Honors Analysis Notes

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Preliminaries

This document is the notes based on Professor Monica Visan's teaching at UCLA's 131AH and 131BH in winter and spring 2021. The corresponding textbook is Baby Rudin.

1 Lecture 1: Statements

In Rubin's notation, natural numbers start with 1, i.e. $\mathbb{N} = \{1, 2, \cdots\}$.

Let A and B be two statements. We use the following notations:

- We write "A" if A is true.
- We write "not A" if A is false.
- We write "A and B" if both A and B are true.
- We write "A or[1] B" if A is true or B is true or both A and B are true.
- We write "$A \Rightarrow B$ if "A and B" or "not A". We read this as "A implies B" or "if A then B". In this case, B is at least as true as A. In particular, A, a false statement A can imply anything.

We usually write shorthand notation "T" and "F" to represent "true" and "false".

Example 1.1. Consider the following statement:

$$\text{If } x \text{ is a natural number, i.e. } x \in \mathbb{N} = \{1, 2, 3, \cdots\}, \text{ then } x \geq 1.$$

In this case, A is the statement "x is a natural number" and B is the statement "$x \geq 1$".

[1] The notation "or" in mathematics is inclusive. We distinguish it from the exclusive or, usually called "xor", which means "either A or B"

- Taking $x = 3$, we get $T \Rightarrow T$.
- Taking $x = \pi$, we get $F \Rightarrow T$.
- Taking $x = 0$, we get $F \Rightarrow F$.

Example 1.2. Consider the statement:

If a number is less than 10, then it is less than 20.

The statement is of the form "if... then...", where A is the statement "a number is less than 10", and B is the statement "it is less than 20".

- Taking a number 5, we get $T \Rightarrow T$.
- Taking a number 15, we get $F \Rightarrow T$.
- Taking a number 25, we get $F \Rightarrow F$.

We also write "$A \iff B$" if A and B are true together or false together. We read this as "A is equivalent to B" or "A if and only if B".

We can now compare these notions in logic to similar ones from set theory. Let X be an ambient space. Let A and B be subsets of X. Then

- ${}^cA = \{x \in X : x \notin A\}$.
- $A \cap B = \{x \in X : x \in A \text{ and } x \in B\}$.
- $A \cup B = \{x \in X : x \in A \text{ or } x \in B \text{ or } x \in A \cap B\}$.
- $A \subseteq B$ corresponds to $A \Rightarrow B$.
- $A = B$ corresponds to $A \iff B$.

We now can use truth tables to check the statements.

A	B	not A	A and B	A or B	$A \Rightarrow B$	$A \iff B$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Example 1.3. We can use the truth table to show that $A \Rightarrow B$ is logically equivalent to (not A) or B. Indeed, by considering the following truth table,

A	B	$A \Rightarrow B$	not A	(not A) or B
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

we realize that the column of $A \Rightarrow B$ and (not A) or B are the same.

Exercise 1.4. Use the truth table to prove De Morgan's laws:

$$\begin{aligned}\text{not (A and B)} &= \text{(not A) or (not B)}\\ \text{not (A or B)} &= \text{(not A) and (not B)}\end{aligned}$$

One can compare these statements to

$$\begin{aligned}{}^c(A \cap B) &= {}^cA \cup {}^cB\\ {}^c(A \cup B) &= {}^cA \cap {}^cB\end{aligned}$$

Example 1.5. Negative the following statement:

If A then B.

Note that the negation is "not ($A \Rightarrow B$)", then it is equivalent to not ((not A) or B), which is equivalent to [not(not A)] and (not B), and that is just A and (not B).

Therefore, the negation is "A is true and B is false".

Example 1.6. Negate the following statement:

If I speak in front of the class, I am nervous.

That would be I speak in front of the class and I am not nervous.

We now introduce quantifiers.

- $\forall$ reads "for all " or "for any".
- $\exists$ reads "there is" or "there exists".
- The negation of "$\forall A$, B is true" is "$\exists A$ such that B is false".
- The negation of "$\exists A$ such that B is true" is "$\forall A$, B is false".

Example 1.7. Negate the following:

Every student had coffee or is late for class.

This statement is represented as

$$\forall \text{ student (had coffee) or (is late for this)}$$

and so the negation would be

$$\exists \text{ student such that not (had coffee) and not (is late for class)}$$

Writing this out, we get "there is a student that did not have coffee and is not late for class".

2 Lecture 2: Peano Axiom and Mathematical Induction

Definition 2.1 (Peano Axiom)**.** The natural numbers $\mathbb{N} = \{1, 2, 3, \cdots\}$ satisfy the Peano axioms:

1. $1 \in \mathbb{N}$.
2. If a number $n \in \mathbb{N}$, then its successor $n + 1 \in \mathbb{N}$.
3. 1 is not the successor of any natural number.
4. If two numbers $n, m \in \mathbb{N}$ are such that they have the same successor, i.e. $n+1 = m+1$, then they are the same, i.e. $n = m$.
5. Let $S \subseteq \mathbb{N}$. Assume that S satisfies the following two conditions:
   (i) $1 \in S$,
   (ii) and if $n \in S$ then $n + 1 \in S$,

   then $S = \mathbb{N}$.

Axiom number 5 forms the basis for mathematical induction.

Definition 2.2 (Mathematical Induction)**.** Assume we want to prove that a property $P(n)$ holds for all $n \in \mathbb{N}$. Then it suffices to verify two steps:

- Step 1 (Base Step): $P(1)$ holds.
- Step 2 (Inductive Step): If $P(n)$ is true for some $n \geq 1$, then $P(n + 1)$ is true, i.e. $P(n) \Rightarrow P(n + 1)\ \forall n \geq 1$.

Indeed, if we let

$$S = \{n \in \mathbb{N} : P(n) \text{ holds}\},$$

then Step 1 implies $1 \in S$ and Step 2 implies if $n \in S$ then $n+1 \in S$. By axiom 5, we deduce that $S = \mathbb{N}$.

Example 2.3. Prove that

$$1^2 + 2^2 + \cdots + n^2 = \tfrac{n(n+1)(2n+1)}{6} \quad \forall n \in \mathbb{N}.$$

We argue that mathematical induction. For $n \in \mathbb{N}$, let $P(n)$ denote the statement

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Step 1 (Base Step): $P(1)$ is the statement $1^2 = \frac{1 \cdot 2 \cdot 3}{6}$, which is true, so $P(1)$ holds.

Step 2 (Inductive Step): Assume that $P(n)$ holds for some $n \in \mathbb{N}$, we want to show that $P(n+1)$ holds. We know

$$1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

then we have

$$\begin{aligned} 1^2 + \cdots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= (n+1)[\frac{n(2n+1}{6} + n + 1] \\ &= (n+1) \cdot \frac{2n^2 + 7n + 6}{6} \\ &= \frac{(n+1) \cdot [2n(n+2) + 3n + 6]}{6} \\ &= \frac{(n+1)(n+2)(2n+3}{6} \end{aligned}$$

So $P(n+1)$ holds.

Collecting the two steps, we conclude $P(n)$ holds $\forall n \in \mathbb{N}$.

Example 2.4. Prove that $2^n > n^2$ for all $n \geq 5$.

We argue by mathematical induction. For $n \geq 5$, let $P(n)$ denote the statement $2^n > n^2$.

Step 1 (Base Step): $P(5)$ is the statement

$$32 = 2^5 > 5^2 = 25$$

which is true. So $P(5)$ holds.

Step 2 (Inductive Step): Assume $P(n)$ is true for some $n \geq 5$ and we want to prove $P(n+1)$. We know $2^n > n^2$, then

$$\begin{aligned} 2^{n+1} &> 2n^2 \\ &= (n+1)^2 + n^2 - 2n - 1 \\ &= (n+1)^2 + (n-2)^2 - 2 \end{aligned}$$

For $n \geq 5$, we have $(n-1)^2 - 2 \geq 4^2 - 2 = 14 \geq 0$, so we know $2^{n+1} > (n+1)^2$. Therefore, $P(n+1)$ holds.

Collecting the two steps, we conclude $P(n)$ holds $\forall n \geq 5$.

Remark 2.5. Each of the two steps are essential when arguing by induction. Note that $P(1)$ is true. However, our proof of the second step fails if $n = 1$: $(1-1)^2 - 2 = -2 < 0$. Also note that our proof of the second step is valid as soon as

$$(n-1)^2 - 2 \geq 0 \iff (n-1)^2 \geq 2 \iff n - 1 \geq 2 \iff n \geq 3.$$

However, $P(3)$ fails.

Example 2.6. Prove by mathematical induction that the number $4^n + 15n - 1$ is divisible by 9 for all $n \geq 1$.

We will argue by induction. For $n \geq 1$, let $P(n)$ denote the statement that "$4^n + 15n - 1$ is divisible by 9". We write this as $9 \mid (4^n + 15n - 1)$.

Step 1: $4^1 + 15 \cdot 1 - 1 = 18 = 9 \cdot 2$. This is divisible by 9, so $P(1)$ holds.

Step 2: Assume $P(n)$ is true for some $n \geq 1$, we want to show $P(n+1)$ holds.

$$\begin{aligned} 4^{n+1} + 15(n+1) - 1 &= 4 \cdot (4^n + 15n - 1) - 60n + 4 + 15n + 14 \\ &= 4 \cdot (4^n + 15n - 1) - 45n + 18 \\ &= 4 \cdot (4^n + 15n - 1) - 9 \cdot (5n - 2). \end{aligned}$$

By the inductive hypothesis, $9 \mid (4^n + 15n - 1)$ implies $9 \mid 4 \cdot (4^n + 15n - 1)$. Also we know $9 \mid 9 \cdot (5n - 2)$ since $5n - 2 \in \mathbb{N}$. Therefore, we know $9 \mid [4 \cdot (4^n + 15n - 1) - 9 \cdot (5n - 2)]$. Hence, $9 \mid [4 \cdot (4^n + 15n - 1) - 9 \cdot (5n - 2)]$, so $P(n+1)$ holds.

Collecting the two steps, we conclude $P(n)$ holds $\forall n \in \mathbb{N}$.

Example 2.7. Compute the following sum and then use mathematical induction to prove your answer: for $n \geq 1$,

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)}.$$

Note that $\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}[\frac{1}{2n-1} - \frac{1}{2n+1}]$ for all $n \geq 1$. So

$$\begin{aligned} \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} &= \frac{1}{2}(\frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{2n-1} - \frac{1}{2n+1}) \\ &= \frac{1}{2} \cdot \frac{2n}{2n+1} \\ &= \frac{n}{2n+1}. \end{aligned}$$

For $n \geq 1$, let $P(n)$ denote the statement

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.$$

Step 1: $P(1)$ becomes $\frac{1}{1 \cdot 3} = \frac{1}{3}$, which is true. So $P(1)$ holds.

Step 2: Assume $P(n)$ holds for some $n \geq 1$. We want to show $P(n+1)$. We know

$$\frac{1}{1 \cdot 3} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1},$$

and so

$$\begin{aligned} \frac{1}{1 \cdot 3} + \cdots + \frac{1}{(2n+1)(2n+3)} &= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} \\ &= \frac{2n^2+3n+1}{(2n+1)(2n+3)} \\ &= \frac{(n+1)(2n+1)}{(2n+1)(2n+3)} \\ &= \frac{n+1}{2n+3}. \end{aligned}$$

So $P(n+1)$ holds.

Collecting the two steps, we conclude $P(n)$ holds $\forall n \geq 1$.

3 Homework 1

Exercise 3.1. Negate the following sentences:

- For every complex problem there is an answer that is clear, simple, and wrong.
- If that plane leaves and you are not on it, you will regret it.
- Not all those who wander are lost.

Exercise 3.2. Let X, Y, and Z be statements. Negate the following sentences:

- At least one of X and Y are true.
- Both X and Y are false.
- At least two of X, Y, and Z are false.

Exercise 3.3. Let $P(x)$ be a statement about elements $x \in X$. Negate the following sentences:

- $P(x)$ is true for all $x \in X$.
- For every x in X, there is a $y \in X$ not equal to x, for which $P(y)$ is true.
- If $P(x)$ and $P(y)$ are both true, then $x = y$.

Exercise 3.4. Let $P(n, m)$ be a statement about two integers n and m. Negate the following sentences:

- There exists an integer n such that $P(n, m)$ is true for all integers $m \geq n$.
- For some integer n, and every integer m, the property $P(n, m)$ is false.
- For every integer m, there exists an integer $n \geq m$ such that $P(n, m)$ is false.

Exercise 3.5. Let X and Y be statements. If we know that X implies Y, which one of the following can we conclude?

(a) X cannot be false.

(b) X is true, and Y is also true.

(c) If Y is false, then X is false.

(d) Y cannot be false.

(e) If X is false, then Y is false.

(f) If Y is true, then X is true.

(g) At least one of X and Y is true.

Exercise 3.6. Let X, Y, and Z be statements. Suppose we know that "X is true implies Y is true", and "X is false implies Z is true". If we know that Z is false, then which one of the following can we conclude?

(a) X is false.

(b) X is true.

(c) Y is true.

(d) (b) and (c).

(e) (a) and (c).

(f) (a), (b), and (c).

(g) None of the above conclusions can be drawn.

Exercise 3.7. Prove the following statement by induction:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2 \text{ for all } n \geq 1.$$

Exercise 3.8. Prove by induction that the sum of the cubes of any three consecutive natural numbers is divisible by 9.

Exercise 3.9. We define the Fibonacci numbers as follows:

$$F_1 = F_2 = 1 \text{ and } F_{n+2} = F_{n+1} + F_n \text{ for all } n \geq 1.$$

Prove the following statements by induction:

$$\begin{aligned} F_n^2 + F_{n+1}^2 &= F_{2n+1} \\ 2F_nF_{n+1} + F_{n+1}^2 &= F_{2n+2}. \end{aligned}$$

4 Lecture 3: Equivalence Relation

We now extend $\mathbb{N}$ and construct the set of integers $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$.

Definition 4.1 (Equivalence Relation)**.** An equivalence relation $\sim$ on a non-empty set A satisfies the following three properties:

1. Reflexivity: $a \sim a \ \forall a \in A$.
2. Symmetry: If $a, b \in A$ are such that $a \sim b$, then $b \sim a$.
3. Transitivity: If $a, b, c \in A$ are such that $a \sim b$ and $b \sim c$, then $a \sim c$.

Example 4.2. The equal relation $=$ is an equivalence relation on $\mathbb{Z}$.

Example 4.3. Let $q \in \mathbb{N}$ and $q > 1$. For $a, b \in \mathbb{Z}$ we write $a \sim b$ if $q \mid (a - b)$. This is an equivalence relation on $\mathbb{Z}$. Indeed, it suffices to check the three properties:

- Reflexivity: If $a \in \mathbb{Z}$, then $a - a = 0$, which is divisible by q. So $q \mid (a - a)$, by definition we know $a \sim a$.
- Symmetry: Let $a, b \in \mathbb{Z}$ such that $a \sim b$, then by definition we know $q \mid (a - b)$. Therefore, there exists some $k \in \mathbb{Z}$ such that $a - b = kq$, so $b - a = (-k) \cdot q$. Note that $-k \in \mathbb{Z}$, so $q \mid (b - a)$, and by definition we know $b \sim a$.

- Transivitity: Let $a, b, c \in \mathbb{Z}$ such that $a \sim b$ and $b \sim c$. Now $a \sim b$ indicates $q \mid (a-b)$, so there exists $n \in \mathbb{Z}$ such that $a - b = qn$. Similarly there exists $m \in \mathbb{Z}$ such that $b - c = qm$. Therefore, $a - c = q(n+m)$, where $n + m \in \mathbb{Z}$. Therefore, $q \mid (a-c)$, so by definition $a \sim c$.

Definition 4.4 (Equivalence Class)**.** Let $\sim$ denote an equivalence relation on a non-empty set A. The equivalence class of an element $a \in A$ is given by

$$C(a) = \{b \in A : a \sim b\}.$$

Proposition 4.5 (Properties of Equivalence Classes)**.** Let $\sim$ denote an equivalence relation on a non-empty set A. Then

1. $a \in C(a)$ for all $a \in A$.
2. If $a, b \in A$ are such that $a \sim b$, then $C(a) = C(b)$.
3. If $a, b \in A$ are such that $a \not\sim b$, then $C(a) \cap C(b) = \varnothing$.
4. $A = \bigcup_{a \in A} C(a)$.

Proof. 1. By reflexivity, $a \sim a$ for all $a \in A$, then $a \in C(a)$ for all $a \in A$.

2. Assume $a, b \in A$ with $a \sim b$. Let us show $C(a) \subseteq C(b)$. Let $c \in C(a)$ be arbitrary, then $a \sim c$. Because $a \sim b$, by symmetry we have $b \sim a$, then by transitivity we know $b \sim c$, and so $c \in C(b)$. This proves that $C(a) \subseteq C(b)$. A similar argument shows that $C(b) \subseteq C(a)$, and so $C(a) = C(b)$.

3. We argue by contradiction. Assume that $a, b \in A$ are such that $a \not\sim b$, but $C(a) \cap C(b) \neq \varnothing$. Let $c \in C(a) \cap C(b)$, then $c \in C(a)$ and $c \in C(b)$. The first property implies $a \in c$, and the second property implies $b \sim c$, so $c \sim b$, and therefore by transitivity we have $a \sim b$. This contradicts the hypothesis $a \not\sim b$. Therefore, if $a \not\sim b$, then $C(a) \cap C(b) = \varnothing$.

4. Clearly, as $C(a) \subseteq A$ for all $a \in A$, we get $\bigcup_{a \in A} C(a) \subseteq A$. Then conversely, $A = \bigcup_{a \in A} \{a\} \subseteq \bigcup_{a \in A} C(a)$, and therefore $A = \bigcup_{a \in A} C(a)$.

□

Example 4.6. Take $q = 2$ in our previous example: for $a, b \in \mathbb{Z}$, we write $a \sim b$ if $2 \mid (a-b)$. The equivalence classes are

$$C(0) = \{a \in \mathbb{Z} : 2 \mid (a-0)\} = \{2n : n \in \mathbb{Z}\}$$
$$C(1) = \{a \in \mathbb{Z} : 2 \mid (a-1)\} = \{2n+1 : n \in \mathbb{Z}\}$$

and $\mathbb{Z} = C(0) \cup C(1)$.

Example 4.7. Let $F = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : b \neq 0\}$. If $(a,b),(c,d) \in F$ we write $(a,b) \sim (c,d)$ if $ad = bc$. Then for example, we have $(1,2) \sim (2,4) \sim (3,6) \sim (-4,-8)$.

Lemma 4.8. $\sim$ is an equivalence relation on F.

Proof. We have to check the three properties.

Reflexivity: Fix $(a,b) \in F$. As $ab = ba$, we have $(a,b) \sim (b,a)$.

Symmetry: Let $(a,b),(c,d) \in F$ such that $(a,b) \sim (c,d)$, then by definition we know $ad = bc$, and so $cb = da$, and by definition $(c,d) \sim (a,b)$.

Transitivity: Let $(a,b),(c,d),(e,f) \in F$ such that $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$. Now $(a,b) \sim (c,d)$ implies $ad = bc$, then $adf = bcf$. Similarly, $cfb = deb$. Therefore, $adf = deb$. Now $d(af - be) = 0$, and because $d \neq 0$ by definition, we know $af = be$, and by definition we have $(a,b) \sim (e,f)$ as desired. $\square$

For $(a,b) \in F$, we denote its equivalence class by $\frac{a}{b}$. We define addition and multiplication of equivalence classes as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$
$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

We have to check that these operations are well-defined. Specifically, if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$, then we should have

$$\begin{cases} (ad + bc, bd) \sim (a'd' + b'c', b'd') \\ (ac, bd) \sim (a'c', b'd') \end{cases}$$

We now check the first property and left the second property as an exercise to the readers. We want to show $(ad + bc)b'd' = bd(a'd' + b'c')$. We know that $(a,b) \sim (a'b')$, so $ab' = ba'$, and therefore $ab'dd' = badd'$. Similarly we know $(c,d) \sim c'd'$, so $cd' = dc'$, and therefore $cd'bb' = dc'bb'$. Now we get

$$ab'dd' + cd'bb' = ba'dd' + dc'bb',$$

and so

$$(ad + bc)b'd' = bd(a'd' + b'c').$$

This proves addition is well-defined.

Now the set of rational numbers is exactly the set of equivalence classes on F, i.e.

$$\mathbb{Q} = \{\frac{a}{b} : (a,b) \in F\}.$$

5 Lecture 4: Field

Definition 5.1 (Field)**.** A field is a set F with at least two elements equipped with two operations: addition (denoted $+$) and multiplication (denoted $\cdot$) that satisfies the following:

1. $(A1)$ Closure: if $a, b \in F$, then $a + b \in F$.
2. $(A2)$ Commutativity: if $a, b \in F$, then $a + b = b + a$.
3. $(A3)$ Associativity: if $a, b, c \in F$, then $(a + b) + c = a + (b + c)$.
4. $(A4)$ Identity: $\exists 0 \in F$ such that $a + 0 = 0 + a = a \ \forall a \in F$.
5. $(A5)$ Inverse: $\forall a \in F$, $\exists(-a) \in F$ such that $a + (-a) = -a + a = 0$.
6. $(M1)$ Closure: if $a, b \in F$, then $a \cdot b \in F$.
7. $(M2)$ Commutativity: if $a, b \in F$, then $a \cdot b = b \cdot a$.
8. $(M3)$ Associativity: if $a, b, c \in F$, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
9. $(M4)$ Identity: $\exists 1 \in F$ such that $a \cdot 1 = 1 \cdot a = a \ \forall a \in F$.
10. $(M5)$ Inverse: $\forall a \in F \backslash \{0\}$, $\exists a^{-1} \in F$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.
11. (D) Distributivity: if $a, b, c \in F$, then $(a + b) \cdot c = a \cdot c + b \cdot c$.

Example 5.2. $(\mathbb{N}, +, \cdot)$ is not a field because (A_4) fails.

Example 5.3. $(\mathbb{Z}, +, \cdot)$ is not a field because (M_5) fails.

Example 5.4. $(\mathbb{Q}, +, \cdot)$ is a field.

Recall $\mathbb{Q} = \{\frac{a}{b} : (a, b) \in \mathbb{Z} \times (\mathbb{Z}\backslash\{0\}\}$ where $\frac{a}{b}$ denotes the equivalence class of $(a, b) \in \mathbb{Z} \times (\mathbb{Z}\backslash\{0\})$ with respect to the equivalence relation $\sim$, where $(a, b) \sim (c, d)$ if and only if $a \cdot d = b \cdot c$. We defined two operations

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$
$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Then the additive identity $\frac{0}{1}$ is the equivalence class of $(0, 1)$, and the multiplicative identity $\frac{1}{1}$ is the equivalence class of $(1, 1)$.

The additive inverse of $\frac{a}{b} \in \mathbb{Q}$ is given by $\frac{-a}{b}$, and for $\frac{a}{b} \in \mathbb{Q}\backslash\{\frac{0}{1}\}$, the multiplicative inverse is given by $\frac{b}{a}$.

Proposition 5.5. Let $(F, +, \cdot)$ be a field. Then

1. The additive and multiplicative identities are unique.

2. The additive and multiplicative inverses are unique.

3. If $a, b, c \in F$ such that $a + b = a + c$, then $b = c$. In particular, if $a + b = a$, then $b = 0$.

4. If $a, b, c \in F$ such that $a \neq 0$ and $a \cdot b = a \cdot c$, then $b = c$. In particular, if $a \neq 0$ and $a \cdot b = a$, then $b = 1$.

5. $a \cdot 0 = 0 \cdot a = 0 \ \forall a \in F$.

6. If $a, b \in F$, then $(-a) \cdot b = a \cdot (-b) = -(a \cdot b)$.

7. If $a, b \in F$, then $(-a) \cdot (-b) = a \cdot b$.

8. If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

Proof.

1. We will show the additive identity is unique. Assume $\exists 0, 0' \in F$ such that $a + 0 = 0 + a = a$ and $a + 0' = 0' + a = a$ for all $a \in F$. Take $a = 0'$ in the first equation and $a = 0$ in the second equation yields $0' + 0 = 0'$ and $0' + 0 = 0$, so $0 = 0'$.

2. We will show that the additive inverse is unique. Let $a \in F$. Assume there exists $-a, a' \in F$ such that $-a + a = a + (-a) = 0$ and $a' + a = a + a' == 0$. Because $a' + a = 0$, then $(a' + a) + (-a) = 0 + (-a)$, so $a' + (a + (-a)) = -a$, but that means $a' + 0 = -a$, so $a' = -a$.

3. Assume $a + b = a + c$. Then $-a + (a + b) = -a + (a + c)$. Therefore, $(-a + a) + b = (-a + a) + c$, so $0 + b = 0 + c$, which means $b = c$. So if $a + b = a = a + 0$, then $b = 0$.

4. We have a proof similar as above.

5. $a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$, so $a \cdot 0 = 0$. Similarly, $0 \cdot a = (0 + 0) \cdot a = 0 \cdot a + 0 \cdot a$, we have $0 \cdot a = 0$.

6. $(-a) \cdot b + a \cdot b = (-a + a) \cdot b = 0 \cdot b = 0$, and so $(-a) \cdot b = -(a \cdot b)$. Similarly, we have $a \cdot (-b) = -(a \cdot b)$.

7. $(-a) \cdot (-b) + [-(a \cdot b)] = (-a) \cdot (-b) + (-a) \cdot b = (-a)(-b + b) = (-a) \cdot 0 = 0$. Therefore, $(-a) \cdot (-b) = a \cdot b$.

8. Assume $a \cdot b = 0$. Assume $a \neq 0$, then $\exists a^{-1} \in F$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$. Now because $a \cdot b = 0$, then $a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0$, and so $(a^{-1} \cdot a) \cdot b = 0$, then $1 \cdot b = 0$, so $b = 0$.

□

Definition 5.6 (Order Relation)**.** An order relation $<$ on a non-empty set A satisfies the following properties:

- Trichotomy: If $a, b \in A$, then one and only one of the following statements holds: $a < b$, or $a = b$, or $b < a$.
- Transitivity: If $a, b, c \in A$ such that $a < b$ and $b < c$, then $a < c$.

Example 5.7. For $a, b \in \mathbb{Z}$, we write $a < b$ if $b - a \in \mathbb{N}$. This is an order relation.

We write $a > b$ if $b < a$, we write $a \leq b$ if $[a < b$ or $a = b]$, and we write $a \geq b$ if $b \leq a$.

Definition 5.8 (Ordered Field)**.** Let $(F, +, \cdot)$ be a field. We say $(F, +, \cdot)$ is an ordered field if it is equipped with an order relation $<$ that satisfies the following:

- $(O1)$: If $a, b, c \in F$ such that $a < b$, then $a + c < b + c$.
- $(O2)$: If $a, b, c \in F$ such that $a < b$ and $0 < c$, then $a \cdot c < b \cdot c$.

6 Lecture 5: Ordered Field

Proposition 6.1. Let $(F, +, \cdot, <)$ be an ordered field. Then,

1. $a > 0 \iff -a < 0$.
2. if $a, b, c \in F$ are such that $a < b$ and $c < 0$, then $a \cdot c > b \cdot c$.
3. if $a \in F \backslash \{0\}$, then $a^2 = a \cdot a > 0$. In particular, $1 > 0$.
4. if $a, b \in F$ are such that $0 < a < b$, then $0 < b^{-1} < a^{-1}$.

Proof. 1. $(\Rightarrow)$: assume $a > 0$, then $a + (-a) > 0 + (-a)$, so $0 > -a$.

$(\Leftarrow)$: assume $-a < 0$, then $-a + a < 0 + a$, then $0 < a$.

2. Assume $a < b$ and $c < 0$, then $-c > 0$, so $a \cdot (-c) < b \cdot (-c)$, which means $-a \cdot c < -b \cdot c$. Therefore, $-ac + (ac + bc) < -bc + (ac + bc)$. We then see $(-ac + ac) + bc < -bc + (bc + ac)$, so $0 + bc < (-bc + bc) + ac$, and so $bc < 0 + ac$, which means $bc < ac$.
3. By trichotomy, exactly one of the following holds:

- if $a > 0$, then $a \cdot a > 0 \cdot a$, so $a^2 > 0$.
- if $a < 0$, then $a \cdot a > 0 \cdot a$, so $a^2 > 0$.

4. First we show that if $a > 0$ then $a^{-1} > 0$. Let us argue by contradiction. Assume $\exists a \in F$ such that $a > 0$ but $a^{-1} \leq 0$. Note $a^{-1} \neq 0$ since a^{-1} has a multiplicative inverse a. Since $a > 0$ and $a^{-1} < 0$, then $a \cdot a^{-1} < 0$, so $1 < 0$. This contradicts the previous part. So if $a > 0$, then $a^{-1} > 0$. Because $0 < a < b$, then $0 \cdot (a^{-1} \cdot b^{-1}) < a \cdot (a^{-1} \cdot b^{-1}) < b \cdot (a^{-1} \cdot b^{-1})$, and so $0 < (a \cdot a^{-1}) \cdot b^{-1} < b \cdot (b^{-1} \cdot a^{-1})$, therefore $0 < 1 \cdot b^{-1} < (b \cdot b^{-1}) \cdot a^{-1}$. Then we have $0 < b^{-1} < 1 \cdot a^{-1}$, therefore $0 < b^{-1} < a^{-1}$.

□

Theorem 6.2. Let $(F, +, \cdot)$ be a field. The following are equivalent:

1. F is an ordered field.
2. There exists $P \subseteq F$that satisfies the following properties:
   - $(O1')$: For every $a \in F$, one and only one of the following statements holds: $a \in P$, or $a = 0$, or $-a \in P$.
   - $(O2')$: If $a, b \in P$, then $a + b \in P$, and $a \cdot b \in P$.

Proof. Let us show that (1) ⇒ (2). Define $P = \{a \in F : a > 0\}$. Let us check $(O1')$. Fix $a \in F$. By trichotomy for the order relation on F, we get that exactly one of the following statements is true: $a > 0$, which implies $a \in P$, or $a = 0$, or $a < 0$, which implies $-a > 0$, so $-a \in P$. We can now check $(O2')$. Fix $a, b \in P$. Because $a \in P$, then $a > 0$, and similarly $b > 0$. Therefore, $a + b > 0 + b = b > 0$, so $a + b \in P$. Also, we know $a \cdot b > 0 \cdot b = 0$, so $a \cdot b \in P$.

We now show that (2) ⇒ (1). For $a, b \in F$, we write $a < b$ if $b - a \in P$. Let us check that this is an order relation.

Trichotomy: fix $a, b \in F$. By $(O1')$, exactly one of the following hold: $b - a \in P$, which means $a < b$, or $b - a = 0$, which means $a = b$, or $-(b - a) \in P$, which means $a - b \in P$ and so $b < a$.

Transitivity: assume $a, b, c \in F$ such that $a < b$ and $b < c$. Therefore, $b - a \in P$ and $c - b \in P$, so $(b - a) + (c - b) = c - a \in P$, and so $a < c$.

We now check that with this order relation, F is an ordered field. We have to check $(O1)$ and $(O2)$.

$(O1)$: fix $a, b, c \in F$ such that $a < b$, then $b - a \in P$, so $(b + c) - (a + c) \in P$, which means $a + c < b + c$.

$(O2)$: fix $a, b, c \in F$ such that $a < b$ and $0 < c$. Because $a < b$, then $b - a \in P$, and because $0 < c$, then $c - 0 = c \in P$. Therefore, $(b-a) \cdot c \in P$, and so $b \cdot c - a \cdot c \in P$, therefore $a \cdot c < b \cdot c$. $\square$

We extend the order relation $<$ from $\mathbb{Z}$ to the field $(\mathbb{Q}, +, \cdot)$ br writing $\frac{a}{b} > 0$ ikf $a \cdot b > 0$.

Let us show that this is well-defined. Specifically, we need to show that if $\frac{a}{b} = \frac{c}{d}$, i.e. $(a, b) \sim (c, d)$, and $a \cdot b > 0$, then $c \cdot d > 0$. Now if $(a, b) \sim (c, d)$, then $a \cdot d = b \cdot c$, so $0 < (ad)^2 = (a \cdot b) \cdot (c \cdot d)$.[2] Therefore, $0 < (ab) \cdot (cd)$ and because $0 < ab$, so $cd > 0$, and therefore $\frac{c}{d} > 0$.

Let $P = \{\frac{a}{b} \in \mathbb{Q} : \frac{a}{b} > 0\}$. By the theorem, to prove that $\mathbb{Q}$ is an ordered field, it suffices to show that P satisfies $(O1')$ and $(O2')$, which is left as an exercise to the readers.

7 Homework 2

Exercise 7.1. Prove that $\sqrt{27}$ is an irrational number.

Exercise 7.2. Computer the following sum

$$\sum_{k=1}^{n} \frac{1}{(k+1)\sqrt{k} + k\sqrt{k+1}}$$

Use mathematical induction to prove your answer holds for all $n \in \mathbb{N}$.

Exercise 7.3. Prove that for every $n \in \mathbb{N}$ there exists a polynomial P_n of degree n so that

$$\cos(n\theta) = P_n(\cos(\theta)) \text{ for all } \theta \in \mathbb{R}.$$

Hint: Find and prove a recurrence relation between P_{n-1}, P_n, and P_{n+1}.

Exercise 7.4. Let $(F, +, \cdot, <)$ be an ordered field with at elast two elements and let 1 denote the identity for multiplication. Show that the equation

$$x^2 = 1$$

has exactly two solutions in F.

Exercise 7.5. Let $(F, +, \cdot)$ be a field with exactly four distinct elements $F = \{0, 1, a, b\}$ where 0 and 1 denote the identities for $+$ and $\cdot$, respectively, and a, b denote the remaining two elements of F. Fill in the addition and multiplication tables below. Use the axioms to justify your answer. (Note that for each table entry there is a unique correct solution.)

[2] Note that $a \cdot d \neq 0$ since $d \neq 0$ and $a \cdot b > 0$, and so $a \neq 0$.

+	0	1	a	b
0				
1				
a				
b				

$\cdot$	0	1	a	b
0				
1				
a				
b				

Hint:

1. Show that in the addition table each row and each column contain every element of F exactly once (as in Sudoku). Specifically, for every $x \in F$, the function

$$f_x : F \to F, \quad f_x(y) = x + y \text{ is one-to-one and onto.}$$

2. Show that the same is true for the rows and columns of the multiplication table that are not identically zero. Specifically, for every $x \in F \backslash \{0\}$, the function

$$g_x : F \to F, \quad g_x(y) = x \cdot y \text{ is one-to-one and onto.}$$

Exercise 7.6. Let $q \geq 2$ be a prime number. Recall the equivalence relation on $\mathbb{Z}$ defined as follows: for $m, n \in \mathbb{Z}$, we write $m \sim n$ if and only if $q \mid (m - n)$. For $n \in \mathbb{Z}$, denote by $C(n)$ the equivalence class of n. Let $\mathbb{Z}/q\mathbb{Z}$ denote the set of equivalence classes. We define addition and multiplication on $\mathbb{Z}/q\mathbb{Z}$ as follows:

$$C(n) + C(m) = C(n + m) \text{ and } C(n) \cdot C(m) = C(nm).$$

1. Prove that addition and multiplication are well defined, that is, the result is independent of the representatives chosen from the equivalence classes.

2. Verify that with these operations $\mathbb{Z}/q\mathbb{Z}$ is a field.

3. Show that there is no order relation on $\mathbb{Z}/q\mathbb{Z}$ that makes it an ordered field.

Exercise 7.7. Define two internal laws of composition on $F = \mathbb{R} \times \mathbb{R}$ as follows:

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$
$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1 - a_2 b_2, a_1 b_2 + a_2 b_1).$$

1. Show that with these operations F is a field.

2. Show that there is no order relation on F that makes F an ordered field.

Exercise 7.8. Define two internal laws of composition on $F = \mathbb{Q} \times \mathbb{Q}$ as follows

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$
$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 b_1 + 2a_2 b_2, a_1 b_2 + a_2 b_1).$$

1. Show that with these operations F is a field.
2. For $(a_1, a_2), (b_1, b_2) \in F$ we write $(a_1, a_2) < (b_1, b_2)$ if $a_1 + a_2\sqrt{2} < b_1 + b_2\sqrt{2}$ in the usual sense on $\mathbb{R}$. Prove that this is an order relation on F and that with it, F is an ordered field.

Remark: Recall that $\sqrt{2}$ is an irrational number. For a proof, see Example 1.1 on page 2 in Rudin.

8 Lecture 6: Bounds

Definition 8.1. Let $(F, +, \cdot, <)$ be an ordered field. Let $\varnothing \neq A \subseteq F$.

- We say that A is bounded above if $\exists M \in F$ such that $a \leq M$ $\forall a \in A$. Then M is called an upper bound for A. If moreover, $M \in A$, then we say that M is the maximum of A.
- We say that A is bounded below if $\exists m \in F$ such that $m \leq a$ $\forall a \in A$. Then m is called a lower bound for A. If moreover, $m \in A$, then we say that m is the minimum of A.
- We say that A is bounded if A is bounded both above and below.

Example 8.2.

- $A = \{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\}$ is a bounded set. 3 is an upper bound for A, $\frac{3}{2}$ is the maximum of A, 0 is a lower bound for A, and 0 is the minimum of A.
- $A = \{x \in \mathbb{Q} : 0 < x^4 \leq 16\}$ is a bounded set. 2 is the maximum of A, and -2 is the minimum of A.
- $A = \{x \in \mathbb{Q} : x^2 < 2\}$ is a bounded set. 2 is an upper bound for A, and -2 is a lower bound for A. But A does not have a maximum. Indeed, let $x \in A$. We will construct $y \in A$ such that $y > x$.

  Define $y = x + \frac{2-x^2}{2+x}$. Because $x \in A$, then $x \in \mathbb{Q}$, so $2 - x^2, 2 + x \in \mathbb{Q}$. Moreover, because $x \in A$, then $2 + x > 0$, and so $\frac{1}{2+x} \in \mathbb{Q}$. Therefore, $\frac{2-x^2}{2+x} \in \mathbb{Q}$. Hence, we know $y \in \mathbb{Q}$.

Also note that $2 - x^2 > 0$ since $x \in A$, and $2 + x > 0$ indicates $\frac{1}{2+x} > 0$, so $\frac{2-x^2}{2+x} > 0$. Therefore, $y = x + \frac{2-x^2}{2+x} > x$.

Let us compute y^2. Note that

$$\begin{aligned} y^2 &= \frac{2x + x^2 + 2 - {x^2}^2}{2+x} \\ &= \frac{4(x+1)^2}{(2+x)^2} \\ &= \frac{4x^2 + 8x + 4}{x^2 + 4x + 4} \\ &= \frac{2(x^2 + 4x + 4) + 2x^2 - 4}{x^2 + 4x + 4} \\ &= 2 + \frac{2 \cdot (x^{-}2)}{(x+2)^2} \\ &< 2. \end{aligned}$$

Collecting the properties above, we constructed $y \in A$ and $y > x$ as desired.

Exercise 8.3. Show that the maximum and minimum of a set are unique, if they exist.

Definition 8.4. Let $(F, +, \cdot, <)$ be an ordered field. Let $\varnothing \neq A \subseteq F$ and assume A is bounded above. We say that L is the least upper bound of A if it satisfies:

1. L is an upper bound of A.
2. If M is an upper bound of A, then $L \leq M$.

We write $L = \sup(A)$ and we say L is the supremum of A.

Lemma 8.5. The least upper bound of a set is unique, if it exists.

Proof. Say that a set A, satisfies $\varnothing A \subseteq F$ and is bounded above, admits two least upper bounds L and M. Because L is a least upper bound, then L is an upper bound for A. But because M is a least upper bound for A, we have $M \leq L$. Similarly we conclude that $L \leq M$, and so $L = M$. □

Definition 8.6. Let $(F, +, \cdot, <)$ be an ordered field. Let $\varnothing \neq A \subseteq F$ and assume A is bounded below. We say that l is the greatest lower bound of A if it satisfies:

1. l is a lower bound of A.
2. If m is a lower bound of A then $m \leq l$.

We write $l = \inf(A)$ and we say l is the infimum of A.

Exercise 8.7. Show that the greatest lower bound of a set is unique, if it exists.

Definition 8.8. Let $(F, +, \cdot, <)$ be an ordered field. Let $\varnothing \neq S \subseteq F$.

We say that S has the least upper bound property if its satisfies the following: for any non-empty subset A of S that is bounded above, there exists a least upper bound of A and $\sup(A) \in S$.

We say that S has the greatest lower bound property if it satisfies the following: $\forall \varnothing \neq A \subseteq S$ with A bounded below, $\exists \inf(A) \in S$.

Example 8.9. $(\mathbb{Q}, +, \cdot, <)$ is an ordered field. Note that

1. Consider $\varnothing \neq\subseteq \mathbb{Q}$, $\mathbb{N}$ has the least upper bound property. Indeed, if $\varnothing \neq A \subseteq N$, A bounded above, then the largest element in A is the least upper bound of A and $\sup(A) \in \mathbb{N}$. $\mathbb{N}$ also has the greatest lower bound property.

2. Consider $\varnothing \neq \mathbb{Q} \subseteq \mathbb{Q}$, but $\mathbb{Q}$ does not have the least upper bound property. Indeed, $\varnothing \neq A = \{x \in \mathbb{Q} : x \geq 0, x^2 < 2\} \subseteq \mathbb{Q}$. Note that A is bounded above by 2. However, $\sup(A) = \sqrt{2} \notin \mathbb{Q}$.

Proposition 8.10. Let $(F, +, \cdot, <)$ be an ordered field. Then F has the least upper bound property if and only if it has the greatest lower bound property.

Proof. We will only prove the $(\Rightarrow)$ direction: the opposite direction has a similar proof.

Assume F has the least upper bound property. Let $\varnothing \neq A \subseteq F$ bounded below. We want to show that $\exists \inf(A) \in F$. Because A is bounded below, then $\exists m \in F$ such that $m \leq a$ $\forall a \in A$. Let $B = \{b \in F : b \text{ is a lower bound for } A\}$. Note $B \neq \varnothing$ because $m \in B$, and we know $B \subseteq F$, and B is bounded above (in fact, every element in A is an upper bound for B), and F has the least upper bound property. Therefore, $\exists \sup(B) \in F$.

Claim 8.11. $\sup(B)$ is a lower bound for A.

Subproof. Indeed, let $a \in A$. We know $a \geq b$ $\forall b \in B$, and $\sup(B)$ is the least upper bound for B, so $a \geq \sup(B)$. As $a \in A$ was arbitrary, we conclude that $\sup(B) \leq a$ $\forall a \in A$, and so $\sup(B)$ is a lower bound for A. ■

Claim 8.12. If l is a lower bound for A, then $l \leq \sup(B)$.

Subproof. Because l is a lower bound for A, then $l \in B$. Also, because $\sup(B)$ is an upper bound for B, we know $l \leq \sup(B)$. ■

Using the two claims above, we find that $\inf(A) = \sup(B)$. □

9 Lecture 7: Archimedean Property

We present an alternative proof of Proposition 8.10.

Remark 9.1 (Alternative Proof)**.** Let $\varnothing \neq A \subseteq F$ be such that A is bounded below. Let $B = \{-a : a \in A\}$. Note $B \subseteq F$ by (A5), and $B \neq \varnothing$ because $A \neq \varnothing$, and B Is bounded above: indeed, if m is a lower bound for A, then $-m$ is an upper bound for B.[3] Also note that F has the least upper bound property. Collecting these properties above, we know $\exists \sup(B) \in F$. The reader can easily show that $-\sup(B) = \inf(A) \in F$.

Theorem 9.2. There exists an ordered field with the least upper bound property. We denote it $\mathbb{R}$ and we call it the set of real numbers. $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield. (We will prove this statement in Theorem 11.4.) Moreover, we have the following uniqueness property: if $(F, +, \cdot, <)$ is an ordered field with the least upper bound property, then F is order isomorphic with $\mathbb{R}$, that is, there exist a bijection $\varphi : \mathbb{R} \to F$ such that

(i) $\varphi(x + y) = \varphi(x) + \varphi(y)$.

(ii) $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$.

(iii) if $x < y$, then $\varphi(x) < \varphi(y)$.

Theorem 9.3. $\mathbb{R}$ has the Archimedean property, that is, $\forall x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ such that $x < n$.

Proof. We argue by contradiction. Assume $\exists x_0 \in \mathbb{R}$ such that $x_0 \geq n \ \forall n \in \mathbb{N}$. Then we know $\varnothing \neq \mathbb{N} \subseteq \mathbb{R}$, $\mathbb{N}$ is bounded above by x_0, and $\mathbb{R}$ has the least upper bound property. Therefore, $\exists L = \sup(\mathbb{N}) \in \mathbb{R}$.

Now we know $L = \sup(\mathbb{N})$ and $L - 1 < L$, so $L - 1$ is not an upper bound for $\mathbb{N}$. That means $\exists n_0 \in \mathbb{N}$ such that $n_0 > L - 1$, so $\sup(\mathbb{N}) = L < n_) + 1 \in \mathbb{N}$. We therefore have a contradiction. $\square$

Remark 9.4. $\mathbb{Q}$ has the Archimedean property. If $r \in \mathbb{Q}$ is such that $r \leq 0$, then choose $n = 1$. If $r \in \mathbb{Q}$ is such that $r > 0$, then write $r = \frac{p}{q}$ for $p, q \in \mathbb{N}$, and we can choose $n = p+1$ since $\frac{p}{q} < p + 1$.

Corollary 9.5. If $a, b \in \mathbb{R}$ are such that $a > 0, b > 0$, then there exists $n \in \mathbb{N}$ such that $n \cdot a > b$.

Proof. Apply the Archimedean property to $x = \frac{b}{a}$. $\square$

[3] Note that $m \leq a \ \forall a \in A$ implies $-m \geq -a \ \forall a \in A$.

Corollary 9.6. If $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.

Proof. Apply the Archimedean property to $x = \frac{1}{\varepsilon}$. □

Lemma 9.7. For any $a \in \mathbb{R}$ there exists $N \in \mathbb{Z}$ such that $N \leq a < N + 1$.

Proof. If $a = 0$, then we can just take $N = 0$.

If $a > 0$. Consider $A = \{n \in \mathbb{Z} : n \leq a\} \subseteq \mathbb{R}$. Obviously $A \neq \varnothing$, as $0 \in A$. We also know A is bounded above by a, and $\mathbb{R}$ has the least upper bound property. Therefore, there exists $L = \sup(A) \in \mathbb{R}$. Now consider $L - 1 < L = \sup(A)$, then $L - 1$ is not an upper bound for A, so there exists $N \in A$ such that $L - 1 < N$, and so $L < N + 1$. But $L = \sup(A)$, so $N + 1 \notin A$. Therefore, $N \in A$, so $N \leq a$, and as $N + 1 \notin A$, then $N + 1 > a$. Therefore, $N \leq a < N + 1$.

If $a < 0$, then $-a > 0$. Then by the case $a > 0$, $\exists n \in \mathbb{Z}$ such that $n \leq -a < n + 1$, so $-n - 1 < a \leq -n$. If $a = -n$, let $N = -n$ and so $N \leq a < N + 1$. If $a < -n$, let $N = -n - 1$, and so $N \leq a < N + 1$. Either way, we conclude the proof. □

Definition 9.8 (Dense)**.** We say that a subset A of $\mathbb{R}$ is dense in $\mathbb{R}$ if for every $x, y \in \mathbb{R}$ such that $x < y$, there exists $a \in A$ such that $x < a < y$.

Lemma 9.9. $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$ such that $x < y$. Since $y - x > 0$, by Corollary 9.6, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < y - x$, so $\frac{1}{n} + x < y$.

Consider $nx \in \mathbb{R}$. By Lemma 9.7, $\exists m \in \mathbb{Z}$ such that $m \leq nx < m + 1$, so $\frac{m}{n} \leq x < \frac{m+1}{n}$. Therefore,

$$x < \frac{m+1}{n} = \frac{m}{n} + \frac{1}{n} \leq x + \frac{1}{n} < y.$$

□

10 Homework 3

Throughout this homework, $\mathbb{R}$ denotes the field of real numbers. It is the unique (up to order isomorphism) *ordered field that satisfies the least upper bound property* — we will prove this in class next week. You may use these properties of $\mathbb{R}$ to solve the exercises below.

Exercise 10.1. Let S be a non-empty bounded subset of $\mathbb{R}$.

1. Prove that $\inf S \leq \sup S$.

2. What can you say about S if $\inf S = \sup S$?

Exercise 10.2. Let S and T be two non-empty bounded subsets of $\mathbb{R}$.

1. Prove that if $S \subseteq T$, then $\inf T \leq \inf S \leq \sup S \leq \sup T$.

2. Prove that $\sup(S \cup T) = \sup\{\sup S, \sup T\}$.

Exercise 10.3. Let A be a non-empty subset of $\mathbb{R}$ which is bounded below and let

$$-A = \{-a : a \in A\}.$$

Prove that $\inf A = -\sup(-A)$.

Exercise 10.4. Let A and B be two non-empty bounded subsets of $\mathbb{R}$ and let

$$S = \{a + b : a \in A \text{ and } b \in B\}.$$

1. Prove that $\sup S = \sup A + \sup B$.

2. Prove that $\inf S = \inf A + \inf B$.

Exercise 10.5. Show that

$$\sup\{r \in \mathbb{Q} : r < a\} = a \text{ for all } a \in \mathbb{R}.$$

Exercise 10.6. Let A and B be two non-empty bounded sets of *positive* real numbers and let

$$C = \{a \cdot b : a \in A \text{ and } b \in B\}.$$

Prove that $\sup C = \sup A \cdot \sup B$.

Exercise 10.7. Let $F_+ = \{\alpha : \alpha \text{ is a cut with } \alpha > \mathbf{0}\}$ be the set of positive Dedekind cuts, where we recall

$$\mathbf{0} = \{q \in \mathbb{Q} : q < 0\}.$$

We define the product of two elements $\alpha, \beta \in F_+$ via

$$\alpha \cdot \beta = \{r \in \mathbb{Q} : r < p \cdot q \text{ for some } 0 < p \in \alpha \text{ and } 0 < q \in \beta\}$$

Prove that this operation satisfies M1 through M5 on F_+.

Remark: You might want to consult the Appendix on page 17 in Rudin for definitions.

11 Lecture 8: Construction of Real Numbers

Remark 11.1. For any two rational numbers $r_1, r_2 \in \mathbb{Q}$ such that $r_1 < r_2$, there exists $s \in \mathbb{Q}$ such that $r_1 < s < r_2$. Indeed, if $r_1 < 0 < r_2$, then we may take $s = 0 \in \mathbb{Q}$. Assume $0 < r_1 < r_2$, write $r_1 = \frac{a}{b}$ and $r_2 = \frac{c}{d}$ with $a, b, c, d \in \mathbb{N}$. Take $s = \frac{ad+bc}{2bd} \in \mathbb{Q}$. Note $r_1 < s < r_2$:

$$r_1 < s \iff \frac{a}{b} < \frac{ad+bc}{2bd} \iff 2ad < ad + bc \iff ad < bc \iff \frac{a}{b} < \frac{c}{d} \iff r_1 < r_2.$$

We leave the construction of s in the remaining cases as an exercise to the readers.

Lemma 11.2. $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$ such that $x < y$, then $x + \sqrt{2} < y + \sqrt{2}$. Because we know $\mathbb{Q}$ is dense in $\mathbb{R}$, we know $\exists q \in \mathbb{Q}$ such that $x + \sqrt{2} < q < y + \sqrt{2}$, so $x < q - \sqrt{2} < y$. It now suffices to prove the following claim.

Claim 11.3. $q - \sqrt{2} \in \mathbb{R}\backslash\mathbb{Q}$.

Subproof. Otherwise, $\exists r \in \mathbb{Q}$ such that $q - \sqrt{2} = r$, so $\sqrt{2} = q - r \in \mathbb{Q}$, contradiction. ∎

□

Theorem 11.4. There exists an ordered field with the least upper bound property. We denote it $\mathbb{R}$ and call it the set of real numbers. $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

Remark 11.5. The rest of the statement in Theorem 9.2 is left as an exercise for the readers.

Proof. We will construct an ordered field with the least upper bound property using Dedekind cuts.

The element of the field are certain subsets of $\mathbb{Q}$ called cuts.

Definition 11.6 (Cut)**.** A cut is a set $\alpha \subseteq \mathbb{Q}$ that satisfies

(i) $\varnothing \neq \alpha \neq \mathbb{Q}$,

(ii) if $q \in \alpha$ and $p \in \mathbb{Q}$ such that $p < q$, then $p \in \alpha$.

(iii) for every $q \in \alpha$, there exists $r \in \alpha$ such that $r > q$, i.e. α has no maximum.

Intuitively, we think of a cut as $\mathbb{Q} \cap (-\infty, a)$.[4] Note that if $\mathbb{Q} \ni q \notin \alpha$, then $q > p$ for all $p \in \alpha$. Indeed, otherwise, if $\exists p_0 \in \alpha$ such that $q \leq p_0$, then by (ii) we would have $q \in \alpha$, contradiction.

[4] Of course, at this point we have not yet constructed $\mathbb{R}$.

We define
$$F = \{\alpha : \alpha \text{ is a cut}\}$$
and we will show that F is an ordered field with the least upper bound property.

Subproof on Order. We first show that there is an order relation on F. For $\alpha, \beta \in F$, we write $\alpha < \beta$ if α is a proper subset of β, i.e. $\alpha \subsetneq \beta$.

- Transitivity: if $\alpha, \beta, \gamma \in F$ are such that $\alpha < \beta$ and $\beta < \gamma$, then $\alpha \subsetneq \beta \subsetneq \gamma$, and so $\alpha \subsetneq \gamma$, so $\alpha < \gamma$.
- Trichotomy: first note that at most one of the following holds: $\alpha < \beta$, or $\alpha = \beta$, or $\beta < \alpha$.

  To prove trichotomy, it thus suffices to show that at least one of the following holds: $\alpha < \beta$, $\alpha = \beta$, or $\alpha < \beta$. We show this by contradiction. Assume that $\alpha < \beta$, $\alpha = \beta$, $\beta < \alpha$ all fail. Then we know that α is not a proper subset of β, $\alpha \neq \beta$, and β is not a proper subset of α, which means $\exists p \in \alpha \backslash \beta$ and $\exists q \in \beta \backslash \alpha$. Therefore, $p > r$ for all $r \in \beta$ and $q > s$ for all $s \in \alpha$. Taking $r = q$ and $s = p$, we get $p > q > p$, which is a contradiction.

Therefore, $<$ defines an order relation on F. ■

We now show that F has the least upper bound property. Let $\varnothing \neq A \subseteq F$ be bounded above by $\beta \in F$. Define $\gamma = \bigcup_{\alpha \in A} \alpha$.

Claim 11.7. $\gamma \in F$.

Subproof of Claim.

- $\gamma \neq \varnothing$ because $A \neq \varnothing$ and $\varnothing \neq \alpha \in A$.
- β being an upper bound for A indicates $\beta \geq \alpha$ for all $\alpha \in A$, and so $\beta \supseteq \alpha$ for all $\alpha \in A$, and therefore $\beta \supseteq \bigcup_{\alpha \in A} \alpha = \gamma$, but since $\beta \neq \mathbb{Q}$, we know that $\gamma \neq \mathbb{Q}$.
- Let $q \in \gamma$ and let $p \in \mathbb{Q}$ such that $p < q$. As $q \in \gamma$, we know $\exists \alpha \in A$ such that $q \in \alpha$. We also know that $\mathbb{Q} \ni p < q$, so $p \in \alpha$ and therefore $p \in \gamma$.
- Consider $q \in \gamma$, then there exists $\alpha \in A$ such that $q \in \alpha$, which means that there exists $r \in \alpha$ such that $q < r$, so $r \in \gamma$ and $q < r$.

Collecting the properties above, we deduce $\gamma \in F$. ■

Claim 11.8. $\gamma = \sup(A)$.

Subproof of Claim. Note $\alpha \subseteq \gamma$ for all $\alpha \in A$, so $\alpha \leq \gamma$ for all $\alpha \in A$. Therefore, γ is an upper bound for A. Moreover, let δ be an upper bound for A, so $\delta \geq \alpha$ for all $\alpha \in A$, but that means $\delta \supseteq \alpha$ for all $\alpha \in A$, and we can deduce that $\delta \supseteq \bigcup_{\alpha \in A} \alpha = \gamma$. Therefore, $\delta \geq \gamma$. ■

We will continue the proof next time. ■

12 Lecture 9: Construction of Real Numbers, Continued

Proof, Continued. We now define addition on the structure F to be

$$\alpha + \beta = \{p + q : p \in \alpha, q \in \beta\}.$$

We now check the axioms and start by $(A1)$, namely, $\alpha + \beta \in F$.

- Note that $\alpha + \beta \neq \varnothing$ because $\alpha \neq \varnothing$ which means $\exists p \in \alpha$, and $\beta \neq \varnothing$, which means $\exists q \in \beta$, and so there exists $p + q \in \alpha + \beta$.

- Note that $\alpha + \beta \neq \varnothing$. Indeed, $\alpha \neq \mathbb{Q}$, so $\exists r \in \mathbb{Q} \backslash \alpha$, so $r > p$ for all $p \in \alpha$; similarly, because $\beta \neq \mathbb{Q}$, so $\exists s \in \mathbb{Q} \backslash \beta$, so $s > q$ for all $q \in \beta$. Therefore, $r + s > p + q$ for all $p \in \alpha$ and $q \in \beta$, and so $r + s \notin \alpha + \beta$.

- Let $r \in \alpha + \beta$ and $s \in \mathbb{Q}$ such that $s < r$. Because $r \in \alpha + \beta$, we know $r = p + q$ for some $p \in \alpha$ and $q \in \beta$. Because $s < r$, then $s < p + q$, and so $\mathbb{Q} \ni s - p < q \in \beta$, therefore $s - p \in \beta$, which means $s = p + (s - p) \in \alpha + \beta$.

- Let $r \in \alpha + \beta$, and so $r = p + q$ for some $p \in \alpha$ and some $q \in \beta$. Because $\alpha \in F$, so $\exists p' \in \alpha$ such that $p' > p$. Similarly, because $\beta \in F$, so $\exists q' \in \beta$ such that $q' > q$. Therefore, $\alpha + \beta \ni p' + q' > p + q = r$. Therefore, $p' + q' \in \alpha + \beta$ is such that $p' + q' > r$.

Collecting all these properties above, we see that $\alpha + \beta \in F$.

We now check $(A2)$: for $\alpha, \beta \in F$, we have

$$\begin{aligned} \alpha + \beta &= \{p + q : p \in \alpha, q \in \beta\} \\ &= \{q + p : q \in \beta, p \in \alpha\} \\ &= \beta + \alpha. \end{aligned}$$

We now check $(A3)$: for $\alpha, \beta, \gamma \in F$, we have

$$\begin{aligned}(\alpha+\beta)+\gamma &= \{s+r : s \in \alpha+\beta, r \in \gamma\} \\ &= \{(p+q)+r : p \in \alpha, q \in \beta, r \in \gamma\} \\ &= \{p+(q+r) : p \in \alpha, q \in \beta, r \in \gamma\} \\ &= \{p+t : p \in \alpha, t \in \beta+\gamma\} \\ &= \alpha+(\beta+\gamma).\end{aligned}$$

We now check $(A4)$: let $0^* = \{q \in \mathbb{Q} : q < 0\}$.

Claim 12.1. $0^* \in F$.

Subproof.

- Note $p^* \neq \varnothing$ because $-1 \in 0^*$.
- Note that $0^* \neq \mathbb{Q}$ because $2 \notin 0^*$.
- Let $q \in 0^*$ and let $p \in \mathbb{Q}$ and $p < q$. Then $q \in 0^*$ implies that $q < 0$, and because $p < q$, then $p < 0$, so $p \in 0^*$.
- Let $q \in 0^*$, then $q < 0$, so $\exists r \in \mathbb{Q}$ such that $q < r < 0$. Therefore, $r \in 0^*$ and $r > q$.

Collecting all these properties, we get $0^* \in F$. ■

Claim 12.2. $\alpha + 0^* = \alpha \;\; \forall \alpha \in F$.

Proof.

- We check $\alpha + 0^* \subseteq \alpha$. Let $r \in \alpha + 0^*$, so $r = p + q$ for some $p \in \alpha$ and some $q \in 0^*$. Therefore, $q < 0$. So we know $\mathbb{Q} \ni r = p + q < p$, and because $p \in \alpha \in F$, so $r \in \alpha$. As r was arbitrary in $\alpha + 0^*$, we find $\alpha + 0^* \subseteq \alpha$.
- We now check $\alpha \subseteq \alpha + 0^*$. Let $p \in \alpha$, so there exists $r \in \alpha$ such that $r > p$. We now write $p = r + (p - r) \in \alpha + 0^*$. As $p \in \alpha$ was arbitrary, this shows that $\alpha \subseteq \alpha + 0^*$.

Collecting the properties above, we get $\alpha + 0^* = \alpha$. ■

We now check $(A5)$: fix $\alpha \in F$. We now define

$$\beta = \{q \in \mathbb{Q} : \exists r \in \mathbb{Q} \text{ with } r > 0 \text{ such that } -q-r \notin \alpha\}.$$

Claim 12.3. $\beta \in F$.

Subproof.

- Note that $\beta\varnothing$. As $\alpha \neq \varnothing$, there exists $p \in \mathbb{Q}\backslash\alpha$, then $(-\beta+1) \in \beta$ because $-[-(p+1)] - 1 = (p+1) - 1 = p \notin \alpha$.

- Note that $\beta \neq \varnothing$. As $\alpha \neq \varnothing$, there exists $p \in \alpha$. Then $-p \notin \beta$ because $\forall r \in \mathbb{Q}$, $r > 0$, we have $-(-p) - r = p - r < p$, and because $p \in \alpha \in F$. Therefore, $p - r \in \alpha$, and so $-p \notin \beta$.

- Let $q \in \beta$ and let $p \in \mathbb{Q}$ such that $p < q$. Because $q \in \beta$, there exists $r \in \mathbb{Q}$ such that $r > 0$ and $-q - r \notin \alpha$, therefore $-q - r > s$ for all $s \in \alpha$. Hence, $-p - r > -q - r > s$ for all $s \in \alpha$, and so $-p - r \notin \alpha$, which means $p \in \beta$.

- Let $q \in \beta$. We want to find $s \in \beta$ such that $s > q$. Because $q \in \beta$, so there exists $r \in \mathbb{Q}$ such that $r > 0$ and $-q - r \notin \alpha$. Therefore, $-(q + \frac{r}{2}) - \frac{r}{2} = -q - r \notin \alpha$, and so $q + \frac{r}{2} \in \beta$. We then define $s = q + \frac{r}{2}$.

Collecting all the properties, we get $\beta \in F$. ∎

Claim 12.4. $\alpha + \beta = 0^*$.

Subproof.

- We first check $\alpha + \beta \subseteq 0^*$. Let $s \in \alpha + \beta$, then $s = p + q$ with $p \in \alpha$ and $q \in \beta$. Because $q \in \beta$, so there exists $r \in \mathbb{Q}$ with $r > 0$ such that $-q - r \notin \alpha$, so $-q - r > p$, which means $\mathbb{Q} \ni p + q < -r < 0$. Therefore, $s = p + q \in 0^*$, and so $\alpha + \beta \subseteq 0^*$.

- We now check $0^* \subseteq \alpha_\beta$. Let $r \in 0^*$, then $r \in \mathbb{Q}$ and $r < 0$.

  Claim 12.5. $\exists N \in \mathbb{N}$ such that $N \cdot (-\frac{r}{2}) \in \alpha$, but $(N+1)(-\frac{r}{2}) \notin \alpha$.

  Subproof. We prove this by contradiction. Assume

  $$\{n \cdot (-\frac{r}{2}) : n \in \mathbb{N}\} \subseteq \alpha.$$

  We will show that in this case $\mathbb{Q} \subseteq \alpha$ and thus reach a contradiction.

  Fix $q \in \mathbb{Q}$. By the Archimedean property for $\mathbb{Q}$, $\exists n \in \mathbb{N}$ such that $n > q \cdot (-\frac{2}{r}) \in \mathbb{Q}$. Therefore, $n \cdot (-\frac{r}{2}) > q$, and because $n \cdot (-\frac{r}{2}) \in \alpha \in F$, and so $q \in \alpha$.As $q \in \mathbb{Q}$ was arbitrary, this shows $\mathbb{Q} \subseteq \alpha$, contradiction. ∎

  We now write $r = N(-\frac{r}{2}) + (N+2) \cdot \frac{r}{2}$, and note that $(N+2)\frac{r}{2} \in \beta$ since

  $$-(N+2) \cdot \frac{r}{2} - \frac{r}{2} = (N+1) \cdot (-\frac{r}{2}) \notin \alpha.$$

  As $r \in 0^*$ was arbitrary, this shows $0^* \subseteq \alpha_\beta$.

Therefore, $\alpha + \beta = 0^*$. ∎

We now check $(O1)$. If $\alpha, \beta, \gamma \in F$ such that $\alpha < \beta$, so $\alpha \subsetneq \beta$, then $\alpha + \gamma \subsetneq \beta + \gamma$, and so $\alpha + \gamma < \beta + \gamma$.

We define multiplication on F as follows: for $\alpha, \beta \in F$ with $\alpha > 0$ and $\beta > 0$, we define

$$\alpha \cdot \beta = \{q \in \mathbb{Q} : q < r \cdot s \text{ for some } 0 < r \in \alpha \text{ and some } 0 < s \in \beta\}.$$

For $\alpha \in F$, we define $\alpha \cdot 0^* = 0^*$. We define

$$\alpha \cdot \beta = \begin{cases} (-\alpha) \cdot (-\beta), & \text{if } \alpha < 0, \beta < 0 \\ -[(-\alpha) \cdot \beta], & \text{if } \alpha < 0, \beta > 0 \\ -[\alpha \cdot (-\beta)], & \text{if } \alpha > 0, \beta < 0 \end{cases}.$$

We leave the proof of properties $(M1)$ through $(M5)$, as well as (D) and $(O2)$ as an exercise for the readers. □

We identify a rational number $r \in \mathbb{Q}$ with the Dedekind cut

$$r^* = \{q \in \mathbb{Q} : q < r\}.$$

One can check that

$$\begin{aligned} r^* + s^* &= (r+s)^* \\ r^* \cdot s^* &= (r \cdot s)^* \\ r < s &\iff r^* < s^* \end{aligned}$$

13 Lecture 10: Sequences

Definition 13.1 (Sequence)**.** A sequence of real number is a function $f : \{n \in \mathbb{Z} : n \geq m\} \to \mathbb{R}$ where m is a fixed integer[5]. We write the sequence as $f(m), f(m+1), f(m+2), \cdots$ or as $\{f(n)\}_{n \geq m}$ or as $\{f_n\}_{n \geq m}$.

Definition 13.2 (Bounded Sequence)**.** We say that a sequence $\{a_n\}_{n \geq 1}$ of real numbers is bounded below (respectively, bounded above, bounded) if the set $\{a_n : n \geq 1\}$ is bounded below (respectively, bounded above, bounded).

We say that the sequence $\{a_n\}_{n \geq 1}$ is

- (monotonically) increasing if $a_n \leq a_{n+1} \ \ \forall n \geq 1$.
- strictly increasing if $a_n < a_{n+1} \ \ \forall n \geq 1$.

[5] m is usually 1 or 0.

- (monotonically) decreasing if $a_n \geq a_{n+1} \quad \forall n \geq 1$.
- strictly decreasing if $a_n > a_{n+1} \quad \forall n \geq 1$.
- monotone if it is either increasing or decreasing.

Example 13.3.
1. $\{a_n\}_{n\geq 1}$ with $a_n = 3 - \frac{1}{n}$ is bounded and strictly increasing.
2. $\{a_n\}_{n\geq 1}$ with $a_n = (-1)^n$ is bounded but not monotone.
3. $\{a_n\}_{n\geq 0}$ with $a_n = n^2$ is bounded below and strictly increasing.
4. $\{a_n\}_{n\geq 0}$ with $a_n = \cos(\frac{n\pi}{3})$ is bounded but not monotone.

To define the notion of convergence of a sequence, we need a notion of distance between two real numbers.

Definition 13.4 (Absolute Value)**.** For $x \in \mathbb{R}$, the absolute value of x is

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

This function satisfies the following:

1. $|x| \geq 0$ for all $x \in \mathbb{R}$.
2. $|x| = 0 \iff x = 0$.
3. $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.[6]
4. $|x \cdot y| = |x| \cdot |y|$ for all $x, y \in \mathbb{R}$.
5. $\big||x| - |y|\big| \leq |x - y|$ for all $x, y \in \mathbb{R}$.[7]

We think of $|x - y|$ as the distance between $x, y \in \mathbb{R}$.

Definition 13.5 (Converge, Limit, Diverge)**.** We say that a sequence $\{a_n\}_{n\geq 1}$ of real numbers converges if $\exists a \in \mathbb{R}$ such that $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ such that $|a_n - a| < \varepsilon \ \forall n \geq n_\varepsilon$.

If this is the case, we say that a is the limit of $\{a_n\}_{n\geq 1}$ and we write $a = \lim\limits_{n\to\infty} a_n$ or $a_n \xrightarrow[n\to\infty]{} a$.

If the sequence does not converge, we say it diverges.

Lemma 13.6. The limit of a convergent sequence is unique.

Proof. We argue by contradiction. Assume that $\{a_n\}_{n\geq 1}$ is a convergent sequence and assume that there exists $a, b \in \mathbb{R}$ such that $a \neq b$ and $a = \lim\limits_{n\to\infty} a_n$ and $b = \lim\limits_{n\to\infty} a_n$. Let $0 < \varepsilon < \frac{|b-a|}{2}$.[8]

[6] This is known as the triangle inequality.

[7] This is known as the inverse triangle inequality.

[8] We can choose such an ε because $\mathbb{Q}$ is dense in $\mathbb{R}$.

Because $a = \lim_{n\to\infty} a_n$, then there exists $n_1(\varepsilon) \in \mathbb{N}$ such that $|a_n - a| < \varepsilon\ \forall n \geq n_1(\varepsilon)$. Similarly, because $b = \lim_{n\to\infty} a_n$, then there exists $n_2(\varepsilon) \in \mathbb{N}$ such that $|a_n - b| < \varepsilon\ \forall n \geq n_2(\varepsilon)$. Now set $n_\varepsilon = \max\{n_1(\varepsilon), n_2(\varepsilon)\}$. Then for $n \geq n_\varepsilon$, we have

$$|b - a| = |b - a_n + a_n - a| \leq |b - a_n| + |a_n - a| < 2\varepsilon < |b - a|.$$

This is a contradiction. □

Example 13.7. We can show that the sequence given by $a_n = \frac{1}{n}$ for all $n \geq 1$ converges to 0.

Let $\varepsilon > 0$. By the Archimedean property, there exists $n_\varepsilon \in \mathbb{N}$ such that $n_\varepsilon > \frac{1}{\varepsilon}$. Then for $n \geq n_\varepsilon$, we have

$$|0 - \frac{1}{n}| = \frac{1}{n} \leq \frac{1}{n_\varepsilon} < \varepsilon.$$

By definition, $\lim_{n\to\infty} \frac{1}{n} = 0$.

Example 13.8. We can show that the sequence given by $a_n = (-1)^n$ for all $n \geq 1$ does not converge.

We argue by contradiction. Assume $\exists a \in \mathbb{R}$ such that $a = \lim_{n\to\infty}(-1)^n$. Let $0 < \varepsilon < 1$. Then $\exists n_\varepsilon \in \mathbb{N}$ such that $|a - (-1)^n| < \varepsilon$ for all $N \geq n_\varepsilon$. By taking $n = 2n_\varepsilon$, we get $|a - 1| < \varepsilon$, and by taking $n = 2n_\varepsilon + 1$, we get $|a + 1| < \varepsilon$. By the triangle inequality,

$$2 = |1 + 1| = |1 - a + a + 1| \leq |1 - a| + |a - 1| < 2\varepsilon < 2.$$

This is a contradiction.

Lemma 13.9. A convergent sequence is bounded.

Proof. Let $\{a_n\}_{n\geq 1}$ be a convergent sequence and let $a = \lim_{n\to\infty} a_n$. There exists $n_1 \in \mathbb{N}$ such that $|a - a_n| < 1$ for all $n \geq n_1$. So $|a_n| \leq |a_n - a| + |a| < 1 + |a|$ for all $n \geq n_1$. Let $M = \max\{1 + |a|, |a_1|, |a_2|, \cdots, |a_{n_1-1}|\}$. Clearly, $|a_n| \leq M$ for all $n \geq 1$, so $\{a_n\}_{n\geq 1}$ is bounded. □

Theorem 13.10. Let $\{a_n\}_{n\geq 1}$ be a convergent sequence and let $a = \lim_{n\to\infty} a_n$. Then for any $k \in \mathbb{R}$, the sequence $\{ka_n\}_{n\geq 1}$ converges and $\lim_{n\to\infty} ka_n = ka$.

Proof. If $k = 0$, then $ka_n = 0$ for all $n \geq 1$, and so $\lim_{n\to\infty} ka_n = 0 = ka$.

If $k \neq 0$, let $\varepsilon > 0$. As $a = \lim_{n\to\infty} a_n$, there exists $n_{\varepsilon,k} \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{|k|}$ for all $n \geq n_{\varepsilon,k}$. Therefore, $|ka_n - ka| = |k| \cdot |a_n - a| < |k| \cdot \frac{\varepsilon}{|k| = \varepsilon}$ for all $n \geq n_{\varepsilon,k}$. By definition, $\lim_{n\to\infty} ka_n = ka$. □

Remark 13.11. The idea is that we want to find $n_\varepsilon \in \mathbb{N}$ such that $\forall n \geq n_\varepsilon$, $|ka_n - ka| < \varepsilon$. But that is equivalent to having $|a_n - a| < \frac{\varepsilon}{|k|}$.

14 Homework 4

Exercise 14.1. Let $(F, +, \cdot, <)$ be an ordered field. Let $1 \in F$ denote the multiplicative identity. We define the set of natural numbers $\mathbb{N}_F$ in the field F recursively via

$$1 \in \mathbb{N}_F \quad \text{and} \quad \text{if } \mathbf{n} \in \mathbb{N}_F \text{ then } \mathbf{n} + 1 \in \mathbb{N}_F.$$

Here $\mathbf{n} = 1 + \ldots + 1$, where the right-hand side contains n summands. Prove that $\mathbb{N}_F$ satisfies the Peano Axioms.

Remark 14.2. Let $(F, +, \cdot, <)$ be an ordered field. Building on the previous exercise, we can define the set of integers $\mathbb{Z}_f$ in the field F via

$$\mathbb{Z}_F = \mathbb{N}_F \cup \{0\} \cup \{-\mathbf{n} : \mathbf{n} : \mathbb{N}_F\},$$

where 0 denotes the additive identity in the field.

Moreover, we can define an equivalence relation on the set of pairs $\mathbb{Z}_F \times (\mathbb{Z}_F \backslash \{0\})$ via $(a, b) \sim (c, d)$ if and only if $a \cdot d = b \cdot c$. Then proceeding as in lecture, we can define the set of rational numbers $\mathbb{Q}_F$ in the field F as the set of all equivalence classes:

$$\mathbb{Q}_F = \{\frac{a}{b} : \frac{a}{b} \text{ is the equivalence class of a pair } (a, b) \in \mathbb{Z}_F \times (\mathbb{Z}_F \backslash \{0\})\}$$

Continuing as in lecture, we can define the operations of addition and multiplication, as well as an order relation on $\mathbb{Q}_F$ with respect to which $\mathbb{Q}_F$ is an ordered field.

Exercise 14.3. Let F be an ordered field with the least upper bound property. Prove that there is a unique function $\phi : \mathbb{Q} \to \mathbb{Q}_F$ that satisfies the following properties:

$$\phi(p + q) = \phi(p) + \phi(q), \ \ \phi(p \cdot q) = \phi(p) \cdot \phi(q), \quad \text{if } p < q \text{ then } \phi(p) < \phi(q) \text{ for any } p, q \in \mathbb{Q}.$$

Hint: When constructing ϕ, work your way up from $\mathbb{N}$, to $\mathbb{Z}$, and then to $\mathbb{Q}$.

Exercise 14.4. Let F be an ordered field with the least upper bound property and let $\phi : \mathbb{Q} \to F$ be given by the map constructed in Exercise 2. For $x \in \mathbb{R}$, let

$$A_x = \{\phi(r) : r \in \mathbb{Q} \text{ with } r < x\}.$$

1. Show that A_x is a non-empty subset of F and is bounded above. We define

$$\phi(x) = \sup A_x.$$

2. Show that this extension of ϕ from $\mathbb{Q}$ to $\mathbb{R}$ satisfies the following: for any $x, y \in \mathbb{R}$,

$$\phi(x + y) = \phi(x) + \phi(y), \ \ \phi(x \cdot y) = \phi(x) \cdot \phi(y), \ \text{if } x < y \text{ then } \phi(x) < \phi(y).$$

3. Show that $\phi : \mathbb{R} \to F$ is bijective.

Hint: When proving surjectivity, for $z \in F$ consider the set

$$B_z = \{r \in \mathbb{Q} : \phi(r) < z\}$$

Show that B_z is a non-empty subset of $\mathbb{R}$, which is bounded above. As $\mathbb{R}$ has the least upper bound property, there exists $x \in \mathbb{R}$ such that $x = \sup B_z$. Show that $\phi(x) = z$.

15 Lecture 11: Sequences, Continued

Theorem 15.1. Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two convergent sequences of real numbers and let $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. Then

1. the sequence $\{a_n + b_n\}_{n\geq 1}$ converges and $\lim_{n\to\infty}(a_n + b_n) = a + b$.
2. the sequence $\{a_n \cdot b_n\}_{n\geq 1}$ converges and $\lim_{n\to\infty}(a_n b_n) = a \cdot b$.
3. if $a \neq 0$ and $a_n \neq 0$ for all $n \geq 1$, then $\{\frac{1}{a_n}\}_{n\geq 1}$ converges and $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$.
4. if $a \neq 0$ and $a_n \neq 0$ for all $n \geq 1$, then $\{\frac{b_n}{a_n}\}_{n\geq 1}$ converges and $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{b}{a}$.

Proof. 1. Let $\varepsilon > 0$. We want to find $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$,

$$|(a+b) - (a_n + b_n)| < \varepsilon.$$

Then it suffices to find large enough n such that $|a - a_n| < \frac{\varepsilon}{2}$ and $|b - b_n| < \frac{\varepsilon}{2}$, which means

$$|(a+b) - (a_n + b_n)| < |a - a_n| + |b - b_n| < \varepsilon.$$

As $\lim_{n\to\infty} a_n = a$, then there exists $n_1(\varepsilon) \in \mathbb{N}$ such that $|a - a_n| < \frac{\varepsilon}{2}$ for all $n \geq n_1(\varepsilon)$. Similarly, as $\lim_{n\to\infty} b_n = b$, then there exists $n_2(\varepsilon) \in \mathbb{N}$ such that $|b - b_n| < \frac{\varepsilon}{2}$ for all $n \geq n_2(\varepsilon)$.

Now let $n_\varepsilon = \max\{n_1(\varepsilon), n_2(\varepsilon)\}$. Then for $n \geq n_\varepsilon$, we have

$$|(a+b) - (a_n + b_n)| \leq |a - a_n| + |b - b_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

By definition, $\lim_{n\to\infty}(a_n + b_n) = a + b$.

2. Let $\varepsilon > 0$. As $\{a_n\}_{n\geq 1}$ converges, it is bounded. Let $M > 0$ be such that $|a_n| \leq M$ for all $n \geq 1$.

   We want to find $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$, $|ab - a_n b_n| < \varepsilon$. To find such n_ε, it suffices to make it large enough so that $|a - a_n| \cdot |b| < \frac{\varepsilon}{2}$ and $|a_n| \cdot |b - b_n| < \frac{\varepsilon}{2}$, then we know that

   $$|ab - a_n b_n| = |(a - a_n) \cdot b + a_n(b - b_n)| \leq |a - a_n| \cdot |b| + |a_n| \cdot |b - b_n| < \varepsilon.$$

   To do so, it suffices to take $|a - a_n| < \frac{\varepsilon}{2(|b|+1)}$ and $|b - b_n| < \frac{\varepsilon}{2M}$, where $M > 0$ is such that $|a_n| \leq M$ for all $n \geq 1$.[9]

   As $\lim\limits_{n\to\infty} a_n = a$, there exists $n_1(\varepsilon) \in \mathbb{N}$ such that $|a - a_n| < \frac{\varepsilon}{2(|b|+1)}$ for all $n \geq n_1(\varepsilon)$. Similarly, as $\lim\limits_{n\to\infty} b_n = b$, there exists $n_2(\varepsilon) \in \mathbb{N}$ such that $|b - b_n| < \frac{\varepsilon}{2M}$ for all $n \geq n_2(\varepsilon)$. Set $n_\varepsilon = \max\{n_1(\varepsilon), n_2(\varepsilon)\}$. For $n \geq n_\varepsilon$, we have

   $$\begin{aligned}
   |ab - a_n b_n| &= |(a - a_n)b + a_n(b - b_n)| \\
   &\leq |a - a_n| \cdot |b| + |a_n| \cdot |b - b_n| \\
   &< \frac{\varepsilon}{2(|b|+1)} \cdot |b| + M \cdot \frac{\varepsilon}{2M} \\
   &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
   &= \varepsilon.
   \end{aligned}$$

   By definition, $\lim\limits_{n\to\infty}(a_n b_n) = ab$.

3. Let $\varepsilon > 0$. We want to find $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$, $|\frac{1}{a} - \frac{1}{a_n}| < \varepsilon$. Note that

   $$\left|\frac{1}{a} - \frac{1}{a_n}\right| = \frac{|a_n - a|}{|a| \cdot |a_n|} < \varepsilon$$

   and so we want $|a_n - a| < \varepsilon |a| \cdot |a_n|$.

   As $a = \lim\limits_{n\to\infty} a_n$, there exists $n_1(a) \in \mathbb{N}$ such that $|a - a_n| < \frac{|a|}{2}$ for all $n \geq n_1(a)$. Then, for all $n \geq n_1$, we have

   $$|a_n| \geq |a| - |a - a_n| > |a| - \frac{|a|}{2} = \frac{|a|}{2}.$$

   Moreover, there exists $n_2(\varepsilon, a) \in \mathbb{N}$ such that $|a - a_n| < \frac{\varepsilon |a|^2}{2}$ for all $n \geq n_2(\varepsilon, a)$.

[9] While the obvious choice for $|b - b_n|$ is to bound it by $\frac{\varepsilon}{|a_n|}$, note that this does not guarantee us to shrink to less than $\frac{\varepsilon}{2}$.

Now let $n_\varepsilon = \max\{n_1(a), n_2(\varepsilon, a)\}$. For $n \geq n_\varepsilon$, we have

$$\left|\frac{1}{a} - \frac{1}{a_n}\right| = \frac{|a - a_n|}{|a| \cdot |a_n|} < \frac{\varepsilon |a|^2}{2|a|} \cdot \frac{2}{|a|} = \varepsilon.$$

By definition, $\lim\limits_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$.

4. We leave this as an exercise.

□

Example 15.2.

$$\begin{aligned}\lim_{n\to\infty} \frac{n^3 + 5n + 8}{3n^3 + 2n^2 + 7} &= \lim_{n\to\infty} \frac{1 + \frac{5}{n^2} + \frac{8}{n^3}}{3 + \frac{2}{n} + \frac{7}{n^3}} \\ &= \frac{1 + 5 \cdot \lim\limits_{n\to\infty} \frac{1}{n^2} + 8 \cdot \lim\limits_{n\to\infty} \frac{1}{n^3}}{3 + 2 \cdot \lim\limits_{n\to\infty} \frac{1}{n} + 7 \cdot \lim\limits_{n\to\infty} \frac{1}{n^3}} \\ &= \frac{1 + 5 \cdot 0 + 8 \cdot 0}{3 + 2 \cdot 0 + 7 \cdot 0} \\ &= \frac{1}{3}.\end{aligned}$$

Theorem 15.3. Every bounded monotone sequence converges.

Proof. We will show that an increasing sequence bounded above converges. A similar argument can be used to show that a decreasing sequence bounded below converges.

Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers that is bounded above and $a_{n+1} \geq a_n$ for all $n \geq 1$. As $\varnothing \neq \{a_n : n \geq 1\} \subseteq \mathbb{R}$ is bounded above and $\mathbb{R}$ has the least upper bound property, there exists $a \in \mathbb{R}$ such that $a = \sup\{a_n : n \geq 1\}$. It now suffices to prove that this number is the point of convergence we want.

Claim 15.4. $a = \lim\limits_{n\to\infty} a_n$.

Subproof. Let $\varepsilon > 0$. Then $a - \varepsilon$ is not an upper bound for $\{a_n : n \geq 1\}$. Therefore, there exists $n_\varepsilon \in \mathbb{N}$ such that $a - \varepsilon < a_{n_\varepsilon}$. Therefore, for $n \geq n_\varepsilon$, we have

$$a - \varepsilon < a_{n_\varepsilon} \leq a_n \leq a < a + \varepsilon,$$

which means $|a_n - a| < \varepsilon$. This proves the claim. ■

□

Definition 15.5. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers.

We write $\lim\limits_{n\to\infty} a_n = \infty$ and say that $\{a_n\}_{n\geq 1}$ diverges to $+\infty$ if $\forall M > 0$, $\exists n_M \in \mathbb{N}$ such that $a_n > M$ for all $n \geq n_M$.

We write $\lim\limits_{n\to\infty} a_n = -\infty$ and say that $\{a_n\}_{n\geq 1}$ diverges to $-\infty$ if $\forall M < 0$, $\exists n_M \in \mathbb{N}$ such that $a_n < M$ for all $n \geq n_M$.

Exercise 15.6.

1. Show that $\lim_{n\to\infty}(\sqrt[3]{n}+1)=\infty$.
2. Show that the sequence given by $a_n=(-1)^n n$ for all $n\geq 1$ does not diverge to ∞ or to $-\infty$.
3. Let $\{a_n\}_{n\geq 1}$ be a sequence of positive real numbers. Show that

$$\lim_{n\to\infty} a_n=\infty \iff \lim_{n\to\infty}\frac{1}{a_n}=0.$$

16 Lecture 12: Cauchy Sequence

Example 16.1. We can show that $\lim_{n\to\infty}\frac{n^2+1}{n+3}=\infty$.

Let $M>0$. We want to find $n_M\in\mathbb{N}$ such that for all $n\geq n_M$ we have $\frac{n^2+1}{n+3}>M$. Note that it suffices to ask $\frac{n}{4}>M$, and then

$$\frac{n^2+1}{n+3}>\frac{n^2}{n+3}>\frac{n^2}{4n}=\frac{n}{4}>M.$$

By the Archimedean property, there exists $n_M\in\mathbb{N}$ such that $n_M>4M$, then for $n\geq n_M$, we have the desired equation above. By the definition, $\lim_{n\to\infty}\frac{n^2+1}{n+3}=\infty$.

Definition 16.2. We say that a sequence of real numbers $\{a_n\}_{n\geq 1}$ is a Cauchy sequence if

$$\forall\varepsilon>0\ \exists n_\varepsilon\in\mathbb{N} \text{ such that } |a_n-a_m|<\varepsilon\ \forall n,m\geq n_\varepsilon.$$

Theorem 16.3 (Cauchy Criterion)**.** A sequence of real numbers is Cauchy if and only if it converges.

We will split the proof of this theorem into various lemmas and properties.

Proposition 16.4. Any convergent sequence is a Cauchy sequence.

Proof. Let $\{a_n\}_{n\geq 1}$ be a convergent sequence and let $a=\lim_{n\to\infty}a_n$. Let $\varepsilon>0$. As $a_n\xrightarrow[n\to\infty]{a}$, there exists $n_\varepsilon\in\mathbb{N}$ such that $|a-a_n|<\frac{\varepsilon}{2}$ for all $n\geq n_\varepsilon$. Then for $n,m\geq n_\varepsilon$, we have

$$|a_n-a_m|\leq|a_n-a|+|a-a_m|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

□

Lemma 16.5. A Cauchy sequence is bounded.

Proof. Let $\{a_n\}_{n\geq 1}$ be a Cauchy sequence. Then there exists $n_1 \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $n, m \geq n_1$. So taking $m = n_1$, we get

$$|a_n| \leq |a_{n_1}| + |a_n - a_{n_1}| < |a_{n_1}| + 1$$

for all $n \geq n_1$. Now let $M = \max\{|a_1|, |a_2|, \cdots, |a_{n_1-1}|, |a_{n_1}| + 1\}$. Clearly, $|a_n| \leq M$ for all $n \geq 1$. $\square$

Definition 16.6 (Subsequence)**.** Let $\{k_n\}_{n\geq 1}$ be a sequence of natural numbers such that $k_1 \geq 1$ and $k_{n+1} > k_n$ for all $n \geq 1$. Using induction, it is easy to see that $k_n \geq n$ for all $n \geq 1$. If $\{a_n\}_{n\geq 1}$ is a sequence, we say that $\{a_{k_n}\}_{n\geq 1}$ is a subsequence of $\{a_n\}_{n\geq 1}$.

Example 16.7. The following are subsequences of $\{a_n\}_{n\geq 1}$:

- $\{a_{2n}\}_{n\geq 1}$.
- $\{a_{2n-1}\}_{n\geq 1}$.
- $\{a_{n^2}\}_{n\geq 1}$.
- $\{a_{p_n}\}_{n\geq 1}$ where p_n denotes the nth prime.

Theorem 16.8. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Then $\lim\limits_{n\to\infty} a_n = a \in \mathbb{R} \cup \{\pm\infty\}$ if and only if every subsequence $\{a_{k_n}\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$ satisfies $\lim\limits_{n\to\infty} a_{k_n} = a$.

Proof. We will consider $a \in \mathbb{R}$. The cases $a \in \{\pm\infty\}$ can be handled by an analogous argument.

($\Leftarrow$): Take $k_n = n$ for all $n \geq 1$.

($\Rightarrow$): Assume $\lim\limits_{n\to\infty} a_n = a$ and let $\{a_{k_n}\}_{n\geq 1}$ be a subsequence of $\{a_n\}_{n\geq 1}$. Let $\varepsilon > 0$. As $a_n \xrightarrow[n\to\infty]{} a$, $\exists n_\varepsilon \in \mathbb{N}$ such that $|a - a_n| < \varepsilon$ for all $n \geq n_\varepsilon$. Recall that $k_n \geq n$ for all $n \geq 1$. So for $n \geq n_\varepsilon$ we have $k_n \geq n \geq n_\varepsilon$ and so $|a - a_{k_n}| < \varepsilon$ for all $n \geq n_\varepsilon$. By definition, $\lim\limits_{n\to\infty} a_{k_n} = a$. $\square$

Proposition 16.9. Every sequence of real numbers has a monotone subsequence.

Proof. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. We say that the nth term term is dominant if $a_n > a_m$ for all $m > n$. We distinguish two cases:

Case 1: There are infinitely many dominant terms. Then a subsequence formed by these dominant terms is strictly decreasing.

Case 2: There are none of finitely many dominant terms. Let N be larger that the largest index of the dominant terms. So for all $n \geq N$, a_n is not dominant. Set $k_1 = N$, $a_{k_1} = a_N$. Because a_{k_1} is not dominant, there exists $k_2 > k_1$ such that $a_{k_2} \geq a_{k_1}$. Now $k_2 > k_1 = N$, then a_{k_2} is not dominant, so there exists $k_3 > k_2$ such that $a_{k_3} \geq a_{k_2}$. Proceeding inductively, we construct a subsequence $\{a_{k_n}\}_{n\geq 1}$ such that $a_{k_{n+1}} \geq a_{k_n}$ for all $n \geq 1$. $\square$

Theorem 16.10 (Bolzano-Weierstrass)**.** Any bounded sequence has a convergent subsequence.

Proof. Let $\{a_n\}_{n\geq 1}$ be a bounded sequence. By the previous proposition, there exists $\{a_{k_n}\}_{n\geq 1}$ monotone subsequence of $\{a_n\}_{n\geq 1}$. As $\{a_n\}_{n\geq 1}$ is bounded, so is $\{a_{k_n}\}_{n\geq 1}$. As bounded monotone sequences converge, $\{a_{k_n}\}_{n\geq 1}$ converges. □

Corollary 16.11. Every Cauchy sequence has a convergent subsequence.

Lemma 16.12. A Cauchy sequence with a convergent subsequence converges.

Proof. Let $\{a_n\}_{n\geq 1}$ be a Cauchy sequence such that $\{a_{k_n}\}_{n\geq 1}$ is a convergent subsequence. Let $a = \lim\limits_{n\to\infty} a_{k_n}$. Let $\varepsilon > 0$. As $a_{k_n} \xrightarrow[n\to\infty]{} a$, there exists $n_1(\varepsilon)$ such that $|a - a_{k_n}| < \frac{\varepsilon}{2}$ for all $n \geq n_1(\varepsilon)$. As $\{a_n\}_{n\geq 1}$ is Cauchy, there exists $n_2(\varepsilon)$ such that $|a_n - a_m| < \frac{\varepsilon}{2}$ for all $n, m \geq n_2(\varepsilon)$. Let $n_\varepsilon = \max\{n_1(\varepsilon), n_2(\varepsilon)\}$. Then for $n \geq n_\varepsilon$, we have

$$|a - a_n| \leq |a - a_{k_n}| + |a_{k_n} = a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

because $k_n \geq n \geq n_\varepsilon$. By definition, $\lim\limits_{n\to\infty} a_n = a$. □

Combining the last two results, we see that a Cauchy sequence of real numbers converges.

17 Lecture 13: Limit Superior and Limit Inferior

Let $\{a_n\}_{n\geq 1}$ be a bounded sequence of real numbers (convergent or not). The asymptotic behavior of $\{a_n\}_{n\geq 1}$ depends on sets of the form $\{a_n : n \geq N\}$ for $N \in \mathbb{N}$.

As $\{a_n\}_{n\geq 1}$ bounded, the set $\{a_n : n \geq N\}$ (where $N \in \mathbb{N}$ is fixed) is a non-empty bounded subset of $\mathbb{R}$.

As $\mathbb{R}$ has the least upper bound property (and so also the greatest lower bound property), the set $\{a_n : n \geq N\}$ has an infimum and a supremum in $\mathbb{R}$.

For $N \geq 1$, let $u_N = \inf\{a_n : n \geq N\}$ and $v_N = \sup\{a_n : n \geq N\}$. Clearly, $u_N \leq v_N$ for all $N \geq 1$.

Notice that for $N \geq 1$, we have $\{a_n : n \geq N\} \supseteq \{a_n : n \geq N+1\}$, therefore

$$\begin{cases} \inf\{a_n : n \geq N\} \leq \inf\{a_n : n \geq N+1\} \\ \sup\{a_n : n \geq N\} \geq \sup\{a_n : n \geq N+1\} \end{cases}$$

So $u_N \leq u_{N+1}$ and $v_{N+1} \leq v_N$ for all $N \geq 1$. Thus, $\{u_N\}_{N\geq 1}$ is increasing and $\{v_N\}_{N\geq 1}$ is decreasing. Moreover, for all $N \geq 1$, we have

$$u_1 \leq u_2 \leq \cdots \leq u_N \leq v_N \leq \cdots \leq v_2 \leq v_1$$

So the two sequences are bounded. As monotone bounded sequences converge, we know the two sequences must converge.

Let

$$u = \lim_{N\to\infty} u_N = \sup\{u_N : N \geq 1\} =: \sup_N u_N$$

and

$$v = \lim_{N\to\infty} v_N = \sup\{v_N : N \geq 1\} =: \inf_N v_N$$

Because of the boundedness, we see that $u_M \leq v_N$ for all $M, N \geq 1$, and so $\lim\limits_{M\to\infty} u_M \leq v_N$ for all $N \geq 1$. Therefore, $u \leq v_N$ for all $N \geq 1$, and therefore $u \leq \lim\limits_{N\to\infty} v_N$, which means $u \leq v$.

Moreover, if $\lim\limits_{n\to\infty} a_n$ exists, then for all $N \geq 1$, we have

$$u_N = \inf\{a_n : n \geq N\} \leq a_n \leq \sup\{a_n : n \geq N\} = v_N$$

for all $n \geq N$. Therefore, $u_N \leq \lim\limits_{n\to\infty} a_n \leq v_N$, and so

$$u = \lim_{N\to\infty} u_N \leq \lim_{n\to\infty} a_n \leq \lim_{N\to\infty} v_N = v.$$

Definition 17.1. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. We define

$$\limsup_{n\to\infty} a_n = \lim_{N\to\infty} \sup\{a_n : n \geq N\} = \lim_{N\to\infty} v_N = \inf_N v_N = \inf_N \sup_{n\geq N} a_n$$

and

$$\liminf_{n\to\infty} a_n = \lim_{N\to\infty} \inf\{a_n : n \geq N\} = \lim_{N\to\infty} u_N = \sup_N u_N = \sup_N \inf_{n\geq N} a_n$$

with the convention that if $\{a_n\}_{n\geq 1}$ is unbounded above, then $\limsup\limits_{n\to\infty} a_n = \infty$ and if $\{a_n\}_{n\geq 1}$ is unbounded below then $\liminf\limits_{n\to\infty} a_n = -\infty$.

Remark 17.2. We have

$$\inf\{a_n : n \geq 1\} \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq \sup\{a_n : n \geq 1\}.$$

Note that $\liminf\limits_{n\to\infty} a_n$ is the smallest value that infinitely many a_n get close to, and $\limsup\limits_{n\to\infty} a_n$ is the largest value that infinitely many a_n get close to.

Example 17.3. Consider $a_n = 3 + \frac{(-1)^n}{n}$, then $\lim\limits_{n\to\infty} a_n = 3$, and therefore $\liminf\limits_{n\to\infty} a_n = \limsup\limits_{n\to\infty} a_n = 3$. Observe that $\inf\{a_n : n \geq 1\} = 2 \neq 3$ and $\sup\{a_n : n \geq 1\} = \frac{7}{2} \neq 3$.

Theorem 17.4. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers.

1. If $\lim\limits_{n\to\infty} a_n$ exists in $\mathbb{R} \cup \{\pm\infty\}$, then $\liminf\limits_{n\to\infty} a_n = \limsup\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_n$.

2. If $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n \in \mathbb{R} \cup \{\pm\infty\}$, then $\lim_{n\to\infty} a_n$ exists and

$$\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

Proof. 1. We distinguish three cases.

- Case 1: $\lim_{n\to\infty} a_n = -\infty$. It is enough to show $\limsup_{n\to\infty} a_n = -\infty$ since $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$.
  Fix $M < 0$. As $\lim_{n\to\infty} a_n = -\infty$, there exists $n_M \in \mathbb{N}$ such that $a_n < M$ for all $n \geq n_M$, then for $N \geq n_M$, we have $v_N = \sup\{a_n : n \geq N\} \leq M$.[10] Now by definition, $\limsup_{n\to\infty} a_n = \lim_{N\to\infty} v_N = -\infty$.
- Case 2: $\lim_{n\to\infty} a_n = \infty$. The proof is essentially the same as above, and we leave this as an exercise.
- Case 3: $\lim_{n\to\infty} a_n = a \in \mathbb{R}$. Fix $\varepsilon > 0$. Then $\exists n_\varepsilon \in \mathbb{N}$ such that $|a - a_n| < \frac{\varepsilon}{2}$ for all $n \geq n_\varepsilon$. So we know
  $$a - \frac{\varepsilon}{2} < a_n < a + \frac{\varepsilon}{2}$$
  for all $n \geq n_\varepsilon$. Thus, for $N \geq n_\varepsilon$, we have
  $$a - \frac{\varepsilon}{2} \leq \inf\{a_n : n \geq N\} \leq \sup\{a_n : n \geq N\} \leq a + \frac{\varepsilon}{2}$$
  which means $a - \frac{\varepsilon}{2} \leq u_N \leq v_N \leq a + \frac{\varepsilon}{2}$.
  Therefore, for all $N \geq n_\varepsilon$, we have $|u_N - a| \leq \frac{\varepsilon}{2} < \varepsilon$ and $|v_N - a| \leq \frac{\varepsilon}{2} < \varepsilon$ for all $N \geq n_\varepsilon$. By definition, that means $\liminf_{n\to\infty} a_n = \lim_{N\to\infty} u_N = a$ and $\limsup_{n\to\infty} a_n = \lim_{N\to\infty} v_N = a$.

2. Again, we distinguish three cases.

- Case 1: $\liminf_{n\to\infty} = \limsup_{n\to\infty} a_n = -\infty$. We will use $\limsup_{n\to\infty} a_n = -\infty$. Fix $M < 0$. Then since $\limsup_{n\to\infty} a_n = \lim_{N\to\infty} v_N = -\infty$, then there exists $N_M \in \mathbb{N}$ such that $v_N < M$ for all $N \geq N_M$. In particular, $v_{N_M} = \sup\{a_n : n \geq N_M\} < M$, which means $a_n < M$ for all $n \geq N_M$. By definition, that means $\lim_{n\to\infty} a_n = -\infty$.
- Case 2: $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \infty$. The proof is essentially the same as above, and we leave this as an exercise.

[10] Note that when taking supremum, the $<$ sign can be changed to $\leq$. For example, $a_n = 3 - \frac{1}{n}$ has the property of $a_n < 3$ for all $n \geq 1$, but $\sup_{n\geq 1} a_n = 3$.

- Case 3: $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = a \in \mathbb{R}$. Fix $\varepsilon > 0$. Because $a = \liminf_{n\to\infty} = \lim_{N\to\infty} u_N$, then there exists $N_1(\varepsilon) \in \mathbb{N}$ such that $|u_N - a| < \varepsilon$ for all $N \geq N_1$. Therefore, $a - \varepsilon < u_{N_1} = \inf\{a_n : n \geq N_1\} < a + \varepsilon$, and we have $a - \varepsilon < a_n$ for all $n \geq N_1$.

  Similarly, considering the limit supremum, there exists $N_2(\varepsilon) \in \mathbb{N}$ such that $|v_N - a| < \varepsilon$ for all $N \geq N_2$, and so $a - \varepsilon < v_{N_2} = \inf\{a_n : n \geq N_2\} < a + \varepsilon$, which means $a_n < a + \varepsilon$ for all $n \geq N_2$.

  Thus, for $n \geq \max\{N_1, N_2\}$, we have $a - \varepsilon < a_n < a + \varepsilon$, which means $|a_n - a| < \varepsilon$. By definition, $\lim_{n\to\infty} a_n = a$.

$\square$

18 Homework 5

Exercise 18.1. (i) Show that for any two real numbers x and y we have

$$\big||x| - |y|\big| \leq |x - y|.$$

(ii) Show that if a sequence $\{a_n\}_{n\in\mathbb{N}}$ of real numbers converges to a, then the sequence $\{|a_n|\}_{n\in\mathbb{N}}$ converges to $|a|$. Show (via an example) that the converse is not true.

Exercise 18.2. Let $\{a_n\}_{n\geq 1}, \{b_n\}_{n\geq 1}$ and $\{c_n\}_{n\geq 1}$ be three convergent sequences of real numbers such that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n \quad \text{and} \quad a_n \leq b_n \leq c_n \text{ for all } n \geq 1.$$

Show that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$.

Exercise 18.3. Prove that

$$\lim_{n\to\infty} \sqrt{4n^2 + n} - 2n = \frac{1}{4}.$$

Exercise 18.4. Let $\{a_n\}_{n\geq 1}$ be a convergent sequence of real numbers.

1. Show that if for all but finitely many a_n we have $a_n \geq a$, then $\lim_{n\to\infty} a_n \geq a$.
2. Show that if for all but finitely many a_n we have $a_n \leq b$, then $\lim_{n\to\infty} a_n \leq b$.
3. Conclude that if all but finitely many a_n belong to the interval $[a, b]$, then $\lim_{n\to\infty} a_n \in [a, b]$.

Exercise 18.5. Let $\{a_n\}_{n\geq 1}$ be a convergent sequence of real numbers and let $a \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n > a$. Show that there exists $n_0 \in \mathbb{N}$ such that $a_n > a$ for all $n \geq n_0$.

Exercise 18.6. Let $\{a_n\}_{n\geq 1}$ be a Cauchy sequence of real numbers. Show that $\{a_n^2\}_{n\geq 1}$ is also a Cauchy sequence.

Exercise 18.7. (In this exercise you will see a Cauchy sequence of rational numbers converging to an irrational number.) Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence defined by the following rule:

$$a_1 = 3 \text{ and } a_{n+1} = \frac{a_n}{2} + \frac{3}{2a_n} \text{ for all } n \geq 1.$$

1. Show that the sequence is bounded below.
2. Show that this is a sequence of rational numbers.
3. Prove that the sequence is monotonically decreasing.
4. Deduce that $\{a_n\}_{n\in\mathbb{N}}$ converges and find its limit.

Exercise 18.8. Consider the following sequence:

$$a_1 = \sqrt{2} \text{ and } a_{n+1} = \sqrt{2 + a_n} \text{ for all } n \geq 1.$$

1. Show that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is bounded above.
2. Prove that the sequence is monotonically increasing.
3. Deduce that $\{a_n\}_{n\in\mathbb{N}}$ converges and find its limit.

Exercise 18.9. Let a_1, b_1 be two real numbers such that $0 < a_1 < b_1$. For $n \geq 1$, we define

$$a_{n+1} = \sqrt{a_n b_n} \text{ and } b_{n+1} = \frac{a_n + b_n}{2}.$$

1. Prove that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is monotonically increasing and that the sequence $\{b_n\}_{n\in\mathbb{N}}$ is monotonically decreasing.
2. Show that the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are bounded.
3. Deduce that the two sequences converge and prove that they converge to the same limit.

19 Lecture 14: Limit Superior and Limit Inferior, Continued

Theorem 19.1. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Then there exists a monotonic subsequence of $\{a_n\}_{n\geq 1}$ whose limit is $\limsup_{n\to\infty} a_n$. Also, there exists a monotonic subsequence of $\{a_n\}_{n\geq 1}$ whose limit is $\liminf_{n\to\infty} a_n$.

Proof. We will prove the statement about $\limsup_{n\to\infty} a_n$. One can use a similar argument to show the statement about $\liminf_{n\to\infty} a_n$.

Note that if suffices to find a subsequence $\{a_{k_n}\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$ such that $\lim_{n\to\infty} a_{k_n} = \limsup_{n\to\infty} a_n$. As every sequence has a monotone subsequence, $\{a_{k_n}\}_{n\geq 1}$ has a monotone subsequence $\{a_{p_{k_n}}\}_{n\geq 1}$. Then as $\lim_{n\to\infty} a_{k_n}$ exists, $\lim_{n\to\infty} a_{p_{k_n}}$ exists and

$$\lim_{n\to\infty} a_{p_{k_n}} = \lim_{n\to\infty} a_{k_n} = \limsup_{n\to\infty} a_n.$$

Finally, note that $\{a_{p_{k_n}}\}_{n\geq 1}$ is a subsequence of $\{a_n\}_{n\geq 1}$.

Let us find a subsequence of $\{a_n\}_{n\geq 1}$ whose limit is $\limsup_{n\to\infty} a_n$.

Case 1: $\limsup_{n\to\infty} a_n = -\infty$. We showed that in this case, $\lim_{n\to\infty} a_n = -\infty$. Choose $\{a_{k_n}\}_{n\geq 1}$ to be $\{a_n\}_{n\geq 1}$.

Case 2: $\limsup_{n\to\infty} a_n = a \in \mathbb{R}$. By definition, $a = \limsup_{n\to\infty} a_n = \lim_{N\to\infty} v_N$, then $\exists N_1 \in \mathbb{N}$ such that $|a - v_N| < 1$ for all $N \geq N_1$. In particular, $a - 1 < v_{N_1} < a + 1$, and note that $a - 1 < \sup\{a_n : n \geq N_1\}$ and there exists $k_1 \geq N_1$ such that $a - 1 < a_{k_1}$. Therefore, $a - 1 < a_{k_1} \leq v_{N_1} < a + 1$. Hence, $|a - a_{k_1}| < 1$.

Similarly, as $a = \lim_{N\to\infty} v_N$, there exists $N_2 \in \mathbb{N}$ such that $|a - v_N| < \frac{1}{2}$ for all $N \geq N_2$. Let $\tilde{N}_2 = \max\{N_2, k_1 + 1\}$, then in particular, $a - \frac{1}{2} < v_{\tilde{N}_2} < a + \frac{1}{2}$. Then we know $a - \frac{1}{2} < \sup\{a_n : n \geq \tilde{N}_2\}$, and because there exists $k_2 \geq \tilde{N}_2 > k_1$ such that $a - \frac{1}{2} < a_{k_2}$, we conclude that $a - \frac{1}{2} < a_{k_2} \leq v_{N_2} < a + \frac{1}{2}$. Hence, $|a - a_{k_2}| < \frac{1}{2}$.

To construct our subsequence, we proceed inductively. Assume we have found $k_1 < k_2 < \cdots < k_n$ and $a_{k_1}, \cdots, a_{k_n}$ such that $|a - a_{k_j}| < \frac{1}{j}$ for all $1 \leq j \leq n$. As $a = \lim_{N\to\infty} v_N$, there exists $N_{n+1} \in \mathbb{N}$ such that $|a - v_N| < \frac{1}{n+1}$ for all $N \geq N_{n+1}$. Now we can let $\tilde{N_{n+1}} = \max\{N_{n+1}, k_n + 1\}$. Then $a - \frac{1}{n+1} < v_{\tilde{N_{n+1}}} < a + \frac{1}{n+1}$. Therefore, we have $a - \frac{1}{n+1} < \sup\{a_n : n \geq \tilde{N_{n+1}}\}$, and there exists $k_{n+1} \geq \tilde{N_{n+1}} > k_n$ such that $a - \frac{1}{n+1} < a_{k_{n+1}}$. Therefore, $a - \frac{1}{n+1} < a_{k_{n+1}} \leq v_{\tilde{N_{n+1}}} < a + \frac{1}{n+1}$, and so $|a_{k_{n+1}} - a| < \frac{1}{n+1}$.

Case 3: $\limsup n \to \infty a_n = \infty$. We leave this as an exercise. □

Definition 19.2 (Subsequential Limit)**.** Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. A subsequential limit of $\{a_n\}_{n\geq 1}$ is any $a \in \mathbb{R} \cup \{\pm\infty\}$ that is the limit of a subsequence of $\{a_n\}_{n\geq 1}$.

Example 19.3.

1. For $a_n = n(1 + (-1)^n)$, the subsequential limits are $0 = \lim_{n\to\infty} a_{2n+1}$ and $\infty = \lim_{n\to\infty} a_{2n}$.
2. For $a_n = \cos(\frac{n\pi}{3})$. The subsequential limits are $1 = \lim_{n\to\infty} a_{6n}$, $\frac{1}{2} = \lim_{n\to\infty} a_{6n+1} = \lim_{n\to\infty} a_{6n+5}$, $-\frac{1}{2} = \lim_{n\to\infty} a_{6n+2} = \lim_{n\to\infty} a_{6n+4}$, and $-1 = \lim_{n\to\infty} a_{6n+3}$.

Theorem 19.4. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers and let A denote its set of subsequential limits:

$$A = \{a \in \mathbb{R} \cup \{\pm\infty\} : \exists \{a_{k_n}\}_{n\geq 1} \text{ subsequence of } \{a_n\}_{n\geq 1} \text{ such that } \lim_{n\to\infty} a_{k_n} = a\}.$$

Then

1. $A \neq \varnothing$.
2. $\lim_{n\to\infty} a_n$ exists in $\mathbb{R} \cup \{\pm\infty\}$ if and only if A has exactly one element.
3. $\inf(A) = \liminf_{n\to\infty} a_n$ and $\sup(A) = \limsup_{n\to\infty} a_n$.

Proof. 1. By Theorem 19.1, $\liminf_{n\to\infty} a_n, \limsup_{n\to\infty} a_n \in A$. Therefore, $A \neq \varnothing$.

2. ($\Rightarrow$): Assume $\lim_{n\to\infty} a_n$ exists. Then if $\{a_{k_n}\}_{n\geq 1}$ is a subsequence of $\{a_n\}_{n\geq 1}$, we have $\lim_{n\to\infty} a_{k_n} = \lim_{n\to\infty} a_n$. So $A = \{\lim_{n\to\infty} a_n\}$.
   ($\Leftarrow$): If A has a single element, then $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$ and so $\lim_{n\to\infty} a_n$ exists.

3. It suffices to prove the following claim.

   Claim 19.5. $\liminf_{n\to\infty} a_n \leq a \leq \limsup_{n\to\infty} a_n \quad \forall a \in A$.

   Assuming the claim, we can first see how to finish the proof. The claim implies

   - Because $\liminf_{n\to\infty} a_n$ is a lower bound for A, so $\liminf_{n\to\infty} a_n \geq \inf(A)$. On the other hand, $\liminf_{n\to\infty} a_n \in A$, and so $\liminf_{a_n} \geq \inf(A)$. Therefore, $\liminf_{n\to\infty} a_n = \inf(A)$.
   - Similarly, we can show that $\limsup_{n\to\infty} a_n = \sup(A)$.

   We now prove the claim.

   Subproof. Fix $a \in A$, then there exists a subsequence $\{a_{k_n}\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$ such that $\lim_{n\to\infty} a_{k_n} = a$. Because of the nature of the subsequence, we know there is

   $$\inf\{a_n : n \geq N\} \leq \inf\{a_{k_n} : n \geq N\} \leq \sup\{a_{k_n} : n \geq N\} \leq \sup\{a_n : n \geq N\}$$

   where the first two sequences are increasing and the last two sequences are decreasing. By taking the limit, we know

   $$\begin{aligned}\lim_{N\to\infty} \inf\{a_n : n \geq N\} &\leq \lim_{N\to\infty} \inf\{a_{k_n} : n \geq N\} \\ &\leq \lim_{N\to\infty} \sup\{a_{k_n} : n \geq N\} \\ &\leq \lim_{N\to\infty} \sup\{a_n : n \geq N\},\end{aligned}$$

which means

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} a_{k_n} \leq \limsup_{n\to\infty} a_{k_n} \leq \limsup_{n\to\infty} a_n.$$

Because the subsequence converges, we have $a = \lim_{a_{k_n}} = \liminf_{n\to\infty} a_{k_n} = \limsup_{n\to\infty} a_{k_n}$. Therefore,

$$\liminf_{n\to\infty} a_n \leq a \leq \limsup_{n\to\infty} a_n.$$

■

□

20 Lecture 15: Cesaro-Stolz Theorem, Series and Convergence Tests

Theorem 20.1 (Cesaro-Stolz)**.** Let $\{a_n\}_{n\geq 1}$ be a sequence of non-zero real numbers. Then

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \leq \liminf_{n\to\infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

In particular, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists, then $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ exists and

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Example 20.2. We can apply this theorem to find $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}}$.

If we let $a_n = n$, then $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{n} \xrightarrow[n\to\infty]{} 1$. By Cesaro-Stolz, we get $\{\sqrt[n]{n}\}_{n\geq 1}$ converges and

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Proof. It suffices to prove the last inequality, i.e.

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

One can prove the first inequality with a similar proof.

Let $l = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \geq 0$ and $L = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \geq 0$. We want to show $l \leq L$. If $L = \infty$, then it is clear. Henceforth, we assume $L \in \mathbb{R}$. We will prove the following claim.

Claim 20.3. l is the lower bound for the set

$$(L, \infty) = \{M \in \mathbb{R} : M > L\}.$$

Assuming the claim for now, we can see how to finish the proof. Note (L, ∞) is a non-empty subset of $\mathbb{R}$ which is bounded below by L. As $\mathbb{R}$ has the least upper bound property, $\inf(L, \infty)$ exists in $\mathbb{R}$. In fact, $\inf(L, \infty) = L$. As l is a lower bound for (L, ∞), we must have $l \leq L$. We now prove the claim.

Subproof. Fix $M \in (L, \infty)$. We will show $l \leq M$. We have $M > L = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \inf_N \sup_{n\geq N} \left|\frac{a_{n+1}}{a_n}\right|$. Therefore, there exists $N_0 \in \mathbb{N}$ such that $\sup_{n\geq N_0} \left|\frac{a_{n+1}}{a_n}\right| < M$, and so $\left|\frac{a_{n+1}}{a_n}\right| < M$ for all $n \geq N_0$. Therefore, $|a_{n+1}| < M \cdot |a_n|$ for all $n \geq N_0$.

A simple inductive argument then yields

$$|a_n| < M^{n-N_0}|a_{N_0}| \;\; \forall n \geq N_0,$$

so $|a_n|^{\frac{1}{n}} < M\left(\frac{|a_{N_0}|}{M^{N_0}}\right)^{\frac{1}{n}}$ for all $n > N_0$. We can conclude that

$$l = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty} M \cdot \left(\frac{|a_{N_0}|}{M^{N_0}}\right)^{\frac{1}{n}} = M \cdot \limsup_{n\to\infty} \left(\frac{|a_{N_0}|}{M^{N_0}}\right)^{\frac{1}{n}}.$$

We need to apply the following claim to the inequality above.

Claim 20.4. For $r > 0$, we have $\lim_{n\to\infty} r^{\frac{1}{n}} = 1$.

Subproof. Indeed, if $r \geq 1$, we have

$$0 \leq r^{\frac{1}{n}} - 1 = \frac{r-1}{r^{n-1} + r^{n-2} + \cdots + 1} \leq \frac{r-1}{n} \xrightarrow[n\to\infty]{} 0.$$

If $r < 1$, then $r^{\frac{1}{n}} = \frac{1}{(\frac{1}{r})^{\frac{1}{n}}} \xrightarrow[n\to\infty]{} \frac{1}{1} = 1$. ■

We now take $r = \frac{|a_{N_0}|}{M^{N_0}}$ in the inequality, then $l \leq M$. ■

□

Definition 20.5. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. For $n \geq 1$, we define the partial sum $s_n = a_1 + \cdots + a_n$.

The series $\sum_{n=1}^{\infty} a_n$, sometimes denoted $\sum_{n\geq 1} a_n$, is said to converge if $\{s_n\}_{n\geq 1}$ converges.

We say that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely if the series $\sum_{n=1}^{\infty} |a_n|$ converges.[11]

[11] Note that $\sum_{n=1}^{\infty} |a_n|$ either converges or it diverges to ∞.

Theorem 20.6 (Cauchy Criterion)**.** A series $\sum_{n\geq 1} a_n$ converges if and only if

$$\forall \varepsilon > 0 \;\; \exists n_\varepsilon \in \mathbb{N} \text{ such that } \left| \sum_{k=n+1}^{n+p} a_k \right| < \varepsilon \;\; \forall n \geq n_\varepsilon \;\; \forall p \in \mathbb{N}.$$

Proof. Note that

$$\begin{aligned} \text{the series } \sum_{n\geq 1} a_n \text{ converges} &\iff \text{the sequence } \{s_n\}_{n\geq 1} \text{ converges} \\ &\iff \{s_n\}_{n\geq 1} \text{ is Cauchy} \\ &\iff \forall \varepsilon > 0 \;\; \exists n_\varepsilon \in \mathbb{N} \text{ such that } |s_m - s_n| < \varepsilon \;\; \forall m, n \geq n_\varepsilon. \end{aligned}$$

Without loss of generality, we may assume $m > n$ and write $m = n + p$ for $p \in \mathbb{N}$. Note

$$|s_m - s_n| = \left| \sum_{k=1}^{n+p} a_k - \sum_{k=1}^{n} a_k \right| = \left| \sum_{k=n+1}^{n+p} a_k \right|,$$

so $\sum_{n\geq 1} a_n$ converges if and only if

$$\forall \varepsilon > 0 \;\; \exists n_\varepsilon \in \mathbb{N} \text{ such that } \left| \sum_{k=n+1}^{n+p} a_k \right| < \varepsilon \;\; \forall n \geq n_\varepsilon \;\; \forall p \in \mathbb{N}.$$

□

Corollary 20.7. If $\sum_{n\geq 1} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof. Taking $p = 1$, we find $\sum_{n\geq 1} a_n$ converges implies

$$\forall \varepsilon > 0 \;\; \exists n_\varepsilon \in \mathbb{N} \text{ such that } |a_{n+1}| < \varepsilon \;\; \forall n \geq n_\varepsilon.$$

□

Corollary 20.8. If $\sum_{n\geq 1} a_n$ converges absolutely, then it converges.

Proof. If $\sum_{n\geq 1} a_n$ converges absolutely, $\sum_{n\geq 1} |a_n|$ converges. By definition,

$$\forall \varepsilon > 0 \;\; \exists n_\varepsilon \in \mathbb{N} \text{ such that } \sum_{k=n+1}^{n+p} |a_k| < \varepsilon \;\; \forall n \geq n_\varepsilon \;\; \forall p \in \mathbb{N}.$$

Note that

$$\left| \sum_{k=n+1}^{n+p} a_k \right| \leq \sum_{k=n+1}^{n+p} |a_k| < \varepsilon \;\; \forall n \geq n_\varepsilon \;\; \forall p \in \mathbb{N}.$$

Therefore, $\sum_{n\geq 1} a_n$ converges by the Cauchy criterion. □

Theorem 20.9 (Comparison Test)**.** Let $\sum\limits_{n\geq 1} a_n$ be a series of real numbers with $a_n \geq 0\ \ \forall n \geq 1$.

1. If $\sum\limits_{n\geq 1} a_n$ converges and $|b_n| \leq a_n\ \ \forall n \geq 1$, then $\sum\limits_{n\geq 1} b_n$ converges.
2. If $\sum\limits_{n\geq 1} a_n$ diverges and $b_n \geq a_n\ \ \forall n \geq 1$, then $\sum\limits_{n\geq 1} b_n$ diverges.

Proof. 1. Because $\sum\limits_{n\geq 1} a_n$ converges, then

$$\forall \varepsilon > 0\ \ \exists n_\varepsilon \in \mathbb{N} \text{ such that } \left|\sum_{k=n+1}^{n+p} a_k\right| < \varepsilon\ \ \forall n \geq n_\varepsilon\ \ \forall p \in \mathbb{N}.$$

Then

$$\left|\sum_{k=n+1}^{n+p} b_k\right| \leq \sum_{k=n+1}^{n+p} |b_k| \leq \sum_{k=n+1}^{n+p} a_k < \varepsilon\ \ \forall n \geq n_\varepsilon\ \ \forall p \in \mathbb{N}.$$

Therefore, by the Cauchy criterion, $\sum\limits_{n\geq 1} b_n$ converges.

2. Note that $b_1 + \cdots + b_n \geq a_1 + \cdots + a_n \xrightarrow[n\to\infty]{} \infty$, and so $\sum\limits_{n\geq 1} b_n$ diverges.

□

Lemma 20.10. Let $r \in \mathbb{R}$. The series $\sum\limits_{n\geq 0} r^n$ converges if and only if $|r| < 1$. If $|r| < 1$, then $\sum\limits_{n\geq 0} r^n = \frac{1}{1-r}$.

Proof. First note that if $|r| \geq 1$, then $|r^n| = |r|^n \geq 1$, therefore $r^n \not\to 0$ as $n \to \infty$. By Corollary 20.7, $\sum\limits_{n\geq 0} r^n$ does not converge. Assume now that $|r| < 1$, then $|r^n| = |r|^n \xrightarrow[n\to\infty]{} 0$. Also note that $\sum\limits_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r} \xrightarrow[n\to\infty]{} \frac{1}{1-r}$. □

21 Homework 6

Exercise 21.1. Let

$$a_1 = 1 \text{ and } a_{n+1} = \left[1 - \frac{1}{(n+1)^2}\right] a_n \text{ for all } n \geq 1.$$

1. Show that the sequence $\{a_n\}_{n\geq 1}$ converges.
2. Find its limits.

Exercise 21.2. Let A be a non-empty bounded subset of $\mathbb{R}$ and suppose $\sup A \notin A$. Show that there exists an increasing sequence of points $\{a_n\}_{n\geq 1}$ in A such that $\lim\limits_{n\to\infty} a_n = \sup A$.

Exercise 21.3. Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two bounded sequences. Show that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Exercise 21.4. Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be two bounded sequences of non-negative numbers. Show that

$$\limsup_{n\to\infty}(a_n \cdot b_n) \leq \limsup_{n\to\infty} a_n \cdot \limsup_{n\to\infty} b_n.$$

Exercise 21.5. Show that a sequence of real numbers $\{a_n\}_{n\geq 1}$ is bounded if and only if $\limsup |a_n| < \infty$.

Exercise 21.6. Let A denote the set of subsequential limits of a sequence $\{a_n\}_{n\geq 1}$. Suppose that $\{b_n\}_{n\geq 1}$ is a subsequence in $A \cap \mathbb{R}$ such that $\lim\limits_{n\to\infty} b_n$ exists in $\mathbb{R} \cup \{\pm\infty\}$. Show that $\lim\limits_{n\to\infty} b_n$ belongs to A.

Exercise 21.7. Let $\{a_n\}_{n\geq 1}$ be a sequence of non-negative numbers. For $n \geq 1$, define

$$s_n = \frac{a_1 + \ldots + a_n}{n}.$$

(i) Show that

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} a_n.$$

(ii) Conclude that if $\lim\limits_{n\to\infty} a_n$ exists, then $\lim\limits_{n\to\infty} s_n$ exists and $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} a_n$.

Exercise 21.8. Let $\{a_n\}_{n\geq 1}$ be a bounded sequence of real numbers. Prove that $L = \limsup a_n$ has the following properties:

(i) For every $\varepsilon > 0$ there are only finitely many n for which $a_n > L + \varepsilon$.

(ii) For every $\varepsilon > 0$ there are infinitely many n for which $a_n > L - \varepsilon$.

Exercise 21.9. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Prove that there can be at most one real number L with the following two properties:

(i) For every $\varepsilon > 0$ there are only finitely many n for which $a_n > L + \varepsilon$, and

(ii) For every $\varepsilon > 0$ there are infinitely many n for which $a_n > L - \varepsilon$.

The following exercise need not be turned in. Its purpose is to provide another construction of an ordered field with the least upper bound property. The fact that F in Exercise 10 has the least upper bound property is quite involved and is not assigned as part of the exercise.

Exercise 21.10. Let $\mathcal{C}$ be the set of Cauchy sequences of rational numbers. Define the relation $\sim$ as follows: if $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1} \in \mathcal{C}$, we write $\{a_n\}_{n\geq 1} \sim \{b_n\}_{n\geq 1}$ if and only if the sequence $\{a_n - b_n\}_{n\geq 1}$ converges to zero.

1. Prove that $\sim$ is an equivalence relation.

2. For $\{a_n\}_{n\geq 1} \in \mathcal{C}$, we denote its equivalence class by $[a_n]$. Let F denote the set of equivalence classes in $\mathcal{C}$. We define addition and multiplication on F as follows:

$$[a_n] + [b_n] = [a_n + b_n] \text{ and } [a_n] \cdot [b_n] = [a_n b_n].$$

Show that these internal laws of composition are well defined and that F together with these operations is a field.

3. We define a relation on F as follows: we write $[a_n] < [b_n]$ if $[a_n] \neq [b_n]$ and there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $a_n < b_n$. Prove that this relation is well defined. Show that the set of positive elements in F, that is,

$$P = \{[a_n] \in F : [a_n] > 0\}$$

satisfies the following properties:

01') For every $[a_n] \in F$, exactly one of the following holds:

$$[a_n] = [0], \quad [a_n] \in P, \quad -[a_n] \in P$$

where $[-]$ denotes the equivalence class of the sequence identically equal to zero.

02') For every $[a_n], [b_n] \in P$, we have $[a_n] + [b_n] \in P$ and $[a_n] \cdot [b_n] \in P$.

4. Conclude that F is an ordered field.

Caution: For Exercise 10, you may not use that Cauchy sequences converge in $\mathbb{R}$ (which is a consequence of the fact that $\mathbb{R}$ has the least upper bound property), since the aim of the exercise is to build such an ordered field with the least upper bound property.

22 Lecture 16: Convergence Tests, Continued

Proposition 22.1 (The Dyadic Criterion)**.** Let $\{a_n\}_{n\geq 1}$ be a decreasing sequence of real numbers with $a_n \geq 0$ for all $n \geq 1$. Then the series $\sum\limits_{n\geq 1} a_n$ converges if and only if the series $\sum\limits_{n\geq 0} 2^n a_{2^n}$ converges.

Proof. For $n \geq 1$, let $s_n = \sum_{k=1}^{n} a_k = a_1 + \cdots + a_n$, and let $t_n = \sum_{k=0}^{n} 2^k a_{2^k} = a_1 + 2a_2 + \cdots + 2^n a_{2^n}$.

Note that both sequences are increasing, thus $\sum_{n\geq 1} a_n$ converges if and only if $\{s_n\}_{n\geq 1}$ is bounded, and $\sum_{n\geq 0} 2^n a_{2^n}$ converges if and only if $\{t_n\}_{n\geq 0}$ is bounded. It now suffices to prove that $\{s_n\}_{n\geq 1}$ is bounded if and only if $\{s_n\}_{n\geq 1}$ is bounded.

Consider the summation $\sum_{l=2^k+1}^{2^{k+1}} a_l$. Because $\{a_n\}_{n\geq 1}$ is decreasing, we know that

$$\frac{1}{2}(2^{k+1}a_{2^{k+1}}) = 2^k a_{2^{k+1}} \leq \sum_{l=2^k+1}^{2^{k+1}} a_l \leq 2^k a_{2^k+1} \leq 2^k a_{2^k}$$

and therefore

$$\frac{1}{2}\sum_{k=0}^{n} 2^{k+1}a_{2^{k+1}} \leq \sum_{k=0}^{n}\sum_{l=2^k+1}^{2^{k+1}} a_l \leq \sum_{k=0}^{n} 2^k a_{2^k},$$

and so $\frac{1}{2}\sum_{l=1}^{n+1} 2^l a_{2^l} \leq \sum_{l=2}^{2^{n+1}} a_l \leq t_n$. That is to say, $\frac{1}{2}(t_{n+1} - a_1) \leq s_{2^{n+1}} - a_1 \leq t_n$. We conclude that $t_{n+1} \leq 2s_{2^{n+1}} - a_1$ and $s_n \leq s_{2^{n+1}} \leq t_n + a_1$ since $n \leq 2^{n+1}$ for all $n \geq 1$.

In particular, if $\{s_n\}_{n\geq 1}$ is bounded, then there exists $M > 0$ such that $|s_n| \leq M$ for all $n \geq 1$, and so $t_{n+1} \leq 2M + a_1$ for all $n \geq 1$. Similarly, if $\{t_n\}$ is bounded, then there exists $L > 0$ such that $|t_n| \leq L$ for all $n \geq 0$, which is to say $s_n \leq L + a_1$ for all $n \geq 1$. $\square$

Corollary 22.2. The series $\sum_{n\geq 1} \frac{1}{n^\alpha}$ converges if and only if $\alpha > 1$.

Proof. If $\alpha \leq 0$, then $\frac{1}{n^\alpha} = n^{-\alpha} \geq 1$ for all $n \geq 1$. In particular, $\frac{1}{n^\alpha} \not\xrightarrow[n\to\infty]{} 0$ so $\sum_{n\geq 1} \frac{1}{n^\alpha}$ cannot converge. Assume $\alpha > 0$, then $\{\frac{1}{n^\alpha}\}_{n\geq 1}$ is a decreasing sequence of positive real numbers. By the dyadic criterion, $\sum_{n\geq 1} \frac{1}{n^\alpha}$ converges if and only if $\sum_{n\geq 0} 2^n \frac{1}{(2^n)^\alpha}$ converges. Note that $\sum_{n\geq 0} \frac{2^n}{(2^n)^\alpha} = \sum_{n\geq 0} (2^{1-\alpha})^n = \sum_{b\geq 0} r^n$ where $r = 2^{1-\alpha}$, and this term converges if and only if $r < 1$ if and only if $2^{1-\alpha} < 1$ if and only if $1 - \alpha < 0$ if and only if $\alpha > 1$. $\square$

Theorem 22.3 (The Root Test)**.** Let $\sum_{n\geq 1}$ be a series of real numbers.

1. If $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} < 1$, then $\sum_{n\geq 1} a_n$ converges absolutely.

2. If $\liminf_{n\to\infty} |a_n|^{\frac{1}{n}} < 1$, then $\sum_{n\geq 1} a_n$ diverges.

3. The test is inconclusive if $\liminf_{n\to\infty} |a_n|^{\frac{1}{n}} \leq 1 \leq \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$.

Proof. 1. Let $L = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$. Since $L < 1$, then $1 - L < 0$, and because $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < 1 - L$, and so $L < L + \varepsilon + 1$. Therefore, $L + \varepsilon > L = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \inf_{N\to\infty} \sup_{n\geq N} |a_n|^{\frac{1}{n}}$. In particular, there exists $N_0 \in \mathbb{N}$ such that $\sup_{n\geq N_0} |a_n|^{\frac{1}{n}} < L+\varepsilon$. Therefore, $|a_n|^{\frac{1}{n}} < L+\varepsilon$ for all $n \geq N_0$, and so $|a_n| < (L+\varepsilon)^n$ for all $n \geq N_0$.

As $L + \varepsilon < 1$, when denote $n = N_0 + k$, we have the series $\sum_{n\geq N_0} (L + \varepsilon)^n = \sum_{k\geq 0}(L + \varepsilon)^{N_0+k} = (L + \varepsilon)^{N_0} \sum_{k\geq 0}(L + \varepsilon)^k = (L + \varepsilon)^{N_0} \cdot \frac{1}{1-(L+\varepsilon)}$. By the comparison test, $\sum_{n\geq N_0} a_n$ converges absolutely. Note that $|a_1| + \cdots + |a_{N_0-1}| \in \mathbb{R}$. Therefore, $\sum_{n\geq 1} a_n$ converges absolutely.

2. Let $\{a_{k_n}\}_{n\geq 1}$ be a subsequence of $\{a_n\}_{n\geq 1}$ such that $\lim_{n\to\infty} |a_{k_n}|^{\frac{1}{k_n}} = \liminf_{n\to\infty} |a_n|^{\frac{1}{n}} > 1$. Therefore, there exists $n_0 \in \mathbb{N}$ such that $|a_{k_n}|^{\frac{1}{k_n}} > 1$ for all $n \geq n_0$. Therefore, $|a_{k_n}| > 1$ for all $n \geq n_0$. That is to say, $|a_{k_n}| > 1$ for all $n \geq n_0$. In particular, $a_{k_n} \underset{n\to\infty}{\not\longrightarrow} 0$, that is to say $a_n \underset{n\to\infty}{\not\longrightarrow} 0$, and so $\sum_{n\geq 1} a_n$ diverges.

3. Consider $a_n = \frac{1}{n}$ for all $n \geq 1$. The series $\sum_{n\geq 1} a_n = \sum_{n\geq 1} \frac{1}{n}$ diverges. However, by Cesaro-Stolz theorem,

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{n}} = \frac{1}{\lim_{n\to\infty} \frac{n+1}{n}} = 1.$$

Therefore, $\liminf_{n\to\infty} \sqrt[n]{a_n} = \limsup_{n\to\infty} \sqrt[n]{a_n} = 1$.

Now consider $a_n = \frac{1}{n^2}$ for all $n \geq 1$. The series $\sum_{n\geq 1} a_n = \sum_{n\geq 1} \frac{1}{n^2}$ converges. However, by Cesaro-Stolz theorem,

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{n^2}} = \frac{1}{\lim_{n\to\infty} \frac{(n+1)^2}{n^2}} = 1.$$

Therefore, $\liminf_{n\to\infty} \sqrt[n]{a_n} = \limsup_{n\to\infty} \sqrt[n]{a_n} = 1$.

□

Theorem 22.4 (The Ratio Test)**.** Let $\sum_{n\geq 1} a_n$ be a series of non-zero real numbers.

1. If $\limsup_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$, the series $\sum_{n\geq 1} a_n$ converges absolutely.

2. If $\liminf_{n\to\infty} |\frac{a_{n+1}}{a_n}| > 1$, the series $\sum_{n\geq 1} a_n$ diverges.

3. The test is inconclusive if $\liminf_{n\to\infty} |\frac{a_{n+1}}{a_n}| \leq 1 \leq \limsup_{n\to\infty} |\frac{a_{n+1}}{a_n}|$.

Proof. The first two conclusions follow from the root test and the Cesaro-Stolz theorem:

$$\liminf_{n\to\infty} |\frac{a_{n+1}}{a_n}| \leq \liminf_{n\to\infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty} |\frac{a_{n+1}}{a_n}|.$$

The last conclusion is true by applying the same examples as in the theorem above. $\square$

Theorem 22.5 (The Abel Criterion)**.** Let $\{a_n\}_{n\geq 1}$ be a decreasing sequence with $\lim_{n\to\infty} a_n = 0$. Let $\{b_n\}_{n\geq 1}$ be a sequence so that $\{\sum_{k=1}^{n} b_k\}_{n\geq 1}$ is bounded. Then $\sum_{n\geq 1} a_n b_n$ converges.

Corollary 22.6 (The Leibniz Criterion)**.** Let $\{a_n\}_{n\geq 1}$ be a decreasing sequence with $\lim_{n\to\infty} a_n = 0$. Then $\sum_{n\geq 1} (-1)^n a_n$ converges.

Proof of the Abel Criterion. Let $t_n = \sum_{k=1}^{n} b_k$ for $n \geq 1$. As $\{t_n\}_{n\geq 1}$ is bounded, there exists $M > 0$ such that $|t_n| \leq M$ for all $n \geq 1$. We will use the Cauchy criterion to prove convergence of $\sum_{n\geq 1} a_n b_n$. Let $\varepsilon > 0$. As $\lim_{n\geq\infty} a_n = 0$, then there exists $n_\varepsilon \in \mathbb{N}$ such that $|a_n| < \frac{\varepsilon}{2M}$ for all $n \geq n_\varepsilon$. For $n \geq n_\varepsilon$ and $p \in \mathbb{N}$, we have

$$\begin{aligned}
\left|\sum_{k=n+1}^{n+p} a_k b_k\right| &= \left|\sum_{k=n+1}^{n+p} a_k (t_k - t_{k-1})\right| \\
&= \left|\sum_{k=n+1}^{n+p} a_k t_k - \sum_{k=n+1}^{n+p} a_k t_{k-1}\right| \\
&= \left|\sum_{k=n+1}^{n+p} a_k t_k - \sum_{k=n}^{n+p-1} a_{k+1} t_k\right| \\
&= \left|\sum_{k=n}^{n+p} t_k (a_k - a_{k+1}) - a_n t_n + a_{n+p+1} t_{n+p}\right| \\
&\leq \sum_{k=n}^{n+p} |t_k||a_k - a_{k-1}| + |a_n| \cdot |t_n| + |a_{n+p+1}| \cdot |t_{n+p}| \\
&\leq \sum_{k=n}^{n+p} M(a_k - a_{k-1}) + a_n M + a_{n+p+1} M \\
&= M(a_n - a_{n+p-1}) + a_n M + a_{n+p+1} M \\
&= 2M a_n \\
&< \varepsilon.
\end{aligned}$$

$\square$

23 Lecture 17: Rearrangement

Definition 23.1 (Rearrangement)**.** Let $k:\mathbb{N}\to\mathbb{N}$ be a bijective function. For a sequence $\{a_n\}_{n\geq 1}$ of real numbers, we denote $\tilde{a_n}=a_{k(n)}=a_{k_n}$. Then $\sum\limits_{n\geq 1}\tilde{a_n}$ is called a rearrangement of $\sum\limits_{n\geq 1}a_n$.

Example 23.2. Consider $a_n=\frac{(-1)^{n-1}}{n}$ for all $n\geq 1$. The series looks like

$$\sum_{n\geq 1}a_n=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$$

Note that the sequence $\{\frac{1}{n}\}_{n\geq 1}$ is decreasing and $\lim\limits_{n\to\infty}\frac{1}{n}=0$. Thus, by the Leibniz criterion, $\sum\limits_{n\geq 1}a_n$ converges.

However, we can also write the series as follows:

$$\sum_{n\geq 1}a_n=1-\frac{1}{2}+\frac{1}{3}-\sum_{k\geq 2}(\frac{1}{2k}-\frac{1}{2k+1})$$

Note that for $k\geq 2$, we have $0<\frac{1}{2k}-\frac{1}{2k+1}=\frac{1}{2k(2k+1)}<\frac{1}{4k^2}$.

Recall that the series $\sum\limits_{k\geq 2}\frac{1}{4k^2}$ converges by the dyadic criterion. By the comparison test, the series $0<\sum\limits_{k\geq 2}\left(\frac{1}{2k}-\frac{1}{2k+1}\right)$ converges. So $\sum\limits_{n\geq 1}a_n<1-\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$. Consider next the following rearrangement:

$$\frac{1}{1}+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}-\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\cdots=\sum_{k\geq 1}\left(\frac{1}{4k-3}+\frac{1}{4k-1}-\frac{1}{2k}\right).$$

Now we have

$$\begin{aligned}
0&<\frac{1}{4k-3}+\frac{1}{4k-1}-\frac{1}{2k}\\
&=\frac{8k^2-2k+8k^2-6k-(16k^2-16k+3)}{(4k-3)(4k-1)\cdot 2k}\\
&=\frac{8k-3}{(4k-3)(4k-1)\cdot 2k}\\
&<\frac{8k}{k\cdot 3k\cdot 2k}\\
&=\frac{4}{3k^2}.
\end{aligned}$$

As the series $\sum\limits_{k\geq 1}\frac{4}{3k^2}$ converges, we deduce that the series $\sum\limits_{k\geq 1}\left(\frac{1}{4k-3}+\frac{1}{4k-1}-\frac{1}{2k}\right)$ converges.

Moreover,

$$\sum_{k\geq 1}\left(\frac{1}{4k-3}+\frac{1}{4k-1}-\frac{1}{2k}\right)=1+\frac{1}{3}-\frac{1}{2}+\sum_{k\geq 2}\left(\frac{1}{4k-3}+\frac{1}{4k-1}-\frac{1}{2k}\right)>1+\frac{1}{3}-\frac{1}{2}=\frac{5}{6}.$$

Therefore, we see that the two series converge to two different numbers.

Theorem 23.3 (Riemann)**.** Let $\sum\limits_{n\geq 1} a_n$ be a series that converges, but it does not converge absolutely. Let $-\infty\leq\alpha\leq\beta\leq\infty$. Then there exists a rearrangement $\sum\limits_{n\geq 1}\tilde{a_n}$ with partial sums $\tilde{s_n}=\sum\limits_{k=1}^{n}\tilde{a_k}$ such that $\liminf\limits_{n\to\infty}\tilde{s_n}=\alpha$ and $\limsup\limits_{n\to\infty}\tilde{s_n}=\beta$.

Proof. For $n\geq 1$, let $b_n=\frac{|a_n|+a_n}{2}$ and $c_n=\frac{|a_n|-a_n}{2}$. This means that $b_n=\begin{cases}a_n, & a_n\geq 0\\ 0, & a_n<0\end{cases}$ and $c_n=\begin{cases}0, & a_n\geq 0\\ -a_n, & a_n<0\end{cases}$. Therefore, $b_n\geq 0$ for all n, and $c_n\geq 0$ for all n.

Claim 23.4. The series $\sum\limits_{n\geq 1} b_n$ and $\sum\limits_{c_n}$ both diverge.

Subproof. Note $\sum\limits_{k=1}^{n} b_k-\sum\limits_{k=1}^{n} c_k=\sum\limits_{k=1}^{n}(b_k-c_k)=\sum\limits_{k=1}^{n} a_k$. Therefore, $\sum\limits_{k=1}^{n} b_k=\sum\limits_{k=1}^{n} c_k+\sum\limits_{k=1}^{n} a_k$. Because $\sum\limits_{k=1}^{n} a_k$ converges as $n\to\infty$, then we know $\{\sum\limits_{k=1}^{n} b_k\}_{n\geq 1}$ converges if and only if $\{\sum\limits_{k=1}^{n} c_k\}_{n\geq 1}$ converges.

On the other hand, if $\sum\limits_{n\geq 1} b_n$ and $\sum\limits_{n\geq 1} c_n$ both converged, then

$$\sum_{k=1}^{n} b_k+\sum_{k=1}^{n} c_k=\sum_{k=1}^{n}(b_k+c_k)=\sum_{k=1}^{n}|a_k|$$

which diverges as $n\to\infty$. However, we know the sum of these two series would converge as $n\to\infty$, contradiction. Therefore, both $\sum\limits_{n\geq 1} b_n$ and $\sum\limits_{n\geq 1} c_n$ diverge to ∞. ■

Note that $\sum\limits_{n\geq 1} a_n$ converges, to $\lim\limits_{n\to\infty} a_n=0$ and so $\lim\limits_{n\to\infty} b_n=\lim\limits_{n\to\infty} c_n=0$.

Let $B_1,B_2,B_3,\cdots$ denote the non-negative terms in $\{a_n\}_{n\geq 1}$ in the order in which they appear. Let $C_1,C_2,C_3,\cdots$ denote the absolute values of the negative terms in $\{a_n\}_{n\geq 1}$ in the order in which they appear.

Note $\sum\limits_{n\geq 1} B_n$ differs $\sum\limits_{n\geq 1} b_n$ only by the terms that are zero. Therefore, $\sum\limits_{n\geq 1} B_n=\infty$. Similarly, $\sum\limits_{n\geq 1} C_n$ differs $\sum\limits_{n\geq 1} c_n$ only by the terms that are zero. Therefore, $\sum\limits_{n\geq 1} C_n=\infty$.

We now choose sequences $\{\alpha_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 1}$ so that $\alpha_n \xrightarrow[n\to\infty]{} \alpha$, $\beta_n \xrightarrow[n\to\infty]{} \beta$, $\alpha_n < \beta_n$ for all $n \geq 1$, and $\beta_1 > 0$.[12] Next, we construct increasing sequences $\{k_n\}_{n\geq 1}$ and $\{j_n\}_{n\geq 1}$ as follows:

1. Choose k_1 and j_1 to be the smallest natural numbers so that $x_1 = B_1 + \cdots + B_{k_1} > \beta_1$ and $y_1 = B_1 + \cdots + B_{k_1} - C_1 - C_2 - \cdots - C_{j_1} < \alpha_1$. Note that both choices are possible because the series summation goes to infinity.

2. Choose k_2 and j_2 to be the smallest natural numbers so that $x_2 = B_1 + \cdots + B_{k_1} - C_1 - C_2 - \cdots - C_{j_1} + B_{k_1+1} + \cdots + B_{k_2} > \beta_2$, and $y_2 = B_1 + \cdots + B_{k_1} - C_1 - \cdots - C_{j_1} + B_{k_1+1} + \cdots + B_{k_2} - C_{j_1+1} - \cdots - C_{j_2} < \alpha_2$.

3. Proceed inductively.

Note that by definition, $x_n - B_{k_n} \leq \beta_n$, and so $\beta_n - B_{k_n} < \beta_m < x_n \leq \beta_n + B_{k_n}$. In particular, $|x_n - \beta_n| \leq B_{k_n} \xrightarrow[n\to\infty]{} 0$ as $\beta_n \xrightarrow[n\to\infty]{} \beta$. Therefore, $\lim\limits_{n\to\infty} x_n = \beta$. Similarly, we have $y_n + C_{j_n} \geq \alpha_n$, and so $\alpha_n - C_{j_n} \leq y_n < \alpha_n < \alpha_n + C_{j_n}$, therefore $|y_n - \alpha_n| \leq C_[j_n] \xrightarrow[n\to\infty]{} 0$ as $\alpha_n \xrightarrow[n\to\infty]{} \alpha$. We conclude that $\lim\limits_{n\to\infty} y_n = \alpha$.

Finally, note that x_n and y_n are partial sums in the rearrangement

$$B_1 + B_2 + \cdots + B_{k+1} - C_1 - \cdots - C_{j_1} + B_{k_1+1} + \cdots + B_{k_2} - C_{j_1+1} - \cdots - C_{j_2} + \cdots$$

By construction, no number less than α or larger than β can occur as a subsequential limit of the partial sums. $\square$

Theorem 23.5. If a series $\sum\limits_{n\geq 1} a_n$ converges absolutely, then any rearrangement $\sum\limits_{n\geq 1} \tilde{a_n}$ converges to $\sum\limits_{n\geq 1} a_n$.

Proof. For $n \geq 1$ let $s_n = \sum\limits_{k=1}^{n} a_k$ and $\tilde{s_n} = \sum\limits_{k=1}^{n} \tilde{a_k}$. As $\sum\limits_{n\geq 1} a_n$ converges absolutely, $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ such that

$$\sum_{k=n+1}^{n+p} |a_k| < \varepsilon \quad \forall n \geq n_\varepsilon, \forall p \in \mathbb{N}.$$

Choose N_ε sufficiently large so that $a_1, \cdots, a_{n_\varepsilon}$ belong to the set $\{\tilde{a_1}, \cdots, \tilde{a_N}\}$. Then for $n > N_\varepsilon$, the terms $a_1, \cdots, a_{n_\varepsilon}$ cancel in $s_n - \tilde{s_n}$, then

$$|s_n - \tilde{s_n}| \leq \sum_{k=n_\varepsilon+1}^{n} |a_k| + \sum_{\substack{1\leq k\leq n \\ \tilde{a_k}\notin\{a_1,\cdots,a_{n_\varepsilon}\}}} |\tilde{a_k}| < \varepsilon$$

[12] This requirement says that α_i's approach α from the left, β_i's approach β from the right, and β_i's start from somewhere large enough.

because the sum holds finitely many terms and all indices are greater than n_ε. Therefore, as $\lim_{n\to\infty} s_n = s \in \mathbb{R}$, we deduce that $\lim_{n\to\infty} \tilde{s_n} = s$. $\square$

24 Lecture 18: Functions, Cardinality

Definition 24.1 (Function)**.** Let A, B be two non-empty sets. A function $f : A \to B$ is a way of associating to each element $a \in A$ exactly one element in B denoted $f(a)$.

We say A is the domain of f and B is the range (alternatively, codomain) of f. The set $f(A) = \{f(a) : a \in A\}$ is the image of A under f.

If $A' \subseteq A$, then $f(A') = \{f(a) : a \in A'\}$ is called the image of A' under f. If $f(A) = B$, then we say that f is surjective, or onto. In this case, $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$. We say that f is injective if it satisfies: if $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$, then $a_1 = a_2$. We say that f is bijective if f is injective and surjective.

Remark 24.2. Injectivity and surjectivity of a function depend not only on the law f , but also on the domain and the codomain.

Example 24.3. Consider $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(n) = 2n$. This function is injective but not surjective.

Consider $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = 2x$, then g is a bijection.

Example 24.4. Consider $f : [0, \infty) \to [0, \infty)$ such that $f(x) = x^2$. This function is a bijection.

Consider $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = x^2$. This function is surjective but not injective.

Definition 24.5 (Composition)**.** Let A, B, C be non-empty sets and $f : A \to B$ and $g : B \to C$ be two functions. The composition of g with f is a function $g \circ f : A \to C$ defined by $(g \circ f)(a) = g(f(a))$.

Remark 24.6. The composition of two functions need not be commutative. For example, consider $f, g : \mathbb{Z} \to \mathbb{Z}$ where $f(n) = 2n$ and $g(n) = n + 1$. Then $g \circ f : \mathbb{Z} \to \mathbb{Z}$ is defined by $(g \circ f)(n) = g(f(n)) = g(2n) = 2n + 1$, and $f \circ g : \mathbb{Z} \to \mathbb{Z}$ is defined by $(f \circ g)(n) = f(g(n)) = f(n + 1) = 2n + 2$.

Exercise 24.7. The composition of functions is associative: if $f : A \to B$, $g : B \to C$ and $h : C \to D$ are three functions, then $(h \circ g) \circ f = h \circ (g \circ f)$.

Definition 24.8 (Inverse)**.** Let $f : A \to B$ be a bijective function. The inverse of f is a function $f^{-1} : B \to A$ defined as follows: if $b \in B$ then $f^{-1}(b) = a$ where a is the unique element in A such that $f(a) = b$. The existence of a is generated by surjectivity and the uniqueness by injectivity. Therefore, $f \circ f^{-1} : B \to B$ is defined by $(f \circ f^{-1})(b) = b$, and $f^{-1} \circ f : A \to A$ is defined by $(f^{-1} \circ f)(a) = a$.

Exercise 24.9. Let $f : A \to B$ and $g : B \to C$ be two bijective functions. Then $g \circ f : A \to C$ is a bijection and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Definition 24.10 (Preimage)**.** Let $f : A \to B$ be a function. If $B' \subseteq B$ then the preimage of B' is $f^{-1}(B') = \{a \in A : f(a) \in B'\}$. The preimage of a set is well-defined whether or not f is bijective. In fact, if $B' \subseteq B$ such that $B' \cap f(A) = \varnothing$, then $f^{-1}(B') = \varnothing$.

Exercise 24.11. Let $f : A \to B$ be a function and let $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$. Then

1. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
2. $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.
3. $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.
4. $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.
5. The following are equivalent:
    - f is injective.
    - $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ for all subsets $A_1, A_2 \subseteq A$.

Definition 24.12 (Cardinality)**.** We say that two sets A and B have the same cardinality (or the same cardinal number) if there exists a bijection $f : A \to B$. In this case, we write $A \sim B$.

Exercise 24.13. Show that $\sim$ is an equivalence relation on sets.

Definition 24.14 (Finite, Countable, At Most Countable)**.** We say that a set A if finite if $A = \varnothing$ (in which case we say that it has cardinality 0) or $A \sim \{1, \cdots, n\}$ for some $n \in \mathbb{N}$ (in which case we say that A has cardinality n).

We say that A is countable if $A \sim \mathbb{N}$. In this case, we say that A has cardinality $\aleph_0$.

We say that A is at most countable if A is finite or countable. If A is not at most countable we say that A is uncountable.

Lemma 24.15. Let A be a finite set and let $B \subseteq A$. Then B is finite.

Proof. If $B = \varnothing$, then B is finite. Assume now that $B \neq \varnothing$, then $A \neq \varnothing$. As A is finite, $\exists n \in \mathbb{N}$, and $\exists f : A \to \{1, \cdots, n\}$ a bijection, then $f \mid_B : B \to f(B)$ is a bijection. We merely have to relabel the elements in $f(B)$. Let $b_1 \in B$ be such that $f(b_1) = \min(f(B))$. Define $g(b_1) = 1$. If $B \backslash \{b_1\} \neq \varnothing$, let $b_2 \in B$ be such that $f(b_2) = \min(f(B \backslash \{b_1\}))$. Define $g(b_2) = 2$, and proceed by induction. The process terminates in at most n steps. $\square$

Example 24.16. Consider $f : \mathbb{N} \cup \{0, \cdots, -k\} \to \mathbb{N}$ where $k \in \mathbb{N}$ and $f(n) = n + l + 1$ is bijective. So the cardinality of $\mathbb{N} \cup \{0, -1, \cdots, -k\}$ is $\aleph_0$.

Example 24.17. Consider $f : \mathbb{Z} \to \mathbb{N}$ where $f(n) = \begin{cases} 2n+2, & n \geq 0 \\ -2n-1, & n < 0 \end{cases}$ is bijective. So the cardinality of $\mathbb{Z}$ is $\aleph_0$.

Example 24.18. Consider $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $f(n, m) = \frac{(n+m-1)(n+m-2)}{2} + n$ is bijective, so the cardinality of $\mathbb{N}$ is $\aleph_0$. This can be done by labeling the following n/m value table with a diagonal argument.

$$\begin{tikzcd}
(1,1) \arrow[r] & (1,2) \arrow[dl] & (1,3) \arrow[dl] & (1,4) \arrow[dl] \\
(2,1) \arrow[urr] & (2,2) \arrow[dl] & (2,3) \arrow[dl] & (2,4) \\
(3,1) \arrow[uurrr] & (3,2) \arrow[dl] & (3,3) & (3,4) \\
(4,1) & (4,2) & (4,3) & (4,4)
\end{tikzcd}$$

25 Homework 7

Exercise 25.1. Let $\{a_n\}_{n \geq 1}$ be a sequence such that $\liminf_{n \to \infty} |a_n| = 0$. Prove that there is a subsequence $\{a_{k_n}\}_{n \geq 1}$ such that the series $\sum_{n=1}^{\infty} a_{k_n}$ converges.

Exercise 25.2. Determine which of the following series converge. Justify your answers.

1. $\sum_{n \geq 1} \frac{n^4}{2^n}$
2. $\sum_{n \geq 1} \frac{2^n}{n!}$
3. $\sum_{n \geq 1} (-1)^n$

4. $\sum\limits_{n\geq 0} \sin(\frac{n\pi}{3})$

Exercise 25.3. Study the convergence of the series:

1. $\sum\limits_{n\geq 2} \frac{1}{[n+(-1)^n]^2}$
2. $\sum\limits_{n\geq 1} [\sqrt{n+1}-\sqrt{n}]$
3. $\sum\limits_{n\geq 1} \frac{n!}{n^n}$

Exercise 25.4. Study the convergence of the series:

1. $\sum\limits_{n\geq 2} \frac{n^{\ln n}}{(\ln n)^n}$
2. $\sum\limits_{n\geq 1} \frac{1}{(\ln n)^{\ln n}}$
3. $\sum\limits_{n\geq 1} \frac{(-1)^n n!}{2^n}$

Exercise 25.5. 1. Give an example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges.

2. Show that if $\sum a_n$ is absolutely convergent, then the series $\sum a_n^2$ also converges.
3. Given an example of a convergent series $\sum a_n$, for which $\sum a_n^2$ diverges.

Exercise 25.6. Prove that

$$\sum_{n\geq 1} \frac{1}{n(n+1)} = 1.$$

Exercise 25.7. 1. Prove that

$$\sum_{n\geq 1} \frac{n-1}{2^{n+1}} = \frac{1}{2}.$$

2. Use part (1) to calculate

$$\sum_{n\geq 1} \frac{n}{2^n}.$$

Hint: Note that $\frac{n-1}{2^{n+1}} = \frac{n}{2^n} - \frac{n+1}{2^{n+1}}$.

Exercise 25.8. Let $\{a_n\}_{n\geq 1}$ be a sequence of positive numbers such that $\sum\limits_{n\geq 1} a_n$ diverges. For $n \geq 1$, let $s_n = a_1 + \ldots, a_n$.

1. Prove that the series

$$\sum_{n\geq 1} \frac{a_n}{a_n+1}$$

diverges.

2. Prove that for all $N \geq 1$ and $n \geq 1$,
$$\sum_{k=1}^{n} \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+n}}.$$
Deduce that the series $\sum \frac{a_n}{s_n}$ diverges.

3. Prove that for all $n \geq 2$,
$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}.$$
Deduce that the series $\sum \frac{a_n}{s_n^2}$ converges.

Exercise 25.9. Let $\{a_n\}_{n\geq 1}$ be a decreasing sequence of non-negative numbers such that $\sum\limits_{n\geq 1} a_n < \infty$. Show that
$$\lim_{n\to\infty} na_n = 0.$$

26 Lecture 19: Cardinality, Continued

Example 26.1 (Continued)**.** We prove that f is surjective by induction. For $k \in \mathbb{N}$, let $P(k)$ denote the statement
$$\exists (n,m) \in \mathbb{N} \times \mathbb{N} \text{ such that } f(n,m) = k.$$

Base Case: Note that $f(1,1) = \frac{1\cdot 0}{2} + 1 = 1$, so $P(1)$ holds.

Inductive Step: Fix $k \geq 1$ and assume that $P(k)$ holds. Then $\exists (n,m) \in \mathbb{N} \times \mathbb{N}$ such that $f(n,m) = k$. Therefore,
$$\frac{(n+m-1)(n+m-2)}{2} + n + 1 = k + 1.$$
Therefore, $\frac{[(n+1)(m+1)-1][(n+1)+(m-1)-2]}{2} + n + 1 = k + 1$, so $f(n+1, m-1) = k+1$. This works if $(n+1, m-1) \in \mathbb{N} \times \mathbb{N}$ if and only if $m - 1 \in \mathbb{N}$ and if and only if $m \geq 2$.

Hence, if $m \geq 2$, we found $(n+1, m-1) \in \mathbb{N} \times \mathbb{N}$ such that $f(n+1, m-1) = k+1$. Assume now $m = 1$. Then $f(n,1) = k$ if and only if $\frac{n(n-1)}{2} + n = k$ if and only if $\frac{(n+1)n}{2} = k$. We now know $\frac{(n+1)n}{2} + 1 = k + 1$. We now have
$$\frac{[1 + (n+1) - 1][1 + (n+1) - 2]}{2} + 1 = k + 1,$$
so $f(1, n+1) = k + 1$. Therefore, if $m = 1$, we found $(1, n+1) \in \mathbb{N} \times \mathbb{N}$ such that $f(1, n+1) = k + 1$. This proves $P(k+1)$ holds. By induction, $\forall k \in \mathbb{N}$, $\exists (n,m) \in \mathbb{N} \times \mathbb{N}$ such that $f(n,m) = k$, i.e. f is surjective.

Let $(n,m),(a,n) \in \mathbb{N} \times \mathbb{N}$ such that $f(n,m) = f(a,b)$. We want to show that $(n,m) = (a,b)$, thus proving that f is injective.

Case 1: $\frac{(n+m-1)(n+m-2)}{2} = \frac{(a+b-1)(a+b-2)}{2}$, then because $f(n,m) = f(a,b)$, so $n = a$. Then $(n+m-1)(n+m-2) = (n+b-1)(n+b-2)$. By simplification, we have $(m-b)(2n+m+b-3) = 0$, but note that $2n+m+b-3 \geq 2+1+1-3 \geq 1$, then $m = b$.

Case 2: $\frac{(n+m-1)(n+m-2)}{2} = \frac{(a+b-1)(a+b-2)}{2} + r$ for some $r \in \mathbb{N}$.

Exercise 26.2. Show that this cannot occur.

Lemma 26.3. Let A be a countable set. Let B be an infinite subset of A. Then B is countable.

Proof. Since A is countable, then $\exists f : \mathbb{N} \to A$ is a bijection. This means we may enumerate the elements of A, i.e. denote $A = \{a_1, a_2, a_3, \cdots\}$ where $f(n) = a_n$. Let $k_1 = \min\{n : a_n \in B\}$. Define $g(1) = a_{k_1}$. Then $B \backslash \{a_{k_1}\} \neq \varnothing$. Let $k_2 = \min\{n : a_n \in B \backslash \{a_{k_1}\}\}$. Define $g(2) = a_{k_2}$.

We proceed inductively. Assume we found $k_1 < \cdots < k_j$ such that $a_{k_1}, \cdots, a_{k_j} \in B$, and $g(1) = a_{k_1}, \cdots, g(j) = a_{k_j}$. Then $B \backslash \{a_{k_1}, \cdots, a_{k_j}\} \neq \varnothing$. Let $k_{j+1} = \min\{n : a_n \in B \backslash \{a_{k_1}, \cdots, a_{k_j}\}\}$. Define $g(j+1) = a_{k_{j+1}}$. By construction, $g : \mathbb{N} \to B$ is bijective. $\square$

Lemma 26.4. Let A be a finite set and let B be a proper subset of A. Then A and B are not equipotent, that is, there is no bijective function $f : A \to B$.

Proof. Suppose $B = \varnothing$, then $A \neq \varnothing$. There is no function $f : A \to B$. Assume $B \neq \varnothing$. Assume towards a contradiction that there exists a bijection $f : A \to B$. As $B \subsetneq A$, $\exists a_0 \in A \backslash B$. For $n \geq 1$, let $a_n = f^n(a_0) = (f \circ f \circ \cdots f)(a_0)$. Note $a_{n+1} = f(a_n)$ for all $n \geq 0$, and that $a_n \in B$ for all $n \geq 1$. We will show the following claim.

Claim 26.5. $a_n \neq a_m$ for $n \neq m$.

If the claim holds, then B (and so A) would contain countably many elements. This is a contradiction since A is finite.

Subproof. Assume that there exists $n, k \in \mathbb{N}$ such that $a_{n+k} = a_n$. We write $a_{n+k} = f^n(a_k)$ and $a_n = f^n(a_0)$, then since f is injective, so f^n is injective, then $a_k = a_0$. However, $a_k \in B$ and $a_0 \in A \backslash B$, contradiction. $\blacksquare$

This proves the claim and completes the proof of the lemma. $\square$

Lemma 26.6. Every infinite set has a countable subset.

Proof. Let A be an infinite set, then $A \neq \varnothing$, so $\exists a_1 \in A$. Then $A\{a_1\} \neq \varnothing$, so $\exists a_2 \in A \backslash \{a_1\}$. We proceed inductively. Having found $a_1, \cdots, a_n \in A$ distinct elements, then $A \backslash \{a_1, \cdots, a_n\} \neq \varnothing$, so there exists $a_{n+1} \in A \backslash \{a_1, \cdots, a_n\}$. This gives a sequence $\{a_n\}_{n \geq 1}$ of distinct elements in A. □

Theorem 26.7. A set A is infinite if and only if there is a bijection between A and a proper subset of A.

Proof. ($\Leftarrow$): Assume that there is a bijection $f : A \to B$ where $B \subsetneq A$. By Lemma 26.4, A must be infinite.

($\Rightarrow$): Assume that A is infinite, then by Lemma 26.6, there exists a countable subset B of A. Write $B = \{a_1, a_2, \cdots\}$ with $a_n \neq a_m$ if $n \neq m$. Then $A \backslash \{a_1\}$ is a proper subset of A.

Define $f : A \to A \backslash \{a_1\}$ via

$$f(a) = \begin{cases} a, & \text{if } a \in A \backslash B \\ a_{j+1}, & \text{if } a = a_j \text{ for some } j \geq 1 \end{cases}$$

This is a bijective function. Assume $f(a) = f(b)$.

Case 1: $a, b \in A \backslash B$. Then $f(a) = a$ and $f(b) = b$, and so $f(a) = f(b)$ implies $a = b$.

Case 2: $a, b \in B$, then $\exists i, j \in \mathbb{N}$ such that $a = a_i$ and $b = a_j$. Then $f(a) = f(b)$ implies $a_{i+1} = a_{j+1}$, so $i + 1 = j + 1$, and therefore $i = j$ and so $a = b$.

Case 3: $a \in A \backslash B$ and $b \in B$. Then $f(a) \in A \backslash B$, and $f(b) \in B$, which cannot occur.

Case 4: $a \in B$ and $b \in A \backslash B$. Argue as for case 3.

Exercise 26.8. f is surjective.

□

Theorem 26.9 (Schroder-Bernstein)**.** Assume that A and B are two sets such that there exists two injective functions $f : A \to B$ and $g : B \to A$. Then A and B are equipotent.

Example 26.10. Suppose $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ with $f(n) = (1, n)$, then f is injective. Suppose $g : \mathbb{N} \times \mathbb{N}$ with $g(n, m) = 2^n \cdot 3^m$, then g is injective. Therefore, by Schroder-Bernstein Theorem, $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$.

27 Lecture 20: Cardinality, Continued

We first prove Schroder-Bernstein Theorem.

Proof. We will decompose each of the sets A and B into disjoint subsets:

$$A = A_1 \cup A_2 \cup A_3$$

with $A_i \cap A_j = \varnothing$ if $i \neq j$, and

$$B = B_1 \cup B_2 \cup B_3$$

with $B_i \cap B_j = \varnothing$ if $i \neq j$, and we will show that $f : A_1 \to B_1$, $f : A_2 \to B_2$ and $g : B_3 \to A_3$ are bijections.

If this is the case, then $h : A \to B$ given by

$$h(a) = \begin{cases} f(a), & \text{if } a \in A_1 \cup A_2 \\ (g\mid_{B_3})^{-1}(a), & \text{if } a \in A_3 \end{cases}$$

is a bijection. We leave this as an exercise. For $a \in A$, we consider the set

$$S_a = \{a, g^{-1}(a), f^{-1} \circ g^{-1}(a), g^{-1} \circ f^{-1} \circ g^{-1}(a), \cdots\}$$

where elements are in A and in B, alternatively. Note that the preimage under f or g is either $\varnothing$ or it contains exactly one point (since f and g are injective). There are three posibilities.

1. The process defining S_a does not terminate. We can always find a preimage.
2. The process defining S_a terminates in A, that is, the last element $x \in S_a$ is $x = a$ or $x = f^{-1} \circ g^{-1} \circ \cdots \circ g^{-1}(a)$ and $g^{-1}(x) = \varnothing$.
3. The process defining S_a terminates in B, that is, the last element $x \in S_a$ is $x = g^{-1}(a)$ or $x = g^{-1} \circ f^{-1} \circ g^{-1} \circ \cdots \circ g^{-1}(a)$ and $f^{-1}(x) = \varnothing$.

We define the sets A_1, A_2, A_3 as the set of elements $a \in A$ where the process defining S_a does not terminate, where the process defining S_a terminates in A, and where the process defining S_a terminates in B, respectively.

Similarly, for $b \in B$, we define the set

$$T_b = \{b, f^{-1}(b), g^{-1} \circ f^{-1}(b), f^{-1} \circ g^{-1} \circ f^{-1}(b), \cdots\}$$

where elements are in B and in A, alternatively. As before, we define the sets B_1, B_2, B_3 as the set of elements $b \in B$ where the process defining T_b does not terminate, where the process defining T_b terminates in A, and where the process defining T_b terminates in B, respectively.

We now show that $f : A_1 \to B_1$ is a bijection. Injectivtiy is inherited from $f : A \to B$ is injective. Let $b \in B_1$. Then the process defining $T_b = \{b, f^{-1}(b), g^{-1} \circ f^{-1}(b), f^{-1} \circ g^{-1} \circ$

$f^{-1}(b), \cdots\}$ does not terminate. In particular, $\exists a \in A$ such that $f^{-1}(b) = a$. Note that $S_a = \{a, g^{-1}(a), f^{-1} \circ g^{-1}(a), g^{-1} \circ f^{-1} \circ g^{-1}(a), \cdots\} = \{f^{-1}(b), g^{-1} \circ f^{-1}(b), f^{-1} \circ g^{-1} \circ f^{-1}(b), \cdots\}$ does not terminate, so $a \in A_1$. This proves $f : A_1 \to B_1$ is surjective.

We now show that $f : A_2 \to B_2$ is a bijection. Again, injectivity is inherited from $f : A \to B$ is injective. Let $b \in B_2$. Then the process defining $T_b = \{b, f^{-1}(b), g^{-1} \circ f^{-1}(b), f^{-1} \circ g^{-1} \circ f^{-1}(b), \cdots\}$ terminates in A. In particular, $\exists a \in A$ such that $f^{-1}(b) = a$. Note that $S_a = \{a, g^{-1}(a), f^{-1} \circ g^{-1}(a), g^{-1} \circ f^{-1} \circ g^{-1}(a), \cdots\} = \{f^{-1}(b), g^{-1} \circ f^{-1}(b), f^{-1} \circ g^{-1} \circ f^{-1}(b), \cdots\}$ terminates in A, so $a \in A_2$. Therefore, $f : A_2 \to B_2$ is surjective.

Exercise 27.1. $g : B_3 \to A_3$ is bijective.

□

Remark 27.2. Note that the proof above does not make use of the axiom of choice. The proof is actually trivial with the axiom of choice as an assumption, since it can be proven using the well-ordering principle.

Theorem 27.3. Let $\{A_n\}_{n\geq 1}$ be a sequence of countable sets. Then $\bigcup\limits_{n\geq 1} A_n = \{a : a \in A_n \text{ for some } n \geq 1\}$ is countable.

Proof. We define $B_1 = A_1$ and $B_{n+1} = A_{n+1} \setminus \bigcup\limits_{k=1}^{n} A_k$ for all $n \geq 1$. By construction, we have $B_n \cap B_m = \varnothing$ for all $n \neq m$, and $\bigcup\limits_{n\geq 1} B_n = \bigcup\limits_{n\geq 1} A_n$. Note that each B_n is at most countable. Let $I = \{n \in \mathbb{N} : B_n \neq \varnothing\}$. Then $\bigcup\limits_{n\geq 1} B_n = \bigcup\limits_{n\in I} B_n$. For $n \in I$, let $f_n : B_n \to J_n$ bijection where J_n is an at most countable subset of $\mathbb{N}$. In particular, $f_1 : B_1 \to \mathbb{N}$ bijective and therefore $f_1^{-1} : \mathbb{N} \to B_1$ is bijective.

To show $\bigcup\limits_{n\in I} B_n$ is countable, we will use the Schroder-Bernstein Theorem. Let $g : \mathbb{N} \to \bigcup\limits_{n\in I} B_n$ be defined by $g(n) = f_1^{-1}(n) \in B_1 \subseteq \bigcup\limits_{n\in I} B_n$, then it is injective. Let $h : \bigcup\limits_{n\in I} B_n \to \mathbb{N} \times \mathbb{N}$ be defined as follows: if $b \in \bigcup\limits_{n\in I} B_n$, then there exists a unique $n \in I$ such that $b \in B_n$. Define $h(b) = (n, f_n(b))$. Note that h is injective. Indeed, if $h(b_1) = h(b_2)$, then $(n_1, f_{n_1}(b_1)) = (n_2, f_{n_2}(b_2))$, then $n_1 = n_2$, and $f_{n_1}(b_1) = f_{n_2}(b_2)$. Because f_{n_1} is injective, then $b_1 = b_2$. Recall that there exists a bijection $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, so $\varphi \circ h : \bigcup\limits_{n\in I} B_n \to \mathbb{N}$ is injective. By Schroder-Bernstein theorem, $\bigcup\limits_{n\in I} B_n = \bigcup\limits_{n\geq 1} A_n \sim \mathbb{N}$. □

Proposition 27.4. Let $\{A_n\}_{n\geq 1}$ be a sequence of sets such that for each $n \geq 1$, A_n has at least two elements. Then $\prod\limits_{n\geq 1} A_n = \{\{a_n\}_{n\geq 1} : a_n \in A_n \quad \forall n \geq 1\}$ is uncountable.

Proof. We argue by contradiction. Assume that $\prod_{n\geq 1} A_n$ is countable. Thus, we may enumerate the elements of $\prod_{n\geq 1} A_n$:

$$\begin{aligned} a_1 &= (a_{11}, a_{12}, a_{13}, \cdots) \\ a_2 &= (a_{21}, a_{22}, a_{23}, \cdots) \\ &\vdots \\ a_n &= (a_{n1}, a_{n2}, a_{n3}, \cdots) \\ &\vdots \end{aligned}$$

Let $x = \{x_n\}_{n\geq 1} \in \prod_{n\geq 1} A_n$ such that $x_n \in A_n \backslash \{a_{nn}\}$, as we look at the diagonal elements. Then $x \neq a_n$ for all $n \geq 1$ since $x_n \neq a_{nn}$. This gives a contradiction. □

Remark 27.5. The same argument using binary expansion shows that the set $(0, 1)$ is uncountable.

28 Lecture 21: Cardinality, Continued, Metric Spaces

Proposition 28.1. Let $\{A_n\}_{n\geq 1}$ be a sequence of sets such that $\forall n \geq 1$, the set A_n has at least two elements. Then $\prod_{n\geq 1} A_n$ is uncountable.

Remark 28.2.
1. The Cantor diagonal argument can be used to show that the set $(0, 1)$ is uncountable. This can be proved by binary expansion.
2. We can identify

$$\begin{aligned} &\left\{ \{a_n\}_{n\geq 1} : a_n \in \{0, 1\}, \forall n \geq 1 \right\} \\ &= \{f : \mathbb{N} \to \{0, 1\} : f \text{ function}\} \\ &= \{0, 1\} \times \{0, 1\} \times \cdots \\ &= \{0, 1\}^{\mathbb{N}}. \end{aligned}$$

By Proposition 28.1, this set is uncountable. We say it has cardinality $2^{\aleph_0}$.

Theorem 28.3. Let A be any set. Then there exists no bijection between A and the power set of A, $\mathcal{P}(A) = \{B : B \subseteq A\}$.

Proof. If $A = \varnothing$, then $\mathcal{P}(A) = \{\varnothing\}$. So the cardinality of A is 0, but the cardinality of $\mathcal{P}(A)$ is 1. Thus, A is not equipotent with $\mathcal{P}(A)$.

Assume $A \neq \varnothing$. We argue by contradiction. Assume that there exists $f : A \to \mathcal{P}(A)$ a bijection. Let $B = \{a \in A : a \notin f(a)\} \subseteq A$. Because f is a surjection, then there exists $b \in A$ such that $f(b) = B$. We now distinguish two cases:

- Case 1: $b \in B = f(b)$, then $b \notin B$, we have a contradiction.
- Case 2: $b \notin B = f(b)$, then $b \in B$, we have a contradiction.

Therefore, A is not equipotent to $\mathcal{P}(A)$. □

Theorem 28.4. The set $[0, 1)$ has cardinality $2^{\aleph_0}$.

Proof. We write $x \in [0, 1)$ using the binary expansion $x = 0.x_1x_2x_3\cdots$ with $x_n \in \{0, 1\}$ for all $n \geq 1$. Therefore, we can write $x = \frac{x_1}{2} + \frac{x_2}{2^2} + \frac{x_3}{2^3} + \cdots = \sum_{n\geq 1} \frac{x_n}{2^n}$, with the convention that no expansion ends in all ones. For example, for $x = 0.x_1x_2x_3\cdots x_n0111\cdots = \frac{x_1}{2} + \cdots + \frac{x_n}{2^n} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+3}} + \cdots = \frac{x_1}{2} + \cdots + \frac{x_n}{2^n} + \frac{1}{2^{n+2}} \cdot \frac{1}{1-\frac{1}{2}} = \frac{x_1}{2} + \cdots + \frac{x_n}{2^n} + \frac{1}{2^{n+1}} = 0.x_1x_2\cdots x_n1000\cdots$. Note that we can identify $[0, 1)$ with $\mathcal{F} = \{f : \mathbb{N} \to \{0, 1\} : \forall n \in \mathbb{N}, \exists\ m > n \text{ such that } f(m) = 0\} \subseteq \{f : \mathbb{N} \to \{0, 1\} : f \text{ function}\}$.

In particular, we have an injection $\varphi : [0, 1) \to \{f : \mathbb{N} \to \{0, 1\}\}$. To prove the theorem, by Schroder-Bernstein theorem, it suffices to construct an injective function $\psi : \{f : \mathbb{N} \to \{0, 1\}\} \to [0, 1)$. For $f : \mathbb{N} \to \{0, 1\}$, we define $\psi(f) = 0.0f(1)0f(2)0f(3)\cdots = \frac{f(1)}{2^2} + \frac{f(2)}{2^4} + \frac{f(3)}{2^6} + \cdots = \sum_{n\geq 1} \frac{f(n)}{2^{2n}}$.

We now show that ψ is injective. Let $f_1, f_2 : \mathbb{N} \to \{0, 1\}$ such that $f_1 \neq f_2$. Let $n_0 = \min\{n : f_1(n) \neq f_2(n)\}$. Say, $f_1(n_0) = 1$ and $f_2(n_0) = 0$. Now

$$
\begin{aligned}
&\psi(f_1) - \psi(f_2) \\
&= \sum_{n\geq 1} \frac{f_1(n)}{2^{2n}} - \sum_{n\geq 1} \frac{f_2(n)}{2^{2n}} \\
&= \frac{f_1(n_0) - f_2(n_0)}{2^{2n_0}} + \sum_{n\geq n_0+1} \frac{f_1(n) - f_2(n)}{2^{2n}} \\
&\geq \frac{1}{2^{2n_0}} - \sum_{n\geq n_0+1} \frac{1}{2^{2n}} \\
&= \frac{1}{2^{2n_0}} - \frac{1}{2^{2(n_0+1)}} \cdot \frac{1}{1-\frac{1}{2}} \\
&= \frac{1}{2^{2n_0}} - \frac{1}{2^{2n_0+1}} \\
&= \frac{1}{2^{2n_0+1}} \\
&> 0,
\end{aligned}
$$

therefore $\psi(f_1) > \psi(f_2)$. Hence, ψ is an injective function. By Schroder-Bernstein, $[0,1) \sim \{f : \mathbb{N} \to \{0,1\}\}$ and so it has cardinality $2^{\aleph_0}$. $\square$

Definition 28.5 (Metric Space)**.** Let X be a non-empty set. A metric on X is a map $d : X \times X \to \mathbb{R}$ such that

1. $d(x,y) \geq 0 \ \ \forall x,y \in X$.
2. $d(x,y) = 0$ if and only if $x = y$.
3. $d(x,y) = d(y,x) \ \ \forall x,y \in X$.
4. $d(x,y) \leq d(x,z) + d(z,y) \ \ \forall x,y,z \in X$.

Then we say (X,d) is a metric space.

Example 28.6.

1. $X = \mathbb{R}$, $d(x,y) = |x-y|$.
2. $X = \mathbb{R}^n$, $d_2(x,y) = \sqrt{\sum\limits_{k=1}^{n} |x_k - y_k|^2}$ is a metric, called the l^2 norm.
3. Let X be any non-empty set, the discrete metric is defined by $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$.
4. Let (x,d) be a metric space. Then $\tilde{d} : X \times X \to \mathbb{R}$ defined by $\tilde{d}(x,y) = \frac{d(x,y)}{1+d(x,y)}$ is a metric.

   We now show that this metric satisfies the fourth property. Since d is a metric, fix $x,y,z \in X$, then $d(x,y) \leq d(x,z) + d(z,y)$. Note $a \mapsto \frac{a}{1+a} = 1 - \frac{1}{1+a}$ is increasing on $[0,\infty)$. Then $\tilde{d}(x,y) = \frac{d(x,y)}{1+d(x,y)} \leq \frac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)} \leq \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)} = \tilde{d}(x,z) + \tilde{d}(z,y)$.

Definition 28.7 (Bounded)**.** We say that a metric space (X,d) is bounded if $\exists M > 0$ such that $d(x,y) \leq M \ \ \forall x,y \in X$.

If (X,d) is not bounded, we say that it is unbounded.

Remark 28.8. If (X,d) is an unbounded metric space, then $(X,\tilde{d})$ is a bounded metric space where $\tilde{d}(x,y) = \frac{d(x,y)}{1+d(x,y)}$.

Definition 28.9 (Distance)**.** Let (X,d) be a metric space and let $A,B \subseteq X$. The distance between A and B is $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$.

Remark 28.10. This does not define a metric on subsets of X. In fact, $d(A, B) = 0$ does not even imply $A \cap B \neq \varnothing$.

For example, $(X, d) = (\mathbb{R}, |\cdot|)$, let $A = (0, 1)$, $B = (-1, 0)$, now $d(A, B) = 0$ but $A \cap B = \varnothing$.

Definition 28.11 (Distance)**.** Let (X, d) be a metric space, $A \subseteq X$, $x \in X$. The distance from x to A is $d(x, A) = \inf\{d(x, a) : a \in A\}$. Again, $d(x, A) = 0$ does not imply $x \in A$.

29 Homework 8

Exercise 29.1. Show that the following two statements are equivalent:

(i) The function $f : A \to B$ is surjective.

(ii) For every set C and any functions $g : B \to C$ and $h : N \to C$ such that $g \circ f = h \circ f$, we have $g = h$.

Exercise 29.2. Show that the following two statements are equivalent:

(i) The function $f : B \to C$ is injective.

(ii) For every set A and any functions $g : A \to B$ and $h : A \to B$ such that $f \circ g = f \circ h$, we have $g = h$.

Exercise 29.3. If the set A has n elements and the set B has m elements, show that there are m^n many functions from A to B.

Exercise 29.4. Fix $n \geq 1$. Show that if $A_1, A_2, \ldots, A_n$ are countable sets, then the cartesian product $A_1 \times A_2 \times \ldots \times A_n$ is countable.

Exercise 29.5. If the sets A and B are equipotent ($A \sim B$), show that $\mathcal{P}(A) \sim \mathcal{P}(B)$.

Exercise 29.6. Prove that $\mathcal{P}(\mathbb{N})$ is equipotent with the set of functions

$$2^{\mathbb{N}} = \{f : \mathbb{N} \to \{0, 1\} : f \text{ is a function}\}.$$

In particular, the cardinality of $\mathcal{P}(\mathbb{N})$ is $2^{\aleph_0}$.

Exercise 29.7. Show that $\mathbb{N}^{\mathbb{N}} \sim 2^{\mathbb{N}}$, that is, the set of sequences with values in $\mathbb{N}$ is equipotent with the set of sequences with values in $\{0, 1\}$.

Exercise 29.8. Fix $n \geq 1$ and let $\mathcal{P}$ denote the set of polynomials of degree n with integer coefficients, that is,

$$\mathcal{P} = \{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 : a_i \in \mathbb{Z} \text{ for all } 1 \leq i \leq n \text{ and } a_n \neq 0\}.$$

Show that the set of all real roots of all polynomials in $\mathcal{P}$, that is,

$$\mathcal{A} = \{x \in \mathbb{R} : \text{ there exists } p \in \mathcal{P} \text{ such that } p(x) = 0\}$$

is countable.

Exercise 29.9. Fix $n \geq 1$. Show that the set of all subsets of $\mathbb{N}$ with n distinct elements is countable.

Exercise 29.10. Prove that the set of irrational numbers has the cardinality of $\mathbb{R}$.

30 Lecture 22: Hölder's Inequality, Basic Topology

Proposition 30.1 (Hölder's Inequality)**.** Fix $1 \leq p \leq \infty$ and let q denote the dual of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$. Let $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and let $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$. Then

$$\sum_{k=1}^{n} |x_k y_k| \leq (\sum_{k=1}^{n} |x_k|^p)^{\frac{1}{p}} (\sum_{k=1}^{n} |y_k|^q)^{\frac{1}{q}}$$

with the convention that if $p = \infty$, then $(\sum_{k=1}^{n} |x_k|^p)^{\frac{1}{p}} = \sup_{1 \leq k \leq n} |x_k|$.

Remark 30.2. If $p = 2$ (and so $q = 2$), we call this the Cauchy-Schwarz inequality.

Proof. If $p = 1$, then $q = \infty$. Now we have

$$\sum_{k=1}^{n} |x_k y_k| \leq \sum_{k=1}^{n} |x_k| \cdot \sup_{1 \leq l \leq n} |y_l| \leq (\sum_{k=1}^{n} |x_k|) \cdot \sup_{1 \leq l \leq n} |y_l|.$$

If $p = \infty$, then $q = 1$, a similar argument yields the claim.

We now assume $1 < p < \infty$. We will use the fact that $f(0, \infty) \to \mathbb{R}$, $f(x) = \log(x)$ is a concave function. This tells us that for any $a > b$, at the point $b < b+t(a-b) = ta(1-t)b < a$, we have $f(b+t(a-b)) > f(b)+t(f(a)-f(b))$. Therefore, for any $(a, b) \in (0, \infty)$ and $t \in (0, 1)$, we have $tf(a) + (1-t)f(b) \leq f(ta + (1-t)b)$. Therefore, we have $t\log(a) + (1-t)\log(b) \leq \log(ta + (1-t)b)$, and so $\log(a^t b^{1-t}) \leq \log(ta + (1-t)b)$, and so $a^t b^{1-t} \leq ta + (1-t)b$.

We now apply this inequality with $a = \frac{|x_k|^p}{\sum\limits_{l=1}^n |x_l|^p}$ and $b = \frac{|y_k|^q}{\sum\limits_{l=1}^n |y_l|^q}$, now $t = \frac{1}{p}$, so $1 - t = 1 - \frac{1}{p} = \frac{1}{q}$. We get

$$\frac{|x_k|}{(\sum\limits_{l=1}^n |x_l|^p)^{\frac{1}{p}}} \cdot \frac{|y_k|}{(\sum\limits_{l=1}^n |y_l|^q)^{\frac{1}{q}}} \leq \frac{1}{p} \frac{|x_k|^p}{\sum\limits_{l=1}^n |x_l|^p} + \frac{1}{q} \frac{|y_k|^q}{\sum\limits_{l=1}^n |y_l|^q}$$

and by summing over $1 \leq k \leq n$,

$$\sum_{k=1}^n \frac{|x_k|}{(\sum\limits_{l=1}^n |x_l|^p)^{\frac{1}{p}}} \cdot \frac{|y_k|}{(\sum\limits_{l=1}^n |y_l|^q)^{\frac{1}{q}}} \leq \frac{1}{p} \sum_{k=1}^n \frac{1}{p} \frac{|x_k|^p}{\sum\limits_{l=1}^n |x_l|^p} + \frac{1}{q} \sum_{k=1}^n \frac{1}{p} \frac{|y_k|^p}{\sum\limits_{l=1}^n |y_l|^p} = \frac{1}{p} + \frac{1}{q} = 1.$$

Therefore,

$$\sum_{k=1}^n |x_k y_k| \leq (\sum_{k=1}^n |x_k|^p)^{\frac{1}{p}} (\sum_{k=1}^n |y_k|^q)^{\frac{1}{q}}.$$

□

Corollary 30.3 (Minkowski's Inequality)**.** Fix $1 \leq p \leq \infty$ and let $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$. Then

$$\left(\sum_{k=1}^n |x_k + y_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^n |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^n |y_k|^p\right)^{\frac{1}{p}}$$

Proof. If $p = 1$, this follows from the triangle inequality: indeed, the left-hand side is just $\sum\limits_{k=1}^{\infty} |x_k + y + k|$, and that is bounded above by $\sum\limits_{k=1}^{n} |x_k| + |y_k|$, which is the right-hand side.

If $p = \infty$, then the left-hand side is just $\sup\limits_{1 \leq k \leq n} |x_k + y_k|$, which is bounded above by $\sup\limits_{1 \leq k \leq n} |x_k| + \sup\limits_{1 \leq k \leq n} |y_k|$, which is the right-hand side.

From now on, we can just assume $1 < p < \infty$. We observe that

$$\begin{aligned}
\sum_{k=1}^{n}|x_k+y_k|^p &= \sum_{k=1}^{n}|x_k+y_k||x_k+y_k|^{p-1} \\
&\leq \sum_{k=1}^{n}\left(|x_k|+|y_k|\right)|x_k+y_k|^{p-1} \\
&= \sum_{k=1}^{n}|x_k|\cdot|x_k+y_k|^{p-1}+\sum_{k=1}^{n}|y_k||x_k+y_k|^{p-1} \\
&\leq \left(\sum_{k=1}^{n}|x_k|^p\right)^{\frac{1}{p}}\cdot\left(\sum_{k=1}^{n}|x_k+y_k|^{(p-1)\cdot q}\right)^{\frac{1}{q}} \\
&+ \left(\sum_{k=1}^{n}|y_k|^p\right)^{\frac{1}{p}}\cdot\left(\sum_{k=1}^{n}|x_k+y_k|^{(p-1)\cdot q}\right)^{\frac{1}{q}} \\
&= \left[\left(\sum_{k=1}^{n}|x_k|^p\right)^{\frac{1}{p}}+\left(\sum_{k=1}^{n}|y_k|^p\right)^{\frac{1}{p}}\right]\cdot\left(\sum_{k=1}^{n}|x_k+y_k|^{(p-1)\cdot q}\right)^{\frac{1}{q}}
\end{aligned}$$

where the last inequality follows from Hölder's inequality. Because $\frac{1}{p}+\frac{1}{q}=1$, then $\frac{1}{q}=\frac{p-1}{p}$, so $q=\frac{p}{p-1}$. We then get that

$$\sum_{k=1}^{n}|x_k+y_k|^p \leq \left[\left(\sum_{k=1}^{n}|x_k|^p\right)^{\frac{1}{p}}+\left(\sum_{k=1}^{n}|y_k|^p\right)^{\frac{1}{p}}\right]\cdot\left(\sum_{k=1}^{n}|x_k+y_k|^p\right)^{1-\frac{1}{p}}$$

and therefore

$$\left(\sum_{k=1}^{n}|x_k+y_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n}|x_k|^p\right)^{\frac{1}{p}}+\left(\sum_{k=1}^{n}|y_k|^p\right)^{\frac{1}{p}}.$$

$\square$

Corollary 30.4. For $1 \leq p < \infty$, let $d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be such that $d_p(x,y) = \left(\sum_{k=1}^{n}|x_k-y_k|^p\right)^{\frac{1}{p}}$. For $p=\infty$, let $d_\infty : \mathbb{R}^n \times \mathbb{R}^{\to}\mathbb{R}$ be such that $d_\infty(x,y) = \sup_{1\leq k\leq n}|x_k-y_k|$. Then d_p is a metric on $\mathbb{R}^n$ for all $1 \leq p \leq \infty$.

Proof. The triangle inequality follows from Minkowski's inequality. $\square$

Remark 30.5. Hölder's and Minkowski's inequalities generalize to sequences. For example, say $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ are sequences of real numbers such that $\left(\sum_{n\geq 1}|x_n|^p\right)^{\frac{1}{p}} < \infty$ and

$\left(\sum_{n\geq 1}|y_n|^q\right)^{\frac{1}{q}} < \infty$. Then for each fixed $N \geq 1$, we have

$$\sum_{n=1}^{N}|x_ky_k| \leq \left(\sum_{n=1}^{N}|x_n|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{n=1}^{N}|y_n|^q\right)^{\frac{1}{q}} \leq \left(\sum_{n\geq 1}|x_n|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{n\geq 1}|y_n|^q\right)^{\frac{1}{q}} < \infty.$$

Note that this is an increasing sequence indexed by N. So

$$\sum_{n\geq 1}|x_ky_k| \leq \left(\sum_{n\geq 1}|x_n|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{n\geq 1}|y_n|^q\right)^{\frac{1}{q}}.$$

A similar argument gives Minkowski for sequences.

Definition 30.6 (Neighborhood)**.** Let (X, d) be a metric space. A neighborhood of a point $a \in X$ is $B_r(a) = \{x \in X : d(a, x) < r\}$ for some $r > 0$.

Example 30.7.

1. $(\mathbb{R}^2, d_2)$ is a metric space. The ball of radius 1 is represented by $B_1(0) = \{(x, y) \in \mathbb{R}^2 : d_2((x, y), (0, 0))\} < 1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, which is just the unit disc.

2. $(\mathbb{R}^2, d_1)$ is a metric space. The ball of radius 1 is represented by $B_1(0) = \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\}$, which is a square of side length $\sqrt{2}$.

3. $(\mathbb{R}^2, d_\infty)$ is a metric space. The ball of radius 1 is represented by $B_1(0) = \{(x, y) \in \mathbb{R}^2 : \max\{|x|, |y|\} < 1\}$, which is a square of side length 2.

Definition 30.8 (Interior, Open)**.** Let (X, d) be a metric space and let $\varnothing \neq A \subseteq X$. We say that a point $a \in X$ is an interior point of A if $\exists r > 0$ such that $B_r(a) \subseteq A$. The set of all interior points of A is denoted by $\mathring{A}$ and is called the interior of A. We say that A is open if $A = \mathring{A}$.

Example 30.9.

1. $\varnothing, X$ are open sets.

2. $B_r(a)$ is an open set for all $a \in X$, for all $r > 0$. Indeed, let $x \in B_r(a)$, then $d(x, a) < r$ by definition. Hence, $l = r - d(x, a) > 0$.

   Claim 30.10. $B_l(x) \subseteq B_r(a)$ and so $x \in \mathring{B_r}(a)$.

   Subproof. Let $y \in B_l(x)$, then $d(x, y) < l$. Now $d(y, a) \leq d(y, x) + d(x, a) < l + d(x, a) = r$, then $y \in B_r(a)$ by definition. $\square$

Remark 30.11. $\mathring{A} \subseteq A$. To prove A is open, it suffices to show $A \subseteq \mathring{A}$.

31 Lecture 23: Basic Topology, Continued

Proposition 31.1. Let (X, d) be a metric space and let $A, B \subseteq X$. Then

1. If $A \subseteq B$ then $\mathring{A} \subseteq \mathring{B}$.

2. $\mathring{A} \cup \mathring{B} \subseteq \overset{\circ}{(A \cup B)}$.

3. $\mathring{A} \cap \mathring{B} = \overset{\circ}{(A \cap B)}$.

4. $\mathring{\mathring{A}} = \mathring{A}$. In particular, $\mathring{A}$ is an open set.

5. $\mathring{A}$ is the largest open set contained in A.

6. A finite intersection of open sets is an open set.

7. An arbitrary union of open sets is an open set.

Remark 31.2. An arbitrary intersection of open sets need not be open. For example, consider $\bigcap\limits_{n \geq 1} (-\frac{1}{n}, \frac{1}{n}) = \{0\} = B_{\frac{1}{n}}(0)$ in $(\mathbb{R}, |\cdot|)$. Note that $\{0\}$ is not an open set because it does not contain any neighborhood of 0.

Proof.

1. If $\mathring{A} = \varnothing$, then this is clear. Assume $\mathring{A} \neq \varnothing$. Let $a \in \mathring{A}$, then there exists $r > 0$ such that $B_r(a) \subseteq A$, and because $A \subseteq B$, then $B_r(a) \subseteq B$. Hence, $a \in \mathring{B}$.

2. Because $A \subseteq A \cup B$, then by (1), $\mathring{A} \subseteq \overset{\circ}{(A \cup B}$. Similarly, $\mathring{B} \subseteq \overset{\circ}{(A \cup B)}$. Therefore, $\mathring{A} \cup \mathring{B} \subseteq \overset{\circ}{(A \cup B)}$.

3. Because $A \cap B \subseteq A$, then $\overset{\circ}{(A \cap B)} \subseteq \mathring{A}$. Similarly, $\overset{\circ}{(A \cap B)} \subseteq \mathring{B}$, and so $\overset{\circ}{(A \cap B)} \subseteq \mathring{A} \cap \mathring{B}$. Now let $x \in \mathring{A} \cap \mathring{B}$, then there exists $r_1, r_2 > 0$ such that $B_{r_1}(x) \subseteq A$ and $B_{r_2}(x) \subseteq B$. Let $r = \min\{r_1, r_2\} > 0$, then $B_r(x) \subseteq B_{r_1}(x) \cap B_{r_2}(x) \subseteq A \cap B$, therefore $x \in \overset{\circ}{(A \cap B)}$. Hence, $\mathring{A} \cap \mathring{B} \subseteq \overset{\circ}{(A \cap B)}$.

4. Because $\mathring{A} \subseteq A$, then $\mathring{\mathring{A}} \subseteq \mathring{A}$. Let $x \in \mathring{A}$, then there exists $r > 0$ such that $B_r(x) \subseteq A$. By (1), $B_r(x) = \overset{\circ}{B_r}(x) \subseteq \mathring{A}$, so $x \in \mathring{\mathring{A}}$. Therefore, $\mathring{A} \subseteq \mathring{\mathring{A}}$.

5. By (4), $\mathring{A}$ is an open set contained in A. Let $B \subseteq A$ be an open set, then by (1), $B = \mathring{B} \subseteq \mathring{A}$.

6. Using (3) and (4), we see that if $A = \mathring{A}$ and $B = \mathring{B}$, then $A \cap B = \overset{\circ}{(A \cap B)}$ is an open set. A simple inductive argument then yields the claim.

7. Let $\{A_i\}_{i\in I}$ be a family of open sets. Let us show $\overset{\circ}{\left(\bigcup_{i\in I} A_i\right)} = \bigcup_{i\in I} A_i$. Let $x \in \bigcup_{i\in I} A_i$, then there exists $i_0 \in I$ such that $x \in A_{i_0}$. But $A_{i_0} = \mathring{A}_{i_0}$, then there exists $r > 0$ such that $B_r(x) \subseteq A_{i_0}$. Therefore, $B_r(x) \subseteq \bigcup_{i\in I} A_i$, so $x \in \overset{\circ}{\left(\bigcup_{i\in I} A_i\right)}$. Thus, $\overset{\circ}{\left(\bigcup_{i\in I} A_i\right)} \supseteq \bigcup_{i\in I} A_i$, which concludes the proof.

$\square$

Definition 31.3 (Closed)**.** Let (X, d) be a metric space. A set $A \subseteq X$ is closed if ${}^cA$ is open.

Example 31.4. 1. $\varnothing, X$ are closed.

2. If $a \in X$, $r < 0$, then ${}^cB_r(a) = \{x \in X : d(a, x) \geq r\}$ is a closed set.

3. If $a \in X$, $r < 0$, then $K_r(a) = \{x \in X : d(a, x) \leq r\}$ is a closed set. Indeed, let us show that ${}^cK_r(a) = \{x \in X : d(a, x) > r\}$ is open. Let $x \in {}^cK_r(a)$, then $d(a, x) > r$. Now let $l = d(a, x) - r > 0$.

   Claim 31.5. $B_l(x) \subseteq {}^cK_r(a)$.

   Subproof. Let $y \in B_l(x)$, then $d(x, y) < l$. By triangle inequality, $d(a, y) \geq d(a, x) - d(x, y) > d(a, x) - l = r$. Therefore, $y \in cK_r(a)$. This concludes the proof. $\square$

   By definition, $x \in ({}^c\mathring{K_r}(a))$. Thus, ${}^cK_r(a)$ is an open set.

4. There are sets that are neither open nor closed. For example, $(0, 1]$.

Definition 31.6 (Adherent Point, Closure, Isolation, Accumulation Point)**.** Let (X, d) be a metric space and let $A \subseteq X$. A point $a \in X$ is an adherent point for A if for all $r > 0$, we have $B_r(a) \cap A \neq \varnothing$. The set of all adherent points of A is denoted $\bar{A}$ and is called the closure of A.

An adherent point is called isolated if $\exists r > 0$ such that $B_r(a) \cap A = \{a\}$. In particular, $a \in A$. If every point in A is an isolated point of A, then A is called an isolated set.

An adherent point a of A that is not isolated is called an accumulation point for A. The set of accumulation points of A is denoted A'.

Note that $a \in A'$ if and only if $\forall r > 0$, $B_r(a) \cap A \backslash \{a\} \neq \varnothing$.

Example 31.7. Consider $(\mathbb{R}, |\cdot|)$ with $A = \{\frac{1}{n} : n \geq 1\}$. Now A is isolated. Indeed, $B_{\frac{1}{n(n+1)}}(\frac{1}{n}) \cap A = \{\frac{1}{n}\}$, and $A' = \{0\}$ since $\forall r > 9$, $B_r(0) = (-r, r)$ intersects $A \backslash \{0\} = A$.

Remark 31.8. 1. $A \subseteq \bar{A}$.

2. $\bar{A} = A' \cup A$.

32 Lecture 24: Basic Topology, Continued; Complete Metric Space

Proposition 32.1. Let (X, d) be a metric space and let $A, B \subseteq X$. Then

1. ${}^c(\bar{A}) = ({}^c\mathring{A})$.
2. ${}^c(\mathring{A}) = \overline{{}^cA}$.
3. A is a closed set if and only if $A = \bar{A}$.
4. If $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$.
5. $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$.
6. $\bar{A} \cup \bar{B} = \overline{A \cup B}$.
7. $\bar{\bar{A}} = \bar{A}$. In particular, $\bar{A}$ is a closed set.
8. $\bar{A}$ is the smallest closed set containing A.
9. A finite union of closed sets is a closed set.
10. An arbitrary intersection of closed sets is a closed set.

Remark 32.2. An arbitrary union of closed sets need not be a closed set. For example, $\bigcup\limits_{n \geq 1} [\frac{1}{n}, 1] = (0, 1]$.

Proof.

1. $x \in {}^c(\bar{A})$ if and only if $x \notin \bar{A}$, if and only if $\exists r > 0$ such that $B_r(x) \cap A = \varnothing$, if and only if $\exists r > 0$ such that $B_r(x) \subseteq {}^cA$, if and only if $x \in {}^c\mathring{A}$.
2. Apply part (1) to ${}^cA$.
3. A is closed if and only if ${}^cA$ is open if and only if ${}^cA = {}^c\mathring{A}$ if and only if (by part 1) that ${}^cA = {}^c(\bar{A})$, if and only if $A = \bar{A}$.
4. If $\bar{A} = \varnothing$, then clearly $\bar{A} \subseteq \bar{B}$. Assume $\bar{A} \neq \varnothing$. Let $a \in \bar{A}$, then for all $r > 0$ we have $B_r(a) \cap A \neq \varnothing$, but since $A \subseteq B$, then $B_r(a) \cap B \neq \varnothing$ for all $r > 0$ as well. hence, $a \in \bar{B}$. Therefore, $\bar{A} \subseteq \bar{B}$.
5. Since $A \cap B \subseteq A$, by part (4) we know $\overline{A \cap B} \subseteq \bar{A}$, and similarly we know $\overline{A \cap B} \subseteq \bar{B}$. Therefore, $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$.

6. Because of part (1), ${}^c(\overline{A \cup B}) = {}^c(\mathring{A \cup B}) = {}^cA \mathring{\cap} {}^cB = {}^c\mathring{A} \cap {}^c\mathring{B}$, then by part (1) we know that ${}^c(\bar{A}) \cap {}^c(\bar{B})$, which is just ${}^c(\bar{A} \cup \bar{B})$. Therefore, $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

7. Clearly $A \subseteq \bar{A}$, then by part (4), we have $\bar{A} \subseteq \bar{\bar{A}}$. We want to show the other inclusion $\bar{\bar{A}} \subseteq \bar{A}$. Let $a \in \bar{\bar{A}}$. We show that for all $r > 0$, $B_r(a) \cap A \neq \varnothing$.

   Fix $r > 0$. Since $a \in \bar{\bar{A}}$, then $B_r(a) \cap \bar{A} \neq \varnothing$. Let $x \in B_r(a) \cap \bar{A}$, then $x \in \bar{A}$, so for any $l > 0$, $B_l(x) \cap A \neq \varnothing$. We now choose $l = r - d(a, x) > 0$, then $B_l(x) \subseteq B_r(a)$. But since $B_l(x) \cap A \neq \varnothing$, then $B_r(a) \cap A \neq \varnothing$. Therefore, $a \in \bar{A}$.

8. Note that $\bar{A}$ is a closed set containing A. Let B be a closed set containing A, then if $A \subseteq B$, then $\bar{A} \subseteq \bar{B} = B$, according to part (3).

9. Let $\{A_n\}_{n=1}^N$ be closed sets. Then ${}^cA_n$ is an open set for all $1 \leq n \leq N$. Therefore, $\bigcap_{n=1}^{N} {}^cA_n$ is an open set. Now $\bigcup_{n=1}^{N} {}^cA_n = {}^c(\bigcup_{n=1}^{N} A_n)$ is open, therefore $\bigcup_{n=1}^{N} A_n$ is closed.

10. Let $\{A_i\}_{i \in I}$ be a family of closed sets. Then ${}^cA_i$ is open for all $i \in I$. Therefore, $\bigcup_{i \in I} {}^cA_i = {}^c(\bigcap_{i \in I} A_i)$ is open, and so $\bigcap_{i \in I} A_i$ is closed.

□

Definition 32.3 (Induced Metric, Subspace)**.** Let (X, d) be a metric space and let $\varnothing \neq Y \subseteq X$. Then $d_1 : Y \times Y \to \mathbb{R}$ defined by $d_1(x, y) = d(x, y)$ for all $x, y \in Y$ is a metric on Y, and is called the induced metric on Y. Now (Y, d_1) is called a subspace of (X, d).

Proposition 32.4. Let (X, d) be a metric space and let $\varnothing \neq Y \subseteq X$ equipped with the induced metric d_1.

1. A set $D \subseteq Y$ is open in (Y, d_1) if and only if there exists $O \subseteq X$ open in (X, d) such that $D = O \cap Y$.

2. A set $F \subseteq Y$ is closed in (Y, d_1) if and only if there exists $C \subseteq X$ closed in (X, d) such that $F = C \cap Y$.

Proof. 1. ($\Rightarrow$): Let $D \subseteq Y$ be open in (Y, d_1). Then for all $a \in D$, there exists $r_a > 0$ such that $B_{r_1}^Y(a) = \{y \in Y : d(a, y) < r_1\} \subseteq B$. Note that $B_{r_a}^Y(a) = B_{r_a}^X(a) \cap Y$. Therefore, $D = \bigcup_{a \in D} B_{r_a}^Y(a) = \bigcup_{a \in D} \left[B_{r_a}^X(a) \cap Y\right] = \left(\bigcup_{a \in D} B_{r_a}^X(a)\right) \cap Y$, with $\bigcup_{a \in D} B_{r_a}^X(a)$ open in (X, d).

   ($\Leftarrow$): Assume that $D = O \cap Y$ for O open in (X, d). Let $a \in D \subseteq O$, then there exists $r > 0$ such that $B_r^X(a) \subseteq O$. Therefore, $B_r^Y(a) = B_r^X(a) \cap Y \subseteq O \cap Y = D$. Therefore, a is an interior point of D in (Y, d_1). Therefore, D is open in (Y, d_1).

2. Note that $F \subseteq Y$ is closed in (Y, d_1) if and only if $Y \backslash F$ is open in (Y, d_1), if and only if (by part (1)) there exists an open set O in (X, d) such that $Y \backslash F = O \cap Y$. But $F = Y \backslash (Y \backslash F) = Y \backslash (O \cap Y) = Y \cap {}^c(O \cap Y) = Y \cap ({}^cO \cup {}^cY) = (Y \cap {}^cO) \cup (Y \cap {}^cY = Y \cap {}^cO$, where ${}^cO$ is closed in (X, d).

□

Example 32.5. 1. $[0, 1)$ is not an open set in $(\mathbb{R}, |\cdot|)$, but it is open in $([0, 2), |\cdot|)$. Say $[0, 1) = (-1, 1) \cap [0, 2)$.

2. $(0, 1]$ is not a closed set in $(\mathbb{R}, |\cdot|)$, but it is closed in $((0, 2), |\cdot|)$. Say $(0, 1] = [-1, 1] \cap (0, 2)$.

Proposition 32.6. Let (X, d) be a metric space and let $\varnothing \neq Y \subseteq X$ equipped with the induced metric. The following are equivalent:

1. Any $A \subseteq Y$ that is open (respectively, closed) in Y is also open (respectively, closed) in X.

2. Y is open (respectively, closed) in X.

Proof. (1) $\Rightarrow$ (2): Take $A = Y$.

(2) $\Rightarrow$ (1): Assume Y is closed in X. Let $A \subseteq Y$ be open in Y, then there exists an open set O in X such that $A = O \cap Y$, but both sets are open in X, then A is also open in X. □

Proposition 32.7. Let (X, d) be a metric space and let $\varnothing \neq Y \subseteq X$ equipped with the induced metric. For a set $A \subseteq Y$, $\bar{A}^Y = \bar{A}^X \cap Y$.

Proof. $a \in \bar{A}^Y$ if and only if for all $r > 0$, $B_r^(a) \cap A \neq \varnothing$, if and only if for all $r > 0$, $B_r^X(a) \cap Y \cap A \neq \varnothing$, if and only if $a \in \bar{A}^X \cap Y$. □

Definition 32.8 (Limit)**.** Let (X, d) be a metric space and let $\{x_n\}_{n \geq 1} \subseteq X$. We say $\{x_n\}_{n \geq 1}$ converges to a point $x \in X$ if for all $\varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq n_\varepsilon$. Then x is called the limit of $\{x_n\}_{n \geq 1}$ and we write $x = \lim\limits_{n \to \infty} x_n$ or $x_n \xrightarrow[n \to \infty]{d} x$.

Exercise 32.9. The limit of a convergent sequence is unique.

Exercise 32.10. A sequence $\{x_n\}_{n \geq 1}$ converges to $x \in X$ if and only if every subsequence of $\{x_n\}_{n \geq 1}$ converges to x.

Remark 32.11. If $x_n \xrightarrow[n \to \infty]{d} x$ and $y_n \xrightarrow[n \to \infty]{d} y$, then $d(x_n, y_n) \xrightarrow[n \to \infty]{} d(x, y)$. Indeed, $|d(x_n, y_n) - d(x, y)| \leq |d(x_n, y_n) - d(x_n, y)| + |d(x_n, y) - d(x, y)| \leq d(y_n, y) + d(x_n, x) \xrightarrow[n \to \infty]{} 0$.

Definition 32.12 (Cauchy Sequence)**.** Let (X, d) be a metric space. We say that $\{x_n\}_{n\geq 1} \subseteq X$ is Cauchy if for all $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $d(x_n, x_m) M\varepsilon$ for all $n, m \geq n_\varepsilon$.

Exercise 32.13. Every convergent sequence is Cauchy.

Remark 32.14. Not every Cauchy sequence is convergent in an arbitrary metric space.

Example 32.15. $(X, d) = ((0,1), |\cdot|)$, $x_n = \frac{1}{n}$ for all $n \geq 2$ is Cauchy but does not converge in X.

Example 32.16. $(X, d) = (\mathbb{Q}, |\cdot|)$, $x_1 = 3$, $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ for all $n \geq 1$. Then $\{x_n\}_{n\geq 1}$ is Cauchy, but does not converge in X.

Definition 32.17 (Complete Metric Space)**.** A metric space (X, d) is complete if every Cauchy sequence in X converges in X.

Example 32.18. $(\mathbb{R}, |\cdot|)$ is a complete metric space.

Exercise 32.19. Show that a Cauchy sequence with a convergent subsequence converges.

33 Homework 9

Exercise 33.1. Let (X, d_1) be a metric space and let $d_2 : X \times X \to \mathbb{R}$ be the metric defined as follows: for any $x, y \in X$,

$$d_2(x, y) = \frac{d_1(x, y)}{1 + d_1(x, y)}.$$

Prove that a subset A of X is open with respect to the metric d_1 if and only if it is open with respect to the metric d_2.

Exercise 33.2. Let $1 \leq p, q \leq \infty$ and consider the two metrics on $\mathbb{R}^n$ given by

$$d_p(x, y) = \left(\sum_{k=1}^{n} |x_k - y_k|^p\right)^{\frac{1}{p}} \quad \text{and} \quad d_q(x, y) = \left(\sum_{k=1}^{n} |x_k - y_k|^q\right)^{\frac{1}{q}},$$

with the usual convention if p or q are infinity. Prove that a set $A \subseteq \mathbb{R}^n$ is open with respect to the metric d_p if and only if it is open with respect to the metric d_q.

Exercise 33.3. Let (X, d) be a metric space and let A be a non-empty subset of X. Prove that A is open if and only if it can be written as the union of a family of open balls of the form $B_r(x) = \{y \in X : d(x, y) < r\}$.

Exercise 33.4. Let X be a non-empty set and let $d : X \times X \to \mathbb{R}$ be the discrete metric on X defined as follows: for any $x, y \in X$,

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y. \end{cases}$$

Find the open and the closed subsets of this metric space.

Exercise 33.5. Let (X, d) be a metric space. The *diameter* of a set $\varnothing \neq A \subseteq X$ is given by

$$\delta(A) = \sup\{d(x, y) : x, y \in A\},$$

with the convention that $\delta(A) = \infty$ if the set $\{d(x, y) : x, y \in A\}$ is unbounded.

1. Assume that $\delta(A) < r$ for some $r > 0$ and that $A \cap B_r(a) \neq \varnothing$ for some $a \in X$. Show that $A \subseteq B_{2r}(a)$.
2. Show that the diameter of a set $\varnothing \neq B \subseteq X$ is equal to the diameter of the closure of B, that is, $\delta(B) = \delta(\bar{B})$.

Exercise 33.6. Let (X, d) be a metric space and let A be a subset of X and O be an open subset of X. Prove that

$$P \cap \bar{A} \subseteq \overline{O \cap A} \text{ and } \overline{O \cap \bar{A}} = \overline{O \cap A}.$$

Conclude that if $O \cap A = \varnothing$, then $O \cap \bar{A} = \varnothing$.

Definition 33.7 (Frontier)**.** Let (X, d) be a metric space. The frontier of a set $A \subseteq X$ is given by

$$\mathrm{Fr}(A) = \bar{A} \cap \overline{{}^cA}.$$

Exercise 33.8. Let (X, d) be a metric space and let A be a subset of X. Prove that A is open if and only if $\mathrm{Fr}(A) \cap A = \varnothing$.

Exercise 33.9. Let (X, d) be a metric space and let A be a subset of X. Prove that A is closed if and only if $\mathrm{Fr}(A) \subseteq A$.

Exercise 33.10. Let (X, d) be a metric space and let A, B be two subsets of X. Prove that

$$\mathrm{Fr}(A \cup B) \subseteq \mathrm{Fr}(A) \cup \mathrm{Fr}(B).$$

Show also that if $\bar{A} \cap \bar{B} = \varnothing$, then $\mathrm{Fr}(A \cup B) = \mathrm{Fr}(A) \cup \mathrm{Fr}(B)$.

34 Lecture 25: Complete Metric Space, Continued

Lemma 34.1. Let (X,d) be a metric space and let $\varnothing \neq F \subseteq X$. The following are equivalent:

1. $a \in \bar{F}$.
2. There exists $\{a_n\}_{n\geq 1} \subseteq F$ such that $a_n \xrightarrow[n\to\infty]{d} a$.

Proof. (1) $\Rightarrow$ (2): Assume $a \in \bar{F}$. Then for all $r > 0$, $B_r(a) \cap F \neq \varnothing$. For $n \geq 1$, take $r = \frac{1}{n}$. Then $B_{\frac{1}{n}}(a) \cap F \neq \varnothing$. Let $a_n \in B_{\frac{1}{n}}(a) \cap F$. Consider $\{a_n\}_{n\geq 1} \subseteq F$. We have for all $n \geq 1$ that $d(a_n, a) < \frac{1}{n} \xrightarrow[n\to\infty]{} 0$, and so $a_n \xrightarrow[n\to\infty]{d} a$.

(2) $\Rightarrow$ (1): Assume there exists a sequence $\{a_n\}_{n\geq 1}$ of F such that $a_n \xrightarrow[n\to\infty]{d} a$. Fix $r > 0$. Then there exists $n_r \in \mathbb{N}$ such that $d(a_n, a) < r$ for all $n \geq n_r$. In particular, for all $n \geq n_r$, $a_n \in B_r(a) \cap F$, so $B_r(a) \cap F \neq \varnothing$. Since r is arbitrary, we get $a \in \bar{F}$. $\square$

Theorem 34.2. Let (X,d) be a metric space. The following are equivalent:

1. (X,d) is a complete metric space.
2. For every sequence $\{F_n\}_{n\geq 1}$ of non-empty closed subsets of X that is nested, i.e., $F_{n+1} \subseteq F_n\ \forall n \geq 1$), and satisfies $\delta(F_n) \xrightarrow[n\to\infty]{} 0$, we have $\bigcap_{n\geq 1} F_n = \{a\}$ for some $a \in X$.

Proof. (1) $\Rightarrow$ (2): Assume (X,d) is complete. As $F_n \neq \varnothing$ for all $n \geq 1$, there exists $a_n \in F_n$.

Claim 34.3. $\{a_n\}_{n\geq 1}$ is Cauchy.

Subproof. Let $\varepsilon > 0$. As the diameter $\delta(F_n) \xrightarrow[n\to\infty]{} 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\delta(F_n) < \varepsilon$ for all $n \geq n_\varepsilon$. Let $m, n \geq n_\varepsilon$, then since $\{F_n\}_{n\geq 1}$ is nested, $F_n \subseteq F_{n_\varepsilon}, F_m \subseteq F_{n_\varepsilon}$, so $d(a_n, a_m) \leq \delta(F_{n_\varepsilon}) < \varepsilon$. $\blacksquare$

As (X,d) is complete, there exists $a \in X$ such that $a_n \xrightarrow[n\to\infty]{d} a$. But because for all $n \geq 1$ we have $\{a_m\}_{m\geq n} \subseteq F_n$, then $a \in \bar{F_n} = F_n$. Therefore, $a \in \bigcap_{n\geq 1} F_n$.

It remains to show that a is the only point in $\bigcap_{n\geq 1} F_n$. Assume, towards contradiction, that $\exists y \neq a$ such that $y \in \bigcap_{n\geq 1} F_n$. Then $y \in F_n$ for all $n \geq 1$, so $d(y,a) \leq \delta(F_n) \xrightarrow[n\to\infty]{} 0$, so $y = a$, contradiction.

(2) $\Rightarrow$ (1): We want to show that (X,d) is complete. Let $\{x_n\}_{n\geq 1} \subseteq X$ be a Cauchy sequence. To prove that $\{x_n\}_{n\geq 1}$ converges in X, it suffices to show that $\{x_n\}_{n\geq 1}$ admits a subsequence that converges in X. Because the sequence is Cauchy, then there exists $n_1 \in \mathbb{N}$ such that $d(x_n, x_m) < \frac{1}{2^2}$ for all $n, m \geq n_1$. Now let $k_1 = n_1$ and select x_{k_1}. For

similar reasons, we can find $n_2 \in \mathbb{N}$ such that $d(x_n, x_m) < \frac{1}{2^3}$ for all $n, m \geq n_2$. Now let $k_2 = \max\{n_2, k_1 + 1\}$ and we select x_{k_2}.

Proceeding inductively, we find a strictly increasing sequence $\{k_n\}_{n\geq 1} \subseteq \mathbb{N}$ such that $d(x_l, x_m) < \frac{1}{2^{n+1}}$ for all $l, m \geq k_n$. For $n \geq 1$, let $F_n = K_{\frac{1}{2^n}} = \{x \in X : d(x, x_{k_n}) \leq \frac{1}{2^n}\}$. Note $\varnothing \neq F_n = \bar{F}_n$, and $\delta(F_n) \leq 2 \cdot \frac{1}{2^n} \xrightarrow[n\to\infty]{} 0$.

Claim 34.4. $F_{n+1} \subseteq F_n$ for all $n \geq 1$.

Subproof. Let $y \in F_{n+1}$, then $d(y, x_{k_{n+1}}) \leq \frac{1}{2^{n+1}}$. By the triangle inequality, $d(y, x_{k_n}) \leq d(y, x_{k_{n+1}}) + d(x_{k_{n+1}}, x_{k_n}) \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n}$. Therefore, $y \in F_n$. As $y \in F_{n+1}$ was arbitrary, we get $F_{n+1} \subseteq F_n$. ■

By hypothesis, $\bigcap_{n\geq 1} F_n = \{a\}$ for some $a \in X$.

As for all $n \geq 1$, $a \in F_n$, we have $d(a, x_{k_n}) \leq \frac{1}{2^n} \xrightarrow[n\to\infty]{} 0$, then $x_{k_n} \xrightarrow[n\to\infty]{d} a$, and because $\{x_n\}_{n\geq 1}$ is Cauchy, then $x_n \xrightarrow[n\to\infty]{d} a$. □

We now look at some examples of complete metric spaces. For instance, we know $(\mathbb{R}, |\cdot|)$ is a complete metric space.

Lemma 34.5. Assume (A, d_1) and (B, d_2) are complete metric spaces. We define $d : (A \times B) \times (A \times B) \to \mathbb{R}$ via $d((a_1, b_1), (a_2, b_2)) = \sqrt{d_1^2(a_1, a_2) + d_2^2(b_1, b_2)}$. Then $(A \times B, d)$ is a complete metric space.

Proof.

Exercise 34.6. Show that d is a metric on $A \times B$.

Let us show that $A \times B$ is complete. Let $\{(a_n, b_n)\}_{n\geq 1} \subseteq A \times B$ be a Cauchy sequence. Fix $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $d((a_n, b_n), (a_m, b_m)) < \varepsilon$ for all $n, m \geq n_\varepsilon$. Therefore, for all $n, m \geq n_\varepsilon$, we have $\sqrt{d_1^2(a_n, a_m) + d_2^2(b_n, b_m)} < \varepsilon$. Therefore, $d_1(a_n, a_m), d_2(b_n, b_m) < \varepsilon$ for all $n, m \geq n_\varepsilon$. Therefore, $\{a_n\}_{n\geq 1}$ is a Cauchy sequence in A, and $\{b_n\}_{n\geq 1}$ is a Cauchy sequence in B. As A and B are complete metric spaces, there exists $a \in A, b \in B$ such that $a_n \xrightarrow[n\to\infty]{d_1} a$ and $b_n \xrightarrow[n\to\infty]{d_2} b$.

Claim 34.7. $(a_n, b_n) \xrightarrow[n\to\infty]{d} (a, b)$.

Subproof. Indeed, $d((a_n, b_n), (a, b)) = \sqrt{d_1^2(a_n, a) + d_2^2(b_n, b)} \leq d_1(a_n, a) + d_2(b_n, b) \xrightarrow[n\to\infty]{} 0$. Therefore, $(a_n, b_n) \xrightarrow[n\to\infty]{d} (a, b)$. ■

□

Corollary 34.8. For $n \geq 2$, $(\mathbb{R}^n, d_2)$ is a complete metric space.

Proof. Make use of induction. Left as an exercise. □

Exercise 34.9. Show that for all $n \geq 2$, $(\mathbb{R}^n, d_p)$ is a complete metric space for all $1 \leq p \leq \infty$.

Example 34.10. We define

$$l^2 = \{\{x_n\}_{n\geq 1} \subseteq \mathbb{R} : \sum_{n\geq 1} |x_n|^2 < \infty\}.$$

We define a metric on l^2 as follows: for $x = \{x_n\}_{n\geq 1} \in l^2$ and $y = \{y_n\}_{n\geq 1} \in l^2$,

$$d_2(x, y) = \sqrt{\sum_{n\geq 1} |x_n - y_n|^2} < \varepsilon.$$

The fact that this is a metric follows from Minkowski's inequality.

35 Lecture 26: Complete Metric Space, Continued; Separation

Claim 35.1. (l^2, d_2) is a complete metric space.

Proof. Let $\{x^{(k)}\}_{k\geq 1}$ be a Cauchy sequence in l^2. Then we denote $x^{(n)} = \{x_1^{(n)}, x_2^{(n)}, \cdots\}$.

By definition, for all $\varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}$ such that $d_2(x^{(k)}, x^{(l)}) < \varepsilon$ for all $k, l \geq k_\varepsilon$. Therefore,

$$d_2(x^{(k)}, x^{(l)}) = \sqrt{\sum_{n\geq 1} |x_n^{(k)} - x_n^{(l)}|^2} < \varepsilon$$

for all $k, l \geq k_\varepsilon$. Hence, $\sum\limits_{n\geq 1} |x_n^{(k)} - x_n^{(l)}|^2 < \varepsilon^2$ for all $k, l \geq k_\varepsilon$. That is, for all $n \geq 1$, we have $|x_n^{(k)} - x_n^{(l)}| < \varepsilon$ for all $k, l \geq k_\varepsilon$.

So whenever $n \geq 1$, the sequence $\{x_n^{(k)}\}_{k\geq 1}$ is Cauchy in $(\mathbb{R}, |\cdot|)$. Since $(\mathbb{R}, |\cdot|)$ is complete, then there exists $x_n \in \mathbb{R}$ such that $x_n^{(k)} \xrightarrow[k\to\infty]{\mathbb{R}} x_n$. Let $x = \{x_n\}_{n\geq 1}$.

Claim 35.2. $x \in l^2$ and $x^{(k)} \xrightarrow[k\to\infty]{l^2} x$.

Subproof. Note $d_2(x^{(k)}, x) = \sqrt{\sum\limits_{n\geq 1} |x_n^{(k)} - x_n|^2}$. While $|x_n^{(k)} - x_n| \xrightarrow[k\to\infty]{0}$ for all $n \geq 1$, the limit theorems do not apply to yield $\sum\limits_{n\geq 1} |x_n^{(k)} - x_n|^2 \xrightarrow[k\to\infty]{} 0$. Instead, we argue as follows.

Fix $\varepsilon > 0$. As $\{x^{(k)}\}_{k\geq 1}$ is Cauchy in l^2, there exists $k_\varepsilon \in \mathbb{N}$ such that $d_2(x^{(k)}, x^{(l)}) < \varepsilon$ for all $k, l \geq k_\varepsilon$. In particular, $\sum\limits_{n\geq 1} |x_n^{(k)} - x_n^{(l)}|^2 < \varepsilon^2$ for all $k, l \geq k_\varepsilon$. So for each fixed $N \in \mathbb{N}$ we have

$$\sum_{n=1}^{N} |x_n^{(k)} - x_n^{(l)}|^2 < \varepsilon^2$$

for all $k, l \geq k_\varepsilon$. Note $\lim_{l\to\infty} |x_n^{(k)} - x_n^{(l)}| = |x_n^{(k)} - x_n|$ for all $n \geq 1$ and all $k \geq k_\varepsilon$. By the limit theorems, $\lim_{l\to\infty} \sum_{n=1}^{N} |x_n^{(k)} - x_n^{(l)}|^2 \leq \varepsilon^2$ for all $k \geq k_\varepsilon$. Therefore, $\sum_{n=1}^{N} |x_n^{(k)} - x_n|^2 \leq \varepsilon^2$ for all $k \geq k_\varepsilon$.

Note that $\{\sum_{n=1}^{N} |x_n^{(k)} - x_n|^2\}_{N\geq 1}$ is an increasing sequence bounded above by ε^2. So $\sum_{n\geq 1}\sum_{n=1}^{N} |x_n^{(k)} - x_n|^2 \leq \varepsilon^2$ for all $k \geq k_\varepsilon$. That is to say, $d_2(x^{(k)}, x) \leq \varepsilon$ for all $k \geq k_\varepsilon$.

Finally, $x \in l^2$ if and only if $d_2(x, 0) < \infty$. But $d_2(x, 0) \leq d_2(x, x^{(k)}) + d_2(x^{(k)}, 0) < \infty$, where the first term is bounded above by ε for all $k \geq k_\varepsilon$, and the second term is finite since $x^{(k)} \in l^2$. ■

□

Exercise 35.3. 1. Fix $1 \leq p < \infty$ and let

$$l^p = \{\{x_n\}_{n\geq 1} \subseteq \mathbb{R} : \sum_{n\geq 1} |x_n|^p < \infty\}.$$

We define $d_p : l^p \times l^p \to \mathbb{R}$ via

$$d_p(\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1}) = \left(\sum_{n\geq 1} |x_n - y_n|^p\right)^{\frac{1}{p}}.$$

Then (l^p, d_p) is a complete metric space.

2. Define

$$l^\infty = \{\{x_n\}_{n\geq 1} \subseteq \mathbb{R} : \sup_{n\geq 1} |x_n| < \infty\}.$$

We define $d_\infty : l^\infty \times l^\infty \to \mathbb{R}$ via

$$d_\infty(\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1}) = \sup_{n\geq 1} |x_n - y_n|.$$

Then (l^∞, d_∞) is a complete metric space.

Definition 35.4 (Separated)**.** Let (X, d) be a metric space and let $A, B \subseteq X$. We say that A and B are separated if $\bar{A} \cap B = \varnothing$ and $A \cap \bar{B} = \varnothing$.

Remark 35.5. Separated sets are disjoint: $A \cap B \subseteq \bar{A} \cap B = \varnothing$. But disjoint sets need not be separated: consider $(X, d) = (\mathbb{R}, |\cdot|)$ where $A = (-1, 0)$ and $B = [0, 1)$. Then $A \cap B = \varnothing$, but $\bar{A} \cap B = \{0\} \neq \varnothing$, so A and B are not separated.

Remark 35.6. If A and B are separated and $A_1 \subseteq A$ and $B_1 \subseteq B$, then A_1 and B_1 are separated.

Lemma 35.7. Let (X,d) be a metric space and let $A, B \subseteq X$. If $d(A,B) > 0$, then A and B are separated.

Proof. Assume, towards contradiction, that A and B are not separated. Then $\bar{A} \cap B \neq \varnothing$ or $A \cap \bar{B} \neq \varnothing$. Say $\bar{A} \cap B \neq \varnothing$, then let $a \in \bar{A} \cap B$. Therefore, $a \in B$ and $a \in \bar{A}$, so $d(a, A) = 0$, and so $d(A,B) = 0$, contradiction. □

Remark 35.8. Two sets A and B can be separated even if $d(A,B) = 0$. For example, let $A = (0,1)$ and $B = (1,2)$ in the usual metric space $\mathbb{R}$. The two sets are separated, but $d(A,B) = 0$.

Proposition 35.9.

1. Two closed sets A and B are separated if and only if $A \cap B = \varnothing$.
2. Two open sets A and B are separated if and only if $A \cap B = \varnothing$.

Proof. Two separated sets are disjoint. Therefore, we only have to prove the $(\Leftarrow)$ direction in both statements.

1. Assume $A \cap B = \varnothing$, then since A is closed, $\bar{A} = A$, so $\bar{A} \cap B = A \cap B = \varnothing$. Similarly, since B is closed, then $\bar{B} = B$ and so $\bar{B} \cap A = B \cap A = \varnothing$. Therefore, A and B are separated.
2. Assume $A \cap B = \varnothing$, then $A \subseteq {}^cB$, where ${}^cB$ is closed. Therefore, $\bar{A} \subseteq \overline{{}^cB} = {}^cB$, so $\bar{A} \cap B = \varnothing$. A similar argument shows that $\bar{B} \cap A = \varnothing$, and so A and B are separated.

□

Proposition 35.10.

1. If an open set D is the union of two separated sets A and B, then A and B are both open.
2. If a closed set F is the union of two separated sets A and B, then A and B are both closed.

Proof.

1. If $A = \varnothing$, then since $D = A \cup B$, we have $B = D$ and so both A and B are open. Assume $A \neq \varnothing$. We want to show that A is open, which is equivalent to the statement $A = \mathring{A}$.

   Let $a \in A \subseteq D$, since D is open, then there exists $r_1 > 0$ such that $B_{r_1}(a) \subseteq D$. As A and B are separated, then $A \cap \bar{B} = \varnothing$, so $a \in A \subseteq {}^c(\bar{B}) = {}^c\mathring{B}$, so there exists $r_2 > 0$ such that $B_{r_2}(a) \subseteq {}^cB$. Let $r = \min\{r_1, r_2\}$, then $B_r(a) \subseteq D \cap {}^cB = (A \cup B) \cap {}^cB = A$, and so $a \in \mathring{A}$. This shows that A is open. A similar argument shows that B is open.

2. Let us show that A is closed, which is to show $\bar{A} = A$. We have $A \subseteq F$, and since F is closed, i.e., $F = \bar{F}$, then $\bar{A} \subseteq \bar{F} = F$, so $\bar{A} = \bar{A} \cap F = \bar{A} \cap (A \cup B) = (\bar{A} \cap A) \cup (\bar{A} \cap B) = A$ since A and B are separated. Similarly, one shows that $\bar{B} = B$, and so B is closed.

□

36 Lecture 27: Connectedness

Definition 36.1 (Disconnected, Connected)**.** Let (X, d) be a metric space and let $A \subseteq X$. We say that A is disconnected if it can be written as the union of two non-empty separated sets, that is, $\exists B, C \subseteq X$ such that $B, C \neq \varnothing$, $\bar{B} \cap C = \bar{C} \cap B = \varnothing$, and $A = B \cup C$.

We say that A is connected if it is not disconnected.

Lemma 36.2. Let (X, d) be a metric space and let $Y \subseteq X$ be equipped with the induced metric d_1. Then Y is connected in (Y, d_1) if and only if Y is connected in (X, d).

Proof. ($\Rightarrow$): Assume that Y is connected in (Y, d_1). We argue by contradiction. Assume that Y is not connected in (X, d). Then $\exists A, B \subseteq X$, $A, B \neq \varnothing$, $\bar{A}^X \cap B = \bar{B}^X \cap A = \varnothing$, and $Y = A \cup B$.

Claim 36.3. A and B are separated in (Y, d_1).

If the claim is true, then $Y = A \cup B$ is disconnected in (Y, d_1), contradiction.

Subproof. Note that $\bar{A}^Y \cap B = (\bar{A}^X \cap Y) \cap B = \bar{A}^X \cap (Y \cap B) = \bar{A}^X \cap B = \varnothing$, and similarly $\bar{B}^Y \cap A = (\bar{B}^X \cap Y) \cap A = \bar{B}^X \cap (Y \cap A) = \bar{B}^X \cap A = \varnothing$. Therefore, A and B are separated in (Y, d_1). ■

($\Leftarrow$): Assume Y is connected in (X, d). We argue by contradiction. Assume that Y is disconnected in (Y, d_1). So $\exists A, B \subseteq Y$, $A, B \neq \varnothing$, $\bar{A}^Y \cap B = \bar{B}^Y \cap A = \varnothing$, $Y = A \cup B$.

Claim 36.4. $A.B$ are separated in (X, d).

If this is true, then $Y = A \cup B$ is disconnected in (X, d), contradiction.

Subproof. Indeed, $\bar{A}^X \cap B = \bar{A}^X \cap (Y \cap B) = (\bar{A}^X \cap Y) \cap B = \bar{A}^Y \cap B = \varnothing$, and $\bar{B}^X \cap A = \bar{B}^X \cap (Y \cap A) = (\bar{B}^X \cap Y) \cap A = \bar{B}^Y \cap A = \varnothing$. So A and B are separated in (X, d). ■

□

Proposition 36.5. Let (X, d) be a metric space. Then X is connected if and only if the only subsets of X that are both open and closed are $\varnothing$ and X.

Proof. ($\Rightarrow$): Assume X is connected. We argue by contradiction. Assume $\exists \varnothing \neq A \subsetneq X$ such that A is both open and closed. Let $B = X \backslash A$. Then $\varnothing \neq B \neq X$, and B is both open and closed. Since A and B are both closed and $A \cap B = A \cap (X \backslash A) = \varnothing$, we have that A and B are separated. So $X = A \cup (X \backslash A) = A \cup B$. But because A and B are non-empty and are separated, then X is disconnected, contradiction.

($\Leftarrow$): Assume that the only subsets of x that are both open and closed in (X, d) are $\varnothing$ and X. We argue by contradiction. Assume that X is disconnected, then $\exists A, B \subseteq X$ such that $A, B \neq \varnothing$ and $\bar{A} \cap B = \bar{B} \cap A = \varnothing$, and $X = A \cup B$. As X is open, we get that A and B are both open. As X is closed, A and B are both closed. So A and B are both open and closed, and they are non-empty, then $A = B = X$. But then $\bar{A} \cap B = \bar{X} \cap X = X \cap X = X \neq \varnothing$, contradiction. □

Corollary 36.6. Let (X, d) be a metric space and let $\varnothing \neq A \subseteq X$. The following are equivalent:

1. A is disconnected.
2. $A \subseteq D_1 \cup D_2$ with D_1, D_2 open in (X, d), $A \cap D_1 \neq \varnothing$, $A \cap D_2 \neq \varnothing$, and $A \cap D_1 \cap D_2 = \varnothing$.
3. $A \subseteq F_1 \cup F_2$ with F_1, F_2 closed in (X, d), $A \cap F_1 \neq \varnothing$, $A \cap F_2 \neq \varnothing$, and $A \cap F_1 \cap F_2 = \varnothing$.

Proof. We will show that (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (3): Assume A is disconnected. By the proposition, there exists $\varnothing \neq B \subsetneq A$ such that B is both open and closed in A. Let $C = A \backslash B$. Then $C \neq \varnothing$ and $C \neq A$, and C is both open and closed in A. Because B is closed in A, then there exists $F_1 \subseteq X$ closed in (X, d) such that $B = A \cap F_1 \neq \varnothing$, and similarly since C is closed in A, then there exists $F_2 \subseteq X$ closed in (X, d) such that $C = A \cap F_2 \neq \varnothing$. Note that $A \cap F_1 \cap F_2 = (A \cap F_1) \cap (A \cap F_2) = B \cap C = B \cap (A \backslash B) = \varnothing$.

(3) $\Rightarrow$ (2): Assume $A \subseteq F_1 \cup F_2$ where F_1, F_2 closed in (X, d), $A \cap F_1 \neq \varnothing$, $A \cap F_2 \neq \varnothing$, and $A \cap F_1 \cap F_2 = \varnothing$. Define $D_1 = {}^cF_1$ open in (X, d) and $D_2 = {}^cF_2$ open in (X, d). Then $A \subseteq F_1 \cup F_2 = {}^cD_1 \cup {}^cD_2 = {}^c(D_1 \cap D_2)$, then $A \cap (D_1 \cap D_2) = \varnothing$. But $\varnothing = A \cap F_1 \cap F_2 = A \cap ({}^cD_1 \cap {}^cD_2) = A \cap ({}^c(D_1 \cup D_2))$, then $A \subseteq D_1 \cup D_2$.

We show that $A \cap D_1 \neq \varnothing$. We argue by contradiction. Assume $A \cap D_1 = \varnothing$, then $A \subseteq {}^cD_1 = F_1$, but then $\varnothing = A \cap F_1 \cap F - 2 = A \cap F_2 \neq \varnothing$, contradiction. This shows that $A \cap D_1 \neq \varnothing$. A similar argument shows that $A \cap D_2 \neq \varnothing$.

(2) $\Rightarrow$ (1): Assume $A \subseteq D_1 \cup D_2$ where D_1, D_2 are open in (X, d), and $A \cap D_1 \neq \varnothing$, $A \cap D_2 \neq \varnothing$, and $A \cap D_1 \cap D_2 = \varnothing$. Let $B = A \cap D_1 \neq \varnothing$, then B is open in A since D_1 is open in X. Similarly, let $C = A \cap D_2 \neq \varnothing$, then C is open in A since D_2 is open in X. Now

$B \cap C = (A \cap D_1) \cap (A \cap D_2) = A \cap D_1 \cap D_2 = \varnothing$. Therefore, B and C are separated in A. But because $A \subseteq D_1 \cup D_2$, so $A = (D_1 \cup D_2) \cap A = (D_1 \cap A) \cup (D_2 \cap A) = B \cup C$, and since $B, C \neq \varnothing$, we know A is disconnected in A, and therefore A is disconnected in X. $\square$

37 Homework 10

Exercise 37.1. Let $\mathbb{R}^n$ be endowed with the Euclidean metric d_2. Let S be a non-empty subset of $\mathbb{R}^n$; in particular, $(S, d_2 |_{S \times S})$ is a metric space.

1. Given $x \in S$, is the set $\{y \in S : d_2(x, y) \geq r\}$ closed in S?
2. Given $x \in S$, is the set $\{y \in S : d_2(x, y) \geq r\}$ contained in the closure of $\{y \in S : d_2(x, y) > r\}$ in S?

Exercise 37.2. Let (X, d) be a metric space and let $A \subseteq X$ be complete. Show that A is closed.

Exercise 37.3. Let (X, d) be a complete metric space and let $F \subseteq X$ be a closed set. Show that F is complete.

Exercise 37.4. Let

$$l^\infty = \{\{x_n\}_{n\geq 1} \subseteq \mathbb{R} : \sup_{n\geq 1} |x_n| < \infty\}.$$

Define $d_\infty : l^\infty \times l^\infty \to \mathbb{R}$ as follows: for any $x = \{x_n\}_{n\geq 1} \in l^\infty$, $y = \{y_n\}_{n\geq 1} \in l^\infty$,

$$d_\infty(x, y) = \sup_{n\geq 1} |x_n - y_n|.$$

Show that (l^∞, d_∞) is a complete metric space.

Exercise 37.5. Let $A = \{\{x_n\}_{n\geq 1} : x_n \in \mathbb{R} \text{ for all } n \geq 1\}$.

1. Show that $d : A \times A \to \mathbb{R}$ given by
$$d(\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1}) = \sum_{n\geq 1} \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1 + |y_n - y_n|}$$
is a metric on A.
2. Show that (A, d) is a complete metric space.

Exercise 37.6. Consider the metric space $(X, d) = (\mathbb{R}, |\cdot|)$. For each of the following subsets of $\mathbb{R}$ decide if they are open, closed, or not open and not closed, connected or not connected. Also, in each case write down the set of accumulation points. Justify your answers.

1. $A = \mathbb{Q}$.
2. $A = \mathbb{Q} \cap [0,1]$.
3. $A = \{(-1)^n(1+\frac{1}{n})\}$.
4. $A = \bigcup_{n\in\mathbb{N}} [n, n+\frac{1}{n}]$.
5. $A = \bigcup_{n\in\mathbb{N}} [\frac{1}{2^{n+1}}, \frac{1}{2^n}]$.

Exercise 37.7. Given an example of a set $\varnothing \neq A \subsetneq \mathbb{Q}$ that is both open and closed in $\mathbb{Q}$. Justify your answer.

Exercise 37.8. Assume that the sets A and B are separated and that the sets A and C are separated. Prove that the sets A and $B \cup C$ are separated.

Exercise 37.9. Let (X,d) be a connected metric space and let A be a connected subset of X. Assume that the complement of A is the union of two separated sets B and C. Prove that $A \cup B$ and $A \cup C$ are connected. Prove also that if A is closed, then so are $A \cup B$ and $A \cup C$.

Exercise 37.10. Let (X,d) be a metric space and let A, B be two closed subsets of X such that $A \cup B$ and $A \cap B$ are connected. Prove that A is connected.

38 Lecture 28: Connectedness, Continued

Proposition 38.1. Let (X,d) be a metric space and let $A \subseteq X$ be disconnected. Let F_1, F_2 be closed in X such that $A \subseteq F_1 \cup F_2$, $A \cap F_1 \neq \varnothing$, $A \cap F_2 \neq \varnothing$, $A \cap F_1 \cap F_2 = \varnothing$. Let $B \subseteq A$ be connected. Then $B \subseteq F_1$ or $B \subseteq F_2$.

Proof. We argue by contradiction. Assume $B \nsubseteq F_1$ and $B \nsubseteq F_2$. Then we know $B \subseteq A \subseteq F_1 \subseteq F_2$, but $B \nsubseteq F_1$, then $B \cap F_2 \neq \varnothing$, and similarly $B \cap F_1 \neq \varnothing$. But $B \cap F_1 \cap F_2 \subseteq A \cap F_1 \cap F_2 = \varnothing$, and $B \subseteq F_1 \cup F_2$, then B is disconnected, contradiction. $\square$

Remark 38.2. One can replace the closed sets (in X) F_1 and F_2 by open sets (in X) D_1 and D_2 and the same conclusion holds.

Proposition 38.3. Let (X,d) be a metric space and let $A \subseteq X$ be connected. Then if $A \subseteq B \subseteq \bar{A}^X$, then B is connected.

Proof. We argue by contradiction. Assume B is disconnected. Then $\exists F_1, F_2 \subseteq X$ closed in X such that $B \subseteq F_1 \cup F_2$, $B \cap F_1 \neq \varnothing$, $B \cap F_2 \neq \varnothing$, and $B \cap F_1 \cap F_2 = \varnothing$. Because $A \subseteq B \subseteq F_1 \subseteq F_2$ and A is connected, then $A \subseteq F_1$ or $A \subseteq F_2$. Without loss of generality, say $A \subseteq F_1$, then $B \subseteq \bar{A}^X \subseteq \bar{F}_1^X = F_1$, then $\varnothing = B \cap F_1 \cap F_2 = B \cap F_2 \neq \varnothing$, contradiction. $\square$

Proposition 38.4. Let (X, d) be a metric space and let $\{A_i\}_{i\in I}$ be a family of connected subsets of X. Assume that each two of these sets are not separated, that is, $\forall i, j \in I$, $i \neq j$, we have $\bar{A}_i \cap A_j \neq \varnothing$ or $A_i \cap \bar{A}_j \neq \varnothing$. Then $\bigcup_{i\in I} A_i$ is connected.

Proof. We argue by contradiction. Assume $\bigcup_{i\in I} A_i$ is disconnected, then there exists B, C non-empty separated sets such that $\bigcup_{i\in I} A_i = B \cup C$. Fix $i \in I$, then $A_i \subseteq B \cup C$, but $A_i = (B\cup C)\cap A_i = (B\cap A_i)\cup(C\cap A_i)$. Because B, C are separated, then $B\cap A_i$ and $C\cap A_i$ are also separated, and because A_i is connected, then either $B \cap A_i = \varnothing$ or $C \cap A_i = \varnothing$. Because $A_i \subseteq B \cup C$, we know that either $A_i \subseteq C$ or $A_i \subseteq B$. So for each $i \in I$, the set A_i satisfies $A_i \subseteq B$ or $A_i \subseteq C$. Therefore, because $\bigcup_{i\in I} A_i = B \cup C$, then there exists $i, j \in I$ such that $A_i \cap B \neq \varnothing$ and $A_j \cap C \neq \varnothing$. Therefore, $A_i \subseteq B$ and $A_j \subseteq C$, but B and C are separated, so A_i and A_j are separated, contradiction. $\square$

Corollary 38.5. Let (X, d) be a metric space and let $\{A_i\}_{i\in I}$ be connected subsets of X. Assume for all $i \neq j$ we have $A_i \cap A_j \neq \varnothing$. Then $\bigcup_{i\in I} A_i$ is connected.

Proposition 38.6. $\mathbb{R}$ is connected.

Proof. Assume, towards a contradiction, that $\mathbb{R}$ is disconnected. Then there exists non-empty subsets A, B of $\mathbb{R}$, both open and closed in $\mathbb{R}$, disjoint, such that $\mathbb{R} = A \cup B$. Because the sets are not empty, then there exists $a_1 \in A$ and $b_1 \in B$. Let $\alpha_1 = \frac{a_1+b_1}{2} \in \mathbb{R} = A\cup B$, then $\alpha_1 \in A$ or $\alpha_1 \in B$. If $\alpha_1 \in A$, let $(a_2, b_2) = (\alpha_1, b_1)$; if $\alpha_1 \in B$, let $(a_2, b_2) = (a_1, \alpha_1)$. Now let $\alpha_2 = \frac{a_2+b_2}{2} \in \mathbb{R} = A \cup B$, then either $\alpha_2 \in A$ or $\alpha_2 \in B$. If $\alpha_2 \in A$, let $(a_3, b_3) = (\alpha_2, b_2)$; if $\alpha_2 \in B$, let $(a_3, b_3) = (a_2, \alpha_2)$. Continuing this process, we find

- an increasing sequence $\{a_n\}_{n\geq 1} \subseteq A$ bounded above by b_1, and
- a decreasing sequence $\{b_n\}_{n\geq 1} \subseteq B$ bounded below by a_1.

Therefore, both sequences converge in $\mathbb{R}$. Let $a = \lim_{n\to\infty} a_n \in \bar{A} = A$, $b = \lim_{n\to\infty} b_n \in \bar{B} = B$. Note that by construction, $b_{n+1} - a_{n+1} = \frac{b_n-a_n}{2}$ for all $n \geq 1$, so $|b_{n+1} - a_{n+1}| = \frac{|b_n-a_n|}{2} = \cdots = \frac{|b_1-a_1|}{2} \xrightarrow[n\to\infty]{} 0$. Hence, $|b - a| = 0$. Therefore, $a = b \in A \cap B = \varnothing$, contradiction. $\square$

Proposition 38.7. The only non-empty connected subsets of $\mathbb{R}$ are intervals.

Proof. The argument in the previous proof extends easily to show that the intervals are connected subsets of $\mathbb{R}$. It remains to show that if $\varnothing \neq A \subseteq \mathbb{R}$ is connected, then A is an interval. Let $\alpha = \inf A$ (where $\alpha = -\infty$ if A is unbounded below), and $\beta = \sup A$ (where $\beta = \infty$ if A is unbounded above). We claim that $(\alpha, \beta) \subseteq A$. This shows A is an interval.

We argue by contradiction. Assume $\exists c \in (\alpha, \beta)\backslash A$. Let $D_1 = (-\infty, c)$ and $D_2 = (c, \infty)$ be open in $\mathbb{R}$. Now $A \subseteq \mathbb{R}\backslash\{c\} = D_1 \cup D_2$ and $A \cap D_1 \cap D_2 = \varnothing$. Note that we also know $A \cap D_1 \neq \varnothing$ because $\inf A = \alpha < c$ and $A \cap D_2 \neq \varnothing$ because $\sup A = \beta > c$. Therefore, A is disconnected, contradiction. $\square$

Proposition 38.8. Let (X, d) be a metric space. Assume that for every pair of points in X, there exists a connected subset of X that contains them. Then X is connected.

Proof. Assume, towards a contradiction, that X is disconnected. Then there exists two non-empty separated sets $A, B \subseteq X$ such that $X = A \cup B$. Because A and B are non-empty, there exists $a \in A$ and $b \in B$. Therefore, there exists a connected subset $C \subseteq X$ such that $\{a, b\} \subseteq C$. Because $C \subseteq X = A \cup B$, C is connected, and A, B are closed since X is closed, then either $C \subseteq A$ or $C \subseteq B$. Therefore, either a or b is contained in $A \cap B$, but we know $A \cap B = \varnothing$, then we have a contradiction. $\square$

Remark 38.9. Let (X, d) be a metric space. For $a, b \in X$, we write $a \sim b$ if there exists a connected subset of X, $A_{ab} \subseteq X$, such that $\{a, b\} \subseteq A_{ab}$. One can easily show that $\sim$ defines an equivalence relation of X. Now for $a \in X$, let C_a denote the equivalence class of a. The following exercises show that we can decompose $X = \bigcup\limits_{a \in X} C_a$ as a union of connected components.

Exercise 38.10.
1. C_a is a connected subset of X.
2. C_a is the largest connected set containing a.
3. C_a is closed in X.
4. If $a \not\sim b$ then C_a and C_b are separated.

39 Lecture 29: Compactness

Definition 39.1 (Open Cover)**.** Let (X, d) be a metric space and let $A \subseteq X$. An open cover of A is a family $\{G_i\}_{i \in I}$ of open sets in X such that $A \subseteq \bigcup\limits_{i \in I} G_i$. The open cover is called finite if the cardinality of I is finite. If it is not finite, the open cover is called infinite.

Definition 39.2 (Compact)**.** Let (X, d) be a metric space and let $K \subseteq X$.

1. We say that K is a compact set if every open cover $\{G_i\}_{i\in I}$ of K admits a finite subcover, that is, $\exists n \geq 1$ and $\exists i_1, \cdots, i_n \in I$ such that $K \subseteq \bigcup_{j=1}^{n} G_{i_j}$.
2. We say that a set $A \subseteq X$ is precompact if $\bar{A}$ is compact.

Lemma 39.3. Let (X,d) be a metric space and let $\varnothing \neq Y \subseteq X$. We equip Y with the induced metric $d_1 : Y \times Y \to \mathbb{R}$ with $d_1(y_1,y_2) = d(y_1,y_2)$. Let $K \subseteq Y \subseteq X$. The following are equivalent:

1. K is compact in (X,d).
2. K is compact in (Y,d_1).

Proof. (1) $\Rightarrow$ (2): Assume K is compact in (X,d). Let $\{V_i\}_{i\in I}$ be a family of open sets in (Y,d_1) such that $K \subseteq \bigcup_{i\in I} V_i$. For $i \in I$ fixed, V_i is open in (Y,d_1), then there exists $G_i \subseteq X$ open in (X,d) such that $V_i = G_i \cap Y$. Then $K \subseteq \bigcup_{i\in I} V_i \subseteq \bigcup_{i\in I} G_i$. But K is compact in (X,d), so there exists some $n \geq 1$ and some $i_1, \cdots, i_n \in I$ such that $K \subseteq \bigcup_{j=1}^{n} G_{i_j}$, but $K \subseteq Y$, then we know $K \subseteq \left(\bigcup_{j=1}^{n} G_{i_j}\right) \cap Y = \bigcup_{j=1}^{n} (G_{i_j} \cap Y) = \bigcup_{j=1}^{n} V_{i_j}$. Therefore, K is compact in (Y,d_1).

(2) $\Rightarrow$ (1): Assume K is compact in (Y,d_1). Let $\{G_i\}_{i\in I}$ be a family of open sets in (X,d) such that $K \subseteq \bigcup_{i\in I} G_i$, but since $K \subseteq Y$, then $K \subseteq \left(\bigcup_{i\in I} G_i\right) \cap Y = \bigcup_{i\in I}(G_i \cap Y)$, but since $G_i \cap Y$ is open in Y for all i, and K is compact in (Y,d_1), then there exists $n \geq 1$ and $i_1, \cdots, i_n \in I$ such that $K \subseteq \bigcup_{j=1}^{n} (G_{i_j} \cap Y) \subseteq \bigcup_{j=1}^{n} G_{i_j}$. $\square$

Proposition 39.4. Let (X,d) be a metric space and let $K \subseteq X$ be compact. Then K is closed and bounded.

Proof. We first prove K is closed by proving ${}^cK$ is open. If ${}^cK = \varnothing$, then this is open. Therefore, we may assume that ${}^cK \neq \varnothing$. Let $x \in {}^cK$. For $y \in K$, let $r_y = \frac{d(x,y)}{2}$. Note that $r_y > 0$ since $x \in {}^cK$ and $y \in K$, and note that $K \subseteq \bigcup_{y\in K} B_{r_y}(y)$, which is open. Because K is compact, then there exists $n \geq 1$ and $y_1, \cdots, y_n \in K$ such that $K \subseteq \bigcup_{j=1}^{n} B_{r_j}(y_j)$, where we use the shorthand notation $r_j = r_{y_j}$.

Let $r = \min_{1\leq j\leq n} r_j > 0$. By construction, $B_r(x) \cap B_{r_j}(y_j) = \varnothing$ for all $1 \leq j \leq n$. Then $B_r(x) \subseteq {}^cB_{r_j}(y_j)$ for all $1 \leq j \leq n$. Therefore, $B_r(x) \subseteq \bigcap_{j=1}^{n} {}^cB_{r_j}(y_j) = {}^c\left(B_{r_j}(y_j)\right) \subseteq {}^cK$. This forces $x \in {}^c\mathring{K}$, but because our choice of x is arbitrary, then ${}^cK = {}^c\mathring{K}$. Hence, K is closed.

We now show K is bounded. Note that $K \subseteq \bigcup_{y \in K} B_1(y)$, and since K is compact, then there exists some $n \geq 1$ and $y_1, \cdots, y_n \in K$ such that $K \subseteq \bigcup_{j=1}^{n} B_1(y_j)$. For $2 \leq j \leq n$, let $r_j = d(y_1, y_j) + 1$.

Claim 39.5. $B_1(y_j) \subseteq B_{r_j}(y_1)$.

In particular, if this claim is true, then set $r = \max_{2 \leq j \leq n} r_j$, and we have $K \subseteq \bigcup_{j=1}^{n} B_1(y_j) \subseteq B_r(y_1)$. Therefore, it suffices to prove the claim.

Indeed, if $x \in B_1(y_j)$, then $d(x, y_j) < 1$. By the triangle inequality,

$$d(y_1, x) \leq d(y_j, x) + d(y_1, y_j) < 1 + d(y_1, y_j) = r_j,$$

so $x \in B_{r_j}(y_1)$. $\square$

Proposition 39.6. Let (X, d) be a metric space and let $F \subseteq K \subseteq X$ such that F is closed in X and K is compact, then F is compact.

Proof. Let $\{G_i\}_{i \in I}$ be a family of open sets in X such that $F \subseteq \bigcup_{i \in I} G_i$. Note that $K \subseteq F \cup {}^cF \subseteq \bigcup_{i \in I} G_i \cup {}^cF$, where ${}^cF$ is open in X, then because K is compact, we know there exists some $n \geq 1$ and $i_1, \cdots, i_n \in I$ such that $K \subseteq \bigcup_{j=1}^{n} G_{i_j} \cup {}^cF$, and since $F \subseteq K$, then $F = \left(\bigcup_{j=1}^{n} G_{i_j} \cup {}^cF \right) \cap F \subseteq \bigcup_{j=1}^{n} G_{i_j}$. Therefore, F is compact. $\square$

Corollary 39.7. Let (X, d) be a metric space and let $F \subseteq X$ be closed and let $K \subseteq X$ be compact. Then $K \cap F$ is compact.

Proof. Since K is compact, then K is closed, and since F is closed, so $K \cap F$ is closed. Since $K \cap F \subseteq K$, which is compact, then $K \cap F$ is compact. $\square$

Definition 39.8 (Sequentially Compact)**.** Let (X, d) be a metric space. A set $K \subseteq X$ is called sequentially compact if every sequence $\{x_n\}_{n \geq 1} \subseteq K$ admits a subsequence that converges in K.

Theorem 39.9 (Bolzano-Weierstrass)**.** An infinite set $K \subseteq X$ is sequentially compact if and only if every infinite set $A \subseteq K$ has an accumulation point in K, that is, $A' \cap K \neq \varnothing$.

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Proof. Suppose K is sequentially compact. Let $A \subseteq K$ be infinite. As every infinite set has a countable subset, we can find a sequence $\{a_n\}_{n\geq 1} \subseteq A$ such that $a_n \neq a_m$ for all $n \neq m$. As K is sequentially compact, there exists a subsequence $\{a_{k_n}\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$ such that $a_{k_n} \xrightarrow[n\to\infty]{d} a \in K$.

The idea is that this point a is what we want. We already know the following claim would be true simply by looking at the definition of accumulation points.

Claim 40.1. $a \in A'$ if and only if for all $r > 0$, $B_r(a) \cap A \backslash \{a\} \neq \varnothing$.

Indeed, fix $r > 0$. Because $a_{k_n} \xrightarrow[n\to\infty]{d} a \in K$, then there exists $n_r \in \mathbb{N}$ such that $d(a, a_{k_n}) < r$ for all $n \geq n_r$. As $a_n \neq a_m$ for all $n \neq m$, there exists some $n_0 \geq n_r$ such that $a_{k_{n_0}} \neq a$. Then $a_{k_{n_0}} \in B_r(a) \cap A \backslash \{a\}$. We then get $a \in A' \cap K$.

We now suppose for every infinite set $A \subseteq K$ we have $A' \cap K \neq \varnothing$. Let $\{a_n\}_{n\geq 1} \subseteq K$. We distinguish two cases.

Case 1: The sequence $\{a_n\}_{n\geq 1}$ contains a constant subsequence. This means the subsequence converges to an element in K.

Case 2: $\{a_n\}_{n\geq 1}$ does not contain a constant subsequence. Then $A = \{a_n : n \geq 1\}$ is infinite and $A \subseteq K$. So $A' \cap K \neq \varnothing$. Let $a \in A' \cap K$, then there exists a subsequence $\{a_{k_n}\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$ such that $a_{k_n} \xrightarrow[n\to\infty]{d} a$. This works out because we know $B_1(a) \cap A \backslash \{a\} \neq \varnothing$, and we pick such a_{k_1}, then we restrict the radius of the ball to $\min\{\frac{1}{2}, d(a, a_{k_1})\}$, the intersection would still be non-empty, so we can pick such $k_2 > k_1$, and so on, until we get a subsequence. □

Theorem 40.2. Let (X, d) be a metric space and let $K \subseteq X$ be compact, then K is sequentially compact.

Proof. If K is finite, then any sequence $\{x_n\}_{n\geq 1} \subseteq K$ will have a constant subsequence. Therefore, we can now assume K is infinite. We will use the Bolzano-Weierstrass theorem. It suffices to prove that for any infinite $A \subseteq K$ we have $A' \cap K \neq \varnothing$.

Note $A \subseteq K$, so $A' \subseteq K'$. Because K is compact, then K is closed, and so $K' \subseteq K$, therefore $A' \subseteq K$, and therefore $A' \cap K = A'$. We now argue by contradiction. Assume $A' = \varnothing$, then for $x \in K$ we have $x \notin A'$, then there exists $r_x > 0$ such that $B_{r_x}(x) \cap A \backslash \{x\} = \varnothing$. Therefore, $K \subseteq \bigcup\limits_{x\in K} B_{r_x}(x)$. Because K is compact, then there exists $n \geq 1$ and $x_1, \cdots, x_n \in K$ such that $K \subseteq \bigcup\limits_{j=1}^{n} B_{r_j}(x_j)$ where $r_j = r_{x_j}$. In particular, $A = \left(\bigcup\limits_{j=1}^{n} B_{r_j}(x_j)\right) \cap A =$

$\bigcup_{j=1}^{n} \left[B_{r_j}(x_j) \cap A\right]$. Since by construction we get $B_{r_j}(x_j) \cap A \subseteq \{x_j\}$, then $A \subseteq \bigcup_{j=1}^{n} \{x_j\}$. We get our contradiction because we have an infinite set contained in a finite set. Therefore, $A' \neq \varnothing$. □

Proposition 40.3. Let (X, d) be a metric space and let $K \subseteq X$ be sequentially compact. Then K is closed and bounded.

Proof. We first show that K is closed, that is, to show $K = \bar{K}$. It suffices to show that $\bar{K} \subseteq K$. Let $x \in \bar{K}$, then there exists $\{x_n\}_{n\geq 1} \subseteq K$ such that $x_n \xrightarrow[n\to\infty]{d} x$. Because K is sequentially compact, then there exists a subsequence $\{x_{k_n}\}_{n\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that $x_{k_n} \xrightarrow[n\to\infty]{d} y \in K$. Because x_n already converges to x, then so does the subsequnece, and so $x = y \in K$ because of the uniqueness of limit for convergent sequences. Because $x \in \bar{K}$ was arbitrary, we see $\bar{K} \subseteq K$ as desired.

We now show that K is bounded. We argue by contradiction and assume that K is not bounded. Let $a_1 \in K$. Because K is not bounded, then $K \not\subseteq B_1(a_1)$, and so there exists $a_2 \in K$ such that $d(a_1, a_2) \geq 1$. Similarly, we know $K \not\subseteq B_{1+d(a_1,a_2)}(a_1)$, then there exists $a_3 \in K$ such that $d(a_1, a_3) \geq 1 + d(a_1, a_2)$. Proceeding inductively, we find a sequence $\{a_n\}_{n\geq 1} \subseteq K$ such that $d(a_1, a_{n+1}) \geq 1 + d(a_1, a_n)$. By construction, $|d(a_1, a_m) - d(a_1, a_n)| \geq |n - m|$ for all $n, m \geq 1$. By the triangle inequality, we see that

$$d(a_n, a_m) \geq |d(a_1, a_n) - d(a_1, a_m)| \geq |n - m|$$

for all $n, m \geq 1$. This sequence cannot have a convergent (Cauchy) subsequence, thus contradicting the hypothesis that K is sequentially compact, so K is bounded. □

Definition 40.4 (Totally Bounded)**.** Let (X, d) be a metric space. A set $A \subseteq X$ is totally bounded if for every $\varepsilon > 0$, A can be covered by finitely many balls of radius ε.

Remark 40.5.

1. A totally bounded implies A bounded.

   Indeed, taking $\varepsilon = 1$, there exists $n \geq 1$ and $x_1, \cdots, x_n \in X$ such that $A \subseteq \bigcup_{j=1}^{n} B_1(x_j) \subseteq B_r(x_1)$ where $r = 1 + \max_{2\leq j\leq n} d(x_1, x_j)$.

2. However, A being bounded does not imply A being totally bounded. Consider $\mathbb{N}$ equipped with the discrete metric d, i.e., $d(n, m)$ is 0 if $n = m$ and is 1 if $n \neq m$. Then $\mathbb{N} = B_2(1)$, but $\mathbb{N}$ cannot be covered by finitely many balls of radius $\frac{1}{2}$ since $B_{\frac{1}{2}}(n) = \{n\}$.

3. However, on the metric space $(\mathbb{R}^n, d_2)$, A bounded implies A totally bounded. Indeed, if A is bounded, then $A \subseteq B_R(0)$ for some $R > 0$. Then $B_R(0)$ can be covered by $10^6 \left(\frac{R}{\varepsilon}\right)^n$ many balls of radius ε.

41 Homework 11

Exercise 41.1. (a) Use mathematical induction to prove that for all $n \geq 1$,

$$\sin(1) + \sin(2) + \ldots + \sin(n) = \frac{\sin(\frac{n}{2})}{\sin(\frac{1}{2})} \sin\left(\frac{n+1}{2}\right).$$

(b) Show that the series

$$\sum_{n \geq 1} \frac{\sin(n)}{n}$$

converges.

Exercise 41.2. Let $\{a_n\}_{n \geq 1}$ be a sequence of positive real numbers so that the series

$$\sum_{n \geq 1} \frac{a_n}{n}$$

converges. Show that the sequence $\{x_n\}_{n \geq 1}$ converges to zero, where

$$x_n = \frac{1}{n} \sum_{k=1}^{n} a_k$$

for all $n \geq 1$.

Exercise 41.3. Suppose $\{a_n\}_{n \geq 1}$ is a sequence of non-negative real numbers such that $s = \sum_{n \geq 1} a_n < \infty$. For $k \geq 1$, let N_k denote the cardinality of the set $\{n \in \mathbb{N} : a_n \geq 2^{-k}\}$. Show that

$$\limsup_{k \to \infty} 2^{-k} N_k = 0.$$

Exercise 41.4. To an equivalence relation $\sim$ on $\mathbb{N}$, we associate its graph

$$\Gamma_{\sim} = \{(a, b) \in \mathbb{N} \times \mathbb{N} : a \sim b\}.$$

Show that the set of equivalence relations on $\mathbb{N}$ has the cardinality of $2^{\mathbb{N}}$.

Exercise 41.5. Consider the space

$$\ell^1 = \left\{ \{x_n\}_{n \geq 1} \subseteq \mathbb{R} : \sum_{n \geq 1} |x_n| < \infty \right\}$$

equipped with the following metric: for two points $x = \{x_n\}_{n \geq 1} \in \ell^1$ and $y = \{y_n\}_{n \geq 1} \in \ell^1$, the distance is given by

$$d_1(x, y) = \sum_{n \geq 1} |x_n - y_n|.$$

(a) Prove that the set
$$A = \left\{ x \in \ell^1 : \sum_{n \geq 1} n|x_n| \leq 1 \right\}$$
is a closed subset of ℓ^1.

(b) Show that the zero sequence does not belong to the interior of A.

Exercise 41.6. Let X be the space of sequences that take values in $\{0, 1\}$, namely,
$$X = \{\{x_n\}_{n \geq 1} : x_n \in \{0, 1\} \text{ for all } n \geq 1\}.$$
For two points $x = \{x_n\}_{n\geq 1}$ and $y = \{y_n\}_{n\geq 1}$ in X, we define
$$d(x, y) = \sum_{n\geq 1} \frac{1}{2^n}|x_n - y_n|.$$

(a) Show that $d : X \times X \to \mathbb{R}$ is a metric.

(b) Show that (X, d) is a complete metric space.

Exercise 41.7. Consider $\mathbb{R}^2$ equipped with the Euclidean metric d_2. Let A be a non-empty subset of $\mathbb{R}^2$, that is bounded and closed in $(\mathbb{R}^2, d_2)$. Let
$$S = \{x^2 + y^2 : (x, y) \in A\} \subset \mathbb{R}.$$

(a) Show that S is a bounded subset of $(\mathbb{R}, |\cdot|)$.

(b) Show that S is a closed subset of $(\mathbb{R}, |\cdot|)$.

Exercise 41.8. Let (X, d) be a metric space with X being an infinite countable set. Show that X is not connected.

Exercise 41.9. Consider the metric space $(\mathbb{R}^2, d_2)$ where d_2 denotes the Euclidean distance. Let A be a non-empty connected subset of $\mathbb{R}^2$. Show that the projection of A onto the first coordinate
$$A_1 = \{x \in \mathbb{R} : \text{ there exists } y \in \mathbb{R} \text{ such that } (x, y) \in A\}$$
is a connected subset of $(\mathbb{R}, |\cdot|)$.

42 Lecture 31: Heine-Borel Theorem

Theorem 42.1. Let (X, d) be a metric space and let $K \subseteq X$. The following are equivalent:

1. K is sequentially compact.

2. K is complete and totally bounded.

Proof. Suppose K is sequentially compact. We first show that K is complete. Let $\{x_n\}_{n\geq 1}$ be a Cauchy sequence with $x_n \in K$ for all $n \geq 1$. Since K is sequentially compact, then there exists a subsequence $\{x_{k_n}\}_{n\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that $x_{k_n} \xrightarrow[n\to\infty]{d} y \in K$. Because $\{x_n\}_{n\geq 1}$ is Cauchy, then $x_n \xrightarrow[n\to\infty]{d} y \in K$. As the sequence $\{x_n\}_{n\geq 1}$ is arbitrary, we get that K is complete.

We now show that K is totally bounded. Fix $\varepsilon > 0$ and $a_1 \in K$. If $K \subseteq B_\varepsilon(a_1)$, then K is totally bounded. If $K \not\subseteq B_\varepsilon(a_1)$, then there exists $a_2 \in K$ such that $d(a_1, a_2) \geq \varepsilon$. If $K \subseteq B_\varepsilon(a_1) \cup B_\varepsilon(a_2)$, then K is totally bounded. If $K \not\subseteq B_\varepsilon(a_1) \cup B_\varepsilon(a_2)$, then there exists $a_3 \in K$ such that $d(a_1, a_3) \geq \varepsilon$ and $d(a_2, a_3) \geq \varepsilon$. Continuing inductively, we distinguish two cases.

Case 1: The process terminates in finitely many steps, then K is totally bounded.

Case 2: The process does not terminate in finitely many steps. Then we find $\{a_n\}_{n\geq 1} \subseteq K$ such that $d(a_n, a_m) \geq \varepsilon$ for all $n \neq m$. The sequence does not admit a convergent subsequence, contradicting the fact that K is sequentially compact.

Now suppose K is complete and totally bounded. Let $\{a_n\}_{n\geq 1} \subseteq K$. Since K is totally bounded, then there exists J_1 finite and $\{x_j^{(1)}\}_{j\in J_1} \subseteq X$ such that $K \subseteq \bigcup_{j\in J_1} B_1(x_j^{(1)})$. Because $\{a_n\}_{n\geq 1} \subseteq K$, then there exists some $j_1 \in J_1$ such that $\left|\{n : a_n \in B_1(x_{j_1}^{(1)})\}\right| = \aleph_0$. We now obtain a corresponding subsequence $\{a_n^{(1)}\}_{n\geq 1}$.

Again, because K is totally bounded, then there exists J_2 finite and $\{x_j^{(2)}\}_{j\in J_2} \subseteq X$ such that $K \subseteq \bigcup_{j\in J_2} B_{\frac{1}{2}}(x_j^{(2)})$. Now because $\{a_n^{(1)}\}_{n\geq 1}$ is in K, then there exists some $j_2 \in J_2$ such that $\left|\{n : a_n^{(1)} \in B_{\frac{1}{2}}(x_{j_2}^{(2)})\}\right| = \aleph_0$. Let $\{a_n^{(2)}\}_{n\geq 1}$ denote the corresponding subsequence.

We proceed inductively. We find that for all $k \geq 1$,

- $\{a_n^{(k+1)}\}_{n\geq 1}$ subsequence of $\{a_n^{(k)}\}_{n\geq 1}$, and
- $\{a_n^{(k)}\}_{n\geq 1} \subseteq B_{\frac{1}{k}}(x_{j_k}^{(k)})$ for some $x_{j_k}^{(k)} \in X$.

We consider the subsequence $\{a_n^{(n)}\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$. The ith term of this subsequence is only contained in the sequence $\{a_n^{(i)}\}_{n\geq 1}$. For $n, m \geq k$, the $a_n^{(n)}, a_m^{(m)}$ belong to the subsequence

$\{a_n^{(k)}\}_{n\geq 1}$. In particular,

$$d(a_n^{(n)}, a_m^{(m)}) \leq d(a_n^{(n)}, x_{j_k}^{(k)}) + d(a_m^{(m)}, x_{j_k}^{(k)}) < \frac{2}{k}$$

for all $n, m \geq k$. This shows $\{a_n^{(n)}\}_{n\geq 1}$ is Cauchy. Since K is complete, then $a_n^{(n)} \xrightarrow[n\to\infty]{d} a \in K$. As $\{a_n\}_{n\geq 1}$ was arbitrary, we get that K is sequentially compact. $\square$

Lemma 42.2. Let (X, d) be a sequentially compact metric space. Let $\{G_i\}_{i\in I}$ be an open cover of X. Then there exists $\varepsilon > 0$ such that every ball of radius ε is contained in at least one G_i.

Proof. We argue by contradiction. Then for all $n \geq 1$, there exists $a_n \in X$ such that $B_{\frac{1}{n}}(a_n)$ is not contained in any G_i. Since X is sequentially compact, then there exists a subsequence $\{a_{k_n}\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$ such that $a_{k_n} \xrightarrow[n\to\infty]{d} a \in X = \bigcup\limits_{i\in I} G_i$, then there exists $i_0 \in I$ such that $a \in G_{i_0}$. Since G_{i_0} is open, then there exists $r > 0$ such that $B_r(a) \subseteq G_{i_0}$. Because the subsequence converges to a, then there exists $n_1(r) \in \mathbb{N}$ such that $d(a, a_{k_n}) < \frac{r}{2}$ for all $n \geq n_1$. Let $n_2(r)$ such that $n_2 > \frac{2}{r}$.

Claim 42.3. For all $n \geq n_r = \max\{n_1, n_2\}$, we have $B_{\frac{1}{k_n}}(a_{k_n}) \subseteq B_r(a) \subseteq G_{i_0}$.

Note that this gives the contradiction we needs.

The claim is true: fix $x \in B_{\frac{1}{k_n}}(a_{k_n})$, then

$$d(a, x) \leq d(x, a_{k_n}) + d(a_{k_n}, a) < \frac{1}{k_n} + \frac{r}{2} < r.$$

$\square$

Theorem 42.4. A sequentially compact metric space (X, d) is compact.

Proof. Let $\{G_i\}_{i\in I}$ be an open cover of X. Let ε be given by Lemma 42.2. Because X is sequentially compact, then X is totally bounded, so there exists $n \geq 1$ and $x_1, \cdots, x_n \in X$ such that $X = \bigcup\limits_{j=1}^{n} B_\varepsilon(x_j)$. Note that for all $1 \leq j \leq n$, there exists $i_j \in I$ such that $B_\varepsilon(x_j) \subseteq G_{i_j}$, then $X = \bigcup\limits_{j=1}^{n} G_{i_j}$. $\square$

Collecting our results so far, we obtain the Heine-Borel theorem.

Theorem 42.5 (Heine-Borel)**.** Let (X, d) be a metric space and let $K \subseteq X$. The following are equivalent:

1. K is compact.

2. K is sequentially compact.

3. K is complete and totally bounded.

4. every infinite subset of K has an accumulation point in K.

Remark 42.6. In $\mathbb{R}^n$, K is compact if and only if K is closed and bounded.

Definition 42.7 (Finite Intersection Property)**.** An infinite family $\{F_i\}_{i\in I}$ of closed sets is said to have the finite intersection property if for all finite subset $J \subseteq X$ we have $\bigcap_{j\in J} F_j \neq \varnothing$.

Theorem 42.8. A metric space (X, d) is compact if and only if every infinite family $\{F_i\}_{i\in I}$ of closed sets with the finite intersection property satisfies $\bigcap_{i\in I} F_i \neq \varnothing$.

Proof. ($\Rightarrow$): We argue by contradiction. Assume there exists a sequence $\{F_i\}_{i\in I}$ closed sets with the finite intersection property such that $\bigcap_{i\in I} F_i = \varnothing$. Therefore, $X = {}^c\left(\bigcap_{i\in I} F_i\right) = \bigcup_{i\in I} {}^cF_i$, which is a union of open sets. Because X is compact, then there exists a finite subset $J \subseteq I$ such that $X = \bigcup_{j\in J} {}^cF_j$, so $\varnothing = {}^c\left(\bigcup_{j\in J} {}^cF_j\right) = \bigcap_{j\in J} F_j$. We reach a contradiction.

($\Leftarrow$): We argue by contradiction. Assume there exists an open cover $\{G_i\}_{i\in I}$ of X that does not admit a finite subcover. Therefore, for all finite subset $J \subseteq I$, we have $X \neq \bigcup_{j\in J} G_j$, then $\varnothing \neq \bigcap_{j\in J} {}^cG_j$, an intersection of closed sets. Therefore, $\{{}^cG_i\}_{i\in I}$ is a family of closed sets with the finite intersection property. Then $\bigcap_{i\in I} {}^cG_i \neq \varnothing$, then $\bigcup_{i\in I} G_i \neq X$. We reach a contradiction. $\square$

43 Lecture 32: Continuity

Definition 43.1 (Continuous)**.** Let (X, d_X) and (Y, d_Y) be two metric spaces. We say that a function $f : X \to Y$ is continuous at a point $x_0 \in X$ if for all $\varepsilon > 0$, $\exists \delta > 0$ such that $d_X(x, x_0) < \delta$, then $d_Y(f(x), f(x_0)) < \varepsilon$.

We say f is continuous (on X) if f is continuous at every point in X.

Remark 43.2. $f : X \to Y$ is continuous at every isolated point in X. Indeed, if $x_0 \in X$ is isolated, then there exists $\delta > 0$ such that $B_\delta^X(x_0) = \{x_0\}$. Now, for all $d_X(x, x_0) < \delta$, $d_Y(f(x), f(x_0)) = 0$.

Proposition 43.3. Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \to Y$ be a function. The following are equivalent:

1. f is continuous at $x_0 \in X$.

2. For any $\{x_n\}_{n\geq 1} \subseteq X$ such that $x_n \xrightarrow[n\to\infty]{d_X} x_0$, we have $f(x_n) \xrightarrow[n\to\infty]{d_Y} f(x_0)$.

Proof. (1) $\Rightarrow$ (2): Let $\{x_n\}_{n\geq 1} \subseteq X$ be such that $x_n \xrightarrow[n\to\infty]{d_X} x_0$. Let $\varepsilon > 0$. Since f is continuous at x_0, then there exists $\delta > 0$ such that $d_X(x, x_0) < \varepsilon$ implies $d_Y(f(x), f(x_0)) < \varepsilon$. Because the sequence converges to x_0, then there exists $n_\delta \in \mathbb{N}$ such that $d_X(x_n, x_0) < \delta$ for all $n \geq n_\varepsilon$. Therefore, $d_X(f(x_n), f(x_0)) < \varepsilon$ for all $n \geq n_\varepsilon$.

(2) $\Rightarrow$ (1): We argue by contradiction. Assume there exists $\varepsilon_0 > 0$ such that for all $\delta > 0$, there exists $x_\delta \in X$ such that $d_X(x_\delta, x_0) < \varepsilon$, but $d_Y(f(x_\delta), f(x_0)) \geq \varepsilon_0$. Letting $\delta = \frac{1}{n}$, we find $\{x_n\}_{n\geq 1} \subseteq X$ such that $d_X(x_n, x_0) < \frac{1}{n}$, but $d_Y(f(x_n), f(x_0)) \geq \varepsilon_0$, contradiction. $\square$

Theorem 43.4. Let (X, d_X), (Y, d_Y) be two metric spaces and let $f : X \to Y$ be a function. The following are equivalent:

1. f is continuous,

2. for any G open in Y, $f^{-1}(G) = \{x \in X : f(x) \in G\}$ is open in X,

3. for any F closed in Y, $f^{-1}(F)$ is closed in X,

4. for any $B \subseteq Y$, $\overline{f^{-1}(B)} \subseteq f^{-1}(\bar{B})$,

5. for any $A \subseteq X$, $f(\bar{A}) \subseteq \overline{f(A)}$.

Proof. We will show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): Let $G \subseteq Y$ be open. Let $x_0 \in f^{-1}(G)$, then $f(x_0) \in G$, and since G is open in Y, then there exists $\varepsilon > 0$ such that $B_\varepsilon^Y(f(x_0)) \subseteq G$. Because f is continuous, then there exists $\delta > 0$ such that $f(B_\delta^X(x_0)) \subseteq B_\varepsilon^Y(f(x_0)) \subseteq G$. Therefore, $B_\delta^X(x_0) \subseteq f^{-1}(G)$, hence $x_0 \in \overset{\circ}{f^{-1}}(G)$ and so $f^{-1}(G)$ is open in X.

(2) $\Rightarrow$ (3): Let $F \subseteq Y$ be closed, then ${}^cF = Y\backslash F$ is open in Y. By the assumption, $f^{-1}({}^cF)$ is open in X. Because $f^{-1}({}^cF) = {}^c[f^{-1}(F)] = X\backslash f^{-1}(F)$, then we know $f^{-1}(F)$ is closed in X. Now we conclude that $f^{-1}(Y\backslash F) = f^{-1}(Y)\backslash f^{-1}(F) = X\backslash f^{-1}(F)$, which is open.

(3) $\Rightarrow$ (4): Let $B \subseteq Y$, then $\bar{B}$ is closed in Y. By assumption, $f^{-1}(\bar{B})$ is closed in X, but because $f^{-1}(\bar{B}) \supseteq f^{-1}(B)$, then $\overline{f^{-1}(B)} \subseteq \overline{f^{-1}(B)} = f^{-1}(\bar{B})$.

(4) $\Rightarrow$ (5): Let $A \subseteq X$. Apply $B = f(A)$ to the hypothesis, we have $\bar{A} \subseteq \overline{f^{-1}(f(A))} \subseteq f^{-1}(\overline{f(A)})$, and therefore $f(\bar{A}) \subseteq \overline{f(A)}$.

(5) $\Rightarrow$ (1): We argue by contradiction. Assume there exists $x_0 \in X$ such that f is not continuous at x_0. Then there exists $\varepsilon_0 > 0$ and $x_n \xrightarrow[n\to\infty]{d_X} x_0$, but $d_Y(f(x_n), f(x_0)) \geq \varepsilon_0$.

Let $A = \{x_n : n \geq 1\}$. Then $x_0 \in \bar{A}$, but $f(x_0) \notin \overline{\{f(x_n) : n \geq 1\}} = \overline{f(A)}$. On the other hand, we must have $f(\bar{A}) \subseteq \overline{f(A)}$, but since $x_0 \in \bar{A}$, then $f(x_0) \in \overline{f(A)}$. We reach a contradiction. $\square$

Proposition 43.5. Let (X, d_X), (Y, d_Y), (Z, d_Z) be metric spaces and assume $f : X \to Y$ is continuous at $x_0 \in X$ and $g : Y \to Z$ is continuous at $f(x_0) \in Y$, then $g \circ f : X \to Z$ is continuous at x_0.

Proof. Fix $\varepsilon > 0$. Because g is continuous at $f(x_0)$, then there exists $\delta > 0$ such that $d_Y(y, f(x_0)) < \delta$ implies $d_Z(g(y), g(f(x_0))) < \varepsilon$. Similarly, because f is continuous at x_0, then there exists $\eta > 0$ such that $d_X(x, x_0) < \eta$ implies $d_Y(f(x), f(x_0)) < \delta$. Therefore, if $d_X(x, x_0) < \delta$, then $d_Z(g(f(x)), g(f(x_0))) < \varepsilon$. $\square$

Exercise 43.6. Let (X, d) be a metric space and let $f, g : X \to \mathbb{R}$ be continuous at $x_0 \in X$. Then $f \pm g$ and $f \cdot g$ are continuous at x_0. If $g(x_0) \neq 0$, then $\frac{f}{g} : X \to \mathbb{R}$ is continuous at x_0.

Exercise 43.7. Let (X, d) be a metric space and let $f_1, \cdots, f_n : X \to \mathbb{R}$. Then $f = (f_1, \cdots, f_n) : X \to \mathbb{R}^n$ is continuous at $x_0 \in X$ if and only if $f_1, \cdots, f_n$ are continuous at x_0.

Hint: $|f_i(x) - f_i(x_0)| \leq d_2(f(x), f(x_0)) = \sqrt{\sum_{j=1}^{n} |f_j(x) - f_j(x_0)|^2}$.

Theorem 43.8. Let (X, d_X), (Y, d_Y) be metric spaces and let $f : X \to Y$ be continuous. If K is compact in X, then $f(K)$ is compact in Y.

Proof. We will prove the theorem in two ways.

Let $\{G_i\}_{i \in I}$ be a family of open sets in Y such that $f(JK) \subseteq \bigcup_{i \in I} G_i$, then $K \subseteq f^{-1}(\bigcup_{i \in I} G_i) = \bigcup_{i \in I} f^{-1}(G_i)$, which is open in X since G_i is open in Y.

Since K is compact, then there exists $n \geq 1$ and $i_1, \cdots, i_n \in I$ such that $K \subseteq \bigcup_{j=1}^{n} f^{-1}(G_{i_j}) = f^{-1}(\bigcup_{j=1}^{n} G_{i_j})$, and therefore $f(K) \subseteq \bigcup_{j=1}^{n} G_{i_j}$.

Alternatively, let us show $f(K)$ is sequentially compact. Let $\{y_n\}_{n \geq 1} \subseteq f(K)$. For $y_n \in f(K)$, we know there exists $x_n = f^{-1}(y_n) \in K$. Since K is sequentially compact, then there exists a subsequence $\{x_{k_n}\}_{n \geq 1}$ of $\{x_n\}_{n \geq 1}$ such that $x_{k_n} \xrightarrow[n \to \infty]{d_X} x_0 \in K$. Because f is continuous, then $y_{k_n} = f(x_{k_n}) \xrightarrow[n \to \infty]{d_Y} f(x_0) \in f(K)$. $\square$

44 Lecture 33: Continuity, Compactness, and Connectedness

Corollary 44.1. Let (X, d_X) be a compact metric space and let $f : X \to \mathbb{R}^n$ be continuous. Then $f(X)$ is closed and bounded.

Corollary 44.2. Let (X, d_X) be a compact metric space and let $f : X \to \mathbb{R}$ be continuous. Then there exists $x_1, x_2 \in X$ such that $f(x_1) = \inf\{f(x) : x \in X\}$ and $f(x_2) = \sup\{f(x) : x \in X\}$.

Proof. Note that $f(X)$ is closed and bounded. Because of boundedness, then $\inf(f(X))$ and $\sup(f(X))$ are well-defined. Because of closedness, then the two values are contained in $\overline{f(X)}$, which is just $f(X)$. $\square$

Proposition 44.3. Let (X, d_X) and (Y, d_Y) be metric spaces such that X is compact. Let $f : X \to Y$ be bijective and continuous, then $f^{-1} : Y \to X$ is continuous.

Proof. It suffices to show that for every closed set $F \subseteq X$, we have $(f^{-1})^{-1}(F) = \{y \in Y : f^{-1}(y) \in F\}$ is closed in Y. But $(f^{-1})^{-1}(F) = f(F)$. Since F is closed in X, which is compact, then F is compact. Moreover, because $f : X \to Y$ is continuous, then $f(F)$ is compact, and therefore $f(F)$ is closed. $\square$

Definition 44.4 (Uniform Continuous)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. We say that a function $f : X \to Y$ is uniformly continuous if for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$.

Remark 44.5. One may want to compare this with the continuous condition. We say $g : X \to Y$ is continuous if for all $x \in X$ and for all $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, x)$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$. We have the following observations.

1. Continuity is defined pointwise. Uniform continuity is a property of a function on a set.

2. Uniform continuity implies continuity.

3. There are continuous functions that are not uniformly continuous. Denote $f : \mathbb{R} \to \mathbb{R}$ to be the function $f(x) = x^2$. Let $x_n = n + \frac{1}{n}$, and $y_n = n$, and we see $|x_n - y_n| = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$. But $|f(x_n) - f(y_n)| = (n + \frac{1}{n})^2 - n^2 = 2 + \frac{1}{n^2} > 2$.

Theorem 44.6. Let (X, d_X) and (Y, d_Y) be metric spaces with X compact. Let $f : X \to Y$ be continuous, then f is uniformly continuous.

Proof. We argue by contradiction. Assume f is not uniformly continuous, then there exists some $\varepsilon_0 > 0$ such that for all $\delta > 0$, there exists $x_\delta, y_\delta \in X$ such that $d_X(x_\delta, y_\delta) < \delta$ but $d_Y(f(x_\delta), f(y_\delta)) \geq \varepsilon_0$. In particular, let $\delta = \frac{1}{n}$ and we can get $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ in X such that $d_X(x_n, y_n) < \frac{1}{n}$, but $d_Y(f(x_n), f(y_n)) \geq \varepsilon_0$. Because X is compact, then there exists a subsequence $\{x_{k_n}\}_{n\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that $x_{k_n} \xrightarrow[n\to\infty]{d_X} x_0 \in X$.

By the triangle inequality, $d(y_{k_n}, x_0) \leq d(x_{k_n}, y_{k_n}) + d(x_{k_n}, x_0)$. Note that the first term is bounded above by $\frac{1}{k_n}$, which is bounded above by $\frac{1}{n}$, which converges to 0 as n goes to infinity. The second term obviously converges to 0 as n goes to infinity. Therefore, the distance itself converges to 0 as n goes to infinity. Therefore, $y_{k_n} \xrightarrow[n\to\infty]{d_X} x_0$.

Now, because f is continuous, then $f(x_{k_n}) \xrightarrow[n\to\infty]{d_Y} f(x_0)$ and $f(y_{k_n}) \xrightarrow[n\to\infty]{d_Y} f(x_0)$. But then $\varepsilon_0 \leq d_Y(f(x_{k_n}), f(y_{k_n})) \leq d_Y(f(x_{k_n}), f(x_0)) + d_Y(f(x_0), f(y_{k_n}))$, and the right-hand side goes to 0 as n goes to infinity. We reach a contradiction. $\square$

Theorem 44.7. Let (X, d_X) and (Y, d_Y) be metric spaces such that X is connected. Let $f : X \to Y$ be continuous. Then $f(X)$ is connected.

Proof. We can show the theorem in two ways.

We now abuse the notation and write $f : X \to f(X)$. It suffices to show that if $\varnothing \neq B \subseteq f(X)$ is both open and closed in $f(X)$, then $B = f(X)$. As f is continuous, $f^{-1}(B) \neq \varnothing$ is both open and closed in X. But X is connected, which implies that $f^{-1}(B) = X$, i.e., $f(X) = B$.

Alternatively, we assume that $f(X)$ is not connected. Then there exists $\varnothing \neq B_1 \subseteq$ and $\varnothing \neq B_2 \subseteq Y$ such that $f(X) \subseteq B_1 \cup B_2$ and $\bar{B}_1 \cap B_2 = \varnothing = B_1 \cap \bar{B}_2$. Let $A_1 = f^{-1}(B_1) \neq \varnothing$ and $A_2 = f^{-1}(B_2) \neq \varnothing$. Now $f(X) \subseteq B_1 \cup B_2$, then $X \subseteq f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2) = A_1 \cup A_2$.

$\bar{A}_1 \cap A_2 = \overline{f^{-1}(B_1)} \cap f^{-1}(B_2) \subseteq f^{-1}(\bar{B}_1) \cap f^{-1}(B_2) = f^{-1}(\bar{B}_1 \cap B_2) = f^{-1}(\varnothing) = \varnothing$. Similarly, $\bar{A}_2 \cap A_1 = \varnothing$. This contradicts the fact that X is connected. $\square$

Corollary 44.8 (Darboux's Property)**.** Let (X, d_X) be a metric space and let $f : X \to \mathbb{R}$ be continuous. If $A \subseteq X$ is connected, then $f(A)$ is an interval in $\mathbb{R}$.

In particular, if $X = \mathbb{R}$, and $a, b \in \mathbb{R}$ such that $a < b$, and y_0 lies between $f(a)$ and $f(b)$, then there exists $x_0 \in (a, b)$ such that $f(x_0) = y_0$.

Remark 44.9. There are functions that have the Darboux property, but are not continuous. For example, define $f : [0, \infty) \to \mathbb{R}$ by $f(x) = \begin{cases} \sin(\frac{1}{x}), & x \neq 0 \\ c, & x = 0 \end{cases}$ where $c \in [-1, 1]$.

Note that f is continuous on $(0, \infty)$ implies f has the Darboux property on $(0, \infty)$. However, f has the Darboux property on $[0, \infty)$, but is not continuous at $x = 0$.

45 Homework 12

Exercise 45.1. Let $\{A_i\}_{i \in I}$ be an infinite family of closed sets with the finite intersection property. Assuming that one member of this family is compact, show that $\bigcap\limits_{i \in I} A_i \neq \varnothing$.

Exercise 45.2. (a) Show that the closed unit ball in ℓ^2, namely,

$$A = \{x \in \ell^2 : \sum_{n=1}^{\infty} |x_n^2| \leq 1\}$$

is not compact in ℓ^2.

(b) Define $B \subseteq \ell^2$ by

$$B = \{x \in \ell^2 : \sum_{n=1}^{\infty} n|x_n|^2 \leq 1\}.$$

Show that B is compact.

Exercise 45.3. Let A be a subset of a complete metric space. Assume that for all $\varepsilon > 0$, there exists a compact set A_ε so that for all $x \in A$, $d(x, A_\varepsilon) < \varepsilon$. Show that $\bar{A}$ is compact.

Exercise 45.4. Let (X, d_1) and (Y, d_2) be two compact metric spaces. Show that the space $X \times Y$ endowed with the "Euclidean" distance

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{[d_1(x_1, x_2)]^2 + [d_2(y_1, y_2)]^2}$$

is a compact metric space.

Exercise 45.5. Show that a totally bounded metric space contains a countable dense subset.

Exercise 45.6. Let (X, d) be a metric space and let $A \subseteq X$ be a compact subset. Show that

(a) For any $y \in X$, there exists $a \in A$ so that $d(y, A) = d(y, a)$.

(b) If $B \subseteq X$ and $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\} = 0$, then $A \cap \bar{B} \neq \varnothing$.

Exercise 45.7. Let (X, d) be a metric space. If A and B are non-empty subsets of X, we define their Hausdorff distance via

$$d_H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$

Let $\mathcal{F}(X) = \{A \subseteq X : A \text{ is compact and non-empty}\}$. Show that

(a) $(\mathcal{F}(X), d_H)$ is a metric space.

(b) For $A, B \in \mathcal{F}(X)$ and $\varepsilon > 0$, show that $d_H(A, B) \leq \varepsilon$ if and only if $A \subseteq B^\varepsilon = \{x \in X : d(x, B) \leq \varepsilon\}$ and $B \subseteq A^\varepsilon = \{x \in X : d(x, A) \leq \varepsilon\}$.

(c) If (X, d) is compact, then so is $(\mathcal{F}(X), d_H)$.

Hint: Prove that $(\mathcal{F}(X), d_H)$ is totally bounded and complete. To prove completeness, for a Cauchy sequence $\{A_n\}_{n\geq 1} \subseteq \mathcal{F}(X)$, let

$$A = \bigcap_{n\geq 1} \overline{\bigcup_{m\geq n} A_m};$$

show that $A \in \mathcal{F}(X)$. To show that A is the limit of the sequence $\{A_n\}_{n\geq 1}$, use part (b).

46 Lecture 34: Path

Proposition 46.1. Let (X, d_X) and (Y, d_Y) be two connected metric spaces. Then $(X\times Y, d)$ where

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2}$$

is a connected metric space.

Remark 46.2. One could replace the distance d by

$$d_1((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

or

$$d_\infty((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

Proof. We will use the fact that a metric space is connected if and only if any two points are obtained in a connected subset of the metric space. So to show $X \times Y$ is connected if suffices to show that if $(a, b), (c, d) \in X \times Y$, then there exists $C \subseteq X \times Y$ connected such that $(a, b), (c, d) \in C$.

Let $f : X \to X \times Y$ to be $f(x) = (x, b)$.

Claim 46.3. f is continuous.

Subproof. Note that the definition of the function tells us that $d(f(x_1), f(x_2)) = d_X(x_1, x_2)$. Take $\delta = \varepsilon$ in the definition of continuity. ■

Now since X is connected, $f(X) = X \times \{b\}$ is connected. Similarly, define $g : Y \to X \times Y$ to be $g(y) = (c, y)$, then we can say g is continuous, and since Y is connected, then $g(Y) = \{c\} \times Y$ is connected as well. Finally, $f(X) \cap g(Y) \ni (c, b)$ and so $f(X), g(Y)$ are not separated. As the union of two connected not separated sets is connected, we get $f(X) \cup g(Y)$ is connected. Note that $(a, b), (c, d) \in f(X) \cup g(Y)$. □

Definition 46.4 (Path)**.** Let (X, d) be a metric space. A path is a continuous function $\gamma : [0, 1] \to X$. $\gamma(0)$ is called the origin of the path and $\gamma(1)$ is called the end of the path.

Remark 46.5. As $[0,1]$ is compact and connected and γ is continuous, $\gamma([0,1])$ is compact and connected.

Given $\gamma:[0,1]\to X$ a path, we define $\gamma^-:[-,1]\to X$ with $\gamma^-(t)=\gamma(1-t)$ as a path.

Given $\gamma_1,\gamma_2:[0,1]\to X$ paths such that $\gamma_1(1)=\gamma_2(0)$, we define $\gamma_1\vee\gamma_2:[0,1]\to X$ via

$$\gamma_1\vee\gamma_2(t)=\begin{cases}\gamma_1(2t), & \text{if } 0\leq t\leq\frac{1}{2}\\ \gamma_2(2t-1), & \text{if } \frac{1}{2}\leq t\leq 1\end{cases}$$

Note $\gamma_1\vee\gamma_2$ is a path.

Proposition 46.6. Let (X,d) be a metric space and let $A\subseteq X$. Consider the following statements:

1. There exists $a\in A$ such that for all $x\in A$, there exists a path $\gamma_x:[0,1]\to A$ such that $\gamma_x(0)=a$ and $\gamma_x(1)=x$.
2. For all $x,y\in A$, there exists a path $\gamma_{x,y}:[0,1]\to A$ such that $\gamma_{x,y}(0)=x$ and $\gamma_{x,y}(1)=y$.
3. A is connected.

Then (1) $\iff$ (2) $\implies$ (3).

Proof. (1) $\Rightarrow$ (2): Let $x,y\in A$. By the hypothesis, there exists paths $\gamma_x,\gamma_y:[0,1]\to A$ such that $\gamma_x(0)=\gamma_y(0)=a$, $\gamma_x(1)=x$, $\gamma_y(1)=y$. Then $\gamma_x^-\vee\gamma_y:[0,1]\to A$ is the desired path.

(2) $\Rightarrow$ (1): Choose $a\in A$ arbitrary.

(1) $\Rightarrow$ (3): Given $x\in A$, let $A_x=\gamma_x([0,1])$ connected. Note $\bigcap\limits_{x\in A}A_x\notin a$, then no two sets A_x and A_y are separated. Then $A=\bigcup\limits_{x\in A}A_x$ is connected. □

Definition 46.7 (Path-connected)**.** If either (1) or (2) holds in Proposition 46.6, we say that A is path-connected.

Remark 46.8. Path-connected implies connected.

Example 46.9. $\mathbb{R}^2\backslash\mathbb{Q}^2$ is path-connected. We will show that any $(x,y)\in\mathbb{R}^2\backslash\mathbb{Q}^2$ can be joined via path in $\mathbb{R}^2\backslash\mathbb{Q}^2$ to $(\sqrt{2},\sqrt{2})$.

For $(x,y)\in\mathbb{R}^2\backslash\mathbb{Q}^2$, then either $x\notin\mathbb{Q}$ or $y\notin\mathbb{Q}$. Say $x\notin\mathbb{Q}$. Then $\{x\}\times\mathbb{R}\subseteq\mathbb{R}^2\backslash\mathbb{Q}^2$. Note also that $\mathbb{R}\times\{\sqrt{2}\}\subseteq\mathbb{R}^2\backslash\mathbb{Q}^2$. Let $\gamma:[0,1]\to\mathbb{R}^2\backslash\mathbb{Q}^2$ for $\gamma=\gamma_1\vee\gamma_2$, where the path $\gamma_1:[0,1]\to\mathbb{R}^2\backslash\mathbb{Q}^2$ defined by $\gamma_1(t)=(\sqrt{2}+t(x-\sqrt{2}),\sqrt{2})$ and $\gamma_2:[0,1]\to\mathbb{R}^2\backslash\mathbb{Q}^2$ defined by $\gamma_2(t)=(x,\sqrt{2}+t(y-\sqrt{2}))$.

Example 46.10 (A connected set which is not path connected)**.** Let $f : [0, \infty) \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} \sin(\frac{1}{x}), & x \neq 0 \\ a, & x = 0 \end{cases}$$

where $a \in [-1, 1]$ is fixed. Then $\Gamma_f = \{(x, f(x)) : x \in [0, \infty)\}$ is connected, but not path connected.

Let us show Γ_f to be connected. The function $g : [0, \infty) \to \mathbb{R}^2$ defined by $g(x) = (x, f(x))$ is continuous on $(0, \infty)$, then $g((0, \infty))$ is connected. Also, $g(\{0\}) = \{(0, a)\}$ is connected. We will show that $(0, a) \in \overline{g((0, \infty))}$ and so $\{(0, a)\}$ and $g((0, \infty))$ are not separated. Then $\Gamma_f = g([0, \infty)) = g(\{0\}) \cup g((0, \infty))$ is connected. To see $(0, a) \in \overline{g(0, \infty)}$, we need to find $x_n \searrow 0$ such that $\sin(\frac{1}{x_n}) = a$. Take $x_n = \frac{1}{\arcsin(a) + 2n\pi}$ where $\arcsin(a) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Now let us show Γ_f is not path-connected. Assume towards a contradiction that there exists a path $\gamma : [0, 1] \to \Gamma_f$ such that $\gamma(0) = (0, a)$ and $\gamma(1) = (\frac{1}{\pi}, 0)$. Note $\pi_1 \circ \gamma : [0, 1] \to \mathbb{R}$ is continuous, with $(\pi_1 \circ \gamma)(0) = 0$ and $(\pi_1 \circ \gamma)(1) = \frac{1}{\pi}$. Let $b \in [-1, 1] \backslash \{a\}$. By the Darboux property, there exists $t_n \in (0, \frac{1}{\pi})$ such that $(\pi_1 \circ \gamma)(t_n) = \frac{1}{\arcsin(b) + 2n\pi}$ where $\arcsin(b) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. As $[0, 1]$ is compact, there exists a subsequence $\{t_{k_n}\}_{n \geq 1}$ that converges to $t_\infty \in [0, 1]$. Because γ is continuous, then $\gamma(t_{k_n})$ converges to $\gamma(t_\infty)$. But $\gamma(t_{k_n}) = (\frac{1}{\arcsin(b) + 2k_n\pi}, b) \xrightarrow[n \to \infty]{} (0, b)$, then $\gamma(t_\infty) = (0, b) \notin \Gamma_f$.

47 Lecture 35: Convergent Sequences of Functions

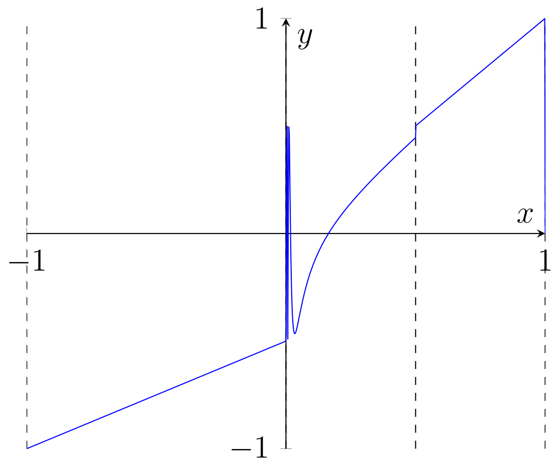
Example 47.1. Consider two connected sets $A, B \subseteq [-1, 1] \times [-1, 1]$ such that $(-1, -1), (1, 1) \in A$, $(-1, 1), (1, -1) \in B$, and $A \cap B = \varnothing$.

Let $f : [-1, 1] \to [-1, 1]$ defined by

$$f(x) = \begin{cases} \frac{x-1}{2}, & -1 \leq x \leq 0 \\ x - \frac{1}{2}\sin(\frac{\pi}{x}), & 0 < x \leq \frac{1}{2} \\ x, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

and $g : [-1, 1] \to [-1, 1]$ defined by

$$g(x) = \begin{cases} \frac{1-x}{2}, & -1 \leq x \leq 0 \\ -x - \frac{1}{2}\sin(\frac{\pi}{x}), & 0 < x \leq \frac{1}{2} \\ -x, & \frac{1}{2} \leq x \leq 1 \end{cases}$$



Now let $A = \Gamma_f = \{(x, f(x)) : x \in [-1, 1]\}$ and $B = \Gamma_g = \{(x, g(x)) : x \in [-1, 1]\}$. We can now prove that $A \cap B = \varnothing$.

- If $-1 \leq x \leq 0$, then $f(x) = g(x)$ if and only if $\frac{x-1}{2} = \frac{1-x}{2}$ if and only if $x = 1$, contradiction.
- If $0 < x \leq \frac{1}{2}$, then $f(x) = g(x)$ if and only if $x = 0$, contradiction.
- $\frac{1}{2} \leq x \leq 1$, then $f(x) = g(x)$ if and only if $x = 0$, contradiction.

Also, $f(-1) = -1$, so $(-1, -1) \in A$; $f(1) = 1$, so $(1, 1) \in A$; $g(-1) = 1$, so $(-1, 1) \in B$; $g(1) = -1$, so $(1, -1) \in B$.

Let us show that A is connected. A similar argument can be used to prove that B is connected.

We write $A = A_1 \cup A_2$ where $A_1 = \{(x, f(x)) : -1 \leq x \leq 0\}$ and $A_2 = \{(x, f(x)) : 0 < x \leq 1\}$. Note that $h : [-1, 1] \to \mathbb{R}^2$ defined by $h(x) = (x, f(x))$ is continuous on $[-1, 0]$ and on $(0, 1]$. So since $[-1, 0]$ and $(0, 1]$ are connected sets, we get that $h([-1, 0]) = A$ and $h((0, 1]) = A_2$ are connected. To show that $A = A_1 \cup A_2$ is connected, it suffices to show that A_1 and A_2 are not separated.

We will show $(0, -\frac{1}{2}) \in A_1 \cap \bar{A}_2$. Clearly $f(0) = -\frac{1}{2}$, then $(0, -\frac{1}{2}) \in A_1$. To show that $(0, -\frac{1}{2}) \in \bar{A}_2$, we need to find $x_n \searrow 0$ such that $f(x_n) = x_n - \frac{1}{2}\sin(\frac{\pi}{x_n}) \xrightarrow[n\to\infty]{} -\frac{1}{2}$. We take x_n such that $\sin(\frac{\pi}{x_n}) = 1$, which is true if and only if $\frac{\pi}{x_n} = \frac{\pi}{2} + 2n\pi$, if and only if $x_n = \frac{2}{4n+1} \searrow 0$. Note $f(x_n) = \frac{2}{4n+1} - \frac{1}{2} \xrightarrow[n\to\infty]{} -\frac{1}{2}$.

Definition 47.2 (Pointwise Convergence)**.** Let (X, d_X) and (Y, d_Y) be two metric spaces and let $f_n : X \to Y$ be a sequence of functions. We say that $\{f_n\}_{n\geq 1}$ converges pointwise if for all $x \in X$ the sequence $\{f_n(x)\}_{n\geq 1}$ converges in Y. The limit $\lim\limits_{n\to\infty} f_n(x) = f(x)$ defines a function $f : X \to Y$.

Remark 47.3. $\{f_n\}_{n\geq 1}$ converges pointswise to f if for all $x \in X$ and $\varepsilon > 0$, there exists $n(\varepsilon, x) \in \mathbb{N}$ such that $d_Y(f_n(x), f(x)) < \varepsilon$ for all $n \geq n(\varepsilon, x)$.

Note that for $\varepsilon > 0$ fixed, $n(\varepsilon, \cdot)$ can be bounded or unbounded. If it is bounded, we get the following definition.

Definition 47.4 (Uniform Convergence)**.** Let (X, d_X) and (Y, d_Y) be metric spaces and let $f_n : X \to Y$ be a sequence of functions. We say that $\{f_n\}_{n\geq 1}$ converges uniformly to a function $f : X \to Y$ (and we write $f_n \xrightarrow[n\to\infty]{u} f$ if for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $d_Y(f(x), f_n(x)) < \varepsilon$ for all $n \geq n_\varepsilon$ and all $x \in X$.

Remark 47.5. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $B(X, Y) = \{f : X \to Y : f \text{ is bounded}\}$, and let $d : B(X, Y) \times B(X, Y) \to \mathbb{R}$ be defined via $d(f, g) = \sup\limits_{x\in X} d_Y(f(x), g(x))$.

Exercise 47.6. Show that $(B(X, Y), d)$ is a metric space.

Remark 47.7. Note that $f_n \xrightarrow[n\to\infty]{u} f$ if and only if $M_n = d(f_n, f) \xrightarrow[n\to\infty]{} 0$. We may now show this.

($\Leftarrow$): For all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $M_n < \varepsilon$ for all $n \geq n_\varepsilon$. Then $d(f_n, f) = \sup\limits_{x\in X} d_Y(f_n(x), f(x)) < \varepsilon$ for all $n \geq n_\varepsilon$, and so $d_Y(f_n(x), f(x)) < \varepsilon$ for all $n \geq n_\varepsilon$ and for all $x \in X$.

($\Rightarrow$): Since $f_n \xrightarrow[n\to\infty]{u} f$, then for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $d_Y(f_n(x), f(x) < \frac{\varepsilon}{2}$ for all $n \geq n_\varepsilon$ and for all $x \in X$. Therefore, $\sup\limits_{x\in X} d_Y(f_n(x), f(x)) \leq \frac{\varepsilon}{2} < \varepsilon$ for all $n \geq n_\varepsilon$. Note that this implies $M_n = d(f_n, f) < \varepsilon$ for all $n \geq n_\varepsilon$ as well.

Remark 47.8.

1. Uniform convergence implies pointwise convergence.
2. Pointwise convergence does not imply uniform convergence.

   Define $f_n : [0, 1] \to \mathbb{R}$ by $f_n(x) = x^n$. Note that $\{f_n\}_{n\geq 1}$ converges pointwise:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

Let $f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$. Note f_n does not converge to f uniformly since $d(f_n, f) = \sup\limits_{x\in[0,1]} |f_n(x) - f(x)| = \sup\limits_{x\in[0,1)} |x^n| = 1 \not\to 0$.

Theorem 47.9 (Weierstrass)**.** Let (X, d_X) and (Y, d_Y) be metric spaces and let $f_n : X \to Y$ be a sequence of functions that converges uniformly to a function $f : X \to Y$. If for all $n \geq 1$, f_n is continuous at $x_0 \in X$, then f is continuous at x_0.

Corollary 47.10. A uniform limit of continuous functions is a continuous function.

Proof of Theorem. Fix $\varepsilon > 0$. Because $f_n \xrightarrow[n\to\infty]{u} f$, then there exists $n_\varepsilon \in \mathbb{N}$ such that $d_Y(f_n(x), f(x)) < \frac{\varepsilon}{3}$ for all $n \geq n_\varepsilon$ and for all $x \in X$. Now fix $n_0 \geq n_\varepsilon$.

Since f_{n_0} is continuous at x_0, then by definition there exists $\delta > 0$ such that if $d_X(x_0, x) < \delta$, then $d_Y(f_{n_0}(x_0), f_{n_0}(x)) < \frac{\varepsilon}{3}$. Then for $x \in B_\delta(x_0)$, we have

$$\begin{aligned} d_Y(f(x), f(x_0))) &\leq d_Y(f(x), f_{n_0}(x)) + d(f_{n_0}(x), f_{n_0}(x_0)) + d(f_{n_0}(x_0), f(x_0)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

By definition, f is continuous at x_0. □

48 Lecture 36: Dini's Theorem and Function Space

Theorem 48.1 (Dini)**.** Let (X, d) be a compact metric space and let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions that converges pointwise to a continuous function $f : X \to \mathbb{R}$. Assume that $\{f_n\}_{n\geq 1}$ is monotone (in the sense that $\{f_n(x)\}_{n\geq 1}$ is either increasing for all $x \in X$, or is decreasing for all $x \in X$), then $f_n \xrightarrow[n\to\infty]{u} f$, i.e., $d(f_n, f) = \sup_{x\in X} |f_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0$.

Proof. Assume that $\{f_n\}_{n\geq 1}$ is increasing, then $\{f - f_n\}_{n\geq 1}$ is decreasing, and for all $x \in X$, we have $\lim_{n\to\infty} [f(x) - f_n(x)] = \inf_{n\to\infty} [f(x) - f_n(x)] = 0$. Therefore, for all $\varepsilon > 0$, there exists $n(\varepsilon, x) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon, x)$, we have

$$0 \leq f(x) - f_n(x) \leq f(x) - f_{n_{\varepsilon,x}}(x) < \varepsilon.$$

As $f - f_{n_{\varepsilon,x}}$ is continuous at x, there exists $\delta(\varepsilon, x) > 0$ such that whenever $d(x, y) < \delta_{\varepsilon,x}$, we have $|[f(x) - f_{n_{\varepsilon,x}}(x)] - [f(y) - f_{n_{\varepsilon,x}}(y)]| < \varepsilon$. By the triangle inequality, we get

$$0 \leq f(y) - f_{n_{\varepsilon,x}}(y) \leq |[f(x) - f_{n_{\varepsilon,x}}(x)] - [f(y) - f_{n_{\varepsilon,x}}(y)]| + f(x) - f_{n_{\varepsilon,x}}(x) < \varepsilon + \varepsilon = 2\varepsilon$$

whenever $y \in B_{\delta_{\varepsilon,x}}(x)$. In particular,

$$0 \leq f(y) - f_n(y) \leq f(y) - f_{n_{\varepsilon,x}}(y) < 2\varepsilon$$

for all $n \geq n_{\varepsilon,x}$ and all $y \in B_{\delta_{\varepsilon,x}}(x)$.

Note that X is compact, and since $X = \bigcup_{x\in X} B_{\delta_{\varepsilon,x}}(x)$, then there exists a finite set $J \subseteq \mathbb{N}$ and $\{x_j\}_{j\in J} \in X$ such that $X = \bigcup_{j\in J} B_{\delta_j}(x_j)$, where $\delta_j = \delta(\varepsilon, x_j)$.

Let $n_\varepsilon = \max_{j\in J} n(\varepsilon, x_j)$, fix $n \geq n_\varepsilon$ and $x \in X$. Since $x \in X = \bigcup_{j\in J} B_{\delta_j}(x_j)$, then there exists $j \in J$ such that $x \in B_{\delta_j}(x_j)$. Because of the bound we got earlier, $0 \leq f(x) - f_n(x) < 2\varepsilon$. Since $x \in X$ is arbitrary, then $d(f, f_n) \leq 2\varepsilon$ for all $n \geq n_\varepsilon$.[13] $\square$

Remark 48.2. The compactness of X is necessary in Dini's theorem.

Example 48.3. Consider $f_n : (0,1) \to \mathbb{R}$ defined by $f_n(x) = x^n$, which is a continuous function. Note that $f_{n+1}(x) \leq f_n(x)$ for all $n \geq 1$ and for all $x \in (0,1)$, and $f_n(x) \xrightarrow[n\to\infty]{} 0$ for all $x \in (0,1)$. We also define $f : (0,1) \to \mathbb{R}$ by $f(x) = 0$ for all $x \in (0,1)$. It is a continuous function. But $d(f_n, f) = \sup_{x\in(0,1)} |x^n| = 1$, which does not converge to 0, therefore, f_n does not converge to f in a uniform sense. Note that $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$ is continuous, decreasing, and converging pointwise to $f : [0,1] \to \mathbb{R}$, where $f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$, which is not continuous. Again, f_n does not converge to f in a uniform manner. This also shows that the continuity of the limit function is necessary in Dini's theorem.

Remark 48.4. Monotonicity is necessary in Dini's theorem.

Example 48.5. Note that $f_n : [0,1] \to \mathbb{R}$ defined by the function connecting the points $(\frac{1}{n+2}, 0)$, $(\frac{1}{n+1}, 1)$, and $(\frac{1}{n}, 0)$, is continuous, and it converges pointwise to $f : [0,1] \to \mathbb{R}$ where $f(x) = 0$ for all $x \in [0,1]$. Note that f is also continuous. But $d(f_n, f) = \sup_{x\in[0,1]} |f_n(x)| = 1$, which does not converge to 0, and therefore $\{f_n\}_{n\geq 1}$ does not converge to f uniformly. Note that the issue here is that $\{f_n\}_{n\geq 1}$ is not monotone.

Definition 48.6 (Function Space)**.** Fix $a, b \in \mathbb{R}$ where $a < b$. We define

$$C([a,b]) = \{f : [a,b] \to \mathbb{R} : f \text{ is continuous}\}.$$

We equip $C([a,b])$ with the metric $d : C([a,b]) \times C([a,b]) \to \mathbb{R}$ given by $d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$. Then $(C([a,b]), d)$ defines a metric space.

Remark 48.7 (Completeness)**.** Let $\{f_n\}_{n\geq 1} \subseteq C([a,b])$ be a Cauchy sequence, so for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $d(f_n, f_m) < \varepsilon$ for all $n, m \geq n_\varepsilon$. Therefore,

$$|f_n(x) - f_m(x)| < \varepsilon$$

for all $n, m \geq n_\varepsilon$ and all $x \in [a,b]$. Therefore, the sequence $\{f_n\}_{n\geq 1}$ is Cauchy for all $x \in [a,b]$. Since $\mathbb{R}$ is complete, then for all $x \in [a,b]$, we have $f_n(x) \xrightarrow[n\to\infty]{} f(x) \in \mathbb{R}$. This defines a function $f : [a,b] \to \mathbb{R}$.

[13] Note that we can shrink the coefficient by a certain scale so that it satisfies the definition. In general, it does not matter whether we ask for ε or 2ε, or asking for a non-strict inequality instead of a strict one.

Recall that for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $n, m \geq n_\varepsilon$ and all $x \in [a, b]$. Letting $m \to \infty$ (with n fixed) we get $|f_n(x) - f(x)| \leq \varepsilon$ for all $n \geq n_\varepsilon$ and all $x \in [a, b]$. That is, we see $d(f_n, f) \leq \varepsilon$ for all $n \geq n_\varepsilon$, so $f_n \xrightarrow[n\to\infty]{u} f$. By the Weierstrass Theorem, $f \in C([a, b])$. Then $(C([a, b]), d)$ is a complete metric space.

Remark 48.8 (Compactness)**.** Note that $(C([a, b]), d)$ is not bounded, and so not compact. For example, consider $f_n : [a, b] \to \mathbb{R}$ defined by $f_n(x) = n$ for all $x \in [a, b]$.

Remark 48.9 (Connectedness)**.** $(C([a, b]), d)$ is path-connected and so connected.

Let $f, g \in C([a, b])$, define $\gamma : [0, 1] \to C([a, b])$ by $\gamma(t) = f + t(g - f)$. Note that for all $t \in [0, 1]$, $\gamma(t) \in C([a, b])$, and $\gamma(0) = f$ and $\gamma(1) = g$. To see that γ is a path, we compute

$$\begin{aligned} d(\gamma(t), \gamma(s)) &= \sup_{x\in[a,b]} |\gamma(t; x) - \gamma(s; x)| \\ &= \sup_{x\in[a,b]} |t - s| \cdot |g(x) - f(x)| \\ &= |t - s| \cdot d(g, f), \end{aligned}$$

and note that this value converges to 0 as $|t - s|$ converges to 0, since $d(g, f) \in \mathbb{R}$ already. Therefore, γ is a continuous function and so a path.

49 Homework 13

Exercise 49.1. Let (X, d_X) be a compact metric space.

(a) Verify that

$$d_Y(f, g) = \sum_{n\in\mathbb{N}} 2^{-n} d_X(f(n), g(n))$$

defines a metric on $Y = \{f : \mathbb{N} \to X\}$.

(b) Show that Y is compact.

Exercise 49.2. Consider the Cantor set

$$K = \{x \in [0, 1] : x = \sum_{n=1}^{\infty} a_n 3^{-n} \text{ with all } a_n \in \{0, 2\}\}.$$

For example, $1 \in K$ because it is represented by setting all $a_n = 2$.

(a) Show that K is uncountable.

(b) Show that K is compact.

(c) Show that no connected subset of K contains more than one point.

Exercise 49.3. Let $f : [0, \infty) \to [0, \infty)$ be a continuous function with $f(0) = 0$. Show that if

$$f(t) \leq 1 + \frac{1}{10} f(t)^3 \quad \text{for all } t \in [0, \infty),$$

then f is uniformly bounded throughout $[0, \infty)$.

Exercise 49.4. Show that the function

$$H(x, y) = x^2 + y^2 + |x - y|^{-1}$$

achieves its global minimum somewhere on the set $\{(x, y) \in \mathbb{R}^2 : x \neq y\}$.

Exercise 49.5. Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : [a, b] \to [a, b]$ be continuous. Show that there exists $x_0 \in [a, b]$ such that $f(x_0) = x_0$.

Exercise 49.6. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R} \backslash \mathbb{Q} \\ \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ with } (p, q) = 1. \end{cases}$$

Prove that f is continuous on $\mathbb{R} \backslash \mathbb{Q}$ and discontinuous on $\mathbb{Q}$.

Exercise 49.7. Let (X, d) be a metric space and let $f, g : X \to \mathbb{R}$ be two continuous functions.

(a) Prove that the set $\{x \in X : f(x) < g(x)\}$ is open.

(b) Prove that if the set $\{x \in X : f(x) < g(x)\}$ is dense in X, then $f(x) \leq g(x)$ for all $x \in X$.

Exercise 49.8. Let $a, b \in \mathbb{R}$ with $a < b$. Show that a function $f : (a, b) \to \mathbb{R}$ is uniformly continuous on (a, b) if and only if it can be extended to a continuous function $\tilde{f}$ on $[a, b]$.

50 Lecture 37: Arzela-Ascoli Theorem

Definition 50.1 (Equicontinuous)**.** We say that a set $\mathcal{F} \subseteq C([a, b])$ is equicontinuous if for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in [a, b]$ with $|x - y| < \delta(\varepsilon)$ and for all $f \in \mathcal{F}$.

Remark 50.2. Note that for a fixed function $f \in \mathcal{F} \subseteq C([a,b])$, we have that f is uniformly continuous (since f is continuous on compact set $[a,b]$), then for all $\varepsilon > 0$, there exists $\delta(\varepsilon, f) > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in [a,b]$ with $|x-y| < \delta(\varepsilon, f)$.

Note that for an equicontinuous family $\mathcal{F}$, $\delta(\varepsilon)$ can be chosen uniformly for $f \in \mathcal{F}$.

Definition 50.3 (Uniformly Bounded)**.** We say that a set $\mathcal{F} \subseteq C([a,b])$ is uniformly bounded if there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$ for all $f \in \mathcal{F}$.

Remark 50.4. Note that for a fixed $f \in \mathcal{F} \subseteq C([a,b])$ we have that $f([a,b])$ is bounded (since f continuous and $[a,b]$ compact implies $f([a,b])$ is compact and so bounded). Therefore, there exists $M_f > 0$ such that $|f(x)| \leq M_f$ for all $x \in [a,b]$.

For a uniformly bounded family $\mathcal{F}$, we can choose the bound M uniformly for $f \in \mathcal{F}$.

Theorem 50.5 (Arzela-Ascoli)**.** Let $\mathcal{F} \subseteq C([a,b])$. The following are equivalent:

1. $\mathcal{F}$ is uniformly bounded and equicontinuous.
2. Every sequence in $\mathcal{F}$ admits a convergent subsequence.

Remark 50.6 (Caution)**.** We cannot guarantee that the limit of the convergent subsequence belongs to $\mathcal{F}$, unless $\mathcal{F}$ is closed in $C([a,b])$. If $\mathcal{F}$ is closed in $C([a,b])$, then the theorem says that the following two statements are equivalent:

1. $\mathcal{F}$ is compact.
2. $\mathcal{F}$ is uniformly bounded and equicontinuous.

Proof. (2) $\Rightarrow$ (1):

Claim 50.7. $\mathcal{F}$ is totally bounded.

Subproof. Fix $\varepsilon > 0$, let $f_1 \in \mathcal{F}$.

- If $\mathcal{F} \subseteq B_\varepsilon(f_1)$, then $\mathcal{F}$ is totally bounded.
- If $\mathcal{F} \not\subseteq B_\varepsilon(f_1)$, then there exists $f_2 \in \mathcal{F}$ such that $d(f_1, f_2) \geq \varepsilon$.
- If $\mathcal{F} \subseteq B_\varepsilon(f_1) \cup B_\varepsilon(f_2)$, then $\mathcal{F}$ is totally bounded.
- If $\mathcal{F} \not\subseteq B_\varepsilon(f_1) \cup B_\varepsilon(f_2)$, then there exists $f_3 \in \mathcal{F}$ such that $d(f_1, f_3) \geq \varepsilon$ and $d(f_2, f_3) \geq \varepsilon$.

If the process terminates in finitely many steps, then $\mathcal{F}$ is totally bounded. Otherwise, we find $\{f_n\}_{n \geq 1} \subseteq \mathcal{F}$ such that $d(f_n, f_m) \geq \varepsilon$ for all $n \neq m$. This sequence does not admit a convergent subsequence, leading to a contradiction. ■

We now show that $\mathcal{F}$ is uniformly bounded. Since $\mathcal{F}$ is totally bounded, then there exists $n \geq 1$ and $f_1, \cdots, f_n \in \mathcal{F}$ such that

$$\mathcal{F} \subseteq \bigcup_{j=1}^{n} B_1(f_j) \subseteq B_r(f_1)$$

where $r = 1 + \max_{2 \leq j \leq n} d(f_1, f_j)$. In particular, for all $f \in \mathcal{F}$, $d(f, f_1) < r$. Note that f_1 is continuous on compact $[a, b]$, then there exists $M_{f_1} > 0$ such that $|f_1(x)| \leq M_{f_1}$ for all $x \in [a, b]$. Therefore, for $f \in \mathcal{F}$,

$$|f(x)| \leq |f(x) - f_1(x)| + |f_1(x)| \leq d(f, f_1) + M_{f_1} < r + M_{f_1}$$

for all $x \in [a, b]$. Therefore, $\mathcal{F}$ is uniformly bounded.

We now show that $\mathcal{F}$ is equicontinuous. Let $\varepsilon > 0$. As $\mathcal{F}$ is totally bounded, there exists $n \geq 1$ and $f_1, \cdots, f_n \in \mathcal{F}$ such that $\mathcal{F} \subseteq \bigcup_{j=1}^{n} B_{\frac{\varepsilon}{3}}(f_j)$. For each $1 \leq j \leq n$, f_j is uniformly continuous on $[a, b]$. Therefore, there exists $\delta_j(\varepsilon) > 0$ such that $|f_j(x) - f_j(y)| < \frac{\varepsilon}{3}$ for all $x, y \in [a, b]$ with $|x - y| < \delta_j(\varepsilon)$. Let $\delta_\varepsilon = \min_{1 \leq j \leq n} \delta_j(\varepsilon) > 0$. Fix $f \in \mathcal{F}$, then there exists $1 \leq j \leq n$ such that $f \in B_{\frac{\varepsilon}{3}}(f_j)$. Then for $x, y \in [a, b]$ with $|x - y| < \frac{\delta}{\varepsilon}$, we have

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| \\
&\leq 2d(f, f_j) + |f_j(x) - f_j(y)| \\
&< \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

This shows $\mathcal{F}$ is equicontinuous.

(1) $\Rightarrow$ (2): Let $\{f_n\}_{n \geq 1} \subseteq \mathcal{F}$. Since $\mathcal{F}$ is uniformly bounded, there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$ and all $f \in \mathcal{F}$. In particular, $|f_n(x)| \leq M$ for all $x \in [a, b]$ and all $n \geq 1$.

Let $\{r_n\}_{n \geq 1}$ denote an ennumeration of the rationals in $[a, b]$. Because $\{f_n(r_1)\}_{n \geq 1} \subseteq \mathbb{R}$ is bounded by M, then there exists a subsequence $\{f_n^{(1)}\}_{n \geq 1}$ of $\{f_n\}_{n \geq 1}$ such that $\{f_n^{(1)}(r_1)\}_{n \geq 1}$ converges. Similarly, because $\{f_n(r_2)\}_{n \geq 1} \subseteq \mathbb{R}$ is bounded by M, then there exists a subsequence $\{f_n^{(2)}\}_{n \geq 1}$ of $\{f_n\}_{n \geq 1}$ such that $\{f_n^{(2)}(r_2)\}_{n \geq 1}$ converges. Proceeding inductively, we find for every $k \geq 1$ that $\{f_n^{(k+1)}\}_{n \geq 1}$ is a subsequence of $\{f_n^{(k)}\}_{n \geq 1}$, and $\{f_n^{(k)}(r_k)\}_{n \geq 1}$ converges. We consider $\{f_n^{(n)}\}_{n \geq 1}$ as a subsequence of $\{f_n\}_{n \geq 1}$. For $n, m \geq k$, $f_n^{(n)}$ and $f_m^{(m)}$ are elements in $\{f_n^{(k)}\}_{n \geq 1}$, so $\{f_n^{(n)}\}_{n \geq 1}$ converges at r_x.[14]

[14] Note that the convergence is not uniform in k.

Fix $\varepsilon > 0$. As $\mathcal{F}$ is equicontinuous, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{3}$$

for all $x, y \in [a, b]$ such that $|x - y| < \delta$ and all $f \in \mathcal{F}$. In particular, we note that

$$|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$$

for all $x, y \in [a, b]$ such that $|x - y| < \delta$ and all $n \geq 1$.

Let $r_1, \cdots, r_N \in \mathbb{Q} \cap [a, b]$ such that $a = r_0 < r_1 < \cdots < r_N < r_{N+1} = b$, and $|r_{j+1} - r_j| < \delta$ for all $0 \leq j \leq N$. Note $N \sim \frac{|a-b|}{\delta}$. For each $1 \leq j \leq N$, there exists $n_j(\varepsilon) \in \mathbb{N}$ such that

$$|f_n^{(n)}(r_j) - f_m^{(m)}(r_j)| < \frac{\varepsilon}{3}$$

for all $n, m \geq n_j(\varepsilon)$. Let $n_\varepsilon = \max_{1 \leq j \leq N} n_j(\varepsilon)$. Note

$$|f_n^{(n)}(r_j) - f_m^{(m)}(r_j)| < \frac{\varepsilon}{3}$$

for all $n, m \geq n_\varepsilon$ and all $1 \leq j \leq N$. Now let $x \in [a, b]$, then there exists $1 \leq j \leq N$ such that $|x - r_j| < \delta$. Then

$$\begin{aligned}
|f_n^{(n)}(x) - f_m^{(m)}(x)| &\leq |f_n^{(n)}(x) - f_n^{(n)}(r_j)| + |f_n^{(n)}(r_j) - f_m^{(m)(r_j)}| + |f_m^{(m)}(r_j) - f_m^{(m)}(x)| \\
&< 2 \cdot \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

for all $n, m \geq n_\varepsilon$. Therefore, $\{f_n^{(n)}\}_{n \geq 1}$ is uniformly Cauchy and so uniformly convergent. $\square$

Definition 50.8 (Uniformly Cauchy)**.** We say a sequence of functions $\{f_n\}_{n \geq 1}$ is uniformly Cauchy in a metric space (X, d) if for any $\varepsilon > 0$, there exists $N > 0$ such that for all $x \in X$ we have $d(f_n(x), f_m(x)) < \varepsilon$ for all $n, m > N$.

Remark 50.9. One can replace $[a, b]$ by any other compact metric space (X, d).

51 Lecture 38: Remarks on Arzela-Ascoli Theorem, Oscillation of a Real Function

Remark 51.1. The compactness of the set on which the functions are defined is necessary in the Arzela-Ascoli theorem.

Example 51.2. Define

$$\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R} : |f(x) - f(y)| \le |x - y| \ \forall x, y \in \mathbb{R}, \sup_{x\in\mathbb{R}} |f(x)| \le 1\}.$$

Note that $\mathcal{F}$ is equicontinuous and uniformly bounded. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{1}{1+x^2}$.

Claim 51.3. $f \in \mathcal{F}$.

Subproof. Indeed, $\sup_{x\in\mathbb{R}} |f(x)| = \sup_{x\in\mathbb{R}} \frac{1}{1+x^2} = 1$. Moreover, for $x, y \in \mathbb{R}$,

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{1}{1+x^2} - \frac{1}{1+y^2} \right| \\ &= \frac{|x^2 - y^2|}{(1+x^2)(1+y^2)} \\ &= |x-y| \cdot \frac{|x+y|}{(1+x^2)(1+y^2)} \\ &\le |x-y| \left(\frac{|x|}{1+x^2} + \frac{|y|}{1+y^2} \right) \\ &\le |x-y| \end{aligned}$$

Therefore, $f \in \mathcal{F}$. ■

For $n \ge 1$, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = f(x-n)$. Note $f_n \in \mathcal{F}$ since $\sup_{x\in\mathbb{R}} |f_n(x)| = \sup_{x\in\mathbb{R}} \frac{1}{1+(x-n)^2} = 1$, and $|f_n(x) - f_n(y)| = |f(x-n) - f(y-n)| \le |(x-n) - (y-n)| = |x-y|$. Also note that $\{f_n\}_{n\ge1}$ converge pointwise to $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$ since $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{(x-n)^2} = 0$. However, $\{f_n\}_{n\ge1}$ does not admit a subsequence that converges uniformly since for all $n \ge 1$, we know $d(f_n, f) = \sup_{x\in\mathbb{R}} |f_n(x)| = 1$, which does not converge to 0.

Remark 51.4. Uniform boundedness is necessary in the Arzela-Ascoli theorem.

Example 51.5. Define

$$\mathcal{F} = \{f : [0,1] \to \mathbb{R}; f \text{ is a constant function}\}.$$

Note $\mathcal{F}$ is equicontinuous. For $n \ge 1$, let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = n$. This sequence

- shows that $\mathcal{F}$ is not uniformly bounded, and
- does not admit a convergent subsequence.

Remark 51.6. Equicontinuity is necessary in the Arzela-Ascoli theorem.

Example 51.7. Define

$$\mathcal{F} = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous and } \sup_{x\in[0,1]} |f(x)| \leq 1\}.$$

Note that the condition implies that $\mathcal{F}$ is uniformly bounded over the compact domain $[0,1]$.

Claim 51.8. $\mathcal{F}$ is not equicontinuous.

Subproof. For $n \geq 1$, let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = \sin(nx)$. Note $f_n \in \mathcal{F}$. Let $x_n = \frac{3\pi}{2n}$ and $y_n = \frac{\pi}{2n}$. Then $|x_n - y_n| = \frac{\pi}{n} \xrightarrow[n\to\infty]{} 0$, but $|f_n(x_n) - f_n(y_n)| = 2$. Therefore, $\{f_n\}_{n\geq 1}$ is not equicontinuous, and so $\mathcal{F}$ is not equicontinuous. ■

Claim 51.9. $\{f_n\}_{n\geq 1}$ does not admit a convergent subsequence.

Subproof. Assume, towards contradiction, that there exists a subsequence $\{f_{k_n}\}_{n\geq 1}$ of $\{f_n\}_{n\geq 1}$ that converges uniformly to $f : [0,1] \to \mathbb{R}$.

By the Weierstrass theorem, $f \in C([0,1])$. Also, since $f_{k_n}(0)$ for all $n \geq 1$ and $f_{k_n}(0) \xrightarrow[n\to\infty]{} f(0)$, then $f(0) = 0$, so for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x)| < \varepsilon$ for all $0 < x < \delta$ and all $n \geq n_\varepsilon$.

Now, since the subsequence converges to f uniformly, then there exists $n_\varepsilon \in \mathbb{N}$ such that $d(f_{k_n}, f) < \varepsilon$ for all $n \geq n_\varepsilon$. In particular, for $0 < x < \delta$, and $n \geq n_\varepsilon$, we have

$$f_{k_n}(x) \leq |f_{k_n}(x) - f(x)| + |f(x)| < d(f_{k_n}, f) + \varepsilon < 2\varepsilon.$$

Choosing $\varepsilon \leq \frac{1}{2}$ and large N, so that $N \geq n_{\varepsilon=\frac{1}{2}}$ and $\frac{\pi}{2N} < \delta_{\varepsilon=\frac{1}{2}}$, we find

$$1 = |f_{k_N}(\frac{\pi}{2N})| < 2\varepsilon \leq 1,$$

contradiction. ■

Definition 51.10 (Oscillation)**.** Let (X,d) be a metric space and let $f : X \to \mathbb{R}$ be a function. For $\varnothing \neq A \subseteq X$, the oscillation of f on A is

$$\omega(f,A) = \sup_{x\in A} f(x) - \inf_{x\in A} f(x) = \sup_{x,y\in A} [f(x) - f(y)] \geq 0.$$

Note that if $A \subseteq B$, then $\omega(f,A) \leq \omega(f,B)$.

For $x_0 \in X$, the oscillation of f at x_0 is given by

$$\omega(f,x_0) = \inf_{\delta>0} \omega(f, B_\delta(x_0)).$$

Proposition 51.11. Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be a function. Then f is continuous at a point $x_0 \in X$ if and only if $\omega(f, x_0) = 0$.

Proof. ($\Rightarrow$): Fox $\varepsilon > 0$. Since f is continuous at x_0, then there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \frac{\varepsilon}{4}$ for all $x \in B_\delta(x_0)$. Therefore,

$$f(x) - f(y) \leq |f(x) - f(x_0)| + |f(x_0) - f(y)| < \frac{\varepsilon}{2}$$

for all $x, y \in B_\delta(x_0)$. Therefore,

$$\omega(f, B_\delta(x_0)) = \sup_{x,y \in B_\delta(x_0)} [f(x) - f(y)] \leq \frac{\varepsilon}{2} < \varepsilon,$$

and so $\omega(f, x_0) \leq \omega(f, B_\delta(x_0)) < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, then $\omega(f, x_0) = 0$.

($\Leftarrow$): Fix $\varepsilon > 0$, then $\omega(f, x_0) = 0 < \varepsilon$, so there exists $\delta > 0$ such that $\omega(f, B_\delta(x_0)) < \varepsilon$. Therefore, $|f(x) - f(y)| < \varepsilon$ for all $, xy \in B_\delta(x_0)$, which is to say $|f(x) - f(x_0)| < \varepsilon$ for all $x \in B_\delta(x_0)$. Hence, f is continuous at x_0. □

Lemma 51.12. Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be a function. Then for any $\alpha > 0$,

$$\{x \in X : \omega(f, x) < \alpha\}$$

is open in X.

Proof. Fix $\alpha > 0$ and let $A = \{x \in X : \omega(f, x) < \alpha\}$. Fix $x_0 \in A$, then $\omega(f, x_0) = \inf_{\delta > 0} \omega(f, B_\delta(x_0)) < \alpha$, therefore there exists $\delta > 0$ such that $\omega(f, B_\delta(x_0)) < \alpha$.

Claim 51.13. $B_\delta(x_0) \subseteq A$. Consequentially, $x_0 \in \mathring{A}$ and so $A = \mathring{A}$, implying the set is open.

Subproof. Let $x \in B_\delta(x_0)$, then $r - \delta - d(x, x_0) > 0$ and $B_r(x) \subseteq B_\delta(x_0)$. Therefore, $\omega(f, B_r(x)) \leq \omega(f, B_\delta(x_0)) < \alpha$, so $\omega(f, x) \leq \omega(f, B_r(x)) < \alpha$, and so $x \in A$. ■

□

Remark 51.14. Let (X, d) be a metric space and let $f : X \to \mathbb{R}$ be a function, then

$$\begin{aligned}\{x \in X : f \text{ is continuous at } x\} &= \{x \in X : \omega(f, x) = 0\} \\ &= \bigcap_{n \geq 1} \{x \in X : \omega(f, x) < \frac{1}{n}\}.\end{aligned}$$

Let us define $G_n = \{x \in X : \omega(f, x) < \frac{1}{n}\}$. By the lemma, $G_n = \mathring{G}_n$ for all $n \geq 1$, also $G_{n+1} \subseteq G_n$ for all $n \geq 1$.

This observation allows us to prove that there are no functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous at every rational point and discontinuous at every irrational point.

52 Lecture 39: Weierstrass Approximation Theorem

We now give a proof sketch for Remark 51.14.

Proof Sketch. Assume, towards contradiction, that $f : \mathbb{R} \to \mathbb{R}$ is such a function. Then $\mathbb{Q} = \{x \in \mathbb{R} : f \text{ is continuous at } x\} = \bigcap\limits_{n\geq 1} G_n$ with G_n open in $\mathbb{R}$. Let $\{q_n\}_{n\geq 1}$ be an enumeration of $\mathbb{Q}$. For each $n \geq 1$, let $H_n = \mathbb{R}\backslash\{q_n\} = (-\infty, q_n) \cup (q_n, \infty)$. Note H_n is open and dense in $\mathbb{R}$ since $\bar{H}_n = \mathbb{R}$. Also, $\bigcap\limits_{n\geq 1} H_n = \mathbb{R}\backslash\mathbb{Q}$. Therefore, $\bigcap\limits_{n\geq 1} G_n \cap \bigcap\limits_{n\geq 1} H_n = \mathbb{Q} \cap \mathbb{R}\backslash\mathbb{Q} = \varnothing$. This contradicts the following property of $\mathbb{R}$ (as we take $\{A_n : n \geq 1\} = \{G_n : n \geq 1\} \cup \{H_n : n \geq 1\}$). $\square$

Exercise 52.1. If $\{A_n\}_{n\geq 1}$ is a countable collection of open and dense subsets of $\mathbb{R}$, then $\overline{\bigcap\limits_{n\geq 1} A_n} = \mathbb{R}$.

Theorem 52.2 (Weierstrass Approximation Theorem)**.** Fix $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{R}$ be a continuous function, then there exists a sequence of polynomials $\{P_n\}_{n\geq 1}$ with $\deg(P_n) \leq n$ for all $n \geq 1$ such that $P_n \xrightarrow[n\to\infty]{u} f$ on $[a, b]$.

Proof. First, we reduce to the case when $[a, b]$ to $[0, 1]$. Let $\varphi : [0, 1] \to [a, b]$ be defined by $\varphi(t) = a + t(b - a)$. Note φ is a continuous bijection with the inverse $\varphi^{-1} : [a, b] \to [0, 1]$ defined by $\varphi^{-1}(x) = \frac{x-a}{b-a}$, which is also a continuous function. Since $f : [a, b] \to \mathbb{R}$ is continuous, $f \circ \varphi : [0, 1] \to \mathbb{R}$ is continuous. If $\{P_n\}_{n\geq 1}$ is a sequence of polynomials with $\deg(P_n) \leq n$ such that $P_n \xrightarrow[n\to\infty]{u} f \circ \varphi$ on $[0, 1]$, then $P_n \circ \varphi^{-1} \xrightarrow[n\to\infty]{u} f$ on $[a, b]$. Indeed, by taking $x = \varphi(t)$, then

$$\sup_{x\in[a,b]} |(P_n \circ \varphi^{-1})(x) - f(x)| = \sup_{t\in[0,1]} |P_n(t) - (f \circ \varphi)(t)| \xrightarrow[n\to\infty]{} 0.$$

Therefore, we may assume $f : [0, 1] \to \mathbb{R}$ is continuous.

Define the Bernstein polynomials via

$$P_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1 - x)^{n-k},$$

and note that $\deg(P_n) \leq n$. Note that if f is a constant, say $f(x) = c$ for all $x \in [0, 1]$, then $P_n(x) = c \sum\limits_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} = c(x + 1 - x)^n = c$ for all $x \in [0, 1]$ and $n \geq 1$. We want to show $P_n \xrightarrow[n\to\infty]{u} f$ on $[0,1]$. Fix $x \in [0,1]$, consider

$$\begin{aligned}
|f(x)-P_n(x)| &= \left| f(x)\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} - \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}\right| \\
&= \left|\sum_{k=0}^{n}\left[f(x)-f\left(\frac{k}{n}\right)\right]\binom{n}{k}x^k(1-x)^{n-k}\right| \\
&\leq \sum_{k=0}^{n}\left|f(x)-f\left(\frac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^{n-k}.
\end{aligned}$$

To estimate the sum, we use the following ideas:

- when $\frac{k}{n}$ is close to x, we use the continuity of f,
- when $\frac{k}{n}$ is far from x, we use the fact that $g : x \mapsto x^k(1-x)^{n-k}$ has a local maximum at $x = \frac{k}{n}$, and note that

$$\begin{aligned}
g'(x) &= kx^{k-1}(1-x)^{n-k} - (n-k)x^k(1-x)^{n-k-1} \\
&= x^{k-1}(1-x)^{n-k-1}\{k(1-x)-(n-k)x\} \\
&= x^{k-1}(1-x)^{n-k-1}\{k-nx\},
\end{aligned}$$

  and this is positive if $x < \frac{k}{n}$, is 0 if $x = \frac{k}{n}$, and is negative if $x > \frac{k}{n}$.

Because $f : [0,1] \to \mathbb{R}$ is continuous, then f is uniformly continuous. Fix $\varepsilon > 0$, then there exists $\delta > 0$ such that $|f(x)-f(y)| < \varepsilon$ whenever $x, y \in [0,1]$ satisfies $|x-y| < \delta$. Again, since f is continuous, then f is bounded. Let $M > 0$ be such that $|f(x)| \leq M$ for all $x \in [0,1]$. We can estimate

$$\begin{aligned}
|f(x)-P_n(x)| \leq &\sum_{\substack{0\leq k\leq n \\ |x-\frac{k}{n}|<\delta}}\left|f(x)-f\left(\frac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^{n-k} \\
&+ \sum_{\substack{0\leq k\leq n \\ |x-\frac{k}{n}|\geq\delta}}\left|f(x)-f\left(\frac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^{n-k} \\
&\leq \varepsilon\sum_{0\leq k\leq n}\binom{n}{k}x^k(1-x)^{n-k} + 2M\sum_{0\leq k\leq n}\frac{(x-\frac{k}{n})^2}{\delta^2}\binom{n}{k}x^k(1-x)^{n-k} \\
&\leq \varepsilon + \frac{2M}{n^2\delta^2}\sum_{k=0}^{n}(nx-k)^2\binom{n}{k}x^k(1-x)^{n-k}.
\end{aligned}$$

To estimate this term, we see that

$$\begin{aligned}
\sum_{k=0}^{n}(nx-k)^2\binom{n}{k}x^k(1-x)^{n-k} &= n^2x^2\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}+2nx\sum_{k=0}^{n}\frac{kn!}{k!(n-k)!}x^k(1-x)^{n-k}\\
&\quad+\sum_{k=0}^{n}k^2\frac{n!}{k!(n-k)!}x^k(1-x)^{n-k}\\
&= n^2x^2+2nx\sum_{k=0}^{n}\frac{n!}{(k-1)!(n-k)!}x^k(1-x)^{n-k}\\
&\quad+\sum_{k=0}^{n}\frac{kn!}{(k-1)!(n-k)!}x^k(1-x)^{n-k},
\end{aligned}$$

which can be evaluated by (taking $l=k-1$)

$$\begin{aligned}
\sum_{k=0}^{n}\frac{n!}{(k-1)!(n-k)!}x^k(1-x)^{n-k} &= x\sum_{k=1}^{n}\frac{n!}{(k-1)!(n-k)!}x^{k-1}(1-x)^{n-k}\\
&= nx\sum_{l=0}^{n-1}\frac{(n-1)!}{l!(n-1-l)!}x^l(1-x)^{n-1-l}\\
&= nx(x+1-x)^{n-1}\\
&= nx
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{n}\frac{kn!}{(k-1)!(n-k)!}x^k(1-x)^{n-k} &= nx\sum_{k=1}^{n}\frac{k(n-1)!}{(k-1)!(n-k)!}x^{k-1}(1-x)^{n-k}\\
&= nx\sum_{k=1}^{n}\frac{(k-1+1)(n-1)!}{(k-1)!(n-k)!}x^{k-1}(1-x)^{n-k}\\
&= n(n-1)x^2\sum_{k=2}^{n}\frac{(n-2)!}{(k-2)!(n-k)!}x^{k-2}(1-x)^{n-k}\\
&\quad+nx\sum_{k=1}^{n}\frac{(n-1)!}{(k-1)!(n-k)!}x^{k-1}(1-x)^{n-k}\\
&= n(n-1)x^2+nx.
\end{aligned}$$

Therefore,

$$\sum_{k=0}^{n}(nx-k)^2\binom{n}{k}x^k(1-x)^{n-k}=n^2x^2-2n^2x^2+n(n-1)x^2+nx=nx(1-x).$$

We get

$$\begin{aligned}|f(x) - P_n(x)| &\leq \varepsilon + \frac{2M}{n^2\delta^2} \cdot nx(1-x) \\ &\leq \varepsilon + \frac{2M}{n\delta^2} \cdot \sup_{x\in[0,1]} x(1-x) \\ &\leq \varepsilon + \frac{M}{2\delta^2 n} \\ &< 2\varepsilon\end{aligned}$$

provided $n > \frac{<}{2\delta^2\varepsilon}$, so the convergence goes uniformly in x. We conclude that $P_n \xrightarrow[n\to\infty]{u} f$ on $[0,1]$. $\square$

53 Homework 14

Exercise 53.1. For $n \geq 1$, let $f_n : [1,2] \to \mathbb{R}$ be defined as follows: for any $x \in [1,2]$, $f_1(x) = 0$ and $f_{n+1}(x) = \sqrt{x + f_n(x)}$ for $n \geq 1$. Prove that $\{f_n\}_{n\geq 1}$ converges uniformly to $f(x) = \frac{1+\sqrt{1+4x}}{2}$.

Exercise 53.2. For $n \geq 1$, let $f_n : [0,\infty) \to \mathbb{R}$ be defined by $f_n(x) = \frac{nx^2+1}{nx+1}$. Study the pointwise and uniform convergence of f_n on each of the intervals $[0,\infty)$, $(0,\infty)$, and $[1,\infty)$.

Exercise 53.3. Let $f : [0,1] \to \mathbb{R}$. We say that f is Hölder continuous of order $\alpha \in (0,1)$ and write $f \in C^\alpha([0,1])$ if the value $||f||_{C^\alpha} := \sup\{|f(x)| : x \in [0,1]\} + \sup\{\frac{|f(x)-f(y)|}{|x-y|^\alpha} : x, y \in [0,1] \text{ with } x \neq y\} < \infty$. For $f, g \in C^\alpha([0,1])$, we define $d(f,g) = ||f-g||_{C^\alpha}$.

(a) Show that $(C^\alpha([0,1]), d)$ is a complete metric space.

(b) Prove that any bounded sequence in $C^{\frac{1}{2}}([0,1])$ admits a subsequence that converges in $C^{\frac{1}{3}}([0,1])$.

Exercise 53.4. Let $f : [1,\infty) \to \mathbb{R}$ be a continuous function such that $\lim_{x\to\infty} |f(x)| = 0$. For $n \geq 1$, let $g_n : [1,\infty) \to \mathbb{R}$ be given by $g_n(x) = f(nx)$. Show that $\{f_n\}_{n\geq 1}$ is equicontinuous on $[1,\infty)$.

Exercise 53.5. For $n \geq 1$, let $f_n : [0,1] \to \mathbb{R}$ be given by $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$. Show that $\{f_n\}_{n\geq 1}$ is equicontinuous on $[0,1]$.

Exercise 53.6. Let $f : [0,1] \to \mathbb{R}$ be a function with Darboux's property such that for any $y \in \mathbb{R}$, the set $f^{-1}(\{y\})$ is closed. Prove that f is continuous.

Exercise 53.7. Let $f, g : [a, b] \to [a, b]$ be two continuous functions that satisfy $f \circ g = g \circ f$. Show that there exists $x_0 \in [a, b]$ such that $f(x_0) = g(x_0)$.

Hint: Use the fact that Exercise 49.5 guarantees the existence of a point $x_1 \in [a, b]$ such that $f(x_1) = x_1$. Show that $g(x_1), g \circ g(x_1), \cdots$ form a sequence of fixed points for f.

54 Lecture 40: Algebra, Stone-Weierstrass Theorem

Corollary 54.1. Let $M > 0$, then there exists a sequence of polynomials $\{P_n\}_{n\geq 1}$ such that

- $\deg(P_n) \leq n$ for all $n \geq 1$,
- $P_n(0) = 0$ for all $n \geq 1$,
- and $P_n \xrightarrow[n\to\infty]{u} |x|$ on $[-M, M]$.

Proof. Let $f : [-M, M] \to \mathbb{R}$ be defined by $f(x) = |x|$. Then f is continuous and $[-M, M]$ compact. By the Weierstrass approximation theorem, there exists $\{Q_n\}_{n\geq 1}$ sequence of polynomials such that $\deg(Q_n) \leq n$ for all $n \geq 1$ and $Q_n \xrightarrow[n\to\infty]{u} f$ on $[-M, M]$. Note that this implies $Q_n(0) \xrightarrow[n\to\infty]{} f(0) = 0$. Let $P_n = Q_n(x) - Q_n(0)$, then $\deg(P_n) \leq n$ and $P_n(0) = 0$ for all $n \geq 1$. For $x \in [M, M]$, note $|P_n(x) - f(x)| \leq |Q_n(x) - f(x)| + |Q_n(0)| \leq d(Q_n, f) + |Q_n(0)|$, so $d(P_n, f) \leq d(Q_n, f) + |Q_n(0)| \xrightarrow[n\to\infty]{} 0$. □

Definition 54.2 (Algebra)**.** Let (X, d) be a metric space and let $\mathcal{A} \subseteq \{f : X \to \mathbb{R}\}$ to be a set of functions. We say that $\mathcal{A}$ is an algebra if

1. $f + g \in \mathcal{A}$ for all $f, g \in \mathcal{A}$,
2. $fg \in \mathcal{A}$ for all $f, g \in \mathcal{A}$,
3. $\lambda f \in \mathcal{A}$ for all $f \in \mathcal{A}$ and all $\lambda \in \mathbb{R}$.[15]

We say that the algebra $\mathcal{A}$ separates points if whenever $x, y \in X$ with $x \neq y$, then there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

We say that the algebra $\mathcal{A}$ vanishes at no point in X if for all $x \in X$, there exists $f \in \mathcal{A}$ such that $f(x) \neq 0$.

Lemma 54.3. Let (X, d) be a compact metric space and let $\mathcal{A} \subseteq C(X)$ be an algebra. Then its closure $\bar{\mathcal{A}}$ with respect to the uniform topology (i.e., the topology of uniform convergence) is also an algebra.

[15]Sometimes we change all the $\mathbb{R}$'s above to $\mathbb{C}$, and the definition still stands.

Proof. Let $f, g \in \mathcal{A}$. Then there exists $f_n \in \mathcal{A}$ such that $f_n \xrightarrow[n\to\infty]{u} f$ on X and there exists $g_n \in \mathcal{A}$ such that $g_n \xrightarrow[n\to\infty]{u} g$ on X. Now $d(f_n + g_n, f + g) \leq d(f_n, f) + d(g_n, g) \xrightarrow[n\to\infty]{} 0$ and $f_n + g_n \in \mathcal{A}$ since this is an algebra, and so $f + g \in \bar{\mathcal{A}}$. Similarly, for $\lambda \in \mathbb{R}$, $d(\lambda f_n, \lambda f) \leq |\lambda| d(f_n, f) \xrightarrow[n\to\infty]{} 0$, and since $\lambda f_n \in \mathcal{A}$, we know $\lambda f \in \bar{\lambda A}$.

We now have

$$\begin{aligned} d(f_n g_n, fg) &= \sup_{x\in X} |f_n(x)g_n(x) - f(x)g(x)| \\ &\leq \sup_{x\in X} [|f_n(x) - f(x)| \cdot |g_n(x)| + |f(x)| \cdot |g_n(x) - g(x)|] \\ &\leq d(f_n, f) \sup_{x\in X} |g_n(x)| + d(g_n, g) \cdot \sup_{x\in X} |f(x)|. \end{aligned}$$

Because $f_n \xrightarrow[n\to\infty]{u} f$ on X, and $f_n \in C(X)$, then by Weierstrass theorem, $f \in C(X)$, and since X is compact, then there exists $M_1 > 0$ such that $\sup_{x\in X} |f(x)| \leq M_1$. Similarly, since $g \in C(X)$, we conclude there exists $M_2 > 0$ such that $\sup_{x\in X} |g(x)| \leq M_2$. We now know $d(g_n, 0) \leq d(g_n, g) + d(g, 0) \leq 1 + M_2$ for n large enough, i.e., $n \geq n_1$. Now let $M_3 = \max\{1 + M_2, d(g_1, 0), \cdots, d(g_{n_1}, 0)\}$, and note that all values are finite since $g_1, \cdots, g_{n_1} \in C(X)$, therefore $d(g_n, 0) \leq M_3$ for all $n \geq 1$. Thus $d(f_n \cdot g_n, fg) \leq d(f_n, f) \cdot M_3 + d(g_n, g) \cdot M_1 \xrightarrow[n\to\infty]{} 0$, and since $f_n \cdot g_n \in \mathcal{A}$, then $fg \in \bar{\mathcal{A}}$. $\square$

Lemma 54.4. Let (X, d) be a compact metric space and let $\mathcal{A} \subseteq C(X)$ be an algebra that separates points and vanishes at no point in X. Then for all $\alpha, \beta \in \mathbb{R}$ and all distinct points $x_1 \neq x_2 \in X$, there exists $f \in \mathcal{A}$ such that $f(x_1) = \alpha$ and $f(x_2) = \beta$.

Proof. Fix α, β, x_1, x_2 as described in the lemma. We would like $f(x) = \alpha \cdot \frac{u(x)}{u(x_1)} + \beta \cdot \frac{v(x)}{v(x_2)}$ for $u, v \in \mathcal{A}$ such that $u(x_1) \neq 0$, $u(x_2) = 0$, and $v(x_1) = 0$, $v(x_2) \neq 0$. Then because $\mathcal{A}$ is an algebra, then $f \in \mathcal{A}$ is the desired function. As $\mathcal{A}$ separates points, there exists $g \in \mathcal{A}$ such that $g(x_1) \neq g(x_2)$. As $\mathcal{A}$ vanishes at no point in x, there exists $h \in \mathcal{A}$ such that $h(x_1) \neq 0$ and there exists $k \in \mathcal{A}$ such that $k(x_2) \neq 0$. Then we define $u(x) = [g(x) - g(x_2)] \cdot h(x) \in \mathcal{A}$ and $v(x) = [g(x) - g(x_1)] \cdot k(x) \in \mathcal{A}$. $\square$

Theorem 54.5 (Stone-Weierstrass)**.** Let (X, d) be a compact metric space and let $\mathcal{A} \subseteq C(X)$ be an algebra that separates points and vanishes at no point in X. Then $\mathcal{A}$ is dense in $C(X)$, i.e., $\bar{\mathcal{A}} = C(X) = \{f : X \to \mathbb{R} : f \text{ continuous}\}$.

Proof. We want to show that for all $f \in C(X)$ and all $\varepsilon > 0$, there exists $g \in \mathcal{A}$ such that $d(f, g) < \varepsilon$.

Claim 54.6. If $f \in \bar{\mathcal{A}}$, then $|f| \in \bar{\mathcal{A}}$.

Subproof. Let $f \in \bar{\mathcal{A}}$, then there exists $f_n \in \mathcal{A}$ such that $f_n \xrightarrow[n\to\infty]{u} f$ on X, and since $f_n \in C(X)$, then $f \in C(X)$. Since X is compact, there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in X$. By Corollary 54.1, there exists a sequence of polynomials $\{P_n\}_{n\geq 1}$ with $\deg(P_n) \leq n$ for all $n \geq 1$ such that $P_n \xrightarrow[n\to\infty]{u} |x|$ on $[-M, M]$, and $P_n(0) = 0$. Therefore, $P_n(f) \xrightarrow[n\to\infty]{u} |f|$ on X. Now, if $P_n(x) = \sum_{k=1}^{n} c_k x^k$, then $P_n(f) = \sum_{k=1}^{n} c_k f^k \in \mathcal{A}$, then we know $|f| \in \bar{\mathcal{A}}$. ■

Claim 54.7. If $f, g \in \bar{\mathcal{A}}$, then $\max\{f, g\}, \min\{f, g\} \in \bar{\mathcal{A}}$.

Subproof. Indeed, $\max\{f, g\} = \frac{f+g}{2} + \frac{|f+g|}{2} \in \bar{\mathcal{A}}$, using the fact that $\bar{\mathcal{A}}$ is an algebra and Claim 54.6. Similarly, we conclude that $\min\{f, g\} = \frac{f+g}{2} - \frac{|f-g|}{2} \in \bar{\mathcal{A}}$. ■

Remark 54.8. In fact, $\bar{\mathcal{A}}$ separates points and vanishes at no point in X.

Claim 54.9. For any $f \in C(X)$, fixed point $x \in X$, and $\varepsilon > 0$, there exists $g \in \bar{\mathcal{A}}$ such that $g(x) = f(x)$ and $g(y) > f(y) - \varepsilon$ for all $y \in X$.

Subproof. For any $y \in X$, there exists $h_y \in \bar{\mathcal{A}}$ such that $h_y(x) = f(x)$ and $h_y(y) = f(y)$. As $h_y \in \bar{\mathcal{A}}$, h_y is continuous. Thus, $h_y - f$ is continuous at y. Therefore, there exists $\delta_y > 0$ such that $||h_y(z) - f(z)| < \varepsilon$ for all $z \in B_{\delta_y}(y)$. In particular, $h_y(z) > f(z) - \varepsilon$ for all $z \in B_{\delta_y}(y)$. Note that the compact space $X = \bigcup_{y\in X} B_{\delta_y}(y)$, then there exists $N \geq 1$ and $y_1, \cdots, y_N \in X$ such that $X = \bigcup_{n=1}^{N} B_{\delta_n}(y_n)$ where $\delta_n = \delta_{y_n}$. By Claim 54.7, we can take $g = \max\{h_{y_1}, \cdots, h_{y_N}\} \in \bar{\mathcal{A}}$. By construction, $g(x) = f(x)$. Also, if $y \in X$, there exists $1 \leq n \leq N$ such that $y \in B_{\delta_n}(y_n)$, so $g(y) \geq h_{y_n}(y) > f(y) - \varepsilon$. ■

Claim 54.10. For all $f \in C(X)$ and $\varepsilon > 0$, there exists $g \in \bar{\mathcal{A}}$ such that $d(f, g) < \varepsilon$.

Subproof. Fix $f \in C(X)$ and $\varepsilon > 0$. For $x \in X$, let $g_x \in \bar{\mathcal{A}}$ be the function given by Claim 54.9. In particular, $g_x(x) = f(x)$, and $g_x(y) > f(y) - \varepsilon$ for all $y \in X$. As $g_x \in \bar{\mathcal{A}}$, the function $g_x - f$ is continuous at x. Therefore, there exists $\delta_x > 0$ such that $|g_x(y) - f(y)| < \varepsilon$ for all $y \in B_{\delta_x}(x)$. In particular, $g_x(y) < f(y) + \varepsilon$ for all $y \in B_{\delta_x}(x)$. Again, because the compact space $X = \bigcup_{x\in X} B_{\delta_x}(x)$, then there exists $N \geq 1$ and $x_1, \cdots, x_N \in X$ such that $X = \bigcup_{n=1}^{N} B_{\delta_n}(x_n)$ where $\delta_n = \delta_{x_n}$. Again, by Claim 54.7, we can take $g = \min\{g_{x_1}, \cdots, g_{x_N}\} \in \bar{\mathcal{A}}$. Now for $y \in X$, there exists $1 \leq n \leq N$ such that $y \in B_{\delta_n}(x_n)$ and so $g(y) \leq g_{x_n}(y) < f(y) + \varepsilon$. Moreover, as $g_{x_n}(y) > f(y) - \varepsilon$ for all $y \in X$ and all $1 \leq n \leq N$, we have $g(y) > f(y) - \varepsilon$ for all $y \in X$. This shows $C(X) \subseteq \bar{\bar{\mathcal{A}}} = \bar{\mathcal{A}} \subseteq C(X)$. ■

□

55 Lecture 41: Differentiation

Definition 55.1 (Limit)**.** Let (X, d_X) and (Y, d_Y) be metric space, and let $\varnothing \neq A \subseteq X$, let $f : A \to Y$. For $x_0 \in A'$ and $y_0 \in Y$, we write $f \xrightarrow[x \to x_0]{} y_0$, or $\lim_{x \to x_0} f(x) = y_0$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), y_0) < \varepsilon$ whenever $0 < d_X(x, x_0) < \delta$.

Equivalently, we say $\lim_{x \to x_0} f(x) = y_0$ if $\lim_{n \to \infty} f(x_n) = y_0$ for every sequence $\{x_n\}_{n \geq 1} \subseteq A \backslash \{x_0\}$ such that $x_n \xrightarrow[n \to \infty]{d_X} x_0$.

Note also that if $x_0 \in A' \cap A$, then f is continuous at x_0 if and only if $\lim_{x \to x_0} f(x) = f(x_0)$.

Exercise 55.2. Let (X, d) be a metric space, $\varnothing \neq A \subseteq X$, $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be functions. Assume that at a point $x_0 \in A'$ we have $\lim_{x \to x_0} f(x) = \alpha$ and $\lim_{x \to x_0} g(x) = \beta$. Then

1. $\lim_{x \to x_0} (\lambda f(x)) = \lambda \alpha$ for all $\lambda \in \mathbb{R}$,
2. $\lim_{x \to x_0} (f(x) + g(x)) = \alpha + \beta$,
3. $\lim_{x \to x_0} (f(x) \cdot g(x)) = \alpha\beta$,
4. if $\beta \neq 0$, then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\alpha}{\beta}$.

Definition 55.3 (Differentiable)**.** Let I be an open interval and let $f : I \to \mathbb{R}$ be a function. We say that f is differentiable at $a \in I$ if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists and is finite, in which case we denote it $f'(a)$.

Example 55.4. Fix $n \geq 1$ and let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^n$. For $a \in \mathbb{R}$ and $x \neq a$,

$$\frac{f(x) - f(a)}{x - a} = \frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2} + \cdots + a^{n-1} \xrightarrow[x \to a]{} na^{n-1}.$$

Therefore, f is differentiable at a and $f'(a) = na^{n-1}$.

Theorem 55.5. Let I be an open interval and let $f : I \to \mathbb{R}$ be differentiable at $a \in I$. Then f is continuous at a.

Proof. For $x \in I \backslash \{a\}$, we write $f(x) = \frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a)$. Because f is differentiable at a, then $\frac{f(x) - f(a)}{x - a} \xrightarrow[x \to a]{} f'(a)$, and since $(x - a) \xrightarrow[x \to a]{} 0$, $f(a) \xrightarrow[x \to a]{} f(a)$, then $f(x) \xrightarrow[x \to a]{} f(a)$. $\square$

Theorem 55.6. Let I be an open interval and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be two functions differentiable at $a \in I$. Then

1. for all $\lambda \in \mathbb{R}$, λf is differentiable at a and $(\lambda f)'(a) = \lambda f'(a)$,

2. $f+g$ is differentiable at a and $(f+g)'(a)=f'(a)+g'(a)$,

3. $f\cdot g$ is differentiable at a and $(f\cdot g)'(a)=f'(a)g(a)+f(a)g'(a)$,

4. $\frac{f}{g}$ is differentiable at a if $g(a)\neq 0$ and $\left(\frac{f}{g}\right)'(a)=\frac{f'(a)g(a)-f(a)g'(a)}{g^2(a)}$.

Proof. 1. For $x\neq a$, we have

$$\frac{\lambda f(x)-\lambda f(a)}{x-a}=\lambda\cdot\frac{f(x)-f(a)}{x-a}\xrightarrow[x\to a]{}\lambda f'(a).$$

2. For $x\neq a$, we have

$$\frac{(f(x)+g(x))=(f(a)+g(a))}{x-a}=\frac{f(x)-f(a)}{x-a}+\frac{g(x)-g(a)}{x-a}\xrightarrow[x\to a]{}f'(a)+g'(a).$$

3. For $x\neq a$, we have

$$\frac{f(x)g(x)-f(a)g(a)}{x-a}=\frac{f(x)-f(a)}{x-a}\cdot g(x)+f(a)\cdot\frac{g(x)-g(a)}{x-a}.$$

As $x\to a$, the four terms converge to $f'(a)$, $g(a)$, $f(a)$, and $g'(a)$, respectively, since g is continuous at a. Therefore, the entire expression converges to $f'(a)g(a)+f(a)g'(a)$ as x converges to a.

4. For $x\neq a$, we have

$$\frac{\frac{f(x)}{g(x)}-\frac{f(a)}{g(a)}}{x-a}=\frac{f(x)-f(a)}{x-a}\cdot\frac{1}{g(x)}+f(a)\cdot\frac{g(a)-g(x)}{x-a}\cdot\frac{1}{g(x)}\cdot\frac{1}{g(a)}.$$

Now, note that for $x\to a$, we have $\frac{f(x)-f(a)}{x-a}\to f'(a)$, $\frac{1}{g(x)}\to\frac{1}{g(a)}$, $\frac{g(a)-g(x)}{x-a}\to -g'(a)$, and $\frac{1}{g(x)}\to\frac{1}{g(a)}$. Combining these expressions above, we conclude that

$$\frac{\frac{f(x)}{g(x)}-\frac{f(a)}{g(a)}}{x-a}\xrightarrow[x\to a]{}\frac{f'(a)}{g(a)}-\frac{g'(a)f(a)}{g^2(a)}=\frac{f'(a)g(a)-f(a)g'(a)}{g^2(a)}.$$

□

56 Lecture 42: Chain Rule, Rolle's Theorem, Mean Value Theorem

Theorem 56.1 (Chain Rule). Let I and J be two open intervals and let $f:I\to\mathbb{R}$ and $g:J\to\mathbb{R}$ be two functions. Assume that f is differentiable at $a\in I$ and that g is differentiable at $f(a)\in J$, then $g\circ f$ is well-defined on a neighborhood of a, $g\circ f$ is differentiable at a, and $(g\circ f)'(a)=g'(f(a))\cdot f'(a)$.

Proof. Since $f(a) \in J$ and J is open, then there exists $\varepsilon > 0$ such that $(f(a)-\varepsilon, f(a)+\varepsilon) \subseteq J$. Since f is differentiable at a, then f is continuous at a, and so there exists $\delta > 0$ such that $f((a-\delta, a+\delta) \cap I) \subseteq (f(a)-\varepsilon, f(a)+\varepsilon)$. As $a \in I$ and I is open, shrinking δ if necessary, we may assume that $(a-\delta, a+\delta) \subseteq I$. Then $g \circ f$ is well-defined on $(a-\delta, a+\delta)$. In particular, we see

$$(a-\delta, a+\delta) \subseteq I \xrightarrow{f} (f(a)-\varepsilon, f(a)+\varepsilon) \subseteq J \xrightarrow{g} \mathbb{R}.$$

Remark 56.2. One may consider the following argument, which is incorrect:

$$\frac{g(f(x)) - g(f(a))}{x-a} = \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x-a}.$$

Note that (because f is continuous at a) the first term converges to $g'(f(a))$ when x approaches a, and the second term converges to $f'(a)$ when x approaches a.

This argument is incorrect because the oscillation of g around a is unaccounted for.

Instead, we argue as follows. Define $h : J \to \mathbb{R}$ by

$$h(y) = \begin{cases} \frac{g(y)-g(f(a))}{y-f(a)}, & \text{if } y \in J \backslash \{f(a)\} \\ g'(f(a)), & \text{if } y = f(a) \end{cases}$$

then since g is differentiable at $f(a)$, then h is continuous at $f(a)$. Moreover, we can write $g(y) - g(f(a)) = h(y) \cdot (y - f(a))$ for all $y \in J$. For $x \in (a-\delta, a+\delta)$, we have $f(x) \in J$, so for $x \in (a-\delta, a+\delta) \backslash \{a\}$, we have

$$\frac{g(f(x)) - g(f(a))}{x-a} = h(f(x)) \cdot \frac{f(x) - f(a)}{x-a}$$

and note that the first term converges to $h(f(a))$ for $x \to a$, and the second term converges to $f'(a)$ for $x \to a$, therefore

$$\lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x-a} = h(f(a)) \cdot f'(a) = g'(f(a)) \cdot f'(a).$$

$\square$

Lemma 56.3. Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. If f is increasing, then $f'(x) \geq 0$ for all $x \in (a, b)$; if f is decreasing, then $f'(x) \leq 0$ for all $x \in (a, b)$.

Proof. Without loss of generality, assume f is increasing.[16] Fix $x \in (a, b)$ and let $\{x_n\}_{n \geq 1}$ be an increasing sequence from (a, b) with $\lim\limits_{n \to \infty} x_n = x$, then $f'(x) = \lim\limits_{n \to \infty} \frac{f(x_n) - f(x)}{x_n - x} \geq 0$. $\square$

[16] The argument for decreasing function f is similar, just by replacing f with $-f$ in the following arguments.

Theorem 56.4. Let $f : (a, b) \to \mathbb{R}$ be a function. Assume that $x_0 \in (a, b)$ is a point of local maximum or local minimum for f. Assume also that f is differentiable at x_0, then $f'(x_0) = 0$.

Proof. Assume that x_0 is a point of local maximum for f.[17] Now, there exists $\delta > 0$ such that $f(x) \le f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap (a, b)$. For $x_n \in (x_0 - \delta, x_0) \cap (a, b)$ such that $x_n \xrightarrow[n\to\infty]{} x_0$, we have $f'(x_0) = \lim_{n\to\infty} \frac{f(x_n)-f(x_0)}{x_n - x_0} \ge 0$. Similarly, for $y_n \in (x_0, x_0 + \delta) \cap (a, b)$ such that $y_n \xrightarrow[n\to\infty]{} x_0$, we have $f'(x_0) = \lim_{n\to\infty} \frac{f(y_n)-f(x_0)}{y_n - x_0} \le 0$. Combining the two expressions, we conclude that $f'(x_0) = 0$. $\square$

Theorem 56.5 (Rolle)**.** Let $f : [a, b] \to \mathbb{R}$ be a function which is continuous on $[a, b]$, differentiable on (a, b), and such that $f(a) = f(b)$. Then there exists some $x \in (a, b)$ such that $f'(x) = 0$.

Proof. Because $f : [a, b] \to \mathbb{R}$ is continuous on the compact interval $[a, b]$, then there exists some $x_0, y_0 \in [a, b]$ such that $f(x_0) = \sup_{x\in[a,b]} f(x)$ and $f(y_0) = \inf_{x\in[a,b]} f(x)$, and so $f(y_0) \le f(x) \le f(x_0)$ for all $x \in [a, b]$.

Suppose $\{x_0, y_0\} \subseteq \{a, b\}$, then because $f(a) = f(b)$, we conclude that $f(x_0) = f(y_0)$, and so f must be a constant function, therefore $f'(x) = 0$ for all $x \in (a, b)$.

Suppose $\{x_0, y_0\} \nsubseteq \{a, b\}$, then either $x_0 \notin \{a, b\}$ or $y_0 \notin \{a, b\}$. Say it is $x_0 \notin \{a, b\}$, then $x_0 \in (a, b)$. By Theorem 56.4, we have $f'(x_0) = 0$. $\square$

Theorem 56.6 (Mean Value Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a.b]$ and differentiable on (a, b). Then there exists some $y \in (a, b)$ such that $f'(y) = \frac{f(b)-f(a)}{b-a}$.

Remark 56.7. The mean value theorem implies Rolle's theorem. We will see from the proof that Rolle's theorem implies the mean value theorem, so the two are equivalent.

Proof. We define $l : [a, b] \to \mathbb{R}$ by $l(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x - a)$. Note that l is continuous on $[a, b]$, differentiable on (a, b), and $l'(x) = \frac{f(b)-f(a)}{b-a}$ for all $x \in (a, b)$. Let $g : [a, b] \to \mathbb{R}$ be defined by $g(x) = f(x) - l(x)$, then g is continuous on $[a, b]$, differentiable on (a, b), and $g(a) = 0 = g(b)$. Now Rolle's theorem implies that there exists $y \in (a, b)$ such that $g'(y) = 0$, and so $f'(y) - l'(y) = 0$, therefore

$$f'(y) = \frac{f(b) - f(a)}{b - a}.$$

$\square$

[17] Again, if x_0 is a local minimum instead, we can replace f by $-f$ in the following argument.

Corollary 56.8. If $f : (a, b) \to \mathbb{R}$ is differentiable and $f'(x) = 0$ for all $x \in (a, b)$, then f is a constant.

Proof. Assume f is not a constant, then there exists $a < x_1 < x_2 < b$ such that $f(x_1) \neq f(x_2)$, so f is continuous on $[x_1, x_2]$, differentiable on (x_1, x_2). By the Mean Value Theorem, there exists $y \in (x_1, x_2)$ such that

$$f'(y) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} \neq 0,$$

contradiction. $\square$

Corollary 56.9. If $f, g : (a, b) \to \mathbb{R}$ are differentiable such that $f'(x) = g'(x)$ for all $x \in (a, b)$, then there exists $c \in \mathbb{R}$ such that $f(x) = g(x) + c$ for all $x \in (a, b)$.

57 Homework 15

Exercise 57.1. Prove that a polynomial of degree n is uniformly continuous on $\mathbb{R}$ if and only if $n = 0$ or $n = 1$.

Exercise 57.2. Let

$$\mathcal{F} = \{f \in C(\mathbb{R}) : \lim_{|x| \to \infty} f(x) = 0\}.$$

Show that $\mathcal{F}$ is closed in $C(\mathbb{R})$.

Exercise 57.3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^{-x^2}$. Find

(a) an open set $D \subseteq \mathbb{R}$ such that $f(D)$ is not open;

(b) a closed set $F \subseteq \mathbb{R}$ such that $f(F)$ is not closed;

(c) a set $A \subseteq \mathbb{R}$ such that $f(\bar{A}) \neq \overline{f(A)}$.

Exercise 57.4. Let $f : [0, 1] \to [0, 1]$ be a continuous function such that $f(0) = 0$ and $f(1) = 1$. Consider the sequence of functions $f_n : [0, 1] \to [0, 1]$ defined as follows:

$$f_1 = f \text{ and } f_{n+1} = f \circ f_n \text{ for } n \geq 1.$$

Prove that if $\{f_n\}_{n \geq 1}$ converges uniformly, then $f(x) = x$ for all $x \in [0, 1]$.

Exercise 57.5. Let (X, d) be a metric space with at least two points and let $\mathcal{A} \subseteq C(X)$ be an algebra that is dense in the metric space $C(X)$.

(a) Show that $\mathcal{A}$ separates points on X.

(b) Show that $\mathcal{A}$ vanishes at no point in X.

Exercise 57.6. (a) Show that given any continuous function $f : [0,1] \times [0,1] \to \mathbb{R}$ and any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ and functions $g_1, \ldots, g_n, h_1, \ldots, h_n \in C([0,1])$ such that

$$\left| f(x,y) - \sum_{k=1}^{n} g_k(x)h_k(y) \right| < \varepsilon \text{ for all } (x,y) \in [0,1] \times [0,1].$$

(b) If $f(x,y) = f(y,x)$ for all $(x,y) \in [0,1] \times [0,1]$, can this be done with $g_k = h_k$ for each $1 \leq k \leq n$? Justify your answer.

Exercise 57.7. Let (X,d) be a compact metric space and let

$$\mathcal{A} \subseteq \mathbb{C}(X;\mathbb{C}) = \{f : X \to \mathbb{C}; f \text{ is continuous}\}$$

be an algebra that separates points and vanishes at no point in X. Assume additionally that $\mathcal{A}$ is self-adjoint, that is, for every $f \in \mathcal{A}$, its complex conjugate $\bar{f}$ is also in $\mathcal{A}$. Show that $\mathcal{A}$ is dense in $C(X;\mathbb{C})$.

58 Lecture 43: Intermediate Value Theorem for Derivatives, Inverse Function Theorem

The following more general statement will be used in the proof of l' Hôpital's rule.

Theorem 58.1. Let $f : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists some $c \in (a,b)$ such that

$$f'(c)[g(b) - g(a)] = g'(c) \cdot [f(b) - f(a)].$$

Remark 58.2. Taking $g(x) = x$, we recover the Mean Value Theorem. In fact, the two results are equivalent, as can be seen from the proof.

Proof. We define $h : [a,b] \to \mathbb{R}$ by $h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)]$. Note that h is continuous on $[a,b]$ and differentiable on (a,b). Moreover,

$$h(a) = f(a)[g(b) - g(a)] - g(a)[f(b) - f(a)] = f(a)g(b) - g(a)f(b)$$

and similarly $h(b) = -f(b)g(a) + g(b)f(a)$, and so $h(a) = h(b)$. By Rolle's theorem, there exists $c \in (a,b)$ such that $h'(c) = 0$. $\square$

Corollary 58.3. Let $f : (a,b) \to \mathbb{R}$ be differentiable.

1. If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing.
2. If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is increasing.
3. If $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing.
4. If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is decreasing.

Proof. We only show the details for (1). Fix $1 < x_1 < x_2 < b$, then since f is differentiable on (a, b), then f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). By the Mean Value Theorem, there exists $c \in (x_1, x_2)$ such that

$$0 < f'(c) = \frac{f'(x_2) - f'(x_1)}{x_2 - x_1},$$

then $f(x_1) < f(x_2)$. As $a < x_1 < x_2 < b$ were arbitrary, f is strictly increasing. □

Example 58.4. The derivative of a differentiable function need not be continuous. Consider $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

then f is continuous on $\mathbb{R}\backslash\{0\}$. To see that it is continuous at 0, $|f(x)-f(0)| = |x^2 \sin(\frac{1}{x})| \leq x^2 \xrightarrow[x\to 0]{} 0$. Also, f is differentiable on $\mathbb{R}\backslash\{0\}$. To see that it is differentiable at 0, we compute for $x \neq 0$, we have

$$\frac{f(x) - f(0)}{x - 0} = x \sin(\frac{1}{x}) \xrightarrow[x\to\infty]{} 0$$

and so $f'(0) = 0$. Therefore,

$$f'(x) = \begin{cases} 2x \sin(\frac{1}{x}) + x^2 \cos(\frac{1}{x}) \cdot (-\frac{1}{x^2}), & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Therefore f' is continuous on $\mathbb{R}\backslash\{0\}$, but f' is not continuous at 0. While $\lim_{x\to 0} 2x \sin(\frac{1}{x}) = 0$, for all $\lambda \in [-1, 1]$ there exists $x_n(\lambda) \xrightarrow[n\to\infty]{} 0$ such that $\cos(\frac{1}{x_n(\lambda)}) = \lambda$.

Nevertheless, the derivative of a differentiable function has the Darboux property.

Theorem 58.5 (Intermediate Value Theorem for Derivatives)**.** Let $f : [a, b] \to \mathbb{R}$ be differentiable. Then f' has the Darboux property, that is, if $a < x_1 < x_2 < b$ and λ lies between $f'(x_1)$ and $f'(x_2)$, then there exists $c \in (x_1, x_2)$ such that $f'(c) = \lambda$.

Proof. Let $g : (a, b) \to \mathbb{R}$ be defined by $g(x) = f(x) - \lambda x$. Then g is differentiable on (a, b), and so g is continuous on (a, b). Fix $a < x_1 < x_2 < b$ and assume without loss of generality that $f'(x_1) < \lambda < f'(x_2)$, then $g'(x_1) = f'(x_1) - \lambda < 0$, and $g'(x_2) = f'(x_2) - \lambda > 0$. Therefore, g is continuous on $[x_1, x_2]$, and so there exists $c \in [x_1, x_2]$ such that $g(c) = \inf_{x\in[x_1,x_2]} g(x)$. If we can prove that $c \in (x_1, x_2)$, then $g'(c) = 0$.

To see that $c \neq x_1$, we argue as follows: $0 > g'(x_1) = \lim_{x\to x_1} \frac{g(x)-g(x_1)}{x-x_1}$, then there exists $\delta_1 > 0$ such that if $0 < |x - x_1| < \delta_1$, then $\frac{g(x)-g(x_1)}{x-x_1} < 0$. In particular, for $x \in (x_1, x_1 + \delta_1)$, we have

$$\frac{g(x) - g(x_1)}{x - x_1} < 0,$$

and so $g(x) < g(x_1)$, therefore g cannot attain its minimum at x_1.

Similarly, to see $c \neq x_2$, note that $0 < g'(x_2) = \lim_{x\to x_2} \frac{g(x)-g(x_2)}{x-x_2}$, then there exists $\delta_2 > 0$ such that if $0 < |x - x_2 < \delta_2|$, then $\frac{g(x)-g(x_2)}{x-x_2} > 0$. In particular, if $x \in (x_2 - \delta_2, x_2)$, then

$$\frac{g(x) - g(x_2)}{x - x_2} > 0,$$

therefore $g(x) < g(x_2)$, so g cannot attain its minimum at x_2. □

Theorem 58.6 (Inverse Function Theorem)**.** Let I be an open interval and let $f : I \to \mathbb{R}$ be continuous and injective. Then $f(I) = J$ is bijective. If f is differentiable at $x_0 \in I$ and $f'(x_0) \neq 0$, then $f^{-1} : J \to I$ is differentiable at $y_0 = f(x_0)$, and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}.$$

Proof. The proof uses the following two exercises:

Exercise 58.7. Let I be an interval and let $f : I \to \mathbb{R}$ be continuous and injective, then f is strictly monotone.

Exercise 58.8. Let I be an interval and let $f : I \to \mathbb{R}$ be strictly increasing and so that $f(I)$ is an interval, then f is continuous.

Using Exercise 58.7, we find that f is strictly monotone. Assume f is strictly increasing, then f^{-1} is strictly increasing.

Using Exercise 58.8 with $g = f^{-1} : J \to I$, we find that f^{-1} is continuous.

Claim 58.9. J is an open interval.

Subproof. Assume, towards a contradiction, that $\inf(J) \in J = f(I)$, then there exists $a \in I$ such that $f(a) = \inf(J)$. Because I is open, then there exists $\delta > 0$ such that $(a-\delta, a+\delta) \subseteq I$, but since f is strictly increasing, we know that $J = f(I) \ni f(a - \frac{\delta}{2}) < f(a) = \inf(J)$, contradiction.

Similarly, one can show that $\sup(J) \notin J$, and thus conclude the proof. ■

Because f is differentiable atx_0, then $f'(x_0) = \lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$, and because $f'(x_0) \neq 0$ and $f(x) \neq f(x_0)$ for all $x \neq x_0$, we conclude that

$$\lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)}.$$

In particular, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |x - x_0| < \delta$ implies

$$\left| \frac{x - x_0}{f(x) - f(x_0)} - \frac{1}{f'(x_0)} \right| < \varepsilon.$$

Because f^{-1} is continuous at y_0, then there exists $\eta > 0$ such that $0 < |y - y_0| < \eta$ implies $0 < |f^{-1}(y) - f^{-1}(y_0)| < \delta$. Therefore, for $0 < |y - y_0| < \eta$, we get

$$\left| \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} - \frac{1}{f'(x_0)} \right| < \varepsilon.$$

Therefore,

$$(f^{-1})'(y_0) = \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{f'(x_0)}.$$

□

59 Lecture 44: L' Hôpital's rule, Taylor Series

Definition 59.1. Let $-\infty \leq a < b \leq \infty$ and let $f : (a, b) \to \mathbb{R}$ be a function.

For $c \in (a, b) \cup \{a\}$, we write

$$\lim_{x \to c+} f(x) = L \in \mathbb{R} \cup \{\pm\infty\}$$

if for every sequence $\{x_n\}_{n \geq 1} \subseteq (c, b)$ such that $\lim\limits_{n \to \infty} x_n = c$, we have $\lim\limits_{n \to \infty} f(x_n) = L$.

For $c \in (a, b) \cup \{b\}$, we write

$$\lim_{x \to c-} f(x) = M \in \mathbb{R} \cup \{\pm\infty\}$$

if for every sequence $\{x_n\}_{n \geq 1} \subseteq (a, c)$ such that $\lim\limits_{n \to \infty} x_n = c$, we have $\lim\limits_{n \to \infty} f(x_n) = M$.

Remark 59.2. In general, if $c \in (a, b)$, we have

$$f(c) \neq \lim_{x \to c-} f(x) \neq \lim_{x \to c+} f(x) \neq f(c).$$

For example, consider the function

$$f(x) = \begin{cases} e^x, & x < 0 \\ 0, & x = 0 \\ e^{-x}, & x > 0 \end{cases}.$$

Theorem 59.3 (L' Hôpital)**.** Let $-\infty \leq a < b \leq \infty$ and let $f, g : (a, b) \to \mathbb{R}$ be diffentiable. Assume that $g'(x) \neq 0$ for all $x \in (a, b)$ and that

$$\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R} \cup \{\pm\infty\}.$$

Assume also that either

1. $\lim\limits_{x \to a+} f(x) = \lim\limits_{x \to a+} g(x) = 0$, or
2. $\lim\limits_{x \to a+} |g(x)| = \infty$,

then $\lim\limits_{x \to a+} \frac{f(x)}{g(x)} = L$.

Remark 59.4. We can replace $\lim\limits_{x \to a+}$ by $\lim\limits_{x \to b-}$ or by $\lim\limits_{x \to c}$ for some $c \in (a, b)$.

Proof. We will present the details for $L \in \mathbb{R}$. We will prove

Claim 59.5. For all $\varepsilon > 0$, there exists $\delta_1(\varepsilon) > 0$ such that $\frac{f(x)}{g(x)} < L + \varepsilon$ for all $x \in (a, a + \delta_1)$.

Claim 59.6. For all $\varepsilon > 0$, there exists $\delta_2(\varepsilon) > 0$ such that $L - \varepsilon < \frac{f(x)}{g(x)}$ for all $x \in (a, a + \delta_2)$.

Then taking $\delta(\varepsilon) = \min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\}$, we get

$$\left| \frac{f(x)}{g(x)} - L \right| < \varepsilon$$

for all $x \in (a, a + \delta)$, and so $\lim\limits_{x \to a+} \frac{f(x)}{g(x)} = L$.

Remark 59.7. Note that if $L = -\infty$, then it suffices to prove Claim 59.5 with $L + \varepsilon$ replaced by $M < 0$; if $L = \infty$, then it suffices to prove Claim 59.6 with $L - \varepsilon$ replaced by $M > 0$.

Now by the assumption, $g'(x) \neq 0$ for all $x \in (a, b)$, then since g is differentiable on (a, b), g' has the Darboux property, and so either $g'(x) < 0$ for all $x \in (a, b)$ or $g'(x) > 0$ for all $x \in (a, b)$. We now assume $g'(x) < 0$ for all $x \in (a, b)$, so g is strictly decreasing on (a, b). In the first case, $\lim\limits_{x \to a+} g(x) = 0$, and as g is strictly decreasing, we get $g(x) < 0$ for all $x \in (a, b)$; in the second case, $\lim\limits_{x \to a+} |g(x)| = \infty$, and as g is strictly decreasing, we get $\lim\limits_{x \to a+} g(x) = \infty$, and so there exists $c \in (a, b)$ such that $g(x) > 0$ for all $x \in (a, c)$. In particular, in both cases $g(x) \neq 0$ for all $x \in (a, c)$.

Proof of Claim 59.5. Fix $\varepsilon > 0$. As $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L$, there exists $\delta_1(\varepsilon) > 0$ such that $\frac{f'(x)}{g'(x)} < L + \frac{\varepsilon}{2}$ for all $x \in (a, a+\delta_1)$. Fix $a < x < y < \min(a+\delta_1, c)$. By the Mean Value Theorem[18], there exists $z \in (x, y)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(z)}{g'(z)} < L + \frac{\varepsilon}{2}.$$

In the first case, take the limit $x \to a+$ in the equation above, and we get

$$\frac{f(y)}{g(y)} \leq L + \frac{\varepsilon}{2} < L + \varepsilon$$

for all $a < y < \min(a + \delta_1, c)$. In the second case, we write

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(y)}{g(x) - g(y)} \cdot \frac{g(x) - g(y)}{g(x)} + \frac{f(y)}{g(x)},$$

and because we know there exists $c \in (a, b)$ such that $g(x) > 0$ for all $x \in (a, c)$ already, then we have $g(x) > g(y) > 0$, and so $\frac{g(x)-g(y)}{g(x)} > 0$. In particular, we have

$$\begin{aligned}
\frac{f(x)}{g(x)} &< \left(L + \frac{\varepsilon}{2}\right) \cdot \frac{g(x) - g(y)}{g(x)} + \frac{f(y)}{g(x)} \\
&= \left(L + \frac{\varepsilon}{2}\right)\left(1 - \frac{g(y)}{g(x)}\right) + \frac{f(y)}{g(x)} \\
&= L + \frac{\varepsilon}{2} + \frac{f(y) - \left(L + \frac{\varepsilon}{2}\right) g(y)}{g(x)}.
\end{aligned}$$

Note that for y fixed, $\lim_{x\to a+} \frac{f(y) - \left(L+\frac{\varepsilon}{2}\right)g(y)}{g(x)} = 0$, then there exists $\tilde{\delta}_1(\varepsilon) > 0$ such that

$$\left| \frac{f(y) - \left(L + \frac{\varepsilon}{2}\right) g(y)}{g(x)} \right| < \frac{\varepsilon}{2}$$

for all $x \in (a, a + \tilde{\delta}_1)$. In particular,

$$\frac{f(x)}{g(x)} < L + \varepsilon$$

for all $a < x < \min\{a + \delta_1, a + \tilde{\delta}_1, c\}$. ■

Exercise 59.8. Prove Claim 59.6.

□

[18]Note that we are applying an equivalent formulation of the theorem stated in Theorem 56.6.

Definition 59.9 (Taylor Expansion, Remainder)**.** Let I be an open interval and let $f : I \to \mathbb{R}$ be differentiable of any order. For $x_0 \in I$, the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

is called the Taylor expansion of f about x_0.

For $n \geq 1$, we define the remainder

$$R_n(x) := f(x) - \sum_{n=0}^{k-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

Theorem 59.10 (Taylor)**.** Let $n \geq 1$ and assume $f : (a,b) \to \mathbb{R}$ is n times differentiable. Let $x_0 \in (a,b)$, then for any $x \in (a,b)\backslash\{x_0\}$, there exists y between x and x_0 such that

$$R_n(x) = \frac{f^{(n)(y)}}{n!}(x-x_0)^n.$$

In particular,

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n)}(y)}{n!}(x-x_0)^n.$$

Proof. Fix $x \in (a,b)\backslash\{x_0\}$. Define $M \in \mathbb{R}$ to be the unique solution to the equation

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + M \cdot \frac{(x-x_0)^n}{n!}.$$

We want to show that there exists y between x and x_0 such that $M = f^{(n)}(y)$. Let $g : (a,b) \to \mathbb{R}$ be defined by

$$g(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(t-x_0)^k - M \cdot \frac{(t-x_0)^n}{n!}.$$

Note that g is n times differentiable. For $1 \leq l \leq n-1$, we have

$$g^{(l)}(t) = f^{(l)}(t) - \sum_{k \geq l}^{n-1} \frac{f^{(k)}(x_0)}{(k-l)!}(t-x_0)^{k-l} - M \cdot \frac{(t-x_0)^{n-l}}{(n-l)!}$$

and

$$g^{(n)}(t) = f^{(n)}(t) - M.$$

In particular, if $0 \leq l \leq n-1$,

$$g^{(l)}(x_0) = f^{(l)}(x_0) - f^{(l)}(x_0) = 0,$$

and also $g(x) = 0$ by construction. Now g is continuous on $[x, x_0]$, differentibale on (x, x_0), and $g(x) = g(x_0) = 0$, so there exists $x_1 \in (x, x_0)$ such that $g'(x_1) = 0$. By Rolle's theorem, there exists $x_2 \in (x_1, x_0)$ such that $g''(x_2) = 0$, and continuing inductively, we can find $x_n \in (x_{n-1}, x_0)$ such that $g^{(n)}(x_n) = 0$. We now set $y = x_n$. $\square$

60 Lecture 45: Taylor Series, Continued

Corollary 60.1. Fix $a > 0$ and let $f : (-a, a) \to \mathbb{R}$ be a function differentiable of any order. Assume that all derivatives of f are uniformly bounded on $(-a, a)$, that is, there exists $M > 0$ such that

$$|f^{(n)}(x)| \leq M$$

for all $x \in (-a, a)$, for all $n \geq 1$, then

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k \xrightarrow[n\to\infty]{u} 0$$

on $(-a, a)$.

Proof. Fix $x \in (-a, a) \backslash \{0\}$. By Taylor's theorem, there exists y between x and 0 such that

$$R_n(x) = \frac{f^{(n)}(y)}{n!} x^n,$$

then

$$|R_n(x)| \leq M \cdot \frac{|x|^n}{n!} \leq M \cdot \frac{a^n}{n!},$$

and

$$\sup_{x\in(-a,a)} |R_n(x)| \leq M \cdot \frac{a^n}{n!} \xrightarrow[n\to\infty]{} 0.$$

□

Example 60.2. $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \cos(x)$, so

$$f^{(n)}(x) = \begin{cases} -\sin(x), & n = 4k+1 \\ -\cos(x), & n = 4k+2 \\ \sin(x), & n = 4k+3 \\ cos(x), & n = 4k \end{cases}$$

for $k \geq 0$. So $|f^{(n)}(x)| \leq 1$ for all $x \in \mathbb{R}$ for all $n \geq 0$. We get

$$f(x) = u - \lim_{N\to\infty} \sum_{n=0}^{N} \frac{f^{(N)}(0)}{n!} x^n$$

on $(-a, a)$ for any $a > 0$. For $n = 2l$, we have

$$f^{(n)}(0) = \begin{cases} -1, & \text{if } l \text{ odd} \\ 1, & \text{if } l \text{ even} \end{cases} = (-1)^l$$

Therefore,

$$f(x) = \sum_{n\geq 0} \frac{f^{(n)}(0)}{n!}x^n = \sum_{l\geq 0} \frac{(-1)^l}{(2l)!}x^{2l}.$$

A similar argument (needs to check) gives

$$\sin(x) = \sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Example 60.3. $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, x \neq 0 \\ 0, \qquad\qquad x = 0 \end{cases}$$

Note f is differentiable of any order on $\mathbb{R}$. Clearly, this holds on $\mathbb{R}\backslash\{0\}$. In fact, for $x \in \mathbb{R}\backslash\{0\}$,

$$f^{(n)}(x) = P_n(\frac{1}{x})e^{-\frac{1}{x^2}}$$

where

$$P_n(\frac{1}{x}) = (\frac{2}{x^3})^n + \cdots$$

To see that f is differntiable at 0 we compute

$$\lim_{x\to 0+} \frac{f(x)}{x} = \lim_{x\to 0+} \frac{\frac{1}{x}}{e^{\frac{1}{x^2}}} = \lim_{t\to\infty} \frac{t}{e^{t^2}} = \lim_{t\to\infty} \frac{1}{2te^{t^2}} = 0,$$

and

$$\lim_{x\to 0-} \frac{f(x)}{x} = \lim_{t\to\infty} \frac{t}{e^{t^2}} = 0,$$

so $f'(0) = 0$. Proceeding inductively, we can prove that f is differentiable of any order at 0, and $f^{(n)}(0) = 0$. Now

$$\lim_{x\to 0+} \frac{f^{(n)}(x)}{x} = \lim_{x\to 0+} \frac{P_n(\frac{1}{x})e^{-\frac{1}{x^2}}}{x} = \lim_{t\to\infty} \frac{tP_n(t)}{e^{t^2}} = 0$$

and $\lim\limits_{x\to 0-} \frac{f^{(n)}(x)}{x} = 0$. Thus,

$$\sum_{n\geq 0} \frac{f^{(n)}(0)}{n!}x^N \equiv 0.$$

At leading order as $x \to 0$, then

$$f^{(n)}(x) \sim 2^n \cdot (\frac{1}{x^2})^{\frac{3n}{2}} e^{-\frac{1}{x^2}} \sim 2^n e^{-\frac{1}{x^2} + \frac{3n}{2}\ln(\frac{1}{x^2})}.$$

The function $g : (0, \infty) \to \mathbb{R}$ defined by $g(t) = -t + \frac{3n}{2}\ln(t)$ achieves its maximum at

$$g'(t) = 0 \iff -1 + \frac{3n}{2t} = 0 \iff t = \frac{3n}{2},$$

so

$$f^{(n)}\left(\sqrt{\frac{2}{3n}}\right) \sim 2^n e^{-\frac{3n}{2}+\frac{3n}{2}\ln(\frac{3n}{2})} \sim 2^n e^{\frac{3n}{2}\ln(\frac{3n}{2e})} \sim 2^n (\frac{3n}{2e})^{\frac{3n}{2}} \xrightarrow[n\to\infty]{} \infty.$$

Theorem 60.4. Assume that $f_n : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b). Assume also that

1. $\{f_n'\}_{n\geq 1}$ converges uniformly on (a,b),
2. $\{f_n'\}_{n\geq 1}$ converges at some $x_0 \in [a,b]$,

then $\{f_n\}_{n\geq 1}$ converges uniformly on $[a,b]$ to some function f. Moreover, f is differentiable on (a,b) and $f'(x) = \lim_{n\to\infty} f_n'(x)$.

Remark 60.5. We can reinstate the conclusion as follows:

$$\lim_{y\to x}\lim_{n\to\infty}\frac{f_n(y)-f_n(x)}{y-x} = \lim_{y\to x}\frac{f(y)-f(x)}{y-x} = f'(x) = \lim_{n\to\infty}\lim_{y\to x}\frac{f_n(y)-f_n(x)}{y-x}.$$

Proof. Let us prove that $\{f_n\}_{n\geq 1}$ converges uniformly on $[a,b]$. Fix $\varepsilon > 0$, since $\{f_n'\}_{n\geq 1}$ converges uniformly on (a,b), then $\{f_n'\}_{n\geq 1}$ is uniformly Cauchy on (a,b), then there exists $n_1(\varepsilon) \in \mathbb{N}$ such that

$$|f_n'(x) - f_m'(x)| < \varepsilon$$

for all $n,m \geq n_1(\varepsilon)$ and for all $x \in (a,b)$. Now because $\{f_n(x_0)\}_{n\geq 1}$ converges, then $\{f_n(x_0)\}_{n\geq 1}$ is Cauchy, so there exists $n_2(\varepsilon) \in \mathbb{N}$ such that $|f_n(x_0) - f_m(x_0)| < \varepsilon$ for all $n,m \geq n_2(\varepsilon)$. For $x \in [a,b]\backslash\{x_0\}$, we have

$$|f_n(x) - f_m(x)| \leq |f_n(x_0) - f_m(x_0)| + |[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)]|$$

By the Mean Value Theorem, there exists y between x and x_0 such that

$$|[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)]| = |f_n'(y) - f_m'(y)| \cdot |x - x_0| < \varepsilon(b-a)$$

using the first inequality we derived. Now, for $n,m \geq n(\varepsilon) = \max\{n_1(\varepsilon), n_2(\varepsilon)\}$, we get

$$|f_n(x) - f_m(x)| \leq |f_n(x_0) - f_m(x_0)| + \varepsilon(b-a) \leq \varepsilon(1+b-a),$$

and so we know

$$\sup_{x\in[a,b]} |f_n(x) - f_m(x)| \leq \varepsilon(1+b-a)$$

for all $n,m \geq n(\varepsilon)$. Therefore, $\{f_n\}_{n\geq 1}$ is uniformly Cauchy on $[a,b]$ and so converge to a function $f = \lim_{n\to\infty} f_n$. It remains to show that f is differentiable on (a,b) and $f'(x) = \lim_{n\to\infty} f_n'(x)$. $\square$

61 Homework 16

Exercise 61.1. Let I be an interval and let $f : I \to \mathbb{R}$ be a strictly increasing function such that $f(I)$ is an interval. Show that f is continuous on I.

Exercise 61.2. Let I be an interval and let $f : I \to \mathbb{R}$ be a continuous, injective function. Show that f is strictly monotone on I.

Exercise 61.3. Assume $f : [a, b] \to \mathbb{R}$ is a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b) with $f(a) = f(b) = 0$. Prove that for every $\lambda \in \mathbb{R}$ there exists $x_0 \in (a, b)$ such that $f'(x_0) = \lambda f(x_0)$.

Exercise 61.4. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function on the closed interval $[0, 1]$ and differentiable on the open interval $(0, 1)$. Assume that $f(0) = 0$ and f' is an increasing function on $(0, 1)$. Show that

$$g(x) = \frac{f(x)}{x}$$

is an increasing function on $(0, 1)$.

Exercise 61.5. Let $f : [a, b] \to \mathbb{R}$ be a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Assume that f' is strictly increasing. Show that for any $c \in (a, b)$ such that $f'(c) = 0$ there exists $x_1, x_2 \in [a, b]$, $x_1 < c < x_2$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Exercise 61.6. Let $f : (0, 1) \to \mathbb{R}$ be a differentiable function such that $|f'(x)| < 1$ for all $x \in (0, 1)$. For $n \geq 2$, let $a_n = f\left(\frac{1}{n}\right)$. Show that $\lim\limits_{n\to\infty} a_n$ exists.

Exercise 61.7. Let $f : (a, b) \to \mathbb{R}$ be differentiable and let $c \in (a, b)$. Suppose that $\lim\limits_{x\to c} f'(x)$ exists and is finite. Show this limit must be $f'(c)$.

Exercise 61.8. If f has a finite third derivative f''' in $[a, b]$ and $f(a) = f'(a) = f(b) = f'(b) = 0$, then there exists $c \in (a, b)$ such that $f'''(c) = 0$.

Exercise 61.9. Compute $\lim\limits_{x\to 0}(1 + 2x)^{\frac{1}{x}}$.

62 Lecture 46: Darboux Integral

Proof, Continued. Last time we showed that $\{f_n\}_{n\geq 1}$ converges uniformly on $[a, b]$. Fix $x \in (a, b)$, we want to show f is differentiable at x and $f'(x) = \lim\limits_{n\to\infty} f'_n(x)$. We define

$g : [a, b]\backslash\{x\} \to \mathbb{R}$ by $g(y) = \frac{f(y)-f(x)}{y-x}$, and define $g_n : [a, b]\backslash\{x\} \to \mathbb{R}$ by $g_n(y) = \frac{f_n(y)-f_n(x)}{y-x}$. Since $f_n \xrightarrow[n\to\infty]{u} f$ we have $\lim\limits_{n\to\infty} g_n(y) = g(y)$. Since f_n is differentiable at x, then $\lim\limits_{y\to x} g_n(y) = f_n'(x)$. Let $L(x) = \lim\limits_{n\to\infty} f_n'(x)$, we want to show that for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|g(y) - L(x)| < \varepsilon$ whenever $0 < |y - x| < \delta$ for $y \in [a, b]$.

Fix $\varepsilon > 0$. By the triangle inequality,

$$|g(y) - L(x) \leq |g(y) - g_n(y)| + |g_n(y) - f_n'(x)| + |f_n'(x) - L(x)|.$$

We have $\{f_n'\}_{n\geq 1}$ converges uniformly on (a, b), so $\{f_n'\}_{n\geq 1}$ is uniformly Cauchy on (a, b), then there exists $n_1(\varepsilon) \in \mathbb{N}$ such that

$$|f_n'(z) - f_m'(z)| < \varepsilon$$

for all $n, m \geq n_1(\varepsilon)$ and for all $z \in (a, b)$. Letting $m \to \infty$ we find

$$|f_n'(z) - L(z)| \leq \varepsilon$$

for all $n \geq n_1(\varepsilon)$ and for all $z \in (a, b)$. For $y \in [a, b]\backslash\{x\}$, by the Mean Value Theorem we can find a point z between x and y so that

$$\begin{aligned} |g_n(y) - g_m(y)| &= \left| \frac{f_n(y) - f_n(x)}{y - x} - \frac{f_m(y) - f_m(x)}{y - x} \right| \\ &= \frac{|[f_n(y) - f_m(y)] - [f_n(x) - f_m(x)]|}{|y - x|} \\ &= |f_n'(z) - f_m'(z)| \\ &< \varepsilon \end{aligned}$$

for all $n, m \geq n_1(\varepsilon)$.

Letting $m \to \infty$, we find

$$|g_n(y) - g(y)| \leq \varepsilon$$

for all $n \geq n_1(\varepsilon)$ and for all $y \in [a, b]\backslash\{x\}$. Fix $n \geq n_1(\varepsilon)$. As f_n is differentiable at x, we find $\delta = \delta(\varepsilon, n) > 0$ such that

$$|g_n(y) - f_n'(x)| < \varepsilon$$

for all $0 < |y - x| < \delta$ for all $y \in [a, b]$. Thus, for this $n \geq n_1(\varepsilon)$ and $0 < |y - x| < \delta$, we have

$$\begin{aligned} |g(y) - L(x)| &\leq |g(y) - g_n(y)| + |g_n(y) - f_n'(x)| + |f_n'(x) - L(x)| \\ &\leq 3\varepsilon. \end{aligned}$$

□

Example 62.1. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{x}{1+nx^2}$. Note that f_n is differentiable and

$$f_n'(x) = \frac{1}{1+nx^2} - \frac{x \cdot 2nx}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}.$$

Now $f_n \xrightarrow[n\to\infty]{u} f \equiv 0$, and

$$f_n'(x) \xrightarrow[n\to\infty]{} \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}.$$

Note that f_n' do not converge uniformly since their limit is not continuous:

$$\lim_{n\to\infty} \lim_{y\to 0} \frac{f_n(y) - f_n(0)}{y - 0} = \lim_{n\to\infty} f_n'(0) = 1,$$

but

$$\lim_{y\to 0} \lim_{n\to\infty} \frac{f_n(y) - f_n(0)}{y - 0} = \lim_{y\to 0} 0 = 0.$$

Definition 62.2 (Bounded Function)**.** Let $f : [a, b] \to \mathbb{R}$ be a bounded function. If $S \subseteq [a, b]$, we denote $M(f; S) = \sup_{x\in S} f(X)$, and $m(f; S) = \inf_{x\in S} f(x)$.

Definition 62.3 (Partition, Darboux Sum)**.** A partition of $[a, b]$ is a finite ordered set $P \subseteq [a, b]$. We write

$$P = \{a = t_0 < t_1 < \ldots < t_n = b\}$$

for some $n \geq 1$.

The upper Darboux sum of f with respect to P is

$$U(f; P) = \sum_{k=1}^{n} M(f; [t_{k-1}, t_k])(t_k - t_{k-1}),$$

and the lower Darboux sum of f with respect to P is

$$L(f; P) = \sum_{k=1}^{n} m(f; [t_{k-1}, t_k])(t_k - t_{k-1}).$$

Remark 62.4. Note that

$$m(f; [a, b])(b - a) \leq L(f; P) \leq U(f; P) \leq M(f; [a, b])(b - a),$$

so

$$\{L(f; P) : P \text{ partition of } [a, b]\}$$

is bounded above, and

$$\{U(f; P) : P \text{ partition of } [a, b]\}$$

is bounded below.

Definition 62.5 (Darboux Integral, Darboux Integrable)**.** The upper Darboux integral of f on $[a, b]$ is

$$U(f) = \inf\{U(f; P) : P \text{ partition of } [a, b]\},$$

and the lower Darboux integral of f on $[a, b]$ is

$$L(f) = \sup\{L(f; P) : P \text{ partition of } [a, b]\}.$$

We say that f is Darboux integrable on $[a, b]$ if $U(f) = L(f)$. In this case, we write

$$\int_a^b f(x)dx = U(f) = L(f).$$

Example 62.6. Let $f : [0, M] \to \mathbb{R}$ be defined by $f(x) = x^3$. Then f is Darboux integrable. Let $P = \{0 = t_0 < \ldots < t_n = M\}$ be a partition of $[0, M]$, then

$$\begin{aligned} U(f; P) &= \sum_{k=1}^{n} M(f; [t_{k-1}, t_k])(t_k - t_{k-1}) \\ &= \sum_{k=1}^{n} t_k^3 (t_k - t_{k-1}). \end{aligned}$$

Similarly,

$$\begin{aligned} L(f; P) &= \sum_{k=1}^{n} m(f; [t_{k-1}, t_k])(t_k - t_{k-1}) \\ &= \sum_{k=1}^{n} t_{k-1}^3 (t_k - t_{k-1}). \end{aligned}$$

Take $t_k = \frac{kM}{n}$ for $0 \leq k \leq n$, then

$$U(f; P) = \sum_{k=1}^{n} \left(\frac{kM}{n}\right)^3 \cdot \frac{M}{n} = \frac{M^4}{n^4} \cdot \sum_{k=1}^{n} k^3 = \frac{M^4}{n^4} \cdot \left[\frac{n(n+1)}{2}\right]^2 \xrightarrow[n\to\infty]{} \frac{M^4}{4},$$

and

$$L(f; P) = \sum_{k=1}^{n} \left(\frac{(k-1)M}{n}\right)^3 \cdot \frac{M}{n} = \frac{M^4}{n^4} \cdot \sum_{k=1}^{n-1} k^3 = \frac{M^4}{n^4} \cdot \left[\frac{n(n-1)}{2}\right]^2 \xrightarrow[n\to\infty]{} \frac{M^4}{4},$$

so $U(f) \leq \frac{M^4}{4}$ and $L(f) \geq \frac{M^4}{4}$.

It remains to show that $L(f) \leq U(f)$ (which we will show later in Corollary 63.2), then we conclude that $U(f) = L(f) = \frac{M^4}{4}$. Therefore, f is Darboux integrable and

$$\int_0^M f(x)dx = \frac{M^4}{4}.$$

Example 62.7. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \in [0,1] \cap \mathbb{Q} \\ 0, & x \in [0,1] \cap \mathbb{Q} \end{cases}.$$

Now f is not Darboux integrable. For any partition P, $U(f;P) = 1$ and $L(f;P) = 0$, so $U(f) = 1$ and $L(f) = 0$.

63 Lecture 47: Mesh

Proposition 63.1. Let $f : [a,b] \to \mathbb{R}$ be bounded and let P and Q be two partitions of $[a.b]$ such that $P \subseteq Q$, then

$$L(f;P) \leq L(f;Q) \leq U(f;Q) \leq U(f;P).$$

Proof. We will prove the first inequality. The first inequality follows from a similar argument.

Arguing by induction, it suffices to prove the claim when the partition Q containing exactly one extra point compared to the partition P. Say $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ and $Q = \{a = t_0 < \ldots M t_{l-1} < s < t_l < \ldots < t_n = b\}$ for some $1 \leq l \leq n$. For

$$\begin{aligned} U(f;Q) = &\sum_{k=1}^{l-1} M(f;[t_{k-1},t_k])(t_k - t_{k-1}) \\ &+ M(f;[t_{l-1},s])(s - t_{k-1}) + M(f;[s,t_l])(t_l - s) \\ &+ \sum_{k=l+1}^{n} M(f;[t_{k-1},t_k])(t_k - t_{k-1}). \end{aligned}$$

Clearly, $M(f;[t_{l-1},s]) \leq M(f;[t_{l-1},t_l])$, and $M(f;[s,t_l]) \leq M(f;t_{k-1},t_l)$. Therefore,

$$U(f;Q) \leq \sum_{k=1}^{n} M(f;[t_{k-1},t_k])(t_k - t_{k-1}) = U(f;P).$$

$\square$

Corollary 63.2. Let $f : [a,b] \to \mathbb{R}$ be bounded and let P and Q be two partitions of $[a,b]$, then $L(f;P) \leq U(f;Q)$. Consequently, $L(f) \leq U(f)$.

Proof. Consider the partition $P \cup Q$. We have

$$L(f;P) \leq L(f;P \cup Q) \leq U(f;P \cup Q) \leq U(f;Q),$$

then $L(f) = \sup_P L(f;P) \leq U(f;Q)$, so $L(f) \leq \inf_Q U(f;Q) = U(f)$. $\square$

Theorem 63.3. Let $f : [a, b] \to \mathbb{R}$ be bounded, then f is Darboux integrable if and only if for all $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(f; P) - L(f; P) < \varepsilon$.

Proof. ($\Leftarrow$): Fix $\varepsilon > 0$, then there exists a partition P of $[a, b]$ such that $U(f; P) - L(f; P) < \varepsilon$, and so

$$U(f) \leq U(f; P) < L(f; P) + \varepsilon \leq L(f) + \varepsilon.$$

Therefore, $U(f) < L(f) + \varepsilon$, and since $\varepsilon > 0$ is arbitrary, we have $U(f) \leq L(f)$ and $L(f) \leq U(f)$, so $U(f) = L(f)$. Therefore, f is Darboux integrable.

($\Rightarrow$): Fix $\varepsilon > 0$. Since f is Darboux integrable, then $U(f) = L(f)$. Now $U(f) = \inf_P U(f; P)$, so there exists a partition P_1 of $[a, b]$ such that $U(f; P_1) < U(f) + \frac{\varepsilon}{2}$. Similarly, since $L(f) = \sup_P L(f; P)$, there exists a partition P_2 of $[a, b]$ such that $L(f; P_2) > L(f) - \frac{\varepsilon}{2}$.

Consider the partition $P_1 \cup P_2$, then $L(f; P_2) \leq L(f; P_1 \cup P_2) \leq U(f; P_1 \cup P_2) \leq U(f; P_1)$. Therefore,

$$U(f; P_1 \cup P_2) - L(f; P_1 \cup P_2) < U(f) + \frac{\varepsilon}{2} - (L(f) - \frac{\varepsilon}{2}) = \varepsilon.$$

□

Definition 63.4 (Mesh)**.** Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ be a partition of $[a, b]$. The mesh of P is given by

$$\text{mesh}(P) = \max_{1 \leq k \leq n} (t_k - t_{k-1}).$$

Theorem 63.5. Let $f : [a, b] \to \mathbb{R}$ be bounded, then f is Darboux integrable if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if P is a partition of $[a, b]$ with $\text{mesh}(P) < \delta$, then $U(f; P) - L(f; P) < \varepsilon$.

Proof. ($\Leftarrow$): By Theorem 63.3, it suffices to show that for all $\varepsilon > 0$, there exists a partition P of $[a, b]$ with $\text{mesh}(P) < \delta$. For $\delta > 0$, let $P = \{a = t_0 < \ldots < t_n = b\}$ where $t_k = a + k \cdot \frac{\delta}{2}$ for $0 \leq k \leq \lfloor \frac{2(b-a)}{\delta} \rfloor = n - 1$, and $t_n = b$. Clearly, $\text{mesh}(P) = \frac{\delta}{2} < \delta$.

($\Rightarrow$): Fix $\varepsilon > 0$. By Theorem 63.3, as f is Darboux integrable, there exists a partition $P_0 = \{a = s_0 < \ldots < s_m = b\}$ of $[a, b]$ such that $U(f; P_0) - L(f; P_0) < \frac{\varepsilon}{2}$. Let $0 < \delta < \text{mesh}(P_0)$ to be chosen later and let $P = \{a = t_0 < \ldots < t_n = b\}$ be a partition of $[a, b]$ with $\text{mesh}(P) < \delta$. Now

$$\begin{aligned} U(f; P) - L(f; P) &\leq U(f; P) - U(f; P_0) + U(f; P_0) - L(f; P_0) + L(f; P_0) - L(f; P) \\ &< \frac{\varepsilon}{2} + U(f; P) - U(f; P_0) + L(f; P_0) - L(f; P). \end{aligned}$$

Consider the partition $P \cup P_0$, then

$$U(f; P) - U(f; P_0) \leq U(f; P) - U(f; P \cup P_0).$$

As $\text{mesh}(P) < \delta < \text{mesh}(P_0)$, there must be at most one point from P_0 in each $[t_{k-1}, t_k]$. Only subintervals $[t_{k-1}, t_k]$ with an $s_j \in P_0 \cap [t_{k-1}, t_k]$ contribute to $U(f; P) - U(f; P_0 \cup P)$. There are only m many such intervals. The contribution of one such interval to $U(f; P) - U(f; P_0 \cup P)$ is

$$M(f; [t_{k-1}, t_k])(t_k - t_{k-1}) - M(f; [t_{k-1}, s_j])(s_j - t_{k-1}) - M(f; [s_j, t_k])(t_k - s_j).$$

Since f is bounded, then there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$. Note $M(f; [t_{k-1}, t_k] \leq M$ and $M(f; [t_{k-1}, s_j]) \geq -M$, and $M(f; [s_j, t_k]) \geq -M$. Therefore,

$$\begin{aligned}
&M(f; [t_{k-1}, t_k])(t_k - t_{k-1}) - M(f; [t_{k-1}, s_j])(s_j - t_{k-1}) - M(f; [s_j, t_k])(t_k - s_j) \\
&\leq M(t_k - t_{k-1}) - (-M)[(s_j - t_{k-1}) + (t_k - s_j)] \\
&= 2M(t_k - t_{k-1}) \\
&< 2M \cdot \text{mesh}(P).
\end{aligned}$$

Thus, $U(f; P) - U(f; P_0) < m \cdot 2M \cdot \text{mesh}(P)$, and similarly $L(f; P_0) - L(f; P) < m \cdot 2M \cdot \text{mesh}(P)$. It now suffices to make our choice of δ to be such that $4Mm \cdot \text{mesh}(P) < \frac{\varepsilon}{2}$, i.e.,

$$\text{mesh}(P) < \frac{\varepsilon}{8Mm}.$$

In particular, we let

$$\delta < \min\left\{\frac{\varepsilon}{8Mm}, \text{mesh}(P_0)\right\}.$$

□

64 Lecture 48: Riemann Integral

Definition 64.1 (Riemann Sum, Riemann Integrable, Riemann Integral)**.** Let $f : [a, b] \to \mathbb{R}$ be a function and let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ be a partition of $[a, b]$. A Riemann sum of f associated to P is a sum of the form $S = \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1})$ where $x_k \in [t_{k-1}, t_k]$ for all $1 \leq k \leq n$. Note that if S is a Riemann sum associated with a partition P of $[a, b]$, then $L(f; P) \leq S \leq U(f; P)$.

We say that f is Riemann integrable if there exists $r \in \mathbb{R}$ such that for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|S - r| < \varepsilon$ for any Riemann sum S of f associated to a partition P with $\text{mesh}(P) < \delta$. Then r is called the Riemann integral of f and we write

$$r = \mathcal{R}\int_a^b f(x)dx.$$

Lemma 64.2. If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then f is bounded.

Proof. Let $r = \mathcal{R}\int_a^b f(x)dx$. Taking $\varepsilon = 1$, we find $\delta > 0$ such that $|S - r| < 1$ for any Riemann sum S of f associated to a partition P with $\text{mesh}(P) < \delta$. Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ with $\text{mesh}(P) < \delta$. Fix $1 \leq k \leq n$, fix $x_l \in [t_{l-1}, t_l]$ for $1 \leq k \leq n$ and $l \neq k$. For $x \in [t_{k-1}, t_k]$, we have that

$$|\sum_{l\neq k} f(x_l)(t_l - t_{l-1}) + f(x)(t_k - t_{k-1}) - r| < 1,$$

and so

$$\frac{r - 1 - \sum\limits_{l\neq k} f(x_l)(t_l - t_{l-1})}{t_k - t_{k-1}} < f(x) < \frac{1 + r - \sum\limits_{l\neq k} f(x_l)(t_l - t_{l-1})}{t_k - t_{k-1}},$$

but since $x \in [t_{k-1}, t_k]$ is arbitrary, we know f is bounded on $[t_{k-1}, t_k]$ for any choice of $1 \leq k \leq n$, and therefore f is bounded on $[a, b]$. □

Theorem 64.3. Let $f : [a, b] \to \mathbb{R}$. The following are equivalent:

1. f is Riemann integrable,
2. f is bounded and Darboux integrable.

If either condition holds, then the integrals agree.

Proof. (2) $\Rightarrow$ (1): Fix $\varepsilon > 0$. Since f is Darboux integrable, then there exists $\delta > 0$ such that $U(f; P) - L(f; P) < \varepsilon$ for any partition P with $\text{mesh}(P) < \delta$. Let P be a partition of $[a, b]$ with $\text{mesh}(P) < \delta$. If S is a Riemann sum of f associated to P, then

$$S \leq U(f; P) < L(f; P) + \varepsilon \leq L(f) + \varepsilon = \int_a^b f(x)dx + \varepsilon,$$

and

$$S \geq L(f; P) < U(f; P) - \varepsilon \geq U(f) - \varepsilon = \int_a^b f(x)dx - \varepsilon,$$

so

$$|S - \int_a^b f(x)dx| < \varepsilon.$$

By definition, f is Riemann integrable and $\mathcal{R}\int_a^b f(x)dx = \int_a^b f(x)dx$.

(1) $\Rightarrow$ (2): By Lemma 64.2, f is bounded. Fix $\varepsilon > 0$ and let $r = \mathcal{R}\int_a^b f(x)dx$, then there exists $\delta > 0$ such that $|S - r| < \frac{\varepsilon}{2}$ for any Riemann sum S of f associated with a partition P with $\text{mesh}(P) < \delta$. Fix $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ be a partition with $\text{mesh}(P) < \delta$, then there exists $x_k, y_k \in [t_{k-1}, t_k]$ such that $f(x_k) > M(f; [t_{k-1}, t_k]) - \frac{\varepsilon}{2(b-a)}$ and $f(y_k) < m(f; [t_{k-1}, t_k]) + \frac{\varepsilon}{2(b-a)}$. Therefore,

$$S_1 = \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1}) > U(f; P) - \frac{\varepsilon}{2(b-a)} \sum_{k=1}^{n}(t_k - t_{k-1}) = U(f; P) - \frac{\varepsilon}{2}$$

and

$$S_2 = \sum_{k=1}^{n} f(y_k)(t_k - t_{k-1}) < L(f;P) + \frac{\varepsilon}{2(b-a)} \sum_{k=1}^{n} (t_k - t_{k-1}) = L(f;P) + \frac{\varepsilon}{2}.$$

However, $|S_1 - r| < \frac{\varepsilon}{2}$ and $|S_2 - r| < \frac{\varepsilon}{2}$, so $U(f;P) - \frac{\varepsilon}{2} < S_1 < r + \frac{\varepsilon}{2}$, and so $U(f) \leq U(f;P) < r + \varepsilon$; similarly, we know $r - \frac{\varepsilon}{2} < S_2 < L(f;P) + \frac{\varepsilon}{2}$, then $r - \varepsilon < L(f;P) \leq L(f)$, then $r - \varepsilon < L(f) \leq U(f) < r + \varepsilon$, but because $\varepsilon > 0$ is arbitrary, then f is Darboux integrable and $\int_a^b f(x)dx = r$. □

Theorem 64.4. Let $f : [a,b] \to \mathbb{R}$ be monotonic, then f is integrable.

Proof. Without loss of generality, assume f is increasing. Then $f(a) \leq f(x) \leq f(b)$ for all $x \in [a,b]$, and so f is bounded. Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ with $\operatorname{mesh}(P) < \delta$ for δ to be chosen later, then

$$\begin{aligned} U(f;P) - L(f;P) &= \sum_{k=1}^{n} \left[M(f;[t_{k-1},t_k]) - m(f;[t_{k-1},t_k])\right](t_k - t_{k-1}) \\ &= \sum_{k=1}^{n} \left[f(t_k) - f(t_{k-1})\right](t_k - t_{k-1}) \\ &\leq \operatorname{mesh}(P) \cdot \sum_{k=1}^{n} [f(t_k) - f(t_{k-1})] \\ &< \delta \cdot [f(b) - f(a)]. \end{aligned}$$

Taking $\delta < \frac{\varepsilon}{f(b)-f(a)+1}$, we see that f is Darboux integrable. □

Theorem 64.5. Let $f : [a,b] \to \mathbb{R}$ be continuous, then f is integrable.

Proof. Because $f : [a,b] \to \mathbb{R}$ is continuous on a compact domain, then f is bounded. Fix $\varepsilon > 0$, as f is continuous on a compact domain, f is uniformly continuous, so there exists $\delta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$ for all $x, y \in [a,b]$ with $|x - y| < \delta$. Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ with $\operatorname{mesh}(P) < \delta$, then

$$U(f;P) - L(f;P) = \sum_{k=1}^{n} \left[M(f;[t_{k-1},t_k]) - m(f;[t_{k-1},t_k]\right](t_k - t_{k-1}).$$

Now since f is continuous on $[t_{k-1}, t_k]$ is compact, then there exists $x_k, y_k \in [t_{k-1}, t_k]$ such that $f(x_k) = M(f;[t_{k-1},t_k])$ and $f(y_k) = m(f;[t_{k-1},t_k])$. Therefore, $U(f;P) - L(f;P) = \sum_{k=1}^{n} [f(x_k) - f(y_k)](t_k - t_{k-1}) < \sum_{k=1}^{n} \frac{\varepsilon}{b-a}(t_k - t_{k-1}) = \varepsilon$. Therefore, f is Darboux integrable. □

65 Homework 17

Exercise 65.1. Assume $f : (a, b) \to \mathbb{R}$ is a twice differentiable function. Show that for any $x \in (a, b)$, the limit

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}$$

exists and equals $f''(x)$.

Exercise 65.2. Assume $f : (1, \infty) \to \mathbb{R}$ is differentibale. If $\lim\limits_{x\to\infty} f(x) = 1$ and $\lim\limits_{x\to\infty} f'(x) = c$, prove that $c = 0$.

Exercise 65.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $f(x) \geq 0$ and $f''(x) \leq 0$ for all $x \in \mathbb{R}$. Show that f is constant.

Exercise 65.4. We say a function $f : [a, b] \to \mathbb{R}$ is a convex function if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for all $x, y \in [a, b]$ and for all $t \in [0, 1]$. Show that for any $x \in (a, b)$ the one-sided limits $\lim\limits_{y\searrow x} \frac{f(y)-f(x)}{y-x}$ and $\lim\limits_{y\nearrow x} \frac{f(y)-f(x)}{y-x}$ exist and are finite.

Hint: Show that for all $1 \leq x < y < z \leq b$, we have

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y}.$$

Exercise 65.5. Let $f : [a, b] \to \mathbb{R}$ be a function such that $L(x) = \lim\limits_{y\to x} f(x)$ is well-defined and finite for all $x \in [a, b]$ (with one-sided limits at $x = a, b$).

(a) Show that L is continuous on $[a, b]$.

(b) Show that the set $\{x \in [a, b] : f(x) \neq L(x)\}$ is at most countable.

Exercise 65.6. Let (X, d) be a complete metric space and let $f : X \to X$ be a function. Writing f^n for the nth iterate of f, let $c_n = \sup\limits_{x\neq y} \frac{d(f^n(x), f^n(y))}{d(x,y)}$. Assume $\sum\limits_{n\geq 1} c_n < \infty$. Show that f has a fixed point in X and that this fixed point is unique.

Exercise 65.7. Let $f_n : [-1, 1] \to [0, 1]$ be continuous functions. Assume that for every $x \in [-1, 1]$, the sequence $\{f_n(x)\}_{n\geq 1}$ is decreasing and $\lim\limits_{n\to\infty} f_n(x) = 0$. For $n \geq 1$ and $x \in [-1, 1]$, let

$$g_n(x) = \sum_{m=1}^{n} (-1)^m f_m(x).$$

(a) Show that $\{g_n(x)\}_{n\geq 1}$ converges to some $g(x) \in \mathbb{R}$ for all $x \in [-1, 1]$.

(b) Show that the function g is continuous on $[-1, 1]$.

66 Lecture 49: Riemann Integral, Continued

Theorem 66.1. Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable.

1. For any $\alpha \in \mathbb{R}$, αf is Riemann integrable and
$$\int_a^b (\alpha f)(x)dx = \alpha \int_a^b f(x)dx.$$
2. $f + g$ is Riemann integrable and
$$\int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

Proof. 1. If $\alpha = 0$, this is obvious. We now suppose $\alpha > 0$. For any $S \subseteq [a, b]$, we have $M(\alpha f; S) = \alpha M(f; S)$ and $m(\alpha f; S) = \alpha m(f; S)$. Now for any partition P of $[a, b]$, $U(\alpha f; P) = \alpha U(f; P)$, so $U(\alpha f) = \sup_P U(\alpha f; P) = \sup_P[\alpha \cdot U(f; P)] = \alpha \sup_P U(f; P) = \alpha U(f)$. Similarly, we conclude that $L(\alpha f) = \alpha L(f)$, and since $L(f) = U(f)$, we know αf is Darboux integrable and $\int_a^b (\alpha f)(x)dx = \alpha \int_a^b f(x)dx$.

Now suppose $\alpha < 0$. For $S \subseteq [a, b]$, we have $M(\alpha f; S) = \alpha m(f; S)$ and $m(\alpha f; S) = \alpha M(f; S)$. If P is a partition of $[a, b]$, then $U(\alpha f; P) = \alpha L(f; P)$, and $L(\alpha f; P) = \alpha U(f; P)$. Thus, $U(\alpha f) = \inf_P U(\alpha f; P) = \inf_P \alpha L(f; P) = \alpha \sup_P L(f; P) = \alpha L(f)$, and similarly $L(\alpha f) = \alpha U(f)$. Moreover, because f is Riemann integrable, then f is bounded and $L(f) = U(f) = \int_a^b f(x)dx$. Therefore, αf is bounded and $L(\alpha f) = U(\alpha f) = \alpha \int_a^b f(x)dx$, so αf is Riemann integrable and $\int_a^b (\alpha f)(x)dx = \alpha \int_a^b f(x)dx$.

2. Since f and g are Riemann integrable, $f+g$ is bounded and f, g are Darboux integrable. Fix $\varepsilon > 0$. Since f is Darboux integrable, then there exists a partition P_1 of $[a, b]$ such that $U(f; P_1) - L(f; P_1) < \frac{\varepsilon}{2}$. Similarly, since g is Darboux integrable, then there exists a partition P_2 of $[a, b]$ such that $U(g; P_2) - L(g; P_2) < \frac{\varepsilon}{2}$. Let $P = P_1 \cup P_2$, we have $U(f; P) - L(f; P) < \frac{\varepsilon}{2}$ and $U(g; P) - L(g; P) < \frac{\varepsilon}{2}$. For $S \subseteq [a, b]$, $M(f + g; S) \leq M(f; S) + M(g; S)$, and $m(f + g; S) \geq m(f; S) + m(g; S)$. Therefore, $U(f + g; P) \leq U(f; P) + U(g; P)$, and $L(f + g; P) \geq L(f; P) + L(g; P)$, hence
$$U(f + g; P) - L(f + g; P) \leq U(f; P) - L(f; P) + U(g; P) - L(g; P) < \varepsilon.$$
Now, we know $f + g$ is Darboux integrable, and since $f + g$ is bounded, then $f + g$ is Riemann integrable.

Moreover,

$$\begin{aligned} U(f+g) &\leq U(f+g;P) \\ &\leq U(f;P)+U(g;P) \\ &< L(f;P)+L(g;P)+\varepsilon \\ &\leq L(f)+L(g)+\varepsilon \\ &= \int_a^b f(x)dx + \int_a^b g(x)dx + \varepsilon \end{aligned}$$

and

$$\begin{aligned} L(f+g) &\geq L(f+g;P) \\ &\geq L(f;P)+L(g;P) \\ &> U(f;P)+U(g;P)-\varepsilon \\ &\geq U(f)+U(g)-\varepsilon \\ &= \int_a^b f(x)dx + \int_a^b g(x)dx - \varepsilon. \end{aligned}$$

Now take $\varepsilon \to 0$, we see

$$\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

□

Theorem 66.2. Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable. Assume $f(x) \leq g(x)$ for all $x \in [a,b]$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Proof. By Theorem 66.1, $h : [a,b] \to \mathbb{R}$ defined by $h(x) = (g-f)(x)$ is Riemann integrable. Moreover, since $h \geq 0$, we have

$$\int_a^b h(x)dx = L(h) = \sup_P L(h;P) \geq 0,$$

and so by Theorem 66.1 we have

$$0 \leq \int_a^b h(x)dx = \int_a^b (g-f)(x)dx = \int_a^b g(x)dx - \int_a^b f(x)dx.$$

□

Theorem 66.3. Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Then $|f|$ is Riemann integrable and

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$$

Proof. Since f is Riemann integrable, then f is bounded and Darboux integrable, therefore $|f|$ is bounded. For $S \subseteq [a, b]$, we have

$$\begin{aligned}
M(|f|; S) - m(|f|; S) &= \sup_{x\in S} |f(x)| - \inf_{y\in S} |f(y)| \\
&= \sup_{x\in S} |f(x)| + \sup_{y\in S} -|f(y)| \\
&= \sup_{x,y\in S} [|f(x)| - |f(y)|] \\
&\leq \sup_{x,y\in S} |f(x) - f(y)| \\
&= \sup_{x,y\in S} [f(x) - f(y)] \\
&= \sup_{x\in S} f(x) - \inf_{y\in S} f(y) \\
&= M(f; S) - m(f; S).
\end{aligned}$$

Therefore, for any partition P of $[a, b]$ we have

$$U(|f|; P) - L(|f|; P) \leq U(f; P) - L(f; P).$$

Since f is Darboux integrable, for any $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that $U(f; P) - L(f; P) < \varepsilon$. That is to say, for any $\varepsilon > 0$, ther eexists a partition P of $[a, b]$ such that $U(|f|; P) - L(|f|; P) < \varepsilon$. Hence, $|f|$ is Darboux integrable, and since $|f|$ is bounded, then $|f|$ is Riemann integrable.

We now have $-|f(x)| \leq f(x) \leq |f(x)|$ for all $x \in [a, b]$, then by Theorem 66.2,

$$-\int_a^b |f(x)|dx = \int_a^b -|f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx,$$

and in particular

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$$

□

Theorem 66.4. Let $f : [a, b] \to \mathbb{R}$ be a function and let $a < c < b$. Assume f is Riemann integrable on $[a, c]$ and on $[c, b]$. Then f is Riemann integrable on $[a, b]$ and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Proof. Since f is Riemann integrable on $[a,c]$ and on $[c,b]$, then f is bounded on $[a,c]$ and on $[c,b]$. Therefore, f is bounded on $[a,b]$. Let $\varepsilon > 0$. Since f is Riemann integrable on $[a,c]$, then f is Darboux integrable on $[a,c]$ and there exists a partition P_1 of $[a,c]$ such that $U_a^c(f;P_1) - L_a^c(f;P_1) < \frac{\varepsilon}{2}$. Similarly, there exist a partition P_2 of $[c,b]$ such that $U_c^b(f;P_1) - L_c^b(f;P_1) < \frac{\varepsilon}{2}$. Now let $P = P_1 \cup P_2$ be a partition on $[a,b]$, and note that $U(f;P) = U_a^c(f;P_1) + U_b^c(f;P_2)$, and $L(f;P) = L_a^c(f;P_1) + L_b^c(f;P_2)$. Therefore, $U(f;P) - L(f;P) < \varepsilon$. Hence, f is Darboux integrable on $[a,b]$, and since it is bounded on $[a,b]$, we know f is Riemann integrable on $[a,b]$. Moreover,

$$\begin{aligned}
U(f) &\leq U(f;P) \\
&= U_a^c(f;P_1) + U_c^b(f;P_2) \\
&< L_a^c(f;P_1) + L_c^b(f;P_2) + \varepsilon \\
&\leq \int_a^c f(x)dx + \int_c^b f(x)dx + \varepsilon.
\end{aligned}$$

Similarly, $L(f) \geq \int_a^c f(x)dx + \int_c^b f(x)dx - \varepsilon$. Therefore, since $\varepsilon > 0$ is arbitrary,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

□

Lemma 66.5. Let $f, g : [a,b] \to \mathbb{R}$ be functions such that f is Riemann integrable and $g(x) = f(x)$ except at finitely many points in $[a,b]$. Then g is Riemann integrable and

$$\int_a^b g(x)dx = \int_a^b f(x)dx.$$

Proof. Arguing by induction, we may assume that there exists exactly one point $x_0 \in [a,b]$ such that $f(x_0) \neq g(x_0)$. Let $B > 0$ such that $|f(x)| \leq B$ and $|g(x)| \leq B$ for all $x \in [a,b]$. Let $P = \{a = t_0 < \ldots < t_n = b\}$. We consider $U(f;P) - U(g;P)$ and $L(f;P) - L(g;P)$. The largest contribution occurs when $x_0 = t_k$ for some $1 \leq k \leq n-1$. Now

$$|M(f;[t_{k-1},t_k]) - M(g;[t_{k-1},t_k])|] \leq [B - (-B)](t_k - t_{k-1}) \leq 2B \cdot \text{mesh}(P).$$

Hence, $|U(f;P) - U(g;P)| \leq 4B \cdot \text{mesh}(P)$. Similarly, $|m(f;[t_{k-1},t_k]) - m(g;[t_{k-1},t_k])| \leq 2B \cdot \text{mesh}(P)$, and therefore $|L(f;P) - L(g;P)| \leq 4B \cdot \text{mesh}(P)$. Thus,

$$\begin{aligned}
U(g;P) - L(g;P) &\leq U(f;P) - L(f;P) + |U(f;P) - U(g;P)| + |L(f;P) - L(g;P)| \\
&\leq U(f;P) - L(f;P) + 8B \cdot \text{mesh}(P).
\end{aligned}$$

Since f is Darboux integrable, then for all $\varepsilon > 0$ there exists some $\delta > 0$ such that $U(f;P) - L(f;P) < \frac{\varepsilon}{2}$ for all partitions P with $\mathrm{mesh}(P) < \delta$. We can pick a δ smaller if necessary, so that $8B\delta < \frac{\varepsilon}{2}$, i.e., $\delta < \frac{\varepsilon}{16B}$. Then $U(g;P) - L(g;P) < \varepsilon$ for all partitions P with $\mathrm{mesh}(P) < \delta$. Hence, g is Darboux integrable, and since g is bounded, we conclude that g is Riemann integrable.

Exercise 66.6.

$$\int_a^b g(x)dx = \int_a^b f(x)dx.$$

□

67 Lecture 50: Intermediate Value Property, Fundamental Theorem of Calculus, Integration by Parts

Definition 67.1 (Piecewise Monotone, Piecewise Continuous)**.** We say that a function $f : [a,b] \to \mathbb{R}$ is piecewise monotone if there exists a partition $P = \{a = t_0 < \ldots < t_n = b\}$ such that f is monotone on (t_{k-1}, t_k) for each $1 \leq k \leq n$.

We say that $f : [a,b] \to \mathbb{R}$ is piecewise continuous if there exists a partition $P = \{a = t_0 < \ldots < t_n = b\}$ such that f is uniformly continuous on (t_{k-1}, t_k) for each $1 \leq k \leq n$.

Theorem 67.2. Let $f : [a,b] \to \mathbb{R}$ be a function that satisfies

1. f is bounded and piecewise monotone, or
2. f is piecewise continuous,

then f is Riemann integrable.

Proof. Let $P = \{a = t_0 < \ldots < t_n = b\}$ be a partition of $[a,b]$ such that 1) f is monotone or 2) f is uniformly continuous on (t_{k-1}, t_k) for all $1 \leq k \leq n$.

If f is monotone on (t_{k-1}, t_k), then f can be extended to a monotone function f_k on $[t_{k-1}, t_k]$. For example, if f is increasing on (t_{k-1}, t_k), we define

$$f_k(t) = \begin{cases} \inf\limits_{t \in (t_{k-1}, t_k)} f(t), & t = t_{k-1} \\ f(t), & t \in (t_{k-1}, t_k) \\ \sup\limits_{t \in (t_{k-1}, t_k)} f(t), & t = t_k \end{cases}.$$

As f_k is monotone on $[t_{k-1}, t_k]$, f_k is Riemann integrable on $[t_{k-1}, t_k]$. As f differs from f_k at at most two points, f is Riemann integrable on $[t_{k-1}, t_k]$ and

$$\int_{t_{k-1}}^{t_k} f(t)dx = \int_{t_{k-1}}^{t_k} f_k(t)dt.$$

If f is uniformly continuous on (t_{k-1}, t_k), then f admits a continuous extension f_k to $[t_{k-1}, t_k]$, then f_k is Riemann integrable on $[t_{k-1}, t_k]$, and so f is Riemann integrable on $[t_{k-1}, t_k]$, and

$$\int_{t_{k-1}}^{t_k} f(t)dx = \int_{t_{k-1}}^{t_k} f_k(t)dt.$$

By Theorem 66.4, we conclude that f is Riemann integrable on $[a, b]$ and

$$\int_a^b f(t)dt = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f(t)dt.$$

□

Theorem 67.3 (Intermediate Value Property for Integrals)**.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function, then there exists $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx.$$

Proof. Since f is continuous on a compact set $[a, b]$, then there exists $x_0, y_0 \in [a, b]$ such that $f(x_0) = \inf_{x\in[a,b]} f(x)$ and $f(y_0) = \sup_{x\in[a,b]} f(x)$. Therefore,

$$(b-a)f(x_0) = \int_a^b f(x_0)dx \leq \int_a^b f(x)dx \leq \int_a^b f(y_0)dx = (b-a)f(y_0).$$

Therefore, $f(x_0) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f(y_0)$. Now since f is continuous, then f has the Darboux property, so now there exists c between x_0 and y_0 such that $f(c) = \frac{1}{b-a}\int_a^b f(x)dx$. □

Definition 67.4 (Riemann Integrable)**.** We say that a function $f : (a, b) \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if every extension of f to $[a, b]$ is Riemann integrable. In this case, $\int_a^b f(t)dt$ does not depend on the values of the extension at a and at b.

Theorem 67.5 (Fundamental Theorem of Calculus)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on $[a, b]$. If f' is Riemann integrable on $[a, b]$, then

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Proof. Fix $\varepsilon > 0$. Since f' is Riemann integrable on $[a, b]$, then there exists $P = \{a = t_0 < \dots < t_n = b\}$ such that $U(f'; P) - L(f'; P) < \varepsilon$. Since f is continuous on $[t_{k-1}, t_k]$ and differentiable on (t_{k-1}, t_k), then by the Mean Value Theorem, there exists $x_k \in (t_{k-1}, t_k)$ so that

$$f'(x_k) = \frac{f(t_k) - f(t_{k-1})}{t_k - t_{k-1}}.$$

In particular,

$$\sum_{k=1}^{n} f'(x_k)(t_k - t_{k-1}) = \sum_{k=1}^{n}[f(t_k) - f(t_{k-1})] = f(b) - f(a)$$

is a Riemann sum of f' associated to the partition P. Moreover, we note that

$$L(f';P) \leq f(b) - f(a) \leq U(f';P) < L(f';P) + \varepsilon$$

and

$$L(f';P) \leq \int_a^b f'(x)dx \leq U(f';P) < L(f';P) + \varepsilon,$$

so

$$\left|\int_a^b f'(x)dx - [f(b) - f(a)]\right| < 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then

$$\int_a^b f'(x)dx = f(b) - f(a).$$

□

Theorem 67.6 (Integration by Parts)**.** Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If f' and g' are Riemann integrable on $[a,b]$, then

$$\int_a^b f(x)g'(x)dx + \int_a^b f'(x)g(x)dx = f(b)g(b) - f(a)g(a).$$

Proof. By Exercise 69.1, the product of two Riemann integrable functions is Riemann integrable. In particular, $f'g$ and fg' are Riemann integrable. Let $h : [a,b] \to \mathbb{R}$ by defined by $h(x) = f(x) \cdot g(x)$. Then h is continuous on $[a,b]$ and differentiable on (a,b), and

$$h'(x) = f'(x)g(x) + f(x)g'(x),$$

so h' is Riemann integrable on $[a,b]$. By the Fundamental Theorem of Calculus,

$$\int_a^b h'(x)dx = h(b) - h(a),$$

and so

$$\int_a^b f(x)g'(x)dx + \int_a^b f'(x)g(x)dx = f(b)g(b) - f(a)g(a).$$

□

Theorem 67.7 (Fundamental Theorem of Calculus)**.** Let $f:[a,b]\to\mathbb{R}$ be Riemann integrable. For $x\in[a,b]$, we define

$$F(x)=\int_a^x f(t)dt,$$

then F is continuous on $[a,b]$. Moreover, if f is continuous at a point $x_0\in(a,b)$, then F is differentibale at x_0 and $F'(x_0)=f(x_0)$.

Proof. For $a\le x<y\le b$, we have

$$\begin{aligned}F(y)-F(x)&=\int_a^y f(t)dt-\int_a^x f(t)dt\\&=\int_a^x f(t)dt+\int_x^y f(t)dt-\int_a^x f(t)dt\\&=\int_x^y f(t)dt.\end{aligned}$$

Since f is Riemann integrable, so f is bounded, and there exists $M>0$ such that $|f(x)|\le M$ for all $x\in[a,b]$. So

$$|F(y)-F(x)|\le\int_x^y|f(t)|dt\le M|y-x|.$$

This shows F is uniformly continuous on $[a,b]$: for any $\varepsilon>0$, if $|y-x|<\frac{\varepsilon}{M}$, then $|F(y)-F(x)|<\varepsilon$. Assume f is continuous at $x_0\in(a,b)$. For $x\in[a,b]\backslash\{x_0\}$,

$$\begin{aligned}\frac{F(x)-F(x_0)}{x-x_0}-f(x_0)&=\frac{1}{x-x_0}\int_{x_0}^x f(t)dt-f(x_0)\\&=\frac{1}{x-x_0}\int_{x_0}^x f(t)dt-\frac{1}{x-x_0}\int_{x_0}^x f(x_0)dt\\&=\frac{1}{x-x_0}\int_{x_0}^x[f(t)-f(x_0)]dt.\end{aligned}$$

Fix $\varepsilon>0$. Since f is continuous at x_0, then there exists $\delta>0$ such that $|f(x)-f(x_0)|<\varepsilon$ for all $x\in[a,b]$ such that $|x-x_0|<\delta$. Hence, for $x\in[a,b]$ with $0<|x-x_0|<\delta$, we have

$$\begin{aligned}\left|\frac{F(x)-F(x_0)}{x-x_0}-f(x_0)\right|&\le\frac{1}{|x-x_0|}\int_{x_0}^x|f(t)-f(x_0)|dt\\&<\frac{1}{|x-x_0|}\int_{x_0}^x\varepsilon dt\\&=\varepsilon.\end{aligned}$$

But since $\varepsilon>0$ is arbitrary, we know F is differentiable at x_0 and $F'(x_0)=f(x_0)$. □

68 Lecture 51: Change of Variables, Zero Outer Measure

Theorem 68.1 (Change of Variables)**.** Let J be an open interval in $\mathbb{R}$ and let $u : J \to \mathbb{R}$ be differentiable with u' continuous on J. Let I be an open interval in $\mathbb{R}$ such that $u(J) \subseteq I$ and let $f : I \to \mathbb{R}$ be continuous. Then $f \circ u : J \to \mathbb{R}$ is continuous, and for any $a, b \in J$ with $a < b$, we have

$$\int_a^b f(u(x)) \cdot u'(x)dx = \int_{u(a)}^{u(b)} f(y)dy.$$

Proof. As $f \circ u$ and u' are continuous on $[a, b]$, the function $x \mapsto (f \circ u)(x) \cdot u'(x)$ is continuous on $[a, b]$ and so it is Riemann integrable on $[a, b]$.

Fix $c \in I$ and consider $F(x) = \int_c^x f(t)dt$. By the Fundamental Theorem of Calculus, F is differentiable on I (since f is continuous on I) and $F'(x) = f(x)$ for all $x \in I$.

Consider $x \mapsto (F \circ u)(x)$ is differentiable on J, and $(F \circ u)'(x) = f(u(x)) \cdot u'(x)$ for all $x \in J$. By the Fundamental Theorem of Calculus,

$$\int_a^b (F \circ u)'(x)dx = (F \circ u)(b) - (F \circ u)(a).$$

Hence,

$$\int_a^b f(u(x)) \cdot u'(x)dx = \int_c^{u(b)} f(y)dy - \int_c^{u(a)} f(y)dy = \int_{u(a)}^{u(b)} f(y)dy.$$

$\square$

Exercise 68.2. Let I be an open interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be injective and differentiable with f' continuous on I. Then $J = f(I)$ is an open interval and $f^{-1} : J \to I$ is differentiable. Then for any $a, b \in I$ with $a < b$, we have

$$\int_a^b f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(y)dy = bf(b) - af(a).$$

Proof. Denote $\Gamma_f = \{(x, f(x)) : a \leq x \leq b\} = \{(f^{-1}(y), y) : y \text{ between } f(a) \text{ and } f(b)\}$. We perform a change of variables: let $y = f(x)$, so $dy = f'dx$, then

$$\begin{aligned}
\int_{f(a)}^{f(b)} f^{-1}(y)dy &= \int_a^b f^{-1}(f(x))f'(x)dx \\
&= \int_a^b xf'(x)dx \\
&= xf(x) \,\Big|_{x=a}^{x=b} - \int_a^b f(x)dx \\
&= bf(b) - af(a) - \int_a^b f(x)dx.
\end{aligned}$$

$\square$

Theorem 68.3. Let $f_n : [a,b] \to \mathbb{R}$ be Riemann integrable such that $f_n \xrightarrow[n\to\infty]{u} f$ on $[a,b]$. Then f is Riemann integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx.$$

Proof. For $n \geq 1$ let $d_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$. As $f_n \xrightarrow[n\to\infty]{u} f$ on $[a,b]$, we have $d_n \xrightarrow[n\to\infty]{} 0$. In particular, $f_n(x) - d_n \leq f(x) \leq f_n(x) + d_n$ for all $x \in [a,b]$, and so f is bounded.

For any partition P of $[a,b]$, we have

$$U(f_n; P) - d_n(b-a) \leq U(f; P) \leq U(f_n; P) + d_n(b-a)$$

and

$$L(f_n; P) - d_n(b-a) \leq L(f; P) \leq L(f_n; P) + d_n(b-a).$$

Therefore, $U(f;P) - L(f;P) \leq U(f_n;P) - L(f_n;P) + 2d_n(b-a)$. Fix $\varepsilon > 0$. As $d_n \xrightarrow[n\to\infty]{} 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $d_n < \frac{\varepsilon}{4(b-a)}$ for all $n \geq n_\varepsilon$. Then for each $n \geq n_\varepsilon$ (fixed), there exists a partition $P = P(\varepsilon, n)$ of $[a,b]$ such that $U(f_n;P) - L(f_n;P) < \frac{\varepsilon}{2}$. For $n \geq n_\varepsilon$ and $P = P(\varepsilon, n)$ as above, we get $U(f;P) - L(f;P) < \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, this shows that f is Riemann integrable (since it is Darboux integrable and bounded). Moreover,

$$\begin{aligned}
\int_a^b f(x)dx &\leq U(f;P) \\
&\leq U(f_n;P) + d_n(b-a) \\
&< L(f_n;P) + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} \\
&\leq \int_a^b f_n(x)dx + \frac{3}{4}\varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
\int_a^b f(x)dx &\geq L(f;P) \\
&\geq L(f_n;P) - d_n(b-a) \\
&> U(f_n;P) - \frac{\varepsilon}{2} - \frac{\varepsilon}{4} \\
&\geq \int_a^b f_n(x)dx - \frac{3}{4}\varepsilon,
\end{aligned}$$

so for all $n \geq n_\varepsilon$, we have

$$\left| \int_a^b f(x)dx - \int_a^b f_n(x)dx \right| < \frac{3}{4}\varepsilon,$$

thus

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

$\square$

Definition 68.4 (Zero Outer Measure)**.** A set $A \subseteq \mathbb{R}$ is said to have zero outer measure if for every $\varepsilon > 0$, there exists a countable collection of open intervals $\{(a_n, b_n)\}_{n\geq 1}$ such that $A \subseteq \bigcup_{n\geq 1}(a_n, b_n)$ and $\sum_{n\geq 1}(b_n - a_n) < \varepsilon$.

Remark 68.5.

1. If $A \subseteq \mathbb{R}$ has zero outer measure and $B \subseteq A$, then B has zero outer measure.

2. If $\{A_n\}_{n\geq 1}$ is a sequence of zero outer measure sets, then $\bigcup_{n\geq 1} A_n$ has zero outer measure.

   Indeed, fix $\varepsilon > 0$. For each $n \geq 1$, let $\{(a_n^{(n)}, b_m^{(n)}\}_{m\geq 1}$ be open intervals such that $A_n \subseteq \bigcup_{m\geq 1}(a_m^{(n)}, b_m^{(n)})$ and $\sum_{m\geq 1}(b_m^{(n)} - a_m^{(n)}) < \frac{\varepsilon}{2^n}$, then $\{(a_n^{(n)}, b_m^{(n)}\}_{m,n\geq 1}$ is a countable collection of open intervals such that

   $$\bigcup_{n\geq 1} A_n \subseteq \bigcup_{n,m\geq 1}(a_m^{(n)}, b_m^{(n)})$$

   and

   $$\sum_{n\geq 1}\sum_{m\geq 1}(b_m^{(n)} - a_m^{(n)}) < \sum_{n\geq 1}\frac{\varepsilon}{2^n} = \varepsilon.$$

3. If A is a set that is at most countable, then A has zero outer measure.

69 Homework 8

Exercise 69.1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $M > 0$ be such that $|f(x)| \leq M$ for all $x \in [a, b]$.

(a) Show that if P is a partition of $[a, b]$, then

$$U(f^2; P) - L(f^2; P) \leq 2M[U(f; P) - L(f; P)].$$

(b) Deduce that if f is integrable on $[a, b]$, then f^2 is also integrable on $[a, b]$.

(c) Prove that if f and g are two integrable functions on $[a, b]$, then the product fg is integrable on $[a, b]$.

Exercise 69.2. Let $f, g : [a, b] \to \mathbb{R}$ be two Riemann integrable functions such that the set $\{x \in [a, b] : f(x) = g(x)\}$ is dense in $[a, b]$. Show that

$$\int_a^b f(x)dx = \int_a^b g(x)dx.$$

Exercise 69.3. Suppose $f : [1, \infty) \to \mathbb{R}$ is Riemann integrable on $[1, a]$ for all $a > 1$. If

$$\lim_{a\to\infty} \int_1^a f(x)dx$$

exists and is finite, we say the integral $\int_1^\infty f(x)dx$ converges and we write

$$\int_1^\infty f(x)dx = \lim_{a\to\infty} \int_1^a f(x)dx.$$

Now assume $f : [1, \infty) \to \mathbb{R}$ is non-negative and decreasing. Show that $\int_1^\infty f(x)dx$ converges if and only if $\sum_{n\geq 1} f(n)$ converges.

Exercise 69.4. Let $f : [1, \infty) \to \mathbb{R}$ be a Riemann integrable function such that $f \geq 0$ and $\int_a^b f(x)dx = 0$. Show that if $x \in [a, b]$ is a point of continuity for f, then $f(x) = 0$.

Exercise 69.5. Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function such that

$$\int_a^b x^n f(x)dx = 0$$

for all $n \geq 0$. Show that if $x \in [a, b]$ is a point of continuity for f, then $f(x) = 0$.

Exercise 69.6. Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable functions such that g is monotone. Show that there exists $x_0 \in [a, b]$ such that

$$\int_a^b f(x)g(x)dx = g(a) \int_a^{x_0} f(x)dx + g(b) \int_{x_0}^b f(x)dx.$$

Hint: Show that if g is monotonically decreasing on $[a, b]$ with $g(b) = 0$, then

$$g(a) \inf_{x\in[a,b]} \int_a^x f(t)dt \leq \int_a^b f(x)g(x)dx \leq g(a) \sup_{x\in[a,b]} \int_a^x f(t)dt.$$

Exercise 69.7. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and define $F : \mathbb{R} \to \mathbb{R}$ via

$$F(x) = \int_{x-1}^{x+1} f(t)dt.$$

Show that F is differentiable and compute its derivative.

Exercise 69.8. For $n \geq 1$, let $f_n : [0,1] \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} n, & \text{if } 0 \leq x \leq \frac{1}{n} \\ 0, & \text{if } \frac{1}{n} < x \leq 1 \end{cases}.$$

(a) Show that $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in (0,1]$.

(b) Show that for each $n \geq 1$, f_n is Riemann integrable and satisfies

$$\int_0^1 f_n(x)dx = 1.$$

70 Lecture 52: Lebesgue Criterion, Improper Riemann Integral

Theorem 70.1 (Lebesgue Criterion)**.** Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable if and only if the set

$$\mathcal{D}_f = \{x \in [a,b] : f \text{ is discontinuous at } x\}$$

has zero outer measure.

Proof. ($\Rightarrow$): Assume that f is Riemann integrable. We write

$$\begin{aligned}\mathcal{D}_f &= \{x \in [a,b] : \omega(f,x) > 0\} \\ &= \bigcup_{n\geq 1}\{x \in [a,b] : \omega(f,x) \geq \frac{1}{n}\}.\end{aligned}$$

For $n \geq 1$, let $F_n = \{x \in [a,b] : \omega(f,x) \geq \frac{1}{n}\}$. In view of Remark 68.5, to show that $\mathcal{D}_f$ has zero outer measure, it suffices to prove that F_n has zero outer measure for all $n \geq 1$.

Fix $N \geq 1$ and $\varepsilon > 0$. As f is Riemann integrable, there exists a partition $P = \{a = t_0 < \ldots < t_n = b\}$ such that $U(f;P) - L(f;P) < \frac{\varepsilon}{N}$. Let $I = \{1 \leq k \leq n : F_N \cap (t_{k-1}, t_k) \neq \varnothing\}$, then $F_N \subseteq \bigcup_{k\in I}(t_{k-1}, t_k) \cup P$. Since P is finite, it has zero outer measure. Thus, it suffices to show that

$$\sum_{k\in I}(t_k - t_{k-1}) < \varepsilon.$$

Now note that

$$\begin{aligned}
\frac{\varepsilon}{N} &> U(f;P)-L(f;P)\\
&=\sum_{k=1}^{n}[M(f;[t_{k-1},t_k])-m(f;[t_{k-1},t_k])](t_k-t_{k-1})\\
&\geq \sum_{k\in I}\omega(f;[t_{k-1},t_k])(t_k-t_{k-1})\\
&\geq \frac{1}{N}\sum_{k\in I}(t_k-t_{k-1}),
\end{aligned}$$

and therefore $\sum\limits_{k\in I}(t_k-t_{k-1})<\varepsilon$.

($\Leftarrow$): Assume that $\mathcal{D}_f$ has zero outer measure. Since f is bounded, then there exists $M>0$ such that $|f(x)|\leq M$ for all $x\in[a,b]$. Fix $\varepsilon>0$ and let $\alpha>0$ to be chosen later. Consider $F_\alpha=\{x\in[a,b]:\omega(f,x)\geq\alpha\}\subseteq\mathcal{D}_f$, and since $\mathcal{D}_f$ has zero outer measure, F_α has zero outer measure as well. Thus, there exists $\{(a_n,b_n)\}_{n\geq 1}$ such that $F_\alpha\subseteq\bigcup\limits_{n\geq 1}(a_n,b_n)$ and $\sum\limits_{n\geq 1}(b_n-a_n)<\varepsilon$.

Let $A=[a,b]\backslash F_\alpha$. For any $x\in A$, $\omega(f,x)<\alpha$, there exists a neighborhood (c_x,d_x) of x such that $\omega(f;[c_x,d_x])<\alpha$. So $[a,b]=F_\alpha\cup A\subseteq\bigcup\limits_{n\geq 1}(a_n,b_n)\cup\bigcup\limits_{x\in A}(c_x,d_x)$, but since $[a,b]$ is compact, then there exists $n_0\in\mathbb{N}$ and a finite subset $J\subseteq A$ such that

$$[a,b]\subseteq\bigcup_{k=1}^{n_0}(a_k,b_k)\cup\bigcup_{x\in J}(c_x,d_x).$$

Let P be a partition of $[a,b]$ formed by the points

$$\left(\{a,b\}\cup\bigcup_{k=1}^{n_0}\{a_k,b_k\}\cup\bigcup_{x\in J}\{c_x,d_x\}\right)\cap[a,b],$$

and say $P=\{a=t_0<\ldots<t_n=b\}$. For any $1\leq l\leq n$, we have $[t_{l-1},t_l]\subseteq[a_k,b_k]$ for some $1\leq k\leq n_0$, or $[t_{l-1},t_l]\subseteq[c_x,d_x]$ for some $x\in J$.

Let $I_1=\{1\leq l\leq n:[t_{l-1},t_l]\subseteq[a_k,b_k]\text{ for some }1\leq k\leq n_0\}$, and $I_2=\{1,\ldots,n\}\backslash I_1$. Note that

- $\sum\limits_{l\in I_1}(t_l-t_{l-1})\leq\sum\limits_{k=1}^{n_0}(b_k-a_k)<\varepsilon$, and
- for $l\in I_2$, $\omega(f;[t_{l-1},t_l])\leq\omega(f;[c_x,d_x])<\alpha$.

Now

$$\begin{aligned} U(f;P)-L(f;P) &= \sum_{k=1}^{n}[M(f;[t_{l-1},t_l])-m(f;[t_{l-1},t_l])](t_l-t_{l-1}) \\ &= \sum_{l\in I_1}[M(f;[t_{l-1},t_l])-m(f;[t_{l-1},t_l])](t_l-t_{l-1}) \\ &+ \sum_{l\in I_2}\omega(f;[t_{l-1},t_l])\cdot(t_l-t_{l-1}). \end{aligned}$$

In particular, we have

$$\begin{aligned} \sum_{k=1}^{n}[M(f;[t_{l-1},t_l])-m(f;[t_{l-1},t_l])](t_l-t_{l-1}) &\leq 2M\sum_{l\in I_1}(t_l-t_{l-1}) \\ &< 2M\varepsilon, \end{aligned}$$

and

$$\begin{aligned} \sum_{l\in I_2}\omega(f;[t_{l-1},t_l])\cdot(t_l-t_{l-1}) &< \alpha\sum_{l\in I_2}(t_l-t_{l-1}) \\ &\leq \alpha\sum_{l=1}^{n}(t_l-t_{l-1}) \\ &= \alpha(b-a). \end{aligned}$$

We now choose $\alpha < \frac{\varepsilon}{b-a}$ to get

$$U(f;P)-L(f;P) < 2M\varepsilon+\varepsilon.$$

Since ε is arbitrary, this shows that f is Darboux integrable, and so Riemann integrable. $\square$

Definition 70.2 (Improper Riemann Integral)**.** Let $-\infty < a < b \leq \infty$. We say that $f:[a,b)\to\mathbb{R}$ is locally Riemann integrable if f is integrable on $[a,c]$ for any $c\in(a,b)$. If in addition,

$$\lim_{c\to b}\int_a^c f(x)dx$$

exists in $\mathbb{R}$, we denote it $\int_a^b f(x)dx$ and we call it the improper Riemann integral of f. In this case, we say that the improper Riemann integral of f converges.

If

$$\lim_{c\to b}\int_a^c f(x)dx = \pm\infty,$$

then we write $\int_a^b f(x)dx=\pm\infty$ and we say that the improper Riemann integral of f diverges to $\pm\infty$.

Remark 70.3. One can make as similar definition if $-\infty \leq a < b < \infty$ and $f : (a, b] \to \mathbb{R}$, or if $-\infty \leq a < b \leq \infty$, and $f : (a, b) \to \mathbb{R}$.

Theorem 70.4. Let $-\infty < a < b < \infty$ and let $f : [a, b) \to \mathbb{R}$ be locally Riemann integrable and bounded. Then the improper Riemann integral $\int_a^b f(x)dx$ converges. Moreover, any extension $\tilde{f} : [a, b] \to \mathbb{R}$ of f to $[a, b]$ is Riemann integrable and

$$\int_a^b \tilde{f}(x)dx = \int_a^b f(x)dx.$$

Proof. Let $\tilde{f} : [a, b] \to \mathbb{R}$ be an extension of f to $[a, b]$. As f is bounded, there exists $M > 0$ such that $|\tilde{f}(x)| \leq M$ for all $x \in [a, b]$. For $c \in (a, b)$, we write

$$U_a^b(\tilde{f}) = U_a^c(\tilde{f}) + U_c^b(\tilde{f}) = \int_a^c f(x)dx + U_c^b(\tilde{f}),$$

and

$$L_a^b(\tilde{f}) = L_a^c(\tilde{f}) + L_c^b(\tilde{f}) = \int_a^c f(x)dx + L_c^b(\tilde{f}).$$

Therefore, $U_a^b(\tilde{f}) - L_a^b(\tilde{f}) = U_c^b(\tilde{f}) - L_c^b(\tilde{f})$. Note that $U_c^b(\tilde{f}) \leq M(b - c)$, and $|L_c^b(\tilde{f})| \leq M(b - c)$, then

$$U_a^b(\tilde{f}) - L_a^b(\tilde{f}) \leq 2M(b - c) \xrightarrow[c \to b]{} 0.$$

This shows that $\tilde{f}$ is Riemann integrable. Moreover, we note that

$$\int_a^b \tilde{f}(x)dx = \lim_{c \to b} \int_a^c f(x)dx,$$

then the improper Riemann integral of f converges and

$$\int_a^b f(x)dx = \int_a^b \tilde{f}(x)dx.$$

$\square$

71 Lecture 53: Improper Riemann Integral, Continued

Proposition 71.1.

72 Lecture 54: Improper Riemann Integral, Continued

73 Homework 9

74 Lecture 55: Continuous 1-periodic Functions, Fourier Series

75 Lecture 56: Fejér Kernel, Baire Property

76 Lecture 57: Baire Property, Continued

77 Homework 10