Motivic Cohomology Notes

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0 Introduction

Let $X \in \mathrm{Sm}/k$ be a smooth separated scheme over a field k. The study of motivic cohomology started with the hope that

Conjecture 0.1 (Beilinson and Lichtenbaum, 1982-1987)**.** There exists some complexes $\mathbb{Z}(n)$ for $n \in \mathbb{N}$ of sheaves in Zariski topology on Sm/k such that

1. $\mathbb{Z}(0)$ is (quasi-isomorphic to) the constant sheaf $\mathbb{Z}$, i.e., the complex
$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$
concentrated at degree 0;
2. $\mathbb{Z}(1)$ is the complex $\mathcal{O}^*[-1]$, i.e., the complex
$$\cdots \longrightarrow 0 \longrightarrow \mathcal{O}^* \longrightarrow 0 \longrightarrow \cdots$$
concentrated at degree 1;
3. for every field F/k, the hypercohomology over Zariski topology satisfies[1]
$$\mathbb{H}^n_{\mathrm{Zar}}(F, \mathbb{Z}(n)) = H^n(\mathbb{Z}(n)(\mathrm{Spec}(F))) = K_n^M(F),$$
where $K_n^M(F)$ is the nth Milnor K-theory of a field F, given by the quotient of the tensor algebra $T(F^*)/\{x \otimes (1-x) : x \in F^*\}$ over $\mathbb{Z}$; (lecture 5 of [MVW06], page 29)

   Example 0.2.
   a. $K_0^M(F) = K_0(F) = \mathbb{Z}$;
   b. $K_1^M(F) = K_1(F) = F^\times$;
   c. $K_2^M(F) = K_2(F)$.

4. $\mathbb{H}^{2n}_{\mathrm{Zar}}(X, \mathbb{Z}(n)) = \mathrm{CH}^n(X)$ (lecture 17 of [MVW06], page 135), where the nth classical Chow group $\mathrm{CH}^n(X)$ is the free group given by
$$\mathrm{CH}^n(X) = \mathbb{Z}\{\text{cycles of codimension } n\}/\text{rational equivalences};$$

[1] Here we use the convention that the (hyper)cohomology of F should be interpreted as of $\mathrm{Spec}(F)$, the corresponding space.

5. there is a natural Atiyah–Hirzebruch spectral sequence
$$E_2^{p,q} = \mathbb{H}^p_{\text{Zar}}(X, \mathbb{Z}(q)) \Rightarrow K_{2q-p}(X).$$
Moreover, tensoring with $\mathbb{Q}$, the spectral sequence degenerates and one has
$$\mathbb{H}^i_{\text{Zar}}(X, \mathbb{Z}(n))_{\mathbb{Q}} = \text{gr}^n_\gamma(K_{2n-i}(X)_{\mathbb{Q}})$$
where gr^n_γ's are the quotients (graded pieces) of γ-filtration. ([Lev94]; [Lev99], Theorem 11.7)

Remark 0.3. Such choice of complexes $\mathbb{Z}(q)$ exists, and is called the motivic complex. For a clear definition of these complexes, see Lecture 3 of [MVW06]. Moreover, by convention $\mathbb{Z}(q) = 0$ for $q < 0$.

Definition 0.4. The motivic cohomology of X is defined by $H^{p,q}(X, \mathbb{Z}) = \mathbb{H}^p_{\text{Zar}}(X, \mathbb{Z}(q))$, the hypercohomology of the motivic complexes with respect to the Zariski topology.

Remark 0.5. In general, a motivic cohomology with coefficient in an abelian group A is a family of contravariant functors $H^{p,q}(-, A) : \text{Sm}/k \to \text{Ab}$.

Remark 0.6. The motivic cohomology of X satisfies the cancellation property: set $\mathbb{G}_m = \mathbb{A}^1 \backslash \{0\}$, then
$$H^{p,q}(X \times \mathbb{G}_m, \mathbb{Z}) = H^{p,q}(X, \mathbb{Z}) \oplus H^{p-1,q-1}(X, \mathbb{Z}).$$

Remark 0.7. It turns out that the group remains unchanged if we replace the Zariski topology by Nisnevich topology.[2] If one uses étale topology instead, we retrieve Lichtenbaum motivic cohomology $H^{p,q}_L(X, \mathbb{Z})$. If $\text{char}(k) \nmid n$, it admits the comparison
$$H^{p,q}_L(X, \mathbb{Z}/n\mathbb{Z}) = H_{\text{étale}}(X, \mathbb{Z}/n\mathbb{Z}(q)),$$
where $\mathbb{Z}/n\mathbb{Z}(q)$ is the q-twist $\mu_n^{\otimes q}$.

We may compare Lichtenbaum motivic cohomology with motivic cohomology by the following theorem, formerly known as Beilinson-Lichtenbaum Conjecture[3]:

Theorem 0.8 ([Voe11])**.** The natural map
$$H^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) \to H^{p,q}_L(X, \mathbb{Z}/n\mathbb{Z})$$
is an isomorphism if $p \leqslant q$, is a monomorphism if $p = q + 1$, and gives a spectral sequence for any pair of p, q.

Corollary 0.9. For $p \leqslant q$, we have
$$H^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) = H^p_{\text{étale}}(X, \mathbb{Z}/n\mathbb{Z}(q)).$$

In particular, for $X = \text{Spec}(k)$ as a point, this is the theorem formerly known as Milnor conjecture:

Corollary 0.10 ([Voe97], [Voe03a], [Voe03b])**.**

- $H^{p,p}(k, \mathbb{Z}/n\mathbb{Z}) = K^M_p(k)/n = H^p_{\text{étale}}(X, \mathbb{Z}/n\mathbb{Z}(p))$ as the Galois cohomology;
- in general,
$$H^{p,q}(k, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} 0, & p > q \\ H^{p,p}(k, \mathbb{Z}/n\mathbb{Z}) \cdot \tau^{q-p}, & p \leqslant q \end{cases}$$
where $\tau \in \mu_n(k) = H^{0,1}(k, \mathbb{Z})$ is a primitive nth root of unity.

Remark 0.11. Unlike the case with finite coefficients, $H^{p,q}(k, \mathbb{Z})$ is quite hard to compute for small $p < q$; for $p \geqslant q$, this is 0.

[2] Recall that the Nisnevich topology is a Grothendieck topology on the category of schemes that is finer than the Zariski topology but coarser than the étale topology.

[3] This is also known as the norm residue isomorphism theorem, or (formerly) Bloch-Kato conjecture.

A current long-standing conjecture is

Conjecture 0.12 (Beilinson-Soulé Vanishing Conjecture, [Lev93])**.** $H^{p,q}(k, \mathbb{Z}) = 0$ if $p < 0$.

Remark 0.13. Here are a few known cases:

- for $\operatorname{char}(k) = 0$, this is known for number fields ([Bor74]), function fields of genus 0 ([Dég08]), curves over number fields, and their inductive limits (more precise references required); ([DG05])
- for $\operatorname{char}(k) > 0$, this is known for finite fields ([Qui72]) and global fields ([Har77]).

Remark 0.14. The motivic cohomology could be realized in a tensor triangulated category, namely the category of effective motives $\mathrm{DM}(k)$. For any pair of p, q, we can find an Eilenberg-Maclane space and a corresponding representable functor so that

$$H^{p,q}(X, \mathbb{Z}) = \mathrm{Hom}_{DM}(\mathbb{Z}(X), \mathbb{Z}(q)[p])$$

where $\mathbb{Z}(X)$ is the motive of X and $\mathbb{Z}(q)[p] = \mathbb{G}_m^{\wedge q}[p-q]$.[4]

Remark 0.15. Dually, we can define the motivic homology by

$$H_{p,q}(X, \mathbb{Z}) = \mathrm{Hom}_{DM}(\mathbb{Z}(q)[p], \mathbb{Z}(X)).$$

Remark 0.16 ([MVW06] Properties 14.5, page 110)**.** By taking the hom functor from the aspect of motives, we can derive theorems for all (co)homologies which can be represented in DM. The main derives are the following:

1. If $E \to X$ is an $\mathbb{A}^n$-bundle, then motives $\mathbb{Z}(E) = \mathbb{Z}(X)$ in DM.
2. If $\{U, V\}$ is a Zariski open covering of X, we have a Mayer-Vietoris sequence

$$\mathbb{Z}(U \cap V) \longrightarrow \mathbb{Z}(U) \oplus \mathbb{Z}(V) \longrightarrow \mathbb{Z}(X) \longrightarrow \mathbb{Z}(U \cap V)[1]$$

   in the form of a distinguished triangle in DM.
3. If $Y \subseteq X$ is a closed embedding of codimension c in Sm/k, then we have a Gysin triangle

$$\mathbb{Z}(X \backslash Y) \longrightarrow \mathbb{Z}(X) \longrightarrow \mathbb{Z}(Y)(c)[2c] \longrightarrow \mathbb{Z}(X \backslash Y)[1]$$

   which is a distinguished triangle where $\mathbb{Z}(Y)(c)[2c] := \mathbb{Z}(Y) \otimes \mathbb{Z}(c)[2c]$.
4. For any vector bundle of rank n on X, we have the projective bundle formula

$$\mathbb{Z}(\mathbb{P}(E)) = \bigoplus_{i=0}^{n} \mathbb{Z}(X)(i)[2i]$$

   which defines the Chern class of E.
5. Let X be a proper smooth scheme and let d_X be its dimension, then $\mathbb{Z}(X)$ has a strong dual $\mathbb{Z}(X)(-d_X)[-2d_X]$ in DM by stabilization. This gives a Poincaré duality[5]

$$H^{p,q}(X, \mathbb{Z}) \cong H_{2d_X - p, d_X - q}(X, \mathbb{Z}).$$

[4] Again, this notation goes back to the concise definition of the motivic complexes: see Lecture 3 from [MVW06] as well as the concept of presheaves with transfers.

[5] We can use cohomology with compact support for this.

1 Intersection Theory

1.1 Cycles of Scheme

Definition 1.1. Let X be a scheme of finite type over k. We define the ith cycle on the scheme X to be a free abelian group

$$Z_i(X) = \bigoplus_{\substack{\text{irreducible closed } c\subseteq X\\ \text{with } \dim(c)=i}} \mathbb{Z}\cdot c$$

and set $Z(X) = \bigoplus_i Z_i(X)$. Define a set $K_i(X)$ to be the set of coherent sheaves $\mathcal{F}$ on X with $\dim(\operatorname{supp}(F)) \leqslant i$.[6]

Remark 1.2. Let $(A, \mathfrak{m})$ be a Noetherian local ring and M be an A-module, then by the dimension theorem, we know $\dim(M) = d(M) = \dim(\operatorname{supp}(M))$, where $d(M)$ is the degree of the Hilbert-Samuel polynomial $P_{\mathfrak{m}}(M, n)$.

Definition 1.3. Let $X \in \mathrm{Sm}/k$ and let $U, V \subseteq X$ be irreducible and closed. Suppose $W \subseteq U \cap V$ is a irreducible and closed component. If $\dim(W) = \dim(U) + \dim(V) - \dim(X)$, i.e., $\operatorname{codim}(W) = \operatorname{codim}(U) + \operatorname{codim}(V)$, we say that U and V intersect properly at W.

Remark 1.4. This condition is weaker than saying they intersect transversely: we do not require information about tangent spaces.

Theorem 1.5. Let $A \supseteq k$ be a Noetherian regular ring, M, N be finitely-generated A-modules, and suppose $\ell(M\otimes_A N) < \infty$, then

1. $\ell(\operatorname{Tor}_i^A(M, N)) < \infty$ for all $i \geqslant 0$;
2. the Euler-Poincaré characteristic $\chi(M, N) := \sum_{i=0}^{\dim(A)} (-1)^i \ell(\operatorname{Tor}_i^A(M, N)) \geqslant 0$;
3. by Remark 1.2, we have $\dim(M) + \dim(N) \leqslant \dim(A)$;
4. in particular, we have $\dim(M) + \dim(N) < \dim(A)$ if and only if $\chi(M, N) = 0$.

Proof. See [Ser12], page 106. □

Remark 1.6. Part 3. from Theorem 1.5 implies that $\dim(W) \geqslant \dim(U) + \dim(V) - \dim(X)$, i.e., $\operatorname{codim}(W) \leqslant \operatorname{codim}(U) + \operatorname{codim}(V)$ in the notation of Definition 1.3.

Definition 1.7. Let X, U, V, W be as in Definition 1.3, then we define the intersection multiplicity $m_W(U, V)$ of U and V at W by

$$m_W(U, V) = \chi^{\mathcal{O}_{X,W}}(\mathcal{O}_{X,W}/P_U, \mathcal{O}_{X,W}/P_V)$$

where P_U and P_V are prime ideals defining U and V, respectively.

Remark 1.8. By Theorem 1.5, we know $m_W(U, V) \geqslant 0$, and $m_W(U, V) = 0$ if and only if U and V do not intersect properly at W.

1.2 Intersection Product and Cross Product

Definition 1.9. Let $X \in \mathrm{Sm}/k$, and let $U \in Z_a(X)$ and $V \in Z_b(X)$. If U and V intersect properly at every component, then we define the intersection product to be the cycle

$$U \cdot V = \sum_{\substack{W\subseteq U\cap V\\ \dim(W)=a+b-d_X}} m_W(U, V) \cdot W \in Z_{a+b-d_X}(X).$$

Example 1.10. Let X be a smooth projective surface, and let C and D be divisors on X. For any point $x \in C \cap D$, locally we think of $C = \{f = 0\}$ and $D = \{g = 0\}$ around x, then $m_x(C, D) = \ell_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(f, g))$.

[6] Despite the notation, this has nothing to do with a K-theory.

Definition 1.11. Suppose X is a scheme of finite type over k, and $\mathcal{F} \in K_n(X)$ is a coherent sheaf, then we define $Z_a(\mathcal{F}) = \sum_{\dim(\bar{\eta})=a} (\mathcal{O}_{X,\eta}(\mathcal{F}_\eta) \cdot \bar{\eta}) \in Z_a(X)$.

Therefore, we define the cycle of F as an element of the cycle of X.

Definition 1.12 ([Har13], Exercise III.6.9)**.** Every coherent sheaf $\mathcal{F}$ on $X \in \mathrm{Sm}/k$ has a resolution

$$0 \longrightarrow E_k \longrightarrow E_{k-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

where E_i's are locally free of finite rank. Therefore, for any coherent sheaf $\mathcal{G}$, we can define the Tor functor[7] of coherent sheaves by

$$\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) = H_i(E_* \otimes_{\mathcal{O}_X} \mathcal{G}).$$

Proposition 1.13. Let $X \in \mathrm{Sm}/k$. Suppose $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(X)$ intersect properly, then

$$Z_a(\mathcal{F}) \cdot Z_b(\mathcal{G}) = \sum_{i=0}^{d_X} (-1)^i \cdot Z_{a+b-d_X}(\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F},\mathcal{G})).$$

Proof. We only have to do it locally, so we can assume X to be affine, and count the coefficients of $\bar{\xi}$ where $\dim(\xi) = a+b-d_X$. It sufficess to show that the stalks at ξ satisfies

$$\chi(F_\xi, G_\xi) = \sum_{\substack{\dim(\bar{\lambda})=a \\ \dim(\beta\eta)=b \\ \xi\in\bar{\lambda}\cap\bar{\eta}}} \ell(\mathcal{F}_\lambda) \cdot \ell(G_\eta) \cdot m_{\bar{\xi}}(\bar{\lambda},\bar{\eta}).$$

Because our ring is Noetherian, then $\mathcal{F}$ admits a filtration

$$0 = M_0 \subseteq \cdots \subseteq M_d = \mathcal{F}$$

such that $M_i/M_{i-1} \cong \mathcal{O}_X/\mathcal{I}$ is coherent for prime ideal $\mathcal{I}$. By the additivity of both sides of the isomorphism, we may assume $\mathcal{F} = \mathcal{O}_X/\mathfrak{p}$ with dimension at most a, where $\mathfrak{p} \sim \lambda \in X$. Similarly, we may assume $\mathcal{G} = \mathcal{O}_X/\mathfrak{q}$ with dimension at most b, where $\mathfrak{q} \sim \eta \in X$. Moreover, set $\xi \in \bar{\lambda} \cap \bar{\eta}$. By definition, we now have $\chi(\mathcal{F}_\xi, \mathcal{G}_\xi) = m_{\bar{\xi}}(\bar{\lambda},\bar{\eta})$.

- If $\dim(\bar{\lambda}) = a$ and $\dim(\bar{\eta}) = b$, then the equality follows from the fact that $\ell(\mathcal{F}_\lambda) = \ell(\mathcal{G}_\eta) = 1$.
- If not, then either $\dim(\bar{\lambda}) < a$ or $\dim(\bar{\eta}) < b$, then $\bar{\lambda}$ and $\bar{\eta}$ do not intersect properly at $\bar{\xi}$, so both the left-hand side and the right-hand side become 0.

□

Proposition 1.14. The intersection product is commutative.

Proof. This is obvious since the Tor functor is commutative. □

Proposition 1.15. The intersection product is associative.

Proof. Suppose we pick $\mathcal{F} \in K_a(X)$, $\mathcal{G} \in K_b(X)$, and $\mathcal{H} \in K_c(X)$ with support dimension at most a, b, c, respectively, and they intersect properly. Let L_* and M_* be free resolutions of $\mathcal{F}$ and $\mathcal{H}$, respectively. Define a double complex $N_{ij} = L_i \otimes \mathcal{G} \otimes M_j$, then the associativity of tensor product allows us to calculate triple Tor

$$H_i(L_i \otimes H_j(\mathcal{G}) \otimes M_j)) \cong \mathrm{Tor}_i(\mathcal{F},\mathcal{G},\mathcal{H}) \cong H_i(H_j(L_i \otimes \mathcal{G}) \otimes M_j)$$

as the homology of two (tensor) double complexes. We obtain two spectral sequences

$${}^{I}E^2_{p,q} = \mathrm{Tor}_p(\mathcal{F}, \mathrm{Tor}_q(\mathcal{G},\mathcal{H})) \Rightarrow \mathrm{Tor}_{p+q}(\mathcal{F},\mathcal{G},\mathcal{H})$$

[7] Since we are working over sheaves of $\mathcal{O}_X$-modules, using the same argument on the level of modules shows that the Tor functor is independent from the choice of resolution.

$$^{II}E^2_{p,q} = \mathrm{Tor}_p(\mathrm{Tor}_q(\mathcal{F},\mathcal{G}),\mathcal{H}) \Rightarrow \mathrm{Tor}_{p+q}(\mathcal{F},\mathcal{G},\mathcal{H}).$$

Recall Euler-Poincaré characteristic is invariant with respect to taking spectral sequence $(*)$, then

$$\begin{aligned}
Z_a(\mathcal{F}) \cdot ((Z_b\mathcal{G}) \cdot Z_c(\mathcal{H})) &= Z_a(\mathcal{F}) \cdot \sum_q (-1)^q Z_{b+c-d_X}(\mathrm{Tor}_q(\mathcal{G},\mathcal{H})) \text{ by Proposition 1.13} \\
&= \sum_{p,q} (-1)^{p+q} Z_{a+b+c-2d_X}(^{I}E^2_{p,q}) \text{ by Proposition 1.13} \\
&= \sum_i (-1)^i Z_{a+b+c-2d_X}(\mathrm{Tor}_i(\mathcal{F},\mathcal{G},\mathcal{H})) \text{ by } (*) \\
&= \sum_{p,q} (-1)^{p+q} Z_{a+b+c-2d_X}(^{II}E^2_{p,q}) \text{ by } (*) \\
&= \sum_p Z_{a+b-d_X}(\mathrm{Tor}_p(\mathcal{F},\mathcal{G})) \cdot Z_c(\mathcal{H}) \text{ by Proposition 1.13} \\
&= (Z_a(\mathcal{F}) \cdot Z_b(\mathcal{G})) \cdot Z_c(\mathcal{H}) \text{ by Proposition 1.13.}
\end{aligned}$$

□

Definition 1.16. Suppose $X_1, X_2 \in \mathrm{Sm}/k$, with $\mathcal{F}_1 \in K_{a_1}(X_1)$ and $\mathcal{F}_2 \in K_{a_2}(X_2)$. We define the cross product of cycles to be

$$Z_{a_1}(\mathcal{F}_1) \times Z_{a_2}(\mathcal{F}_2) = Z_{a_1+d_{X_2}}(p_1^*\mathcal{F}_1) \cdot Z_{a_2+d_{X_1}}(p_2^*\mathcal{F}_2),$$

where $p_i : X_1 \times X_2 \to X_i$ is the projection for $i = 1, 2$.

Exercise 1.17. One should check that this is well-defined.

Remark 1.18. Suppose $X_1, X_2 \in \mathrm{Sm}/k$, with $\mathcal{F}_1 \in K_{a_1}(X_1)$, $\mathcal{F}_2 \in K_{b_1}(X_1)$, $\mathcal{G}_1 \in K_{a_2}(X_2)$ and $\mathcal{G}_2 \in K_{a_2}(X_2)$. Suppose $Z_{a_1}(\mathcal{F}_1) \cdot Z_{a_2}(\mathcal{G}_1)$ and $Z_{b_1}(\mathcal{F}_2) \cdot Z_{b_2}(\mathcal{G}_2)$ are defined, then

- $Z_{a_1}(\mathcal{F}_1) \times Z_{a_2}(\mathcal{G}_1)$ and $Z_{b_1}(\mathcal{F}_2) \times Z_{b_2}(\mathcal{G}_2)$ intersect properly on $X_1 \times X_2$, and
- $(Z_{a_1}(\mathcal{F}_1) \times Z_{a_2}(\mathcal{G}_1)) \cdot (Z_{b_1}(\mathcal{F}_2) \times Z_{b_2}(\mathcal{G}_2)) = (Z_{a_1}(\mathcal{F}_1) \cdot Z_{b_1}(\mathcal{F}_2)) \times (Z_{a_2}(\mathcal{G}_1) \cdot Z_{b_2}(\mathcal{G}_2))$.

1.3 Pushout and Pullback

Definition 1.19. Suppose X, Y are schemes of finite type over k, and let $f : X \to Y$ be a proper map. For every irreducible closed subset $c \subseteq X$ of dimension a, we define the direct image to be

$$f_*c = \begin{cases} [k(c) : k(f(c))] \cdot f(c) \in Z_a(Y), & \dim(f(c)) = a \\ 0, & \dim(f(c)) < a \end{cases}$$

to be the direct image of c under f.

Lemma 1.20. Suppose X and Y are schemes of finite type over k of the same dimension n, and that $f : X \to Y$ is proper, then there exists an open subset $U \subseteq Y$ such that $\dim(Y \backslash U) < n$ and $f : f^{-1}(U) \to U$ is a finite morphism.

Proof. Suppose $\xi \in Y$ has $\dim(\bar{\xi}) = n$. We can find $U \ni \xi$ such that $f|_U$ has finite fibers by Exercise II.3.7 from [Har13]. By Exercise III.11.2 in [Har13], such f is finite. □

Proposition 1.21. Let $f : X \to Y$ be a proper morphism between schemes over k of finite type, and let $\mathcal{F} \in K_a(X)$, then

1. $f_*\mathcal{F} \in K_a(Y)$ and the right derived $R^i f_*\mathcal{F} \in K_{a-1}(Y)$ for $i > 0$.
2. $f_*Z_a(\mathcal{F}) = Z_a(f_*\mathcal{F})$.

Proof. 1. By Theorem III.8.8 from [Har13], $R^i f_* \mathcal{F}$ is coherent for all $i \geqslant 0$. We have $\operatorname{supp}(R_i f_* \mathcal{F}) \subseteq \operatorname{supp}(\mathcal{F})$. If f is finite, then f_* is exact, so $R^i f_* \mathcal{F} = 0$ for $i > 0$. For general cases, we may assume $\dim(f(\operatorname{supp}(\mathcal{F}))) = a$ and set $W = \operatorname{supp}(\mathcal{F})$. We have a commutative diagram

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, "i"'] & f(W) \arrow[d, "j"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where h is also proper. By Lemma 1.20, there exists $V \subseteq f(W)$ such that $\dim(f(W) \backslash V) < a$ and $h|_V$ is finite. Let $\mathcal{J}$ be the ideal sheaf of W, then $\mathcal{J}^s \mathcal{F} / \mathcal{J}^{s+1} \mathcal{F} = i_* i^* \mathcal{J}^s \mathcal{F} / \mathcal{J}^{s+1} \mathcal{F}$. By the long exact sequence, it suffices to prove the case for $\mathcal{F} = i_* \mathcal{G}$. Then

$$(R^k f_*) i_* \mathcal{G} = R^k (fi)_* \mathcal{G} = j_* R^k h_* \mathcal{G}.$$

It suffices to consider h, but

$$(R^k h_* \mathcal{G}) V = R^k h(\mathcal{G}|_{f^{-1}(V)}) = 0$$

for $k > 0$, so $\operatorname{supp}(R^k h_* \mathcal{G}) \subseteq f(W) \backslash V$ if $k > 0$.

2. If f is finite, let us write down the coefficients of ξ of dimension a on both sides, namely

$$\ell((f_* \mathcal{F})_\xi) = \sum_{\substack{\eta \in f^{-1}(\xi) \\ \dim(\overline{\eta}) = a}} \ell(F_\eta) \cdot [k(\overline{\eta}) : k(\overline{f(\eta)})].$$

By additivity, one reduces to the case when X is affine and $F = \mathcal{O}_X / \mathfrak{p}$. For the general case, use Lemma 1.20, and the case where f is finite.

□

Definition 1.22. Suppose $f : X \to Y$ where $Y \in \mathrm{Sm}/k$ and X is closed in $Z \in \mathrm{Sm}/k$. Define $j : X \to Z \times Y$ to be the graph map. For any $C \in Z_a(X)$ and $D \in Z_b(Y)$ such that C and $f^{-1}(D)$ intersect properly, define the intersection cycle to be

$$C \cdot_f D = j_*^{-1}(j(C) \cdot (Z \times D)) \in Z_{a+b-d_Y}(X)$$

In particular, $f^*(D) = X \cdot_f D$ for $C = X$.

Proposition 1.23. Using the notation above, for $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(Y)$, if $\mathcal{F}$ and $f^* \mathcal{G}$ intersect properly, we have

$$Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G}) = \sum_{i=0}^{d_Y} (-1)^i Z_{a+b-d_Y}(L_i(\mathcal{F} \otimes f^*)\mathcal{G})$$

Proof. Denote $p_2 : Z \times Y \to Y$ to be the projection onto the second coordinate. By linearity, $Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G}) = j_*^{-1}(Z_a(j_* \mathcal{F}) \cdot Z_{b+d_Z}(p_2^* \mathcal{G})$ for $j : X \to Z \times Y$. Suppose $L_* \to \mathcal{G}$ is the locally free resolution of $\mathcal{G}$. Note that for all $i \geqslant 0$, we have

$$j^*(j_* \mathcal{F} \otimes p_2^* L_i) + F \otimes f^* L_i,$$

which induces an isomorphism

$$j_* \mathcal{F} \otimes p_2^* L_i = j_*(\mathcal{F} \otimes f^* L_i).$$

Hence $\operatorname{Tor}_i^{\mathcal{O}_{Z \times Y}}(j_* \mathcal{F}, p_2^* \mathcal{G}) = j_* L_i(F \otimes f^*) \mathcal{G}$. So

$$j_*^{-1} Z_{a+b-d_Y}(\operatorname{Tor}_i^{\mathcal{O}_{Z \times Y}}(j_* \mathcal{F}, p_2^* \mathcal{G})) = Z_{a+b-\dim(Y)}(L_i(F \otimes f^*) \mathcal{G}).$$

Therefore the statement follows. □

Proposition 1.24. Let $X \in \mathrm{Sm}/k$, $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(X)$ such that $\mathcal{F}$ and $\mathcal{G}$ intersect properly. Let $\Delta : X \to X \times X$ be the diagonal map, then

$$\Delta^*(Z_a(\mathcal{F}) \times Z_b(\mathcal{G})) = Z_a(\mathcal{F}) \cdot Z_b(\mathcal{G}).$$

Proof. See page 115 of [Ser12]. $\square$

Proposition 1.25. f^* is compatible with intersection product, and $f^*g^* = (gf)^*$.

Proof. See page 119 of [Ser12]. $\square$

Lemma 1.26. Let $\mathcal{A}$ be an abelian category with enough projectives (respectively, injectives) and F be a right (respectively, left) exact functor from $\mathcal{A}$. Suppose C is chain complex in $\mathcal{A}$, then there exists a double complex $M_{*,*}$ in $\mathcal{A}$ such that

$$^{I}E^2_{p,q} = L_p F H_q(C) \quad (\text{respectively, } R^{-p}F(H_q(C))).$$

Proof. To do this when F is right exact, use the Cartan-Eilenberg resolution[8] $C_* \to C$ and consider the double complex FC_*. $\square$

Proposition 1.27. Suppose $f : X \to Y$ is in Sm/k, suppose $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(Y)$, then

$$Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G}) = Z_a(\mathcal{F}) \cdot f^* Z_b(\mathcal{G})$$

if both sides are defined.

Proof. We may assume X is affine. Let $L_* \to \mathcal{G}$ be a free resolution and apply Lemma 1.26 to f^*L_* and $F \otimes -$, then we find a double complex such that

$$\begin{aligned} ^{I}E^2_{p,q} &= \mathrm{Tor}_p(\mathcal{F}, L_q f^*\mathcal{G}) \\ ^{II}E^2_{p,q} &= L_p(F \otimes f^*)\mathcal{G}. \end{aligned}$$

$\square$

Proposition 1.28. Let $X \subseteq Z$ and $Y, Z \in \mathrm{Sm}/k$ and $f : X \to Y$ be proper. Suppose $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(Y)$, and suppose $\mathcal{F}$ and $f^*\mathcal{G}$ intersect properly, then

$$f_*(Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G})) = (f_* Z_a(\mathcal{F})) \cdot Z_b(\mathcal{G}).$$

Proof. Pick $L_* \to \mathcal{G}$ to be a resolution and apply Lemma 1.26 to $F \otimes f^*L_*$ and f_*, then we have a double complex $M_{*,*}$ such that

$$^{I}E^2_{p,q} = R^{-p} f_* L_q(F \otimes f^*)\mathcal{G}).$$

On the other hand, $H_q(M_{*,n}) = R^{-q} f_*(F \otimes f^* L_n) = (R^{-q} f_* \mathcal{F}) \otimes L_n.$, therefore

$$^{II}E^2_{p,q} = \mathrm{Tor}_p(R^{-q} f_* \mathcal{F}, \mathcal{G}).$$

$\square$

Corollary 1.29. Under the same hypothesis as Proposition 1.28, we have

$$f_*(Z_a(\mathcal{F}) \cdot f^*(Z_b(\mathcal{G}))) = f_*(Z_a(\mathcal{F})) \cdot Z_b(\mathcal{G}).$$

[8] See Proposition 11 on page 210 of [GM13].

2 Sheaves with Transfers

We fix a base scheme $S \in \mathrm{Sm}/k$.

2.1 Algebra of Correspondences

Definition 2.1. Let $X, Y \in \mathrm{Sm}/S$, then we define the group of finite correspondences

$$\mathrm{Cor}_S(X,Y) = \mathbb{Z}\{\text{irreducible closed } C \subseteq X \times_S Y \mid C \to X \text{ finite}, \dim(C) = \dim(X)\}$$

to be the free abelian group generated by elementary correspondences from X to Y.

Example 2.2. For any $f : X \to Y$, the graph $\Gamma_f = (x, f(x)) \subseteq X \times_S Y$ is a finite correspondence from $X \to Y$.

Example 2.3. If $f : X \to Y$ is finite and $\dim(X) = \dim(Y)$, then the graph Γ_f is also a finite correspondence from $Y \to X$.

Definition 2.4. Define an additive category Cor_S whose objects are the same as Sm/S, and the hom sets defined as $\mathrm{Hom}_{\mathrm{Sm}/S}(X,Y) = \mathrm{Cor}_S(X,Y)$ as in Definition 2.1. The contravariant additive functors

$$F : \mathrm{Cor}_S^{\mathrm{op}} \to \mathrm{Ab}$$

are called presheaves with transfers. The corresponding category is denoted by $\mathrm{PSh}(S) = \mathrm{PSh}(\mathrm{Cor}_S)$, which is abelian with enough injectives and projectives. We have a functor $\gamma : \mathrm{Sm}/S \to \mathrm{Cor}_S$ by Example 2.3.

Remark 2.5. For any additive F and $X, Y \in \mathrm{Sm}/S$, there is a pairing

$$\mathrm{Cor}_S(X,Y) \otimes F(Y) \to F(X).$$

Restricting to Sm/S over Cor_S, we note that F is a presheaf of abelian groups over Sm/S with transfer map $F(Y) \to F(X)$ indexed by finite correspondences from X to Y.

Example 2.6. Every $X \in \mathrm{Sm}/S$ gives an element $\mathbb{Z}(X) \in \mathrm{PSh}(S)$ defined by $\mathbb{Z}(X)(Y) = \mathrm{Cor}_S(Y,X)$. Therefore, we say $\mathbb{Z}(X)$ is the presheaf with transfers represented by X. By Yoneda Lemma we know there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{PST}}(\mathbb{Z}(X), F) \cong F(X).$$

Moreover, representable functors give embeddings of Sm/S and Cor_S into $\mathrm{PSh}(S)$ via

$$\begin{array}{ccccc} \mathrm{Sm}/S & \xrightarrow{\gamma} & \mathrm{Cor}_S & \longrightarrow & \mathrm{PSh}(S) \\ X & \longmapsto & X & \longmapsto & \mathbb{Z}(X) \end{array}$$

In particular, $\mathbb{Z}(S) = \mathbb{Z}$.

Example 2.7. The presheaves $\mathcal{O}$ and $\mathcal{O}^*$ are in $\mathrm{PSh}(S)$. For any $C \in \mathrm{Cor}_S(X,Y)$ and $f \in \mathcal{O}(Y)$ (respectively, $\mathcal{O}^*(Y)$), we have a diagram

$$\begin{tikzcd} C \arrow[r, "i"] & X \times_S Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \\ & X & \end{tikzcd}$$

and can define $\mathcal{O}(C)(f) = \mathrm{Tr}_{C/X}((p_2 \circ i)^*(f))$ (respectively, $\mathcal{O}^*(C)(f) = \mathrm{N}_{C/X}((p_2 \circ i)^*(f))$).

We study the properties of finite correspondence through Chapter 16.1 in [Ful13].

Definition 2.8. Let us describe the composition in Cor_S. Suppose $f \in \mathrm{Cor}_S(X, Y)$ and $g \in \mathrm{Cor}_S(Y, Z)$, then from the diagram

$$\begin{tikzcd} X \times_S Z & \\ X \times_S Y \times_S Z \arrow[u, "p_{13}"] \arrow[r, "p_{23}"] \arrow[d, "p_{12}"] & Y \times_S Z \\ X \times_S Y & \end{tikzcd}$$

we define the composition $g \circ f = p_{13*}(p_{23}^*(g)p_{12}^*(f))$.

Exercise 2.9. One should check that all intersections are proper.

Remark 2.10. Using this language, given a correspondence $\alpha \in \mathrm{Cor}_S(X, Y)$, we can define pullbacks and pushouts on the cycles as homomorphisms

$$\begin{aligned} \alpha_* : Z(X) &\to Z(Y) \\ x &\mapsto p_{Y*}^{XY}(\alpha \cdot p_X^{XY*}(x)) \end{aligned}$$

and

$$\begin{aligned} \alpha^* : Z(Y) &\to Z(X) \\ y &\mapsto p_{X*}^{XY}(\alpha \cdot p_Y^{XY*}(y)) \end{aligned}$$

Remark 2.11 ([Ful13], Proposition 1.7, Base-change Formula)**.** Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

be a fiber square where f is proper and g is flat, then f' is proper and g' is flat, and that $f'_* g'^* = g^* f_*$ over Y'.

Proposition 2.12 ([Ful13], Proposition 16.1.1)**.** The composition law is associative.

Proof. Suppose

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

are morphisms in Cor_S, then we have two Cartesian squares

$$\begin{tikzcd} X \times_S Y \times_S Z \times_S W \arrow[r] \arrow[d] & X \times_S Z \times_S W \arrow[d] \\ X \times_S Y \times_S Z \arrow[r] & X \times_S Z \end{tikzcd}$$

and

$$\begin{tikzcd} X \times_S Y \times_S Z \times_S W \arrow[r] \arrow[d] & X \times_S Y \times_S W \arrow[d] \\ Y \times_S Z \times_S W \arrow[r] & Y \times_S W \end{tikzcd}$$

Now using the base-change formula, we know

$$\begin{aligned} h \circ (g \circ f) &= p_{XW*}^{XZW}(p_{ZW}^{XZW*}(h) p_{XZ}^{XZW*} p_{XZ*}^{XYZ}(p_{YZ}^{XYZ*}(g) p_{XY}^{XYZ*}(f))) \\ &= p_{XW*}^{XZW}(p_{ZW}^{XZW*}(h) p_{XZW*}^{XYZW} p_{XYZ}^{XYZW*}(p_{YZ}^{XYZ*}(g) p_{XY}^{XYZ*}(f))) \\ &= p_{XW*}^{XZW}(p_{ZW}^{XZW*}(h) p_{XZW*}^{XYZW}(p_{YZ}^{XYZW*}(g) p_{XY}^{XYZW*}(f))) \end{aligned}$$

$$
\begin{aligned}
&= p^{XZY}_{XW*} p^{XYZW}_{XZW*}(p^{XYZW*}_{ZW}(h) p^{XYZW*}_{YZ}(g) p^{XYZW}_{XY}(f)) \\
&= p^{XYW}_{XW*} p^{XYZW}_{XYW*}(p^{XYZW*}_{ZW}(h) p^{XYZW*}_{YZ}(g) p^{XYZW}_{XY}(f)) \\
&= p^{XYW}_{XW*}(p^{XYZW}_{XYW*}(p^{XYZW*}_{ZW}(h) p^{XYZW*}_{YZ}(g)) p^{XYW*}_{XY}(f)) \\
&= p^{XYW}_{XW*}(p^{XYZW}_{XYW*} p^{XYZW*}_{YZW}(p^{YZW*}_{ZW}(h) p^{YZW}_{YZ}(g)) p^{XYW*}_{XY}(f)) \\
&= p^{XYW}_{XW*}(p^{XYW*}_{YW} p^{YZW}_{YW*}(p^{YZW*}_{ZW}(h) p^{YZW}_{YZ}(g)) p^{XYW*}_{XY}(f)) \\
&= p^{XYW}_{XW*}(p^{XYW*}_{YW} p^{YZW}_{YW*}(p^{YZW*}_{ZW}(h) p^{YZW}_{YZ}(g)) p^{XYW*}_{XY}(f)) \\
&= h \circ g \circ f.
\end{aligned}
$$

□

Theorem 2.13. We have $\mathcal{O}(g \circ f) = \mathcal{O}(f) \circ \mathcal{O}(g)$ and $\mathcal{O}^*(g \circ f) = \mathcal{O}^*(f) \circ \mathcal{O}^*(g)$.

Proof. We sketch the proof for $\mathcal{O}$. Pick $X \in \mathrm{Sm}/k$. For every $a \in \mathbb{N}$, define $\mu_a(x) = \bigoplus_{\dim(\bar{V})=a} K(V)$. Therefore, we have a pairing

$$
\begin{aligned}
\mathcal{O}(X) \times Z_a(X) &\to \mu_a(X) \\
(s, V) &\mapsto s|_V
\end{aligned}
$$

by restricting the regular function on the closed subset. For any map $f: X \to Y$ where X contains irreducible and closed C, suppose C is finite over Y and $s \in K(C)$, then we define $f_*(s) = \mathrm{Tr}_{K(C)/K(f(C))}(s)$.[9] Therefore, for any finite correspondence $C \in \mathrm{Cor}(Y, X)$ and $s \in \mathcal{O}(X)$, we have

$$
\begin{tikzcd}
C \arrow[r, hook] & X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \\
& X &
\end{tikzcd}
$$

and thus $\mathcal{O}(C)(s) = p_{2*}(p_1^*(s)|_C)$.

Now suppose we have closed subsets $C \subseteq X$ and $D \subseteq Y$, with

$$
\begin{tikzcd}
X \arrow[r, "f"] & Y \\
C \arrow[u] \arrow[ur, "\text{finite}"'] &
\end{tikzcd}
$$

and that C and $f^{-1}(D)$ intersect properly, then one can show that

$$f_*(s|_C)|_D = f_*(s|_{C \cdot_f D})$$

by Tor formula. Moreover, for diagrams like

$$
\begin{tikzcd}
X \times_S Y \times_S Z \arrow[r, "p_{23}"] \arrow[d, "p_{12}"'] & YZ \arrow[d, "p_1"] \\
X \times_S Y \arrow[r, "p_2"'] & Y
\end{tikzcd}
$$

where $C \subseteq YZ$ and C is finite over Y, then one can show that for all $s \in \mathcal{O}(Y \times_S Z)$ and C finite over Y, we have

$$p_2^* p_{1*}(s|_C) = p_{12*}(p_{23}^*(s)|_{p_{23}^*(C)}).$$

We finish the proof by working with formal calculation. □

Remark 2.14 ([Ful13], Proposition 16.1.2)**.** For $\alpha \in \mathrm{Cor}_S(X, Y)$ and $\beta \in \mathrm{Cor}_S(Y, Z)$, we have

$$(\beta \circ \alpha)_* = \beta_* \circ \alpha_*$$

and

$$(\beta \circ \alpha)^* = \alpha^* \circ \beta^*.$$

[9] Here $f(C)$ is closed since f is finite.

2.2 Operations on Presheaves with Transfers

Definition 2.15. Suppose $\mathcal{F}_1, \mathcal{F}_2, \mathcal{G} \in \mathrm{PSh}(S)$ be presheaves with transfers. A bilinear function $\varphi : \mathcal{F}_1 \times \mathcal{F}_2 \to \mathcal{G}$ is a collection of bilinear maps

$$\varphi_{x_1,x_2} : \mathcal{F}_1(x_1) \times \mathcal{F}_2(x_2) \to \mathcal{G}(x_1 \times_S x_2)$$

for every $x_1, x_2 \in \mathrm{Sm}/S$ any any morphisms $f_i \in \mathrm{Cor}_S(x_i, x_i')$ for $i = 1, 2$, such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{F}_1(x_1') \times \mathcal{F}_2(x_2) \arrow[r, "\varphi_{x_1',x_2}"] \arrow[d, "\mathcal{F}_1(f_1)\times\mathrm{id}"'] & \mathcal{G}(x_1' \times_S x_2) \arrow[d, "(f_1\times\mathrm{id})"] \\
\mathcal{F}_1(x_1) \times \mathcal{F}_2(x_2) \arrow[r, "\varphi_{x_1,x_2}"'] & \mathcal{G}(x_1 \times_S x_2)
\end{tikzcd}$$

for f_1 and similarly there is a diagram that commutes for f_2.

Definition 2.16. Define the tensor product $\mathcal{F}_1 \otimes \mathcal{F}_2$ to be the presheaf such that for every $\mathcal{G}$, the hom set $\mathrm{Hom}(\mathcal{F}_1 \otimes \mathcal{F}_2, G)$ is the same as the collection of bilinear functions $\mathcal{F}_1 \times \mathcal{F}_2 \to \mathcal{G}$.

Proposition 2.17. The tensor product $\mathcal{F}_1 \otimes \mathcal{F}_2$ exists.

Proof. For every $Z \in \mathrm{Sm}/S$, define

$$(\mathcal{F}_1 \otimes \mathcal{F}_2)(Z) = \bigoplus_{X,Y \in \mathrm{Sm}/S} \mathcal{F}_1(X) \otimes_{\mathbb{Z}} \mathcal{F}_2(Y) \otimes_{\mathbb{Z}} \mathrm{Cor}_S(Z, X \times_S Y)/\sim$$

where $\sim$ is the subgroup generated by the relations $\varphi \otimes \psi(f \times \mathrm{id}_Y) \circ h = f^*(\varphi) \otimes \psi \otimes h$ where $f \in \mathrm{Cor}_S(X', X)$, $\varphi \in \mathcal{F}_1(X)$, $\psi \in \mathcal{F}_2(Y)$, $h \in \mathrm{Cor}_S(Z, X' \times Y)$, and the relations $\varphi \otimes \psi \otimes (\mathrm{id}_X \times g) \circ h = \varphi \otimes g^*(\psi) \otimes h$ where $g \in \mathrm{Cor}_S(Y', Y)$, $\varphi \in \mathcal{F}_1(X)$, $\psi \in \mathcal{F}_2(Y)$, $h \in \mathrm{Cor}_S(Z, X \times Y')$. $\square$

Definition 2.18. A pointed presheaf $(\mathcal{F}, x)$ is a split injective map given by the constant presheaf $x : \mathbb{Z} \to \mathcal{F}$ for some $\mathcal{F} \in \mathrm{PSh}(S)$. We set $\mathcal{F}^{\wedge 1} = \mathcal{F}/x$. For any two pointed presheaves $(\mathcal{F}_1, x_1)$ and $(\mathcal{F}_2, x_2)$, define $\mathcal{F}_1 \wedge \mathcal{F}_2 = (\mathcal{F}_1 \otimes \mathcal{F}_2)/((\mathcal{F}_1 \otimes x_2) \oplus (x_1 \otimes \mathcal{F}_2))$. This allows us to define $\mathcal{F}^{\wedge n}$ inductively as a cokernel, c.f., Definition 2.12 from [MVW06].

Proposition 2.19.

- $\mathbb{Z}(X) \otimes \mathbb{Z}(Y) = \mathbb{Z}(X \times Y)$;
- $\mathcal{F}^{\wedge 1} \otimes \mathcal{G}^{\wedge 1} = \mathcal{F} \wedge \mathcal{G}$.

Definition 2.20. For any $\mathcal{F} \in \mathrm{PSh}(S)$ and $X \in \mathrm{Sm}/S$, define $\mathcal{F}^{\times} \in \mathrm{PSh}(S)$ by $\mathcal{F}^{\times}(Y) = \mathcal{F}(X \times_S Y)$. For any $\mathcal{F}, \mathcal{G} \in \mathrm{PSh}(S)$, define $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}) \in \mathrm{PSh}(S)$ by $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})(x) = \mathrm{Hom}(\mathcal{F}, \mathcal{G}^{\times})$.

Proposition 2.21. We have a tensor-hom adjunction

$$\mathrm{Hom}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \cong \mathrm{Hom}(\mathcal{F}, \underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{H})).$$

2.3 Nisnevich Topology

Let us give a brief introduction to Nisnevich topology, c.f., section 3 and 4 from Chapter I of [Mil80].

Definition 2.22. Suppose $f : Y \to X$ is a morphism between schemes that are locally of finite type.

1. It is called unramified if for all $y \in Y$, the maximal ideals satisfy $\mathfrak{m}_{f(y)}\mathcal{O}_{Y,y} = \mathfrak{m}_y$, and $k(y)/k(f(y))$ is a finite separable field extension of function fields.
2. It is called étale if it is both flat and unramified.
3. It is called Nisnevich if for all $x \in X$, there is some $y \in Y$ such that $f(y) = x$, $k(y) = k(x)$, and f is étale.

Definition 2.23. A morphism $f : Y \to X$ is called a Nisnevich covering if f is Nisnevich and surjective.

Definition 2.24. Suppose $\mathcal{F} \in \mathrm{PSh}(S)$. We say that it is a Nisnevich sheaf with transfers if for any $X \in \mathrm{Sm}/S$ and Nisnevich covering $\pi : Y \to X$, the sequences

$$0 \longrightarrow \mathcal{F}(X) \xrightarrow{\pi^*} \mathcal{F}(Y) \xrightarrow{p_1^* - p_2^*} \mathcal{F}(Y \times_X Y) \xrightarrow{p_1} Y$$

$$0 \longrightarrow \mathcal{F}(X) \xrightarrow{\pi^*} \mathcal{F}(Y) \xrightarrow{p_1^* - p_2^*} \mathcal{F}(Y \times_X Y) \xrightarrow{p_2} Y$$

are exact. The category of Nisnevich sheaves with transfers is denoted by $\mathrm{Sh}(S)$.

Definition 2.25. A local ring is called Hensalian if for any monic polynomial $f \in A[t]$ such that its image $\bar{f}$ in the residue field satisfies $\bar{f} = g_0 h_0$ in $k(A)[T]$ where g_0, h_0 are monic and relatively prime, there are monic $g, h \in A[T]$ such that $\bar{g} = g_0, \bar{h} = h_0$ in the residue fields, and $f = gh$.

Example 2.26. Complete local rings are Henselian.

Theorem 2.27 ([Mil80], Theorem I.4.2)**.** Let A be a local ring, $X = \mathrm{Spec}(A)$, and $x \in X$ be the closed point, then the following are equivalent:

1. A is Henselian;
2. any finite A-algebra B is a direct product of local rings $B \cong \prod_{i \in I} B_i$, where each B_i is of the form $B_{\mathfrak{m}_i}$ for some maximal ideal $\mathfrak{m}_i$ of B;
3. if $f : Y \to X$ has finite fibers and is separated, then $Y = \coprod_{i=0}^{n} Y_i$ where $X \notin f(Y_0)$, and for $i \geqslant 1$, Y_i is finite over X and is the spectrum of a local ring;
4. if $f : Y \to X$ is étale and there exists $y \in Y$ such that $f(y) = x$ and $k(y) = k(x)$, then f has a section $s : X \to Y$ such that $f \circ s = \mathrm{id}_X$.

Now let A be a Noetherian ring and $\mathfrak{p} \in \mathrm{Spec}(A)$. Consider the set I whose elements are pairs $(B, \mathfrak{q})$, where B is a connected étale A-algebra, $\mathfrak{q} \in \mathrm{Spec}(B)$, $\mathfrak{q} \cap A = \mathfrak{p}$, i.e., $\mathfrak{q}$ lies over $\mathfrak{p}$, and $k(\mathfrak{p}) = k(\mathfrak{q})$. We say that $(B_1, \mathfrak{q}_1) \preceq (B_2, \mathfrak{q}_2)$ if there is an A-morphism $f : B_1 \to B_2$ such that $f^{-1}(\mathfrak{q}_2) = \mathfrak{q}_1$. This gives a poset structure.

Proposition 2.28. The set I is a directed set and the ring $\varinjlim_{(B,\mathfrak{q})} B = A_{\mathfrak{p}}^h$, i.e., the Henselization of $A_{\mathfrak{p}}$, is Henselian and admits the following universal property: for any Henselian A-algebra C such that $\mathfrak{m}_C \cap A = \mathfrak{p}$, there is a unique morphism $\varphi : A_{\mathfrak{p}}^h \to C$ (as a local homomorphism) such that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \\ A_{\mathfrak{p}}^h \arrow[ur, dashed, "\exists!\varphi"'] & \end{tikzcd}$$

Proof. This makes use of Lemma I.4.8 from [Mil80]. □

Let φ_X be the smallest Nisnevich site on X. Suppose X is Noetherian, pick $x \in X$, and $\mathcal{F} \in \mathrm{PSh}(\varphi_X)$. We write $\mathcal{F}_x = \mathcal{F}(\mathcal{O}_{X,x}^h) = \varinjlim_{(V,u)} \mathcal{F}(V)$ as the stalk of F at x, taking all the pairs (V, u) with étale morphism

$$V \to X$$
$$u \mapsto x$$

with $k(u) = k(x)$.

Proposition 2.29. Let

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

be a complex in $\mathrm{Sh}(\varphi_X)$. The following are equivalent:

1. the complex is exact;
2. for every $x \in X$, the complex
$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{H}_x \longrightarrow 0$$
is exact.

Proof. This mimics the idea in the usual sheaf theory with Zariski topology. To do so, we need to construct a sheafification in the sense of Nisnevich, explained as follows: suppose $\mathcal{F} \in \mathrm{PSh}(\varphi_X)$, define $\mathcal{F}^+$ as the following: for every Nisnevich covering $\{V_i\}$ of U, define
$$\mathcal{F}(U) = \{(s_i) \in \prod_i \mathcal{F}(V_i) : s_i|_{V_i \times_X V_j} = s_j|_{V_i \times_S V_j}\}.$$
Now let $\mathcal{F}^+(U) = \varinjlim_{V \supseteq U} \mathcal{F}(V)$, then $\mathcal{F}^{++}$ is a Nisnevich sheaf with the same stalks as $\mathcal{F}$, with a map $\mathcal{F} \to \mathcal{F}^{++}$.

If the complex is exact, then the sequence of stalks is also exact because the direct limit functor is exact. Conversely, if we have an exact sequence of stalks, then we prove that the given sequence is exact using the usual proof in the Zariski case. □

For any Noetherian scheme X with $\dim(X) < \infty$, we define the cochain to be
$$C^p(X) = \{Y \subseteq X \mid \mathrm{codim}(Y) \geqslant p\} = \bigoplus_{\substack{y \in X \\ \mathrm{codim}(\bar{y}) \geqslant p}} \mathbb{Z} \cdot \bar{y}.$$

Fir $\mathcal{F} \in \mathrm{Sh}(\varphi_X)$. For closed subschemes $Z \subseteq W$ of X where $Z \in C^{p+1}(X)$ and $W \in C^p(X)$, we have a long exact sequence
$$\cdots \longrightarrow H^i_Z(X, \mathcal{F}) \longrightarrow H^i_W(X, \mathcal{F}) \longrightarrow H^i_{W \backslash Z}(X \backslash Z, \mathcal{F}) \longrightarrow H^{i+1}_Z(X, \mathcal{F}) \longrightarrow \cdots$$
with supports specified as subscripts, using the exactness of
$$0 \longrightarrow \mathcal{F}_Z(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}(X \backslash Z)$$
and defining $H^i_Z = R^i\Gamma_Z(X, -) : D(X_{\text{étale}}) \to D(\mathrm{Ab})$ as the right exact functor, where
$$\Gamma_Z(X, \mathcal{F}) = \{s \in \mathcal{F}(X) \mid \mathrm{supp}(s) \subseteq Z\}$$
for closed subscheme $Z \subseteq X$. Now define $H^i(C^p(X), \mathcal{F}) = \varinjlim_{Z \in C^p(X)} H^i_Z(X, \mathcal{F})$, then
$$H^i(C^p(X)/C^{p+1}(X), \mathcal{F}) = \varinjlim_{\substack{Z \subseteq W \\ W \in C^p, Z \in C^{p+1}}} H^i_{W \backslash Z}(X \backslash Z, \mathcal{F}).$$

Taking limit with respect to pairs $Z \subseteq W$ where $W \in C^p(X)$ and $Z \in C^{p+1}(X)$, we get a long exact sequence
$$\cdots \longrightarrow H^i(C^{p+1}(X), \mathcal{F}) \longrightarrow H^i(C^p(X), \mathcal{F}) \longrightarrow H^i(C^p(X)/C^{p+1}(X), \mathcal{F}) \longrightarrow H^{i+1}(C^{p+1}(X), \mathcal{F}) \longrightarrow \cdots$$

Set the pth filtration to be $F^pH^i(X, \mathcal{F}) = \mathrm{im}(H^i(C^p(X), \mathcal{F}) \to H^i(X, \mathcal{F}))$, then we obtain the Coniveau spectral sequence
$$E_1^{p,q} = H^{p+q}(C^p(X)/C^{p+1}(X), \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

Remark 2.30. $E_1^{p,q} = 0$ if $p > \dim(X)$ and $q > 0$.

Definition 2.31. Suppose $x \in X$. Define the local cohomology
$$H^i_x(X, \mathcal{F}) = \varinjlim_{\text{open } x \in V \subseteq X} H^i_{\bar{x} \cap V}(V, \mathcal{F}).$$

This allows us to calculate $E_1^{p,q}$ as
$$E_1^{p,q} = \bigoplus_{\mathrm{codim}(\bar{x})=p} H_x^{p+q}(X,\mathcal{F}).$$

Proposition 2.32 (Étale Excision)**.** Suppose $\varphi : Y \to X$ is a étale morphism of sheaves, and suppose $Z \subseteq X$ is a closed subset such that $\varphi^{-1}(Z) = Z$. For any $\mathcal{F} \in \mathrm{Sh}(\varphi_X)$, we have
$$H_Z^i(Y,\varphi^*\mathcal{F}) = H_Z^i(X,\mathcal{F}).$$

Proof. The morphism
$$Y \coprod (X\backslash Z) \to X$$
is a Nisnevich covering, but by the (Nisnevich) sheaf condition on $\mathcal{F}$, we have a Cartesian square
$$\begin{tikzcd} \mathcal{F}(X) \arrow[r] \arrow[d] & \mathcal{F}(Y) \arrow[d] \\ \mathcal{F}(X\backslash Z) \arrow[r] & \mathcal{F}(Y\backslash Z) \end{tikzcd}$$
which shows the result for $i = 0$. The map φ^* is exact and has a left adjoint $\varphi_!$, namely the extension by zero, which is the sheafification of the presheaf defined by
$$(\varphi_!\mathcal{F})(U) = \begin{cases} \mathcal{F}(U), & U \subseteq Y \\ 0, & U \nsubseteq Y \end{cases}$$
In particular, φ^* preserves injective objects. Using the case where $i = 0$ and the δ-functor, we prove that the case for $i > 0$ follows. $\square$

Corollary 2.33. The local cohomology (think of $x \in X$ as a point) agrees with the supported cohomology (think of $x \in X$ as a maximal ideal in $\mathrm{Spec}(\mathcal{O}_{X,x}^h)$), i.e.,
$$H_x^i(X,\mathcal{F}) \cong H_x^i(\mathrm{Spec}(\mathcal{O}_{X,x}^h),\mathcal{F}_x).$$

Theorem 2.34. For all $n > \dim(X)$, $H^n(X,\mathcal{F}) = 0$.

Proof. We proceed by induction on $\dim(X)$. If $\dim(X) = 0$, then X is a disjoint union of spectra of Henselian rings[10], but over each Henselian, the higher cohomology vanishes since henselization is an exact functor. so the statement holds. Now suppose the statement is true for any scheme Y such that $\dim(Y) < \dim(X)$, then we have a long exact sequence
$$\cdots \to H^i(\mathrm{Spec}(\mathcal{O}_{X,x}^h),\mathcal{F}_x) \to H^i(\mathrm{Spec}(\mathcal{O}_{X,x}^h\backslash\{x\}),\mathcal{F}_x) \to H_x^{i+1}(\mathrm{Spec}(\mathcal{O}_{X,x}^h),\mathcal{F}_x) \to H^{i+1}(\mathrm{Spec}(\mathcal{O}_{X,x}^h),\mathcal{F}_x) \to \cdots$$
For $i > 0$, we know that $H^i(\mathrm{Spec}(\mathcal{O}_{X,x}^h),\mathcal{F}_x) = H^{i+1}(\mathrm{Spec}(\mathcal{O}_{X,x}^h),\mathcal{F}_x) = 0$, therefore
$$H^i(\mathrm{Spec}(\mathcal{O}_{X,x}^h\backslash\{x\}),\mathcal{F}_x) \cong H_x^{i+1}(\mathrm{Spec}(\mathcal{O}_{X,x}^h),\mathcal{F}_x)$$
for $i > 0$. By induction, $H^{n-1}(\mathrm{Spec}(\mathcal{O}_{X,x}^h\backslash\{x\}),\mathcal{F}_x) = 0$ if $n > \dim(\bar{x})$,[11] therefore $H_x^n(\mathrm{Spec}(\mathcal{O}_{X,x}^h,\mathcal{F}_x) = 0$ if $n > \dim(\bar{x})$. This tells us that the Coniveau spectral sequence satisfies
$$E_1^{p,q} \cong \bigoplus_{\mathrm{codim}(\bar{x})=p} H_x^{p+q}(\mathrm{Spec}(\mathcal{O}_{X,x}^h),\mathcal{F}_x) = 0$$
when $p + q > \dim(X)$ (since $n > \dim(\bar{x})$). Therefore the spectral sequence collapses, i.e., $H^n(X,\mathcal{F}) = 0$ for $n > \dim(X)$. $\square$

[10] Being Artinian local rings, they should be complete and therefore Henselian.

[11] Note that removing the closure of the point (as a maximal ideal) reduces the length by 1, therefore drops the dimension by 1, so the inductive hypothesis still works.

Theorem 2.35. Let $X, U \in \mathrm{Sm}/S$ and $p : U \to X$ be a Nisnevich covering. Denote the n-fold product $A \times_B A \times_B \cdots \times_B A$ by A_B^n, then the Čech complex of sheaves (associated to the complex over Sm/S)

$$\check{C}(U/X) = (\cdots \longrightarrow \mathbb{Z}(U_X^n) \xrightarrow{d_n} \cdots \longrightarrow \mathbb{Z}(U \times_X U) \xrightarrow{d_2} \mathbb{Z}(U) \longrightarrow \mathbb{Z}(X) \longrightarrow 0)$$

is exact[12], where $d_n = \sum_i (-1)^{i-1}\mathbb{Z}(p_i)$ for ith omission map $p_i : U_X^n \to U_X^{n-1}$.

Proof. It suffices to show exactness stalkwise, so to do things locally, we suppose $Y = \mathrm{Spec}(A)$ where A is Henselian, regular and local, and $a \in \mathrm{Cor}_S(Y, U_X^n) = \mathbb{Z}(U_X^n)(Y)$ such that $d_n(a) = 0$. Define $T = \mathrm{supp}(a)$ and $R = T \times_{X\times Y} (U \times Y)$. Since U is Nisnevich over X, then R is Nisnevich over T. Since a is a finite correspondence, and $T \subseteq Y \times_S U_X^n$ is a closed subset, then T is finite over Y. But Y is Henselian, then T is the spectrum of a disjoint union of Henselian rings by Theorem 2.27. Since R is a Nisnevich covering of T, so the map $R \to T$ admits a section $s : T \to R$,[13] where s is both an open immersion and a closed immersion, i.e., T is clopen in R. This gives a diagram of Cartesian squares

$$\begin{tikzcd}
R_T^n \arrow[r] \arrow[d, "\mathrm{id}^n \times s"'] & (U \times Y)_{X\times Y}^n \times_{X\times Y} ((U \times Y)\backslash(R\backslash T)) \arrow[d, "j_{n+1}"] \\
R_T^{n+1} \arrow[r] \arrow[d, "p_{n+1}"'] & (U \times Y)_{X\times Y}^{n+1} \arrow[d, "p_{n+1}"] \\
R_T^n \arrow[r] & (U \times Y)_{X\times Y}^n
\end{tikzcd}$$

where j_{n+1} is a closed immersion. But note that the composition of the left column is just identity, so we define

$$b = (j_{n+1*}(p_{n+1} \circ j_{n+1})^*)(a) \in \mathrm{Cor}_S(Y, U_X^{n+1}).$$

By intersection theory, one can check that $d_{n+1}(b) = a$. $\square$

Theorem 2.36. There is a unique sheafification function $a : \mathrm{PSh}(S) \to \mathrm{Sh}(S)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{PSh}(S) \arrow[r, "a"] \arrow[d] & \mathrm{Sh}(S) \arrow[d] \\
\mathrm{PSh}(\mathrm{Sm}/S) \arrow[r, "+"'] & \mathrm{Sh}(\mathrm{Sm}(S))
\end{tikzcd}$$

Proof. Take $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Sh}(S)$. We first prove uniqueness. Suppose $\mathcal{F}_1|_{\mathrm{Sm}/S} = \mathcal{F}_2|_{\mathrm{Sm}/S} = (\mathcal{F}|_{\mathrm{Sm}/S})^+$, set $s \in \mathcal{F}_1(Y) = \mathcal{F}_2(Y)$ and $T \in \mathrm{Cor}_S(X, Y)$ where X is Henselian, then there is a Nisnevich covering $p : U \to Y$ such that $s|_U = t^+$ where $t \in \mathcal{F}(U)$. Consider the Cartesian square

$$\begin{tikzcd}
T_U \arrow[r] \arrow[d] & X \times U \arrow[d] \\
T \arrow[r] & X \times Y
\end{tikzcd}$$

then since T is irreducible so T is the spectrum of some Henselian ring, which gives a section s of the map $T_U \to T$. Denote $D = \mathrm{im}(s)$, then $D \in \mathrm{Cor}_S(X, U)$. Therefore $p \circ D = T$, so we have a commutative diagram

$$\begin{tikzcd}
\mathcal{F}_1(X) \arrow[r, equal] & \mathcal{F}_2(X) \\
\mathcal{F}_1(U) \arrow[r, equal] \arrow[u, "\mathcal{F}_1(D)"] & \mathcal{F}_2(U) \arrow[u, "\mathcal{F}_2(D)"'] \\
\mathcal{F}_1(Y) \arrow[r, equal] \arrow[u, "\mathcal{F}_1(p)"] \arrow[uu, bend left=60, "\mathcal{F}_1(T)"] & \mathcal{F}_2(Y) \arrow[u, "\mathcal{F}_2(p)"'] \arrow[uu, bend right=60, "\mathcal{F}_2(T)"']
\end{tikzcd}$$

[12]To be precise, we consider this sequence to be the sheafification of Nisnevich presheaves restricted on Nisnevich sites.

[13]We have an étale morphism $R \to T$ that is Nisnevich at the maximal ideal of T, so we admit a section by Theorem 2.27.

In particular, $\mathcal{F}_1 = \mathcal{F}_2$, so we have uniqueness. To prove existence, we make $(\mathcal{F}|_{\mathrm{Sm}/S})^+$ a sheaf with transfers. Suppose $y \in (\mathcal{F}|_{\mathrm{Sm}/S})^+(Y)$, and $y|_U = Z^+$, where $p : U \to Y$ is a Nisnevich covering and $Z \in \mathcal{F}(U)$ (and so Z^+ is the image of Z over sheafification). By shrinking U, we allow Z to agree on the intersection, i.e., we may assume that Z is mapped to 0 in $\mathcal{F}(U \times_Y U)$. This gives a sequence

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}(Y), (\mathcal{F}|_{\mathrm{Sm}/S})^+ \longrightarrow \mathrm{Hom}(\mathbb{Z}(0), (\mathcal{F}|_{\mathrm{Sm}/S})^+ \longrightarrow \mathrm{Hom}(\mathbb{Z}(U \times_X U), (\mathcal{F}|_{\mathrm{Sm}/S})^+)$$

which is exact by Theorem 2.35. We know that $p^*(Z) = 0$, so there exists $[y] : \mathbb{Z}(Y) \to (\mathcal{F}|_{\mathrm{Sm}/S})^+$ such that $[y]|_U = y|_U$. Take $f \in \mathrm{Cor}_S(X, Y)$, then by Yoneda lemma we know the composition

$$\mathbb{Z}(X) \xrightarrow{f} \mathbb{Z}(Y) \xrightarrow{[y]} (\mathcal{F}|_{\mathrm{Sm}/S})^+$$

of Nisnevich sheaves produces the transfer of y with respect to f. □

Remark 2.37. The category $\mathrm{Sh}(S)$ is an abelian category, then the statement in Proposition 2.29 holds for $\mathrm{Sh}(S)$.

Proposition 2.38. Suppose $X \in \mathrm{Sm}/S$ and $\{U_1, U_2\}$ is a Zariski covering of X, then we have an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}(U_1 \cap U_2) & \longrightarrow & \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2) & \longrightarrow & \mathbb{Z}(X) & \longrightarrow & 0 \\ & & s & \longmapsto & (s|_{U_1}, -s|_{U_2}) & & & & \\ & & & & (s_1, s_2) & \longmapsto & s_1 + s_2 & & \end{array}$$

Proof. Note that $U_1 \coprod U_2$ is a Nisnevich covering of X. Applying the Čech complex of X in Theorem 2.35, we obtain an exact sequence

$$\mathbb{Z}(U_1) \oplus \mathbb{Z}(U_1 \cap U_2)^{\oplus 2} \oplus \mathbb{Z}(U_2) \xrightarrow{d} \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2) \xrightarrow{+} \mathbb{Z}(X) \longrightarrow 0$$

where $d(x, y, a, b) = (a - y, y - a)$. □

Definition 2.39. Define Sim to be the category of simplicial sets $[n] = \{0, \cdots, n\}$ for $n \in \mathbb{N}$, where $\mathrm{Hom}_{\mathrm{Sim}}([n], [m])$ is the set of non-decreasing simplicial maps $[n] \to [m]$.

For any category $\mathscr{C}$, we define a simplicial (respectively, cosimplicial) object in $\mathscr{C}$ to be a functor $\mathrm{Sim}^{\mathrm{op}} \to \mathscr{C}$ (respectively, $\mathrm{Sim} \to \mathscr{C}$).

For any $n \in \mathbb{N}$, we define a scheme $\Delta^n = \mathrm{Spec}(k[x_0, \ldots, x_n])/\{(x_0, \ldots, x_n) : \sum_{i=0}^{n} x_i = 1\}$ that is isomorphic to $\mathbb{A}^n$. This is a cosimplicial object in Sm/k. For any $f : [n] \to [m]$, we have

$$\Delta(f)(x_i) = (y_j)$$

where $y_j = \sum_{i \in f^{-1}(j)} x_i$.

Definition 2.40. For any $F \in \mathrm{PSh}(S)$, we define a simplicial object

$$(C_* F)_n = F^{\Delta^n}$$

which associates to the Suslin complex of F

$$C_* F : \cdots \longrightarrow F^{\Delta^n} \xrightarrow{d_n} F^{\Delta^{n-1}} \longrightarrow \cdots \longrightarrow F^{\Delta^1} \xrightarrow{d_1} F \longrightarrow 0$$

with $d_n = \sum_i (-1)^{i-1} \partial_i$, where $\partial_i : \Delta^{n-1} \to \Delta^n$ is the ith face map.

Remark 2.41. In Theorem 2.34, we showed that the cohomological dimension of Nisnevich topology on X is just $\dim(X)$, so for every bounded-above (cochain) complex $C \in C^{-}(\mathrm{Sh}(S))$, we could find a quasi-isomorphism $i: C \to I^*$ where $H^n(X, I^m) = 0$ for any m and any $n > 0$. Therefore, we can define the nth hypercohomology of C with respect to X as

$$\mathbb{H}^n(X, C) = H^n(I^*(X)).$$

It is a standard argument to show that $\mathbb{H}^n(X, C)$ is independent of I^*.

Definition 2.42. For every $q \in \mathbb{N}$, we define the motivic complex to be

$$\mathbb{Z}(q) = C_*(\mathbb{Z}(\mathbb{G}_m^{\wedge q})[-q]),$$

given by the augmentation of the smashing with a shifting by $-q$, where $\mathbb{Z}(\mathbb{G}_m^{\wedge q}) = (\mathbb{Z}(\mathbb{G}_m), 1)^{\wedge q}$, and $\mathbb{Z}(q)^i = C_{q-i}\mathbb{Z}(\mathbb{G}_m^{\wedge q})$ in the Suslin complex for $q \geqslant 0$. For $q < 0$, we define $\mathbb{Z}(q) = 0$.

For any group A, we write $\mathbb{Z}(q) \otimes_{\mathbb{Z}}^{L} A$ as $A(q)$.

Example 2.43. $\mathbb{Z}(0)$ is the constant sheaf $\mathbb{Z}$.

Definition 2.44. For every $X \in \mathrm{Sm}/k$, we define the motivic cohomology to be the hypercohomology with respect to Nisnevich topology

$$H^{p,q}(X, A) = \mathbb{H}^p_{\mathrm{Nis}}(X, A(q))$$

with coefficients in A.

Remark 2.45. It turns out that this is equivalent to giving the hypercohomology the Zariski topology instead.

Proposition 2.46. For any $X \in \mathrm{Sm}/k$, we have

$$H^{p,q}(X, A) = 0$$

if $p > \dim(X) + q$. In particular, if A is a field, then $H^{p,q}(X, A) = 0$ if $p > q$.

Proof. Using Lemma 1.26, we obtain a spectral sequence

$$H^s(X, H^t(A(q))) \Rightarrow \mathbb{H}^{s+t}(X, A(q)) = H^{s+t,q}(X, A)$$

Let $p = s + t$. If $p > \dim(X) + q$, then either $t > q$ or $s > \dim(X)$. This gives $H^s(X, H^t(A(q))) = 0$. $\square$

3 Milnor K-theory

3.1 K-theory of Residue Field

Definition 3.1. For any field F, define the Milnor K-theory to be the graded algebra

$$K_*^M(F) = T(F^\times)/\{x \otimes (1 - x) : x \in F\backslash\{0, 1\}\},$$

defined as the tensor algebra of $F^\times$ quotient by the Steinberg relation.

Example 3.2.

- $K_0^M(F) = \mathbb{Z}$;
- $K_1^M(F) = F^\times$.

Proposition 3.3. For any $x \in F^\times$, let $[x] \in K_1^M(F)$ be its representative, then obviously $[xy] = [x] + [y]$. Moreover,

1. $[x][y] + [y][x] = 0$ for all $x, y \in F^\times$;
2. $[x][x] = [x][-1]$ for all $x \in F^\times$.

Proof.

1. We have
$$\begin{aligned}[x][-x] &= [x]\left[\frac{1-x}{1-x^{-1}}\right]\\ &= [x][1-x] + [x^{-1}][1-x^{-1}]\\ &= 0+0\\ &= 0,\end{aligned}$$
therefore
$$\begin{aligned}[x][y]+[y][x] &= [x][-x]+[x][y]+[y][x]+[y][-y]\\ &= [x][-xy]+[y][-xy]\\ &= [xy][-xy]\\ &= 0.\end{aligned}$$

2. Using the previous part, we know
$$\begin{aligned}[x][x] &= [x][-1]+[x][-x]\\ &= [x][-1].\end{aligned}$$

□

Proposition 3.4 ([Hes05], Proposition 1)**.** Let k be a field and ν be a normalized discrete valuation on k. We define the residue field of k with respect to ν as $k(\nu) = \mathcal{O}_\nu/\mathfrak{m}_\nu$, then there exists a unique homomorphism (known as the Milnor residue map)
$$\partial_\nu : K_n^M(k) \to K_{n-1}^M(k(\nu))$$
such that for all $u_1, \dots, u_{n-1} \in \mathcal{O}_\nu^\times$ and $x \in k^\times$,
$$\partial_\nu([x][u_1]\cdots[u_{n-1}]) = \nu(x)\cdot[\bar{u}_1]\cdots[\bar{u}_{n-1}]$$
where $\bar{u}_i \in k(\nu)^\times$ is the image of u_i in the residue field.

Proof. The uniqueness is obvious by the universal property, so we shall prove existence. We choose a uniformizer π, and define a graded ring morphism
$$\begin{aligned}\theta_\pi : K_*^M(k) &\to K_*^M(k(\nu))[\varepsilon]/(\varepsilon^2 - \varepsilon[-1])\\ [\pi^i u] &\mapsto [\bar{u}] + i\varepsilon\end{aligned}$$
for $u \in \mathcal{O}_\nu^\times$ and some variable ε of degree 1, then this morphism satisfies the Steinberg relation. Now if we decompose it into
$$\theta_\pi(z) = s_\pi(z) + \partial_\nu(z)\varepsilon,$$
then
$$\begin{aligned}\theta_\pi([\pi^i u][u_1]\cdots[u_{n-1}]) &= ([u]+i\varepsilon)[\bar{u}_1]\cdots[\bar{u}_{n-1}]\\ &= [\bar{u}][\bar{u}_1]\cdots[\bar{u}_{n-1}] + i[\bar{u}_1]\cdots[\bar{u}_{n-1}]\varepsilon.\end{aligned}$$
In particular, the ∂_ν map does what we want. □

Theorem 3.5 ([Hes05], Theorem 5)**.** There is a split exact sequence
$$0 \longrightarrow K_*^M(k) \xrightarrow{i} K_*^M(k(T)) \xrightarrow{(\partial_p)} \bigoplus_{\text{irreducible monic } p} K_{*-1}^M(k[T]/p) \longrightarrow 0$$
where each ∂_p is given by evaluation of p using the partial map defined in Proposition 3.4.

Proof. It is easy to see that this is an exact sequence, and that we have $s_\pi \circ i = \mathrm{id}$. Now we want to construct an isomorphism

$$\tau_{n,p} : \bigoplus_{\text{irreducible monic } p} K_n^M(k[T]/p) \to K_{n+1}^M(k(T))/K_{n+1}^M(k)$$

with inverse $(\partial_p)_p$. We define $\tau_{n,p}$ inductively on $\deg(p)$. Suppose $p = T - \lambda$, then we define $\tau_{n,p}$ as the composite

$$K_n^M(k[T]/p) \xrightarrow{\text{ev}} K_n^M(k) \xrightarrow{[p]} K_{n+1}^M(k(T))/K_{n+1}^M(k).$$

Let $f_i \in k[T]$ for each i, then this composite maps $[\bar{f}_1] \cdots [\bar{f}_n]$ to $[p][f_1(\lambda)] \cdots [f_n(\lambda)]$. Moreover,

$$\partial_q \circ \tau_{n,p} = \begin{cases} \mathrm{id}, & q = p \\ 0, & q \neq p. \end{cases}$$

For general polynomial p and general $f_1, \ldots, f_n \in k[T]$ such that $\deg(f_i) < \deg(p)$ for all i[14], then we define

$$\tau_{n,p}([\bar{f}_1] \cdots [\bar{f}_n]) = [\bar{p}] \cdot [f_1] \cdots [f_n] - \sum_{\substack{\text{irreducible monic } q \\ \text{such that } \deg(q) < \deg(p)}} \tau_{n,q}(\partial_q([p][f_1] \cdots [f_n])),$$

then by inductive hypothesis we are done. It remains to check that this is well-defined, and that

- $$\partial_q \circ \tau_{n,p} = \begin{cases} \mathrm{id}, & q = p \\ 0, & q \neq p. \end{cases}$$
- $\sum_{\deg(q) < \deg(p)} \tau_{n,q}(\partial_q(x)) = x$ if $x = \sum_{\substack{i \\ \deg(f_{ij}) < \deg(p)}} [f_{i1}] \cdots [f_{in}]$.

□

Remark 3.6. The map $-d_\infty$ from

$$\begin{tikzcd} K_*^M(k(T)) \arrow[r, "-\partial_\infty"] & K_{*-1}^M(k) \\ K_*^M(k) \arrow[u, "i"] \arrow[ur, "0"'] & \end{tikzcd}$$

and the exact sequence from Theorem 3.5 together induce a norm map

$$(N_p) : \bigoplus_p K_*^M(k[T]/p) \to K_*^M(k)$$

with $N_\infty = \mathrm{id}$.

Definition 3.7. Suppose $k(a)/k$ is a finite simple extension and the minimal polynomial of a is p. Define the norm

$$N_{a/k} : K_*^*M(k(a)) \to K_*^M(k)$$

to be N_p. In general, suppose K/k is a finite extension where $K = k(a_1, \ldots, a_n)$, then define the norm map to be

$$N_{a_1,\ldots,a_r/k} = N_{a_1/k} \circ N_{a_1/k(a_1)} \circ \cdots \circ N_{a_r/k(a_1,\ldots,a_{r-1})}.$$

Theorem 3.8. The norm map $N_{a_1,\ldots,a_r/k}$ is independent from the choices of $a_1, \ldots, a_r$. In particular, this gives rise a well-defined norm map

$$N_{K/k} : K_*^M(K) \to K_*^M(k)$$

on all finite extensions K/k.

[14] This makes sense since we can pick it in the residue field.

3.2 Proof of Theorem 3.8

Proposition 3.9 ([Hes05], Lemma 10)**.** Let k be a field and p be a prime, then there exists an algebraic extension L/k such that every finite extension of L has order a power of p, and localization at p gives a map

$$K_*^M(k)_{(p)} \to K_*^M(L)_{(p)}$$

is injective.

Proof. Recall an ordinal W is a limit ordinal if and only if $W = \bigcup_{\alpha < W} \alpha$. Define a poset

$$S = \{(\alpha, \{L_\beta \mid \beta \leqslant \alpha\}) : \alpha \text{ ordinal number}, p \nmid [L_\beta : k] < \infty, [L_{\beta+1} : L_\beta] > 1, L_W = \bigcup_{\alpha < W} L_\alpha \text{ for limit ordinal } W\}$$

for some field extensions $k \subseteq L_\beta \subseteq \bar{k}$ in the algebraic closure $\bar{k}$. The partial order on S is given by $(\alpha, \{L_\beta \mid \beta \leqslant \alpha\}) \leqslant (\alpha', \{L_{\beta'} \mid \beta' \leqslant \alpha'\}$ if and only if $\alpha \leqslant \alpha'$, $L_\beta = L_{\beta'}$, $\beta \leqslant \alpha$. We note that $\operatorname{card}(\alpha) \leqslant \operatorname{card}(\bar{k})$, so S must be a set. Every totally ordered subset of S has a maximal element by taking the union, therefore there is a maximal element $(\alpha, \{L_\beta \mid \beta \leqslant \alpha\})$ in S. Now $L = L_\alpha$ does not have an extension with order prime to p, hence every finite extension of L has order a power of p. For any simple extension $k(a)/k$, the composite

$$K_*^M(k) \longrightarrow K_*^M(k(a)) \xrightarrow{N_{a/k}} K_*^M(k)$$

is the multiplication by $[k(a) : k]$ by direct computation. Therefore, for any $\beta \leqslant \alpha$, the composite

$$K_*^M(L_\beta)_{(p)} \longrightarrow K_*^M(L_{\beta+1})_{(p)} \xrightarrow{N} K_*^M(L_\beta)_{(p)}$$

is an injection, hence $K_*^M(k)_{(p)} \to K_*^M(L)_{(p)}$ is also injective by transfinite induction. □

Proposition 3.10 ([Hes05], Lemma 2)**.** Suppose k'/k is a field extension and ν (respectively, ν') is a discrete valuation on k (respectively, k') such that $\nu'|_k = \nu$. Then there is a commutative diagram

$$\begin{tikzcd}
K_*^M(k) \arrow[r, "\partial_\nu"] \arrow[d] & K_{*-1}^M(k(\nu)) \arrow[d] \\
K_*^M(k') \arrow[r, "e\partial_{\nu'}"'] & K_{*-1}^M(k(\nu'))
\end{tikzcd}$$

where e is the ramification index, i.e., $\pi_\nu = u \cdot \pi_{\nu'}^e$ for some uniformizer $u \in \mathcal{O}_{\nu'}^*$.

Proposition 3.11 ([Hes05], Lemma 11)**.** Let $k' = k(a)$ be a finite extension of k, and let p be the minimal polynomial of a over k. Let L/k be a field extension and suppose $p = \prod_i p_i^{e_i}$ is the prime decomposition for some polynomials p_i in L, then for each i we define $L_i' \supseteq k'$ to be $L[t]/p_i$, and set $a_i = \bar{t} \in L_i'$, then we have a commutative diagram

$$\begin{tikzcd}
K_*^M(k') \arrow[r, "(e_i)"] \arrow[d, "N_{a/k}"'] & \bigoplus_i K_*^M(L_i') \arrow[d, "\sum_i N_{a_i/L}"] \\
K_*^M(k) \arrow[r, "\text{base-change}"'] & K_*^M(L)
\end{tikzcd}$$

where e_i is the (multiplication of) ramification index of L_i' over k'.

Proof. Let $f_1, \ldots, f_n \in k[i]$ be prime to p, then $\partial_{p_i}([p][f_1] \cdots [f_n]) = e_i[\bar{f}_1] \cdots [\bar{f}_n]$. Therefore, there is a commutative diagram

$$\begin{tikzcd}
K_*^M(k(t)) \arrow[r] \arrow[d] & K_*^M(L(t)) \arrow[d] \\
\bigoplus_R K_{*-1}^M(k[T]/R) \arrow[r, "\varphi_{R,Q}"'] & \bigoplus_Q K_{*-1}^M(L[T]/Q)
\end{tikzcd}$$

where

$$\varphi_{R,Q} = \begin{cases} \operatorname{ord}_Q(p), & R = p \\ 0, & R \neq p \end{cases}$$

The statement follows from the definition of the map $(N_p) : \bigoplus_p K_*^M(k[T]/p) \to K_*^M(k)$. □

Proposition 3.12 ([Hes05], Proposition 13)**.** Let k be a field and set $k' = k(a)$ to be such that the extension $k(a)/k$ has prime degree, then the map

$$N_{a/k} : K_n^M(k') \to K_n^M(k)$$

is independent of the choice of the generator a.

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