MATH 518 Notes

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August 28, 2023

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Definition 1.1. Let M be a topological space. An *atlas* on M is a collection $\{\varphi_\alpha : U_\alpha \to W_\alpha\}_{\alpha \in A}$ of homeomorphisms called *coordinate charts*, so that

1. $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M,

2. for all $\alpha \in A$, W_α is an open subset of some $\mathbb{R}^{n_\alpha}$,

3. for all $\alpha, \beta \in A$, the induced map $\varphi_\beta \circ \varphi_\alpha^{-1}|_{U_\alpha \cap U_\beta}$ is C^∞, i.e., smooth.
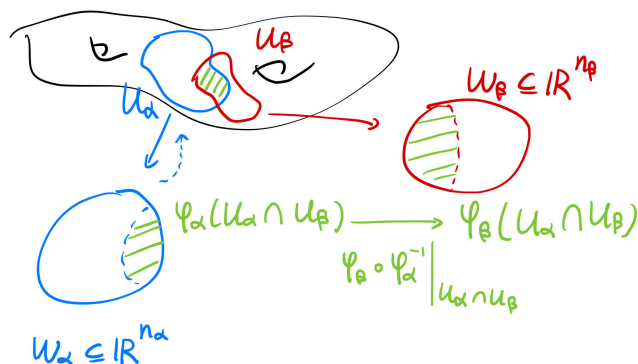

Figure 1: Atlas and Coordinate Chart

Example 1.2. Let $M = \mathbb{R}^n$ be equipped with standard topology, and let $A = \{*\}$, so $U_* = \mathbb{R}^n$ is the open cover of itself. Now the identity map

$$\varphi_* : U_* \to \mathbb{R}^n$$
$$u \mapsto u$$

is an atlas on $\mathbb{R}^n$.

Example 1.3. Let $M = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be equipped with subspace topology. Let $U_\alpha = S^1 \backslash \{(1, 0)\}$ and $U_\beta = S^1 \backslash \{(-1, 0)\}$, and let $A = \{\alpha, \beta\}$. Let $W_\alpha = (0, 2\pi)$ and $W_\beta = (-\pi, \pi)$. We define $\varphi_\alpha^{-1}(\theta) = (\cos(\theta), \sin(\theta))$ and $\varphi_\beta^{-1}(\theta) = (\cos(\theta), \sin(\theta))$, then

$$(\varphi_\beta \circ \varphi_\alpha^{-1})(\theta) = \begin{cases} \theta, 0 < \theta < \pi \\ \theta - 2\pi, \pi < \theta < 2\pi \end{cases}$$

is smooth.

Example 1.4. Let X be a topological space with discrete topology, and let $A = X$, then $\{\varphi_x : \{x\} \to \mathbb{R}^0\}_{x \in X}$ gives an atlas.

Example 1.5. Let V be a finite-dimensional real vector space of dimension n. Pick a basis $\{v_1, \ldots, v_n\}$ of V, then there is a linear bijection φ with inverse

$$\varphi^{-1} : \mathbb{R}^n \to V$$

$$(x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i v_i.$$

The topology on V needs to make φ^{-1} a homeomorphism, and the obvious choice is just the collection of preimages, namely

$$\mathcal{T} = \{\varphi^{-1}(W) \mid W \subseteq \mathbb{R}^n \text{ open}\},$$

then $\varphi : V \to \mathbb{R}^n$ becomes an atlas.

Definition 1.6. Two atlases $\{\varphi_\alpha : U_\alpha \to W_\alpha\}_{\alpha \in A}$ and $\{\psi_\beta : V_\beta \to O_\beta\}_{\beta \in B}$ on a topological space M are *equivalent* if for all $\alpha \in A$ and $\beta \in B$,

$$\psi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap V_\beta) \subseteq \mathbb{R}^{n_\alpha} \to \psi_\beta(U_\alpha \cap V_\beta) \subseteq \mathbb{R}^{n_\beta}$$

is always C^∞, with C^∞-inverses. Such continuous maps are called *diffeomorphisms*. Alternatively, the two atlases are equivalent if their union $\{\varphi_\alpha\}_{\alpha \in A} \cup \{\psi_\beta\}_{\beta \in B}$ is always an atlas.

Exercise 1.7. Equivalence of atlases is an equivalence condition.

Definition 1.8. A (smooth) *manifold* is a topological space together with an equivalence class of atlases.

Convention. All manifolds are assumed to be smooth of C^∞, but not necessarily *Haudorff* and/or *second countable*.

Example 1.9. Continuing from Example 1.5, now suppose $\{w_1, \ldots, w_n\}$ gives another basis of V, with

$$\psi^{-1} : \mathbb{R}^n \to V$$

$$(y_1, \ldots, y_n) \mapsto \sum_{i=1}^n y_i w_i.$$

This gives a change-of-basis matrix, so it is automatically C^∞ as a multiplication of invertible matrices. Therefore, the topology here does not depend on the chosen basis.

Recall. A topological space X is *Hausdorff* if for all distinct points $x, y \in X$, there exists open neighborhoods $U \ni x$ and $V \ni y$ such that $U \cap V = \varnothing$.
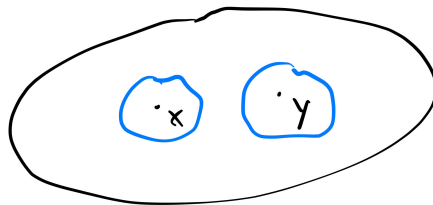

Figure 2: Hausdorff Condition

Convention. Via our definition (Definition 1.8), not all manifolds are Hausdorff.

Example 1.10. Let $Y = \mathbb{R} \times \{0, 1\}$, i.e., a space with two parallel lines, with a fixed topology. Define $\sim$ to be the smallest equivalence relation on Y such that $(x, 0) \sim (x, 1)$ for $x \neq 0$, and define $X = Y/\sim$. X is called the *line with two origins*, and it is second countable but not Hausdorff.
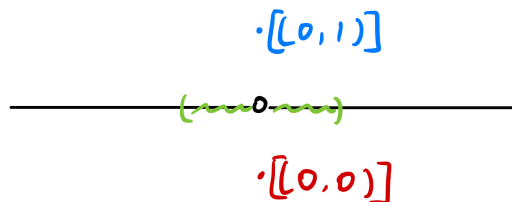
Figure 3: Line with Two Origins

Example 1.11. Take charts

$$\{\varphi : M = \mathbb{R} \to \mathbb{R}\}$$
$$x \mapsto x$$

and

$$\{\psi : M = \mathbb{R} \to \mathbb{R}\}$$
$$x \mapsto x^3$$

on $M = \mathbb{R}$, then

$$\varphi \circ \psi^{-1} : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^{\frac{1}{3}}$$

is not C^∞, so φ and ψ are two different charts, hence give two different manifolds.

Definition 1.12. A map $F : M \to N$ between two manifolds is *smooth* if

1. F is continuous, and

2. for all charts $\varphi : U \to \mathbb{R}^m$ on M and charts $\psi : V \to \mathbb{R}^n$ on N, $\psi^{-1} \circ F \circ \varphi|_{\varphi(U \cap F^{-1}(V))}$ is C^∞.

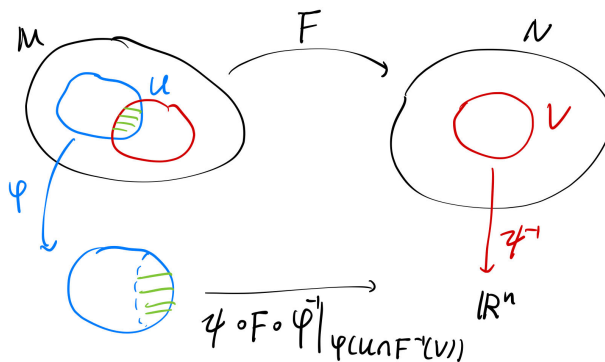Figure 4: Smooth Map between Manifolds

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Exercise 2.1.   1. $\mathrm{id} : M \to M$ is smooth.

2. If $f : M \to N$ and $g : N \to Q$ are smooth maps between manifolds, then so is $gf : M \to Q$.

Punchline. The manifolds and the smooth maps between manifolds form a category.

Recall. A smooth map $f : M \to N$ is called a *diffeomorphism*, as seen in Definition 1.6, if it has a smooth inverse. This is the notion of an isomorphism in the category of manifolds.

Warning.   1. Following Example 1.11,

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^3$$

has an inverse

$$f^{-1} : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^{\frac{1}{3}},$$

but f^{-1} is not differentiable at $x = 0$. Hence, f is not a diffeomorphism.

2. Take $\mathbb{R}$ with discrete topology, then all singletons are open sets, then the map

$$f : \mathbb{R}_{\text{dis}} \to \mathbb{R}_{\text{std}}$$
$$x \mapsto x$$

is a smooth bijection, but f^{-1} is not continuous.

Example 2.2. Consider $M = (\mathbb{R}, \{\psi = \text{id} : \mathbb{R} \to \mathbb{R}\})$ and $N = (\mathbb{R}, \{\psi : \mathbb{R} \to \mathbb{R}, x \mapsto x^3\})$ as two manifolds on $\mathbb{R}$ with standard topology. To see that they are equivalent, consider the homeomorphism

$$f : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^{\frac{1}{3}},$$

then $(\psi \circ f \circ \varphi^{-1})(x) = \psi(f(x)) = (x^{\frac{1}{3}})^3 = x$, so f is smooth, and $(\psi \circ f \circ \varphi^{-1})^{-1} = \varphi \circ f^{-1} \circ \psi^{-1} = \text{id}$, therefore f^{-1} is also smooth. Hence, f is a diffeomorphism.

We will now consider the real projective space $\mathbb{R}P^{n-1}$ and the quotient map $\pi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}P^{n-1}$.

Definition 2.3. Define a binary relation on $\mathbb{R}^n \setminus \{0\}$ by $v_1 \sim v_2$ if and only if there exists $\lambda \neq 0$ such that $v_1 = \lambda v_2$. This is an equivalence relation, and we identify the equivalence class $[v]$ of $v \in \mathbb{R}^n \setminus \{0\}$ as a line $\mathbb{R}v = \text{span}_{\mathbb{R}}\{v\}$ through v. Then we define the *real projective space* $\mathbb{R}P^{n-1} = (\mathbb{R}^n \setminus \{0\})/ \sim$.

The natural topology on $\mathbb{R}P^{n-1}$ is the quotient topology, where $\pi : \mathbb{R}^n \setminus \{0\} \twoheadrightarrow \mathbb{R}P^{n-1}$ is surjective and continuous, so we define $U \subseteq \mathbb{R}P^{n-1}$ to be open if and only if $\pi^{-1}(U)$ is open in $\mathbb{R}^n \setminus \{0\}$.
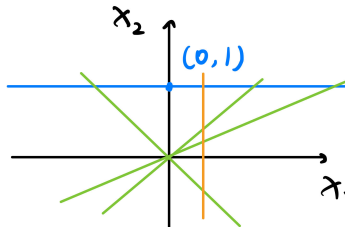

Figure 5: Stereographical Projection

Claim 2.4. $\mathbb{R}P^{n-1}$ is a manifold.

Proof. Define

$$\varphi_i : U_i \to \mathbb{R}^{n-1}$$
$$[v_1, \ldots, v_n] \mapsto \left(\frac{v_1}{v_i}, \ldots, \frac{v_{i-1}}{v_i}, \frac{v_{i+1}}{v_i}, \ldots, \frac{v_n}{v_i} \right),$$

then

$$\varphi_i^{-1} : \mathbb{R}^{n-1} \mapsto U_i$$
$$(x_1, \ldots, x_{n-1}) \mapsto [(x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_{n-1})],$$

therefore

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$
$$(x_1, \ldots, x_{n-1}) \mapsto \varphi_j([(x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_{n-1})])$$
$$= \begin{cases} \left(\frac{x_1}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_{n-1}}{x_j} \right), & j < i \\ (x_1, \ldots, x_{n-1}), & j = i \\ \left(\frac{x_1}{x_{j-1}}, \ldots, \frac{1}{x_{j-1}}, \ldots, \frac{x_{j-2}}{x_{j-1}}, \frac{x_j}{x_{j-1}}, \ldots, \frac{x_{n-1}}{x_{j-1}} \right), & j > i \end{cases}$$

Therefore, this is C^∞ as a rational map on $\varphi_i(U_i \cap U_j)$, and so this gives an atlas, hence $\mathbb{R}P^{n-1}$ is a manifold. $\qquad\square$

Claim 2.5. $\pi : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}P^{n-1}$ is smooth.

Proof. Note that

$$\psi : \mathbb{R}^n \backslash \{0\} \hookrightarrow \mathbb{R}^n$$
$$x \mapsto x$$

is an atlas on $\mathbb{R}^n \backslash \{0\}$, and

$$\varphi_i \circ \pi \circ \varphi_i^{-1} : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^{n-1}$$
$$(v_1, \ldots, v_n) \mapsto \varphi_i([(v_1, \ldots, v_n)])$$
$$= \left(\frac{v_1}{v_i}, \ldots, \frac{v_{i-1}}{v_i}, \frac{v_{i+1}}{v_i}, \ldots, \frac{v_n}{v_i} \right).$$

This is C^∞ on $\pi^{-1}(U_i) = \{(v_1, \ldots, v_n) \mid v_i \neq 0\}$, so π is smooth. $\qquad\square$

Definition 2.6. A *smooth function* on a manifold M is a function $f : M \to \mathbb{R}$ so that for any coordinate chart $\varphi : U \to \varphi(U)$ open in $\mathbb{R}^m$, the function $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is smooth.

Remark 2.7. $f : M \to \mathbb{R}$ is smooth if and only if $f : M \to (\mathbb{R}, \{\mathrm{id} : \mathbb{R} \to \mathbb{R}\})$, usually called the *standard manifold structure on* $\mathbb{R}$, is smooth.

Notation. We denote $C^\infty(M)$ to be the set of all smooth functions $f : M \to \mathbb{R}$.

Remark 2.8. $C^\infty(M)$ is a smooth $\mathbb{R}$-vector space, that is, for all $\lambda, \mu \in \mathbb{R}$ and $f, g \in C^\infty(M)$,

- $(\lambda f + \mu g)(x) = \lambda f(x) + \mu g(x)$ for all $x \in M$,

- $(f \cdot g)(x) = f(x)g(x)$ for all $x \in M$.

Therefore, $C^\infty(M)$ becomes a (commutative, associative) $\mathbb{R}$-algebra.

Fact. Connecting manifolds have the notion of dimension. That is, the dimensions of open subsets induced by coordinate charts are the same.

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Definition 3.1. Let M be a manifold, then for every point $q \in M$, there exists a well-defined non-negative integer $\dim_M(q)$, so that for any coordinate chart $\varphi : U \to \mathbb{R}^m$ for $U \ni q$, we have $\dim_M(q) = m$ for some non-negative integer m that only depend on M. Consequently, $\dim_M : M \to \mathbb{Z}^{\geq 0}$ is a locally constant function. This integer m is called the *dimension* of M.

Proof. Indeed, say $\psi : V \to \mathbb{R}^n$ is another chart with $U \cap V \ni q$, then $\psi \circ \varphi^{-1}|_{\varphi(U \cap V)} : \varphi(U \cap V) \subseteq \mathbb{R}^m \to \psi(U \cap V) \subseteq \mathbb{R}^n$ is a diffeomorphism, therefore the Jacobian $D(\psi \circ \varphi^{-1})(\varphi(a)) : \mathbb{R}^m \to \mathbb{R}^n$ is a linear isomorphism, thus $m = n$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Definition 3.2. Suppose $(M, \{\varphi_\alpha : U_\alpha \to \mathbb{R}^m\}_{\alpha \in A})$ and $(N, \{\psi_\alpha : V_\beta \to \mathbb{R}^n\}_{\beta \in B})$ are two manifolds. One can give a manifold structure to the product set $M \times N$, called the *product manifold*, as follows:

- give $M \times N$ the product topology,

- let $\{\varphi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n\}_{(\alpha, \beta) \in A \times B}$ to be the atlas on $M \times N$. This is well-defined since the transition maps of $\alpha, \alpha' \in A$ and $\beta, \beta' \in B$ are over $(U_\alpha \times V_\beta) \cap U_{\alpha'} \times V_{\beta'} = (U_\alpha \cap U_{\alpha'}) \times (V_\beta \cap V_{\beta'})$ with $(\varphi_{\alpha'} \times \psi_{\beta'}) \circ (\varphi_\alpha \times \psi_\beta)^{-1} = (\varphi_{\alpha'} \circ \varphi_\alpha^{-1}, \psi_{\beta'} \circ \psi_\beta^{-1})$. This is smooth since products of smooth maps are smooth.

Punchline. The product construction of manifolds gives the categorical product in the category of manifolds.

Property.   1. The projection maps

$$p_M : M \times N \to M$$
$$(m, n) \mapsto m$$

and

$$p_N : M \times N \to N$$
$$(m, n) \mapsto n$$

are C^∞.

2. *Universal Property of Product*: for any manifold Q and smooth maps $f_M : Q \to M$ and $f_N : Q \to N$, there exists a unique map

$$g : Q \to M \times N$$
$$q \mapsto (f(q), g(q))$$

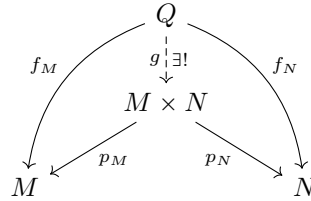such that $p_M \circ g = f_M$, and $p_N \circ g = f_N$.

Figure 6: Universal Property of Product

Recall.   • A topological space X is *second countable* if the topology has a countable basis: there exists a collection $\mathcal{B} = \{B_i\}_{i \in \mathbb{N}}$ of open sets so that any open set of X is a union of some B_i's.

- A cover $\{U_\alpha\}_{\alpha \in A}$ of a topological space is *locally finite* if for all $x \in X$, there exists a neighborhood N of X such that $N \cap U_\alpha = \varnothing$ for all but finitely many α's.

Example 3.3. Let $X = \mathbb{R}$, then

- $\{U_n = (-n, n)\}_{n \geqslant 0}$ is an open cover, but is not locally finite,

- $\{U_n = (n, n + 2)\}_{n \in \mathbb{Z}}$ is a locally finite open cover of $\mathbb{R}$,

- $\{U_n = (n, n + 2]\}_{n \in \mathbb{Z}}$ is a locally finite cover of $\mathbb{R}$, but is not an open cover.

Recall. An (open) cover $\{V_\beta\}_{\beta \in B}$ is a *refinement* of a cover $\{U_\alpha\}_{\alpha \in A}$ if for all β, there exists $\alpha = \alpha(\beta)$ such that $V_\beta \subseteq U_{\alpha(\beta)}$.

Definition 3.4. A Hausdorff topological space is *paracompact* if every open cover has a locally finite open refinement.

Fact. A connected Hausdorff manifold is paracompact if and only if it is second countable.

Corollary 3.5. A Haudorff manifold is paracompact if and only if its connected components are second countable.

Example 3.6. $\mathbb{R}$ with discrete topology is paracompact but not second countable.

Convention. Usually, we assume manifolds are paracompact, except when we need a non-Haudorff manifold. This condition is required for the existence of *partition of unity* (i.e., constant function id).

Recall. If X is a space, and $Y \subseteq X$ is a subset, then the *closure* $\bar{Y}$ of Y is the smallest closed set containing Y.

Definition 3.7. Given a topological space X and a function $f : X \to \mathbb{R}$, the *support* of f over X is

$$\mathrm{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}.$$

Example 3.8. The function

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leqslant 0 \end{cases}$$

is C^∞, with support $\overline{(0, \infty)} = [0, \infty)$.

Definition 3.9. Let M be a topological space and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover. A *partition of unity* subordinate to the cover is a collection of continuous functions $\{\psi_\alpha : M \to [0, 1]\}_{\alpha \in A}$ such that

1. $\mathrm{supp}(\psi_\alpha) \subseteq U_\alpha$ for all $\alpha \in A$,

2. $\{\mathrm{supp}(\psi_\alpha)\}_{\alpha \in A}$ is a locally finite closed cover of M,

3. $\sum\limits_{\alpha \in A} \psi_\alpha(x) = 1$ for all $x \in M$.

Remark 3.10. For all $x \in M$, there exists $\alpha_1, \ldots, \alpha_n$ such that $x \in \mathrm{supp}(\psi_{\alpha_i})$. Hence, for $\alpha \neq \alpha_1, \ldots, \alpha_n, \psi_\alpha(x) = 0$. Therefore, the summation in Definition 3.9 is finite.

Theorem 3.11. Let M be a paracompact manifold with open cover $\{U_\alpha\}_{\alpha \in A}$, then there exists a partition of unity $\{\psi_\alpha : U_\alpha \to [0, 1]\}_{\alpha \in A} \subseteq C^\infty(M)$ subordinate to the cover.

Example 3.12. Let $M = \mathbb{R}$ and consider for $n > 0$ the open sets $\{U_n = (-n, n)\}_{n \in \mathbb{N}}$. This is not locally finite at one point.

Example 3.13. Let $M = \mathbb{R}^n$, then for all $x \in \mathbb{R}^n$ and for $r > 0$, we have $B_r(x) = \{x' \in \mathbb{R}^n \mid ||x - x'|| < r\}$ and so $\{B_r(x)\}_{r > 0, x \in \mathbb{R}^n}$ is an open cover, but this is not locally finite everywhere.

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We will start to talk about tangent vectors.

Recall. For any point $q \in \mathbb{R}^n$ and any vector $v \in \mathbb{R}^n$, and any $f \in C^\infty(\mathbb{R}^n)$, the *directional derivative* of q in direction v with respect to f is

$$D_v f(q) = \frac{d}{dt}|_0 f(q + tv).$$

This gives a map $D_v(-)(q) : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ which is

- linear, and

- Leibniz rule holds, i.e.,

$$D_v(fg)(q) = D_v(f)(q) \cdot g(q) + f(q)D_v(g)(q).$$

In other words, $D_v(-)(q) : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation.

Definition 4.1. Let q be a point of a manifold M. A *tangent vector* to M at q is an $\mathbb{R}$-linear map $v : C^\infty(M) \to \mathbb{R}$ such that for all $f, g \in C^\infty(M)$,

$$v(fg) = v(f)g(q) + f(q)v(g).$$

Remark 4.2. v gives smooth vector fields over M an $C^\infty(M)$-module structure via evaluation.

Lemma 4.3. The set T_qM of all tangent vectors to M at q is an $\mathbb{R}$-vector space.

Lemma 4.4. Suppose $c \in C^\infty(M)$ is a constant function, then for all q and all $v \in T_qM$, $v(c) = 0$.

Proof. We have $v(1) = v(1 \cdot 1) = 1(q)v(1) + v(1)1(q) = 2v(1)$, so $v(1) = 0$. For a constant function c, we have

$$v(c) = v(c \cdot 1) = cv(1) = c(0) = 0.$$

$\square$

Lemma 4.5 (Hadamard). For any $f \in C^\infty(\mathbb{R}^n)$, there exists $g_1, \ldots, g_n \in C^\infty(\mathbb{R}^n)$ such that

- $f(x) = f(0) + \sum\limits_{i=1}^{n} x_i g_i(x)$, and

- $g_i(0) = \left(\frac{\partial}{\partial x_i} f \right)(0)$.

Proof. We have

$$
\begin{aligned}
f(x) - f(0) &= \int_0^1 \frac{d}{dt}(f(tx))dt \\
&= \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(tx) \cdot x_i dt \\
&= \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt \\
&= \sum_{i=1}^{n} x_i g_i(x).
\end{aligned}
$$

Therefore, $g_i(0) = \int_0^1 \frac{\partial f}{\partial x_i}(t \cdot 0)dt = \frac{\partial f}{\partial x_i}(0)$. $\square$

Remark 4.6. For $1 \leqslant i \leqslant n$, we have canonical tangent vectors to $\mathbb{R}^n$ at 0 given by

$$\frac{\partial}{\partial x_i}|_0 : C^\infty(\mathbb{R}^n) \to \mathbb{R}$$

$$f \mapsto \frac{\partial f}{\partial x_i}(0).$$

Lemma 4.7. $\left\{ \frac{\partial}{\partial x_1}|_0, \ldots, \frac{\partial}{\partial x_n}|_0 \right\}$ is a basis of $T_0\mathbb{R}^n$.

Proof. Suppose $\sum c_i \frac{\partial}{\partial x_i}|_0 = 0$, then

$$0 = \left(\sum_i c_i \frac{\partial}{\partial x_i}|_0 \right)(x_j) = \sum_i c_i \delta_{ij} = c_j.$$

Therefore, $c_j = 0$ for all j, thus we have linear independence. For all $v \in T_0 \mathbb{R}^n$, i.e., $v : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation, then $v = \sum_i v(x_i) \frac{\partial}{\partial x_i}|_0$. Let $f \in C^\infty(\mathbb{R}^n)$, then $f(X) = f(0) + \sum x_i g_i(x)$, thus

$$
\begin{aligned}
v(f) &= v(f(0)) + \sum_{i=1}^{n} v(x_i g_i(x)) \\
&= \sum_{i=1}^{n} v(x_i g_i(x)) \\
&= \sum_{i=1}^{n} (v(x_i) g_i(0) + x_i(0) v(g_i)) \\
&= \sum_{i=1}^{n} v(x_i) g_i(0) \\
&= \sum_{i=1}^{n} v(x_i) \frac{\partial f}{\partial x_i}(0).
\end{aligned}
$$

$\square$

Remark 4.8. This shows $\dim(T_0 \mathbb{R}^n) = n$ with the basis above.

Now let V be a finite-dimensional vector space with a basis $e_1, \ldots, e_n$, then

$$
\varphi : \mathbb{R}^n \to V
$$

$$
(t_1, \ldots, t_n) \mapsto \sum_{i=1}^{n} t_i e_i
$$

is a linear bijection, with linear inverse

$$
\psi : V \to \mathbb{R}^n
$$

$$
v \mapsto (\psi_1(v), \ldots, \psi_n(v))
$$

where $\psi_i(v)$'s are linear maps. To describe this with a basis, we have $\psi(\sum_i a_i e_i) = (a_1, \ldots, a_n)$, i.e., $\psi_i(e_j) = \delta_{ij}$.

Claim 4.9. $\{\psi_1, \ldots, \psi_n\}$ is a basis of $V^* = \operatorname{Hom}(V, \mathbb{R})$, called the *dual basis* of $\{e_1, \ldots, e_n\}$, denoted $e_j^* = \psi_j$.

Proof. Linear independence follows from $e_j^*(e_i) = \delta_{ij}$. Given $\ell : V \to \mathbb{R}$ to be a linear map, then $\ell = \sum \ell(e_i) e_i^*$ since $\left(\sum_i \ell(e_i) e_i^*\right)(e_j) = \ell(e_j)$. Given $v \in T_0 \mathbb{R}^n$, $v(f) = \sum a_i \left(\frac{\partial}{\partial x_i}|_0 f\right)$ for all $f \in C^\infty(\mathbb{R}^n)$. Note that $\frac{\partial}{\partial x_i}|_0(x_j) = \delta_{ij}$, so $v(x_j) = \sum a_i \frac{\partial}{\partial x_i}|_0(x_j) = \sum_i a_i \delta_{ij} = a_j$. Therefore, we have $a_i = v(x_i)$ for all i, thus $v(f) = \sum v(x_i) \left(\frac{\partial}{\partial x_i}|_0 f\right)$. Thus, the dual basis to $\frac{\partial}{\partial x_1}|_0, \ldots, \frac{\partial}{\partial x_n}|_0$ is $\{d(x_i)_0\}_{i=1}^{n}$ where $(dx_i)_0(v) = v(x_i)$ for all i. Hence, we have $v = \sum (dx_i)_0(v) \frac{\partial}{\partial x_i}|_0$. $\square$

Remark 4.10. Via a change of basis, this works at every point q on the local chart, so we can describe the tangent space on any point on a local chart.