Motivic Cohomology Notes

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0 Introduction

Let $X \in \mathrm{Sm}/k$ be a smooth separated scheme over a field k. The study of motivic cohomology started with the hope that

Conjecture 0.1 (Beilinson and Lichtenbaum, 1982-1987)**.** There exists some complexes $\mathbb{Z}(n)$ for $n \in \mathbb{N}$ of sheaves in Zariski topology on Sm/k such that

1. $\mathbb{Z}(0)$ is (quasi-isomorphic to) the constant sheaf $\mathbb{Z}$, i.e., the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$

   concentrated at degree 0;

2. $\mathbb{Z}(1)$ is the complex $\mathcal{O}^*[-1]$, i.e., the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{O}^* \longrightarrow 0 \longrightarrow \cdots$$

   concentrated at degree 1;

3. for every field F/k, the hypercohomology over Zariski topology satisfies[1]

$$\mathbb{H}^n_{\mathrm{Zar}}(F, \mathbb{Z}(n)) = H^n(\mathbb{Z}(n)(\mathrm{Spec}(F))) = K_n^M(F),$$

   where $K_n^M(F)$ is the nth Milnor K-theory of a field F, given by the quotient of the tensor algebra $T(F^*)/\{x \otimes (1-x) : x \in F^*\}$ over $\mathbb{Z}$; (lecture 5 of [MVW06], page 29)

   Example 0.2.

   a. $K_0^M(F) = K_0(F) = \mathbb{Z}$;
   b. $K_1^M(F) = K_1(F) = F^\times$;
   c. $K_2^M(F) = K_2(F)$.

4. $\mathbb{H}^{2n}_{\mathrm{Zar}}(X, \mathbb{Z}(n)) = \mathrm{CH}^n(X)$ (lecture 17 of [MVW06], page 135), where the nth classical Chow group $\mathrm{CH}^n(X)$ is the free group given by

$$\mathrm{CH}^n(X) = \mathbb{Z}\{\text{cycles of codimension } n\}/\text{rational equivalences};$$

[1] Here we use the convention that the (hyper)cohomology of F should be interpreted as of $\mathrm{Spec}(F)$, the corresponding space.

5. there is a natural Atiyah–Hirzebruch spectral sequence
$$E_2^{p,q} = \mathbb{H}^p_{\text{Zar}}(X, \mathbb{Z}(q)) \Rightarrow K_{2q-p}(X).$$
Moreover, tensoring with $\mathbb{Q}$, the spectral sequence degenerates and one has
$$\mathbb{H}^i_{\text{Zar}}(X, \mathbb{Z}(n))_{\mathbb{Q}} = \text{gr}^n_\gamma(K_{2n-i}(X)_{\mathbb{Q}})$$
where gr^n_γ's are the quotients (graded pieces) of γ-filtration. ([Lev94]; [Lev99], Theorem 11.7)

Remark 0.3. Such choice of complexes $\mathbb{Z}(q)$ exists, and is called the motivic complex. For a clear definition of these complexes, see Lecture 3 of [MVW06]. Moreover, by convention $\mathbb{Z}(q) = 0$ for $q < 0$.

Definition 0.4. The motivic cohomology of X is defined by $H^{p,q}(X, \mathbb{Z}) = \mathbb{H}^p_{\text{Zar}}(X, \mathbb{Z}(q))$, the hypercohomology of the motivic complexes with respect to the Zariski topology.

Remark 0.5. In general, a motivic cohomology with coefficient in an abelian group A is a family of contravariant functors $H^{p,q}(-, A) : \text{Sm}/k \to \text{Ab}$.

Remark 0.6. The motivic cohomology of X satisfies the cancellation property: set $\mathbb{G}_m = \mathbb{A}^1 \backslash \{0\}$, then
$$H^{p,q}(X \times \mathbb{G}_m, \mathbb{Z}) = H^{p,q}(X, \mathbb{Z}) \oplus H^{p-1,q-1}(X, \mathbb{Z}).$$

Remark 0.7. It turns out that the group remains unchanged if we replace the Zariski topology by Nisnevich topology.[2] If one uses étale topology instead, we retrieve Lichtenbaum motivic cohomology $H^{p,q}_L(X, \mathbb{Z})$. If $\text{char}(k) \nmid n$, it admits the comparison
$$H^{p,q}_L(X, \mathbb{Z}/n\mathbb{Z}) = H_{\text{étale}}(X, \mathbb{Z}/n\mathbb{Z}(q)),$$
where $\mathbb{Z}/n\mathbb{Z}(q)$ is the q-twist $\mu_n^{\otimes q}$.

We may compare Lichtenbaum motivic cohomology with motivic cohomology by the following theorem, formerly known as Beilinson-Lichtenbaum Conjecture[3]:

Theorem 0.8 ([Voe11])**.** The natural map
$$H^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) \to H^{p,q}_L(X, \mathbb{Z}/n\mathbb{Z})$$
is an isomorphism if $p \leqslant q$, is a monomorphism if $p = q + 1$, and gives a spectral sequence for any pair of p, q.

Corollary 0.9. For $p \leqslant q$, we have
$$H^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) = H^p_{\text{étale}}(X, \mathbb{Z}/n\mathbb{Z}(q)).$$

In particular, for $X = \text{Spec}(k)$ as a point, this is the theorem formerly known as Milnor conjecture:

Corollary 0.10 ([Voe97], [Voe03a], [Voe03b])**.**

- $H^{p,p}(k, \mathbb{Z}/n\mathbb{Z}) = K^M_p(k)/n = H^p_{\text{étale}}(X, \mathbb{Z}/n\mathbb{Z}(p))$ as the Galois cohomology;
- in general,
$$H^{p,q}(k, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} 0, & p > q \\ H^{p,p}(k, \mathbb{Z}/n\mathbb{Z}) \cdot \tau^{q-p}, & p \leqslant q \end{cases}$$
where $\tau \in \mu_n(k) = H^{0,1}(k, \mathbb{Z})$ is a primitive nth root of unity.

Remark 0.11. Unlike the case with finite coefficients, $H^{p,q}(k, \mathbb{Z})$ is quite hard to compute for small $p < q$; for $p \geqslant q$, this is 0.

[2] Recall that the Nisnevich topology is a Grothendieck topology on the category of schemes that is finer than the Zariski topology but coarser than the étale topology.

[3] This is also known as the norm residue isomorphism theorem, or (formerly) Bloch-Kato conjecture.

A current long-standing conjecture is

Conjecture 0.12 (Beilinson-Soulé Vanishing Conjecture, [Lev93])**.** $H^{p,q}(k,\mathbb{Z}) = 0$ if $p < 0$.

Remark 0.13. Here are a few known cases:

- for $\operatorname{char}(k) = 0$, this is known for number fields ([Bor74]), function fields of genus 0 ([Dég08]), curves over number fields, and their inductive limits (more precise references required); ([DG05])
- for $\operatorname{char}(k) > 0$, this is known for finite fields ([Qui72]) and global fields ([Har77]).

Remark 0.14. The motivic cohomology could be realized in a tensor triangulated category, namely the category of effective motives $\mathrm{DM}(k)$. For any pair of p, q, we can find an Eilenberg-Maclane space and a corresponding representable functor so that

$$H^{p,q}(X,\mathbb{Z}) = \mathrm{Hom}_{DM}(\mathbb{Z}(X), \mathbb{Z}(q)[p])$$

where $\mathbb{Z}(X)$ is the motive of X and $\mathbb{Z}(q)[p] = \mathbb{G}_m^{\wedge q}[p-q]$.[4]

Remark 0.15. Dually, we can define the motivic homology by

$$H_{p,q}(X,\mathbb{Z}) = \mathrm{Hom}_{DM}(\mathbb{Z}(q)[p], \mathbb{Z}(X)).$$

Remark 0.16 ([MVW06] Properties 14.5, page 110)**.** By taking the hom functor from the aspect of motives, we can derive theorems for all (co)homologies which can be represented in DM. The main derives are the following:

1. If $E \to X$ is an $\mathbb{A}^n$-bundle, then motives $\mathbb{Z}(E) = \mathbb{Z}(X)$ in DM.
2. If $\{U, V\}$ is a Zariski open covering of X, we have a Mayer-Vietoris sequence
$$\mathbb{Z}(U \cap V) \longrightarrow \mathbb{Z}(U) \oplus \mathbb{Z}(V) \longrightarrow \mathbb{Z}(X) \longrightarrow \mathbb{Z}(U \cap V)[1]$$
in the form of a distinguished triangle in DM.
3. If $Y \subseteq X$ is a closed embedding of codimension c in Sm/k, then we have a Gysin triangle
$$\mathbb{Z}(X \backslash Y) \longrightarrow \mathbb{Z}(X) \longrightarrow \mathbb{Z}(Y)(c)[2c] \longrightarrow \mathbb{Z}(X \backslash Y)[1]$$
which is a distinguished triangle where $\mathbb{Z}(Y)(c)[2c] := \mathbb{Z}(Y) \otimes \mathbb{Z}(c)[2c]$.
4. For any vector bundle of rank n on X, we have the projective bundle formula
$$\mathbb{Z}(\mathbb{P}(E)) = \bigoplus_{i=0}^{n} \mathbb{Z}(X)(i)[2i]$$
which defines the Chern class of E.
5. Let X be a proper smooth scheme and let d_X be its dimension, then $\mathbb{Z}(X)$ has a strong dual $\mathbb{Z}(X)(-d_X)[-2d_X]$ in DM by stabilization. This gives a Poincaré duality[5]
$$H^{p,q}(X,\mathbb{Z}) \cong H_{2d_X-p, d_X-q}(X,\mathbb{Z}).$$

[4] Again, this notation goes back to the concise definition of the motivic complexes: see Lecture 3 from [MVW06] as well as the concept of presheaves with transfers.

[5] We can use cohomology with compact support for this.

1 Intersection Theory

1.1 Cycles of Scheme

Definition 1.1. Let X be a scheme of finite type over k. We define the ith cycle on the scheme X to be a free abelian group

$$Z_i(X) = \bigoplus_{\substack{\text{irreducible closed } c\subseteq X\\ \text{with } \dim(c)=i}} \mathbb{Z}\cdot c$$

and set $Z(X) = \bigoplus_i Z_i(X)$. Define a set $K_i(X)$ to be the set of coherent sheaves $\mathcal{F}$ on X with $\dim(\operatorname{supp}(F)) \leqslant i$.[6]

Remark 1.2. Let $(A, \mathfrak{m})$ be a Noetherian local ring and M be an A-module, then by the dimension theorem, we know $\dim(M) = d(M) = \dim(\operatorname{supp}(M))$, where $d(M)$ is the degree of the Hilbert-Samuel polynomial $P_{\mathfrak{m}}(M, n)$.

Definition 1.3. Let $X \in \mathrm{Sm}/k$ and let $U, V \subseteq X$ be irreducible and closed. Suppose $W \subseteq U \cap V$ is a irreducible and closed component. If $\dim(W) = \dim(U) + \dim(V) - \dim(X)$, i.e., $\operatorname{codim}(W) = \operatorname{codim}(U) + \operatorname{codim}(V)$, we say that U and V intersect properly at W.

Remark 1.4. This condition is weaker than saying they intersect transversely: we do not require information about tangent spaces.

Theorem 1.5. Let $A \supseteq k$ be a Noetherian regular ring, M, N be finitely-generated A-modules, and suppose $\ell(M \otimes_A N) < \infty$, then

1. $\ell(\operatorname{Tor}_i^A(M, N)) < \infty$ for all $i \geqslant 0$;
2. the Euler-Poincaré characteristic $\chi(M, N) := \sum_{i=0}^{\dim(A)} (-1)^i \ell(\operatorname{Tor}_i^A(M, N)) \geqslant 0$;
3. by Remark 1.2, we have $\dim(M) + \dim(N) \leqslant \dim(A)$;
4. in particular, we have $\dim(M) + \dim(N) < \dim(A)$ if and only if $\chi(M, N) = 0$.

Proof. See [Ser12], page 106. □

Remark 1.6. Part 3. from Theorem 1.5 implies that $\dim(W) \geqslant \dim(U) + \dim(V) - \dim(X)$, i.e., $\operatorname{codim}(W) \leqslant \operatorname{codim}(U) + \operatorname{codim}(V)$ in the notation of Definition 1.3.

Definition 1.7. Let X, U, V, W be as in Definition 1.3, then we define the intersection multiplicity $m_W(U, V)$ of U and V at W by

$$m_W(U, V) = \chi^{\mathcal{O}_{X,W}}(\mathcal{O}_{X,W}/P_U, \mathcal{O}_{X,W}/P_V)$$

where P_U and P_V are prime ideals defining U and V, respectively.

Remark 1.8. By Theorem 1.5, we know $m_W(U, V) \geqslant 0$, and $m_W(U, V) = 0$ if and only if U and V do not intersect properly at W.

1.2 Intersection Product and Cross Product

Definition 1.9. Let $X \in \mathrm{Sm}/k$, and let $U \in Z_a(X)$ and $V \in Z_b(X)$. If U and V intersect properly at every component, then we define the intersection product to be the cycle

$$U \cdot V = \sum_{\substack{W\subseteq U\cap V\\ \dim(W)=a+b-d_X}} m_W(U, V) \cdot W \in Z_{a+b-d_X}(X).$$

Example 1.10. Let X be a smooth projective surface, and let C and D be divisors on X. For any point $x \in C \cap D$, locally we think of $C = \{f = 0\}$ and $D = \{g = 0\}$ around x, then $m_x(C, D) = \ell_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(f, g))$.

[6] Despite the notation, this has nothing to do with a K-theory.

Definition 1.11. Suppose X is a scheme of finite type over k, and $\mathcal{F} \in K_n(X)$ is a coherent sheaf, then we define $Z_a(\mathcal{F}) = \sum_{\dim(\bar{\eta})=a} (\mathcal{O}_{X,\eta}(\mathcal{F}_\eta) \cdot \bar{\eta}) \in Z_a(X)$.

Therefore, we define the cycle of F as an element of the cycle of X.

Definition 1.12 ([Har13], Exercise III.6.9)**.** Every coherent sheaf $\mathcal{F}$ on $X \in \mathrm{Sm}/k$ has a resolution

$$0 \longrightarrow E_k \longrightarrow E_{k-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

where E_i's are locally free of finite rank. Therefore, for any coherent sheaf $\mathcal{G}$, we can define the Tor functor[7] of coherent sheaves by

$$\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = H_i(E_* \otimes_{\mathcal{O}_X} \mathcal{G}).$$

Proposition 1.13. Let $X \in \mathrm{Sm}/k$. Suppose $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(X)$ intersect properly, then

$$Z_a(\mathcal{F}) \cdot Z_b(\mathcal{G}) = \sum_{i=0}^{d_X} (-1)^i \cdot Z_{a+b-d_X}(\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})).$$

Proof. We only have to do it locally, so we can assume X to be affine, and count the coefficients of $\bar{\xi}$ where $\dim(\xi) = a + b - d_X$. It sufficess to show that the stalks at ξ satisfies

$$\chi(F_\xi, G_\xi) = \sum_{\substack{\dim(\bar{\lambda})=a \\ \dim(\beta\eta)=b \\ \xi \in \bar{\lambda} \cap \bar{\eta}}} \ell(\mathcal{F}_\lambda) \cdot \ell(G_\eta) \cdot m_{\bar{\xi}}(\bar{\lambda}, \bar{\eta}).$$

Because our ring is Noetherian, then $\mathcal{F}$ admits a filtration

$$0 = M_0 \subseteq \cdots \subseteq M_d = \mathcal{F}$$

such that $M_i/M_{i-1} \cong \mathcal{O}_X/\mathcal{I}$ is coherent for prime ideal $\mathcal{I}$. By the additivity of both sides of the isomorphism, we may assume $\mathcal{F} = \mathcal{O}_X/\mathfrak{p}$ with dimension at most a, where $\mathfrak{p} \sim \lambda \in X$. Similarly, we may assume $\mathcal{G} = \mathcal{O}_X/\mathfrak{q}$ with dimension at most b, where $\mathfrak{q} \sim \eta \in X$. Moreover, set $\xi \in \bar{\lambda} \cap \bar{\eta}$. By definition, we now have $\chi(\mathcal{F}_\xi, \mathcal{G}_\xi) = m_{\bar{\xi}}(\bar{\lambda}, \bar{\eta})$.

- If $\dim(\bar{\lambda}) = a$ and $\dim(\bar{\eta}) = b$, then the equality follows from the fact that $\ell(\mathcal{F}_\lambda) = \ell(\mathcal{G}_\eta) = 1$.
- If not, then either $\dim(\bar{\lambda}) < a$ or $\dim(\bar{\eta}) < b$, then $\bar{\lambda}$ and $\bar{\eta}$ do not intersect properly at $\bar{\xi}$, so both the left-hand side and the right-hand side become 0.

□

Proposition 1.14. The intersection product is commutative.

Proof. This is obvious since the Tor functor is commutative. □

Proposition 1.15. The intersection product is associative.

Proof. Suppose we pick $\mathcal{F} \in K_a(X)$, $\mathcal{G} \in K_b(X)$, and $\mathcal{H} \in K_c(X)$ with support dimension at most a, b, c, respectively, and they intersect properly. Let L_* and M_* be free resolutions of $\mathcal{F}$ and $\mathcal{H}$, respectively. Define a double complex $N_{ij} = L_i \otimes \mathcal{G} \otimes M_j$, then the associativity of tensor product allows us to calculate triple Tor

$$H_i(L_i \otimes H_j(\mathcal{G}) \otimes M_j)) \cong \mathrm{Tor}_i(\mathcal{F}, \mathcal{G}, \mathcal{H}) \cong H_i(H_j(L_i \otimes \mathcal{G}) \otimes M_j)$$

as the homology of two (tensor) double complexes. We obtain two spectral sequences

$${}^I E^2_{p,q} = \mathrm{Tor}_p(\mathcal{F}, \mathrm{Tor}_q(\mathcal{G}, \mathcal{H})) \Rightarrow \mathrm{Tor}_{p+q}(\mathcal{F}, \mathcal{G}, \mathcal{H})$$

[7] Since we are working over sheaves of $\mathcal{O}_X$-modules, using the same argument on the level of modules shows that the Tor functor is independent from the choice of resolution.

$$^{II}E^2_{p,q} = \mathrm{Tor}_p(\mathrm{Tor}_q(\mathcal{F}, \mathcal{G}), \mathcal{H}) \Rightarrow \mathrm{Tor}_{p+q}(\mathcal{F}, \mathcal{G}, \mathcal{H}).$$

Recall Euler-Poincaré characteristic is invariant with respect to taking spectral sequence $(*)$, then

$$\begin{aligned}
Z_a(\mathcal{F}) \cdot ((Z_b\mathcal{G}) \cdot Z_c(\mathcal{H})) &= Z_a(\mathcal{F}) \cdot \sum_q (-1)^q Z_{b+c-d_X}(\mathrm{Tor}_q(\mathcal{G}, \mathcal{H})) \text{ by Proposition 1.13} \\
&= \sum_{p,q} (-1)^{p+q} Z_{a+b+c-2d_X}({}^{I}E^2_{p,q}) \text{ by Proposition 1.13} \\
&= \sum_i (-1)^i Z_{a+b+c-2d_X}(\mathrm{Tor}_i(\mathcal{F}, \mathcal{G}, \mathcal{H})) \text{ by } (*) \\
&= \sum_{p,q} (-1)^{p+q} Z_{a+b+c-2d_X}({}^{II}E^2_{p,q}) \text{ by } (*) \\
&= \sum_p Z_{a+b-d_X}(\mathrm{Tor}_p(\mathcal{F}, \mathcal{G})) \cdot Z_c(\mathcal{H}) \text{ by Proposition 1.13} \\
&= (Z_a(\mathcal{F}) \cdot Z_b(\mathcal{G})) \cdot Z_c(\mathcal{H}) \text{ by Proposition 1.13.}
\end{aligned}$$

□

Definition 1.16. Suppose $X_1, X_2 \in \mathrm{Sm}/k$, with $\mathcal{F}_1 \in K_{a_1}(X_1)$ and $\mathcal{F}_2 \in K_{a_2}(X_2)$. We define the cross product of cycles to be

$$Z_{a_1}(\mathcal{F}_1) \times Z_{a_2}(\mathcal{F}_2) = Z_{a_1+d_{X_2}}(p_1^*\mathcal{F}_1) \cdot Z_{a_2+d_{X_1}}(p_2^*\mathcal{F}_2),$$

where $p_i : X_1 \times X_2 \to X_i$ is the projection for $i = 1, 2$.

Exercise 1.17. One should check that this is well-defined.

Remark 1.18. Suppose $X_1, X_2 \in \mathrm{Sm}/k$, with $\mathcal{F}_1 \in K_{a_1}(X_1)$, $\mathcal{F}_2 \in K_{b_1}(X_1)$, $\mathcal{G}_1 \in K_{a_2}(X_2)$ and $\mathcal{G}_2 \in K_{a_2}(X_2)$. Suppose $Z_{a_1}(\mathcal{F}_1) \cdot Z_{a_2}(\mathcal{G}_1)$ and $Z_{b_1}(\mathcal{F}_2) \cdot Z_{b_2}(\mathcal{G}_2)$ are defined, then

- $Z_{a_1}(\mathcal{F}_1) \times Z_{a_2}(\mathcal{G}_1)$ and $Z_{b_1}(\mathcal{F}_2) \times Z_{b_2}(\mathcal{G}_2)$ intersect properly on $X_1 \times X_2$, and
- $(Z_{a_1}(\mathcal{F}_1) \times Z_{a_2}(\mathcal{G}_1)) \cdot (Z_{b_1}(\mathcal{F}_2) \times Z_{b_2}(\mathcal{G}_2)) = (Z_{a_1}(\mathcal{F}_1) \cdot Z_{b_1}(\mathcal{F}_2)) \times (Z_{a_2}(\mathcal{G}_1) \cdot Z_{b_2}(\mathcal{G}_2))$.

1.3 Pushout and Pullback

Definition 1.19. Suppose X, Y are schemes of finite type over k, and let $f : X \to Y$ be a proper map. For every irreducible closed subset $c \subseteq X$ of dimension a, we define the direct image to be

$$f_*c = \begin{cases} [k(c) : k(f(c))] \cdot f(c) \in Z_a(Y), & \dim(f(c)) = a \\ 0, & \dim(f(c)) < a \end{cases}$$

to be the direct image of c under f.

Lemma 1.20. Suppose X and Y are schemes of finite type over k of the same dimension n, and that $f : X \to Y$ is proper, then there exists an open subset $U \subseteq Y$ such that $\dim(Y \backslash U) < n$ and $f : f^{-1}(U) \to U$ is a finite morphism.

Proof. Suppose $\xi \in Y$ has $\dim(\bar{\xi}) = n$. We can find $U \ni \xi$ such that $f|_U$ has finite fibers by Exercise II.3.7 from [Har13]. By Exercise III.11.2 in [Har13], such f is finite. □

Proposition 1.21. Let $f : X \to Y$ be a proper morphism between schemes over k of finite type, and let $\mathcal{F} \in K_a(X)$, then

1. $f_*\mathcal{F} \in K_a(Y)$ and the right derived $R^i f_*\mathcal{F} \in K_{a-1}(Y)$ for $i > 0$.
2. $f_*Z_a(\mathcal{F}) = Z_a(f_*\mathcal{F})$.

Proof. 1. By Theorem III.8.8 from [Har13], $R^i f_* \mathcal{F}$ is coherent for all $i \geqslant 0$. We have $\operatorname{supp}(R_i f_* \mathcal{F}) \subseteq \operatorname{supp}(\mathcal{F})$. If f is finite, then f_* is exact, so $R^i f_* \mathcal{F} = 0$ for $i > 0$. For general cases, we may assume $\dim(f(\operatorname{supp}(\mathcal{F}))) = a$ and set $W = \operatorname{supp}(\mathcal{F})$. We have a commutative diagram

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, "i"'] & f(W) \arrow[d, "j"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where h is also proper. By Lemma 1.20, there exists $V \subseteq f(W)$ such that $\dim(f(W) \backslash V) < a$ and $h|_V$ is finite. Let $\mathcal{J}$ be the ideal sheaf of W, then $\mathcal{J}^s \mathcal{F} / \mathcal{J}^{s+1} \mathcal{F} = i_* i^* \mathcal{J}^s \mathcal{F} / \mathcal{J}^{s+1} \mathcal{F}$. By the long exact sequence, it suffices to prove the case for $\mathcal{F} = i_* \mathcal{G}$. Then

$$(R^k f_*) i_* \mathcal{G} = R^k (fi)_* \mathcal{G} = j_* R^k h_* \mathcal{G}.$$

It suffices to consider h, but

$$(R^k h_* \mathcal{G}) V = R^k h(\mathcal{G}|_{f^{-1}(V)}) = 0$$

for $k > 0$, so $\operatorname{supp}(R^k h_* \mathcal{G}) \subseteq f(W) \backslash V$ if $k > 0$.

2. If f is finite, let us write down the coefficients of ξ of dimension a on both sides, namely

$$\ell((f_* \mathcal{F})_\xi) = \sum_{\substack{\eta \in f^{-1}(\xi) \\ \dim(\overline{\eta}) = a}} \ell(F_\eta) \cdot [k(\overline{\eta}) : k(\overline{f(\eta)})].$$

By additivity, one reduces to the case when X is affine and $F = \mathcal{O}_X / \mathfrak{p}$. For the general case, use Lemma 1.20, and the case where f is finite.

$\square$

Definition 1.22. Suppose $f : X \to Y$ where $Y \in \mathrm{Sm}/k$ and X is closed in $Z \in \mathrm{Sm}/k$. Define $j : X \to Z \times Y$ to be the graph map. For any $C \in Z_a(X)$ and $D \in Z_b(Y)$ such that C and $f^{-1}(D)$ intersect properly, define the intersection cycle to be

$$C \cdot_f D = j_*^{-1}(j(C) \cdot (Z \times D)) \in Z_{a+b-d_Y}(X)$$

In particular, $f^*(D) = X \cdot_f D$ for $C = X$.

Proposition 1.23. Using the notation above, for $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(Y)$, if $\mathcal{F}$ and $f^* \mathcal{G}$ intersect properly, we have

$$Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G}) = \sum_{i=0}^{d_Y} (-1)^i Z_{a+b-d_Y}(L_i(\mathcal{F} \otimes f^*)\mathcal{G})$$

Proof. Denote $p_2 : Z \times Y \to Y$ to be the projection onto the second coordinate. By linearity, $Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G}) = j_*^{-1}(Z_a(j_* \mathcal{F}) \cdot Z_{b+d_Z}(p_2^* \mathcal{G})$ for $j : X \to Z \times Y$. Suppose $L_* \to \mathcal{G}$ is the locally free resolution of $\mathcal{G}$. Note that for all $i \geqslant 0$, we have

$$j^*(j_* \mathcal{F} \otimes p_2^* L_i) + F \otimes f^* L_i,$$

which induces an isomorphism

$$j_* \mathcal{F} \otimes p_2^* L_i = j_*(\mathcal{F} \otimes f^* L_i).$$

Hence $\operatorname{Tor}_i^{\mathcal{O}_{Z \times Y}}(j_* \mathcal{F}, p_2^* \mathcal{G}) = j_* L_i(F \otimes f^*) \mathcal{G}$. So

$$j_*^{-1} Z_{a+b-d_Y}(\operatorname{Tor}_i^{\mathcal{O}_{Z \times Y}}(j_* \mathcal{F}, p_2^* \mathcal{G})) = Z_{a+b-\dim(Y)}(L_i(F \otimes f^*)\mathcal{G}).$$

Therefore the statement follows. $\square$

Proposition 1.24. Let $X \in \mathrm{Sm}/k$, $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(X)$ such that $\mathcal{F}$ and $\mathcal{G}$ intersect properly. Let $\Delta : X \to X \times X$ be the diagonal map, then

$$\Delta^*(Z_a(\mathcal{F}) \times Z_b(\mathcal{G})) = Z_a(\mathcal{F}) \cdot Z_b(\mathcal{G}).$$

Proof. See page 115 of [Ser12]. □

Proposition 1.25. f^* is compatible with intersection product, and $f^*g^* = (gf)^*$.

Proof. See page 119 of [Ser12]. □

Lemma 1.26. Let $\mathcal{A}$ be an abelian category with enough projectives (respectively, injectives) and F be a right (respectively, left) exact functor from $\mathcal{A}$. Suppose C is chain complex in $\mathcal{A}$, then there exists a double complex $M_{*,*}$ in $\mathcal{A}$ such that

$$^{I}E^2_{p,q} = L_pFH_q(C) \quad \text{(respectively, } R^{-p}F(H_q(C))\text{)}.$$

Proof. To do this when F is right exact, use the Cartan-Eilenberg resolution[8] $C_* \to C$ and consider the double complex FC_*. □

Proposition 1.27. Suppose $f : X \to Y$ is in Sm/k, suppose $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(Y)$, then

$$Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G}) = Z_a(\mathcal{F}) \cdot f^* Z_b(\mathcal{G})$$

if both sides are defined.

Proof. We may assume X is affine. Let $L_* \to \mathcal{G}$ be a free resolution and apply Lemma 1.26 to f^*L_* and $F \otimes -$, then we find a double complex such that

$$\begin{aligned} ^{I}E^2_{p,q} &= \mathrm{Tor}_p(\mathcal{F}, L_q f^*\mathcal{G}) \\ ^{II}E^2_{p,q} &= L_p(F \otimes f^*)\mathcal{G}. \end{aligned}$$

□

Proposition 1.28. Let $X \subseteq Z$ and $Y, Z \in \mathrm{Sm}/k$ and $f : X \to Y$ be proper. Suppose $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(Y)$, and suppose $\mathcal{F}$ and $f^*\mathcal{G}$ intersect properly, then

$$f_*(Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G})) = (f_* Z_a(\mathcal{F})) \cdot Z_b(\mathcal{G}).$$

Proof. Pick $L_* \to \mathcal{G}$ to be a resolution and apply Lemma 1.26 to $F \otimes f^* L_*$ and f_*, then we have a double complex $M_{*,*}$ such that

$$^{I}E^2_{p,q} = R^{-p} f_* L_q(F \otimes f^*)\mathcal{G}).$$

On the other hand, $H_q(M_{*,n}) = R^{-q} f_*(F \otimes f^* L_n) = (R^{-q} f_* \mathcal{F}) \otimes L_n.$, therefore

$$^{II}E^2_{p,q} = \mathrm{Tor}_p(R^{-q} f_* \mathcal{F}, \mathcal{G}).$$

□

Corollary 1.29. Under the same hypothesis as Proposition 1.28, we have

$$f_*(Z_a(\mathcal{F}) \cdot f^*(Z_b(\mathcal{G}))) = f_*(Z_a(\mathcal{F})) \cdot Z_b(\mathcal{G}).$$

[8] See Proposition 11 on page 210 of [GM13].

2 Sheaves with Transfers

We fix a base scheme $S \in \mathrm{Sm}/k$.

2.1 Algebra of Correspondences

Definition 2.1. Let $X, Y \in \mathrm{Sm}/S$, then we define the group of finite correspondences

$$\mathrm{Cor}_S(X,Y) = \mathbb{Z}\{\text{irreducible closed } C \subseteq X \times_S Y \mid C \to X \text{ finite}, \dim(C) = \dim(X)\}$$

to be the free abelian group generated by elementary correspondences from X to Y.

Example 2.2. For any $f : X \to Y$, the graph $\Gamma_f = (x, f(x)) \subseteq X \times_S Y$ is a finite correspondence from $X \to Y$.

Example 2.3. If $f : X \to Y$ is finite and $\dim(X) = \dim(Y)$, then the graph Γ_f is also a finite correspondence from $Y \to X$.

Definition 2.4. Define an additive category Cor_S whose objects are the same as Sm/S, and the hom sets defined as $\mathrm{Hom}_{\mathrm{Sm}/S}(X,Y) = \mathrm{Cor}_S(X,Y)$ as in Definition 2.1. The contravariant additive functors

$$F : \mathrm{Cor}_S^{\mathrm{op}} \to \mathrm{Ab}$$

are called presheaves with transfers. The corresponding category is denoted by $\mathrm{PSh}(S) = \mathrm{PSh}(\mathrm{Cor}_S)$, which is abelian with enough injectives and projectives. We have a functor $\gamma : \mathrm{Sm}/S \to \mathrm{Cor}_S$ by Example 2.3.

Remark 2.5. For any additive F and $X, Y \in \mathrm{Sm}/S$, there is a pairing

$$\mathrm{Cor}_S(X,Y) \otimes F(Y) \to F(X).$$

Restricting to Sm/S over Cor_S, we note that F is a presheaf of abelian groups over Sm/S with transfer map $F(Y) \to F(X)$ indexed by finite correspondences from X to Y.

Example 2.6. Every $X \in \mathrm{Sm}/S$ gives an element $\mathbb{Z}(X) \in \mathrm{PSh}(S)$ defined by $\mathbb{Z}(X)(Y) = \mathrm{Cor}_S(Y,X)$. Therefore, we say $\mathbb{Z}(X)$ is the presheaf with transfers represented by X. By Yoneda Lemma we know there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{PST}}(\mathbb{Z}(X), F) \cong F(X).$$

Moreover, representable functors give embeddings of Sm/S and Cor_S into $\mathrm{PSh}(S)$ via

$$\begin{array}{ccccc} \mathrm{Sm}/S & \xrightarrow{\gamma} & \mathrm{Cor}_S & \longrightarrow & \mathrm{PSh}(S) \\ X & \longmapsto & X & \longmapsto & \mathbb{Z}(X) \end{array}$$

In particular, $\mathbb{Z}(S) = \mathbb{Z}$.

Example 2.7. The presheaves $\mathcal{O}$ and $\mathcal{O}^*$ are in $\mathrm{PSh}(S)$. For any $C \in \mathrm{Cor}_S(X,Y)$ and $f \in \mathcal{O}(Y)$ (respectively, $\mathcal{O}^*(Y)$), we have a diagram

$$\begin{tikzcd} C \arrow[r, "i"] & X \times_S Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \\ & X & \end{tikzcd}$$

and can define $\mathcal{O}(C)(f) = \mathrm{Tr}_{C/X}((p_2 \circ i)^*(f))$ (respectively, $\mathcal{O}^*(C)(f) = \mathrm{N}_{C/X}((p_2 \circ i)^*(f))$).

We study the properties of finite correspondence through Chapter 16.1 in [Ful13].

Definition 2.8. Let us describe the composition in Cor_S. Suppose $f \in \mathrm{Cor}_S(X,Y)$ and $g \in \mathrm{Cor}_S(Y,Z)$, then from the diagram

$$\begin{tikzcd} X \times_S Z & \\ X \times_S Y \times_S Z \arrow[u, "p_{13}"] \arrow[r, "p_{23}"] \arrow[d, "p_{12}"] & Y \times_S Z \\ X \times_S Y & \end{tikzcd}$$

we define the composition $g \circ f = p_{13*}(p_{23}^*(g)p_{12}^*(f))$.

Exercise 2.9. One should check that all intersections are proper.

Remark 2.10. Using this language, given a correspondence $\alpha \in \mathrm{Cor}_S(X,Y)$, we can define pullbacks and pushouts on the cycles as homomorphisms

$$\begin{aligned} \alpha_* : Z(X) &\to Z(Y) \\ x &\mapsto p_{Y*}^{XY}(\alpha \cdot p_X^{XY*}(x)) \end{aligned}$$

and

$$\begin{aligned} \alpha^* : Z(Y) &\to Z(X) \\ y &\mapsto p_{X*}^{XY}(\alpha \cdot p_Y^{XY*}(y)) \end{aligned}$$

Remark 2.11 ([Ful13], Proposition 1.7, Base-change Formula)**.** Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

be a fiber square where f is proper and g is flat, then f' is proper and g' is flat, and that $f'_*g'^* = g^*f_*$ over Y'.

Proposition 2.12 ([Ful13], Proposition 16.1.1)**.** The composition law is associative.

Proof. Suppose

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

are morphisms in Cor_S, then we have two Cartesian squares

$$\begin{tikzcd} X \times_S Y \times_S Z \times_S W \arrow[r] \arrow[d] & X \times_S Z \times_S W \arrow[d] \\ X \times_S Y \times_S Z \arrow[r] & X \times_S Z \end{tikzcd}$$

and

$$\begin{tikzcd} X \times_S Y \times_S Z \times_S W \arrow[r] \arrow[d] & X \times_S Y \times_S W \arrow[d] \\ Y \times_S Z \times_S W \arrow[r] & Y \times_S W \end{tikzcd}$$

Now using the base-change formula, we know

$$\begin{aligned} h \circ (g \circ f) &= p_{XW*}^{XZW}(p_{ZW}^{XZW*}(h)p_{XZ}^{XZW*}p_{XZ*}^{XYZ}(p_{YZ}^{XYZ*}(g)p_{XY}^{XYZ*}(f))) \\ &= p_{XW*}^{XZW}(p_{ZW}^{XZW*}(h)p_{XZW*}^{XYZW}p_{XYZ}^{XYZW*}(p_{YZ}^{XYZ*}(g)p_{XY}^{XYZ*}(f))) \\ &= p_{XW*}^{XZW}(p_{ZW}^{XZW*}(h)p_{XZW*}^{XYZW}(p_{YZ}^{XYZW*}(g)p_{XY}^{XYZW*}(f))) \end{aligned}$$

$$
\begin{aligned}
&= p^{XZY}_{XW*}p^{XYZW}_{XZW*}(p^{XYZW*}_{ZW}(h)p^{XYZW*}_{YZ}(g)p^{XYZW}_{XY}(f)) \\
&= p^{XYW}_{XW*}p^{XYZW}_{XYW*}(p^{XYZW*}_{ZW}(h)p^{XYZW*}_{YZ}(g)p^{XYZW}_{XY}(f)) \\
&= p^{XYW}_{XW*}(p^{XYZW}_{XYW*}(p^{XYZW*}_{ZW}(h)p^{XYZW*}_{YZ}(g))p^{XYW*}_{XY}(f)) \\
&= p^{XYW}_{XW*}(p^{XYZW}_{XYW*}p^{XYZW*}_{YZW}(p^{YZW*}_{ZW}(h)p^{YZW}_{YZ}(g))p^{XYW*}_{XY}(f)) \\
&= p^{XYW}_{XW*}(p^{XYW*}_{YW}p^{YZW}_{YW*}(p^{YZW*}_{ZW}(h)p^{YZW}_{YZ}(g))p^{XYW*}_{XY}(f)) \\
&= p^{XYW}_{XW*}(p^{XYW*}_{YW}p^{YZW}_{YW*}(p^{YZW*}_{ZW}(h)p^{YZW}_{YZ}(g))p^{XYW*}_{XY}(f)) \\
&= h \circ g \circ f.
\end{aligned}
$$

□

Theorem 2.13. We have $\mathcal{O}(g \circ f) = \mathcal{O}(f) \circ \mathcal{O}(g)$ and $\mathcal{O}^*(g \circ f) = \mathcal{O}^*(f) \circ \mathcal{O}^*(g)$.

Proof. We sketch the proof for $\mathcal{O}$. Pick $X \in \mathrm{Sm}/k$. For every $a \in \mathbb{N}$, define $\mu_a(x) = \bigoplus_{\dim(\bar{V})=a} K(V)$. Therefore, we have a pairing

$$
\begin{aligned}
\mathcal{O}(X) \times Z_a(X) &\to \mu_a(X) \\
(S, V) &\mapsto S|_V
\end{aligned}
$$

by restricting the regular function on the closed subset. For any map $f : X \to Y$ where X contains irreducible and closed C, suppose C is finite over Y and $s \in K(C)$, then we define $f_*(s) = \mathrm{Tr}_{K(C)/K(f(C))}(s)$.[9] Therefore, for any finite correspondence $C \in \mathrm{Cor}(Y, X)$ and $s \in \mathcal{O}(X)$, we have

$$
\begin{tikzcd}
C \arrow[r, hook] & X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \\
& X &
\end{tikzcd}
$$

and thus $\mathcal{O}(C)(s) = p_{2*}(p_1^*(s)|_C)$.

Now suppose we have closed subsets $C \subseteq X$ and $D \subseteq Y$, with

$$
\begin{tikzcd}
X \arrow[r, "f"] & Y \\
C \arrow[u] \arrow[ur, "\text{finite}"'] &
\end{tikzcd}
$$

and that C and $f^{-1}(D)$ intersect properly, then one can show that

$$
f_*(s|_C)|_D = f_*(s|_{C \cdot_f D})
$$

by Tor formula. Moreover, for diagrams like

$$
\begin{tikzcd}
X \times_S Y \times_S Z \arrow[r, "p_{23}"] \arrow[d, "p_{12}"'] & YZ \arrow[d, "p_1"] \\
X \times_S Y \arrow[r, "p_2"'] & Y
\end{tikzcd}
$$

where $C \subseteq YZ$ and C is finite over Y, then one can show that for all $s \in \mathcal{O}(Y \times_S Z)$ and C finite over Y, we have

$$
p_2^* p_{1*}(s|_C) = p_{12*}(p_{23}^*(s)|_{p_{23}^*(C)}).
$$

We finish the proof by working with formal calculation. □

Remark 2.14 ([Ful13], Proposition 16.1.2)**.** For $\alpha \in \mathrm{Cor}_S(X, Y)$ and $\beta \in \mathrm{Cor}_S(Y, Z)$, we have

$$
(\beta \circ \alpha)_* = \beta_* \circ \alpha_*
$$

and

$$
(\beta \circ \alpha)^* = \alpha^* \circ \beta^*.
$$

[9]Here $f(C)$ is closed since f is finite.

2.2 Operations on Presheaves with Transfers

Definition 2.15. Suppose $\mathcal{F}_1, \mathcal{F}_2, \mathcal{G} \in \mathrm{PSh}(S)$ be presheaves with transfers. A bilinear function $\varphi : \mathcal{F}_1 \times \mathcal{F}_2 \to \mathcal{G}$ is a collection of bilinear maps

$$\varphi_{x_1,x_2} : \mathcal{F}_1(x_1) \times \mathcal{F}_2(x_2) \to \mathcal{G}(x_1 \times_S x_2)$$

for every $x_1, x_2 \in \mathrm{Sm}/S$ any any morphisms $f_i \in \mathrm{Cor}_S(x_i, x_i')$ for $i = 1, 2$, such that the following diagram commutes

$$\begin{tikzcd} \mathcal{F}_1(x_1') \times \mathcal{F}_2(x_2) \arrow[r, "\varphi_{x_1',x_2}"] \arrow[d, "\mathcal{F}_1(f_1)\times\mathrm{id}"'] & \mathcal{G}(x_1' \times_S x_2) \arrow[d, "(f_1\times\mathrm{id})"] \\ \mathcal{F}_1(x_1) \times \mathcal{F}_2(x_2) \arrow[r, "\varphi_{x_1,x_2}"'] & \mathcal{G}(x_1 \times_S x_2) \end{tikzcd}$$

for f_1 and similarly there is a diagram that commutes for f_2.

Definition 2.16. Define the tensor product $\mathcal{F}_1 \otimes \mathcal{F}_2$ to be the presheaf such that for every $\mathcal{G}$, the hom set $\mathrm{Hom}(\mathcal{F}_1 \otimes \mathcal{F}_2, G)$ is the same as the collection of bilinear functions $\mathcal{F}_1 \times \mathcal{F}_2 \to \mathcal{G}$.

Proposition 2.17. The tensor product $\mathcal{F}_1 \otimes \mathcal{F}_2$ exists.

Proof. For every $Z \in \mathrm{Sm}/S$, define

$$(\mathcal{F}_1 \otimes \mathcal{F}_2)(Z) = \bigoplus_{X,Y \in \mathrm{Sm}/S} \mathcal{F}_1(X) \otimes_{\mathbb{Z}} \mathcal{F}_2(Y) \otimes_{\mathbb{Z}} \mathrm{Cor}_S(Z, X \times_S Y)/\sim$$

where $\sim$ is the subgroup generated by the relations $\varphi \otimes \psi(f \times \mathrm{id}_Y) \circ h = f^*(\varphi) \otimes \psi \otimes h$ where $f \in \mathrm{Cor}_S(X', X)$, $\varphi \in \mathcal{F}_1(X)$, $\psi \in \mathcal{F}_2(Y)$, $h \in \mathrm{Cor}_S(Z, X' \times Y)$, and the relations $\varphi \otimes \psi \otimes (\mathrm{id}_X \times g) \circ h = \varphi \otimes g^*(\psi) \otimes h$ where $g \in \mathrm{Cor}_S(Y', Y)$, $\varphi \in \mathcal{F}_1(X)$, $\psi \in \mathcal{F}_2(Y)$, $h \in \mathrm{Cor}_S(Z, X \times Y')$. $\square$

Definition 2.18. A pointed presheaf $(\mathcal{F}, x)$ is a split injective map given by the constant presheaf $x : \mathbb{Z} \to \mathcal{F}$ for some $\mathcal{F} \in \mathrm{PSh}(S)$. We set $\mathcal{F}^{\wedge 1} = \mathcal{F}/x$. For any two pointed presheaves $(\mathcal{F}_1, x_1)$ and $(\mathcal{F}_2, x_2)$, define $\mathcal{F}_1 \wedge \mathcal{F}_2 = (\mathcal{F}_1 \otimes \mathcal{F}_2)/((\mathcal{F}_1 \otimes x_2) \oplus (x_1 \otimes \mathcal{F}_2))$. This allows us to define $\mathcal{F}^{\wedge n}$ inductively as a cokernel, c.f., Definition 2.12 from [MVW06].

Proposition 2.19.

- $\mathbb{Z}(X) \otimes \mathbb{Z}(Y) = \mathbb{Z}(X \times Y)$;
- $\mathcal{F}^{\wedge 1} \otimes \mathcal{G}^{\wedge 1} = \mathcal{F} \wedge \mathcal{G}$.

Definition 2.20. For any $\mathcal{F} \in \mathrm{PSh}(S)$ and $X \in \mathrm{Sm}/S$, define $\mathcal{F}^{\times} \in \mathrm{PSh}(S)$ by $\mathcal{F}^{\times}(Y) = \mathcal{F}(X \times_S Y)$. For any $\mathcal{F}, \mathcal{G} \in \mathrm{PSh}(S)$, define $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}) \in \mathrm{PSh}(S)$ by $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})(x) = \mathrm{Hom}(\mathcal{F}, \mathcal{G}^{\times})$.

Proposition 2.21. We have a tensor-hom adjunction

$$\mathrm{Hom}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \cong \mathrm{Hom}(\mathcal{F}, \underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{H})).$$

2.3 Nisnevich Topology

Let us give a brief introduction to Nisnevich topology, c.f., section 3 and 4 from Chapter I of [Mil80].

Definition 2.22. Suppose $f : Y \to X$ is a morphism between schemes that are locally of finite type.

1. It is called unramified if for all $y \in Y$, the maximal ideals satisfy $\mathfrak{m}_{f(y)}\mathcal{O}_{Y,y} = \mathfrak{m}_y$, and $k(y)/k(f(y))$ is a finite separable field extension of function fields.
2. It is called étale if it is both flat and unramified.
3. It is called Nisnevich if for all $x \in X$, there is some $y \in Y$ such that $f(y) = x$, $k(y) = k(x)$, and f is étale.

Definition 2.23. A morphism $f : Y \to X$ is called a Nisnevich covering if f is Nisnevich and surjective.

Definition 2.24. Suppose $\mathcal{F} \in \mathrm{PSh}(S)$. We say that it is a Nisnevich sheaf with transfers if for any $X \in \mathrm{Sm}/S$ and Nisnevich covering $\pi : Y \to X$, the sequences

$$0 \longrightarrow \mathcal{F}(X) \xrightarrow{\pi^*} \mathcal{F}(Y) \xrightarrow{p_1^* - p_2^*} \mathcal{F}(Y \times_X Y) \xrightarrow{p_1} Y$$

$$0 \longrightarrow \mathcal{F}(X) \xrightarrow{\pi^*} \mathcal{F}(Y) \xrightarrow{p_1^* - p_2^*} \mathcal{F}(Y \times_X Y) \xrightarrow{p_2} Y$$

are exact. The category of Nisnevich sheaves with transfers is denoted by $\mathrm{Sh}(S)$.

Definition 2.25. A local ring is called Hensalian if for any monic polynomial $f \in A[t]$ such that its image $\bar{f}$ in the residue field satisfies $\bar{f} = g_0 h_0$ in $k(A)[T]$ where g_0, h_0 are monic and relatively prime, there are monic $g, h \in A[T]$ such that $\bar{g} = g_0$, $\bar{h} = h_0$ in the residue fields, and $f = gh$.

Example 2.26. Complete local rings are Henselian.

Theorem 2.27 ([Mil80], Theorem I.4.2.)**.** Let A be a local ring, $X = \mathrm{Spec}(A)$, and $x \in X$ be the closed point, then the following are equivalent:

1. A is Henselian;
2. any finite A-algebra B is a direct product of local rings $B \cong \prod_{i \in I} B_i$, where each B_i is of the form $B_{\mathfrak{m}_i}$ for some maximal ideal $\mathfrak{m}_i$ of B;
3. if $f : Y \to X$ has finite fibers and is separated, then $Y = \coprod_{i=0}^{n} Y_i$ where $X \notin f(Y_0)$, and for $i \geqslant 1$, Y_i is finite over X and is the spectrum of a local ring;
4. if $f : Y \to X$ is étale and there exists $y \in Y$ such that $f(y) = x$ and $k(y) = k(x)$, then f has a section $s : X \to Y$ such that $f \circ s = \mathrm{id}_X$.

Now let A be a Noetherian ring and $\mathfrak{p} \in \mathrm{Spec}(A)$. Consider the set I whose elements are pairs $(B, \mathfrak{q})$, where B is a connected étale A-algebra, $\mathfrak{q} \in \mathrm{Spec}(B)$, $\mathfrak{q} \cap A = \mathfrak{p}$, i.e., $\mathfrak{q}$ lies over $\mathfrak{p}$, and $k(\mathfrak{p}) = k(\mathfrak{q})$. We say that $(B_1, \mathfrak{q}_1) \preceq (B_2, \mathfrak{q}_2)$ if there is an A-morphism $f : B_1 \to B_2$ such that $f^{-1}(\mathfrak{q}_2) = \mathfrak{q}_1$. This gives a poset structure.

Proposition 2.28. The set I is a directed set and the ring $\varinjlim_{(B,\mathfrak{q})} B = A_{\mathfrak{p}}^h$, i.e., the Henselization of $A_{\mathfrak{p}}$, is Henselian and admits the following universal property: for any Henselian A-algebra C such that $\mathfrak{m}_C \cap A = \mathfrak{p}$, there is a unique morphism $\varphi : A_{\mathfrak{p}}^h \to C$ (as a local homomorphism) such that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \\ A_{\mathfrak{p}}^h \arrow[ur, dashed, "\exists!\varphi"'] & \end{tikzcd}$$

Proof. This makes use of Lemma I.4.8 from [Mil80]. □

Let φ_X be the smallest Nisnevich site on X. Suppose X is Noetherian, pick $x \in X$, and $\mathcal{F} \in \mathrm{PSh}(\varphi_X)$. We write $\mathcal{F}_x = \mathcal{F}(\mathcal{O}_{X,x}^h) = \varinjlim_{(V,u)} \mathcal{F}(V)$ as the stalk of F at x, taking all the pairs (V, u) with étale morphism

$$V \to X$$
$$u \mapsto x$$

with $k(u) = k(x)$.

Proposition 2.29. Let

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

be a complex in $\mathrm{Sh}(\varphi_X)$. The following are equivalent:

1. the complex is exact;
2. for every $x \in X$, the complex
$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{H}_x \longrightarrow 0$$
is exact.

Proof. This mimics the idea in the usual sheaf theory with Zariski topology. To do so, we need to construct a sheafification in the sense of Nisnevich, explained as follows: suppose $\mathcal{F} \in \mathrm{PSh}(\varphi_X)$, define $\mathcal{F}^+$ as the following: for every Nisnevich covering $\{V_i\}$ of U, define
$$\mathcal{F}(U) = \{(s_i) \in \prod_i \mathcal{F}(V_i) : s_i|_{V_i \times_X V_j} = s_j|_{V_i \times_S V_j}\}.$$
Now let $\mathcal{F}^+(U) = \varinjlim_{V \supseteq U} \mathcal{F}(V)$, then $\mathcal{F}^{++}$ is a Nisnevich sheaf with the same stalks as $\mathcal{F}$, with a map $\mathcal{F} \to \mathcal{F}^{++}$. □

For any Noetherian scheme X with $\dim(X) < \infty$, we define the cochain to be
$$C^p(X) = \{Y \subseteq X \mid \mathrm{codim}(Y) \geqslant p\}.$$
Fir $\mathcal{F} \in \mathrm{Sh}(\varphi_X)$. For a closed subscheme $Z \subseteq W$ where $Z \in C^{p+1}(X)$ and $W \in C^p(X)$, we have a long exact sequence
$$\cdots \longrightarrow H^i_Z(X, \mathcal{F}) \longrightarrow H^i_W(X, \mathcal{F}) \longrightarrow H^i_{W \backslash Z}(X \backslash Z, \mathcal{F}) \longrightarrow H^{i+1}_Z(X, \mathcal{F}) \longrightarrow \cdots$$
with supports specified as subscripts, using the exactness of
$$0 \longrightarrow \mathcal{F}_Z(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}(X \backslash Z)$$
This induces a right derived functor $H^i_Z = R^i\Gamma_Z(X, -) : D(X_{\text{étale}}) \to D(\mathrm{Ab})$, where
$$\Gamma_Z(X, \mathcal{F}) = \{s \in \mathcal{F}(X) \mid \mathrm{supp}(s) \subseteq Z\}$$
for closed subscheme $Z \subseteq X$. Now define $H^i(C^p(X), \mathcal{F}) = \varinjlim_{Z \in C^p(X)} H^i_Z(X, \mathcal{F})$, then
$$H^i(C^p(X)/C^{p+1}(X), \mathcal{F}) = \varinjlim_{\substack{Z \subseteq W \\ W \in C^p, Z \in C^{p+1}}} H^i_{W \backslash Z}(X \backslash Z, \mathcal{F}).$$
Taking limit with respect to pairs $Z \subseteq W$ where $W \in C^p(X)$ and $Z \in C^{p+1}(X)$, we get a long exact sequence
$$\cdots \longrightarrow H^i(C^{p+1}(X), \mathcal{F}) \longrightarrow H^i(C^p(X), \mathcal{F}) \longrightarrow H^i(C^p(X)/C^{p+1}(X), \mathcal{F}) \longrightarrow H^{i+1}(C^{p+1}(X), \mathcal{F}) \longrightarrow \cdots$$
Set $F^pH^i(X, \mathcal{F}) = \mathrm{im}(H^i(C^p(X), \mathcal{F}) \to H^i(X, \mathcal{F}))$, then we obtain the Coniveau spectral sequence
$$E_1^{p,q} = H^{p+q}(C^p(X)/C^{p+1}(X), \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

Definition 2.30. Suppose $x \in X$. Define the local cohomology
$$H^i_x(X, \mathcal{F}) = \varinjlim_{\text{open } x \in V \subseteq X} H^i_{\bar{x} \cap V}(V, \mathcal{F}).$$

This allows us to calculate $E_1^{p,q}$ as
$$E_1^{p,q} = \bigoplus_{\mathrm{codim}(\bar{x}) = p} H^{p+q}_x(X, \mathcal{F}).$$

Proposition 2.31 (Excision)**.** Suppose $\varphi : Y \to X$ is a étale morphism of sheaves, and suppose $Z \subseteq X$ is a closed subset such that $\varphi^{-1}(Z) = Z$. For any $\mathcal{F} \in \mathrm{Sh}(\varphi_X)$, we have
$$H^i_Z(Y, \varphi^*\mathcal{F}) = H^i_Z(X, \mathcal{F}).$$

Proof. The morphism
$$Y\coprod(X\backslash Z)\to X$$
is a Nisnevich covering, but by the (Nisnevich) sheaf condition on $\mathcal{F}$, we have a Cartesian square
$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r] \arrow[d] & \mathcal{F}(Y) \arrow[d] \\
\mathcal{F}(X\backslash Z) \arrow[r] & \mathcal{F}(Y\backslash Z)
\end{tikzcd}$$
which shows the result for $i=0$. The map φ^* is exact and has a left adjoint $\varphi_!$, namely the extension by zero. In particular, φ^* preserves injective objects, and the statement for $i>0$ follows. □

Corollary 2.32. The local cohomology agrees with the supported cohomology, i.e.,
$$H_x^i(X,\mathcal{F})\cong H_x^i(\mathrm{Spec}(\mathcal{O}_{X,x}^h),\mathcal{F}_x).$$

Theorem 2.33. For all $n>\dim(X)$, $H^n(X,\mathcal{F})=0$.

Proof. We proceed by induction on $\dim(X)$. If $\dim(X)=0$, then X is a disjoint union of spectra of Henselian rings, so the statement holds. Now suppose the statement is true for any scheme Y such that $\dim(Y)<\dim(X)$, then the sequence
$$H^i(\mathrm{Spec}(\mathcal{O}_{X,x}^h),\mathcal{F}_x)\longrightarrow H^i(\mathrm{Spec}(\mathcal{O}_{X,x}^h\backslash\{x\}),\mathcal{F}_x)\longrightarrow H_x^{i+1}(\mathrm{Spec}(\mathcal{O}_{X,x}^h),\mathcal{F}_x)\longrightarrow H^{i+1}(\mathrm{Spec}(\mathcal{O}_{X,x}^h),\mathcal{F}_x)$$
has first term and last term as zero, therefore the two terms in the middle agree for $i>0$, i.e.,
$$H^i(\mathrm{Spec}(\mathcal{O}_{X,x}^h\backslash\{x\}),\mathcal{F}_x)\cong H_x^{i+1}(\mathrm{Spec}(\mathcal{O}_{X,x}^h),\mathcal{F}_x)$$
for $i>0$. By induction, $H^{n-1}(\mathrm{Spec}(\mathcal{O}_{X,x}^h\backslash\{x\}),\mathcal{F}_x)=0$ if $n>\dim(\bar{x})$,[10] then
$$E_1^{p,q}\cong\bigoplus_{\mathrm{codim}(\bar{x})=p}H_x^{p+q}(\mathrm{Spec}(\mathcal{O}_{X,x}^h,\mathcal{F}_x)=0$$
when $p+q>\dim(X)$. Therefore, $H^n(X,\mathcal{F})=0$ for $n>\dim(X)$. □

Theorem 2.34. Let $X,U\in\mathrm{Sm}/S$ and $p:U\to X$ be a Nisnevich covering. Denote the n-fold product $A\times_B A\times_B\cdots\times_B A$ by A_B^n, then the Čech complex of sheaves (associated to the complex over Sm/S)
$$\check{C}(U/X)\longrightarrow\cdots\longrightarrow\mathbb{Z}(U_X^n)\xrightarrow{d_n}\cdots\longrightarrow\mathbb{Z}(U\times_X U)\xrightarrow{d_2}\mathbb{Z}(U)\longrightarrow\mathbb{Z}(X)\longrightarrow 0$$
is exact, where $d_n=\sum_i(-1)^{i-1}\mathbb{Z}(p_i)$ for ith omission map $p_i:U_X^n\to U_X^{n-1}$.

Proof. It suffices to show exactness stalkwise, so to do things locally, we suppose $Y=\mathrm{Spec}(A)$ where A is Henselian, regular and local, and $a\in\mathrm{Cor}_S(Y,U_X^n)$ such that $d_n(A)=0$. Define $T=\mathrm{supp}(a)$ and $R=T\times_{X\times Y}(U\times Y)$. Since T is a finite correspondence, so by definition it is finite over Y, but Y is Henselian, so T is the spectrum of a disjoint union of Henselian rings by Theorem 2.27. Now R is a Nisnevich covering over T, so the map $R\to T$ admits a section $s:T\to R$, where s is both an open immersion and a closed immersion. This gives a diagram of Cartesian squares
$$\begin{tikzcd}
R_T^n \arrow[r] \arrow[d, "\mathrm{id}^n\times s"'] & (U\times Y)_{X\times Y}^n\times_{X\times Y}((U\times Y)\backslash(R\backslash T)) \arrow[d, "j_{n+1}"] \\
R_T^{n+1} \arrow[r] \arrow[d, "p_{n+1}"'] & (U\times Y)_{X\times Y}^{n+1} \arrow[d, "p_{n+1}"] \\
R_T^n \arrow[r] & (U\times Y)_{X\times Y}^n
\end{tikzcd}$$

[10]Note that removing the closure of the point reduces the length by 1.

where j_{n+1} is a closed immersion. But note that the composition of the left column is just identity, so we define

$$b = (j_{n+1*}(p_{n+1} \circ j_{n+1})^*)(a) \in \mathrm{Cor}_S(Y, U_X^{n+1}).$$

By intersection theory, one can check that $d_{n+1}(b) = a$. □

Theorem 2.35. There is a unique sheafification function $a : \mathrm{PSh}(S) \to \mathrm{Sh}(S)$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{PSh}(S) \arrow[r, "a"] \arrow[d] & \mathrm{Sh}(S) \arrow[d] \\ \mathrm{PSh}(\mathrm{Sm}/S) \arrow[r, "+"'] & \mathrm{Sh}(\mathrm{Sm}(S)) \end{tikzcd}$$

Proof. Take $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Sh}(S)$. We first prove uniqueness. Suppose $\mathcal{F}_1|_{\mathrm{Sm}/S} = \mathcal{F}_2|_{\mathrm{Sm}/S} = (\mathcal{F}|_{\mathrm{Sm}/S})^+$, set $s \in \mathcal{F}_1(Y) = \mathcal{F}_2(Y)$ and $T \in \mathrm{Cor}_S(X, Y)$ where X is Henselian, then there is a Nisnevich covering $p : U \to Y$ such that $s|_U = t^+$ where $t \in \mathcal{F}(U)$. Consider the Cartesian square

$$\begin{tikzcd} T_U \arrow[r] \arrow[d] & X \times U \arrow[d] \\ T \arrow[r] & X \times Y \end{tikzcd}$$

then since T is irreducible so T is the spectrum of some Henselian ring, which gives a section s of the map $T_U \to T$. Denote $D = \mathrm{im}(s)$, then $D \in \mathrm{Cor}_S(X, U)$. Therefore $p \circ D = T$, so we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}_1(X) \arrow[r, equal] & \mathcal{F}_2(X) \\ \mathcal{F}_1(U) \arrow[r, equal] \arrow[u, "\mathcal{F}_1(D)"] & \mathcal{F}_2(U) \arrow[u, "\mathcal{F}_2(D)"'] \\ \mathcal{F}_1(Y) \arrow[r, equal] \arrow[u, "\mathcal{F}_1(p)"] \arrow[uu, bend left=60, "\mathcal{F}_1(T)"] & \mathcal{F}_2(Y) \arrow[u, "\mathcal{F}_2(p)"'] \arrow[uu, bend right=60, "\mathcal{F}_2(T)"'] \end{tikzcd}$$

In particular, $\mathcal{F}_1 = \mathcal{F}_2$, so we have uniqueness. To prove existence, we make $(\mathcal{F}|_{\mathrm{Sm}/S})^+$ a sheaf with transfers. Suppose $y \in (\mathcal{F}|_{\mathrm{Sm}/S})^+(Y)$, and $y|_U = Z^+$, where $p : U \to Y$ is a Nisnevich covering and $Z \in \mathcal{F}(U)$ (and so Z^+ is the image of Z over sheafification). By shrinking U, we allow Z to agree on the intersection, i.e., we may assume that Z is mapped to 0 in $\mathcal{F}(U \times_Y U)$. This gives a sequence

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}(Y), (\mathcal{F}|_{\mathrm{Sm}/S})^+ \longrightarrow \mathrm{Hom}(\mathbb{Z}(0), (\mathcal{F}|_{\mathrm{Sm}/S})^+ \longrightarrow \mathrm{Hom}(\mathbb{Z}(U \times_X U), (\mathcal{F}|_{\mathrm{Sm}/S})^+)$$

which is exact by Theorem 2.34. We know that $p^*(Z) = 0$, so there exists $[y] : \mathbb{Z}(Y) \to (\mathcal{F}|_{\mathrm{Sm}/S})^+$ such that $[y]|_U = y|_U$. Take $f \in \mathrm{Cor}_S(X, Y)$, then by Yoneda lemma we know the composition

$$\mathbb{Z}(X) \xrightarrow{f} \mathbb{Z}(Y) \xrightarrow{[y]} (\mathcal{F}|_{\mathrm{Sm}/S})^+$$

of Nisnevich sheaves produces the transfer of y with respect to f. □

Remark 2.36. The category $\mathrm{Sh}(S)$ is an abelian category, then the statement in Proposition 2.29 holds for $\mathrm{Sh}(S)$.

Proposition 2.37. Suppose $X \in \mathrm{Sm}/S$ and $\{U_1, U_2\}$ is a Zariski covering of X, then we have an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}(U_1 \cap U_2) & \longrightarrow & \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2) & \longrightarrow & \mathbb{Z}(X) & \longrightarrow & 0 \\ & & s & \longmapsto & (s|_{U_1}, -s|_{U_2}) & & & & \\ & & & & (s_1, s_2) & \longmapsto & s_1 + s_2 & & \end{array}$$

Proof. Note that $U_1 \coprod U_2$ is a Nisnevich covering of X. Applying the Čech complex of X in Theorem 2.34, we obtain an exact sequence

$$\mathbb{Z}(U_1) \oplus \mathbb{Z}(U_1 \cap U_2)^{\oplus 2} \oplus \mathbb{Z}(U_2) \xrightarrow{d} \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2) \xrightarrow{+} \mathbb{Z}(X) \longrightarrow 0$$

where $d(x, y, a, b) = (a - y, y - a)$. □

Definition 2.38. Define Sim to be the category of simplicial sets $[n] = \{0, \cdots, n\}$ for $n \in \mathbb{N}$, where $\mathrm{Hom}_{\mathrm{Sim}}([n], [m])$ is the set of non-decreasing simplicial maps $[n] \to [m]$.

For any category $\mathscr{C}$, we define a simplicial (respectively, cosimplicial) object in $\mathscr{C}$ to be a functor $\mathrm{Sim}^{\mathrm{op}} \to \mathscr{C}$ (respectively, $\mathrm{Sim} \to \mathscr{C}$).

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