trivial, if $n \geq 4$.*

(b) Let $g \in S_n$ be an odd transformation. Show that the map $f : A_n \to A_n$, given by $f(x) = gxg^{-1}$ is an automorphism. Prove that f is not an inner automorphism if $n \geq 3$.

Proof. (a) We first show that A_n has trivial center for $n \geq 4$. In particular, we will show that $\forall \mathbf{id} \neq \sigma \in A_n$, there exists $\tau \in A_n$ such that $\sigma\tau \neq \tau\sigma$. By definition, such σ maps some element a to $b \neq a$. Pick c, d distinct from a, b, then we have $\sigma(b\ c\ d)$ that maps a to b, but $(b\ c\ d)\sigma$ that maps a to c. Therefore, σ does not commute with $(b\ c\ d)$. Therefore, $Z(A_n)$ has to be trivial by definition for $n \geq 4$.

Denote $C(A_n)$ as the centralizer of the group A_n in S_n. Consider arbitrary $n \geq 4$, then $C(A_n) \cap A_n = Z(A_n) = \{e\}$ by definition. Since $A_n \lhd S_n$, then we have $C(A_n)A_n/A_n \cong C(A_n)/(C(A_n) \cap A_n) = C(A_n)$. However, $C(A_n)A_n \subseteq S_n$, and S_n/A_n is of order 2, then $C(A_n)$ has order 1 or 2. Suppose, towards contradiction, that $C(A_n)$ has order 2, then there is an element $e \neq \sigma \in C(A_n)$ that is not in A_n. In particular, this element σ has order 2 in the subgroup, but that means the permutation has to be a transposition, namely (a, b). One can easily show that $(a\ b)(a\ b\ c) = (b\ c) \neq (a\ c)(a\ b\ c)(a\ b)$ for any arbitrary $c \neq a, b$ (which exists because $n \geq 4$), contradiction. Hence, the centralizer has to be trivial for $n \geq 4$.

(b) We first show that $f : A_n \to A_n$ is an automorphism. It suffices to check that f is an isomorphism. Consider arbitrary $x, y \in A_n$. $f(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = f(x)f(y)$. Therefore, f is a homomorphism. We know that f is surjective because for arbitrary $x \in A_n$, we have $f(g^{-1}xg) = x$, where $g^{-1}xg \in A_n$ because g is odd and x is even. Also, f is injective because suppose $f(x) = f(y)$, then $gxg^{-1} = gyg^{-1}$, which means $x = y$. Therefore, f is an isomorphism and therefore an automorphism.

We now show that f is not an inner automorphism for $n \geq 3$. Suppose that f is an inner automorphism, then for some $h \in A_n$, we have $hxh^{-1} = gxg^{-1}$ for all $x \in A_n$. In particular, $h^{-1}g \in C(A_n)$. However, if $n \geq 4$, we know $g = h$ since it is trivial, which is a contradiction since g has to be even then. If $n = 3$, note that A_3 is Abelian, then $\mathrm{Inn}(A_3)$ is trivial, so f is trivial. Therefore, $gx = xg$ for all $x \in A_n$. Hence, $g \in S_n$ is a centralizer of A_n. However, the centralizer of A_n is A_n itself, contradiction.

□

Problem 4.8. *Prove that every automorphism of S_3 is inner and $Aut(S_3)$ is isomorphic to S_3.*

Proof. Take arbitrary $f \in \mathrm{Aut}(S_3)$. Since S_3 is generated by $(1\ 2)$ and $(1\ 2\ 3)$, then f is determined by the mapping on $(1\ 2)$ and $(1\ 2\ 3)$. Note that an automorphism preserves the order of a cycle, which means it sends a cycle of order 2 to another cycle of order 2, and a cycle of order 3 to another cycle of order 3. Recall that S_3 has three cycles of order 2 and two cycles of order 3, then there are 3×2 choices. We then have at most 6 automorphisms. We can easily check that S_3 has trivial center, and recall that $S_3/Z(S_3) \cong \mathrm{Inn}(S_3)$, which means $S_3 \cong \mathrm{Inn}(S_3)$. However, $\mathrm{Inn}(S_3) \subseteq \mathrm{Aut}(S_3)$, then there are exactly six automorphisms, which are all inner automorphisms by the cardinality argument. In particular, all automorphisms are inner automorphisms, and $\mathrm{Aut}(S_3) \cong S_3$. □

Problem 4.9. *Describe all Sylow subgroups in S_n for $n \geq 5$.*

Proof. Recall that $S_1 = \{e\}$ is trivial, which has no Sylow subgroups. $S_2 \cong \mathbb{Z}/2\mathbb{Z}$, then the only Sylow subgroup is itself. Recall that the Sylow subgroups of S_3 are the non-trivial subgroups.

Sylow subgroups of S_4 are Sylow 2-groups and Sylow 3-groups. Note that D_8 is a Sylow 2-subgroup, and since these are groups are up to conjugations, then there are 3 Sylow 2-subgroups. Also note that A_3 is a Sylow 3-subgroup, then similarly there are 4 Sylow 3-subgroups because of conjugations. This is all of them by Sylow theorems.

Sylow subgroups of S_5 are Sylow p-groups with $p = 2, 3, 5$. By the embedding through S_4, Sylow 2-subgroups of S_5 are the Sylow 2-subgroups of S_4, which is of the form D_8. This is generated by a 4-cycle $(a\ b\ c\ d)$ and a 2-cycle $(a\ c)$, then there are 15 unique 4-cycles because $\frac{5!}{4} = 30$ to get rid of groups like $(b\ c\ d\ a), (c\ d\ a\ b)$, etc, and $30/2 = 15$ for $(a\ b\ c\ d)$ is the inverse of $(d\ c\ b\ a)$. Therefore, there are 15 Sylow 2-subgroups. The Sylow 3-subgroups are cyclic and are generated by 3-cycles, where there are $2 \times \binom{5}{3}$ of them. Since each cycle contains two 3-cycles as inverses, then there are 10 3-subgroups. The Sylow 5-subgroups must be cyclic, and generated by 5-cycles. There are exactly 6 of them by Sylow theorem. ☐

Problem 4.10. *(a) Show that for any $n \geq 1$, there is an injective homomorphism $S_n \to A_{n+2}$.*

(b) Prove that every finite group is isomorphic to a subgroup of a finite simple group.

Proof. (a) Define $\varphi : S_n \to A_{n+2}$ as $\varphi(\sigma) = \begin{cases} \sigma \text{ if even } \sigma \\ \sigma\tau \text{ if odd } \sigma \end{cases}$, where $\tau = (n+1\ n+2)$. The map is well-defined because $\sigma\tau$ would be even if σ is odd. We show that φ is an injective homomorphism.

Consider arbitrary $\alpha, \beta \in S_n$. Suppose they are both odd, then $\alpha\beta$ is even, and $\varphi(\alpha\beta) = \alpha\beta = \alpha\beta\tau^2 = (\alpha\tau)(\beta\tau) = \varphi(\alpha)\varphi(\beta)$. Suppose one of them is odd, without loss of generality α is odd, then $\alpha\beta$ is odd, and we have $\varphi(\alpha\beta) = \alpha\beta\tau = \alpha\tau\beta = \varphi(\alpha)\varphi(\beta)$. Finally, suppose both are even, then the composition is even, and we have $\varphi(\alpha\beta) = \alpha\beta = \varphi(\alpha)\varphi(\beta)$. Hence, φ is a homomorphism indeed. Consider arbitrary even $\alpha \in S_n$, then $\varphi(\alpha) = \alpha$, so we have $\varphi(e) = e$. Suppose odd $\alpha \in S_n$, then $\varphi(\alpha) = \alpha\tau$. However, $\alpha\tau$ is never trivial since α and τ are disjoint. Therefore, the kernel of φ is the identity. In particular, φ is injective.

(b) Recall that a finite group G can be embedded in some S_n. Therefore, we have $G \hookrightarrow S_n \hookrightarrow A_{n+2}$. Note that if $n+2 \leq 5$, we have $A_{n+2} \hookrightarrow A_5$, which is a finite simple group; if $n+2 > 5$, we have A_{n+2} itself as a finite simple group. Indeed, A_n is simple for $n \geq 5$.

☐

5 MATH 210A Homework 5

Problem 5.1. *Let $\sigma = (1\ 2\ 3\ \cdots\ n) \in S_n$. Show that the conjugacy class of σ has $(n-1)!$ elements. Show that the centralizer of σ is the cyclic subgroup generated by σ.*

Proof. Consider $\sigma = (1\ 2\ \cdots\ n)$. Recall that two elements in S_n are conjugates if and only if they have the same type. Therefore, the conjugacy class of σ is exactly the elements in S_n of type (n). It suffices to find all the n-cycles in S_n. Take an arbitrary element $\tau \in S_n$, then WLOG we can make the first element as 1 since $(a_1\ a_2\ \cdots\ a_n) = (a_2\ a_3\ \cdots\ a_1)$. Therefore, $\tau = (1\ a_2\ a_3\ \cdots\ a_n)$ where a_i are elements in $\{2, \cdots, n\}$. Consider the set of elements Ω of the form τ, then there is a bijection $\Omega \to S_{n-1}$ with the map $\tau = (1\ a_2\ a_3\ \cdots\ a_n) \mapsto s = (a_2\ a_3\ \cdots\ a_n)$. In particular, the set Ω has $(n-1)!$ elements, which means the conjugacy class must have $(n-1)!$ elements.

Consider the centralizer $C(\sigma)$, then by definition it is the set of elements $\tau \in S_n$ such that $\sigma\tau = \tau\sigma$. We first show that $\langle\sigma\rangle \subseteq C(\sigma)$. An arbitrary element in the cyclic group is σ^k, then $\sigma^k\sigma = \sigma\sigma^k = \sigma^{k+1}$. On the other hand we show $C(\sigma) \subseteq \langle\sigma\rangle$. Take arbitrary $\tau \in C(\sigma)$. Then

$\sigma\tau = \tau\sigma$. Therefore, by example in class we have $(1\ 2\ \cdots\ n) = \sigma = \tau\sigma\tau^{-1} = (\tau(1)\ \tau(2)\ \cdots\ \tau(n))$. Therefore, τ has to be an n-cycle, and by the equation above we know $\tau(i) = i+k$ for some constant k. In particular, $\tau = \sigma^k$ for some constant k. In particular, the centralizer is the cyclic subgroup indeed. $\square$

Problem 5.2. *Prove the following Useful Counting Result. Let H be a subgroup of a finite group G with $H \neq G$. Suppose that $|G|$ does not divide $[G : H]!$. Then G contains a proper normal subgroup N such that N is a subgroup of H. In particular, G is not simple.*

Proof. Denote X as the cosets of G/H. Consider the map $f : G \to \sum(X)$ defined by $f(g)(aH) = gaH$, then f is clearly a homomorphism. Let $\ker(f) = N$. Observe that elements of N are stabilizers of H: $f(n)(H) = H = nH$, so $n \in H$. However, recall that the stabilizers of H are exactly elements of H, then $N \subseteq H$. In particular, N is a proper subgroup of G. Furthermore, $N = \ker(f) \lhd G$ by definition. Collecting these properties, we know G is not simple. This concludes the proof. $\square$

Problem 5.3. *Prove that all groups of order $2p^n$ and $4p^n$ are not simple (p is a prime integer).*

Proof. Consider a group G of order $2p^n$. Suppose $p > 2$, then note that G has a Sylow p-group P of order p^n. Then $[G : P] = 2$, which means $P \lhd G$. However, P is not $\{e\}$ or G, then G is not simple. If $p = 2$, then G is a p-group,then G has to be solvable. If G is Abelian, since G has order at least p^2, then G is not isomorphic to any cyclic group $\mathbb{Z}/q\mathbb{Z}$ for some prime q, then G is not simple by remark in class. If G is non-Abelian, then since G is solvable, G is not simple by remark in class.

Consider a group G of order $4p^n$. Suppose $p \geq 5$, then by Sylow's Theorem, the number of Sylow p-subgroups $n_p \equiv 1 \pmod p$, and $n_p \mid 4$. Therefore, $n_p = 1$, which means there is a unique Sylow p-subgroup P. In particular, $P \lhd G$. Therefore, G is not simple.

Suppose $p = 2$, then this is exactly the case for order $2p^n$, then G is not simple.

Suppose $p = 3$. By Sylow's Theorem, there exists a Sylow 3-subgroup P of G of order 3^n. Note that $[G : P] = 4$, then $[G : P]! = 24$. Suppose $n \geq 2$, then $|G| \nmid 24$, then by problem 2 we have some $N \lhd G$ which means G is not simple. Therefore, consider the case where $n = 1$. In particular, G is a group of order 12. Since $12 = 2^2 \cdot 3$, then $n_3 \equiv 1 \pmod 3$, and $n_3 \mid 4$, then $n_3 = 1$ or 4. If $n_3 = 1$ then there is a unique Sylow 3-group, which is normal in G then G is not simple. If $n_3 = 4$, then consider $n_2 \equiv 1 \pmod 2$ and $n_2 \mid 3$, then $n_2 = 1$ or 3. However, if $n_2 = 1$, similar as above we have a non-trivial normal subgroup of G which means G is not simple; if $n_2 = 3$, then since $n_3 = 4$, we have at least $(3-1) \times 4 + (2^2-1) \times 3 > 11$ non-trivial elements, contradiction. Therefore, this case is not possible. Hence, G has to be not simple. This concludes the proof. $\square$

Problem 5.4. *Let G be a non-Abelian group of order p^3 (p is a prime integer). Prove that the center $Z(G)$ of G coincides with the commutator subgroup $[G, G]$.*

Proof. We want to show that $Z(G) = [G, G]$.

Since G is a p-group, then $Z(G) \neq \{e\}$ as proven in class. Furthermore, $Z(G) \neq G$ since G is non-Abelian. Therefore, since $Z(G)$ is a subgroup of G, then $Z(G)$ is of order p or p^2. If $Z(G)$ has order p^2, then since $Z(G) \lhd G$, $G/Z(G)$ has order p and must be cyclic, which means G is Abelian by Homework 2 problem 5, contradiction. Therefore, $Z(G)$ has to have order p. Similarly, since $Z(G) \lhd G$, then $G/Z(G)$ is of order p^2, which is Abelian by Homework 3 Problem 5. However, recall that $H = [G, G]$ is the smallest subgroup such that G/H is Abelian, then $[G, G] \subseteq Z(G)$. In particular, either $[G, G]$ is trivial or $[G, G] = Z(G)$. Suppose $[G, G]$ is trivial, then G is Abelian, contradiction. Hence, $[G, G] = Z(G)$. □

Problem 5.5. *Let G be a semidirect product of a cyclic normal subgroup N of order n and an Abelian group K. Show that if $|K|$ is relatively prime to $\varphi(n)$ (φ is the Euler function), then G is Abelian.*

Proof. Recall $G = N \rtimes K$, then there is an homomorphism $f : K \to \mathrm{Aut}(N)$. Since N is finite, then by Homework 2 problem 9, $\mathrm{Aut}(N)$ has $\varphi(n)$ elements where φ is the Euler function. By the first isomorphism theorem, we have $K/\ker(f) \cong \mathrm{im}(f)$, but the image is a subgroup of $\mathrm{Aut}(N)$, then $|\mathrm{im}(f)| \mid \varphi(n)$, and so $\frac{|K|}{|\ker(f)|} | \varphi(n)$. Howver, since $|K|$ and $\varphi(n)$ are relatively prime, then $|K| = |\ker(f)|$, which means f is trivial. By remark in class, $G = N \times K$ as the direct product. Since N is cyclic, then it is Abelian, and since K is Abelian, we know G has to be Abelian. □

Problem 5.6. *Determine the center of the dihedral group D_{2n}.*

Proof. By classification of small groups, $D_2 \cong \mathbb{Z}/2\mathbb{Z}$ with the group itself as the center, and $D_4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ also has the full group as the center.

For $n \geq 3$, consider $D_{2n} = \{\langle r, s \rangle \,|\, r^n = s^2 = (rs)^2 = e\}$. For any element $r^i s$, we have $r(r^i s)r^{-1} = r^i(rsr^{-1}) = r^{i+2}s \neq r^i s$ since $r^2 \neq e$. For any element r^i, we have $r(r^i)r^{-1} = r^i$ and $sr^i s^{-1} = sr^i s = (srs)^i = r^{-i}$. In particular, note that $r^i = r^{-i}$ if and only if n is even and $i = \frac{n}{2}$. Therefore, for $n \geq 3$, if n is odd, then $Z(D_{2n}) = e$ and if n is even, then $Z(D_{2n}) = \{e, r^{\frac{n}{2}}\}$. □

Problem 5.7. *Let K and H be two groups and $f : K \to Aut(H)$ a group homomorphism. Prove that for every $g \in Aut(K)$ the external semidirect products $H \rtimes K$ with respect to f and $f \circ g$ are isomorphic.*

Proof. We show that $H \rtimes_f K \cong H \rtimes_{f \circ g} K$. Define $h : H \rtimes_f K \to H \rtimes_{f \circ g} K$ by $h(h_1, k_1) = (h_1, g^{-1}(k_1))$. Note that

$$\begin{aligned} h((h_1, k_1) \cdot_f (h_2, k_2)) &= h(h_1 f(k_1) h_2, k_1 k_2) \\ &= (h_1 f(k_1) h_2, g^{-1}(k_1 k_2)) \\ &= (h_1 f(k_1) h_2, g^{-1}(k_1) g^{-1}(k_2)) \\ &= (h_1, g^{-1}(k_1)) \cdot_{fg} (h_2, g^{-1}(k_2)) \\ &= h(h_1, k_1) \cdot_{fg} h(h_2, k_2) \end{aligned}$$

Therefore, h is a homomorphism. We now show that this is an isomorphism. Suppose $h(h_1, k_1) = h(h_2, k_2)$, then $(h_1, g^{-1}(k_1)) = (h_2, g^{-1}(k_2))$, so $h_1 = h_2$, and since g is an isomorphism, then $g^{-1}(k_1) = g^{-1}(k_2)$ indicates $k_1 = k_2$. Hence, h is injective. Furthermore, for arbitrary $(h_1, k_1) \in H \rtimes_{f \circ g} K$, we have $h(h_1, g(k_1)) = (h_1, k_1)$. Therefore, h is surjective, so h is an isomorphism. This concludes the proof. $\square$

Problem 5.8. *Let K and H be two groups and $f : K \to Aut(H)$ a group homomorphism. Prove that for every $s \in Aut(H)$ the external semidirect products $H \rtimes K$ with respect to f and $f' : K \to Aut(H)$, defined by $f'(k) = s \circ f(k) \circ s^{-1}$ for $k \in K$, are isomorphic.*

Proof. We show that $H \rtimes_f K \cong H \rtimes_{f'} K$. Define $h : H \rtimes_f K \to H \rtimes_{f'} K$ by $h(h_1, k_1) = (s(h_1), k_1)$. Note that

$$\begin{aligned}
h((h_1, k_1) \cdot_f (h_2, k_2)) &= h(h_1 f(k_1) h_2, k_1 k_2) \\
&= (s(h_1 f(k_1) h_2), k_1 k_2) \\
&= (s(h_1)(s \circ f(k_1))(h_2), k_1 k_2) \\
&= (s(h_1) f'(k_1) s(h_2), k_1 k_2) \\
&= (s(h_1), k_1) \cdot_{f'} (s(h_2), k_2) \\
&= h(h_1, k_1) \cdot_{f'} h(h_2, k_2)
\end{aligned}$$

Therefore h is a homomorphism. We now show that h is an isomorphism. Suppose $h(h_1, k_1) = h(h_2, k_2)$, then $(s(h_1), k_1) = (s(h_2), k_2)$, then $k_1 = k_2$ and $s(h_1) = s(h_2)$, but since s is an automorphism, then $h_1 = h_2$, therefore h is injective. For arbitrary $(h_1, k_1) \in H \rtimes_{f'} K$, we have $h(s^{-1}(h_1), k_1) = (h_1, k_1)$. Hence, h is surjective, and h is an isomorphism. This concludes the proof. $\square$

Problem 5.9. *Let G be the external semidirect product of $\mathbb{Z}/n\mathbb{Z}$ and $(\mathbb{Z}/n\mathbb{Z})^\times$ with respect to the isomorphism $(\mathbb{Z}/n\mathbb{Z})^\times \to Aut(\mathbb{Z}/n\mathbb{Z})$. Prove that G is isomorphic to a subgroup of S_n.*

Proof. Denote $G = \mathbb{Z}/n\mathbb{Z} \rtimes (\mathbb{Z}/n\mathbb{Z})^\times$. G is called the holomorph of $\mathbb{Z}/n\mathbb{Z}$. By considering the short exact sequence $1 \to \mathbb{Z}/n\mathbb{Z} \to G \to (\mathbb{Z}/n\mathbb{Z})^\times \to 1$, we note that there is a left translation action f of G acting on X, the set of left cosets of $(\mathbb{Z}/n\mathbb{Z})^\times$ with respect to G, which contains n cosets: $f(g)(a(\mathbb{Z}/n\mathbb{Z})^\times) = ga(\mathbb{Z}/n\mathbb{Z})^\times$, for $g \in G$ and $a \in \mathbb{Z}/n\mathbb{Z}$. This action induces a homomorphism $h : G \to S_n$, where $h(g) = f(g) = f_g$ as an action that permutes the n cosets, defined as the action of left multiplication by $g \in G$. Furthermore, this homomorphism is an injection because if $h(g_1) = h(g_2)$, then $f_{g_1} = f_{g_2}$, which means the left translation by g_1 and g_2 are the identical action, and then $g_1 = g_2$. Therefore, this shows there is an isomorphism from G to a subset of S_n by embedding. $\square$

Problem 5.10. *Let H be a group and $\varphi \in Aut(H)$. Prove that there is a group G containing H as a subgroup and an element $g \in G$ such that $\varphi(h) = ghg^{-1}$ for all $h \in H$, i.e., φ extends to an inner automorphism of G.*

Proof. Define G as the holomorph of H, i.e. $G = H \rtimes \mathrm{Aut}(H)$, as the operation is defined as $(h_1, k_1) \cdot (h_2, k_2) = (h_1 k_1(h_2), k_1 k_2)$. Take $g = (\mathrm{id}_H, \varphi)$, note that $g \in G$ by definition. Consider the inner automorphism θ in G, that is defined by conjugation by g. For arbitrary $(h, h_1) \in G$, define $\theta(h, h_1) = g(h, h_1)g^{-1} = (\mathrm{id}_H, \varphi)(h, h_1)(\mathrm{id}_H, \varphi^{-1}) = (\mathrm{id}_H, \varphi)(h, h_1\varphi^{-1}) = (\varphi(h), \varphi h_1 \varphi^{-1})$. In particular, the restriction of θ on H is exactly the map φ. In particular, φ extends to an inner automorphism of G. □

6 MATH 210A Homework 6

Problem 6.1. *Let $1 \to H \to G \to F \to 1$ be an exact sequence of finite groups such the F is a cyclic group and the integers $|H|$ and $|F|$ are relatively prime. Prove that the sequence is split.*

Proof. Since the sequence is exact, it suffices to show it splits. Denote $s : H \to G$ and $t : G \to F$.

Without loss of generality say $|H| = n$ and $|F| = m$, where $\gcd(n, m) = 1$. Since F is cyclic, then $F \cong \mathbb{Z}/m\mathbb{Z}$, and there is an element x of order m, i.e. $\langle x \rangle = F$. Since the sequence is exact, then t is surjective, hence there exists some $y \in G$ such that $t(y) = x$. Since t is a homomorphism, then $t(y^m) = x^m = e$, and it is obvious that y has order m. Now note that $z = y^n$ also has order m since $\gcd(n, m) = 1$. Notice that given by the map, we have $t(z) = x^n = w$. Note that w is still a generator of F because it has order m.

Define a group homomorphism $v : F \to G$ by $v(w) = z$. This defines the whole structure since w generates F. Note that this is well-defined because $v(e) = v(w^m) = z^m = e$. We claim that $t \circ v = \mathbf{id}_F$. Indeed, $t(v(w)) = t(z) = w$ by definition, and therefore $t \circ v = \mathbf{id}_F$ as desired. Hence, the sequences is split by definition. □

Problem 6.2. *For every two nonzero integers n and m construct an exact sequence*

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/nm\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0.$$

For which n and m is the sequence split?

Proof. Define $s : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/nm\mathbb{Z}$ where $[a]_n \mapsto [am]_{nm}$, and define $t : \mathbb{Z}/nm\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ as $[a]_{nm} \mapsto [a]_m$. One can check that s is injective and t is surjective. This defines the exact sequence as claimed in the problem.

We now show that the sequence splits if and only if $\gcd(n, m) = 1$. Suppose $\gcd(n, m) = 1$, then by Chinese Remainder Theorem, there is a decomposition $\mathbb{Z}/nm\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$. In particular, there is an embedded subgroup $K \subseteq \mathbb{Z}/nm\mathbb{Z}$ where $K \cong \mathbb{Z}/m\mathbb{Z}$, which can be constructed by the set $\{[n]_{nm}, [2n]_{nm}, \cdots, [mn]_{nm} = e\}$. Note that K is cyclic and is generated by $[n]_{nm}$. Now, the restricted map $t|_K$ maps $[n]_{nm} \mapsto [n]_m$, which maps the generator of a cyclic group of to a generator of another cyclic group (since $\gcd(n, m) = 1$). Furthermore, the two cyclic groups have the same order, which gives an isomorphism. By definition, the exact sequence splits.

Now suppose the exact sequence splits, then $\mathbb{Z}/nm\mathbb{Z} = \mathbb{Z}/n\mathbb{Z} \rtimes K$ for some subgroup $K \subseteq G$. However, since the order of semi-direct product is the product of n and order of K, then K has order m. Note that $\mathbb{Z}/nm\mathbb{Z} = H' \rtimes K'$ as an internal product for some $H' \cong \mathbb{Z}/n\mathbb{Z}$ and $K' \cong K$. In particular, $K' \subseteq \mathbb{Z}/nm\mathbb{Z}$ is a subgroup, but $\mathbb{Z}/nm\mathbb{Z}$ is cyclic, so K' has to be cyclic, which means K is cyclic, and in particular $K = \mathbb{Z}/m\mathbb{Z}$. Suppose, towards contradiction, that there is $\varphi : \mathbb{Z}/m\mathbb{Z} \to \operatorname{Aut}(\mathbb{Z}/n\mathbb{Z})$ that is not trivial. In particular, take φ to be the conjugation action by fixing some $k \in \mathbb{Z}/m\mathbb{Z}$, by remark in class. Therefore, for some $h \in \mathbb{Z}/n\mathbb{Z}$, $khk^{-1} \neq h$, so $kh \neq hk$. By considering $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}$ as embedded in $\mathbb{Z}/nm\mathbb{Z}$, we have $kh \neq hk$ in $\mathbb{Z}/nm\mathbb{Z}$. However, $\mathbb{Z}/nm\mathbb{Z}$ is Abelian, contradiction. Hence, φ has to be trivial, then by remark in class we know the semidirect product is direct. However, this is only the case when $\gcd(n, m) = 1$. This concludes the proof. $\square$

Problem 6.3. *Let G be a group of order 8 and $H \subseteq G$ a cyclic subgroup of order 4. Show that the exact sequence*

$$1 \to H \to G \to G/H \to 1$$

is split if $G = D_4$ and it is not split if $G = Q_8$.

Proof. Note that G/H is always defined as a group since it has index 2. Furthermore, $G/H \cong \mathbb{Z}/2\mathbb{Z}$. Since H is a cyclic group of order 4, then it is essentially generated by an element $x \in H$ of order 4.

Suppose $G = D_4$. By remark in class, $G = H \rtimes K$ where K is a cyclic group generated by some $y \in G \backslash H$ of order 2. Therefore, $G/H \cong K$. Define $v : G/H \to G$ by $v(yH) = y$ and $v(H) = e$. Note that $t : G \to G/H$ is the canonical surjective homomorphism, then we have $t \circ v(yH) = t(y) = yH$ and $t \circ v(H) = t(e) = H$. Hence, the exact sequence is split by definition.

Suppose $G = Q_8$. Suppose, towards contradiction, that Q_8 has a split short exact sequence, then there is a subgroup $K \subseteq G$ such that for the canonical surjective homomorphism $t : G \to G/H$, the restricted map $t|_K$ is an isomorphism towards G/H. Note that for this to happen, K has to be of order 2. Denote $G = \{e, x, x^2, x^3, y, xy, x^2y, x^3y\}$ where $H = \langle x \rangle$. However, G only has one element $x^2 = y^2$ of order 2, then $K = \{e, x^2\}$. Now notice that $t|_K(e) = H$ and $t|_K(x^2) = x^2H = H$. In particular, this is not an isomorphism, contradiction. Therefore, Q_8 does not split over the sequence. $\square$

Problem 6.4. *Prove that the infinite dihedral group D_∞ is generated by two elements of order 2.*

Proof. Recall from class that $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. It suffices to show that D_∞ has a presentation $\langle x, y \mid x^2, y^2 \rangle$. Note that $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ are cyclic, then pick generators $\sigma \in \mathbb{Z}$ and $\tau \in \mathbb{Z}/2\mathbb{Z}$ with $D_\infty = \langle \sigma \rangle \rtimes \langle \tau \rangle$. Note that $\tau\sigma\tau^{-1} = \sigma^{-1}$, and $\tau^2 = e$, so $(\tau\sigma)^2 = e$. Define a map $f : \langle x, y \mid x^2, y^2 \rangle \to$

D_∞ generated by $x \mapsto \sigma\tau$ and $y \mapsto \tau$. This is a homomorphism because one can extend it from the two homomorphisms $f_1 : \langle x \mid x^2 \rangle \to D_\infty$ by $x \mapsto \sigma\tau$ and $f_2 : \langle y \mid y^2 \rangle \to D_\infty$ by $y \mapsto \tau$. On the other hand, define a homomorphism $g : D_\infty \to \langle x, y \mid x^2, y^2 \rangle$ generated by $\sigma \mapsto xy$ and $\tau \mapsto y$, as every element of D_∞ is of the form $\sigma^n \tau^m$. We now show that f and g are inverses. Note that $fg(\sigma) = f(xy) = \sigma\tau^2 = \sigma$ and $fg(\tau) = f(y) = \tau$. On the other hand, $gf(x) = g(\sigma\tau) = xy^2 = x$ and $gf(y) = g(\tau) = y$. Hence, f and g are inverse, so this defines an isomorphism. In particular, D_∞ has the presentation defined above. $\square$

Problem 6.5. *Prove that every group generated by two elements of order* 2 *is isomorphic to a factor group of* D_∞.

Proof. We first show that a group G generated by two elements x, y of order 2 is isomorphic to D_{2n} for some n. Let n be the order of element xy. We define a map $f : G \to D_{2n}$ that sends $xy \mapsto r$ (rotation operation) and $y \mapsto s$ (reflection operator). One can define a homomorphism based on these mappings since G is generated by x and y and D_{2n} is generated by rs and s (which generates r and s). We now show that this homomorphpsim f is an isomorphism. Indeed, first notice that f is surjective as $r^m s^j = (rs)^m s^{j-m} = f(x^m y^{j-m})$. Also note that both groups have $2n$ elements by checking the definition. Therefore, f must be an isomorphism, so $G \cong D_{2n}$.

We now show that D_{2n} is always a factor group of D_∞. Define $f : D_\infty \to \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ by $f(a, b) = ([a]_n, b)$. This is well-defined since $D_\infty = \mathbb{Z} \rtimes_\varphi \mathbb{Z}/2\mathbb{Z}$ for some homomorphism $\varphi : \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z})$. Note that there are two such maps since $\mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$ has two elements. Now $f((a, b) \cdot (c, d)) = f(a + (-1)^b c, b + d) = ([a + (-1)^b c]_n, b + d) = ([a]_n, b) \cdot ([c]_n, d)$. Therefore, f is a homomorphism, and is obviously surjective. Therefore, $D_\infty / \ker(f) \cong D_{2n}$. Since the kernel is non-trivial, then D_{2n} is always a factor group. This concludes the proof. $\square$

Problem 6.6. *Let X be a subset in a group G. Prove that $\langle\langle X \rangle\rangle = \langle Y \rangle$, where Y is the union of gXg^{-1} for all $g \in G$. (Here $\langle\langle X \rangle\rangle$ denotes the normal subgroup generated by X.)*

Proof. Recall that normal subgroups of G are closed under conjugation action of G, then we have $Y \subseteq \langle\langle X \rangle\rangle$. Therefore, $\langle Y \rangle \subseteq \langle\langle X \rangle\rangle$. Now observe that for every element $y = g_1 x_1 g_1^{-1} \cdots g_n x_n g_n^{-1}$ of $\langle Y \rangle$, applying a conjugation by $g \in G$ would result in $(gg_1)x_1(gg_1)^{-1}(gg_2)x_2(gg_2)^{-1} \cdots (gg_n)x_n(gg_n)^{-1} \in \langle Y \rangle$. Therefore, $\langle Y \rangle$ is a normal subgroup of G as well. Furthermore, $\langle Y \rangle$ contains X. Recall that $\langle\langle X \rangle\rangle$ is the smallest normal subgroup of G that contains X by construction. Therefore, $\langle\langle X \rangle\rangle \subseteq \langle Y \rangle$. Therefore, $\langle\langle X \rangle\rangle = \langle Y \rangle$. $\square$

Problem 6.7. *Let G be the group defined by generators a, b and relations $w^3 = e$ for all words w in a and b. Show that G is finite and find $|G|$. (Hint: Show that ab and ba commute and the subgroup generated by ab and ba is normal.)*

Proof. By definition $(ab^2)^3 = e$, which means $ab^2ab^2ab^2 = e$. Since $w^3 = e$ for all words, then $a^3 = e$ and $b^3 = e$, and we have $ab^2ab^{-1}a^{-2}b^{-1} = e$, which means $ab^2a = ba^2b$. Therefore, ab and ba commute.

Consider the subgroup H of G generated by ab and ba. (One can check that one of the two terms cannot induce the other.) Since ab and ba commutes, then every element in H can be written of the form $(ab)^m(ba)^n$ for some $0 \leq m, n < 3$. In particular, there are 9 elements in H. Since every element has order 3, then this group must be $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ by group classification, and the group is Abelian.

We now show that $H \lhd G$. Note that $a(ab)a^{-1} = a^2ba^2$, where $(ba)(a^2ba^2)(ab) = b^3 = e$, which means $a(ab)a^{-1} \in H$. Similarly, $a(ba)a^{-1} = ab \in H$. In a symmetrical argument, $b(ab)b^{-1}, b(ba)b^{-1} \in H$. In particular, since every element in H is of the form $(ab)^m(ba)^n$, then we can rewrite the conjugation in the form $a(ab)^m(ba)^na^{-1} = (a(ab)a^{-1} \cdots a(ab)a^{-1})(a(ba)a^{-1} \cdots a(ba)a^{-1}) \in H$, and similarly for conjugation by b. Therefore, $aHa^{-1}, bHb^{-1} \subseteq H$. Since a, b generates G, then $gHg^{-1} \subseteq H$ for all $g \in G$. Hence, $H \lhd G$.

Since $H \lhd G$, then one can consider the factor group G/H. We claim that the group only has three elements: H, aH and bH. Note that $a^2 = b \cdot (ab)^{-1} \in bH$, and similarly $b^2 \in aH$. Furthermore, clearly $a \notin H$, otherwise $a \in H$ indicates $b \in H$, so $G = H$ since they generate the group, contradiction. Similarly, $b \notin H$. Finally, $a \notin bH$ because otherwise $a^{-1}b \in H$ whcih means $a^2b \in H$ so $a \in H$, same contradiction; similarly $b \notin aH$, which means these three classes are distinct. In particular, $[G : H] = 3$, which means $|G| = 27$. □

Problem 6.8. *Prove that if the free groups $F(X)$ and $F(Y)$ for finite sets X and Y are isomorphic, then $|X| = |Y|$.*

Proof. The problem is essentially saying that if a free group F is generated by a finite basis, then all bases have the same size. It suffices to show that if F has a basis of order n, then $F/[F, F]$ is a free Abelian group with basis of order n, which must be isomorphic to $\mathbb{Z}^n$. However, $\mathbb{Z}^m \cong \mathbb{Z}^n$ if and only if $m = n$ (based on ring and module theory knowledge), which means all bases have the same size. This proof is modified from a proof in Rotman's Advanced Modern Algebra.

Let F be a free group with basis $X = \{x_1, \cdots, x_n\}$. We claim that the induced set $X' = \{x_1[F, F], \cdots, x_n[F, F]\}$ is a basis of $F/[F, F]$. Obviously X' generates $F/[F, F]$ by correspondence. Furthermore, $F/[F, F]$ is obviously Abelian by the Abelianization.

We now prove that for every Abelian group G and every function $\gamma : X' \to G$, there exists a unique homomorphism $g' : F/[F, F] \to G$ with $g'(x_i[F, F]) = \gamma(x_i[F, F])$ for all $x_i[F, F] \in X'$. As a result, we would show that $F/[F, F]$ is an Abelian group of rank n. Consider the following figure:

$$\begin{tikzcd}
F \arrow[rr, "\pi"] \arrow[dr, dashed, "g"] & & F/[F,F] \arrow[dl, dashed, "g'"'] \\
& G & \\
X \arrow[uu, "p"] \arrow[ur, "\gamma v"] \arrow[rr, "v"] & & X' \arrow[uu, "p'"'] \arrow[ul, "\gamma"']
\end{tikzcd}$$

Take arbitrary Abelian group G and function $\gamma : X' \to G$. Let $p : X \to F$ and $p' : X' \to F/[F,F]$ be inclusion maps and $\pi : F \to F/[F,F]$ be the canonical surjective homomorphism, and $v : X \to X'$ be the obvious correspondence $x \mapsto x[F,F]$. The setting induces a function $\gamma v : X \to G$ by composition. By applying the universal property of free group F, this induces a unique homomorphismm $g : F \to G$ such that $gf = \gamma v$. As a result one can induce the map $g' : F/[F,F] \to G$ by $y[F,F] \mapsto g(y)$. Note that g' is well-defined because for $y[F,F] = y'[F,F]$, $y'^{-1}y \in [F,F]$, so we have $g'(y[F,F]) = g(y) = g(y') = g'(y'[F,F])$. Note that the equality in the middle holds because $\mathrm{im}(g) \subseteq G$ is an Abelian subgroup, but then $F/\ker(g)$ has to be Abelian, however, $[F,F]$ is the smallest subgroup H such that F/H is Abelian, which means $[F,F] \subseteq \ker(g)$, and so $y'^{-1}y \in \ker(g)$, which means the equality in the middle holds. In particular, $g'p'v = g'\pi p = gp = \gamma v$, but v is a canonical surjection map, which means right cancellation holds, so $g'p' = \gamma$. Note that g' is the unique map that satisfies this relation, otherwise if $g''p' = \gamma$, then both g' and g'' map the embedded set $X' \subseteq F/[F,F]$ to the same generating set into G, which shows they are the same.

Now, $F/[F,F]$ is an Abelian group with a generating subset X' of size n and $F/[F,F]$ satisfies that $\forall G$ group and $\forall \gamma : X' \to G$, there is a unique homomorphism $g' : F/[F,F] \to G$ such that $g'(x_i) = \gamma(x_i)$ for all $x_i \in X'$ (by embedding). We now show that $F/[F,F]$ is isomorphic to $\mathbb{Z}^n$, which means $F/[F,F]$ is a free Abelian group of rank n. By the unique property we just proved, take $p : X' \to F/[F,F]$ and $q : X' \to \mathbb{Z}^n$ as the inclusion maps, then there is a unique homomorphism $g' : F/[F,F] \to \mathbb{Z}^n$ such that $g(x_i) = q(x_i)$ for all $x_i \in X'$. In particular, the property says that $g'p = q$. On the other hand, the universal property of free Abelian group tells us that there is a unique homomorphism $h : \mathbb{Z}^n \to F/[F,F]$ such that $hq = p$. Note that $hg'p = hq = p$, and hg is the unique homomorphism that satisfies this relation. One can conclude that $hg' = \mathbf{id}_{F/[F,F]}$. In a similar fashion, we can also show that $g'h = \mathbf{id}_{\mathbb{Z}^n}$, and therefore g is an isomorphism. In particular, $F/[F,F] \cong \mathbb{Z}^n$. It then follows from the first paragraph that X and Y must be of the same size. □

Problem 6.9. *Find a presentation of the quaternion group Q_8 by generators and relations.*

Proof. Denote $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. Denote $G = \langle x, y, c \mid c^2, x^2c, y^2c, xyxyc \rangle$. We claim that $Q_8 = G$. Note that by corresponding i to x, j to y and -1 to c, then Q_8 satisfies $(-1)^2 = 1$, $i^2(-1) = 1$, $j^2(-1) = 1$, $ijijc = k^2c = 1$. Therefore, Q_8 must be a quotient of G. It suffices to show that G has order 8. Since $c^2 = e$, then $x^2 = y^2 = c^{-1} = c$. Hence, c must commute with all generators. Therefore, $c \in Z(G)$. In particular, $\langle c \rangle$ is a normal subgroup of G, with $G/\langle c \rangle = \langle x, y, c \mid c, c^2, x^2c, y^2c, xyxyc \rangle$, therefore, $G/\langle c \rangle = \langle x, y, c \mid x^2, y^2, xyxy \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, which has order 4. Therefore, G has order 8, so $G \cong Q_8$. □

Problem 6.10. *Determine the order of the group $\langle a, b \mid a^5, b^4, aba^{-1}b \rangle$.*

Proof. Note that each element in the group can be represented by $a^{\alpha_1}b^{\beta_1}a^{\alpha_2}b^{\beta_2}\cdots a^{\alpha_k}b^{\alpha_k}$ for some k and some $0 \leq \alpha_i < 5$ and $0 \leq \beta_i < 4$. Furthermore, we can rewrite the third relation as $aba^{-1} = b^3$ and/or $bab = a$. Therefore, by induction, one can rewrite the arbitrary word above into the form

$a^\alpha b^\beta$ for some $0 \le \alpha < 5$ and $0 \le \beta < 4$. In particular, the group has at most 12 elements. However, one element a has order 5, while element b has order 2 or 4, then by Lagrange Theorem the only choice would be a group of order 10. $\square$

7 MATH 210A Homework 7

Problem 7.1. *Let $\mathscr{C}$ be a category. Consider the category $\mathbf{Ar}(\mathscr{C})$ of all diagrams of the shape $\bullet \to \bullet$. Determine all initial and final objects in $\mathbf{Ar}(\mathbf{Set})$.*

Proof. The initial object in $\mathbf{Ar}(\mathbf{Set})$ is the identity morphism $\mathbf{id}_\varnothing$ from $\varnothing$ to itself. Pick arbitrary object $A \in \mathbf{Ar}(\mathbf{Set})$, then A is a morphism $a : C \to D$ $\mathbf{Set}$ in for some $C, D \in \mathbf{Set}$. Recall that $\varnothing$ is the initial object in $\mathbf{Set}$, then by definition there exists a unique morphism $c : \varnothing \to C$ and a unique morphism $d : \varnothing \to D$ in $\mathbf{Set}$. Therefore, $a \circ c = d$ since they are both morphisms from $\varnothing$ to D, and therefore $a \circ c = d \circ \mathbf{id}_\varnothing$. Therefore, by definition, (c, d) is a unique pair of morphisms such that the diagram commutes, which means $\mathbf{id}_\varnothing$ is the initial object of $\mathbf{Ar}(\mathbf{Set})$.

The final objects in $\mathbf{Ar}(\mathbf{Set})$ are all the morphisms in $\mathbf{Set}$ between singleton sets. Take arbitrary morphism $f : C \to D$ where $C = \{*\}$ and $D = \{\star\}$. Observe that f is the unique morphism from C to D since they are final objects in the category of sets. Take arbitrary morphism $g : E \to F$ between some objects $E, F \in \mathbf{Set}$. Recall that singleton sets are the final objects in $\mathbf{Set}$, then there exists a unique morphism $h_1 : E \to C$ and unique morphism $h_2 : F \to D$. Recall that D is a final objet in $\mathbf{Set}$, then there is a unique morphism from E to D in $\mathbf{Set}$, which has to be $f \circ h_1 : E \to D$. However, $h_2 \circ g$ is also a morphism from E to D, then $h_2 \circ g = f \circ h_1$, so the square commutes, which means $f : C \to D$ is a final object indeed. Since f is arbitrary, then all morphisms in $\mathbf{Set}$ between singleton sets are final objects in $\mathbf{Ar}(\mathbf{Set})$. $\square$

Problem 7.2. *Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two categories with initial objects A_1 and A_2 respectively. Prove that (A_1, A_2) is an initial object in $\mathscr{C}_1 \times \mathscr{C}_2$.*

Proof. Consider arbitrary object $(B_1, B_2) \in \mathscr{C}_1 \times \mathscr{C}_2$, then $B_1 \in \mathscr{C}_1$ and $B_2 \in \mathscr{C}_2$. Since $A_1 \in \mathscr{C}_1$ is initial, then there exists a unique morphism $f_1 : A_1 \to B_1$ in $\mathscr{C}_1$, and similarly there is a unique morphism $f_2 : A_2 \to B_2$ in $\mathscr{C}_2$. In particular, $(f_1, f_2) : (A_1, A_2) \to (B_1, B_2)$ is the unique morphism between the two objects in $\mathscr{C}_1 \times \mathscr{C}_2$. Therefore, (A_1, A_2) is an initial object in $\mathscr{C}_1 \times \mathscr{C}_2$ by definition. $\square$

Problem 7.3. *Give an example of a category $\mathscr{C}$ and a full subcategory $\mathscr{C}'$ such that the initial objects of $\mathscr{C}$ and $\mathscr{C}'$ are different.*

Proof. Consider the following category $\mathscr{C}_1$ induced from a poset category, and notice that $1 \in \mathscr{C}_1$ is initial because for all objects O in $\mathscr{C}_1$, there is a unique morphism $1 \to O$. Also note that 2 is not initial because the arrow from 2 to 1 is not unique.

$$\begin{tikzcd}
1 \arrow[loop above] \arrow[r] \arrow[rr, bend left=50] \arrow[rrr, bend left=60] & 2 \arrow[loop above] \arrow[l, bend right] \arrow[l, bend left] \arrow[r] \arrow[rr, bend right=40] & 3 \arrow[loop above] \arrow[r] & 4 \arrow[loop above]
\end{tikzcd}$$

We induce a new category $\mathscr{C}_2$ by deleting object 1 and all associating morphisms, we then have the following category:

$$\begin{tikzcd}
2 \arrow[loop above] \arrow[r] \arrow[rr, bend right=40] & 3 \arrow[loop above] \arrow[r] & 4 \arrow[loop above]
\end{tikzcd}$$

Recall that deleting a class of objects and all associated morphisms would induce a full subcategory, then $\mathscr{C}_2 \subseteq \mathscr{C}_1$ is full. Also note that 2 is now the initial object by definition.

The reason to add morphisms from 2 to 1 is to avoid having the situation where deleting 4 is "isomorphic" to deleting 1, which makes the initial objects in two categories "the same" in some sense. $\square$

Problem 7.4. *A morphism $f : A \to B$ in a category $\mathscr{C}$ is called monomorphism if for every two morphisms $g, h : C \to A$, the equality $fg = fh$ implies $g = h$. Show that the composition of two monomorphisms is a monomorphism. Determine all monomorphisms in* **Set** *and* **Grp**. *Define the dual notion of an epimorphism and determine all epimorphisms in* **Set**.

Proof. Without loss of generality, let $f : A \to B$ and $g : B \to C$ be monomorphisms in some category $\mathscr{C}$. We claim that $g \circ f$ is another monomorphism. Take arbitrary morphisms $\alpha, \beta : D \to A$ such that $(g \circ f) \circ \alpha = (g \circ f) \circ \beta$. By the composition we have $g \circ (f \circ \alpha) = g \circ (f \circ \beta)$, so this implies $f \circ \alpha = f \circ \beta$ since g is a monomorphism. In a similar fashion we have $\alpha = \beta$ since f is a monomorphism. As a whole, we showed that $(g \circ f) \circ \alpha = (g \circ f) \circ \beta$ would imply $\alpha = \beta$, which means $g \circ f$ is a monomorphism by definition.

We claim that all monomorphisms in **Set** are the injective functions. Consider arbitrary injective function $f : A \to B$, suppose there is $g, h : C \to A$ such that $fg = fh$. Suppose, towards contradiction, that for some $x \in C$ there is $a = g(x) \neq h(x) = b$, then $f(a) \neq f(b)$ since f is injective, which means $fg(x) \neq fh(x)$, contradiction. Therefore, $g(x) = h(x)$ for all $x \in C$, which means $g = h$, so injective functions are monomorphisms in **Set**.

We claim that all monomorphisms in **Grp** are the injective homomorphisms. Consider arbitrary injective homomorphism $f : G \to G'$. Consider arbitrary group homomorphisms $h, k : G'' \to G$ such that $fh = fk$. In a similar fashion, suppose $h(g) \neq k(g)$ for some $g \in G''$, then $f(h(g)) \neq f(k(g))$ since f is injective, contradiction to the fact that $fh = fk$. Hence, $h = k$. Therefore, injective homomorphisms are the monomorphisms in **Grp**.

In light of the dual notion, we define $f : A \to B$ in a category $\mathscr{C}$ to be an epimorphism if for every two morphisms $g, h : B \to C$ such that $gf = hf$, we have $g = h$. The epimorphisms in

Set are exactly the surjective maps of **Set**. Pick arbitrary surjective map $f : A \to B$. Suppose there are $g, h : B \to C$ such that $gf = hf$. Suppose, towards contradiction, that some $g(x) \neq h(x)$ for some $x \in B$. Since f is surjective, then there exists some $y \in A$ such that $f(y) = x$, so $gf(y) = g(x) \neq h(x) = hf(y)$, contradiction. Hence, $g = h$, which means f is an epimorphism. $\square$

Problem 7.5. *An object P of a category $\mathscr{C}$ is called projective, if for every epimorphism $f : B \to C$ and every morphism $g : P \to C$ there is a morphism $h : P \to B$ such that $fh = g$. Show that the free groups in* **Grp** *are projective objects. (You may use without proof the fact that epimorphisms in* **Grp** *are surjective homomorphisms.)*

Proof. Since P is a free group, then there exists some set of elements S such that $P = F(S)$, i.e. freely generates P. Consider arbitrary group homomorphism $f : B \to C$ and a corresponding homomorphism $g : P \to C$. We construct a set map $\bar{h} : S \to B$ by mapping each $s \in S$ to some element $b \in B$ such that $g(s) = f(b)$. Note that such $b \in B$ always exists since $f : B \to C$ is a surjective homomorphism. By the universal property of free groups, the map $\bar{h} : S \to B$ extends to a unique group homomorphism $h : P \to B$. By definition, $fh(s) = f(\bar{h}(s)) = f(b) = g(s)$ for all $s \in S$. However, since S generates the free group P, then correspondingly $fh(p) = g(p)$ for all elements $p \in P$, so $fh = g$. Therefore, P is projective by definition. Therefore, free groups in **Grp** are projective objects. $\square$

Problem 7.6. *Let X be an object of a category $\mathscr{C}$. Consider a new category $\mathscr{C}/X$ with objects the morphisms $f : Y \to X$ for Y in $\mathscr{C}$ and morphisms between $f : Y \to X$ and $g : Z \to X$ being morphisms $h : Y \to Z$ such that $gh = f$. The product of two objects $f : Y \to X$ and $g : Z \to X$ in $\mathscr{C}/X$ is called the fiber product of Y and Z over X, denoted $Y \times_X Z$. Show that fiber products exist in $\mathscr{C}/X =$* **Set**$/X$ *and* **Grp**$/X$.

Proof. We first find the fiber products in **Set**$/X$. Pick arbitrary objects $f : Y \to X$ and $g : Z \to X$ in **Set**$/X$. We want to define $Y \times_X Z$ such that for all object $u : U \to X$ in **Set**$/X$, there exists some unique $k : U \to Y \times_X Z$ that makes the following diagram commute:

$$\begin{tikzcd}
 & U \arrow[dl, "p_2"'] \arrow[d, dashed, "\exists! k"] \arrow[dr, "q_2"] \arrow[ddd, bend right, "u"'] & \\
 & Y \times_X Z \arrow[dl, "p_1"'] \arrow[dr, "q_1"] \arrow[dd, "f \times g"] & \\
Y \arrow[dr, "f"'] & & Z \arrow[dl, "g"] \\
 & X &
\end{tikzcd}$$

Define $Y \times_X Z = \{(y, z) : f(y) = g(z)\} \in$ **Set** and let $f \times g : Y \times_X Z \to X$ be the set map that sends the pair $(y, z) \in Y \times_X Z$ to $x \in X$ such that $x = f(y) = g(z)$. Note that $f \times g$ is an object in **Set**$/X$ by definition. Moreover, there exists projection maps $p_1 : Y \times_X Z \to Y$ and $q_1 : Y \times_X Z \to Z$

such that $fp_1 = f \times g = gq_1$. By definition, we may consider $p_1 : f \times g \to f$ and $p_2 : f \times g \to g$ as morphisms in $\mathbf{Set}/X$. We claim that $Y \times_X Z \in \mathbf{Ob}(\mathbf{Set}/X)$ along with $p_1, p_2 \in \mathbf{Mor}(\mathbf{Set}/X)$ is the product construction we want.

Take arbitrary object $u : U \to X$ in $\mathbf{Set}/X$ for some $U \in \mathbf{Ob}(\mathbf{Set})$, along with morphisms $p_2 : U \to Y$ and $q_2 : U \to Z$, i.e. $fp_2 = u = gq_2$. It suffices to show that there exists a unique morphism $k : U \to Y \times_X Z$, i.e. $u = (f \times g)k$, as well as $p_2 = p_1 k$, $q_2 = q_1 k$.

Define $k : U \to Y \times_X Z$ by the set map $k(a) = (p_2(a), q_2(a))$. Notice that for arbitrary $a \in U$, there is $(f \times g)(k(a)) = (f \times g)(p_2(a), q_2(a)) = b$ for some $b = fp_2(a) = gq_2(a)$. In particular, $(f \times g)k = fp_2 = gq_2 = u$. We now show that $p_2 = p_1 k$. For arbitrary $a \in U$, there is $p_1(k(a)) = p_1(p_2(a), q_2(a)) = p_2(a)$. Therefore, $p_1 k = p_2$. Similarly, one can show that $q_2 = q_1 k$.

Finally, we show that such construction of k is unique. Suppose we also have another morphism $k' : U \to Y \times_X Z$, i.e. $u = (f \times g)k'$, such that $p_2 = p_1 k'$ and $q_2 = q_1 k'$. Suppose, towards contradiction, that $k'(a) \neq k(a)$ for some $a \in U$, then either $p_1(k'(a)) \neq p_1(k(a))$ or $q_1(k'(a)) \neq q_1(k(a))$. Without loss of generality, let us assume it is the former, but that means $p_2(a) \neq p_2(a)$, contradiction. Hence, $k' = k$. This shows that k is unique, and that concludes the proof: the construction above defines fiber product in $\mathbf{Set}/X$.

We extend this construction to $\mathbf{Grp}/X$ analogously. The only thing we have to check is that our construction of $Y \times_X Z$ is a group with respect to the elementwise multiplication, and our object $f \times g : Y \times_X Z \to X$ is a group homomorphism.

First, notice that elementwise multiplication is closed in $Y \times_X Z$, as $(y_1, z_1) \cdot (y_2, z_2) = (y_1 y_2, z_1 z_2)$, and $f(y_1 y_2) = f(y_1) f(y_2) = g(z_1) g(z_2) = g(z_1 z_2)$ since $f(y_1) = g(z_1)$ and $f(y_2) = g(z_2)$. Furthermore, notice that (e_Y, e_Z) is the identity, and $(e_Y, e_Z) \in Y \times_X Z$ because $f(e_Y) = g(e_Z) = e_X$ by definition. Also, for arbitrary $(y_1, z_1), (y_2, z_2), (y_3, z_3) \in Y \times_X Z$, we have

$$
\begin{aligned}
((y_1, z_1) \cdot (y_2, z_2)) \cdot (y_3, z_3) &= (y_1 y_2, z_1 z_2) \cdot (y_3, z_3) \\
&= (y_1 y_2 y_3, z_1 z_2 z_3) \\
&= (y_1, z_1) \cdot (y_2 y_3, z_2, z_3) \\
&= (y_1, z_1) \cdot ((y_2, z_2) \cdot (y_3, z_3))
\end{aligned}
$$

which shows associativity. Finally, for arbitrary element $(y, z) \in Y \times_X Z$, there is the inverse $(y^{-1}, z^{-1}) \in Y \times_X Z$ since $f(y^{-1}) = f(y)^{-1} = g(z)^{-1} = g(z^{-1})$, and $(y, z) \cdot (y^{-1}, z^{-1}) = (yy^{-1}, zz^{-1}) = (e_Y, e_Z) = (y^{-1}, z^{-1}) \cdot (y, z)$. Therefore, our construction $Y \times_X Z$ is a group indeed.

We now show that $f \times g$ is a group homomorphism. Indeed, note that for arbitrary $(y_1, z_2), (y_2, z_2) \in Y \times_X Z$, we have

$$\begin{aligned}(f \times g)((y_1, z_1) \cdot (y_2, z_2)) &= (f \times g)(y_1 y_2, z_1 z_2) \\ &= x = f(y_1 y_2) = g(z_1 z_2)\end{aligned}$$

In particular, $x = f(y_1)f(y_2) = g(z_1)g(z_2)$, and $x_1 = f(y_1) = g(z_1)$, $x_2 = f(y_2) = g(z_2)$ by definition. In particular, $x = x_1 x_2$. Therefore, we have

$$\begin{aligned}(f \times g)((y_1, z_1) \cdot (y_2, z_2)) &= x \\ &= x_1 x_2 \\ &= (f \times g)(y_1, z_1) \cdot (f \times g)(y_2, z_2)\end{aligned}$$

by definition, which means $f \times g$ is a group homomorphism indeed. Following the argument above, then our construction can be extended to a notion of fiber product in **Grp**$/X$ as well. □

Problem 7.7. *The equalizer of a pair of morphisms $f, g : X \to Y$ in a category $\mathscr{C}$ is a morphism $h : Z \to X$ such that $fh = gh$ and for any morphism $i : T \to X$ in $\mathscr{C}$ such that $fi = gi$ there exists a unique morphism $j : T \to Z$ such that $hj = i$. Show that the equalizers exist in* **Set** *and* **Grp***.*

Proof. Consider the category **Set**. Pick arbitrary set functions $f, g : X \to Y$. Define $Z = \{z \in X \mid f(z) = g(z)\} \in \mathbf{Ob}(\mathbf{Set})$. We claim that the equalizer of the pair of morphisms f, g is a morphism $h : Z \to X$, defined as an inclusion. It automatically follows that $fh = gh$. Consider arbitrary morphism $i : T \to X$ in **Set** such that $fi = gi$. By definition, the image of i must be a subset of Z. In particular, this induces a function $j : T \to Z$ as $j(t) = i(t)$ for all $t \in T$, then it is obvious that $i = hj$. Furthermore, such function j is unique. Suppose there is also some $j' : T \to Z$ such that $hj' = i$. However, notice that $h : Z \to X$ is an injective set map, which means it is a monomorphism in **Set**. By problem 4, since we have $hj' = i = hj$, then $j' = j$. Therefore, our function j is unique. Hence, this function $h : Z \to X$ is an equialızer for the pair of functions $f, g : X \to Y$, which means equalizers exist in **Set**.

Again, we can extend the construction to the category **Grp**. We first show that $Z \subseteq X$ is a subgroup of X. Note that f, g are group homomorphisms. Take arbitrary $z_1, z_2 \in Z$, then $f(z_1 z_2) = f(z_1)f(z_2) = g(z_1)g(z_2) = g(z_1 z_2)$ since $f(z_1) = g(z_1)$ and $f(z_2) = g(z_2)$. Therefore, $z_1 z_2 \in Z$. Note that $e_X \in Z$ obviously. Finally, for arbitrary $z \in Z$, $f(z^{-1}) = f(z)^{-1} = g(z)^{-1} = g(z^{-1})$. Therefore, $z^{-1} \in Z$. Hence, Z is a subgroup of X and is therefore a group. Then define the equalizer of f, g as $h : Z \to X$ which is a group homomorphism, defined as the inclusion. Following the exact same argument as above, $h : Z \to X$ is an equalizer of **Grp**. Therefore, equalizers exist in **Grp**. □

Problem 7.8. *Prove that if X is a set of n elements, then $F(X) \cong \mathbb{Z} * \cdots * \mathbb{Z}$ (n times). Here $*$*

denotes the coproduct of groups.

Proof. We show that for a set of 2 elements, the free group generated by the set is isomorphic to $\mathbb{Z} * \mathbb{Z}$. The rest follows from an induction argument. Recall that coproduct in **Grp** is exactly $G * H = \langle X \coprod Y \mid R \cup S \rangle$ where $G = \langle X \mid R \rangle$ and $H = \langle Y \mid S \rangle$. Therefore, $\mathbb{Z} * \mathbb{Z} = \langle \mathbb{Z} * \mathbb{Z} \mid \varnothing \rangle$.

By the universal property of free group, take $A = \{*\}$, then every map $f : A \to \underline{G}$ is extended uniquely to a group homomorphism $f' : F(A) \to G$, where $\underline{G}$ is the underlying set of a group G. Note that $F(A) = \mathbb{Z}$. Therefore, this induces a correspondence $\mathbf{Hom}(\mathbb{Z}, G) \cong \underline{G}$.

Recall the universal property of coproduct, then $\mathbf{Hom}(\coprod_{i \in I} X_i, Z) \cong \prod_{i \in I} \mathbf{Hom}(X_i, Z)$, and in particular $\mathbf{Hom}(X * Y, Z) \cong \mathbf{Hom}(X, Z) \times \mathbf{Hom}(Y, Z)$. Take arbitrary group G, then $\mathbf{Hom}(\mathbb{Z} * \mathbb{Z}, G) \cong \mathbf{Hom}(\mathbb{Z}, G) \times \mathbf{Hom}(\mathbb{Z}, G)$, then $\mathbf{Hom}(\mathbb{Z} * \mathbb{Z}, G) \cong \underline{G} \times \underline{G}$.

Consider a set $B = \{1, 2\}$, then the universal property of free group induces a group homomorphism $g' : F(B) \to G$ from set map $g : B \to \underline{G}$. Therefore, there is a correspondence between $\mathbf{Hom}(F(B), G)$ and $\underline{G \times G}$ by taking each homomorphism to the pair $(g'(1), g'(2))$, which means $\mathbf{Hom}(F(B), G) \cong \underline{G \times G} = \underline{G} \times \underline{G}$. In particular, $\mathbf{Hom}(F(B), G) \cong \mathbf{Hom}(\mathbb{Z} * \mathbb{Z}, G)$. Since this is true for arbitrary group G, then $\mathbf{Hom}(F(B), -) \cong \mathbf{Hom}(\mathbb{Z} * \mathbb{Z}, -)$. By notation in class, $R^{F(B)} \cong R^{\mathbb{Z} * \mathbb{Z}}$. By remark in class, there is an isomorphism $R^\alpha : R^{F(B)} \to R^{\mathbb{Z} * \mathbb{Z}}$, which is given by an isomorphism $\mathbb{Z} * \mathbb{Z} \to F(B)$. In particlar, $F(\{1, 2\}) \cong \mathbb{Z} * \mathbb{Z}$. Therefore, the free group on two elements is just the coproduct $\mathbb{Z} * \mathbb{Z}$. □

Problem 7.9. *Prove that* $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) \cong D_\infty$.

Proof. In Homework 6 we showed that D_∞ has the presentation $\langle x, y \mid x^2, y^2 \rangle$. Note that $\mathbb{Z}/2\mathbb{Z} = \langle x \mid x^2 \rangle$, then $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle x \mid x^2 \rangle * \langle y \mid y^2 \rangle = \langle x, y \mid x^2, y^2 \rangle$, since it is given by disjoint union of generators as well as disjoint union of their relations. Therefore, $D_\infty \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ since they have the same presentation. □

Problem 7.10. *Prove that the group* $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/5\mathbb{Z})$ *is not solvable.*

Proof. It suffices to show that there is a surjective homomorphism $f : (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/n\mathbb{Z}) \to S_n$. If this is true, then there is a surjective homomorphsim from $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/5\mathbb{Z})$ to S_5. In the dual notion, there is an injective homomorphism from S_5 to $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/5\mathbb{Z})$. In particular, S_5 can be considered as a subgroup embedded in $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/5\mathbb{Z})$. Therefore, since S_5 is not solvable, then $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/5\mathbb{Z})$ cannot be solvable.

Take a set map $f : \{\bar{1}, \bar{2}\} \to S_n$ by mapping $f(\bar{1}) = (1\ 2)$ and $f(\bar{2}) = (1\ 2\ \cdots\ n)$. Recall from previous homework that the two permutations generate S_n. By the universal property of free group, this extends to a unique group homomorphism $\bar{f} : F(\{\bar{1}, \bar{2}\}) \to S_n$, i.e. $\bar{f}(\bar{1}) = (1\ 2)$ and $\bar{f}(\bar{2}) = (1\ 2\ \cdots\ n)$. Note that $\bar{f}$ is surjective because S_n is generated by the two elements and $F(\{\bar{1}, \bar{2}\})$ is a free group. Observe that $\bar{f}(\bar{2}^n) = e = \bar{f}(\bar{1}^2)$. Therefore, $\ker(\bar{f})$ includes $\bar{1}^2, \bar{2}^n$. By the third isomorphism theorem, $\Big(F(\{\bar{1}, \bar{2}\}) / \langle\langle \bar{1}^2, \bar{2}^n \rangle\rangle\Big) \Big/ \Big(\ker(f) / \langle\langle \bar{1}^2, \bar{2}^n \rangle\rangle\Big) \cong F(\{\bar{1}, \bar{2}\}) / \ker(f) \cong$

S_n. Therefore, we can still define a surjective map $g : F(\{\bar{1}, \bar{2}\})/\left\langle\left\langle\bar{1}^2, \bar{2}^n\right\rangle\right\rangle \to S_n$. Notice that $F(\{\bar{1}, \bar{2}\})/\left\langle\left\langle\bar{1}^2, \bar{2}^n\right\rangle\right\rangle$ gives a presentation $\left\langle x, y \mid x^2, y^n\right\rangle$, which is isomorphic to $\left\langle x \mid x^2\right\rangle * \langle y \mid y^n\rangle$ by definition of coproduct in groups, which is exactly isomorphic to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}$. Hence, $g : \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/n\mathbb{Z} \to S_n$ is the surjective homomorphism we want. It then follows from the first paragraph that $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/5\mathbb{Z})$ cannot be solvable. $\square$

8 MATH 210A Homework 8

Problem 8.1. *Let X be a final object of a category $\mathscr{C}$. Prove that $X \times Y \cong Y$ for every object Y in $\mathscr{C}$.*

Proof. Take arbitrary object $Y \in \mathscr{C}$. Since X is a final object, there exists a morphism $f : Y \to X$, then taking object Y as well as morphisms $f : Y \to X$, $\mathbf{id}_Y : Y \to Y$, according to the universal property of product, we have the following diagram:

$$\begin{tikzcd}
Y \arrow[dr, dashed, "\exists ! g"] \arrow[drr, bend left, "\mathbf{id}_Y"] \arrow[ddr, bend right, "f"'] & & \\
& X \times Y \arrow[r, "q"'] \arrow[d, "p"] & Y \\
& X &
\end{tikzcd}$$

Note that this diagram commutes with a unique morphism $g : Y \to X \times Y$. Since the diagram commutes, we have $q \circ g = \mathbf{id}_Y$ where q is the canonical projection onto Y. It suffices to show that $g \circ q = \mathbf{id}_{X \times Y}$. We also have the following diagram:

$$\begin{tikzcd}
X \times Y \arrow[dr, dashed, "\mathbf{id}_{X \times Y}"] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "p"'] & & \\
& X \times Y \arrow[r, "q"'] \arrow[d, "p"] & Y \\
& X &
\end{tikzcd}$$

Since $q \circ g = \mathbf{id}_Y$, then $q \circ g \circ q = q$. Moreover, $p \circ g \circ q$ is a morphism between $X \times Y$ and X by definition, but since X is a final object, then such morphism must be unique, so $p \circ g \circ q = p$. However, that means morphism $g \circ q : X \times Y \to X \times Y$ can also make the diagram above commute. However, by the universal property, $g \circ q = \mathbf{id}_{X \times Y}$. On the other hand, since $q \circ g = \mathbf{id}_Y$, then by definition $X \times Y \cong Y$. This concludes the proof. $\square$

Problem 8.2. *Let $F : \mathscr{C} \to$ **Set** be a functor. Consider a new category $\mathscr{D}$ with objects the pairs (X, u), where X is an object in $\mathscr{C}$ and $u \in F(X)$. A morphism between (X, u) and (X', u') in $\mathscr{D}$ is* *a morphism $f : X \to X'$ in $\mathscr{C}$ such that $F(f)(u) = u'$. Prove that if (X, u) is an initial object in $\mathscr{D}$, then the functor F is represented by X.*

Proof. It suffices to show that F is isomorphic to the hom functor $\mathbf{Hom}(X, -)$, which is to show there is a natural isomorphism $\alpha : \mathbf{Hom}(X, -) \Rightarrow F$. Let us define $\alpha : \mathbf{Hom}(X, -) \Rightarrow F$ such that $\alpha_Y : \mathbf{Hom}(X, Y) \to FY$ is defined by $f \mapsto Ff(u)$ for arbitrary object $Y \in \mathscr{C}$.

We first show that the following diagram commutes for arbitrary morphism $f : Y \to Y'$ in $\mathscr{C}$:

$$\begin{tikzcd}
\mathbf{Hom}(X,Y) \arrow[r, "\alpha_Y"] \arrow[d, "\mathbf{Hom}(X,f)"'] & FY \arrow[d, "Ff"] \\
\mathbf{Hom}(X,Y') \arrow[r, "\alpha_{Y'}"'] & FY'
\end{tikzcd}$$

Note that $f_* = \mathbf{Hom}(X, f)$ is the left composition by f. Take arbitrary $g \in \mathbf{Hom}(X, Y)$. Notice that

$$\begin{aligned}
(Ff \circ \alpha_Y)(g) &= Ff(\alpha_Y(g)) \\
&= Ff(Fg(u)) \\
&= F(f \circ g)(u) \\
&= \alpha_{Y'}(f \circ g) \\
&= \alpha_{Y'}(f_*(g)) \\
&= (\alpha_{Y'} \circ f_*)(g)
\end{aligned}$$

Therefore, this diagram commutes, and so α is a natural transformation. We now show that α_Y is a bijection for arbitrary object $Y \in \mathscr{C}$.

Take arbitrary object $a \in FY$. Since (X, u) is initial in $\mathscr{D}$, then there is always a unique morphism k between (X, u) and (Y, a), which is just a morphism $k : X \to Y$ in $\mathscr{C}$ such that $Fk(u) = a$. Therefore, picking $k : X \to Y$ would map to $a \in FY$ through α_Yl so α_Y is surjective.

Suppose $\alpha_Y(f_1) = \alpha_Y(f_2)$ for some morphisms $f_1, f_2 \in \mathbf{Hom}(X, Y)$, then $Ff_1(u) = Ff_2(u) = v$. Therefore, f_1 and f_2 give two morphisms between (X, u) and (Y, v). However, since (X, u) is initial, then they must be the same morphism, hence $f_1 = f_2$. Therefore, α_Y is injective.

Collecting the properties above, we know α_Y is bijective, since this is true for arbitrary object $Y \in \mathscr{C}$, and α is a natural transformation, then α is a natural isomorphism. That concludes the proof. □

Problem 8.3. *Prove that the category* $\mathbf{Set}^\circ$ *is not equivalent to* $\mathbf{Set}$.

Proof. Suppose, towards contradiction, that there is an equivalence between $\mathbf{Set}$ and its opposite category. In particular, there exists a fully faithful functor $F : \mathbf{Set}^\circ \to \mathbf{Set}$. In particular, for arbitrary objects $X, Y \in \mathbf{Set}^\circ$, there is a bijection $\mathbf{Mor}_{\mathbf{Set}^\circ}(X, Y) \xrightarrow{\sim} \mathbf{Mor}_{\mathbf{Set}}(FX, FY)$. By considering the dual notion of a category, there is a bijection $\mathbf{Mor}_{\mathbf{Set}}(Y, X) \xrightarrow{\sim} \mathbf{Mor}_{\mathbf{Set}}(FX, FY)$. We show that this is impossible. In particular, consider the following two morphisms:

- $\mathbf{Mor}_{\mathbf{Set}}(\{1,2\}, \varnothing) \xrightarrow{\sim} \mathbf{Mor}_{\mathbf{Set}}(F(\varnothing), F(\{1,2\}))$, where both sets must be empty;
- $\mathbf{Mor}_{\mathbf{Set}}(\{1\}, \{1,2\}) \xrightarrow{\sim} \mathbf{Mor}_{\mathbf{Set}}(F(\{1,2\}), F(\{1\}))$, where both sets must contain 2 morphism.

Notice that the first statement holds if and only if $F(\{1,2\})$ is $\varnothing$, because otherwise the set has at least one element. We can also deduce from the second statement that $F(\{1,2\})$ cannot be $\varnothing$, otherwise since $\varnothing$ is the initial object, the set must have size 1, contradiction. This leads to another contradiction, so there cannot be such equivalence between the two categories. $\square$

Problem 8.4. *Construct a functor $F : \mathbf{Grp} \to \mathbf{Set}$ that assigns to each group the set of all its subgroups. Is this functor represented?*

Proof. We first construct this functor $F : \mathbf{Grp} \to \mathbf{Set}$. The problem already provides us with the object map, so it suffices to construct the morphism map: for arbitrary group homomorphism $f : G \to G'$, we map it to a map Ff between the set of subgroups of G and the set of subgroups of G', induced from the homomorphism f. i.e. if $H \in FG$, then Ff maps H to $f(H) \in FG'$.

We now check the two properties of F. Take arbitrary morphisms $f' : G' \to G''$ and $f : G \to G'$, then there is a composition $f' \circ f : G \to G''$. Now take arbitrary subgroup $H \subseteq G$, we have

$$\begin{aligned} F(f' \circ f)(H) &= f'(f(H)) \\ &= Ff'(Ff(H)) \\ &= (Ff' \circ Ff)(H) \end{aligned}$$

Furthermore, for the identity homomorphism $\mathbf{id}_G : G \to G$ and subgroup $H \subseteq G$, there is

$$\begin{aligned} F\mathbf{id}_G(H) &= \mathbf{id}_G(H) \\ &= H \\ &= \mathbf{id}_{FG}(H) \end{aligned}$$

Since this is true for arbitrary subgroup $H \subseteq G$, then $F\mathbf{id}_G = \mathbf{id}_{FG}$. Collecting the properties above, we have a construction of functor F.

We now show that this construction of functor F is not representable. Suppose, towards contradiction, that there is $X \in \mathbf{Grp}$ such that $\alpha : \mathbf{Hom}(X, -) \to F$ is a natural isomorphism, which means $\mathbf{Hom}(X, Y) \xrightarrow{\sim} FY$ is a set bijection for all objects $Y \in \mathbf{Grp}$.

In particular, take $Y = \mathbb{Z}/3\mathbb{Z}$, then there is $\mathbf{Hom}(X, \mathbb{Z}/3\mathbb{Z}) \xrightarrow{\sim} F\mathbb{Z}/3\mathbb{Z}$, so $\mathbf{Hom}(X, \mathbb{Z}/3\mathbb{Z})$ has size 2. Obviously there is a trivial homomorphism, and we define $\varphi : X \to \mathbb{Z}/3\mathbb{Z}$ to be the other homomorphism. However, let $\psi : \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$ be the map generated by $[1]_3 \mapsto [2]_3$, then ψ is a homomorphism. However, the composition $\psi \circ \varphi$ has to be another homomorphism between X and $\mathbb{Z}/3\mathbb{Z}$. Note that $\varphi \neq \psi \circ \varphi$ since ψ is not the identity map, and $\psi \circ \varphi$ is not trivial because that would imply φ is trivial, contradiction. Therefore, we have found the third homomorphism, contradiction. Hence, F is not representable. $\square$

Problem 8.5. *Prove that if a functor $F : \mathscr{C} \to \mathscr{D}$ is an equivalence of categories, then X and Y are isomorphic in $\mathscr{C}$ if and only if $F(X)$ and $F(Y)$ are isomorphic in $\mathscr{D}$.*

Proof. Suppose $X \cong Y$, then there is an isomorphism $f : X \to Y$, along with an inverse $f^{-1} : Y \to X$. Note that $Ff \circ Ff^{-1} = F(f \circ f^{-1}) = F(\mathbf{id}_Y) = \mathbf{id}_{FY}$, and $Ff^{-1} \circ Ff = F(f^{-1} \circ f) = F(\mathbf{id}_X) = \mathbf{id}_{FX}$. By definition, $Ff : FX \to FY$ is an isomorphism with inverse Ff^{-1}. By definition, $FX \cong FY$.

Suppose $FX \cong FY$, then there is an isomorphism $g : FX \to FY$ along with an inverse $g^{-1} : FY \to FX$. By definition, since F is an equivalence of categories, then F is fully faithful, which means $\mathbf{Mor}_{\mathscr{C}}(X, Y) \xrightarrow{\sim} \mathbf{Mor}_{\mathscr{D}}(FX, FY)$ is a bijection. In particular, for $g \in \mathbf{Mor}_{\mathscr{D}}(FX, FY)$, there exists some $h \in \mathbf{Mor}_{\mathscr{C}}(X, Y)$ such that $Fh = g$. We claim that h is the isomorphism we want.

Similar as above, since F is fully faithful, then we may obtain a bijection $\mathbf{Mor}_{\mathscr{C}}(Y, X) \xrightarrow{\sim} \mathbf{Mor}_{\mathscr{D}}(FY, FX)$. In particular, for $g^{-1} \in \mathbf{Mor}_{\mathscr{D}}(FY, FX)$, there is some $h' \in \mathbf{Mor}_{\mathscr{C}}(Y, X)$ such that $Fh' = g^{-1}$. Notice that $Fh \circ Fh' = g \circ g^{-1} = \mathbf{id}_{FY}$, so $F(h \circ h') = \mathbf{id}_{FY}$. However, since F is fully faithful, then $\mathbf{Mor}_{\mathscr{C}}(Y, Y) \xrightarrow{\sim} \mathbf{Mor}_{\mathscr{D}}(FY, FY)$ is a bijection. In particular, $F(h \circ h') = \mathbf{id}_{FY} = F(\mathbf{id}_Y)$, which means $h \circ h' = \mathbf{id}_Y$. Similarly, one can show that $h' \circ h = \mathbf{id}_X$. By definition, h is an isomorphism with inverse h', which means $X \cong Y$. $\square$

Problem 8.6. *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Suppose that F has a left adjoint functor. Prove that F takes the terminal objects of $\mathscr{C}$ to the terminal objects of $\mathscr{D}$.*

Proof. Let $G : \mathscr{D} \to \mathscr{C}$ be the left adjoint of F. Take arbitrary terminal object $C \in \mathscr{C}$. It suffices to show that FC is terminal in $\mathscr{D}$.

Pick arbitrary $D \in \mathscr{D}$. By definition, $\mathbf{Mor}_{\mathscr{C}}(GD, C) \cong \mathbf{Mor}_{\mathscr{D}}(D, FC)$. However, since C is terminal, then $\mathbf{Mor}_{\mathscr{C}}(GD, C)$ has one morphism in it, which means $\mathbf{Mor}_{\mathscr{D}}(D, FC)$ also contains one morphism as well. Since this is true for arbitrary D, then $\mathbf{Mor}_{\mathscr{D}}(D, FC)$ always have one morphism for all all objects $D \in \mathscr{D}$. In particular, FC is terminal in $\mathscr{D}$ by definition. $\square$

Problem 8.7. *Determine the limit and colimit of the diagram*

$$X \rightrightarrows Y$$

in **Set**.

Proof. Without loss of generality denote pair of morphisms as $f, g : X \to Y$. Denote it as diagram $F : I \to \mathscr{C}$ where $I = \bullet \rightrightarrows \bullet$.

The limit and colimit of the diagram are precisely the equalizer and coequalizer in $\mathscr{C}$, respectively, which are denoted in the following two diagrams.

$$\begin{tikzcd} Z \arrow[d, dashed, "\exists! a"'] \arrow[dr, "k"] & & \\ W \arrow[r, "h"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"] \arrow[dr, "k"'] & W \arrow[d, dashed, "\exists! a"] \\ & & Z \end{tikzcd}$$

To show that the equalizer is the limit, it suffices to show that for arbitrary object $Z \in \mathscr{C}$, $\mathbf{Mor}_{\mathscr{C}}(Z, W) \cong \mathbf{Mor}_{\mathbf{Func}}(c_Z, F)$. Take arbitrary object $Z \in \mathscr{C}$. Notice that $\mathbf{Mor}_{\mathbf{Func}}(c_Z, F)$ is exactly the collection of all natural transformations from c_Z to F. Every element α in the collection have the property that the diagram

$$\begin{tikzcd} Z = c_Z i \arrow[r, "\alpha_i"] \arrow[d, "\mathbf{id}_Z"'] & Fi = X \arrow[d, "f,g"] \\ Z = c_Z j \arrow[r, "\alpha_j"] & Fj = Y \end{tikzcd}$$

commutes for all $i, j \in I$ (here we take i as the first bullet and j as the second bullet in particular). In particular, $f \circ \alpha_i = \alpha_j = g \circ \alpha_i$. Therefore, there is a correspondence between a morphism $k : Z \to X$ and a natural transformation α. However, by the universal property of equalizer, there is a correspondence between a morphism $k : Z \to X$ and a morphism $a : Z \to W$. Hence, there is a correspondence between a natural transformation α and a morphism $a : Z \to W$, which means $\mathbf{Mor}_{\mathscr{C}}(Z, W) \cong \mathbf{Mor}_{\mathbf{Func}}(c_Z, F)$ as a bijection.

Using the dual notion, we can prove that the coequalizer is the colimit in a similar fashion.

We now consider the equalizer and coequalizer under the category **Set**. Therefore, the diagram is essentially a pair of set functions $f, g : X \to Y$, and the limit of the diagram would be a set $W = \{x \in X : f(x) = g(x)\} \subseteq X$ along with an inclusion function $h : W \to X$. Therefore, $fh = gh$. In particular, this construction satisfies the universal property: take arbitrary set Z and function $k : Z \to X$ such that $fk = gk$. Therefore, $\mathbf{im}(k) \subseteq \{x \in X : f(x) = g(x)\} = W \subseteq X$. This induces an inclusion map $a : Z \to W$ such that $ha = k$. Suppose there is also a map $b : Z \to W$ such that $hb = k$, then $ha = hb$, but h is an injection, which is a monomorphism in **Set**, so $a = b$. Therefore, the inclusion map a is unique. This proves the universal property.

Also, the colimit of the diagram would be a set $W = Y/\sim$ based on the relation $\sim$, which is the smallest equivalence relation on Y such that $f(x) \sim g(x)$ for all $x \in X$, along with a set map $h : Y \to Y/\sim$ as the canonical surjection. In particular, $hf(x) = hg(x)$ for all $x \in X$ since $f(x) \sim g(x)$ are in the same equivalence class. Note that this construction satisfies the universal property: take arbitrary set Z and a set map $k : Y \to Z$ such that $kf = kg$. Let function

$a : W \to Z$ be defined as the restriction map $a([y]_\sim) = k(y)$. This map is well-defined because for some distinct x, y with $[x]_\sim = [y]_\sim$, there exists some z such that $x = f(z)$ and $y = g(z)$. In particular, $k(x) = kf(z) = kg(z) = k(y)$, which is well-defined. Now, we also have that $ah = k$ because $ah(y) = a([y]_\sim) = k(y)$ for all $y \in Y$. Finally, we check that this set function a is unique. For every set function $b : W \to Z$ such that $bh = k$, there is $bh = k = ah$, but h is a canonical surjection, which is an epimorphism in the category **Set**, hence we have $a = b$. Therefore, a is unique, and that concludes the proof. □

Problem 8.8. *Prove that all (small) limits and colimits exist in* **Set** *and* **Grp**.

Proof. Let $F : I \to \mathbf{Set}$ be a functor for some small category I. Since I is small, we may index the objects as $\{O_i\}_{i \in I}$. Note that products exist in **Set**, then we may define an object $Z = \prod_{i \in I} F(O_i)$ for all objects $O_i \in I$ along with projection morphisms $\pi_i : Z \to F(O_i)$. We define the limit in **Set** over F as the following set (as summit of cone):

$$\lim F = \{z \in Z \mid F(f)(\pi_i(z)) = \pi_j(z)\ \forall i, j, \forall f : O_i \to O_j\}$$

To show that this is a limit over F, it suffices to show that $\lim F$ is a final object in the category of cones. [5]

We first show that $\lim F$ is a cone indeed. Note that the construction above induces natural transformation $\bar{\pi} : \mathbf{id}_{\lim F} \to F$ such that

$$\begin{tikzcd} & \lim F \arrow[dl, "\bar{\pi}_j"'] \arrow[dr, "\bar{\pi}_k"] & \\ F(O_j) \arrow[rr, "Ff"'] & & F(O_k) \end{tikzcd}$$

where $\bar{\pi}_j = \pi_j \circ i$ for inclusion $i : \lim F \to Z$, then this is the desired cone structure we want.

We now show that $\lim F$ is a final object in the category of cones. Suppose there is another cone with summit L (as a set) and legs $p_j : L \to F$ such that $p_k = Ff \circ p_j$ for all $f : O_j \to O_k$. By the universal property of product, this induces a unique morphism $p : L \to Z$ such that $\pi_j \circ p = p_j$ for all index j. However, if this is the case, then $\pi_k \circ p = Ff \circ \pi_j \circ p$. Since this is true for all index j, k, we know that $\mathbf{im}(p)$ satisfies the property of $\lim F$, which means $\mathbf{im}(p) \subseteq \lim F$. Therefore, to make all the related diagrams commute, p is essentially an inclusion functor into $\lim F$, which is unique by construction as a set map. In particular, this means $\lim F$ is the terminal cone we want, which means it is the limit over F indeed.

We now consider the colimit of F. In a similar fashion as described in the footnote, the colimit is just the initial object in the category of cones. We define the colimit (summit of cone) $\mathrm{colim} F$ as the following:

[5]Note that this is an equivalent definition of a limit, as described in Emily Rihel's book *Category Theory in Context*. The idea is that given by the universal property in class, the cone functor $\mathbf{Cone}(-, F) \cong \mathbf{Mor}_{\mathscr{C}}(-, \lim F)$ is representable, then one can show that this means this contravariant functor's category of element has a terminal object, which is just $\lim F$ in the category of cones.

$$\text{colim}F = \coprod_{i \in I} F(O_i)/\sim$$

where $\sim$ is the smallest equivalence relation on the coproduct that is generated by the following relation: $(a, j) \sim (a', k) \in \coprod_{i \in I} F(O_i)$ if there exists a morphism $f : O_j \to O_k$ such that $Ff(a) = a'$. Similar as above, we need to show that this construction is a cocone and is initial in the category of cocones. Consider the following diagram for a given morphism $f : O_j \to O_k$:

$$\begin{tikzcd}
FO_j \arrow[rr, "Ff"] \arrow[dr, "\pi \circ i_j"'] & & FO_k \arrow[dl, "\pi \circ i_k"] \\
& \text{colim}F &
\end{tikzcd}$$

where i is the inclusion map into the coproduct and π is the quotient map. Consider arbitrary $a \in O_j$ with $f : O_j \to O_k$ such that $f(a) = a'$. Now we have a corresponding $F(a) \in F(O_j)$. We then have $\pi \circ i_k \circ Ff(F(a)) = \pi \circ i_k(F(a')) = \pi(F(a'), k)$, and $\pi \circ i_j(F(a)) = \pi(F(a), j)$. However, $\pi(F(a'), k) = \pi(F(a), j)$ because $(F(a), j) \sim (F(a'), k)$. Therefore, this diagram commutes and so colim$F$ is a cocone. Suppose we have another cone with nadir C (basically a cocone) and legs $p_j : F \to C$ for arbitrary j in the index set such that $p_k \circ Ff = p_j$ for all index j, k. By the universal property of coproduct, there is a morphism $p : \coprod_{i \in I} FO_i \to C$ such that $p \circ i_j = p_j$ for all $j \in I$. However, since the colimit is a quotient set from the coproduct, then by the universal property of quotient set, there is a unique map $k : \text{colim}F \to C$ that makes the diagram commute. In particular, colim$F$ is initial in the category of cones.

We extend the construction for limit in **Set** to the limit in **Grp**. (Consider $F : I \to \mathbf{Grp}$ where I is a small category.) Notice that the only we we have the check is that the limit object is still a group, i.e. $\lim F$ is a subgroup of Z. Note that $e_Z \in \lim F$ because $\pi_k(e) = e = Ff(\pi_i(e))$. Also, for $z, z' \in \lim F$, $zz' \in \lim F$ by property of group homomorphisms, which are exactly the morphisms in **Grp**. Finally, for $z \in \lim F$, we have $z^{-1} \in \lim F$ again by properties of group homomorphisms. Therefore, $\lim F \subseteq Z$ is a group. Therefore, $\lim F$ is a limit in the category **Grp**.

Similarly, we may extend the construction for colimit in **Set** to the colimit in **Grp**. In this case, let the colimit object be $\coprod_{i \in I} F(O_i)/\sim$ as the quotient of an equivalence relation over the coproduct of groups. In this case, in addition to the relation given from the colimit in **Set**, we want the quotient to be a normal closure based on this relation. In particular, recall that a normal closure corresponds to relations in a group presentation. In such sense, the quotient creates relations on the free product (the coproduct), which is just a factor group. Therefore, our colimit object is a group indeed. Furthermore, this structure is a cone because we can use the same argument as in the set colimit case. Finally, one can see that the cone is initial by a similar argument as the case for colimits in **Set**, using the universal property of factor groups. □

Problem 8.9. *Let $\mathcal{I}$ be a small category with an initial object i and let $F : \mathcal{I} \to \mathscr{C}$ be a functor. Prove that* $\lim F = F(i)$.

Proof. We first show that $F(i)$ is a cone over F. Since i is initial in $\mathcal{I}$, then for all objects $j \in I$, there is a unique morphism $f_j : i \to j$. Construct a natural transformation $f : \mathbf{id}_{F(i)} \to F$ with summit $F(i)$ and legs $\{Ff_j\}_{j\in\mathcal{I}}$. For arbitrary morphism $g : j \to k$, this induces the following cone:

$$\begin{tikzcd} & F(i) \arrow[dl, "Ff_j"'] \arrow[dr, "Ff_k"] & \\ F(j) \arrow[rr, "Fg"'] & & F(k) \end{tikzcd}$$

Note that the diagram commutes: since i is initial, then $f_k : i \to k$ is unique, which means $g \circ f_j : i \to k$ is equivalent to f_k. In particular, $Fg \circ Ff_j = F(g \circ f_j) = Ff_k$. Therefore, this is a cone by definition.

We now show that $F(i)$ is the terminal object in the category of cones. Consider arbitrary cone $\tau : \mathbf{id}_c \to F$. In particular, $\tau_i : c \to Fi$ is a morphism such that the following diagram commutes:

$$\begin{tikzcd} & c \arrow[dl, "\tau_i"'] \arrow[dr, "\tau"] & \\ F(i) \arrow[rr, "f"'] & & F \end{tikzcd}$$

In particular, τ_i is the unique morphism for this property because for arbitrary morphism p that has this property, when taking $i \in \mathcal{I}$, we have $\mathbf{id}_{F(i)} \circ p = \tau_i$, which means $p = \tau_i$. This shows that $F(i)$ is a terminal object in the category of cones, i.e. $F(i) \cong \lim F$.

□

Problem 8.10. *Let $\mathcal{I}$ be a small category and $\mathscr{C}$ a category. Prove that the functor*

$$\mathscr{C} \to \mathbf{Functors}(\mathcal{I}, \mathscr{C}),$$

taking an object X to the constant functor c_X, has a left and right adjoint functors. Determine these functors.

Proof. (I assume this is under the assumption that $\mathscr{C}$ has a limit for each I-shaped diagram as functor from I to $\mathscr{C}$.)

We claim that the limit functor $\mathbf{lim} : \mathbf{Functors}(I, \mathscr{C}) \to \mathscr{C}$ is the right adjoint to the functor $F : \mathscr{C} \to \mathbf{Functors}(I, \mathscr{C})$ given in the problem, and the colimit functor $\mathbf{colim} : \mathbf{Functors}(I, \mathscr{C}) \to \mathscr{C}$ is the left adjoint to the functor F. We will give a detailed proof for the first statement, and the second statement may be induced in the dual notion.

Lemma 8.11. *Let $G, H : I \to \mathscr{C}$ be two I-shaped diagram, and $\eta : G \to H$ is a natural transformation, and consider limits $(\lim G, \lambda)$ for G and $(\lim H, \mu)$ for H (as a limit object with natural tranformations as cones), then there is a unique morphism $\lim \eta : \lim G \to \lim H$ such that the diagram*

$$\begin{array}{ccc} \lim G & \xrightarrow{\lim \eta} & \lim H \\ \lambda_i \downarrow & & \downarrow \mu_i \\ G_i & \xrightarrow[\eta_i]{} & H_i \end{array}$$

commutes for all $i \in I$.

Proof. The vertical composite $\eta \circ \lambda : \lim G \to G \to H$ is a cone over H, so by the universal property of $\lim H$, there is a unique $\lim \eta : \lim G \to \lim H$ such that $\mu_i \circ \lim \eta = (\eta \circ \lambda)_i = \eta_i \circ \lambda_i$ for all $i \in I$. □

Proposition 8.12. *For each functor $G : I \to \mathscr{C}$, we choose a limit object $\lim G$ along with a cone $\lambda_G : \lim G \to G$, then one can extend this to a functor $\lim$: **Functors**$(I, \mathscr{C}) \to \mathscr{C}$ that sends a natural transformation $\eta : G \to H$ to the unique morphism $\lim \eta : \lim G \to \lim H$ given by the previous lemma.*

Remark 8.13. *This induced functor is the limit functor we define.*

Proof. By definition, this mapping **lim** preserves domain and codomain, so it suffices to check its preservation of identity and composition.

Take $G : I \to \mathscr{C}$ functor and $\mathbf{id}_G : G \to G$. Then $\lim \mathbf{id}_G$ is the unique morphism such that

$$\begin{array}{ccc} \lim G & \xrightarrow{\lim \mathbf{id}_G} & \lim G \\ (\lambda_G)_i \downarrow & & \downarrow (\lambda_G)_i \\ G_i & \xrightarrow[\mathbf{id}_{G_i}]{} & G_i \end{array}$$

commutes for all $i \in I$. However, $\mathbf{id}_{\lim G}$ also satisfies this universal property, so they must be identical.

Consider $\eta : G \to H$, $\theta : H \to K$ are natural transformations of functors from I to $\mathscr{C}$. Then the diagram

$$\begin{array}{ccccc} \lim G & \xrightarrow{\lim \eta} & \lim H & \xrightarrow{\lim \theta} & \lim K \\ \downarrow (\lambda_G)_i & & \downarrow (\lambda_H)_i & & \downarrow (\lambda_K)_i \\ G_i & \xrightarrow{\eta_i} & H_i & \xrightarrow{\theta_j} & K_i \end{array}$$

commutes for all $i \in I$, so $(\lambda_K)_i \circ (\lim \theta \circ \lim \eta) = (\theta \circ \eta)_i \circ (\lambda_G)_i$ for all $i \in I$, but $\lim(\theta \circ \eta) L \lim G \to \lim K$ is the unique morphism with this property, so $\lim(\theta \circ \eta) = \lim \theta \circ \lim \eta$. We may then conclude that this is a functor indeed.

□

We now show that the limit functor is a right adjoint. Pick arbitrary I-shaped diagram $G : I \to \mathscr{C}$ and a limit functor defined accordingly. In particular, we have a limit $(\lim G, \lambda^G : \lim G \to G)$ for each diagram $G : I \to \mathscr{C}$, and for any natural transformation $\eta : G \to G'$ in **Functors**$(I, \mathscr{C})$, let $\lim \eta : \lim G \to \lim G'$ be the unique morphism such that

$$\begin{array}{ccc} \lim G & \xrightarrow{\lim \eta} & \lim G' \\ \downarrow \lambda_i^G & & \downarrow \lambda_i^{G'} \\ G_i & \xrightarrow{\eta_i} & G'_i \end{array}$$

commutes for all $i \in I$. Observe that for diagram $G : I \to \mathscr{C}$ and $C \in \mathscr{C}$, we have $\mathbf{Mor_{Functors}}(FC = c_C, G) = \mathbf{Cone}(C, G)$. However, by the universal property of limit, there is $\mathscr{C}(C, \lim G) \cong \mathbf{Mor_{Functor}}(FC = c_C, G)$, taking $g : C \to \lim G$ to $\lambda^G \circ F$, which is natural in both object C and functor G. This induces the adjunction $F \dashv \lim$, where lim is the right adjoint of our functor F.

We may define a colimit functor and check the naturality using a dual proof. By taking J-shaped diagram $F : J \to \mathscr{C}$ and a colimit functor $\mathbf{Mor_{Func}}(J, \mathscr{C}) \to \mathscr{C}$, which is constructed in a similar fashion as above, if we pick a colimit $(\mathrm{colim}F, \iota : F \to \mathrm{colim}F)$ for each diagram $F : J \to \mathscr{C}$ and if for all natural transformations $\eta : F \to F'$, we set $\mathrm{colim}\eta : \mathrm{colim}F \to \mathrm{colim}F'$ to be the unique morphism such that the diagram

$$\begin{array}{ccc} Fj & \xrightarrow{\eta_j} & F'j \\ \downarrow \iota_j & & \downarrow \iota'_j \\ \mathrm{colim}F & \xrightarrow{\mathrm{colim}\eta} & \mathrm{colim}F' \end{array}$$

commutes for all $j \in J$, then the universal property of the colimits gives a bijection $\mathscr{C}(\mathrm{colim}F, c) \cong \mathbf{Mor_{Func}}(J, \mathscr{C})(F, \Delta c)$ by taking $f : \mathrm{colim}F \to c$ to $\Delta f \circ \iota : F \to \Delta c$ that is natural in F and c. Therefore, we have a desired adjunction $\mathrm{colim} \dashv \Delta$. □

9 MATH 210A Homework 9

Problem 9.1. *Give definition of a functor commuting with limits. Prove that a functor that admits a left adjoint functor commutes with limits.*

Proof. We say a functor F commutes with limits of D if the following holds: for a functor $D : \mathcal{I} \to \mathscr{C}$ with limit $\lim D$, then $F(\lim D) \cong \lim FD$.

Notice that it suffices to show the following: let $F \dashv G$ be an adjunction with $F : \mathscr{D} \to \mathscr{C}$ and $G : \mathscr{C} \to \mathscr{D}$, then for functor $D : \mathcal{I} \to \mathscr{C}$ with limit $\lim D$ and legs $\lambda : \mathbf{id}_{\lim D} \to D$, we have $\mathbf{Mor}_{\mathscr{D}}(X, G \lim D) \cong \mathbf{Mor_{Functor}}(c_X, GD)$. Then by the defining property of limits, $G(\lim D) \cong \lim GD$ with the legs $G\lambda : \mathbf{id}_{G \lim D} \to GD$ is the limit of GD.

We have the following derivation in terms of natural isomorphisms:

$$\begin{aligned}
\mathbf{Mor}_{\mathbf{Functor}}(c_X, GD) &\cong \mathbf{Mor}_{\mathbf{Functor}}(Fc_X, D) \\
&\cong \mathbf{Mor}_{\mathbf{Functor}}(c_{FX}, D) \\
&\cong \mathbf{Mor}_{\mathscr{C}}(FX, \lim D) \\
&\cong \mathbf{Mor}_{\mathscr{D}}(X, G\lim D)
\end{aligned}$$

This is always natural in both slots since we are always taking natural isomorphisms.

Therefore, that concludes the proof. $\square$

Problem 9.2. *Prove that every representable functor $\mathscr{C} \to$ **Set** commutes with limits.*

Proof. Note that every representable functor $\mathscr{C} \to \mathbf{Set}$ is isomorphic to the hom functor $\mathbf{Hom}_{\mathscr{C}}(X, -)$ for some object $X \in \mathscr{C}$. Therefore, it suffices to show that the hom functor $\mathbf{Hom}_{\mathscr{C}}(X, -)$ commutes with limits. This is well-defined because we showed in homework 8 that **Set** admits all small limits.

Consider a diagram $F : \mathcal{I} \to \mathscr{C}$ with limit $(\lim F, \lambda : \mathbf{id}_{\lim F} \to F)$. By applying the hom functor, we get a cone $\mathscr{C}(X, \lim F)$ with legs $\mathscr{C}(X, \lambda_j) = (\lambda_j)_* : \mathscr{C}(X, \lim F) \to \mathscr{C}(X, Fj)$. We want to show that this is a limit cone by showing it satisfies the universal property.

Take arbitrary cone $(T, \tau : \mathbf{id}_T \to \mathscr{C}(X, F-))$. For arbitrary $i \in \mathcal{I}$ and $t \in T$, we have $\tau_i(t) : X \to Fi$, and for all morphisms $f : i \to j$ in $\mathcal{I}$, the following triangle commutes:

$$\begin{tikzcd}
 & T \arrow[dl, "\tau_i"'] \arrow[dr, "\tau_j"] & \\
\mathscr{C}(X, Fi) \arrow[rr, "Ff_*"'] & & \mathscr{C}(X, Fj)
\end{tikzcd}$$

Therefore, we have $Ff \circ \tau_i(t) = Ff_*(\tau_i(t)) = \tau_j(t)$, then for all $f : i \to j$ in $\mathcal{I}$, the following diagram commutes:

$$\begin{tikzcd}
 & X \arrow[dl, "\tau_i(t)"'] \arrow[dr, "\tau_j(t)"] & \\
Fi \arrow[rr, "Ff"'] & & Fj
\end{tikzcd}$$

We may conclude that $(X, (\tau_i(t) : X \to Fi)_{i \in \mathcal{I}})$ forms a cone over F for all $t \in T$.

Now since $(X, (\tau_i(t))_{i \in \mathcal{I}})$ forms a cone over F for all $t \in T$, define $\tau(t) : X \to \lim F$ as the unique map given by the universal property of the limit. Therefore, we have a function $\tau : T \to \mathscr{C}(X, \lim F)$, and for all $i \in \mathcal{I}$, we have $((\lambda_i)_* \circ \tau)(t) = \lambda_i(\tau(t)) = \tau_i(t)$. In particular, $(\lambda_i)_* \circ \tau = \tau_i$. Then we have the following diagram:

$$\begin{tikzcd}
T \arrow[rr, "\exists \tau"] \arrow[dr, "\tau_i"'] & & \mathscr{C}(X, \lim F) \arrow[dl, "(\lambda_i)_*"] \\
 & Fi &
\end{tikzcd}$$

Finally, it suffices to show that $\tau : T \to \mathscr{C}(X, \lim F)$ is the unique morphism such that the diagram above commutes. Notice that for all $t \in T$, we have $\tau(t) : X \to \lim F$ and $\tau_i(t) = (\lambda_i)_* \circ (\tau(t)) = \lambda_i(\tau(t))$. Therefore, $\tau(t)$ is the unique morphism into $\lim F$ induced by the cone $(X, (\tau_i(t))_{i \in \mathcal{I}})$. Therefore, our construction of $\tau : T \to \mathscr{C}(X, \lim F)$ has to be unique. This concludes the proof.

Therefore, the covariant hom functor commutes with the limit, and since there is a natural isomorphism from every representable functor $\mathscr{C} \to \mathbf{Set}$ to the hom functor $\mathbf{Hom}_{\mathscr{C}}(X, -)$ for some object $X \in \mathscr{C}$, then every representable functor $\mathscr{C} \to \mathbf{Set}$ commutes with limits. □

Problem 9.3. *Let I be a small category and let X be an object of a category $\mathscr{C}$. Prove that the functor* $\mathbf{Functor}(I, \mathscr{C}) \to \mathbf{Set}$ *taking a functor $F : I \to \mathscr{C}$ to* $\mathbf{Mor}_{\mathscr{C}}(X, \lim F)$ *is represented.*

Proof. We denote the functor in the problem as $G : \mathbf{Functor}(I, \mathscr{C}) \to \mathbf{Set}$. Now take the constant functor $c_X : I \to \mathscr{C}$ that sends all objects in I to X and all morphisms to $\mathbf{id}_X$. We claim that $G \cong R^{c_X}$. Indeed, for arbitrary functor $K : I \to \mathscr{C}$, we have $GK = \mathbf{Mor}_{\mathscr{C}}(X, \lim K) \cong \mathbf{Mor}_{\mathbf{Functor}}(c_X, K) = R^{c_X}(K)$ by the limit's representation property. By definition, our functor G is represented by the hom functor. □

Problem 9.4. *Prove that the product of two Abelian categories is Abelian.*

Proof. Let $\mathscr{A}, \mathscr{B}$ be Abelian categories. Then $\mathscr{A} \times \mathscr{B}$ is a category with objects (A, B) where $A \in \mathscr{A}, B \in \mathscr{B}$, and morphisms $f \times g : (A_1, A_2) \to (B_1, B_2)$ where $f : A_1 \to B_1$ and $g : A_2 \to B_2$.

We first show that $\mathscr{A} \times \mathscr{B}$ is pre-additive. Observe that for arbitrary objects (A_1, B_1), (A_2, B_2) in the category, we have an Abelian group structure on the set of morphisms $\mathbf{Mor}_{\mathscr{A} \times \mathscr{B}}((A_1, B_1), (A_2, B_2)) \cong \mathbf{Mor}_{\mathscr{A}}(A_1, A_2) \times \mathbf{Mor}_{\mathscr{B}}(B_1, B_2)$ because the product of Abelian group structures is still an Abelian group structure, defined by $(f_1, f_2) + (f_1', f_2') = (f_1 + f_1', f_2 + f_2')$. This derives the following:

$$
\begin{aligned}
((f_1, f_2) + (f_1', f_2')) \circ (g_1, g_2) &= (f_1 + f_1', f_2 + f_2') \circ (g_1, g_2) \\
&= (f_1 + f_1', f_2 + f_2') \circ (g_1, g_2) \\
&= ((f_1 + f_1') \circ g_1, (f_2 + f_2') \circ g_2) \\
&= (f_1 \circ g_1 + f_1' \circ g_1, f_2 \circ g_2 + f_2' \circ g_2) \\
&= (f_1 \circ g_1, f_2 \circ g_2) + (f_1' \circ g_1 + f_2' \circ g_2) \\
&= (f_1, f_2) \circ (g_1, g_2) + (f_1', f_2') \circ (g_1, g_2)
\end{aligned}
$$

Similarly we can check the other direction, so this operation is bilinear. In particular, the zero morphism is structured as the product of zero morphisms in each category. Therefore, the product category is pre-Abelian.

We now show that the category is additive. Denote the object $(0_{\mathscr{A}}, 0_{\mathscr{B}}) \in \mathscr{A} \times \mathscr{B}$ where $0_{\mathscr{A}}$ is the zero object in $\mathscr{A}$ and $0_{\mathscr{B}}$ is the zero object in $\mathscr{B}$. According to homework 7, problem 2, $(0_{\mathscr{A}}, 0_{\mathscr{B}}) \in \mathscr{A} \times \mathscr{B}$ is initial since both $0_{\mathscr{A}}$ and $0_{\mathscr{B}}$ are initial. In a dual argument, one can show that $(0_{\mathscr{A}}, 0_{\mathscr{B}})$ is terminal. Therefore, it is a zero object by definition. We can also show that the category has finite product. We claim that $(\prod_{i\in I} A_i, \prod_{i\in I} B_i)$ is the product of $(A_i, B_i) \in \mathscr{A} \times \mathscr{B}$ with projections $\pi_i : (\prod_{i\in I} A_i, \prod_{i\in I} B_i) \to (A_i, B_i)$ given by the projection maps of each product $\prod_{i\in I} A_i, \prod_{i\in I} B_i$'s projection. This object satisfies the necessary universal properties because for arbitrary object (X, Y) and morphisms $p_i : (X, Y) \to (A_i, B_i)$, the universal property of each product gives a unique morphism $f : X \to \prod_{i\in I} A_i$ and $g : Y \to \prod_{i\in I} B_i$, which is constructed as a unique morphism from (X, Y) to $(\prod_{i\in I} A_i, \prod_{i\in I} B_i)$ that satisfies the properties. Therefore, the category is additive.

We then show that the category is pre-Abelian. Take arbitrary morphism $f \times g : (A_1, B_1) \to (A_2, B_2)$. The kernel of this morphism $f \times g$ is the object $(\ker(f), \ker(g))$ with necessary morphisms (and forms the equalizer of f and zero morphism, and the equalizer of g and zero morphism, respectively), which is in the product category by definition. One can check that it satisfies the necessary universal property:

$$\begin{tikzcd}
(\ker(f), \ker(g)) \arrow[r, "i_{\mathscr{A}} \times i_{\mathscr{B}}"] & (A_1, B_1) \arrow[r, "f \times g"] & (A_2, B_2) \\
(X, Y) \arrow[u, "\exists! s \times t"] \arrow[ur, "x \times y"'] & &
\end{tikzcd}$$

where $s : X \to \ker(f)$ is given by the universal property of equalizer of f and zero morphism, and $t : Y \to \ker(g)$ is given by the universal property of equalizer of g and zero morphism. Note that these equalizers exists because $\mathscr{A}$ and $\mathscr{B}$ themselves are Abelian categories. In a dual argument, one can show that the cokernel exists in $\mathscr{A} \times \mathscr{B}$ because Abelian categories $\mathscr{A}$ and $\mathscr{B}$ also have them. Therefore, the product category is a pre-Abelian category.

Finally, we check that this product category is an Abelian category. Take arbitrary morphism $f \times g : (A_1, A_2) \to (B_1, B_2)$ in the product category. It suffices to show that $s : \mathbf{coim}(f \times g) \to \mathbf{im}(f \times g)$ is an isomorphism. By unpacking, notice that $\mathbf{coim}(f \times g) = (A_1, A_2)/\ker(f \times g) = (A_1, A_2)/(\ker(f), \ker(g)) = (A_1/\ker(f), A_2/\ker(g))$, where the last equality holds because we can create a canonical surjection so that $(\ker(f), \ker(g))$ is the kernel. Since we have two isomorphisms $s_1 : A_1/\ker(f) \to \mathbf{im}(f)$ and $s_2 : A_2/\ker(g) \to \mathbf{im}(g)$, then we have another isomorphism $s : (A_1/\ker(f), A_2/\ker(g)) \to (\mathbf{im}(f), \mathbf{im}(g))$, which is isomorphic to $s : \mathbf{coim}(f \times g) \to \mathbf{im}(f \times g)$ by definition. Therefore, the product category is also an Abelian category. ☐

Problem 9.5. *Prove that an object X in an additive category is a zero object if and only if $0_X = 1_X$.*

Proof. Suppose X is a zero object in an additive category. Therefore, X is a final object in the category, then 0_X and 1_X are morphisms from X to itself, but since X is terminal, then $0_X = 1_X$.

Suppose $0_X = 1_X$ for some X in additive category, then for all morphisms $f : X \to Y$, we have $f = f \circ 1_X = f \circ 0_X = 0$, which means all morphisms in $\mathbf{Mor}(X, Y)$ are zero morphisms. Hence, X is initial. On the other hand, for all morphisms $g : Y \to X$, we have $g = 1_X \circ g = 0_X \circ g = 0$, which means all morphisms in $\mathbf{Mor}(Y, X)$ are zero morphisms. Hence, X is terminal. By definition, X is a zero object since it is both intial and terminal. $\square$

Problem 9.6. *Let $\mathscr{C}$ be an Abelian category. Prove that the category of presheaves of Abelian groups on $\mathscr{C}$ is also Abelian.*

Proof. By definition, the category of presheaves of Abelian groups on $\mathscr{C}$ is the functor category $\mathbf{Functor}(\mathscr{C}^\circ, \mathbf{Ab})$. By example in class, $\mathbf{Functor}(\mathscr{C}^\circ, \mathbf{Ab})$ is additive since $\mathscr{C}$ is additive. We now show that the functor category is pre-Abelian. For arbitrary morphism (natural transformation) $\eta : F \to G$ in the functor category $\mathbf{Functor}(\mathscr{C}^\circ, \mathbf{Ab})$, we construct the kernel pointwise. Pick arbitrary element $X \in \mathscr{C}^\circ$, we have $\eta_X : FX \to GX$ as a morphism in $\mathbf{Ab}$. Since $\mathbf{Ab}$ is an Abelian category, there is an equalizer E_X with inclusion morphism into FX that satisfies the universal property. Since this is true for all elements $X \in \mathscr{C}^\circ$, then this collection of objects with morphisms induces a functor $E : \mathscr{C}^\circ \to \mathbf{Ab}$ taking object X to the equalizer of the morphism η_X, and the properties of a functor because this is essentially a limit. Similarly, we may construct a coequalizer because $\mathbf{Ab}$ is Abelian. Note that equalizers and coequalizers of the diagram are essentially the kernels and cokernels, which means the functor category is pre-Abelian.

Finally, we show that the functor category is Abelian. Take arbitrary natural transformation $\eta : F \to G$. The coimage is $F/\ker(\eta)$. Note that by applying arbitrary element onto the coimage and the image, we have that $\mathbf{coim}(\eta)(X) \cong \mathbf{im}(\eta)(X)$ because they are elements in $\mathbf{Ab}$. This induces a map $\mu : \mathbf{coim}(\eta) \to \mathbf{im}(\eta)$, and we can check that this is a natural transformation for $f : X \to Y$ in $\mathscr{C}^\circ$.

$$\begin{tikzcd}
\mathbf{coim}(\eta)(Y) \arrow[r, "\mu_Y"] \arrow[d, "Ff/\ker(\eta)(f)"'] & \mathbf{im}(\eta)(Y) \arrow[d, "Gf"] \\
\mathbf{coim}(\eta)(X) \arrow[r, "\mu_X"'] & \mathbf{im}(\eta)(X)
\end{tikzcd}$$

Therefore, we have a natural isomorphism $\mathbf{coim}(\eta) \cong \mathbf{im}(\eta)$. $\square$

Problem 9.7. *Prove that the category of short exact sequences in* $\mathbf{Ab}$ *is additive but not Abelian.*

Proof. Take arbitrary exact sequences in $\mathbf{Ab}$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

$$0 \longrightarrow A' \longrightarrow B' \longrightarrow C' \longrightarrow 0$$

The morphisms between the two sequences are tuples of group homomorphisms (f, g, h) where $f : A \to A'$, $g : B \to B'$, $h : C \to C'$ such that the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

One can define the Abelian structure on the morphisms between short exact sequences as the folllowing: for $(f, g, h), (f', g', h')$ in the set of morphisms between the two exact sequences above, we say $(f, g, h) + (f', g', h') = (f + f', g + g', h + h')$, where $(f + f')(x) = f(x) + f'(x)$ is defined as a group homomorphism as well, and similarly for other pairs. Moreover, this structure is bilinear because for $(f, g, h), (f', g', h') : A \to A', (x, y, z) : A' \to A''$, we have

$$\begin{aligned}((f, g, h) + (f', g', h')) \circ (x, y, z) &= (f + f', g + g', h + h') \circ (x, y, z)\\ &= ((f + f') \circ x, (g + g') \circ y, (h + h') \circ z)\\ &= (f \circ x, g \circ y, h \circ z) + (f' \circ x, g' \circ y, h' \circ z)\\ &= (f, g, h) \circ (x, y, z) + (f', g', h') \circ (x, y, z)\end{aligned}$$

and similarly one can show the other direction. In particular, there is the zero object is the tuple of trivial homomorphisms between the groups. Therefore, the category of short exact sequences in **Ab** is pre-additive.

Note that the initial and final objects in **Ab** are both the trivial group, then one can easily deduce that the trivial short exact sequence is both initial and final in the category of short exact sequences in **Ab**:

$$0 \longrightarrow e \longrightarrow e \longrightarrow e \longrightarrow 0$$

Therefore, this is the zero object in the category. Furthermore, there is a notion of finite product in the category, that is, for two short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

$$0 \longrightarrow A' \xrightarrow{f'} B' \xrightarrow{g'} C' \longrightarrow 0$$

the product is another short exact sequence

$$0 \longrightarrow A \times A' \xrightarrow{f \times f'} B \times B' \xrightarrow{g \times g'} C \times C' \longrightarrow 0$$

This is clearly true because all the necessary properties hold componentwise. One can also check that the universal property holds. Suppose there is another short exact sequence with morphisms to the original two sequences, then one can define a morphism from this exact sequence to the product by taking componentwise maps. Furthermore, this construction is unique because of the universal property of group product on each component. Therefore, the category is additive.

We first show that if a morphism $f : A \to B$ is a monomorphism in an Abelian category, then it is the kernel of $g : B \to \mathbf{coker}(f)$ (canonical surjective homomorphism) and zero morphism. Indeed, $g \circ f = 0 \circ f = 0$ by definition. Also, it satisfies the universal property because suppose there is some $k : C \to B$ satisfies the same property $g \circ k = 0$. By definition, the image of k is contained in the image of f. By the lemma, there is $\mathbf{im}(f) \cong A$. In particular, there is some inverse $f' : B \to A$ of f. Therefore, the image of k is contained in A. Therefore, let $h : C \to A$ be defined as taking c to $f'(k(c)) \in A$. Therefore, we have $f \circ h = ff'k = k$ by definition. Note that since f is a monomorphism, so by left cancellation h is unique.

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "g"] & B/\mathbf{im}(f) \\
C \arrow[u, dashed, "h"] \arrow[ur, "k"'] & &
\end{tikzcd}$$

Therefore, f is the kernel indeed.

We now show that for an Abelian category, if a morphism is both a monomorphism and an epimorphism, then it is an isomorphism. Let $f : A \to B$ be a monomorphism and an epimorphism. Since it is a monomorphism, then by the fact we just proved, it is the kernel/equalizer of some $g, h : B \to C$. However, as f is an epimorphism and is an equalizer, by definition $g = h$. In particular, the equalizer of two identical morphisms is clearly the isomorphism from B to itself, as it satisfies the universal property. By definition, f is an isomorphism indeed.

Given by the properties above, to show that the category is not Abelian, it suffices to show that there is some morphism that is both monic and epic, but is not an isomorphism.

Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r, "0"] \arrow[d, "0"] & A \arrow[r, "\mathbf{id}"] \arrow[d, "\mathbf{id}"] & A \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & A \arrow[r, "\mathbf{id}"] & A \arrow[r, "0"] & 0 \arrow[r] & 0
\end{tikzcd}$$

One can easily check that both sequences are exact. We now show that $(0, \mathbf{id}, 0)$ is a monomorphism. Consider another morphism (a, b, c) such that $(0, \mathbf{id}, 0) \circ (a, b, c) = 0$. Note that $a = 0$ because we have the zero object, and $b = 0$ because it is composed to the identity. Finally, $\mathbf{id} \circ b = 0$, which means $c = 0$ has to be true so that the square can commute. Similarly, $(0, \mathbf{id}, 0)$ is an epimorphism. However, it is not an isomorphism because there is clearly no inverse morphism in this case. Therefore, the category is not Abelian by the argument above.

Final note: considering the diagram below, one can show that an epimorphism in this category is when f_2, f_3 are epimorphisms, and a monomorphism in this category is when f_1, f_2 are monomorphisms.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f_1"] & B \arrow[r] \arrow[d, "f_2"] & C \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

□

Problem 9.8. *Show that an additive functor $F : \mathscr{C} \to \mathscr{D}$ between Abelian categories is exact if and only if it carries all short exact sequences in $\mathscr{C}$ to short exact sequences in $\mathscr{D}$.*

Proof. Suppose the additive functor $F : \mathscr{C} \to \mathscr{D}$ of Abelian categories is exact. Consider the following arbitrary short exact sequence in $\mathscr{C}$

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

By definition, since F is exact, then it is both left exact and right exact, which means

$$0 \longrightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \longrightarrow 0$$

in $\mathscr{D}$ is also a short exact sequence because $\ker(Fg) = FA$, $\mathbf{coker}(Ff) = FC$, where Ff is a monomorphism and Fg is an epimorphism. Therefore, F carries all short exact sequences in $\mathscr{C}$ to short exact sequences in $\mathscr{D}$.

Conversely, suppose all short exact sequences in $\mathscr{C}$ are mapped to short exact sequences. Take arbitrary short exact sequence in $\mathscr{C}$ as

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

Therefore,

$$0 \longrightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \longrightarrow 0$$

is also a short exact sequence, then by definition both $0 \to FA \xrightarrow{Ff} FB \xrightarrow{Fg} C$ and $FA \xrightarrow{Ff} FB \xrightarrow{Fg} C \to 0$ are exact. By definition, F is both left exact and right exact, then by definition F is exact. ☐

Problem 9.9. *Let $\mathscr{C}$ be the category of pairs (A, B) of Abelian groups such that $A \subset B$. A morphism between (A, B) and (A', B') is a group homomorphism $f : B \to B'$ such that $f(A) \subseteq A'$. Prove that $\mathscr{C}$ is an additive category admitting kernels and cokernels, but $\mathscr{C}$ is not Abelian.*

Proof. Take arbitrary elements $(A, B), (C, D)$ in $\mathscr{C}$. There is an Abelian group structure on $\mathbf{Mor}((A, B), (C, D))$ in the following sense: take arbitrary $f, f' : (A, B) \to (C, D)$ and $g : (C, D) \to (E, F)$ is another morphism in $\mathscr{C}$. These morphisms are basically the following: $f, f' : B \to D$ satisfies $f(A) \subseteq C$ and $f'(A) \subseteq C$, and $g : D \to F$ satisfies $g(C) \subseteq E$. Define $f + f'$ as the morphism such that $(f + f')(x) = f(x) + f'(x)$ for all elements in B. In particular, if $x \in A$, then $(f + f')(x) = f(x) + f'(x) \in C$. Note that this structure is essentially embedded in $\mathbf{Ab}$, which has an Abelian structure itself. Therefore, $f + f' \in \mathbf{Mor}((A, B), (C, D))$. One can check the following derivation:

$$\begin{aligned} g \circ (f + f')(x) &= g(f(x) + f'(x)) \\ &= g(f(x)) + g(f'(x)) \\ &= (g \circ f)(x) + (g \circ f')(x) \\ &= (g \circ f + g \circ f')(x) \end{aligned}$$

Therefore, $g\circ(f+f') = g\circ f+g\circ f'$. One can also check the other property in a similar fashion, then this Abelian structure is bilinear indeed. Note that there is a zero morphism in $\mathbf{Mor}((A,B),(C,D))$, given by the zero morphism from B to D. Therefore, the structure is pre-additive.

We now show that $\mathscr{C}$ is additive. Note that the zero object is (e,e) where e is the trivial group (which is Abelian) and is the zero object of the category of Abelian groups, which is still both initial and terminal. Finally, for arbitrary objects (A,B) and (A',B'), there is a product $(A\times A', B\times B')$. This is well-defined because by definition $A\times A'\subseteq B\times B'$. We can also check the universal property as follows:

$$\begin{tikzcd}
(C,D) \arrow[dr, dashed, "\exists! f"] \arrow[drr, bend left, "q_1"] \arrow[ddr, bend right, "q_2"'] & & \\
& (A\times A', B\times B') \arrow[r, "p_1"] \arrow[d, "p_2"] & (A,B) \\
& (A',B') &
\end{tikzcd}$$

Here p_1 are canonical surjective homomorphism defined from $B\times B'$ to B, and satisfies the property that $p(A\times A')\subseteq A$. Similarly we may define p_2. Let (C,D) be some arbitrary object with morphisms $q_1 : (C,D)\to(A,B)$ and $q_2 : (C,D)\to(A',B')$. Then we define $f : (C,D)\to(A\times A', B\times B')$ by the homomorphism $f(d) = (b,b')$ where $b = q_1(d)$ and $b' = q_2(d)$. Note that for $c\in C\subseteq D$, we have $f(c) = (a,a')$ where $a = q_1(c)\in A$ and $a' = q_2(c)\in A'$ by definition. Moreover, we have $p_1\circ f(d) = p_1(q_1(d), q_2(d)) = q_1(d)$, which means $p_1\circ f = q_1$. Similarly, $p_2\circ f = q_2$. Therefore, f satisfies the universal property. We can show that this morphism is unique. Suppose there is some other morphism g that satisfies this property, then $p_1\circ g = p_1\circ f$. However, p_1 is an epimorphism, so by left cancellation we have $g = f$. Therefore, this ensures f to be unique. Hence, this construction is the finite product indeed. By definition, $\mathscr{C}$ is an additive category.

Now we show that the category admits kernels. Take arbitrary $f : (A,B)\to(A',B')$. We construct the kernel of f in the traditional sense: consider an object (C,D) such that C is the kernel of homomorphism f under the mapping $A\to A'$, and D is the kernel of homomorphism f under the mapping $B\to B'$. In particular, $C\subseteq A$ and $D\subseteq B$, so there is an inclusion morphism $i : (C,D)\to(A,B)$. Moreover, both C and D are Abelian groups by definition. Also, this satisfies the property that $0\circ i = f\circ i$ by definition. Finally, we show that this construction (C,D) satisfies the universal property: take (C',D') with morphism $p : (C',D')\to(A,B)$ such that $0\circ p = f\circ p$. We have the following diagram:

$$\begin{tikzcd}
(C,D) \arrow[r, "i"] & (A,B) \arrow[r, "f"] & (A',B') \\
(C',D') \arrow[u, dashed, "\exists! g"] \arrow[ur, "p"'] & &
\end{tikzcd}$$

Since p satisfies $f \circ p = 0 \circ p$, then by definition p sends elements in D' to elements in the kernel of $f : B \to B'$ (which is just D), especially send elements in C' to the kernel of $f : A \to A'$ (which is just C). We construct $g : (C', D') \to (C, D)$ by sending $d' \in D'$ to $p(d') \in D$, and sending $c' \in C'$ to $p(c') \in C$. In particular, we have $i(g(d')) = i(p(d')) = p(d')$, which means $i \circ g = p$ under domain D'. Similarly, we have $i \circ g = p$ under domain C'. Note that such morphism g is unique, by the universal property given by equalizer for $f : B \to B'$ and zero morphism, and the universal property given by equalizer for $f : A \to A'$ and zero morphism. Therefore, the category admits kernels, and in a dual argument one can check that it also admits cokernels.

Finally, we show that $\mathscr{C}$ is not Abelian. Suppose, towards contradiction, that it is Abelian. We first show that if $f : A \to B$ is a monomorphism in an Abelian category, then it is the kernel of $g : B \to \mathbf{coker}(f)$ (canonical surjective homomorphism) and zero morphism. Indeed, $g \circ f = 0 \circ f = 0$ by definition. Also, it satisfies the universal property because suppose there is some $k : C \to B$ satisfies the same property $g \circ k = 0$. By definition, the image of k is contained in the image of f. By the lemma, there is $\mathbf{im}(f) \cong A$. In particular, there is some inverse $f' : B \to A$ of f. Therefore, the image of k is contained in A. Therefore, let $h : C \to A$ be defined as taking c to $f'(k(c)) \in A$. Therefore, we have $f \circ h = ff'k = k$ by definition. Note that since f is a monomorphism, so by left cancellation h is unique.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & B/\mathbf{im}(f) \\ C \arrow[u, dashed, "h"] \arrow[ur, "k"'] & & \end{tikzcd}$$

Therefore, f is the kernel indeed.

We now show that for an Abelian category, if a morphism is both a monomorphism and an epimorphism, then it is an isomorphism. Let $f : A \to B$ be a monomorphism and an epimorphism. Since it is a monomorphism, then by the fact we just proved, it is the kernel/equalizer of some $g, h : B \to C$. However, as f is an epimorphism and is an equalizer, by definition $g = h$. In particular, the equalizer of two identical morphisms is clearly the isomorphism from B to itself, as it satisfies the universal property. By definition, f is an isomorphism indeed.

We now construct a morphism that is both a monomorphism and an epimorphism but is not an isomorphism. In particular, that means $\mathscr{C}$ is not an Abelian category. Consider $f : (0, \mathbb{Z}) \to (\mathbb{Z}, \mathbb{Z})$, where $f : \mathbb{Z} \to \mathbb{Z}$ is given by the identity morphism. Note that this is a monomorphism: consider $g : (A, B) \to (0, \mathbb{Z})$ such that $f \circ g = 0$. By definition, $g(B) = f(g(B)) = 0$, and $g(A) = 0$. Therefore, g is the zero morphism. Hence, by definition f is a monomorphism. On the other hand, this is also an epimorphism: consider $g : (\mathbb{Z}, \mathbb{Z}) \to (A, B)$ such that $g \circ f = 0$. By definition, $g(\mathbb{Z}) = g(f(\mathbb{Z})) = 0$, which means it is also the zero morphism. Hence, f is also an epimorphism. However, f is obviously not an isomorphism because we cannot create an inverse map. Therefore, by the contrapositive statement of the result above, $\mathscr{C}$ is not Abelian. □

Problem 9.10. *Give a proof of the Snake Lemma.*

Proof. Consider the following diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[r, "g"] \arrow[d, "v"] & C \arrow[r] \arrow[d, "w"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0 \end{tikzcd}$$

then we may derive the following sequence

$$\begin{tikzcd} & 0 \arrow[r] & \ker(u) \arrow[r, "\bar{f}"] & \ker(v) \arrow[r, "\bar{g}"] & \ker(w) \arrow[dlll, "\delta"] \\ & \mathbf{coker}(u) \arrow[r, "\bar{f}'"'] & \mathbf{coker}(v) \arrow[r, "\bar{g}'"'] & \mathbf{coker}(w) \arrow[r] & 0 \end{tikzcd}$$

Note that $\bar{f}, \bar{g}, \bar{f}', \bar{g}'$ are defined as restrictions of f, g, f', g'. We now show the exactness one by one. Observe that by restriction, $\bar{f}$ is still a monomorphism and $\bar{g}'$ is still an epimorphism, then by theorem in class, the sequence is exact at $\ker(u)$ and at $\mathbf{coker}(w)$ respectively.

Since $g \circ f = 0$, then $\bar{g} \circ \bar{f} = 0$, which means $\mathbf{im}(\bar{f}) \subseteq \ker(\bar{g})$. Take $\alpha \in \ker(\bar{g})$, then $\bar{g}(\alpha) = 0$, so $g(\alpha) = 0$, which means $\alpha \in \ker(g) = \mathbf{im}(f)$. We pull it back to some $\beta \in A$ with $f(\beta) = \alpha$. Note that $\beta \in \ker(u)$ because $f'(u(\beta)) = v(f(\beta)) = v(\alpha) = 0$, but f' is a monomorphism, so $u(\beta) = 0$. Therefore, $\alpha \in \mathbf{im}(\bar{f})$. Therefore, $\ker(\bar{g}) \subseteq \mathbf{im}(\bar{f})$ and so $\ker(\bar{g}) = \mathbf{im}(\bar{f})$. Therefore, the sequence is exact at $\ker(v)$.

We show that $\delta \circ \bar{g} = 0$, then by definition $\mathbf{im}(\bar{g}) \subseteq \ker(\delta)$. Take arbitrary $a \in \ker(v)$, then $v(a) = 0$, which means $v(a) = f'(0)$. In particular, 0 is the lift of $\bar{g}(a)$, which means $\delta(\bar{g}(a)) = 0 + \mathbf{im}(u)$. By definition, $\delta \circ \bar{g} = 0$. We now show that $\ker(\delta) \subseteq \mathbf{im}(\bar{g})$. Take $b \in \ker(\delta)$, which means $b \in \ker(w)$ as well. Also, since $b \in \ker(\delta)$, then $\delta(b) \in \mathbf{im}(u)$. Therefore, by pulling b back, there is $g(b') = b$, with $v(b') = f'(a)$ for some $a \in \mathbf{im}(u)$. Hence, there is $u(a') = a$. Now $v(f(a')) = f'(u(a')) = v(b')$, so $b' - f(a') \in \ker(v)$. Now, $\bar{g}(b' - f(a')) = \bar{g}(b') - \bar{g}(f(a')) = b - g(f(a')) = b - 0 = b$. Therefore, $b \in \mathbf{im}(\bar{g})$. Therefore, $\ker(\delta) \subseteq \mathbf{im}(\bar{g})$, so $\ker(\delta) = \mathbf{im}(\bar{g})$, and the sequence is exact at $\ker(w)$.

Now we show that $\bar{f}' \circ \delta = 0$. Take arbitrary $a \in \ker(w)$. There is a pullback $a' \in B$ such that $g(a') = a$. Furthermore, $v(a') = f'(b)$ for some $b \in A'$ pullback by the exactness. By definition, there is $\delta(a) = b + \mathbf{im}(u)$. Now, $\bar{f}'(\delta(a)) = \bar{f}'(b + \mathbf{im}(u)) = f'(b) + \mathbf{im}(v)$, which is just 0 since $f'(b) = v(a') \in \mathbf{im}(v)$. Therefore, $\mathbf{im}(\delta) \subseteq \ker(\bar{f}')$. We now show the other direction. Take $a + \mathbf{im}(u) \in \mathbf{coker}(\bar{f}')$. By definition, $\bar{f}'(a) \in \mathbf{im}(v)$. In particular, take $\bar{f}'(a) = v(b)$. Now $w(g(b)) = g'(v(b)) = g'(f'(a)) = 0$. Therefore, $g(b) \in \ker(w)$. By construction, $\delta(g(b)) = a + \mathbf{im}(u)$, so $a + \mathbf{im}(u) \in \mathbf{im}(\delta)$. Hence, $\mathbf{im}(\delta) = \ker(\bar{f}')$, which means the sequence is exact at $\mathbf{coker}(u)$.

Finally, we show that the sequence is exact at $\mathbf{coker}(v)$. By definition, $\bar{g}' \circ \bar{f}' = 0$, so $\mathbf{im}(\bar{f}') \subseteq \ker(\bar{g}')$. Now take $a + \mathbf{im}(v) \in \ker(\bar{g}')$, so it is in the cokernel of v. By exactness, $g'(a) \in \mathbf{im}(w)$. Denote $g'(a) = w(b)$. There is now some $c \in B$ such that lifts $g(c) = b$. Now $g'(a - v(c)) = g'(a) - g'(v(c)) = g'(a) - w(g(c)) = 0$ by definition. Since we started with $a + \mathbf{im}(v) \in \ker(\bar{g}')$, we can replace it with $a - v(c) + \mathbf{im}(v) \in \ker(\bar{g}')$ without loss of generality as the new representation of the coset, then we take $g'(a) = 0$. In particular, $a \in \ker(g') = \mathbf{im}(f')$. Let $a = f'(x)$ for some x. Therefore, $a + \mathbf{im}(v) = f'(x + \mathbf{im}(u))$. By definition, $\ker(\bar{g}') \subseteq \mathbf{im}(\bar{f}')$. Therefore, $\ker(\bar{g}') = \mathbf{im}(\bar{f}')$, which means the sequence is exact at $\mathbf{coker}(v)$. This concludes the proof. □

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