MATH 545 Notes

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These notes were (mostly)[1] live-texed from a harmonic analysis class (MATH 545) taught by Professor X. Li in Fall 2024 at University of Illinois. Any mistakes and inaccuracies would be my own. Although this class does not have a designated textbook, important references have been included in the bibliography section. Finally, an older version of notes from the same course can be found here.

Contents

[1] I would like to thank several other students from this class for providing notes, commenting on and clarifying the existing content, as well as editing the document.

1 Marcinkiewicz Interpolation Theorem

Let us recall a few concepts and results from measure theory that one is expected to understand before the start of the course, c.f., [Fol99] and/or course notes from a class taught by the same professor.

Definition 1.1. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f : X \to \mathbb{C}$ be a function. For any $0 < p < \infty$, there is an associated L^p-norm

$$||f||_p = \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}}.$$

For $p = \infty$, we define the ∞-norm by

$$\operatorname{esssup}_{x\in X} |f(x)| = \inf\{M \in \mathbb{R} : \mu(\{x \in X : |f(x)| > M\}) = 0\}.$$

The L^p-space of X is defined by

$$L^p(X) = \{f : ||f||_p < \infty\}$$

for $0 < p \leqslant \infty$. A weak L^p-norm is

$$||f||_{p,\infty} = \sup_{\lambda>0} (\lambda^p \mu(\{x \in X : |f(x)| > \lambda\}))^{\frac{1}{p}}$$

for $0 < p < \infty$. For $p = \infty$, this coincides with the L^∞-norm. There is then a corresponding notion of weak L^p-space. Recall that the L^p-space $L^{p,\infty}(X)$ is contained in the weak L^p-space $L^p(X)$.

Theorem 1.2. For any $0 < p \leqslant \infty$, $L^p(X) \subseteq L^{p,\infty}(X)$.

Definition 1.3. Let T be an operator from $(X, \mathcal{A}, \mu)$ to a space of measurable functions on $(Y, \mathcal{B}, \nu)$.

1. If $T(f_1 + f_2) = T(f_1) + T(f_2)$ for all $f_1, f_2 \in L^p(X, \mathcal{A}, \mu)$, and $T(\lambda f) = \lambda T(f)$ for all $f \in L^p(X, \mathcal{A}, \mu)$, then T is called a linear operator.

2. If $|T(f_1 + f_2)| \leqslant |T(f_1)| + |T(f_2)|$ for all f_1, f_2, and $|T(\lambda f)| = |\lambda||T(f)|$ for all f and all $\lambda \in \mathbb{C}$, then T is called a sublinear operator.

3. If $||T(f)||_{L^q(Y,\mathcal{B},\nu)} \leqslant C||f||_{L^p(X,\mathcal{A},\mu)}$ for some constant C independent of f for all $f \in L^p(X, \mathcal{A}, \mu)$, then T is called a (strong) (p, q) operator.

   Remark 1.4. An equality of the form $||T(f)||_{L^q(Y,\mathcal{B},\nu)} \leqslant C||f||_{L^p(X,\mathcal{A},\mu)}$ is called a (p, q)-type inequality.

   Remark 1.5. When $p = q$, we say the operator T is bounded.

4. If $||T(f)||_{L^{q,\infty}(Y,\mathcal{B},\nu)} \leqslant C_{p,q}||f||_{L^p(X,\mathcal{A},\mu)}$ for all $f \in L^p$, then T is called a weak (p, q) operator.

Theorem 1.6.

$$||f||_p^p = p\int_0^\infty \lambda^{p-1}\mu(\{x \in X : |f(x)| > \lambda\})d\lambda.$$

Theorem 1.7 (Riesz-Thorin Interpolation Theorem)**.** Suppose that $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are measure spaces and let $p_0, p_1, q_0, q_1 \in [1, \infty]$. In the case where $q_0 = q_1 = \infty$, we should assume in addition that ν is semi-finite. If T is a linear operator such that T is strong (p_0, q_0) and strong (p_1, q_1), i.e., $||T(f)||_{q_0} \leqslant M_0||f|_{p_0}$ for all $f \in L^{p_0}$, and $||T(f)||_{q_1} \leqslant M_1||f||_{p_1}$ for all $f \in L^{p_1}$, then for any $0 < \theta < 1$,

$$||T(f)||_{q_\theta} \leqslant M_0^{1-\theta} M_1^\theta ||f||_{p_\theta}$$

for all $f \in L^{p_\theta}$, where p_θ and q_θ satisfy $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Remark 1.8. To interpret this, let us say $(\frac{1}{a}, \frac{1}{b})$ is a good point if T is strong (a, b). The theorem then says that if (p_0, q_0) and (p_1, q_1) are good, then any point along the line connecting these two points is also good.

Problem 1. Prove Theorem 1.7.

A proof can be found in Theorem V.1.3 of [SW71].

Theorem 1.9 (Marcinkiewicz Interpolation Theorem)**.** Suppose that $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are measure spaces, and let $p_0, p_1, q_0, q_1 \in [1, \infty]$, such that $p_0 \leqslant q_0$, $p_1 \leqslant q_1$, and that $q_0 \neq q_1$. Let $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and $\frac{1}{q_0} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, where $0 < \theta < 1$. If T is a sublinear operator and is weak (p_0, q_0) and weak (p_1, q_1), then T is strong (p_θ, q_θ).

Again, there is a geometric interpretation via interpolation, as in Remark 1.8.

Proof. We split the proof into cases.

Case 1: $p_0 = q_0$, $p_1 = q_1$, and $p_0 \neq p_1$. For simplicity, we assume the measure is σ-finite. Set $p = p_\theta$, then we want to construct a decomposition of f via level sets and then $||T(f)||_{L^p(Y)} \leqslant C_p ||f||_{L^p(X)}$ for all $f \in L^p$. Let $\lambda > 0$, and $C > 0$ be a constant that we will choose later. We give a decomposition $f = f_0 + f_1$, where $f_0 = f\chi_{\{x \in X : |f(x)| > C\lambda\}}$ is associated to p_0 and $f_1 = f\chi_{\{x \in X : |f(x)| \leqslant C\lambda\}}$ is associated to p_1. Since T is sublinear, then $|T(f)| \leqslant |T(f_0)| + |T(f_1)|$. Now

$$\nu(\{x : |Tf(x)| > \lambda\}) \leqslant \nu(\{x : |Tf_0(x)| > \frac{\lambda}{2}\}) + \nu(\{x : |Tf_1(x)| > \frac{\lambda}{2}\}).$$

Subcase 1: Assume $p_1 = \infty$. Therefore, $||T(f)||_{p_0,\infty} \leqslant A_0 ||f||_{p_0}$ and $||T(f)||_\infty \leqslant A_1 ||f||_\infty$. In particular, $\lambda > 0$, $\nu(\{x : |Tf(x)| > \lambda\}) \leqslant \frac{A_0^{p_0} ||f||_{p_0}^{p_0}}{\lambda^{p_0}}$. Moreover, we know that $||T(f_1)||_\infty \leqslant A_1 ||f_1||_\infty \leqslant CA_1\lambda$. Take $C = \frac{1}{2A_1}$, then $||T(f_1)||_\infty < \frac{\lambda}{2}$, therefore $\nu(\{x : |Tf_1(x)| > \frac{\lambda}{2}\}) = 0$, and by Theorem 1.6 and Fubini theorem we have

$$\begin{aligned}
||T(f)||_p^p &= p \int\limits_0^\infty \lambda^{p-1} \nu(\{x : |Tf(x)| > \lambda\}) d\lambda \\
&\leqslant p \int\limits_0^\infty \lambda^{p-1} \nu(\{x : |Tf_0(x)| > \lambda\}) d\lambda \\
&\leqslant p \int\limits_0^\infty \lambda^{p-1} \frac{(2A_0)^{p_0} ||f_0||_{p_0}^{p_0}}{\lambda^{p_0}} d\lambda \\
&\leqslant p(2A_0)^{p_0} \int\limits_0^\infty \lambda^{p-p_0-1} \int\limits_{\{x : |f(x)| > C\lambda\}} |f(x)|^{p_0} d\mu d\lambda \\
&= p(2A_0)^{p_0} \int\limits_X |f(x)|^{p_0} \int\limits_0^{\frac{|f(x)|}{C}} \lambda^{p-p_0-1} d\lambda d\mu \\
&= \frac{p}{p-p_0} (2A_0)^{p_0} (2A_1)^{p-p_0} ||f||_p^p.
\end{aligned}$$

Subcase 2: Assume $1 \leqslant p_1 < \infty$. Using the very same idea, we can find

$$||T(f)||_p \leqslant 2p^{\frac{1}{p}} \left(\frac{1}{p-p_0} + \frac{1}{p_1 - p} \right)^{\frac{1}{p}} A_0^{1-\theta} A_1^\theta ||f||_p.$$

Case 2: One can finish the proof using the same technical idea.

□

Problem 2. Finish the proof of Theorem 1.9.

Problem 3. Let $p_0, p_1, q_0, q_1 \in [1, \infty]$, and suppose $T : L^p(X) \to L^q(Y)$ is a sublinear operator. Suppose that $||T\chi_E||_{L^{q_0}} \leqslant C_0\mu(E)^{\frac{1}{p_0}}$,[2] and $||T\chi_E||_{L^{q_1}} \leqslant C_1\mu(E)^{\frac{1}{p_1}}$ for all measurable set $E \subseteq X$. Prove that there exists $C_{p,q} > 0$ such that for all $f \in L^p$, $||T(f)||_q \leqslant C_{p,q}||f||_p$ where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ for any $\theta \in (0, 1)$.

[2] This can be generalized as $||T(F)||_{L^{q_0}(\nu)} \leqslant C_0||f||_{L^{p_0}(\mu)}$ for all $f \in L^{p_0}$.

2 Approximation to the Identity

Definition 2.1. Let $\varphi \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \varphi dx = 1$ via the Lebesgue measure. For any $\varepsilon > 0$, let $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$ be the dilation of φ by ε.[3] The sequence $\{\varphi_\varepsilon\}_{\varepsilon>0}$ is called an approximation to the identity.

Example 2.2. Set $\varphi(x) = e^{-\pi|x|^2}$ where $|x|$ is the Euclidean distance. One can show that $\int_{\mathbb{R}^n} \varphi(x)dx = 1$ via polar coordinates. By definition, set $\varphi_\varepsilon(x) = \varepsilon^{-n}e^{-\frac{\pi}{\varepsilon^2}|x|^2}$ gives a sequence $\{\varphi_\varepsilon\}_{\varepsilon>0}$ as an approximation to the identity. The graph of this function is of bell-shaped such that as $\varepsilon \to 0$, the mass is concentrated at 0.

$$\varphi_\varepsilon(x) \to \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

One can also say $\varphi_\varepsilon \to \delta$ as $\varepsilon \to 0$, converging to the dirac mass.

Definition 2.3. Let f and g both be integrable, then the function

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$$

is called the convolution of f and g whenever the integral exists.

Example 2.4. Let f be a "nice" function, i.e., continuous with compact support, or of C^∞, then

$$\lim_{\varepsilon\to 0}(\varphi_\varepsilon * f)(x) = \int \delta(x-y)f(y)dy = f(x).$$

Definition 2.5. Let $f \in C^\infty(\mathbb{R}^n)$. If

$$M := \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)| < \infty,$$

for any $\alpha, \beta \in \mathbb{N}_0^n$, then we say f is a Schwartz-function. We call $\alpha \in \mathbb{N}_0^n$ the multi-index, and for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we define $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and similarly $D^\beta = \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}$. We also denote $\mathcal{S}(\mathbb{R}^n)$ to be the collections of Schwartz function.

Remark 2.6. For large enough x, $|D^\beta f(x)| \leqslant \frac{M}{|x|^\alpha}$ decays rapidly.

Example 2.7. The Gaussian kernel is a Schwartz function. In fact, $\mathcal{S}(\mathbb{R}^n)$ is dense in the L^p-space.

Lemma 2.8. If $f \in \mathcal{S}(\mathbb{R}^n)$, then $|D^\beta f(x)| \leqslant \frac{C_{N,\beta}}{(1+|x|)^N}$ for any β, N, x.

Proof. Let $C_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)|$, then set $C_{N,\beta} = \max\{C_{\alpha,\beta} : \alpha \in \mathbb{N}_0^n, |\alpha| = \alpha_1 + \cdots + \alpha_n \leqslant N\}$. □

Remark 2.9. Lemma 2.8 is equivalent to Definition 2.5.

Theorem 2.10. Let $\{\varphi_\varepsilon\}_{\varepsilon>0}$ be an approximation to the identity, then

$$\lim_{\varepsilon\to 0}(\varphi_\varepsilon * f)(x) = f(x)$$

for any $x \in \mathbb{R}^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. To simplify the convolution a little bit, note that

$$(\varphi_\varepsilon * f)(x) = \int \varphi(y)f(x-\varepsilon y)dy$$

[3] By taking $\varepsilon^{-1}(x)$ we are able to normalize the function.

by a change of variables. Taking the limit, we get

$$\lim_{\varepsilon\to 0}(\varphi_\varepsilon * f)(x) = \lim_{\varepsilon\to 0}\int \varphi(y)f(x-\varepsilon y)dy.$$

To enlarge the integrand, we note that

$$|\varphi(y)f(x-\varepsilon y)| \leqslant |\varphi(y)|||f||_\infty \in L^1(\mathbb{R}^n)$$

since $f \in \mathcal{S}(\mathbb{R}^n)$. By Dominant Convergence Theorem, we know

$$\begin{aligned}
\lim_{\varepsilon\to 0}(\varphi_\varepsilon * f)(x) &= \int \varphi(y)\lim_{\varepsilon\to 0} f(x-\varepsilon y)dy \\
&= \int \varphi(y) f(\lim_{\varepsilon\to 0} x-\varepsilon y)dy \\
&= \int \varphi(y)f(x)dy \\
&= f(x)\int \varphi(y)dy \\
&= f(x)
\end{aligned}$$

since f is continuous. □

We now try to pass this conclusion to the L^p-space. Note that $\mathcal{S}(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$, and although the pointwise convergence may not hold, the L^p-convergence still holds.

Lemma 2.11 (Minkowski)**.** For any $1 \leqslant p \leqslant \infty$, we have

$$\left(\int\limits_{\mathbb{R}^n}\left(\int\limits_{\mathbb{R}^n}|f(x,y)|dy\right)^p dx\right)^{\frac{1}{p}} \leqslant \int\limits_{\mathbb{R}^n}\left(\int\limits_{\mathbb{R}^n}|f(x,y)|^p dx\right)^{\frac{1}{p}} dy.$$

Remark 2.12. For any $1 \leqslant p < \infty$, the Minkowski inequality $||f+g||_p \leqslant ||f||_p + ||g||_p$ which is the triangle inequality in L^p-space. The Minkowski inequality above is a continuous analogue of the result we have seen before.

Proof. Recall that for any $1 \leqslant p < \infty$, we have

$$||F||_p = \sup\left\{\left|\int\limits_{\mathbb{R}^n} Fgdx\right| : g \in L^{p'}(\mathbb{R}^n), ||g||_{p'} = 1\right\}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Now

$$\left(\int\limits_{\mathbb{R}^n}\left(\int\limits_{\mathbb{R}^n}|f(x,y)|dy\right)^p dx\right)^{\frac{1}{p}} = \sup\left\{\left|\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}|f(x,y)|dyg(x)dx\right| : g \in L^{p'}(\mathbb{R}^n), ||g||_{p'} = 1\right\},$$

but

$$\begin{aligned}
\left|\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}|f(x,y)|dyg(x)dx\right| &\leqslant \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}|f(x,y)|dy|g(x)|dx \\
&= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}|f(x,y)||g(x)|dxdy
\end{aligned}$$

$$\leqslant \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x,y)|^p dx \right)^{\frac{1}{p}} dy ||g(x)||_{p'}$$
$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x,y)|^p dx \right)^{\frac{1}{p}} dy$$

by Fubini theorem and Hölder inequality. □

Theorem 2.13. Let $1 \leqslant p < \infty$, and $\{\varphi_\varepsilon\}_{\varepsilon>0}$ be an approximation to the identity, then for any $f \in L^p(\mathbb{R}^n)$, we have

$$\lim_{\varepsilon \to 0} ||\varphi_\varepsilon * f - f||_p = 0,$$

or equivalently, $\lim_{\varepsilon \to 0}(\varphi_\varepsilon * f) =_{L^p} f$.

Remark 2.14. This conclusion does not hold for $p = \infty$. Over the supremum norm, we ignore the contribution of the null set, therefore $\varphi_\varepsilon * f \xrightarrow{L^\infty} f$ for $f \in L^\infty$ is a uniform convergence, which forces f to be continuous. However, L^∞-functions cannot be continuous, contradiction.

Proof of Theorem 2.13. First, we have the following conclusion.

Problem 4. Suppose $K \in L^1(\mathbb{R}^n)$, prove that $||K * f||_p \leqslant ||K||_1 ||f||_p$ for any $f \in L^p$ and any $p \in [1,\infty]$. (Hint: use Minkowski or interpolation.)

By Problem 4, $\varphi_\varepsilon * f \in L^p$ since $\varphi_\varepsilon \in L^1$ and $f \in L^p$. We have

$$\begin{aligned} f * \varphi_\varepsilon(x) - f(x) &= \int f(x-y)\varphi_\varepsilon(y)dy - \int f(x)\varphi_\varepsilon(y)dy \text{ since } \int \varphi_\varepsilon = 1 \\ &= \int_{\mathbb{R}^n} (f(x-y) - f(x))\varphi_\varepsilon(y)dy \text{ by setting } \varphi_\varepsilon(y) = \varepsilon^{-n}\varphi(\frac{y}{\varepsilon}) \\ &= \int_{\mathbb{R}^n} (f(x-\varepsilon y) - f(x))\varphi(y)dy \text{ by taking } y \to \varepsilon y \end{aligned}$$

where $dy = d\mathfrak{m}$. By Lemma 2.11, we have

$$||f * \varphi_\varepsilon - f||_p \leqslant \int |\varphi(y)| ||f(x-\varepsilon y) - f(x)||_{L^p(dx)} dy.$$

Problem 5. For any $y \in \mathbb{R}^n$, $||f(\cdot - \varepsilon y) - f(\cdot)||_{L^p(\mathbb{R}^n)} \to 0$ as $\varepsilon \to 0$. (Hint: use the fact that $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.)

We now know that

$$|\varphi(y)| ||f(x-\varepsilon y) - f(x)||_{L^p(dx)} \leqslant |\varphi(y)|(||f||_p + ||f||_p) \in L^1(dy),$$

then taking the limit, we have

$$\begin{aligned} \lim_{\varepsilon \to 0} ||f * \varphi_\varepsilon - f||_p &\leqslant \lim_{\varepsilon \to 0} \int |\varphi(y)| ||f * \varphi_\varepsilon - f||_p dy \\ &= \int |\varphi(y)| \lim_{\varepsilon \to 0} ||f(x-\varepsilon y) - f(x)||_p dy \\ &= 0 \end{aligned}$$

by Problem 5. □

Corollary 2.15. $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ if $1 \leqslant p < \infty$.

Proof. Note that the set $L^p_c = \{f \in L^p : f \text{ has compact support}\}$ is dense in L^p: for large enough value M, we know the ball B_M satisfies $f\chi_{B_M} \in L^p$. For any $g \in L^p_c$, we take $\varphi(x) = e^{-\pi|x|^2}$, then $\varphi_\varepsilon * g \xrightarrow{L^p} g$ as $\varepsilon \to 0$. But one can check that $\varphi_\varepsilon * g \in \mathcal{S}(\mathbb{R}^n)$, so this shows denseness. □

Problem 6. Let $\varphi \in C_c^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

a. Show that $\varphi * f \in C^\infty(\mathbb{R}^n)$ and $D^\alpha(\varphi * f) = (D^\alpha\varphi) * f$ for multi-index $\alpha \in \mathbb{N}_0^n$. (Hint: apply DCT.)

b. If $f, g \in \mathcal{S}(\mathbb{R}^n)$, then $f * g \in \mathcal{S}(\mathbb{R}^n)$.

Theorem 2.16. Let $\varphi \in L^1$ such that $\int\limits_{\mathbb{R}^n} \varphi = 1$. We define the least decreasing radial majorant of φ to be $\psi(x) = \sup\limits_{|y|\geqslant|x|} |\varphi(y)|$.[4] Suppose that $\psi \in L^1(\mathbb{R}^n)$ and $\int\limits_{\mathbb{R}^n} \psi(x)dx = A$, then

a. $\sup\limits_{\varepsilon>0} |f * \varphi_\varepsilon(x) \leqslant AMf(x)$ almost everywhere, where $Mf(x)$ is the Hardy-Littlewood maximal function;

b. for any $1 \leqslant p < \infty$, $\lim\limits_{\varepsilon\to 0} f * \varphi_\varepsilon(x) = f(x)$ almost everywhere for all $f \in L^p(\mathbb{R}^n)$.

Remark 2.17.

1. The proof of statement a. requires applying the polar coordinate formula.

2. The proof of statement b. mimics the proof of Lebesgue differentiation theorem. It is also true even if $p = \infty$. However, since our proof uses the denseness of Schwartz functions in L^p space, this would not work in $p = \infty$.

Proof.

a. By the translation and dilation invariance, it suffices to prove that $|f*\varphi_1(0)| = |f*\varphi(0)| \leqslant AMf(0)$. It suffices to show that $f * \psi(0) \leqslant AMf(0)$ for all $f \in L^+(\cap L^1_{\text{loc}})$, then since $|\varphi(x)| \leqslant \psi(x)$, we have $|f * \varphi(0)| \leqslant AMf(0)$, and therefore gives the statement. Recall the polar coordinate formula

$$\int\limits_{\mathbb{R}^n} f(x)dx = \int\limits_0^\infty \int\limits_{S^{n-1}} f(rx')dx' r^{n-1} dr$$

where (r, x') is the polar coordinate of x, i.e., $r = |x|$ and $x' = \frac{x}{|x|} \in S^{n-1}$.

Remark 2.18. For $E^* = E \cap S^{n-1}$ and $dx' = d\sigma(x')$ given by the surface measure σ induced by $\mathfrak{m}$, then $\sigma(E^*) = \mathfrak{m}(E)$. Indeed, for $\Sigma_r = B^n(0, r) \subseteq \mathbb{R}^n$, then

$$\int\limits_{\mathbb{R}^n} = \int\limits_0^\infty \int\limits_{\Sigma_r} d\sigma_1$$

where $d\sigma_r = r^{n-1}d\sigma_1$. This can be interpreted as Fubini theorem.

We calculate

$$\begin{aligned} f * \psi(0) &= \int\limits_{\mathbb{R}^n} f(x)\psi(-x)dx \\ &= \int\limits_{\mathbb{R}^n} f(x)\psi(|x|)dx \\ &= \int\limits_0^\infty \int\limits_{S^{n-1}} f(rx')\psi(r)dx'dr \end{aligned}$$

[4] We say a function $f : \mathbb{R}^n \to \mathbb{C}$ is radial if for any $x \in \mathbb{R}^n$, $f(x) = f(|x|)$, i.e., the value of f only depend on the direction of x, but not by the magnitude from the origin.

$$= \int_0^\infty \psi(r) r^{n-1} \int_{S^{n-1}} f(rx')dx'dr.$$

Set $F(r) = \int_{S^{n-1}} f(rx')dx'$, then

$$\begin{aligned} G(x) &:= \int_{B^n(0,r)} f(x)dx \\ &= \int_{|x|\leqslant r} f(x)dx \\ &= \int_0^r t^{n-1} \int_{S^{n-1}} f(rx')dx'dt \\ &= \int_0^r t^{n-1} F(t)dt. \end{aligned}$$

By the fundamental theorem of calculus, $G'(r) = r^{n-1}F(r)$. On the other hand,

$$\begin{aligned} G(r) &= \mathfrak{m}(B^n(0,r)) \cdot \frac{1}{\mathfrak{m}(B^n(0,r))} \int_{B^n(0,r)} f(x)dx \\ &\leqslant \mathfrak{m}(B^n(0,r)) \cdot Mf(0) \\ &\leqslant C_n \cdot r^n Mf(0) \end{aligned}$$

by the Hardy-Littlewood maximal function.

Recall that

$$\begin{aligned} f * \psi(0) &= \int_0^\infty r^{n-1} F(r)\psi(r)dr \\ &= \int_0^\infty G'(r)\psi(r)dr \\ &= \psi(r)G(r)\big|_{r=0}^{\infty} - \int_0^\infty G(r)d\psi(r) \\ &\quad \text{by integration by parts, since } \psi'(r)dr = d\psi(r) \text{ is differentiable almost everywhere} \\ &= \lim_{r\to\infty} \psi(r)G(r) - \lim_{r\to 0} \psi(r)G(r) - \int_0^\infty G(r)d\psi(r) \text{ assuming the limits exist.} \end{aligned}$$

Let us show that the limits exist.

Claim 2.19.

$$\lim_{r\to 0} \psi(r)G(r) = \lim_{r\to\infty} \psi(r)G(r) = 0.$$

Subproof. We have

$$\begin{aligned} |\psi(r)G(r)| &\leqslant \psi(r)|G(r)| \\ &\leqslant C_n r^n \psi(r) Mf(0). \end{aligned}$$

It remains to show $r^n\psi(r) \to 0$ as $r \to 0$ or $r \to \infty$. We have

$$\begin{aligned}
r^n\psi(r) &= c_n \int\limits_{\frac{r}{2}\leqslant|x|\leqslant r} dx\psi(r) \\
&\leqslant c_n \int\limits_{\frac{r}{2}\leqslant|x|\leqslant r} \psi(x)dx \text{ since } \psi \text{ is decreasing} \\
&\to 0
\end{aligned}$$

as $r \to 0$ or $r \to \infty$, since $\psi \in L^1$. ■

Now

$$\begin{aligned}
f * \psi(0) &= 0\int_0^\infty G(r)d\psi(r) \\
&= \int_0^\infty G(r)d(\psi(r)) \\
&\leqslant C_nMf(0)\int_0^\infty r^n d(-\psi(r)) \\
&= nC_nMf(0)\int_0^\infty \psi(r)r^{n-1}dr \text{ by integral by parts} \\
&= Mf(0)\int\limits_{\mathbb{R}^n} \psi(x)dx \\
&= AMf(0).
\end{aligned}$$

b.

Lemma 2.20. Let $\{T_\varepsilon\}_{\varepsilon>0}$ be a family of linear operators on $L^p(\mathbb{R}^n)$ for $1 \leqslant p \leqslant \infty$. Define $T^*f(x) = \sup\limits_{\varepsilon>0} |T_\varepsilon f(x)|$ for all $x \in \mathbb{R}^n$. If T^* is of weak (p,p), then

$$\{f \in L^p(\mathbb{R}^n) : \lim_{\varepsilon\to 0} T_\varepsilon f(x) = f(x) \text{ almost everywhere}\}$$

is closed in $L^p(\mathbb{R}^n)$. That is, for any family $\{f_k\}$ in L^p with $||f_k - f||_p \to 0$ as $k \to \infty$, and $\lim\limits_{\varepsilon\to 0} T_\varepsilon f_k(x) = f_k(x)$ almost everywhere, then $\lim\limits_{\varepsilon\to 0} T_\varepsilon f(x) = f(x)$ almost everywhere.

Subproof. Consider the level set $\{x \in X : \lim\limits_{\varepsilon\to 0} |T_\varepsilon f(x) - f(x)| > \lambda\}$. Now

$$\mu(\{x \in X : \limsup_{\varepsilon\to 0} |T_\varepsilon f(x) - f(x)| > \lambda\}) = \mu(\{x \in X : \limsup_{\varepsilon\to 0} |T_\varepsilon(f - f_k)(x) - (f - f_k)(x)| > \lambda\}),$$

but

$$|T_\varepsilon(f - f_k)(x) - (f - f_k)(x)| \leqslant T^*(f - f_k)(x) + |(f - f_k)(x)|$$

gives a uniform upper bound, then

$$\begin{aligned}
\mu(\{x \in X : \limsup_{\varepsilon\to 0} |T_\varepsilon f(x) - f(x)| > \lambda\}) &= \mu(\{x \in X : \limsup_{\varepsilon\to 0} |T_\varepsilon(f - f_k)(x) - (f - f_k)(x)| > \lambda\}) \\
&\leqslant \mu(\{x \in X : T^*(f - f_k)(x) > \frac{\lambda}{2}\})
\end{aligned}$$

$$
\begin{aligned}
&+\mu(\{x \in X : |(f-f_k)(x)| > \frac{\lambda}{2}\}) \\
&\leqslant \frac{C_p\|f-f_k\|_p^p}{\lambda^p} \\
&\to 0
\end{aligned}
$$

as $k \to \infty$. Since $\mu(\{x \in X : \limsup_{\varepsilon\to 0} |T_\varepsilon f(x) - f(x)| > \lambda\})$ is independent from f_k, then this squeezes $\mu(\{x \in X : \limsup_{\varepsilon\to 0} |T_\varepsilon f(x) - f(x)| > \lambda\}) = 0$ for all $\lambda > 0$. By writing

$$
\begin{aligned}
\mu(\{x \in X : \limsup_{\varepsilon\to 0} |T_\varepsilon f(x) - f(x)| > 0\}) &\leqslant \mu\left(\bigcup_{k=1}^{\infty}\left\{x \in X : \limsup_{\varepsilon\to 0} |T_\varepsilon f(x) - f(x)| > \frac{1}{k}\right\}\right) \\
&\leqslant \sum_{k=1}^{\infty} \mu\left(\left\{x \in X : \limsup_{\varepsilon\to 0} |T_\varepsilon f(x) - f(x)| > \frac{1}{k}\right\}\right) \\
&= 0
\end{aligned}
$$

as the limit of partial sums. In particular, this forces

$$\mu(\{x \in X : \limsup_{\varepsilon\to 0} |T_\varepsilon f(x) - f(x)| > 0\}) = 0.$$

Therefore, $\limsup_{\varepsilon\to 0} |T_\varepsilon f(x) - f(x)| = 0$ almost everywhere in x, and hence that means the limit $\lim_{\varepsilon\to 0} |T_\varepsilon f(x) - f(x)| = 0$ exists. That is, $\lim_{\varepsilon\to 0} T_\varepsilon f(x) = f(x)$ almost everywhere. ■

We now want to show that $\lim_{\varepsilon\to 0} f * \varphi_\varepsilon(x) = f(x)$ almost everywhere on x for all $f \in L^p$ and $1 \leqslant p < \infty$. We know this is true if $f \in \mathcal{S}(\mathbb{R}^n)$, a dense collection in L^p-space. By Lemma 2.20, we just need to show that $\sup_{\varepsilon\to 0} |f * \varphi_\varepsilon| = T^* f$ defines a weak (p,p) operator T^*. By part a., we know

$$\sup_{\varepsilon\to 0} |f * \varphi_\epsilon(x)| \leqslant AMf(x)$$

for some finite number A, then T^* is weak (p,p) since M is of strong (p,p).

□

Example 2.21. Let $\varphi(x) = (4\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4}}$ for all $x \in \mathbb{R}^n$. Let $\varepsilon = \sqrt{t}$, then let

$$\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} =: W_t(x),$$

which is the Gauss-Weierstrass kernel. Consider the heat equation

$$
\begin{cases}
\Delta_x u = \frac{\partial u}{\partial t} \ \forall (x,t) \in \mathbb{R}_+^{n+1} \\
u(x,0) := \lim_{t\to 0} u(x,t) = f(x) \in L^p(\mathbb{R}^n), \ 1 \leqslant p < \infty
\end{cases}
\tag{2.22}
$$

with respect to the Laplacian $\Delta_x = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Here the complex-valued function u is defined in the upper half plane $\mathbb{R}_+^{n+1} = \{(x,t) : x \in \mathbb{R}^n, t > 0\}$. By solving Equation (2.22), we obtain $u(x,t) = W_t * f(x)$, where W_t is a fundamental solution to the heat equation.

Example 2.23. Consider a complex-valued function $u : \mathbb{R}_+^{n+1} \to \mathbb{C}$, and we have the Laplacian $\Delta_{x,t} = \Delta_x + \partial_t^2$ and a PDE

$$
\begin{cases}
\Delta_{x,t} u = 0 \ \forall (x,t) \in \mathbb{R}_+^{n+1} \\
u(x,0) = f(x) \in L^p(\mathbb{R}^n), \ 1 \leqslant p < \infty
\end{cases}
\tag{2.24}
$$

To solve this, we define $\varphi(x) = \frac{1}{(1+|x|^2)^{\frac{n+1}{2}}}$, and set $\varepsilon = t$, therefore we have the Poisson kernel

$$\varphi_t(x) = \frac{t}{(t^2+|x|^2)^{\frac{n+1}{2}}} =: P_t(x).$$

By Theorem 2.16, we know $u(x,t) = P_t * f(x)$ solves Equation (2.24).

3 Fourier Transforms

Definition 3.1. Let $f \in L^1(\mathbb{R}^n)$ be a function, then we define the Fourier transform to be the Lebesgue integral

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} dx$$

for all $\xi \in \mathbb{R}^n$, where $\xi \cdot x = x_1\xi_1 + \cdots + x_n\xi_n$ for $\xi \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Therefore, $\hat{f}$ is integrable.

Proposition 3.2. Let $f \in L^1(\mathbb{R}^n)$, then

a. $||\hat{f}||_\infty \leqslant ||f||_1$;

b. $\hat{f}$ is uniformly continuous on $\mathbb{R}^n$;

c. $\lim\limits_{|\xi| \to \infty} \hat{f}(\xi) = 0$;

d. $\widehat{f * g} = \hat{f}\hat{g}$ for all $f, g \in L^1$.

Problem 7. Verify parts a., b., d.

Proof of part c. of Proposition 3.2. We know that

$$\begin{aligned}
\hat{f}(\xi) &= \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} dx \\
&= \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} e^{-2\pi i \xi \cdot \frac{\xi}{2|\xi|^2}} (-1) dx \\
&= -\int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot \left(x + \frac{\xi}{2|\xi|^2}\right)} dx \\
&= -\int_{\mathbb{R}^n} f\left(y - \frac{\xi}{2|\xi|^2}\right) e^{-2\pi i \xi \cdot y} dy
\end{aligned}$$

by a change of variable $y = x + \frac{\xi}{2|\xi|^2}$. By comparing this with the definition, then we have

$$\left| \int_{\mathbb{R}^n} f(x) e^{-2\pi i \xi \cdot x} dx - \int_{\mathbb{R}^n} f\left(x - \frac{\xi}{2|\xi|^2}\right) e^{-2\pi i \xi \cdot x} dx \right| = \left| 2\hat{f}(\xi) \right|.$$

Note that the left-hand side is bounded above by $||f(\cdot) - f(\cdot - \frac{\xi}{2|\xi|^2})||_1 \to 0$ as $|\xi| \to \infty$, by the continuity condition. Therefore, $\lim\limits_{|\xi| \to \infty} |2\hat{f}(\xi)| = 0$. □

Problem 8. A sequence of functions $\{f_k\}_{k \in \mathbb{N}} \subseteq \mathcal{S}(\mathbb{R}^n)$ converges in $\mathcal{S}(\mathbb{R}^n)$ to $f \in \mathcal{S}(\mathbb{R}^n)$ if $\lim\limits_{k \to \infty} ||f_k - f||_{\alpha,\beta} = 0$ for all $\alpha, \beta \in \mathbb{N}_0^n$. Here $||f||_{\alpha,\beta} = \sup\limits_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)|$. Prove that for all $f \in \mathcal{S}(\mathbb{R}^n)$, there exists a sequence $\{f_k\}_{k \in \mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^n)$ such that $\{f_k\}_{k \geqslant 1}$ converges to f in $\mathcal{S}(\mathbb{R}^n)$. That is, $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$.

Hint: take $\varphi : \mathbb{R}^n \to \mathbb{R}$ to be a C^∞-function satisfying

1. φ being radial,
2. $0 \leqslant \varphi \leqslant 1$,
3. $\varphi(x) = 1$ whenever $|x| \leqslant 1$ and $\varphi(x) = 0$ whenever $|x| \geqslant 2$.

Note that φ is a bump function. Now for any $k \in \mathbb{N}$, set $f_k(x) = f(x)\varphi\left(\frac{x}{k}\right)$, then $f_k \in C_c^\infty(\mathbb{R}^n)$. You can prove that $f_k \to f$ in $\mathcal{S}(\mathbb{R}^n)$ as $k \to \infty$. Use Lebniz's rule to show that

$$D^\alpha(fg) = \sum_{\substack{\beta \in \mathbb{N}_0^n \\ \beta \leqslant \alpha}} C_{\alpha,\beta} D^{\alpha-\beta} f D^\beta f$$

for $C_{\alpha,\beta} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Note that $\beta \leqslant \alpha$ if and only if $\beta_j \leqslant \alpha_j$ for all $1 \leqslant j \leqslant n$, once we write $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$. Now

$$D^\alpha(fg) = \sum_{\beta+\gamma=\alpha} C_{\beta\gamma} D^\beta f D^\gamma g.$$

Also note that $D^\beta\left(\varphi\left(\frac{x}{k}\right)\right) \leqslant \frac{C}{k}$ if $|\beta| > 0$ where $\beta \in \mathbb{N}_0^n$.

Proposition 3.3. Let $f \in L^1(\mathbb{R}^n)$. For $\hat{f}(\xi) = \int\limits_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}$, we have

1. $(\widehat{f(\cdot - b)})(\xi) = e^{-2\pi i \xi \cdot b}\hat{f}(\xi)$ for all $b \in \mathbb{R}^n$;
2. $(\widehat{e^{2\pi i x\cdot h} f(x)})(\xi) = \hat{f}(\xi - h)$ for all $h \in \mathbb{R}^n$;
3. $(\widehat{t^{-n} f\left(\frac{\cdot}{t}\right)})(\xi) = \hat{f}(t\xi)$ for all $t \in \mathbb{R}$;
4. let ρ be an orthogonal transform on $\mathbb{R}^n$, that is, $\rho : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transform preserving the inner product $\rho(x) \cdot \rho(y) = x \cdot y$ for all $x, y \in \mathbb{R}^n$, then $(\widehat{f \circ \rho})(\xi) = \hat{f} \circ \rho(\xi)$ for all $\xi \in \mathbb{R}^n$;
5. if f is radial, then $\hat{f}$ is radial as well.

Problem 9. Prove Part 1-3 and 5.

Proof of Part 4. Set $y = \rho x$, and note that this is equivalent to having $x = \rho^{-1}y$, and in particular $\det(|A|) = 1$ of the corresponding matrix. Now

$$\begin{aligned}
(\widehat{f \circ \rho})(\xi) &= \int\limits_{\mathbb{R}^n} f(\rho(x))e^{-2\pi i x\cdot\xi} dx \\
&= \int\limits_{\mathbb{R}^n} f(y)e^{-2\pi i \rho^{-1} y \cdot \xi} |\det(A)| dy \\
&= \int\limits_{\mathbb{R}^n} f(y)e^{-2\pi i y\cdot\rho(\xi)} dy \\
&= \hat{f}(\rho\xi) \\
&= \hat{f} \circ \rho(\xi).
\end{aligned}$$

□

Theorem 3.4. Let $f \in L^1(\mathbb{R}^n)$, then

1. if $x_k f \in L^1(\mathbb{R}^n)$, then

$$\frac{\partial \hat{f}(\xi)}{\partial \xi_k} = (\widehat{-2\pi i x_k f})(\xi)$$

for all $\xi \in \mathbb{R}^n$, where $\xi = (\xi_1, \ldots, \xi_n)$ and $x = (x_1, \ldots, x_n)$;

2. if $\frac{\partial f}{\partial x_k} \in L^1$, then $\left(\widehat{\frac{\partial f}{\partial x_k}}\right)(\xi) = 2\pi i \xi_k \hat{f}(\xi)$.

Remark 3.5. To get an intuition, note that for nice enough functions, we have

$$\begin{aligned}
\partial \xi_k \hat{f} = \partial \xi_k \int f(x) e^{-2\pi i x \cdot \xi} dx \\
&= \int \partial_{\xi_k} f(x) e^{-2\pi i x \cdot \xi} dx \\
&= \int f(x) \partial_{\xi_k} e^{-2\pi i x \cdot \xi} dx \\
&= \int f(x) \cdot (-2\pi i x_k) e^{-2\pi i x \cdot \xi} dx \\
&= \widehat{(-2\pi i x_k f)}(\xi),
\end{aligned}$$

and similarly for the second formula.

Proof. Let us prove the first part. Set $h = (0, \dots, 0, h_k, 0, \dots, 0) \in \mathbb{R}^n$. Now

$$\begin{aligned}
\partial_{\xi_k} \hat{f}(\xi) &= \lim_{h_k \to 0} \frac{\hat{f}(\xi + h_k) - \hat{f}(\xi)}{h_k} \\
&= \lim_{h_k \to 0} \int \frac{e^{-2\pi i x_k h_k} - 1}{h_k} f(x) e^{-2\pi i \xi \cdot x} dx \\
&=: \lim_{h_k \to 0} \int I dx.
\end{aligned}$$

Now by Dominated Convergence Theorem, we know $I \leqslant C|x_k f(x)| \in L^1$, and by the inequality $|e^{i\theta} - 1| \leqslant C|\theta|$, we have

$$\begin{aligned}
\left| \frac{e^{-2\pi i x_k h_k} - 1}{h_k} \right| \leqslant C \frac{|x_k h_k|}{|h_k|} \\
&= \int \lim_{h_k \to 0} I dx \\
&= \int_{\mathbb{R}^n} (-2\pi i x_k \cdot f_k) \cdot e^{-2\pi i \xi \cdot x} dx.
\end{aligned}$$

□

Corollary 3.6. Let $P(x) = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| \leqslant d}} a_\alpha x^\alpha$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $a_\alpha \in \mathbb{C}$. Define the differential operator $P(D) = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| \leqslant d}} a_\alpha D^\alpha$. (Recall that $D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$.) Then for any $f \in \mathcal{S}(\mathbb{R}^n)$, we have $P(D)\hat{f}(\xi) = \widehat{(P(-2\pi i \cdot) f(\cdot))}(\xi)$, and $\widehat{(P(D)f)}(\xi) = P(2\pi i \xi) \hat{f}(\xi)$.

Definition 3.7. For any $g \in L^1(\mathbb{R}^n)$, we define the inverse Fourier transform of g to be $\check{g}(x) = \int g(\xi) e^{2\pi i \xi \cdot x} d\xi = \hat{g}(-x)$.

Lemma 3.8. For any $f, g \in L^1$, we have $\int \hat{f} g = \int f \hat{g}$.

Proof. By Fubini theorem, we know

$$\begin{aligned}\int \hat{f}g &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\xi)e^{-2\pi i\xi x}d\xi g(x)dx\\ &= \int_{\mathbb{R}^n} f(\xi)\int_{\mathbb{R}^n} g(x)e^{-2\pi i\xi x}dxd\xi\\ &= \int f\hat{g}.\end{aligned}$$

□

Lemma 3.9. $\widehat{(e^{-\pi|x|^2})}(\xi) = e^{-\pi|\xi|^2}$ for $x, \xi \in \mathbb{R}^n$.

Proof. It suffices to the case where $n = 1$: in general, we have iterated integrals

$$\begin{aligned}\widehat{(e^{-\pi|\cdot|^2})} &= \int_{\mathbb{R}^n} e^{-\pi(x_1^2+\cdots+x_n^2)}e^{-2\pi i(x_1\xi_1+\cdots+x_n\xi_n)}dx_1\cdots dx_n\\ &= \prod_{j=1}^n \int_{\mathbb{R}} e^{-\pi x_j^2}e^{-2\pi i x_j\xi_j}dx_j\\ &= \prod_{j=1}^n \widehat{(e^{-\pi x_j^2})}(\xi_j).\end{aligned}$$

It remains to show that

$$\widehat{(e^{-\pi x^2})}(\xi) = e^{-\pi\xi^2}$$

for $\xi, x \in \mathbb{R}$.

Consider the following ODE problem

$$\begin{cases} u' + 2\pi xu = 0\\ u(0) = 1\end{cases}$$

for function $u : \mathbb{R} \to \mathbb{C}$. It is obvious that this ODE has a unique solution $u(x) = e^{-\pi x^2}$. It suffices to show that the Fourier transform $\hat{u}$ satisfies the same ODE. We have $\hat{u'} + \widehat{2\pi xu} = 0$, and therefore $2\pi i\xi\hat{u}(\xi) + i\hat{u}'(\xi) = 0$. This gives

$$\hat{u'} + 2\pi\xi\hat{u} = 0.$$

The corresponding boundary value is $\hat{u}(0) = \int u(x)e^{-\pi i\cdot 0\cdot x}dx = \int u(x)dx = \int e^{-\pi x^2}dx = 1$. Therefore, $\hat{u}$ satisfies the same ODE, and so $\hat{u} = \hat{f} = e^{-\pi\xi^2}$, as desired. □

The following conclusion now follows by dilating the result above.

Corollary 3.10. $\widehat{(e^{-4\pi^2|x|^2})}(\xi) = (4\pi)^{-\frac{n}{2}}e^{-\frac{|\xi|^2}{4}}$.

Definition 3.11. Let $g \in L^1(\mathbb{R}^n)$. The Gaussian mean of g is

$$G_\varepsilon(g) = \int_{\mathbb{R}^n} g(\xi)e^{-4\pi^2\varepsilon^2|\xi|^2}d\xi.$$

Remark 3.12. By Dominated Convergence Theorem, we have $\lim_{\varepsilon\to 0} G_\varepsilon(g) = ||g||_1$.

Lemma 3.13. Let $f \in L^1(\mathbb{R}^n)$, then

$$\lim_{\varepsilon \to 0} \left\| \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} e^{-4\pi^2 \varepsilon^2 |\xi|^2} d\xi - f(x) \right\|_{L^1(\mathbb{R}^n)} = 0.$$

Proof. Let $g = e^{2\pi i x \cdot \xi} e^{-4\pi^2 \varepsilon^2 |\xi|^2}$ be a function, then

$$\begin{aligned}
\int_{\mathbb{R}^n} f(y)(\widehat{e^{2\pi i x \cdot \xi} e^{-4\pi^2 \varepsilon^2 |\xi|^2}})(y))dy &= \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} e^{-4\pi^2 \varepsilon^2 |\xi|^2} d\xi \text{ by Lemma 3.8} \\
&= \int_{\mathbb{R}^n} f(y) \varepsilon^{-n} (\widehat{e^{-4\pi^2 |\cdot|^2}})(\varepsilon^{-1}(x-y))dy \\
&= f * \varphi_\varepsilon, \text{ by Corollary 3.10}
\end{aligned}$$

which converges to f in the L^1-sense. Here $\{\varphi_\varepsilon\}_{\varepsilon > 0}$ is an approximation to the identity of $\varphi(x) = (\widehat{e^{-4\pi^2 |\cdot|^2}})(x) = (4\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4}}$, so $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(\varepsilon^{-1} x)$. □

Theorem 3.14 (Fourier Inversion Theorem)**.** Suppose $f \in L^1$ and $\hat{f} \in L^1$, then $\check{\hat{f}} = f$.

Proof. By Lemma 3.13, there exists a sequence $\{\varepsilon_k\}_{k \in \mathbb{N}}$ such that

- $\lim\limits_{k \to \infty} \varepsilon_k = 0$,
- $\lim\limits_{k \to \infty} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{-2\pi i x \cdot \xi} e^{-4\pi^2 \varepsilon_k^2 |\xi|^2} d\xi = f(x)$ almost everywhere for x.

By Dominant Convergence Theorem, we know

$$\begin{aligned}
f &= \lim_{k \to \infty} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} e^{-4\pi^2 \varepsilon_k^2 |\xi|^2} d\xi \\
&= \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} \lim_{k \to \infty} e^{-4\pi^2 \varepsilon_k^2 |\xi|^2} d\xi \\
&= \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi \\
&= \check{\hat{f}}.
\end{aligned}$$

□

4 Fourier Transforms on $L^p(\mathbb{R}^n)$ for $1 \leqslant p \leqslant 2$

Theorem 4.1. $f \in \mathcal{S}(\mathbb{R}^n)$ if and only if $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$.

Proof. ($\Rightarrow$): we show that $\sup_{\xi \in \mathbb{R}^n} |(2\pi i \xi)^\alpha D^\beta \hat{f}(\xi)| < \infty$ for all $\alpha, \beta \in \mathbb{N}_0^n$. We know

$$
\begin{aligned}
(2\pi i \xi)^\alpha D^\beta \hat{f}(\xi) &= (2\pi i \xi)^\alpha \widehat{((-2\pi i x)^\beta f(x))}(\xi) \\
&= \widehat{(D^\alpha((2\pi i x)^\beta f(x)))}(\xi) \\
&= \int D^\alpha((-2\pi i x)^\beta f(x)) e^{-2\pi i \xi \cdot x} dx.
\end{aligned}
$$

Since $f \in \mathcal{S}(\mathbb{R}^n)$, then $|D^\alpha((-2\pi i x)^\beta f(x))| \leqslant \frac{C_{N,\alpha,\beta}}{(1+|x|)^N} \in L^1$. This shows the statement.

($\Leftarrow$): suppose $\hat{f} \in \mathcal{S}(\mathbb{R}^n) \subseteq L^1$, and we want to show that $f \in \mathcal{S}(\mathbb{R}^n)$. By a similar argument on $\hat{f}$, we know that $\check{\hat{f}} \in \mathcal{S}(\mathbb{R}^n) \subseteq L^1(\mathbb{R}^n)$. By Theorem 3.14, $f = \check{\hat{f}} \in L^1(\mathbb{R}^n)$. □

Lemma 4.2. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$, then $\langle f, g \rangle = \left\langle \hat{f}, \hat{g} \right\rangle$, where $\langle f, g \rangle = \int_{\mathbb{R}^n} f \bar{g} dx$. In particular, $||f||_2 = ||\hat{f}||_2$ for all $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. We have

$$
\begin{aligned}
\left\langle \hat{f}, \hat{g} \right\rangle &= \int_{\mathbb{R}^n} \hat{f}(x) \overline{\hat{g}(x)} dx \\
&= \int_{\mathbb{R}^n} f(x) \widehat{\overline{\hat{g}(x)}} dx \\
&= \int_{\mathbb{R}^n} f \bar{g} dx \\
&= \langle f, g \rangle
\end{aligned}
$$

by Lemma 3.8 and Theorem 3.14. □

We now extend the theory to $L^2(\mathbb{R}^n)$. For any $f \in L^2(\mathbb{R}^n)$, there exists a sequence $\{f_k\}_{k \geqslant 1}$ in $\mathcal{S}(\mathbb{R}^n)$ such that $\lim_{k \to \infty} ||f_k - f||_2 = 0$, i.e., $\lim_{k \to 0} f_k = f$ in L^2-sense. Therefore, we define $\hat{f}$ of f in $L^2(\mathbb{R}^n)$ to be the limit $\lim_{k \to \infty} \hat{f}_k$.

Lemma 4.3. The limit $\lim_{k \to \infty} \hat{f}_k$ exists.

Proof. Since $L^2(\mathbb{R}^n)$ is complete, then $\{f_k\}_{k \geqslant 1}$ is Cauchy, thus $||f_k - f_j||_2 \to 0$ as $k, j \to \infty$. Therefore, this is equivalent to the fact that for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that $||f_k - f_j||_2 < \varepsilon$ for all $k, j \geqslant N$. By Lemma 4.2, we know

$$
\begin{aligned}
||f_k - f_j||_2 &= ||\widehat{f_k - f_j}||_2 \\
&= ||\hat{f}_k - \hat{f}_j||_2
\end{aligned}
$$

which converges to) as $j, k \to \infty$. Therefore, $\{\hat{f}_k\}_{k \geqslant 1}$ is Cauchy in $L^2(\mathbb{R}^n)$. Now there exists $g \in L^2$ such that $\lim_{k \to \infty} ||\hat{f}_k - g||_2 = 0$, that is, $g = \lim_{k \to \infty} \hat{f}_k$ in the L^2-sense. □

Therefore, the definition we want of $\hat{f}$ of f in $L^2(\mathbb{R}^n)$ is $\hat{f} = g$ in the sense above. We just need to show that this is well-defined.

Lemma 4.4. The choice of g above is independent of the choice of $\{f_k\}_{k\geqslant 1}$.

Proof. Take another sequence $\tilde{f}_k$ in $L^2(\mathbb{R}^n)$ such that $\lim\limits_{k\to\infty} \tilde{f}_k = f$ in L^2-sense, and that $\lim\limits_{k\to\infty} \hat{\tilde{f}}_k = \tilde{g}$. It suffices to show that $\tilde{g} = g$. Consider a new sequence $\{h_k\}_{k\geqslant 1}$ where $h_k = f_n$ if $k = 2n-1$, and $h_k = \tilde{f}_n$ if $k = 2n$, i.e., $f_1, \tilde{f}_1, f_2, \tilde{f}_2, \ldots$. Therefore, $\lim\limits_{k\to 0} h_k = \lim\limits_{k\to\infty} f_k = f$ in the L^2-sense, so $\{\hat{h}_k\}_{k\geqslant 1}$ is Cauchy in L^2, so there exists $h \in L^2$ such that $h = \lim\limits_{k\to\infty} \hat{h}_k$ in L^2-sense. Therefore, in sense of L^2, we know

$$\tilde{g} = \lim_{k\to\infty} \hat{\tilde{f}}_k = \lim_{k\to\infty} \hat{h}_k = \lim_{k\to\infty} \hat{f}_k = g,$$

thus $\tilde{g} = g = h$. □

Theorem 4.5 (Plancherel)**.** Let $f \in L^2$, then $\hat{f} \in L^2$ and is an isometry, i.e., $||\hat{f}||_2 = ||f||_2$.

Proof. Let $f_k \in \mathcal{S}(\mathbb{R}^n)$ such that $f =_{L^2} \lim\limits_{k\to\infty} f_k$. By definition, $\hat{f} =_{L^2} \lim\limits_{k\to\infty} \hat{f}_k \in L^2$ by the completeness of L^2. Therefore, $||f_k||_2 = ||\hat{f}_k||_2$ for all $k \in \mathbb{N}$, then taking the limit on both sides, we see that

$$||f||_2 = \lim_{k\to\infty} ||f_k||_2 = \lim_{k\to\infty} ||\hat{f}_k||_2 = ||\hat{f}||_2.$$

□

Definition 4.6. A unitary operator on a Hilbert space H is a linear operator that is an isometry and "onto".

Theorem 4.7. The Fourier transform on $L^2(\mathbb{R}^n)$ is a unitary operator on $L^2(\mathbb{R}^n)$.

Proof. It remains to show that the Fourier transform is "onto". That is, for any $g \in L^2$, there exists $f \in L^2$ such that $\hat{f} = g$. Since $\mathcal{S}(\mathbb{R}^n)$ is dense in L^2, then there exists $g_k \in \mathcal{S}(\mathbb{R}^n)$ such that $g =_{L^2} \lim\limits_{k\to\infty} g_k$. Let $f =_{L^2} \lim\limits_{k\to\infty} \check{g}_k \in L^2$, so it suffices to show that $\hat{f} = g$. We know

$$\hat{f}_{L^2} = \lim_{k\to\infty} (\hat{\check{g}}_k) =_{L^2} \lim_{k\to\infty} g_k =_{L^2} = g.$$

□

Definition 4.8. For any $f \in L^2$, we define the inverse Fourier transform $\check{f} =_{L^2} \lim\limits_{k\to\infty} \check{f}_k$ if $f_k \in \mathcal{S}(\mathbb{R}^n)$ and $f =_{L^2} \lim\limits_{k\to\infty} f_k$.

Theorem 4.9 (Inverse Theorem on $L^2(\mathbb{R}^n)$)**.** For any $f \in L^2$, we have $(\hat{\check{f}}) = f$.

Proof. Let U be defined by $Uf = \hat{f}$ for any $f \in L^2$. For unitary operator U on Hilbert space H, there exists operator U^* such that $\langle Ux, y\rangle = \langle x, U^*y\rangle$ for any $x, y \in H$. We say U^* is the adjoint operator, and we will show that is just the inverse Fourier transform.

Claim 4.10. The adjoint operator U^* satisfies $U^*f = \check{f}$ for any $f \in L^2(\mathbb{R}^n)$, i.e., U^* is the inverse Fourier transform.

Subproof. For any $f, g \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned}
\langle U^*f, g\rangle &= \langle f, Ug\rangle \\
&= \langle f, \hat{g}\rangle \\
&= \int f(x)\overline{\hat{g(x)}}dx \\
&= \int f(x)\overline{\int g(\xi)e^{-2\pi i x\cdot\xi}d\xi}dx \\
&= \int f(x)\int \bar{g}(\xi)e^{-2\pi i x\cdot\xi}d\xi dx
\end{aligned}$$

$$
\begin{aligned}
&= \int \bar{g}(\xi) \int f(x) e^{-2\pi i x\cdot\xi} dx d\xi \\
&= \left\langle \check{f}, g \right\rangle.
\end{aligned}
$$

Therefore,

$$\left\langle U^* f - \check{f}, g \right\rangle = 0$$

for any $g \in \mathcal{S}(\mathbb{R}^n)$, hence $U^* f \equiv \check{f}$ almost everywhere.

In general, take any $f \in L^2(\mathbb{R}^n)$, then for every $k \geqslant 1$, there exists $f_k \in \mathcal{S}(\mathbb{R}^n)$ such that $f =_{L^2} \lim\limits_{k\to\infty} f_k$. For any $g \in L^2(\mathbb{R}^n)$, we know

$$
\begin{aligned}
\langle U^* f, g \rangle &= \langle f, \hat{g} \rangle \\
&= \langle f - f_k, \hat{g} \rangle + \langle f_k, \hat{g} \rangle \\
&= \langle f - f_k, \hat{g} \rangle + \langle U^* f_k, g \rangle.
\end{aligned}
$$

Recall that $\langle U^*(f - f_k), g \rangle = \langle f - f_k, \hat{g} \rangle$, therefore

$$|\langle U^*(f - f_k), g \rangle| = |\langle f - f_k, \hat{g} \rangle| \leqslant ||f - f_k||_2 ||\hat{g}||_2 \to 0$$

as $k \to \infty$. Therefore,

$$\lim_{k\to\infty} |\langle U^*(f - f_k), g \rangle| = 0$$

for any $g \in L^2(\mathbb{R}^n)$. Now

$$||U^*(f - f_k)||_2 = \sup_{g\in L^2} |\langle U^*(f - f_k), g \rangle|,$$

therefore

$$\lim_{k\to\infty} ||U^*(f - f_k)||_2 = 0.$$

Hence, in the L^2-sense, we know

$$
\begin{aligned}
U^* f &= \lim_{k\to\infty} U^* f_k \\
&= \lim_{k\to\infty} \check{f}_k \\
&= \check{f}.
\end{aligned}
$$

■

Claim 4.11. If U is a unitary operator on a Hilbert space H, then $U^* = U^{-1}$.

Subproof. For any $x \in H$, we have

$$
\begin{aligned}
\langle U^* U x, y \rangle &= \langle Ux, Uy \rangle \\
&= \langle x, y \rangle.
\end{aligned}
$$

Therefore, $\langle U^* U x - x, y \rangle = 0$ for any $x, y \in H$. Hence, $U^* U = I$ is the identity operator, so $U^* = U^{-1}$. ■

This shows that

$$
\begin{aligned}
\check{\hat{f}} &= U^* \hat{f} \\
&= U^*(Uf) \\
&= f.
\end{aligned}
$$

□

Let $1 \leqslant p \leqslant 2$. For any $f \in L^p$, one can show that $f = f_1 + f_2$ where $f_1 \in L^1$ and $f_2 \in L^2$. For instance, let $f_1 = f \mathbb{1}_{\{x:|f(x)|\geqslant 1\}}$ and $f_2 = f \mathbb{1}_{\{x:|f(x)|\leqslant 1\}}$. Correspondingly, we have $\hat{f} := \hat{f}_1 + \hat{f}_2$. Alternatively, we can define $\hat{f} =_{L^p} \lim\limits_{k\to\infty} f_k$ where $f_k \xrightarrow{L^2} f$ as $k \to \infty$, and $f_k \in \mathcal{S}(\mathbb{R}^n)$.

Theorem 4.12 (Hausdorff-Young)**.** Let $f \in L^p(\mathbb{R}^n)$ with $1 \leqslant p \leqslant 2$. Then $\hat{f} \in L^{p'}(\mathbb{R}^n)$ and $||\hat{f}||_{p'} \leqslant ||f||_p$ where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. When $p = 1$, then $||\hat{f}||_\infty \leqslant ||f||_1$ by the usual properties. When $p = 2$, then $||\hat{f}||_2 = ||f||_2 \leqslant ||f||_2$. By Theorem 1.7, $||\hat{f}||_{p'} \leqslant ||f||_p$. □

Theorem 4.13 (Young's Inequality)**.** We have

$$||f * g||_r \leqslant ||f||_p ||g||_q$$

for all $f \in L^p$, $g \in L^q$, where $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$.

Proof. Fix $f \in L^p$ and consider $T_f g = f * g$ as an operator. Now $||f * g||_p \leqslant ||f||_p ||g||_1$ and $||f * g||_\infty \leqslant ||f||_p ||g||_{p'}$ by Minkowski inequality and Holder inequality for $g \in L^1$ and $g \in L^{p'}$, respectively. By Theorem 1.7, $||f * g||_r \leqslant ||f||_p ||g||_q$ for $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. □

Problem 10. Show that $\int \hat{f} g = \int f \hat{g}$ for all $f, g \in L^2$.

Problem 11. Let $f \in L^1$ and $g \in L^p$ for $1 \leqslant p \leqslant 2$. Prove that $\widehat{f * g} = \hat{f}\hat{g}$ almost everywhere.

Recall that for any $f \in \mathcal{S}(\mathbb{R}^n)$, we have $||f||_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)|$ for any $\alpha, \beta \in \mathbb{N}_0^n$. Recall that we define the convergence of functions as $f_k \to f$ in $\mathcal{S}(\mathbb{R}^n)$ if $\lim_{k \to \infty} ||f_k - f||_{\alpha,\beta} = 0$ for any α, β.

Definition 4.14. Let $L : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ be a linear functional. We say L is continuous if $\lim_{k \to \infty} L(f_k) = 0$ as $f_k \to 0$ in $\mathcal{S}(\mathbb{R}^n)$. We denote $\mathcal{S}'(\mathbb{R}^n)$ to be the set of all continuous linear functionals $L : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$, which is called the space of tempered distributions.

Definition 4.15. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Define $\hat{L}(\varphi) = L(\hat{\varphi})$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Definition 4.16. A function $f : \mathbb{R}^n \to \mathbb{C}$ is called a tempered function if there exists $N \geqslant 1$ such that $\int_{\mathbb{R}^n} (1 + |x|)^{-N} |f(x)| dx$ is finite.

Remark 4.17. Let $\mathcal{F} = \{f : \mathbb{R}^n \to \mathbb{C} : f \text{ tempered}\}$, then $L^p \subseteq \mathcal{F}$ for $p \geqslant 1$.

Definition 4.18. Let $f \in \mathcal{F}$. If there exists a function $g : \mathbb{R}^n \to \mathbb{C}$ such that

$$\int_{\mathbb{R}^n} f \varphi dx = \int_{\mathbb{R}^n} g \hat{\varphi} dx$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then we may define $\hat{f} = g$ to be the Fourier transform for tempered functions.

Example 4.19. Let μ be a finite Borel measure on $\mathbb{R}^n$, then

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} d\mu.$$

Let δ be the dirac function

$$\delta(E) = \begin{cases} 1, & 0 \in E \\ 0, & o \notin E \end{cases}$$

for any $E \in \mathcal{B}(\mathbb{R}^n)$. Its Fourier transform is

$$\begin{aligned} \hat{\delta}(\xi) &= \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} d\delta \\ &= \int_{\mathbb{R}^n \setminus \{0\}} e^{-2\pi i i \xi \cdot x} d\delta + \int_{\{0\}} e^{-2\pi i \xi \cdot x} d\delta \\ &= 0 + \delta(\{0\}) \\ &= 1. \end{aligned}$$

5 Singular Integrals

Let $f \in \mathcal{S}(\mathbb{R}^n)$, then we want to understand the integral $Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy$ for some kernel function K, which should be understood as a distribution.

Definition 5.1 (Calderón-Zygmund Kernel)**.** We say $K \in \mathcal{S}'(\mathbb{R}^n \times \mathcal{R}^n)$ is a Calderón-Zygmund kernel if K is a complex-valued function on $(\mathbb{R}^n \times \mathbb{R}^n)\backslash\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$ such that it satisfies

1. size condition: $|K(x,y)| \leqslant \frac{C}{|x-y|^n}$ if $x \neq y$;
2. smoothness condition: there exists $\varepsilon_1 > 0$ such that $|K(x,y) - K(x,y')| \leqslant \frac{C_n|y-y'|^{\varepsilon_1}}{|x-y|^{n+\varepsilon_1}}$ whenever $|x-y| > 2|y-y'|$;
3. smoothness condition: there exists $\varepsilon_2 > 0$ such that $|K(x,y)-K(x',y)| \leqslant \frac{C|x-x'|_2^{\varepsilon}}{|x-y|^{n+\varepsilon_2}}$ whenever $|x-y| > 2|x-x'|$.

Definition 5.2 (Singular Integral Operator)**.** Let $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be continuous in $\mathcal{S}$, that is, $\lim_{k\to\infty} T\varphi_k(\psi) = T\varphi(\psi)$ for all $\psi \in \mathcal{S}(\mathbb{R}^n)$ as $\varphi_k \to \varphi$ in $\mathcal{S}(\mathbb{R}^n)$. We say T is a singular integral operator associated to a kernel K if

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} K(x,y)\varphi(x)\psi(y)dxdy = \int_{\mathbb{R}^n} T\varphi(x)\psi(x)dx.$$

If K is a Calderón-Zygmund kernel, then we say T is a Calderón-Zygmund singular integral operator.

Remark 5.3. We may understand the integral in the definition above as follows,

$$\langle K, \psi \otimes \varphi\rangle = \langle T\varphi, \psi\rangle \in S'(\mathbb{R}^n),$$

where $(\psi \otimes \varphi)(x,y) = \psi(x)\varphi(y) \in \mathcal{S}(\mathbb{R}^n \times \mathcal{R}^n)$.

Remark 5.4. Suppose $T \in L^p(\mathbb{R}^n)$. For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we know that $||T\varphi||_p \leqslant C_p||\varphi||_p$ for any $1 < p < \infty$.

Theorem 5.5 (Calderón-Zygmund)**.** Let T be a Calderón-Zygmund singular integral operator. If $||T\varphi||_2 \leqslant C||\varphi||_2$ for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then we may extend T to a bounded operator on $L^p(\mathbb{R}^n)$ for any $1 < p < \infty$.

To prove this theorem, we need to show that a Calderón-Zygmund operator can be extended to a bounded operator in L^2.

Definition 5.6. The Hilbert transform of a function $f \in C_c^1(\mathbb{R}^n)$ is

$$Hf(x) = \frac{1}{\pi}\lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \frac{1}{x-y}f(y)dy,$$

where $K(x,y) = \frac{1}{x-y}$ is given in terms of its principal value, and is fact a Calderón-Zygmund kernel.

Example 5.7. Let $f \in C_c^\infty$, then we may bound

$$\int_{\{y:|x-y|>\varepsilon\}} K(x,y)f(y)dy = \int_{\{y:1>|x-y|>\varepsilon\}} K(x,y)f(y)dy + \int_{\{y:|x-y|\geqslant 1\}} K(x,y)f(y)dy =: I_\varepsilon + J$$

We bound $J \leqslant \int_{\mathbb{R}} |f(y)|dy < \infty$. Notice that

$$\int_{\{y:1>|x-y|>\varepsilon\}} K(x,y)dy = \int_{\{y:1>|x-y|>\varepsilon\}} \frac{1}{y}dy = 0.$$

Therefore

$$\begin{aligned}|I_\varepsilon| &= \left|\int\limits_{\{y:1>|x-y|>\varepsilon\}} K(x,y)f(y)dy\right| \\ &= \left|\int\limits_{\{y:1>|x-y|>\varepsilon\}} K(x,y)(f(y)-f(x))dy\right| \\ &\leqslant \int\limits_{\{y:1>|x-y|>\varepsilon\}} \frac{|f(y)-f(x)|}{|y-x|}dy \\ &\leqslant \int\limits_{\{y:1>|x-y|>\varepsilon\}} ||f'||_\infty \frac{|y-x|}{|y-x|}dy \\ &\leqslant ||f'||_\infty.\end{aligned}$$

By dominant convergence theorem, we know $\lim\limits_{\varepsilon\to 0} I_\varepsilon$ exists.

Example 5.8. Consider the Riesz transform in $\mathbb{R}^n$ for $n \geqslant 2$. For any $1 \leqslant j \leqslant n$ for $x \in \mathbb{R}^n$, we define it to be

$$R_j f(x) = C_n \lim_{\varepsilon\to 0} \int\limits_{\{y\in\mathbb{R}^n:|x-y|>\varepsilon\}} \frac{x_j-y_j}{|x-y|^{n+1}} f(y)dy.$$

Set $K_j(x,y) = \frac{x_j-y_j}{|x-y|^{n+1}}$ given in terms of the principal values, then they are the Calderón-Zygmund kernels. With this, we can write

$$R_j f(x) = \int\limits_{\{y\in\mathbb{R}^n:|x-y|>\varepsilon\}} k_j(x,y)f(y)dy.$$

Example 5.9. Suppose $\Omega : \mathbb{R}^n \to \mathbb{C}$ satisfies

- $\Omega(\lambda x) = \Omega(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n$;
- $\Omega \in L^1(S^{n-1})$;
- $\int\limits_{S^{n-1}} \Omega(x)d\sigma = 0$,

then $T_n f(x) = \int\limits_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy$ given in terms of its principal values gives a Hilbert transform as well.

Example 5.10. Let us consider the Cauchy integral along Lipschitz curves. Let γ be a Lipschitz curve in the complex plane $\mathbb{C}$, i.e., γ is the graph

$$\{(x, A(x)) \in \mathbb{C}\}$$

such that A is Lipschitz with $||A'||_\infty < \infty$ for $A : \mathbb{R} \to \mathbb{R}$, then we write down the Calderón-Zygmund singular integral operator C_γ as

$$C_\gamma f(z) = \int\limits_\gamma \frac{1}{z-\xi} f(\xi)d\xi$$

where $\xi = \xi_1 + i\xi - 2$ then $ds = d\xi_1 + id\xi_2$. Therefore the shifting gives $z \to x + iA(x)$, $\xi \to y + iA(y)$, and $d\xi \to (1 + iA'(y))dy$. Using this, we can write

$$C\tilde{f}(x) = \int\limits_{\mathbb{R}} \frac{\tilde{f}(y)}{x-y+i(A(x)-A(y))}dy$$

where $\tilde{f}(y) = f(y + iA(y))(1 + iA'(y))$.

Theorem 5.11 (Calderón-Zygmund)**.** Let T be a Calderón-Zygmund singular integral operator, and suppose T is L^2-bounded, then it is L^p-bounded for any $1 < p < \infty$.

Claim 5.12. It suffices to show that T being L^2-bounded implies T is of the weak $(1,1)$ type, then by Theorem 1.9, we know T is of (p,p) type for any $1 < p < \infty$. In particular, by duality, since T is of (p,p) for any $2 < p < \infty$, then T^* is of type (q,q) for any $1 < q < 2$.

Let us start by proving that

Lemma 5.13 (Calderón-Zygmund Decomposition)**.** Let $f \in L^1(\mathbb{R}^n)$. For any given $\lambda > 0$, there exists a collection of non-overlapping cubes $\{Q_j\}_{j \geqslant 1}$ with $|Q| = \mathfrak{m}(Q)$, such that

- $\lambda < \frac{1}{|Q_j|} \int\limits_{Q_j} |f| \leqslant 2^n \lambda$;
- $|f(x)| \leqslant \lambda$ almost everywhere for $x \in \mathbb{R}^n \backslash \bigcup\limits_{j \geqslant 1} Q_j$;
- $\left| \bigcup\limits_{j \geqslant 1} Q_j \right| = \sum\limits_{j \geqslant 1} |Q_j| \leqslant \frac{||f||_1}{\lambda}$.

Proof. This is a classical proof strategy known as the stopping time argument. We divide $\mathbb{R}^n$ into a union of non-overlapping cubes Q's of the same size, such that $\frac{1}{Q} \int\limits_Q |f| \leqslant \lambda$. Now let $\mathcal{D}$ be all cubes Q that satisfy the said inequality. If Q satisfy such inequality, then we divide it into 2^n smaller cubes Q' of the same size, with side length $\ell(Q') = \frac{1}{\ell}(Q)$. If Q' is such that $\frac{1}{|Q'|} \int\limits_{Q'} |f| > \lambda$, then it satisfies

$$\lambda < \frac{1}{|Q'|} \int\limits_{Q'} |f| \leqslant \frac{2^n}{|Q|} \int\limits_{Q} |f|,$$

so we include Q' into the family; if Q' is such that $\frac{1}{|Q'|} \int\limits_{Q'} |f| \leqslant \lambda$, then we divide Q' into smaller cubes in the same fashion, and we repeat this procedure. Eventually, we obtain a sequence $\{Q_j\}_{j \in \mathbb{N}}$ that satisfies the first condition.

For any $x \notin \bigcup\limits_{j \geqslant 1} Q_j$, there exists a subsequence $\{Q_k\}_{k \geqslant 1}$ such that $\lim\limits_{k \to \infty} |Q_k| = 0$, $x \in Q_k$ for all $k \in \mathbb{N}$, and that $\frac{1}{|Q_k|} \int\limits_{Q_k} |f| \leqslant \lambda$. By Lebesgue differentiation theorem,

$$\lambda \geqslant \lim_{k \to \infty} \frac{1}{|Q_k|} \int\limits_{Q_k} |f| d\mathfrak{m} = f(x)$$

for almost all $x \notin \bigcup\limits_{j \geqslant 1} Q_j$, hence $|f(x)| \leqslant \lambda$ for almost all $x \in \mathbb{R}^n \backslash \bigcup\limits_{j \geqslant 1} Q_j$, hence we have the second condition.

To verify the last condition, we note that

$$\begin{aligned} \sum_j |Q_j| &\leqslant \sum_j \frac{1}{\lambda} \int\limits_{Q_j} |f| \\ &= \frac{||f||_1}{\lambda}. \end{aligned}$$

□

Lemma 5.14. Let $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$, then $f = g + b$ such that

- $g \in L^2(\mathbb{R}^n)$ and $||g||_2^2 \leqslant C\lambda ||f||_1$;
- $b(x) = \sum\limits_{j \geqslant 1} b_j(x)$, where each b_j is supported in a cube Q_j, such that Q_j's are non-overlapping;

- $\sum\limits_{j\geqslant 1}|Q_j| \leqslant \frac{||f||_1}{\lambda}$, $\int\limits_{Q_j} b_j = 0$, and $\sum\limits_{j\geqslant 1}||b_j||_1 \leqslant 2||f||_1$.

Proof. Let $\{Q_j\}_{j\geqslant 1}$ be the collection of cubes in Lemma 5.13. For any $j \in \mathbb{N}$, we know

$$b_j(x) = \left(f(x) - \frac{1}{|Q_j|}\int\limits_{Q_j} f\right)\chi_{Q_j}(x),$$

then $\int b_j = 0$. Define $b(x) = \sum\limits_{j\geqslant 1} b_j(x)$ and $g(x) = f(x) - b(x)$. The only non-trivial thing we need to verify is the first condition. Note that

$$g(x) = f(x)\chi_{\left(\bigcup\limits_{j\geqslant 1} Q_j\right)^c}(x) + \sum_{j\geqslant 1}\left(\frac{1}{|Q_j|}\int\limits_{Q_j} f\right)\chi_{Q_j}(x),$$

therefore

$$\begin{aligned} ||g||_\infty &\leqslant ||f||_{L^\infty\left(\mathbb{R}^n\setminus \bigcup\limits_{j\geqslant 1} Q_j\right)} + \sup_{j\geqslant 1}\frac{1}{|Q_j|}\int\limits_{Q_j}|f| \\ &< \lambda + 2^n\lambda \\ &= C_n\lambda \end{aligned}$$

for some constant C_n depending on n. On the other hand, we have

$$\begin{aligned} ||g||_1 &= ||f-b||_1 \\ &\leqslant ||f||_1 + ||b||_1 \\ &\leqslant ||f||_1 + \sum_{j\geqslant 1}||b_j||_{L^1(Q_j)} \\ &\leqslant ||f||_1 + 2\sum_{j\geqslant 1}\int\limits_{Q_j}|f| \\ &\leqslant 3||f||_1. \end{aligned}$$

By Hölder inequality (or interpolation theorem), we have

$$\begin{aligned} ||g||_2 &\leqslant (3||f||_1)^{\frac{1}{2}}(C_n\lambda)^{\frac{1}{2}} \\ &\leqslant \tilde{C}_n\lambda^{\frac{1}{2}}||f||_1^{\frac{1}{2}}, \end{aligned}$$

as desired. □

Proof of Theorem 5.11. Recall from Claim 5.12 that it suffices to show T satisfies the weak $(1,1)$ estimate, that is, for any $\lambda > 0$,

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \leqslant \frac{C}{\lambda}||f||_1$$

for any $f \in L^1(\mathbb{R}^n)$. By Lemma 5.14, let us write $f = g + b$ where $g \in L^2$ and $b \in L^1$, then

$$\begin{aligned} |\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| &\leqslant \left|\left\{x \in \mathbb{R}^n : |Tg(x)| > \frac{\lambda}{2}\right\}\right| + \left|\left\{x \in \mathbb{R}^n : |Tb(x)| > \frac{\lambda}{2}\right\}\right| \\ &=: I_g + I_b. \end{aligned}$$

We can bound

$$I_g \leqslant \frac{C}{\lambda^2}||g||_2^2$$

$$\begin{aligned}
&\leqslant \frac{C'}{\lambda^2}\lambda||f||_1\\
&= \frac{C'}{\lambda}||f||_1
\end{aligned}$$

since T is of strong $(2,2)$ type. It remains to show that $I_b \leqslant \frac{C||f||_1}{\lambda}$. Let us write

$$I_b = \left|\left\{x \in \mathbb{R}^n \setminus \bigcup_{j\geqslant 1} 5Q_j : |Tb(x)| > \frac{\lambda}{2}\right\}\right| + \left|\left\{x \in \bigcup_{j\geqslant 1} 5Q_j : |Tb(x)| > \frac{\lambda}{2}\right\}\right|$$

where $5Q_j$ is the dilation of Q_j by 5 times, then

$$\begin{aligned}
\left|\left\{x \in \bigcup_{j\geqslant 1} 5Q_j : |Tb(x)| > \frac{\lambda}{2}\right\}\right| &\leqslant \left|\bigcup_{j\geqslant 1} 5Q_j\right|\\
&\leqslant \sum_{j\geqslant 1} |5Q_j|\\
&\leqslant \frac{C}{\lambda}||f||_1.
\end{aligned}$$

It then suffices to bound the first term. Since the support of b_j is contained in Q_j, then whenever $x \notin 5Q_j$ with $y \in Q_j$, we may have $K(x,y)$ treated by the usual complex-valued function dominated by $\frac{1}{x-y}$. Let y_j be the center of Q_j, then since $\int b_j = 0$, we know that $\int K(x,y_j)b_j = 0$ as well. Therefore, by Chebyshev inequality,

$$\begin{aligned}
\left|\left\{x \in \mathbb{R}^n \setminus \bigcup_{j\geqslant 1} 5Q_j : |Tb(x)| > \frac{\lambda}{2}\right\}\right| &\leqslant \frac{2}{\lambda} \int\limits_{\left(\bigcup_{j\geqslant 1} 5Q_j\right)^c} |Tb(x)|dx\\
&\leqslant \frac{2}{\lambda}\sum_{j\geqslant 1} \int\limits_{\left(\bigcup_{j\geqslant 1} 5Q_j\right)^c} |Tb_j(x)|dx\\
&\leqslant \frac{2}{\lambda}\sum_{j\geqslant 1} \int\limits_{(5Q_j)^c} \left|\int K(x,y)b_j(y)dy\right|dx\\
&= \frac{2}{\lambda}\sum_{j\geqslant 1} \int\limits_{(5Q_j)^c} \left|\int K(x,y)b_j(y)dy - \int K(x,y_j)b_j(y)dy\right|dx\\
&= \frac{2}{\lambda}\sum_{j\geqslant 1} \int\limits_{(5Q_j)^c} \left|\int (K(x,y) - K(x,y_j))b_j(y)dy\right|dx.
\end{aligned}$$

Recall that $|K(x,y) - K(x,y_j)| \leqslant C\frac{|y-y_j|^\varepsilon}{|x-y|^{n+\varepsilon}}$ for some constant C whenever $|x-y| > 2|y-y_j|$. Since x is outside of $5Q_j$ while y and y_j are inside Q_j, then x and y satisfy the bound indeed. By Fubini theorem,

$$\begin{aligned}
\left|\left\{x \in \mathbb{R}^n \setminus \bigcup_{j\geqslant 1} 5Q_j : |Tb(x)| > \frac{\lambda}{2}\right\}\right| &\leqslant \frac{C}{\lambda}\sum_{j\geqslant 1} \int\limits_{(5Q_j)^c}\int\limits_{Q_j} \frac{|y-y_j|^\varepsilon}{|x-y|^{n+\varepsilon}}|b_j(y)|dydx\\
&\leqslant \frac{C}{\lambda}\sum_{j\geqslant 1} \int\limits_{Q_j} |b_j(y)| \int\limits_{\{x\in\mathbb{R}^n : |x-y|\geqslant 2|y-y_j|\}} \frac{|y-y_j|^\varepsilon}{|x-y|^{n+\varepsilon}}dxdy
\end{aligned}$$

for some other constant C. Let $I = \int\limits_{\{x\in\mathbb{R}^n : |x-y|\geqslant 2|y-y_j|\}} \frac{|y-y_j|^\varepsilon}{|x-y|^{n+\varepsilon}}dx$, then

$$I = \int\limits_{\{x\in\mathbb{R}^n : |x-y|\geqslant 2|y-y_j|\}} \frac{|y-y_j|^\varepsilon}{|x-y|^{n+\varepsilon}}dx$$

$$
\begin{aligned}
&= |y-y_j|^{\varepsilon} \int\limits_{|x| \geqslant 2|y-y_j|} \frac{1}{|x|^{n+\varepsilon}} dx \\
&\leqslant C_{\varepsilon,n}
\end{aligned}
$$

using polar coordinates, where $C_{\varepsilon,n}$ is independent of x, y, and y_j. Thus,

$$
\begin{aligned}
\left|\left\{x \in \mathbb{R}^n \backslash \bigcup_{j \geqslant 1} 5Q_j : |Tb(x)| > \frac{\lambda}{2}\right\}\right| &\leqslant \frac{C}{\lambda} \sum_{j \geqslant 1} \int\limits_{Q_j} |b_j(y)| \int\limits_{\{x \in \mathbb{R}^n : |x-y| \geqslant 2|y-y_j|\}} \frac{|y-y_j|^{\varepsilon}}{|x-y|^{n+\varepsilon}} dx dy \\
&\leqslant \frac{C}{\lambda} \sum_{j \geqslant 1} \int\limits_{Q_j} |b_j(y)| C_{\varepsilon,n} dy \\
&=: \frac{\tilde{C}}{\lambda} \sum_{j \geqslant 1} \int\limits_{Q_j} |b_j(y)| dy \\
&\leqslant \frac{\tilde{C}}{\lambda} ||f||_1.
\end{aligned}
$$

□

Problem 12. Show that Theorem 5.11 still holds if the second condition of the Calderón-Zygmund kernel K is replaced by the Hörmander condition

$$
\int\limits_{|x-y| > 2|y-y'|} |K(x,y) - K(x,y')| dx \leqslant C.
$$

6 Hilbert Transforms

Recall that the Hilbert transform is defined by

$$Hf(x) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int\limits_{|y|>\varepsilon}\frac{f(x-y)}{y}dy = K * f(x),$$

which is well-defined for any integrable function. One may replace the kernel $K(x)$ to be the principal values of $\frac{1}{x}$.

Definition 6.1. Let $x, t \in \mathbb{R}$ for $t > 0$. The Poisson kernel is defined by

$$P_t(x) = \frac{1}{\pi}\frac{t}{t^2 + x^2}.$$

Let us define $u(x,t) = P_t * f(x) = \int_{\mathbb{R}} P_t(x-y)f(y)dy$, then u is a solution to

$$\begin{cases}\Delta u(x,t) = 0\ \forall (x,t) \in \mathbb{R}^2_+ \\ u(x,0) = \lim\limits_{t\to 0^+} u(x,t) = f(x) \in L^p\end{cases}$$

for $1 \leqslant p < \infty$ for almost all x, on the upper half plane $\mathbb{R}^2_+ = \{(x,t) \in \mathbb{R}^2 : t > 0\}$. Instead of the real space, let us consider it as a complex plane for $z \in \mathbb{C}$ such that $z = \mathrm{Re}(z) + i\,\mathrm{im}(z)$ where $\mathrm{im}(z) > 0$. Therefore, z corresponds to a pair $(\mathrm{Re}(z), \mathrm{im}(z)) \in \mathbb{R}^2_+$. For simplicity, let $f \in L^1$ (while the following statements still hold for general L^p functions). Define

$$F(z) = 2\int_0^\infty \hat{f}(\xi)e^{2\pi i\xi z}d\xi,$$

then this is well-defined since $\hat{f}$ is bounded. If we write

$$e^{2\pi i\xi z} = e^{2\pi i\xi\,\mathrm{Re}(z)} \cdot e^{-2\pi\xi\,\mathrm{Im}(z)},$$

we see that the function decays fast enough, thus $F(z)$ is analytic in $\mathbb{R}^2_+$. Let us assume that f is real-valued by considering its real part and imaginary part, then we may write

$$F(z) = \left(\int_0^\infty \hat{f}(\xi)e^{2\pi i\xi z}d\xi + \int_{-\infty}^0 \hat{f}(\xi)e^{2\pi i\xi\bar{z}}d\xi\right) + \left(\int_0^\infty \hat{f}(\xi)e^{2\pi i\xi z}d\xi - \int_{-\infty}^0 \hat{f}(\xi)e^{2\pi i\xi\bar{z}}d\xi\right)$$

to give us the real and imaginary part of F. Since f is of real-valued, then the first term is a real-valued function; note that the second term is complex-valued, so it is i multiplied by some real-valued function. Therefore, let us write $F(z) = u+iv$. In fact, both u and v are related to the Hilbert transform. To see this, note that $\Delta u = \Delta v = 0$ if $(x,t) \in \mathbb{R}^2_+$, so the boundary values are given by

$$\begin{aligned}\lim_{t\to 0^+} u(x,t) &= \int_0^\infty \hat{f}(\xi)e^{2\pi ix\cdot\xi}d\xi + \int_{-\infty}^0 \hat{f}(\xi)e^{2\pi ix\xi}d\xi \\ &= \int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi ix\cdot\xi}d\xi \\ &= f(x).\end{aligned}$$

by dominant convergence theorem and the inversion formula. Therefore, u should satisfy

$$\begin{cases}\Delta u(x,t) = 0, x, t \in \mathbb{R}^2_+ \\ u(x,0) = f(x)\end{cases}$$

which gives $u(x,t) = P_t * f(x)$. Also, we have

$$v(z) = \int_{-\infty}^{\infty} -i\,\mathrm{sgn}(\xi) e^{-2\pi\,\mathrm{Im}(z)|\xi|} \hat{f}(\xi) e^{2\pi i\,\mathrm{Re}(z)\xi} d\xi$$

where

$$\mathrm{sgn}(\xi) = \begin{cases} 1, & \xi \geqslant 0 \\ -1, & \xi < 0 \end{cases}$$

is the signal function. Set $z = x + it$, then we represent

$$v(x+it) = \int_{-\infty}^{\infty} -i\,\mathrm{sgn}(\xi) e^{-2\pi t|\xi|} \hat{f}(\xi) e^{2\pi i x\xi} d\xi.$$

Let $Q_t(x) = \frac{1}{\pi}\frac{x}{t^2+x^2}$ and recall $P_t(x) = \frac{1}{\pi}\frac{t}{t^2+x^2}$, then

$$P_t + iQ_t = \frac{1}{\pi}\frac{t+ix}{t+x^2} = \frac{1}{\pi}\cdot\frac{i}{z}$$

is analytic on $\mathbb{R}^2_+$ where $z = x + it$.

Claim 6.2. $v(x,t) = v(x+it) = Q_t * f(x)$ for integrable real-valued function f.

Proof. It suffices to show that

$$F(z) = P_t * f(x) + iQ_t * f(x)$$

where $z = x + it \in \mathbb{R}^2_+$. To show this, we have

$$\begin{aligned} F(z) &= 2\int_0^{\infty} \hat{f}(\xi) e^{2\pi i \xi z} d\xi \\ &= 2\int_0^{\infty} \left(\int_{\mathbb{R}} f(y) e^{-2\pi i \xi y} dy \right) e^{2\pi i \xi z} d\xi \\ &= 2\int_{\mathbb{R}} f(y) \left(\int_0^{\infty} e^{-2\pi i \xi (z-y)} d\xi \right) dy \\ &= \int_{\mathbb{R}} f(y) \frac{i}{\pi(x-y+it)} dy \\ &= (P_t + iQ_t) * f(x). \end{aligned}$$

□

Theorem 6.3. Let $f \in \mathcal{S}(\mathbb{R})$ or $C_c^{\infty}(\mathbb{R})$, then

$$\lim_{t\to 0} Q_t * f(x) = \frac{1}{\pi}\lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} \frac{f(x-y)}{y} dy = Hf(x)$$

almost everywhere.

Remark 6.4. This is true for $f \in L^p(\mathbb{R}^1)$ for $1 \leqslant p < \infty$.

Proof. Let $\psi_t(x) = \frac{1}{\pi x}\chi_{\{|x|>t\}}$, then the Hilbert transform $Hf(x) = \lim_{\varepsilon\to 0}\psi_\varepsilon * f(x)$. By dominant convergence theorem,

$$\begin{aligned}\lim_{\varepsilon\to 0}\left((Q_\varepsilon - \psi_\varepsilon) * f\right) &= \lim_{\varepsilon\to 0}(Q_\varepsilon - \psi_\varepsilon) * f \\ &= 0\end{aligned}$$

□

Remark 6.5. Note that $\sup_{\varepsilon>0}|(Q_\varepsilon - \psi_\varepsilon) * f| \leqslant CMf(x)$ since $|(Q_\varepsilon - \psi_\varepsilon)(y)| \leqslant \frac{1}{\varepsilon}\frac{1}{(1+\left(\frac{y}{\varepsilon}\right)^2}$.

Let us now verify the boundedness of Hilbert transform on L^2 space.

Theorem 6.6. $\widehat{Hf}(\xi) = -i\operatorname{sgn}(\xi)\hat{f}(\xi)$ for $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. We know

$$\begin{aligned}\widehat{Hf}(\xi) &= \int Hf(x)e^{-2\pi i x\xi}dx \\ &= \int \lim_{t\to 0} Q_t * f(x)e^{-2\pi i x\xi}dx \\ &= \lim_{t\to 0}\int Q_t * f(X)e^{-2\pi i x\xi}dx \\ &= \lim_{t\to 0}\widehat{Q_t * f}(\xi).\end{aligned}$$

By the inversion formula for Fourier transform, we know

$$\begin{aligned}v(x,t) &= Q_t * f(x) \\ &= \int_{-\infty}^{\infty} -i\operatorname{sgn}(\xi)e^{-2\pi t|\xi|}\hat{f}(\xi)e^{2\pi i x\xi}d\xi,\end{aligned}$$

therefore

$$\widehat{Q_t * f} = -i\operatorname{sgn}(\xi)e^{-2\pi t|\xi|}.$$

Hence,

$$\begin{aligned}\widehat{Hf}(\xi) &= \lim_{t\to 0}\widehat{Q_t * f}(\xi) \\ &= \lim_{t\to 0} -i\operatorname{sgn}(\xi)e^{-2\pi t|\xi|}\hat{f}(\xi) \\ &= -i\operatorname{sgn}(\xi)\hat{f}(\xi).\end{aligned}$$

□

Corollary 6.7. $\widehat{Hf}(\xi) = -i\operatorname{sgn}(\xi)\hat{f}(\xi)$ for $f \in L^2$.

Proof. For $f_k \in \mathcal{S}$ such that $f_k \xrightarrow{L^2} f$, we know

$$\begin{aligned}\widehat{Hf}(\xi) &=_{L^2} \lim_{k\to\infty}\widehat{Hf_k}(\xi) \\ &= \lim_{k\to\infty}(-i\operatorname{sgn}(\xi))\hat{f}_k(\xi) \\ &=_{L^2} (-i\operatorname{sgn}(\xi))\hat{f}(\xi),\end{aligned}$$

therefore

$$\widehat{Hf}(\xi) = (-i\operatorname{sgn}(\xi))\hat{f}(\xi)$$

almost everywhere. □

Corollary 6.8. $||Hf||_2 = ||f||_2$ for all $f \in L^2$.

Proof. We have

$$\begin{aligned}||Hf||_2 &= ||\widehat{Hf}||_2 \\ &= ||-i\,\mathrm{sgn}(\xi)\hat{f}(\xi)||_2 \\ &= ||\hat{f}||_2 \\ &= ||f||_2.\end{aligned}$$

□

Corollary 6.9. For any $f \in L^p$ with $1 < p < \infty$, we have $||Hf||_p \leqslant C_p||f||_p$. Therefore, the Hilbert transform is of type weak $(1, 1)$.

Theorem 6.10. Let $H^*f(x) = \sup_{\varepsilon>0}\left|\frac{1}{\pi}\int_{|y|>\varepsilon} f(x-y)\frac{1}{y}dy\right|$, then $||H^*f||_p \leqslant C_p||f||_p$ for any $f \in L^p$ where $1 < p < \infty$.

Proof.

Lemma 6.11. $H^*f(x) \leqslant M(Hf)(x) + CMf(x)$ almost everywhere for $x \in \mathbb{R}$.

Subproof. Let $\psi_\varepsilon(x) = \frac{1}{\pi x}\chi_{\{|x|>\varepsilon\}}$, then

$$\frac{1}{\pi}\int_{|y|>\varepsilon} f(x-y)\frac{1}{y}dy = \psi_\varepsilon * f(x).$$

Let $\varphi \in \mathcal{S}(\mathbb{R})$ be a non-negative even and decreasing function on $(0, \infty)$, supported on $[-\frac{1}{2}, \frac{1}{2}]$ and $\int \varphi = 1$. Now set $\varphi_\varepsilon(x) = \varepsilon^{-1}\varphi(\frac{x}{\varepsilon})$, then

$$\psi_\varepsilon * f(x) = [\psi_\varepsilon * f(x) - \varphi_\varepsilon * (Hf)(x)] + \varphi_\varepsilon * (Hf)(x),$$

so

$$|\varphi_\varepsilon * (Hf)(x)| \leqslant M(Hf)(x),$$

since $|\varphi_\varepsilon(x)| = \varepsilon^{-1}|\varphi\left(\frac{x}{\varepsilon}\right)| \leqslant \varepsilon^{-1}\frac{C_N}{(1+|\frac{x}{\varepsilon}|)^N}$ for any N. In particular, if $N = 2$, then we have

$$|\varphi_\varepsilon(x)| \leqslant \frac{C\varepsilon^{-1}}{(1+\frac{|x|}{\varepsilon})^2} = \frac{C\varepsilon}{(\varepsilon+|x|)^2} < \frac{C\varepsilon}{\varepsilon^2+|x|^2}.$$

Now Lemma 6.11 follows from the following two claims. ■

Claim 6.12. We have

$$\int_{\mathbb{R}} \frac{\varepsilon}{|\varepsilon|^2+|y|^2}|f(x-y)|dy \leqslant CMf(x)$$

almost everywhere on x. Here C is independent of ε and x.

Subproof. We should start by decomposing

$$\mathbb{R} = (-\varepsilon, \varepsilon) \cup \left(\bigcup_{j\geqslant 1}(2^j\varepsilon, 2^{j+1}\varepsilon) \cup (-2^{j+1}\varepsilon, -2^j\varepsilon)\right).$$

And the claim easily follows. ■

Claim 6.13. We have

$$|\psi_\varepsilon * f(x) - \varphi_\varepsilon * (Hf)(x)| \leqslant CMf(x).$$

Subproof. Note that

$$\begin{aligned} |\psi_\varepsilon * f(x) - \varphi_\varepsilon * (Hf)(x)| &= \left|\int \left[\psi_\varepsilon(y) - \frac{1}{\pi}\,\mathrm{p.\,v.}\int \varphi_\varepsilon(z)\frac{1}{y-z}dz\right] f(x-y)dy\right| \\ &= \int \left|\psi_\varepsilon(y) - \frac{1}{\pi}\,\mathrm{p.\,v.}\int \varphi_\varepsilon(z)\frac{1}{y-z}dz\right| \cdot |f(x-y)|dy, \end{aligned}$$

but

$$\left|\psi_\varepsilon(y) - \frac{1}{\pi}\,\mathrm{p.\,v.}\int \varphi_\varepsilon(z)\frac{1}{y-z}\right| \leqslant \frac{C\varepsilon}{\varepsilon^2+y^2}, \tag{6.14}$$

so by Claim 6.12, we may prove the claim. ■

□

Problem 13. Prove Claim 6.12.

Problem 14. Prove Equation (6.14).

Conjecture 6.15. Let $f \in L^2$, is

$$\lim_{R\to\infty} \int\limits_{\{\xi\in\mathbb{R}^2, |\xi|<R\}} \hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi = f(x)$$

almost everywhere? Note that one can define

$$C^* f(x) = \sup_{R>0}\left|\int\limits_{\{\xi\in\mathbb{R}^2:|\xi|<R\}} \hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi\right|,$$

so we should ask, is C^* of type weak $(2,2)$?

7 Riesz Transforms

Definition 7.1. Let us define $R_jf(x) = C_n \lim\limits_{\varepsilon\to 0} \int\limits_{\{y:|x-y|>\varepsilon\}} \frac{x_j-y_j}{|x-y|^{n+1}} f(y)dy$, where x_j and y_j are given by the jth coordinate of $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Now set $K_j(x,y) = \text{p. v.}\, \frac{x_j-y_j}{|x-y|^{n+1}}$ to be the principal values as a Calderón-Zygmund kernel, and let $\tilde{K}_j(x) = \text{p. v.}\, \frac{x_j}{|x|^{n+1}}$, then $R_jf = \tilde{K}_j * f$.

Remark 7.2. Recall that $Hf(x) = K * f(x)$ where $K = \frac{1}{\pi}\,\text{p. v.}\,\frac{1}{x}$, then $\widehat{Hf} = \hat{K}\hat{f}$, where $\hat{K}(\xi) = \widehat{\frac{1}{\pi}\,\text{p. v.}\,\frac{1}{x}}(\xi) = -i\,\text{sgn}(\xi)$, therefore

$$||\hat{H}f||_2 = ||\hat{K}\hat{f}||_2 \leqslant ||\hat{K}||_\infty ||\hat{f}||_2.$$

Definition 7.3. We define $T_\Omega f(x) = \text{p. v.} \int\limits_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy$, where Ω is a function that satisfies

1. $\Omega(\lambda x) = \Omega(x)$ for all $\lambda > 0$,
2. $\Omega \in L^1(S^{n-1})$,
3. $\int\limits_{S^{n-1}} \Omega d\sigma = 0$. (This allows the limit in principal values to exist.)

Problem 15. Let $\Omega \in L^1(S^{n-1})$ and $\Omega(\lambda x) = \Omega(x)$ for all $\lambda > 0$. Suppose

$$\lim_{\varepsilon\to 0} \int\limits_{|x-y|>\varepsilon} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy$$

exists in $\mathbb{R}$ almost everywhere for all $f \in C_c^\infty(\mathbb{R}^n)$. Show that $\int\limits_{S^{n-1}} \Omega d\sigma = 0$.

Example 7.4. Set $\Omega(x) = \frac{x_j}{x}$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

Theorem 7.5. We have

$$\hat{K}_\Omega(\xi) = \int\limits_{S^{n-1}} \Omega(y')\left(\log\frac{1}{|y'\cdot\xi'|} - \frac{i\pi}{2}\,\text{sgn}(y'\cdot\xi')\right)d\sigma(y')$$

where $\xi' = \frac{\xi}{|\xi|} \in S^{n-1}$, in the sense of distributions.

Proof. For any $\varepsilon > 0$, let $K_\varepsilon(x) = \frac{\Omega(x)}{|x|^n} \cdot \chi_{\{\varepsilon<|x|<\frac{1}{\varepsilon}\}} \in L^1$, then define $\hat{K}_\Omega(\xi) = \lim\limits_{\varepsilon\to 0} \hat{K}_\varepsilon(\xi)$. Then

$$\begin{aligned}
\hat{K}_\varepsilon(\xi) &= \int\limits_{\mathbb{R}^n} K_\varepsilon(x)e^{-2\pi i x\cdot\xi}dx \\
&= \int\limits_{\{\varepsilon<|x|<\frac{1}{\varepsilon}\}} \frac{\Omega(x')}{|x|^n} e^{-2\pi i x\cdot\xi}dx \\
&= \int\limits_{S^{n-1}} \int\limits_{\varepsilon}^{\frac{1}{\varepsilon}} \frac{\Omega(y')}{r^n} e^{-2\pi i r|\xi|(y'\cdot\xi')} r^{n-1} dr d\sigma(y') \\
&= \int\limits_{S^{n-1}} \Omega(y')\left(\int\limits_{\varepsilon}^{\frac{1}{\varepsilon}} e^{-2\pi i r|\xi|(y'\cdot\xi')}\frac{dr}{r}\right)d\sigma(y') \\
&= \int\limits_{S^{n-1}} \Omega(y')\left(\int\limits_{\varepsilon}^{1} e^{-2\pi i r|\xi|(y'\cdot\xi')}\frac{dr}{r} + \int\limits_{1}^{\frac{1}{\varepsilon}} e^{-2\pi i r|\xi|(y'\cdot\xi')}\frac{dr}{r}\right)d\sigma(y')
\end{aligned}$$

$$
\begin{aligned}
&= \int\limits_{S^{n-1}} \Omega(y') \left(\int\limits_{\varepsilon}^{1} \left(e^{-2\pi i r|\xi|(y'\cdot\xi')} - 1 \right) \frac{dr}{r} \right) d\sigma(y') + \int\limits_{S^{n-1}} \Omega(y') \left(\int\limits_{1}^{\frac{1}{\varepsilon}} e^{-2\pi i r|\xi|(y'\cdot\xi')} \frac{dr}{r} \right) d\sigma(y') \\
&= \int\limits_{S^{n-1}} \Omega(y') \left(\int\limits_{\varepsilon}^{1} \left(\cos(2\pi r|\xi|y'\xi') - 1 \right) \frac{dr}{r} \right) d\sigma(y') + \int\limits_{S^{n-1}} \Omega(y') \left(\int\limits_{1}^{\frac{1}{\varepsilon}} \cos(2\pi r|\xi|y'\xi') \frac{dr}{r} \right) d\sigma(y') \\
&- i \int\limits_{S^{n-1}} \Omega(y') \int\limits_{\varepsilon}^{\frac{1}{\varepsilon}} \sin(2\pi r|\xi|y'\xi') \frac{dr}{r} d\sigma(y') \\
&=: I_1 + iI_2.
\end{aligned}
$$

Set $S = 2\pi r|\xi| \cdot M' \cdot |y'\xi'|$, then

$$
I_2 = \int\limits_{S^{n-1}} \left(\int\limits_{2\pi|\xi||y'\cdot\xi'|\varepsilon}^{2\pi|\xi||y'\cdot\xi'|\frac{1}{\varepsilon}} (\sin(S)) \operatorname{sgn}(y'\cdot\xi') \frac{dS}{S} \right) d\sigma(y'),
$$

then for $\varepsilon \to 0$, we have

$$
\begin{aligned}
I_2 &\to \int\limits_{S^{n-1}} \Omega(y') \operatorname{sgn}(y'\cdot\xi')) \int\limits_{0}^{\infty} \frac{\sin(S)}{S} dS d\sigma(y') \\
&= \frac{\pi}{2} \int\limits_{S^{n-1}} \Omega(y') \operatorname{sgn}(y'\cdot\xi') d\sigma(y').
\end{aligned}
$$

Similarly, we have

$$
I_1 = \int\limits_{S^{n-1}} \Omega(y') \int\limits_{2\pi|\xi||y'\cdot\xi'|\cdot\varepsilon}^{2\pi|\xi||y'\cdot\xi'|} \frac{\cos(S)-1}{S} dS d\sigma(y') + \int\limits_{S^{n-1}} \Omega(y') \int\limits_{2\pi|\xi|\cdot|y'\cdot\xi'|}^{2\pi|\xi|\cdot|y'\cdot\xi'|\frac{1}{\varepsilon}} \frac{\cos(S)}{S} dS d\sigma(y').
$$

For $\varepsilon \to 0$, this time

$$
I_1 \to \int\limits_{S^{n-1}} \Omega(y') \int\limits_{0}^{2\pi|\xi|\cdot|y'\cdot\xi'|} \frac{\cos(S)-1}{S} dS d\sigma(y') + \int\limits_{S^{n-1}} \Omega(y') \int\limits_{2\pi|\xi|\cdot|y'\cdot\xi'|}^{\infty} \frac{\cos(S)}{S} dS d\sigma(y'),
$$

therefore

$$
\begin{aligned}
\lim_{\varepsilon\to 0} I_1 &= \int\limits_{S^{n-1}} \Omega(y) \int\limits_{0}^{2\pi|\xi|} \frac{\cos(S)-1}{S} dS + \int\limits_{2\pi|\xi|}^{\infty} \frac{\cos(S)}{S} dS + \int\limits_{2\pi|\xi|\cdot|\xi'\cdot y'|}^{2\pi|\xi|} \frac{dS}{S} d\sigma \\
&= \int\limits_{S^{n-1}} \Omega(y') \int\limits_{2\pi|\xi|\cdot|y'\cdot\xi'|}^{2\pi|\xi|} \frac{dS}{S} d\sigma(y') \\
&= \int\limits_{S^{n-1}} \Omega(y') \log \frac{1}{|y'\cdot\xi'|} d\sigma(y').
\end{aligned}
$$

□

Remark 7.6. If Ω is odd, then $\hat{K}_\Omega(\xi) = -\int\limits_{S^{n-1}} \Omega(y') \frac{i\pi}{2} \operatorname{sgn}(y'\cdot\xi') d\sigma(y')$ which is bounded above by $||\Omega||_{L^1(S^{n-1})}$.

Corollary 7.7. Since $\widehat{\mathrm{p.v.}(\frac{x_j}{|x|^{n+1}})}$ is bounded, then k_j is bounded on L^2.

Remark 7.8. If Ω is even, then $\hat{K}_\Omega(\xi) = \int\limits_{S^{n-1}} \Omega(y') \log \frac{1}{|y' \cdot \xi'|} d\sigma(y')$.

Definition 7.9. Let us define $\Omega_e(y') = \frac{1}{2}(\Omega(y') + \Omega(-y')$, and $\Omega_o(y') = \frac{1}{2}(\Omega(y') - \Omega(-y'))$, then $\Omega = \Omega_e + \Omega_0$, Moreover, define $L \log L(S^{n-1}) = \{\Omega : \int\limits_{S^{n-1}} |\Omega(y')| \log^+ |\Omega(y')| d\sigma(y') < \infty\}$, where $\log^+(t) = \max\{0, \log(t)\}$.

Proposition 7.10. $L \log L(S^{n-1}) \supseteq L^q(S^{n-1})$ for all $q > 1$.

Theorem 7.11. Suppose Ω satisfies property 1 and 3 in Definition 7.3, and suppose $\Omega_0 \in L^1(S^{n-1})$ and $\Omega_e \in L \log L(S^{n-1}$, then $\widehat{\mathrm{p.v.}\, \frac{\Omega(x)}{|x|^n}}$ is a bounded function.

This can be done by setting $2^{-k-1} \leqslant |y'\xi'| \leqslant 2^{-k}$ for all $k > 0$.

Remark 7.12.

1. Note that $K(x-y) = \mathrm{p.v.}\, \frac{\Omega(x-y)}{(x-y)^n}$ is not a standard Calderón-Zygmund kernel, unless Ω is smooth enough.
2. If $\Omega \in L \log L(S^{n-1})$, then T_Ω is of type weak $(1,1)$.
3. Here is an open problem: let $\Omega \in L^1(S^{n-1})$ and suppose Ω satisfies property 1 and 3 in Definition 7.3, and is an odd function. Does $T_\Omega f(x) = \mathrm{p.v.} \int \frac{\Omega(x-y)}{|x-y|^n} f(y) dy$ define a weak $(1,1)$ type operator?

Problem 16. Show that
$$L^q(S^{n-1}) \subseteq L \log L(S^{n-1}) \subseteq L^1(S^{n-1})$$
for any $1 < q < \infty$.

8 Method of Rotation

Recall that in Definition 7.3 we defined $T_\Omega f(x) = \text{p. v.} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy$, where $f \in C_c^\infty(\mathbb{R}^n)$, where Ω satisfies

1. $\Omega(\lambda x) = \Omega(x)$ for all $\lambda > 0$ and all $x \in \mathbb{R}^n$,
2. $\Omega \in L^1(S^{n-1})$, and
3. $\int\limits_{S^{n-1}} \Omega d\sigma = 0$.

When Ω is odd, we know the Fourier transform $\left|\widehat{\text{p. v.} \frac{\Omega(\cdot)}{|\cdot|^n}}\right| \leqslant C||\Omega||_{L^1(S^{n-1})}$ is bounded. This suggests the following corollary.

Corollary 8.1. T_Ω can be extended to an operator bounded on $L^2(\mathbb{R}^n)$.

Note that we cannot apply Calderón-Zygmund theorem directly which gives a bounded operator in any L^p-space, but we may still prove the following result.

Theorem 8.2. If Ω is odd and satisfies the three properties above, then $||T_\Omega f||_p \leqslant C_p||f||_p$ for any $f \in C_c^\infty(\mathbb{R}^n)$ and any $1 < p < \infty$.

To apply the method of rotation, we decompose $\mathbb{R}^n$ into $W \times W^\perp$ where $W \cong \mathbb{R}^1$. On W, we treat the operator as a Hilbert transform, which allows the estimate in L^p-sense.

Proof. Let $f \in C_c^\infty(\mathbb{R}^n)$, then

$$\begin{aligned}
Tf(x) &= \lim_{\varepsilon\to 0} \int\limits_{\{y\in\mathbb{R}^n:|y|>\varepsilon\}} \frac{\Omega(y)}{|y|^n} f(x-y)dy \text{ by definition} \\
&= \lim_{\varepsilon\to 0} \int\limits_{S^{n-1}} \Omega(y') \int_\varepsilon^\infty f(x-ry')\frac{dr}{r} d\sigma(y') \text{ by polar coordinate formula} \\
&= \frac{1}{2}\lim_{\varepsilon\to 0} \int\limits_{S^{n-1}} \Omega(y') \int\limits_{\{r\in\mathbb{R}:|r|>\varepsilon\}} f(x-ry')\frac{dr}{r} d\sigma(y') \text{ since } \Omega \text{ is odd} \\
&= \frac{1}{2}\lim_{\varepsilon\to 0} \int\limits_{S^{n-1}} \Omega(y') \left(\int\limits_{\varepsilon<|r|<1} f(x-ry')\frac{dr}{r} - \int\limits_{\varepsilon<|r|<1} f(x)\frac{dr}{r} \right) d\sigma(y') \\
&+ \frac{1}{2} \int\limits_{S^{n-1}} \Omega(y') \int\limits_{|r|>1} f(x-ry')\frac{dr}{r} d\sigma(y') \text{ as } \int\limits_{S^{n-1}} \Omega = 0 \\
&= \frac{1}{2} \int\limits_{S^{n-1}} \Omega(y') \lim_{\varepsilon\to 0} \left(\int\limits_{\varepsilon<|r|<1} f(x-ry')\frac{dr}{r} - \int\limits_{\varepsilon<|r|<1} f(x)\frac{dr}{r} \right) d\sigma(y') \\
&+ \frac{1}{2} \int\limits_{S^{n-1}} \Omega(y') \int\limits_{|r|>1} f(x-ry')\frac{dr}{r} d\sigma(y') \\
&= \frac{1}{2} \int\limits_{S^{n-1}} \Omega(y') \lim_{\varepsilon\to 0} \left(\int\limits_{\varepsilon<|r|<1} f(x-ry')\frac{dr}{r} - f(x) \int\limits_{\varepsilon<|r|<1} \frac{dr}{r} \right) d\sigma(y') \\
&+ \frac{1}{2} \int\limits_{S^{n-1}} \Omega(y') \int\limits_{|r|>1} f(x-ry')\frac{dr}{r} d\sigma(y')
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} \int\limits_{S^{n-1}} \Omega(y') \lim_{\varepsilon \to 0} \left(\int\limits_{\varepsilon < |r| < 1} f(x - ry') \frac{dr}{r} \right) d\sigma(y') \\
&+ \frac{1}{2} \int\limits_{S^{n-1}} \Omega(y') \int\limits_{|r|>1} f(x - ry') \frac{dr}{r} d\sigma(y') \\
&= \frac{1}{2} \int\limits_{S^{n-1}} \Omega(y') \lim_{\varepsilon \to 0} \int\limits_{|r|>\varepsilon} f(x - ry') \frac{dr}{r} d\sigma(y').
\end{aligned}
$$

Now for any $y' \in S^{n-1}$, we have $H_{y'} f(x) = \lim\limits_{\varepsilon \to 0} \int\limits_{|r|>\varepsilon} f(x - ry') \frac{dr}{r}$.

Problem 17. Prove that

$$
||H_{y'} f||_p \leqslant C_p ||f||_p
$$

for any $f \in C_c^\infty(\mathbb{R}^n)$ or $L^p(\mathcal{S}^{\mathbb{R}^n})$ and any $1 < p < \infty$.

Now $T_n f(x) = \frac{1}{2} \int\limits_{S^{n-1}} \Omega(y') H_{y'} f(x) d\sigma(y')$, hence

$$
||T_n f||_p \leqslant \int\limits_{S^{n-1}} |\Omega(y')| ||H_{y'} f||_p d\sigma(y').
$$

Problem 17 concludes the proof. □

Recall that the Riesz transform is given by

$$
R_j f(x) = C_n \,\mathrm{p.v.} \int \frac{y_j}{|y|^{n+1}} f(x - y) dy.
$$

Lemma 8.3. We have $\widehat{\mathrm{p.v.}\, C_n \frac{y_j}{|y|^{n+1}}} = -i \frac{\xi_j}{|\xi|}$.

Proof. Observe that $\frac{1}{1-n} \frac{\partial}{\partial x_j} \left(\frac{1}{|x|^{n-1}} \right) = \frac{x_j}{|x|^{n+1}}$ for $n > 1$. Therefore,

$$
\begin{aligned}
\widehat{\mathrm{p.v.}\, C_n \frac{y_j}{|y|^{n+1}}} &= \frac{1}{1-n} \widehat{\frac{\partial}{\partial x_j} \frac{1}{|x|^{n-1}}}(\xi) \\
&= \frac{1}{1-n} 2\pi i \xi_j \widehat{\frac{1}{|x|^{n-1}}}(\xi).
\end{aligned}
$$

Claim 8.4.

$$
\widehat{\frac{1}{|x|^{n-1}}}(\xi) = C(n) \frac{1}{|\xi|}
$$

where $C(n)$ depends on the volume of the unit ball.

Subproof. Note that $\frac{1}{|x|^{n-1}}$ is regular, so its Fourier transform $\widehat{\frac{1}{|x|^{n-1}}}$ is radial. Moreover, it is homogeneous of degree -1: when we dilate by $\lambda > 0$, we get $\widehat{\frac{1}{|x|^{n-1}}}(\lambda \xi) = \lambda^{-n} \widehat{\frac{1}{|\frac{x}{\lambda}|^{n-1}}}(\xi) = \lambda^{-1} \widehat{\frac{1}{|x|^{n-1}}}(\xi)$. Therefore,

$$
\widehat{\frac{1}{|x|^{n-1}}}(\xi) = C(n) \frac{1}{|\xi|}
$$

for some constant $C(n)$. ■

Now

$$\widehat{\mathrm{p.\,v.}\, C_n \frac{y_j}{|y|^{n+1}}} = \frac{1}{1-n} 2\pi i C(n) \frac{\xi_j}{|\xi|},$$

and make a choice of $C(n)$ in terms of C_n. □

Corollary 8.5. $\widehat{R_j f(\xi)} = \widehat{\mathrm{p.\,v.}\, C_n \frac{x_j}{|x|^{n+1}}}(\xi)\hat{f}(\xi) = -i\frac{\xi_j}{|\xi|}\hat{f}(\xi)$.

Corollary 8.6. $\sum\limits_{j=1}^{n} R_j^2 = I$, that is, $\sum\limits_{j=1}^{n} R_j^2 f = f$ for all $f \in \mathcal{S}(\mathbb{R}^n)$ where $R_j^2 = R_j \circ R_j$.

Theorem 8.7. For $1 \leqslant j, k \leqslant n$ and any $1 < p < \infty$,

$$\left|\left| \frac{\partial^2}{\partial x_k \partial x_j} u \right|\right|_p \leqslant C_p ||\Delta u||_p$$

where Δ is the Laplacian operator.

Proof.

Claim 8.8. $\frac{\partial^2 u}{\partial x_k \partial x_j} = -R_j R_k \Delta u$.

Subproof. We may prove that $\widehat{\frac{\partial^2 u}{\partial x_k \partial x_j}} = \widehat{-R_j R_k \Delta u}$. Indeed,

$$\begin{aligned}
\widehat{\frac{\partial^2 u}{\partial x_k \partial x_j}}(\xi) &= (2\pi i \xi_k)(2\pi i \xi_j)\hat{u}(\xi) \\
&= -4\pi^2 \xi_k \xi_j \hat{u}(\xi) \\
&= (-\frac{i\xi_j}{|\xi|})(-\frac{i\xi_k}{|\xi|})4\pi^2 |\xi|^2 \hat{u}(\xi) \\
&= \widehat{-R_j R_k \Delta u}.
\end{aligned}$$

■

Therefore,

$$\left|\left| \frac{\partial^2}{\partial x_k \partial x_j} u \right|\right|_p \leqslant ||R_j R_k \Delta u||_p \leqslant C_p ||\Delta u||_p.$$

□

9 Littlewood-Paley Theory

Let $\Delta_j = \{x \in \mathbb{R} : 2^j|x| < 2^{j+1}\}$ for $j \in \mathbb{Z}$. We define $\widehat{S_j f}(\xi) = \chi_{\Delta_j}(\xi)\hat{f}(\xi)$ for $f \in L^2$, then $S_j f(\xi) = \widecheck{\widehat{S_j f}}(\xi)$. Now let $Sf(x) = \left(\sum_{j\in\mathbb{Z}} |\delta_j f(x)|^2\right)^{\frac{1}{2}}$, then

$$\begin{aligned}
||Sf||_2 &= \left|\left|\left(\sum_{j\in\mathbb{Z}} |S_j f(x)|^2\right)^{\frac{1}{2}}\right|\right|_2 \\
&= \left(\sum_{j\in\mathbb{Z}} \int |S_j f(x)|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{j\in\mathbb{Z}} ||S_j f||_2^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{j\in\mathbb{Z}} ||\widehat{S_j f}||_2^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{j\in\mathbb{Z}} ||\chi_{\Delta_j}\hat{f}||_2^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{j\in\mathbb{Z}} \int_{\Delta_j} |\hat{f}|^2\right)^{\frac{1}{2}} \\
&= ||\hat{f}||_2 \\
&= ||f||_2.
\end{aligned}$$

We now partition $\mathbb{R} = \bigcup_{j\in\mathbb{Z}} \Delta_j$ into a union of disjoint subsets, with $||Sf||_2 = ||f||_2$ for all $f \in L^2$.

Theorem 9.1 (Littlewood-Paley). Let $1 < p < \infty$, then there exists $C_1, C_2 \in \mathbb{R}$ such that for all $f \in L^p(\mathbb{R})$, we have

$$C_2||f||_p \leqslant ||Sf||_p \leqslant C_1||f||_p.$$

Let $\psi \in \mathcal{S}(\mathbb{R})$ be a non-negative bump function such that

- $\operatorname{supp}(\psi) \subseteq \{\frac{1}{2} \leqslant |x| \leqslant 4\}$, and
- $\psi(x) = 1$ if $1 \leqslant |x| \leqslant 2$,

then let $\psi_j(\xi) = \psi(2^{-j}\xi)$. In this new language, define $\widehat{S_j^* f}(\xi) = \psi_j(\xi)\hat{f}(\xi)$, then $S_j^* f(x) = \check{\psi}_j * f(x)$, and define $S^* f(x) = \left(\sum_j |S_j^* f(\xi)|^2\right)^{\frac{1}{2}}$, and $K_j = \check{\psi}_j \in L^1$.

Theorem 9.2. For any $f \in L^p$,

$$C_1||f||_p \leqslant ||S^* f||_p \leqslant C_2||f||_p.$$

Proof. Note that $\{S_j^* f\} = \{S_1^* f, S_{-1}^* f, S_2^* f, S_{-2}^* f, \cdots\}$, then set $\vec{T}f(x) = \{S_j f(x)\}_{j\in\mathbb{Z}}$. Now for a sequence $\{a_j\}_{j\in\mathbb{Z}}$ we define $||\{a_j\}_{j\in\mathbb{Z}}||_{L^2} = \left(\sum_j |a_j|^2\right)^{\frac{1}{2}}$, then

$$\vec{T}f(x) = \{S_j^* f(x)\}_{j\in\mathbb{Z}} = \{K_j * f(x)\}_{j\in\mathbb{Z}}$$

and define $\vec{K} = \{K_j\}_{j\in\mathbb{Z}}$ with $\vec{K} * f = \{K_j * f\}_{j\in\mathbb{Z}}$. Therefore

$$||\vec{T}f(x)||_{L^2} = \left(\sum_j |S_j^* f(x)|^2\right)^{\frac{1}{2}} = S^* f(x)$$

and

$$||S^* f||_p = ||||\vec{T}f||_{L^2}||_p = ||\vec{T}f||_{L^p(\ell^2)}.$$

When $p = 2$, this can be done by using

Theorem 9.3 (Calderón-Zygmund)**.** Let $\vec{T}f(x) = \vec{K} * f(x)$ such that for some $\varepsilon > 0$, we have

$$||\vec{K}(x-y) - \vec{K}(x-y')||_{L^2} \leqslant C\frac{|y-y'|^\varepsilon}{|x-y|^{1+\varepsilon}}$$

whenever $|x-y| > 2|y-y'|$. If $||\vec{K} * f||_{L^2(\ell^2)} \leqslant C||f||_2$ for all $f \in L^2$, then $||\vec{K} * f||_{L^p(\ell^2)} \leqslant C_p||f||_p$ for all $f \in L^p$ where $1 < p < \infty$.

Remark 9.4. For any $\lambda > 0$ and any $f \in L^1$, we have

$$|\{x : ||\vec{K} * f(x)||_{\ell^2} > \lambda\}| \leqslant \frac{C||f||_1}{\lambda}.$$

It then remains to show that the kernel $\vec{K} = \{\check{\psi}_j\}_{j\in\mathbb{Z}}$ is Calderón-Zygmund. Most importantly, we verify that there exists some $\varepsilon > 0$ such that

$$||\vec{K}(x-y) - \vec{K}(x-y')||_{\ell^2} \leqslant C \cdot \frac{|y-y'|^\varepsilon}{|x-y|^{1+\varepsilon}}$$

whenever $|x-y| > 2|y-y'|$. By definition, it suffices to show that

$$||\vec{K}(x-y) - \vec{K}(x-y')||_{\ell^2} \leqslant \left(\sum_{j=-\infty}^{\infty} |\check{\psi}_j(x-y) - \check{\psi}_j(x-y')|^2\right)^{\frac{1}{2}}.$$

By Mean Value Theorem, there exists some η between $x-y$ and $x-y'$ such that

$$|\check{\psi}_j(x-y) - \check{\psi}_j(x-y')| = |\left(\check{\psi}_j\right)'(\eta)||y-y'|.$$

Therefore,

$$\begin{aligned}
\check{\psi}_j(x) &= \int \psi_j(\xi)e^{2\pi i\xi\cdot x}d\xi \\
&= \int \psi(2^{-j}\xi)e^{2\pi i\xi\cdot x}d\xi \\
&= 2^j\int \psi(\xi)e^{2\pi i\xi\cdot(2^jx)}d\xi \\
&= 2^j\check{\psi}(2^jx)
\end{aligned}$$

by a change of variables, and so

$$\begin{aligned}
\left|\left(\check{\psi}_j\right)'(x)\right| &= \left|2^j \cdot 2^j\left(\check{\psi}\right)'(2^jx)\right| \\
&\leqslant \frac{C_N 2^{2j}}{(1+2^j|x|)^N}
\end{aligned}$$

for all $N \geqslant 2$. Let us write

$$|\eta| = |\theta(x-y) + (1-\theta)(x-y')|$$

$$\begin{aligned}
&= |(x-y') - \theta(y-y')| \\
&= |(x-y) + (1-\theta)(y-y')| \\
&\geqslant |x-y| - (1-\theta)|y-y'| \\
\geqslant \frac{1}{2}|x-y|
\end{aligned}$$

for some $\theta \in [0,1]$ and using our assumption on the distance. Therefore, by substitution,

$$\begin{aligned}
|\check{\psi}_j(x-y) - \check{\psi}_j(x-y')| &\leqslant \frac{C_N 2^{2j}}{(1+2^j|\eta|)^N}|y-y'| \\
&\leqslant C_N \left(\sum_{j\in\mathbb{Z}} \frac{2^{4j}|y-y'|^2}{(1+2^j|x-y|)^{2N}}\right)^{\frac{1}{2}} \\
&= C_N|y-y'| \left(\sum_{j\in\mathbb{Z}} \frac{2^{4j}}{(1+2^j|x-y|)^{2N}}\right)^{\frac{1}{2}} \\
&= C_N|y-y'| \left(\sum_{1\geqslant 2^j|x-y|} \frac{2^{4j}}{(1+2^j|x-y|)^{2N}} + \sum_{1<2^j|x-y|} \frac{2^{4j}}{(1+2^j|x-y|)^{2N}}\right)^{\frac{1}{2}} \\
&\leqslant C_N|y-y'| \left(\sum_{1\geqslant 2^j|x-y|} 2^{4j} + \sum_{1<2^j|x-y|} \frac{2^{4j}}{(2^j|x-y|)^{2N}}\right)^{\frac{1}{2}} \\
&\leqslant C_N|y-y'| \left(\sum_{1\geqslant 2^j|x-y|} 2^{2j} + \sum_{1<2^j|x-y|} \frac{2^{2j}}{(2^j|x-y|)^{N}}\right).
\end{aligned}$$

For N large enough, we can bound both terms, for instance the second term is bounded above by $|x-y|^{-2}$. □

Lemma 9.5 (Khinchin's Inequality)**.** Let $\{\omega_n\}_{n=1}^N$ be independent random variables taking values in $\{\pm 1\}$ with equal probabilities, then $\mathbb{E}(|\sum_{n=1}^N a_n\omega_n|^p) \sim \left(\sum_{n=1}^N |a_n|^2\right)^{\frac{p}{2}}$ for any $0 < p < \infty$. Here we use the notation that $A \sim B$ if and only if there exists $C_1, C_2 \in \mathbb{R}$ such that $C_1 B \leqslant A \leqslant C_2 B$.

Proof. Let us prove the case where $1 < p < \infty$. We know that $\frac{1}{2}(e^x + e^{-x}) \leqslant e^{\frac{x^2}{2}}$ for all $x \in \mathbb{R}$. Assume $a_n \in \mathbb{R}$ for all $n \in \{1, \dots, N\}$, and let $\mu > 0$, then

$$\begin{aligned}
\int_\Omega e^{\mu \sum_n a_n\omega_n} dP &= \mathbb{E}\left(e^{\mu \sum_{n=1}^N a_n\omega_n}\right) \\
&= \mathbb{E}\left(\prod_{n=1}^N e^{\mu a_n\omega_n}\right) \\
&= \prod_{n=1}^N \mathbb{E}\left(e^{\mu a_n\omega_n}\right) \\
&= \prod_{n=1}^N \frac{1}{2}\left(e^{\mu a_n} + e^{-\mu a_n}\right) \\
&\leqslant \prod_{n=1}^N e^{\frac{\mu^2 a_n^2}{2}}
\end{aligned}$$

For any $\lambda > 0$ and $\mu > 0$, we know

$$P(\{\sum_n a_n\omega_n \geqslant \lambda\}) = \prod_{n=1}^{N} e^{\frac{\mu^2 a_n^2}{2}} e^{-\mu\lambda}.$$

In particular, take $\mu = \frac{\lambda}{\sum_n a_n^2}$, then

$$P(\{\sum_n a_n\omega_n \geqslant \lambda\}) \leqslant e^{-\frac{\lambda^2}{2\sum_n a_n^2}}.$$

Similarly, we have that

$$P(\{\sum_n a_n\omega_n \leqslant -\lambda\}) \leqslant e^{-\frac{\lambda^2}{2\sum_n a_n^2}}.$$

Therefore,

$$P(\{|\sum_n a_n\omega_n| \leqslant \lambda\}) \leqslant 2e^{-\frac{\lambda^2}{2\sum_n a_n^2}}.$$

This gives

$$\begin{aligned}
\mathbb{E}[|\sum_n a_n\omega_n|^p] &= \int_\Omega |\sum_n a_n\omega_n|^p dP \\
&= p\int_0^\infty \lambda^{p-1} P(\{|\sum_n a_n\omega_n| > \lambda\}) d\lambda \\
&\leqslant 2p\int_0^\infty \lambda^{p-1} e^{-\frac{\lambda^2}{2\sum_n a_n^2}} d\lambda \\
&\xrightarrow{\lambda\to(\sum_n a_n^2)^{\frac{1}{2}}\lambda} 2p(\sum_n a_n^2)^{\frac{p}{2}} \int_0^\infty \lambda^{p-1} e^{-\frac{\lambda^2}{2}} d\lambda \\
&= 2pC_p(\sum_n a_n^2)^{\frac{p}{2}}.
\end{aligned}$$

by Fubini theorem.

Conversely,

$$\begin{aligned}
\sum_n |a_n|^2 &= \mathbb{E}[|\sum_n a_n\omega_n|^2] \\
&= \int_\Omega |\sum_n a_n\omega_n||\sum_n a_n\omega_n| dP \\
&\leqslant \mathbb{E}[|\sum_n a_n\omega_n|^p]^{\frac{1}{p}} \mathbb{E}[|\sum_n a_n\omega_n|^{p'}]^{\frac{1}{p'}} \\
&\leqslant C_p\mathbb{E}[|\sum_n a_n\omega_n|^p]^{\frac{1}{p}} (\sum_n |a_n|^2)^{\frac{1}{2}}
\end{aligned}$$

by Hölder inequality. In particular,

$$\left(\sum_n |a_n|^2\right)^{\frac{1}{2}} \leqslant C_p\mathbb{E}[|\sum_n a_n\omega_n|^p]^{\frac{1}{p}}$$

and therefore

$$\left(\sum_n |a_n|^2\right)^{\frac{p}{2}} \leqslant C_p\mathbb{E}[|\sum_n a_n\omega_n|^p].$$

□

Theorem 9.6. Let T be a linear operator such that $||Tf||_p \leqslant C_p||f||_p$ for any $f \in L^p$ and $1 < p < \infty$, then $||(\sum\limits_{j\in\mathbb{Z}} |Tf_j|^2)^{\frac{1}{2}}||_p \leqslant \tilde{C}_p||(\sum\limits_{j\in\mathbb{Z}} |f_j|^2)^{\frac{1}{2}}||_p$.

Proof. We may assume the sum is finite, and later taking a limit to prove the general case. By Lemma 9.5, we have

$$\begin{aligned}(\sum_j |Tf_j|^2)^{\frac{p}{2}} &\sim \mathbb{E}(|\sum_j Tf_j\omega_j|^p) \\ &= \int\limits_\Omega |T(\sum_j f_j\omega_j)|^p dP.\end{aligned}$$

Therefore, by Fubini theorem,

$$\begin{aligned}||(\sum_{j\in\mathbb{Z}} |Tf_j|^2)^{\frac{1}{2}}||_p &\leqslant \int\limits_X\int\limits_\Omega |T(\sum_j f_j\omega_j)|^p dPdx \\ &= \int\limits_\Omega\int\limits_X |T(\sum_j f_j\omega_j)|^p dxdP \\ &\leqslant C_p^p \int\limits_\Omega |\sum_j f_j\omega_j|^p dxdP \\ &= C_p^p \int\limits_X \mathbb{E}[|\sum_j f_j\omega_j|^p]dx \\ &\leqslant \tilde{C}_p||(\sum_j |f_j|^2)^{\frac{1}{2}}||_p^p.\end{aligned}$$

□

Lemma 9.7. Let $\widehat{S_{[a,b)}f}(\xi) = \chi_{[a,b)}(\xi)\hat{f}(\xi)$, then

$$S_{[a,b)} = \frac{i}{2}(M_aHM_{-a} - M_bHM_{-b}),$$

where H represents the Hilbert transform, and M_a is defined by $M_af(x) = e^{2\pi iax}f(x)$.

Proof. This is because $\hat{S}_{[a,b)} = \frac{i}{2}[\widehat{M_aHM_{-a} - M_bHM_{-b}}]$. □

Proof of Theorem 9.1. Note that $\widehat{S_jf}(\xi) = \chi_{\Delta_j}(\xi)\hat{f}(\xi)$ and $\widehat{S_j^*f}(\xi) = \psi_j(\xi)\hat{f}(\xi)$. We know that

$$||(\sum_j |S_j^*f|^2)^{\frac{1}{2}}||_p \leqslant C_p||f||_p$$

for any $1 < p < \infty$ and any function f. Since $S_jS_j^*f = S_jf$, then

$$||(\sum_j |S_jf|^2)^{\frac{1}{2}}||_p = ||(\sum_j |S_jS_j^*f|^2)^{\frac{1}{2}}||_p.$$

By Lemma 9.7,

$$\begin{aligned}S_j &= \frac{i}{2}[M_{2^j}HM_{-2^j} - M_{2^{j+1}}HM_{-2^{j+1}}] + \frac{i}{2}[M_{-2^{j+1}}HM_{2^{j+1}} - M_{-2^j}HM_{2^j}] \\ &= \frac{i}{2}M_{a_j}HM_{-a_j}.\end{aligned}$$

Claim 9.8. We have
$$||(\sum_j |M_{a_j} H M_{-a_j} S_j^* f|^2)^{\frac{1}{2}}||_p \leqslant C_p ||f||_p.$$

Subproof.
$$\begin{aligned}||(\sum_j |M_{a_j} H M_{-a_j} S_j^* f|^2)^{\frac{1}{2}}||_p &\leqslant ||(\sum_j |H(M_{-a_j} S_j^* f)|^2)^{\frac{1}{2}}|| \\ &\leqslant ||(\sum_j |M_{-a_j} S_j^* f|^2)^{\frac{1}{2}}||_p \\ &= ||(\sum_j |S_j^* f|^2)^{\frac{1}{2}}||_p \\ &\leqslant C_p ||f||_p\end{aligned}$$

by Theorem 9.6. ■

In particular, this shows that
$$\left|\left|\left(\sum_j |S_j f|^2\right)^{\frac{1}{2}}\right|\right|_p \leqslant C_p ||f||_p,$$

so
$$\begin{aligned}\int_{\mathbb{R}} \sum_j S_j f \overline{S_j g} &= \sum_j \langle S_j f, S_j g\rangle \\ &= \sum_j \left\langle \widehat{S_j f}, \widehat{S_j g}\right\rangle \\ &= \sum_j \left\langle \chi_{\Delta_j} \hat{f}, \chi_{\Delta_j} \hat{g}\right\rangle \\ &= \sum_j \int_{\Delta_j} \hat{f}\bar{\hat{g}} \\ &= \int_{\mathbb{R}} \hat{f}\bar{\hat{g}} \\ &= \left\langle \hat{f}, \hat{g}\right\rangle \\ &= \langle f, g\rangle .\end{aligned}$$

For any $1 < p < \infty$, let p' be the conjugate of p, then
$$\begin{aligned}||f||_p &= \sup_{\substack{g \in L^{p'} \\ ||g||_{p'}=1}} |\langle f, g\rangle| \\ &= \sup_{\substack{g \in L^{p'} \\ ||g||_{p'}=1}} \int_{\mathbb{R}} \sum_j S_j f \overline{S_j g} \\ &\leqslant \int (\sum_j |S_j f|^2)^{\frac{1}{2}} (\sum_j |S_j g|^2)^{\frac{1}{2}} \\ &\leqslant \left|\left|\left(\sum_j |S_j f|^2\right)^{\frac{1}{2}}\right|\right|_p C_p \left|\left|\left(\sum_j |S_j g|^2\right)^{\frac{1}{2}}\right|\right|_{p'}\end{aligned}$$

$$\leqslant \left|\left|\left(\sum_j |S_j f|^2\right)^{\frac{1}{2}}\right|\right|_p C_p ||g||_{p'}$$
$$= C_p \left|\left|\left(\sum_j |S_j f|^2\right)^{\frac{1}{2}}\right|\right|_p.$$

□

Problem 18. Prove Theorem 9.9.

Theorem 9.9. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\psi(0) = 0$. For each $j \in \mathbb{Z}$, let S_j be given by $(\widehat{S_j f})(\xi) = \psi(2^{-j}\xi)\hat{f}(\xi)$, then for any $1 < p < \infty$,

$$\left|\left|\left(\sum_j |S_j f|^2\right)^{\frac{1}{2}}\right|\right|_p \leqslant C_p ||f||_p.$$

However, if $\sum_j |\psi(2^{-j}\xi)|^2$ is a constant for every $\xi \neq 0$, then

$$||f||_p \leqslant C_p \left|\left|\left(\sum_j |S_j f|^2\right)^{\frac{1}{2}}\right|\right|_p.$$

10 Multipliers

For any $f \in L^2 \cap L^p$, let $\widehat{Tf(\xi)} = m(\xi)\hat{f}(\xi)$, where $\xi \in \mathbb{R}^n$ and m is a measurable function.

Definition 10.1. Suppose T is such that for some $p \in [1, \infty]$, $||Tf||_p \leqslant C_p||f||_p$ for any $f \in L^p$, then we say m is an L^p-multiplier.

If m is an L^p-multiplier, then T can be extended to an operator which bounded on L^p.

Now define $T = T^{\text{ext}}$ to be the extension. For any $f \in L^p$ where $1 \leqslant p < \infty$, there exists a sequence $\{f_k\}_{k\geqslant 1} \subseteq \mathcal{S}(\mathbb{R}^n)f$ such that $f_k \xrightarrow{L^p} f$, where $\{Tf_k\}_{k\geqslant 1}$ is Cauchy in L^p. Therefore, there exists $g \in L^p$ such that $g =_{L^p} \lim\limits_{k\to\infty} Tf_k$, or equivalently, $||Tf_k - g||_p \to 0$ as $k \to \infty$. We define $Tf = T^{\text{ext}}f = g$.

Let $D = \{\xi \in \mathbb{R}^2 : |\xi| \leqslant 1\}$, then we may define

$$\widehat{T_D f}(\xi) = \chi_D(\xi)\hat{f}(\xi)$$

for $f \in \mathcal{S}(\mathbb{R}^2)$, therefore

$$||T_D f||_2 \leqslant ||f||_2$$

for all $f \in \mathcal{S}(\mathbb{R}^2)$. However, it is not true that $||T_D f||_p \leqslant ||f||_p$ for all $f \in \mathcal{S}(\mathbb{R}^2)$ if $p \neq 2$.

Theorem 10.2. m is an L^2-multiplier if and only if $m \in L^\infty$.

Proof. Note that for any $f \in L^2$, we have

$$\begin{aligned} ||Tf||_2 &= ||\widehat{Tf}||_2 \\ &= ||m\hat{f}||_2 \\ &\leqslant ||m||_\infty ||\hat{f}||_2 \\ &= ||m||_\infty ||f||_2 \\ &\leqslant C||f||_2. \end{aligned}$$

Conversely, suppose T is an L^2-multiplier, then we define $||T|| = ||T||_{L^2\to L^2}$ via $\sup\limits_{0\neq f\in L^2} \frac{||Tf||_2}{||f||_2} < \infty$. Assume $||T|| \neq 0$, otherwise we have $||Tf||_2 = 0$ for all $f \in L^2$, thus $m \equiv 0$ almost everywhere, which means $m \in L^\infty$.

Claim 10.3. $|m(\xi)| \leqslant 2||T||$ for almost every $\xi \in \mathbb{R}^n$. Equivalently, $m(\{\xi : |m(\xi)| \geqslant 2||T||\}) = 0$.

Subproof. Let $E_k = \{\xi \in \mathbb{R}^n : 2^k \leqslant |\xi| \leqslant 2^{k+1}\}$, then $\{\xi : |m(\xi)| > 2||T||\} = \bigcup\limits_{k\in\mathbb{Z}} E_k$. We will show that $|E_k| = 0$ for all $k \in \mathbb{Z}$ for Lebesgue measure $|\cdot|$. Suppose not, then there exists $k \in \mathbb{Z}$ such that $|E_k| > 0$, then let $\hat{g} = \chi_{E_n}$, then

$$\begin{aligned} 4||T||^2|E_k| &\leqslant \int\limits_{E_k} |m|^2 \\ &= \int |m|^2|\hat{g}(\xi)|^2 \\ &= ||m\hat{g}||_2^2 \\ &= ||Tg||_2^2 \\ &\leqslant ||T||^2||\hat{g}||_2^2 \\ &= ||T||^2|E_k|, \end{aligned}$$

therefore $4||T||^2 \leqslant ||T||^2$, which means $||T|| = 0$, contradiction. ■

Problem 19. Prove that if m is a L^2-multiplier, then $||m||_\infty = ||T||$.

□

Definition 10.4. We define the Sobolev space $L^2_\alpha(\mathbb{R}^n) = \{f : (1+|\xi|^2)^{\frac{\alpha}{2}}\hat{f}(\xi) \in L^2\} \subseteq L^2(\mathbb{R}^n)$. The Sobolev norm is defined by

$$||f||_{L^2_\alpha} = \left(\int\limits_{\mathbb{R}^n} (1+|\xi|^2)^\alpha |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}.$$

We may also defined general Sobolev space as $L^p_\alpha(\mathbb{R}^n) = \{f : (1+|\xi|^2)^{\frac{\alpha}{2}}\hat{f}(\xi) \in L^p\}$.

Lemma 10.5. If $\alpha > \frac{n}{2}$ and $f \in L^2_\alpha(\mathbb{R}^n)$, then $\hat{f} \in L^1(\mathbb{R}^n)$. In particular, f is continuous and bounded.

Proof. Note that

$$\begin{aligned}\int\limits_{\mathbb{R}^n} |\hat{f}(\xi)| d\xi &= \int\limits_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^{\frac{\alpha}{2}}}(1+|\xi|^2)^{\frac{\alpha}{2}}|\hat{f}(\xi)|d\xi \\ &\leqslant \left(\int\limits_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^\alpha} d\xi\right) ||f||_{L^2_\alpha}\end{aligned}$$

by Cauchy-Schwartz. When $\alpha > \frac{n}{2}$, the integral is bounded since $\frac{1}{(1+|\xi|^2)^\alpha} \sim \frac{1}{|\xi|^{2\alpha}} \in L^1(\mathbb{R}^n \backslash B(1))$ whenever $|\xi| > 1$. This gives

$$\int\limits_{\mathbb{R}^n} |\hat{f}(\xi)| d\xi = C_{n,\alpha}||f||_{L^2_\alpha}.$$

□

Theorem 10.6. Let $m \in L^2_\alpha$ with $\alpha > \frac{n}{2}$, then m is an L^p-multiplier for any $1 \leqslant p \leqslant \infty$.

Proof. Recall that $\widehat{Tf} = m\hat{f}$, then by Lemma 10.5, $\check{m} \in L^1(\mathbb{R}^n)$, therefore $Tf = \check{m} * f$, where $\check{m}(x) = \int m(\xi)e^{2\pi i\xi\cdot x}d\xi$. Therefore,

$$\begin{aligned}||Tf||_1 &= ||\check{m} * f||_1 \\ &\leqslant ||\check{m}||_1||f||_1 \\ &\leqslant C||f||_1\end{aligned}$$

for any $f \in L^1 \cap L^2$. Moreover,

$$\begin{aligned}||Tf||_2 &\leqslant ||f||_\infty||\check{m}||_1 \\ &\leqslant C||f||_\infty.\end{aligned}$$

By the interpolation theorem, $||Tf||_p \leqslant C_p||f||_p$ for any $f \in L^p \cap L^2$. □

Lemma 10.7. Let $m \in L^2_\alpha(\mathbb{R}^n)$ with $\alpha > \frac{n}{2}$. For any $\lambda > 0$, we define T_λ by $\widehat{T_\lambda f}(\xi) = m(\lambda\xi) = \hat{f}(\xi)$ for any $f \in L^2 \cap L^p$. Then

$$\int\limits_{\mathbb{R}^n} |T_\lambda f(x)|^2 u(x)dx \leqslant C \int\limits_{\mathbb{R}^n} |f(x)|^2 Mu(x)dx,$$

where M is the Hardy-Littlewood maximal function, $u \geqslant 0$ is a measurable function, and C is a constant independent of u, f, and λ. Here we may define a new measure $d\mu = u(x)dx$.

Proof. Let $K = \check{m}$, i.e., $\hat{K} = m$. Since $m \in L^2_\alpha$, then $(1+|\xi|^2)^{\frac{\alpha}{2}}\hat{m}(\xi) \in L^2$, that is, $(1+|\xi|^2)^{\frac{\alpha}{2}}\check{m}(\xi) \in L^2$. Now $\check{m}(\xi) = \hat{m}(-\xi)$, so $||m||_{L^2_\alpha} = ||(1+|\xi|^2)^{\frac{\alpha}{2}}K(\xi)||_{L^2}$. Now $T_\lambda f(x) = K_\lambda * f(x)$, where $K_\lambda(x) = \lambda^{-n}K(\lambda^{-1}x)$. Now

$$\int\limits_{\mathbb{R}^n} |T_\lambda f(x)|^2 u(x)dx \leqslant \int\limits_{\mathbb{R}^n}\left|\int\limits_{\mathbb{R}^n} \lambda^{-n}K(\lambda^{-1}(x-y))f(y)dy\right|^2 udx$$

$$\begin{aligned}
&\leqslant \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \lambda^{-n} K(\lambda^{-1}(x-y)) \frac{1+|\lambda^{-1}(x-y)|^2]^{\frac{\alpha}{2}}}{1+|\lambda^{-1}(x-y)|^2]^{\frac{\alpha}{2}}} f(y)dy \right|^2 udx \\
&\leqslant \int \mathbb{R}^n \left(\int_{\mathbb{R}^n} \left|K(\lambda^{-1}(x-y))[1+|\lambda^{-1}(x-y)|^2]^{\frac{\alpha}{2}}\right|^2 dy \right) \int_{\mathbb{R}^n} \frac{\lambda^{-2n}|f(y)|^2}{[1+|\lambda^{-1}(x-y)|^2]^{\alpha}} dyu(x)dx \\
&\leqslant ||m||_{L^2_\alpha} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\lambda^{-n}|f(y)|^2}{[1+\lambda^{-1}(x-y)|]^{\alpha}} dyu(x)dx \\
&= C_\alpha \int_{\mathbb{R}^n} |f(y)|^2 \left(\int_{\mathbb{R}^n} \frac{\lambda^{-n}}{(1+|\lambda^{-1}(x-y)|^2)^{\alpha}} dx \right) dy \\
&\leqslant C_\alpha \int_{\mathbb{R}^n} |f(y)|^2 Mu(y)dy
\end{aligned}$$

by Cauchy-Schwartz. □

Problem 20. Let $m \in \mathcal{S}(\mathbb{R}^n)$. Prove that m is an L^p-multiplier for any $1 \leqslant p \leqslant \infty$.

Problem 21. Let $1 \leqslant p \leqslant \infty$. Prove that m is an L^p-multiplier if and only if m is an $L^{p'}$-multiplier.

Problem 22. Prove that $L^2_\alpha(\mathbb{R}^n) \subseteq L^2_\beta(\mathbb{R}^n)$ if $\alpha \geqslant \beta$.

Theorem 10.8 (Hörmender Multiplier Theorem)**.** Let $\psi \in C^\infty$ be a radial function supported on $\{\xi : \frac{1}{2} \leqslant |\xi| \leqslant 2\}$ such that $\sum_{j\in\mathbb{Z}} |\psi(2^{-j}\xi)|^2 = 1$ for any $\xi \neq 0$. Let $\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$ such that $\sup_{j\in\mathbb{Z}}(||m(2^j\cdot)\psi(\cdot)||_{L^2_\alpha} < \infty$ for some $\alpha > \frac{n}{2}$. Then M is an L^p-multiplier for any $1 < p < \infty$. That is, $||Tf||_p \leqslant C_p||f||_p$ for any $f \in L^2 \cap L^p$.

Proof. We have $\widehat{S_jf}(\xi) = \psi(2^{-j}\xi)\hat{f}(\xi)$ for all $j \in \mathbb{Z}$, and $|| \left(L\sum_j |S_jf|^2 \right)^{\frac{1}{2}} ||_p \sim ||f||_p$ for all $1 < p < \infty$. Define $\psi'(\xi) = 1$ if $\frac{1}{2} \leqslant |\xi| \leqslant 2$, with $\operatorname{supp}(\psi') \subseteq \{\frac{1}{2} \leqslant |\xi| \leqslant 4\}$. We have set $\widehat{S'_jf}(\xi) = \psi'(2^{-j}\xi)\hat{f}(\xi)$ and that $\psi(2^{-j}\xi)\psi'(2^{-j}\xi) = \psi(2^{-j}\xi)$. Therefore, $S_jT_jS'_j = S_jT$, which is equivalent to saying that $\widehat{S_jT_jS_j}(f) = \widehat{S_jT}(f)$. By Theorem 9.1,

$$||Tf||_p \leqslant C_p|| \sum_j \left(|S_jT_jS'_jf|^2\right)^{\frac{1}{2}} ||_p.$$

Let $g_j = S'_jf$, then $S_jT_jS'_jf = S_jTg_j$, and $\widehat{S_jTf}(\xi) = \psi(2^{-j}\xi)m(\xi)\hat{f}(\xi)$. By Lemma 10.7,

$$\int_{\mathbb{R}^n} |S_jTf(x)|^2u(x)dx \leqslant C \int_{\mathbb{R}^n} |f|^2Mu(x)dx.$$

We may assume that $p > 2$, since the case where $1 < p < 2$ follows easily. By Hölder inequality, we have

$$\begin{aligned}
||(\sum_{j\in\mathbb{Z}} |S_jT_jg_j|^2)^{\frac{1}{2}}||_p &= \left(\int_{\mathbb{R}^n} (\sum_j |S_jTg_j|^2)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\
&= \sup_{||h||_{(\frac{p}{2})'}=1} \left(\int_{\mathbb{R}^n} \sum_j |S_jT_jg_j|^2 h(x)dx \right)^{\frac{1}{2}} \\
&\leqslant C \sup_{||h||_{(\frac{p}{2})'}=1} \left(\sum_j |g_j(x)|^2 Mh(x)dx \right)^{\frac{1}{2}}
\end{aligned}$$

$$\leqslant C \sup_{||h||_{(\frac{p}{2})'}=1} \left|\left| \left(\sum_j |g_j(x)|^2 \right)^{\frac{1}{2}} \right|\right|_p ||Mh||_{(\frac{p}{2})'}.$$

Here notice that $||Mh||_{(\frac{p}{2})'} \leqslant C_p ||h||_{(\frac{p}{2})'}$, therefore we have

$$\begin{aligned} ||(\sum_{j\in\mathbb{Z}} |S_jT_jg_j|^2)^{\frac{1}{2}}||_p &\leqslant C \sup_{||h||_{(\frac{p}{2})'}=1} \left|\left| \left(\sum_j |g_j(x)|^2 \right)^{\frac{1}{2}} \right|\right|_p ||Mh||_{(\frac{p}{2})'} \\ &\leqslant C_p \left|\left| \left(\sum_j |g_j(x)|^2 \right)^{\frac{1}{2}} \right|\right|_p \\ &= C_p \left|\left| \left(\sum_j |S_j' f|^2 \right)^{\frac{1}{2}} \right|\right|_p . \end{aligned}$$

□

Corollary 10.9. Let $\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$. Let $m \in \mathbb{C}^k$ be away from the origin for $k = \left[\frac{n}{2}\right] + 1$. If for any $\beta \in \mathbb{N}_0^n$ with $|\beta| \leqslant k$, we have

$$\sup_{R>0} R^{|\beta|} \left(\frac{1}{k^n} \int_{R<|\xi|<2R} |D^\beta m(\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty,$$

then $||Tf||_p \leqslant C_p ||f||_p$ for all $f \in L^2 \cap L^p$ for all $1 < p < \infty$. In particular, if $|D^\beta m(\xi)| \leqslant C_{\beta,n}|\xi|^{-|\beta|}$ for all $|\beta| \leqslant k$ any all $\xi \neq 0$, then m is an L^p-multiplier.

Proof. We perform a change of variables from ξ to $R\xi$. Now the given condition

$$\sup_{R>0} R^{|\beta|} \left(\frac{1}{k^n} \int_{R<|\xi|<2R} |D^\beta m(\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty$$

becomes

$$\sup_{R>0} \left(\int_{1<|\xi|<2} |D^\beta m(R\cdot)(\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty$$

with $D^\beta m(R\cdot) = D^\beta m_R$ where $m_R(x) = m(Rx)$. Let ψ be the function in Theorem 10.8, then it suffices to show that

$$\sup_{j\in\mathbb{Z}} ||m(2^j\cdot)\psi(\cdot)||_{L_R^2} < \infty.$$

Indeed, $||m(2^j\cdot)\psi(\cdot)||_{L_R^2} \leqslant \sum_{|\beta|\leqslant R} ||D^\beta(m(2^j\cdot)\psi(\cdot)||_2$, so

$$D^\beta(m(2^j\cdot)\psi(\cdot)) = \sum_{|\gamma|\leqslant|\beta|} C_{\gamma,\beta} D^\gamma m(2^j\cdot)(\xi) D^{\beta-\gamma}\psi(\xi)$$

for $|\beta| \leqslant R$. Therefore,

$$\sum_{|\beta|\leqslant R} ||D^\beta(m(2^j\cdot)\psi(\cdot)||_2 \leqslant \sum_{|\beta|\leqslant R} \sum_{|\gamma|\leqslant|\beta|} |C_{\gamma,\beta}| \left(\int |D^\gamma m(2^j\cdot)(\xi)|^2 d\xi \right)^{\frac{1}{2}} < C_k < \infty,$$

which completes the proof. □

11 Fractional Integrals

Let Δ be the Laplacian, and recall that

$$\widehat{(-\Delta f)}(\xi) = 4\pi^2|\xi|^2\hat{f}(\xi)$$

for any function $f \in \mathcal{S}(\mathbb{R}^n)$. Let $\alpha \in \mathbb{R}$ and define $(-\Delta)^{\frac{\alpha}{2}}$ to be the operator

$$\left(\widehat{(-\Delta)^{\frac{\alpha}{2}}f}\right)(\xi) = (2\pi|\xi|)^\alpha \hat{f}(\xi).$$

Remark 11.1. If $\alpha > 0$, then $(-\Delta)^{\frac{\alpha}{2}} \sim D^\alpha$. If $\alpha = 0$, the operator is identity.

Remark 11.2. Let $-n < \alpha < 0$, then we denote $I_{-\alpha} = (-\Delta)^{\frac{\alpha}{2}}$, as an integration operator of order α.

Definition 11.3. Let $0 < \alpha < n$, then we define I_α to be the fractional integral operator, characterized by the fact that the Fourier transform $\widehat{I_\alpha f}(\xi) = (2\pi|\xi|)^{-\alpha}\hat{f}(\xi)$ for all $f \in \mathcal{S}(\mathbb{R}^n)$. Then $I_\alpha f = K * f$ for $K(x) = C|x|^{\alpha-n}$.

Proposition 11.4. Let $0 < \alpha < n$, then $(\widehat{|x|^{\alpha-n}})(\xi) = C_0|\xi|^{-\alpha}$ in the sense that

$$\int\limits_{\mathbb{R}^n} |x|^{\alpha-n}\hat{\varphi}(x)dx = C_0 \int\limits_{\mathbb{R}^n} |\xi|^{-\alpha}\varphi(\xi)d\xi$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Here $C_0 = \pi^{\frac{n}{2}-\alpha}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}$ where

$$\Gamma(z) = \int\limits_0^\infty x^{z-1}e^{-x}dx$$

for $z \in \mathbb{C}$.

Proof. Consider the standard Gauss kernel

$$\begin{aligned}\int\limits_{\mathbb{R}^n} e^{-\pi\delta|x|^2}\hat{\varphi}(x)dx &= \int\limits_{\mathbb{R}^n} \widehat{e^{-\pi\delta|x|^2}}(\xi)\varphi(\xi)d\xi \\ &= \delta^{-\frac{n}{2}}\int\limits_{\mathbb{R}^n} e^{-\frac{\pi}{\delta}|\xi|^2}\varphi(\xi)d\xi.\end{aligned}$$

Multiplying both sides by $\delta^{\beta-1}$ with $\beta = \frac{n-\alpha}{2}$, and taking the integral in terms of δ, then

$$\begin{aligned}\int\limits_0^\infty \delta^{\beta-1-\frac{n}{2}}\int\limits_{\mathbb{R}^n} e^{-\frac{\pi}{\delta}|\xi|^2}\varphi(\xi)d\xi d\delta &= \int\limits_0^\infty \delta^{\beta-1}\int\limits_{\mathbb{R}^n} e^{-\pi\delta|x|^2}\hat{\varphi}(x)dxd\delta \\ &= \int\limits_{\mathbb{R}^n}\hat{\varphi}(x)\int\limits_0^\infty \delta^{\beta-1}e^{-\pi\delta|x|^2}d\delta dx \\ &\xrightarrow{\delta\to\frac{\delta}{\pi|x|^2}} \pi^{-\beta}\int\limits_{\mathbb{R}^n}\hat{\varphi}(x)|x|^{-2\beta}\Gamma(\beta)dx \\ &= \pi^{-\frac{n-\alpha}{2}}\Gamma(\frac{n-\alpha}{2})\int\limits_{\mathbb{R}^n}|x|^{\alpha-n}\hat{\varphi}(x)dx.\end{aligned}$$

Similarly,

$$C_0\int\limits_{\mathbb{R}^n}|\xi|^{-\alpha}\varphi(\xi)d\xi = \Gamma(\frac{\alpha}{2})\pi^{-\frac{\alpha}{2}}\int\limits_{\mathbb{R}^n}\varphi(\xi)|\xi|^{-\alpha}d\xi.$$

Therefore,

$$\int_{\mathbb{R}^n} |x|^{\alpha-n}\hat{\varphi}(x)dx = \pi^{\frac{n}{2}-\alpha}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}\int_{\mathbb{R}^n} |\xi|^{-\alpha}\varphi(\xi)d\xi.$$

□

For any $f \in \mathcal{S}(\mathbb{R}^n)$ and $0 < \alpha < n$, we have

$$I_\alpha f(x) = C_{\alpha,n}\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}dy.$$

Remark 11.5. For any $f \in L^p(\mathbb{R}^n)$ with $1 < p < \frac{n}{\alpha}$, then

$$C_{\alpha,n}\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}dy$$

converges absolutely, i.e.,

$$\left|\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}dy\right| < \infty$$

almost everywhere for x.

Let $K(x) = \frac{1}{|x|^{n-\alpha}}$, then $K = K_0 + K_\infty$ where

$$K_0 = K\chi_{\{|x|\leqslant 1\}} \qquad K_\infty = K\chi_{\{|x|>1\}}$$

then

$$|K * f| \leqslant |K_0 * f| + |K_\infty * f|.$$

Notice that $K_0 \in L^1$ since $0 < \alpha < n$, therefore

$$||K_0 * f||_p \leqslant ||K_0||_1||f||_p < \infty,$$

and we also have

$$||K_\infty * f|| \leqslant ||K_\infty||_{p'}||f||_p < \infty$$

since $K_\infty \in L^{p'}$ because $(n-\alpha)p' > n$.

Proposition 11.6.

i. $I_\alpha I_\beta = I_{\alpha+\beta}$ where $0 < \alpha, \beta < n$ and $\alpha + \beta < n$;

ii. $\Delta I_\alpha = I_{\alpha-2}$ for $2 < \alpha < n$;

iii. $(-\Delta)^{\frac{\beta}{2}} I_\alpha = I_{\alpha-\beta}$, where $n > \alpha > \beta > 0$;

iv. $-I_2 f$ is the solution of $\Delta u = f$, that is, I_2 is the Fourier solution of $(-\Delta)$.

Problem 23. Verify Proposition 11.6.

Problem 24. Let μ be a probability measure on a compact subset $E \subseteq \mathbb{R}^n$, and suppose $0 < \alpha < n$. Prove that

$$\int_E\int_E |x-y|^{-\alpha}d\mu(x)d\mu(y) = C_\alpha \int |\hat{\mu}(\xi)|^2|\xi|^{-(n-\alpha)}d\xi$$

where $\hat{\mu}(\xi) = \int_E e^{-2\pi i\xi\cdot x}d\mu(x)$.

Hint: first verify that this identity for μ with smooth density, i.e., $d\mu(x) = \varphi(x)dx$ for $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Hint: let $\varphi(x) = e^{\pi|x|^2}\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(\varepsilon^{-1}x)$, then $\varphi_\varepsilon * \mu = \int_E \varphi_\varepsilon d\mu(y) \in \mathcal{S}(\mathbb{R}^n)$. Now apply the previous hint to μ^ε defined by $d\mu^\varepsilon = \varphi_\varepsilon * \mu dx$. If both parts converge to real numbers, then apply dominant convergence theorem; if at least one part converges to ∞, then apply Fatou's lemma. Also, one may refer to [Wol03].

Conjecture 11.7 (Falconer Conjecture)**.** Let $E \subseteq \mathbb{R}^n$ with Hausdorff dimension $\dim_H(E) > \frac{d}{2}$. Set $\Delta(E) = \{|x-y| : x \in E, y \in E\}$, is $|\Delta(E)| > 0$?

Theorem 11.8 (Hardy-Littlewood-Sobolev)**.** Let $0 < \alpha < n$ and $1 \leqslant p < q < \infty$ where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

i. If $p > 1$, then $||I_\alpha f||_q \leqslant C_{p,q}||f||_p$ for any $f \in \mathcal{S}(\mathbb{R}^n)$ or $L^p(\mathbb{R}^n)$.

ii. If $p = 1$, $|\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}| \leqslant \left(\frac{C||f||_1}{\lambda}\right)^q$.

Proof. We have $||I_\alpha f||_q \leqslant C_{p,q}||f||_p$ where $p > 1$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, then

$$\begin{aligned}
I_\alpha f(x) &= C_{\alpha,n} \int\limits_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y)dy \\
&= C_{\alpha,n} \int\limits_{|x-y|\leqslant R} |x-y|^{\alpha-n} f(y)dy + C_{\alpha,n} \int\limits_{|x-y|>R} |x-y|^{\alpha-n} f(y)dy \\
&=: I_{(1)} + I_{(2)}.
\end{aligned}$$

We use the annuli to approximate the center x via

$$\begin{aligned}
I_{(1)} &\leqslant \sum_{k=0}^{\infty} \int\limits_{2^{-k-1}R<|x-y|\leqslant 2^{-k}R} \frac{C_{\alpha,n}}{(2^{-k}R)^{n-\alpha}} |f(y)|dy \\
&\leqslant C_{\alpha,n} R^{-(n-\alpha)} \sum_{k=0}^{\infty} 2^{k(n-\alpha)} \int\limits_{|x-y|\leqslant 2^{-k}R} |f(y)|dy \\
&= C_{\alpha,n} R^{\alpha} \sum_{k=0}^{\infty} 2^{-\alpha k} \frac{1}{|B(x, 2^{-k}R)|} \int\limits_{B(x,2^{-k}R)} |f(y)|dy \\
&\leqslant C_{\alpha,n} R^{\alpha} Mf(x) \sum_{k=0}^{\infty} 2^{-\alpha k} \\
&\leqslant \tilde{C}_{\alpha,n} R^{\alpha} Mf(x)
\end{aligned}$$

since $C_\alpha := \sum\limits_{k=0}^{\infty} 2^{-\alpha k}$ defines on α. Moreover,

$$\begin{aligned}
I_{(2)} &\leqslant C_{\alpha,n} \left(\int\limits_{|x-y|>R} |x-y|^{(\alpha-n)p'} \right)^{\frac{1}{p'}} ||f||_p \\
&= C_{\alpha,n} \left(\int\limits_{|y|>R} |y|^{(\alpha-n)p'} dy \right)^{\frac{1}{p'}} ||f||_p \\
&= C_{\alpha,n} \left(\int_R^{\infty} \frac{r^{n-1}}{r^{p'(n-\alpha)}} \right)^{\frac{1}{p'}} ||f||_p \\
&= \tilde{C}_{\alpha,n} R^{-\frac{n}{q}} ||f||_p.
\end{aligned}$$

Let us denote $A \lesssim_{\alpha,n} B$ if and only if there exists $C_{\alpha,n} \in \mathbb{R}$ such that $A \leqslant C_{\alpha,n}B$, then

$$I_{(1)} + I_{(2)} \lesssim_{\alpha,n} R^{\alpha} Mf(x) + R^{-\frac{n}{q}} ||f||_p$$

for all $R > 0$. Let us choose $R^{-\frac{n}{p}} = \frac{Mf(x)}{||f||_p}$, then $R^{\alpha}Mf(x) = R^{-\frac{n}{q}}||f||_p$, hence

$$\begin{aligned}|I_\alpha f(x)| &\leqslant |I_{(1)} + I_{(2)}| \\ &\lesssim_{\alpha,n} ||f||_p^{\frac{\alpha-p}{n}} Mf(x)^{\frac{p}{q}}.\end{aligned}$$

- Suppose $p > 1$, then

$$\begin{aligned}||I_\alpha||_q &\lesssim_{\alpha,n} ||f||_p^{\frac{\alpha p}{n}} ||(Mf)^{\frac{p}{q}}||_q \\ &\lesssim_{\alpha,n} ||f||_p^{\frac{\alpha p}{n}} \left(\int |Mf|^p\right)^{\frac{1}{q}} \\ &\lesssim_{\alpha,n} ||f||_p^{\frac{\alpha p}{n}} ||f||_p^{\frac{p}{q}} \\ &\lesssim_{\alpha,n} ||f||_p.\end{aligned}$$

- Suppose $p = 1$, then

$$\begin{aligned}|\{x : |I_\alpha f(x)| > \lambda\}| &\leqslant |\{x : Mf(x) \geqslant C_{\alpha,n}||f||_p^{-\frac{\alpha q}{n}} \lambda^{\frac{q}{p}}\}| \\ &\lesssim_{\alpha,n} \frac{||f||_1}{||f||_1^{-\frac{\alpha q}{n}} \lambda^q} \\ &= \frac{||f||_1^q}{\lambda^p}.\end{aligned}$$

$\square$

Problem 25. Let $0 < \alpha < n$ and ε be a small positive number. Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} |x|^{-\alpha}(\log \frac{1}{|x|})^{-\frac{\alpha}{n}(1+\varepsilon)}, & |x| \leqslant \frac{1}{2} \\ 0, & |x| > \frac{1}{2} \end{cases}$$

then f is measurable. Prove that $f \in L^{\frac{n}{\alpha}}(\mathbb{R}^n)$, but $I_\alpha f \notin L^\infty$ as long as $\frac{\alpha}{n}(1+\varepsilon) \leqslant 1$. Therefore, $||I_\alpha f||_\infty \not\lesssim_{\alpha,n} ||f||_{\frac{n}{\alpha}}$.

12 Continuous Littlewood-Paley Theorem

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be radial and $\int_{\mathbb{R}^n} \psi(x)dx = 0$, or equivalently $\hat{\psi}(0) = 0$.

Remark 12.1. In practice, we take ψ to be a real-valued function. If ψ is radial and real-valued, then $\hat{\psi}$ must be real-valued as well.

Definition 12.2. For any $t > 0$, we denote $\psi_t(x) = t^{-n}\psi(t^{-1}x)$. Define $Q_t f(x) = \psi_t * f(x)$.

Claim 12.3. $\int_0^\infty |\hat{\psi}(t)|^2 \frac{dt}{t} < \infty$.

Proof. We have

$$\begin{aligned}\int_0^\infty |\hat{\psi}(t)|^2 \frac{dt}{t} &= \int_0^1 |\hat{\psi}(t)|^2 \frac{dt}{t} + \int_1^\infty |\hat{\psi}(t)|^2 \frac{dt}{t} \\ &=: I_1 + I_2,\end{aligned}$$

where

$$\begin{aligned}I_2 &\lesssim_N \int_1^\infty \frac{1}{(1+t)^N} \frac{dt}{t} \\ &< \infty\end{aligned}$$

since $\hat{\psi} \in \mathcal{S}(\mathbb{R}^n)$ which has polynomial decay as well. Moreover,

$$\begin{aligned}I_1 &= \int_0^1 |\hat{\psi}(t) - \hat{\psi}(0)|^2 \frac{dt}{t} \\ &\leqslant \int_0^1 |\nabla \hat{\psi}(\eta)|^2 t^2 \frac{dt}{t} \\ &= ||\nabla \hat{\psi}||_{L^\infty([0,1])} \\ &< \infty\end{aligned}$$

for some η between 0 and t, by the Mean Value Theorem. □

Denote $C := \int_0^\infty |\hat{\psi}(t)|^2 \frac{dt}{t}$, then we may normalize ψ so that we may assume

$$\int_0^\infty |\hat{\psi}(t)|^2 \frac{dt}{t} = 1.$$

Theorem 12.4 (Calderón Reproducing Formula)**.** For any $f \in L^2(\mathbb{R}^n)$, we may write $f(x) = \int_0^\infty Q_t^2 f(x) \frac{dt}{t} = \int_0^\infty \psi_t * f(x) \frac{dt}{t}$ in L^2 sense. That is, $||\int_\varepsilon^R Q_t^2 f(x)\frac{dt}{t} - f||_2 \to 0$ as $\varepsilon \to 0$ and $R \to \infty$.

Proof.

$$||\int_\varepsilon^R Q_t^2 f(x) \frac{dt}{t} - f||_2 = ||\widehat{\int_\varepsilon^R Q_t^2 f(x) \frac{dt}{t} - f}||_2$$

$$
\begin{aligned}
&= ||\int_{\varepsilon}^{R} \widehat{Q_t f}\frac{dt}{t} - \hat{f}||_2 \\
&= ||\int_{\varepsilon}^{R} (\hat{\psi}(t|\xi|))^2 \hat{f}(\xi)\frac{dt}{t} - \hat{f}(\xi)||_2 \\
&= ||\hat{f}(\cdot)[\int_{\varepsilon}^{R} (\hat{\psi}(t|\xi|))^2\frac{dt}{t} - 1]||_2 \\
&\to 0
\end{aligned}
$$

by dominated convergence theorem, where $\varepsilon \to 0$ and $R \to \infty$. □

Definition 12.5. We define the Littlewood-Paley g-function to be

$$g(f)(x) = \left(\int_0^\infty |Q_t f(x)|\frac{dt}{t}\right)^{\frac{1}{2}}.$$

Theorem 12.6. For any $f \in L^2$, we have $||g(f)||_2 = ||f||_2$.

Proof. We have

$$
\begin{aligned}
||g(f)||_2^2 &= \int_{\mathbb{R}^n}\int_0^\infty |Q_t f(x)|^2\frac{dt}{t}dx \\
&= \int_{\mathbb{R}^n}\int_0^\infty |\hat{\psi}(t|\xi|)\hat{f}(\xi)|^2\frac{dt}{t}d\xi \\
&= ||\hat{f}||_2^2 \\
&= ||f||_2^2.
\end{aligned}
$$

□

Theorem 12.7. Denote $A \sim B$ if there exists C such that $A \leqslant CB$ and $CA \leqslant B$. Then $||g(f)||_p \sim ||f||_p$ for any $1 < p < \infty$ and $f \in L^p$.

Remark 12.8. Set $p(x) = \frac{C_n}{(1+|x|^2)^{\frac{n+1}{2}}}$ for all $x \in \mathbb{R}^n$, and let $p_t(x) = t^{-n}p(t^{-1}x)$, then

$$g(f)(x) = \left(\int_0^\infty |t\frac{\partial}{\partial t}p_t) * f(x)|^2 dt\right)^{\frac{1}{2}}$$

where $||g(f)||_p \sim ||f||_p$.

13 A Weak Version of T1 Theorem

For any $f \in \mathcal{S}(\mathbb{R}^n)$ and $x \notin \operatorname{supp}(f)$, we have $Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy$. Let $T : \mathcal{S} \to \mathcal{S}'$ be continuous and linear such that $\langle T\varphi, \psi\rangle = \langle K, \varphi \otimes \psi\rangle$, where K is a Calderón-Zygmund kernel.

Definition 13.1 (Weak-boundedness Property)**.** We say T satisfies the weak-boundedness property (WBP) if $|\langle T\varphi,\psi\rangle| \lesssim R^n(||\varphi||_\infty + R||\nabla\varphi||_\infty)\cdot(||\psi||_\infty + R||\nabla\psi||_\infty)$ for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ supported in a ball of radius $R > 0$.

Lemma 13.2. If T can be extended to a bounded operator on L^2, then T satisfies the WBP.

Proof. Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ be supported in B_R, then

$$\begin{aligned}
|\langle T\varphi,\psi\rangle| &\leqslant ||T\varphi||_2||\psi||_2 \\
&\lesssim ||\varphi||_2||\psi||_2 \\
&\lesssim R^n||\varphi||_\infty||\psi||_\infty.
\end{aligned}$$

□

Definition 13.3. For an operator T, we define its adjoint operator T^* via

$$\langle \psi, T^*\varphi\rangle = \int_{\mathbb{R}^n} T^*\varphi(x)\psi(x)dx = \int_{\mathbb{R}^n\times\mathbb{R}^n} \overline{K(y,x)}\varphi(x)\psi(y)dxdy = \langle T\psi,\varphi\rangle$$

for all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$.

Definition 13.4. Let $\mathcal{S}_0(\mathbb{R}^n) = \{\varphi \in C_c^\infty(\mathbb{R}^n) : \int\varphi = 0\}$. Let $\varphi \in \mathcal{S}_0(\mathbb{R}^n)$, then there exists a ball B in $\mathbb{R}^n$ such that $\varphi(x) = 0$ for all $x \in B^c$. One can then define a function η in $C_c^\infty(\mathbb{R}^n)$, taking value 1 on $3B$.

We can now define a $T1$ operator to be such that, for any $\varphi \in \mathcal{S}_0(\mathbb{R}^n)$, $\langle T_1, \varphi\rangle = \langle T\eta, \varphi\rangle + \langle 1-\eta, T^*\varphi\rangle$.

Remark 13.5. The term $\langle 1-\eta, T^*\varphi\rangle$ converges, i.e., it is finite.

Assuming φ is a real-valued function, we have

$$\langle 1-\eta, T^*\varphi\rangle = \int (1-\eta)(x)\left(\int \overline{K^*(x,y)\varphi(y)}dy\right)dx$$

for $K^*(x,y) = \overline{K(x,y)}$. Therefore,

$$|x-y| \geqslant |x-x_0| - |x_0-y| \geqslant 5r(B) - r(B) = 4r(B) \geqslant 2|y-x_0|,$$

where x_0 is the center of the ball B. We thereby obtain a bound of

$$|K(y,x) - K(x_0,x)| \lesssim \frac{|x-x_0|^\varepsilon}{|x-y|^{n+\varepsilon}},$$

so

$$\begin{aligned}
\int \overline{K^*(x,y)\varphi(y)}dy &= \int K(x,y)\varphi(y)dy \\
&= \int [K(y,x) - K(x_0,x)]\varphi(y)dy \\
&\lesssim \int_B \frac{|y-x_0|^\varepsilon}{|x-y|^{n+\varepsilon}}||\varphi||_\infty dy.
\end{aligned}$$

Therefore,

$$|\langle 1-\eta, T^*\varphi\rangle| \leqslant ||\varphi||_\infty \int_{(5B)^c}\int_B \frac{|y-x_0|^\varepsilon}{|x-y|^{n+\varepsilon}}dydx \lesssim ||\varphi||_B \cdot r(B)^n < \infty.$$

Problem 26. Show that
$$\int_{(5B)^c}\int_B \frac{|y-x_0|^\varepsilon}{|x-y|^{n+\varepsilon}}dydx \leqslant C\cdot r(B)^n.$$

As an extra exercise, one should show that the definition of $\langle T,\varphi\rangle$ is independent of choice of η.

Theorem 13.6. Let T be a singular integral operator associated with a Calderón-Zygmund kernel. Suppose that T satisfies the WBP, $T(1)=0$, and $T^*(1)=0$, then T extends to a bounded operator on L^2.

Proof. Let $\varphi\in C_c^\infty(\mathbb{R}^b)$ such that $\int\varphi=1$ and φ is radial, then in particular φ is even. Moreover, we know $\nabla\hat{\varphi}(0)=0$: this is because $\partial_j\hat{\varphi}(\xi)=-2\pi i\int\varphi(x)x_je^{-2\pi x\cdot\xi}dx$, so $\partial_j\hat{\varphi}(0)=-2\pi i\int\varphi(x)x_jdx=0$. Now define P_t by $P_tf(x)=\varphi_t*f(x)$ where $\varphi_t(x)=t^{-n}\varphi(t^{-1}x)$, so $P_t(P_tf)=P_t^2f$, with $P_t^*=P^t$. One can then verify that $T=\lim\limits_{t\to 0}P_t^2TP_t^2$.

Lemma 13.7. Suppose that T satisfies the WBP, then for any $\varphi,\psi\in C_c^\infty(\mathbb{R}^n)$, we have
$$\langle T\varphi,\psi\rangle=\lim_{t\to 0}\left\langle P_t^2TP_t^2\varphi,\psi\right\rangle.$$

Subproof. Assume that φ and ψ are supported in a ball B_R of radius $R>0$. Assume t is very small, so that $P_t^2\varphi$ and $P_t^2\psi$ are supported in B_R as well. Let $||f||=||f||_\infty+R||\nabla f||_\infty$ if $\operatorname{supp}(f)\subseteq B_R$. We want to show that
$$\lim_{t\to 0}|\left\langle P_t^2TP_t^2\varphi,\psi\right\rangle-\langle T\varphi,\psi\rangle|=0.$$

We now have
$$\begin{aligned}|\left\langle TP_t^2\varphi,P_t^2\psi\right\rangle-\langle T\varphi,\psi\rangle| &\leqslant |\left\langle T(P_t^2\varphi-\varphi),P_t^2\psi\right\rangle|+|\left\langle T\varphi,P_t^2\psi-\psi\right\rangle|\\ &\lesssim R^n(||P_t^2\varphi-\varphi||\cdot||P_t^2\psi||+||\varphi||\cdot||P_t^2\psi-\psi||)\end{aligned}$$

by WBP. Since $\int\varphi=1$, then $||P_t^2f||\leqslant||f||$. On the other hand, since for $f\in\mathcal{S}(\mathbb{R}^n)$, we know $||\hat{f}||_\infty||f||_1$ by definition. Therefore, $||f||_\infty\leqslant||\hat{f}||_1$,hence
$$||P_t^2\varphi-\varphi||\leqslant||\widehat{P_t^2\varphi-\varphi}||_1+R\sum_{j=1}^n||\xi_j\left(\widehat{P_t^2\varphi-\varphi}\right)(\xi)||_1,$$

and thus we conclude
$$\widehat{P_t^2\varphi-\varphi}(\xi)=((\hat{\varphi}(t\xi))^2-1)\hat{\varphi}(\xi),$$

and thus
$$\lim_{t\to 0}||\widehat{P_t^2\varphi-\varphi}||_1=\int\lim_{t\to 0}|\hat{\varphi}(t\xi)^2-1|\cdot|\hat{\varphi}(\xi)|d\xi=0.$$

Similarly,
$$\lim_{t\to 0}\sum_{j=1}^n||\xi_j(\widehat{P_t^2\varphi-\varphi})(\xi)||_1=0.$$

Finally, taking $t\to 0$, we conclude that
$$\begin{aligned}||P_t^2\varphi-\varphi||\cdot||P_t^2\psi||+||\varphi||\cdot||P_t^2\psi-\psi|| &\leqslant ||P_t^2\varphi-\varphi||\cdot||\psi||+||\varphi||\cdot||P_t^2\psi-\psi||\\ &\to 0.\end{aligned}$$

■

The proof of Lemma 13.8 below can be done in a similar fashion.

Lemma 13.8. Let T satisfy the WBP, then
$$\lim_{t\to\infty}\left\langle P_t^2TP_t^2\varphi,\psi\right\rangle=0$$
for all $\varphi,\psi\in C_c^\infty(\mathbb{R}^n)$.

Problem 27. Verify Lemma 13.8.

By Lemma 13.7 and Lemma 13.8, for any $\varphi,\psi\in C_c^\infty(\mathbb{R}^n)$, we have
$$\langle T\varphi,\psi\rangle=\lim_{\varepsilon\to 0}\left\langle P_\varepsilon^2TP_\varepsilon^2\varphi-P_{\frac{1}{\varepsilon}}^2TP_{\frac{1}{\varepsilon}}^2\varphi,\psi\right\rangle.$$
To prove that T extends to a bounded operator on L^2, we need to show that
$$\lim_{\varepsilon\to 0}||P_\varepsilon^2TP_\varepsilon^2\varphi-P_{\frac{1}{\varepsilon}}^2TP_{\frac{1}{\varepsilon}}^2\varphi||_2\lesssim||\varphi||_2$$
for all $\varphi\in C_c^\infty$. By the fundamental theorem of calculus, we have
$$(P_\varepsilon^2TP_\varepsilon^2-P_{\frac{1}{\varepsilon}}^2TP_{\frac{1}{\varepsilon}}^2)\varphi=-\int_\varepsilon^{\frac{1}{\varepsilon}}\partial_t(P_t^2TP_t^2\varphi)dt.$$
By product rule, we get
$$\partial_t(P_t^2TP_t^2\varphi)=(\partial_tP_t^2)TP_t^2\varphi+P_t^2\partial_t(TP_t^2)\varphi.$$
Note that we can express $\partial_tP_t^2f(x)=\partial_t(\phi_t*\phi_t)*f(x)$ since P_t^2 is a convolution-type operator, and the second term is similar to the first one by taking the adjoint operator. To see this, we note that
$$P_t^2\partial_t(TP_t^2)=P_t^2T(\partial_tP_t^2)$$
whose adjoint operator is
$$(\partial_tP_t^2)T^*P_t^2$$
as P_t^2 and $\partial_tP_t^2$ are self-adjoint. Therefore, it suffices to estimate the first term, then the estimation for the second term follows similarly. That is, it remains to show that
$$\lim_{\varepsilon\to 0}\left|\left|\int_\varepsilon^{\frac{1}{\varepsilon}}(\partial_tP_t^2)TP_t^2\varphi dt\right|\right|_2\lesssim||\varphi||_2, \tag{13.9}$$
and we may estimate the second term by
$$\lim_{\varepsilon\to 0}\left|\left|\int_\varepsilon^{\frac{1}{\varepsilon}}P_t^2\partial_t(TP_t^2)dt\right|\right|_2\lesssim||\varphi||_2,$$
in a similar fashion.

We set $Q_tf(x)=t(\partial_tP_t^2)f(x)$, so
$$\int_\varepsilon^{\frac{1}{\varepsilon}}(\partial_tP_t^2)TP_t^2\varphi dt=\int_\varepsilon^{\frac{1}{\varepsilon}}Q_tTP_t^2\varphi\frac{dt}{t}.$$
To construct the G-functions, we take the Fourier transform. We have
$$\widehat{Q_tf}(\xi)=t\partial_t\left(\widehat{P_t^2f}(\xi)\right)$$

$$
\begin{aligned}
&= t\partial_t \left(\hat{\phi}^2(t\xi)\right)\hat{f}(\xi)\\
&= 2t\hat{\phi}(t\xi)\xi\cdot(\nabla\hat{\phi})(t\xi)\hat{f}(\xi).
\end{aligned}
$$

Let $\psi_t^{(1)}(x) = \frac{i}{\pi}t^{-n}(\nabla\phi)(t^{-1}x)$ and $\psi_t^{(2)}(x) = -2\pi i t^{-n}\phi(\frac{x}{t})\frac{x}{t}$ for $t > 0$ and $x \in \mathbb{R}^n$. To see why this is useful, suppose $F = (f_1, \ldots, f_n)$ is a complex-valued function, where each f_j is a function on $\mathbb{R}^n$ that is real-valued or complex-valued, then we define its Fourier transform to be

$$\hat{F} = (\hat{f}_1, \ldots, \hat{f}_n).$$

Using this definition, we have

$$\widehat{\psi_t^{(1)}}(\xi) = 2t\hat{\phi}(t\xi)\xi$$

for all $\xi \in \mathbb{R}^n$, and

$$\widehat{\psi_t^{(2)}}(\xi) = (\nabla\hat{\phi})(t\xi).$$

By these estimates, we have

$$
\begin{aligned}
\widehat{Q_t f}(\xi) &= 2t\hat{\phi}(t\xi)\xi\cdot(\nabla\hat{\phi})(t\xi)\hat{f}(\xi)\\
&= \widehat{\psi_t^{(1)}}(\xi)\widehat{\psi_t^{(2)}}(\xi)\hat{f}(\xi).
\end{aligned}
$$

For $F = (f_1, \ldots, f_n)$, we set $F * g = (f_1 * g, \ldots, f_n * g)$. Define vector-valued functions $\vec{Q}_t^{(1)} f(x) = \psi_t^{(1)} * f(x)$ and $\vec{Q}_t^{(2)} f(x) = \psi_t^{(2)} * f(x)$. For $F = (f_1, \ldots, f_n)f$ and $G = (g_1, \ldots, g_n)$, we define their inner product to be $\langle F, G\rangle = \sum_{j=1}^n \langle f_j, g_j\rangle$. For any $f, g \in L^2$, or $\mathcal{S}(\mathbb{R}^n)$, or C_c^∞, we may represent

$$\langle Q_t f, g\rangle = \left\langle \vec{Q}_t^{(2)} f, \vec{Q}_t^{(1)} g\right\rangle$$

because we may take Fourier transform on every term and use the fact that $\widehat{Q_t f} = \widehat{\psi_t^{(1)}} \cdot \widehat{\psi_t^{(2)}}\hat{f}$. One can show that

$$\left|\left|\left(\int_0^\infty |\vec{Q}_t^{(j)} f|^2 \frac{dt}{t}\right)^{\frac{1}{2}}\right|\right|_2 \lesssim ||f||_2$$

which is independent of f. Therefore,

$$
\begin{aligned}
\left|\left\langle \int_\varepsilon^{\frac{1}{\varepsilon}} Q_t T P_t^2 \varphi \frac{dt}{t}, \psi\right\rangle\right| &= \left|\int_\varepsilon^{\frac{1}{\varepsilon}} \left\langle Q_t T P_t^2 \varphi, \psi\right\rangle \frac{dt}{t}\right|\\
&= \left|\int_\varepsilon^{\frac{1}{\varepsilon}} \left\langle \vec{Q}_t^{(2)} T P_t^2 \varphi, Q_t^{(1)}\psi\right\rangle \frac{dt}{t}\right|\\
&\lesssim \left|\left|\left(\int_\varepsilon^{\frac{1}{\varepsilon}} |\vec{Q}_t^{(2)} T P_t^2 \varphi|^2 \frac{dt}{t}\right)^{\frac{1}{2}}\right|\right| \cdot \left|\left|\left(\int_\varepsilon^{\frac{1}{\varepsilon}} |\vec{Q}_t^{(1)}\psi|^2 \frac{dt}{t}\right)^{\frac{1}{2}}\right|\right|\\
&\leqslant \left|\left|\left(\int_\varepsilon^{\frac{1}{\varepsilon}} |\vec{Q}_t^{(2)} T P_t^2 \varphi|^2 \frac{dt}{t}\right)^{\frac{1}{2}}\right|\right| \cdot ||\psi||^2
\end{aligned}
$$

by Cauchy-Schwartz. It then suffices to show

Proposition 13.10. There exists some constant C independent of ε such that

$$\int_{\mathbb{R}^n}\int_{\varepsilon}^{\frac{1}{\varepsilon}} |\vec{Q}_t^{(2)} T P_t^2 \varphi|^2 \frac{dt}{t} dx \leqslant C||\varphi||_2^2$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$.

Proof of Proposition 13.10. Define $\vec{\mathcal{L}}_t = \vec{Q}_t^{(2)} T P_t$ to be a vector-valued singular integral operator associated to a vector-valued kernel L_t. For $F = (f_1, \ldots, f_n)$ and a function g defined on $\mathbb{R}^n$ with values in $\mathbb{R}$ or $\mathbb{C}$, we denote $\langle F, g\rangle = (\langle f_1, g\rangle, \ldots, \langle f_n, g\rangle)$, then

$$\left\langle \vec{\mathcal{L}}_t \varphi, \psi \right\rangle = \int_{\mathbb{R}^n\times\mathbb{R}^n} L_t(x,y)\varphi(y)\psi(x)dxdy.$$

On the other hand, we have

$$\begin{aligned}\left\langle \vec{\mathcal{L}}_t \varphi, t \right\rangle &= \left\langle \vec{Q}_t^{(2)} T P_t \varphi, \psi \right\rangle \\ &= \left\langle T P_t \varphi, \vec{Q}_t^{(2)} \psi \right\rangle \\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n} K(x,y) P_t\varphi(y) \vec{Q}_t^{(2)}\psi(x) dxdy.\end{aligned}$$

Problem 28. We have

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} K(x,y) P_t\varphi(y) \vec{Q}_t^{(2)}\psi(x) dxdy = \int_{\mathbb{R}^n\times\mathbb{R}^n} \left\langle T\varphi_t^y, \psi_t^{(2),x}\right\rangle \varphi(y)\psi(x)dxdy$$

where $\varphi_t^y(z) = \varphi_t(z-y)$ for all z, and $\psi_t^{(2),x}(z) = \psi_t^{(2)}(z-x)$. *Hint*: by weak boundedness of the operator, we may interchange the integral and arrive at this identity.

Therefore, the kernel in the sense of this distribution is

$$L_t(x,y) = \left\langle T\varphi_t^y, \psi_t^{(2),x}\right\rangle.$$

Lemma 13.11. There exists $\sigma \in (0,1]$ such that for any $x, y \in \mathbb{R}^n$,

$$|L_t(x,y)| \leqslant \frac{Ct^\sigma}{(t+|x-y|)^{n+\sigma}}.$$

Proof of Lemma 13.11.

- Suppose $|x-y| < 10t$. Now

$$\begin{aligned}|L_t(x,y)| &= |\left\langle T\phi_t^y, \Psi_t^{(2),x}\right\rangle| \\ &\leqslant t^n(||\Psi_t^{(2),x}||_\infty + t||\nabla\Psi_t^{(2),x}||_\infty)(||\phi_t^y||_\infty + t||\nabla\phi_t^y||_\infty) \\ &= t^n(||\Psi_t^{(2)}||_\infty + t||\nabla\Psi_t^{(2)}||_\infty)(||\psi_t||_\infty + t||\nabla\phi_t||_\infty),\end{aligned}$$

  but since $\max\{||\Psi_t^{(2)}||_\infty, ||\phi_t||_\infty\} \lesssim t^{-n}$, we note $\max\{||\nabla\Psi_t^{(2)}||_\infty, ||\nabla\phi_t||_\infty\} \lesssim t^{-n-1}$, so combining them altogether, we get

$$|L_t(x,y)| \lesssim t^{-n} \lesssim \frac{t^\sigma}{(t+|x-y|)^{n+\sigma}}.$$

- Suppose $|x-y| \geqslant 10t$. In this case, we have

$$\begin{aligned} L_t(x,y) &= \left\langle T\phi_t^y, \Psi_t^{(2),x} \right\rangle \\ &= \int\limits_{\mathbb{R}^n \times \mathbb{R}^n} K(u,v)\phi_t(v-y)\Psi_t^{(2)}(u-x)dudv. \end{aligned}$$

If $u = v$, then $K(u,v)$ is a distribution. From the support condition for ϕ_t and $\Psi_t^{(2)}$, we get $|v-y| \leqslant t$ and $|u-x| \leqslant t$, but since we need these conditions to be true to not evaluate as zero, we must have $u \neq v$:

$$\begin{aligned} |u-v| &= |(u-x)+(x-y)+(y-v)| \\ &\geqslant |x-y| - |u-x| - |y-v| \\ &\geqslant |x-y| - 2t \\ &\geqslant 8t, \end{aligned}$$

so $|u-v| \geqslant 8t \geqslant 8|u-x|$. By Fubini theorem, we have

$$\begin{aligned} |L_t(x,y)| &= \left| \int\limits_{\mathbb{R}^n} \left(\int\limits_{\mathbb{R}^n} K(u,v)\Psi_t^{(2)}(u-x)du \right) \phi_t(v-y)dv \right| \\ &= \left| \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} (K(u,v)-K(x,v))\Psi_t^{(2)}(u-x)du\phi_t(v-y)dv \right| \\ &\lesssim \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{|u-x|^\sigma}{|u-v|^{n+\sigma}} |\Psi_t^{(2)}(u-x)|du|\phi_t(v-y)|dv \\ &\lesssim \int\limits_{B_t(y)} \int\limits_{B_t(x)} \frac{|u-x|^\sigma}{|u-v|^{n+\sigma}} \frac{1}{t^n} \frac{|u-x|}{t} \frac{1}{\left(1+\frac{|u-x|}{t}\right)^N} \frac{1}{t} \frac{1}{\left(1+\frac{|v-y|}{t}\right)^N} dudv. \end{aligned}$$

Since

$$\frac{1}{|u-v|^{n+\sigma}} = \frac{1}{t^{n+\sigma}} \frac{1}{\left|\frac{u-v}{t}\right|^{n+\sigma}} \sim \frac{1}{t^{n+\sigma}} \frac{1}{\left(1+\frac{|u-v|}{t}\right)^{n+\sigma}},$$

and

$$\frac{1}{1+|a|}\frac{1}{1+|b|} \leqslant \frac{1}{1+|a-b|},$$

then

$$|L_t(x,y)| \lesssim \frac{1}{t^n\left(1+\frac{|x-y|}{t}\right)^{n+\sigma}}.$$

∎

By Lemma 13.11, we know $\int\limits_{\mathbb{R}^n} L_t(x,y)dy$ converges absolutely. Since this is an integrable function, we may represent the kernel as an integrable one. In particular, for any $f \in C_c^\infty(\mathbb{R}^n)$, we have

$$\vec{\mathcal{L}}_t f(x) = \int\limits_{\mathbb{R}^n} L_t(x,y)f'(y)dy.$$

Note that the right-hand side is well-defined when $f = 1$, so in particular we get

$$\vec{\mathcal{L}}_t = \int\limits_{\mathbb{R}^n} L_t(x,y)dy.$$

Claim 13.12. If $T1 = 0$, then $\vec{\mathcal{L}}_t 1 = 0$.

Proof of Claim 13.12. For any $\varphi \in C_c^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}\left\langle \vec{\mathcal{L}}_t 1, \varphi \right\rangle &= \left\langle Q_t^{(2)} T P_t 1, \varphi \right\rangle \\ &= \left\langle T1, Q_t^{(2)} \varphi \right\rangle.\end{aligned}$$

By definition, we have

$$\begin{aligned}\int_{\mathbb{R}^n} Q_t^{(2)} \varphi(x) dx &= \int \psi_t^{(2)} * \varphi(x) dx \\ &= \left(\int \varphi\right)\left(\int \psi_t^{(2)}\right) \\ &= 0.\end{aligned}$$

Since $Q_t^{(2)} \varphi \in \mathcal{S}_0(\mathbb{R}^n)$, then this shows that

$$\left\langle \vec{\mathcal{L}}_t 1, \varphi \right\rangle = \left\langle T1, Q_t^{(2)} \varphi \right\rangle = 0.$$

■

Therefore, we know $\int_{\mathbb{R}^n} \vec{\mathcal{L}}_t(x,y) dy = 0$. It remains to show that

$$\int_{\mathbb{R}^n} \int_{\varepsilon}^{\frac{1}{\varepsilon}} |\vec{\mathcal{L}}_t(P_t, \varphi)|^2 \frac{dt}{t} \lesssim ||\varphi||_2^\sigma$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$. Note that

$$\begin{aligned}\int_{\mathbb{R}^n} \int_{\varepsilon}^{\frac{1}{\varepsilon}} |\vec{\mathcal{L}}_t(P_t, \varphi)|^2 \frac{dt}{t} &= \int_{\mathbb{R}^n} \int_{\varepsilon}^{\frac{1}{\varepsilon}} |\vec{\mathcal{L}}_t(P_t\varphi(y) - P_t\varphi(x)) dy|^2 \frac{dt}{t} dx \\ &\lesssim \int_{\mathbb{R}^n} \int_{\varepsilon}^{\frac{1}{\varepsilon}} \left(\int \frac{t^\sigma}{(t+|x-y|)^{n+\sigma}} |P_t\varphi(y) - P_t\varphi(x)| dy\right)^2 \frac{dt}{t} dx\end{aligned}$$

where $\int \frac{t^\sigma}{(t+|x-y|)^{n+\sigma}} |P_t\varphi(y) - P_t\varphi(x)| = \left(\frac{t^\sigma}{(t+|x-y|)^{n+\sigma}}\right)^{\frac{1}{2}} \left(\frac{t^\sigma}{(t+|x-y|)^{n+\sigma}}\right)^{\frac{1}{2}} |P_t\varphi(y) - P_t\varphi(x)|$. Therefore, by Cauchy-Schwartz, we know

$$\begin{aligned}\int_{\mathbb{R}^n} \int_{\varepsilon}^{\frac{1}{\varepsilon}} |\vec{\mathcal{L}}_t(P_t, \varphi)|^2 \frac{dt}{t} &\lesssim \int_{\mathbb{R}^n} \int_{\varepsilon}^{\frac{1}{\varepsilon}} \left(\int \frac{t^\sigma}{(t+|x-y|)^{n+\sigma}} |P_t\varphi(y) - P_t\varphi(x)| dy\right)^2 \frac{dt}{t} dx \\ &\lesssim \int_{\mathbb{R}^n} \int_0^\infty \left(\int \frac{t^\sigma}{(t+|x-y|)^{n+\sigma}} dy\right) \left(\int \frac{t^\sigma}{(t+|x-y|)^{n+\sigma}} |P_t\varphi(y) - P_t\varphi(x)|^2 dy\right) \frac{dt}{t} dx.\end{aligned}$$

Since $\int \frac{t^\sigma}{(t+|x-y|)^{n+\sigma}} dy = C_{n,\sigma}$ by a change of variables, we know that

$$\int_{\mathbb{R}^n} \int_{\varepsilon}^{\frac{1}{\varepsilon}} |\vec{\mathcal{L}}_t(P_t, \varphi)|^2 \frac{dt}{t} \lesssim \int_{\mathbb{R}^n} \int_0^\infty \left(\int \frac{t^\sigma}{(t+|x-y|)^{n+\sigma}} dy\right) \left(\int \frac{t^\sigma}{(t+|x-y|)^{n+\sigma}} |P_t\varphi(y) - P_t\varphi(x)|^2 dy\right) \frac{dt}{t} dx$$

$$
\begin{aligned}
&\lesssim_{n,\sigma} \int_{\mathbb{R}^n} \int_0^\infty \int \frac{t^\sigma}{(t+|x-y|)^{n+\sigma}} |P_t\varphi(y) - P_t\varphi(x)|^2 dx \frac{dt}{t} dy \\
&\lesssim_{n,\sigma} \int_{\mathbb{R}^n} \int_0^\infty \int \frac{t^\sigma}{(t+|x-y|)^{n+\sigma}} |P_t\varphi(y) - P_t\varphi(u+y)|^2 du \frac{dt}{t} dy \\
&= \int_{\mathbb{R}^n} \int_0^\infty \frac{t^\sigma}{(t+|u|)^{n+\sigma}} \left(\int_{\mathbb{R}^n} |P_t\varphi(y) - P_t\varphi(u+y)|^2 dy \right) \frac{dt}{t} du.
\end{aligned}
$$

By Theorem 4.5, we know that

$$
\int_{\mathbb{R}^n} |P_t\varphi(y) - P_t\varphi(u+y)|^2 dy = \int_{\mathbb{R}^n} |e^{2\pi i u\cdot\xi} - 1|^2 |\hat{\varphi}(t\xi)|^2 |\hat{\varphi}(\xi)|^2 d\xi,
$$

therefore

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \int_0^\infty \frac{t^\sigma}{(t+|u|)^{n+\sigma}} \left(\int_{\mathbb{R}^n} |P_t\varphi(y) - P_t\varphi(u+y)|^2 dy \right) \frac{dt}{t} du \\
&= \int_{\mathbb{R}^n} |\hat{\varphi}(\xi)| \left(\int_{\mathbb{R}^n} \int_0^\infty |e^{2\pi i u\cdot\xi} - 1| \cdot |\hat{\varphi}(t\xi)| \frac{t^\sigma}{(t+|u|)^{n+\sigma}} \frac{dt}{t} du \right) d\xi.
\end{aligned}
$$

Finally, it suffices to show that

Lemma 13.13. There exists some constant C independent of ξ such that

$$
\int_{\mathbb{R}^n} \int_0^\infty |e^{2\pi i u\cdot\xi} - 1| \cdot |\hat{\varphi}(t\xi)| \frac{t^\sigma}{(t+|u|)^{n+\sigma}} \frac{dt}{t} du \leqslant C.
$$

Subproof. Without loss of generality, assume that $\xi \neq 0$. Now take $\delta = \frac{\sigma}{2} > 0$ and $\varepsilon = \frac{\delta}{2}$, then we have

$$
\begin{aligned}
\int_{\mathbb{R}^n} \int_0^\infty |e^{2\pi i u\cdot\xi} - 1| \cdot |\hat{\varphi}(t\xi)| \frac{t^\sigma}{(t+|u|)^{n+\sigma}} \frac{dt}{t} du &\lesssim \int_{\mathbb{R}^n} \int_0^\infty |u\cdot\xi|^\delta |\hat{\varphi}(t\xi)|^2 \frac{t^\sigma}{(t+|u|)^{n+\sigma}} \frac{dt}{t} du \\
&= \int_{\mathbb{R}^n} \int_0^\infty |u\cdot\xi|^\delta |\hat{\varphi}(t|\xi|)|^2 \frac{t^\sigma}{(t+|u|)^{n+\sigma}} \frac{dt}{t} du \\
&\quad \text{by a change of variable } u \mapsto tu, \\
&= \int_{\mathbb{R}^n} \int_0^\infty t^\delta |u|^\delta |\xi|^\sigma |\hat{\varphi}(t\xi)|^2 \frac{t^\sigma}{(t+|tu|)^{n+\sigma}} \frac{dt}{t} \cdot t^n du \\
&= \left(\int_{\mathbb{R}^n} \frac{|u|^\delta}{(1+|u|)^{n+\sigma}} du \right) \left(\int_0^\infty (t|\xi|)^\delta |\hat{\varphi}(t|\xi|)|^2 \frac{dt}{t} \right) \\
&\leqslant \left(\int_{\mathbb{R}^n} \frac{(1+|u|)^\delta}{(1+|u|)^{n+\delta+\delta}} du \right) \left(\int_0^\infty (t|\xi|)^\delta |\hat{\varphi}(t|\xi|)|^2 \frac{dt}{t} \right) \\
&\lesssim_{n,\sigma} \int_0^\infty |t|^\delta |\hat{\varphi}(t)|^2 \frac{dt}{t}
\end{aligned}
$$

$$\lesssim_{n,\sigma} C.$$

$\blacksquare$

$\blacksquare$

$\square$

Problem 29. Let K be a Calderón-Zygmund kernel that is anti-symmetric, i.e., $K(x,y) = -K(y,x)$. Let T be the singular integral operator associated with K. Prove that T satisfies the WBP condition.

Problem 30. Prove that

i. for any $\delta > 0$, $\int\limits_{\mathbb{R}^n} e^{-\pi\delta|\xi|^2} e^{-2\pi i x\cdot\xi} d\xi = \delta^{-\frac{n}{2}} e^{-\pi|x|^2/\delta}$;

ii. for any $\gamma > 0$, $e^{-\gamma} = \frac{1}{\sqrt{\pi}} \int\limits_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\frac{\gamma^2}{4u}} du$. *Hint*: note that $e^{-\gamma} = \frac{1}{\pi} \int\limits_{-\infty}^\infty \frac{e^{i\gamma z}}{1+x^2} dx$ and $\frac{1}{1+x^2} = \int\limits_0^\infty e^{-(1+x^2)u} du$;

iii. $\widehat{e^{-2\pi t|\cdot|}}(x) = \frac{C_n t}{(|x|^2+t^2)^{\frac{n+1}{2}}}$.

14 Bounded Mean Oscillation and Sharp Functions

Definition 14.1. Let f be a locally integrable function on $\mathbb{R}^n$, and let Q be a cube in $\mathbb{R}^n$, then we define $f_Q = \frac{1}{|Q|}\int\limits_Q f(x)dx$, and the bounded mean oscillation (BMO) of f is defined by

$$||f||_{\text{BMO}} = \sup_{\text{cube } Q \text{ in } \mathbb{R}^n} \frac{1}{|Q|}\int\limits_Q |f(x) - f_Q(x)|dx.$$

Moreover, we define the collection of functions with bounded mean oscillation on $\mathbb{R}^n$ to be $\text{BMO}(\mathbb{R}^n) = \{f \in L^1_{\text{loc}}(\mathbb{R}^n) : ||f||_{\text{BMO}} < \infty\}$.

Remark 14.2.

- $L^\infty(\mathbb{R}^n) \subseteq \text{BMO}(\mathbb{R}^n)$;
- if f is a constant function, then $||f||_{\text{BMO}} = 0$;
- suppose f and g are functions such that $f - g$ is a constant function, then $||f||_{\text{BMO}} = ||g||_{\text{BMO}}$. In particular, in the function space $\text{BMO}(\mathbb{R}^n)$, this implies $f = g$ in $\text{BMO}(\mathbb{R}^n)$.

Lemma 14.3. $||f||_{\text{BMO}} \sim \sup\limits_{\text{cube } Q \text{ in } \mathbb{R}^n} \inf\limits_{c\in\mathbb{C}} \frac{1}{|Q|}\int\limits_Q |f(x) - c|dx.$

Problem 31. Prove Lemma 14.3.

Theorem 14.4 (John-Nirenberg)**.** There exists $C_1, C_2 > 0$ such that for any $f \in \text{BMO}(\mathbb{R}^n)$ and any cube $Q \subseteq \mathbb{R}^n$ and any $\lambda > 0$, we have

$$|\{x \in Q : f(x) - f_Q > \lambda\}| \leqslant e^{-\frac{c_2\lambda}{||f||_{\text{BMO}}}}|Q|.$$

To prove the theorem, we need a few lemmas.

Lemma 14.5. Let $Q \subseteq \mathbb{R}^n$ be a cube and $\lambda > 0$. Suppose $f \in L^1(Q)$ and $\frac{1}{|Q|}\int\limits_Q |f(x)|dx < \lambda$, then there exists a sequence $\{Q_j\}_{j\geqslant 1}$ of pairwise disjoint[5] sub-cubes of Q, such that

1. $|f(x)| \leqslant \lambda$ almost everywhere for $Q\setminus\bigcup\limits_{j\geqslant 1} Q_j$, and
2. $\lambda \leqslant \frac{1}{|Q_j|}\int\limits_{Q_j} |f| < 2^n\lambda$.

Problem 32. Prove Lemma 14.5.

Hint: use a stopping time argument.

Lemma 14.6. Let $f \in \text{BMO}(\mathbb{R}^n)$ with $||f||_{\text{BMO}} = 1$, and let $Q \subseteq \mathbb{R}^n$ be a cube, then there exists a sequence $\{Q_j\}_{j\geqslant 1}$ of pairwise disjoint sub-cubes of Q such that

1. $|f(x) - f_Q| \leqslant \frac{3}{2}$ almost everywhere for $x \in Q\setminus\bigcup\limits_j Q_j$,
2. $\sum\limits_j |Q_j| \leqslant \frac{2}{3}|Q|$, and
3. $\frac{1}{|Q_j|}\int\limits_{Q_j} |f(x) - f_Q| < 3 \cdot 2^{n-1}$.

[5] By pairwise disjoint, we mean the borders may touch.

Proof. Apply Lemma 14.5 to the function $f - f_Q$ with $\lambda = \frac{3}{2}$, which can be done since

$$\begin{aligned}\frac{1}{|Q|}\int_Q |(f - f_Q)(x)|dx &\leqslant ||f||_{\text{BMO}} \\ &\leqslant 1 \\ &< \frac{3}{2},\end{aligned}$$

so there exists a sequence $\{Q_j\}_{j\geqslant 1}$ of pairwise disjoint sub-cubes of Q, such that

1. $|(f - f_Q)(x)| \leqslant \frac{3}{2}$ almost everywhere for $Q \backslash \bigcup\limits_j Q_j$, and
2. $\frac{3}{2} \leqslant \frac{1}{|Q_j|}\int\limits_{Q_j} |f - f_Q| < 3 \cdot 2^{n-1}$.

It suffices to show that $\sum\limits_j |Q_j| \leqslant \frac{2}{3}|Q|$. Since $\frac{3}{2} \leqslant \frac{1}{|Q_j|}\int\limits_{Q_j} |f - f_Q| < 3 \cdot 2^{n-1}$, then $|Q_j| \leqslant \frac{2}{3}\int\limits_{Q_j} |f - f_Q|$, therefore

$$\begin{aligned}\sum_j |Q_j| &\leqslant \frac{2}{3}\sum_j \int_{Q_j} |f - f_Q| \\ &\leqslant \frac{2}{3}\int_Q |f - f_Q| \\ &\leqslant \frac{2}{3}|Q| \cdot ||f||_{\text{BMO}}.\end{aligned}$$

□

Proof of Theorem 14.4. Without loss of generality, we may assume that $||f||_{\text{BMO}} = 1$, since we can apply a dilation argument for the general case. We will show that the level set

$$|\{x \in Q : |f(x) - f_Q| > \lambda\}| \leqslant C_1 e^{-C_2\lambda}|Q|$$

by applying Lemma 14.6 repeatedly. Let us first apply Lemma 14.6 for the given cube Q and function f, then we get a sequence $\{Q_j^{(1)}\}_{j\geqslant 1}$ of disjoint sub-cubes $Q_j^{(1)} \subseteq Q$ such that

- $|f(x) - f_Q| \leqslant \frac{3}{2}$ almost everywhere for $x \in Q \backslash \bigcup\limits_j Q_j^{(1)}$,
- $\sum\limits_j |Q_j^{(1)}| \leqslant \frac{2}{3}|Q|$, and
- $\frac{1}{|Q_j^{(1)}|}\int\limits_{Q_j^{(1)}} |f(x) - f_Q| < 3 \cdot 2^{n-1}$.

Define $J^{(1)} = \{Q_j^{(1)} : j \in \mathbb{N}\}$ to be the set of all such cubes. For each cube $Q^{(1)}$ in $J^{(1)}$, we apply Lemma 14.6 again, then we get a sequence $\{Q_j^{(2)}\}_{j\geqslant 1}$ of sub-cubes of $Q^{(1)}$ such that

- $|f(x) - f_{Q^{(1)}}| \leqslant \frac{3}{2}$ almost everywhere for $x \in Q^{(1)} \backslash \bigcup\limits_j Q_j^{(2)}$,
- $\sum\limits_j |Q_j^{(2)}| \leqslant \frac{2}{3}|Q^{(1)}|$, and
- $\frac{1}{|Q_j^{(2)}|}\int\limits_{Q_j^{(2)}} |f(x) - f_{Q^{(1)}}| < 3 \cdot 2^{n-1}$.

Define $J^{(2)} = \{Q_j^{(2)} : j \in \mathbb{N}\}$ to be the set of all such cubes. We have

$$\bigcup_{j\in\mathbb{N}} Q_j^{(2)} = \bigcup_{Q^{(1)}\subseteq J^{(1)}} \bigcup_{j\in J^{(1)}(Q^{(1)})} Q_j^{(2)},$$

therefore

$$\begin{aligned}\sum_j |Q_j^{(2)}| &\leqslant \frac{2}{3}\sum_{Q^{(1)}} |Q^{(1)}| \\ &\leqslant \left(\frac{2}{3}\right)^2 |Q|.\end{aligned}$$

Moreover, we claim that $|f(x) - f_Q| \leqslant \frac{3}{2} + 3\cdot 2^{2-1}$ almost everywhere for $x \in Q \setminus \bigcup_{j\in\mathbb{N}} Q_j^{(2)}$. This can be done by considering two cases:

- if x does not belong to any cube of the form $Q^{(1)}$, then $|f(x) - f_Q| \leqslant \frac{3}{2}$;
- if $x \in Q^{(1)}$ for some cube $Q^{(1)} \in J^{(1)}$, then

$$\begin{aligned}|f(x) - f_Q| &\leqslant |f(x) - f_{Q^{(1)}}| + |f_{Q^{(1)}} - f_Q| \\ &\leqslant |f(x) - f_{Q^{(1)}}| + \frac{1}{|Q^{(1)}|}\int_{Q^{(1)}} |f - f_Q|\end{aligned}$$

  by triangle inequality.

By applying this argument repeatedly, at the Nth step we obtain a sequence $\{Q_j^{(N)}\}_{j\geqslant 1}$ of disjoint sub-cubes of Q such that

- $|f(x) - f_Q| \leqslant \frac{3}{2} + 3(N-1)2^{n-1} \leqslant 3N2^{n-1}$ almost everywhere for $x \in Q\setminus\bigcup_j Q_j^{(N)}$, and
- $\sum_j |Q_j^{(N)}| \leqslant \left(\frac{2}{3}\right)^N |Q|$.

If $\lambda < 3\cdot 2^{n-1}$, the conclusion is trivial. For any $\lambda \geqslant 3\cdot 2^{n-1}$, there exists some $N \in \mathbb{N}$ such that $3N2^{n-1} \leqslant \lambda < 3(N+1)2^{n-1}$, then

$$\begin{aligned}|\{x \in Q : |f(x) - f_Q| > \lambda\}| &= |\{x \in \bigcup_j Q_j^{(N)} : |f(x) - f_Q| > \lambda\}| \\ &\leqslant \sum_j |Q_j^{(N)}| \\ &\leqslant \left(\frac{2}{3}\right)^N |Q| \\ &< e^{-c_2\lambda}|Q|\end{aligned}$$

where $c_2 = \frac{\log\left(\frac{3}{2}\right)}{3\cdot 2^{n-1}}$. □

Definition 14.7. For $1 \leqslant p < \infty$, we define $||f||_{\mathrm{BMO},p} = \sup_{\text{cube } Q \text{ in } \mathbb{R}^n} \left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^p dx\right)^{\frac{1}{p}}$. Under this notation, $||f||_{\mathrm{BMO}} = ||f||_{\mathrm{BMO},1}$.

Corollary 14.8. For any $1 \leqslant p < \infty$, $||f||_{\mathrm{BMO},p} \sim ||f||_{\mathrm{BMO}}$.

Proof. We need to show that $||f||_{\text{BMO},p} \lesssim_p ||f||_{\text{BMO}}$. To calculate the L^p-norm of the difference, we have

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |f(x)-f_Q|^p dx &= \frac{p}{|Q|}\int_0^\infty \lambda^{p-1}|\{x \in Q : |f(x)-f_Q| > \lambda\}|d\lambda \\
&\lesssim_p \int_0^\infty \lambda^{p-1} e^{-\frac{c\lambda}{||f||_{\text{BMO}}}} d\lambda \text{ by Theorem 14.4} \\
&= ||f||_{\text{BMO}}^p \int_0^\infty \lambda^{p-1}e^{-\lambda}d\lambda \text{ by changing } \lambda \to \frac{||f||_{\text{BMO}}}{c}\lambda.
\end{aligned}$$

□

Definition 14.9. Given a function f, we define the sharp function of f to be $f^\#(x) = \sup_{\text{cube } x\in Q \text{ in } \mathbb{R}^n} \frac{1}{|Q|}\int_Q |f(x)-f_Q|dx$.

Remark 14.10. Since $f^\#(x) \lesssim Mf(x)$, then $||f^\#||_\infty \lesssim ||Mf||_\infty \lesssim ||f||_\infty$. Based on the same observation, we have we have $||f^\#||_p \lesssim ||f||_p$ for any $1 < p \leqslant \infty$. For the rest of the section, we will show that the reverse inequality still holds.

Definition 14.11. Let $k \in \mathbb{Z}$. We define a dyadic cube to be $\mathscr{D}_k = \left\{\prod_{j=1}^n [2^{-k}n_j, 2^{-k}(n_j+1)) : n_j \in \mathbb{Z}\right\}$. The collection of dyadic cubes is defined by $\mathscr{D} = \bigcup_{k\in\mathbb{Z}} \mathscr{D}_k$.

The dyadic cubes define a grid structure: for any $Q_1, Q_2 \in \mathscr{D}$, either $Q_1 \cap Q_2 = \varnothing$, or $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$. Let us define

$$M_d f(x) = \sup_{x\in Q\in\mathscr{D}} \frac{1}{|Q|}\int_Q |f(y)|dy.$$

Obviously $M_d f(x) \leqslant Mf(x)$, and conversely $M_d f(x) \gtrsim Mf(x)$.

Remark 14.12. It is not true that $M_d f(x) \lesssim f^\#(x)$.

However, even though we don't have a pointwise estimate, we may estimate it in the sense of distributions.

Theorem 14.13 (Good-λ Inequality)**.** For any $\gamma > 0$ and any $\lambda > 0$, we have the following level set estimate:

$$|\{x \in \mathbb{R}^n : M_d f(x) > 2\lambda, f^\#(x) < \gamma\lambda\}| \leqslant 2^n\gamma|\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}|.$$

Proof. By Lemma 14.5, we may write $\{x \in \mathbb{R}^n : M_d f(x) > \lambda\} = \left(\bigsqcup_Q Q\right) \cup N$ as a disjoint union of cubes along with a null set N. Therefore, it remains to show that for any maximal[6] dyadic cube Q in $\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}$, we have

$$|\{x \in Q : M_d f(x) > 2\lambda, f^\#(x) < \gamma\lambda\}| \leqslant 2^n\gamma|Q|. \tag{14.14}$$

Problem 33. For any maximal dyadic cube Q in $\{x : M_d f(x) > \lambda\}$, if $x \in Q$ and $M_d f(x) > 2\lambda$, then

$$M_d(f\chi_Q)(x) > 2\lambda.$$

Given a dyadic cube Q, suppose Q^* is its unique parent Q. By maximality of Q, then $Q^* \nsubseteq \{x : M_d f(x) > \lambda\}$, therefore

$$\frac{1}{|Q^*|}\int_{Q^*} |f(x)|dx \leqslant \lambda.$$

[6] Here Q is called a maximal cube in E if $Q \subseteq E$ but $2Q \nsubseteq E$.

For $f_{Q*} = \frac{1}{|Q*|}\int\limits_{Q*} f$, we have

$$\begin{aligned} M_d(f_{Q*}\chi_Q)(x) &= |M_d(f_{Q*}\chi_Q)(x)| \\ &\leqslant |f_{Q*}|M_d(\chi_Q)(x) \\ &\leqslant \lambda||\chi_Q||_\infty \\ &\leqslant \lambda. \end{aligned}$$

Using this estimate, we bound

$$\begin{aligned} M_d((f-f_{Q*})\chi_Q) &\geqslant M_d(f\chi_Q) - M_d(f_{Q*}\chi_Q) \\ &\geqslant M_d(f\chi_Q) - \lambda \\ &> 2\lambda - \lambda \text{ by Problem 33 as } x \in Q \text{ and } M_d f(x) > 2\lambda \\ &= \lambda. \end{aligned}$$

Therefore,

$$\{x \in Q : M_d f(x) > 2\lambda, f^{\#}(x) < \gamma\lambda\} \subseteq \{x \in Q : M_d((f-f_{Q*})\chi_Q)(x) > \lambda\}.$$

By the fact that M_d is of type weak $(1,1)$, we note that

$$\begin{aligned} |\{x \in Q : M_d((f-f_{Q*})\chi_Q)(x) > \lambda\}| &\leqslant \frac{\int\limits_Q |f-f_{Q*}|}{\lambda} \\ &\leqslant \frac{2^n|Q|}{\lambda}\frac{1}{|Q*|}\int\limits_{Q*}|f-f_{Q*}| \\ &\leqslant \frac{2^n|Q|}{\lambda}\inf_{x\in Q*} f^{\#}(x) \\ &\leqslant \frac{2^n|Q|}{\lambda}\inf_{x\in Q} f^{\#}(x). \end{aligned}$$

If $\{x \in Q : f^{\#}(x) < 2\lambda\} = \varnothing$, then the statement is true trivially, so suppose $\{x \in Q : f^{\#}(x) < 2\lambda\} \neq \varnothing$, then

$$\begin{aligned} |\{x \in Q : M_d((f-f_{Q*})\chi_Q)(x) > \lambda\}| &\leqslant \frac{2^n|Q|}{\lambda}\inf_{x\in Q} f^{\#}(x) \\ &\leqslant \frac{2^n|Q|}{\lambda}\gamma\lambda \\ &= 2^n\gamma|Q|, \end{aligned}$$

as desired. ∎

Theorem 14.15. Let $p \in [1,\infty)$. Suppose that $f \in L^{p_0}$ for some $p_0 \in [1,p]$, then there exists a constant $C_{p,n}$ such that

$$(||f||_p \lesssim)||M_d f||_p \leqslant C_{p,n}||f^{\#}||_p.$$

Proof. We have

$$\begin{aligned} ||M_d f||_p^p &= p\int_0^\infty \lambda^{p-1}|\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}|d\lambda \text{ which converges under assumption} \\ &= p2^p\int_0^\infty \lambda^{p-1}|\{x \in \mathbb{R}^n : M_d f(x) > 2\lambda\}|d\lambda \text{ by a change of variables } \lambda \to 2\lambda \\ &\lesssim_p \int_0^\infty \lambda^{p-1}|\{x \in \mathbb{R}^n : M_d f(x) > 2\lambda, f^{\#}(x) < \gamma\lambda\}|d\lambda + \int_0^\infty \lambda^{p-1}|\{x \in \mathbb{R}^n : f^{\#}(x) > \gamma\lambda\}|d\lambda \end{aligned}$$

$$\lesssim \gamma \int_0^\infty \lambda^{p-1} |\{x \in \mathbb{R}^n : M_d f(x) > \lambda\}| d\lambda + \frac{1}{\gamma^p} ||f^\#||_p^p$$
$$\lesssim \gamma ||M_d f||_p^p + \frac{1}{\gamma^p} ||f^\#||_p^p$$

for any $\gamma > 0$. Let us choose γ small enough such that γ multiplied by the hidden coefficients is still less than $\frac{1}{2}$, then this gives $||M_d f||_p^p \lesssim ||f^\#||_p^p$. □

Theorem 14.16. Let $p_0 \in (1, \infty)$, and let T be a linear operator satisfying

$$||Tf||_{p_0} \lesssim ||f||_{p_0}$$

for all $f \in L^{p_0}$. Suppose $||Tf||_{\mathrm{BMO}} \lesssim ||f||_\infty$ for any $f \in L^\infty$, then $||Tf||_p \lesssim ||f||_p$ for any $f \in L^p$ and $p_0 < p < \infty$.

Remark 14.17. This is a weaker interpolation result since we replaced $||Tf||_\infty$ by $||Tf||_{\mathrm{BMO}}$.

Proof. Define $T^\# f(x) = (Tf)^\#(x)$, then $T^\#$ is a sublinear operator. We have

$$||T^\# f||_{p_0} = ||(Tf)^\#||_{p_0} \lesssim ||Tf||_{p_0} \lesssim ||f||_{p_0}.$$

But by definition we have $||T^\# f||_\infty = ||Tf||_{\mathrm{BMO}} \lesssim ||f||_\infty$, then $||Tf||_p \lesssim ||T^\# f||_p \lesssim ||f||_p$ for any $p \in (p_0, \infty)$ for all $f \in L^p$. □

15 Carleson Measures

Let us denote $\mathbb{R}^{n+1}_+$ to be the upper half plane $\{(x,t)\in\mathbb{R}^n\times\mathbb{R}: t>0\}$.

Definition 15.1. Let Q be a cube in $\mathbb{R}^n$ with side length $\ell(Q)$, then we define a Carleson box $\hat{Q}$ by

$$\hat{Q}=\{(x,t)\in\mathbb{R}^{n+1}_+: x\in Q, 0\leqslant t<\ell(Q)\}.$$

A Borel measure μ of domain $\mathcal{B}_{\mathbb{R}^{n+1}_+}$ is called a Carleson measure if $\mu(\hat{Q})\leqslant C|Q|$ for all cube $Q\subseteq\mathbb{R}^n$. The norm of μ is defined by

$$||\mu||=\sup_Q\frac{\mu(\hat{Q})}{|Q|}.$$

Let f be a measurable function on $\mathbb{R}^{n+1}_+$, then we define the non-tangential maximal function $\mathcal{N}^+f(x)=\sup_{(y,t)\in\Gamma(x)}|f(y,t)|$, where $\Gamma(x)$ is a cone generated by $x\in\mathbb{R}^n$, to be

$$\Gamma(x)=\{(y,t)\in\mathbb{R}^{n+1}_+: |y-x|<t\}.$$

Theorem 15.2. Let f be a continuous function on $\mathbb{R}^{n+1}_+$ and μ be a Carleson measure, then

$$\int_{\mathbb{R}^{n+1}_+}|f(x,t)|^p d\mu\lesssim||\mu||\int_{\mathbb{R}^n}|\mathcal{N}^*f(x)|^p dx$$

for any $0<p<\infty$. Alternatively, we may write this inequality as follows:

$$||f||_{L^p(\mathbb{R}^{n+1}_+,d\mu)}\lesssim||\mu||^{\frac{1}{p}}||\mathcal{N}^*f||_{L^p(\mathbb{R}^n)}.$$

Theorem 15.3 (Whitney Decomposition)**.** Let Ω be an open set in $\mathbb{R}^n$ and $\Omega^c\neq\varnothing$, then there is a collection of non-overlapping cubes $\{Q_j\}_{j\in\mathbb{N}}$ such that

i. $\Omega=\bigcup_j Q_j$, and

ii. there exists constants $c_1(\Omega), c_2(\Omega)$ independent of Q such that

$$c_1\ell(Q)\leqslant\operatorname{dist}(Q,\Omega^c)\leqslant c_2\ell(Q).$$

Proof of Theorem 15.3. Recall that for any $k\in\mathbb{Z}$, we defined the dyadic cube to be

$$\mathscr{D}_k=\left\{\prod_{j=1}^n\left[2^{-k}n_j,2^{-k}(n_j+1)\right): n_j\in\mathbb{Z}\right\}.$$

For any $k\in\mathbb{Z}$, we define

$$\Omega_k=\left\{x\in\Omega: 3\sqrt{n}\cdot2^{-k}<\operatorname{dist}(x,\Omega^c)\leqslant3\sqrt{n}\cdot2^{1-k}\right\}.$$

Therefore, these are the points $x\in\Omega$ such that $\operatorname{dist}(x,\Omega^c)$ is comparable to 2^{-k}. In particular, we have a partition $\Omega=\bigcup_{k\in\mathbb{Z}}\Omega_k$. Let us now define

$$\mathcal{J}_k=\{Q\in\mathscr{D}_k: Q\cap\Omega_k\neq\varnothing\},$$

and define $\mathcal{J}=\bigcup_{k\in\mathbb{Z}}\mathcal{J}_k$. To finish the proof, it suffices to prove the following statement.

Problem 34. Prove that $\Omega=\bigcup_{Q\in\mathcal{J}}Q$.

□

Proof of Theorem 15.2. Let us define the level sets

$$E_\lambda = \{(x,t) \in \mathbb{R}^{n+1}_+ : |f(x,t)| > \lambda\} \qquad \text{and} \qquad E^*_\lambda = \{x \in \mathbb{R}^n : \mathcal{N}^* f(x) > \lambda\}.$$

It now suffices to show that

Claim 15.4. $\mu(E_\lambda) \lesssim ||\mu|||E^*_\lambda|$.

Indeed, recall that

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+} |f(x,t)|^P d\mu &= p\int_0^\infty \lambda^{p-1}\mu(E_\lambda)d\lambda \\ &\lesssim_p ||\mu|| \int_0^\infty \lambda^{p-1}|E^*_\lambda| d\lambda \\ &\lesssim_p ||\mu|| \int_{\mathbb{R}^n} |\mathcal{N}^* f|^p dx\end{aligned}$$

To prove Claim 15.4, one can assume that $|E^*_\lambda| < \infty$, so that its complement $(E^*_\lambda)^c \neq \varnothing$. By Theorem 15.3, we may represent $E^*_\lambda = \bigcup_j Q_j$ and

$$c_1 \ell(Q_j) \leqslant \operatorname{dist}(Q_j, (E^*_\lambda)^c) \leqslant c_2 \ell(Q_j).$$

Lemma 15.5. There is an absolute constant α such that

$$E_\lambda \subseteq \bigcup_j \widehat{\alpha Q_j},$$

where αQ_j is the dilation of Q_j by α, with the center fixed.

Let us show that Lemma 15.5 implies Claim 15.4. By Lemma 15.5, we have

$$\begin{aligned}\mu(E_\lambda) &\leqslant \mu\left(\bigcup_j \widehat{\alpha Q_j}\right) \\ &\leqslant \sum_j \mu\left(\widehat{\alpha Q_j}\right) \\ &\lesssim_\alpha ||\mu|| \sum_j |Q_j| \\ &\lesssim ||\mu|| \cdot |E^*_\lambda|.\end{aligned}$$

Therefore, to finish the proof of Theorem 15.2, it suffices to show Lemma 15.5.

Subproof of Lemma 15.5. For any ball or cube B in $\mathbb{R}^n$, a tent based on B is given by

$$T(B) = \{(y,t) \in \mathbb{R}^{n+1}_+ : B(y,t) \subseteq B\}.$$

Claim 15.6. For any $(y,t) \in E_\lambda$, then $B(y,t) \subseteq E^*_\lambda$.

Subproof of Claim 15.6. Note that $x \in B(y,t)$ if and only if $(y,t) \in \Gamma(x)$. Now

$$\begin{aligned}\mathcal{N}^* f(x) &= \sup_{(y',t)\in\Gamma(x)} |f(y',t)| \\ &\geqslant |f(y,t)| \\ &> \lambda\end{aligned}$$

since $(y,t) \in E_\lambda$. ■

Now set $\alpha = 100c_2$.

Claim 15.7. We have $E_\lambda \subseteq \bigcup_j T(\alpha Q_j)$.

Subproof of Claim 15.7. For any $(y,t) \in E_\lambda$, we know $B(y,t) \subseteq E_\lambda^* = \bigcup_j Q_j$ by Claim 15.6. We have two possible cases:

- Case 1: every Q_j such that $Q_j \cap B(y,t) \neq \varnothing$ satisfies $\ell(Q_j) \leqslant \frac{4t}{\alpha}$. We claim that this would never happen: suppose it happens, then there exists some cube Q_{j_0} such that $y \in Q_{j_0}$. Therefore, we have $\ell(Q_{j_0}) \leqslant \frac{4t}{\alpha}$, so $8c_2Q_{j_0} \subseteq B(y,t)$. We also know that $\operatorname{dist}(Q_{j_0}, (E_\lambda^*)^c) \leqslant C_2\ell(Q_j)$, thus we know that $B(y,t) \cap (E_\lambda^*)^c \neq \varnothing$. This implies $E_\lambda^* \cap (E_\lambda^*)^c \neq \varnothing$, which is a contradiction.
- Case 2: at least one of Q_j such that $Q_j \cap B(y,t) \neq \varnothing$ satisfies $\ell(Q_j) > \frac{4t}{\alpha}$. Let us pick such Q_j, then $B(y,t) \subseteq \alpha Q_j$, but having one base covering the other implies one tent covers the other: $T(B(y,t)) \subseteq T(\alpha Q_j)$. In particular, the vertex (y,t) of $T(B(y,t))$ is contained in $T(\alpha Q_j)$. Since $(y,t) \in E_\lambda$ is arbitrary, this implies that $E_\lambda \subseteq \bigcup_j T(\alpha Q_j)$.

■

This proves Lemma 15.5, as desired. ■

□

Problem 35. Suppose that φ is a function on $\mathbb{R}^n$ satisfying

$$|\varphi(x)| \leqslant \frac{c_1}{(1+|x|)^{n+\varepsilon}}$$

where $\varepsilon \in (0,1]$ and c_1 is a constant independent of x. Prove that

$$\sup_{(y,t)\in\Gamma(x)} |\varphi_t * f(y)| \lesssim Mf(x)$$

where M is independent of f, t, and x. Moreover, prove that for any $p \in (1,\infty)$,

$$\left(\int_{\mathbb{R}_+^{n+1}} |\varphi_t * f(x)|^p d\mu\right)^{\frac{1}{p}} \lesssim ||\mu||^{\frac{1}{p}} ||f||_{L^p(\mathbb{R}^n)}$$

if μ is a Carleson measure.

Definition 15.8. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $Q_tb(x) = \psi_t * b(x)$, where $\psi_t = t^{-n}\psi(\frac{x}{t})$ and ψ is radial such that $\int_{\mathbb{R}^n} \psi = 0$ and

$$|\psi(x)| + |\nabla\psi(x)| \leqslant \frac{C}{(1+|x|)^{n+\varepsilon}}. \tag{15.9}$$

For any Borel set $E \subseteq \mathbb{R}_+^{n+1}$, we define

$$\mu(E) = \int_E |\psi_t * b(x)|^2 \frac{dxdt}{t}.$$

Theorem 15.10. $\mu(E)$ defined above gives a Carleson measure, and $||\mu|| \lesssim ||b||_{\mathrm{BMO}}^2$.

Proof. Let $Q \subseteq \mathbb{R}^n$, then it suffices to show that $\mu(\hat{Q}) \lesssim ||b||_{\mathrm{BMO}}^2|Q|$. We may write

$$b = b_1 + b_2 + b_3$$

where $b_1 := (b - b_{2Q})\chi_{2Q}$, $b_2 := (b - b_{2Q})\chi_{(2Q)^c}$, and $b_3 := b_{2Q}$. Notice that $\psi_t * b_3(x) = b_{2Q} \int \psi_t = b_{2Q} \int \psi = 0$. By triangle inequality, we have that

$$\mu(\hat{Q}) \lesssim \int_{\hat{Q}} |\psi_t * b_1|^2 \frac{dxdt}{t} + \int_{\hat{Q}} |\psi_t * b_2|^2 \frac{dxdt}{t}.$$

We denote $I_1 = \int_{\hat{Q}} |\psi_t * b_1|^2 \frac{dxdt}{t}$ and $I_2 = \int_{\hat{Q}} |\psi_t * b_2|^2 \frac{dxdt}{t}$.

Problem 36. Suppose ψ is radial, $\int_{\mathbb{R}^n} \psi = 0$, and satisfies Equation (15.9). We may prove that

$$\int_{\mathbb{R}^{n+1}_+} |\psi_t * f|^2 \frac{dxdt}{t} \lesssim ||f||_2^2$$

for any $f \in L^2(\mathbb{R}^n)$.

Hint: note that $|e^{i\theta} - 1| \lesssim |\theta|^\delta$ for any $0 < \delta$, and apply Theorem 4.5.

By Problem 36, we have

$$\begin{aligned} I_1 &\leqslant \int_{\mathbb{R}^{n+1}_+} |\psi_t * b_1|^2 \frac{dxdt}{t} \\ &\lesssim \int_{\mathbb{R}^n} |b_1|^2 \\ &\lesssim \int_{2Q} |b - b_{2Q}|^2 \\ &\lesssim ||b||_{\mathrm{BMO}}^2 |Q|. \end{aligned}$$

For I_2, we have

$$\begin{aligned} |\psi_t * b_2(x)| &\leqslant \frac{1}{t^n} \int \left|\psi\left(\frac{x-y}{t}\right)\right| |b_2(y)| dy \\ &\lesssim \frac{1}{t^n} \int_{(2Q)^c} \frac{|b(y) - b_{2Q}|}{(1 + t^{-1}|x-y|)^{n+\varepsilon}} dy \\ &\lesssim \int_{(2Q)^c} \frac{t^\varepsilon |b(y) - b_{2Q}|}{(t + |x-y|)^{n+\varepsilon}} dy. \end{aligned}$$

When $(x,t) \in \hat{Q}$ and $y \notin 2Q$, we have that

$$\begin{aligned} |x - y| &\geqslant |y - c(Q)| - |x - c(Q)| \\ &\geqslant \frac{1}{2} |y - c(Q)|, \end{aligned}$$

therefore

$$\begin{aligned} |\psi_t * b_2(x)| &\lesssim \int_{(2Q)^c} \frac{t^\varepsilon |b(y) - b_{2Q}|}{(t + |x-y|)^{n+\varepsilon}} dy \\ &\lesssim t^\varepsilon \int_{(2Q)^c} \frac{|b(y) - b_{2Q}|}{|y - c(Q)|^{n+\varepsilon}} dy. \end{aligned}$$

Problem 37. Prove that
$$\int_{(2Q)^c} \frac{|b(y)-b_{2Q}|}{|y-c(Q)|^{n+\varepsilon}} \lesssim \frac{||b||_{\mathrm{BMO}}}{\ell(Q)^{\varepsilon}}$$
whenever $b \in \mathrm{BMO}(\mathbb{R}^n)$.

By Problem 37, we note that
$$\begin{aligned}|\psi_t * b_2(x)| &\lesssim t^{\varepsilon} \int_{(2Q)^c} \frac{|b(y)-b_{2Q}|}{|y-c(Q)|^{n+\varepsilon}} dy \\ &\lesssim \frac{t^{\varepsilon}}{\ell(Q)^{\varepsilon}} ||b||_{\mathrm{BMO}}.\end{aligned}$$
Therefore, we may bound
$$\begin{aligned}I_2 &\lesssim ||b||_{\mathrm{BMO}}^2 \int_Q \int_0^{\ell(Q)} \frac{t^{2\varepsilon-1}}{\ell(Q)^{2\varepsilon}} dt dx \\ &\lesssim ||b||_{\mathrm{BMO}}^2 |Q|,\end{aligned}$$
and this finishes the proof. □

Problem 38. Let φ be a bounded integrable function and $\varphi > 0$. Suppose that
$$\left(\int_{\mathbb{R}_+^{n+1}} |\varphi_t * f(x)|^p d\mu \right)^{\frac{1}{p}} \lesssim ||f||_{L^p(\mathbb{R}^n)}$$
for any $f \in L^p$ and some $p \in [1, \infty)$, then show that μ is a Carleson measure.

16 T1 Theorem in Strong Form

Theorem 16.1 (David and Journé)**.** Suppose that T is a singular integral operator associated to a Calderón-Zygmund kernel, then T extends to a bounded operator on $L^2(\mathbb{R}^n)$ if and only if

- T satisfies the WBP, and
- $T1 \in \mathrm{BMO}$ and $T^*1 \in \mathrm{BMO}$.

Remark 16.2. Recall that $\mathcal{S}_0(\mathbb{R}^n) = \{\psi \in C_c^\infty(\mathbb{R}^n) : \int \psi = 0\}$. Using this notation, we note that $T1 \in \mathrm{BMO}$ if and only if there exists $b \in \mathrm{BMO}$ such that $\langle T1, \psi\rangle = \langle b, \psi\rangle = \int\limits_{\mathbb{R}^n} b\overline{\psi}dx$.

Let us first verify the only-if part of Theorem 16.1.

Lemma 16.3. Let T be a Calderón-Zygmund singular integral operator which is L^2-extendable, i.e., can be extended to a bounded operator on L^2. Let f be any bounded function with compact support, then $||Tf||_{\mathrm{BMO}} \lesssim ||f||_\infty$, up to an independent constant.

Proof. Let Q be any cube in $\mathbb{R}^n$, then define a_Q to be the integral

$$a_Q = \int\limits_{\mathbb{R}^n} K(c(Q), y)f(y)\chi_{(5Q)^c}(y)dy = T(f\chi_{(5Q)^c})(c(Q)),$$

where $c(Q)$ is the center of the cube, and K is the standard Calderón-Zygmund kernel. We find that

$$\frac{1}{|Q|}\int\limits_Q |Tf - a_Q|dx \leqslant \frac{1}{|Q|}\int\limits_Q |T(f\chi_{5Q})|dx + \frac{1}{|Q|}\int\limits_Q |T(f\chi_{(5Q)^c})(x) - a_Q|dx$$

By Cauchy-Schwartz Theorem,

$$\begin{aligned}\frac{1}{|Q|}\int\limits_Q |T(f\chi_{5Q})|dx &\lesssim \left(\frac{1}{|Q|}\int\limits_Q |T(f\chi_{5Q})|^2dx\right)^{\frac{1}{2}}\\ &\lesssim \left(\frac{1}{|Q|}\int\limits_{5Q} |f(x)^2|dx\right)^{\frac{1}{2}} \text{ since } T \text{ is bounded on } L^2\\ &\lesssim ||f||_\infty.\end{aligned}$$

By the smoothness condition on K, we have

$$\begin{aligned}\frac{1}{|Q|}\int\limits_Q |T(f\chi_{(5Q)^c})(x) - a_Q|dx &\lesssim \frac{1}{|Q|}\int\limits_Q\int\limits_{(5Q)^c} |K(x,y) - K(c(Q),y)|\\ &\lesssim \frac{1}{|Q|}\int\limits_Q\int\limits_{(5Q)^c} \frac{|x-c(Q)|^n}{|x-y|^{n+\varepsilon}}|f(y)|dydx\\ &\lesssim \frac{||f||_\infty}{|Q|}\int\limits_Q\int\limits_{(5Q)^c} \frac{|x-c(Q)|^n}{|x-y|^{n+\varepsilon}}dydx\\ &\lesssim ||f||_\infty\end{aligned}$$

by Problem 37. □

Theorem 16.4. Let T be an L^2-extendable Calderón-Zygmund singular integral operator, then T extends to a bounded operator from L^∞ to BMO.

Proof. For any $j \in \mathbb{Z}$, let $B_j = B(0, 2^j)$. For any $f \in L^\infty$, any B_j with $j \geqslant 0$, and any $x \in B_j$, we define $T_{B_j} f(x) = T(f\chi_{5B_j})(x) + \int_{\mathbb{R}^n} (K(x,y) - K(0,y))f(y)\chi_{(5B_j)^c}(y)dy$ which is well-defined: the first term is well-defined according to Lemma 16.3, and the BMO norm of the second term is bounded above by $||f||_\infty$. We now show that $||Tf||_{\text{BMO}} \lesssim ||f||_\infty$. We know $||T(f\chi_{5B_j})||_{\text{BMO}} \lesssim ||f||_\infty$ by Lemma 16.3, and that

Claim 16.5.

$$\left|\int_{\mathbb{R}^n} (K(x,y) - K(0,y))f(y)\chi_{(5B_j)^c}(y)dy\right| \lesssim ||f||_\infty.$$

Subproof. We note that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} (K(x,y) - K(0,y))f(y)\chi_{(5B_j)^c}(y)dy\right| &\lesssim \int_{(5B_j)^c} |K(x,y) - K(0,y)|dy||f||_\infty \\ &\leqslant C||f||_\infty.\end{aligned}$$

■

□

Proof of only-if part of Theorem 16.1. To verify the only-if part of Theorem 16.1, we may assume T is an L^2-extendable Calderón-Zygmund singular integral operator, then we want to find $b \in \text{BMO}$ such that for any $\psi \in \mathcal{S}_0(\mathbb{R}^n)$, $\langle T1, \psi\rangle = \langle b, \psi\rangle$ for any $x \in B_j$. Let us define

$$b(x) = T(\chi_{5B_j})(x) + g(x)$$

where

$$g(x) = \int_{\mathbb{R}^n} (K(x,y) - K(0,y))\chi_{(5B_j)^c}(y)dy.$$

By Theorem 16.4, we know that $b(x) \in \text{BMO}$. For any $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ supported in B_J for some large $J \in \mathbb{Z}$, then

$$\langle T1, \psi\rangle = \langle T(\chi_{5B_J}), \psi\rangle + \left\langle \chi_{(5B_j)^c}, T^*\psi\right\rangle.$$

It remains to verify that $\left\langle \chi_{(5B_j)^c}, T^*\psi\right\rangle = \langle g, \psi\rangle$. We recall that

$$\begin{aligned}\left\langle \chi_{(5B_j)^c}, T^*\psi\right\rangle &= \int \chi_{(5B_j)^c}(x) \int (K(y,x) - K(0,x))\overline{\psi(y)}dydx \text{ since } \int \psi = 0 \\ &= \langle g, \psi\rangle \text{ by Fubini Theorem.}\end{aligned}$$

Therefore, $T1 \in \text{BMO}$. Similarly, one can show that $T^*1 \in \text{BMO}$ as well. □

We now start proving the if part of Theorem 16.1. Suppose T satisfies the WBP and that $T1, T^*1 \in \text{BMO}$, then we want to show that T extends to a bounded operator on L^2. This requires the simple version of T1 theorem.

Let φ, ψ be radial Schwartz functions on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} \varphi = 1$ and $\int_{\mathbb{R}^n} \psi = 0$, and that $\int_0^\infty |\hat{\psi}(t)|^2 \frac{dt}{t} = 1$. Moreover, we may assume that ψ is real-valued. We define $Q_t f(x) = \psi_t * f(x)$ to be a convolution-type operator, where $\psi_t(x) = \frac{1}{t^n}\psi(\frac{x}{t})$ is the dilation. Moreover, let us define $P_t f(x) = \varphi_t * f(x)$ to be a convolution-type operator, motivated by the Poisson kernel.

Definition 16.6. For any $\varepsilon > 0$ and $b \in \text{BMO}(\mathbb{R}^n)$, we define the paraproduct operator to be

$$\prod_{b,\varepsilon} f(x) = \int_\varepsilon^{\frac{1}{\varepsilon}} Q_t(Q_t b P_t f)(x)\frac{dt}{t},$$

and that

$$\prod_b = \lim_{\varepsilon \to 0} \prod_{b,\varepsilon}.$$

Remark 16.7. We have $\left\langle \prod_b \varphi, \psi \right\rangle = \lim_{\varepsilon \to 0} \left\langle \prod_{b,\varepsilon} \varphi, \psi \right\rangle$.

Remark 16.8. For any $b \in \mathrm{BMO}(\mathbb{R}^n)$, $\prod_b 1$ is a linear functional on $\mathcal{S}_0(\mathbb{R}^n)$, defined by

$$\left\langle \prod_b 1, \varphi \right\rangle = \lim_{\varepsilon \to 0} \left\langle \prod_{b,\varepsilon} 1, \varphi \right\rangle$$

for any $\varphi \in \mathcal{S}_0(\mathbb{R}^n)$.

Lemma 16.9. $\prod_b 1 = b$.

Proof. We have

$$\begin{aligned} \prod_{b,\varepsilon} 1 &= \int_{\varepsilon}^{\frac{1}{\varepsilon}} Q_t(Q_T b P_t 1)(x) \frac{dt}{t} \\ &= \int_{\varepsilon}^{\frac{1}{\varepsilon}} Q_t(Q_t b)(x) \frac{dt}{t}, \end{aligned}$$

so in the sense of distribution,

$$\begin{aligned} \prod_b 1 &= \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\frac{1}{\varepsilon}} Q_t(Q_t b)(x) \frac{dt}{t} \\ &= \int_0^{\infty} Q_t(Q_t b)(x) \frac{dt}{t}. \end{aligned}$$

To show that this integral is just b, we may apply an extended version of Theorem 12.4 in $L^p(\mathbb{R}^n)$. In fact, it is sufficient to show that

$$\begin{aligned} \hat{b} &= \int_0^{\infty} \widehat{Q_t(Q_t b)} \frac{dt}{t} \\ &= \int_0^{\infty} |\hat{\psi}(|\xi|t)|^2 \hat{b}(\xi) \frac{dt}{t} \\ &= \hat{b} \int_0^{\infty} |\hat{\psi}(t)|^2 \frac{dt}{t} \text{ by normalization} \\ &= 1. \end{aligned}$$

$\square$

Definition 16.10. We define

$$\prod_{b,\varepsilon}^* f(x) = \int_{\varepsilon}^{\frac{1}{\varepsilon}} P_t^*(Q_t b Q_t^* f)(x) \frac{dt}{t}$$

and

$$\prod_b^* = \lim_{\varepsilon \to 0} \prod_{b,\varepsilon}^*.$$

Lemma 16.11. We have $\prod_b {}^*1 = 0$.

Proof. Since $Q_t^* = Q_t$ and $Q_t 1 = 0$, then

$$\begin{aligned}\prod_{b,\varepsilon} {}^*1 &= \int_\varepsilon^{\frac{1}{\varepsilon}} P_t^*(0)\frac{dt}{t}\\ &= \int_\varepsilon^{\frac{1}{\varepsilon}} 0\frac{dt}{t}\\ &= 0,\end{aligned}$$

and therefore $\prod_b {}^*1 = 0$. □

Lemma 16.12. Let $b \in \mathrm{BMO}$, then $\prod_b$ is a bounded operator on $L^2(\mathbb{R}^n)$.

Proof. We want to show that

$$||\prod_b f||_2 \lesssim ||f||_2$$

for any $f \in L^2$, so it suffices to show that

$$||\prod_{b,\varepsilon} f||_2 \lesssim ||f||_2$$

for any $f \in L^2$. For any $f, g \in L^2(\mathbb{R}^n)$, we find that the inner product

$$\begin{aligned}\left|\left\langle \prod_{b,\varepsilon} f, g\right\rangle\right| &= \left|\int_\varepsilon^{\frac{1}{\varepsilon}}\int_{\mathbb{R}^n} Q_t(Q_t b P_t f)(x) g(x)\frac{dt}{t}\right|\\ &= \left|\int_\varepsilon^{\frac{1}{\varepsilon}}\int_{\mathbb{R}^n} Q_t b(x) P_t f(x) Q_t^*(\bar{g})(x)\frac{dxdt}{t}\right|\\ &\leqslant \left(\int_\varepsilon^{\frac{1}{\varepsilon}}\int_{\mathbb{R}^n} |P_t f(x)|^2 |Q_t b(x)|^2 \frac{dxdt}{2}\right)^{\frac{1}{2}}\left(\int_\varepsilon^{\frac{1}{\varepsilon}}\int_{\mathbb{R}^n} |Q_t^*(\bar{g})(x)|^2 \frac{dxdt}{2}\right)^{\frac{1}{2}} \text{ by Cauchy-Schwartz}\\ &\lesssim ||g||_2 \left(\int_0^\infty\int_{\mathbb{R}^n} |P_t f(x)|^2 |Q_t b(x)|^2 \frac{dxdt}{2}\right)^{\frac{1}{2}}.\end{aligned}$$

Set $d\mu = |Q_t b(x)|^2 \frac{dxdt}{t}$, then since $b \in \mathrm{BMO}$, we know $d\mu$ is a Carleson measure. By Theorem 15.2, we have

$$\begin{aligned}\left|\left\langle \prod_{b,\varepsilon} f, g\right\rangle\right| &\lesssim ||g||_2 \left(\int_{\mathbb{R}^n} \sup_{(y,t)\in\Gamma(x)} |P_t f(y)|^2 dx\right)^{\frac{1}{2}}\\ &\lesssim ||g||_2 ||Mf||_2 \text{ from homework problem}\\ &\lesssim ||f||_2 ||g||_2.\end{aligned}$$

□

Lemma 16.13. Let $b \in \mathrm{BMO}$, then $\prod_b$ is a Calderón-Zygmund singular integral operator.

We will use these results to finish the proof of if part of Theorem 16.1.

Proof of if part of Theorem 16.1. We have $T1 = b_1$ and $T^*1 = b_2$ for $b_1, b_2 \in$ BMO. Define $T_0 = T - \prod_{b_1} - \prod_{b_2}^*$, then

$$\begin{aligned} T_0 1 &= T1 - \prod_{b_1} 1 - \prod_{b_2}^* 1 \\ &= b_1 - b_1 - 0 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} T_0^* 1 &= b_2 - b_2 \\ &= 0. \end{aligned}$$

By Lemma 16.13, T_0 is a difference of Calderón-Zygmund singular integral operator, then it is also a Calderón-Zygmund singular integral operator, which satisfies the WBP. By the simple version of T1 theorem, c.f., Theorem 13.6, we know T_0 extends to a bounded operator on L^2, and now $T = T_0 + \prod_{b_1} + \prod_{b_2}^*$ is a sum of bounded operators on L^2, therefore T is also a bounded operator on L^2. □

Proof of Lemma 16.13. Let

$$K_t(x, y) = \frac{1}{t^{2n}} \int_{\mathbb{R}^n} \psi(\frac{x-z}{t})\varphi(\frac{z-y}{t}) Q_t b(z) dz$$

then

$$Q_t(Q_t b P_t f)(x) = \int_{\mathbb{R}^n} K_t(x, y) f(y) dy.$$

By Fubini theorem,

$$\prod_{b,\varepsilon} f(x) = \int_{\mathbb{R}^n} \int_{\varepsilon}^{\frac{1}{\varepsilon}} K_t(x, y) \frac{dt}{t} f(y) dy,$$

therefore the kernel of $\prod_b$ is

$$K(x, y) = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\frac{1}{\varepsilon}} K_t(x, y) \frac{dt}{t}.$$

We now need to show that $K(x, y)$ is a Calderón-Zygmund singular integral operator. This follows from Claim 16.15.

Claim 16.14. We have

$$||Q_t b||_\infty \lesssim ||b||_{\text{BMO}}$$

and

$$||\nabla_x Q_t b||_\infty \lesssim \frac{1}{t} ||b||_{\text{BMO}}.$$

Proof. We will prove the first inequality, and the second inequality can be proven in a similar fashion. We note that

$$Q_t b(x) = \int_{\mathbb{R}^n} \psi_t(x - y)(b(y) - b_{Q(x,t)}) dy,$$

where $Q(x, t)$ is a cube centered at x of side length t. Therefore,

$$Q_t b(x) \lesssim \int \frac{1}{t^n \left(1 + \frac{|x-y|}{t}\right)^{n+1}} |b(y) - b_{Q(x,t)}| dy$$

$$
\begin{aligned}
&= \int\limits_{2Q(x,t)} \frac{1}{t^n\left(1+\frac{|x-y|}{t}\right)^{n+1}}|b(y)-b_{Q(x,t)}|dy + \int\limits_{(2Q(x,t))^c} \frac{1}{t^n\left(1+\frac{|x-y|}{t}\right)^{n+1}}|b(y)-b_{Q(x,t)}|dy \\
&\lesssim \frac{1}{|Q(x,t)|}\int\limits_{2Q(x,t)} |b(y)-b_{Q(x,t)}|dy + \int\limits_{(2Q(x,t))^c} \frac{t|b(y)-b_{Q(x,t)}|}{|y-x|^{n+1}}dy \\
&\lesssim ||b||_{\mathrm{BMO}} + ||b||_{\mathrm{BMO}} \text{ by Problem 37.}
\end{aligned}
$$

■

Claim 16.15.

$$K(x,y) \lesssim \frac{||b||_{\mathrm{BMO}}}{|x-y|^n}$$

and

$$|\nabla K(x,y)| \lesssim \frac{||b||_{\mathrm{BMO}}}{|x-y|^{n+1}}.$$

Subproof. The two inequalities follow from the two inequalities in Claim 16.14, respectively. ■

□

17 Fourier Restriction Problems

Recall we have

$$\hat{f}d\sigma = \int_{S^{n-1}} f(\xi)e^{-2\pi i x\cdot\xi}d\sigma(\xi)$$

where we may take $-2\pi ix$ instead of the usual $2\pi ix$, and we may then restrict ξ to a hypersurface of S^{n-1} that satisfies nice properties.

Conjecture 17.1 (Stein)**.** Suppose f is integrable, then for what kind of function f can we guarantee the integrability of the Fourier transform of the restricted function?

Remark 17.2 (Kakeya Needle Problem)**.** Suppose we have a unit line segment on a plane, and we move the segment continuously on the plane until it points towards the opposite direction. What is the smallest possible area covered by the continuous movement of the segment? In fact, there is no such minimal area: the area can be arbitrary small. This is due to the existence of Besicovitch sets $B(\mathbb{R}^n)$. This is a set of Lebesgue measure zero, but it contains unit line segments that point in each direction. There are at least two known constructions of the Besicovitch set.

- One construction is to decompose the space into small triangles, and squeeze all triangles altogether, then these small triangles may point in all possible directions.
- Another construction is given by Cantor sets. Consider two parallel line segments with separation distance approximately 1, then we construct the Cantor sets in each line segment, and then connecting them together in some way, we do get a Besicovitch set, then one may verify that the Cantor sets point in all possible directions. In particular, the Cantor set has Lebesgue measure zero, therefore we are done.

We may then ask about the Hausdorff dimension of the Besicovitch set.

Conjecture 17.3 (Kakeya Conjecture)**.** The Hausdorff dimension $\dim_H(B(\mathbb{R}^n)) = n$.

It turns out that this conjecture is related to the question in Conjecture 17.1. Moreover, we may restate Conjecture 17.1 in terms of concepts in number theory. In particular, Conjecture 17.4 implies Conjecture 17.3.

Conjecture 17.4 (Stein, Fourier Restriction Conjecture)**.** Let $a_j \in \mathbb{C}$ be with $|a_j| = 1$ for $j = 1, \dots, M \sim R^{n-1}$, and let $F(x) = \sum_{j=1}^{M} a_j e^{2\pi i\omega_j\cdot x}$ for any $x \in \mathbb{R}^n$, where the frequencies ω_j's are evenly distributed on the sphere

$$\left\{\xi \in \mathbb{R}^n : |\xi| = \sqrt{\xi_1^2 + \cdots + \xi_n^2} = R\right\}$$

with separation ~ 1, i.e., comparable to 1. Then

$$\left(\int_{[0,1]^n} |F(x)|^p dx\right)^{\frac{1}{p}} \lesssim M^{\frac{1}{2}} + R^{-\frac{n}{p}}M \tag{17.5}$$

for $2 \leqslant p < \infty$.

The most interesting case is when we have the critical points, i.e., when $M^{\frac{1}{2}} \sim R^{-\frac{n}{p}}M$, we get

$$p_c = \frac{2n}{n-1} > 2.$$

Therefore, we get an estimation

$$\left(\int_{[0,1]^n} |F(x)|^{p_c}\right)^{\frac{1}{p_c}} \lesssim M^{\frac{1}{2}}.$$

Therefore, Conjecture 17.4 really asks for maximal p such that we get square root consolation. For $p = 2$, this is obvious. However, for $p > 2$, especially $p = p_c$, this is much harder. Again, we may rephrase this question back to Fourier transform, so we can apply harmonic analysis.

Remark 17.6. Conjecture 17.4 is solved for $n = 2$, but not for $n > 2$.

First, we describe a kind of extension operator.

Definition 17.7. For any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we define the extension Ef of f to be

$$Ef(x) = \int_{\mathbb{R}^{n-1}} f(\xi_1, \ldots, \xi_{n-1}) e^{2\pi i \left(\xi_1 x_1 + \cdots + \xi_{n-1} x_{n-1} + \sqrt{1-\xi_1^2 - \cdots - \xi_{n-1}^2} x_n\right)} d\xi_1 \cdots d\xi_{n-1}.$$

Here we assume that $\operatorname{supp}(f) \subseteq B^{n-1}(0,1)$.

Remark 17.8. The extension operator is the dual of the restriction operator $Rf = \hat{f} d\sigma$ to the hypersurface of unit sphere S^{n-1}.

We may now reform Conjecture 17.4 in terms of a harmonic analysis question, as in Conjecture 17.9.

Conjecture 17.9 (Stein, Fourier Restriction Conjecture)**.** For any $R \in \mathbb{R}$, and any $p \geqslant 2$, we have

$$||Ef||_{L^p(B^n(R))} \lesssim \left(R^{\frac{n}{p} - \frac{n-1}{2}} + 1\right) ||f||_\infty.$$

This ball satisfies some translation invariance, therefore we do not need to specify the center of the ball.

One can show that Conjecture 17.4 and Conjecture 17.9 are equivalent. Here we give the proof for one direction.

Theorem 17.10. Conjecture 17.9 implies Conjecture 17.4.

Proof. By a change of variables, we note that Equation (17.5) is equivalent to

$$\left(\int_{B^n(0,R)} \left|\sum_{j=1}^M a_j e^{2\pi i \frac{\omega_j}{R} \cdot x}\right|^p dx\right)^{\frac{1}{p}} \lesssim R^{\frac{n}{p}} M^{\frac{1}{2}} + M \lesssim R^{\frac{n}{p}} R^{\frac{n-1}{2}} + R^{n-1}.$$

We will show that Conjecture 17.9 implies this new inequality, so we need to represent the exponential sums $\sum\limits_{j=1}^M a_j e^{2\pi i \frac{\omega_j}{R} \cdot x}$ in terms of integrals, using the extension operator. Eventually, $Ef \approx \sum\limits_{j=1}^M a_j e^{2\pi i \frac{\omega_j}{R} \cdot x}$.

Note that $\frac{\omega_j}{R} \in S^{n-1} \subseteq \mathbb{R}^n$ is a vector. Let $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, so let $y^* = (y_1, \ldots, y_{n-1})$. For $\xi = (\xi_1, \ldots, \xi_{n-1})$, then let

$$f(\xi_1, \ldots, \xi_{n-1}) = \sum_{j=1}^M a_j \psi\left(\frac{\xi - \frac{\omega_j^*}{R}}{\frac{1}{R}}\right),$$

where

$$\psi(y) = \psi(y_1, \ldots, y_{n-1}) = \psi_1(y_1) \cdots \psi_{n-1}(y_{n-1})$$

where each ψ_j is a (smooth) bump function supported in $[-1, 1]$. Therefore, ψ can be thought of as a (smooth) bump function supported in $[-1, 1]^n$. Therefore, the value of $\xi - \frac{\omega_j^*}{R}$ is close to $\frac{1}{R}$. Now

$$Ef(x) = \sum_{j=1}^M a_j \int \psi\left(\frac{\xi - \frac{\omega_j^*}{R}}{\frac{1}{R}}\right) e^{2\pi i \left(\xi_1 x_1 + \cdots + \xi_{n-1} x_{n-1} + \sqrt{1-\xi_1^2 - \cdots - \xi_{n-1}^2} x_n\right)} d\xi_1 \cdots d\xi_{n-1}.$$

To finish the proof rigorously, we need Taylor expansion. The idea being, since this problem is translation invariant, so we may assume $B^n(R)$ is centered at 0, therefore for $x \in B^n(R)$, we know $|x| \leqslant R$. Now, for example, we can rewrite

$$e^{i\xi_1 x_1} = e^{2\pi i \left(\xi_1 - \frac{(\omega_j^*)_1}{R}\right) x_1} e^{2\pi i \frac{(\omega_j^*)_1}{R} x_1}$$

$$\approx e^{2\pi i \frac{(\omega_j^*)_1}{R} x_1}$$

by Taylor expansion at 0, then we may replace this for the first $n-1$ terms in the exponent. Finally, for the last term $\sqrt{1-\xi_1^2-\cdots-\xi_{n-1}^2}$, we replace each ξ_j by $\frac{(\omega_j)^*}{R}$, then we note that

$$Ef(x) \approx \frac{1}{R^{n-1}} \sum_j a_j e^{2\pi i \frac{\omega_j}{R} \cdot x},$$

so we may write

$$\sum_j a_j e^{2\pi i \frac{\omega_j}{R} \cdot x} \approx R^{n-1} Ef(x).$$

□

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