MATH 131H Notes

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Preliminaries

This document is the notes based on Professor Monica Visan's teaching in the MATH 131AH and 131BH course in winter and spring 2021. The corresponding textbook is Baby Rudin.

1 Lecture 1: Statements

In Rubin's notation, natural numbers start with 1, i.e. $\mathbb{N} = \{1, 2, \cdots\}$.

Let A and B be two statements. We use the following notations:

- We write "A" if A is true.
- We write "not A" if A is false.
- We write "A and B" if both A and B are true.
- We write "A or[1] B" if A is true or B is true or both A and B are true.
- We write "$A \Rightarrow B$ if "A and B" or "not A". We read this as "A implies B" or "if A then B". In this case, B is at least as true as A. In particular, A, a false statement A can imply anything.

We usually write shorthand notation "T" and "F" to represent "true" and "false".

Example 1.1. Consider the following statement:

$$\text{If } x \text{ is a natural number, i.e. } x \in \mathbb{N} = \{1, 2, 3, \cdots\}, \text{ then } x \geq 1.$$

In this case, A is the statement "x is a natural number" and B is the statement "$x \geq 1$".

- Taking $x = 3$, we get $T \Rightarrow T$.
- Taking $x = \pi$, we get $F \Rightarrow T$.
- Taking $x = 0$, we get $F \Rightarrow F$.

[1] The notation "or" in mathematics is inclusive. We distinguish it from the exclusive or, usually called "xor", which means "either A or B"

Example 1.2. Consider the statement:

If a number is less than 10, then it is less than 20.

The statement is of the form "if... then...", where A is the statement "a number is less than 10", and B is the statement "it is less than 20".

- Taking a number 5, we get $T \Rightarrow T$.
- Taking a number 15, we get $F \Rightarrow T$.
- Taking a number 25, we get $F \Rightarrow F$.

We also write "$A \iff B$" if A and B are true together or false together. We read this as "A is equivalent to B" or "A if and only if B".

We can now compare these notions in logic to similar ones from set theory. Let X be an ambient space. Let A and B be subsets of X. Then

- ${}^cA = \{x \in X : x \notin A\}$.
- $A \cap B = \{x \in X : x \in A \text{ and } x \in B\}$.
- $A \cup B = \{x \in X : x \in A \text{ or } x \in B \text{ or } x \in A \cap B\}$.
- $A \subseteq B$ corresponds to $A \Rightarrow B$.
- $A = B$ corresponds to $A \iff B$.

We now can use truth tables to check the statements.

A	B	not A	A and B	A or B	$A \Rightarrow B$	$A \iff B$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Example 1.3. We can use the truth table to show that $A \Rightarrow B$ is logically equivalent to (not A) or B. Indeed, by considering the following truth table,

A	B	$A \Rightarrow B$	not A	(not A) or B
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

we realize that the column of $A \Rightarrow B$ and (not A) or B are the same.

Exercise 1.4. Use the truth table to prove De Morgan's laws:

not (A and B) = (not A) or (not B)
not (A or B) = (not A) and (not B)

One can compare these statements to

$$
\begin{aligned}
{}^c(A \cap B) &= {}^cA \cup {}^cB \\
{}^c(A \cup B) &= {}^cA \cap {}^cB
\end{aligned}
$$

Example 1.5. Negative the following statement:

If A then B.

Note that the negation is "not ($A \Rightarrow B$)", then it is equivalent to not ((not A) or B), which is equivalent to [not(not A)] and (not B), and that is just A and (not B).

Therefore, the negation is "A is true and B is false".

Example 1.6. Negate the following statement:

If I speak in front of the class, I am nervous.

That would be I speak in front of the class and I am not nervous.

We now introduce quantifiers.

- $\forall$ reads "for all " or "for any".
- $\exists$ reads "there is" or "there exists".
- The negation of "$\forall A$, B is true" is "$\exists A$ such that B is false".
- The negation of "$\exists A$ such that B is true" is "$\forall A$, B is false".

Example 1.7. Negate the following:

Every student had coffee or is late for class.

This statement is represented as

$\forall$ student (had coffee) or (is late for this)

and so the negation would be

$\exists$ student such that not (had coffee) and not (is late for class)

Writing this out, we get "there is a student that did not have coffee and is not late for class".

2 Lecture 2: Peano Axiom and Mathematical Induction

Definition 2.1 (Peano Axiom)**.** The natural numbers $\mathbb{N} = \{1, 2, 3, \cdots\}$ satisfy the Peano axioms:

1. $1 \in \mathbb{N}$.
2. If a number $n \in \mathbb{N}$, then its successor $n + 1 \in \mathbb{N}$.
3. 1 is not the successor of any natural number.
4. If two numbers $n, m \in \mathbb{N}$ are such that they have the same successor, i.e. $n+1 = m+1$, then they are the same, i.e. $n = m$.
5. Let $S \subseteq \mathbb{N}$. Assume that S satisfies the following two conditions:
   (i) $1 \in S$,
   (ii) and if $n \in S$ then $n + 1 \in S$,

   then $S = \mathbb{N}$.

Axiom number 5 forms the basis for mathematical induction.

Definition 2.2 (Mathematical Induction)**.** Assume we want to prove that a property $P(n)$ holds for all $n \in \mathbb{N}$. Then it suffices to verify two steps:

- Step 1 (Base Step): $P(1)$ holds.
- Step 2 (Inductive Step): If $P(n)$ is true for some $n \geq 1$, then $P(n + 1)$ is true, i.e. $P(n) \Rightarrow P(n + 1)$ $\forall n \geq 1$.

Indeed, if we let

$$S = \{n \in \mathbb{N} : P(n) \text{ holds}\},$$

then Step 1 implies $1 \in S$ and Step 2 implies if $n \in S$ then $n + 1 \in S$. By axiom 5, we deduce that $S = \mathbb{N}$.

Example 2.3. Prove that

$$1^2 + 2^2 + \cdots + n^2 = \tfrac{n(n+1)(2n+1)}{6} \quad \forall n \in \mathbb{N}.$$

We argue that mathematical induction. For $n \in \mathbb{N}$, let $P(n)$ denote the statement

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}.$$

Step 1 (Base Step): $P(1)$ is the statement $1^2 = \frac{1 \cdot 2 \cdot 3}{6}$, which is true, so $P(1)$ holds.

Step 2 (Inductive Step): Assume that $P(n)$ holds for some $n \in \mathbb{N}$, we want to show that $P(n + 1)$ holds. We know

$$1^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$$

then we have

$$\begin{aligned}
1^2 + \cdots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= (n+1)[\frac{n(2n+1}{6} + n + 1] \\
&= (n+1) \cdot \frac{2n^2 + 7n + 6}{6} \\
&= \frac{(n+1) \cdot [2n(n+2) + 3n + 6]}{6} \\
&= \frac{(n+1)(n+2)(2n+3}{6}
\end{aligned}$$

So $P(n+1)$ holds.

Collecting the two steps, we conclude $P(n)$ holds $\forall n \in \mathbb{N}$.

Example 2.4. Prove that $2^n > n^2$ for all $n \geq 5$.

We argue by mathematical induction. For $n \geq 5$, let $P(n)$ denote the statement $2^n > n^2$.

Step 1 (Base Step): $P(5)$ is the statement

$$32 = 2^5 > 5^2 = 25$$

which is true. So $P(5)$ holds.

Step 2 (Inductive Step): Assume $P(n)$ is true for some $n \geq 5$ and we want to prove $P(n+1)$. We know $2^n > n^2$, then

$$\begin{aligned}
2^{n+1} &> 2n^2 \\
&= (n+1)^2 + n^2 - 2n - 1 \\
&= (n+1)^2 + (n-2)^2 - 2
\end{aligned}$$

For $n \geq 5$, we have $(n-1)^2 - 2 \geq 4^2 - 2 = 14 \geq 0$, so we know $2^{n+1} > (n+1)^2$. Therefore, $P(n+1)$ holds.

Collecting the two steps, we conclude $P(n)$ holds $\forall n \geq 5$.

Remark 2.5. Each of the two steps are essential when arguing by induction. Note that $P(1)$ is true. However, our proof of the second step fails if $n = 1$: $(1-1)^2 - 2 = -2 < 0$. Also note that our proof of the second step is valid as soon as

$$(n-1)^2 - 2 \geq 0 \iff (n-1)^2 \geq 2 \iff n - 1 \geq 2 \iff n \geq 3.$$

However, $P(3)$ fails.

Example 2.6. Prove by mathematical induction that the number $4^n + 15n - 1$ is divisible by 9 for all $n \geq 1$.

We will argue by induction. For $n \geq 1$, let $P(n)$ denote the statement that "$4^n + 15n - 1$ is divisible by 9". We write this as $9 \mid (4^n + 15n - 1)$.

Step 1: $4^1 + 15 \cdot 1 - 1 = 18 = 9 \cdot 2$. This is divisible by 9, so $P(1)$ holds.

Step 2: Assume $P(n)$ is true for some $n \geq 1$, we want to show $P(n+1)$ holds.

$$\begin{aligned} 4^{n+1} + 15(n+1) - 1 &= 4 \cdot (4^n + 15n - 1) - 60n + 4 + 15n + 14 \\ &= 4 \cdot (4^n + 15n - 1) - 45n + 18 \\ &= 4 \cdot (4^n + 15n - 1) - 9 \cdot (5n - 2). \end{aligned}$$

By the inductive hypothesis, $9 \mid (4^n + 15n - 1)$ implies $9 \mid 4 \cdot (4^n + 15n - 1)$. Also we know $9 \mid 9 \cdot (5n - 2)$ since $5n - 2 \in \mathbb{N}$. Therefore, we know $9 \mid [4 \cdot (4^n + 15n - 1) - 9 \cdot (5n - 2)]$. Hence, $9 \mid [4 \cdot (4^n + 15n - 1) - 9 \cdot (5n - 2)]$, so $P(n+1)$ holds.

Collecting the two steps, we conclude $P(n)$ holds $\forall n \in \mathbb{N}$.

Example 2.7. Compute the following sum and then use mathematical induction to prove your answer: for $n \geq 1$,

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)}.$$

Note that $\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}[\frac{1}{2n-1} - \frac{1}{2n+1}]$ for all $n \geq 1$. So

$$\begin{aligned} \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} &= \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= \frac{1}{2} \cdot \frac{2n}{2n+1} \\ &= \frac{n}{2n+1}. \end{aligned}$$

For $n \geq 1$, let $P(n)$ denote the statement

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.$$

Step 1: $P(1)$ becomes $\frac{1}{1 \cdot 3} = \frac{1}{3}$, which is true. So $P(1)$ holds.

Step 2: Assume $P(n)$ holds for some $n \geq 1$. We want to show $P(n+1)$. We know

$$\frac{1}{1 \cdot 3} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1},$$

and so

$$\begin{aligned} \frac{1}{1 \cdot 3} + \cdots + \frac{1}{(2n+1)(2n+3)} &= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} \\ &= \frac{2n^2 + 3n + 1}{(2n+1)(2n+3)} \\ &= \frac{(n+1)(2n+1)}{(2n+1)(2n+3)} \\ &= \frac{n+1}{2n+3}. \end{aligned}$$

So $P(n+1)$ holds.

Collecting the two steps, we conclude $P(n)$ holds $\forall n \geq 1$.

3 Lecture 3: Equivalence Relation

We now extend $\mathbb{N}$ and construct the set of integers $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$.

Definition 3.1 (Equivalence Relation)**.** An equivalence relation $\sim$ on a non-empty set A satisfies the following three properties:

1. Reflexivity: $a \sim a \; \forall a \in A$.

2. Symmetry: If $a, b \in A$ are such that $a \sim b$, then $b \sim a$.

3. Transitivity: If $a, b, c \in A$ are such that $a \sim b$ and $b \sim c$, then $a \sim c$.

Example 3.2. The equal relation $=$ is an equivalence relation on $\mathbb{Z}$.

Example 3.3. Let $q \in \mathbb{N}$ and $q > 1$. For $a, b \in \mathbb{Z}$ we write $a \sim b$ if $q \mid (a - b)$. This is an equivalence relation on $\mathbb{Z}$. Indeed, it suffices to check the three properties:

- Reflexivity: If $a \in \mathbb{Z}$, then $a - a = 0$, which is divisible by q. So $q \mid (a - a)$, by definition we know $a \sim a$.

- Symmetry: Let $a, b \in \mathbb{Z}$ such that $a \sim b$, then by definition we know $q \mid (a - b)$. Therefore, there exists some $k \in \mathbb{Z}$ such that $a - b = kq$, so $b - a = (-k) \cdot q$. Note that $-k \in \mathbb{Z}$, so $q \mid (b - a)$, and by definition we know $b \sim a$.

- Transivitity: Let $a, b, c \in \mathbb{Z}$ such that $a \sim b$ and $b \sim c$. Now $a \sim b$ indicates $q \mid (a - b)$, so there exists $n \in \mathbb{Z}$ such that $a - b = qn$. Similarly there exists $m \in \mathbb{Z}$ such that $b - c = qm$. Therefore, $a - c = q(n + m)$, where $n + m \in \mathbb{Z}$. Therefore, $q \mid (a - c)$, so by definition $a \sim c$.

Definition 3.4 (Equivalence Class)**.** Let $\sim$ denote an equivalence relation on a non-empty set A. The equivalence class of an element $a \in A$ is given by

$$C(a) = \{b \in A : a \sim b\}.$$

Proposition 3.5 (Properties of Equivalence Classes)**.** Let $\sim$ denote an equivalence relation on a non-empty set A. Then

1. $a \in C(a)$ for all $a \in A$.

2. If $a, b \in A$ are such that $a \sim b$, then $C(a) = C(b)$.

3. If $a, b \in A$ are such that $a \not\sim b$, then $C(a) \cap C(b) = \varnothing$.

4. $A = \bigcup_{a \in A} C(a)$.

Proof. 1. By reflexivity, $a \sim a$ for all $a \in A$, then $a \in C(a)$ for all $a \in A$.

2. Assume $a, b \in A$ with $a \sim b$. Let us show $C(a) \subseteq C(b)$. Let $c \in C(a)$ be arbitrary, then $a \sim c$. Because $a \sim b$, by symmetry we have $b \sim a$, then by transitivity we know $b \sim c$, and so $c \in C(b)$. This proves that $C(a) \subseteq C(b)$. A similar argument shows that $C(b) \subseteq C(a)$, and so $C(a) = C(b)$.

3. We argue by contradiction. Assume that $a, b \in A$ are such that $a \not\sim b$, but $C(a) \cap C(b) \neq \varnothing$. Let $c \in C(a) \cap C(b)$, then $c \in C(a)$ and $c \in C(b)$. The first property implies $a \in c$, and the second property implies $b \sim c$, so $c \sim b$, and therefore by transitivity we have $a \sim b$. This contradicts the hypothesis $a \not\sim b$. Therefore, if $a \not\sim b$, then $C(a) \cap C(b) = \varnothing$.

4. Clearly, as $C(a) \subseteq A$ for all $a \in A$, we get $\bigcup\limits_{a \in A} C(a) \subseteq A$. Then conversely, $A = \bigcup\limits_{a \in A} \{a\} \subseteq \bigcup\limits_{a \in A} C(a)$, and therefore $A = \bigcup\limits_{a \in A} C(a)$.

□

Example 3.6. Take $q = 2$ in our previous example: for $a, b \in \mathbb{Z}$, we write $a \sim b$ if $2 \mid (a - b)$. The equivalence classes are

$$\begin{aligned} C(0) &= \{a \in \mathbb{Z} : 2 \mid (a - 0)\} = \{2n : n \in \mathbb{Z}\} \\ C(1) &= \{a \in \mathbb{Z} : 2 \mid (a - 1)\} = \{2n + 1 : n \in \mathbb{Z}\} \end{aligned}$$

and $\mathbb{Z} = C(0) \cup C(1)$.

Example 3.7. Let $F = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : b \neq 0\}$. If $(a, b), (c, d) \in F$ we write $(a, b) \sim (c, d)$ if $ad = bc$. Then for example, we have $(1, 2) \sim (2, 4) \sim (3, 6) \sim (-4, -8)$.

Lemma 3.8. $\sim$ is an equivalence relation on F.

Proof. We have to check the three properties.

Reflexivity: Fix $(a, b) \in F$. As $ab = ba$, we have $(a, b) \sim (b, a)$.

Symmetry: Let $(a, b), (c, d) \in F$ such that $(a, b) \sim (c, d)$, then by definition we know $ad = bc$, and so $cb = da$, and by definition $(c, d) \sim (a, b)$.

Transitivity: Let $(a, b), (c, d), (e, f) \in F$ such that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Now $(a, b) \sim (c, d)$ implies $ad = bc$, then $adf = bcf$. Similarly, $cfb = deb$. Therefore, $adf = deb$. Now $d(af - be) = 0$, and because $d \neq 0$ by definition, we know $af = be$, and by definition we have $(a, b) \sim (e, f)$ as desired. □

For $(a, b) \in F$, we denote its equivalence class by $\frac{a}{b}$. We define addition and multiplication of equivalence classes as follows:

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} &= \frac{ad + bc}{bd} \\ \frac{a}{b} \cdot \frac{c}{d} &= \frac{ac}{bd} \end{aligned}$$

We have to check that these operations are well-defined. Specifically, if $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, then we should have

$$\begin{cases} (ad + bc, bd) \sim (a'd' + b'c', b'd') \\ (ac, bd) \sim (a'c', b'd') \end{cases}$$

We now check the first property and left the second property as an exercise to the readers. We want to show $(ad + bc)b'd' = bd(a'd' + b'c')$. We know that $(a, b) \sim (a'b')$, so $ab' = ba'$, and therefore $ab'dd' = badd'$. Similarly we know $(c,d) \sim c'd'$, so $cd' = dc'$, and therefore $cd'bb' = dc'bb'$. Now we get

$$ab'dd' + cd'bb' = ba'dd' + dc'bb',$$

and so

$$(ad + bc)b'd' = bd(a'd' + b'c').$$

This proves addition is well-defined.

Now the set of rational numbers is exactly the set of equivalence classes on F, i.e.

$$\mathbb{Q} = \{\frac{a}{b} : (a,b) \in F\}.$$

4 Lecture 4: Field

Definition 4.1 (Field)**.** A field is a set F with at least two elements equipped with two operations: addition (denoted $+$) and multiplication (denoted $\cdot$) that satisfies the following:

1. $(A1)$ Closure: if $a, b \in F$, then $a + b \in F$.
2. $(A2)$ Commutativity: if $a, b \in F$, then $a + b = b + a$.
3. $(A3)$ Associativity: if $a, b, c \in F$, then $(a + b) + c = a + (b + c)$.
4. $(A4)$ Identity: $\exists 0 \in F$ such that $a + 0 = 0 + a = a$ $\forall a \in F$.
5. $(A5)$ Inverse: $\forall a \in F$, $\exists(-a) \in F$ such that $a + (-a) = -a + a = 0$.
6. $(M1)$ Closure: if $a, b \in F$, then $a \cdot b \in F$.
7. $(M2)$ Commutativity: if $a, b \in F$, then $a \cdot b = b \cdot a$.
8. $(M3)$ Associativity: if $a, b, c \in F$, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
9. $(M4)$ Identity: $\exists 1 \in F$ such that $a \cdot 1 = 1 \cdot a = a$ $\forall a \in F$.
10. $(M5)$ Inverse: $\forall a \in F \backslash \{0\}$, $\exists a^{-1} \in F$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.
11. (D) Distributivity: if $a, b, c \in F$, then $(a + b) \cdot c = a \cdot c + b \cdot c$.

Example 4.2. $(\mathbb{N}, +, \cdot)$ is not a field because (A_4) fails.

Example 4.3. $(\mathbb{Z}, +, \cdot)$ is not a field because (M_5) fails.

Example 4.4. $(\mathbb{Q}, +, \cdot)$ is a field.

Recall $\mathbb{Q} = \{\frac{a}{b} : (a,b) \in \mathbb{Z} \times (\mathbb{Z}\backslash\{0\}\}$ where $\frac{a}{b}$ denotes the equivalence class of $(a,b) \in \mathbb{Z} \times (\mathbb{Z}\backslash\{0\})$ with respect to the equivalence relation $\sim$, where $(a,b) \sim (c,d)$ if and only if $a \cdot d = b \cdot c$. We defined two operations

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$
$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Then the additive identity $\frac{0}{1}$ is the equivalence class of $(0,1)$, and the multiplicative identity $\frac{1}{1}$ is the equivalence class of $(1,1)$.

The additive inverse of $\frac{a}{b} \in \mathbb{Q}$ is given by $\frac{-a}{b}$, and for $\frac{a}{b} \in \mathbb{Q}\backslash\{\frac{0}{1}\}$, the multiplicative inverse is given by $\frac{b}{a}$.

Proposition 4.5. Let $(F, +, \cdot)$ be a field. Then

1. The additive and multiplicative identities are unique.
2. The additive and multiplicative inverses are unique.
3. If $a, b, c \in F$ such that $a + b = a + c$, then $b = c$. In particular, if $a + b = a$, then $b = 0$.
4. If $a, b, c \in F$ such that $a \neq 0$ and $a \cdot b = a \cdot c$, then $b = c$. In particular, if $a \neq 0$ and $a \cdot b = a$, then $b = 1$.
5. $a \cdot 0 = 0 \cdot a = 0 \; \forall a \in F$.
6. If $a, b \in F$, then $(-a) \cdot b = a \cdot (-b) = -(a \cdot b)$.
7. If $a, b \in F$, then $(-a) \cdot (-b) = a \cdot b$.
8. If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

Proof.

1. We will show the additive identity is unique. Assume $\exists 0, 0' \in F$ such that $a + 0 = 0 + a = a$ and $a + 0' = 0' + a = a$ for all $a \in F$. Take $a = 0'$ in the first equation and $a = 0$ in the second equation yields $0' + 0 = 0'$ and $0' + 0 = 0$, so $0 = 0'$.
2. We will show that the additive inverse is unique. Let $a \in F$. Assume there exists $-a, a' \in F$ such that $-a + a = a + (-a) = 0$ and $a' + a = a + a' == 0$. Because $a' + a = 0$, then $(a' + a) + (-a) = 0 + (-a)$, so $a' + (a + (-a)) = -a$, but that means $a' + 0 = -a$, so $a' = -a$.
3. Assume $a + b = a + c$. Then $-a + (a + b) = -a + (a + c)$. Therefore, $(-a + a) + b = (-a + a) + c$, so $0 + b = 0 + c$, which means $b = c$. So if $a + b = a = a + 0$, then $b = 0$.
4. We have a proof similar as above.
5. $a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$, so $a \cdot 0 = 0$. Similarly, $0 \cdot a = (0 + 0) \cdot a = 0 \cdot a + 0 \cdot a$, we have $0 \cdot a = 0$.
6. $(-a) \cdot b + a \cdot b = (-a + a) \cdot b = 0 \cdot b = 0$, and so $(-a) \cdot b = -(a \cdot b)$. Similarly, we have $a \cdot (-b) = -(a \cdot b)$.
7. $(-a) \cdot (-b) + [-(a \cdot b)] = (-a) \cdot (-b) + (-a) \cdot b = (-a)(-b + b) = (-a) \cdot 0 = 0$. Therefore, $(-a) \cdot (-b) = a \cdot b$.
8. Assume $a \cdot b = 0$. Assume $a \neq 0$, then $\exists a^{-1} \in F$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$. Now because $a \cdot b = 0$, then $a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0$, and so $(a^{-1} \cdot a) \cdot b = 0$, then $1 \cdot b = 0$, so $b = 0$.

$\square$

Definition 4.6 (Order Relation)**.** An order relation $<$ on a non-empty set A satisfies the following properties:

- Trichotomy: If $a, b \in A$, then one and only one of the following statements holds: $a < b$, or $a = b$, or $b < a$.
- Transitivity: If $a, b, c \in A$ such that $a < b$ and $b < c$, then $a < c$.

Example 4.7. For $a, b \in \mathbb{Z}$, we write $a < b$ if $b - a \in \mathbb{N}$. This is an order relation.

We write $a > b$ if $b < a$, we write $a \leq b$ if $[a < b$ or $a = b]$, and we write $a \geq b$ if $b \leq a$.

Definition 4.8 (Ordered Field)**.** Let $(F, +, \cdot)$ be a field. We say $(F, +, \cdot)$ is an ordered field if it is equipped with an order relation $<$ that satisfies the following:

- $(O1)$: If $a, b, c \in F$ such that $a < b$, then $a + c < b + c$.
- $(O2)$: If $a, b, c \in F$ such that $a < b$ and $0 < c$, then $a \cdot c < b \cdot c$.

5 Lecture 5: Ordered Field

Proposition 5.1. Let $(F, +, \cdot, <)$ be an ordered field. Then,

1. $a > 0 \iff -a < 0$.
2. if $a, b, c \in F$ are such that $a < b$ and $c < 0$, then $a \cdot c > b \cdot c$.
3. if $a \in F \backslash \{0\}$, then $a^2 = a \cdot a > 0$. In particular, $1 > 0$.
4. if $a, b \in F$ are such that $0 < a < b$, then $0 < b^{-1} < a^{-1}$.

Proof.

1. $(\Rightarrow)$: assume $a > 0$, then $a + (-a) > 0 + (-a)$, so $0 > -a$.

   $(\Leftarrow)$: assume $-a < 0$, then $-a + a < 0 + a$, then $0 < a$.

2. Assume $a < b$ and $c < 0$, then $-c > 0$, so $a \cdot (-c) < b \cdot (-c)$, which means $-a \cdot c < -b \cdot c$. Therefore, $-ac + (ac + bc) < -bc + (ac + bc)$. We then see $(-ac + ac) + bc < -bc + (bc + ac)$, so $0 + bc < (-bc + bc) + ac$, and so $bc < 0 + ac$, which means $bc < ac$.

3. By trichotomy, exactly one of the following holds:
   - if $a > 0$, then $a \cdot a > 0 \cdot a$, so $a^2 > 0$.
   - if $a < 0$, then $a \cdot a > 0 \cdot a$, so $a^2 > 0$.

4. First we show that if $a > 0$ then $a^{-1} > 0$. Let us argue by contradiction. Assume $\exists a \in F$ such that $a > 0$ but $a^{-1} \leq 0$. Note $a^{-1} \neq 0$ since a^{-1} has a multiplicative inverse a. Since $a > 0$ and $a^{-1} < 0$, then $a \cdot a^{-1} < 0$, so $1 < 0$. This contradicts the previous part. So if $a > 0$, then $a^{-1} > 0$. Because $0 < a < b$, then $0 \cdot (a^{-1} \cdot b^{-1}) < a \cdot (a^{-1} \cdot b^{-1}) < b \cdot (a^{-1} \cdot b^{-1})$, and so $0 < (a \cdot a^{-1}) \cdot b^{-1} < b \cdot (b^{-1} \cdot a^{-1})$, therefore $0 < 1 \cdot b^{-1} < (b \cdot b^{-1}) \cdot a^{-1}$. Then we have $0 < b^{-1} < 1 \cdot a^{-1}$, therefore $0 < b^{-1} < a^{-1}$.

$\square$

Theorem 5.2. Let $(F, +, \cdot)$ be a field. The following are equivalent:

1. F is an ordered field.

2. There exists $P \subseteq F$that satisfies the following properties:

   - $(O1')$: For every $a \in F$, one and only one of the following statements holds: $a \in P$, or $a = 0$, or $-a \in P$.
   - $(O2')$: If $a, b \in P$, then $a + b \in P$, and $a \cdot b \in P$.

Proof. Let us show that $(1) \Rightarrow (2)$. Define $P = \{a \in F : a > 0\}$. Let us check $(O1')$. Fix $a \in F$. By trichotomy for the order relation on F, we get that exactly one of the following statements is true: $a > 0$, which implies $a \in P$, or $a = 0$, or $a < 0$, which implies $-a > 0$, so $-a \in P$. We can now check $(O2')$. Fix $a, b \in P$. Because $a \in P$, then $a > 0$, and similarly $b > 0$. Therefore, $a + b > 0 + b = b > 0$, so $a + b \in P$. Also, we know $a \cdot b > 0 \cdot b = 0$, so $a \cdot b \in P$.

We now show that $(2) \Rightarrow (1)$. For $a, b \in F$, we write $a < b$ if $b - a \in P$. Let us check that this is an order relation.

Trichotomy: fix $a, b \in F$. By $(O1')$, exactly one of the following hold: $b - a \in P$, which means $a < b$, or $b - a = 0$, which means $a = b$, or $-(b - a) \in P$, which means $a - b \in P$ and so $b < a$.

Transitivity: assume $a, b, c \in F$ such that $a < b$ and $b < c$. Therefore, $b - a \in P$ and $c - b \in P$, so $(b - a) + (c - b) = c - a \in P$, and so $a < c$.

We now check that with this order relation, F is an ordered field. We have to check $(O1)$ and $(O2)$.

$(O1)$: fix $a, b, c \in F$ such that $a < b$, then $b - a \in P$, so $(b + c) - (a + c) \in P$, which means $a + c < b + c$.

$(O2)$: fix $a, b, c \in F$ such that $a < b$ and $0 < c$. Because $a < b$, then $b - a \in P$, and because $0 < c$, then $c - 0 = c \in P$. Therefore, $(b - a) \cdot c \in P$, and so $b \cdot c - a \cdot c \in P$, therefore $a \cdot c < b \cdot c$. ∎

We extend the order relation $<$ from $\mathbb{Z}$ to the field $(\mathbb{Q}, +, \cdot)$ br writing $\frac{a}{b} > 0$ ikf $a \cdot b > 0$.

Let us show that this is well-defined. Specifically, we need to show that if $\frac{a}{b} = \frac{c}{d}$, i.e. $(a, b) \sim (c, d)$, and $a \cdot b > 0$, then $c \cdot d > 0$. Now if $(a, b) \sim (c, d)$, then $a \cdot d = b \cdot c$, so $0 < (ad)^2 = (a \cdot b) \cdot (c \cdot d)$.[2] Therefore, $0 < (ab) \cdot (cd)$ and because $0 < ab$, so $cd > 0$, and therefore $\frac{c}{d} > 0$.

Let $P = \{\frac{a}{b} \in \mathbb{Q} : \frac{a}{b} > 0\}$. By the theorem, to prove that $\mathbb{Q}$ is an ordered field, it suffices to show that P satisfies $(O1')$ and $(O2')$, which is left as an exercise to the readers.

6 Lecture 6: Bounds

Definition 6.1. Let $(F, +, \cdot, <)$ be an ordered field. Let $\varnothing \neq A \subseteq F$.

[2] Note that $a \cdot d \neq 0$ since $d \neq 0$ and $a \cdot b > 0$, and so $a \neq 0$.

- We say that A is bounded above if $\exists M \in F$ such that $a \leq M$ $\forall a \in A$. Then M is called an upper bound for A. If moreover, $M \in A$, then we say that M is the maximum of A.
- We say that A is bounded below if $\exists m \in F$ such that $m \leq a$ $\forall a \in A$. Then m is called a lower bound for A. If moreover, $m \in A$, then we say that m is the minimum of A.
- We say that A is bounded if A is bounded both above and below.

Example 6.2.

- $A = \{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\}$ is a bounded set. 3 is an upper bound for A, $\frac{3}{2}$ is the maximum of A, 0 is a lower bound for A, and 0 is the minimum of A.
- $A = \{x \in \mathbb{Q} : 0 < x^4 \leq 16\}$ is a bounded set. 2 is the maximum of A, and -2 is the minimum of A.
- $A = \{x \in \mathbb{Q} : x^2 < 2\}$ is a bounded set. 2 is an upper bound for A, and -2 is a lower bound for A. But A does not have a maximum. Indeed, let $x \in A$. We will construct $y \in A$ such that $y > x$.

  Define $y = x + \frac{2-x^2}{2+x}$. Because $x \in A$, then $x \in \mathbb{Q}$, so $2 - x^2, 2 + x \in \mathbb{Q}$. Moreover, because $x \in A$, then $2 + x > 0$, and so $\frac{1}{2+x} \in \mathbb{Q}$. Therefore, $\frac{2-x^2}{2+x} \in \mathbb{Q}$. Hence, we know $y \in \mathbb{Q}$.

  Also note that $2 - x^2 > 0$ since $x \in A$, and $2 + x > 0$ indicates $\frac{1}{2+x} > 0$, so $\frac{2-x^2}{2+x} > 0$. Therefore, $y = x + \frac{2-x^2}{2+x} > x$.

  Let us compute y^2. Note that

$$
\begin{aligned}
y^2 &= \frac{2x + x^2 + 2 - x^2{}^2}{2+x} \\
&= \frac{4(x+1)^2}{(2+x)^2} \\
&= \frac{4x^2 + 8x + 4}{x^2 + 4x + 4} \\
&= \frac{2(x^2 + 4x + 4) + 2x^2 - 4}{x^2 + 4x + 4} \\
&= 2 + \frac{2 \cdot (x^- 2)}{(x+2)^2} \\
&< 2.
\end{aligned}
$$

  Collecting the properties above, we constructed $y \in A$ and $y > x$ as desired.

Exercise 6.3. Show that the maximum and minimum of a set are unique, if they exist.

Definition 6.4. Let $(F, +, \cdot, <)$ be an ordered field. Let $\varnothing \neq A \subseteq F$ and assume A is bounded above. We say that L is the least upper bound of A if it satisfies:

1. L is an upper bound of A.

2. If M is an upper bound of A, then $L \leq M$.

We write $L = \sup(A)$ and we say L is the supremum of A.

Lemma 6.5. The least upper bound of a set is unique, if it exists.

Proof. Say that a set A, satisfies $\varnothing A \subseteq F$ and is bounded above, admits two least upper bounds L and M. Because L is a least upper bound, then L is an upper bound for A. But because M is a least upper bound for A, we have $M \leq L$. Similarly we conclude that $L \leq M$, and so $L = M$. $\square$

Definition 6.6. Let $(F, +, \cdot, <)$ be an ordered field. Let $\varnothing \neq A \subseteq F$ and assume A is bounded below. We say that l is the greatest lower bound of A if it satisfies:

1. l is a lower bound of A.

2. If m is a lower bound of A then $m \leq l$.

We write $l = \inf(A)$ and we say l is the infimum of A.

Exercise 6.7. Show that the greatest lower bound of a set is unique, if it exists.

Definition 6.8. Let $(F, +, \cdot, <)$ be an ordered field. Let $\varnothing \neq S \subseteq F$.

We say that S has the least upper bound property if its satisfies the following: for any non-empty subset A of S that is bounded above, there exists a least upper bound of A and $\sup(A) \in S$.

We say that S has the greatest lower bound property if it satisfies the following: $\forall \varnothing \neq A \subseteq S$ with A bounded below, $\exists \inf(A) \in S$.

Example 6.9. $(\mathbb{Q}, +, \cdot, <)$ is an ordered field. Note that

1. Consider $\varnothing \neq\subseteq \mathbb{Q}$, $\mathbb{N}$ has the least upper bound property. Indeed, if $\varnothing \neq A \subseteq N$, A bounded above, then the largest element in A is the least upper bound of A and $\sup(A) \in \mathbb{N}$. $\mathbb{N}$ also has the greatest lower bound property.

2. Consider $\varnothing \neq \mathbb{Q} \subseteq \mathbb{Q}$, but $\mathbb{Q}$ does not have the least upper bound property. Indeed, $\varnothing \neq A = \{x \in \mathbb{Q} : x \geq 0, x^2 < 2\} \subseteq \mathbb{Q}$. Note that A is bounded above by 2. However, $\sup(A) = \sqrt{2} \notin \mathbb{Q}$.

Proposition 6.10. Let $(F, +, \cdot, <)$ be an ordered field. Then F has the least upper bound property if and only if it has the greatest lower bound property.

Proof. We will only prove the $(\Rightarrow)$ direction: the opposite direction has a similar proof.

Assume F has the least upper bound property. Let $\varnothing \neq A \subseteq F$ bounded below. We want to show that $\exists \inf(A) \in F$. Because A is bounded below, then $\exists m \in F$ such that $m \leq a$ $\forall a \in A$. Let $B = \{b \in F : b \text{ is a lower bound for } A\}$. Note $B \neq \varnothing$ because $m \in B$, and we know $B \subseteq F$, and B is bounded above (in fact, every element in A is an upper bound for B), and F has the least upper bound property. Therefore, $\exists \sup(B) \in F$.

Claim 6.11. $\sup(B)$ is a lower bound for A.

Subproof. Indeed, let $a \in A$. We know $a \geq b \ \forall b \in B$, and $\sup(B)$ is the least upper bound for B, so $a \geq \sup(B)$. As $a \in A$ was arbitrary, we conclude that $\sup(B) \leq a \ \forall a \in A$, and so $\sup(B)$ is a lower bound for A. ■

Claim 6.12. If l is a lower bound for A, then $l \leq \sup(B)$.

Subproof. Because l is a lower bound for A, then $l \in B$. Also, because $\sup(B)$ is an upper bound for B, we know $l \leq \sup(B)$. ■

Using the two claims above, we find that $\inf(A) = \sup(B)$. □

7 Lecture 7: Archimedean Property

We present an alternative proof of Proposition 6.10.

Remark 7.1 (Alternative Proof)**.** Let $\varnothing \neq A \subseteq F$ be such that A is bounded below. Let $B = \{-a : a \in A\}$. Note $B \subseteq F$ by (A5), and $B \neq \varnothing$ because $A \neq \varnothing$, and B Is bounded above: indeed, if m is a lower bound for A, then $-m$ is an upper bound for B.[3] Also note that F has the least upper bound property. Collecting these properties above, we know $\exists \sup(B) \in F$. The reader can easily show that $-\sup(B) = \inf(A) \in F$.

Theorem 7.2. There exists an ordered field with the least upper bound property. We denote it $\mathbb{R}$ and we call it the set of real numbers. $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield. (We will prove this statement in Theorem 8.4.) Moreover, we have the following uniqueness property: if $(F, +, \cdot, <)$ is an ordered field with the least upper bound property, then F is order isomorphic with $\mathbb{R}$, that is, there exist a bijection $\varphi : \mathbb{R} \to F$ such that

(i) $\varphi(x + y) = \varphi(x) + \varphi(y)$.

(ii) $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$.

(iii) if $x < y$, then $\varphi(x) < \varphi(y)$.

Theorem 7.3. $\mathbb{R}$ has the Archimedean property, that is, $\forall x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ such that $x < n$.

Proof. We argue by contradiction. Assume $\exists x_0 \in \mathbb{R}$ such that $x_0 \geq n \ \forall n \in \mathbb{N}$. Then we know $\varnothing \neq \mathbb{N} \subseteq \mathbb{R}$, $\mathbb{N}$ is bounded above by x_0, and $\mathbb{R}$ has the least upper bound property. Therefore, $\exists L = \sup(\mathbb{N}) \in \mathbb{R}$.

Now we know $L = \sup(\mathbb{N})$ and $L - 1 < L$, so $L - 1$ is not an upper bound for $\mathbb{N}$. That means $\exists n_0 \in \mathbb{N}$ such that $n_0 > L - 1$, so $\sup(\mathbb{N}) = L < n_) + 1 \in \mathbb{N}$. We therefore have a contradiction. □

Remark 7.4. $\mathbb{Q}$ has the Archimedean property. If $r \in \mathbb{Q}$ is such that $r \leq 0$, then choose $n = 1$. If $r \in \mathbb{Q}$ is such that $r > 0$, then write $r = \frac{p}{q}$ for $p, q \in \mathbb{N}$, and we can choose $n = p+1$ since $\frac{p}{q} < p + 1$.

Corollary 7.5. If $a, b \in \mathbb{R}$ are such that $a > 0, b > 0$, then there exists $n \in \mathbb{N}$ such that $n \cdot a > b$.

[3] Note that $m \leq a \ \forall a \in A$ implies $-m \geq -a \ \forall a \in A$.

Proof. Apply the Archimedean property to $x = \frac{b}{a}$. $\square$

Corollary 7.6. If $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.

Proof. Apply the Archimedean property to $x = \frac{1}{\varepsilon}$. $\square$

Lemma 7.7. For any $a \in \mathbb{R}$ there exists $N \in \mathbb{Z}$ such that $N \leq a < N + 1$.

Proof. If $a = 0$, then we can just take $N = 0$.

If $a > 0$. Consider $A = \{n \in \mathbb{Z} : n \leq a\} \subseteq \mathbb{R}$. Obviously $A \neq \varnothing$, as $0 \in A$. We also know A is bounded above by a, and $\mathbb{R}$ has the least upper bound property. Therefore, there exists $L = \sup(A) \in \mathbb{R}$. Now consider $L - 1 < L = \sup(A)$, then $L - 1$ is not an upper bound for A, so there exists $N \in A$ such that $L - 1 < N$, and so $L < N + 1$. But $L = \sup(A)$, so $N + 1 \notin A$. Therefore, $N \in A$, so $N \leq a$, and as $N + 1 \notin A$, then $N + 1 > a$. Therefore, $N \leq a < N + 1$.

If $a < 0$, then $-a > 0$. Then by the case $a > 0$, $\exists n \in \mathbb{Z}$ such that $n \leq -a < n + 1$, so $-n - 1 < a \leq -n$. If $a = -n$, let $N = -n$ and so $N \leq a < N + 1$. If $a < -n$, let $N = -n - 1$, and so $N \leq a < N + 1$. Either way, we conclude the proof. $\square$

Definition 7.8 (Dense)**.** We say that a subset A of $\mathbb{R}$ is dense in $\mathbb{R}$ if for every $x, y \in \mathbb{R}$ such that $x < y$, there exists $a \in A$ such that $x < a < y$.

Lemma 7.9. $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$ such that $x < y$. Since $y - x > 0$, by Corollary 7.6, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < y - x$, so $\frac{1}{n} + x < y$.

Consider $nx \in \mathbb{R}$. By Lemma 7.7, $\exists m \in \mathbb{Z}$ such that $m \leq nx < m + 1$, so $\frac{m}{n} \leq x < \frac{m+1}{n}$. Therefore,

$$x < \frac{m+1}{n} = \frac{m}{n} + \frac{1}{n} \leq x + \frac{1}{n} < y.$$

$\square$

8 Lecture 8: Construction of Real Numbers

Remark 8.1. For any two rational numbers $r_1, r_2 \in \mathbb{Q}$ such that $r_1 < r_2$, there exists $s \in \mathbb{Q}$ such that $r_1 < s < r_2$. Indeed, if $r_1 < 0 < r_2$, then we may take $s = 0 \in \mathbb{Q}$. Assume $0 < r_1 < r_2$, write $r_1 = \frac{a}{b}$ and $r_2 = \frac{c}{d}$ with $a, b, c, d \in \mathbb{N}$. Take $s = \frac{ad+bc}{2bd} \in \mathbb{Q}$. Note $r_1 < s < r_2$:

$$r_1 < s \iff \frac{a}{b} < \frac{ad + bc}{2bd} \iff 2ad < ad + bc \iff ad < bc \iff \frac{a}{b} < \frac{c}{d} \iff r_1 < r_2.$$

We leave the construction of s in the remaining cases as an exercise to the readers.

Lemma 8.2. $\mathbb{R} \backslash \mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$ such that $x < y$, then $x + \sqrt{2} < y + \sqrt{2}$. Because we know $\mathbb{Q}$ is dense in $\mathbb{R}$, we know $\exists q \in \mathbb{Q}$ such that $x + \sqrt{2} < q < y + \sqrt{2}$, so $x < q - \sqrt{2} < y$. It now suffices to prove the following claim.

Claim 8.3. $q - \sqrt{2} \in \mathbb{R}\backslash\mathbb{Q}$.

Subproof. Otherwise, $\exists r \in \mathbb{Q}$ such that $q - \sqrt{2} = r$, so $\sqrt{2} = q - r \in \mathbb{Q}$, contradiction. ■

□

Theorem 8.4. There exists an ordered field with the least upper bound property. We denote it $\mathbb{R}$ and call it the set of real numbers. $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

Remark 8.5. The rest of the statement in Theorem 7.2 is left as an exercise for the readers.

Proof. We will construct an ordered field with the least upper bound property using Dedekind cuts.

The element of the field are certain subsets of $\mathbb{Q}$ called cuts.

Definition 8.6 (Cut)**.** A cut is a set $\alpha \subseteq \mathbb{Q}$ that satisfies

(i) $\varnothing \neq \alpha \neq \mathbb{Q}$,

(ii) if $q \in \alpha$ and $p \in \mathbb{Q}$ such that $p < q$, then $p \in \alpha$.

(iii) for every $q \in \alpha$, there exists $r \in \alpha$ such that $r > q$, i.e. α has no maximum.

Intuitively, we think of a cut as $\mathbb{Q} \cap (-\infty, a)$.[4] Note that if $\mathbb{Q} \ni q \notin \alpha$, then $q > p$ for all $p \in \alpha$. Indeed, otherwise, if $\exists p_0 \in \alpha$ such that $q \leq p_0$, then by (ii) we would have $q \in \alpha$, contradiction.

We define

$$F = \{\alpha : \alpha \text{ is a cut}\}$$

and we will show that F is an ordered field with the least upper bound property.

Subproof on Order. We first show that there is an order relation on F. For $\alpha, \beta \in F$, we write $\alpha < \beta$ if α is a proper subset of β, i.e. $\alpha \subsetneq \beta$.

- Transitivity: if $\alpha, \beta, \gamma \in F$ are such that $\alpha < \beta$ and $\beta < \gamma$, then $\alpha \subsetneq \beta \subsetneq \gamma$, and so $\alpha \subsetneq \gamma$, so $\alpha < \gamma$.

- Trichotomy: first note that at most one of the following holds: $\alpha < \beta$, or $\alpha = \beta$, or $\beta < \alpha$.

  To prove trichotomy, it thus suffices to show that at least one of the following holds: $\alpha < \beta$, $\alpha = \beta$, or $\alpha < \beta$. We show this by contradiction. Assume that $\alpha < \beta$, $\alpha = \beta$, $\beta < \alpha$ all fail. Then we know that α is not a proper subset of β, $\alpha \neq \beta$, and β is not a proper subset of α, which means $\exists p \in \alpha\backslash\beta$ and $\exists q \in \beta\backslash\alpha$. Therefore, $p > r$ for all $r \in \beta$ and $q > s$ for all $s \in \alpha$. Taking $r = q$ and $s = p$, we get $p > q > p$, which is a contradiction.

Therefore, $<$ defines an order relation on F. ■

[4] Of course, at this point we have not yet constructed $\mathbb{R}$.

We now show that F has the least upper bound property. Let $\varnothing \neq A \subseteq F$ be bounded above by $\beta \in F$. Define $\gamma = \bigcup_{\alpha \in A} \alpha$.

Claim 8.7. $\gamma \in F$.

Subproof of Claim.

- $\gamma \neq \varnothing$ because $A \neq \varnothing$ and $\varnothing \neq \alpha \in A$.
- β being an upper bound for A indicates $\beta \geq \alpha$ for all $\alpha \in A$, and so $\beta \supseteq \alpha$ for all $\alpha \in A$, and therefore $\beta \supseteq \bigcup_{\alpha \in A} \alpha = \gamma$, but since $\beta \neq \mathbb{Q}$, we know that $\gamma \neq \mathbb{Q}$.
- Let $q \in \gamma$ and let $p \in \mathbb{Q}$ such that $p < q$. As $q \in \gamma$, we know $\exists \alpha \in A$ such that $q \in \alpha$. We also know that $\mathbb{Q} \ni p < q$, so $p \in \alpha$ and therefore $p \in \gamma$.
- Consider $q \in \gamma$, then there exists $\alpha \in A$ such that $q \in \alpha$, which means that there exists $r \in \alpha$ such that $q < r$, so $r \in \gamma$ and $q < r$.

Collecting the properties above, we deduce $\gamma \in F$. ■

Claim 8.8. $\gamma = \sup(A)$.

Subproof of Claim. Note $\alpha \subseteq \gamma$ for all $\alpha \in A$, so $\alpha \leq \gamma$ for all $\alpha \in A$. Therefore, γ is an upper bound for A. Moreover, let δ be an upper bound for A, so $\delta \geq \alpha$ for all $\alpha \in A$, but that means $\delta \supseteq \alpha$ for all $\alpha \in A$, and we can deduce that $\delta \supseteq \bigcup_{\alpha \in A} \alpha = \gamma$. Therefore, $\delta \geq \gamma$. ■

We will continue the proof next time. ■

9 Lecture 9: Construction of Real Numbers, Continued

Proof, Continued. We now define addition on the structure F to be

$$\alpha + \beta = \{p + q : p \in \alpha, q \in \beta\}.$$

We now check the axioms and start by $(A1)$, namely, $\alpha + \beta \in F$.

- Note that $\alpha + \beta \neq \varnothing$ because $\alpha \neq \varnothing$ which means $\exists p \in \alpha$, and $\beta \neq \varnothing$, which means $\exists q \in \beta$, and so there exists $p + q \in \alpha + \beta$.
- Note that $\alpha + \beta \neq \varnothing$. Indeed, $\alpha \neq \mathbb{Q}$, so $\exists r \in \mathbb{Q} \backslash \alpha$, so $r > p$ for all $p \in \alpha$; similarly, because $\beta \neq \mathbb{Q}$, so $\exists s \in \mathbb{Q} \backslash \beta$, so $s > q$ for all $q \in \beta$. Therefore, $r + s > p + q$ for all $p \in \alpha$ and $q \in \beta$, and so $r + s \notin \alpha + \beta$.
- Let $r \in \alpha + \beta$ and $s \in \mathbb{Q}$ such that $s < r$. Because $r \in \alpha + \beta$, we know $r = p + q$ for some $p \in \alpha$ and $q \in \beta$. Because $s < r$, then $s < p + q$, and so $\mathbb{Q} \ni s - p < q \in \beta$, therefore $s - p \in \beta$, which means $s = p + (s - p) \in \alpha + \beta$.
- Let $r \in \alpha + \beta$, and so $r = p + q$ for some $p \in \alpha$ and some $q \in \beta$. Because $\alpha \in F$, so $\exists p' \in \alpha$ such that $p' > p$. Similarly, because $\beta \in F$, so $\exists q' \in \beta$ such that $q' > q$. Therefore, $\alpha + \beta \ni p' + q' > p + q = r$. Therefore, $p' + q' \in \alpha + \beta$ is such that $p' + q' > r$.

Collecting all these properties above, we see that $\alpha + \beta \in F$.

We now check $(A2)$: for $\alpha, \beta \in F$, we have

$$\begin{aligned}\alpha + \beta &= \{p + q : p \in \alpha, q \in \beta\}\\ &= \{q + p : q \in \beta, p \in \alpha\}\\ &= \beta + \alpha.\end{aligned}$$

We now check $(A3)$: for $\alpha, \beta, \gamma \in F$, we have

$$\begin{aligned}(\alpha + \beta) + \gamma &= \{s + r : s \in \alpha + \beta, r \in \gamma\}\\ &= \{(p + q) + r : p \in \alpha, q \in \beta, r \in \gamma\}\\ &= \{p + (q + r) : p \in \alpha, q \in \beta, r \in \gamma\}\\ &= \{p + t : p \in \alpha, t \in \beta + \gamma\}\\ &= \alpha + (\beta + \gamma).\end{aligned}$$

We now check $(A4)$: let $0^* = \{q \in \mathbb{Q} : q < 0\}$.

Claim 9.1. $0^* \in F$.

Subproof. • Note $p^* \neq \varnothing$ because $-1 \in 0^*$.

- Note that $0^* \neq \mathbb{Q}$ because $2 \notin 0^*$.

- Let $q \in 0^*$ and let $p \in \mathbb{Q}$ and $p < q$. Then $q \in 0^*$ implies that $q < 0$, and because $p < q$, then $p < 0$, so $p \in 0^*$.

- Let $q \in 0^*$, then $q < 0$, so $\exists r \in \mathbb{Q}$ such that $q < r < 0$. Therefore, $r \in 0^*$ and $r > q$.

Collecting all these properties, we get $0^* \in F$. ■

Claim 9.2. $\alpha + 0^* = \alpha \ \ \forall \alpha \in F$.

Proof. • We check $\alpha + 0^* \subseteq \alpha$. Let $r \in \alpha + 0^*$, so $r = p + q$ for some $p \in \alpha$ and some $q \in 0^*$. Therefore, $q < 0$. So we know $\mathbb{Q} \ni r = p + q < p$, and because $p \in \alpha \in F$, so $r \in \alpha$. As r was arbitrary in $\alpha + 0^*$, we find $\alpha + 0^* \subseteq \alpha$.

- We now check $\alpha \subseteq \alpha + 0^*$. Let $p \in \alpha$, so there exists $r \in \alpha$ such that $r > p$. We now write $p = r + (p - r) \in \alpha + 0^*$. As $p \in \alpha$ was arbitrary, this shows that $\alpha \subseteq \alpha + 0^*$.

Collecting the properties above, we get $\alpha + 0^* = \alpha$. ■

We now check $(A5)$: fix $\alpha \in F$. We now define

$$\beta = \{q \in \mathbb{Q} : \exists r \in \mathbb{Q} \text{ with } r > 0 \text{ such that } -q - r \notin \alpha\}.$$

Claim 9.3. $\beta \in F$.

Subproof. • Note that $\beta \varnothing$. As $\alpha \neq \varnothing$, there exists $p \in \mathbb{Q} \backslash \alpha$, then $(-\beta + 1) \in \beta$ because $-[-(p + 1)] - 1 = (p + 1) - 1 = p \notin \alpha$.

- Note that $\beta \neq \varnothing$. As $\alpha \neq \varnothing$, there exists $p \in \alpha$. Then $-p \notin \beta$ because $\forall r \in \mathbb{Q}$, $r > 0$, we have $-(-p) - r = p - r < p$, and because $p \in \alpha \in F$. Therefore, $p - r \in \alpha$, and so $-p \notin \beta$.

- Let $q \in \beta$ and let $p \in \mathbb{Q}$ such that $p < q$. Because $q \in \beta$, there exists $r \in \mathbb{Q}$ such that $r > 0$ and $-q - r \notin \alpha$, therefore $-q - r > s$ for all $s \in \alpha$. Hence, $-p - r > -q - r > s$ for all $s \in \alpha$, and so $-p - r \notin \alpha$, which means $p \in \beta$.

- Let $q \in \beta$. We want to find $s \in \beta$ such that $s > q$. Because $q \in \beta$, so there exists $r \in \mathbb{Q}$ such that $r > 0$ and $-q - r \notin \alpha$. Therefore, $-(q + \frac{r}{2}) - \frac{r}{2} = -q - r \notin \alpha$, and so $q + \frac{r}{2} \in \beta$. We then define $s = q + \frac{r}{2}$.

Collecting all the properties, we get $\beta \in F$. ■

Claim 9.4. $\alpha + \beta = 0^*$.

Subproof.

- We first check $\alpha + \beta \subseteq 0^*$. Let $s \in \alpha + \beta$, then $s = p + q$ with $p \in \alpha$ and $q \in \beta$. Because $q \in \beta$, so there exists $r \in \mathbb{Q}$ with $r > 0$ such that $-q - r \notin \alpha$, so $-q - r > p$, which means $\mathbb{Q} \ni p + q < -r < 0$. Therefore, $s = p + q \in 0^*$, and so $\alpha + \beta \subseteq 0^*$.

- We now check $0^* \subseteq \alpha_\beta$. Let $r \in 0^*$, then $r \in \mathbb{Q}$ and $r < 0$.

  Claim 9.5. $\exists N \in \mathbb{N}$ such that $N \cdot (-\frac{r}{2}) \in \alpha$, but $(N+1)(-\frac{r}{2}) \notin \alpha$.

  Subproof. We prove this by contradiction. Assume

  $$\{n \cdot (-\frac{r}{2}) : n \in \mathbb{N}\} \subseteq \alpha.$$

  We will show that in this case $\mathbb{Q} \subseteq \alpha$ and thus reach a contradiction.

  Fix $q \in \mathbb{Q}$. By the Archimedean property for $\mathbb{Q}$, $\exists n \in \mathbb{N}$ such that $n > q \cdot (-\frac{2}{r}) \in \mathbb{Q}$. Therefore, $n \cdot (-\frac{r}{2}) > q$, and because $n \cdot (-\frac{r}{2}) \in \alpha \in F$, and so $q \in \alpha$.As $q \in \mathbb{Q}$ was arbitrary, this shows $\mathbb{Q} \subseteq \alpha$, contradiction. ■

  We now write $r = N(-\frac{r}{2}) + (N+2) \cdot \frac{r}{2}$, and note that $(N+2)\frac{r}{2} \in \beta$ since

  $$-(N+2) \cdot \frac{r}{2} - \frac{r}{2} = (N+1) \cdot (-\frac{r}{2}) \notin \alpha.$$

  As $r \in 0^*$ was arbitrary, this shows $0^* \subseteq \alpha_\beta$.

Therefore, $\alpha + \beta = 0^*$. ■

We now check $(O1)$. If $\alpha, \beta, \gamma \in F$ such that $\alpha < \beta$, so $\alpha \subsetneq \beta$, then $\alpha + \gamma \subsetneq \beta + \gamma$, and so $\alpha + \gamma < \beta + \gamma$.

We define multiplication on F as follows: for $\alpha, \beta \in F$ with $\alpha > 0$ and $\beta > 0$, we define

$$\alpha \cdot \beta = \{q \in \mathbb{Q} : q < r \cdot s \text{ for some } 0 < r \in \alpha \text{ and some } 0 < s \in \beta\}.$$

For $\alpha \in F$, we define $\alpha \cdot 0^* = 0^*$. We define

$$\alpha \cdot \beta = \begin{cases} (-\alpha) \cdot (-\beta), & \text{if } \alpha < 0, \beta < 0 \\ -[(-\alpha) \cdot \beta], & \text{if } \alpha < 0, \beta > 0 \\ -[\alpha \cdot (-\beta)], & \text{if } \alpha > 0, \beta < 0 \end{cases}.$$

We leave the proof of properties $(M1)$ through $(M5)$, as well as (D) and $(O2)$ as an exercise for the readers. □

We identify a rational number $r \in \mathbb{Q}$ with the Dedekind cut

$$r^* = \{q \in \mathbb{Q} : q < r\}.$$

One can check that

$$r^* + s^* = (r+s)^*$$
$$r^* \cdot s^* = (r \cdot s)^*$$
$$r < s \iff r^* < s^*$$

10 Lecture 10: Sequences

Definition 10.1 (Sequence)**.** A sequence of real number is a function $f : \{n \in \mathbb{Z} : n \geq m\} \to \mathbb{R}$ where m is a fixed integer[5]. We write the sequence as $f(m), f(m+1), f(m+2), \cdots$ or as $\{f(n)\}_{n \geq m}$ or as $\{f_n\}_{n \geq m}$.

Definition 10.2 (Bounded Sequence)**.** We say that a sequence $\{a_n\}_{n\geq 1}$ of real numbers is bounded below (respectively, bounded above, bounded) if the set $\{a_n : n \geq 1\}$ is bounded below (respectively, bounded above, bounded).

We say that the sequence $\{a_n\}_{n\geq 1}$ is

- (monotonically) increasing if $a_n \leq a_{n+1} \ \ \forall n \geq 1$.
- strictly increasing if $a_n < a_{n+1} \ \ \forall n \geq 1$.
- (monotonically) decreasing if $a_n \geq a_{n+1} \ \ \forall n \geq 1$.
- strictly decreasing if $a_n > a_{n+1} \ \ \forall n \geq 1$.
- monotone if it is either increasing or decreasing.

Example 10.3.

1. $\{a_n\}_{n\geq 1}$ with $a_n = 3 - \frac{1}{n}$ is bounded and strictly increasing.
2. $\{a_n\}_{n\geq 1}$ with $a_n = (-1)^n$ is bounded but not monotone.
3. $\{a_n\}_{n\geq 0}$ with $a_n = n^2$ is bounded below and strictly increasing.
4. $\{a_n\}_{n\geq 0}$ with $a_n = \cos(\frac{n\pi}{3})$ is bounded but not monotone.

[5] m is usually 1 or 0.

To define the notion of convergence of a sequence, we need a notion of distance between two real numbers.

Definition 10.4 (Absolute Value)**.** For $x \in \mathbb{R}$, the absolute value of x is

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

This function satisfies the following:

1. $|x| \geq 0$ for all $x \in \mathbb{R}$.
2. $|x| = 0 \iff x = 0$.
3. $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.[6]
4. $|x \cdot y| = |x| \cdot |y|$ for all $x, y \in \mathbb{R}$.
5. $\big||x| - |y|\big| \leq |x - y|$ for all $x, y \in \mathbb{R}$.[7]

We think of $|x - y|$ as the distance between $x, y \in \mathbb{R}$.

Definition 10.5 (Converge, Limit, Diverge)**.** We say that a sequence $\{a_n\}_{n\geq 1}$ of real numbers converges if $\exists a \in \mathbb{R}$ such that $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ such that $|a_n - a| < \varepsilon \ \forall n \geq n_\varepsilon$.

If this is the case, we say that a is the limit of $\{a_n\}_{n\geq 1}$ and we write $a = \lim\limits_{n\to\infty} a_n$ or $a_n \xrightarrow[n\to\infty]{} a$.

If the sequence does not converge, we say it diverges.

Lemma 10.6. The limit of a convergent sequence is unique.

Proof. We argue by contradiction. Assume that $\{a_n\}_{n\geq 1}$ is a convergent sequence and assume that there exists $a, b \in \mathbb{R}$ such that $a \neq b$ and $a = \lim\limits_{n\to\infty} a_n$ and $b = \lim\limits_{n\to\infty} a_n$. Let $0 < \varepsilon < \frac{|b-a|}{2}$.[8] Because $a = \lim\limits_{n\to\infty} a_n$, then there exists $n_1(\varepsilon) \in \mathbb{N}$ such that $|a_n - a| < \varepsilon \ \forall n \geq n_1(\varepsilon)$. Similarly, because $b = \lim\limits_{n\to\infty} a_n$, then there exists $n_2(\varepsilon) \in \mathbb{N}$ such that $|a_n - b| < \varepsilon \ \forall n \geq n_2(\varepsilon)$. Now set $n_\varepsilon = \max\{n_1(\varepsilon), n_2(\varepsilon)\}$. Then for $n \geq n_\varepsilon$, we have

$$|b - a| = |b - a_n + a_n - a| \leq |b - a_n| + |a_n - a| < 2\varepsilon < |b - a|.$$

This is a contradiction. ☐

Example 10.7. We can show that the sequence given by $a_n = \frac{1}{n}$ for all $n \geq 1$ converges to 0.

Let $\varepsilon > 0$. By the Archimedean property, there exists $n_\varepsilon \in \mathbb{N}$ such that $n_\varepsilon > \frac{1}{\varepsilon}$. Then for $n \geq n_\varepsilon$, we have

$$|0 - \frac{1}{n}| = \frac{1}{n} \leq \frac{1}{n_\varepsilon} < \varepsilon.$$

By definition, $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

[6] This is known as the triangle inequality.

[7] This is known as the inverse triangle inequality.

[8] We can choose such an ε because $\mathbb{Q}$ is dense in $\mathbb{R}$.

Example 10.8. We can show that the sequence given by $a_n = (-1)^n$ for all $n \geq 1$ does not converge.

We argue by contradiction. Assume $\exists a \in \mathbb{R}$ such that $a = \lim_{n\to\infty} (-1)^n$. Let $0 < \varepsilon < 1$. Then $\exists n_\varepsilon \in \mathbb{N}$ such that $|a-(-1)^n| < \varepsilon$ for all $N \geq n_\varepsilon$. By taking $n = 2n_\varepsilon$, we get $|a-1| < \varepsilon$, and by taking $n = 2n_\varepsilon + 1$, we get $|a+1| < \varepsilon$. By the triangle inequality,

$$2 = |1+1| = |1-a+a+1| \leq |1-a| + |a-1| < 2\varepsilon < 2.$$

This is a contradiction.

Lemma 10.9. A convergent sequence is bounded.

Proof. Let $\{a_n\}_{n\geq 1}$ be a convergent sequence and let $a = \lim_{n\to\infty} a_n$. There exists $n_1 \in \mathbb{N}$ such that $|a - a_n| < 1$ for all $n \geq n_1$. So $|a_n| \leq |a_n - a| + |a| < 1 + |a|$ for all $n \geq n_1$. Let $M = \max\{1+|a|, |a_1|, |a_2|, \cdots, |a_{n_1-1}|\}$. Clearly, $|a_n| \leq M$ for all $n \geq 1$, so $\{a_n\}_{n\geq 1}$ is bounded. □

Theorem 10.10. Let $\{a_n\}_{n\geq 1}$ be a convergent sequence and let $a = \lim_{n\to\infty} a_n$. Then for any $k \in \mathbb{R}$, the sequence $\{ka_n\}_{n\geq 1}$ converges and $\lim_{n\to\infty} ka_n = ka$.

Proof. If $k = 0$, then $ka_n = 0$ for all $n \geq 1$, and so $\lim_{n\to\infty} ka_n = 0 = ka$.

If $k \neq 0$, let $\varepsilon > 0$. As $a = \lim_{n\to\infty} a_n$, there exists $n_{\varepsilon,k} \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{|k|}$ for all $n \geq n_{\varepsilon,k}$. Therefore, $|ka_n - ka| = |k| \cdot |a_n - a| < |k| \cdot \frac{\varepsilon}{|k|=\varepsilon}$ for all $n \geq n_{\varepsilon,k}$. By definition, $\lim_{n\to\infty} ka_n = ka$. □

Remark 10.11. The idea is that we want to find $n_\varepsilon \in \mathbb{N}$ such that $\forall n \geq n_\varepsilon$, $|ka_n - ka| < \varepsilon$. But that is equivalent to having $|a_n - a| < \frac{\varepsilon}{|k|}$.

11 Lecture 11: Sequences, Continued

Theorem 11.1. Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two convergent sequences of real numbers and let $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. Then

1. the sequence $\{a_n + b_n\}_{n\geq 1}$ converges and $\lim_{n\to\infty} (a_n + b_n) = a + b$.

2. the sequence $\{a_n \cdot b_n\}_{n\geq 1}$ converges and $\lim_{n\to\infty} (a_n b_n) = a \cdot b$.

3. if $a \neq 0$ and $a_n \neq 0$ for all $n \geq 1$, then $\{\frac{1}{a_n}\}_{n\geq 1}$ converges and $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$.

4. if $a \neq 0$ and $a_n \neq 0$ for all $n \geq 1$, then $\{\frac{b_n}{a_n}\}_{n\geq 1}$ converges and $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{b}{a}$.

Proof. 1. Let $\varepsilon > 0$. We want to find $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$,

$$|(a+b) - (a_n + b_n)| < \varepsilon.$$

Then it suffices to find large enough n such that $|a - a_n| < \frac{\varepsilon}{2}$ and $|b - b_n| < \frac{\varepsilon}{2}$, which means

$$|(a + b) - (a_n + b_n)| < |a - a_n| + |b - b_n| < \varepsilon.$$

As $\lim_{n\to\infty} a_n = a$, then there exists $n_1(\varepsilon) \in \mathbb{N}$ such that $|a - a_n| < \frac{\varepsilon}{2}$ for all $n \geq n_1(\varepsilon)$. Similarly, as $\lim_{n\to\infty} b_n = b$, then there exists $n_2(\varepsilon) \in \mathbb{N}$ such that $|b - b_n| < \frac{\varepsilon}{2}$ for all $n \geq n_2(\varepsilon)$.

Now let $n_\varepsilon = \max\{n_1(\varepsilon), n_2(\varepsilon)\}$. Then for $n \geq n_\varepsilon$, we have

$$|(a + b) - (a_n + b_n)| \leq |a - a_n| + |b - b_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

By definition, $\lim_{n\to\infty} (a_n + b_n) = a + b$.

2. Let $\varepsilon > 0$. As $\{a_n\}_{n\geq 1}$ converges, it is bounded. Let $M > 0$ be such that $|a_n| \leq M$ for all $n \geq 1$.

   We want to find $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$, $|ab - a_n b_n| < \varepsilon$. To find such n_ε, it suffices to make it large enough so that $|a - a_n| \cdot |b| < \frac{\varepsilon}{2}$ and $|a_n| \cdot |b - b_n| < \frac{\varepsilon}{2}$, then we know that

$$|ab - a_n b_n| = |(a - a_n) \cdot b + a_n(b - b_n)| \leq |a - a_n| \cdot |b| + |a_n| \cdot |b - b_n| < \varepsilon.$$

   To do so, it suffices to take $|a - a_n| < \frac{\varepsilon}{2(|b|+1)}$ and $|b - b_n| < \frac{\varepsilon}{2M}$, where $M > 0$ is such that $|a_n| \leq M$ for all $n \geq 1$.[9]

   As $\lim_{n\to\infty} a_n = a$, there exists $n_1(\varepsilon) \in \mathbb{N}$ such that $|a - a_n| < \frac{\varepsilon}{2(|b|+1)}$ for all $n \geq n_1(\varepsilon)$. Similarly, as $\lim_{n\to\infty} b_n = b$, there exists $n_2(\varepsilon) \in \mathbb{N}$ such that $|b - b_n| < \frac{\varepsilon}{2M}$ for all $n \geq n_2(\varepsilon)$.

   Set $n_\varepsilon = \max\{n_1(\varepsilon), n_2(\varepsilon)\}$. For $n \geq n_\varepsilon$, we have

$$\begin{aligned} |ab - a_n b_n| &= |(a - a_n)b + a_n(b - b_n)| \\ &\leq |a - a_n| \cdot |b| + |a_n| \cdot |b - b_n| \\ &< \frac{\varepsilon}{2(|b| + 1)} \cdot |b| + M \cdot \frac{\varepsilon}{2M} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

   By definition, $\lim_{n\to\infty} (a_n b_n) = ab$.

3. Let $\varepsilon > 0$. We want to find $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$, $|\frac{1}{a} - \frac{1}{a_n}| < \varepsilon$. Note that

$$\left|\frac{1}{a} - \frac{1}{a_n}\right| = \frac{|a_n - a|}{|a| \cdot |a_n|} < \varepsilon$$

[9] While the obvious choice for $|b - b_n|$ is to bound it by $\frac{\varepsilon}{|a_n|}$, note that this does not guarantee us to shrink to less than $\frac{\varepsilon}{2}$.

and so we want $|a_n - a| < \varepsilon|a| \cdot |a_n|$.

As $a = \lim\limits_{n\to\infty} a_n$, there exists $n_1(a) \in \mathbb{N}$ such that $|a - a_n| < \frac{|a|}{2}$ for all $n \geq n_1(a)$. Then, for all $n \geq n_1$, we have

$$|a_n| \geq |a| - |a - a_n| > |a| - \frac{|a|}{2} = \frac{|a|}{2}.$$

Moreover, there exists $n_2(\varepsilon, a) \in \mathbb{N}$ such that $|a - a_n| < \frac{\varepsilon|a|^2}{2}$ for all $n \geq n_2(\varepsilon, a)$.

Now let $n_\varepsilon = \max\{n_1(a), n_2(\varepsilon, a)\}$. For $n \geq n_\varepsilon$, we have

$$\left|\frac{1}{a} - \frac{1}{a_n}\right| = \frac{|a - a_n|}{|a| \cdot |a_n|} < \frac{\varepsilon|a|^2}{2|a|} \cdot \frac{2}{|a|} = \varepsilon.$$

By definition, $\lim\limits_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$.

4. We leave this as an exercise.

$\square$

Example 11.2.

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^3 + 5n + 8}{3n^3 + 2n^2 + 7} &= \lim_{n\to\infty} \frac{1 + \frac{5}{n^2} + \frac{8}{n^3}}{3 + \frac{2}{n} + \frac{7}{n^3}} \\
&= \frac{1 + 5 \cdot \lim\limits_{n\to\infty} \frac{1}{n^2} + 8 \cdot \lim\limits_{n\to\infty} \frac{1}{n^3}}{3 + 2 \cdot \lim\limits_{n\to\infty} \frac{1}{n} + 7 \cdot \lim\limits_{n\to\infty} \frac{1}{n^3}} \\
&= \frac{1 + 5 \cdot 0 + 8 \cdot 0}{3 + 2 \cdot 0 + 7 \cdot 0} \\
&= \frac{1}{3}.
\end{aligned}$$

Theorem 11.3. Every bounded monotone sequence converges.

Proof. We will show that an increasing sequence bounded above converges. A similar argument can be used to show that a decreasing sequence bounded below converges.

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers that is bounded above and $a_{n+1} \geq a_n$ for all $n \geq 1$. As $\varnothing \neq \{a_n : n \geq 1\} \subseteq \mathbb{R}$ is bounded above and $\mathbb{R}$ has the least upper bound property, there exists $a \in \mathbb{R}$ such that $a = \sup\{a_n : n \geq 1\}$. It now suffices to prove that this number is the point of convergence we want.

Claim 11.4. $a = \lim\limits_{n\to\infty} a_n$.

Subproof. Let $\varepsilon > 0$. Then $a - \varepsilon$ is not an upper bound for $\{a_n : n \geq 1\}$. Therefore, there exists $n_\varepsilon \in \mathbb{N}$ such that $a - \varepsilon < a_{n_\varepsilon}$. Therefore, for $n \geq n_\varepsilon$, we have

$$a - \varepsilon < a_{n_\varepsilon} \leq a_n \leq a < a + \varepsilon,$$

which means $|a_n - a| < \varepsilon$. This proves the claim. $\blacksquare$

$\square$

Definition 11.5. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers.

We write $\lim\limits_{n\to\infty} a_n = \infty$ and say that $\{a_n\}_{n\geq 1}$ diverges to $+\infty$ if $\forall M > 0$, $\exists n_M \in \mathbb{N}$ such that $a_n > M$ for all $n \geq n_M$.

We write $\lim\limits_{n\to\infty} a_n = -\infty$ and say that $\{a_n\}_{n\geq 1}$ diverges to $-\infty$ if $\forall M < 0$, $\exists n_M \in \mathbb{N}$ such that $a_n < M$ for all $n \geq n_M$.

Exercise 11.6.

1. Show that $\lim\limits_{n\to\infty} (\sqrt[3]{n} + 1) = \infty$.
2. Show that the sequence given by $a_n = (-1)^n n$ for all $n \geq 1$ does not diverge to ∞ or to $-\infty$.
3. Let $\{a_n\}_{n\geq 1}$ be a sequence of positive real numbers. Show that

$$\lim_{n\to\infty} a_n = \infty \iff \lim_{n\to\infty} \frac{1}{a_n} = 0.$$

12 Lecture 12: Cauchy Sequence

Example 12.1. We can show that $\lim\limits_{n\to\infty} \frac{n^2+1}{n+3} = \infty$.

Let $M > 0$. We want to find $n_M \in \mathbb{N}$ such that for all $n \geq n_M$ we have $\frac{n^2+1}{n+3} > M$. Note that it suffices to ask $\frac{n}{4} > M$, and then

$$\frac{n^2+1}{n+3} > \frac{n^2}{n+3} > \frac{n^2}{4n} = \frac{n}{4} > M.$$

By the Archimedean property, there exists $n_M \in \mathbb{N}$ such that $n_M > 4M$, then for $n \geq n_M$, we have the desired equation above. By the definition, $\lim\limits_{n\to\infty} \frac{n^2+1}{n+3} = \infty$.

Definition 12.2. We say that a sequence of real numbers $\{a_n\}_{n\geq 1}$ is a Cauchy sequence if

$$\forall \varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N} \text{ such that } |a_n - a_m| < \varepsilon \ \forall n, m \geq n_\varepsilon.$$

Theorem 12.3 (Cauchy Criterion)**.** A sequence of real numbers is Cauchy if and only if it converges.

We will split the proof of this theorem into various lemmas and properties.

Proposition 12.4. Any convergent sequence is a Cauchy sequence.

Proof. Let $\{a_n\}_{n\geq 1}$ be a convergent sequence and let $a = \lim\limits_{n\to\infty} a_n$. Let $\varepsilon > 0$. As $a_n \xrightarrow[n\to\infty]{a}$, there exists $n_\varepsilon \in \mathbb{N}$ such that $|a - a_n| < \frac{\varepsilon}{2}$ for all $n \geq n_\varepsilon$. Then for $n, m \geq n_\varepsilon$, we have

$$|a_n - a_m| \leq |a_n - a| + |a - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\square$

Lemma 12.5. A Cauchy sequence is bounded.

Proof. Let $\{a_n\}_{n\geq 1}$ be a Cauchy sequence. Then there exists $n_1 \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $n, m \geq n_1$. So taking $m = n_1$, we get

$$|a_n| \leq |a_{n_1}| + |a_n - a_{n_1}| < |a_{n_1}| + 1$$

for all $n \geq n_1$. Now let $M = \max\{|a_1|, |a_2|, \cdots, |a_{n_1-1}|, |a_{n_1}| + 1\}$. Clearly, $|a_n| \leq M$ for all $n \geq 1$. □

Definition 12.6 (Subsequence)**.** Let $\{k_n\}_{n\geq 1}$ be a sequence of natural numbers such that $k_1 \geq 1$ and $k_{n+1} > k_n$ for all $n \geq 1$. Using induction, it is easy to see that $k_n \geq n$ for all $n \geq 1$. If $\{a_n\}_{n\geq 1}$ is a sequence, we say that $\{a_{k_n}\}_{n\geq 1}$ is a subsequence of $\{a_n\}_{n\geq 1}$.

Example 12.7. The following are subsequences of $\{a_n\}_{n\geq 1}$:

- $\{a_{2n}\}_{n\geq 1}$.
- $\{a_{2n-1}\}_{n\geq 1}$.
- $\{a_{n^2}\}_{n\geq 1}$.
- $\{a_{p_n}\}_{n\geq 1}$ where p_n denotes the nth prime.

Theorem 12.8. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Then $\lim\limits_{n\to\infty} a_n = a \in \mathbb{R} \cup \{\pm\infty\}$ if and only if every subsequence $\{a_{k_n}\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$ satisfies $\lim\limits_{n\to\infty} a_{k_n} = a$.

Proof. We will consider $a \in \mathbb{R}$. The cases $a \in \{\pm\infty\}$ can be handled by an analogous argument.

($\Leftarrow$): Take $k_n = n$ for all $n \geq 1$.

($\Rightarrow$): Assume $\lim\limits_{n\to\infty} a_n = a$ and let $\{a_{k_n}\}_{n\geq 1}$ be a subsequence of $\{a_n\}_{n\geq 1}$. Let $\varepsilon > 0$. As $a_n \xrightarrow[n\to\infty]{} a$, $\exists n_\varepsilon \in \mathbb{N}$ such that $|a - a_n| < \varepsilon$ for all $n \geq n_\varepsilon$. Recall that $k_n \geq n$ for all $n \geq 1$. So for $n \geq n_\varepsilon$ we have $k_n \geq n \geq n_\varepsilon$ and so $|a - a_{k_n}| < \varepsilon$ for all $n \geq n_\varepsilon$. By definition, $\lim\limits_{n\to\infty} a_{k_n} = a$. □

Proposition 12.9. Every sequence of real numbers has a monotone subsequence.

Proof. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. We say that the nth term term is dominant if $a_n > a_m$ for all $m > n$. We distinguish two cases:

Case 1: There are infinitely many dominant terms. Then a subsequence formed by these dominant terms is strictly decreasing.

Case 2: There are none of finitely many dominant terms. Let N be larger that the largest index of the dominant terms. So for all $n \geq N$, a_n is not dominant. Set $k_1 = N$, $a_{k_1} = a_N$. Because a_{k_1} is not dominant, there exists $k_2 > k_1$ such that $a_{k_2} \geq a_{k_1}$. Now $k_2 > k_1 = N$, then a_{k_2} is not dominant, so there exists $k_3 > k_2$ such that $a_{k_3} \geq a_{k_2}$. Proceeding inductively, we construct a subsequence $\{a_{k_n}\}_{n\geq 1}$ such that $a_{k_{n+1}} \geq a_{k_n}$ for all $n \geq 1$. □

Theorem 12.10 (Bolzano-Weierstrass)**.** Any bounded sequence has a convergent subsequence.

Proof. Let $\{a_n\}_{n\geq 1}$ be a bounded sequence. By the previous proposition, there exists $\{a_{k_n}\}_{n\geq 1}$ monotone subsequence of $\{a_n\}_{n\geq 1}$. As $\{a_n\}_{n\geq 1}$ is bounded, so is $\{a_{k_n}\}_{n\geq 1}$. As bounded monotone sequences converge, $\{a_{k_n}\}_{n\geq 1}$ converges. $\square$

Corollary 12.11. Every Cauchy sequence has a convergent subsequence.

Lemma 12.12. A Cauchy sequence with a convergent subsequence converges.

Proof. Let $\{a_n\}_{n\geq 1}$ be a Cauchy sequence such that $\{a_{k_n}\}_{n\geq 1}$ is a convergent subsequence. Let $a = \lim_{n\to\infty} a_{k_n}$. Let $\varepsilon > 0$. As $a_{k_n} \xrightarrow[n\to\infty]{} a$, there exists $n_1(\varepsilon)$ such that $|a - a_{k_n}| < \frac{\varepsilon}{2}$ for all $n \geq n_1(\varepsilon)$. As $\{a_n\}_{n\geq 1}$ is Cauchy, there exists $n_2(\varepsilon)$ such that $|a_n - a_m| < \frac{\varepsilon}{2}$ for all $n, m \geq n_2(\varepsilon)$. Let $n_\varepsilon = \max\{n_1(\varepsilon), n_2(\varepsilon)\}$. Then for $n \geq n_\varepsilon$, we have

$$|a - a_n| \leq |a - a_{k_n}| + |a_{k_n} = a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

because $k_n \geq n \geq n_\varepsilon$. By definition, $\lim_{n\to\infty} a_n = a$. $\square$

Combining the last two results, we see that a Cauchy sequence of real numbers converges.

13 Lecture 13: Limit Superior and Limit Inferior

Let $\{a_n\}_{n\geq 1}$ be a bounded sequence of real number (convergent or not). The asymptotic behavior of $\{a_n\}_{n\geq 1}$ depends on sets of the form $\{a_n : n \geq N\}$ for $N \in \mathbb{N}$.

As $\{a_n\}_{n\geq 1}$ bounded, the set $\{a_n : n \geq N\}$ (where $N \in \mathbb{N}$ is fixed) is a non-empty bounded subset of $\mathbb{R}$.

As $\mathbb{R}$ has the least upper bound property (and so also the greatest lower bound property), the set $\{a_n : n \geq N\}$ has an infimum and a supremum in $\mathbb{R}$.

For $N \geq 1$, let $u_N = \inf\{a_n : n \geq N\}$ and $v_N = \sup\{a_n : n \geq N\}$. Clearly, $u_N \leq v_N$ for all $N \geq 1$.

Notice that for $N \geq 1$, we have $\{a_n : n \geq N\} \supseteq \{a_n : n \geq N+1\}$, therefore

$$\begin{cases} \inf\{a_n : n \geq N\} \leq \inf\{a_n : n \geq N+1\} \\ \sup\{a_n : n \geq N\} \geq \sup\{a_n : n \geq N+1\} \end{cases}$$

So $u_N \leq u_{N+1}$ and $v_{N+1} \leq v_N$ for all $N \geq 1$. Thus, $\{u_N\}_{N\geq 1}$ is increasing and $\{v_N\}_{N\geq 1}$ is decreasing. Moreover, for all $N \geq 1$, we have

$$u_1 \leq u_2 \leq \cdots \leq u_N \leq v_N \leq \cdots \leq v_2 \leq v_1$$

So the two sequences are bounded. As monotone bounded sequences converge, we know the two sequences must converge.

Let

$$u = \lim_{N\to\infty} u_N = \sup\{u_N : N \geq 1\} =: \sup_N u_N$$

and

$$v = \lim_{N\to\infty} v_N = \sup\{v_N : N \geq 1\} =: \inf_N v_N$$

Because of the boundedness, we see that $u_M \leq v_N$ for all $M, N \geq 1$, and so $\lim_{M\to\infty} u_M \leq v_N$ for all $N \geq 1$. Therefore, $u \leq v_N$ for all $N \geq 1$, and therefore $u \leq \lim_{N\to\infty} v_N$, which means $u \leq v$.

Moreover, if $\lim_{n\to\infty} a_n$ exists, then for all $N \geq 1$, we have

$$u_N = \inf\{a_n : n \geq N\} \leq a_n \leq \sup\{a_n : n \geq N\} = v_N$$

for all $n \geq N$. Therefore, $u_N \leq \lim_{n\to\infty} a_n \leq v_N$, and so

$$u = \lim_{N\to\infty} u_N \leq \lim_{n\to\infty} a_n \leq \lim_{N\to\infty} v_N = v.$$

Definition 13.1. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. We define

$$\limsup_{n\to\infty} a_n = \lim_{N\to\infty} \sup\{a_n : n \geq N\} = \lim_{N\to\infty} v_N = \inf_N v_N = \inf_N \sup_{n\geq N} a_n$$

and

$$\liminf_{n\to\infty} a_n = \lim_{N\to\infty} \inf\{a_n : n \geq N\} = \lim_{N\to\infty} u_N = \sup_N u_N = \sup_N \inf_{n\geq N} a_n$$

with the convention that if $\{a_n\}_{n\geq 1}$ is unbounded above, then $\limsup_{n\to\infty} a_n = \infty$ and if $\{a_n\}_{n\geq 1}$ is unbounded below then $\liminf_{n\to\infty} a_n = -\infty$.

Remark 13.2. We have

$$\inf\{a_n : n \geq 1\} \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq \sup\{a_n : n \geq 1\}.$$

Note that $\liminf_{n\to\infty} a_n$ is the smallest value that infinitely many a_n get close to, and $\limsup_{n\to\infty} a_n$ is the largest value that infinitely many a_n get close to.

Example 13.3. Consider $a_n = 3 + \frac{(-1)^n}{n}$, then $\lim_{n\to\infty} a_n = 3$, and therefore $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = 3$. Observe that $\inf\{a_n : n \geq 1\} = 2 \neq 3$ and $\sup\{a_n : n \geq 1\} = \frac{7}{2} \neq 3$.

Theorem 13.4. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers.

1. If $\lim_{n\to\infty} a_n$ exists in $\mathbb{R} \cup \{\pm\infty\}$, then $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n$.

2. If $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n \in \mathbb{R} \cup \{\pm\infty\}$, then $\lim_{n\to\infty} a_n$ exists and

$$\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

Proof. 1. We distinguish three cases.

- Case 1: $\lim_{n\to\infty} a_n = -\infty$. It is enough to show $\limsup_{n\to\infty} a_n = -\infty$ since $\liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n$.
  Fix $M < 0$. As $\lim_{n\to\infty} a_n = -\infty$, there exists $n_M \in \mathbb{N}$ such that $a_n < M$ for all $n \ge n_M$, then for $N \ge n_M$, we have $v_N = \sup\{a_n : n \ge N\} \le M$.[10] Now by definition, $\limsup_{n\to\infty} a_n = \lim_{N\to\infty} v_N = -\infty$.
- Case 2: $\lim_{n\to\infty} a_n = \infty$. The proof is essentially the same as above, and we leave this as an exercise.
- Case 3: $\lim_{n\to\infty} a_n = a \in \mathbb{R}$. Fix $\varepsilon > 0$. Then $\exists n_\varepsilon \in \mathbb{N}$ such that $|a - a_n| < \frac{\varepsilon}{2}$ for all $n \ge n_\varepsilon$. So we know
  $$a - \frac{\varepsilon}{2} < a_n < a + \frac{\varepsilon}{2}$$
  for all $n \ge n_\varepsilon$. Thus, for $N \ge n_\varepsilon$, we have
  $$a - \frac{\varepsilon}{2} \le \inf\{a_n : n \ge N\} \le \sup\{a_n : n \ge N\} \le a + \frac{\varepsilon}{2}$$
  which means $a - \frac{\varepsilon}{2} \le u_N \le v_N \le a + \frac{\varepsilon}{2}$.
  Therefore, for all $N \ge n_\varepsilon$, we have $|u_N - a| \le \frac{\varepsilon}{2} < \varepsilon$ and $|v_N - a| \le \frac{\varepsilon}{2} < \varepsilon$ for all $N \ge n_\varepsilon$. By definition, that means $\liminf_{n\to\infty} a_n = \lim_{N\to\infty} u_N = a$ and $\limsup_{n\to\infty} a_n = \lim_{N\to\infty} v_N = a$.

2. Again, we distinguish three cases.

   - Case 1: $\liminf_{n\to\infty} = \limsup_{n\to\infty} a_n = -\infty$. We will use $\limsup_{n\to\infty} a_n = -\infty$. Fix $M < 0$. Then since $\limsup_{n\to\infty} a_n = \lim_{N\to\infty} v_N = -\infty$, then there exists $N_M \in \mathbb{N}$ such that $v_N < M$ for all $N \ge N_M$. In particular, $v_{N_M} = \sup\{a_n : n \ge N_M\} < M$, which means $a_n < M$ for all $n \ge N_M$. By definition, that means $\lim_{n\to\infty} a_n = -\infty$.
   - Case 2: $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \infty$. The proof is essentially the same as above, and we leave this as an exercise.
   - Case 3: $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = a \in \mathbb{R}$. Fix $\varepsilon > 0$. Because $a = \liminf_{n\to\infty} = \lim_{N\to\infty} u_N$, then there exists $N_1(\varepsilon) \in \mathbb{N}$ such that $|u_N - a| < \varepsilon$ for all $N \ge N_1$. Therefore, $a - \varepsilon < u_{N_1} = \inf\{a_n : n \ge N_1\} < a + \varepsilon$, and we have $a - \varepsilon < a_n$ for all $n \ge N_1$.
     Similarly, considering the limit supremum, there exists $N_2(\varepsilon) \in \mathbb{N}$ such that $|v_N - a| < \varepsilon$ for all $N \ge N_2$, and so $a - \varepsilon < v_{N_2} = \inf\{a_n : n \ge N_2\} < a + \varepsilon$, which means $a_n < a + \varepsilon$ for all $n \ge N_2$.
     Thus, for $n \ge \max\{N_1, N_2\}$, we have $a - \varepsilon < a_n < a + \varepsilon$, which means $|a_n - a| < \varepsilon$. By definition, $\lim_{n\to\infty} a_n = a$.

$\square$

[10] Note that when taking supremum, the $<$ sign can be changed to $\le$. For example, $a_n = 3 - \frac{1}{n}$ has the property of $a_n < 3$ for all $n \ge 1$, but $\sup_{n\ge1} a_n = 3$.

14 Lecture 14: Limit Superior and Limit Inferior, Continued

Theorem 14.1. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Then there exists a monotonic subsequence of $\{a_n\}_{n\geq 1}$ whose limit is $\limsup_{n\to\infty} a_n$. Also, there exists a monotonic subsequence of $\{a_n\}_{n\geq 1}$ whose limit is $\liminf_{n\to\infty} a_n$.

Proof. We will prove the statement about $\limsup_{n\to\infty} a_n$. One can use a similar argument to show the statement about $\liminf_{n\to\infty} a_n$.

Note that if suffices to find a subsequence $\{a_{k_n}\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$ such that $\lim_{n\to\infty} a_{k_n} = \limsup_{n\to\infty} a_n$. As every sequence has a monotone subsequence, $\{a_{k_n}\}_{n\geq 1}$ has a monotone subsequence $\{a_{p_{k_n}}\}_{n\geq 1}$. Then as $\lim_{n\to\infty} a_{k_n}$ exists, $\lim_{n\to\infty} a_{p_{k_n}}$ exists and

$$\lim_{n\to\infty} a_{p_{k_n}} = \lim_{n\to\infty} a_{k_n} = \limsup_{n\to\infty} a_n.$$

Finally, note that $\{a_{p_{k_n}}\}_{n\geq 1}$ is a subsequence of $\{a_n\}_{n\geq 1}$.

Let us find a subsequence of $\{a_n\}_{n\geq 1}$ whose limit is $\limsup_{n\to\infty} a_n$.

Case 1: $\limsup_{n\to\infty} a_n = -\infty$. We showed that in this case, $\lim_{n\to\infty} a_n = -\infty$. Choose $\{a_{k_n}\}_{n\geq 1}$ to be $\{a_n\}_{n\geq 1}$.

Case 2: $\limsup_{n\to\infty} a_n = a \in \mathbb{R}$. By definition, $a = \limsup_{n\to\infty} a_n = \lim_{N\to\infty} v_N$, then $\exists N_1 \in \mathbb{N}$ such that $|a - v_N| < 1$ for all $N \geq N_1$. In particular, $a - 1 < v_{N_1} < a + 1$, and note that $a - 1 < \sup\{a_n : n \geq N_1\}$ and there exists $k_1 \geq N_1$ such that $a - 1 < a_{k_1}$. Therefore, $a - 1 < a_{k_1} \leq v_{N_1} < a + 1$. Hence, $|a - a_{k_1}| < 1$.

Similarly, as $a = \lim_{N\to\infty} v_N$, there exists $N_2 \in \mathbb{N}$ such that $|a - v_N| < \frac{1}{2}$ for all $N \geq N_2$. Let $\tilde{N}_2 = \max\{N_2, k_1 + 1\}$, then in particular, $a - \frac{1}{2} < v_{\tilde{N}_2} < a + \frac{1}{2}$. Then we know $a - \frac{1}{2} < \sup\{a_n : n \geq \tilde{N}_2\}$, and because there exists $k_2 \geq \tilde{N}_2 > k_1$ such that $a - \frac{1}{2} < a_{k_2}$, we conclude that $a - \frac{1}{2} < a_{k_2} \leq v_{N_2} < a + \frac{1}{2}$. Hence, $|a - a_{k_2}| < \frac{1}{2}$.

To construct our subsequence, we proceed inductively. Assume we have found $k_1 < k_2 < \cdots < k_n$ and $a_{k_1}, \cdots, a_{k_n}$ such that $|a - a_{k_j}| < \frac{1}{j}$ for all $1 \leq j \leq n$. As $a = \lim_{N\to\infty} v_N$, there exists $N_{n+1} \in \mathbb{N}$ such that $|a - v_N| < \frac{1}{n+1}$ for all $N \geq N_{n+1}$. Now we can let $\tilde{N_{n+1}} = \max\{N_{n+1}, k_n + 1\}$. Then $a - \frac{1}{n+1} < v_{\tilde{N_{n+1}}} < a + \frac{1}{n+1}$. Therefore, we have $a - \frac{1}{n+1} < \sup\{a_n : n \geq \tilde{N_{n+1}}\}$, and there exists $k_{n+1} \geq \tilde{N_{n+1}} > k_n$ such that $a - \frac{1}{n+1} < a_{k_{n+1}}$. Therefore, $a - \frac{1}{n+1} < a_{k_{n+1}} \leq v_{\tilde{N_{n+1}}} < a + \frac{1}{n+1}$, and so $|a_{k_{n+1}} - a| < \frac{1}{n+1}$.

Case 3: $\limsup n \to \infty a_n = \infty$. We leave this as an exercise. $\square$

Definition 14.2 (Subsequential Limit)**.** Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. A subsequential limit of $\{a_n\}_{n\geq 1}$ is any $a \in \mathbb{R} \cup \{\pm\infty\}$ that is the limit of a subsequence of $\{a_n\}_{n\geq 1}$.

Example 14.3.

1. For $a_n = n(1 + (-1)^n)$, the subsequential limits are $0 = \lim_{n\to\infty} a_{2n+1}$ and $\infty = \lim_{n\to\infty} a_{2n}$.

2. For $a_n = \cos(\frac{n\pi}{3})$. The subsequential limits are $1 = \lim\limits_{n\to\infty} a_{6n}$, $\frac{1}{2} = \lim\limits_{n\to\infty} a_{6n+1} = \lim\limits_{n\to\infty} a_{6n+5}$, $-\frac{1}{2} = \lim\limits_{n\to\infty} a_{6n+2} = \lim\limits_{n\to\infty} a_{6n+4}$, and $-1 = \lim\limits_{n\to\infty} a_{6n+3}$.

Theorem 14.4. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers and let A denote its set of subsequential limits:

$$A = \{a \in \mathbb{R} \cup \{\pm\infty\} : \exists \{a_{k_n}\}_{n\geq 1} \text{ subsequence of } \{a_n\}_{n\geq 1} \text{ such that } \lim_{n\to\infty} a_{k_n} = a\}.$$

Then

1. $A \neq \varnothing$.
2. $\lim\limits_{n\to\infty} a_n$ exists in $\mathbb{R} \cup \{\pm\infty\}$ if and only if A has exactly one element.
3. $\inf(A) = \liminf\limits_{n\to\infty} a_n$ and $\sup(A) = \limsup\limits_{n\to\infty} a_n$.

Proof. 1. By Theorem 14.1, $\liminf\limits_{n\to\infty} a_n, \limsup\limits_{n\to\infty} a_n \in A$. Therefore, $A \neq \varnothing$.

2. ($\Rightarrow$): Assume $\lim\limits_{n\to\infty} a_n$ exists. Then if $\{a_{k_n}\}_{n\geq 1}$ is a subsequence of $\{a_n\}_{n\geq 1}$, we have $\lim\limits_{n\to\infty} a_{k_n} = \lim\limits_{n\to\infty} a_n$. So $A = \{\lim\limits_{n\to\infty} a_n\}$.

   ($\Leftarrow$): If A has a single element, then $\liminf\limits_{n\to\infty} a_n = \limsup\limits_{n\to\infty} a_n$ and so $\lim\limits_{n\to\infty} a_n$ exists.

3. It suffices to prove the following claim.

   Claim 14.5. $\liminf\limits_{n\to\infty} a_n \leq a \leq \limsup\limits_{n\to\infty} a_n \quad \forall a \in A$.

   Assuming the claim, we can first see how to finish the proof. The claim implies

   - Because $\liminf\limits_{n\to\infty} a_n$ is a lower bound for A, so $\liminf\limits_{n\to\infty} a_n \geq \inf(A)$. On the other hand, $\liminf\limits_{n\to\infty} a_n \in A$, and so $\liminf\limits_{a_n} \geq \inf(A)$. Therefore, $\liminf\limits_{n\to\infty} a_n = \inf(A)$.
   - Similarly, we can show that $\limsup\limits_{n\to\infty} a_n = \sup(A)$.

   We now prove the claim.

   Subproof. Fix $a \in A$, then there exists a subsequence $\{a_{k_n}\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$ such that $\lim\limits_{n\to\infty} a_{k_n} = a$. Because of the nature of the subsequence, we know there is

   $$\inf\{a_n : n \geq N\} \leq \inf\{a_{k_n} : n \geq N\} \leq \sup\{a_{k_n} : n \geq N\} \leq \sup\{a_n : n \geq N\}$$

   where the first two sequences are increasing and the last two sequences are decreasing. By taking the limit, we know

   $$\begin{aligned}\lim_{N\to\infty} \inf\{a_n : n \geq N\} &\leq \lim_{N\to\infty} \inf\{a_{k_n} : n \geq N\} \\ &\leq \lim_{N\to\infty} \sup\{a_{k_n} : n \geq N\} \\ &\leq \lim_{N\to\infty} \sup\{a_n : n \geq N\},\end{aligned}$$

which means

$$\liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} a_{k_n} \le \limsup_{n\to\infty} a_{k_n} \le \limsup_{n\to\infty} a_n.$$

Because the subsequence converges, we have $a = \lim_{a_{k_n}} = \liminf_{n\to\infty} a_{k_n} = \limsup_{n\to\infty} a_{k_n}$. Therefore,

$$\liminf_{n\to\infty} a_n \le a \le \limsup_{n\to\infty} a_n.$$

■

□

15 Lecture 15: Cesaro-Stolz Theorem, Series and Convergence Tests

Theorem 15.1 (Cesaro-Stolz)**.** Let $\{a_n\}_{n\ge1}$ be a sequence of non-zero real numbers. Then

$$\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \le \liminf_{n\to\infty}|a_n|^{\frac{1}{n}} \le \limsup_{n\to\infty}|a_n|^{\frac{1}{n}} \le \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

In particular, if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists, then $\lim_{n\to\infty}|a_n|^{\frac{1}{n}}$ exists and

$$\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

Example 15.2. We can apply this theorem to find $\lim_{n\to\infty}\sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}}$.

If we let $a_n = n$, then $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{n} \xrightarrow[n\to\infty]{} 1$. By Cesaro-Stolz, we get $\{\sqrt[n]{n}\}_{n\ge1}$ converges and

$$\lim_{n\to\infty}\sqrt[n]{n} = 1.$$

Proof. It suffices to prove the last inequality, i.e.

$$\limsup_{n\to\infty}|a_n|^{\frac{1}{n}} \le \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

One can prove the first inequality with a similar proof.

Let $l = \limsup_{n\to\infty}|a_n|^{\frac{1}{n}} \ge 0$ and $L = \limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \ge 0$. We want to show $l \le L$. If $L = \infty$, then it is clear. Henceforth, we assume $L \in \mathbb{R}$. We will prove the following claim.

Claim 15.3. l is the lower bound for the set

$$(L,\infty) = \{M \in \mathbb{R} : M > L\}.$$

Assuming the claim for now, we can see how to finish the proof. Note (L, ∞) is a non-empty subset of $\mathbb{R}$ which is bounded below by L. As $\mathbb{R}$ has the least upper bound property, $\inf(L, \infty)$ exists in $\mathbb{R}$. In fact, $\inf(L, \infty) = L$. As l is a lower bound for (L, ∞), we must have $l \leq L$. We now prove the claim.

Subproof. Fix $M \in (L, \infty)$. We will show $l \leq M$. We have $M > L = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \inf_N \sup_{n\geq N} \left|\frac{a_{n+1}}{a_n}\right|$. Therefore, there exists $N_0 \in \mathbb{N}$ such that $\sup_{n\geq N_0} \left|\frac{a_{n+1}}{a_n}\right| < M$, and so $\left|\frac{a_{n+1}}{a_n}\right| < M$ for all $n \geq N_0$. Therefore, $|a_{n+1}| < M \cdot |a_n|$ for all $n \geq N_0$.

A simple inductive argument then yields

$$|a_n| < M^{n-N_0}|a_{N_0}| \quad \forall n \geq N_0,$$

so $|a_n|^{\frac{1}{n}} < M\left(\frac{|a_{N_0}|}{M^{N_0}}\right)^{\frac{1}{n}}$ for all $n > N_0$. We can conclude that

$$l = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty} M \cdot \left(\frac{|a_{N_0}|}{M^{N_0}}\right)^{\frac{1}{n}} = M \cdot \limsup_{n\to\infty} \left(\frac{|a_{N_0}|}{M^{N_0}}\right)^{\frac{1}{n}}.$$

We need to apply the following claim to the inequality above.

Claim 15.4. For $r > 0$, we have $\lim_{n\to\infty} r^{\frac{1}{n}} = 1$.

Subproof. Indeed, if $r \geq 1$, we have

$$0 \leq r^{\frac{1}{n}} - 1 = \frac{r-1}{r^{n-1} + r^{n-2} + \cdots + 1} \leq \frac{r-1}{n} \xrightarrow[n\to\infty]{} 0.$$

If $r < 1$, then $r^{\frac{1}{n}} = \frac{1}{(\frac{1}{r})^{\frac{1}{n}}} \xrightarrow[n\to\infty]{} \frac{1}{1} = 1$. ■

We now take $r = \frac{|a_{N_0}|}{M^{N_0}}$ in the inequality, then $l \leq M$. ■

□

Definition 15.5. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. For $n \geq 1$, we define the partial sum $s_n = a_1 + \cdots + a_n$.

The series $\sum_{n=1}^{\infty} a_n$, sometimes denoted $\sum_{n\geq 1} a_n$, is said to converge if $\{s_n\}_{n\geq 1}$ converges.

We say that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely if the series $\sum_{n=1}^{\infty} |a_n|$ converges.[11]

Theorem 15.6 (Cauchy Criterion)**.** A series $\sum_{n\geq 1} a_n$ converges if and only if

$$\forall \varepsilon > 0 \;\; \exists n_\varepsilon \in \mathbb{N} \text{ such that } \left|\sum_{k=n+1}^{n+p} a_k\right| < \varepsilon \;\; \forall n \geq n_\varepsilon \;\; \forall p \in \mathbb{N}.$$

[11]Note that $\sum_{n=1}^{\infty} |a_n|$ either converges or it diverges to ∞.

Proof. Note that

$$\begin{aligned}\text{the series } \sum_{n\geq 1} a_n \text{ converges} &\iff \text{the sequence } \{s_n\}_{n\geq 1} \text{ converges}\\ &\iff \{s_n\}_{n\geq 1} \text{ is Cauchy}\\ &\iff \forall \varepsilon > 0 \;\; \exists n_\varepsilon \in \mathbb{N} \text{ such that } |s_m - s_n| < \varepsilon \;\; \forall m, n \geq n_\varepsilon.\end{aligned}$$

Without loss of generality, we may assume $m > n$ and write $m = n + p$ for $p \in \mathbb{N}$. Note

$$|s_m - s_n| = \left|\sum_{k=1}^{n+p} a_k - \sum_{k=1}^{n} a_k\right| = \left|\sum_{k=n+1}^{n+p} a_k\right|,$$

so $\sum_{n\geq 1} a_n$ converges if and only if

$$\forall \varepsilon > 0 \;\; \exists n_\varepsilon \in \mathbb{N} \text{ such that } \left|\sum_{k=n+1}^{n+p} a_k\right| < \varepsilon \;\; \forall n \geq n_\varepsilon \;\; \forall p \in \mathbb{N}.$$

□

Corollary 15.7. If $\sum_{n\geq 1} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof. Taking $p = 1$, we find $\sum_{n\geq 1} a_n$ converges implies

$$\forall \varepsilon > 0 \;\; \exists n_\varepsilon \in \mathbb{N} \text{ such that } |a_{n+1}| < \varepsilon \;\; \forall n \geq n_\varepsilon.$$

□

Corollary 15.8. If $\sum_{n\geq 1} a_n$ converges absolutely, then it converges.

Proof. If $\sum_{n\geq 1} a_n$ converges absolutely, $\sum_{n\geq 1} |a_n|$ converges. By definition,

$$\forall \varepsilon > 0 \;\; \exists n_\varepsilon \in \mathbb{N} \text{ such that } \sum_{k=n+1}^{n+p} |a_k| < \varepsilon \;\; \forall n \geq n_\varepsilon \;\; \forall p \in \mathbb{N}.$$

Note that

$$\left|\sum_{k=n+1}^{n+p} a_k\right| \leq \sum_{k=n+1}^{n+p} |a_k| < \varepsilon \;\; \forall n \geq n_\varepsilon \;\; \forall p \in \mathbb{N}.$$

Therefore, $\sum_{n\geq 1} a_n$ converges by the Cauchy criterion. □

Theorem 15.9 (Comparison Test)**.** Let $\sum_{n\geq 1} a_n$ be a series of real numbers with $a_n \geq 0 \;\; \forall n \geq 1$.

1. If $\sum_{n\geq 1} a_n$ converges and $|b_n| \leq a_n \quad \forall n \geq 1$, then $\sum_{n\geq 1} b_n$ converges.

2. If $\sum_{n\geq 1} a_n$ diverges and $b_n \geq a_n \quad \forall n \geq 1$, then $\sum_{n\geq 1} b_n$ diverges.

Proof. 1. Because $\sum_{n\geq 1} a_n$ converges, then

$$\forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \text{ such that } \left| \sum_{k=n+1}^{n+p} a_k \right| < \varepsilon \quad \forall n \geq n_\varepsilon \quad \forall p \in \mathbb{N}.$$

Then

$$\left| \sum_{k=n+1}^{n+p} b_k \right| \leq \sum_{k=n+1}^{n+p} |b_k| \leq \sum_{k=n+1}^{n+p} a_k < \varepsilon \quad \forall n \geq n_\varepsilon \quad \forall p \in \mathbb{N}.$$

Therefore, by the Cauchy criterion, $\sum_{n\geq 1} b_n$ converges.

2. Note that $b_1 + \cdots + b_n \geq a_1 + \cdots + a_n \xrightarrow[n\to\infty]{} \infty$, and so $\sum_{n\geq 1} b_n$ diverges.

$\square$

Lemma 15.10. Let $r \in \mathbb{R}$. The series $\sum_{n\geq 0} r^n$ converges if and only if $|r| < 1$. If $|r| < 1$, then $\sum_{n\geq 0} r^n = \frac{1}{1-r}$.

Proof. First note that if $|r| \geq 1$, then $|r^n| = |r|^n \geq 1$, therefore $r^n \not\to 0$ as $n \to \infty$. By Corollary 15.7, $\sum_{n\geq 0} r^n$ does not converge. Assume now that $|r| < 1$, then $|r^n| = |r|^n \xrightarrow[n\to\infty]{} 0$. Also note that $\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r} \xrightarrow[n\to\infty]{} \frac{1}{1-r}$. $\square$

16 Lecture 16: Convergence Tests, Continued

Proposition 16.1 (The Dyadic Criterion)**.** Let $\{a_n\}_{n\geq 1}$ be a decreasing sequence of real numbers with $a_n \geq 0$ for all $n \geq 1$. Then the series $\sum_{n\geq 1} a_n$ converges if and only if the series $\sum_{n\geq 0} 2^n a_{2^n}$ converges.

Proof. For $n \geq 1$, let $s_n = \sum_{k=1}^{n} a_k = a_1 + \cdots + a_n$, and let $t_n = \sum_{k=0}^{n} 2^k a_{2^k} = a_1 + 2a_2 + \cdots + 2^n a_{2^n}$.

Note that both sequences are increasing, thus $\sum_{n\geq 1} a_n$ converges if and only if $\{s_n\}_{n\geq 1}$ is bounded, and $\sum_{n\geq 0} 2^n a_{2^n}$ converges if and only if $\{t_n\}_{n\geq 0}$ is bounded. It now suffices to prove that $\{s_n\}_{n\geq 1}$ is bounded if and only if $\{s_n\}_{n\geq 1}$ is bounded.

Consider the summation $\sum_{l=2^k+1}^{2^{k+1}} a_l$. Because $\{a_n\}_{n\geq 1}$ is decreasing, we know that

$$\frac{1}{2}(2^{k+1}a_{2^{k+1}}) = 2^k a_{2^{k+1}} \leq \sum_{l=2^k+1}^{2^{k+1}} a_l \leq 2^k a_{2^k+1} \leq 2^k a_{2^k}$$

and therefore

$$\frac{1}{2}\sum_{k=0}^{n} 2^{k+1}a_{2^{k+1}} \leq \sum_{k=0}^{n}\sum_{l=2^k+1}^{2^{k+1}} a_l \leq \sum_{k=0}^{n} 2^k a_{2^k},$$

and so $\frac{1}{2}\sum_{l=1}^{n+1} 2^l a_{2^l} \leq \sum_{l=2}^{2^{n+1}} a_l \leq t_n$. That is to say, $\frac{1}{2}(t_{n+1} - a_1) \leq s_{2^{n+1}} - a_1 \leq t_n$. We conclude that $t_{n+1} \leq 2s_{2^{n+1}} - a_1$ and $s_n \leq s_{2^{n+1}} \leq t_n + a_1$ since $n \leq 2^{n+1}$ for all $n \geq 1$.

In particular, if $\{s_n\}_{n\geq 1}$ is bounded, then there exists $M > 0$ such that $|s_n| \leq M$ for all $n \geq 1$, and so $t_{n+1} \leq 2M + a_1$ for all $n \geq 1$. Similarly, if $\{t_n\}$ is bounded, then there exists $L > 0$ such that $|t_n| \leq L$ for all $n \geq 0$, which is to say $s_n \leq L + a_1$ for all $n \geq 1$. □

Corollary 16.2. The series $\sum_{n\geq 1} \frac{1}{n^\alpha}$ converges if and only if $\alpha > 1$.

Proof. □

Theorem 16.3 (The Root Test)**.**

Proof. □

Theorem 16.4 (The Ratio Test)**.**

Proof. □

Theorem 16.5 (The Abel Criterion)**.**

Corollary 16.6 (The Leibniz Criterion)**.**

Proof of the Abel Criterion. □