MATH 526 Notes

Jiantong Liu

September 22, 2023

1 Aug 21, 2023

Let X be a topological space with basepoint $x_0 \in X$. We already know two invariants,

- the fundamental group $\pi_1(X, x_0)$, and
- the homology groups $H_n(X)$ for $n \geqslant 0$, which are abelian groups.

We will look at two more invariants,

- the cohomology groups $H^n(X)$ for $n \geqslant 0$, and
- the higher homotopy groups $\pi_n(X, x_0)$ for $n \geqslant 0$.

In particular, $\pi_*(X, x_0)$ is a very good invariant in the following sense:

Theorem 1.1 (Whitehead)**.** If $f : (X, x_0) \to (Y, y_0)$ is a map of CW-complexes, then f is a homotopy equivalence if and only if $\pi_*(f) : \pi_*(X, x_0) \to \pi_*(Y, y_0)$ is an isomorphism.

However, π_* is very hard to compute. On the other hand, $H^*(X)$ is relatively easy to compute, but this is not a complete invariant. For instance, $\mathbb{C}P^2$ and $S^2 \vee S^4$ have isomorphic cohomology groups, but they are not equivalent. $H^*(X)$ is closely related to $H_*(X)$, but $H^*(X)$ is a graded ring structure with cup product. It is contravariant in X, where $H_*(X)$ is covariant. The cup product is defined by the composition of induced diagonal map with an external product:

$$H^i(X) \times H^j(X) \xrightarrow{\times} H^{i+j}(X \times X) \xrightarrow{\Delta^*} H^{i+j}(X)$$

Other things we will talk about include:

- Natural transformations $H^i(-) \to H^j(-)$ encoded by Steenrod operations.
- $H^n(-)$ becomes a representable functor, i.e., $H^n(X) = [X, K(\mathbb{Z}, n)]$, where $K(\mathbb{Z}, n)$ is the Eilenberg-Maclane space, and the bracket indicates the homotopy classes of maps.
- Poincaré duality in $H^*(M)$ for compact manifold M, namely the cup product gives
$$H^i(M) \otimes H^{\dim(M)-i}(M) \xrightarrow{\smile} H^{\dim(M)}(M).$$
- Characteristic classes in $H^*(X)$ associated to vector bundles over X.

Recall for a topological space X, we obtain a collection of (singular) homology groups $H_n(X)$, with $H_*(X) = \bigoplus_{n \geqslant 0} H_n(X)$. The functoriality of morphisms says that $X \xrightarrow{f} Y \xrightarrow{g} Z$ induces $f_* g_* = (fg)_* : H_*(X) \xrightarrow{f_*} H_*(Y) \xrightarrow{g_*} H_*(Z)$. So

$$H_*(-) : \mathbf{Top} \to \mathbf{Ab}$$

is a well-defined functor. This factors into

$$\begin{tikzcd} \mathbf{Top} \arrow[rr, "H_*(-)"] \arrow[dr, "C_*(-)"'] & & \mathbf{Ab} \\ & \mathbf{Ch} \arrow[ur, "H_*(-)"'] & \end{tikzcd}$$

Here $C_*(-)$ is usually the singular chain, given by $\partial : C_n(X) \to C_{n-1}(X)$, where $C_n(X)$ is the free abelian group generated by $\mathrm{Hom}_{\mathbf{Top}}(\Delta^n, X) \cong \bigoplus_{\sigma:\Delta^n\to X} \mathbb{Z}\sigma$. $\Delta^n \subseteq \mathbb{R}^{n+1}$ is the set of tuples $(t_0, \dots, t_n)$ such that the coordinates sum to 1. The boundary is $\partial\sigma = \sum_{0\leqslant i\leqslant n} (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$.

We say $C_*(-)$ is homotopy invariant, i.e., if $f : X \to Y$ is a homotopy equivalence, then the induced map $C_*(X) \to C_*(Y)$ on chain complexes is a chain equivalence.

Remark 1.2. $C_*^{\Delta}(X)$ and $C_*^{\mathrm{CW}}(X)$ are both chain equivalent to $C_*(X)$.

2 Aug 23, 2023

Here is a list of properties of $C_*(-) : \mathbf{Top} \to \mathbf{Ch}$:

- Functoriality: given a continuous map $f : X \to Y$, there is an induced map

$$\begin{aligned} f_* : C_*(X) &\to C_*(Y) \\ (\sigma : \Delta^n \to X) &\mapsto (f\sigma : \Delta^n \to Y) \end{aligned}$$

- Homotopy invariance: given $f, g : X \to Y$ such that $f \simeq g$, i.e., there is $H : X \times [0, 1] \to Y$ such that $H|_0 = f$ and $H|_1 = g$, then $f_* \simeq g_*$ as a chain homotopy equivalence, i.e., there exists maps $h_n : C_n(X) \to C_{n+1}(Y)$ making a diagram

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(X) \arrow[r] \arrow[d, shift right, "g"'] \arrow[d, shift left, "f"] \arrow[dl, "h"] & C_n(X) \arrow[r] \arrow[d, shift right, "g"'] \arrow[d, shift left, "f"] \arrow[dl, "h"] & C_{n-1}(X) \arrow[r] \arrow[d, shift right, "g"'] \arrow[d, shift left, "f"] \arrow[dl, "h"] & \cdots \\ \cdots \arrow[r] & C_{n+1}(Y) \arrow[r] & C_n(Y) \arrow[r] & C_{n-1}(Y) \arrow[r] & \cdots \end{tikzcd}$$

such that $f - g = \partial h + h\partial$. Therefore $f_* = g_* : H_*(X) \to H_*(Y)$.

Remark 2.1. $f : A_* \to B_*$ is a chain equivalence if there exists $g : B_* \to A_*$ and $fg \simeq \mathrm{id}_B$ and $gf \simeq \mathrm{id}_A$, then $f_* : H_*(A_*) \to H_*(B_*)$ is an isomorphism, i.e., f is a quasi-isomorphism.

Example 2.2. The complexes $A : 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to 0$ and $B : 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0$ gives a quasi-isomorphism $f : A \to B$ in the canonical way, but this is not a chain equivalence, since the backwards map has to be zero.

- Additivity: $C_*(\coprod_\alpha X_\alpha) \cong \bigoplus_\alpha C_*(X_\alpha)$.

- Excision: given a pair (X, A) with $Z \subseteq A$ such that $\bar{Z} \subseteq \mathrm{int}(A)$, then we have $C_*(X\backslash Z, A\backslash Z) \cong C_*(X, A)$.

- Mayer-Vietoris: given $A, B \subseteq X$, with $X = \mathrm{int}(A) \cup \mathrm{int}(B)$, then we have a short exact sequence

$$0 \longrightarrow C_*(A \cap B) \longrightarrow C_*(A) \oplus C_*(B) * C_*(X) \longrightarrow 0$$

The cochain complex is obtained via inverting the indices and maps δ from a chain complex. This induces a cohomology $H^*(C^*) = \ker(\delta)/\mathrm{im}(\delta)$ as the quotient of cocycles over coboundaries. Now $f : A^* \to B^*$ is a quasi-isomorphism if $f^* : H^*(A^*) \to H^*(B^*)$ is an isomorphism. Similarly, one can define the cochain homotopy equivalence.

Example 2.3. If $C_* \in \mathbf{Ch}$, and $k \in \mathbf{Ab}$, then we can form cochain complex $C_k^* := \mathrm{Hom}(C_*, k)$, where $C_k^n = \mathrm{Hom}_{\mathbf{Ab}}(C_n, k) \xrightarrow{\delta} C_k^{n+1}$ by sending $f : C_n \to k$ to $f\partial : C_{n+1} \to C_n \to k$.

- $\operatorname{Hom}(-,k) : \mathbf{Ch} \to \mathbf{coCh}$ is a functor.
- The functor preserves quasi-isomorphisms between chain complexes of free abelian groups.

Definition 2.4. For $k \in \mathbf{Ab}$, the singular cochains with coefficients in k is

$$\begin{tikzcd} C^*(-,k) : \mathbf{Top} \arrow[rr] \arrow[dr, "C_*(-)"'] & & \mathbf{coCh} \\ & \mathbf{Ch} \arrow[ur, "\operatorname{Hom}(-,k)"'] & \end{tikzcd}$$

The cohomology of X with coefficients in k is defined by $H^*(X;k) = H^*(C^*X,k)$. We have the convention $C^*(X) = C^*(X,\mathbb{Z})$.

Alternatively, we take the opposite categories $\mathbf{Top}^*$ and $\mathbf{Ch}^*$ so that the functors are viewed as covariant.

The corresponding map $\delta : C^n(X;k) \to C^{n+1}(X;k)$ is given by δf that maps $\sigma \in C_{n+1}(X)$ to $(-1)^{n+1} f(\partial\sigma)$. Although the cochains are in general the dual of chains, the cohomology is not going to be the dual of the homology.

3 Aug 25, 2023

Recall:

$$\begin{tikzcd} \mathbf{Top}^{\mathrm{op}} \arrow[r, "C_*"] \arrow[rrr, bend left, "H^*(-,k)"] & \mathbf{Ch}^{\mathrm{op}} \arrow[r, "\operatorname{Hom}(-,k)"] & \mathbf{coCh} \arrow[r, "H^*"] & \mathbf{GrAb} \end{tikzcd}$$

Properties of $H^*(-,k) : \mathbf{Top} \to \mathbf{GrAb}$:

- Dimension:

  Claim 3.1. $H^i(\{*\},k) = \begin{cases} 0, & i \neq 0 \\ k, & i = 0 \end{cases}$

  Proof. Note that each degree of cohomology is given the free abelian group generated by $\operatorname{Hom}(\Delta^n, \{*\})$, but the singleton set is the terminal object in the category of topological spaces, so there is always a unique generator, thus the chain complex is given by $\mathbb{Z}$'s on each degree $n \geqslant 0$.

  Now the generating map at degree n is $\sigma_n : \Delta^n \to \{*\}$, and see Homework 1 where we proved the homology. Now looking at $C^*(\{*\},k)$, we have

  $$k \xrightarrow{0} k \xrightarrow{\cong} k \xrightarrow{0} k \longrightarrow \cdots$$

  and this gives the cohomology. □

- Homotopy: if $f \simeq g : X \to Y$, then $f^* = g^* : H^*(Y,k) \to H^*(X,k)$.

  Proof. We have $f_* = g_* : C_*X \to C_*Y$, and then $\operatorname{Hom}(f_*,k) \cong \operatorname{Hom}(g_*,k)$, so $H^*(-)$ is invariant under cochain homotopies. □

- Additivity: $H^*(\coprod_\alpha X_\alpha, k) \cong \prod_\alpha H^*(X_\alpha, k)$.

  Proof. We know that for chains there is $C_*(\coprod_\alpha X_\alpha) = \bigoplus_\alpha C_*(X_\alpha)$, so the cochain version says that $C^*(\coprod_\alpha, X_\alpha, k) \cong \operatorname{Hom}(\bigoplus_\alpha C_*(X_\alpha), k) \cong \prod_\alpha \operatorname{Hom}(C_*(X_\alpha), k) \cong \prod_\alpha C^*(X_\alpha)$ and $H^* : \mathbf{coCh} \to \mathbf{GrAb}$ commutes with the product. □

- Exactness: for a pair (X,A), there is a natural long exact sequence

$$\cdots \longrightarrow H^n(X,A;k) \longrightarrow H^n(X;k) \longrightarrow H^n(A;k) \longrightarrow \cdots$$

Proof. We have a short exact sequence

$$0 \longrightarrow C_*A \longrightarrow C_*X \longrightarrow C_*(X,A) \longrightarrow 0$$

where $C_*A \to C_*X$ is an inclusion of summands. Therefore, the quotient $C_*(X,A)$ is also a chain complex of free abelian groups. Therefore, taking the cochains also gives a short exact sequence. We then obtain a short exact sequence of cochain complexes

$$0 \longrightarrow C^*(X,A;k) \longrightarrow C^*(X;k) \longrightarrow C^*(A;k) \longrightarrow 0$$

and can then apply cohomology functor. □

- Excision: given a pair (X,A) and Z such that $\bar{Z} \subseteq \operatorname{int}(A)$, we have $H^*(X,A;k) \cong H^*(X \backslash Z, A \backslash Z; k)$.
- Mayer-Vietoris: given $A,B \subseteq X$ such that $\operatorname{int}(A) \cup \operatorname{int}(B) = X$, then we have a natural long exact sequence

$$\cdots \longrightarrow H^n(X;k) \longrightarrow H^n(A;k) \oplus H^n(B;k) \longrightarrow H^n(A \cap B;k) \longrightarrow \cdots$$

Definition 3.2. A functor $E^* : \mathbf{Top}^{\mathrm{op}} \to \mathbf{GrAb}$ is called a generalized cohomology theory if it satisfies the four middle property (except the dimension property and Mayer-Vietoris).

Remark 3.3. If E^* also satisfies the dimension property, then E^* is naturally isomorphic to the cohomology $H^*(-;k)$. There are also other generalized cohomology theories like K-theory, cobordism, etc.

The Mayer-Vietoris becomes a consequence of the first five properties.

We will now try to use homological algebra to relate $H_*(X) = H_*(CX)$ and $H^*(X;k) = H^*(\operatorname{Hom}(C_*X,k))$.

Definition 3.4. We say $C_*(X;k) \cong C_*(X) \otimes_{\mathbb{Z}} k$ and $H_*(X;k) \cong H_*(C_*X \otimes k)$ gives the singular homology of X with coefficients in k.

Lemma 3.5. $-\otimes k : \mathbf{Ab} \to \mathbf{Ab}$ is a right exact functor. $\operatorname{Hom}(-,k) : \mathbf{Ab}^{\mathrm{op}} \to \mathbf{Ab}$ is left exact.

Proof. Exercise. □

Remark 3.6. The covariant hom functor is also left exact.

Remark 3.7. The left adjoint is right exact, the right adjoint is left exact. In particular, we have the hom-tensor adjunction

$$\operatorname{Hom}(A, \operatorname{Hom}(B,C)) \cong \operatorname{Hom}(A \otimes B, C).$$

Note that

$$\operatorname{Hom}(A, \operatorname{Hom}(B,C)) \cong \operatorname{Hom}(A \otimes B, C) \cong \operatorname{Hom}(B \otimes A, C) \cong \operatorname{Hom}(B, \operatorname{Hom}(A,C))$$

Example 3.8. Consider

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

Tensoring with $\mathbb{Z}/n\mathbb{Z}$, we do not have exactness.

Example 3.9.

$$0 \longrightarrow A \longrightarrow A \oplus C \longrightarrow C \longrightarrow 0$$

is always exact after tensoring $-\otimes k$ or applying the hom functor $\operatorname{Hom}(-,k)$.

Definition 3.10. A short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ is split if any of the following equivalence conditions hold:

(i) p has a section $s : C \to B$ such that $ps = 1$;

(ii) i has a retraction $r : B \to A$ such that $ri = 1$;

(iii) $B \cong A \oplus C$, i.e.,

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & B \arrow[r, "p"] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, hook, "i"] & A \oplus C \arrow[r, two heads, "p"] & C \arrow[r] & 0 \end{tikzcd}$$

4 Aug 28, 2023

We will prove that (ii) implies (iii).

Suppose $b \in B$, then $b = (b - irb) + irb$, which is a decomposition of elements in $\ker(r)$ and in $\operatorname{im}(i)$, respectively. Also, $\ker(r) \cap \operatorname{im}(i) = 0$, therefore $B = \ker(r) \cap \operatorname{im}(i)$. Since i is an inclusion, then $\operatorname{im}(i) \cong A$. Now $p : B \to C$ factors through the projection onto $\ker(r)$ since $ri = 0$. By restricting p onto $\ker(r)$, we see p is also injective, thereby an isomorphism.

Lemma 4.1. If we have a split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, bend left, "i"] & B \arrow[l, bend left, "r"] \arrow[r, bend left, "p"] & C \arrow[l, bend left, "s"] \arrow[r] & 0 \end{tikzcd}$$

then $- \otimes k$ and $\operatorname{Hom}(-, k)$ preserves the split exactness, i.e.,

$$0 \longrightarrow A \otimes k \longrightarrow B \otimes k \longrightarrow C \otimes k \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Hom}(C, k) \longrightarrow \operatorname{Hom}(B, k) \longrightarrow \operatorname{Hom}(A, k) \longrightarrow 0$$

The point is tensors and homs preserve retracts.

Proof.
- $(r \otimes \mathrm{id}_k)(i \otimes \mathrm{id}_k) = ri \otimes \mathrm{id}_k = \mathrm{id}_{A\otimes k}$, so $i \otimes \mathrm{id}_k$ is split injective.
- Similarly, $\operatorname{Hom}(i, \mathrm{id})$ is split surjective.

□

Example 4.2. Given a surjection $B \to C \to 0$ such that C is free abelian, then there is always a section $s : C \to B$ making the exact sequence split. (That is, C is projective.) That is, if $0 \to A \to B \to C \to 0$ is an exact sequence where C is free, then the sequence is split exact.

Definition 4.3. Let $C \in \mathbf{Ab}$. A free resolution of C is a chain complex of free objects

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

and an augmentation $F_0 \to C$, so that

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0$$

is acyclic, i.e., exact everywhere.

Example 4.4.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow 0$$

is a free resolution of $\mathbb{Z}/n\mathbb{Z}$. So is

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

as well as

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\mathrm{id} \oplus (\times n)} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0$$

Lemma 4.5. Any $C \in \mathbf{Ab}$ admits a free resolution, and moreover, it admits a resolution of length 1¡ given by

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0$$

Proof. Choose a surjection $p : F_0 \to C$ from a free abelian group F_0 to C. Let $F_1 = \ker(p)$, then F_1 is free, so we are done. $\square$

Lemma 4.6. Free resolutions are essentially unique, i.e., if $F. \to C$ and $F'_. \to C$ are free resolutions, then there is a quasi-isomorphism $F. \xrightarrow{\sim} F'_.$ which commutes with the augmentations to C.

Definition 4.7. Let $C \in \mathbf{Ab}$ and let $F. \to C$ be a free resolution, then we define the torsion groups to be $\operatorname{Tor}_n^{\mathbb{Z}}(C, k) = H_n(F. \otimes k)$, and the ext groups to be $\operatorname{Ext}_{\mathbb{Z}}^n(C, k) = H^n(\operatorname{Hom}_{\mathbb{Z}}(F., k))$.

Remark 4.8.

- Tor and Ext are independent of the choice of resolutions.
- $\operatorname{Tor}_n^{\mathbb{Z}}$ and $\operatorname{Ext}_{\mathbb{Z}}^n$ are zero for $n > 1$.
- $\operatorname{Tor}_n^{\mathbb{Z}}(C, k) \cong \operatorname{Tor}_n^{\mathbb{Z}}(k, C)$.
- $\operatorname{Tor}_0^{\mathbb{Z}}(C, k) \cong C \otimes k$.
- $\operatorname{Ext}_{\mathbb{Z}}^0(C, k) \cong \operatorname{Hom}(C, k)$.

Example 4.9.

- If C is free, then $\operatorname{Tor}_1(C, k) = \operatorname{Ext}^1(C, k) = 0$.
- $\operatorname{Tor}_1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.
- $\operatorname{Tor}_1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = 0$.
- $\operatorname{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.
- $\operatorname{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.
- $\operatorname{Ext}^1(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = 0$.

Proof. Look at

$$0 \longrightarrow F_1 = \mathbb{Z} \longrightarrow F_0 = \mathbb{Z} \longrightarrow C = \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

then $\operatorname{Tor}_*(\mathbb{Z}/p\mathbb{Z}, k) = H_*(F_1 \otimes k = k \xrightarrow{\times p} F_0 \otimes k = k) = \begin{cases} k[p], * = 0 \\ k/pk, * = 1 \end{cases}$. Here $k[p]$ denotes p-torsion subgroup of k. Moreover, $\operatorname{Ext}^*(\mathbb{Z}/p\mathbb{Z}, k) = H^*(\operatorname{Hom}(F_1, k) = k \xleftarrow{\times p} \operatorname{Hom}(F_0, k) = k) = \begin{cases} k[p], * = 0 \\ k/pk, * = 1 \end{cases}$. $\square$

5 Aug 30, 2023

Recall that cohomology are basically the dual of homology, where the difference originates from the failure of exactness of the hom functor.

Theorem 5.1 (Universal Coefficient Theorem)**.** Let C_* be a chain of free abelian groups and $k \in \mathbf{Ab}$, then there exists a natural short exact sequence

$$0 \longrightarrow \operatorname{Ext}^1(H_{n-1}(C_*), k) \longrightarrow H^n(\operatorname{Hom}(C_*, k)) \xrightarrow{h} \operatorname{Hom}(H_n(C_*), k) \longrightarrow 0$$

that splits in an unnatural sense.

Here we define $h \in \operatorname{Hom}(H^n(\operatorname{Hom}(C_*, k)), \operatorname{Hom}(H_n(C_*), k))$. Note that this hom set is isomorphic to the hom set $\operatorname{Hom}(H^n(\operatorname{Hom}(C_*, k)) \otimes H_n(C_*), k)$ via the tensor-hom adjunction. That is, h is given by a bilinear pairing $H^n(\operatorname{Hom}(C_*, k)) \times H_n(C_*) \to k$. We use the Kronecker pairing $([f], [x]) \mapsto f(x)$. To see this is well-defined, let $f \in \operatorname{Hom}(C_n, k)$ with $\delta f = 0$, for $x \in C_n$, we have $\partial x = 0$. Now replace x by $x + \partial y$, then $f(x + \partial y) = f(x) = f(\partial y) = f(x) \pm (\delta f)(y) = f(x)$. Also, replace f by $f + \delta(g)$ gives $(f + \partial g)(x) = f(x) + (\delta g)(x) = f(x) + g(\delta x) = f(x)$.

Lemma 5.2. h is a split surjection.

Proof. Write $C_k^* = \mathrm{Hom}(C_*, k)$. Now $h : \ker(\delta, C_k^n \to C_k^{n+1}) \to \mathrm{Hom}(H_n(C_*), k)$ via $h : f \mapsto (x \mapsto f(x))$, then we will construct a section of h via $\varphi \mapsto \tilde{\varphi}$. Let $Z_n = \ker(\partial)$ and $B_n = \mathrm{im}(\partial)$, then $H_n(C_*) = Z_n/B_n$, and the short exact sequence of free abelian groups

$$0 \longrightarrow Z_n \xrightarrow{i} C_n \longrightarrow B_{n-1} \longrightarrow 0$$

and this splits so $C_n \cong Z_n \oplus B_{n-1}$. Given $\varphi : H_n(C_*) \to k$, we have

$$C_n \xrightarrow{r} Z_n \longrightarrow Z_n/B_n \xrightarrow{\varphi} k$$

where r is the retraction to i, and we define the composition to be $\tilde{\varphi}$. Now the composition

$$C_{n+1} \xrightarrow{\partial} C_n \longrightarrow Z_n \longrightarrow Z_n/B_n \longrightarrow k$$

is still zero since $C_{n+1} \to Z_n$ is zero, but that means $\delta\tilde{\varphi}$ is also zero. $\square$

We will now prove the universal coefficient theorem.

Proof. Since h is a split surjection, then we know this extends to a short exact sequence, hence we just need to identify the kernel of h, i.e., to show that $\ker(h) \cong \mathrm{Ext}^1(H_{n-1}(C_*), k)$. Given the split short exact sequence

$$0 \longrightarrow Z_n \xrightarrow{i} C_n \xrightarrow{\partial} B_{n-1} \longrightarrow 0$$

we have a diagram

$$\begin{tikzcd}
 & \vdots \arrow[d, "0"] & \vdots \arrow[d, "\delta"] & \vdots \arrow[d, "0"] & \\
0 \arrow[r] & \mathrm{Hom}(B_{n-1}, k) \arrow[r] \arrow[d, "0"] & \mathrm{Hom}(C_n, k) \arrow[r] \arrow[d, "\delta"] & \mathrm{Hom}(Z_n, k) \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \mathrm{Hom}(B_n, k) \arrow[r] \arrow[d, "0"] & \mathrm{Hom}(C_{n+1}, k) \arrow[r] \arrow[d, "\delta"] & \mathrm{Hom}(Z_{n+1}, k) \arrow[r] \arrow[d, "0"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

which is a short exact sequence of complexes. By the snake lemma, we have the long exact sequence of cohomology$\cdots \to H^n(B_k^{*-1}) \to H^n(C_k^*) \to H^n(Z_k^*) \to H^{n+1}(B_k^{*-1}) \to \cdots$. We claim that the connecting homomorphism $H^n(Z_k^*) \to H^{n+1}(B_k^{*-1})$ is $\mathrm{Hom}(B_n \subseteq Z_n, k)$. But $0 \to B^n \to Z^n \to H_n(C_*) \to 0$ is a free resolution of $H_n(C_*)$ of length 1. Then $H^*(\beta : \mathrm{Hom}(Z_n, k) \to \mathrm{Hom}(B_n, k)) = \mathrm{Ext}^*(H_n(C_*), k)$ where β has kernel $\mathrm{Hom}(H_n(C_*), k)$ and cokernel $\mathrm{Ext}^1(H_n(C_*), k)$. Therefore, the long exact sequence of cohomomology is the splicing (as epi-mono factorization) of

$$0 \longrightarrow \mathrm{coker}(\beta_{n-1}) \longrightarrow H_n(C_k^*) \longrightarrow \ker(\beta_n) \longrightarrow 0$$

and by identification we are done. $\square$

Corollary 5.3. If $C_* \to C'_*$ is a quasi-isomorphism, then $\mathrm{Hom}(C'_*, k) \to \mathrm{Hom}(C_*, k)$ is a quasi-isomorphism.

Corollary 5.4. Let $X \in \mathbf{Top}$ and $A \subseteq X$, then there exists a short exact sequence

$$0 \longrightarrow \mathbf{Ext}^1(H_{n-1}(X, A), k) \longrightarrow H^n(X, A; k) \longrightarrow \mathbf{Hom}(H_n(X, A); k) \longrightarrow 0$$

which is natural in (X, A). This also splits in (X, A) in an unnatural way.

Theorem 5.5. If C_* is a chain complex of free abelian groups, then there is a short exact sequence

$$0 \longrightarrow H_n(C_*) \otimes k \longrightarrow H_n(C_* \otimes k) \longrightarrow \mathbf{Tor}_1(H_{n-1}(C_*, k) \longrightarrow 0$$

which is natural. It splits unnaturally.

Corollary 5.6. For any pair (X, A), there is a natural short exact sequence

$$0 \longrightarrow H_n(X, A) \otimes k \longrightarrow H_n(X, A; k) \longrightarrow \mathbf{Tor}_1(H_{n-1}(X, A), k) \longrightarrow 0$$

which splits in an unnatural way.

6 Sept 1, 2023

Example 6.1. Take $X = \mathbb{C}P^2$, then the Tor and Ext terms go away, so the cohomology is equivalent to the homology.

Example 6.2. Take $X = \mathbb{R}P^2$, the Tor term gives $\mathrm{Tor}_1(\mathbb{Z}/2\mathbb{Z}, k) = k/2 \cong k[2]$, as the 2-torsion of k, i.e., the set of $a \in k$ such that $2a = 0$. Also, $\mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z}, k) = k/2k$.

Indeed, the Tor is given by the homology on multiplication by 2 map over k via tensor, and the Ext is given by the cohomology on multiplication by 2 map over k via hom.

Tor stands for torsion and Ext stands for extension.

Went on to talk about the limits and colimits.

Remark 6.3. In many abelian categories (and in particular, the category of abelian groups), we find a short exact sequence

$$0 \longleftarrow \mathrm{colim}_I \longleftarrow \bigoplus_{i \geqslant 0} X_i \longleftarrow \bigoplus_{i \geqslant 0} X_i \longleftarrow 0$$

and note that taking the dual version in the opposite category, we should obtain a sequence in the covariant sense. However, there is an asymmetry given by

$$0 \longrightarrow \lim_{I^{\mathrm{op}}} X \longrightarrow \prod_{i \geqslant 0} X_i \longrightarrow \prod_{i \geqslant 0} X_i \longrightarrow \lim_{I^{\mathrm{op}}}^1 X \longrightarrow 0$$

which is not short anymore. This is called a Milnor sequence.

7 Sept 6, 2023

The colimit of the empty diagram is the initial object; dually, the limit of the empty diagram is the terminal object.

Definition 7.1. We say $X : I \to \mathscr{C}$ is a filtered diagram if

- $\mathrm{Ob}(\mathscr{C}) \neq \varnothing$,
- for all $i, j \in I$, there exists $k \in I$ and morphisms $i \to k$ and $j \to k$, and
- for parallel morphisms $a, b : i \to j$ in I, then there exists coequalizers.

Example 7.2. A poset (as a category) P is a directed set if for any $i, j \in P$, there exists $k \in P$ such that $i \leqslant k$ and $j \leqslant k$.

For a filtered diagram $X : I \to \mathbf{Set}$, the colimit $\mathrm{colim}_I\, X$ exists and is isomorphic to $\coprod_{i \in I} X_i / \sim$, where $x_i \in X_i$ and $x_j \in X_j$ are equivalent if for some $k \in I$, we have $a : i \to k$ and $b : j \to k$ and that $a(x_i) = b(x_j)$

For concrete categories, we forget the additional structure to the category of sets, and find the colimits there, and give it the additional structure we want.

Lemma 7.3. If I is a directed set, then

$$0 \longrightarrow \bigoplus_{i\in I} A_i \longrightarrow \bigoplus_{i\in I} A_i \longrightarrow \operatorname{colim}_{i\in I} A_i \longrightarrow 0$$

$$(a_i)_{i\in I} \longrightarrow (a_j - f_{ij}(a_i))$$

where $f_{ij} : i \to j$.

Example 7.4. The colimit of a sequence given by $A \xrightarrow{\times n} A$ is $A\left[\frac{1}{n}\right]$.

Lemma 7.5. Colimit functor is exact in category of abelian groups.

8 Sept 8, 2023

- For a sequential diagram
$$\cdots \longrightarrow A_2 \longrightarrow A_1 \longrightarrow A_0$$
the limit of A_i's is the terminal cone, and in fact is the kernel of
$$\prod_{i\geqslant 0} A_i \to \prod_{i\geqslant 0} A_i$$
$$(a_i) \mapsto (a_i - f_{i+1}(a_{i+1}))_i$$
However, this sequence is not exact, as we discussed before.

Lemma 8.1. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_i & \longrightarrow & B_i & \longrightarrow & C_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_{i-1} & \longrightarrow & B_{i-1} & \longrightarrow & C_{i-1} & \longrightarrow & 0 \end{array}$$

then we have a long exact sequence

$$0 \longrightarrow \lim A_i \longrightarrow \lim B_i \longrightarrow \lim C_i \longrightarrow \lim^1 A_i \longrightarrow \lim^1 B_i \longrightarrow \lim^1(C_1) \longrightarrow 0$$

Proof. Take the products to get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \prod_i A_i & \longrightarrow & \prod_i B_i & \longrightarrow & \prod_i C_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \prod_i A_{i-1} & \longrightarrow & \prod_i B_{i-1} & \longrightarrow & \prod_i C_{i-1} & \longrightarrow & 0 \end{array}$$

and now use the snake lemma. $\square$

Example 8.2. The p-adic integers $\mathbb{Z}_p = \lim(\cdots \twoheadrightarrow \mathbb{Z}/p^k \twoheadrightarrow \mathbb{Z}/p^{k+1} \twoheadrightarrow \cdots)$ is a limit.

Theorem 8.3 (Mittag-Leffler Condition)**.** If $\{A_{i+1} \to A_i\}$ satisfies for each k, there is $i \geqslant k$ such that $\operatorname{im}(A_i \to A_k) \to \operatorname{im}(A_j \to A_k)$ for all $j \geqslant i \leqslant k$, then $\lim^1(A_i) = 0$.

Example 8.4.

1. This is true if all maps are surjections.
2. This is also true if all A_i's are finite.

Definition 8.5. Recall that a mapping cylinder is $M_f = (X \times s[0,1] \coprod Y)/((x,1) \sim f(x))$, so there is an inclusion $X \hookrightarrow M_f \cong Y$. Now given a sequence with $f_i : X_i \to X_{i+1}$, then the mapping telescope is

$$T = \mathrm{Tel}(X_*) = (\coprod_{n \geqslant 0} X_n \times [0,1])/((n,x,1) \sim (n+1, f_n(x), 0)),$$

with

$$\begin{aligned} i_n : X_n &\to T \\ x &\mapsto (n, x_n, 0) \end{aligned}$$

and homotopies $(i_n \circ f_{n-1}) \cong i_{n-1} : X_{n-1} \to T$. Therefore, the diagrams

$$\begin{tikzcd} H_*(X_{n-1}) \arrow[d, "(f_{n-1})_*"'] \arrow[dr, "(i_{n-1})_*"] & \\ & H_*(T) \\ H_*(X_n) \arrow[ur, "(i_n)_*"'] & \end{tikzcd}$$

commute. This induces a map $\mathrm{colim}_n(H_*(X_n)) \to H_*(T)$. We claim that this is an isomorphism.

Proof. Indeed, consider the refinement

$$\begin{aligned} \lambda : T = \coprod_n X_n \times [0,1]/\sim &\to \mathbb{R}_{\geqslant 0} \\ (n,x,t) &\mapsto n+t \end{aligned}$$

Let $T_{\leqslant a} = \lambda^{-1}([0,a])$ or $T_{<a} = \lambda^{-1}([0,a])$. We observe that $T_{\leqslant n}$ has a homotopy equivalence via $X_n \hookrightarrow T_{\leqslant n}$ with a deformation retraction. But $T_{\leqslant n}$ is also homotopy equivalent to $T_{<n+1}$. The upshot is that it suffices to show that $\mathrm{colim}(H_*(T_{<n})) \to H_*(T)$ is an isomorphism. $\square$

Proposition 8.6. Let Y be a space and let $\mathcal{A}$ be a collection of subspaces forming a direct system under inclusion. Assume that $Y = \bigcup_{A \in \mathcal{A}} A$, and for any compact $K \subseteq Y$, $K \subseteq A$ for some $A \in \mathcal{A}$. Then the map $\mathrm{colim}_{A \in \mathcal{A}} C_*(A \to C_*(Y)$ is an isomorphism, hence induces an isomorphism on the level of homology: $\mathrm{colim}(H_*(A)) \cong H_*(Y)$.

9 Sept 11, 2023

Recall that $H_*(\mathrm{Tel}(X_n)) \cong \mathrm{colim}_n H_*(X_n)$, with the proof replying on $C_*(\mathrm{Tel}(X_n)) \cong \mathrm{colim}_n C_*(X_n)$.

Example 9.1. $\mathrm{Tel}(S^1 \xrightarrow{p} S^1 \xrightarrow{p} \cdots) = T = S^1\left[\frac{1}{p}\right]$. Correspondingly, we have $\mathrm{colim}(H_0(S^1) \cong \mathbb{Z} \xrightarrow{p_*} H_0(S^1) \cong \mathbb{Z} \xrightarrow{p_*} \cdots) = \mathbb{Z}$, where the induced maps are just identities. Also, $\mathrm{colim}(H_1(S^1) \cong \mathbb{Z} \xrightarrow{p_*} H_1(S^1) \cong \mathbb{Z} \xrightarrow{p_*} \cdots) = \mathbb{Z}\left[\frac{1}{p}\right] \cong H_1(T)$, where the induced maps are multiplications by p.

By the Universal Coefficient theorem, we can calculate the cohomology of T as follows:

$$0 \longrightarrow \mathrm{Ext}^1(H_n^1(S^1\left[\tfrac{1}{p}\right], \mathbb{Z}) \longrightarrow H^n(S^1\left[\tfrac{1}{p}\right]) \, \mathrm{Hom}(H_n(S^1\left[\tfrac{1}{p}\right]), \mathbb{Z}) \longrightarrow 0$$

Here

- $H^0 * (S^1\left[\frac{1}{p}\right]) = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$;
- $H^1(S^1\left[\frac{1}{p}\right]) \cong \mathrm{Hom}(\mathbb{Z}\left[\frac{1}{p}\right]) = 0$, since the Ext term is 0;

- Higher homologies are zero, so $H^2(S^1\left[\frac{1}{p}\right]) \cong \operatorname{Ext}(\mathbb{Z}\left[\frac{1}{p}\right], \mathbb{Z}) \cong \mathbb{Z}_p/\mathbb{Z}$, the p-adic integers over $\mathbb{Z}$.

We are interested in calculating $H^*(\mathrm{Tel})$ in terms of $H^*(X_i)$'s. Note that the chain complex $C_*(\mathrm{Tel}(X_i)) \cong \operatorname{colim}_i(C_*X_i)$, so

$$\begin{aligned} C^*(\mathrm{Tel}(X_i)) &= \operatorname{Hom}(\operatorname{colim}_i(C_*X_i), \mathbb{Z}) \\ &= \lim_i(C^*(X_i)). \end{aligned}$$

Therefore, the question becomes, what is $H^*(\lim\limits_i(C_i^*)$?

Theorem 9.2 (Milnor Exact Sequence)**.** Suppose $\{C_i^*\}$ is an inverse system of cochain complexes, such that for each n, $\{C_i^n\}$ is an inverse system that satisfies Mittag-Leffler condition, i.e., we need $\lim^1 = 0$, then we have a short exact sequence

$$0 \longrightarrow \lim_i{}^1(H^{n-1}(C_i^*)) \longrightarrow H^n(\lim_i C_i^*) \longrightarrow \lim_i(H^n(C_n^*)) \longrightarrow 0$$

Proof. We set $B_i^n = \operatorname{im}(\delta : C_i^{n-1} \to C_i^n)$, and $Z_i^n = \ker(\delta : C_i^n \to C_i^{n+1})$. With this notation, we have a system of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & Z_i^n \arrow[r] \arrow[d] & C_i^n \arrow[r, "\delta"] \arrow[d] & B_i^{n+1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Z_{i-1}^n \arrow[r] & C_{i-1}^n \arrow[r, "\delta"'] & B_{i-1}^{n+1} \arrow[r] & 0 \end{tikzcd}$$

Therefore we have a long exact sequence

$$0 \longrightarrow \lim_i Z_i^n \longrightarrow \lim_i C_i^n \longrightarrow \lim_i B_i^{n+1} \longrightarrow \lim_i{}^1 Z_i^n \longrightarrow \lim_i{}^1 C_i^n \longrightarrow \lim_i{}^1 B_i^{n+1} \longrightarrow 0$$

By assumption, $\lim\limits_i{}^1 C_i^n = 0$, so $\lim\limits_i{}^1 B_i^{n+1} = 0$, and we have the sequence

$$0 \longrightarrow \lim_i Z_i^n \longrightarrow \lim_i C_i^n \longrightarrow \lim_i B_i^{n+1} \longrightarrow \lim_i{}^1 Z_i^n \longrightarrow 0$$

Denote $C^* = \lim\limits_i C_i^*$, and $Z^n = \ker(C^n \xrightarrow{\delta} C^{n+1})$, and $B^n = \operatorname{im}(C^{n-1} \to C^n)$. This gives

$$\begin{tikzcd} 0 \arrow[r] & Z^n \arrow[r] & C^n \arrow[dr, two heads, "\delta"'] & & \lim_i B_i^{n+1} \arrow[r] & \lim_i{}^1 Z_i^n \arrow[r] & 0 \\ & & & B_{n+1} \arrow[ur, hook] & & & \end{tikzcd}$$

We know have $0 \subseteq B^{n+1} \subseteq \lim\limits_i B_i^{n+1} \subseteq \lim\limits_i Z_i^{n+1} = Z^{n+1}$, therefore this gives an exact sequence

$$0 \longrightarrow \lim_i B_i^{n+1}/B^{n+1} \longrightarrow Z^{n+1}/B^{n+1} \longrightarrow Z^{n+1}/\lim_i B_i^{n+1} \longrightarrow 0$$

so this is

$$0 \longrightarrow \lim_i{}^1 Z_i^n \longrightarrow H^{n+1}(C^*) \longrightarrow Z^{n+1}/\lim_i B_i^{n+1} \longrightarrow 0$$

From the canonical exact sequence

$$0 \longrightarrow B_i^n \longrightarrow Z_i^n \longrightarrow H^n(C_i^*) \longrightarrow 0$$

we induce

$$0 \longrightarrow \lim_i B_i^n \longrightarrow Z^n \longrightarrow \lim_i H^n(C_i^n) \longrightarrow \lim_i^1 B_i^n \longrightarrow \lim_I^1 Z_i^n \longrightarrow \lim_i^1 H^n(C_i^*) \longrightarrow 0$$

but we have $\lim_i^1 B_i^n = 0$, so $\lim_i^1 Z_i^n \cong \lim_i^1 H^n(C_i^*)$, therefore we identify $Z^{n+}/\lim_i B_i^{n+1} \cong \lim_i H^{n+1}(C_i^*)$. □

Corollary 9.3. Let $X \in$ **Top** and $X = \bigcup_i X_i$ such that if there is compact $K \subseteq X$, then there exists some i such that $K \subseteq X_i$. If this is the case, then we have a short exact sequence in cohomology given by

$$0 \longrightarrow \lim_i^1 H^{n-1}(X_i) \longrightarrow H^n(X) \longrightarrow \lim_i H^n(X_i) \longrightarrow 0$$

Proof. We have $C_*(X) \cong \operatorname{colim}(C_*(X_i))$, and $C^*(X) \cong \lim C^*(X_i)$.

Claim 9.4. $\lim_i^1(C^n(X_i)) = 0$ for all n.

Subproof. We want the open cover of X to be a direct system, i.e., nested in some sense, so that we have a telescope and by the Mittag-Leffler condition we win. For instance, if we have telescopes, then $T = \operatorname{Tel}(X_0 \to X_1 \to \cdots)$, then $\bigcup_n T_{\leqslant n}$ gives $T_{\leqslant 0} \subseteq T_{\leqslant 1} \subseteq \cdots \subseteq T = \bigcup T_{\leqslant i}$. The point being, now we have $T_{\leqslant i} \cong X_i$ by deformation retraction, so we have a Milnor exact sequence on the level of cohomology of T, and we are done. ■

□

Example 9.5.

$$0 \longrightarrow \lim^1 H^1(S^1) \xrightarrow{\cong} H^2(S^1\left[\tfrac{1}{p}\right]) \longrightarrow H^2(S^1) \longrightarrow 0$$

where $\lim^1 H^1(S^1)$ is $\lim^1(\cdots \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \cdots) \cong \mathbb{Z}_p/\mathbb{Z}$.

10 Sept 13, 2023

We now want to define a map on cohomology groups. Let R be a commutative ring, and let $\varphi_i \in C^{n_i}(X, R)$ be with $i = 1, 2$, then we can define the cup product on $\smile$ with

$$\begin{aligned} C^{n_1}(X,R) \times C^{n_2}(X,R) &\to C^{n_1+n_2}(X,R) \\ (\varphi_1 \smile \varphi_2)(\sigma) &= \varphi_1(\sigma|_{[v_0,\dots,v_{n_1}]}\,\varphi_2(\sigma|_{[v_{n_1},\dots,v_{n_2}]} \end{aligned}$$

and we extend it linearly. Note that if $n_1 = 0$, then the map sends σ to $\varphi_1(\sigma|_{v_0})\varphi_2(\sigma)$. Moreover, if $\varphi_1 = e$ is the constant mapping with image 1, then $e \smile \varphi = \varphi = \varphi \smile e$. By associativity, we know $C^*(X, R)$ is a graded ring.

Lemma 10.1. $\smile$ is functorial in X, that is, if $f : X \to Y$, then $f^* : C^*(Y, R) \to C^*(X, R)$ is a ring homomorphism.

Lemma 10.2. $\partial(\varphi_1 \smile \varphi_2) = \partial\varphi_1 \smile \varphi_2 + (-1)^{|\varphi_1|} \smile \partial\varphi_2$.

Corollary 10.3.
- If $\varphi_1, \varphi_2 \in Z^*$ are cocycles, then the cup product $\varphi_1 \smile \varphi_2 \in Z^*$.
- If $\varphi_i \in Z^*$, and one is in B^*, then $\varphi_1 \smile \varphi_2 \in B^*$.

Using these two facts, we know that $\smile$: $H^{n_1}(X, \mathbb{R}) \times H^{n_2}(X, \mathbb{R}) \to H^{n_1+n_2}(X, R)$ is an induced map. In particular, if X is connected, then $H^0(X, R) \cong R$, and the cup product becomes the product on R. This has a graded ring structure.

Theorem 10.4. The cohomology cup product satisfies:

1. naturality in X,

2. $1 \smile \alpha = \alpha = \alpha \smile 1$ for $\alpha \in H^*(X, R)$. This is given by $1 : C_0 X \to R$ with $\sigma : \Delta^0 \to X$ sent to 1. Therefore, $1 = [1]$.
3. $\alpha \smile (\beta \smile \gamma) = (\alpha \smile \beta) \smile \gamma$.
4. $\alpha \smile \beta = (-1)^{|\alpha||\beta|}\beta \smile \alpha$.
5. For any pair (X, A) with $i : A \hookrightarrow X$ with $\delta : H^*(A; R) \to H^{*+1}(X, A; R)$, then for $\alpha \in H^*(A; R)$ and $\beta \in H^*(X; R)$, then $\delta(\alpha \smile i^*\beta) = \delta(\alpha) \smile \beta)$, and $\delta(i^*\beta \smile \alpha) = (-1)^{|\beta|}\beta \smile \delta(\alpha)$.

Remark 10.5. The cup product $\smile$ comes from $C^*(X) \otimes C^*(X) \to C^*(X)$, also regarded as $\mathrm{Hom}(C_*X, R) \otimes \mathrm{Hom}(C_*X, R) \to \mathrm{Hom}(C_*X, R)$, which is given by the factoring via $\mathrm{Hom}(C_*X \otimes C_*X, R)$. This gives a pairing on C^*X if we have a commutative diagram

$$\begin{tikzcd} C_*X \arrow[r] \arrow[d, "\sigma_n \mapsto 0"] & C_*X \otimes C_*X \arrow[d] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z} \end{tikzcd}$$

The map $C_*X \to C_*X \otimes C_*X$ is called the diagonal approximation. More generally, if we think of X and Y, then we have

$$\begin{tikzcd} C_*(X \times Y) \arrow[r] \arrow[d] & C_*X \otimes C_*Y \arrow[d] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}$$

In particular, if $X = Y$, then we have a diagonal mapping $X \to X \times X$, therefore induces $C_*X \to C_*(X \times X)$.

Definition 10.6. The Alexander-Whitney map is given by

$$AW_{X,Y} : C_*(X \times Y) \to C_*X \otimes C_*Y$$

where $C_*X \otimes C_*Y$ is given by total complex of degree n, i.e., $\bigoplus\limits_{i+j=n} C_iX \otimes C_jY$, and differential $\partial(a \otimes b) = \partial a \otimes b + (-1)^{|a|}a \otimes \partial b$.

$$\begin{tikzcd} & X \\ \Delta^n \arrow[ur, "\sigma"] \arrow[r, "(\sigma,\tau)"] \arrow[dr, "\tau"'] & X \times Y \arrow[u, "\pi_X"'] \arrow[d, "\pi_Y"] \\ & Y \end{tikzcd}$$

The Alexander-Whitney map defines $AW(\sigma, \tau) = \sum\limits_{i+j=n} \sigma|_{[v_0,\dots,v_i]} \otimes \tau|_{[v_i,\dots,v_n]}$. On the level of cochains, the cup product is $\mathrm{Hom}(-, R)$ of composition of Alexander-Whitney map and the induced diagonal mapping.

Similarly, we can define the cochain version, with a pair (X, A), then

$$\begin{tikzcd} C_*(X \times Y, A \times Y) \arrow[r, dashed] & C_*(X, A) \otimes C_*Y \\ C_*(X \times Y) \arrow[r, "AW_{X\times Y}"] \arrow[u, two heads] & C_*X \otimes C_*Y \arrow[u, two heads] \\ C_*(A \times Y) \arrow[r, "AW_{A,Y}"'] \arrow[u] & C_*A \otimes C_*Y \arrow[u] \end{tikzcd}$$

We now want $(X, A) \times (Y, B) = (X \times Y, A \times Y \cup X \times B)$ to have the suitable mapping. Naturally, we get the Alexander-Whitney map

$$\begin{tikzcd} C_*(X \times Y)/(C_*(X \times B) + C_*(A \times Y)) \arrow[r] \arrow[d] & C_*(X, A) \otimes C_*(Y, B) \\ C_*(X \times Y)/C_*(A \times Y \cup X \times B) \arrow[ur, dashed] & \end{tikzcd}$$

The summation is not the direct sum but not summation in complex.

11 Sept 15, 2023

Recall that the Alexander-Whitney map is the natural transformation of functors **Top** $\times$ **Top** $\to$ **Ch** via $C_*(X \times Y) \to C_*(X) \to C_*(Y)$, where

$$AW(\sigma, \tau) = \sum_{i+j=n} \sigma|_{[v_0,\dots,v_i]} \otimes \tau|_{[v_i,\dots,v_n]}$$

for $\sigma, \tau : \Delta^n \to X \times Y$. We also note that the cross product is defined as the composition

$$\begin{tikzcd} H^*(\mathrm{Hom}(C_*X, R)) \otimes H^*(\mathrm{Hom}(C_*Y, R)) \arrow[r] \arrow[dr, dashed, "\times"'] & H^*(\mathrm{Hom}(C_*X \otimes C_*Y, R)) \arrow[d, "AW^*"] \\ & H^*(\mathrm{Hom}(C_*(X,Y), R)) \end{tikzcd}$$

where the horizontal map is induced by homological algebra. The cup product is composed by the diagonal inclusion and the cross product:

$$\begin{tikzcd} H^*(X) \otimes H^*(X) \arrow[r, "\times"] \arrow[rr, bend right, "\smile"'] & H^*(X \times X) \arrow[r, "\Delta^*"] & H^*(X) \end{tikzcd}$$

given by

$$(f \smile g)(\sigma) = f(\sigma|_{[v_0,\dots,v_i]}\, g(\sigma|_{[v_i,\dots,v_{i+j}]}$$

for $f \in H^i(X)$, $g \in H^j(X)$, $\sigma : \Delta^{i+j} \to X$.

Remark 11.1.

- If X is connected, then $H^0(X, R) = R$.
- The cup product gives the R-module structure on $H^n(X)$.

Example 11.2. Let $X = S^n$, then

$$H^*(X, R) = \begin{cases} R, & * = 0, n \\ 0, & \text{otherwise} \end{cases}$$

This says that the induced multiplication map $R \otimes R \to R$ on cohomology has the same behavior, i.e., $H^n(S^n; R) \otimes H^n(S^n; R) \to H^{2n}(S^n, R) = 0$. That is, we have $H^*(S^n; R) \cong R[e_n]/e_n^2$.

For the unit interval $I = [0, 1]$, then

$$\tilde{H}^*(S^1) \cong H^*(I, \partial I) = \begin{cases} \mathbb{Z}, & * = 1 \\ 0, & \text{otherwise} \end{cases}$$

Claim 11.3.

$$H^1(I, \partial I) \otimes H^n(Y) \xrightarrow{\times} H^{n+1}(I \times Y, \partial I \times Y)$$

is an isomorphism for any Y.

Corollary 11.4.

$$H^*(S^1) \otimes H^*(Y) \xrightarrow{\times} H^*(S^1 \times Y)$$

is an isomorphism for any space Y.

Example 11.5. Consider the Moore spaces. For any $m \in \mathbb{Z}$, we have $X_m = S^1 \cup_m e^2$, so we have

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, "m"'] & D^2 \arrow[d] \\ S^1 \arrow[r] & X_m \end{tikzcd}$$

We can give this a cell structure, so for instance $m = 2$, we have $X_m = \mathbb{R}P^2$. For general m, we have the cell structure with vertices x and y, $m+1$ edges $a, e_0, \dots, e_m$, and m faces $C_0, \dots, C_{m-1}$, then the boundary map is given by $\partial(a) = 0$, $\partial(e_i) = y - x$, and $\partial(C_i) = a - e_{i+1} + e_i$.

In the case $m = 2$, we have

$$\begin{tikzcd} x \arrow[r, "e_1"] \arrow[d, "e_0"'] & y \\ y \arrow[ur, "a"] & x \arrow[l, "e_1"] \arrow[u, "e_0"'] \end{tikzcd}$$

where the upper triangle is the face C_0 and the bottom triangle is the face C_1. We look at the chain equivalences

$$\begin{tikzcd} C_0X_2 & C_1X_2 \arrow[l] & C_0X_2 \arrow[l] & \cdots \arrow[l] \\ \mathbb{Z}\{x,y\} \arrow[u, hook] & \mathbb{Z}\{\alpha, e_0, e_1\} \arrow[l] \arrow[u, hook] & \mathbb{Z}\{C_0, C_1\} \arrow[l] \arrow[u, hook] & \cdots \arrow[l] \end{tikzcd}$$

The integral cohomology is just the cohomology of the above chain with respect to the dual basis, then checking the kernel and image, we know $\delta(x^\vee) = -e_0^\vee - e_1^\vee$, $\delta(y^\vee) = e_0^\vee + e_1^\vee$, $\delta(a^\vee) = C_0^\vee + C_1^\vee$, and $\delta(e_0^\vee) = C_0^\vee - C_1^\vee$, $\delta(e_1^\vee) = C_1^\vee - C_0^\vee$, therefore $x^\vee + y^\vee$ generates H^0. Similarly, we can show that $H^1 = 0$ and $H^2 = \mathbb{Z}/2$.

12 Sept 18, 2023

We need to prove that $\alpha \smile \beta = (-1)^{|\alpha||\beta|}\beta \smile \alpha$ in $H^*(X;k)$. Define $\rho : \Delta^n \to \Delta^n$ by sending $[v_0, \dots, v_n]$ to $[v_n, \dots, v_0]$. Using this, we can define a map

$$\begin{aligned} \rho : C_*X &\to C_*X \\ \sigma &\mapsto (-1)^{\varepsilon_n}\ \sigma|_{[v_n,\dots,v_0]} \end{aligned}$$

where ε_n is the number of permutations required to permute $(0, \dots, n)$ into $(n, \dots, 0)$. This should just be $\binom{n+1}{2}$.

Exercise 12.1. ρ is a chain map.

This induces $\rho : C^*X \to C^*X$ with $\rho(\alpha)(\sigma) = (-1)^{\varepsilon_i}\alpha(\sigma)_{[v_n,\dots,v_0]}$. Therefore,

$$\begin{aligned} \rho(\alpha \smile \beta)(\sigma) &= (-1)^{\varepsilon_n}(\alpha \smile \beta)(\sigma|_{[v_n,\dots,v_0]}) \\ &= (-1)^{\varepsilon_n}\alpha(\sigma|_{[v_n,\dots,v_j]})\beta(\sigma|_{[v_j,\dots,v_0]}) \\ &= (-1)^{\varepsilon_n}(-1)^{\varepsilon_i}\rho(\alpha)(\sigma|_{[v_j,\dots,v_n]}) \cdot (-1)^{\varepsilon_j}\rho(\beta)(\sigma|_{[v_0,\dots,v_j]}) \\ &= (-1)^{\varepsilon_n+\varepsilon_i+\varepsilon_j}\rho(\beta) \smile \rho(\alpha)(\sigma). \end{aligned}$$

Claim 12.2. $\varepsilon_i + \varepsilon_j - \varepsilon_{i+n} \equiv ij \pmod 2$.

In particular, this proves the claim. Moreover, ρ is a chain equivalence.

Proposition 12.3. If $f, g : C_*X \to C_*X$ are natural transformations of functors **Top** $\to$ **Ch**, such that f_0 and g_0 are naturally isomorphic (as components of the natural transformations), then f and g are naturally equivalent. Here $f_0, g_0 :$ **Top** $\to$ **Ab**.

Theorem 12.4. Given a functor $F : \mathscr{C} \to$ **Ch**, there is an equivalence of categories **Func**$(\mathscr{C},$ **Ch**$) \cong$ **Ch**$($**Func**$(\mathscr{C},$ **Ab**$))$.

To prove the theorem, we introduce acyclic models. Suppose we have a functor $F : \mathscr{C} \to$ **Ch**. We regard $\mathscr{C}$ as **Top**, or **Top** $\times$ **Top**.

Definition 12.5. A functor $F : \mathscr{C} \to$ **Ab** is called free on models M if

- there exists a set $M \subseteq \mathrm{Ob}(\mathscr{C})$ such that F is naturally isomorphic to the functor defined by the mapping $X \mapsto \bigoplus_{A \in M} \mathbb{Z}\{\mathrm{Hom}_{\mathscr{C}}(A, X)\}$.

Remark 12.6. Note that if $G : \mathscr{C} \to \mathbf{Set}$ is representable with respect to $A \in \mathscr{C}$, then the composition of the free functor $\mathbf{Set} \to \mathbf{Ab}$ and $G : \mathscr{C} \to \mathbf{Set}$ is free on model A.

- A functor $F_* : \mathscr{C} \to \mathbf{Ch}$ is free on models $\{M_n\}_{n\in\mathbb{Z}}$ if each $F_n : \mathscr{C} \to \mathbf{Ab}$ is free on M_n.
- Given $M \subseteq \mathrm{Ob}(\mathscr{C})$, a functor $F : \mathscr{C} \to \mathbf{Ch}$ is M-acyclic in positive degrees if for all $A \in M$, $H_q(F(A)) = 0$ for all $q > 0$.

Example 12.7. $C_* : \mathbf{Top} \to \mathbf{Ch}$ is acyclic in positive degrees on $\{\Delta^n\}_{n\in\mathbb{Z}}$.

Example 12.8. Consider $\mathbf{Top}^2 \to \mathbf{Ch}$.

1. If we have $(X, Y) \mapsto C_*(X \times Y)$, then $C_n(- \times -)$ is free on the model $\Delta^n \times \Delta^n$, and $C_*(- \times -)$ is acyclic on $\{\Delta^p \times \Delta^q\}_{p,q\geqslant 0}$.
2. If we have $(X, Y) \mapsto C_*(X) \otimes C_*(Y)$, then $(C_*(-) \otimes C_*(-))_n$ is free on the models $\{\Delta^p \times \Delta^{n-p}\}_p$, which is acyclic in positive degrees on $\{\Delta^p \times \Delta^q\}$.

13 Sept 20, 2023

Theorem 13.1 (Acyclic Models)**.** Suppose $F_*, G_* : \mathscr{C} \to \mathbf{Ch}$ are functors, and assume $F_n = 0 = G_n$ for $n < 0$. Assume

(a) each $F_n : \mathscr{C} \to \mathbf{Ab}$ is free on models $M_n \subseteq \mathrm{ob}(\mathscr{C})$, and

(b) G_* is acyclic in positive degrees on $\bigcup_{n\geqslant 0} M_n$,

then

1. any natural transformation $H_0F_* \to H_0G_*$ of functors $\mathscr{C} \to \mathbf{Ab}$ is induced by a natural transformation $F_* \to G_*$, and
2. if $f, g : F_* \to G_*$ are natural transformations such that $H_0 f = H_0 g$, then there exists a natural chain homotopy $f \simeq g$, and
3. assume, in addition, that G_* is free on some model N, then if $f : F_* \to G_*$ is a natural transformation such that $H_0 f : H_0F_* \to H_0G_*$ is a natural isomorphism, then f is a natural chain equivalence.

Claim 13.2. Any natural transformation $C_*X \to C_*X$ that induces an isomorphism $H_0X \to H_0X$ is a chain equivalence.

Example 13.3. Take $\rho : C_*X \to C_*X$ that inverts orientation, then ρ induces identity on cohomology, so

$$\alpha \smile \beta = \rho(\alpha \smile \beta) = (-1)^{|\beta|\times|\alpha|}\rho(\beta) \smile \rho(\alpha) = (-1)^{|\beta|\times|\alpha|}\beta \smile \alpha$$

Claim 13.4.

$$AW : C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$$

is a natural chain equivalence.

Proof. Apply acyclic models. □

Lemma 13.5 (Yoneda)**.** If $G : \mathscr{C} \to \mathbf{Set}$ is a functor, and let $C \in \mathrm{Ob}(\mathscr{C}($ be a representation of the functor, that is, $F_c(d) = \mathrm{Hom}_{\mathscr{C}}(c, d)$, then there is a natural bijection of sets $\mathrm{Nat}(F_c, G) \cong G(c)$ by $f : F_c \to G \mapsto f(\mathrm{id}_c)$.

Corollary 13.6. If $F : \mathscr{C} \to \mathbf{Ab}$ is free on models M, that is, $F(X) = \mathbb{Z}\left\{\coprod_{A\in M} \mathrm{Hom}_{\mathscr{C}}(A, X)\right\} \cong \bigoplus_{A\in M} \mathbb{Z}\{F_A(X)\}$, which induces

$$F : \mathscr{C} \xrightarrow{\coprod_{A\in M} F_A} \mathbf{Set} \xrightarrow{\text{Free}} \mathbf{Ab}$$

then for any $G : \mathscr{C} \to \mathbf{Ab}$, then we have a natural isomorphism $\mathrm{Nat}(F, G) \cong \prod_{A\in M} G(A)$ given by $(f : F \to G) \mapsto (f(\mathrm{id}_A))_{A\in M}$.

We will now prove the acyclic models theorem.

Proof. 1. Take $F_* \to G_*$, then we are given a natural transformation $\bar{\varphi}_{-1} : H_0F_* \to H_0G_*$ with

$$\begin{tikzcd} 0 & H_0F_* \arrow[l] \arrow[d, "\bar{\varphi}_{-1}"] & F_0 \arrow[l] & F_1 \arrow[l] & \cdots \arrow[l] \\ 0 & H_0G_* \arrow[l] & G_0 \arrow[l] & G_1 \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

We want to lift $\varphi_0 \in \mathrm{Nat}(F_0, G_0)$, we take a look into the commutative diagram

$$\begin{tikzcd} \mathrm{Nat}(F_0, G_0) \arrow[r, "\cong"] \arrow[d, "\partial_G"'] & \prod_{A \in M_0} G_0(A) \arrow[d, two heads] \\ \mathrm{Nat}(F_0, H_0G_0) \arrow[r, "\cong"'] & \prod_{A \in M_0} H_0G_0(A) \end{tikzcd}$$

so we take $\varphi_{-1} \circ \partial_F \in \prod_{A \in M_0} H_0G_0(A)$, then lift it to $\varphi_0 \in \prod_{A \in M_0} G_0(A)$, then we obtain $\varphi_0 : F_0 \to G_0$ as desired. By construction, $\partial_G \varphi_0 = \varphi_{-1}\partial_F$. Proceeding inductively, we complete the diagram.

2. Now given $f, g : F_* \to G_*$, with $H_0f = H_0g$, we want $f \simeq g$. We want $h_i : F_i \to G_i$ to be such that $f_i - g_i = h_{i-1}\partial_F + \partial_F h$.

□

14 Sept 22, 2023

A complex C_* that is chain equivalent to 0 implies it is acyclic, i.e., $H_q(C_*) = 0$ for all q.

Proposition 14.1. If C_* is a complex of free abelian groups with $C_n = 0$ for $n \ll 0$, then C_* is acyclic if and only if it is chain equivalent to 0.

Proof. We can assume $C_n = 0$ for $n < 0$. Now consider $F : \mathscr{C} = \{*\} \to \mathbf{Ch}$ where $F(*) = C_*$. Now F is free and acyclic on models $\{*\}$, then the identity and zero map gives the same map on H_0, and by the acyclic model theorem we are done. □

Example 14.2. If $X \in \mathbf{Top}$, then X is acyclic if $H_*X = \begin{cases} \mathbb{Z}, * = 0 \\ 0, \text{otherwise} \end{cases}$, and so we extend the kernel and get a short exact sequence

$$0 \longrightarrow \tilde{C}_*X \longrightarrow C_*X \twoheadrightarrow C_*\{*\} \longrightarrow 0$$

Note that the last map admits a section with respect to a choice of a point $x_0 \in X$. Therefore, X is acyclic if and only if $\tilde{C}_*X$ is acyclic. Also, $\tilde{C}_*X$ is a complex of free abelian groups, so eX being acyclic implies $\tilde{C}_*X$ is chain equivalent to 0. Therefore, C_*X is chain homotopic to zero, as a complex concentrated at degree 0.

For instance, let $X = \Delta^p$ or $\Delta^p \times \Delta^q$.

Corollary 14.3 (Eilenberg-Zilber)**.** For any $X, Y \in \mathbf{Top}$, $C_*(X \times Y) \cong C_*X \otimes C_*Y$.

Claim 14.4. There is an anti-commutative diagram

$$\begin{tikzcd} H^p(X) \times H^q(Y) \arrow[r, "\times"] \arrow[d, "s"', "\cong"] & H^{p+q}(X \times Y) \arrow[d, "s^*"] \\ H^q(Y) \times H^p(X) \arrow[r, "\times"'] & H^{q+p}(Y \times X) \end{tikzcd}$$

with $\alpha \times \beta = (-1)^{|\alpha||\beta|} s^*(\beta \times \alpha)$.

This follows from

Lemma 14.5.

$$\begin{tikzcd} C_*(X \times Y) \arrow[r, "AW"] \arrow[d, "s_*"'] & C_*(X) \otimes C_*(Y) \\ C_*(Y \times X) \arrow[r, "AW"'] & C_*(Y) \otimes C_*(X) \arrow[u, "T"'] \end{tikzcd}$$

where T is a twist map via $T(y \otimes x) = (-1)^{|x||y|} x \otimes y$.

Theorem 14.6 (Kunneth)**.** Let $C_*, D_* \in \mathbf{Ch}$, say C_* is built out of free abelian groups, then

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(C_*) \otimes H_j(D_*) \longrightarrow H_n(C_* \otimes D_*) \longrightarrow \bigoplus_{i+j=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_i C_*, H_j D_*) \longrightarrow 0$$

Remark 14.7. $\mathrm{Tor}(M, A) \cong \mathrm{Tor}(A, M)$.

Example 14.8. $\mathrm{Tor}(A, \mathbb{Z}) = 0 = \mathrm{Tor}(\mathbb{Z}, A)$, and $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$.