Quillen's Dévissage Theorem and Localization Theorem

Jiantong Liu

December 1, 2023

1 Review: Quillen's Theorem A and B

Recall that we have established the definitions for a nerve and a geometric realization of a category, as well as the notions

Definition (Generalized Slice Category)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and fix $D \in \mathcal{D}$. A generalized slice-over category F/D is a category of pairs (C, v) where $C \in \mathcal{C}$ and $v : FC \to D$ is a morphism in $\mathcal{D}$. A morphism of this category of the form $(C, v) \to (C', v')$ is a map $w : C \to C'$ such that $v = v'F(w)$.

Dually, a generalized slice-under category $D\backslash F$ is a category of pairs (C, v) where $C \in \mathcal{C}$ and $v : D \to FC$ is a morphism in $\mathcal{D}$. A morphism of this category of the form $(C, v) \to (C', v')$ is a map $w : C \to C'$ such that $F(w)v = v'$.

Remark. In particular, if $F = \mathrm{id}$ is the identity functor, then we recover slice categories.

Definition (Fibered Functor)**.** We say $F : \mathcal{E} \to \mathcal{B}$ is pre-fibered if for all $B \in \mathcal{B}$ the inclusion $F^{-1}(B) \hookrightarrow B\backslash F$ has a right adjoint. In particular, the classifying spaces $BF^{-1}(B) \simeq B(B\backslash F)$ are equivalent. Recall that a base-change functor is $f^* : F^{-1}(B') \to F^{-1}(B)$ associated to a morphism $f : B \to B'$ in $\mathcal{B}$, defined by the composition $F^{-1}(B') \hookrightarrow (B\backslash F) \to F^{-1}(B)$.

We say F is fibered if it is pre-fibered and $g^*f^* = (fg)^*$, so F^{-1} gives a contravariant functor from $\mathcal{B}$ to **Cat**.

Remark. Given $X \in \mathcal{A}$, the domain functor

$$\begin{aligned} \mathcal{A}/X &\to \mathcal{A} \\ (f : Y \to X) &\mapsto Y \end{aligned}$$

is a fibered functor.

Dually, there is the notion of a (pre-)cofibered functor. Finally, we proved

Theorem (Quillen's Theorem A)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor such that the classifying space $B(D \downarrow F)$ of the comma category $D \downarrow F$ is contractible for any object $D \in \mathcal{D}$, then F induces a homotopy equivalence $B\mathcal{C} \to B\mathcal{D}$. In particular, the theorem holds if we substitute the comma category $D \downarrow F$ to the slice categories $D\backslash F$ and F/D.

Corollary. Suppose $F : \mathcal{C} \to \mathcal{D}$ to be either pre-fibered or pre-cofibered, and suppose $F^{-1}(D)$ is contractible for all $D \in \mathcal{D}$, then $BF : B\mathcal{C} \to B\mathcal{D}$ is a homotopy equivalence.

Using a similar proof, we have

Theorem (Quillen's Theorem B)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor such that for every morphism $D \to D'$ in $\mathcal{D}$, the induced functor $B(D' \downarrow F) \to B(D \downarrow F)$ is a homotopy equivalence. Then for each $D \in \mathcal{D}$, the geometric realization of

$$D \downarrow F \xrightarrow{j} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is a homotopy fibration sequence. That is, the Cartesian square of categories

$$\begin{tikzcd} D \downarrow F \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\ D \downarrow \mathcal{D} \arrow[r] & \mathcal{D} \end{tikzcd}$$

gives rise to a homotopy-Cartesian of geometric realizations. Therefore, there is a long exact sequence

$$\cdots \longrightarrow \pi_{i+1}(B\mathcal{D}) \xrightarrow{\partial} \pi_i B(D \downarrow F) \xrightarrow{j} \pi_i B\mathcal{C} \xrightarrow{F} \pi_i B\mathcal{D} \xrightarrow{\partial} \cdots$$

In particular, one can replace $D \downarrow F$ by $D\backslash F$ or F/D.

Corollary. Suppose $F : \mathcal{C} \to \mathcal{D}$ is pre-fibered (respectively, pre-cofibered), and that for every arrow $u : Y \to Y'$, the base-change functor $u^* : F^{-1}(Y') \to F^{-1}(Y)$ (respectively, co cobase-change functor $u_* : F^{-1}(Y) \to F^{-1}(Y')$) is a homotopy equivalence. Then for any Y in $\mathcal{D}$, the category $F^{-1}(Y)$ is homotopy equivalent to the homotopy fiber of F over Y. That is, given the inclusion functor $i : F^{-1}(Y) \to \mathcal{C}$, the diagram

$$\begin{tikzcd}
F^{-1}(Y) \arrow[r, "i"] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
* \arrow[r, "Y"'] & \mathcal{D}
\end{tikzcd}$$

is homotopy Cartesian. In particular, for any $X \in F^{-1}(Y)$, we have an exact homotopy sequence

$$\cdots \longrightarrow \pi_{n+1}(\mathcal{D}, Y) \longrightarrow \pi_n(F^{-1}(Y), X) \xrightarrow{i_*} \pi_n(\mathcal{C}, X) \xrightarrow{F_*} \pi_n(\mathcal{D}, Y) \longrightarrow \cdots$$

To understand two other Quillen's theorems, we need to study Quillen exact categories.

2 Quillen Q-construction and K-groups of Quillen Exact Category

Definition 2.1 (Quillen Exact Category)**.** Let $\mathcal{A}$ be an abelian category and let $\mathcal{B} \subseteq \mathcal{A}$ be a full additive subcategory of $\mathcal{A}$ that is closed under extensions in $\mathcal{A}$, i.e., given a short exact sequence $0 \to A \to B \to C \to 0$, $B \in \mathcal{B}$ if $A, C \in \mathcal{B}$.

Alternatively, one can define a Quillen exact category $\mathcal{B}$ independent from the ambient category $\mathcal{A}$, but this requires additional structure on the category. To see how, let $(\mathcal{B}, S)$ be a pair where $\mathcal{B}$ is an additive category, and S is a collection of diagrams based on morphisms in $\mathcal{B}$ of shape

$$A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$$

called admissible exact sequences. Here f is called an admissible monomorphism and g is called an admissible epimorphism, such that

1. replete axiom: $A \twoheadrightarrow A \to 0$ and $0 \to B \rightarrowtail B$ are admissible;
2. they are short exact sequences, i.e., $g \circ f = 0$, $g = \operatorname{coker}(f)$ and $f = \ker(g)$;
3. composition of admissible monomorphisms (respectively, epimorphisms) are admissible monomorphisms (respectively, epimorphisms);
4. pushouts of admissible monomorphisms exist and remain admissible monomorphisms, i.e., for any diagram of shape

$$\begin{tikzcd}
A \arrow[r, tail, "f"] \arrow[d, "u"'] & B \\
A' &
\end{tikzcd}$$

can be completed as a pushout square

$$\begin{tikzcd}
A \arrow[r, tail, "f"] \arrow[d, "u"'] & B \arrow[d, "u'"] \\
A' \arrow[r, tail, "f'"'] & B'
\end{tikzcd}$$

such that f' is an admissible monomorphism;

5. dually, pullbacks of admissible epimorphisms exist and remain admissible epimorphisms.

Remark 2.2. Current literature usually calls this "exact category" for short. A related concept is "Frobenius exact category."

Example 2.3.
- An abelian category is precisely an exact additive category, with admissible short exact sequences given by the short exact sequences.
- Let $\mathcal{A}$ be an additive category and $\mathcal{B} = \mathbf{Ch}(\mathcal{A})$ be the category of chain complexes in $\mathcal{A}$. It is obvious that $\mathcal{B}$ is an additive category. Moreover, $\mathcal{B}$ has an obvious Quillen exact structure $\mathcal{B}$ has a Quillen exact structure where admissible short exact sequences are the short sequences that are split-exact in every degree (without requiring the splitting to be compatible with the differentials, hence not the split-exact structure on the additive category $\mathcal{B}$).
- Let A be a ring, and let $\mathbf{P}(A)$ be the additive category of finitely-generated projective A-modules, then $\mathbf{P}(A)$ is exact with exact sequences as the ones that are exact in the category of all A-modules.

Definition 2.4 (Exact Functor)**.** For an exact category, an exact functor is a functor that preserves the admissible short exact sequences.

We have seen how to build K-theory on category of finitely-generated projective modules. A natural task would be to build K-groups on arbitrary exact categories. This requires understanding Quillen Q-construction.

Definition 2.5 (Quillen Q-construction)**.** Let $\mathcal{B}$ be an exact category, we construct $Q\mathcal{B}$ as follows. $Q\mathcal{B}$ has the same objects as $\mathcal{B}$, and morphisms in $\mathrm{Hom}_{Q\mathcal{B}}(X,Y)$ are the isomorphism classes of zigzag diagrams of the form

$$X \overset{j}{\twoheadleftarrow} Z \overset{i}{\rightarrowtail} Y$$

The isomorphism classes of $\mathrm{Hom}_{Q\mathcal{B}}(X,Y)$ are defined such that two diagrams

$$X \overset{j}{\twoheadleftarrow} Z \overset{i}{\rightarrowtail} Y$$

$$X \overset{j}{\twoheadleftarrow} Z' \overset{i}{\rightarrowtail} Y$$

give an isomorphism $Z \cong Z'$. A composition of morphisms is defined by the pullback, that is, given two morphsims

$$X \overset{j}{\twoheadleftarrow} Z \overset{i}{\rightarrowtail} Y$$

$$Y \overset{j'}{\twoheadleftarrow} Z' \overset{i'}{\rightarrowtail} Y'$$

the composition is the morphism

$$X \overset{j\circ\pi_Z}{\twoheadleftarrow} Z \times_Y Z' \overset{i'\circ\pi_{Z'}}{\rightarrowtail} Y'$$

defined from the pullback diagram

$$\begin{tikzcd} Z \times_Y Z' \arrow[r, tail, "\pi_{Z'}"] \arrow[d, two heads, "\pi_Z"'] & Z' \arrow[r, tail, "i'"] \arrow[d, two heads, "j'"] & Y' \\ Z \arrow[r, tail, "i"'] \arrow[d, two heads, "j"'] & Y & \\ X & & \end{tikzcd}$$

This defines a category $Q\mathcal{B}$.

Remark 2.6. Alternatively, we can define the morphism using the notion of admissible layer. An admissible layer of object $X \in \mathcal{B}$ is a pair of subobjects X_1, X_2 of X, i.e., an isomorphism class (of objects over X) of admissible monomorphisms $X_i \rightarrowtail X$, such that $X_1, X_2/X_1$, and X/X_2 are objects in $\mathcal{B}$. In this sense, a morphism from Y to X is an isomorphism $Y \cong X_2/X_1$ where (X_1, X_2) is an admissible layer of X.

Remark 2.7. Every morphism in $Q\mathcal{B}$ is a monomorphism. Moreover, the slice category $Q\mathcal{B}/X$ is equivalent to the ordered set of admissible layers in X with ordering $(X_0, X_1) \leqslant (X_2, X_3)$ iff $X_2 \leqslant X_0 \leqslant X_1 \leqslant X_3$, where $A \leqslant B$ is the ordering on admissible subobjects of X: the unique map $A \to B$ over X is an admissible monomorphism, i.e., $A \leqslant B$ iff (A, B) is an admissible layer of X.

Remark 2.8. Given admissible monomorphism $i : B' \rightarrowtail B$, there exists an induced morphism $i_! : B' \to B$ in $Q\mathcal{B}$, which we call it to be injective. Dually, we denote $j^! : B' \to B$ to be surjective induced from admissible epimorphism $j : B \to B'$. One should note that they are not actually injective/surjective, i.e., monomorphism/epimorphism in the categorical sense: as noted above, every morphism in $Q\mathcal{B}$ is a monomorphism.

Using these notions, any morphism u in $Q\mathcal{B}$ is given by the unique factorization $u = i_! j^!$ up to unique isomorphism.

We restrict our attention to small exact categories $\mathcal{B}$, so that we get to defined the classifying space $BQ\mathcal{B}$.

Remark 2.9. The classifying space $BQ\mathcal{B}$ is exactly the geometric realization of the semisimplicial set whose n-simplices are chains $M_0 \to M_1 \to \cdots \to M_n$ of arrows in a small category equivalent to $Q\mathcal{B}$. This is equivalent to the geometric realization of the nerve of $Q\mathcal{B}$, denoted $|\mathcal{N}(Q\mathcal{B})[-]|$, which is independent of the basepoint, which we assume to be the zero object 0 of the category.

Definition 2.10. The K-theory space is $K(\mathcal{B}) = \Omega BQ\mathcal{B}$ is an infinite loopspace. The K-groups are defined by $K_i\mathcal{B} = \pi_i(K\mathcal{B}) = \pi_{i+1}(BQ\mathcal{B}, 0)$.

Theorem 2.11 (Quillen, Quillen (1975))**.** $\pi_1(BQ\mathcal{B}, 0) \cong K_0(\mathcal{B})$ canonically.

Theorem 2.12 (Quillen, Quillen (1975))**.** If A is a regular ring, i.e., A is a Noetherian ring such that every module has finite projective dimension, then $K_n(A)$ is isomorphic to the nth K-group of the category of finitely-generated A-modules.

3 Quillen's Dévissage Theorem

Setup. For the rest of the talk, let $\mathcal{A}$ be an abelian category, and let $\mathcal{B}$ be a non-empty full subcategory $\mathcal{A}$ that is closed under taking subobjects, quotients, and finite products in $\mathcal{A}$. Under this setting, $\mathcal{B}$ is abelian as well, and the inclusion functor $\iota : \mathcal{B} \to \mathcal{A}$ is exact, where we regard both categories to be exact in the obvious way, i.e., monomorphisms and epimorphisms are admissible. With this in mind, the Q-construction gives a full subcategory $Q\mathcal{B}$ of $Q\mathcal{A}$.

Theorem 3.1 (Dévissage Theorem)**.** Suppose that every object M of $\mathcal{A}$ has a finite filtration, i.e.,

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

such that $M_j/M_{j-1} \in \mathcal{B}$ for all j. Then the inclusion functor $Q\iota : Q\mathcal{B} \to Q\mathcal{A}$ is a homotopy equivalence.

Proof. By Quillen's Theorem A, it suffices to prove that $Q\iota/M$ is contractible for any object M of $\mathcal{A}$. Here $Q\iota/M$ is thought of as a generalized slice category over $Q\mathcal{B}$ with objects as pairs (N, u) where $N \in Q\mathcal{B}$ is an object and $u : N \to M$ is a morphism in $Q\mathcal{A}$.

If we associate u with the target M, we get an admissible layer (M_0, M_1) of M such that u defines an isomorphism $N \cong M_1/M_0$. Therefore, $Q\iota/M$ is categorically equivalent to the the poset category $J(M)$ of admissible layers (M_0, M_1) in M such that $M_1/M_0 \in \mathcal{B}$, with ordering $(M_0, M_1) \leqslant (M_0', M_1')$ if and only if $M_0' \subseteq M_0 \subseteq M_1 \subseteq M_1'$.

Since M has a finite filtration with quotients in $\mathcal{B}$, then it suffices to show that $i : J(M') \to J(M)$ is a homotopy equivalence whenever $M' \subseteq M$ is such that $M/M' \in \mathcal{B}$. Define

$$\begin{aligned} r : J(M) &\to J(M') \\ (M_0, M_1) &\mapsto (M_0 \cap M', M_1 \cap M') \end{aligned}$$

and

$$\begin{aligned} s : J(M) &\to J(M) \\ (M_0, M_1) &\mapsto (M_0 \cap M', M_1). \end{aligned}$$

We claim that r is the homotopy inverse of i. Note that they are well-defined because $\mathcal{B}$ is closed under subobjects and products, and because $(M_1 \cap M')/(M_0 \cap M') \subseteq M_1/(M_0 \cap M') \subseteq (M_1/M_0) \times (M/M')$. Now $ri = \mathrm{id}_{J(M')}$ and there exists natural transformations $ir \to s \leftarrow \mathrm{id}_{J(M)}$ represented by ordering

$$(M_0 \cap M', M_1 \cap M') \leqslant (M_0 \cap M', M_1) \geqslant (M_0, M_1).$$

It is an easy consequence from the property of geometric realizations we saw that

Lemma 3.2. A natural transformation $\theta : F \to G$ of functors $\mathcal{C} \to \mathcal{D}$ induces a homotopy $B\mathcal{C} \times [0, 1] \to B\mathcal{D}$ between BF and BG.

Therefore r is a homotopy inverse for i. □

Corollary 3.3. $K_n\mathcal{B} \cong K_n\mathcal{A}$ for all $n \geqslant 0$.

Corollary 3.4. If $\mathcal{A}$ is such that every object has finite length, then $K_n\mathcal{A} \cong \coprod_{j\in J} K_n D_j$ where $\{X_j, j \in J\}$ is a set of representatives for the isomorphism classes of simple objects of $\mathcal{A}$, and D_j is the field $\mathrm{End}(X_j)^{\mathrm{op}}$, as an endomorphism ring of a simple module.

Proof. By Corollary 3.3, $K_n\mathcal{B} \cong K_n\mathcal{A}$ for all $n \geqslant 0$, where $\mathcal{B}$ is the subcategory of semisimple objects. Therefore, it suffices to prove the statement assuming every object of $\mathcal{A}$ is semisimple. Note that K-groups commute with products and filtered limits, then we may assume $\mathcal{A}$ has a unique object X up to isomorphism. In this case, the mapping $M \mapsto \mathrm{Hom}(X, M)$ defines a categorical equivalence of $\mathcal{A}$ with $\mathbf{P}\,\mathrm{End}(X)^{\mathrm{op}}$, the additive category of finitely-generated projective modules over $\mathrm{End}(X)^{\mathrm{op}}$. □

Corollary 3.5. If I is a nilpotent two-sided ideal in a Noetherian ring A, then $K_n'(A/I) \cong K_n'(A)$. Here $K_n'(R)$ is the nth K-group of finitely-generated R-modules of a Noetherian ring R.

4 Quillen's Localization Theorem

Definition 4.1 (Serre subcategory)**.** A Serre subcategory $\mathcal{B}$ of $\mathcal{A}$ is a full subcategory that is closed under

- subobjects: suppose $B \rightarrowtail A$ in $\mathcal{A}$ is a subobject and $A \in \mathcal{B}$, then $B \in \mathcal{B}$.
- quotients: suppose $A \twoheadrightarrow B$ in $\mathcal{A}$ and $A \in \mathcal{B}$, then $B \in \mathcal{B}$.
- extensions: suppose $A \rightarrowtail B \twoheadrightarrow A'$ is exact in $\mathcal{A}$ where $A, A' \in \mathcal{B}$, then $B \in \mathcal{B}$.

Remark 4.2. The kernel of an exact functor $F : \mathcal{C} \to \mathcal{D}$ is a Serre subcategory of $\mathcal{C}$.

Definition 4.3 (Gabriel Quotient)**.** Given a Serre subcategory $\mathcal{B}$ of $\mathcal{A}$, the quotient structure $\mathcal{A}/\mathcal{B}$, called the Gabriel Quotient, is a well-defined abelian category as follows: the objects of $\mathcal{A}/\mathcal{B}$ are exactly the objects of $\mathcal{A}$, and the morphisms are given by the direct limit of abelian groups

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X, Y) := \varinjlim \mathrm{Hom}_{\mathcal{A}}(X', Y/Y')$$

for subobjects $X' \subseteq X$ and $Y' \subseteq Y$ such that $X/X' \in \mathcal{B}$ and $Y' \in \mathcal{B}$.

Remark 4.4. There is a canonical exact (quotient) functor $Q : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ such that $Q(\mathcal{B}) = 0$, and Q is initial among exact functors $F : \mathcal{A} \to \mathcal{C}$ such that $F(\mathcal{B}) = 0$, that is,

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{C} \\ \mathcal{A}/\mathcal{B} \arrow[ur, dashed, "\exists!\bar{F}"'] & \end{tikzcd}$$

In particular, $\bar{F}$ is exact.

Theorem 4.5 (Localization Theorem)**.** Let $\mathcal{B}$ be a Serre subcategory of $\mathcal{A}$, then $\mathcal{A}/\mathcal{B}$ is a well-defined quotient category. Let $e : \mathcal{B} \to \mathcal{A}$ be the inclusion functor and let $s : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ be the quotient functor, then there exists a long exact sequence

$$\cdots \xrightarrow{s_*} K_1(\mathcal{A}/\mathcal{B}) \longrightarrow K_0\mathcal{B} \xrightarrow{e_*} K_0\mathcal{A} \xrightarrow{s_*} K_0(\mathcal{A}/\mathcal{B}) \longrightarrow 0$$

In particular, $\mathcal{B} \to \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is a fibration.

Remark 4.6. The argument of the proof connects back to Grothendieck-Riemann-Roch theorem.

The following corollary is more well-known, and fits in the setting of Suslin (1983).

Corollary 4.7. Let A be a Dedekind domain with field of fractions $F = \mathrm{Frac}(A)$, then there exists a long exact sequence

$$\cdots \longrightarrow K_{n+1}F \longrightarrow \coprod_{\text{maximal } \mathfrak{m}} K_n(A/\mathfrak{m}) \longrightarrow K_n(A) \longrightarrow K_n(F) \longrightarrow \cdots$$

Proof. Let $\mathcal{A}$ be the category of finitely-generated A-modules, and $\mathcal{B}$ be the subcategory of torsion A-modules in $\mathcal{A}$. Now the Gabriel quotient $\mathcal{A}/\mathcal{B}$ is $\mathcal{M}(F)$, the category of finitely-generated F-modules, also known as $\mathbf{P}(F)$. By Theorem 2.12, we have $K_n\mathcal{A} = K_nA$, and by Corollary 3.4 we know $K_n\mathcal{B} = \coprod K_n(A/\mathfrak{m})$. Now note that the map $K_nA \to K_nF$ is induced by the transfer map associated to $A \to A/\mathfrak{m}$, and this induces a long exact sequence, c.f., Quillen (1975). ☐

Corollary 4.8. Let A be a discrete valuation ring (DVR), i.e., A is a local Dedekind domain that is not a field. Let $\mathfrak{m}$ be the unique maximal ideal of A, $E = \mathrm{Frac}(A)$ be the field of fractions, and $F = A/\mathfrak{m}$ be the residue field, then there exists a long exact sequence

$$\cdots \longrightarrow K_{n+1}E \longrightarrow K_n(F) \longrightarrow K_n(A) \longrightarrow K_n(E) \longrightarrow \cdots$$

Proof of Theorem 4.5. Let $0 \in \mathcal{A}$ be the zero object, then with an abuse of notation we denote 0 to be the image in $\mathcal{A}/\mathcal{B}$. Therefore $\mathcal{B}$ is exactly the full subcategory of $\mathcal{A}$ of elements M such that $sM \cong 0$. Therefore, the composition $Qe \circ Qs$ of $Qe : Q\mathcal{B} \to Q\mathcal{A}$ and $Qs : Q\mathcal{A} \to Q(\mathcal{A}/\mathcal{B})$ is isomorphic to the constant functor of value 0. Therefore, Qe factors as

$$\begin{array}{ccccc} Q\mathcal{B} & \longrightarrow & 0\backslash Qs & \longrightarrow & Q\mathcal{A} \\ M & \longmapsto & (M, 0 \cong aM) & & \\ & & (N, u) & \longmapsto & N \end{array}$$

By Quillen's Theorem B, it suffices to show

(a) ***For every $u : V' \to V$ in $Q(\mathcal{A}/\mathcal{B})$, the induced map $u^* : V\backslash Qs \to V'\backslash Qs$ is a homotopy equivalence.***
In particular, by Theorem B, we conclude that $(Qs)^{-1}(V)$, which is homotopy equivalent to $V\backslash Qs$ for prefiber Qs, is homotopy equivalent to the homotopy fiber of Qs over V.

(b) ***The functor $Q\mathcal{B} \to 0\backslash Qs$ is a homotopy equivalence.***
In particular, $Q\mathcal{B}$ is homotopy equivalent to the homotopy fiber $(Qs)^{-1}(0)$ over 0, and since the composition is just the constant functor at 0, then by definition $Q\mathcal{B} \to Q\mathcal{A} \to Q(\mathcal{A}/\mathcal{B})$ is a homotopy fibration, and hence gives rise to a long exact sequence of homotopy groups as desired.

To prove (a), since u can be given an epi-mono factorization, then it suffices to prove it in the case where u is either a monomorphism or an epimorphism. However, the K-groups of opposite categories are the same, therefore it suffices to prove (a) assuming u is a monomorphism. Therefore, we write $u = i_!$ for $i : V' \rightarrowtail V$. It then suffices to prove (a) for injective maps $i_{V!}$ for any $V \in \mathcal{A}/\mathcal{B}$. We will postpone the proof (a) for now to tackle (b).

Let $\mathcal{F}_V$ be the full subcategory of $V\backslash Qs$ consisting of pairs (M, u) such that $u : V \to sM$ is an isomorphism. In particular, $\mathcal{F}_0 \cong Q\mathcal{B}$. Therefore, to prove (b), it suffices to show that

Lemma 4.9. The inclusion functor $F : \mathcal{F}_V \to V\backslash Qs$ is a homotopy equivalence.

Subproof. By Quillen's Theorem A, it suffices to show that $F/(M,u)$ is contractible for all (M,u) of $V\backslash Qs$. Let $u: V \to sM$ in $Q(\mathcal{A}/\mathcal{B})$ be represented by isomorphism $V \cong V_1/V_0$, where (V_0, V_1) is an admissible layer in sM. Recall that $F/(M,u)$ is categorically equivalent to the ordered set of layers (M_0, M_1) in M such that $(sM_0, sM_1) = (V_0, V_1)$ with usual ordering. Again, the ordering is directed and non-empty, so $F/(M,u)$ is filtered, thus contractible. Indeed, since $I = F/(M,u)$ is filtered, then I is the inductive limit of the functor $i \mapsto I/i$ where I/i is a slice category with a terminal object, hence contractible. ■

For the rest of the proof, we will argue that $\mathcal{F}_V$ is homotopy equivalent to $Q\mathcal{B}$ for all V, then (a) follows.

To begin with, we fix N to be an object of $\mathcal{A}$, and let $\mathcal{E}_N$ be the category of object pairs (M,h) where $M \in \mathcal{A}$ and $h: M \to N$ is a mod-$\mathcal{B}$ isomorphism, i.e., a morphism in $\mathcal{A}$ with kernel and cokernel in $\mathcal{B}$, or equivalently is an isomorphism in $\mathcal{A}/\mathcal{B}$ through Q. A morphism in this category $\mathcal{E}_N$ of form $(M,h) \to (M',h')$ is a map $u: M \to M'$ in $Q\mathcal{A}$ such that the diagram

$$\begin{tikzcd} M'' \arrow[r, tail, "i"] \arrow[d, two heads, "j"'] & M' \arrow[d, "h'"] \\ M \arrow[r, "h"'] & N \end{tikzcd}$$

commutes if we write down the factorization $u = i_! j^!$. For each (M,h) in $\mathcal{E}_N$, there exists a unique object $\ker(h) \in \mathcal{B}$ up to canonical isomorphism. In particular, this extends to a functor

$$\begin{aligned} k_N : \mathcal{E}_N &\to Q\mathcal{B} \\ (M,h) &\mapsto \ker(h) \end{aligned}$$

which is determined up to canonical isomorphism. The rest of the proof is divided into the following steps.

Step 1 Show that k_N is a homotopy equivalence.

Step 1.1 Let $\mathcal{E}'_N$ be the full subcategory of $\mathcal{E}_N$ consisting of pairs (M,h) such that $h: M \to N$ is an epimorphism, then the restriction $k'_N : \mathcal{E}'_N \to Q\mathcal{B}$ of k_N is a homotopy equivalence.
It suffices to show that for any $T \in Q\mathcal{B}$, k'_N/T is contractible. This uses the universal construction of the kernel on fiber category over $\mathcal{E}'_N$.

Step 1.2 k_N is a homotopy equivalence.
By Step 1.1, it suffices to show that the inclusion $\mathcal{E}'_N \hookrightarrow \mathcal{E}_N$ is a homotopy equivalence. Let $\mathcal{I}$ be the ordered set of subjects I of N such that $N/I \in \mathcal{B}$, and let define

$$\begin{aligned} F : \mathcal{E}_N &\to \mathcal{I} \\ (M,h) &\mapsto \operatorname{im}(h) \end{aligned}$$

Then F is a fibered functor with fiber of I being $\mathcal{E}'_I$, and the base change functor is a homotopy equivalence. By Quillen's Theorem B, $\mathcal{E}'_I$ is homotopy equivalent to the homotopy fiber of F over I. But $\mathcal{I}$ has a terminal object and is therefore contractible, so the inclusion $\mathcal{E}'_I \hookrightarrow \mathcal{E}_N$ is a homotopy equivalence for all I and we are done.

Step 2 The isomorphism $sN \cong V$ gives rise to a homotopy equivalence between $\mathcal{F}_V$ and $\mathcal{E}_N$.
By Step 1.2, we know k_N and $k_{N'}$ are homotopy equivalences, and it is easily check that k_N and $k_{N'}g^*$ are homotopic, therefore g_* is a homotopy equivalence. One can then see that for any isomorphism $\varphi: sN \to V$, the functor

$$\begin{aligned} p_{(N,\varphi)} : \mathcal{E}_N &\to \mathcal{F}_V \\ (M,h) &\mapsto (M, s(h)^{-1}\varphi^{-1} : V \cong sN \cong sM) \end{aligned}$$

gives rise to an equivalence of categories

$$\varinjlim_{\mathcal{I}_V} \{(N,\varphi) \mapsto \mathcal{E}_N\} \simeq \mathcal{F}_V$$

where $\mathcal{I}_V$ is a filtered category of object pairs (N,φ) where $N \in \mathcal{A}$ and $\varphi: sN \cong V$ is an isomorphism in $\mathcal{A}/\mathcal{B}$. Therefore, $p_{(N,\varphi)}$ is a homotopy equivalence.

Step 3 Finish the proof.
It now suffices to show that $(i_{V!})^* : V\backslash Qs \to 0\backslash Qs$ is a homotopy equivalence. Fix a choice of (N, φ) in Step 2, one can check that the diagram

$$\begin{tikzcd} \mathcal{E}_N \arrow[r, "p_{(N,\varphi)}"] \arrow[d, "k_N"'] & \mathcal{F}_V \subseteq V\backslash Qs \arrow[d, "(i_{V!})^*"] \\ Q\mathcal{B} \arrow[r, "\simeq"'] & \mathcal{F}_0 \subseteq 0\backslash Qs \end{tikzcd}$$

is homotopy commutative. By Step 1 and Step 2, we know k_N and $p_{(N,\varphi)}$ are homotopy equivalences as well, then $(i_{V!})^*$ isa homotopy equivalence.

□

Corollary 4.10. Let R be a Noetherian ring, the denote $K_n'(R) := K_n(\mathcal{M}(R))$, where $\mathcal{M}(R)$ is the category of finitely-generated R-modules. Then there are canonical isomorphisms

(a) $K_n'(R[t]) \cong K_n'(R)$;

(b) $K_n'(R[t, t^{-1}]) \cong K_n'(R) \oplus K_{n-1}'(R)$.

Partial Proof. A proof of (a) can be found in Quillen (1975). We will give a proof of (b). Let $\mathcal{B}$ be the category of finitely-generated $R[t]$-modules consisting of modules on which t is nilpotent, then applying Quillen's localization theorem gives

$$\begin{tikzcd} \cdots \arrow[r] & K_n\mathcal{B} \arrow[r] & K_n'(R[t]) \arrow[r] & K_n'(R[t,t^{-1}]) \arrow[r] & \cdots \\ & K_n'(R) \arrow[u, "\cong"] & K_n'(R) \arrow[u, "\cong"] \arrow[ur] & & \end{tikzcd}$$

The first isomorphism is given by applying dévisssage theorem on the embedding of finitely-generated A-module, i.e., finitely-generated $A[t]/tA[t]$-modules, into $\mathcal{B}$. The second isomorphism is from (a). We study the induced composition $K_n'(R) \to K_n'(R[t, t^{-1}])$ from the homomorphism $R[t, t^{-1}] \to R$, which is given by mapping $t \mapsto 1$ which makes R a right module of Tor dimension 1 over $R[t, t^{-1}]$. Therefore, this induces a left inverse $K_n'(R[t, t^{-1}]) \to K_n'(R)$, which means the exact sequence breaks up into split short exact sequences. □

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