# MATH 502 Notes

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References:

- Atiyah and MacDonald, *Commutative Algebra*.
- J.P. Serre, *Local Algebra*.
- Zariski and Samuel, *Commutative Algebra* Volume 1 and 2.
- Matsumura, *Commutative Algebra*.
- Bourbaki, *Commutative Algebra*.

We always assume a ring $R$ has a multiplicative identity and is commutative.

## 0 Noetherian, Artinian, and Localization

**Proposition 0.1.** Let $R$ be a (commutative) ring, and let $M$ be an $A$-module, then the following are equivalent:

(i) Given an infinite increasing chain of submodules of $M$

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq M_{n+1} \subseteq \cdots$$

then there exists some $N \in \mathbb{N}$ such that $M_N = M_{N+1} = \cdots$, i.e., for all $n \geqslant N$, $M_n = M_{n+1}$.

(ii) Every non-empty family of submodules has a maximal element.

(iii) Every submodule of $M$ is finitely-generated.

*Proof.* $(i) \Rightarrow (ii)$: This is a direct result of Zorn's lemma.

$(ii) \Rightarrow (i)$: Obvious.

$(i), (ii) \Rightarrow (iii)$: Take any submodule $N$ of $M$ and take $x_1 \in N$. If $(x_1) \neq N$, then there exists $x_2 \in N \backslash (x_1)$, so $(x_1, x_2) \subseteq N$, now we proceed inductively, but by the given property we know this stops in finite number of steps, hence we have $N = (x_1, \ldots, x_n)$ for some $n \in \mathbb{N}$, thus $N$ is finitely-generated.

$(iii) \Rightarrow (i)$: Note that the property implies $M$ is finitely-generated, but that means the chain of submodules must be finite. $\square$

**Definition 0.2** (Noetherian Module)**.** If any of the conditions in Proposition 0.1 holds, then $M$ is said to be a Noetherian module. Alternatively, we say $M$ satisfies the ascending chain condition.

**Proposition 0.3.** Let $R$ be a (commutative) ring, and let $M$ be an $A$-module, then the following are equivalent:

(i) Given an infinite decreasing chain of submodules of $M$

$$M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq M_{n+1} \supseteq \cdots$$

then there exists some $N \in \mathbb{N}$ such that $M_N = M_{N+1} = \cdots$, i.e., for all $n \geqslant N$, $M_n = M_{n+1}$.

(ii) Every non-empty family of submodules has a minimal element.

*Proof.* Again, Zorn's lemma. □

**Definition 0.4** (Artinian Module)**.** If any of the conditions in Proposition 0.3 holds, then $M$ is said to be a Artinian module. Alternatively, we say $M$ satisfies the descending chain condition.

**Example 0.5.**

- $\mathbb{Z}$ is Noetherian.
- $\mathbb{Q}/\mathbb{Z}$ is not Noetherian.
- Let $p$ be a prime. Let $\mathbb{Z}(p^\infty)$ be the union of chains (as direct limits)

$$\left\langle \frac{\bar{1}}{p} \right\rangle \subseteq \left\langle \frac{\bar{1}}{p^2} \right\rangle \subseteq \cdots \subseteq \left\langle \frac{\bar{1}}{p^n} \right\rangle \subseteq \cdots$$

then there is an embedding $\mathbb{Z}(p^\infty) \subseteq \mathbb{Q}/\mathbb{Z}$, where $\bar{a}$ is the image of $a$ in $\mathbb{Q}/\mathbb{Z}$. With this construction, $\mathbb{Z}(p^\infty)$ is Artinian.

**Exercise 0.6.** Show that $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}(p^\infty)$ where $p$ traverses through all the primes.

**Proposition 0.7.** Let $N$ be a submodule of $M$. Suppose $M$ satisfies ascending (respectively, descending) chain condition, then $N$ and $M/N$ also satisfy ascending (respectively, descending) chain condition. If, for some submodule $N$ of $M$, we know $N$ and $M/N$ satisfy ascending (respectively, descending) chain condition, then $M$ also satisfies ascending (respectively, descending) chain condition.

*Proof.* Suppose $M$ satisfies ascending (respectively, descending) chain condition, and let $N$ be a submodule of $M$. Let $\{N_i\}$ be an increasing (respectively, decreasing) sequence of submodules of $N$, then they can be regarded as submodules of $M$, therefore by the Noetherian (respectively, Artinian) condition, we know $N$ satisfies ascending (respectively, descending) chain condition. Now let $\bar{M} = M/N$, and take $\{\bar{M}_i\}$ be an increasing (respectively, decreasing) sequence of submodules of $\bar{M}$. Let $\pi : M \to M/N$ be the quotient map, then the preimages give an increasing (respectively, decreasing) sequence $\{M_i\}$ of submodules of $M$, where $M_i = \pi^{-1}(\bar{M}_i)$, but by the Notherian (respectively, Artinian) condition, we know the sequence stops in finite steps, therefore the original sequence stops in finite steps as well, hence $\bar{M}$ satisfies the ascending (respectively, descending) chain condition.

Suppose a submodule $N$ of $M$ is such that $N$ and $M/N$ both satisfy ascending chain condition. Take a submodule $T$ of $M$, then we have a short exact sequence

$$0 \longrightarrow T \cap N \hookrightarrow T \longrightarrow T/(T \cap N) \longrightarrow 0$$

Now $T \cap N$ is finitely-generated as $N$ is finitely-generated, therefore we have an embedding $T/T \cap N \hookrightarrow M/N$, thus $T/T \cap N$ is finitely-generated, therefore $T$ is also finitely-generated by a vector space argument.

Suppose we have a decreasing sequence $\{M_n\}$ of $M$, then we have a decreasing sequence $\{N \cap M_n\}$. Let $\bar{M} = M/N$, then $\bar{M}_n := (M_n + N)/N$ defines a decreasing sequence of submodules in $\bar{M}$, but $N$ satisfies the descending chain condition, so the sequence $\{N \cap M_n\}$ stops in finite number of steps, say $n_0$. Moreover, the sequence of $\bar{M}_n$'s also stops in finite number of steps, so by definition the sequence of $(M_n + N)/N$ stops in finite number of steps, say $m_0$, but by the isomorphism theorem this shows that the sequence of $M_n/(N \cap M_n)$ stops in $m_0$ steps. Therefore, whenever $n \geqslant m_0, n_0$, then $N \cap M_n = N \cap M_{n+1}$, hence $M_n = M_{n+1} = \cdots$ for such $n$. □

**Remark 0.8.** The final argument should also work in the Noetherian case.

**Definition 0.9** (Simple Module)**.** An $A$-module $M$ is simple if the submodules of $M$ are either $0$ or $M$.

**Exercise 0.10.** Let $A$ be a commutative ring, and $M$ is an $A$-module, then $M$ is simple if and only if $M \cong A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of $A$.

**Definition 0.11** (Jordan-Hölder Chain)**.** Let $A$ be a commutative ring and $M$ be an $A$-module. We say $M$ has a Jordan-Hölder chain if there exists a decreasing chain of submodules $\{M_i\}$ such that

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_{n-1} \supsetneq M_n = 0$$

such that $M_i/M_{i+1}$ is simple. In such a situation, we know $n$ is the length of the Jordan-Hölder chain, and such $n$ is unique. We say $M$ is a module of finite length, and the length is $\ell_A(M) = n$.

**Exercise 0.12.** Let $A$ be a commutative ring, and let $M$ be an $A$-module, then $M$ is of finite length if and only if $M$ is both Noetherian and Artinian.

**Theorem 0.13.** Let $A$ be a commutative ring, then $A$ is Artinian if and only if $A$ is Noetherian and every prime ideal of $A$ is maximal.

*Proof.* ($\Leftarrow$):

**Lemma 0.14.** Let $A$ be Noetherian, then every ideal of $A$ contains a product of prime ideals.

*Subproof.* Suppose, towards contradiction, that there exists some ideal $I$ of $A$ that does not contain a product of prime ideals. Let $\mathcal{J}$ be the set of such ideals of $A$, then $\mathcal{J} \neq \varnothing$, and we can take a maximal element of $\mathcal{J}$, namely $J$.[1] By definition, $J$ is not prime, therefore there exists $a, b \in A$ such that $a \notin J$ and $b \notin J$, but $ab \in J$. Now $J \subsetneq J + Aa$ and $J \subsetneq J + Ab$, therefore $J + Aa, J + Ab \notin J$, therefore $J + Aa$ and $J + Ab$ both contain product of prime ideals. But now $(J + Aa)(J + Ab)$ should also contain products of prime ideals, but by distribution this is just $J^2 + Ja + Jb + Aab$, which is contained in $J$ because every term is contained in $J$, so $J$ contains a product of prime ideals as well, contradiction. ■

In particular, $(0)$ contains a product of prime ideals, in particular $(0)$ equals to this product, but every prime ideal is maximal, therefore $(0) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$ becomes the product of maximal ideals (which may not necessarily be distinct), hence we have a descending chain of ideals

$$A \supseteq \mathfrak{m}_1 \supseteq \mathfrak{m}_1\mathfrak{m}_2 \supseteq \cdots \supseteq \mathfrak{m}_1 \cdots \mathfrak{m}_n = (0),$$

and in particular $(\mathfrak{m}_1 \cdots \mathfrak{m}_{i-1})/(\mathfrak{m}_1 \cdots \mathfrak{m}_i)$ is a finite-dimensional since $A$ is Noetherian, and it has a natural structure as a $A/\mathfrak{m}_i$-vector space. From the short exact sequence

$$0 \longrightarrow \mathfrak{m}_1 \cdots \mathfrak{m}_i \longrightarrow \mathfrak{m}_1 \cdots \mathfrak{m}_{i-1} \longrightarrow (\mathfrak{m}_1 \cdots \mathfrak{m}_{i-1})/(\mathfrak{m}_1 \cdots \mathfrak{m}_i) \longrightarrow 0$$

we know the two sides of the sequence are Artinian, hence the central term is Artinian. Proceeding inductively, we know that $\mathfrak{m}_1$ is Artinian, and $R/\mathfrak{m}_1$ would also be Artinian, hence $A$ is Artinian.

($\Rightarrow$): Now suppose $A$ is Artinian, and we want to show that every prime ideal is maximal, and $(0)$ is a product of maximal ideals. The result then follows from the argument above.

**Lemma 0.15.** Every Artinian domain is a field.

*Subproof.* Let $0 \neq a \in A$, then consider the chain

$$(a) \supseteq (a^2) \supseteq \cdots \supseteq (a^n) \supseteq \cdots$$

and by the Artinian property, for some large enough $n$ the descending chain stops. Hence, we have $a^n = \lambda a^{n+1}$ for some large enough $n$ and some $\lambda \in A$. Hence, $a^n(1 - \lambda a) = 0$, by the cancellation property of a domain, since $a \neq 0$, we must have $\lambda a = 1$, therefore $a$ is a unit, as desired. ■

**Corollary 0.16.** Let $A$ be Artinian, then every prime ideal of $A$ is maximal.

Finally, it suffices to show that $(0) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$. Let $\mathfrak{J}$ be the set of finite products of maximal ideals, then $\mathfrak{J}$ has a minimal element, and it suffices to show that this element is $(0)$. Suppose not, let $I \neq (0)$ be a minimal element of $R$. For any two ideals $\alpha, \beta$ of $A$, let $(\alpha : \beta) = \{a \in A \mid a\beta \subseteq \alpha\}$. Note that this has a natural structure as an ideal of $A$. Let $J = ((0) : I)$, and suppose $J = A$, then $I = 0$, contradiction, so $J \neq A$ is a proper ideal of $A$, now consider $A/J$ which is Artinian, then let $\mathfrak{G}$ be the set of all non-zero ideals of $A/J$, so $\mathfrak{G}$ has a minimal element as well, call it $\bar{H}$. Let $H = \pi^{-1}(\bar{H})$ where $\pi : A \to A/J$, so we have $J \subsetneq H$, thus let $P = (J : H)$.

**Claim 0.17.** $P$ is a prime ideal.

*Subproof.* Given $c, d \notin P$, we want to show that $cd \notin P$. Indeed, consider $J \subsetneq J + cH \subseteq H$, then since $H$ is minimal, then $J + cH = H$, and similarly we have that $J + dH = H$. Therefore, we have that $J + cdH = J + c(dH + J) = J + cH = H$, hence we know $cd \notin P$, as desired. ■

[1] The existence of this maximal element is the result of Zorn's lemma and ACC condition.

Now $P = (J : H)$ and $J = (0 : I)$, the by definition we have $PHI = (0)$. Since $P$ is a prime ideal, then $P$ is maximal, and now

$$(0 : PI) \supseteq H \supsetneq J = (0 : I)$$

Therefore $PI \subsetneq I$, where $I$ is a minimal element, contradiction, hence (0) is a product of maximal ideals. □

**Definition 0.18** (Short Exact Sequence)**.** Consider the sequence

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} T \longrightarrow 0$$

This is called a short exact sequence if $\ker(f) = 0$, $\operatorname{im}(g) = T$, and $\ker(g) = \operatorname{im}(f)$. In particular, one slot of the sequence is said to be exact if the kernel of the previous map equals to the image of the subsequent map.

**Definition 0.19** (Flat Module)**.** Let $M$ be an $A$-module, then we say $M$ is a flat $A$-module if for every short exact sequence

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

the tensored sequence

$$0 \longrightarrow M \otimes_A N_1 \longrightarrow M \otimes_A N_2 \longrightarrow M \otimes_A N_3 \longrightarrow 0$$

remains exact.

**Remark 0.20.** Recall that the properties of modules have the following implications: free ⇒ projective ⇒ flat ⇒ torsion-free, and in the case of finitely-generated modules, torsion-free ⇒ free.

**Remark 0.21.** We already know that the tensor functor is right exact, namely given the short exact sequence above, then

$$M \otimes_A N_1 \longrightarrow M \otimes_A N_2 \longrightarrow M \otimes_A N_3 \longrightarrow 0$$

is exact.

**Exercise 0.22.** Let $M$ be an $A$-module, and if there exists a short exact sequence of $A$-modules

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

where $N_1$ and $N_2$ are finitely-generated as $A$-modules, and such that tensoring $M$ preserves the short exact sequence, then $M$ is flat.

**Definition 0.23** (Multiplicatively Closed Subset)**.** Let $A$ be a commutative ring and $M$ be an $A$-module. Let $S \subseteq A$ be a subset. We say $S$ is a multiplicatively closed subset of $A$ if $1 \in S$, $0 \notin S$, and whenever $s_1, s_2 \in S$, then $s_1 s_2 \in S$.

**Definition 0.24** (Localization)**.** Let $S \subseteq A$ be a multiplicatively closed subset, and let $M$ be an $A$-module, then $S^{-1}M = (M \times S)/\sim$, where $\sim$ is an equivalence relation defined by the following: $(m_1, s_1) \sim (m_2, s_2)$ if and only if there exists $t \in S$ such that $t(m_1 s_2 - m_2 s_1) = 0$. $S^{-1}M$ is said to be the localization of $M$ at $S$.

Given $(m, s) \in M \times S$, we write $\overline{(m, s)}$ to be the equivalence class in $S^{-1}M$ represented by $(m, s)$.

**Exercise 0.25.** Similarly, one can define the localization $S^{-1}A$ of $A$ at $S$. In fact, $S^{-1}A$ inherits a ring structure from $A$, namely

- $\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{a_1 s_2 + a_2 s_1}{s_1 s_2}$,
- $\frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{a_1 a_2}{s_1 s_2}$,
- $\frac{1}{s} \cdot \frac{s}{1} = \frac{1}{1} = 1$.

**Remark 0.26.** Note that a ring structure does not guarantee every element to have a multiplicative inverse. The localization of $A$ at $S$ ensures that every element of $S$ now becomes invertible in the new ring $S^{-1}A$. In particular, this induces a ring homomorphism

$$\begin{aligned} f : A &\to S^{-1}A \\ a &\mapsto \frac{a}{1} \end{aligned}$$

This homomorphism is injective if $A$ is a domain.

**Remark 0.27.** Let $I$ be an ideal of $A$.

- Consider the ring homomorphism $f : A \to S^{-1}A$ above, then
$$S^{-1}I = IS^{-1}A = f(I)S^{-1}A.$$
In particular, $f^{-1}(IS^{-1}A) \supseteq I$.
- If $I \cap S \neq \varnothing$, then $IS^{-1}A = S^{-1}A$.
- If $P$ is a prime ideal of $A$ such that $P \cap S = \varnothing$, then $f^{-1}(PS^{-1}A) = P$.
- Let $M$ be an $A$-module, then if $N \subseteq M$ is a submodule, then $S^{-1}N \subseteq S^{-1}M$. That is, given an exact sequence
$$0 \longrightarrow N \longrightarrow M$$
then we obtain an exact sequence
$$0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M$$
Indeed, given $0 \to N \xrightarrow{f} M$, say we have it sending $\frac{n}{1} \mapsto \frac{f(n)}{1} = 0$, then there exists $s \in S$ such that $sf(n) = 0$, so $f(sn) = 0$, therefore $sn = 0$ by injection, hence $\frac{n}{1} = 0$ in $S^{-1}N$ as well.

**Exercise 0.28.** The localization functor is exact.

**Lemma 0.29.** Let $A$ be a commutative ring and $S$ be a multiplicatively closed subset of $A$, then $S^{-1}A \otimes_A M \cong S^{-1}M$.

*Proof.* We define

$$\begin{aligned} \varphi : S^{-1}A \otimes_A M &\to S^{-1}M \\ \frac{a}{s} \otimes m &\mapsto \frac{am}{s}. \end{aligned}$$

For any $\frac{m}{s} \in S^{-1}M$, we have $\varphi\left(\frac{1}{s} \otimes m\right) = \frac{m}{s}$, so the map is onto. Now suppose $\varphi\left(\sum_{i=1}^{n} \frac{a_i}{s_i} \otimes m_i\right) = 0$ (since this is a finite sum), then $\varphi\left(\sum_{i=1}^{n} \frac{a_i}{s_i} \otimes m_i\right) = \sum_{i=1}^{n} \frac{a_i m_i}{s_i} = 0$. We make $s = s_1 \cdots s_n$, so

$$\frac{a_i}{s_i} \otimes m_i = \frac{a_i s_1 \cdots s_{i-1} s_{i+1} \cdots s_n}{s} \otimes m_i =: \frac{b_i}{s} \otimes m_i,$$

then $\sum_{i=1}^{n} \frac{a_i}{s_i} \otimes m_i = \sum_{i=1}^{n} \frac{b_i}{s} \otimes m_i$, therefore

$$\varphi\left(\sum_{i=1}^{n} \frac{a_i}{s_i} \otimes m_i\right) = \varphi\left(\sum_{i=1}^{n} \frac{b_i}{s} \otimes m_i\right) = \frac{\sum_{i=1}^{n} b_i m_i}{s} = 0,$$

so there exists $t \in S$ such that $t \sum_{i=1}^{n} b_i m_i = 0$, now

$$\sum_{i=1}^{n} \frac{a_i}{s_i} \otimes m_i = \sum_{i=1}^{n} \frac{b_i}{s} \otimes m_i$$

$$\begin{aligned}
&= \sum_{i=1}^{n} \frac{1}{s} \otimes b_i m_i \\
&= \frac{1}{s} \otimes \sum_{i=1}^{n} b_i m_i \\
&= \frac{t}{ts} \otimes \sum_{i=1}^{n} b_i m_i \\
&= \frac{1}{ts} \otimes t \sum_{i=1}^{n} b_i m_i \\
&= \frac{1}{ts} \otimes 0 \\
&= 0.
\end{aligned}$$

□

**Proposition 0.30.** The map $A \to S^{-1}A$ is $A$-flat, i.e., $S^{-1}A$ is a flat $A$-module.

*Proof.* Consider

$$0 \longrightarrow N \longrightarrow M \longrightarrow T \longrightarrow 0$$

By Lemma 0.29 (since the isomorphism is functorial), it suffices to show the exactness of

$$0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M \longrightarrow S^{-1}T \longrightarrow 0$$

and this follows from Exercise 0.28. □

**Definition 0.31.** Let $A$ be a commutative ring. We say $A$ is quasi-local if $A$ has exactly one maximal ideal. In particular, if $A$ is also Noetherian, then we say $A$ is a local ring.

**Definition 0.32.** Let $A$ be a commutative ring and $\mathfrak{p}$ be a prime ideal of $A$. Note that $S = A \backslash \mathfrak{p}$ is a multiplicatively closed subset, then we write $S^{-1}A = A_\mathfrak{p}$ (in general, we have $S^{-1}M = M_\mathfrak{p}$, where $M \otimes_A A_\mathfrak{p} \cong M_\mathfrak{p}$) to denote the localization of $A$ away from the prime ideal $\mathfrak{p}$.

**Exercise 0.33.** $A_\mathfrak{p}$ is quasi-local with unique maximal ideal $\mathfrak{p}A_\mathfrak{p}$.

**Remark 0.34.** Take $x \in M$, then the following are equivalent:

- $x = 0$;
- $\frac{x}{1} = 0$ in $M_\mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of $A$;
- $\frac{x}{1} = 0$ in $M_\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ of $A$.

*Proof.* We will prove the first two are equivalent. The ($\Rightarrow$) direction is obvious. Conversely, let $I = \{a \in A \mid ax = 0\}$ to be the annihilator of $x$ in $A$. Suppose, towards contradiction, that $I \neq A$, then $I$ is contained in some maximal ideal $\mathfrak{m}$ of $A$, then consider $M_\mathfrak{m}$. Since $\frac{x}{1} = 0$ in $\mathfrak{m}$, then there exists $t \in A \backslash \mathfrak{m}$ such that $tx = 0$, but $I \subseteq \mathfrak{m}$ and $t \notin \mathfrak{m}$, then we reach a contradiction, hence $I = A$, and obviously we are done. □

**Exercise 0.35.** 1. Given the sequence

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} T \longrightarrow 0$$

the following are equivalent:

- the sequence is exact;
- the sequence

$$0 \longrightarrow M_\mathfrak{m} \xrightarrow{f_\mathfrak{m}} N_\mathfrak{m} \xrightarrow{g_\mathfrak{m}} T_\mathfrak{m} \longrightarrow 0$$

is exact for all maximal ideals $\mathfrak{m}$ of $A$;

- the sequence
$$0 \longrightarrow M_{\mathfrak{p}} \xrightarrow{f_{\mathfrak{p}}} N_{\mathfrak{p}} \xrightarrow{g_{\mathfrak{p}}} T_{\mathfrak{p}} \longrightarrow 0$$
is exact for all prime ideals $\mathfrak{p}$ of $A$.

To see this, apply Remark 0.34.

2. Let $A$ be a commutative ring and $M$ be an $A$-module, then the following are equivalent:
   - $M$ is $A$-flat;
   - $M_{\mathfrak{m}}$ is $A_{\mathfrak{m}}$-flat for all maximal ideals $\mathfrak{m}$ of $A$;
   - $M_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-flat for all prime ideals $\mathfrak{p}$ of $A$;

Hence, exactness is a local property.

**Exercise 0.36.** Let $A$ be a commutative ring, then $A$ is Artinian if and only if $A$ as an $A$-module is of finite length, i.e., $\ell_A(A) < \infty$. Indeed, note that $(0) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$, and write down the Jordan-Hölder series.

# 1 Primary Decomposition Theorem

Throughout Section 1, the commutative ring $A$ is always Noetherian. In Section 1.1, $M$ is a finitely-generated $A$-module; in Section 1.2, we drop this assumption.

## 1.1

**Definition 1.1.** We say $M$ is a coprimary module if for all $a \in A$, the left multiplication $m_a : M \to M$ is either injective or nilpotent (i.e., there exists $n > 0$ such that $a^n M = 0$).

**Remark 1.2.** (i) If $M$ is coprimary, then $N$ is coprimary for all $N \subseteq M$.

(ii) If $M$ is coprimary, let $P = \{a \in A \mid a : M \to M \text{ is nilpotent}\}$, then $P$ is a prime ideal of $A$.

*Proof.* For $a, b \notin P$, $a, b : M \to M$ are injective maps, so $ab : M \to M$ is injective, hence $ab \notin P$. □

Hence, we usually say $M$ is $P$-coprimary.

(iii) Let $M$ be $P$-coprimary, then there exists an injection (as $M$-linear map) $A/P \hookrightarrow M$.

*Proof.* Take any $x \neq 0$ in $M$, then consider
$$\begin{aligned} a_x : A &\to M \\ 1 &\mapsto x \end{aligned}$$
Let $I = \ker(a_x)$, then we have
$$\begin{aligned} A/I &\hookrightarrow M \\ \bar{1} &\mapsto x \end{aligned}$$
Now $I \subseteq P$ since $I$ already kills $x$. Since $A$ is Noetherian, $P$ is finitely-generated, thus consider $P = (a_1, \ldots, a_r)$, then $a_i^{t_i} \cdot x = 0$ for all $i$ and some $t_i$'s. Let $t = t_1 + \cdots + t_r$, then $P^t \cdot x = 0$ by binomial theorem, so $P^t \subseteq I \subseteq P$, hence there exists $j$ such that $P^j \subseteq I \subsetneq P^{j-1}$. Take $y \in P^{j-1} \backslash I$, so $\bar{y} \neq 0$ in $A/P$, taking the injection into $M$, then $\mathrm{Ann}_A(\bar{y}) = P$. We now have the composition
$$\begin{aligned} A/P \hookrightarrow A/I &\hookrightarrow M \\ \bar{1} &\mapsto \bar{y} \end{aligned}$$
to be injective. □

(iv) Suppose $M$ is $P$-coprimary, and $Q$ is a prime ideal such that $A/Q \hookrightarrow M$, then $P = Q$.

*Proof.* By definition of $P$, $Q \subseteq P$ is obvious: $Q$ kills elements in $M$, therefore the mapping becomes nilpotent. The other direction is also easy. □

**Definition 1.3.** Let $N \subseteq M$ be a submodule. We say $N$ is a primary submodule of $M$ if $M/N$ is coprimary. If $M/N$ is $P$-coprimary, we say $N$ is $P$-primary.

**Remark 1.4.** Let $\mathfrak{p}$ be a prime ideal of $A$. We claim that $\mathfrak{p}^t$ is $P$-primary. Consider

$$m_x : A/\mathfrak{p}^t \to A/\mathfrak{p}^t$$

then $x^t = 0$ on $A/\mathfrak{p}^t$.

**Example 1.5.** Let $A = k[X,Y,Z]/(Z^2 - XY)$, let $\mathfrak{p} = (x, z)$ where $x = \mathrm{im}(X)$ and $z = \mathrm{im}(Z)$. Now $A/\mathfrak{p} = k[Y]$. $\mathfrak{p}^2$ is not $P$-primary. Indeed, note that $A/\mathfrak{p}^2 = k[X,Y,Z]/(z^2 - xy, x^2, z^2) \cong k[X,Y,Z]/(X^2, XY, Z^2, XZ)$. Now the mapping given by multiplication by $y$ on this map is injective, so $\mathfrak{p}^2$ is not $P$-primary.

In particular, the represented surface is not smooth, since the origin $(0,0,0)$ is a singularity.

**Theorem 1.6** (Primary Decomposition Theorem)**.** By assumption, $A$ is Noetherian and $M$ is finitely-generated. Let $N \subseteq M$ be a submodule, then there exists a decomposition

$$N = \bigcap_{i=1}^{r} N_i$$

where each $N_i$ is $P_i$-primary, and such that

1. all $P_i$'s are distinct, and
2. this decomposition is irredundant, i.e., minimal. In particular, this means removing any of the $N_i$'s gives a different intersection, i.e., $\bigcap_{j \neq i} N_j \nsubseteq N_i$.

This is called a primary decomposition of $N$. Moreover, the primary decomposition is unique up to permutation of modules, that is, if there exists another primary decomposition, i.e., $N = \bigcap_{i=1}^{s} N_i'$ where $N_i'$'s are $P_i'$-primary, then $r = s$ and $\{N_1, \ldots, N_r\} = \{N_1', \ldots, N_s'\}$.

1.2

# 2 Filtered Rings and Modules, Completions

# 3 Dimension Theory

# 4 Integral Extensions

# 5 Noether's Normalization Lemma

# 6 Homological Algebra