

MATH 595 (Group Cohomology) Notes

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September 15, 2023

1 AUG 21, 2023: INTRODUCTION

Group cohomology works over different settings of groups, like finite groups, profinite groups, and topological groups. The course will develop towards

- duality in $H^*(G, -)$, and
- focus on computations, e.g., spectral sequences.

We first establish some notations.

- Let G be a group. If G has a topology, that would also be part of the information of G .
- A (left) G -module is an abelian group M with an action map

$$\begin{aligned} G \times M &\rightarrow M \\ (g, m) &\mapsto g \cdot m = gm \end{aligned}$$

satisfying

- $1 \cdot m = m$,
- $(gh) \cdot m = g \cdot (hm)$,
- $g(m + m') = gm + gm'$.

Remark 1.1. If G is a finite group, then the associated (non-commutative) group ring $\mathbb{Z}[G] := \bigoplus_{g \in G} \mathbb{Z}e_g$, where the multiplication is determined by $e_g e_h = e_{gh}$. Therefore, a G -module is just a $\mathbb{Z}[G]$ -module.

Example 1.2. • Trivial module \mathbb{Z} , or any abelian group with the trivial action $g \cdot a = a$.

- C_2 , or any group with $f : G \twoheadrightarrow C_2$, then G with C_2 as a quotient gives the sign representation \mathbb{Z}_{sgn} , with $g \cdot (a) = (-1)^{\rho(g)} a$.
- $\mathbb{Z}[G]$ is a G -module via the left multiplication action, and/or the conjugation action.

Definition 1.3 (Fixed points/Invariants). The set of fixed points of M over G is $M^G = \{m \in M \mid gm = m \ \forall g \in G\}$.

Definition 1.4 (Orbits/Coinvariants). The set of orbits of M over G is $M_G = M/(gm - m)$.

Example 1.5. If $M = \mathbb{Z}_{\text{sgn}}$, then everything gets multiplied by -1 , so there are no fixed points. The orbits of M over G would be $\mathbb{Z}_{\text{sgn}}/(-2) \cong \mathbb{Z}/2\mathbb{Z}$.

Example 1.6. If $M = \mathbb{Z}[G]$, then the fixed points are $\mathbb{Z} \left\{ \sum_{g \in G} e_g \right\}$.

Thinking in a categorical setting, we have a trivial action function $\mathbb{Z}\text{-Mod} \rightarrow G\text{-Mod}$, sending $ga \mapsto a$ for all $g \in G$ and $a \in A$. This gives an exact functor from \mathbf{Ab} to $G\text{-Mod}$. Then this functor has a right adjoint $()^G : G\text{-Mod} \rightarrow \mathbf{Ab}$, and a left adjoint $()_G : \mathbf{Ab} \rightarrow G\text{-Mod}$. More specifically, M^G becomes the maximal trivial action submodule of M , namely $\text{Hom}_G(\mathbb{Z}, M)$; M_G becomes the largest quotient of M with trivial action, namely $\mathbb{Z} \otimes_{\mathbb{Z}[G]} M$. This simplifies to the tensor-hom adjunction in some sense. For a more detailed derivation of this, see Chapter 6.1 of Weibel.

Remark 1.7. In general, as in the category of G -sets, we have the orbit functor $X \mapsto X/G$ and the fixed point functor $X \mapsto X^G$. The orbit functor is left adjoint to the free G -set functor, and the fixed point functor is the right adjoint of the trivial G -set functor.

Remark 1.8. Read more about the setting in profinite groups with their topologies in Neukirch-Schmidt-Wingberg.

Definition 1.9 (Profinite Group). A profinite group of a collection of groups is $G = \varprojlim_i G_i$ as an inverse limit, where each G_i is a finite group of the form G/U_i for some open U_i . This gives a topology to the profinite group.

Remark 1.10. The groups rings $\mathbb{Z}[[G]] = \varprojlim_i \mathbb{Z}[G_i]$. For instance, let $G = \hat{\mathbb{Z}}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$, then $\mathbb{Z}_p[[G]] = \varprojlim_n \mathbb{Z}_p[\mathbb{Z}/p^n\mathbb{Z}]$, where each $\mathbb{Z}[\mathbb{Z}/p^n\mathbb{Z}] \cong \bigoplus_{i=0}^{n-1} \mathbb{Z}\{e_i\}$ where $e_i \cdot e_j = e_{ij}$. Therefore, $\mathbb{Z}_p[[G]]$ is now equivalent to $\varprojlim_n \mathbb{Z}_p[t]/(t^{p^n} - 1_e)$, and hence becomes a power series.

Remark 1.11. By a change of variables, this becomes $\varprojlim_n \mathbb{Z}_p[x]/(x^{p^n})$, but this only works in the finite group \mathbb{Z}_p case, and not in general for \mathbb{Z} .

Example 1.12. $\mathbb{Z}[C_n] \cong \mathbb{Z}\{e\} \oplus \mathbb{Z}\{g\} \oplus \mathbb{Z}\{g^2\} \oplus \cdots \oplus \mathbb{Z}\{g^{n-1}\} \cong \mathbb{Z}[g]/(g^n - 1_e)$.

2 AUG 23, 2023: COHOMOLOGY OF GROUPS

Definition 2.1. Let G be a group, then we have a diagram

$$EG : \cdots \rightrightarrows G \times G \rightrightarrows G$$

where the arrows are given by

$$EG^n = G^{n+1} \xrightarrow{d_i} G^n$$

for all $0 \leq i \leq n$. In the sense of simplicial sets, we have $d_i(g_0, \dots, g_n) = (g_0, \dots, \hat{g}_i, \dots, g_n)$.

Now let M be a G -module, then we define $X^n = X^n(G, M) = \text{Map}_{\text{Set}}(G^{n+1}, M)$. G now has an action on this set, given by

$$(g \circ f)(g_0, \dots, g_n) = gf(g^{-1}g_0, \dots, g^{-1}g_n).$$

The action on d^i 's are contravariant, namely we obtain $d_i^* : X_n \rightarrow X^{n+1}$ with an inherited structure. Note that M sits inside X^0 , therefore we have a complex $(*)$:

$$0 \longrightarrow M \xleftarrow{\partial_0} X^0 \xrightarrow{\partial_1} X^1 \xrightarrow{\partial_2} X^2 \xrightarrow{\partial_3} \cdots$$

Here ∂_0 includes M as the constant functions into X , namely $\partial_0(m) = f$ for $f(g) = m$, and so on. In general, for $n > 0$, we have

$$\partial_n = \sum_{i=0}^n (-1)^i d_i^*.$$

Lemma 2.2. The complex $(*) : M \rightarrow X^\bullet$ is an exact complex of G -modules, i.e., $\partial^2 = 0$ and $\ker(\partial_{n+1}) = \text{im}(\partial_n)$, and the ∂_i 's preserves the G -action. This is called the standard resolution of M as a G -module.

Proof. Exercise. □

Definition 2.3. The G -fixed points of the X^n 's are defined by $C^n(G, M) = (X^n(G, M))^G$, called the homogeneous n -cochains of G with coefficients in M . Because the complex preserves G -actions, then we obtain a complex of $C^n(G, M)$'s, given by

$$0 \longrightarrow C^0(G, M) \xrightarrow{\partial_0} C^1(G, M) \xrightarrow{\partial_1} \dots$$

Remark 2.4. To see what the induced mapping is, suppose $A \rightarrow B$ is a G -module map, then there is an induced map of fixed points $A^G \rightarrow B^G$ by the restriction. In particular, let $a \in A$ be fixed with $ga = a$ for all $g \in G$, then $f(a) = f(ga) = gf(a)$.

Remark 2.5. In the complex of Definition 2.3, $\partial^2 = 0$ as well, but in general this is not an exact sequence.

Definition 2.6 (Group Cohomology). The group cohomology of G with coefficients in M is the collection

$$\{H^n(G, M)\}_{n \geq 0},$$

where $H^n(G, M) := H^n(C^\bullet(G, M)) = \ker(\partial : C^n \rightarrow C^{n+1}) / \text{im}(\partial : C^{n-1} \rightarrow C^n)$. We usually use the notion of cocycles $Z^n(G, M) = \ker(\partial : C^n \rightarrow C^{n+1})$ and coboundaries $B^n(G, M) = \text{im}(\partial : C^{n-1} \rightarrow C^n)$.

Exercise 2.7. Show that $H^0(G, M)$ is isomorphic to M^G .

Definition 2.8. The inhomogeneous cochains $C_i^n(G, M)$ are given by

- $C_i^0 = M$, and
- for $n > 0$, $C_i^n = \text{Map}(G^n, M)$,

with coboundary maps $\partial^{n+1} : C_i^n \rightarrow C_i^{n+1}$, given by

- $\partial^1(m)(g) = gm - m$,
- $\partial^2(f)(g_1, g_2) = g_1 f(g_2) - f(g_1 g_2) + f(g_1)$, and so on, with
- $\partial^{n+1}(f)(g_1, \dots, g_{n+1}) = g_1 f(g_2, \dots, g_{n+1}) + \sum_{i=1}^n (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) + (-1)^{n+1} f(g_1, \dots, g_n)$.

This gives the inhomogeneous setting of this cochain.

Lemma 2.9. The maps

$$\begin{aligned} C^n(G, M) &\rightarrow C_i^n(G, M) \\ (\varphi : G^{n+1} \rightarrow M) &\mapsto (f : G^n \rightarrow M) \\ f(g_1, \dots, g_n) &:= \varphi(1, g_1, g_1 g_2, \dots, g_1 g_2 \cdots g_n) \end{aligned}$$

give a cochain homotopy equivalence $C^\bullet(G, M) \xrightarrow{\sim} C_i^\bullet(G, M)$, and hence this is a quasi-isomorphism.

Corollary 2.10. The cohomology $H^*(C_i^\bullet(G, M)) \cong H^*(G, M)$.

Remark 2.11. Any cohomology class can be represented by a normalized inhomogeneous cocycle $f : G^n \rightarrow M$, i.e., $f(g_1, \dots, g_n) = 0$ where $g_i = 1$ for some i .

Remark 2.12. Even for $G = C_2$, C_i^n or C^n get large as n grows.

Remark 2.13. • Using homological algebra, we can find other cochain complexes which computes group cohomology $H^*(G, M)$.

- We would also understand $H^*(G, M)$ as the failure of exactness of $()^G : G\text{-Mod} \rightarrow \mathbf{Ab}$. Therefore, when taking the fixed points, the exact sequence may not be mapped to another exact sequence. In particular, if we take an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of G -modules, the induced sequence

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G$$

do not give a surjection at $B^G \rightarrow C^G$. One needs to take higher cohomology to obtain a long exact sequence. Hence, $()^G : G\text{-Mod} \rightarrow \mathbf{Ab}$ is a left exact functor, but not necessarily right exact.

3 AUG 25, 2023: COHOMOLOGY OF GROUPS, CONTINUED

Example 3.1. Let G be C_2 , or any group with a surjection p onto C_2 , then it has an action on \mathbb{Z}_{sgn} given by $g \cdot a = (-1)^{p(g)}a$, therefore we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}_{\text{sgn}} \xrightarrow{\times 2} \mathbb{Z}_{\text{sgn}} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

and taking the fixed point functor we have

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

Remark 3.2. Higher homologies measure the failure of exactness.

Remark 3.3. The collection $\{H^n(G, -)\}_{n \in \mathbb{Z}}$ satisfies

- $H^n(G, -) = 0$ for $n < 0$;
- for short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $G\text{-Mod}$, we have a long exact sequence

$$0 \longrightarrow H^0(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, C) \xrightarrow{\delta} H^1(G, A) \longrightarrow \cdots$$

where δ is the connecting homomorphism.

- the connecting homomorphisms δ are natural, i.e., given a commuting diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \longrightarrow 0 \end{array}$$

the induced diagram

$$\begin{array}{ccc} H^n(G, C) & \xrightarrow{\delta} & H^{n+1}(G, A) \\ \downarrow & & \downarrow \\ H^n(G, C') & \xrightarrow{\delta} & H^{n+1}(G, A') \end{array}$$

also commutes, and $\{H^n(G, -)\}_{n \in \mathbb{Z}}$ is a cohomological δ -functor. Note that a δ -functor is additive, and usually occurs in abelian categories.

Definition 3.4 (δ -functor). A map of δ -functors $T^* \rightarrow F^*$ is a collection of natural transformations $T^n \rightarrow F^n$, commuting with the δ 's, i.e.,

$$\begin{array}{ccc} T^n & \longrightarrow & F^n \\ \delta_T \downarrow & & \downarrow \delta_F \\ T^{n+1} & \longrightarrow & F^{n+1} \end{array}$$

A δ -functor T^* is universal if, given any other δ -functor F^* , a map $T^* \rightarrow F^*$ is uniquely determined by $T^0 \rightarrow F^0$.

Proposition 3.5. $H^*(G, -) : G\text{-Mod} \rightarrow \mathbf{Ab}$ is a δ -functor.

Proof. We need to show:

- each $H^n(G, -)$ is a well-defined functor,
- the connecting homomorphisms δ 's gives a long exact sequence,
- the naturality of δ .

First, let $f : A \rightarrow B$ be in $G\text{-Mod}$, then $C^*(G, A) \rightarrow C^*(G, B)$ is equivalent to $\text{Map}(G^{*+1}, A)^G \rightarrow \text{Map}(G^{*+1}, B)^G$ by composition with f . One can show that this is equivariant, i.e., respects the G -action, so it is well-defined to take the fixed points, and thus commutes with ∂ 's.

Second, we need to apply the snake lemma. Given a short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, we claim:

Claim 3.6. $0 \rightarrow C^*(G, A) \rightarrow C^*(G, B) \rightarrow C^*(G, C) \rightarrow 0$ is a short exact sequence of cochain complexes, i.e., $C^*(G, -) : G\text{-Mod} \rightarrow \mathbf{coCh}$ is an exact functor.

Subproof. Exercise. ■

Now take the complex

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^n(G, A) & \longrightarrow & C^n(G, B) & \longrightarrow & C^n(G, C) \longrightarrow 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\ 0 & \longrightarrow & C^{n+1}(G, A) & \longrightarrow & C^{n+1}(G, B) & \longrightarrow & C^{n+1}(G, C) \longrightarrow 0 \end{array}$$

and quotient the boundaries everywhere (and thus lose the injectivity/surjectivity when applicable)

$$\begin{array}{ccccccc} C^n(G, A)/B^n(G, A) & \longrightarrow & C^n(G, B)/B^n(G, B) & \longrightarrow & C^n(G, C)/B^n(G, C) & \longrightarrow & 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\ 0 & \longrightarrow & Z^{n+1}(G, A) & \longrightarrow & Z^{n+1}(G, B) & \longrightarrow & Z^{n+1}(G, C) \end{array}$$

Taking the kernels and cokernels on ∂ 's, we obtain a complex

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & H^n(G, A) & \longrightarrow & H^n(G, B) & \longrightarrow & H^n(G, C) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & C^n(G, A)/B^n(G, A) & \longrightarrow & C^n(G, B)/B^n(G, B) & \longrightarrow & C^n(G, C)/B^n(G, C) \longrightarrow 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\ 0 & \longrightarrow & Z^{n+1}(G, A) & \longrightarrow & Z^{n+1}(G, B) & \longrightarrow & Z^{n+1}(G, C) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & H^{n+1}(G, A) & \longrightarrow & H^{n+1}(G, B) & \longrightarrow & H^{n+1}(G, C) \end{array}$$

By the snake lemma, we obtain the long exact sequence. □

Proposition 3.7. If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence such that $H^*(G, B) = 0$ for $* > 0$ (or at least $H^n(G, B) = 0 = H^{n+1}(G, B)$), then $\delta : H^n(G, C) \rightarrow H^{n+1}(G, A)$ is an isomorphism.

Definition 3.8 (Acyclic, Cohomologically Trivial). A G -module M is

- acyclic if $H^*(G, M) = 0$ for $* > 0$,
- cohomologically trivial if $H^*(H, M) = 0$ for $* > 0$ and any (closed) subgroup $H \subseteq G$.

Definition 3.9 (Induced Module). Given any G -module M , the induced module $\text{ind}_G(M) = \text{Map}(G, M) = X^0(G, M)$.

Example 3.10. M could have the trivial action.

Exercise 3.11. For any M , the induced module of M over G is isomorphic (under the G -action) to the induced module of module given by forgetful action over G .

Remark 3.12. • $\text{Ind}_G(-) : G\text{-Mod} \rightarrow G\text{-Mod}$ is exact.

- We say A is an induced module if $A \cong \text{Ind}_G(M)$ for some module M . If A is an induced G -module, then A is induced as an H -module for any subgroup $H \subseteq G$.

Lemma 3.13. Induced modules are cohomologically trivial.

Proof. There is an isomorphism

$$C^*(G, \text{Ind}_G(M)) \cong X^*(G, M).$$

□

Remark 3.14. We have an equivariant inclusion of fixed points

$$M \hookrightarrow \text{Ind}_G(M)$$

which is an embedding, and we take $Q \cong \text{Ind}_G(M)/M$, then this extends to a short exact sequence

$$0 \longrightarrow M \hookrightarrow \text{Ind}_G(M) \longrightarrow Q \longrightarrow 0$$

then $H^{n+1}(G, M) \cong H^n(G, Q)$. One say that $H^*(G, -)$ is effaceable. By Tohoku, an effaceable is universal.

4 AUG 28, 2023: FIRST COHOMOLOGY OF GROUPS

There are three ways to think about $H^1(G, M)$.

4.1 CROSSED HOMOMORPHISMS

Recall that $H^1(G, M) = Z^1_i(G, M)/B^1_i(G, M)$ as inhomogeneous cochains, where

- $Z^1_i(G, M) = \ker(\text{Map}(G, M) \rightarrow \text{Map}(G \times G, M))$ where the map sends $f \mapsto (g, h) \mapsto gf(h) - f(gh) + f(g)$. The kernel of this is exactly the maps f such that $f(gh) = gf(h) + f(g)$, and note that this is not a group homomorphism.
- $B^1_i(G, M) = \text{im}(M \rightarrow \text{Map}(G, M))$ given by $m \mapsto (g \mapsto gm - m)$, where the image is called a principal crossed homomorphism.

Exercise 4.1. $B^1_i(G, M) \cong M/M^G$ as an isomorphism of $\mathbb{Z}[G]$ -modules.

Remark 4.2. If the G -action is trivial, then $H^1(G, M) = \text{Hom}_{\text{Grp}}(G, M)$.

Corollary 4.3. If G is a finite group with trivial action, then $H^1(G, \mathbb{Z}) = 0$.

Theorem 4.4 (Hilbert's Theorem 90). Let L/K be a Galois extension with (finite or profinite) Galois group G , then $H^1(G, L^\times) = 0$.

Proof. Let $f : G \rightarrow L^\times$ be a crossed homomorphism. We know the addition is given by $f(gh) = gf(h) + f(g)$, and the multiplication is given by $f(gh) = (g \cdot f(h))f(g)$, where \cdot represents the group action. Now for any $l \in L^\times$, the multiplication with respect to l is given by $m_l = \sum_{h \in G} f(h)(h \cdot l)$. We can first choose l so that $m_l \neq 0$, since the Galois conjugates $h \cdot l$ over $l \in L$ are linearly independent. For $g \in G$, we have

$$\begin{aligned} g \cdot m_l &= \sum_{h \in G} (g \cdot f(h))(gh \cdot l) \\ &= \sum_{h \in G} \frac{f(gh)}{f(g)} (gh \cdot l) \\ &= \frac{1}{f(g)} \sum_{h \in G} f(gh)(gh \cdot l) \\ &= \frac{1}{f(g)} m_l. \end{aligned}$$

Therefore, $f(g) = \frac{m_l}{g \cdot m_l}$. For any crossed homomorphism, there exists $m \in L^\times$ such that $f(g) = \frac{gm}{m}$, so every crossed homomorphism is principal. □

Exercise 4.5. Let G acts over a commutative ring R , then $H^1(G, R^\times)$ classifies invariant R -modules with a compatible G -action.

4.2 NON-ABELIAN H^1 AND TORSORS

Let A be a group with G -action, so let the action $g \cdot a = {}^g a$. Hence, $g \cdot (ab) = {}^g a {}^g b$. Define the G -cocycles to be $f : G \rightarrow A$ such that $f(gh) = f(g) {}^g f(h)$. Two cocycles f and f' are said to be cohomologous as $f \sim f'$ if there exists $a \in A$ such that for all $g \in G$, $f'(g) = a^{-1} f(g) {}^g a$. This becomes an equivalence relation on the set of G -cocycles with coefficients in A , then $H^1(G, A)$ is the set of equivalence classes of G -cocycles. Now the first cohomology $H^1(G, A)$ has only a pointed set structure with distinguished point $f \equiv 1$, the constant function at 1.

Exercise 4.6. This definition is equivalent to the inhomogeneous cochain definition in the abelian case.

Definition 4.7. An A -torsor is a G -set X with action

$$\begin{aligned} X \times A &\rightarrow A \\ (x, a) &\mapsto xa \end{aligned}$$

that is free and transitive, i.e., for any $x, y \in X$, there exists a unique $a \in A$ such that $y = xa$. Moreover, the action $X \times A \rightarrow X$ respects the G -action, i.e., ${}^g(xa) = {}^g x {}^g a$.

Remark 4.8. • A is an A -torsor.

- An isomorphism of A -torsors is a bijection that respects the G - and A - action.
- If $A \subseteq B$ is a sub- G -group, then bA is an A -torsor.
- An A -torsor is a principal A -bundle on the classifying space BG .

Theorem 4.9. There is a canonical bijection of pointed sets

$$H^1(G, A) \cong \text{Torsor}(G, A)$$

Proof. • The backwards map $\lambda : \text{Torsor}(G, A) \rightarrow H^1(G, A)$ is defined as follows: for $x \in \text{Torsor}(G, A)$, we want to define a cocycle $f(X) : G \rightarrow A$. For arbitrary $x \in X$, note that for any $g \in G$, there exists a unique $f_x(g) \in A$ such that ${}^g x = x f_x(g)$ by the simple transitivity of the A -action on X . To see this is well-defined, if we have another $y \in X$, then $y = xb$ for some $b \in A$, then $f_y(g) = b^{-1} f_x(g) {}^g b$, so f_x and f_y are cohomologous and define the same class in $H^1(G, A)$, which is defined to be the image $\lambda(X)$.

- To define $\mu : H^1(G, A) \rightarrow \text{Torsor}(G, A)$, given a cocycle $f : G \rightarrow A$, let X_f be the group A , then the action of A on X_f is by multiplication on the right, and one can twist the G -action on it using cocycle $f : G \rightarrow A$ with ${}^g x = f(g)gx$, which defines an A -torsor. This is well-defined.

□

Remark 4.10. Suppose

$$1 \longrightarrow A \longrightarrow B \xrightarrow{p} C \longrightarrow 1$$

is a short exact sequence of G -groups, i.e., A is a sub- G -group and $C \cong B/A$, then there is a long exact sequence

$$1 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \xrightarrow{\delta} H^1(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, C)$$

where δ is given by $\delta(c) = p^{-1}(c)$. For the exactness in the sense of pointed sets to work, the kernel is the subset mapping to the distinguished element.

4.3 EXTENSION SPLITTING

Consider the a split extension

$$1 \longrightarrow A \longrightarrow E \xrightarrow{p} G \longrightarrow 1$$

That is, E is the direct product $A \times G$ with group action $(a, g)(a', g') = (a {}^g a', gg')$, and by definition E is the semidirect product $A \rtimes G$. Equivalently, there exists a section (as group homomorphism) $s : G \rightarrow E$.

There is an equivalence relation on the set of sections to the projection $p : E \rightarrow G$, where the sections $s, s' : G \rightarrow E$ are conjugates if there exists $a \in A$ such that $s'(g) = a^{-1} s(g) a$. We denote $\text{sec}(E \rightarrow G)$ to be the conjugacy class of sections of p . Note that the class of trivial section $s : g \mapsto (1, g) \in E$ is the distinguished element.

Proposition 4.11. The pointed set $H^1(G, A)$ is isomorphic to $\text{sec}(E \rightarrow G)$.

Proof. Take $\varphi \in \text{sec}(E \rightarrow G)$, then the composition $G \xrightarrow{\varphi} E \xrightarrow{\pi_1} A$, where π_1 is the set-theoretic projection to the first component, defines a cocycle $G \rightarrow A$. Conversely, given a cocycle $f : G \rightarrow A$, the section is given by $g \mapsto (f(g), g)$. \square

Exercise 4.12. Expand the proof above.

Exercise 4.13. Describe $\mathbb{Z} \rtimes C_2$ where C_2 acts on \mathbb{Z} by inversion. How many sections are there of $\mathbb{Z} \rtimes C_2 \rightarrow C_2$?

Exercise 4.14. How many sections are there to the projection $D_{2n} \rightarrow C_2$?

5 AUG 30, 2023: H^2 , ABELIAN EXTENSIONS, AND BRAUER GROUP

Suppose we have an abelian extension, that is, let A be abelian, the short exact sequence of group extensions

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1$$

is such that $E/i(A) \cong G$. Note that A can be regarded as a normal subgroup in E given this notation.

Note that two extensions are equivalent if there exists a group isomorphism $\varphi : E \rightarrow E'$ such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G \longrightarrow 1 \\ & & \parallel & & \downarrow \varphi & & \parallel \\ 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G \longrightarrow 1 \end{array}$$

commutes.

Consider the continuous functions

$$\varphi : G \times G \rightarrow A$$

such that $\varphi(g_1g_2, g_3) + \varphi(g_1, g_2) = \varphi(g_1, g_2g_3) + g_1\varphi(g_2, g_3)$. We know $H^2(G, M)$ is the quotient of all such functions over the coboundaries, i.e., the functions φ such that $\varphi(g_1, g_2) = f(g_1) - f(g_1g_2) + g_1f(g_2)$.

Now $E \cong A \times G$ can be considered as a bijection, so we pick a set-theoretic section $s : G \rightarrow E$ with $s(1) = 1$, and now every element in E is written as $as(g)$ uniquely for some $a \in A$ and $g \in G$, we have

$$s(g)a = s(g)as(g)^{-1}s(g) = {}^g as(g).$$

Note that s may not be a homomorphism, but we have $s(g)s(h) = f(g, h)s(gh)$ since $s(g)s(h)$ and $s(gh)$ are both lifts of gh .

As a consequence, we have

$$(s(g_1)s(g_2))s(g_3) = f(g_1, g_2)s(g_1g_2)s(g_3) = f(g_1, g_2)f(g_1g_2, g_3)s(g_1g_2g_3)$$

and

$$s(g_1)(s(g_2)s(g_3)) = s(g_1)f(g_2, g_3)s(g_2, g_3) = {}^{g_1}f(g_2, g_3)s(g_1)s(g_2g_3) = {}^{g_1}f(g_2, g_3)f(g_1, g_2g_3)s(g_1g_2g_3).$$

In additive notation, we have

$$f(g_1, g_2) + f(g_1g_2, g_3) = g_1f(g_2, g_3) + f(g_1, g_2g_3).$$

Therefore, f becomes an inhomogeneous 2-cocycle.

Proposition 5.1. The induced map $\lambda : \text{ext}(G, A) \rightarrow H^2(G, A)$ is a well-defined bijection between the set of equivalence classes of extensions and $H^2(G, A)$.

Example 5.2. The two elements in $H^2(C_2, \mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ are given by non-split extension of Q_8

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow Q_8 \longrightarrow C_2 \longrightarrow 1$$

and the identity element given by $D_8 \cong \mathbb{Z}/4\mathbb{Z} \rtimes C_2$

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow D_8 \longrightarrow C_2 \longrightarrow 1$$

where D_8 has the action of C_2 over $\mathbb{Z}/4\mathbb{Z}$.

Proposition 5.3. An associative finite-dimensional K -algebra A is a CSA if and only if one of the following equivalent conditions hold:

1. Base-changed to the separable closure \bar{K} of K via $\bar{K} \otimes_K A$, $A \cong M_n(\bar{K})$ for some integer $n \geq 1$.
2. there exists a finite Galois extension L/K such that base-changed to L via $L \otimes_K A$, A becomes isomorphic to a matrix algebra $M_n(L)$ for some integer $n \geq 1$.
3. $A \cong M_n(D)$ matrix algebra for some $m \geq 1$ and some finite division algebra D over K .

A CSA A over K is said to be split over L if the above holds, i.e., $A \otimes_K L \cong M_n(L)$. One can define an equivalence class on CSAs, such that $A \sim B$ if and only if $A \otimes_K M_n(K) \cong B \otimes_K M_m(K)$. Now the Brauer group of K is the abelian group of equivalence classes of CSAs over K equipped with tensor product.

Suppose L/K is an extension, then there exists a homomorphism of base-change of algebras $\text{Br}(K) \rightarrow \text{Br}(L)$. We say the kernel $\text{Br}(L | K)$ is the relative Brauer group of K -CSAs that split over L . The absolute Brauer group is $\text{Br}(\bar{K} | K) = \text{Br}(K)$, then

$$\text{Br}(K) = \bigcup_{L/K \text{ finite}} \text{Br}(L | K).$$

Now let L/K be a finite Galois extension with Galois group G , and we pick a normalized inhomogeneous 2-cycle $\varphi : G \times G \rightarrow L^\times$ as the representative of its class, and we can construct A_φ as a K -CSA, then $A_\varphi = \bigoplus_{g \in G} L e_g$ has dimension $|G|^2$, where e_g 's are the generators, with a multiplication operation $(l e_g)(m e_h) = l(g \cdot m) \varphi(g, h) e_{gh}$ which can be extended via distribution. A_φ is said to be the crossed product of L and G via φ .

Theorem 5.4. 1. A_φ is a split algebra over L .

2. If φ, φ' are two normalized inhomogeneous 2-cocycles, then $A_\varphi \sim A_{\varphi'}$ if and only if $\varphi \sim \varphi'$.
3. $A_{\varphi\varphi'} \sim A_\varphi \otimes_K A_{\varphi'}$.
4. Any K -CSA which is split over L is similar to a crossed product A_φ for some $\varphi : G \times G \rightarrow L^\times$.

Corollary 5.5. $H^2(G, L^\times)$ is isomorphic to $\text{Br}(L | K)$, and $H^2(\text{Gal}(\bar{K}/K), \bar{K}^\times)$ is isomorphic to $\text{Br}(K)$.

6 SEPT 1, 2023: COHOMOLOGY OF CYCLIC AND FREE GROUPS

Recall that we can compute $H^*(G, M)$ using any acyclic resolution of M . We want to describe $H^*(G, M)$ for specific G using nice resolutions.

We have

$$\dots \rightarrow G^3 \xrightarrow{\delta} G^2 \xrightarrow{\delta} G$$

and to obtain $X^*(G, M)$ we map out of the resolution and into M , so $\text{Map}(G, M) \cong \text{Hom}(\mathbb{Z}[G], M)$ as G -modules, and in general we obtain

$$\text{Map}(G^k, M) \cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]^{\otimes k}, M)$$

as \mathbb{Z} -modules.

We denote F^{st} to be the standard free resolution given by

$$\mathbb{Z}[G]^{\otimes k} \xrightarrow{d} \mathbb{Z}[G]^{\otimes(k-1)} \rightarrow \dots \rightarrow \mathbb{Z}[G]^{\otimes 2} \xrightarrow{d_1 - d_0} \mathbb{Z}[G]$$

To obtain $X^*(G, M)$, we can map this into M . Now the standard resolution becomes an augmentation of \mathbb{Z} that makes $X^*(G, M)$ exact, free, and acyclic. The kernel of $\mathbb{Z}[G] \rightarrow \mathbb{Z}$ is the augmentation ideal of G as of $\mathbb{Z}[G]$. Since this is a G -equivariant map, then the augmentation ideal is a G -submodule of $\mathbb{Z}[G]$, as a free abelian group generated by the set $\{(g-1) \mid 1 \neq g \in G\}$.

Lemma 6.1. If $P_* \rightarrow \mathbb{Z}$ is any free resolution of \mathbb{Z} as a G -module, then for a G -module M , we have $H^*(G, M) \cong H^*(\text{Hom}(P_*, M))^G$.

Proof. Since each P_i is free, then $\text{Hom}(P_i, M)$ is an acyclic module, so $M \rightarrow \text{Hom}(P_*, M)$ is an acyclic resolution of M . Now apply Proposition 2.28 in the notes. \square

Remark 6.2. $H^*(G, M) \cong \text{Ext}_{\mathbb{Z}[G]}^*(\mathbb{Z}, M)$ as universal δ -functors.

Now let C_n be the cyclic group of order n , generated by element g , then $\mathbb{Z}[C_n] \cong \mathbb{Z}[g]/(g^n - 1)$, so we have $0 = g^n - 1 = (g-1)N_g$ in $\mathbb{Z}[C_n]$ where N_g is the norm element $N_g = 1 + g + \cdots + g^{n-1}$, so we have a free resolution of \mathbb{Z} :

$$\cdots \longrightarrow \mathbb{Z}[C_n] \xrightarrow{1-g} \mathbb{Z}[C_n] \xrightarrow{N_g} \mathbb{Z}[C_n] \xrightarrow{1-g} \mathbb{Z}[C_n] \xrightarrow{\varepsilon} \mathbb{Z}$$

where augmentation ε sends g to 1. This allows us to compute the cohomology of any C_n -modules.

Proposition 6.3. Let M be an C_n -module, then

$$H^i(G, M) = \begin{cases} M^G, & i = 0 \\ \{m \in M \mid N_g m = 0\}/(1-g)M, & i > 0 \text{ odd} \\ M^G/N_g M, & i > 0 \text{ even} \end{cases}$$

Proof. Taking $\text{Hom}(P_*, M)^G$ gives

$$\cdots \longleftarrow M \xleftarrow{1-g} M \xleftarrow{N_g} M \xleftarrow{1-g} M \longleftarrow \cdots$$

\square

Remark 6.4. If M has trivial action, then

$$H^i(G, M) = \begin{cases} M, & i = 0 \\ M[n], & i > 0 \text{ odd} \\ M/n, & i > 0 \text{ even} \end{cases}$$

where $M[n]$ is the n -torsion in M .

Now if $T = \mathbb{Z}$ be with generator t , then $\mathbb{Z}[T]$ is isomorphic to the Laurent polynomials, so we have a resolution

$$0 \longrightarrow \mathbb{Z}[T] \xrightarrow{1-t} \mathbb{Z}[T] \longrightarrow \mathbb{Z}$$

since $(1-t)$ is not a zero-divisor of $\mathbb{Z}[T]$. Therefore, taking $\text{Hom}(P_*, M)^T$ gives

$$0 \longleftarrow M \xleftarrow{1-t} M$$

$$H^i(T, M) = \begin{cases} M^T, & i = 0 \\ M_T, & i = 1 \\ 0, & \text{otherwise} \end{cases}$$

Now let X be a set, and let G_X be the free group on X .

Proposition 6.5. The augmentation ideal I_X is a free $\mathbb{Z}[G_X]$ -module, generated by the set $\{(x-1) \mid x \in X\}$, and so the exact sequence

$$0 \longrightarrow I_X \longrightarrow \mathbb{Z}[G_X] \longrightarrow \mathbb{Z} \longrightarrow 0$$

is a free resolution of \mathbb{Z} as a G_X -module.

Proof. As \mathbb{Z} -bases of I_X , we have $\{(g-1) \mid g \in G_X\}$, but $\{h(x-1) \mid h \in G, x \in X\}$ is also a \mathbb{Z} -linear basis for I_X . \square

Remark 6.6. Groups are free if and only if they have cohomological dimension 1.

7 SEPT 6, 2023: CUP PRODUCT

Remark 7.1. 1. A crossed homomorphism would be a group homomorphism when G has trivial action on M .

2. If X is an A -torsor, then there is a given G -action and a right A -action so that $X \times A \rightarrow X$ is given by a diagonal action compatible to the G -action. Therefore, ${}^g(x \cdot a) = {}^gx \cdot {}^ga$.

Definition 7.2. Let A and B be G -modules, then there is a notion of tensor product $A \otimes_G B$ as a G -module via the diagonal action $g(a \otimes b) = ga \otimes gb$. On the level of cochain, we have a cup product

$$\begin{aligned} C^p(G, A) \otimes C^q(G, B) &\xrightarrow{\sim} C^{p+q}(G, A \otimes B) \\ (\alpha : G^{p+1} \rightarrow A) \otimes (\beta : G^{q+1} \rightarrow B) &\mapsto (\alpha \smile \beta) \\ (g_0, \dots, g_{p+q}) &\mapsto \alpha(g_0, \dots, g_p) \otimes \beta(g_p, \dots, g_{p+q}) \end{aligned}$$

Proposition 7.3. $\partial(\alpha \smile \beta) = (\partial\alpha) \cup \beta + (-1)^{|\alpha|} \alpha \smile \partial\beta$.

Corollary 7.4. • If α and β are cocycles, then $\alpha \smile \beta$ is also a cocycle.

• If α is a cocycle β is a coboundary, or vice versa, then $\alpha \smile \beta$ is a coboundary. Indeed, if $\beta = \partial\gamma$, then $\partial(\alpha \smile \gamma) = (-1)^{|\alpha|} \alpha \smile \beta$.

Therefore, on the level of cohomology, we have a (bilinear) cup product as well:

$$H^p(G, A) \otimes H^q(G, B) \rightarrow H^{p+q}(G, A \otimes B)$$

Example 7.5. • If $p = q = 0$, then

$$\begin{aligned} H^0(G, A) \otimes H^0(G, B) &\cong A^G \otimes B^G \rightarrow H^0(G, A \otimes B) \cong (A \otimes B)^G \\ a \otimes b &\mapsto a \otimes b \end{aligned}$$

• By extending this property, we get a G -equivariant pairing $A \otimes B \rightarrow C$ and therefore

$$H^p(G, A) \otimes H^q(G, B) \xrightarrow{\sim} H^{p+q}(G, C).$$

Example 7.6. Let R be a commutative ring, and if there is a G -action on R , then the multiplication $m : R \otimes R \rightarrow R$ is G -equivariant, so we have a cup product

$$\smile : H^p(G, R) \otimes H^q(G, R) \rightarrow H^{p+q}(R)$$

This has the following properties:

1. This is natural in A, B , and C .
2. This is compatible with connecting homomorphism and exact sequences, that is,
 - Given short exact sequences

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

and

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

and pairing $A \otimes B \rightarrow C$, then this induces $A \otimes B \rightarrow C'$ and in the quotients we have $A'' \otimes B \rightarrow C''$, so $\delta(\alpha \smile \beta) = \delta\alpha \smile \beta$, so we have a commutative diagram¹

$$\begin{array}{ccccccc} A' \otimes B & \longrightarrow & A \otimes B & \longrightarrow & A'' \otimes B & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' \longrightarrow 0 \end{array}$$

¹This may require the assumption that the modules are flat.

and thus

$$\begin{array}{ccc} H^o(G, A'') \otimes H^q(G, B) & \longrightarrow & H^{p+q}(G, A'' \otimes B) \\ \downarrow \delta \otimes 1 & & \downarrow \delta \\ H^{p+1}(G, A') \otimes H^q(G, B) & \longrightarrow & H^{p+q+1}(G, A' \otimes B) \end{array}$$

• Given

$$0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$$

and

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

and pairings

$$\begin{array}{ccccccc} A \otimes B' & \longrightarrow & A \otimes B & \longrightarrow & A \otimes B'' & & \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' \longrightarrow 0 \end{array}$$

$$\text{so } \delta(\alpha \smile \beta) = (-1)^{|\alpha|} \alpha \smile \delta\beta.$$

Proof. Let $\alpha = [a]$ for $a : G^{p+1} \rightarrow A$ and $\beta = [b]$ for $b : G^{q+1} \rightarrow B''$, then there is a lift $\tilde{b} : G^{q+1} \rightarrow B \rightarrow B''$. Then we have

$$\begin{array}{ccccccc} C^q/B^q(B') & \longrightarrow & C^q/B^q(B) & \longrightarrow & C^q/B^q(B'') & \longrightarrow & 0 \\ \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\ 0 & \longrightarrow & Z^q(B') & \longrightarrow & Z^{q+1}(B) & \longrightarrow & Z^{q+1}(B'') \end{array}$$

and by the snake lemma we have a connecting homomorphism over group cohomologies. \square

8 SEPT 8, 2023: RESTRICTION AND TRANSFER

Recall that we have a chain-level cup product, and we extend it to the level of cohomology. The cup product has the following properties:

1. If $p = q = 0$, then the cup product is the natural composition

$$A^G \otimes B^G \rightarrow (A \otimes B)^G \rightarrow C^G$$

2. Functoriality.

3. We have $\delta(\alpha \smile \beta) = \delta(\alpha) \smile \beta$, and incorporating this with the exact sequence, we have $\delta(\alpha \smile \beta) = (-1)^{|\alpha|} \alpha \smile \delta(\beta)$.

By the universal property of the tensor product, there exists a unique bilinear pairing that also satisfies these properties. To prove this, we use dimension-shifting.

Remark 8.1. Let M be a module, and map it into the induced module with an extended short exact sequence

$$0 \longrightarrow M \longrightarrow \text{Ind}^G(M) = \text{Map}(G, M) \cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], M) \longrightarrow M_1 \longrightarrow 0$$

Taking the fixed points, we have

$$0 \longrightarrow M^G \longrightarrow (\text{Ind}^G(M))^G \longrightarrow (M_1)^G \longrightarrow H^1(G, M) \longrightarrow 0 \longrightarrow \dots$$

$$\dots \longrightarrow 0 \longrightarrow H^k(G, M_1) \xrightarrow{\cong} H^{k+1}(G, M)$$

Here $(M_1)^G \rightarrow H^1(G, M)$ is a surjection. Now we know $\delta : H^i(G, M_1) \rightarrow H^{i+1}(G, M)$ is a surjection for $i = 0$, and is an isomorphism for $i > 0$.

Proceeding inductively, we define

$$0 \longrightarrow M_i \longrightarrow \text{Ind}^G(M) \longrightarrow M_{i+1} \longrightarrow 0$$

If we start with $A \otimes B \rightarrow C$, then use property (3) repeatedly to the short exact sequence above, we get the uniqueness.

Example 8.2. Consider $G = C_2$, and consider the cohomology ring $H^*(C_2, \mathbb{F}_2)$. The action is obviously trivial. This induced the sequence with augmentation

$$0 \longrightarrow \mathbb{F}_2 \longrightarrow \mathbb{F}_2[C_2] \longrightarrow \mathbb{F}_2 \longrightarrow 0$$

The boundary map is $\delta : H^i(C_2, \mathbb{F}_2) \rightarrow H^{i+1}(C_2, \mathbb{F}_2)$ is an isomorphism for all i .

We know $H^i(C_2, \mathbb{F}_2) = \mathbb{F}_2\{x_i\}$, so we can write $x_{i+1} = \delta x_i$. The product $x_i \smile x_j = \delta^i x_0 \smile \delta^j x_0 = \delta^{i+j} x_0 \smile x_0 = \delta^{i+j} x_0 = x_{i+j}$. Hence, $H^*(C_2, \mathbb{F}_2) \cong \mathbb{F}_2[x]$ where $x = |x_1|$.

Note that

$$H^i(C_2, M) = \begin{cases} M^{C_2}, & i = 0 \\ \ker(N)/(\sim), & i \text{ odd} \\ M^{C_2}/N, & i > 0 \text{ even} \end{cases}$$

Remark 8.3. For odd prime p , we want to use the same method to calculate $H^i(C_p, \mathbb{F}_p)$ with trivial action, then this is $\{\mathbb{F}_p, i \geq 0\}$. For instance, if we look at $x_1 \smile x_1$, then this is $(-1)^{|x_1|} x_1 \smile x_1$, so this gives $2x_1 \smile x_1 = 0 \in H^2 = \mathbb{F}_p$, so this gives $x_1 \smile x_1 = 0$. Note that $H^*(C_p, \mathbb{F}_p) \cong \bigwedge(x_1) \otimes \mathbb{F}_p[y]$.

We now talk about the functoriality in G . Given G_1 acting on M_1 and G_2 acting on M_2 , and say $\varphi : G_1 \rightarrow G_2$ is a group homomorphism, and a map of modules $f : M_2 \rightarrow M_1$, then we say φ and f is a compatible pair of morphisms if for any $g \in G_1$, the diagram

$$\begin{array}{ccc} M_2 & \xrightarrow{f} & M_1 \\ \varphi(g) \downarrow & & \downarrow g \\ M_2 & \xrightarrow{f} & M_1 \end{array}$$

This gives a map $C^*(G_2, M_2) \rightarrow C^*(G_1, M_1)$, and hence a map on cohomology $H^*(G_2, M_2) \rightarrow H^*(G_1, M_1)$. For instance, if $\varphi = \text{id}$, we obtain the functoriality in M , as we previously saw. Similarly, if $f = \text{id}$, and $M = M_2$ is a G_2 -module, on which $g_1 \cdot m = \varphi(g_1) \cdot m$.

There are some special situations from the relations above.

1. Conjugation: let $H \subseteq G$ be a subgroup, and we consider A to be a G -module, then there is restriction of G -action on A to H , so A becomes a H -module. Let $B \subseteq A$ be a H -submodule in this sense. This is preserved by the action of A , but not necessarily by the action of G . For any $g \in G$, let the right conjugation be $h^g = g^{-1}hg$ on h , and let ${}^gH = gHg^{-1}$ on subgroup H . The compatible morphisms are now

$$\begin{aligned} {}^gH &\rightarrow H \\ h &\mapsto h^g \end{aligned}$$

and

$$\begin{aligned} B &\rightarrow gB \\ b &\mapsto gb \end{aligned}$$

Therefore, the induced maps on conjugation is given by $(g)_* = H^*(H, B) \rightarrow H^*({}^gH, gB)$. Therefore, $(g_1g_2)_* = (g_1)_*(g_2)_*$.

2. Inflation: suppose $H \triangleleft G$ is a normal subgroup. We have the canonical map $G \rightarrow G/H$. Let A be a G -module, then G/H acts on A^H , and we look at the inclusion $A^H \hookrightarrow A$. Now $\varphi : G \rightarrow G/H$ and $f : A^H \hookrightarrow A$ are compatible, so on the level of cohomology, we get an inflation map

$$\inf_G^{G/H} : H^*(G/H, A^H) \rightarrow H^*(G, A).$$

If we look at $H_1 \subseteq H_2 \triangleleft G$ where $H_i \triangleleft G$, we have $G \rightarrow G/H_1 \rightarrow G/H_2 \cong (G/H_1)/(H_2/H_1)$, then the inflation is

$$\inf_G^{G/H_1} \circ \inf_{G/H_1}^{G/H_2} = \inf_G^{G/H_2}.$$

3. Restriction: Let $\varphi : H \hookrightarrow G$ and consider A as G -module and H -module respectively. There is now a restriction map

$$\text{res}_H^G : H^*(G, A) \rightarrow H^*(H, A)$$

Now if $H_1 \subseteq H_2 \subseteq G$, then

$$\text{res}_{H_1}^G = \text{res}_{H_1}^{H_2} \circ \text{res}_{H_2}^G$$

Inflation and restriction fit in a long exact sequence.

Finally, we discuss corestriction/transfer/norm. Let G be a finite group and let M be a G -module, then we have $M^G \hookrightarrow M$ as inclusion. On the other way around, we have

$$\begin{aligned} \text{tr}/N : M &\rightarrow M^G \\ m &\mapsto \sum_{g \in G} gm. \end{aligned}$$

9 SEPT 11, 2023:

Let $\varphi : G_1 \rightarrow G_2$ and $f : M_2 \rightarrow M_1$ be compatible, then we denote $(\varphi, f)^* = H^*(G_2, M_2) \rightarrow H^*(G_1, M_1)$, with

$$G_1^{\times(*+1)} \longrightarrow G_2^{\times(*+1)} \longrightarrow M_2 \xrightarrow{f} M_1$$

such that it follows composition, and $(\varphi, f)^*$ commutes with δ , i.e.,

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_2' & \longrightarrow & M_2 & \longrightarrow & M_2'' \longrightarrow 0 \\ & & \downarrow f & & \downarrow f & & \downarrow f \\ 0 & \longrightarrow & M_1' & \longrightarrow & M_1 & \longrightarrow & M_1'' \longrightarrow 0 \end{array}$$

and therefore we have a commutative square

$$\begin{array}{ccc} H^k(G, M_2'') & \xrightarrow{\delta} & H^{k+1}(G_2, M_2') \\ (\varphi, f)^* \downarrow & & \downarrow (\varphi, f)^* \\ H^k(G_1, M_1'') & \xrightarrow{\delta} & H^{k+1}(G, M_1') \end{array}$$

For $\alpha \in C^k(M_2'')/B^k$, we trace it back to $\tilde{\alpha} \in C^k(M_2)/B_k$, and α is sent to $Z^{k+1}(M_2'')$, but now that means $\tilde{\alpha}$ lands in the kernel of $Z^{k+1}(M_2) \rightarrow Z^{k+1}(M_2')$, so this is in $Z^{k+1}(M_2')$.

$$\begin{array}{ccccccc} C^k(M_2)/B_k & \longrightarrow & C^k(M_2'')/B_k & \longrightarrow & 0 \\ \partial \downarrow & & \downarrow \partial & & \\ 0 & \longrightarrow & Z^{k+1}(M_2') & \longrightarrow & Z^{k+1}(M_2) & \longrightarrow & Z^{k+1}(M_2'') \end{array}$$

Moreover, we have $(\varphi, f)^*(\alpha \smile \beta) = (\varphi, f)^*\alpha \smile (\varphi, f)^*\beta$, whenever the modules are compatible.

For transfer/corestriction, if $H \subseteq G$ is a subgroup with finite index, and M is a G -module, then we have

$$\begin{aligned} \mathrm{tr}_G^H : M^H &\rightarrow M^G \\ m &\mapsto \sum_{g \in G/H} gm \end{aligned}$$

For instance, we have $\mathrm{tr} : \mathbb{Z}^H = \mathbb{Z} \rightarrow \mathbb{Z}^G = \mathbb{Z}$ is multiplication by $[G : H]$. Note that $H^*(X^*(G, M)^G) = H^*(G, M)$, but $H^*(X^*(G, M)^H) = H^*(H, M)$, and the latter maps to the former cohomology structure via the transfer mapping. Hence, we have $\mathrm{tr}_G^H : X^*(G, M)^H \rightarrow X^*(G, M)^G$ giving $\mathrm{tr}_G^H \equiv \mathrm{cores}_G^H : H^*(H, M) \rightarrow H^*(G, M)$. This is not a ring homomorphism.

Remark 9.1 (Properties). 1. tr commutes with δ , that is, for a short exact sequence of G -modules (hence a short exact sequence of H -modules),

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

then we have

$$\begin{array}{ccc} H^k(H, C) & \xrightarrow{\delta} & H^{k+1}(H, A) \\ \mathrm{tr} \downarrow & & \downarrow \mathrm{tr} \\ H^k(G, C) & \xrightarrow{\delta} & H^{k+1}(G, A) \end{array}$$

2. If $H_1 \subseteq H_2 \subseteq G$ are subgroups with finite indices, then $\mathrm{tr}_G^{H_1} = \mathrm{tr}_G^{H_2} \mathrm{tr}_{H_2}^{H_1}$.

3. $\mathrm{tr}(\mathrm{res}(\alpha) \smile \beta) = \alpha \smile \mathrm{tr}(\beta)$. Now given a pairing $A \otimes B \rightarrow C$ of G -modules, with $H \subseteq G$, then

$$\begin{array}{ccccc} H^i(H, A) & \otimes & H^j(H, B) & \xrightarrow{\smile} & H^{i+j}(H, C) \\ \mathrm{res} \uparrow & & \downarrow \mathrm{tr} & & \downarrow \mathrm{tr} \\ H^i(G, A) & \otimes & H^j(G, B) & \xrightarrow{\smile} & H^{i+j}(G, C) \end{array}$$

Proof Idea. By dimension shifting, we reduce the case H^0 , in which we have an explicit description. We have $A^H \otimes B^H \rightarrow C^H$, so for $\alpha \in A^G$ and $\beta \in B^H$, we have $\mathrm{tr}(\alpha \otimes \beta) = \sum_{g \in G/H} g(\alpha \otimes \beta) = \sum g\alpha \otimes g\beta = \alpha \otimes \sum_{g \in G/H} g\beta$. \square

Example 9.2. Let R be a commutative ring with a G -action, then the restriction $\mathrm{res} : H^*(G, R) \rightarrow H^*(H, R)$ is a ring homomorphism, so $H^*(H, R)$ is a $H^*(G, R)$ -algebra. The opposite side has tr is a map of $H^*(G, R)$ -modules where the cohomology of H is given the module structure from the restriction. This induces the Frobenius reciprocity.

Remark 9.3 (Other compatibilities). Let $K \subseteq H \subseteq G$ be (normal) subgroups, then $G \rightarrow G/K \rightarrow G/H$ are quotient maps. The restrictions of inclusions correspond to inflations of surjections: if $K \triangleleft G$, then $G \rightarrow G/K$ and $H \rightarrow H/K$, so $\mathrm{inf}_H^{H/K} \circ \mathrm{res}_{H/K}^{G/K} = \mathrm{res}_H^G \circ \mathrm{inf}_G^{G/K}$. Note that the maps are contravariants. Moreover, we have $\mathrm{inf}_G^{G/K} \circ \mathrm{cores}_{G/K}^{H/K} = \mathrm{cores}_G^H \circ \mathrm{inf}_H^{H/K}$.

If $H \triangleleft G$, then $\mathrm{res}_H^G \circ \mathrm{cor}_G^H = N_{G/H}$; also, $\mathrm{cor}_G^H \circ \mathrm{res}_H^G = [G : H]$.

10 SEPT 13, 2023: SPECTRAL SEQUENCE

Whenever G is not cyclic or Q_8 , the group cohomology $H^*(G, M)$ would not have a small resolution. We know there is a pullback diagram

$$\begin{array}{ccc} M & \longrightarrow & \prod_p M_p^n \\ \downarrow & & \downarrow \\ M_{\mathbb{Q}} & \longrightarrow & \prod_p (M_p^n)_{\mathbb{Q}} \end{array}$$

Here $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ is the base-change, and $M_p^n = \varprojlim_i M/p^i$ is the completion. For finite group G , we have $H^*(G, M_{\mathbb{Q}}) = M_{\mathbb{Q}}^G$ if $*$ = 0 and is trivial otherwise. Now we have the diagram

$$\begin{array}{ccc} H^*(G, M) & \xrightarrow{\text{res}} & H^*(\{e\}, M) \\ & \searrow |G| & \downarrow \text{tr} \\ & & H^*(G, M) \end{array}$$

where $H^*(\{e\}, M)$ is M if $*$ = 0 and is otherwise trivial. Note that if $*$ > 0, then $H^*(G, M)$ is annihilated by $|G|$. Let $P \subseteq G$ be a Sylow p -subgroup, then if P is normal, then $H^*(G, M_p^n) \cong H^*(P, M_p^n)^{G/P}$. Therefore we have a normal series $\cdots \triangleleft P_2 \triangleleft P_1 \triangleleft P$ with simple enough quotients, e.g., as abelian series. Therefore, we need ways to reassemble the cohomology.

For $H \triangleleft G$ we know there is a G/H -action on $H^*(H, M)$ via conjugation, so we can calculate $H^*(G/H, H^*(H, M))$, hence calculate $H^*(G, M)$ using Lyndon-Hochschild-Serre spectral sequences.

We will first look at Bockstein spectral sequences. We start by looking at the sequence

$$\cdots \subseteq p^2\mathbb{Z} \subseteq p\mathbb{Z} \subseteq \mathbb{Z}$$

and factors each inclusion $p^k\mathbb{Z} \subseteq p^{k-1}\mathbb{Z}$ via $p^k(\mathbb{Z}/p\mathbb{Z})$, then we have cohomology $H^*(G, M/p)[p]$, thus calculate $H^*(G, M)$. (Here the attachment by p is given by tensoring $\mathbb{Z}[v_0]$ with grading p .) In general, we construct the abstract version as filtered cochain complex, with

$$\cdots \subseteq F^{p+1}C^* \subseteq F^pC^* \subseteq \cdots \subseteq C^*$$

so we can map each term to the graded version $\text{gr}^p C^*$. We denote the inclusions by i and the projections to the graded versions by π . The goal is to understand $H^*(C^*)$ from the building blocks $H^*(\text{gr}^* C^*)$. There exists the factoring

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^q(F^{p+2}) & \xrightarrow{i} & H^q(F^{p+1}) & \xrightarrow{i} & H^q(F^p) \longrightarrow \cdots \\ & & \delta \uparrow & \swarrow \pi & \delta \uparrow & \swarrow \pi & \\ & & H^q(\text{gr}^{p+1}) & & H^q(\text{gr}^p) & & \end{array}$$

This is the E_1 -page of the spectral sequence, given by $E_1^{p,q} = H^q(\text{gr}^p)$. We denote $d_1 : H^q(\text{gr}^p) \rightarrow H^{q+1}(\text{gr}^{p+1})$ as the composition. Obviously $d_1^2 = 0$.

Now the E_2 -page is given by $H^*(E_1, d_1)$. For $a \in \ker(d_1)$, the map i induces $\tilde{\delta} \mapsto \delta a$ by lifting, so $\pi(\tilde{\delta}a) \in H^{q+1}(\text{gr}^{p+2}) = E_1^{p+2, q+1}$, with $d_1(\pi(\tilde{\delta}a)) = \pi\delta\pi(\tilde{\delta}a) = 0$. We then define $d_2([a]) = [\pi(\tilde{\delta}a)] \in E_2$. We then proceed inductively and find higher pages. This is usually done by calculating derived pages.

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Recall that: if H is a finite group, A is a finite H -module, then an extension of H by A is a group G such that

$$0 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 1$$

is exact, where the H -module structure on A is realized via conjugation $h \cdot a = hah^{-1} \in G$. We already know that the equivalence classes of extensions of H by A correspond to $H^2(H, A)$, where $A \rtimes H$ corresponds to $0 \in H^2(H, A)$.

Theorem 11.1. Let p be an odd prime, $|G| = p^{n+1}$, and G contains \mathbb{Z}_q for $q = p^n$ as a subgroup. If this is the case, then G is either $\mathbb{Z}_{p^{n+1}}$, $\mathbb{Z}_q \times \mathbb{Z}_p$, or $\mathbb{Z}_q \rtimes \mathbb{Z}_p$, where the generator $e \in H$ acts on $1 \in \mathbb{Z}_q$ by $e1e^{-1} = 1 + p^{n-1}$. We denote $H = \mathbb{Z}_p$ in this case.

Proof. We want to look at the short exact sequence

$$0 \longrightarrow \mathbb{Z}_q \longrightarrow G \longrightarrow H \longrightarrow 1$$

where $H = \mathbb{Z}_p$.

Lemma 11.2. If p is an odd prime, and there exists integer a such that $a^p \equiv 1 \pmod{p^n}$ for $n \geq 2$, then $a \equiv 1 \pmod{p^{n-1}}$.

Subproof. This is trivial if $a = 1$. If $a \neq 1$, let $d(a)$ be the largest possible integer d such that $a \equiv 1 \pmod{p^d}$. It suffices to show that $d(a) \geq n - 1$. By Fermat's Little theorem, we have $d(a) \geq 1$. We now want to show $d(a^p) = d(a) + 1$. Indeed, let $a = 1 + p^d b$, then using the binomial theorem, we have $a^p = (1 + p^d b)^p = 1 + p^{d+1} b + \dots$ where the omitted terms have higher order of p^{d+2} . However, $d(a^p) \geq n$, so $d(a) \geq n - 1$. ■

Now let

$$0 \longrightarrow \mathbb{Z}_q \longrightarrow G \longrightarrow H \longrightarrow 1$$

be the extension with $|H| = p$, then the H -module of \mathbb{Z}_q is given by a map $\varphi : H \rightarrow \text{Aut}(\mathbb{Z}_q) \cong \mathbb{Z}_q^\times$. Since $|H|$ is prime, then φ is either trivial or injective.

If φ is trivial, then $h1h^{-1} = 1$ for all $h \in H$, so G is an abelian group. By the fundamental theorem of abelian groups, we know G is either $\mathbb{Z}_{p^{n+1}}$ or $\mathbb{Z}_q \times \mathbb{Z}_p$.

If φ is injective, then $n \geq 2$, otherwise the size of H is larger than the size of the units. Given some element $h \in H$ such that $h1h^{-1} = k$, then $k^p \equiv 1 \pmod{p^n}$. By Lemma 11.2, $k = 1 + p^{n-1}b$ for some $b \in \mathbb{Z}_p$. Because φ is injective, then the image of φ has size p , but every element in the image has the form of k , therefore the image is just the set of such elements. Let $e \in H$ be a generator such that $e1e^{-1} = 1 + p^{n-1}$. Now let $A = \mathbb{Z}_q$ with this H -module structure, and it suffices to show that $H^2(H, A) = 0$, then we have the semidirect product only.

Since H and A are both cyclic groups, we write down the periodic resolution to be

$$A \xrightarrow{e-1} A \xrightarrow{N} A \xrightarrow{e-1} A \xrightarrow{N} A \longrightarrow \dots$$

where N is the norm element $\sum_{h \in H} h$. We know the action via $e - 1$ on 1 is $(e - 1) \cdot 1 = (1 + p^{n-1}) - 1 = p^{n-1}$, so $\ker(e - 1) = p\mathbb{Z}/q\mathbb{Z}$; the action via N is $N \cdot 1 = \sum_{b \in \mathbb{Z}_p} (1 + p^{n-1}b) \equiv p \pmod{p^n}$, therefore the image of the norm map is $\text{im}(\mathbb{Z}) = p\mathbb{Z}/q\mathbb{Z}$ as well. Therefore, $H^2(H, A) = 0$. □

Corollary 11.3. If we have a p -group G with $p \neq 2$, then there is a unique subgroup of order p and a unique subgroup of index p .

Let H be a normal subgroup of G , then we consider the free $\mathbb{Z}[H]$ -resolution

$$\mathbb{Z} \longleftarrow C_H^0 \longleftarrow C_H^1 \longleftarrow C_H^2 \longleftarrow \dots$$

and we can try turning it into a free G -resolution of $\mathbb{Z}[G/H]$ by taking the tensor via

$$\mathbb{Z} \otimes \mathbb{Z}[G/H] \cong \mathbb{Z}/[G/H] \longleftarrow C_H^* \otimes \mathbb{Z}[G/H]$$

Because $\mathbb{Z}[H] \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G] \cong \mathbb{Z}[G]$, then we have

$$\mathbb{Z}[G/H] \cong \mathbb{Z} \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G] \longleftarrow C_H^* \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G]$$

Now given an arbitrary free $\mathbb{Z}[G/H]$ -resolution and we want to map the given resolution to it.

$$\begin{array}{ccccccc} \mathbb{Z} & \longleftarrow & D_{G/H}^0 \cong \mathbb{Z}[G/H] & \longleftarrow & D_{G/H}^1 \cong \mathbb{Z}[G/H]^m & \longleftarrow & \dots \\ & & \uparrow & & \uparrow & & \\ & & C_H^* \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G] & \longleftarrow & (C_H^* \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G])^m & & \end{array}$$

The vertical maps are resolved as G -modules by using the resolution of $\mathbb{Z}[G/H]$. We claim that there are horizontal maps that gives a double complex whose total complex is a resolution of \mathbb{Z} as a G -module.

Example 11.4. Consider the dihedral group $D_{2n} \triangleright C_n$, so $D_{2n}/C_n \cong C_2$. In particular, say D_{2n} is generated by τ of order n and T of order 2, so C_n is generated by τ and C_2 is generated by T . Consider the resolutions

$$D^* : \mathbb{Z} \longleftarrow \mathbb{Z}[T]/(T^2 - 1) \xleftarrow{T-1} \mathbb{Z}[T]/(T^2 - 1) \xleftarrow{T+1} \mathbb{Z}[T]/(T^2 - 1) \xleftarrow{T-1} \mathbb{Z}[T]/(T^2 - 1) \xleftarrow{T+1} \cdots$$

and

$$C^* : \mathbb{Z} \longleftarrow \mathbb{Z}[\tau]/(\tau^n - 1) \xleftarrow{\tau-1} \mathbb{Z}[\tau]/(\tau^n - 1) \xleftarrow{N_\tau} \mathbb{Z}[\tau]/(\tau^n - 1) \xleftarrow{\tau-1} \cdots$$

and so on. Therefore we have an induced resolution given by

$$\mathbb{Z}[T]/T^2 \longleftarrow \mathbb{Z}[D_{2n}] \xleftarrow{\tau-1} \mathbb{Z}[D_{2n}] \xleftarrow{N_\tau} \mathbb{Z}[D_{2n}] \xleftarrow{\tau-1} \mathbb{Z}[D_{2n}] \xleftarrow{N_\tau} \cdots$$

Now let the sequence of $D_{G/H}^n$'s be of

$$\begin{array}{ccccccc} \mathbb{Z} & \longleftarrow & \mathbb{Z}[T]/T^2 & \xleftarrow{T-1} & \mathbb{Z}[T]/T^2 & \xleftarrow{T+1} & \mathbb{Z}[T]/T^2 & \xleftarrow{T-1} & \mathbb{Z}[T]/T^2 & \xleftarrow{T+1} & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ & & \mathbb{Z}[D_{2n}] & & \mathbb{Z}[D_{2n}] & & \mathbb{Z}[D_{2n}] & & \mathbb{Z}[D_{2n}] & & \\ & & \tau-1 \uparrow & & \tau-1 \uparrow & & \tau-1 \uparrow & & \tau-1 \uparrow & & \\ & & \mathbb{Z}[D_{2n}] & & \mathbb{Z}[D_{2n}] & & \mathbb{Z}[D_{2n}] & & \mathbb{Z}[D_{2n}] & & \\ & & N_\tau \uparrow & & N_\tau \uparrow & & N_\tau \uparrow & & N_\tau \uparrow & & \\ & & \mathbb{Z}[D_{2n}] & & \mathbb{Z}[D_{2n}] & & \mathbb{Z}[D_{2n}] & & \mathbb{Z}[D_{2n}] & & \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ & & \cdots & & \cdots & & \cdots & & \cdots & & \end{array}$$

The horizontal maps are hard to construct, they may look like $\tau - 1$, but we need to introduce signs at certain places.