MATH 518 Notes

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Definition 1.1. Let M be a topological space. An *atlas* on M is a collection $\{\varphi_\alpha : U_\alpha \to W_\alpha\}_{\alpha\in A}$ of homeomorphisms called *coordinate charts*, so that

1. $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M,
2. for all $\alpha \in A$, W_α is an open subset of some $\mathbb{R}^{n_\alpha}$,
3. for all $\alpha, \beta \in A$, the induced map $\varphi_\beta \circ \varphi_\alpha^{-1}|_{U_\alpha\cap U_\beta}$ is C^∞, i.e., smooth.

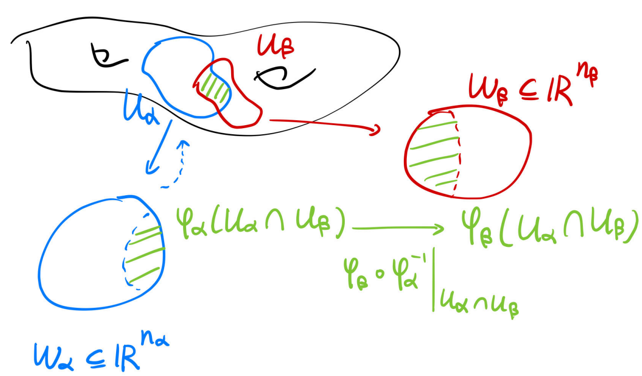

Figure 1: Atlas and Coordinate Chart

Example 1.2. Let $M = \mathbb{R}^n$ be equipped with standard topology, and let $A = \{*\}$, so $U_* = \mathbb{R}^n$ is the open cover of itself. Now the identity map

$$\begin{aligned}\varphi_* : U_* &\to \mathbb{R}^n\\ u &\mapsto u\end{aligned}$$

is an atlas on $\mathbb{R}^n$.

Example 1.3. Let $M = S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$ be equipped with subspace topology. Let $U_\alpha = S^1\backslash\{(1,0)\}$ and $U_\beta = S^1\backslash\{(-1,0)\}$, and let $A = \{\alpha,\beta\}$. Let $W_\alpha = (0,2\pi)$ and $W_\beta = (-\pi,\pi)$. We define $\varphi_\alpha^{-1}(\theta) = (\cos(\theta),\sin(\theta))$ and $\varphi_\beta^{-1}(\theta) = (\cos(\theta),\sin(\theta))$, then

$$(\varphi_\beta\circ\varphi_\alpha^{-1})(\theta) = \begin{cases}\theta, 0<\theta<\pi\\ \theta-2\pi, \pi<\theta<2\pi\end{cases}$$

is smooth.

Example 1.4. Let X be a topological space with discrete topology, and let $A = X$, then $\{\varphi_x : \{x\} \to \mathbb{R}^0\}_{x \in X}$ gives an atlas.

Example 1.5. Let V be a finite-dimensional real vector space of dimension n. Pick a basis $\{v_1, \ldots, v_n\}$ of V, then there is a linear bijection φ with inverse

$$\begin{aligned} \varphi^{-1} : \mathbb{R}^n &\to V \\ (x_1, \ldots, x_n) &\mapsto \sum_{i=1}^{n} x_i v_i. \end{aligned}$$

The topology on V needs to make φ^{-1} a homeomorphism, and the obvious choice is just the collection of preimages, namely

$$\mathcal{T} = \{\varphi^{-1}(W) \mid W \subseteq \mathbb{R}^n \text{ open}\},$$

then $\varphi : V \to \mathbb{R}^n$ becomes an atlas.

Definition 1.6. Two atlases $\{\varphi_\alpha : U_\alpha \to W_\alpha\}_{\alpha \in A}$ and $\{\psi_\beta : V_\beta \to O_\beta\}_{\beta \in B}$ on a topological space M are *equivalent* if for all $\alpha \in A$ and $\beta \in B$,

$$\psi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap V_\beta) \subseteq \mathbb{R}^{n_\alpha} \to \psi_\beta(U_\alpha \cap V_\beta) \subseteq \mathbb{R}^{n_\beta}$$

is always C^∞, with C^∞-inverses. Such continuous maps are called *diffeomorphisms*. Alternatively, the two atlases are equivalent if their union $\{\varphi_\alpha\}_{\alpha \in A} \cup \{\psi_\beta\}_{\beta \in B}$ is always an atlas.

Exercise 1.7. Equivalence of atlases is an equivalence condition.

Definition 1.8. A (smooth) *manifold* is a topological space together with an equivalence class of atlases.

Convention. All manifolds are assumed to be smooth of C^∞, but not necessarily *Haudorff* and/or *second countable*.

Example 1.9. Continuing from Example 1.5, now suppose $\{w_1, \ldots, w_n\}$ gives another basis of V, with

$$\begin{aligned} \psi^{-1} : \mathbb{R}^n &\to V \\ (y_1, \ldots, y_n) &\mapsto \sum_{i=1}^{n} y_i w_i. \end{aligned}$$

This gives a change-of-basis matrix, so it is automatically C^∞ as a multiplication of invertible matrices. Therefore, the topology here does not depend on the chosen basis.

Recall. A topological space X is *Hausdorff* if for all distinct points $x, y \in X$, there exists open neighborhoods $U \ni x$ and $V \ni y$ such that $U \cap V = \varnothing$.

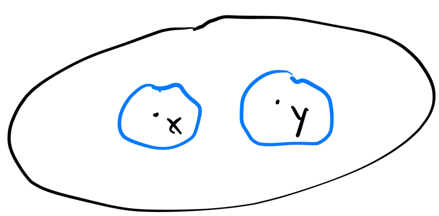

Figure 2: Hausdorff Condition

Convention. Via our definition (Definition 1.8), not all manifolds are Hausdorff.

Example 1.10. Let $Y = \mathbb{R} \times \{0, 1\}$, i.e., a space with two parallel lines, with a fixed topology. Define $\sim$ to be the smallest equivalence relation on Y such that $(x, 0) \sim (x, 1)$ for $x \neq 0$, and define $X = Y/\sim$. X is called the *line with two origins*, and it is second countable but not Hausdorff.

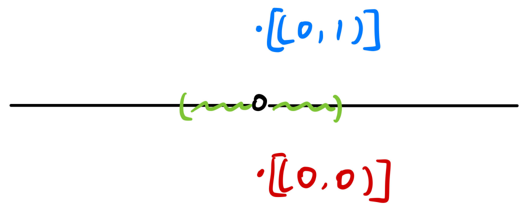

Figure 3: Line with Two Origins

Example 1.11. Take charts

$$\begin{aligned}\{\varphi : M = \mathbb{R} &\to \mathbb{R}\} \\ x &\mapsto x\end{aligned}$$

and

$$\begin{aligned}\{\psi : M = \mathbb{R} &\to \mathbb{R}\} \\ x &\mapsto x^3\end{aligned}$$

on $M = \mathbb{R}$, then

$$\begin{aligned}\varphi \circ \psi^{-1} : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^{\frac{1}{3}}\end{aligned}$$

is not C^∞, so φ and ψ are two different charts, hence give two different manifolds.

Definition 1.12. A map $F : M \to N$ between two manifolds is *smooth* if

1. F is continuous, and
2. for all charts $\varphi : U \to \mathbb{R}^m$ on M and charts $\psi : V \to \mathbb{R}^n$ on N, $\psi^{-1} \circ F \circ \varphi|_{\varphi(U \cap F^{-1}(V))}$ is C^∞.

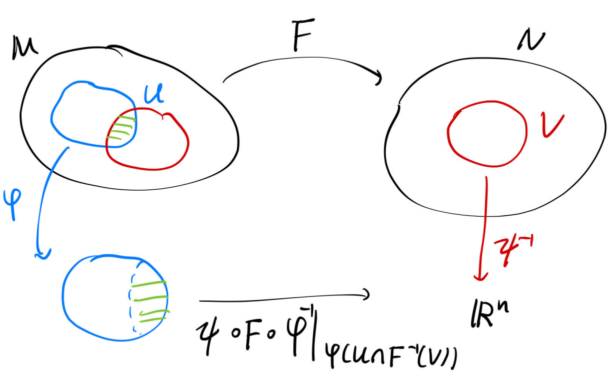

Figure 4: Smooth Map between Manifolds

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Exercise 2.1.
1. $\mathrm{id} : M \to M$ is smooth.
2. If $f : M \to N$ and $g : N \to Q$ are smooth maps between manifolds, then so is $gf : M \to Q$.

Punchline. The manifolds and the smooth maps between manifolds form a category.

Recall. A smooth map $f : M \to N$ is called a *diffeomorphism*, as seen in Definition 1.6, if it has a smooth inverse. This is the notion of an isomorphism in the category of manifolds.

Warning.

1. Following Example 1.11,
$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^3 \end{aligned}$$
has an inverse
$$\begin{aligned} f^{-1} : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^{\frac{1}{3}}, \end{aligned}$$
but f^{-1} is not differentiable at $x = 0$. Hence, f is not a diffeomorphism.

2. Take $\mathbb{R}$ with discrete topology, then all singletons are open sets, then the map
$$\begin{aligned} f : \mathbb{R}_{\text{dis}} &\to \mathbb{R}_{\text{std}} \\ x &\mapsto x \end{aligned}$$
is a smooth bijection, but f^{-1} is not continuous.

Example 2.2. Consider $M = (\mathbb{R}, \{\psi = \mathrm{id} : \mathbb{R} \to \mathbb{R}\})$ and $N = (\mathbb{R}, \{\psi : \mathbb{R} \to \mathbb{R}, x \mapsto x^3\})$ as two manifolds on $\mathbb{R}$ with standard topology. To see that they are equivalent, consider the homeomorphism
$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^{\frac{1}{3}}, \end{aligned}$$
then $(\psi \circ f \circ \varphi^{-1})(x) = \psi(f(x)) = (x^{\frac{1}{3}})^3 = x$, so f is smooth, and $(\psi \circ f \circ \varphi^{-1})^{-1} = \varphi \circ f^{-1} \circ \psi^{-1} = \mathrm{id}$, therefore f^{-1} is also smooth. Hence, f is a diffeomorphism.

We will now consider the real projective space $\mathbb{R}P^{n-1}$ and the quotient map $\pi : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}P^{n-1}$.

Definition 2.3. Define a binary relation on $\mathbb{R}^n\backslash\{0\}$ by $v_1 \sim v_2$ if and only if there exists $\lambda \neq 0$ such that $v_1 = \lambda v_2$. This is an equivalence relation, and we identify the equivalence class $[v]$ of $v \in \mathbb{R}^n\backslash\{0\}$ as a line $\mathbb{R}v = \mathrm{span}_{\mathbb{R}}\{v\}$ through v. Then we define the *real projective space* $\mathbb{R}P^{n-1} = (\mathbb{R}^n\backslash\{0\})/\sim$.

The natural topology on $\mathbb{R}P^{n-1}$ is the quotient topology, where $\pi : \mathbb{R}^n\backslash\{0\} \twoheadrightarrow \mathbb{R}P^{n-1}$ is surjective and continuous, so we define $U \subseteq \mathbb{R}P^{n-1}$ to be open if and only if $\pi^{-1}(U)$ is open in $\mathbb{R}^n\backslash\{0\}$.

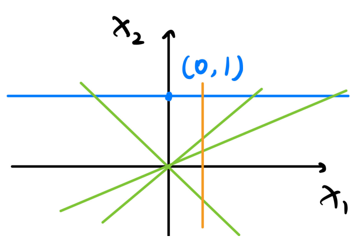

Figure 5: Stereographical Projection

Claim 2.4. $\mathbb{R}P^{n-1}$ is a manifold.

Proof. Define
$$\begin{aligned} \varphi_i : U_i &\to \mathbb{R}^{n-1} \\ [v_1, \dots, v_n] &\mapsto \left(\frac{v_1}{v_i}, \dots, \frac{v_{i-1}}{v_i}, \frac{v_{i+1}}{v_i}, \dots, \frac{v_n}{v_i}\right), \end{aligned}$$

then

$$\begin{aligned}\varphi_i^{-1}: \mathbb{R}^{n-1} &\mapsto U_i \\ (x_1, \dots, x_{n-1}) &\mapsto [(x_1, \dots, x_{i-1}, 1, x_i, \dots, x_{n-1})],\end{aligned}$$

therefore

$$\begin{aligned}\varphi_j \circ \varphi_i^{-1}: \varphi_i(U_i \cap U_j) &\to \varphi_j(U_i \cap U_j) \\ (x_1, \dots, x_{n-1}) &\mapsto \varphi_j([(x_1, \dots, x_{i-1}, 1, x_i, \dots, x_{n-1})]) \\ &= \begin{cases} \left(\frac{x_1}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots, \frac{x_{n-1}}{x_j}\right), & j < i \\ (x_1, \dots, x_{n-1}), & j = i \\ \left(\frac{x_1}{x_{j-1}}, \dots, \frac{1}{x_{j-1}}, \dots, \frac{x_{j-2}}{x_{j-1}}, \frac{x_j}{x_{j-1}}, \dots, \frac{x_{n-1}}{x_{j-1}}\right), & j > i \end{cases}\end{aligned}$$

Therefore, this is C^∞ as a rational map on $\varphi_i(U_i \cap U_j)$, and so this gives an atlas, hence $\mathbb{R}P^{n-1}$ is a manifold. ☐

Claim 2.5. $\pi : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}P^{n-1}$ is smooth.

Proof. Note that

$$\begin{aligned}\psi : \mathbb{R}^n \backslash \{0\} &\hookrightarrow \mathbb{R}^n \\ x &\mapsto x\end{aligned}$$

is an atlas on $\mathbb{R}^n \backslash \{0\}$, and

$$\begin{aligned}\varphi_i \circ \pi \circ \varphi_i^{-1} : \mathbb{R}^n \backslash \{0\} &\to \mathbb{R}^{n-1} \\ (v_1, \dots, v_n) &\mapsto \varphi_i([(v_1, \dots, v_n)]) \\ &= \left(\frac{v_1}{v_i}, \dots, \frac{v_{i-1}}{v_i}, \frac{v_{i+1}}{v_i}, \dots, \frac{v_n}{v_i}\right).\end{aligned}$$

This is C^∞ on $\pi^{-1}(U_i) = \{(v_1, \dots, v_n) \mid v_i \neq 0\}$, so π is smooth. ☐

Definition 2.6. A *smooth function* on a manifold M is a function $f : M \to \mathbb{R}$ so that for any coordinate chart $\varphi : U \to \varphi(U)$ open in $\mathbb{R}^m$, the function $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is smooth.

Remark 2.7. $f : M \to \mathbb{R}$ is smooth if and only if $f : M \to (\mathbb{R}, \{\mathrm{id} : \mathbb{R} \to \mathbb{R}\})$, usually called the *standard manifold structure on* $\mathbb{R}$, is smooth.

Notation. We denote $C^\infty(M)$ to be the set of all smooth functions $f : M \to \mathbb{R}$.

Remark 2.8. $C^\infty(M)$ is a smooth $\mathbb{R}$-vector space, that is, for all $\lambda, \mu \in \mathbb{R}$ and $f, g \in C^\infty(M)$,

- $(\lambda f + \mu g)(x) = \lambda f(x) + \mu g(x)$ for all $x \in M$,
- $(f \cdot g)(x) = f(x)g(x)$ for all $x \in M$.

Therefore, $C^\infty(M)$ becomes a (commutative, associative) $\mathbb{R}$-algebra.

Fact. Connecting manifolds have the notion of dimension. That is, the dimensions of open subsets induced by coordinate charts are the same.

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Definition 3.1. Let M be a manifold, then for every point $q \in M$, there exists a well-defined non-negative integer $\dim_M(q)$, so that for any coordinate chart $\varphi : U \to \mathbb{R}^m$ for $U \ni q$, we have $\dim_M(q) = m$ for some non-negative integer m that only depend on M. Consequently, $\dim_M : M \to \mathbb{Z}^{\geqslant 0}$ is a locally constant function. This integer m is called the *dimension* of M.

Proof. Indeed, say $\psi : V \to \mathbb{R}^n$ is another chart with $U \cap V \ni q$, then $\psi \circ \varphi^{-1}|_{\varphi(U \cap V)} : \varphi(U \cap V) \subseteq \mathbb{R}^m \to \psi(U \cap V) \subseteq \mathbb{R}^n$ is a diffeomorphism, therefore the Jacobian $D(\psi \circ \varphi^{-1})(\varphi(a)) : \mathbb{R}^m \to \mathbb{R}^n$ is a linear isomorphism, thus $m = n$. $\square$

Definition 3.2. Suppose $(M, \{\varphi_\alpha : U_\alpha \to \mathbb{R}^m\}_{\alpha \in A})$ and $(N, \{\psi_\alpha : V_\beta \to \mathbb{R}^n\}_{\beta \in B})$ are two manifolds. One can give a manifold structure to the product set $M \times N$, called the *product manifold*, as follows:

- give $M \times N$ the product topology,
- let $\{\varphi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n\}_{(\alpha,\beta) \in A \times B}$ to be the atlas on $M \times N$. This is well-defined since the transition maps of $\alpha, \alpha' \in A$ and $\beta, \beta' \in B$ are over $(U_\alpha \times V_\beta) \cap U_{\alpha'} \times V_{\beta'} = (U_\alpha \cap U_{\alpha'}) \times (V_\beta \cap V_{\beta'})$ with $(\varphi_{\alpha'} \times \psi_{\beta'}) \circ (\varphi_\alpha \times \psi_\beta)^{-1} = (\varphi_{\alpha'} \circ \varphi_\alpha^{-1}, \psi_{\beta'} \circ \psi_\beta^{-1})$. This is smooth since products of smooth maps are smooth.

Punchline. The product construction of manifolds gives the categorical product in the category of manifolds.

Property. 1. The projection maps

$$\begin{aligned} p_M : M \times N &\to M \\ (m,n) &\mapsto m \end{aligned}$$

and

$$\begin{aligned} p_N : M \times N &\to N \\ (m,n) &\mapsto n \end{aligned}$$

are C^∞.

2. *Universal Property of Product*: for any manifold Q and smooth maps $f_M : Q \to M$ and $f_N : Q \to N$, there exists a unique map

$$\begin{aligned} g : Q &\to M \times N \\ q &\mapsto (f(q), g(q)) \end{aligned}$$

such that $p_M \circ g = f_M$, and $p_N \circ g = f_N$.

$$\begin{tikzcd}
 & Q \arrow[d, dashed, "g"', "\exists!"] \arrow[ddl, bend right, "f_M"'] \arrow[ddr, bend left, "f_N"] & \\
 & M \times N \arrow[dl, "p_M"] \arrow[dr, "p_N"'] & \\
M & & N
\end{tikzcd}$$

Figure 6: Universal Property of Product

Recall.
- A topological space X is *second countable* if the topology has a countable basis: there exists a collection $\mathcal{B} = \{B_i\}_{i \in \mathbb{N}}$ of open sets so that any open set of X is a union of some B_i's.
- A cover $\{U_\alpha\}_{\alpha \in A}$ of a topological space is *locally finite* if for all $x \in X$, there exists a neighborhood N of X such that $N \cap U_\alpha = \varnothing$ for all but finitely many α's.

Example 3.3. Let $X = \mathbb{R}$, then

- $\{U_n = (-n, n)\}_{n \geqslant 0}$ is an open cover, but is not locally finite,
- $\{U_n = (n, n+2)\}_{n \in \mathbb{Z}}$ is a locally finite open cover of $\mathbb{R}$,
- $\{U_n = (n, n+2]\}_{n \in \mathbb{Z}}$ is a locally finite cover of $\mathbb{R}$, but is not an open cover.

Recall. An (open) cover $\{V_\beta\}_{\beta\in B}$ is a *refinement* of a cover $\{U_\alpha\}_{\alpha\in A}$ if for all β, there exists $\alpha = \alpha(\beta)$ such that $V_\beta \subseteq U_{\alpha(\beta)}$.

Definition 3.4. A Hausdorff topological space is *paracompact* if every open cover has a locally finite open refinement.

Fact. A connected Hausdorff manifold is paracompact if and only if it is second countable.

Corollary 3.5. A Haudorff manifold is paracompact if and only if its connected components are second countable.

Example 3.6. $\mathbb{R}$ with discrete topology is paracompact but not second countable.

Convention. Usually, we assume manifolds are paracompact, except when we need a non-Haudorff manifold. This condition is required for the existence of *partition of unity* (i.e., constant function id).

Recall. If X is a space, and $Y \subseteq X$ is a subset, then the *closure* $\bar{Y}$ of Y is the smallest closed set containing Y.

Definition 3.7. Given a topological space X and a function $f : X \to \mathbb{R}$, the *support* of f over X is

$$\operatorname{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}.$$

Example 3.8. The function

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leqslant 0 \end{cases}$$

is C^∞, with support $\overline{(0,\infty)} = [0,\infty)$.

Definition 3.9. Let M be a topological space and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover. A *partition of unity* subordinate to the cover is a collection of continuous functions $\{\psi_\alpha : M \to [0,1]\}_{\alpha\in A}$ such that

1. $\operatorname{supp}(\psi_\alpha) \subseteq U_\alpha$ for all $\alpha \in A$,
2. $\{\operatorname{supp}(\psi_\alpha)\}_{\alpha\in A}$ is a locally finite closed cover of M,
3. $\sum_{\alpha\in A} \psi_\alpha(x) = 1$ for all $x \in M$.

Remark 3.10. For all $x \in M$, there exists $\alpha_1, \dots, \alpha_n$ such that $x \in \operatorname{supp}(\psi_{\alpha_i})$. Hence, for $\alpha \neq \alpha_1, \dots, \alpha_n$, $\psi_\alpha(x) = 0$. Therefore, the summation in Definition 3.9 is finite.

Theorem 3.11. Let M be a paracompact manifold with open cover $\{U_\alpha\}_{\alpha\in A}$, then there exists a partition of unity $\{\psi_\alpha : U_\alpha \to [0,1]\}_{\alpha\in A} \subseteq C^\infty(M)$ subordinate to the cover.

Example 3.12. Let $M = \mathbb{R}$ and consider for $n > 0$ the open sets $\{U_n = (-n, n)\}_{n\in\mathbb{N}}$. This is not locally finite at one point.

Example 3.13. Let $M = \mathbb{R}^n$, then for all $x \in \mathbb{R}^n$ and for $r > 0$, we have $B_r(x) = \{x' \in \mathbb{R}^n \mid ||x - x'|| < r\}$ and so $\{B_r(x)\}_{r>0, x\in\mathbb{R}^n}$ is an open cover, but this is not locally finite everywhere.

4 Aug 28, 2023

We will start to talk about tangent vectors.

Recall. For any point $q \in \mathbb{R}^n$ and any vector $v \in \mathbb{R}^n$, and any $f \in C^\infty(\mathbb{R}^n)$, the *directional derivative* of q in direction v with respect to f is

$$D_v f(q) = \frac{d}{dt}|_0 f(q + tv).$$

This gives a map $D_v(-)(q) : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ which is

- linear, and

- Leibniz rule holds, i.e.,
$$D_v(fg)(q) = D_v(f)(q) \cdot g(q) + f(q)D_v(g)(q).$$

In other words, $D_v(-)(q) : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation.

Definition 4.1. Let q be a point of a manifold M. A *tangent vector* to M at q is an $\mathbb{R}$-linear map $v : C^\infty(M) \to \mathbb{R}$ such that for all $f, g \in C^\infty(M)$,
$$v(fg) = v(f)g(q) + f(q)v(g).$$

Remark 4.2. v gives smooth vector fields over M an $C^\infty(M)$-module structure via evaluation.

Lemma 4.3. The set T_qM of all tangent vectors to M at q is an $\mathbb{R}$-vector space.

Lemma 4.4. Suppose $c \in C^\infty(M)$ is a constant function, then for all q and all $v \in T_qM$, $v(c) = 0$.

Proof. We have $v(1) = v(1 \cdot 1) = 1(q)v(1) + v(1)1(q) = 2v(1)$, so $v(1) = 0$. For a constant function c, we have
$$v(c) = v(c \cdot 1) = cv(1) = c(0) = 0.$$
□

Lemma 4.5 (Hadamard)**.** For any $f \in C^\infty(\mathbb{R}^n)$, there exists $g_1, \ldots, g_n \in C^\infty(\mathbb{R}^n)$ such that

- $f(x) = f(0) + \sum\limits_{i=1}^n x_i g_i(x)$, and
- $g_i(0) = \left(\frac{\partial}{\partial x_i} f\right)(0)$.

Proof. We have
$$\begin{aligned} f(x) - f(0) &= \int_0^1 \frac{d}{dt}(f(tx))dt \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(tx) \cdot x_i dt \\ &= \sum_{i=1}^n x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt \\ &= \sum_{i=1}^n x_i g_i(x). \end{aligned}$$

Therefore, $g_i(0) = \int_0^1 \frac{\partial f}{\partial x_i}(t \cdot 0)dt = \frac{\partial f}{\partial x_i}(0)$. □

Remark 4.6. For $1 \leqslant i \leqslant n$, we have canonical tangent vectors to $\mathbb{R}^n$ at 0 given by
$$\begin{aligned} \frac{\partial}{\partial x_i}|_0 : C^\infty(\mathbb{R}^n) &\to \mathbb{R} \\ f &\mapsto \frac{\partial f}{\partial x_i}(0). \end{aligned}$$

Lemma 4.7. $\left\{\frac{\partial}{\partial x_1}|_0, \ldots, \frac{\partial}{\partial x_n}|_0\right\}$ is a basis of $T_0\mathbb{R}^n$.

Proof. Suppose $\sum c_i \frac{\partial}{\partial x_i}|_0 = 0$, then
$$0 = \left(\sum_i c_i \frac{\partial}{\partial x_i}|_0\right)(x_j) = \sum_i c_i \delta_{ij} = c_j.$$

Therefore, $c_j = 0$ for all j, thus we have linear independence. For all $v \in T_0\mathbb{R}^n$, i.e., $v : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation, then $v = \sum_i v(x_i)\frac{\partial}{\partial x_i}|_0$. Let $f \in C^\infty(\mathbb{R}^n)$, then $f(X) = f(0) + \sum x_i g_i(x)$, thus

$$\begin{aligned} v(f) &= v(f(0)) + \sum_{i=1}^n v(x_i g_i(x)) \\ &= \sum_{i=1}^n v(x_i g_i(x)) \\ &= \sum_{i=1}^n (v(x_i)g_i(0) + x_i(0)v(g_i)) \\ &= \sum_{i=1}^n v(x_i)g_i(0) \\ &= \sum_{i=1}^n v(x_i)\frac{\partial f}{\partial x_i}(0). \end{aligned}$$

□

Remark 4.8. This shows $\dim(T_0\mathbb{R}^n) = n$ with the basis above.

Now let V be a finite-dimensional vector space with a basis $e_1, \dots, e_n$, then

$$\begin{aligned} \varphi : \mathbb{R}^n &\to V \\ (t_1, \dots, t_n) &\mapsto \sum_{i=1}^n t_i e_i \end{aligned}$$

is a linear bijection, with linear inverse

$$\begin{aligned} \psi : V &\to \mathbb{R}^n \\ v &\mapsto (\psi_1(v), \dots, \psi_n(v)) \end{aligned}$$

where $\psi_i(v)$'s are linear maps. To describe this with a basis, we have $\psi(\sum_i a_i e_i) = (a_1, \dots, a_n)$, i.e., $\psi_i(e_j) = \delta_{ij}$.

Claim 4.9. $\{\psi_1, \dots, \psi_n\}$ is a basis of $V^* = \operatorname{Hom}(V, \mathbb{R})$, called the *dual basis* of $\{e_1, \dots, e_n\}$, denoted $e_j^* = \psi_j$.

Proof. Linear independence follows from $e_j^*(e_i) = \delta_{ij}$. Given $\ell : V \to \mathbb{R}$ to be a linear map, then $\ell = \sum \ell(e_i)e_i^*$ since $\left(\sum_i \ell(e_i)e_i^*\right)(e_j) = \ell(e_j)$. Given $v \in T_0\mathbb{R}^n$, $v(f) = \sum a_i \left(\frac{\partial}{\partial x_i}|_0 f\right)$ for all $f \in C^\infty(\mathbb{R}^n)$. Note that $\frac{\partial}{\partial x_i}|_0(x_j) = \delta_{ij}$, so $v(x_j) = \sum a_i \frac{\partial}{\partial x_i}|_0(x_j) = \sum_i a_i\delta_{ij} = a_j$. Therefore, we have $a_i = v(x_i)$ for all i, thus $v(f) = \sum v(x_i)\left(\frac{\partial}{\partial x_i}|_0 f\right)$. Thus, the dual basis to $\frac{\partial}{\partial x_1}|_0, \dots, \frac{\partial}{\partial x_n}|_0$ is $\{d(x_i)_0\}_{i=1}^n$ where $(dx_i)_0(v) = v(x_i)$ for all i. Hence, we have $v = \sum(dx_i)_0(v)\frac{\partial}{\partial x_i}|_0$. □

Remark 4.10. Via a change of basis, this works at every point q on the local chart, so we can describe the tangent space on any point on a local chart.

5 Aug 30, 2023

Let M be a manifold and $x \in M$. Recall that a tangent vector $v : C^\infty(M) \to \mathbb{R}$ is a derivation, i.e., linear map, and the set of tangent vectors at q gives the tangent space.

Example 5.1. Let $M = \mathbb{R}^n$, and $q = 0$, then $\left\{\frac{\partial}{\partial x_1}|_0, \dots, \frac{\partial}{\partial x_n}|_0\right\}$ is a basis of $T_0\mathbb{R}^n$. Moreover, for all $v \in T_0\mathbb{R}^n$, $v = \sum v(x_i)\frac{\partial}{\partial x_i}|_0$, thus $\{v \mapsto v(x_i)\}_{i=1}^n$ is the dual basis, with $v(x_i) = (dx_i)_0(v)$ for all $1 \leqslant i \leqslant n$.

Remark 5.2. The proof used Hadamard's lemma (Lemma 4.5) and the fact that for all $x \in \mathbb{R}^n$ and all $t \in [0,1]$, $f(tx)$ is defined. Thus, the same argument should work for a version of Hadamard's lemma for star-shaped open subsets $U \subseteq \mathbb{R}^n$.

Definition 5.3. We say an open subset $U \subseteq \mathbb{R}^n$ is a *star-shaped domain* if for all $t \in [0,1]$ and all $x \in U$, $tx \in U$.

Definition 5.4. Let $F : M \to N$ be a smooth map between two manifolds, and $q \in M$ is a point, then

$$\begin{aligned} T_qF : T_qM &\to T_qN \\ v(f) &\mapsto v(f \circ F) \end{aligned}$$

via the pullback.

Exercise 5.5. Check that the definition makes sense, in particular:

(i) $(T_qF)(v)$ is a tangent vector to N of $F(q)$, and

(ii) T_qF is a derivation.

Remark 5.6. (a) It is easy to deduce the *chain rule*. That is, given $M \xrightarrow{F} N \xrightarrow{G} Q$ with $q \in M$, then $T_q(G \circ F) = T_{F(q)}G \circ T_qF$ because for all $f \in C^\infty(Q)$ and all $v \in T_qM$, we have

$$(T_q(G \circ F)(v))(f) = v(f \circ (G \circ F))$$

and

$$(T_{F(q)}G(T_qF(v))) = (T_qF)(v)(f \circ G) = v((f \circ G) \circ F).$$

(b) $T_q(\mathrm{id}_M) = \mathrm{id}_{T_qM}$.

As a result, we know T is a functor from the category of pointed manifolds to the category of $\mathbb{R}$-vector spaces.

Corollary 5.7. If $F : M \to N$ is a diffeomorphism, then for all $q \in M$, $T_qF : T_qM \to T_{F(q)}N$ is an isomorphism.

Proof. Since F is a diffeomorphism, then it has a smooth inverse $G : N \to M$, so

$$\mathrm{id}_{T_qM} = T_q(\mathrm{id}_M) = T_q(G \circ F) = T_{F(q)}G \circ T_qF$$

and

$$\mathrm{id}_{T_{F(q)}N} = T_{F(q)}(\mathrm{id}_N) = T_{F(q)}(F \circ G) = T_{F(q)}F \circ T_{F(q)}G.$$

□

We also need to show that $\dim(T_qM) = \dim_q(M)$, which is a result of Lemma 5.8, whose proof will be postponed till next time.

Lemma 5.8. Let M be a manifold and $q \in M$, and let U be an open neighborhood of q in M, and let $i : U \hookrightarrow M$ be an inclusion, then

$$\begin{aligned} I = T_qi : T_qU &\to T_qM \\ v(f) &\mapsto v(f|_U) \end{aligned}$$

is an isomorphism for all $v \in T_qM$ and all $U \subseteq M$.

Notation. We denote $r_1, \ldots, r_n : \mathbb{R}^m \to \mathbb{R}$ to be the standard coordinates on $\mathbb{R}^m$.

Let M be a manifold, $q_0 \in M$, and $\varphi : U \to \mathbb{R}^m$ is a coordinate chart with $q_0 \in U$. Now let $x_i = r_i \circ \varphi$, then $\varphi(q) = (x_1(q), \ldots, x_m(q))$.

We may now assume that

- $\varphi(q_0) = 0$, otherwise, we replace $\varphi(q)$ by $\varphi(q) := \varphi(q) - \varphi(q_0)$, and
- $\varphi(U)$ is an open ball $B_R(0) = \{r \in \mathbb{R}^m \mid ||r|| < R\}$ because there exists $R > 0$ such that $B_R(0) \subseteq \varphi(U)$, and we can then replace U with $\varphi^{-1}(B_R(0))$ and restrict the charts φ to $\varphi|_{\varphi^{-1}(B_R(0))}$.

We now define

$$\begin{aligned}\frac{\partial}{\partial x_j}|_{q_0} : C^\infty(U) &\to \mathbb{R}\\ f &\mapsto \frac{\partial}{\partial r_j}|_0(f\circ\varphi^{-1})\end{aligned}$$

Claim 5.9. $\left\{\frac{\partial}{\partial x_j}|_{q_0}\right\}_{j=1}^m$ is a basis of T_qM and for all $v \in T_{q_0}M$, $v = \sum v(x_j)\frac{\partial}{\partial x_j}|_{q_0}$.

Proof. By Hadamard's lemma Lemma 4.5 on $B_R(0)$, for all $f \in C^\infty(U)$, we have $f\circ\varphi^{-1} \in C^\infty(B_R(0))$, so there exists $g_1,\dots,g_m \in C^\infty(B_R(0))$ such that $(f\circ\varphi^{-1})(r) = f(\varphi^{-1}(0)) + \sum r_i g_i(r)$. Therefore, $f(q) = f(q_0) + \sum(r_i\circ\varphi)(q)(g_i\circ\varphi)(q)$, hence $f = f(q_0) + \sum x_i(g_i\circ\varphi)$, and $(g_i\circ\varphi)(q_0) = g_i(0) = \frac{\partial}{\partial r_i}|_0(f\circ\varphi^{-1}) = \frac{\partial}{\partial x_i}|_0(f)$.

Hence, for all $v \in T_{q_0}(U)$, we know

$$\begin{aligned}v(f) &= v(f(q_0)) + v\left(\sum x_i\cdot(g_i\circ\varphi)\right)\\ &= \sum_i v(x_i)(g_i\circ\varphi)(q_0)\\ &= \sum v(x_i)\frac{\partial}{\partial x_i}|_{q_0}(f).\end{aligned}$$

□

Remark 5.10.

1. The linear functionals

$$\begin{aligned}(dx_i)_{q_0} : T_{q_0}U &\to \mathbb{R}\\ v &\mapsto v(x_i)\end{aligned}$$

is the basis of $(T_{q_0}U)^*$ dual to $\left\{\frac{\partial}{\partial x_i}|_{q_0}\right\}$.

2. $(T_0\varphi^{-1})\left(\frac{\partial}{\partial r_i}|_0\right) = \frac{\partial}{\partial x_i}|_{q_0}$ by definition. Since $\left\{\frac{\partial}{\partial x_i}|_0\right\}_{i=1}^n$ is a basis of $T_0(B_R(0))$, then $\left\{\frac{\partial}{\partial x_i}|_{q_0}\right\}$ has to be a basis.

Lemma 5.11. Let M be a manifold and $q \in M$ a point. Let $U \ni q$ be anopen neighborhood, and $f \in C^\infty(M)$ such that $f|_U = 0$, then for all $v \in T_qM$, we have $v(f) = 0$.

Proof. We have shown the existence of a bump function $\rho \in C^\infty(M)$ in homework 1, that is, $0 \leqslant \rho(x) \leqslant 1$, $\mathrm{supp}(\rho) \subseteq U$ and $\rho \equiv 1$ near q.

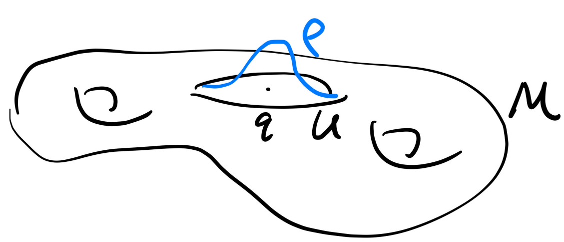


Figure 7: Bump Function

Therefore, $\rho f \equiv 0$, so $v(f) = v(\rho)f(q) + \rho(q)v(f) = v(\rho f) = 0$. □

6 Sept 1, 2023

Recall. Given a coordinate chart $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$, and $q \in U$ with $f(q) = 0$, we defined $\left\{ \frac{\partial}{\partial x_i}|_q \right\}_{i=1}^m \subseteq T_qU$ by $\frac{\partial}{\partial x_i}|_q f = \frac{\partial}{\partial r_i}(f \circ \varphi^{-1})|_{\varphi(q)}$ where $\frac{\partial}{\partial r_i}$'s are the standard partials on $C^\infty(\mathbb{R}^m)$.

We know this is a basis with dual basis

$$\begin{aligned}(dx_i)_q : T_qM &\to \mathbb{R}\\ v &\mapsto v(x_i)\end{aligned}$$

therefore $v = \sum v(x_i)\frac{\partial}{\partial x_i}|_q$ for all v. Note that

$$\begin{aligned}C^\infty(M) &\to C^\infty(U)\\ f &\mapsto f|_U\end{aligned}$$

is not surjective.

Also, we know $v \in T_qM$ is local, if $f, g \in C^\infty(M)$ agree on a neighborhood of q, then $v(f) = v(g)$.

Finally, given $F : M \to N$, this induces

$$\begin{aligned}T_qF : T_qM &\to T_{F(q)}N\\ v &\mapsto v(f \circ F).\end{aligned}$$

Lemma 6.1. Given a manifold M and $q \in M$, open neighborhood $q \in U \subseteq M$ and $i : U \hookrightarrow M$ inclusion, then

$$I \equiv T_qi : T_qU \to T_qM$$

is an isomorphism with $(I(v))(f) = v(f|_U)$ for all $f \in C^\infty(M)$.

Proof. Suppose $v \in \ker(I)$, then $v(f|_U) = 0$ for all $f \in C^\infty(M)$. We want $v(h) = 0$ for all $h \in C^\infty(U)$. We first choose bump function $\rho : M \to [0, 1]$ that is C^∞, and $\rho \equiv 1$ near q, and suppose $\operatorname{supp}(\rho) \subseteq U$, hence $\rho|_{M\setminus U} \equiv 0$. Then define $\rho h \in C^\infty(M)$ via

$$\rho h(x) = \begin{cases}\rho(x)h(x), & x \in U\\ 0, & x \notin U\end{cases}$$

Now $\rho h|_U \equiv h$ near q, i.e., identically 1. Therefore, $v(h) = v(\rho h|_U) = 0$, so $v \equiv 0$.

It remains to show that for all $w \in T_qM$, there exists $v \in T_qU$ such that $I(v) = w$, i.e., for all $f \in C^\infty(M)$, $w(f) = v(f|_U)$. Take the same $\rho \in C^\infty(M, [0.1])$ as above, define $v(h) = w(\rho h)$ for all $h \in C^\infty(M)$, and we can check that

- $v \in T_qM$, and
- for all $f \in C^\infty(M)$, $v(f|_U) = w(f)$.

Note that v is $\mathbb{R}$-linear, and for all $f, g \in C^\infty(W)$ we have $v(fg) = w(\rho fg) = w(\rho^2 fg)$ since $\rho fg = \rho^2 fg$ near q, then we have

$$\begin{aligned}v(fg) &= w(\rho^2 fg)\\ &= w((\rho f)(\rho g))\\ &= v(\rho f) \cdot (\rho g)(g) + \rho(f)(q) \cdot v(\rho g)\\ &= v(f)g(q) + f(q)v(g).\end{aligned}$$

Finally, for all $f \in C^\infty(M)$, we have $v(f|_U) = w(\rho f) = w(f)$ since $\rho f = f$ near q. □

Notation. We now suppress the isomorphisms $I : T_qU \to T_qM$. In particular, given a chart $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$, we view $\left\{ \frac{\partial}{\partial x_i}|_q \right\}_{i=1}^m$ as a basis of T_qM.

Lemma 6.2. Let V be a finite-dimensional vector space with $q \in V$, then

$$\begin{aligned} \varphi : V &\to T_qV \\ v(f) &\mapsto \frac{d}{dt}|_0 f(q+tv) \end{aligned}$$

for all $f \in C^\infty(V)$, is an isomorphism.

Proof. One can see this is linear, so it suffices to show injectivity. We have

$$\ker(\varphi) = \{v \in V \mid \frac{d}{dt}|_0(q+tv) = 0 \,\forall f \in C^\infty(V)\}.$$

If $0 \neq v \in \ker(\varphi)$, then there exists $\ell : V \to \mathbb{R}$ such that $\ell(V) \neq 0$, so

$$0 \neq \frac{d}{dt}|_0(\ell(q+tv)) = \frac{d}{dt}|_0(\ell(q) + t\ell(v)) = \ell(v).$$

□

Definition 6.3. A *curve* through a point $q \in M$ on a manifold M is a C^∞-map $\gamma : (a,b) \to M$ with $0 \in (a,b)$ such that $\gamma(0) = q$.

Definition 6.4. Given $\gamma : (a,b) \to M$ with $\gamma(0) = q$, we define $\dot{\gamma}(0) \in T_qM$ by $\dot{\gamma}(0)f = \frac{d}{dt}|_0 f(\gamma(t)) = \frac{d}{dt}|_0(f \circ \gamma)$ for all $f \in C^\infty(M)$.

Remark 6.5.

$$\begin{aligned} t : (a,b) &\to \mathbb{R} \\ x &\mapsto x \end{aligned}$$

is a coordinate chart on (a,b), where $\frac{d}{dt}|_0 \in T_0(a,b)$ is a basis vector. Since γ is C^∞,

$$\begin{aligned} T_0\gamma : T_0(a,b) &\to T_{\gamma(0)}M \equiv T_qM \\ ((T_0\gamma)(\frac{d}{dt}|_0))f &= \frac{d}{dt}|_0(f \circ \gamma) = \dot{\gamma}(0), \end{aligned}$$

so $\dot{\gamma}(0) = (T_0\gamma)\left(\frac{d}{dt}|_0\right)$.

Let $\mathscr{C} = \{\gamma : I \to M \mid \gamma(0) = q, I \text{ interval depending on } \gamma\}$, then we have a map

$$\begin{aligned} \Phi : \mathscr{C} &\to T_qM \\ \gamma &\mapsto \dot{\gamma}(0) \end{aligned}$$

Note that Φ is not injective. However, there is an equivalence relation $\sim$ on $\mathscr{C}$ defined by $\gamma \sim \sigma$ if and only if $\Phi(\gamma) = \Phi(\sigma)$, so this gives an injection

$$\begin{aligned} \tilde{\Phi} : \mathscr{C}/\sim &\to T_qM \\ [\gamma] &\mapsto \dot{\gamma}(0). \end{aligned}$$

Claim 6.6. $\tilde{\Phi}$ is onto.

Proof. Choose coordinates $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ near q such that $(x_1, \ldots, x_m)(q) = 0$. Now, for all $v \in T_qM$, we have $v = \sum v(x_i) \frac{\partial}{\partial x_i}|_q$. Consider $\gamma(t) = \varphi^{-1}(tv(x_1), \ldots, tv(x_m))$, then $\gamma(0) = \varphi^{-1}(0) = q$ and for any $f \in C^\infty(M)$, we have

$$\dot{\gamma}(0)f = \frac{d}{dt}|_0(f \circ \varphi^{-1})(tv(x_1), \ldots, tv(x_m))$$

$$
\begin{aligned}
&= \sum \frac{\partial}{\partial r_i}(f \circ \varphi^{-1})|_0 \cdot v(x_i) \\
&= \sum v(x_i) \frac{\partial}{\partial x_i}|_q f \\
&= v(f).
\end{aligned}
$$

□

Lemma 6.7. For any smooth map $F : M \to N$ between manifolds, for all $q \in M$, we have

$$T_qF(\dot{\gamma}(0)) = (F \circ \gamma)^{\cdot}(0).$$

Proof.

$$
\begin{aligned}
T_qF(\dot{\gamma}(0)) &= T_qF(T_0\gamma\left(\frac{d}{dt}|_0\right)) \\
&= T_0(F \circ \gamma)\left(\frac{d}{dt}|_0\right) \\
&= (F \circ \gamma)^{\cdot}(0).
\end{aligned}
$$

□

Example 6.8. Let $M = N = \mathbb{C}$ and $F(z) = e^z$. We claim that $(T_zF)(v) = e^z v$, which uses $\mathbb{C} \cong T_w\mathbb{C}$ for all $w \in \mathbb{C}$. Indeed, since $\frac{d}{dt}|_0 e^{tv} = v$, then

$$
\begin{aligned}
(T_zF)(v) &= \frac{d}{dt}|_0 F(z + tv) \\
&= \frac{d}{dt}|_0 e^{z+tv} \\
&= \frac{d}{dt}|_0 (e^z e^{tv}) \\
&= e^z v.
\end{aligned}
$$

Note that T_zF is an isomorphism for all z, given by

$$
\begin{tikzcd}
T_z\mathbb{C} \arrow[r, "T_zF"] \arrow[d, "\cong"] & T_{F(z)}\mathbb{C} \arrow[d, "\cong"] \\
\mathbb{C} \arrow[r, "e^z \cdot -"'] & \mathbb{C}
\end{tikzcd}
$$

Also note that this is not a diffeomorphism, since the inverse is the complex logarithm function, which is only well-behaved on the principal branches.

7 Sept 6, 2023

Definition 7.1. Given a manifold M, $q \in M$, and $f \in C^\infty(M)$, we define the *exact differential* to be a linear map

$$
\begin{aligned}
df_q : T_qM &\to \mathbb{R} \\
v &\mapsto v(f)
\end{aligned}
$$

in $\mathrm{Hom}(T_qM, \mathbb{R}) =: T_q^*M$, the cotangent space.

Exercise 7.2.

- df_q is linear,
- $f \equiv g$ near q, then $df_q = dg_q$.

We have seen differentials before: given a coordinate chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$ is a coordinate chart, then $\{(dx_i)_q\}_{i=1}^m$ is a basis of T_q^*M dual to $\{\frac{\partial}{\partial x_i}|_q\}_{i=1}^m$. Note that for all $\eta \in T_q^*M \equiv (T_qM)^*$, then $\eta = \sum \eta \left(\frac{\partial}{\partial x_i}|_q\right)(dx_i)_q$.

Lemma 7.3. Let M be a manifold, $q \in M$, and $f \in C^\infty(M)$, then the derivative

$$(T_q f)(v) = df_q(v)\frac{d}{dt}|_{f(q)}.$$

Proof. Note that $\{dt_{f(q)}\}$ is a basis of $T^*_{f(q)}\mathbb{R}$, then

$$dt_{f(q)}(T_q f(v)) = (T_q f(v))t = v(t \circ f) = v(f) = df_q(v),$$

so $(T_q f)(v) = df_q(v)\frac{d}{dt}|_{f(q)}$. □

Recall. Let $T : V \to W$ be a linear map, and let $\{e_1, \dots, e_n\}$ be a basis of V, and let $\{f_1, \dots, f_n\}$ be a basis of W, with dual basis $\{f_1^*, \dots, f_n^*\}$ in W^*. Then let $t_{ij} = f_i^*(Te_j)$, then

$$T(e_j) = \sum_i f_i^*(Te_j)f_i = \sum_i t_{ij} f_i.$$

For all $F : \mathbb{R}^m \to \mathbb{R}^n$, consider the coordinates $(x_1, \dots, x_m) : \mathbb{R}^m \to \mathbb{R}$ and $(y_1, \dots, y_n) : \mathbb{R}^n \to \mathbb{R}$, which gives coordinates $\{(\frac{\partial}{\partial x_i}|_q)\}$ and $\{(\frac{\partial}{\partial y_i}|_{F(q)})\}$, respectively. With $T = T_qF$, we have

$$t_{ij} = (dy_i)_{F(q)}(T_qF(\frac{\partial}{\partial x_j}|_q)) = (T_qF(\frac{\partial}{\partial x_j}|_q))y_i = \frac{\partial}{\partial x_j}|_q(y_i \circ F).$$

If we denote $F = (F_1, \dots, F_n)$ where $F_i = y_i \circ F$ then this is just $\frac{\partial F_i}{\partial x_j}(q)$, so $\left(\frac{\partial F_i}{\partial x_j}(q)\right)$ is the matrix of T_qF.

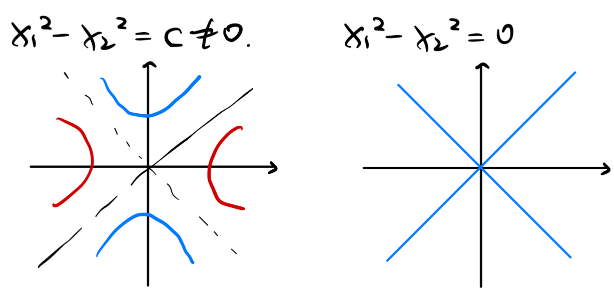
Definition 7.4. Let $F : M \to N$ be a smooth map, we say $c \in N$ is a *regular value* of F if either $F^{-1}(c) = \varnothing$, or for all $q \in F^{-1}(c)$, $T_qF : T_qM \to T_{F(q)}N = T_cN$ is onto.

We say $c \in N$ is a *singular value* if it is not a regular value.

Example 7.5. Consider

$$\begin{aligned} F : \mathbb{R}^2 &\to \mathbb{R} \\ (x_1, x_2) &\mapsto x_1 - x_2^2 \end{aligned}$$

for all $q = (x_1, x_2) \in \mathbb{R}^2$, then T_qF is the matrix $\left(\frac{\partial F}{\partial x_1}(q), \frac{\partial F}{\partial x_1}(q)\right) = (2x_1, 2x_2)$. Hence, $c \neq 0$ is a regular value, and $c = 0$ is a singular value.



Definition 7.6. An *embedded submanifold* (of dimension k) of a manifold M is a subspace $Z \subseteq M$ such that for all $q \in Z$ there exists a coordinate chart $\varphi = (x_1, \dots, x_k, x_{k+1}, \dots, x_m) : U \to \mathbb{R}^m$ with $\varphi(U \cap Z) = \{(r_1, \dots, r_m) \in \varphi(U) \mid r_k = \dots = r_m = 0\}$.

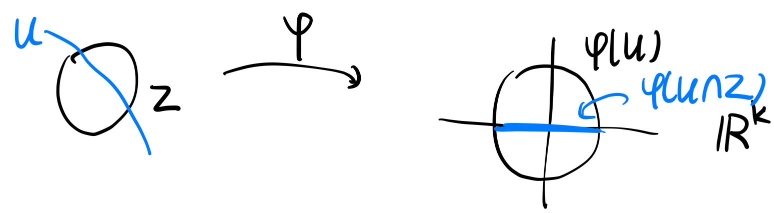


Figure 8: Embedded Submanifold

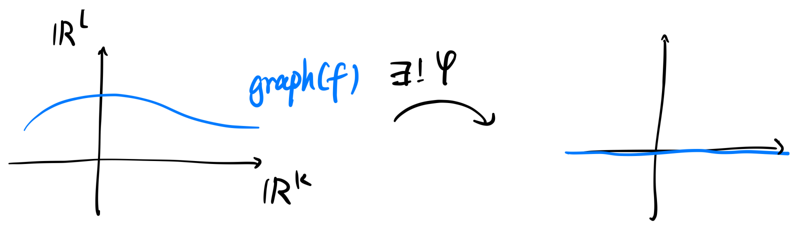
Remark 7.7.

- Any open subset $U \subseteq M$ is an embedded submanifold.
- Any singleton in M is an embedded submanifold.

Example 7.8. Let $f : \mathbb{R}^k \to \mathbb{R}^l$ be C^∞, then the graph of f is

$$\text{graph}(f) = \{(x, f(x) \in \mathbb{R}^k \times \mathbb{R}^l \mid x \in \mathbb{R}^k\}$$

is an embedded submanifold of $\mathbb{R}^k \times \mathbb{R}^l$.



Here $\varphi(x, y) = (x, y - f(x))$ is a coordinate chart of $\mathbb{R}^k \times \mathbb{R}^l$ with inverse $\varphi^{-1}(x, y') = (x, y' + f(x))$.

Theorem 7.9 (Regular Value Theorem)**.** Let $c \in N$ be a regular value of smooth function $F : M \to N$. If $F^{-1}(c) = \varnothing$, then for all $q \in F^{-1}(c)$, $T_qF : T_qM \to T_qN$ is onto, so $F^{-1}(c)$ is an embedded submanifold of M. Moreover, $T_qF^{-1}(c) = \ker(T_qF)$ and $\dim(F^{-1}(c)) = \dim(M) - \dim(N)$.

Example 7.10. Consider

$$\begin{aligned} F : \mathbb{R}^m &\to \mathbb{R} \\ x &\mapsto \sum x_i^2 = ||x||^2 \end{aligned}$$

Now T_qF gives a local chart with $(2x_1, \ldots, 2x_m)$. Any $c \neq 0$ is a regular value. We have $F^{-1}(c) = \{x \mid ||x||^2 = c\}$ is the sphere of radius $\sqrt{c}$ for $c > 0$. Moreover, $F^{-1}(0) = \{0\}$, an embedded submanifold, but $\dim(\{0\}) \neq \dim(\mathbb{R}^m) - \dim(\mathbb{R})$.

8 Sept 8, 2023

Recall. A subset Z of a manifold M is an embedded submanifold (of dimension k and codimension $m - k$ for $m = \dim(M)$) if for all $z \in Z$, there exists a coordinate chart $\varphi : U \to \mathbb{R}^m$ and $z \in U$ which is adapted to Z, i.e., $\varphi(U \cap Z) = \varphi(U) \cap (\mathbb{R}^k \times \{0\})$.

Remark 8.1.

- Submanifolds of codimension 0 are open subsets.
- Submanifolds of codimension $m = \dim(M)$ are discrete sets of points.

We will proceed to prove Theorem 7.9.

Remark 8.2. Once we proved $F^{-1}(c)$ is embedded and $\dim(F^{-1}(c)) = \dim(M) - \dim(N)$, then the last statement follows. Indeed, given $v \in T_q(F^{-1}(c))$, there exists $\gamma : (a,b) \to F^{-1}(c)$ such that $\gamma(0) = q, \gamma'(0) = v$, and $F(\gamma(t)) = c$ for all t. Therefore,

$$0 = \frac{d}{dt}|_0 F(\gamma(t)) = T_qF(\gamma'(0)) = T_qFv,$$

so $v \in \ker(T_qF)$, and so $T_qF^{-1}(c) \subseteq \ker(T_qF)$. By dimension argument, we have equality.

We will introduce inverse function theorem and implicit function theorem.

Theorem 8.3 (Inverse Function Theorem)**.** Let $U \subseteq \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^n$ be C^∞ with $q \in U$ such that $T_qf = Df(q) : T_qU = \mathbb{R}^n \to \mathbb{R}^n = T_{F(q)}\mathbb{R}^n$ is an isomorphism. Then there exists an open neighborhood $q \in V \subseteq U$ and $f(q) \in W$ such that $f : V \to W$ is a diffeomorphism.

Notation. Given $F : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^m$ for $(a,b) \in \mathbb{R}^k \times \mathbb{R}^l$, then we denote

- $\frac{\partial F}{\partial x}(a,b) = T_{(a,b)}F|_{\mathbb{R}^k \times \{0\}} = DF(a,b)|_{\mathbb{R}^k \times \{0\}}$,
- $\frac{\partial F}{\partial y}(a,b) = T_{(a,b)}F|_{\{0\} \times \mathbb{R}^l} = DF(a,b)|_{\{0\} \times \mathbb{R}^l}$.

Theorem 8.4 (Implicit Function Theorem)**.** Let $F : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ be C^∞, let $(a,b) \in \mathbb{R}^k \times \mathbb{R}^l$. Suppose $\frac{\partial F}{\partial y}(a,b) : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, then there exists a neighborhood $W \ni (a,b)$ and $U \ni a$ in $\mathbb{R}^k$, as well as C^∞-map $g : U \to \mathbb{R}^n$ such that $F^{-1}(c) \cap W = \mathrm{graph}(g) \cap W$.

Remark 8.5. inverse function theorem and implicit function theorem are equivalent.

Proof. Consider

$$\begin{aligned} H : \mathbb{R}^k \times \mathbb{R}^n &\to \mathbb{R}^k \times \mathbb{R}^n \\ (x,y) &\mapsto (x, F(x,y)) \end{aligned}$$

then $H(a,b) = (a, F(a,b)) = (a,c)$. The partials give

$$DH(a,b) = \begin{pmatrix} I & 0 \\ \frac{\partial F}{\partial x}(a,b) & \frac{\partial F}{\partial y}(a,b) \end{pmatrix}$$

As $\frac{\partial F}{\partial y}(a,b)$ is invertible, so is $DH(a,b)$, so there exists neighborhoods $(a,b) \in W \subseteq \mathbb{R}^n \times \mathbb{R}^k$ and $a \in U \subseteq \mathbb{R}^k$, $c \in V \subseteq \mathbb{R}^n$, such that $H : W \to U \times V$ is a diffeomorphism. Consider

$$\begin{aligned} G = H^{-1} : U \times V &\to W \subseteq \mathbb{R}^n \times \mathbb{R}^l \\ (u,v) &\mapsto (G_1(u,v), G_2(u,v)) \end{aligned}$$

therefore

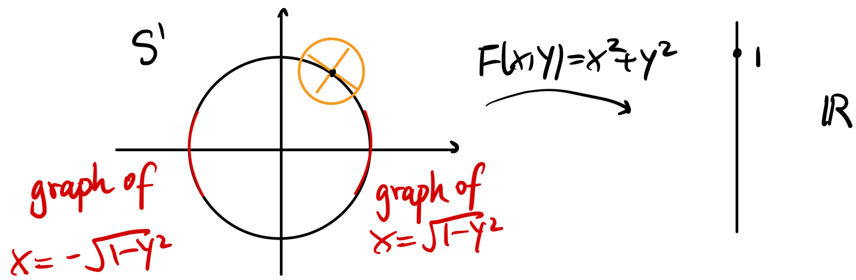
$$(u,v) = H(H^{-1}(u,v)) = H(G_1(u,v), G_2(u,v)) = (G_1(u,v), F(G_1(u,v), G_2(u,v))$$

so $G_1(u,v) = u$, and $v = F(u, G_2(u,v))$ for all u, v, hence $c = F(u, G_2(u,c))$ for all u. Now let $g(u) = G_2(u,c)$, then $F(u, g(u)) = c$ for all u. Hence, $\mathrm{graph}(g) \subseteq F^{-1}(c)$. □

Proof of Regular Value Theorem. Let $F : M \to N$, $c \in N$, $F^{-1}(c) \neq \varnothing$. Now for all $q \in F^{-1}(c)$, then $T_qF : T_qM \to T_qN$ is onto. Given $q \in F^{-1}(c)$, we want a chart T from a neighborhood of q to $\mathbb{R}^m$, adapted to $F^{-1}(c)$. Let $\varphi : U \to \mathbb{R}^m$ and $\psi : V \to \mathbb{R}^m$ be charts such that $q \in U$, $c \in V$, then

$$\tilde{F} = \psi \circ F \circ \varphi^{-1}|_{\varphi(F^{-1}(V) \cap U} : \varphi(F^{-1}(V) \cap U) \subseteq \mathbb{R}^m \to \mathbb{R}^n$$

is C^∞. Now $\psi(c)$ is a regular value in $\tilde{F}$, Let $r = \varphi(q)$, then we have $D\tilde{F}(r) : \mathbb{R}^m \twoheadrightarrow \mathbb{R}^n$. Let $X = \ker(D\tilde{F}(r))$ and Y be a complement in $\mathbb{R}^m$. So $\mathbb{R}^m = X \otimes Y$ and $D\tilde{F}(r)|_Y : Y \to \mathbb{R}^n$ is an isomorphism. Apply inverse function theorem to $\tilde{F}$ from the intersection of $X \times Y$ and the open subset to $\mathbb{R}^n$.



□

Example 8.6. Let $\mathrm{Sym}^2(\mathbb{R}^n)$ be the $n \times n$ symmetric real matrices, also known as $\mathbb{R}^{\frac{n^2-n}{2}+n}$. There is

$$\begin{aligned} F : \mathrm{GL}_(n, \mathbb{R}) &\to \mathrm{Sym}^2(\mathbb{R}^n) \\ A &\mapsto A^T A \\ F^{-1}I = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T A = I\} &\leftarrow I \end{aligned}$$

Remark 8.7. We have $F = F \circ L_A$ for all $A \in O(U)$, then for all A, we have $T_A F$ onto.

Claim 8.8. 1 is a regular value of F, so $O(n)$ is an embedded submanifold of $\mathrm{GL}(n, \mathbb{R})$.

Proof.

$$\begin{aligned} (T_I F)(v) &= \frac{d}{dt}|_0 (I + tv)^T (I + tv) \\ &= \frac{d}{dt}|_0 (I^2 + tv^T + tv + t^2 v^T v) \\ &= v^T + v \end{aligned}$$

and this is surjective since for all $Y \in \mathrm{Sym}^2(\mathbb{R})$, we have $Y = \frac{1}{2}(Y^T + Y)$, so $Y = (T_I F)(\frac{1}{2}Y)$. □

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Recall. Let $F : M \to N$ be C^∞, let $c \in N$ be a regular value such that $F^{-1}(c) \neq \varnothing$. (For all $q \in F^{-1}(c)$, $T_q F : T_q M \to T_q N$ is onto.) Then:

i $F^{-1}(c)$ is an embedded submanifold of M.

ii $\dim(M) = \dim(F^{-1}(c)) = \dim(N)$.

iii for all $q \in F^{-1}(c)$, $T_q F^{-1}(c) = \ker(T_q F)$.

The proof uses inverse function theorem and/or implicit function theorem, and the key is to note that locally $f^{-1}(c)$ is a graph.

Also, $O(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T A = I\}$ is an embedded submanifold.

Definition 9.1. A *Lie group* G is a group and a manifold so that

i the multiplication map

$$\begin{aligned} m : G \times G &\to G \\ (a, b) &\mapsto (a, b) \end{aligned}$$

is C^∞.

ii the inverse map

$$\begin{aligned} \mathrm{inv} : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

is C^∞.

Notation. $e_G = 1_G$ is the identity element.

Example 9.2. $G = \mathbb{R}^n$ with $m(v, w) = v + w$, and $\mathrm{inv}(v) = -v$ gives a Lie group.

Example 9.3. Let $G = \mathrm{GL}(n, \mathbb{R})$ be with $e_G = \mathrm{diag}(1, \dots, 1) = I$, with maps $m(A, B) = AB$ and $\mathrm{inv}(A) = A^{-1}$.

Remark 9.4. One can think of a Lie group G as four pieces of data:

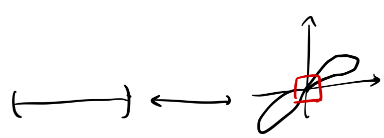
- manifold G,
- map $m : G \times G \to G$,
- map $\mathrm{inv} : G \to G$,
- $e_G \in G$.

Note that a subgroup H of a Lie group G is not necessarily a Lie group. The sufficient condition would be H is an embedded submanifold of G, i.e.,

- $m|_{H\times H} : H \times H \to H$ are C^∞,
- $\mathrm{inv}|_H : H \to H$

are C^∞. Note $m|_{H\times H} : H \times H \to G$ is C^∞ since $i : H \hookrightarrow G$ is C^∞ and $m|_{H\times H} = m(i \times i)$.

Example 9.5. For example, think of the embedding

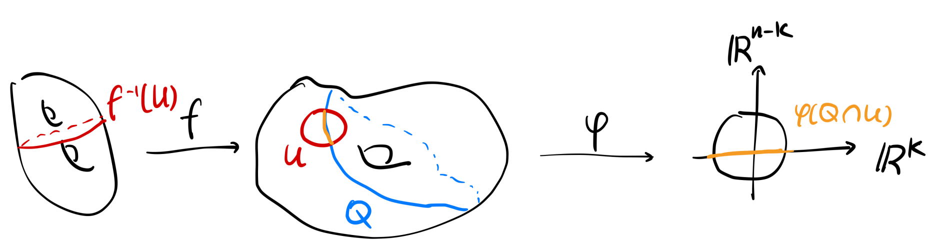


but at the origin the preimage is split into three pieces, because the inverse is not continuous, which does not embed into a submanifold.

Lemma 9.6. If $i : Q \hookrightarrow M$ is an embedded submanifold, and $f : N \to M$ is a smooth map such that $f(N) \subseteq Q$, then $g : N \to Q$ with $g(n) = f(n)$ is C^∞.

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[dr, dashed, "g"'] & M \\ & Q \arrow[u, hook, "i"'] \end{tikzcd}$$

Proof. Since $Q \hookrightarrow M$ is embedded, for all $q \in Q$, there exists an adapted chart $\varphi = (x_1, \dots, x_n, x_{k+1}, \dots, x_m) : U \to \mathbb{R}^m$ such that $Q \cap U = \{x_k = \cdots = x_n = 0\}$. Consider $\varphi \circ f|_{f^{-1}(U)} : f^{-1}(U) \to \mathbb{R}^m$, then $f(f^{-1}(U)) \subseteq Q \cap U$.



Then $\varphi \circ f|_{f^{-1}(U)} = \varphi(U \cap Q) = \{(r_1, \dots, r_k, r_{k+1}, \dots, r_m) \mid r_{k+1} = \cdots = r_n = 0\}$, so $\varphi \circ f = (h_1, \dots, h_k, 0, \dots, 0)$ where $h_1, \dots, h_k \in C^\infty(f^{-1}(U))$. Therefore, $\varphi|_{U\cap Q}\, g|_{f^{-1}(U)} = (h_1, \dots, h_k)$. $\square$

Example 9.7. $O(n) \subseteq \mathrm{GL}(n, \mathbb{R})$ is embedded, thus a Lie group.

Example 9.8. $\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid \det(A) = 1\}$ is also a Lie group.

Claim 9.9. $1 \in \mathbb{R}$ is a regular value of $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$.

Proof. The key fact is that $T_I(\det) : \mathbb{R}^{n^2} \to \mathbb{R}$ is an $(n \times n)$-matrix given by $A \mapsto \mathrm{tr}(A)$. Indeed, note that the trace is the differential of the determinant. □

Definition 9.10. A (real) *Lie algebra* is a (real) vector space $\mathfrak{g}$ with an $\mathbb{R}$-bilinear map

$$\begin{aligned} [\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g} \\ (X, Y) &\mapsto [X, Y] \end{aligned}$$

such that for all $X, Y, Z \in \mathfrak{g}$,

- $[Y, X] = -[X, Y]$,
- $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$.

Example 9.11. Let $\mathfrak{g} = M_n(\mathbb{R})$, $[X, Y] = XY - YX$ is the anti-commutator.

Example 9.12. Let M be a manifold, $\mathfrak{g} = \mathrm{Der}(C^\infty(M)) = \{X : C^\infty(M) \to C^\infty(M) \mid X(fg) = X(f) \cdot g + f \cdot X(g)\}$. Therefore, $\mathfrak{g}$ is a Lie algebra with the bracket $[X, Y](f) = X(Y(f)) - Y(X(f))$ for all $f \in C^\infty(M)$. This is the Lie algebra of vector fields on M.

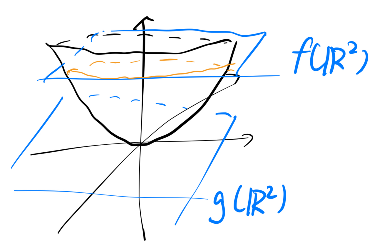
Example 9.13. Let $\mathfrak{g} = \mathbb{R}^3$, then $[v, w] := v \times w$ is a Lie algebra with cross product.

We will see that for all Lie group G, $\mathfrak{g} = \mathrm{Lie}(G) = T_e G$ is naturally a Lie algebra.

Definition 9.14. Let $F : M \to N$ be a C^∞-map, $Z \subseteq N$ be an embedded submanifold. We say F is *transverse* to Z, denoted $F \pitchfork Z$, if for all $x \in F^{-1}(Z)$, $T_x F(T_x M) + T_{F(x)} Z = T_{F(x)} N$.

Example 9.15. If $Z = \{c\}$, then $F \pitchfork c$ if and only if for all $q \in F^{-1}(c)$, $(T_x F)(T_x N) + T_c c = T_c N$, if and only if for all $q \in F^{-1}(c)$, $(T_x F)(T_x N) = T_c N$, if and only if c is a regular value of F.

Example 9.16. Let $M = \mathbb{R}^2$, $N = \mathbb{R}^3$, $Z = \{(x, y, z) \mid z = x^2 + y^2\}$, with $f(x, y) = (x, y, 1)$ and $g(x, y) = (x, y, 0)$, then $f \pitchfork Z$ but $g \not\pitchfork Z$.



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Theorem 10.1. Suppose $f : M \to N$ is transverse to an embedded submanifold $Z \subseteq N$, then

(i) $f^{-1}(z)$ is an embedded submanifold of M.

(ii) If $f^{-1}(z) \neq \varnothing$, then $\dim(M) - \dim(f^{-1}(z)) = \dim(N) - \dim(Z)$, i.e., $\mathrm{codim}(f^{-1}(Z)) = \mathrm{codim}(Z)$.

Proof. Fix $z_0 \in Z$ with $f^{-1}(z_0) \neq \varnothing$, let $\psi : V \to \mathbb{R}^n$ be a coordinate chart on N, adapted to Z such that $\psi(V \cap Z) = \psi(V) \cap (\mathbb{R}^k \backslash \{0\})$. Let $\pi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ be the canonical projection, then

$$(\pi \circ \psi)^{-1}(0) = \psi^{-1}(\pi^{-1}(0)) = \psi^{-1}(\psi(V) \cap (\mathbb{R}^k \times \{0\})) = Z \cap V,$$

therefore

$$(\pi \circ \psi \circ f)^{-1}(0) = f^{-1}(Z \cap V) = f^{-1}(Z) \cap f^{-1}(V).$$

Claim 10.2. 0 is a regular value of $\pi \circ \psi \circ f|_{f^{-1}}(V)$.

Subproof. Take arbitrary $x \in (\pi \circ \psi \circ f)^{-1}(0) = f^{-1}(V) \cap f^{-1}(Z)$, then $T_x f(T_x M) + T_{f(x)} Z = T_{f(x)} N$. Note that $T_x M = T_x(f^{-1}(V))$. Therefore,

$$\mathbb{R}^k \times \mathbb{R}^{n-k} = T_{f(x)}\psi(T_{f(x)}N) = T_{f(x)}\psi(T_x f(T_x f^{-1}(V))) + T_{f(x)}\psi(T_{f(x)}Z)$$

by applying $T_{f(x)}\psi$ on both sides. Now apply $T_{\psi(f(x))}\psi$ on both sides, then $T_{f(x)}\psi(T_{f(x)}Z)$ vanishes, so we get

$$\begin{aligned}\mathbb{R}^{n-k} &= T_{\psi(f(x))}\pi(T_{f(x)}\psi(T_x f(T_x f^{-1}(V)))) \\ &= T_x(\pi \circ \psi \circ f)(T_x f^{-1}(V)).\end{aligned}$$

■

□

Definition 10.3. A C^∞-map $f : Q \to M$ is an *embedding* if

(i) $f(Q) \subseteq M$ is an embedded submanifold, and

(ii) $f : Q \to f(Q)$ is a diffeomorphism.

Remark 10.4. We know $f : Q \to f(Q)$ is C^∞ since $f(Q) \subseteq M$ is embedded and $f : Q \to M$ is given by the composition of $i : f(Q) \hookrightarrow M$ and $f : Q \to f(Q)$.

Remark 10.5.

1. Since $f : Q \to f(Q)$ is a diffeomorphism, then it is a homeomorphism. Thus $f : Q \to M$ is a topological embedding.
2. For all $q \in Q$, then $T_q f : T_q Q \to T_{f(q)} M$ is injective, i.e., $T_q f(T_q Q) = T_{f(q)} f(Q)$.

Example 10.6 (Non-example)**.** Let $Q = \mathbb{R}$ with discrete topology, then Q is a paracompact but not second countable as a 0-dimensional manifold. Consider

$$\begin{aligned} f : Q &\to \mathbb{R}^2 \\ x &\mapsto (x, 0) \end{aligned}$$

be a C^∞-map, then this is not an embedding.

Example 10.7. Let M be a manifold with $f \in C^\infty(M)$, then

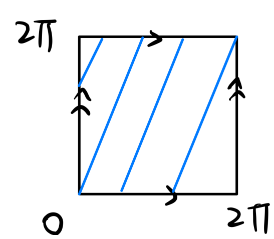
$$\begin{aligned} g : M &\to M \times \mathbb{R} \\ q &\mapsto (q, f(q)) \end{aligned}$$

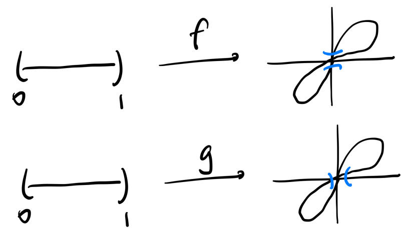
gives an embedding of M into $R \times \mathbb{R}$, as the graph of f.

Definition 10.8. A C^∞-map $f : Q \to M$ is an *immersion* if for all $q \in Q$, $T_q f : T_q Q \to T_{f(q)} M$ is injective.

Example 10.9. Consider

$$\begin{aligned} f : \mathbb{R} &\to S^1 \times S^1 \\ \theta &\mapsto (e^{i\theta}, e^{i\sqrt{2}\theta}) \end{aligned}$$





Example 10.10. Now $g \circ f^{-1} : (0,1) \to (0,1)$ is not an embedding, as it is not continuous.

Definition 10.11. The *rank* of a C^∞-map $f : M \to N$ at a point $q \in M$ is the rank of the linear map $T_q f : T_q M \to T_{f(q)} N$, i.e., $\operatorname{rank}_q(f) = \dim(T_q f(T_q M))$.

Example 10.12. If $f : M \to N$ is an immersion, then $\operatorname{rank}_q(f) = \dim_q(M)$.

Remark 10.13. Immersions are embeddings.

Theorem 10.14 (Rank Theorem)**.** Let $F : M \to N$ be a C^∞-map of constant rank k. Then for all $q \in M$, there exists coordinates $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$ on M with $q \in U$, and $\psi = (y_1, \dots, y_n) : V \to \mathbb{R}^n$ with $F(q) \in V$ such that $(\psi \circ F \circ \varphi^{-1})(r_1, \dots, r_m) = (r_1, \dots, r_k, 0, \dots, 0)$ for all $r = (r_1, \dots, r_m) \in \varphi(F^{-1}(V) \cap U)$.

Notation. Given a collection of sets $\{S_\alpha\}_{\alpha \in A}$, $\coprod\limits_{\alpha \in A} S_\alpha$ is the disjoint union of the collection.

We will give the following construction of a tangent bundle.

Remark 10.15. Given a manifold M, we form a set $TM = \coprod\limits_{q \in M} T_q M$. Given a chart $\varphi = (x_1, \dots, x_n) : U \to \mathbb{R}^m$ on M, the corresponding candidate chart is $\tilde{\varphi} : TU = \coprod\limits_{q \in U} T_q M \to \varphi(U) \times \mathbb{R}^m$. One can check that if $\varphi : U \to \mathbb{R}^m$ and $\psi : V \to \mathbb{R}^m$ are charts on M with $U \cap V \neq \varnothing$, then $\tilde{\psi}\tilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^m \to \psi(U \cap V) \times \mathbb{R}^m$ is C^∞. Now we give TM the topology making $\tilde{\varphi}$'s homeomorphic onto their images, then $\{\tilde{\varphi} : TU \to \varphi(U) \times \mathbb{R}^m\}$ will be an atlas on TM.

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Definition 11.1. A map $f : M \to N$ is a *submersion* if for all $p \in M$, the differential $T_q f : T_q M \to T_{f(q)} N$ is onto.

Remark 11.2. Every value over a submersion is regular.

Recall. For a manifold M, we defined the set $TM = \coprod\limits_{q \in M} T_q M = \bigcup(\{q\} \times T_q M)$, which is a called a tangent bundle, with additional structures. We will show that TM is a manifold, and

$$\begin{aligned} \pi : TM &\to M \\ (q, v) &\mapsto q \end{aligned}$$

is C^∞ and a submersion.

Proof. Let $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$ be a coordinate chart on M. For any $q \in U$, let $\left\{ \frac{\partial}{\partial x_1}\Big|_q, \dots, \frac{\partial}{\partial x_m}\Big|_q \right\}$ be a basis of $T_q M$. The dual basis is $\{(dx_1)_q, \dots, (dx_m)_q\}$. For any $v \in T_q M$, we have $v = \sum v(x_i) \frac{\partial}{\partial x_i}\Big|_q := \sum (dx_i)_q(v) \frac{\partial}{\partial x_i}\Big|_q$, and

$$\begin{aligned} T_q M &\to \mathbb{R} \\ v &\mapsto ((dx_1)_q(v), \dots, (dx_m)_q(v)) \end{aligned}$$

is a linear isomorphism. Define

$$\tilde{\varphi} : TU = \coprod_{q \in M} T_q M \to \mathbb{R}^m \times \mathbb{R}^m$$

$$(q,v) \mapsto (x_1(q),\dots,x_m(q),(dx_1)_q(v),\dots,(dx_m)_q(v)).$$

Suppose $\psi = (y_1,\dots,y_m) : V \to \mathbb{R}^m$ is another chart, we then have

$$\begin{aligned}\tilde{\psi} : TV &\to \mathbb{R}^m \times \mathbb{R}^m\\ (q,v) &\mapsto (y_1(q),\dots,y_m(q),(dy_1)_q(v),\dots,(dy_m)_q(v)).\end{aligned}$$

Claim 11.3. For any $(r,w) \in \varphi(U \cap V) \times \mathbb{R}^m$, we have

$$\begin{aligned}(\tilde{\psi} \circ \tilde{\varphi}^{-1})(r,w) &= ((\psi \circ \varphi^{-1})(r), \sum_j \frac{\partial y_1}{\partial x_j}(\varphi^{-1}(r))w_i, \dots, \sum_j \frac{\partial y_m}{\partial x_j}(\varphi^{-1}(r))w_i)\\ &= \left((\psi \circ \varphi^{-1})(r), \left(\frac{\partial y_i}{\partial x_j}(\varphi^{-1}(r))\right)\begin{pmatrix} w_1 \\ \vdots \\ w_m\end{pmatrix}\right)\end{aligned}$$

Subproof.

Recall. If $T : A \to B$ is a linear map, with $\{e_1,\dots,e_n\}$ basis of A, $\{f_1,\dots,f_n\}$ is a basis of B, with dual basis $\{f_1^*,\dots,f_n^*\}$, then we set $t_{ij} = f_u^*(Te_j)$, i.e.,

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "(t_{ij})"] \arrow[d, "(v_1,\dots,v_n)\mapsto\sum v_i e_i"'] & \mathbb{R}^n \arrow[d] \\ A \arrow[r, "T"'] & B \end{tikzcd}$$

In our case, we have $A = B = T_qM$ with $T = \mathrm{id}$, with basis $\left\{ \frac{\partial}{\partial x_i}\Big|_q \right\}$ of A, $\{f_1,\dots,f_n\} = \left\{ \frac{\partial}{\partial y_1}\Big|_q, \dots, \frac{\partial}{\partial y_m}\Big|_q \right\}$ and dual basis $\{f_1^*,\dots,f_m^*\} = \{(dy_1)_q,\dots,(dy_m)_q\}$, then

$$\begin{aligned} t_{ij} &= (dy_i)_q\left(\frac{\partial}{\partial x_j}\Big|_q\right)\\ &= \frac{\partial}{\partial x_j}(y_i)(q)\\ &= \frac{\partial y_i}{\partial x_j}(\varphi^{-1}(\gamma)).\end{aligned}$$

■

We define the topology on TM to be the topology generated by the sets of form $\tilde{\varphi}^{-1}(W)$ where $\varphi : U \to \mathbb{R}^m$ is a coordinate chart with open subset $W \subseteq \mathbb{R}^m \times \mathbb{R}^m$. Given an atlas $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^m\}$ on M, we get an induced atlas $\{\tilde{\varphi}_\alpha : TU_\alpha \to \mathbb{R}^m \times \mathbb{R}^m\}$ on TM. One can check that the choice of an atlas on M does not matter. □

Exercise 11.4.
- If M is Hausdorff, then so is TM.
- If M is second countable, then so is TM.

Lemma 11.5. The canonical projection $\pi : TM \to M$ is C^∞ and is a submersion.

Proof. Let $\varphi = (x_1,\dots,x_m) : U \to \mathbb{R}^m$ be a coordinate chart, $\tilde{\varphi} : TU \to \mathbb{R}^m \times \mathbb{R}^m$ be the induced chart on TM, then

$$\begin{aligned}(\varphi \circ \pi \circ \tilde{\varphi}^{-1})(r,w) &= \varphi \circ \pi\left(\varphi^{-1}(r), \sum_i w_i \frac{\partial}{\partial x_i}\Big|_q\right)\\ &= \varphi(\varphi^{-1}(r))\\ &= r.\end{aligned}$$

Moreover,

$$\left(T_{(r,w)}(\varphi \circ \pi \circ \tilde{\varphi}^{-1})\right)(v, w') = v$$

where $(v, w') \in T_{(r,w)}(\varphi(U) \times \mathbb{R}^m) \cong \mathbb{R}^n \times \mathbb{R}^m$. Therefore, $T_{(q,v)}\pi : T_{(q,v)}TM \to T_qM$ is onto, hence a submersion. □

Definition 11.6. A *(algebraic) vector field* on a manifold M is a derivation $v : C^\infty(M) \to C^\infty(M)$, i.e., v is $\mathbb{R}$-linear and $v(fg) = v(f)g + fv(g)$ for all $f, g \in C^\infty(M)$.

Definition 11.7. A *(geometric) vector field* on a manifold M is a section of the tangent bundle TM of M, i.e., $X : M \to TM$ is C^∞ with $\pi \circ X = \mathrm{id}_M$. Geometrically, this depicts tangent vectors over a point with directions in $X(q)$.

Notation.

- $\mathrm{Der}(C^\infty(M))$ is the set of all derivations of $C^\infty(M)$.
- $\mathfrak{X}(M) = \Gamma(TM)$ is the set of sections of $\pi : TM \to M$.

Proposition 11.8. Given a section $v : M \to TM$ in $\mathfrak{X}(M)$, we can try and define

$$\begin{aligned} D_v : C^\infty(M) &\to C^\infty(M) \\ (D_v(f))(q) &\mapsto v(q)f \end{aligned}$$

and this assignment $v \mapsto D_v$ is a linear isomorphism.

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Recall. $TM = \coprod_{q\in M} T_qM$ is a manifold. To show this, given chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$ on M, we set

$$\begin{aligned} \tilde{\varphi} = (x_1, \dots, x_m, dx_1, \dots, dx_m) : TU \equiv \coprod_{q\in U} T_qM &\to \mathbb{R}^m \to \mathbb{R}^m \\ (q, v) &\mapsto (\varphi(q), (dx_1)_q(v), \dots, (dx_m)_q(v)) \end{aligned}$$

with inverse

$$\tilde{\varphi}^{-1}(r, u) = (\varphi^{-1}(r), \sum w_i \left.\frac{\partial}{\partial q_i}\right|_{\varphi(r)}.$$

Also,

$$\begin{aligned} \pi : TM &\to M \\ (q, v) &\mapsto q \end{aligned}$$

is a C^∞-submersion.

We defined vector fields in two ways,

- as sections of tangent bundle $\pi : TM \to M$, i.e., as C^∞-maps $X : M \to TM$ such that $\pi X = \mathrm{id}$, i.e., $X(q) \in T_qM$, and
- as derivations $c : C^\infty(M) \to C^\infty(M)$, i.e., as $\mathbb{R}$-linear maps such that $v(fg) = fv(g) + v(f)g$ for all $f, g \in C^\infty(M)$.

Remark 12.1. Both $\Gamma(TM)$ and $\mathfrak{X}(M)$ are $\mathbb{R}$-vector spaces, and $C^\infty(M)$-modules.

We now prove Proposition 11.8.

Proof. Given $v \in \Gamma(TM)$ and $f \in C^\infty(M)$, consider a function

$$\begin{aligned} D_vf : M &\to \mathbb{R} \\ (D_v(f))(q) &= v(q)f \end{aligned}$$

To go back, given $X \in \mathrm{Der}(C^\infty(M))$, for any $q \in M$, we have $\mathrm{ev}_q : C^\infty(M) \to \mathbb{R}$, and then $\mathrm{ev}_q \circ X : C^\infty(M) \to \mathbb{R}$ is a tangent vector. Define $v_X(q) = \mathrm{ev} \circ X$, and we can check other requirements like C^∞ and so on.

Claim 12.2. $D_v f$ is C^∞.

Subproof. Given a chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$, we have

$$\begin{aligned} \tilde{\varphi} : TU &\to \mathbb{R}^m \times \mathbb{R}^m \\ (q, v) &\mapsto (\varphi(q), dx_1(v), \dots, dx_m(v)) \end{aligned}$$

Since v is C^∞, the map $\tilde{\varphi} \circ v|_U : U \to \mathbb{R}^m \times \mathbb{R}^m$, defined by $(\tilde{\varphi} \circ v)(q) = (\varphi(q), (dx_1)_q(v(q)), \dots, (dx_m)_q(v(q)))$, is C^∞. Therefore, the assignment $q \mapsto (dx_i)_q(v(q))$ are C^∞ on U. Hence, $v = \sum v_i \frac{\partial}{\partial x_i}$ where $v_i(q) = (dx_i)_q(v(q))$ for all i. So $(D_v f)|_U = \left(\sum v_i \frac{\partial}{\partial x_i}\right) f = \sum v_i \frac{\partial f}{\partial x_i}$. This concludes the proof. ■

Also, for all $f, g \in C^\infty(M)$ and all q, we have

$$\begin{aligned} (D_v(fg))(q) &= v(q)(fg) \\ &= (v(q)f)g(q) + f(q)(v(q)g) \\ &= ((D_v f)g + f(D_v g))(q). \end{aligned}$$

Recall that derivations are local, i.e., for $X \in \mathrm{Der}(C^\infty(M))$ and $f \in C^\infty(M)$ and $f|_U \equiv 0$, then $Xf|_U \equiv 0$. As a consequence, for $U \subseteq M$ open, define $X|_U : C^\infty(U) \to C^\infty(U)$ such that $(X|_U)(f|_U) = (Xf)|_U$ for all $f \in C^\infty(M)$. Now given a chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$, we know x_i's are in $C^\infty(U)$, then $(X|_U)(x_i)$ is a smooth function on U. Therefore,

$$\begin{aligned} v_X|_U &= \sum (dx_i)(v_X) \frac{\partial}{\partial x_i} \\ &= \sum v_X X(x_i) \frac{\partial}{\partial x_i} \\ &= \sum X|_U (x_i) \frac{\partial}{\partial x_i}, \end{aligned}$$

and thus $v_X|_U : U \to TU$ is C^∞, and since U is arbitrary, then $v_X \in \Gamma(TM)$. □

Recall. For any $X, Y \in \mathrm{Der}(C^\infty(M))$, $[X, Y] \in \mathrm{Der}(C^\infty(M))$. Therefore, $\mathrm{Der}(C^\infty(M))$ is a real Lie algebra with bracket $(X, Y) \mapsto [X, Y]$. Note that $\mathrm{Der}(C^\infty(M)) \subseteq \mathrm{Hom}_\mathbb{R}(C^\infty(M), C^\infty(M))$.

Recall. If $(A, \circ)$ is a real associative algebra, then $[a, b] := a \circ b - b \circ a$ gives A the structure of a Lie algebra, and $\mathrm{Der}(C^\infty(M)) \subseteq \mathrm{Hom}_\mathbb{R}(C^\infty(M), C^\infty(M))$.

Now given a C^∞-map $f : M \to N$ of manifolds, we get a map

$$\begin{aligned} Tf : TM &\to TN \\ (q, v) &\mapsto (f(q), T_q f v) \end{aligned}$$

Exercise 12.3. Tf is C^∞.

Remark 12.4. Given $f : M \to N$ and $v \in \Gamma(TM)$, we may not have a commutative diagram:

$$\begin{tikzcd} TM \arrow[r, "Tf"] & TN \\ M \arrow[u, "v"] \arrow[r, "f"'] & N \arrow[u, dashed, "?"'] \end{tikzcd}$$

Definition 12.5. Let $f : M \to N$ be a smooth map on manifolds, then $v \in \Gamma(TM)$ and $w \in \Gamma(TN)$ are *f-related* if we have a commutative diagram

$$\begin{tikzcd} TM \arrow[r, "Tf"] & TN \\ M \arrow[u, "v"] \arrow[r, "f"'] & N \arrow[u, "w"'] \end{tikzcd}$$

That is, for any $q \in M$, $w(f(q)) = (f(q), T_q f(v(q)))$.

Equivalently, for $f : M \to N$, we say $X \in \mathrm{Der}(C^\infty(M))$ is *f-related* to $Y \in \mathrm{Der}(C^\infty(N))$ if for all $h \in C^\infty(N)$, we have $Y(h) \circ f = X(h \circ f)$ in $C^\infty(M)$.

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Recall. Let M be a manifold, we have a bijection

$$\begin{aligned}\Gamma(TM) &\to \operatorname{Der}(C^\infty(M))\\ v &\mapsto D_v : (Dvf)(q) = v_q(f)\ \forall f, q\end{aligned}$$

with inverse by assignment $X \mapsto v_X$ where $v_X(q)f = (Xf)(q)$.

Lemma 13.1. Let $f : M \to N$, then $v \in \Gamma(TM)$ is f-related to $w \in \Gamma(TN)$ if and only if $D_v \in \operatorname{Der}(C^\infty(M))$ is f-related to $D_w \in \operatorname{Der}(C^\infty(N))$.

Proof. v is f-related to w if and only if $(T_qf)(v(q)) = w(f(q))$ for all q, if and only if $((T_qf)(v(q)))h = (w(f(q)))h$ for all q and all h, if and only if $(D_v(h \circ f))(q) = (D_wh)(f(q))$, if and only if $D_v(h \circ f) = D_w(h \circ f)$. □

Lemma 13.2. Suppose $f : M \to N$, let $X_1, X_2 \in \operatorname{Der}(C^\infty(M))$, and $Y_1, Y_2 \in \operatorname{Der}(C^\infty(N))$ such that X_i is f-related to Y_i for $i = 1, 2$, then $[X_1, X_2]$ is f-related to $[Y_1, Y_2]$.

Proof. For any $h \in C^\infty(N)$, $X_i(h \circ f) = Y_i(h) \circ f$ for $i = 1, 2$. Therefore,

$$\begin{aligned}([X_1, X_2])(h \circ f) &= X_1(X_2(h \circ f)) - X_2(X_1(h \circ f))\\ &= X_1(Y_2(h) \circ f) - X_2(Y_1(h) \circ f)\\ &= Y_1(Y_2(h)) \circ f - Y_2(Y_1(h)) \circ f\\ &= ([Y_1, Y_2](h)) \circ f.\end{aligned}$$

□

Definition 13.3. Let $Q \subseteq M$ be an embedded submanifold. A vector field $Y \in \Gamma(TM)$ is *tangent* to Q if for all $q \in Q$, $Y(q) \in T_qQ$.

Example 13.4. If $M = \mathbb{R}^2$, let $Q = \mathbb{R} \times \{0\}$, then $Y(x_1, x_2) = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$, so $Y(x, 0) = x_1 \frac{\partial}{\partial x_1} + 0 \in T_{(x,0)}Q$. Equivalently, we have $i : Q \hookrightarrow M$ to be an inclusion, so $Ti : TQ \hookrightarrow TM$ is an embedding since i is, as $Y(q) \in T_qQ$ for all $q \in Q$ indicates $(Y \circ i)(Q) \subseteq TQ$:

$$\begin{tikzcd} Q \arrow[r, "i"] \arrow[d, dashed, "Y\circ i"'] & M \arrow[d, "Y"] \\ TQ \arrow[r, hook, "Ti"'] & TM \end{tikzcd}$$

Hence, $Y \circ i : Q \to TQ$ is a vector field on Q, and $Y \circ i$ is i-related to Y.

Lemma 13.5. Let $Q \subseteq M$ be an embedded submanifold, let $Y_1, Y_2 \in \Gamma(TM)$ which are tangent to Q, then $[Y_1, Y_2]$ is tangent to Q.

Proof. Since $Y_i|_Q$ is i-related to Y_i, then $[Y_1, Y_2]|_Q$ is i-related to $[Y_1, Y_2]$. □

Definition 13.6. Let G be a Lie group, then we give T_eG the structure of a Lie algebra. A vector field $X : G \to TG$ is *left-invariant* if for all $a \in G$, $TL_a(X(g)) = X(L_ag)$ for all $g \in G$ and all $a \in G$, that is, X is L_a-related to X where $L_a(g) = ag$ is the left translation.

Recall.
- $(La)^{-1} = L_{a^{-1}}$.
- By Lemma 13.2, if X and Y are left-invariant, then so is $[X, Y]$.

Notation. We denote $\mathfrak{g} = \operatorname{Lie}(G)$ to be the Lie algebra of the left-invariant vector fields.

Lemma 13.7. Let G be a Lie group, let $\mathfrak{g}$ be the space of left-invariant vector fields, then the evaluation map

$$\begin{aligned}\mathrm{ev}_e : \mathfrak{g} &\to T_eG\\ X &\mapsto X(e)\end{aligned}$$

is an $\mathbb{R}$-linear bijection. In particular, they have the same dimension.

Proof. Obviously ev_e is linear. If $X(e) = 0$, then for all $a \in G$, $X(a) = X(L_a e) = (TL_a)_e(X(e)) = 0$, so ev_e is injective. Conversely, given $v \in T_eG$, define

$$\begin{aligned} \tilde{v} : G &\to TG \\ a &\mapsto (TL_a)_e v \end{aligned}$$

then $\tilde{v}$ is left-invariant. We know

$$\begin{aligned} m : G \times G &\to G \\ (a, b) &\mapsto ab \end{aligned}$$

is C^∞, so $T_m : TG \times TG \to TG$ is C^∞. Consider

$$\begin{aligned} f : G &\to TG \times TG \\ a &\mapsto ((a, 0), (e, v)). \end{aligned}$$

Claim 13.8. $(T_m \circ f)(a) = (T_e L_a)(v)$.

Subproof. Pick $\gamma : I \to G$ such that $\gamma(0) = e$ and $\dot{\gamma}(0) = v$, then

$$\begin{aligned} \sigma : I &\to G \times G \\ t &\mapsto (a, \gamma(t)) \end{aligned}$$

is C^∞ where $\sigma(0) = (a, e)$, and $\frac{d}{dt}\big|_0 (a, \gamma(t)) = (0, v) \in T_{(a,e)}(G \times G)$. Now

$$\begin{aligned} T_m(f(a)) &= (T_m)_{(a,e)}(0, v) \\ &= \left.\frac{d}{dt}\right|_0 m(\sigma(t)) \\ &= \left.\frac{d}{dt}\right|_0 a\gamma(t) \\ &= \left.\frac{d}{dt}\right|_0 L_a(\gamma(t)) \\ &= (T_e L_a)(\dot{\gamma}(0)) \\ &= (T_e L_a)(v) \\ &= \tilde{v}(a). \end{aligned}$$

■

□

Therefore, the left-invariant vector field $\mathrm{Lie}(G)$ is isomorphic to T_eG as $\mathbb{R}$-vector spaces.

Definition 13.9. Let $X : M \to TM$ be a vector field. An *integral curve* $\gamma : I \to M$ of X passing through q at $t = 0$ is a C^∞-map $\gamma : I \to M$ such that $\gamma(0) = q$ and $\dot{\gamma}(t) = X(\gamma(t))$ for all $t \in I$. Here $\dot{\gamma}(t) = (T_t\gamma)\left(\frac{d}{dt}\big|_t\right) \in T_{\gamma(t)}M$. Equivalently, $\dot{\gamma}(t)f = X(\gamma(t))f = \frac{d}{dt}\big|_t (f \circ \gamma)$ for all $f \in C^\infty(M)$.

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Remark 14.1. if $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ is a coordinate chart and v is a vector field on U, so $v = \sum v_i \frac{\partial}{\partial x_i}$ for $v_1, \ldots, v_m$ in $C^\infty(U)$. This is a section $q \mapsto \sum v_i(q) \left.\frac{\partial}{\partial x_i}\right|_q \in \Gamma(TU)$ and for all $f \in C^\infty(U)$, $f \mapsto \sum v_i \frac{\partial f}{\partial x_i} \in C^\infty(U)$ which is a derivation.

Recall. An integral curve of $X \in \Gamma(TM)$ is a curve $\gamma : I \to M$ with $\gamma(0) = q$ such that $\left.\frac{d\gamma}{dt}\right|_t = X(\gamma(t))$.

Example 14.2. Let $M = U$ be open in $\mathbb{R}^m$, and $X = \sum x_i \frac{\partial}{\partial r_i}$. Let $\gamma(t) = (\gamma_1(t), \dots, \gamma_m(t))$ for $\gamma_i \in C^\infty(I)$, then $\frac{\partial \gamma}{\partial t}\Big|_t = \sum \gamma_i'(t) \frac{\partial}{\partial \gamma_i}$. Therefore, $\frac{\partial \gamma}{\partial t} = X(\gamma(t))$ amounts to $\sum \gamma_i'(t)\frac{\partial}{\partial \gamma_i} = \sum x_i(\gamma(t))\frac{\partial}{\partial \gamma_i}$. Therefore, $\gamma_i'(t) = x_i(\gamma_1(t), \dots, \gamma_m(t))$.

Hence, γ is an integral curve of X if and only if γ solves such a system of equations with initial condition $\gamma(0) = q$.

Theorem 14.3. Let $U \subseteq \mathbb{R}^m$ be open, $X = (x_1, \dots, x_m) : U \to \mathbb{R}^m$ be C^∞, then for all $q_0 \in U$, there exists an open neighborhood V of q_0 in U and $\varepsilon > 0$, and a C^∞-map $\Phi : V \times (-\varepsilon, \varepsilon) \to U$ such that for all $q \in V$, $\gamma_q(t) := \Phi(q,t)$ solves $\gamma_i'(t) = x_i(\gamma_1(t), \dots, \gamma_m(t))$ with initial condition $\gamma_q(0) = q$. Moreover, such mapping Φ is unique.

Proof. Apply contraction mapping principle. □

Example 14.4. Say $U = (-1,1)$, let

$$\begin{aligned} X : (-1,1) &\to \mathbb{R} \\ x &\mapsto \frac{d}{dx} \end{aligned}$$

with $X(q) = 1$ be the ODE, i.e., $\frac{dX}{dt} = 1$ with $X(0) = q$, then $\Phi(q,t) = q + t$. The domain of definition of Φ is $W = \{(q,t) \mid q \in (-1,1), q + t \in (-1,1)\}$.

Remark 14.5. We need to keep track of the initial conditions. Say $\gamma : (a,b) \to M$ is an integral curve of vector field X on M with $\gamma(0) = q$, then for all $t_0 \in (a,b)$, we know

$$\begin{aligned} \sigma : (a - t_0, b - t_0) &\to M \\ s &\mapsto \gamma(s + t_0) \end{aligned}$$

is also an integral curve. Therefore, γ and σ has the same image.

Proof.

$$\begin{aligned} \frac{d}{dt}\Big|_t \sigma &= \frac{d}{ds}\Big|_t \gamma(s+t_0) \\ &= \frac{d}{du}\Big|_{u=t+t_0} \gamma(u) \\ &= X(\gamma(t+t_0)) \\ &= X(\sigma(t)). \end{aligned}$$

□

Lemma 14.6. Let $X : M \to TM$ be a vector field, $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$ be a coordinate chart and $X = \sum x_i \frac{\partial}{\partial x_i}$ where $x_i \in C^\infty(U)$, then $\gamma : I \to U$ with $\gamma(0) = q$ is an integral curve of X if and only if $(x_1 \circ \gamma, \dots, x_m \circ \gamma) : I \to \mathbb{R}^m$ solves $y_i' = Y_i(Y_1, \dots, y_m)$ with $y_i(0) = x_i(\gamma(0))$. Here $Y_i = X_i \circ \varphi^{-1} \in C^\infty(\varphi^{-1}(U))$.

Proof. We have $\dot\gamma(t) = \sum dx_i(\dot\gamma(t))\frac{\partial}{\partial x_i} = \sum (x_i \circ \gamma)'(t)\frac{\partial}{\partial x_i}$. Therefore, $\dot\gamma(t) = X(\gamma(t))$ if and only if $(X_i \circ \gamma)' = X_i(\gamma(t)) = (X_i \circ \varphi^{-1})(\varphi(\gamma(t))) = Y_i(X_1 \circ \gamma(t), \dots, X_m \circ \gamma(t))$ for all i. □

Corollary 14.7. Let $X : M \to TM$ be a vector field, then for all $q \in M$, there exists an integral curve $\gamma : I \to M$ of X such that $\gamma(0) = q$. Moreover, γ depends smoothly on q, and is locally unique: for all integral curve $\sigma : J \to M$ of X mapping $0 \mapsto q$, there exists $\delta > 0$ such that $(-\delta, \delta) \in I \cap J$ and $\gamma|_{(-\delta,\delta)} = \sigma|_{(-\delta,\delta)}$.

Remark 14.8. It may not be the case that $\gamma|_{I \cap J} = \sigma|_{I \cap J}$. This is true if M is Hausdorff.

Example 14.9. Consider line with two origins in Example 1.10, with translations that agree before the origins.

Lemma 14.10. Suppose $\gamma : I \to M$ and $\sigma : J \to M$ are continuous curves, and M is Hausdorff, then the set $Z = \{t \in I \cap J \mid \gamma(t) = \sigma(t)\}$ is closed in $I \cap J$.

Proof. Note that

$$\begin{aligned}(\gamma,\sigma): I\cap J &\to M\times M\\ t &\mapsto (\gamma(t),\sigma(t))\end{aligned}$$

is continuous, and $Z=(\gamma,\sigma)^{-1}(\Delta_M)$. □

Lemma 14.11. Let $\gamma: I\to M$ and $\sigma: J\to M$ be two integral curves of a vector field X on M with $\sigma(0)=\gamma(0)$, then $W=\{t\in I\cap J\mid \gamma(t)=\sigma(t)\}$ is open in $I\cap J$.

Proof. Given $t_0\in W$, then $t_0\in I\cap J$ and $\sigma(t_0)=\gamma(t_0)$, and we consider $\tilde{\sigma}(t):=\sigma(t+t_0)$ and $\tilde{\gamma}(t)=\gamma(t+t_0)$, then $\tilde{\sigma}(0)=\sigma(t_0)=\gamma(t_0)=\tilde{\gamma}(0)$. Both $\tilde{\gamma}$ and $\tilde{\sigma}$ are integral curves of X with $\tilde{\sigma}(0)=\tilde{\gamma}(0)$, therefore by Corollary 14.7, there exists $\delta>0$ such that $\tilde{\sigma}|_{(-\delta,\delta)}=\tilde{\gamma}|_{(-\delta,\delta)}$, then $t_0+(-\delta,\delta)=(t_0-\delta,t_0+\delta)\subseteq W$. □

Lemma 14.12. Let M be a Hausdorff manifold, $X\in\Gamma(TM)$, $\gamma: I\to M$ and $\sigma: J\to M$ be two integral curves with $\gamma(0)=\sigma(0)$, then $\gamma|_{I\cap J}=\sigma|_{I\cap J}$.

Proof. Since I and J are intervals, then $I\cap J$ is connected. By Lemma 14.11 and Lemma 14.10, $W=\{t\in I\cap J\mid \gamma(t)=\sigma(t)\}$ is clopen, thus $W=I\cap J$. □

15 Sept 25, 2022

Recall. We introduced integral curves of vector fields, and in particular we introduced Lemma 14.12.

Corollary 15.1. For any vector field $X\in\Gamma(TM)$ and any $q\in M$, there exists a unique maximal integral curve $\gamma_q: I_q\to M$ of X with $\gamma_q(0)=q$. Here *maximal* means that if $\sigma: J\to M$ is another integral curve of X with $\sigma(0)=q$, then $J\subseteq I_q$ and $\sigma=\gamma_q|_J$.

Proof. Consider the subset $\Gamma\subseteq\mathbb{R}\times M$ defined as follows: let Y be the set of all integral curves γ of X with $\gamma(0)=q$, then define $\Gamma=\bigcup_{\gamma\in Y}\operatorname{graph}(\gamma)$. By Lemma 14.12, Γ is a graph of a smooth curve, which is the desired maximal integral curve γ_q of X with $\gamma_q(0)=q$. □

Lemma 15.2. Let $f: M\to N$ be a map of manifolds, with $X\in\Gamma(TM)$ and $Y\in\Gamma(TY)$, and $Tf\circ X=Y\circ f$, i.e., X and Y are f-related, then for any integral curve γ of X, $f\circ\gamma$ is an integral curve of Y.

Proof. We have

$$\begin{aligned}\frac{d}{dt}\left.(f\circ\gamma)\right|_t &= T_t(f\circ\gamma)\left(\frac{d}{dt}\right)\\ &= T_{\gamma(t)}f\left(T_t\gamma\left(\frac{d}{dt}\right)\right)\\ &= T_{\gamma(t)}f(X(\gamma(t)))\\ &= Y(f(\gamma(t)))\\ &= Y((f\circ\gamma)(t)).\end{aligned}$$

□

Example 15.3. Let $M=(-1,1)$, $N=\mathbb{R}$, $f:(-1,1)\hookrightarrow\mathbb{R}$ be the inclusion. Let $X=\frac{d}{dt}$ and $Y=\frac{d}{dt}$, then

$$\begin{aligned}\gamma: (-1,1) &\to M\\ t &\mapsto t\end{aligned}$$

is a maximal integral curve of X with $\gamma(0)=0$. Note that it is not a maximal integral curve of Y because $f\circ\gamma$ is not an integral curve of Y that is not maximal.

Example 15.4. Let $M = \mathbb{R}^2$ and $N = \mathbb{R}$, then consider $f(x, y) = x$ with $X = \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$, with $Y(x) = \frac{d}{dx}$, then $\gamma_x(t) = x + t$ is the integral curve of Y with $\gamma_x(0) = x$. it is defined for all $t \in \mathbb{R}$.

To compute integral curves of X, we solve

$$\begin{cases} \dot{x} = 1, x(0) = x_0 \\ \dot{y} = y^2, y(0) = y_0, \end{cases}$$

then $x(t) = x_0 + t$ and $\frac{1}{y}^2 \frac{dy}{dt} = 1$, therefore

$$\int_0^t \frac{1}{y^2} \frac{dy}{dt} dt = \int_0^t dt$$

and so $t = -\left.\frac{1}{y}\right|_0^t = \frac{1}{y_0} - \frac{1}{y(t)}$, hence $y(t) = \frac{y_0}{1 - y_0 t}$. Thus, $t \in (-\infty, \frac{1}{y_0})$. That is, the curve runs off to ∞ in finite time.

Definition 15.5. Let X be a vector field on a (Hausdorff) manifold M, and let $\gamma_q : I_q \to M$ be the unique maximal integral curve with $\gamma_q(0) = q$. Let $W = \bigcup_{q \in M} \{q\} \times I_q \subseteq M \times \mathbb{R}$, then the *(local) flow* of X is the map

$$\begin{aligned} \Phi : W &\to M \\ (q, t) &\mapsto \gamma_q(t) \end{aligned}$$

We say Φ is a *global flow* if $W = M \times \mathbb{R}$, and in this case we say X is *complete*.

Theorem 15.6. Let $\Phi : M \to M$ be a flow of a vector field, then

1. $M \times \{0\} \subseteq W$,
2. W is open, and
3. Φ is C^∞.

Proof. See Lee. □

Example 15.7. Let $X = y^2 \frac{d}{dy} \in \Gamma(\mathbb{R})$, then $W = \{(y, t) \in \mathbb{R} \times \mathbb{R} \mid t < \frac{1}{y}$ when $y > 0, t$ arbitrary when $y = 0, t > \frac{1}{y}$ if $y < 0\}$. The flow is $\Phi(y, t) = \frac{y}{1 - yt}$.

Lemma 15.8. Let $\Phi : W \to M$ be a local flow of a vector field X, then $\Phi(q, s + t) = \Phi(\Phi(q, s), t)$ whenever both sides are defined.

Remark 15.9. Note that if $s = -t$, then the left-hand side is defined, but the right-hand side is not.

Proof. Fix q and fix s such that $(q, s) \in W$. Consider $\sigma(t) = \Phi(q, s + t) = \gamma_q(s + t)$, and $\tau(t) = \Phi(\Phi(q, s), t) = \gamma_{\Phi(q,s)}(t)$, then $\tau(0) = \Phi(q, s) = \gamma_q(s) = \sigma(0)$. Both $\sigma(t)$ and $\tau(t)$ are integral curves, and that they agree at $t = 0$, then $\sigma(t) = \tau(t)$ for all t in the intersection of their domains of definition. Therefore, the two equations agree whenever both sides are defined. □

Definition 15.10. An (left) *action* of a Lie group G on a manifold M is a C^∞-map

$$\begin{aligned} G \times M &\to M \\ (g, q) &\mapsto g \cdot q \end{aligned}$$

such that

1. $e \cdot q = q$ for all q, and
2. $g_1 \cdot (g_2 \cdot q) = (g_1 g_2) \cdot q$.

Claim 15.11. If X is complete, then its flow is an action of the Lie group $(\mathbb{R}, +, \cdot)$.

Proof. Define $t \cdot q = \Phi(q, t)$, then

$$\begin{aligned} t \cdot (s \cdot q) &= \Phi(\Phi(q, s), t) \\ &= \Phi(q, s + t) \\ &= (t + s) \cdot q \end{aligned}$$

and $0 \cdot q = \Phi(q, 0) = q$. □

Remark 15.12. If we have a group action, we determine the groupoid structure, and therefore we recover the groupoid version of the lemma.

Remark 15.13. For a Lie group G, the multiplication $m : G \times G \to G$ is a left action of G on G, with $e \cdot g = g$ and $a \cdot (b \cdot g) = (a \cdot b) \cdot g$.

Remark 15.14. For any manifold, there exists a group $\mathrm{Diff}(M) = \{f : M \to M \mid f \text{ is a diffeomorphism}\}$, where the operation is function composition, and the identity is the identity map.

Exercise 15.15. An (left) action $G \times M \to M$ of a Lie group G on a manifold M gives rise to a homomorphism

$$\begin{aligned} \rho : G &\to \mathrm{Diff}(M) \\ (\rho(g))(q) &\mapsto g \cdot q \end{aligned}$$

In particular, the multiplication $m : G \times G \to G$ gives rise to

$$\begin{aligned} L : G &\to \mathrm{Diff}(G) \\ a &\mapsto L_a \end{aligned}$$

Definition 15.16. An *abstract local flow* on a manifold M is a C^∞-map $\psi : W \to M$, where W is an open neighborhood of $M \times \{0\}$ in $M \times \mathbb{R}$, so that $\psi(q, 0) = q$ for all $q \in M$ and $\psi(q, s + t) = \psi(\psi(q, s), t)$ whenever both sides are defined.

We will show that any abstract local flow is part of a flow on a vector field.

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Recall. Given a vector field X on a manifold M, we define the flow to be $\Phi : W \to \mathbb{R}$ for some open neighborhood of $M \times \{0\}$ in $M \times \mathbb{R}$. The defining property of Φ would be that for every $q \in M$, $W \cap (\{q\} \times \mathbb{R}) = \{q\} \times I_q$ and $I_q \ni t \mapsto \Phi(q, t)$ is the maximal integral curve of X. We also proved that $\Phi(q, t + s) = \Phi(\Phi(q, t), s)$ for all q, t, s such that both sides are defined.

We say the flow is a global flow if $W = M \times \mathbb{R}$, that is, for all $q \in M$, the maximal integral curve $\gamma_q \in I_q \to M$ of X with $\gamma_q(0) = q$ is defined for all $t \in \mathbb{R}$, i.e., $I_q = \mathbb{R}$.

Lemma 16.1. Let M be a manifold, $U \subseteq M \times \mathbb{R}$ be an open neighborhood of $M \times \{0\}$ with $U \cap (\{q\} \times \mathbb{R})$ connected for all $q \in M$, and $\psi : U \to M$ a smooth map such that

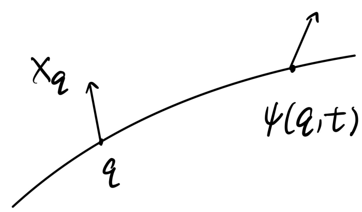
1. $\psi(q, 0) = q$ for all q, and
2. $\psi(q, s + t) = \psi(\psi(q, s), t)$ whenever both sides are defined,

then there exists a vector field X on M such that for all $q \in M$, the assignment $t \mapsto \psi(q, t)$ is an integral (but not necessarily maximal) curve of X with $\psi(q, 0) = q$.

Proof. For all $q \in M$, we define $X(q) = \frac{d}{dt}\big|_0 \psi(q, t)$, then

$$\begin{aligned} \frac{d}{dt}\bigg|_t \psi(q, t) &= \frac{d}{dt}\bigg|_0 \psi(q, t + s) \\ &= \frac{d}{ds}\bigg|_0 \psi(\psi(q, t), s) \end{aligned}$$

$$= X(\psi(q,t)).$$



□

Lemma 16.2. Let $\Phi : W \to M$ be a flow of a vector field X on a manifold M. Suppose there exists $\varepsilon > 0$ such that $M \times [-\varepsilon, \varepsilon] \subseteq W$, then $W = M \times \mathbb{R}$, i.e., the vector field X is complete.

Proof. We want to show that for all $q \in M$, $I_q := \{t \in \mathbb{R} \mid (q,t) \mid W\}$ is $\mathbb{R}$. Since I_q is connected, then it suffices to show that I_q is unbounded. By assumption, $\varphi_\varepsilon(q) := \varphi(q,\varepsilon)$ and $\varphi_{-\varepsilon}(q) := \varphi(q,-\varepsilon)$ are defined for all $q \in M$, since $q = \varphi(q,0) = \varphi(\varphi(q,\varepsilon),-\varepsilon) = \varphi(\varphi(q,-\varepsilon),\varepsilon)$, therefore $(\varphi_\varepsilon)^{-1}$ exists and is just $\varphi_{-\varepsilon}$.

Given $q \in M$, we consider $\mu(t) = \varphi(q, t+\varepsilon) = \gamma_q(\varepsilon + t)$, and it is easy to check that $\mu'(t) = X(\mu(t))$, therefore μ is an integral curve of X with $\mu(0) = \gamma_\varepsilon(\varepsilon)$. Since γ_q is defined on I_q, then μ is defined for all t such that $t + \varepsilon \in I_q$, that is, $t \in I_q - \varepsilon$. Since $\gamma_{\varphi_\varepsilon(q)} : I_{\varphi_\varepsilon(q)} \to M$ is a maximal integral curve of X such that $\gamma_{\varphi_\varepsilon(q)}(0) = \Phi_\varepsilon(q) = \gamma_q(\varepsilon)$, so $I_q - \varepsilon \subseteq I_{\varphi_\varepsilon(q)}$, and similarly $I_q + \varepsilon \subseteq I_{\varphi_{-\varepsilon}(q)}$, therefore $I_{\varphi_\varepsilon(q)} + \varepsilon \subseteq I_{\varphi_{-\varepsilon}}(\varphi_\varepsilon(q)) = I_q$. Therefore, $I_q - \varepsilon = I_{\varphi_\varepsilon(q)}$. By induction, we conclude that for all $n > 0$, $I_q - n\varepsilon = I_{(\varphi_\varepsilon)^n(q)}$. Since $0 \in I_{q'}$ for all q', and $0 \in I_q - n\varepsilon$, so $n\varepsilon \in I_q$ for all $n \in \mathbb{N}$. Similar argument shows that $-n\varepsilon \in I_q$ for all $n \in \mathbb{N}$. That is, I_q is neither bounded above nor bounded below. □

Definition 16.3. The *support* of a vector field $X \in \Gamma(TM)$ is $\operatorname{supp}(X) = \overline{\{q \in M \mid X(q) \neq 0\}}$.

Corollary 16.4. Suppose $X \in \Gamma(TM)$ has compact support, then X is complete: its flow exists for all time.

Proof. Note that $X \equiv 0$ on $M \backslash \operatorname{supp}(X)$, so for all $q \in M \backslash \operatorname{supp}(X)$. Note that $\gamma_q(t) = q$ is the maximal integral curve of X, which exists for all t, so $(M \backslash \operatorname{supp}(X)) \times \mathbb{R} \subseteq W$, which is the domain of the flow φ. Since $\operatorname{supp}(X)$ is compact, then $(\operatorname{supp}(X) \times \{0\}) \subseteq W$ is compact. Since W is open, then by tube lemma, there exists $\varepsilon > 0$ such that $\operatorname{supp}(X) \times (-2\varepsilon, 2\varepsilon) \subseteq W$, hence $\operatorname{supp}(X) \times [-\varepsilon, \varepsilon] \subseteq W$. Therefore,

$$(M \backslash \operatorname{supp}(X)) \times [-\varepsilon, \varepsilon] \subseteq (M \backslash \operatorname{supp}(X)) \times \mathbb{R} \subseteq W,$$

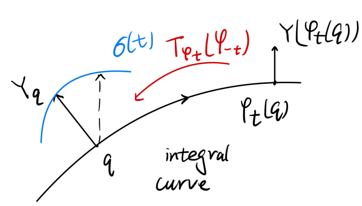
so $M \times [-\varepsilon, \varepsilon] \subseteq W$. Now apply Lemma 16.2. □

We will start talking about Lie derivatives. Let $X, Y \in \Gamma(TM)$ be two vector fields. For simplicity we assume X and Y have global flow $\varphi(q,t) = \varphi_t(q)$, and $\psi(q,t) = \psi_t(q)$, respectively. (It suffices to have the flow maintained for small neighborhood of time.) Fix $q \in M$. Consider

$$\begin{aligned} \sigma : \mathbb{R} &\to T_qM \\ t &\mapsto (T_{\varphi_t(q)}\varphi_{-t})(Y(\varphi_t(q))) \end{aligned}$$

Remark 16.5. For any curve $\gamma : \mathbb{R}$, $\dot\gamma(t) \in T_{\gamma(t)}(T_qM) = T_qM$ since γ_qM is a vector space. In particular,

$$\left.\frac{d\sigma}{dt}\right|_0 = \left.\frac{d}{dt}\right|_0 (T_{\varphi_t(q)}\Phi_{-t}(Y(\varphi_t(q)))) \in T_qM.$$



Definition 16.6. The *Lie derivative* $L_X Y$ of Y with respect to X is defined by

$$(L_X Y)(q) = \left.\frac{d}{dt}\right|_0 T_{\varphi_t(q)}\varphi_{-t}(Y(\varphi_t(q))) = \lim_{t\to 0}\frac{1}{t}\left(T_{\varphi_t(q)}\varphi_{-t}(Y(\varphi_t(q))) - Y_q\right).$$

Theorem 16.7. For any two vector fields $X, Y \in \Gamma(TM)$, $L_X Y = [X, Y]$.

To prove this, we will prove the following.

Lemma 16.8. Let M be a manifold and $\gamma : I \to T_q M$ be a curve. Let $f \in C^\infty(M)$, then

$$\left.\frac{d}{dt}\right|_0 (\gamma(t)f) = \left(\left.\frac{d\gamma}{dt}\right|_0\right) f.$$

Proof. Choose a chart $(x_1, \dots, x_n) : U \to \mathbb{R}^m$ with $q \in U$, then $\gamma(t) = \sum \gamma_i(t) \left.\frac{\partial}{\partial x_i}\right|_q$, where each $\gamma_i : I \to \mathbb{R}$ is C^∞. Now $\left.\frac{d\gamma}{dt}\right|_0 = \sum \gamma_i'(0) \left.\frac{\partial f}{\partial x_i}\right|_q$. We also know that $\gamma(t)f = \sum \gamma_i(t) \left.\frac{\partial f}{\partial x_i}\right|_q$, therefore $\left.\frac{d}{dt}\right|_0 \gamma(t) = \left.\frac{d}{dt}\right|_0 \left(\sum \gamma_i(t) \left.\frac{\partial f}{\partial x_i}\right|_q\right) = \sum \gamma_i'(0) \left.\frac{\partial f}{\partial x_i}\right|_q$ as well. $\square$

Lemma 16.9. Let X and Y be two vector fields with flows $\{\varphi_t\}$ and $\{\psi_t\}$, viewed as family of diffeomorphisms with $\mathbb{R}$-actions. For any $f \in C^\infty(M)$,

$$(L_X Y)(q)f = \left.\frac{\partial^2}{\partial s \partial t}\right|_{(0,0)} (f \circ \varphi_{-t} \circ \psi_s \circ \varphi_t)(q).$$

Proof. We have

$$\begin{aligned}
(L_X Y)(q)f &= \left(\left.\frac{d}{dt}\right|_0 T\varphi_{-t}(Y(\varphi_t(q)))\right) f \\
&= \left.\frac{d}{dt}\right|_0 (T\varphi_{-t}(Y(\varphi_t(q)) f)) \\
&= \left.\frac{d}{dt}\right|_0 Y(\varphi_t(q))(f \circ \varphi_{-t}) \\
&= \left.\frac{d}{dt}\right|_0 \left.\frac{\partial}{\partial s}\right|_0 (f \circ \varphi_{-t})(\psi_s(\varphi_t(q))) \\
&= \left.\frac{\partial^2}{\partial t \partial s}\right|_{(0,0)} (f \circ \varphi_{-t} \circ \psi_s \circ \varphi_t)(q) \\
&= \left.\frac{\partial^2}{\partial s \partial t}\right|_{(0,0)} (f \circ \varphi_{-t} \circ \psi_s \circ \varphi_t)(q).
\end{aligned}$$

$\square$

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Recall. Let $X, Y \in \Gamma(TM)$ be two vector fields, and we assume for simplicity that X, Y have global flows $\{\varphi_t\}_{t\in\mathbb{R}}$ and $\{\psi_s\}_{s\in\mathbb{R}}$. We define the Lie derivative $L_X Y$ of Y with respect to X by

$$(L_X Y)(q) = (L_X Y)(q) = \left.\frac{d}{dt}\right|_0 T_{\varphi_t(q)}\varphi_{-t}(Y(\varphi_t(q))).$$

Theorem 17.1. $L_X Y = [X, Y]$.

Proof. It suffices to show that for all $f \in C^\infty(M)$ and all $q \in M$,

$$((L_X Y)(q))f = ([X,Y](q))f = ([X,Y]f)(q).$$

Consider

$$\begin{aligned} H &: \mathbb{R}^3 \to \mathbb{R} \\ (x,y,z) &\mapsto (f \circ \Phi_x \circ \psi_y \circ \Phi_z)(q), \end{aligned}$$

then by Lemma 16.8,

$$((L_X Y)(q))f = \left.\frac{\partial^2}{\partial t \partial s}\right|_{(0,0)} (H(-t,s,t)) = \left.\frac{d}{ds}\right|_{s=0} \left(\left.\frac{\partial}{\partial t}\right|_{(0,s)} H(-t,s,t) \right),$$

and by the chain rule,

$$\left.\frac{\partial}{\partial t}\right|_{(0,s)} H(-t,s,t) = -\frac{\partial H}{\partial x}(0,s,0) + \frac{\partial H}{\partial z}(0,s,0).$$

Hence,

$$\begin{aligned} ((L_X Y)(q))f &= \left.\frac{d}{ds}\right|_0 \left(-\left.\frac{\partial}{\partial x}\right|_{(0,s)} (f \circ \varphi_x \circ \psi_s \circ \varphi_0)(q) + \left.\frac{\partial}{\partial z}\right|_{(0,s)} (f \circ \psi_s \circ \varphi_z)(q) \right) \\ &= \left.\frac{d}{ds}\right|_0 \left(-\left.\frac{\partial}{\partial x}\right|_{(0,s)} (f \circ \varphi_x \circ \psi_s \circ \varphi_0)(q) + \left.\frac{\partial}{\partial z}\right|_{(0,s)} (f \circ \psi_s \circ \varphi_z)(q) \right) \\ &= \left.\frac{d}{ds}\right|_0 (-(Xf)(\psi_s(q)) + \left.\frac{d}{dz}\right|_0 (Yf)(\varphi_z(q)) \\ &= (-Y(Xf))(q) + (X(Yf))(q) \\ &= ((XY - YX)f)(q) \\ &= ([X,Y](q))f. \end{aligned}$$

□

Corollary 17.2. Let $X, Y \in \Gamma(TM)$ be two complete vector fields with flows $\{\varphi_t\}_{t\in\mathbb{R}}$, $\{\psi_s\}_{s\in\mathbb{R}}$, then $[X,Y] = 0$ if and only if $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$ for all s and t.

Proof. ($\Leftarrow$): Suppose $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$ for all t, s, then for all $f \in C^\infty(M)$, we have

$$\begin{aligned} ([X,Y]f)(q) &= (L_X Y)(q)f \\ &= \left.\frac{\partial^2}{\partial t \partial s}\right|_{(0,0)} (f \circ \varphi_{-t} \circ \psi_s \circ \varphi_t)(q) \\ &= \left.\frac{\partial^2}{\partial s \partial t}\right|_{(0,0)} (f \circ \psi_s \circ \varphi_{-t} \circ \varphi_t)(q) \\ &= \left.\frac{\partial^2}{\partial s \partial t}\right|_{(0,0)} (f \circ \psi_s)(q) \\ &= 0. \end{aligned}$$

($\Rightarrow$): Suppose $0 = [X,Y] = L_X Y$, consider $\sigma(t) = (T\varphi_{-t})_{\varphi_t(q)}(Y(\varphi_t(q)))$, then we have $\sigma(0) = (T\varphi_0)(Y(q)) = Y(q)$, therefore

$$\begin{aligned} \sigma'(t) &= \left.\frac{d}{ds}\right|_{s=0} \sigma(t+s) \\ &= \left.\frac{d}{ds}\right|_0 (T\varphi_{-t-s})(Y(\varphi_s(q))) \end{aligned}$$

$$\begin{aligned}
&= \left.\frac{d}{ds}\right|_{s=0} (T\varphi_{-t})(T\varphi_{-s})(Y(\varphi_s(\varphi_t(q)))) \\
&= (T\varphi_{-t}) \left(\left.\frac{d}{ds}\right|_0 (T\varphi_{-s})_{\varphi_t(q)}(Y(\varphi_s(\varphi_t(q)))) \right) \\
&= (T\varphi_{-t}) \left(\left.\frac{d}{ds}\right|_0 (T\varphi_{-s})_{q'}(Y(\varphi_s(q'))) \right)
\end{aligned}$$

where $(T\varphi_{-s})(Y(\varphi_s(\varphi_t(q))))$ is a path in $T_{\varphi_t(q)}(M)$. Therefore, the expression is just applying a linear map onto $(L_X Y)(q')$, but this term is now just zero.

Therefore, for all t, we know that

$$Y(q) = \sigma(0) = \sigma(t) = (T\varphi_{-t})_{\varphi_t(q)}(Y(\varphi_t(q))),$$

so $(T\varphi_t)_q(Y(q)) = Y(\varphi_t(q))$, therefore $T\varphi_t \circ Y = Y \circ \varphi_t$, therefore this means Y is φ_t-related to Y, that means for all q, we know $\varphi_t(\psi_s(q)) = \psi_s(\varphi_t(q))$ for all s, t. □

We will now talk about linear algebra a bit. The blanket assumption is that all vector spaces are real and has finite dimensions.

Recall. Given vector spaces $V_1, \dots, V_n$ and U, we say $f : V_1 \times \cdots \times V_n \to U$ is multi-linear if it is linear in each slot, that is , for all i, the assignment $v \mapsto f(v_1, \dots, v_{i-1}, v, \dots, v_n)$ is a linear map.

Example 17.3.

$$\begin{aligned}
\det : (\mathbb{R}^n)^n &\to \mathbb{R} \\
(v_1, \dots, v_n) &\mapsto \det(v_1, \dots, v_n)
\end{aligned}$$

is n-linear.

Example 17.4. For any inner product g on a vector space V, the map

$$\begin{aligned}
g : V \to V &\times \mathbb{R} \\
(v_1, v_2) &\mapsto g(v_1, v_2)
\end{aligned}$$

is bilinear.

Example 17.5. If $\mathfrak{g}$ is a Lie algebra, then the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is bilinear.

Notation. We say $\mathrm{Mult}(V_1, \dots, V_n; U)$ is the set of n-linear maps $f : V_1 \times \cdots \times V_n \to U$.

Fact. $\mathrm{Mult}(V_1, \dots, V_n; U)$ is an $\mathbb{R}$-vector space.

Lemma 17.6. Let V, W, U be three vector spaces with bases $\{v_i\}$, $\{w_j\}$, and $\{u_k\}$, respectively, and let $\{v_i^*\}$, $\{w_j^*\}$, and $\{u_k^*\}$ be their duals, respectively. We now define

$$\begin{aligned}
\varphi_{ij}^k : V \times W &\to U \\
(v, w) &\mapsto v_i^*(v) \cdot w_j^*(w) \cdot u_k \\
(-, \cdot) &\mapsto v_i^*(-) \cdot w_j^*(\cdot) u_k,
\end{aligned}$$

then $\{\varphi_{ij}^k\}$ is a basis of $\mathrm{Mult}(V, W; U)$.

Proof. Given a bilinear map $b : V \times W \to U$ with $(x, y) \in V \times W$, then

$$\begin{aligned}
b(x, y) &= b(\sum v_i^*(x) y_j, \sum w_j^*(y) w_j) \\
&= \sum_{i,j} v_i^*(x) w_j^*(y) b(v_i, w_j)
\end{aligned}$$

$$\begin{aligned} &= \sum_{i,j,k} v_j^*(x) w_j^*(y) u_k^*(b(v_i, w_j)) u_k \\ &= \sum_{i,j,k} u_k^*(b(v_i, w_j)) \varphi_{ij}^k(x, y), \end{aligned}$$

therefore $\{\varphi_{ij}^k\}$ spans $\mathrm{Mult}(V, W; U)$.

Suppose $\sum\limits_{i,j,k} c_k^{ij} \varphi_{ij}^k = 0$, then for all r, l, we know $\varphi_{ij}^k(v_r, w_l) = v_i^*(v_r) w_j^*(w_l) u_k = \delta_{ir}\delta_{jl} u_k$, so

$$0 = \sum c_k^{ij} \varphi_{ij}^k(v_r, w_l) = \sum_{i,j,k} c_k^{ij} \delta_{ir}\delta_{il} u_k = \sum_k c_k^{rl} u_k.$$

□

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Definition 18.1. Let V and W be two (finite-dimensional) vector spaces over $\mathbb{R}$. The tensor product $V \otimes W$ of V and W is a vector space together with a unique bilinear map

$$\begin{aligned} \otimes : V \times W &\to V \otimes W \\ (v, w) &\mapsto v \otimes w \end{aligned}$$

with the following universal property: for any bilinear map $b : V \times W \to U$, there exists a unique linear map $\bar{b} : V \otimes W \to U$ so that the diagram

$$\begin{tikzcd} V \otimes W \arrow[r, "\bar{b}"] & U \\ V \times W \arrow[u, "\otimes"] \arrow[ur, "b"'] & \end{tikzcd}$$

commutes, i.e., $b(v, w) = \bar{b}(v \otimes w)$ for all $(v, w) \in V \times W$.

Lemma 18.2. For any two vector spaces V and W, the tensor product $V \otimes W$ with respect to $\otimes : V \times W \to V \otimes W$ exists and is unique up to unique isomorphism.

Corollary 18.3. For any three vector spaces U, V, and W, the map

$$\begin{aligned} \varphi : \mathrm{Hom}(V \otimes W, U) &\to \mathrm{Mul}(V, W; U) \\ A &\mapsto \varphi(A) = A \circ \otimes \end{aligned}$$

is an isomorphism of vector spaces.

Proof. The uniqueness follows from the universal property. To prove existence, recall that for any set X, there is a construction of free vector space which has a copy of X as a basis. Define the tensor product to be the categorical product quotiented out by the obvious equivalence relations, given by additions and scalar multiplications, then this gives a tensor product construction over the free vector space. To prove the universal property, write down the canonical mapping, then the bilinear map $b : V \times W \to U$ induces $\bar{b} : F(V \times W) \to U$, then it satisfies the universal property and we are done. □

Lemma 18.4. For any two finite-dimensional vector spaces V and W, then $V \otimes W$ is a finite-dimensional vector space and $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$.

Proof. We know $\mathrm{Hom}(V \otimes W, \mathbb{R}) = \mathrm{Mult}(V, W; \mathbb{R})$, and we know that $\dim(\mathrm{Mult}(V, W; \mathbb{R})) = \dim(V) \cdot \dim(W) \cdot \dim(\mathbb{R})$, therefore $\dim(\mathrm{Hom}(V \otimes W, \mathbb{R})) < \infty$, so $\dim(V \otimes W) < \infty$, and then $\dim(V \otimes W) = \dim(\mathrm{Hom}(V \otimes W, \mathbb{R})) = \dim(V) \cdot \dim(W)$. □

Corollary 18.5. If $\{v_i\}_{i=1}^n$ is a basis of V and $\{w_j\}_{j=1}^m$ a basis of W, then $\{v_i \otimes w_j\}$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$ is a basis of $V \otimes W$.

Proof. By construction of the tensor product, we know this set spans $V \otimes W$ already. For any element $x \otimes y \in V \otimes W$, then write down each element with respect to the basis, reorder them, then we get a sum with respect to the given basis $\{v_i \otimes w_j\}$, and we know this spans indeed. Moreover, the dimension matches and we are done. $\square$

Lemma 18.6. There exists a unique linear map

$$\begin{aligned} T : V \otimes W &\to W \otimes V \\ v \otimes w &\mapsto w \otimes v \end{aligned}$$

for all $v \in V$ and $w \in W$.

Proof. The uniqueness is easy: this is given by the assignment. To show the existence, consider

$$\begin{aligned} b : V \times W &\to W \otimes V \\ (v, w) &\mapsto w \otimes v \end{aligned}$$

which is a bilinear map and then take the universal property and we are done. $\square$

Remark 18.7. T is an isomorphism, and the tensor product $\otimes$ gives rise to a symmetric monoidal category structure on the category of vector spaces.

Lemma 18.8. For any two finite-dimensional vector space V and W, there exists a unique linear map

$$\begin{aligned} \varphi : V^* \otimes W &\to \operatorname{Hom}(V, W) \\ l \otimes w &\mapsto l(-)w. \end{aligned}$$

Proof. Consider the bilinear map

$$\begin{aligned} b : V^* \times W &\to \operatorname{Hom}(V, W) \\ (l, w) &\mapsto l(-)w \end{aligned}$$

then by the universal property φ is the unique linear map as specified above. This is an isomorphism if we check the basis. $\square$

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Remark 19.1. The universal property of $\otimes$ can be explained by 1) the universal property over bilinear maps; 2) the universal property over categorical product; 3) the natural bijection between bilinear maps to U and homomorphisms to U.

Remark 19.2. If V and W are finite-dimensional, then there exists a natural transformation

$$\begin{aligned} V^* \otimes W^* &\xrightarrow{\sim} \operatorname{Mult}(V, W; \mathbb{R}) \\ l \otimes \eta &\mapsto l(-)\eta(-) \end{aligned}$$

Remark 19.3. Since $\operatorname{Mult}(V, W; \mathbb{R}) \cong \operatorname{Hom}(V \otimes W, \mathbb{R}) = (V \otimes W)^*$, so $(V \otimes W)^* \cong V^* \otimes W^*$.

Recall. An $\mathbb{R}$-algebra is a vector space A with a bilinear map $\circ : A \times A \to A$. An algebra A is associative if $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in A$.

Definition 19.4. An $(\mathbb{Z}_{\geqslant 0})$-*graded vector space* A is a sequence of vector spaces $\{V_i\}_{i \geqslant 0}$. Equivalently, a graded vector space V is a direct sum $V = \bigoplus_{i=0}^{\infty} V_i$.

Recall.

$$\bigoplus_{i=0}^{\infty} V_i = \left\{ \{v_i\}_{i=0}^{\infty} \mid v_i \in V_i, v_i = 0 \text{ for all but finitely many } i \right\}.$$

Definition 19.5. A $(\mathbb{Z}_{\geqslant 0})$-*graded algebra* is a graded vector space $A = \bigoplus_{i\geqslant 0} A_i$ together with a bilinear map $\circ : A \times A \to A$ such that for all i, j, $a_i \in A_i$ and $a_j \in A_j$, $a_i \circ a_j \in A_{i+j}$.

We are mostly interested in two types of graded associative algebras:

- the tensor algebra of a vector space V, given by $\mathcal{T}(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$, and
- the Grassmannian/exterior algebra $\bigwedge^*(V) = \bigoplus_{i=0}^{\infty} \bigwedge^k V$.

Definition 19.6. We define the *exterior algebra* as follows: $V^{\otimes 0} = \mathbb{R}$, $V^{\otimes 1} = V$, and $V^{\otimes 2} = V \otimes V$. For $k > 2$, there exists a unique (up to isomorphism) vector space $V^{\otimes k}$ together with a k-linear map

$$\begin{aligned} \otimes^k : V^k &\to V^{\otimes k} \\ (v_1, \dots, v_k) &\mapsto v_1 \otimes v_2 \otimes \cdots \otimes v_k, \end{aligned}$$

so that it satisfies the following universal property, that is, for any vector space U, we have $\mathrm{Hom}(V^{\otimes k}, U) = \mathrm{Mult}(V^k = (V, \dots, V); U)$. To define each of them, we can

- either define it inductively, using the fact that tensor products are associative up to unique isomorphism, or
- we construct it using the free vector space, that is, $V^{\otimes k} = F(V^k)/S$ where S is an appropriate subspace, imitating the construction of the tensor product. Therefore, we want $\bigotimes^k(v_1, \dots, v_k) = \delta_{(v_1,\dots,v_k)} + S \cdots$

Remark 19.7. Consider the tensor product $\mathbb{R}^2 \otimes \mathbb{R}^2$. We have

$$\begin{aligned} (1,1) \otimes (1,-1) &= ((1,0) + (0,1)) \otimes ((1,0) + (0,-1)) \\ &= (1,0) \otimes (1,0) - (0,1) \otimes (0,1) - (1,0) \otimes (0,1) + (0,1) \otimes (1,0) \\ &= \cdots \end{aligned}$$

Definition 19.8. To make $\mathcal{T}(V) = \bigoplus V^{\otimes k}$ into an (associative) algebra, we need bilinear maps $\circ_{k,l} : V^{\otimes k} \times V^{\otimes l} \to V^{\otimes (k+l)}$. We would want

$$(v_1 \otimes \cdots \otimes v_k) \circ_{k,l} (v_{k+1} \otimes \cdots \otimes v_{k+l}) = v_1 \otimes \cdots \otimes v_k \otimes v_{k+1} \otimes \cdots \otimes v_{k+l}.$$

To start with, we take $k, l \geqslant 1$,

$$\begin{aligned} \varphi : V^k \times V^l &\to V^{\otimes (k+l)} \\ ((v_1, \dots, v_k), (v_{k+1}, \dots, v_{k+l})) &\mapsto v_1 \otimes \cdots \otimes v_k \otimes \cdots \otimes v_{k+l}, \end{aligned}$$

then this is a $(k+l)$-linear map. We now fix $(v_{k+1}, \dots, v_{k+l}) \in V^l$, then

$$\begin{aligned} \varphi_{(v_{k+1},\dots,v_{k+l})} : V^k &\to V^{\otimes (k+l)} \\ (v_1, \dots, v_k) &\mapsto v_1 \otimes \cdots \otimes v_k \otimes v_{k+1} \otimes \cdots \otimes v_{k+l} \end{aligned}$$

which is k-linear, then by universality there exists a unique map $\bar{\varphi}_{(v_{k+1},\dots,v_{k+l})} : V^{\otimes k} \to V^{\otimes (k+l)}$, then for any each fixed t in $V^{\otimes k}$, we get a map

$$\begin{aligned} V^l &\to V^{\otimes (k+l)} \\ (v_{k+1}, \dots, v_{k+l}) &\mapsto \bar{\varphi}_{(v_{k+1},\dots,v_{k+l})}(t) \end{aligned}$$

and therefore we get a bilinear map

$$\circ_{k,l} : V^{\otimes k} \times V^{\otimes l} \to V^{\otimes (k+l)}$$

with $(v_1 \otimes \cdots v_k) \circ_{k,l} (v_{k+1}, \ldots, v_{k+l}) = v_1 \otimes \cdots \otimes v_{k+l}$. It now remains to check that for all k, l, m, we have

$$\begin{tikzcd} & V^{\otimes k} \times V^{\otimes l} \times V^{\otimes m} \arrow[dl, "\circ_{k,l} \times \mathrm{id}"'] \arrow[dr, "\mathrm{id} \times \circ_{l,m}"] & \\ V^{\otimes(k+l)} \times V^{\otimes m} \arrow[dr, "\circ_{k+l,m}"'] & & V^{\otimes k} \times V^{\otimes(l+m)} \arrow[dl, "\circ_{k,l+m}"] \\ & V^{\otimes(k+l+m)} & \end{tikzcd}$$

To show this, we just have to check on the generators, since all maps are already well-defined. It is enough to check on generators, given by

$$\begin{tikzcd} & (v_1 \otimes \cdots \otimes v_k, v_{k+1} \otimes \cdots \otimes v_{k+l}, v_{k+l+1} \otimes \cdots \otimes v_{k+l+m}) \arrow[dl] \arrow[dr] & \\ (v_1 \otimes \cdots \otimes v_{k+l}, v_{k+l+1} \otimes \cdots \otimes v_{k+l+m}) \arrow[dr] & & (v_1 \otimes \cdots \otimes v_k, v_{k+1} \otimes \cdots \otimes v_{k+l+m}) \arrow[dl] \\ & v_1 \otimes \cdots \otimes v_{k+l+m} & \end{tikzcd}$$

Therefore, this proves associativity.

Remark 19.9. We can think of TV as an associative algebra freely generated by elements in degree 1, which is just V.

Definition 19.10. The *Grassmannian/exterior algebra* on a vector space V is a graded-commutative associative algebra $\bigwedge^* V = \bigoplus_{k=0}^{\infty} \bigwedge^k V$ with an injective linear map $i : V \hookrightarrow \bigwedge^* V$ so that $\bigwedge^0 V = \mathbb{R}$, $i(V) = \bigwedge^1 V$, that has the following universal property: for any associative algebra A, for all linear map $j : V \to A$ such that $j(v) \cdot j(v) = 0$ for all $v \in V$, there exists a unique map of algebras (i.e., linear map that preserves multiplications) $\bar{j} : \bigwedge^* V \to A$ such that

$$\begin{tikzcd} \bigwedge^* V \arrow[r, dashed, "\exists ! \bar{j}"] & A \\ V \arrow[u, hook, "i"] \arrow[ur, "j"'] & \end{tikzcd}$$

Remark 19.11. The pair $(\bigwedge^* V, i : V \hookrightarrow \bigwedge^* V)$ is unique up to a unique isomorphism.

20 Oct 6, 2023

Definition 20.1. A graded associative algebra $A = \bigoplus_{k \geqslant 0} A_k$ is *graded-commutative* if for all k, l, $a \in A_k$, $b \in A_l$, then $ab = (-1)^{kl} ba$.

Definition 20.2. Let V be a finite-dimensional vector space, the *Grassmannian/exterior algebra* $\bigwedge^* V = \bigoplus_{k \geqslant 0} \bigwedge^k V$ of V is a graded-commutative algebra freely generated by $\bigwedge^1 V = V$. The term "freely generated" has the following universal property: for any unital associative algebra A and any linear map $j : V \to A$ such that $(j(v))^2 = 0$ for all $v \in V$, then there exists a unique map of algebras $\bar{j} : \bigwedge^* V \to A$ such that the restriction $\bar{j}|_{\bigwedge^1 V = V} = j$. That is, we have a commutative diagram

$$\begin{tikzcd} \bigwedge^* V \arrow[r, dashed, "\exists ! \bar{j}"] & A \\ V \arrow[u, hook] \arrow[ur, "j"'] & \end{tikzcd}$$

Remark 20.3. Analogously, the tensor algebra $\mathcal{T}(V)$ is the associative algebra freely generated by elements in $V^{\otimes 1} = V$.

Remark 20.4.
- Being unital means there exists $1_A \in A$ such that $1_A a = a 1_A = a$ for all $a \in A$.

- $(j(v))^2 = 0$ for all v implies that $j(v_1)j(v_2) = -j(v_2)j(v_1)$ for all $v_1, v_2 \in V$. Indeed, we have

$$\begin{aligned} 0 &= j(v_1+v_2)j(v_1+v_2) \\ &= (j(v_1)+j(v_2))(j(v_1)+j(v_2)) \\ &= (j(v_1))^2 + j(v_2)j(v_1) + j(v_1)j(v_2) + (j(v_2))^2 \\ &= j(v_2)j(v_1) + j(v_1)j(v_2). \end{aligned}$$

Remark 20.5 (Existence of $\bigwedge^* V$)**.** Consider the two-sided ideal I in $\mathcal{T}(V)$ generated by $\{v \otimes v \mid v \in V\}$. Therefore, I is the $\mathbb{R}$-span of elements of the form $a \otimes v \otimes v \otimes b$ where $v \in V$, $a, b \in \mathcal{T}(V)$. Since I is generated by elements of degree 2, then $I = \bigoplus_{k \geqslant 0} I_k$ where $I_k = I \cap V^{\otimes k}$ is a graded ideal of degree k. Note $I_0 = I \cap V^{\otimes 0} = 0$; $I_1 = I \cap V = 0$. We construct $\bigwedge^* V = \mathcal{T}(V)/I$ to be an associative algebra. Denote the multiplication of $\bigwedge^* V$ by $\wedge$ where $(a+I)\wedge(b+I) = a \otimes b + I$ for all $a, b \in V$. In particular, $\bigwedge^k V = V^{\otimes k}/I_k$, and so $\bigwedge^* V = \bigoplus_{k \geqslant 0} \bigwedge^k V$.

Notation. We denote $v_1 \wedge \cdots \wedge v_k := v_1 \otimes \cdots \otimes v_k + I$ for all $v_1, \dots, v_k \in V$. This identifies $v \mapsto v + I$. With this abuse of notation, $v \wedge v + I = 0 + I = 0$. Therefore, $v \wedge w = -w \wedge v$ for all $v, w \in V$, which satisfies graded-commutativity.

Remark 20.6 (Uniqueness of $\bigwedge^* V$)**.** Suppose A is a unital associative algebra, and $j : V \to A$ is a linear map with $(j(v))^2 = 0$ for all $v \in V$. Consider

$$\begin{aligned} V^n &\to A \\ (v_1, \dots, v_n) &\mapsto j(v_1)\cdots j(v_n). \end{aligned}$$

This is n-linear, hence gives rise to a unique linear map $\tilde{j}^n : V^{\otimes n} \to A$ with $\tilde{j}^n(v_1 \otimes \cdots \otimes v_n) = j(v_1)\cdots j(v_n)$, hence we get a morphism $\tilde{j} : \bigoplus_{n \geqslant 0} V^{\otimes n} \to A$ of algebras. For all $v \in V$, $\tilde{j}(v \otimes v) = j(v)j(v) = 0$, so there exists a unique $\bar{j} : \bigoplus_{n \geqslant 0} V^{\otimes n}/I \to A$ such that $\bar{j}(v_1 \wedge \cdots \wedge v_n) = j(v_1)\cdots j(v_n)$, by the first isomorphism theorem.

Remark 20.7. Recall in $\bigwedge^* V$ we have $v \wedge w = (-1)w \wedge v$ for $v, w \in V$ since they have degree 1. In general, we have

$$\begin{aligned} (v_1 \wedge \cdots \wedge v_k) \wedge (v_{k+1} \wedge \cdots \wedge v_{k+l}) &= v_1 \wedge \cdots \wedge v_k \wedge v_{k+1} \wedge \cdots \wedge v_{k+l} \\ &= (-1)^k v_{k+1} \wedge v_1 \wedge \cdots \wedge v_k \wedge v_{k+2} \wedge \cdots \wedge v_{k+l} \\ &= (-1)^{kl}(v_{k+1} \wedge \cdots \wedge v_{k+l}) \wedge (v_1 \wedge \cdots \wedge v_k) \end{aligned}$$

and therefore $\bigwedge^* V$ is graded-commutative.

Recall. The permutation group S_n is generated by transpositions $(i\ j)$ for $1 \leqslant i < j \leqslant n$. In fact, it is generated by $(1\ 2), (2\ 3), \cdots, (n-1\ n)$.

Lemma 20.8. Let V be a finite-dimensional vector space and let $n \geqslant 2$, then take $v_1, \dots, v_n \in V$. For any permutation $\sigma \in S_n$, we have $v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)} = (\operatorname{sgn}(\sigma))v_1 \wedge \cdots \wedge v_n$.

Proof. It suffices to check when $\sigma = (i\ i+1)$, which is obvious. □

Corollary 20.9. Let $v_1, \dots, v_n$ be a basis of a finite-dimensional vector space V, then

1. $\bigwedge^k V = 0$ for $k > n$,
2. elements of kth exterior power $\{v_{i_1} \wedge \cdots \wedge v_{i_k} \mid i_1 < i_2 < \cdots < i_k\}$ spans $\bigwedge^k V$.

Proof. We know $\{v_i \otimes v_j \mid 1 \leqslant i, j \leqslant n\}$ is a basis of $V \otimes V = V^{\otimes 2}$. Proceeding by induction on k, we know $\{v_{i_1} \otimes \cdots \otimes v_{i_k} \mid 1 \leqslant i_1, \dots, i_k \leqslant n\}$ is a basis of $V^{\otimes k}$, therefore $\{v_{i_1} \wedge \cdots \wedge v_{i_k} \mid 1 \leqslant i_1, \dots, i_k \leqslant n\}$ spans $\bigwedge^k V = V^{\otimes n}/I_k$.

If $k > n$, we must have repeated indices in $v_{i_1} \wedge \cdots \wedge v_{i_k}$, therefore this is zero: if we permute the indices, we can ask the two repeated indices stand next to each other, and in particular their wedge is zero, therefore the entire term would be zero. We will prove that $\{v_{i_1} \wedge \cdots \wedge v_{i_k} \mid 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n\}$ is a basis of $\bigwedge^k V$. The key is $v_1 \wedge \cdots \wedge v_n \neq 0$. □

21 Oct 9, 2023

First, we will show that $v_1 \wedge \cdots \wedge v_n \neq 0$.

Definition 21.1. Let V, U be two vector spaces. A k-linear map $f : V^k \to U$ is said to be *alternating* if for all $\sigma \in S_k$, $f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma) f(v_1, \ldots, v_k)$.

Example 21.2. For all $l_1, l_2 : V \to \mathbb{R}$, the map

$$\begin{aligned} f : V \times V &\to \mathbb{R} \\ (v_1, v_2) &\mapsto l_1(v_1)l_2(v_2) - l_1(v_2)l_2(v_1) = \det \begin{pmatrix} l_1(v_1) & l_1(v_2) \\ l_2(v_1) & l_2(v_2) \end{pmatrix} \end{aligned}$$

Notation. We denote $\operatorname{Alt}^n(V; U)$ to be the set of maps $f : V^n \to U$ where f is alternating.

Proposition 21.3. For any $n \geqslant 2$, for all $f \in \operatorname{Alt}^n(V; U)$, there exists a unique linear map $\bar{f} : \bigwedge^n V \to U$ such that $\bar{f}(v_1 \wedge \cdots \wedge v_n) = f(v_1, \ldots, v_n)$.

Proof. Since f is n-linear, there exists a unique linear map $\tilde{f} : V^{\otimes n} \to U$ such that $\tilde{f}(v_1 \otimes \cdots \otimes v_n) = f(v_1, \ldots, v_n)$. Recall that $\bigwedge^n V = V^{\otimes n}/I_n$ where I_n is the intersection of $V^{\otimes n}$ and the ideal generated by $\{v \otimes v \mid v \in V\}$. Since f is alternating, then $\tilde{f}\Big|_{I_n} = 0$, so there exists a linear map $\bar{f} : \bigwedge^n V \to U$ such that $\bar{f}(v_1 \wedge \cdots \wedge v_n) = f(v_1, \ldots, v_n)$ for all $v_1, \ldots, v_n \in U$. Since $\{v_1 \wedge \cdots \wedge v_n \mid v_i \in V\}$ generates $\bigwedge^n V$, then $\bar{f}$ is unique. □

Lemma 21.4. Suppose $\{v_1, \ldots, v_n\}$ is a basis of V, then $v_1 \wedge \cdots \wedge v_n \neq 0$, hence $\dim(\bigwedge^n V) = 1$ and $\bigwedge^n V \cong \mathbb{R}$.

Proof. Take the dual basis $\{v_1^*, \ldots, v_n^*\}$, and consider

$$\begin{aligned} f : V^n &\to \mathbb{R} \\ (x_1, \ldots, x_n) &\mapsto \det \begin{pmatrix} v_1^*(x_1) & \cdots & v_1^*(x_n) \\ \vdots & & \vdots \\ v_n^*(x_1) & \cdots & v_n^*(x_n) \end{pmatrix} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n v_i^*(x_{\sigma(i)}). \end{aligned}$$

Therefore, f is alternating. Hence, there exists a unique

$$\begin{aligned} \bar{f} : \bigwedge^n V &\to \mathbb{R} \\ x_1 \wedge \cdots \wedge x_n &\mapsto \det(v_i^*(x_j)) \end{aligned}$$

and such that $\bar{f}(v_1 \wedge \cdots \wedge v_n) = \det(\operatorname{diag}(1, \ldots, 1)) = 1$, hence $v_1 \wedge \cdots \wedge v_n \neq 0$. □

Corollary 21.5. Let $\{v_1, \ldots, v_n\}$ be a basis of V, then for any $1 \leqslant k \leqslant n$, the generating set $\mathcal{B} = \{v_{i_1} \wedge \cdots \wedge v_{i_k} \mid 1 \leqslant i_1 < \cdots < i_k \leqslant n\}$ is a basis of $\bigwedge^k V$.

Proof. We know $\mathcal{B}$ spans $\bigwedge^k V$. Suppose $\sum_{i_1 < \cdots < i_k} a_{i_1, \ldots, i_k} v_{i_1} \wedge \cdots \wedge v_{i_k} = 0$. Fix $1 \leqslant i_1^\circ < \cdots < i_k^\circ \leqslant n$. Let $1 \leqslant j_{k+1} < \cdots < j_n \leqslant n$ denote the complementary set of the indices, i.e., $\{i_1^\circ, \ldots, i_k^\circ\} \cap \{j_{k+1}, \ldots, j_n\} = \varnothing$, then for all $1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant n$, we have

$$v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge v_{j_{k+1}} \wedge \cdots \wedge v_{j_n} = \begin{cases} 0, & (i_1, \ldots, i_k) \neq (i_1^\circ, \ldots, i_k^\circ) \\ \pm v_1 \wedge \cdots \wedge v^n, & (i_1, \ldots, i_k) = (i_1^\circ, \ldots, i_k^\circ) \end{cases}$$

Therefore, $\left(\sum_{i_1 < \cdots < i_k} a_{i_1, \ldots, i_k} v_{i_1} \wedge \cdots \wedge v_{i_k} \right) \wedge (v_{j_{k+1}} \wedge \cdots \wedge v_{j_n}) = \pm a_{i_1^\circ, \ldots, i_k^\circ} v_1 \wedge \cdots \wedge v_k$, but $v_1 \wedge \cdots \wedge v_k \neq 0$, so $a_{i_1^\circ, \ldots, i_k^\circ} = 0$ since $\{v_1 \wedge \cdots \wedge v_n\}$ is a basis. □

Corollary 21.6. Suppose $\dim(V) = n$, then for all $1 \leqslant k \leqslant n$, $\dim(\bigwedge^k V) = \binom{n}{k}$. Consequently, $\dim(\bigwedge^* V) = 2^n$.

Lemma 21.7. Let $f : V \to W$ be a linear map, then there exists a unique $\bigwedge^*(f) : \bigwedge^* V \to \bigwedge^* W$ of graded commutative algebras so that for all k and for all $v_1, \ldots, v_k \in V$,

$$(\bigwedge^* f)(v_1 \wedge \cdots \wedge v_k) = f(v_1) \wedge \cdots \wedge f(v_k).$$

In particular, note that $\bigwedge^*(f)(\bigwedge^k V) \subseteq \bigwedge^k W$.

Proof. Note that $V \xrightarrow{f} W = \bigwedge^1 W \subseteq \bigwedge^* W$, and for any $v \in V$, $f(v) \wedge f(v) = 0$. Therefore, there exists a unique map $\bigwedge^* f : \bigwedge^* V \to \bigwedge^* W$ such that $\bigwedge^* f\big|_{\bigwedge^1 V} = f$. Moreover, for any k, and any $v_1, \ldots, v_k \in V$, we know

$$\begin{aligned}\bigwedge^* f(v_1 \wedge \cdots \wedge v_k) &= \bigwedge^* f(v_1) \wedge \cdots \wedge \bigwedge^* f(v_k) \\ &= f(v_1) \wedge \cdots \wedge f(v_K).\end{aligned}$$

□

Remark 21.8. Uniqueness of $\bigwedge^* f$ implies that if we have two linear maps

$$V \xrightarrow{f} W \xrightarrow{g} U$$

and then $\bigwedge^*(g \circ f) = \bigwedge^*(g) \circ \bigwedge^*(f)$. Moreover, $\bigwedge^*(\mathrm{id}_V) = \mathrm{id}_{\bigwedge^* V}$. In other words, there is a functor

$$\bigwedge^*(-) : \mathbf{Vect} \to \mathbf{CGA},$$

from the category of finite-dimensional real vector spaces with linear maps as morphisms, to the category of graded commutative algebras over $\mathbb{R}$.

Remark 21.9. The map $V \to \mathcal{T}(V)$ also extends to a functor

$$\mathcal{T}(-) : \mathbf{Vect} \to \mathbf{GAA}$$

from the category of finite-dimensional real vector spaces to the category of graded associative algebras. In particular, it sends $f : V \to W$ to $\mathcal{T} : \mathcal{T}(V) \to \mathcal{T}(W)$ that maps $v_1 \otimes \cdots \otimes v_n$ to $f(v_1) \otimes \cdots \otimes f(v_k)$ for all k and for all $v_1, \ldots, v_K \in V$.

Remark 21.10. For each $k \geqslant 0$, we also have functors

$$\bigwedge^k(-) : \mathbf{Vect} \to \mathbf{Vect}$$

that takes a linear map $f : V \to W$ and sends it to $\bigwedge^k f : \bigwedge^k V \to \bigwedge^k W$, as well as

$$(-)^{\otimes k} : \mathbf{Vect} \to \mathbf{Vect}$$

that sends $f : V \to W$ to $f^{\otimes k} : V^{\otimes k} \to W^{\otimes k}$.

Lemma 21.11. For any two finite-dimensional vector spaces V and U, for all k, we have an isomorphism

$$\begin{gathered}\mathrm{Hom}(\bigwedge^k V, U) \to \mathrm{Alt}^k(V; U) \\ (\varphi : \bigwedge^k V \to U) \mapsto (\varphi \circ i^{(k)} : V^k \to U)\end{gathered}$$

where

$$\begin{gathered}i^{(k)} : V^k \to \bigwedge^k V \\ (v_1, \ldots, v_k) \mapsto v_1 \wedge \cdots \wedge v_k.\end{gathered}$$

Proof. Same as Proposition 21.3. □

22 Oct 13, 2023

Recall. If $\{d_1, \dots, d_n\}$ is a basis of V, then for all $1 \leqslant k \leqslant n$, $\{\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k} \mid 1 \leqslant i_1 < \cdots < i_k \leqslant n$ is a basis$\}$ of $\bigwedge^k V$.

For vector spaces V and U, we have

$$\begin{aligned} i^{(k)} : V^k &\to \bigwedge^k V \\ (v_1, \dots, v_k) &\mapsto v_1 \wedge \cdots \wedge v_k, \end{aligned}$$

then

$$\begin{aligned} \mathrm{Hom}(\bigwedge^k V, U) &\to \mathrm{Alt}^k(V, U) \\ \varphi &\mapsto \varphi \circ i^{(k)} \end{aligned}$$

is an isomorphism. The inverse if $f \mapsto \bar{f}$ where $\bar{f}(v_1 \wedge \cdots \wedge v_k) = f(v_1, \dots, v_k)$ for all v_i's.

Remark 22.1. Lemma 21.11 says that $(\bigwedge^k V)^* = \mathrm{Hom}(\bigwedge^k V, \mathbb{R}) \cong \mathrm{Alt}^k(V, \mathbb{R})$.

Lemma 22.2. let V be a finite-dimensional vector space, then for all $1 \leqslant k \leqslant n$, we have

$$\bigwedge^k V^* \cong (\bigwedge^k V)^* \cong \mathrm{Alt}^k(V; \mathbb{R}).$$

Proof. Consider $\mathrm{Map}(V^k, \mathbb{R})$ be the set of all maps from V^k to $\mathbb{R}$. Note that the multilinear maps $\mathrm{Alt}^k(V, \mathbb{R}) \subseteq \mathrm{Mult}^k(V, \mathbb{R}) \subseteq \mathrm{Map}(V^k, \mathbb{R})$. Consider

$$\begin{aligned} \varphi : (V^*)^k &\to \mathrm{Map}(V^k, \mathbb{R}) \\ (\varphi(l_1, \dots, l_k))(v_1, \dots, v_k) &= l_1(v_1) \cdots l_k(v_k) =: \det(l_i(v_j)). \end{aligned}$$

for all $v_1, \dots, v_k \in V$ and $l_1, \dots, l_k \in V^*$. For fixed $l_1, \dots, l_k$, $\varphi(l_1, \dots, l_k)$ is k-linear and alternating, so $\varphi(l_1, \dots, l_k) \in \mathrm{Alt}^k(V, \mathbb{R})$. Thus we have

$$\begin{aligned} \varphi : (V^*)^k &\to \mathrm{Alt}^k(V; \mathbb{R}) \\ (l_1, \dots, l_k) &\mapsto ((v_1, \dots, v_k) \mapsto \det(l_i(v_j))) \end{aligned}$$

Since φ is k-linear in $l_1, \dots, l_k$, therefore we have another map

$$\begin{aligned} \tilde{\varphi} : (V^*)^{\otimes k} &\to \mathrm{Alt}^k(V, \mathbb{R}) \\ (\tilde{\varphi}(l_1 \otimes \cdots \otimes l_k))(v_1, \dots, v_k) &= \det(l_i(v_j)). \end{aligned}$$

Note that $\tilde{\varphi}$ vanishes if any two l_i's are repeated, so there is a unique map

$$\begin{aligned} \bar{\varphi} : \bigwedge^k V^k &\to \mathrm{Alt}^k(V, \mathbb{R}) \\ \bar{\varphi}(l_1 \wedge \cdots \wedge l_k)(v_1, \dots, v_k) &= \det(l_i(v_j)). \end{aligned}$$

Composing with the isomorphism $\mathrm{Alt}^k(V, \mathbb{R}) \to (\bigwedge^k V)^*$, we get

$$\begin{aligned} \psi : \bigwedge^k (V^k) &\to (\bigwedge^k V)^* \\ (\psi(l_1 \wedge \cdots \wedge l_k))(v_1 \wedge \cdots \wedge v_k) &= \det(l_i(v_j)). \end{aligned}$$

It remains to show that ψ is an isomorphism. Pick a basis $\{\alpha_1, \dots, \alpha_n\}$ of V, with dual basis $\{\alpha_1^*, \dots, \alpha_n^*\}$ of V^*. Let $\mathcal{A} = \{\alpha_{j_1}^* \wedge \cdots \wedge \alpha_{j_k}^* \mid 1 \leqslant j_1 < \cdots < j_k \leqslant n\}$ and let $\mathcal{B} = \{\alpha_{i_1} \wedge \cdots \alpha_{i_k} \mid 1 \leqslant i_1 < \cdots < i_k \leqslant n\}$ of $\bigwedge^k (V)$. We have

$$(\psi(\alpha_{j_1}^* \wedge \cdots \wedge \alpha_{j_k}^*))(\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k}) = \det(\alpha_{j_r}^*(\alpha_{i_s}))_{s,r}$$

$$
\begin{aligned}
&= \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma)\prod_{r=1}^{k}\alpha_{j_r}^*(\alpha_{i_{\sigma(r)}}) \\
&= \begin{cases} 1, & (j_1,\ldots,j_k)=(l_1,\ldots,l_k) \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

Hence, ψ is an isomorphism. □

Remark 22.3. For $\alpha\in\operatorname{Alt}^k(V,\mathbb{R})$ and $\beta\in\operatorname{Alt}^l(V,\mathbb{R})$, then we have

$$
\begin{aligned}
\alpha\beta : V^k\times V^l &\to \mathbb{R} \\
v_1,\ldots,v_k,v_{k+1},\ldots,v_{k+l} &\mapsto \alpha(v_1,\ldots,v_k)\beta(v_{k+1},\ldots,v_{k+l})
\end{aligned}
$$

which is $k+l$-linear but not alternating.

Example 22.4. For $k=l=1$, $\operatorname{Alt}^1(V,\mathbb{R})=V^*$, so $(\alpha\cdot\beta)(v_1,v_2)=\alpha(v_1)\beta(v_2)\neq-\alpha(v_2)\beta(v_1)$. On the other hand, $\operatorname{Alt}^k(V,\mathbb{R})\cong\bigwedge^k(V^*)$, so $\bigwedge_{k\geqslant 0}\operatorname{Alt}^k(V,\mathbb{R})\cong\bigoplus_{k=0}^{\infty}\bigwedge^k(V^*)=\bigwedge^*(V^*)$ which is a graded commutative algebra. (We set $\operatorname{Alt}^0(V,\mathbb{R})=\mathbb{R}$.) Therefore, there is a graded commutative algebra structure on the direct sum of alternating maps.

Remark 22.5. For each $n\geqslant 1$, there exists a projection

$$
\begin{aligned}
\pi : \operatorname{Mult}^n(V,\mathbb{R}) &\to \operatorname{Alt}^n(V,\mathbb{R}) \\
(\pi(\gamma))(v_1,\ldots,v_n) &= \frac{1}{n!}\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\gamma(v_{\gamma(1)},\ldots,v_{\sigma(n)}).
\end{aligned}
$$

Therefore, we could have defined a multiplication

$$
\begin{aligned}
\wedge : \operatorname{Alt}^k(V,\mathbb{R})\times\operatorname{Alt}^l(V,\mathbb{R}) &\to \operatorname{Alt}^k(V,\mathbb{R}) \\
\alpha\wedge\beta &= \pi(\alpha\beta).
\end{aligned}
$$

The issue is, we do not have associativity: $\pi(\pi(\alpha\beta)\gamma)\neq\pi(\alpha\pi(\beta\gamma))$.

Note that for any k,

$$
\begin{aligned}
\bigwedge^k(V^*) &\to \operatorname{Alt}^k(V,\mathbb{R}) \\
l_1\wedge\cdots\wedge l_k &\mapsto k!\pi(k_1(-)l_2(-)\cdots l_k(-))
\end{aligned}
$$

for all $l_1,\ldots,l_k\in V^*$. One should be cautious because

$$
\begin{aligned}
\bigwedge^k(V^*) &\to \operatorname{Alt}^k(V,\mathbb{R}) \\
l_1\wedge\cdots\wedge l_k &\mapsto \pi(k_1(-)l_2(-)\cdots l_k(-))
\end{aligned}
$$

is also used in literature.

We now want to define the cotangent bundle, but first we need to redefine the charts.

Recall. Recall the construction of charts on TM is as follows:

- Given a chart $\varphi=(x_1,\ldots,x_m):U\to\mathbb{R}^m$ on M, we define

$$
\begin{aligned}
\tilde{\varphi} : TU = TM &\to \mathbb{R}^m\times\mathbb{R}^m \\
\tilde{\varphi}(q,v) &= (x_1(q),\ldots,x_n(q),(dx_1)_q(v),\ldots,(dx_n)_q(v))
\end{aligned}
$$

  Given another chart $\psi=(y_1,\ldots,y_m)\to\mathbb{R}^n$, we have

$$
\begin{aligned}
F : \tilde{\psi}\circ(\tilde{\varphi}|_{\varphi(U\cap V)\times\mathbb{R}^m})^{-1} : \varphi(U\cap V)\times\mathbb{R}^m &\to \psi(U\cap V)\times\mathbb{R}^m \\
(a_1,\ldots,a_m,w_1,\ldots,w_m) &\mapsto (\psi(\varphi^{-1}(a_1,\ldots,a_m)),D(\psi\circ\varphi^{-1})(a)\begin{pmatrix} w_1\\ \vdots\\ w_m\end{pmatrix})
\end{aligned}
$$

- From a better point of view, let $\varphi = (x_1, \dots, x_m)$, then the map $\tilde{\varphi} : TU \to \varphi(U) \times \mathbb{R}^m$ "is" $T\varphi : TU \to T(\varphi(U))$. To see this, for all $f \in C^\infty(U)$, we have $\frac{\partial}{\partial x_i}\Big|_q f = \frac{\partial}{\partial r_i}\Big|_{\varphi(q)} (f \circ \varphi^{-1}) = ((T_{\varphi(q)}\varphi^{-1}) \left(\frac{\partial}{\partial r_i}\Big|_{\varphi(q)}\right))f$, that is, $T\varphi^{-1}\left(\frac{\partial}{\partial r_i}\right) = \frac{\partial}{\partial x_i}$, dropping the basepoint. Therefore, $T\varphi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial r_i}$. Hence, $(T_q\varphi)(\sum v_i \frac{\partial}{\partial x_i}\Big|_q) = \sum v_i \frac{\partial}{\partial r_i}\Big|_{\varphi(q)}$, From this point of view, $T\varphi : TU \to T(\varphi(U))$ is $(q, \sum v_i \frac{\partial}{\partial x_i}\Big|_q) \mapsto (\varphi(q), \sum v_i \frac{\partial}{\partial r_i}\Big|_{\varphi(q)})$. Now identify
$$T\varphi(U) \cong \varphi(U) \times \mathbb{R}^m$$
$$(r_1, \dots, r_m, \sum v_i \frac{\partial}{\partial r_i}) \mapsto (r_1, \dots, r_m, v_1, \dots, v_m),$$
and given this we can write $\tilde{\psi}\tilde{\varphi}^{-1} : T\psi \circ (T\varphi)^{-1} = T\psi \circ T\varphi^{-1} = T(\psi \circ \varphi^{-1})$ by the functoriality.

Definition 22.6. The *cotangent bundle* is defined by $T^*M = \coprod_{q \in M} (T_qM)^*$.

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Recall. Let M be a manifold, then the cotangent bundle $T^*M = \coprod_{q\in M} T_q^*M$ where $T_q^*M = \mathrm{Hom}(T_qM, \mathbb{R})$.

Remark 23.1. For a coordinate chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$, then $\left\{\frac{\partial}{\partial x_i}\Big|_q\right\}$ is a basis of $T_qM = (T_q^*M)^*$, so for all $p \in T_q^*M$, we have $p = \sum p\left(\frac{\partial}{\partial x_i}\Big|_q\right)(dx_i)_q$, then this induces
$$\bar{\varphi} : T^*U = \coprod_{q\in U} T_q^*M \to \varphi(U) \times \mathbb{R}^m$$
$$(q, p) \mapsto \left(\varphi(q), p\left(\frac{\partial}{\partial x_1}\Big|_q\right), \dots, p\left(\frac{\partial}{\partial x_m}\Big|_q\right)\right)$$
then given another coordinate chart $\psi = (y_1, \dots, y_m) : V \to \mathbb{R}^m$, we get $\bar{\psi}(q, p) = \left(\psi(q), p\left(\frac{\partial}{\partial y_1}\Big|_q\right), \dots, p\left(\frac{\partial}{\partial y_m}\Big|_q\right)\right)$.

Therefore for $(r, w) \in \varphi(U \cap V) \times \mathbb{R}^m$, we have $(\bar{\psi} \circ \bar{\varphi}^{-1})(r, w) = \left((\psi \circ \varphi^{-1})(r), ((D(\psi \circ \varphi^{-1})(r))^{-1})^T \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}\right)$.

Therefore, T^*M is a manifold and $\pi : T^M \to M$ defined by $(q, p) \mapsto q$ is a surjective submersion. Now define $\bigwedge^k(T^*M) = \coprod_{q\in M} \bigwedge^k(T_q^*M)$. To introduce coordinate charts we need multi-indices. Let $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$ be a chart on U, then $\{(dx_{i_1})_q \wedge \dots \wedge d(x_{i_k})_q \mid 1 \leqslant i_1 < \dots < i_k \leqslant m\}$ is a basis of $\bigwedge^k(T_q^*M)$. For $I = \{1 \leqslant i_1 < \dots < i_k \leqslant m\}$, set $(dx_I)_q = d_{x_1} \wedge \dots \wedge dx_{i_k}$, then the set of $dx_I\big|_q$ where I's are ordered multi-indices is a basis of $\bigwedge^k T^*M$, with dual basis $\left\{\frac{\partial}{\partial x_I}\Big|_q = \frac{\partial}{\partial x_{i_1}}\Big|_q \wedge \dots \wedge \frac{\partial}{\partial x_{i_j}}\Big|_q\right\}$ where I is an ordered multi-index, which gives us coordinate charts $\bigwedge^k(T^*U) \to \varphi(U) \times (\mathbb{R}^m)^{\binom{m}{k}}$.

Remark 23.2. To do this a better way, we saw that given $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$, we can view $\tilde{\varphi} : TU \to \varphi(U) \times \mathbb{R}^m$ as $T\varphi : TU \to T(\varphi(U))$, then given another chart ψ, we get

$$\begin{tikzcd} & T_q(U \cap V) \arrow[dl, "T\varphi"'] \arrow[dr, "T\psi"] & \\ T_{\varphi(q)}\varphi(U \cap V) \arrow[rr, dashed, "T_{\varphi(q)}(\psi\circ\varphi^{-1})"'] & & T_{\psi(q)}(\psi(U \cap V)) \end{tikzcd}$$

where $T\psi \circ T\varphi^{-1} = T_{\varphi(q)}(\psi \circ \varphi^{-1}(U \cap V))$.

Recall. For any linear map $A : V \to W$, we have

$$\begin{aligned} A^* : W^* &\to V^* \\ l &\mapsto l \circ A \end{aligned}$$

and given any composition $V \xrightarrow{A} W \xrightarrow{B} U$, we have

$$(A \circ B)^* l = l \circ (A \circ B) = (l \circ A) \circ B = B^* A^*(l).$$

Applying the contravariant functor, we get

$$\begin{tikzcd} & T_q^*(U \cap V) & \\ T^*_{\varphi(q)}\varphi(U \cap V) \arrow[ur, "(T_q\varphi)^*"] & & T^*_{\psi(q)}(\psi(U \cap V)) \arrow[ul, "(T_q\psi)^*"'] \arrow[ll, "(T_{\varphi(q)}(\psi\circ\varphi^{-1}))^*"] \end{tikzcd}$$

and taking inverses everywhere, we have

$$\begin{tikzcd} & T_q^*(U \cap V) \arrow[dl, "(T_q\varphi^{-1})"'] \arrow[dr, "(T_q\psi^{-1})"] & \\ T^*_{\varphi(q)}\varphi(U \cap V) \arrow[rr, dashed, "(T_{\psi(q)}(\psi\circ\varphi^{-1})^{-1})^*"'] & & T^*_{\psi(q)}(\psi(U \cap V)) \end{tikzcd}$$

as $(T_{\psi(q)}(\psi \circ \varphi^{-1})^{-1})^* = (T_{\varphi(q)}\varphi \circ \psi^{-1})^*$.

Note that $\bar{\psi} \circ \bar{\varphi}^{-1}$ is C^∞ because

$$\begin{aligned} \mathrm{GL}(\mathbb{R}^m) &\to \mathrm{GL}((\mathbb{R}^m)^*) \\ A &\mapsto (A^{-1})^* \end{aligned}$$

is C^∞.

For any k, we have a functor

$$\begin{aligned} \bigwedge^k(-) &: \mathbf{Vect} \to \mathbf{Vect} \\ (T : V \to W) &\mapsto (\bigwedge^k T : \bigwedge^k V \to \bigwedge^k W) \end{aligned}$$

which is defined by $(\bigwedge^k)(v_1 \wedge \cdots \wedge v_k) = (Tv_1) \wedge \cdots \wedge (Tv_k)$. Now given a chart $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ on M, we have $(T_q\varphi^{-1})^* : T_q^* = T_q^*M \cong T^*_{\varphi(q)}\mathbb{R}^m$. Therefore, $\bigwedge^k((T_q\varphi^{-1})^*) : \bigwedge^k(T_q^*M) \to \bigwedge^k(T^*_{\varphi(q)}\mathbb{R}^m) \cong \mathbb{R}^{\binom{m}{k}}$.

If $\psi = (y_1, \ldots, y_m) : V \to \mathbb{R}^n$ is another chart, then for $q \in V \cap U$ we have a commutative diagram

$$\begin{tikzcd} & T_q^*M \arrow[dl, "((T_q\varphi)^{-1})^*"'] \arrow[dr, "((T_q\psi)^{-1})^*"] & \\ T^*_{\varphi(q)}\varphi(U) \arrow[rr, "((T_{\varphi(q)}(\psi\circ\varphi^{-1}))^{-1})^*"'] & & T_{\psi(q)}\psi(U) \end{tikzcd}$$

Applying the functor, we have another commutative diagram

$$\begin{tikzcd} & \bigwedge^k(T_q^*M) \arrow[dl, "\bigwedge^k(((T_q\varphi)^{-1})^*)"'] \arrow[dr, "\bigwedge^k(((T_q\psi)^{-1})^*)"] & \\ \bigwedge^k(T^*_{\varphi(q)}\varphi(U)) \arrow[rr, "\bigwedge^k(((T_{\varphi(q)}(\psi\circ\varphi^{-1}))^{-1})^*)"'] & & \bigwedge^k(T_{\psi(q)}\psi(U)) \end{tikzcd}$$

This gives charts

$$\begin{aligned}\hat{\varphi} : \bigwedge^k (T^*U) &\to \bigwedge^k (T^*\varphi(U)) \\ (q, \alpha) &\mapsto (\varphi(q), \bigwedge^k ((T_q\varphi)^{-1})^*\alpha).\end{aligned}$$

Then corresponding transition maps takes elements of $(r, \beta) \in \bigwedge^k (T^*\varphi(U \cap V))$ to $((\psi \circ \varphi^{-1})(r), \bigwedge^k ((T_r(\psi \circ \varphi^{-1}))^{-1})^*\beta)$. The assignment $r \mapsto \bigwedge^k ((T_r(\psi \circ \varphi^{-1}))^{-1})^*$ is the exterior power of transpose of inverse of the Jacobian $D(\psi \circ \varphi^{-1}(r))$, so it suffices to check that the exterior power map is smooth, as the inverse and the transpose are both smooth. That is, we want to check

$$\begin{aligned}\mathrm{GL}((\mathbb{R}^m)^*) &\to \mathrm{GL}(\bigwedge^k ((\mathbb{R}^m)^*)) \\ B &\mapsto \bigwedge^k B\end{aligned}$$

is C^∞. Choose a basis $f_1, \ldots, f_m$ of $(\mathbb{R}^m)^*$, then $Bf_j = \sum b_{ij} f_j$, so elements of the form $f_I = f_{j_1} \wedge \cdots \wedge f_{j_k}$ is a basis of $\bigwedge^k ((\mathbb{R}^m)^*)$. Therefore,

$$\begin{aligned}(\bigwedge^k B) f_I &= Bf_{j_1} \wedge \cdots \wedge Bf_{j_k} \\ &= \left(\sum_{i_1} b_{i_1 j_1} f_{i_1} \wedge \cdots \wedge \sum_{i_k} b_{i_k j_k} f_{i_k} \right)\end{aligned}$$

which is a summation of products of polynomials in b_{ij}'s with $f_{s_1} \wedge \cdots f_{s_k}$. Therefore, the mapping we want is just a polynomial function in B, and therefore it is smooth.

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Note that $\bigwedge^k T^*M$ comes with a surjective submersion to M given by

$$\begin{aligned}\pi : \bigwedge^k T^*M &\to M \\ (q, \alpha) &\mapsto q\end{aligned}$$

For all $q \in M$, we say $\pi^{-1}(q) = \bigwedge^k (T_q M)$ is the fiber of π at q. This is a $\binom{dim(M)}{k}$-dimensional real vector space. We will see later that $\pi : \bigwedge^k T^M \to M$ is a vector bundle.

Suppose $k = 0$, then $\bigwedge^0 (T_q^* M) = \mathbb{R}$, so $\bigwedge^0 (T^*M) = M \times \mathbb{R}$.

Definition 24.1. A *differential k-form* on a manifold M is a C^∞-map $\omega : M \to \bigwedge^k (T^*M)$ such that $\omega(q) \in \bigwedge^k (T_q^* M)$ for all $q \in M$. Equivalently, $\pi \circ \omega = \mathrm{id}_M$.

Notation. We denote $\omega_q = \omega(q)$. We denote $\Omega^k(M)$ to be the space of all differential k-forms, i.e., the set of $\omega : M \to \bigwedge^k (T^*M)$ such that $\pi \circ \omega = \mathrm{id}_M$. This is a $\mathbb{R}$-vector space.

Example 24.2. $\Omega^0(M)$ is the set of $\omega : M \to M \times \mathbb{R}$ such that $\pi \circ \omega = \mathrm{id}$, i.e., the assignments $q \mapsto (q, f(q))$ where $f : M \to \mathbb{R}$ is C^∞, i.e., this is $C^\infty(M)$. Therefore, we can write $\Omega^* M = \bigoplus_{k \geqslant 0} \Omega^k(M)$ as a graded commutative algebra. For any k, l, for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$, we have $(\alpha \wedge \beta)_q := \alpha_q \wedge \beta_q$ for all q, as a wedge in exterior algebra $\bigwedge^* (T_q^* M)$.

Remark 24.3. Given a coordinate chart $(x_1, \ldots, x_m) : U \to \mathbb{R}^m$ on M, let $\alpha|_U : \sum_{|I|=k} \alpha_I dx_I$ and $\beta|_U : \sum_{|J|=l} \beta_J dx_J$, then $(\alpha \wedge \beta)|_U = \sum \alpha_I \beta_J dx_I \wedge dx_J$, so $\alpha \wedge \beta|_U$ is C^∞ for all U, thus $\alpha \wedge \beta$ is in C^∞.

Recall. For any finite-dimensional vector space V, we have $\bigwedge^k V^* \cong \mathrm{Alt}^k(V;\mathbb{R})$. Therefore, for any $q \in M$, we have $\bigwedge^k T_q^* M \cong \mathrm{Alt}^k(T_q M, \mathbb{R})$.

Remark 24.4 (Differential Form Pullback)**.** Let $F : M \to N$ be a C^∞-map between manifolds. We denote $F^* : \Omega^*(N) \to \Omega^*(M)$ be the pullback map as follows: for any $k \geqslant 0$, any $\alpha \in \Omega^k(N)$, and any $q \in M$, we define $(F^*\alpha)_q = \bigwedge^k((T_qF)^*)\alpha_{F(q)}$. Therefore, for any $T_qF : T_qM \to T_{F(q)}N$, we have $(T_qF)^* : T^*_{F(q)}N \to T^*_q M$, and so there is $\bigwedge^k((T_qF)^*) : T^*_{F(q)}N \ni \alpha_{F(q)} \to \bigwedge^k(T_q^*M)$.

If $k = 0$, $\Omega^0(N) = C^\infty(N)$, so $(F^*\alpha)_q = \alpha_{F(q)} = (\alpha \circ F)(q)$, thus $F^*\alpha = \alpha \circ F$ as a pullback of functions.

Therefore, this definition implies $F^* : \Omega^*(N) \to \Omega^*(M)$ is a map of graded algebras, that means $F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta)$ for all $\alpha, \beta \in \Omega^*(N)$.

If we identify $\bigwedge^k(T_q^*M) = \mathrm{Alt}^k(T_qM, \mathbb{R})$, then we have the pullback as $(F^*\alpha)_q(v_1, \dots, v_k) = \alpha_{F(q)}(T_qFv_1, \dots, T_qFv_k)$ for all $v_1, \dots, v_k \in T_qM$. However, our definition has the advantage that $\wedge$ is preserved automatically.

Recall. For any finite-dimensional vector space V, $\bigwedge^k V^* \cong \mathrm{Alt}^k(V;\mathbb{R})$. Therefore, for any $q \in M$, $\bigwedge^k T_q^*M \cong \mathrm{Alt}^k(T_qM, \mathbb{R})$.

Remark 24.5. Recall that for any $f \in C^\infty(M)$ and $q \in M$, we have

$$\begin{aligned} df_q : T_qM &\to \mathbb{R} \\ v &\mapsto v(f) \end{aligned}$$

Therefore f gives rise to

$$\begin{aligned} df : M &\to T^*M \\ q &\mapsto df_q \end{aligned}$$

This is C^∞ because given coordinates $(x_1, \dots, x_n) : U \to \mathbb{R}^n$, we have $df = \sum \frac{\partial f}{\partial x_i} dx_i$ where each partial differential is $C^\infty(U)$.

Lemma 24.6. For any smooth map $F : M \to N$ and any $f : C^\infty(M)$, $d(F^*f) = F^*(df)$.

Proof. For any $q \in M$, for any tangent vector $v \in T_qM$, we have

$$\begin{aligned} (F^*df)_q(v) &= (df)_{F(q)}(T_qFv) \\ &= ((T_qF)(v))f \\ &= v(f \circ F) \\ &= (d(f \circ F))_q(v) \end{aligned}$$

□

Example 24.7. Given

$$\begin{aligned} F : (0, \infty) \times \mathbb{R} &\to \mathbb{R}^2 \\ (r, \theta) &\mapsto (r\cos(\theta), r\sin(\theta)) \end{aligned}$$

we have

$$\begin{aligned} F^*(dx \wedge dy) &= (F^*dx) \wedge (F^*dy) \\ &= d(F^*x) \wedge d(F^*y) \\ &= d(r\cos(\theta)) \wedge d(r\sin(\theta)) \\ &= (\cos(\theta)dr - r\sin(\theta)d\theta) \wedge (\sin(\theta)dr + r\cos)\theta)d\theta) \\ &= r\cos^2(\theta)dr \wedge d\theta - r\sin^2(\theta)d\theta \wedge dr \\ &= (r\cos^2(\theta) + r\sin^2(\theta))dr \wedge d\theta \\ &= rdr \wedge d\theta. \end{aligned}$$

Proposition 24.8. Let $U \subseteq \mathbb{R}^m$ be open, let $F : U \to \mathbb{R}^m$ be a C^∞-map sending $(x_1, \dots, x_m) \mapsto (y_1, \dots, y_m)$. For any $f \in C^\infty(\mathbb{R}^m)$,

$$F^*(f(y)dy_1 \wedge \cdots \wedge dy_m) = f(F(x)) \det(DF(x))dx_1 \wedge \cdots \wedge dx_m$$

Remark 24.9. Recall that we define an integral of f over $[a, b]$ to be the signed area under the curve, which is the same as $\int_a^b f(x)dx = -\int_b^a f(x)dx$. Therefore the integral is just the integral of a 1-form. In particular, one need to keep track of orientation when thinking about this as manifolds, so this gives a signed determinant in vector calculus.

Proof. Recall that for any linear map $A : V \to V$ with $m = \dim(V)$, we have

$$\begin{aligned} \bigwedge^m A : \bigwedge^m V &\to \bigwedge^m V \\ \eta &\mapsto (\det(A)) \cdot \eta \end{aligned}$$

for any $q \in \bigwedge^m A$, i.e., as multiplication by $\det(A)$. Given $F : U \to \mathbb{R}^m$ with $x \in U$, we have $DF(x) : T_xU = \mathbb{R}^m \to T_{F(x)}\mathbb{R}^m = \mathbb{R}^m$, so $(DF(x))^* : (\mathbb{R}^m)^* \to (\mathbb{R}^m)^*$ and

$$\begin{aligned} \bigwedge^m((DF(x))^*) : \bigwedge^m(\mathbb{R}^m)^* &\to \bigwedge^m((\mathbb{R}^m)^*) \\ e_1 * \wedge \cdots e_m^* &\mapsto \det(DF(x))e_1^* \wedge \cdots e_m^* \end{aligned}$$

For all $q \in M$, we have $(dy_i)_q = e_i^*$ and $(dx_i)_r = e_i^*$, so

$$\begin{aligned} F^*(dy_1 \wedge \cdots \wedge dy_m)_q &= \bigwedge^m(DF(q)^*)(dy_1)_{F(q)} \wedge \cdots \wedge (dy_m)_{F(q)} \\ &= \bigwedge^m(DF(q)^*)(e_1^* \wedge \cdots \wedge e_m^*) \\ &= \det(DF(q)^*)e_1^* \wedge \cdots \wedge e_m^* \\ &= \det(DF(q))(dx_1)_q \wedge \cdots (dx_m)_q. \end{aligned}$$

□

Remark 24.10. To compute F^*, it would be easier to use the definition of $f \circ F$ instead.

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Recall. Let $U \subseteq \mathbb{R}^m$ be open and let $F : U \to \mathbb{R}^m$ be C^∞, and let $f \in C^\infty(\mathbb{R}^m)$, then

$$F^*(f(y) \wedge dy_1 \wedge \cdots \wedge dy_m) = f(F(x)) \det(DF(x)) \cdot dx_1 \wedge \cdots \wedge dx_n.$$

For $[a, b] \subseteq \mathbb{R}$, $f \in C^\infty([a, b])$, i.e., there exists $\varepsilon > 0$ and $h \in C^\infty(a - \varepsilon, b + \varepsilon)$ such that $h|_{[a,b]} = f$, then

$$\int_{[a,b]} f = \int_a^b f(x)dx = -\int_b^a f(x)dx.$$

The first expression is independent of the orientation of $[a, b]$, while the other two are dependent on orientations.

Definition 25.1. The *support of a k-form* $\omega \in \Omega^k(M)$ is

$$\operatorname{supp}(\omega) := \overline{\{q \in M \mid \omega_q \neq 0\}}.$$

Notation. $\Omega_c^k(M) = \{\omega \in \Omega^k(M) \mid \operatorname{supp}(\omega) \text{ is compact}\}$.

Definition 25.2. Let $\mu \in \Omega_c^m(\mathbb{R}^m)$, so $\mu = f dx_1 \wedge \cdots \wedge dx_m$ for $f \in C_c^\infty(\mathbb{R}^m)$. Let U be an open set in $\mathbb{R}^m$, $\operatorname{supp}(\mu) \subseteq U$, we define

$$\int_U f dx_1 \wedge \cdots \wedge dx_n = \int_U \mu := \int_U f = \int_U f dx_1 \cdots dx_n.$$

Definition 25.3. A C^∞ map $f : O \to O'$ for $O, O' \subseteq \mathbb{R}^m$ open is *orientation-preserving* if $\det(DF(x)) > 0$ for all $x \in O$.

Lemma 25.4. Let $M \xrightarrow{F} N \xrightarrow{G} P$ bet two smooth maps between manifolds, then for any k and any $\omega \in \Omega^k(P)$,

$$F^*(G^*\omega) = (G \circ F)^*\omega.$$

Proof. Exercise; taking k-exterior power is a functor. □

Lemma 25.5. Let M be a manifold, let $\varphi, \psi : U \to \mathbb{R}^m$ be two charts so that $\psi \circ \varphi^{-1} : \varphi(U) \to \psi(U)$ is orientation-preserving, then for all $V \in \Omega_c^m(M)$ with $\operatorname{supp}(V) \subseteq U$, we have

$$\int_{\varphi(U)} (\varphi^{-1})^* V = \int_{\psi(U)} (\psi^{-1})^* V,$$

given by

$$\begin{tikzcd} & U \subseteq M \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ \varphi(U) \arrow[rr, "\psi\circ\varphi^{-1}"'] & & \psi(U) \end{tikzcd}$$

Proof. By Lemma 25.4 we have

$$(\varphi^{-1})^* V = (\psi^{-1} \circ \psi \circ \varphi^{-1})^* V = (\psi \circ \varphi^{-1})^* (\psi^{-1})^* V$$

and

$$(\psi^{-1})^* V = f(y) dy_1 \wedge \cdots \wedge dy_m$$

for some $f \in C_c^\infty(\psi(U))$. Let $F = \psi \circ \varphi^{-1}$, then by assumption $\det(DF(x)) > 0$ for all x, thus

$$\begin{aligned}
\int_{\psi(U)} f(y) dy_1 \wedge \cdots \wedge dy_m &= \int_{\psi(U)} f(y) dy_1 \cdots dy_m \\
&= \int_{F(\varphi(U))} f dy_1 \cdots dy_m \\
&= \int_{\varphi(U)} f(F(x)) \det(DF(x)) dx_1 \cdots dx_m \\
&= \int_{\varphi(U)} F^*(f dy_1 \wedge \cdots \wedge dy_m) \\
&= \int_{\varphi(U)} (\psi \circ \varphi^{-1})^* (\psi^{-1})^* V \\
&= \int_{\varphi(U)} (\varphi^{-1})^* V.
\end{aligned}$$

□

Definition 25.6. An *orientation* of a manifold M (if it exists) is an atlas $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^m\}_{\alpha \in A}$ such that for all $\alpha, \beta \in A$, $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ preserves the orientation. Two orientations $\{\varphi_\alpha\}_{\alpha \in A}$ and $\{\psi_\beta\}_{\beta \in B}$ are said to be compatible if $\{\varphi_\alpha\}_{\alpha \in A} \cup \{\psi_\beta\}_{\beta \in B}$ is also an orientation.

Theorem 25.7. Let M be an orientable manifold and let $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^m\}_{\alpha \in A}$ be an orientation, then there exists a non-zero linear map

$$\begin{aligned}
\int_M : \Omega_c^m(M) &\to \mathbb{R} \\
\omega &\mapsto \int_M \omega
\end{aligned}$$

which does not depend on the choice of atlas $\{\varphi_\alpha\}_{\alpha \in A}$. A compatible orientation $\{\psi_\beta : V_\beta \to \mathbb{R}^m\}$ gives rise to the same linear map.

Proof. 1. Fix $\omega \in \Omega_c^m(M)$. Since $\operatorname{supp}(\omega)$ is compact, there exists some k with $\alpha_1, \ldots, \alpha_k$ such that $\operatorname{supp}(\omega) \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_k}$. Let $U_0 = M \setminus \operatorname{supp}(\omega)$. Let $\{\rho_0, \ldots, \rho_k\}$ be a partition of unity such that $\operatorname{supp}(\rho_0) \subseteq U_0$ and $\operatorname{supp}(\rho_i) \subseteq U_{\alpha_i}$ for $i = 1, \ldots, k$. Since $\rho_0|_{\operatorname{supp}(\omega)} \equiv 0$, then $(\rho_1 + \cdots + \rho_k)|_{\operatorname{supp}(\omega)} = 1$. Define

$$\int_M \omega = \sum_{i=1}^k \int_{\varphi_{\alpha_i}(U_{\alpha_i})} (\varphi_{\alpha_i}^{-1})^*(\rho_i \omega).$$

2. We now argue that the sum does not depend on the choices. Let $\{\psi_j : V_j \to \mathbb{R}^n\}_{j=1}^l$ be another collection of charts such that $\operatorname{supp}(\omega) \subseteq V_1 \cup \cdots \cup V_l$, and $\det(D(\psi_j^{-1} \circ \varphi_{\alpha_i})) > 0$, $\det(D(\psi_j^{-1} \circ \psi + i)) > 0$ for all i, j. Let $\{\tau_0, \ldots, \tau_l\}$ be a partition of unity such that $\operatorname{supp}(\tau_0) \subseteq M \setminus \operatorname{supp}(\omega)$ and $\operatorname{supp}(\tau_i) \subseteq V_i$. We have

$$\begin{aligned}
\sum_{i=1}^k \int_{\varphi_{\alpha_i}(U_i)} (\varphi_{\alpha_i}^{-1})^*(\rho_i \omega) &= \sum_{i=1}^k \int_{\varphi_{\alpha_i}(U_i)} (\varphi_{\alpha_i}^{-1})^*(\rho_i \sum_{j=1}^l \tau_j \omega) \\
&= \sum_{i,j} \int_{\varphi_{\alpha_i}(U_{\alpha_i} \cap V_j)} (\varphi_{\alpha_i}^{-1})^*(\rho_i \tau_j \omega) \\
&= \sum_{i,j} \int_{\psi_j(U_{\alpha_i} \cap V_j)} (\psi_j^{-1})^*(\rho_i \tau_j \omega) \\
&= \cdots \\
&= \sum_j \int_{\psi_j(V_j)} (\psi_j^{-1})^*(\tau_j \omega).
\end{aligned}$$

$\square$

26 Oct 23, 2023

Recall. We constructed a non-zero linear map $\int_M : \Omega_C^m(M) \to \mathbb{R}$ where $m = \dim(M)$.

Fact. Let $N \subseteq M$ be a closed embedded submanifold with $\dim(M) - \dim(N) > 0$, or more generally, a subset of measure 0, then for all $\omega \in \Omega_c^m(M)$, we have $\int_M \omega = \int_{M \setminus N} \omega$. See Lee, Proposition 16.8.

Notation. Let $i : M \hookrightarrow N$ be an embedded submanifold, then for any differential form $\omega \in \Omega(N)$, $\omega|_M = i^*\omega$.

Example 26.1. Consider $M = S^1 = \{(x, y) \in \mathbb{R}| x^2 + y^2 = 1\}$, and let $\omega = \left(\frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy\right)\Big|_{S^1}$. To find $\int_{S^1} \omega$, consider

$$\begin{aligned}
\varphi : (0, 2\pi) &\to S^1 \\
\theta &\mapsto (\cos(\theta), \sin(\theta))
\end{aligned}$$

with image $S^1 \setminus \{(1, 0)\}$, then

$$\begin{aligned}
\int_{S^1} \omega &= \int_{(0,2\pi)} \varphi^*\omega \\
&= \int_{(0,2\pi)} \left(\frac{-\sin(\theta)}{\cos^2(\theta) + \sin^2(\theta)} d\cos(\theta) + \frac{\cos(\theta)}{\cos^2(\theta) + \sin^2(\theta)} d\sin(\theta) \right) \\
&= \int_{(0,2\pi)} (\sin^2(\theta) d\theta + \cos^2(\theta) d\theta) \\
&= \int_{(0,2\pi)} d\theta \\
&= 2\pi.
\end{aligned}$$

Remark 26.2. $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is defined on $\mathbb{R}^2\backslash(\{0\}\times\mathbb{R})$, but $d\theta = d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2=y^2}dy$ is defined on $\mathbb{R}^2\backslash\{0\}$.

Recall (Fundamental Theorem of Calculus)**.**

$$\int_a^b f'(x)dx = f(b) - f(a)$$

which is equivalent to

$$\int_{[a,b]} df = \int_{\{b,-a\}} = \int_{\partial[a,b]} f,$$

over the oriented boundary.

Recall (Green's Theorem)**.** Let $D \subseteq \mathbb{R}^2$ be a domain with smooth boundary ∂D, then

$$\int_{\partial D} Pdx + Qdy = \int\int_D \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dxdy$$

where ∂D is oriented. Let $\alpha = Pdx + Qdy$, then $d\alpha = dP \wedge dx + dQ \wedge dy = -\frac{\partial P}{\partial y}dy \wedge dx + \frac{\partial Q}{\partial x}dx \wedge dy = \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dx \wedge dy$. Therefore, this says

$$\int_{\partial D} \alpha = \int_D d\alpha.$$

Note that we have not yet defined the operator d, so we need to make a good definition of it.

Theorem 26.3 ((Generalized) Stokes)**.** Let M be an oriented manifold, let $D \subseteq M$ be a domain with smooth boundary ∂D, then for any compactly supported $\omega \in \Omega_c^{\dim(M)-1}(M)$, the integral of the boundary

$$\int_{\partial D} \omega = \int_D d\omega$$

where ∂D is suitably oriented.

We will construct a sequence of $\mathbb{R}$-linear maps $d_M^i : \Omega^i(M) \to \Omega^{i+1}(M)$ for $0 \leqslant i \leqslant< \infty$ called *exterior derivatives*. We will write $d_M : \Omega^*(M) \to \Omega^{*+1}(M)$ for this sequence, and we think of $d_M = \{d_M^i\}_{i\geqslant 0}$ or $d_M = \bigoplus_i d_M^i : \bigoplus \Omega^i(M) \to \bigoplus \Omega^{i+1}(M)$.

Theorem 26.4. For any manifold M, there exists a unique $\mathbb{R}$-linear map $d_M : \Omega^*(M) \to \Omega^{*+1}(M)$, called the exterior derivatives, such that

- for all $f \in C^\infty(M) = \Omega^0(M)$, $d_M f = df$,
- for any $U \subseteq M$ open, for any $\omega \in \Omega^*(M)$, $(d_M\omega)|_U = d_U(\omega|_U)$,
- for any $\omega \in \Omega^k(M)$ and any $\eta \in \Omega^l(M)$, we have $d_M(\omega \wedge \eta) = (d_M\omega) \wedge \eta + (-1)^k\omega \wedge d_M\eta$,
- $d_M \circ d_M = 0$.

Remark 26.5.
- This is a construction of map between sheaves, and can be generalized on schemes.
- If $f \in C^\infty(M)$, $d_M(f\omega) = df \wedge \omega + (-1)^0 f d_M\omega$.
- Once we prove the theorem, $d = d_M$ for any M.

Proof. We first show uniqueness. Suppose for any M we have $d_M : \Omega^*(M) \to \Omega^{*+1}(M)$ satisfying all four conditions. Fix M, pick a chart $(x_1, \ldots, x_m) : U \to \mathbb{R}^m$, For any $\omega \in \Omega^k(M)$, then

$$\begin{aligned} \omega|_U &= \sum_{|I|=k} a_I dx_I \\ &= \sum_{i_1<\cdots<i_k} a_{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \end{aligned}$$

and we get $d_U : \Omega^*(U) \to \Omega^{*+1}(U)$.

Claim 26.6. $d_U(dx_I) = 0$.

Subproof. We proceed by induction on k. For the base case, we have $d_U(dx_i) = d_U(d_U x_i) = 0$ by the properties above. We have

$$\begin{aligned} d_U(dx_{i_1} \wedge \cdots \wedge dx_{i_{n+1}}) &= d_U(dx_{i_1}) \wedge (dx_{i_2} \wedge \cdots \wedge dx_{i_{n+1}}) + (-1)^1 dx_{i_1} \wedge d_U(dx_{i_2} \wedge \cdots \wedge dx_{i_{n+1}}) \\ &= 0 \end{aligned}$$

since $d_U(dx_{i_1}) = 0$ and $d_U(dx_{i_2} \wedge \cdots \wedge dx_{i_{n+1}}) = 0$. ■

Therefore, by Claim 26.6, we have

$$\begin{aligned} d_U(a_I dx_I) &= da_I \wedge dx_I + a_I d_U(dx_I) \\ &= da_I \wedge dx_I \end{aligned}$$

and so $d_U(\sum a_I dx_I) = \sum da_I \wedge dx_I$. Therefore, for any $\omega \in \Omega^k(M)$ and any U as a domain of a coordinate chart, then $(d_M U)|_U = d_U(\omega|_U) = d_U(\sum a_I dx_I) = \sum da_I \wedge dx_I$, so if d' is another exterior derivative with the four properties above, then $(d'_M\omega)|_U = d'_U(\omega|_U) = d'_U(\sum a_I dx_I) = \sum da_I \wedge dx_I = (d_M\omega)|_U$. This shows uniqueness (of the family $\{d_U\}_{U\subseteq M}$ for U open).

To show existence, we first prove a special case, where we assume there exists a global coordinate chart $(x_1, \ldots, x_m) : M \to \mathbb{R}^m$, then for $\omega \in \Omega^k$, there exists unique $a_I \in C^\infty(M)$ such that $\omega = \sum\limits_{|I|=k} a_I dx_I$. note that if $k = 0$, then $\omega = a \in C^\infty(M)$. We define $d_M\omega := \sum a_I \wedge dx_I$, and we need to check that the four properties holds.

- The first property holds by definition: $d_M a = da$.
- Suppose $W \subseteq M$ is open, then $x_1|_W, \ldots, x_m|_W : W \to \mathbb{R}^m$ is another chart, so
$$\left(\sum a_I dx_I\right)\Big|_W = \sum a_I|_W \, (dx_I)|_W\,,$$
and therefore
$$\begin{aligned} \left(d_M\left(\sum a_I dx_I\right)\right)\Big|_W &= \left(\sum da_I \wedge dx_I\right)\Big|_W \\ &= \sum (da_I)|_W \wedge (dx_I)|_W \\ &= \sum d(a_I|_W) \wedge d(x_I|_W) \\ &= d_W\left(\left(\sum a_I dx_I\right)\Big|_W\right). \end{aligned}$$
- Consider $\omega = a_I dx_I$, $\eta = b_I dx_I$, where $|I| = k$ and $|J| = l$, then $\omega \wedge \eta = a_I b_J dx_I \wedge dx_J$, so
$$\begin{aligned} d_M(\omega \wedge \eta) &= d(a_I b_J) dx_I \wedge dx_J \\ &= (b_J da_I + a_I db_J) \wedge dx_I \wedge dx_J \\ &= (da_I \wedge dx_I) \wedge (b_J dx_J) + (-1)^k (a_I dx_I) \wedge (db_J \wedge dx_J) \\ &= (d_M\omega) \wedge \eta + (-1)^k \omega \wedge d\eta. \end{aligned}$$
- Finally, $d_M(d_M(a_I dx_I)) = d_M(\sum\limits_{i=1}^m \frac{\partial a_I}{\partial x_i} dx_i \wedge dx_I) = \sum\limits_{i,j} \frac{\partial^2 a_I}{\partial x_j \partial x_i} dx_i \wedge dx_i) \wedge dx_I$. Since $\frac{\partial^2 a_I}{\partial x_j \partial x_i} = \frac{\partial^2 a_I}{\partial x_i \partial x_j}$ and $dx_j \wedge dx_i = -dx_i \wedge dx_j$ for all i, j, we know the summation must be 0, thus $d_M \circ d_M = 0$.

For the general case, given a manifold M, we choose an atlas $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^m\}_{\alpha\in A}$. Given $\omega \in \Omega^k(M)$, for any $\alpha \in A$ we have $d_{U_\alpha}(\omega|_{U_\alpha}) \in \Omega^{k+1}(U_\alpha)$. Set $U_{\alpha\beta} := U_\alpha \cap U_\beta$, then for any α, β, since $(\omega|_{U_\alpha})\big|_{U_{\alpha\beta}} = \omega|_{U_{\alpha\beta}} = (\omega|_{U_\beta})\Big|_{U_{\alpha\beta}}$, then $(d_{U_\alpha}(\omega|_{U_\alpha}))\big|_{U_{\alpha\beta}} = d_{U_{\alpha\beta}}(\omega|_{U_{\alpha\beta}}) = d_{U_\beta}\,(\omega|_{U_\beta})\Big|_{U_{\alpha\beta}}$. Therefore, there exists a unique $\eta \in \Omega^{k+1}(M)$ such that $\eta|_{U_\alpha} = d_{U_\alpha}(\omega|_{U_\alpha})$ for all α. Define $d_M\omega = \eta$, then $d_M\omega \in \Omega^{k+1}(M)$ is the unique $k+1$-form such that the differential commutes with restriction on U_α.

Exercise 26.7. d_M, as defined, is the desired map.

□

27 Oct 25, 2023

Example 27.1. For any $P, Q \in C^\infty(\mathbb{R}^2)$, we have $Pdx + Qdy \in \Omega^1(\mathbb{R}^2)$ and $d(Pdx + Qdy) = dP \wedge dx + dQ \wedge dy = \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) d \wedge dy$.

Example 27.2. $\frac{1}{x^2+y^2}(xdy - ydx) \in \Omega^1(\mathbb{R}^2)$, then

$$\begin{aligned} d\left(\frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx\right) &= \left(\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right) dx \wedge dy \\ &= 0. \end{aligned}$$

Example 27.3. $d(xdy - ydx) = dx \wedge dy - dy \wedge dx = 2dx \wedge dy$.

Remark 27.4. There are alternative constructions of the exterior derivative $d : \Omega^*(M) \to \Omega^{*+1}(M)$. For example, given $\omega \in \Omega^n(M)$, we can define

$$\begin{aligned} (d\omega)(x_1, \dots, x_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} X_i(\omega(x_1, \dots, \hat{x}_i, \dots, x_{n+1})) \\ &+ \sum_{i<j} (-1)^{i+j+1} \omega([x_i, x_j], \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+1}). \end{aligned}$$

See Palais (1954).

Lemma 27.5. Exterior derivatives commute wiht pullbacks: given a map $F : M \to N$, $\omega \in \Omega^*(N)$, we have $F^*(d\omega) = d(F^*\omega)$.

To prove this, we need

Lemma 27.6. For all $k \geqslant 0$, and for all $f_0, \dots, f_k \in C^\infty(N)$, we have $d_N(f_0 df_1 \wedge \cdots \wedge df_k) = df_0 \wedge \cdots \wedge df_k$.

Proof. We have seen this is the special case where $f_0, \dots, f_k$ were coordinate functions. If $k = 0$, then $d_N(f_0) = df_0$ by definition of d_N. For the inductive step, suppose this is true for $k = n$, then

$$\begin{aligned} d_N(f_0 df_1 \wedge \cdots \wedge df_n \wedge df_{n+1}) &= d_N(f_0 df_1 \wedge \cdots \wedge df_n) \wedge df_{n+1} \\ &+ (-1)^n (f_0 df_1 \wedge \cdots \wedge df_N) \wedge d_N(df_{n+1}) \\ &= (df_0 \wedge \cdots \wedge df_n) \wedge df_{n+1} + (-1)^n (f_0 df_1 \wedge \cdots \wedge df_n) \wedge 0. \end{aligned}$$

$\square$

Proof of Lemma 27.5. Recall that for all $h \in C^\infty(N)$, $d(F^*h) = f^*dh$. Let $\omega \in \Omega^k(N)$ for $k > 0$, let $(x_1, \dots, x_n) : U \to \mathbb{R}^n$ be a coordinate chart on N, then $\omega|_U = \sum\limits_{|I|=k} a_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ since

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \uparrow & & \uparrow \\ F^{-1}(U) & \longrightarrow & U \end{array}$$

commutes for all $\mu \in \Omega^*(N)$ as $F^*(\mu|_U) = (F^*\mu)|_{F^{-1}(U)}$. Therefore,

$$\begin{aligned} (F^*(d\omega))|_{F^{-1}(U)} &= F^*((d\omega)|_U) = F^*(d(\omega|_U)) \\ &= F^*(d(\sum a_I dx_I)) \\ &= \sum F^*(da_I \wedge dx_I) \\ &= \sum d(F^*a_I) \wedge d(F^*x_{i_1}) \wedge \cdots \wedge d(F^*x_{i_k}) \end{aligned}$$

$$
\begin{aligned}
&= \sum d(F^* a_I \wedge dF^* x_{i_1} \wedge \cdots \wedge dF^* x_{i_k}) \\
&= d(\sum F^*(a_I dx_I)) \\
&= d(F^*(\sum a_I dx_I)) \\
&= d(F^* \left.\omega\right|_{F^{-1}(U)}) \\
&= (d(F^*\omega))_{F^{-1}(U)}
\end{aligned}
$$

Since coordinate charts cover N, their preimages cover M, thus $F^*(d\omega) = d(F^*\omega)$. □

28 Oct 27, 2023

Definition 28.1. Let V be a finite-dimensional vector space, $\eta \in \mathrm{Alt}^k(V;\mathbb{R})$, the alternating k-linear map and $u \in V$ a vector. We define $\iota(u)\eta \in \mathrm{Alt}^{k-1}(V;\mathbb{R})$ by $(\iota(u)\eta)(v_1,\dots,v_{k-1}) = \eta(u, v_1, \dots, v_{k-1})$ for all $v_1,\dots,v_{k-1} \in V$. Therefore, we get a linear map $\iota(u) : \mathrm{Alt}^k(V;\mathbb{R}) \to \mathrm{Alt}^{k-1}(V;\mathbb{R})$.

Example 28.2. For $l_1, l_2 \in V^*$, we have $l_1 \wedge l_2 \in \mathrm{Alt}^2(V;\mathbb{R})$ defined by $(l_1 \wedge l_2)(v_1,v_2) = l_1(v_1)l_2(v_2) - l_1(v_2)l_2(v_1)$, then

$$(\iota(u)(l_1 \wedge l_2))(v) = l_1(u)l_2(v) - l_1(v)l_2(u) = (l_1(u)l_2 - l_2(u)l_1)(v)$$

and then $\iota(u)(l_1 \wedge l_2) = l_1(u)l_2 - l_2(u)l_1 = (\iota(u)l_1)l_2 - (\iota(u)l_2)l_1$. Equivalently, $\bigwedge^k V^* \cong \mathrm{Alt}^k(V;\mathbb{R})$ with $(l_1 \wedge \cdots \wedge l_k)(v_1,\dots,v_k) = \det(l_i(v_j))$. We get a linear map $\iota(w) : \bigwedge^k V^* \to \bigwedge^{k-1} V^*$ for all $k > 0$. Therefore, $\iota(u) : \bigwedge^0 V^* \to \bigwedge^{-1} V^*$ is of the form $\mathbb{R} \to 0$.

Lemma 28.3. Let V be a finite-dimensional vector space and let $u \in V$. For all r, $\alpha \in \bigwedge^r(V^*)$, for all $\beta \in \bigwedge^*(V^*)$, we have

$$\iota(u)(\alpha \wedge \beta) = (\iota(u)\alpha) \wedge \beta + (-1)^r \alpha \wedge (\iota(u)\beta).$$

Remark 28.4. Let $A^* = \bigoplus\limits_{i \geqslant 0} A^i$ be a graded commutative algebra. A graded derivation of A^* of degree $k \in \mathbb{Z}$ is an $\mathbb{R}$-linear map $\delta : A^* \to A^{*+k}$ such that for all $a \in A^j$ and $b \in B^*$ we have $\delta(a \wedge b) = (\delta a) \wedge b + (-1)^{kj} a \wedge (\delta b)$.

Lemma 28.3 says $\iota(u)$ is a graded derivation of degree -1. Then $d : \Omega^*(M) \to \Omega^{*+1}(M)$ is a graded derivation of degree 1.

We will define Lie derivatives $L_X : \Omega^*(M) \to \Omega^*(M)$ for all $x \in M$. They are graded derivations of degree 0.

Lemma 28.5. Let V be a finite-dimensional vector space, $u \in V$, and $l_1, \dots, l_k \in V^*$, then

$$\iota(u)(l_1 \wedge \cdot s \wedge l_k) = \sum_{j=1}^k (-1)^{j-1} \iota(u) l_j \, l_1 \wedge \cdots \wedge \hat{l}_j \wedge \cdots \wedge l_k.$$

Proof. For all $v_1, \dots, v_{k-1} \in V$, we have

$$(\iota(u)(l_1 \wedge \cdots \wedge l_k))(v_1,\dots,v_{k-1}) = (l_1 \wedge \cdots \wedge l_k)(u, v_1, \dots, v_k).$$

This is just

$$
\begin{aligned}
\det \begin{pmatrix} l_1(u) & l_1(v_1) & \cdots & l_1(v_{k+1}) \\ l_2(u) & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ l_k(u) & \cdots & \cdots & l_k(v_{k-1}) \end{pmatrix} &= \sum (-1)^{j-1} l_j(u) \det(l_i(v_j)) j \\
&= \sum (-1)^{j-1} l_j(u) (l_1 \wedge \cdots \wedge \hat{l}_j \wedge \cdots \wedge l_k)(v_1, \dots, v_{k-1}).
\end{aligned}
$$

□

Proof of Lemma 28.3. We may assume $\alpha = l_1 \wedge \cdots l_r$ and $\beta = l_{r+1} \wedge \cdots \wedge l_{r+s}$ for some $l_1, \ldots, l_{r+s} \in V^*$. Therefore

$$\begin{aligned}
\iota(u)(\alpha \wedge \beta) &= \iota(u)(l_1 \wedge \cdots \wedge l_{r+s}) \\
&= \sum_{j=1}^{r} (-1)^{j-1}(\iota(u)l_j)(l_1 \wedge \cdots \wedge \hat{l}_j \wedge \cdots \wedge l_{r+s}) \\
&+ \sum_{j=r+1}^{r+s} (-1)^{j+1}\iota(u)l_j(l_1 \wedge \cdots \wedge \hat{l}_j \wedge \cdots \wedge l_{r+s}) \\
&= \iota(u)\alpha \wedge \beta + \alpha \wedge (-1)^r \sum_{j'=1}^{s} (-1)^{j'-1} l_{r+1} \wedge \cdots \wedge \hat{l}_{j'+r} \wedge \cdots \wedge l_{r+s} \\
&= \iota(u)\alpha + (-1)^r \alpha \wedge \iota(u)\beta.
\end{aligned}$$

□

Definition 28.6. Let M be a manifold, $X \in \mathfrak{X}(M)$. We define $\iota(X) : \Omega^*(M) \to \Omega^{*-1}(M)$ by $(\iota(X)\omega)_q = \iota(X_q)\omega_q$ for all $q \in M$. Note that by definition, over the zero forms, we have $\iota(X) : \Omega^0(M) \to \Omega^{-1}(M) = 0$ is the zero map.

Example 28.7. Let $M = \mathbb{R}^2$, $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and let $\omega = dx \wedge dy$, then $\iota(X)\omega = (\iota(X)dx)dy - (\iota(X)dy)dx = xdy - ydx$.

Definition 28.8. The *Lie derivative* of a differential form $\omega \in \Omega^k(M)$ with respect to $X \in \mathfrak{X}(M)$ is $L_X\omega = \frac{d}{dt}\big|_{t=0} \varphi_t^*\omega$ where $\{\varphi_t\}$ is the flow of X.

Remark 28.9. For a fixed $q \in M$ and small t, $t \mapsto (\varphi_t^*\omega)_q$ is a path in the finite-dimensional vector space $\bigwedge^k(T_q^*M)$ so $(L_X\omega)_q = \frac{d}{dt}\big|_0 (\varphi_t^*\omega)_q$ makes sense.

Theorem 28.10 (Cartan's Magic Formula)**.** For any differential form $\omega \in \Omega^*(M)$ and any vector field $X \in \mathfrak{X}(M)$, we have $L_X\omega = d(\iota(X)\omega) = \iota(X)(d\omega)$.

Remark 28.11. For $k = 0$ this is easy: for $f \in \Omega^0(M)$, we have $(L_Xf)_q = \frac{d}{dt}f(\varphi_t(q)) = X_q(f) = (df_q)(X_q) = (\iota(X)df)_q + 0 = (\iota(X)df)_q + d(\iota(X)f)_q$.

Example 28.12. Let $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $\omega = dx \wedge dy$, we have $\varphi_t(x, y) = e^t(x, y) = (e^tx, e^ty)$ so $\varphi_t^*\omega = d(e^tx) \wedge d(e^ty) = e^{2t}dx \wedge dy$. Therefore,

$$\begin{aligned}
\frac{d}{dt}\bigg|_0 (\varphi_t^*\omega) &= \frac{d}{dt}\bigg|_0 e^{2t}dx \wedge dy \\
&= 2dx \wedge dy
\end{aligned}$$

and

$$\begin{aligned}
(d\iota(X) + \iota(X)d)(dx \wedge dy) &= d(\iota(X)dx \wedge dy) \\
&= d(xdy - ydx \\
&= dx \wedge dy - dy \wedge dx \\
&= 2dx \wedge dy
\end{aligned}$$

as well.

Proof Idea. Let $Q_X = d\iota(X) + \iota(X)d : \Omega^*(M) \to \Omega^*(M)$. We will show that both L_X and Q_X are derivations of degree 0, both commute with d, and behave well with restrictions to open sets, i.e., $(L_X\omega)|_W = L_X(\omega|_W)$. □

29 Oct 30, 2023

Lemma 29.1. $L_X : \Omega^*(M) \to \Omega^*(M)$ is a derivation of degree 0, i.e., L_X is $\mathbb{R}$-linear and $L_X(\alpha \wedge \beta) = L_X(\alpha) \wedge \beta + \alpha \wedge L_X(\beta)$, and $L_X \circ d = d \circ L_X$.

Proof. Since pullback φ_t^* and differentiation $\frac{d}{dt}\big|_0$ are both $\mathbb{R}$-linear, then $L_X = \frac{d}{dt}\big|_0 \varphi_t^*$ is $\mathbb{R}$-linear as well. For any finite-dimensional vector space V, $\wedge : \bigwedge^* V \times \bigwedge^* V \to \bigwedge^* V$ is $\mathbb{R}$-linear, so for any two curves $\gamma, \sigma : I \to \bigwedge^* V$ we have

$$\frac{d}{dt}\bigg|_0 (\gamma \wedge \sigma) = (\frac{d}{dt}\bigg|_0 \gamma) \wedge \sigma(0) + \gamma(0) \wedge \frac{d}{dt}\bigg|_0 \sigma$$

since $\wedge$ is bilinear. Therefore, for any forms $\alpha, \beta \in \Omega^*(M)$ and any $q \in M$,

$$\begin{aligned}
(L_X(\alpha \wedge \beta))_q &= \frac{d}{dt}\bigg|_0 (\varphi_t^*(\alpha \wedge \beta))_q \\
&= \frac{d}{dt}\bigg|_0 ((\varphi_t^*\alpha)_q \wedge (\varphi_t^*\beta)_q) \\
&= (\frac{d}{dt}\bigg|_0 (\varphi_t^*\alpha))_q \wedge \beta_q + \alpha_q \wedge (\frac{d}{dt}\bigg|_0 \varphi_t^*\beta)_q.
\end{aligned}$$

Also,

$$\begin{aligned}
d(L_X\alpha) &= d(\frac{d}{dt}\bigg|_q \varphi_t^*\alpha) \\
&= \frac{d}{dt}\bigg|_0 d(\varphi_t^*\alpha) \\
&= \frac{d}{dt}\bigg|_0 \varphi_t^*(d\alpha) \\
&= L_X(d\alpha).
\end{aligned}$$

□

Lemma 29.2. $Q_X := \iota(X) \circ d + d \circ \iota(X)$ is a derivation of degree 0 that commutes with d.

Proof. We have

$$\begin{aligned}
(Q_X \circ d)(\alpha) &= (\iota(X) \circ d \circ d)(\alpha) + (d \circ \iota(X) \circ d)(\alpha) \\
&= (d \circ \iota(X) \circ d)(\alpha) \\
&= (d \circ \iota(X) \circ d + d \circ d \circ \iota(X))(\alpha) \\
&= (d \circ Q_X)(\alpha).
\end{aligned}$$

Moreover, for any $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^*(M)$,

$$\begin{aligned}
Q_X(\alpha \wedge \beta) &= (d\iota(X) + \iota(X)d)(\alpha \wedge \beta) \\
&= d((\iota(X)\alpha \wedge \beta) + (-1)^k \alpha \wedge \iota(X)\beta) + \iota(X)(d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta) \\
&= (d\iota(X)\alpha) \wedge \beta + (-1)^{k-1}\iota(X)\alpha \wedge d\beta + (-1)^k d\alpha \wedge \iota(X)\beta + (-1)^k(-1)^k \alpha \wedge d\iota(X)\beta \\
&+ (\iota(X)d\alpha) + (-1)^{k+1} d\alpha \wedge \iota(X)\beta + (-1)^k \iota(X)\alpha \wedge d\beta + (-1)^k(-1)^k \alpha \wedge \iota(X)d\beta \\
&= (Q_X\alpha) \wedge \beta + \alpha \wedge (Q_X\beta).
\end{aligned}$$

□

Theorem 29.3 (Cartan's Formula)**.** $L_X = d \circ \iota(X) + \iota(X) \circ d.$

Proof. Recall from last time that for all $f \in C^\infty(M) = \Omega^0(M)$, we have $L_X f = Q_X f$, so

$$L_X(df) = d(L_X f) = d(Q_X f) = Q_X(df),$$

therefore for all $k > 0$ and any $f_0, \dots, f_k \in C^\infty(M)$, we have

$$\begin{aligned} Q_X(f_0 df_1 \wedge \cdots \wedge df_k) &= (Q_X f_0) \wedge df_1 \wedge \cdots \wedge df_k + f_0 \sum_{l=1}^k df_1 \wedge \cdots (Q_X df_i) \wedge \cdots \wedge df_k \\ &= (L_X f_0) df_1 \wedge \cdots \wedge df_k + \sum_{i=1}^k df_1 \wedge \cdots \wedge L_X df_i \wedge \cdots \wedge df_k \\ &= L_X(f_0 df_1 \wedge \cdots \wedge df_k). \end{aligned}$$

Therefore, if we know that $\Omega^k(M) = \operatorname{span}_{C^\infty(M)}\{df_1 \wedge \cdots \wedge df_k \mid f_1, \dots, f_k \in C^\infty(M)\}$, we are done. To see this, recall for any $W \subseteq M$ open and any α, we have

$$(L_X \alpha)|_W = L_X(\alpha|_W)$$

and

$$(Q_X \alpha)|_W = Q_X(\alpha|_W),$$

so it is enough to prove Cartan's formula in a coordinate chart, but a coordinate chart $(x_1, \dots, x_m) : U \to \mathbb{R}^m$ gives $\alpha \in \Omega^k(U)$ as $\alpha = \sum_{|I|=k} a_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ in $\operatorname{span}_{C^\infty(U)}\{dx_{i_1} \wedge \cdots \wedge dx_{i_k} \mid 1 \leqslant i_1 < \cdots < i_k \leqslant m\}$. □

Notation. Given a manifold M of dimension m, we write $\Omega^{\text{top}}(M)$ for $\Omega^m(M)$.

Definition 29.4. A *volume form* on a manifold M, if exists, is $\mu \in \Omega^{\text{top}}(M)$ such that $\mu_q \neq 0$ for all $q \in M$.

Remark 29.5. For all $q \in M$, $\dim(\bigwedge^{\text{top}})(T_q^* M) = 1$. So if a volume form μ exists, $\mu_q \in \bigwedge^{\text{top}}(T_q^* M)$ is a basis. Hence, this defines a map

$$\begin{aligned} M \times \mathbb{R} &\to \bigwedge^{\text{top}}(T^* M) \\ (q, t) &\mapsto (q, t\mu_q) \end{aligned}$$

which is a bijection, and is a linear isomorphism on the fibers. In particular, it is a local diffeomorphism. In coordinates $(x_1, \dots, x_m)$, $\mu = a(x_1, \dots, x_m) dx_1 \wedge \cdots \wedge dx_m$ and $a(x_1, \dots, x_m) \neq 0$ for all $x_1, \dots, x_m$. Therefore the mapping $(x_1, \dots, x_m, t) \mapsto (x_1, \dots, x_m, ta)$ has a backwards mapping $(x_1, \dots, x_m, \frac{\eta}{a}) \leftarrow (x_1, \dots, x_m, \eta)$.

Proposition 29.6.

1. A manifold M is orientable if and only if there exists a volume form μ on M.
2. Two volume forms μ, ν arise from equivalent orientations if and only if there exists $f \in C^\infty(M)$ such that $f > 0$ with $\mu = f\nu$.

Remark 29.7. Equivalently, (2) is true if and only if the top form minus the zero section gives exactly two connected component, which is an algebraic topological criterion.

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Proof. ($\Rightarrow$): Suppose M is orientable, then there exists an atlas $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^m\}_{\alpha \in A}$ so that $\det(D(\varphi_\beta \circ \varphi_\alpha^{-1}) > 0$ for all α, β. Choose a partition of unity $\{\rho_\alpha\}_{\alpha \in A}$ such that $\operatorname{supp}(\rho_\alpha) \subseteq U_\alpha$. Note that $dr_1 \wedge \cdots \wedge dr_m \in \Omega^{\text{top}}(\mathbb{R}^m)$ is a volume form, and let $\mu = \sum_\alpha \rho_\alpha \varphi_\alpha^*(dr_1 \wedge \cdots \wedge dr_m)$, so we need to check that $\mu_q \neq 0$ for all $q \in M$. Fix $q \in M$, then there exists α_1 such that $\rho_{\alpha_1}(q) \neq 0$, therefore $q \in U_{\alpha_1}$. We get to write

$$((\varphi_{\alpha_1}^{-1})^* \mu)_{\varphi_{\alpha_1}(q)} = \sum \rho_\alpha(q) (\varphi_{\alpha_1}^{-1})^* \varphi_\alpha^*(dr_1 \wedge \cdots \wedge dr_m))_{\varphi_{\alpha_1}(q)}.$$

Since $\{\operatorname{supp}(\rho_\alpha)\}_{\alpha\in A}$ is locally finite, then $\rho_\alpha(q) = 0$ except for finitely many indices $\alpha_1, \ldots, \alpha_k$, for which $\rho_{\alpha_i}(q) > 0$. Hence, this becomes a finite sum

$$\begin{aligned}((\varphi_{\alpha_1}^{-1})^*\mu)_{\varphi_{\alpha_1}(q)} &= \sum_{i=1}^{k} \rho_{\alpha_i}(q)(\varphi_{\alpha_i}^{-1})^*\varphi_{\alpha_i}^*(dr_1 \wedge \cdots \wedge dr_m))_{\varphi_{\alpha_1}(q)} \\ &= \left(\sum_{i=1}^{k} \rho_{\alpha_i}(q) \det(D(\varphi_{\alpha_i} \circ \varphi_{\alpha_1}^{-1})(\varphi_{\alpha_1}(q))\right) dr_1 \wedge \cdot dr_m\end{aligned}$$

But note that the summation $\sum_{i=1}^{k} \rho_{\alpha_i}(q) \det(D(\varphi_{\alpha_i} \circ \varphi_{\alpha_1}^{-1}) > 0$, therefore the term above is non-zero, hence $\mu_q \neq 0$.

($\Leftarrow$): Suppose $\mu \in \Omega^{\text{top}}(M)$ is a volume form, so choose an atlas $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^m\}_{\alpha\in A}$ such that each U_α is connected, then $(\psi_\alpha^{-1})^*\mu = f_\alpha dr_1 \wedge \cdots \wedge dr_m$ for some $f_\alpha \in C^\infty(\psi_\alpha(U))$. Since for all $q \in U_\alpha$, $\mu_q \neq 0$, then $f_\alpha(r) \neq 0$ for all $r \in \varphi_\alpha(U_\alpha)$. Since U_α is connected, then $\psi_\alpha(U_\alpha)$ is connected, so f_α is either strictly positive or strictly negative. If $f_\alpha > 0$, let $\varphi_\alpha = \psi_\alpha$; if $f_\alpha < 0$, let $\varphi_\alpha = T \circ \psi_\alpha$ where $T(r_1, \ldots, r_m) = (-r_1, r_2, \ldots, r_m)$, which has $\det(T) < 0$, so $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^m\}_{\alpha\in A}$ is a desired atlas. □

Definition 30.1. Two volume forms $\mu, \nu \in \Omega^{\text{top}}(M)$ are *equivalent* if there exists $f \in C^\infty(M)$ with $f > 0$ so that $\mu = f\nu$.

Lemma 30.2. A connected manifold M is orientable if and only if $\bigwedge^{\text{top}}(T^*M)\backslash M$ has two components. Here $M \hookrightarrow \bigwedge^{\text{top}}(T^*M)$ as the zero section by $q \mapsto (q, 0)$.

Proof. ($\Rightarrow$): Suppose M is orientable, then there exists a volume form $\mu \in \Omega^{\text{top}}(M)$ and

$$\begin{aligned}\psi : M \times \mathbb{R} &\to \bigwedge^{\text{top}}(T^*M) \\ (q, t) &\mapsto (q, t\mu_q)\end{aligned}$$

is a diffeomorphism (as we have seen before). Now $\psi^{-1}(\bigwedge^{\text{top}}(T^*M)\backslash M) = M \times \mathbb{R}\backslash(M \times \{0\}) = M \times (\mathbb{R}\backslash\{0\})$, which has exactly two components, $M \times (0, \infty)$ and $M \times (-\infty, 0)$.

($\Leftarrow$): Suppose $\bigwedge^{\text{top}}(T^*M)\backslash M$ has two components. Choose one and call it W. An sufficiently small open subset $U \subseteq M$ is orientable since it is diffeomorphic to an open subset of $\mathbb{R}^{\dim(M)}$. Therefore, there exists a volume form $\mu_U \in \Omega^{\text{top}}(U)$. Next assume U is connected, then $\mu_U(U) \subseteq \bigwedge^{\text{top}}(T^*M)\backslash M$ is connected, hence either $\mu_U(U) \subseteq W$ or $\mu_U(U) \subseteq \bigwedge^{\text{top}}(T^*M)\backslash W$. If $\mu_U(U) \subseteq U$, we keep μ_U; if not, replace it by $-\mu_U$. We get an open cover $\{U_\alpha\}_{\alpha\in A}$ of M with $\mu_\alpha : U_\alpha \to \bigwedge^{\text{top}}(T^*M)\big|_{U_\alpha}$ such that $\mu_\alpha(U_\alpha) \subseteq W$ for all α. Now choose a partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$ and set $\mu = \sum_\alpha \rho_\alpha\mu_\alpha$. □

Remark 30.3. Let V be an m-dimensional vector space and $0 \neq \mu \in \bigwedge^m(V^*)$, then given an ordered basis $\{v_1, \ldots, v_m\}$ of V, either $\mu(v_1, \ldots, v_m) > 0$ or $\mu(v_1, \ldots, v_m) < 0$. We say $\{v_1, \ldots, v_m\}$ is *positively oriented* with respect to μ if $\mu(v_1, \ldots, \mu_m) > 0$.

Example 30.4. Let $V = \mathbb{R}^m$, $\mu \in \bigwedge^m(V^*) \cong \operatorname{Alt}^m(V; \mathbb{R})$ the determinant, then $\{v_1, \ldots, v_m\}$ is positively oriented if and only if $\det(v_1 \mid \cdots \mid v_m) > 0$.

Definition 30.5. Let $\mathbb{H}^m = \{x \in \mathbb{R}^m \mid x_1 \leqslant 0\}$ be the closed half-space, we define $C^\infty(\mathbb{H}^m)$ be the set of smooth functions $f : \mathbb{H}^m \to \mathbb{R}$ such that for all $q \in \mathbb{H}^m$ there exists an open neighborhood U of q in $\mathbb{R}^m$ and $g_U \in C^\infty(U)$ such that $f|_{U\cap\mathbb{H}} = g|_{U\cap\mathbb{H}}$. This is exactly the set of smooth functions $f : \mathbb{H}^m \to \mathbb{R}$ such taht there exists an open set $W \subseteq \mathbb{R}^m$ with $\mathbb{H} \subseteq W$ and $g_W \in C^\infty(W)$ such that $g|_{W\cap\mathbb{H}} = f$.

Example 30.6. The function

$$f(x) = \begin{cases} e^{\frac{1}{x}}, & x < 0 \\ 0, & x = 0 \end{cases}$$

is in $C^\infty((-\infty, 0]) = C^\infty(\mathbb{H}^1)$; the function $g(x) = \sqrt{-x}$ is not in $C^\infty(\mathbb{H}^1)$.

Similarly, for any open set $U \subseteq \mathbb{R}^m$ we can define $f \in C^\infty(U)$ if there exists $\tilde{U} \subseteq \mathbb{R}^m$ open and $g \in C^\infty(\tilde{U})$ such that $U = \mathbb{H} \cap \tilde{U}$ and $f = g|_U = g|_{\mathbb{H}\cap\tilde{U}}$.

Definition 30.7. For $U, W \subseteq \mathbb{H}^m$ open, we say $F = (F_1, \dots, F_m) : U \to W$ is C^∞ if $F_1, \dots, F_m \in C^\infty(U)$.

Definition 30.8. A *manifold with boundary* M is a (Hausdorff paracompact) topological space M together with an equivalence class of atlases, with each chart modelled on open subsets of $\mathbb{H}^m$ (for some m), i.e., there exists open cover $\{U_\alpha\}_{\alpha\in A}$ of M and a collection of homeomorphisms $\{\varphi_\alpha : U_\alpha \to W_\alpha \subseteq \mathbb{H}^m\}$ with $W_\alpha \subseteq \mathbb{H}^m$ open and $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are C^∞.

Example 30.9. $M = \{x \in \mathbb{R}^m \mid \sum x_i^2 = 1\}$ is a manifold with boundary.

$N = [0,1] \times [0,1]$ is not a manifold with boundary: there are no diffeomorphisms from a neighborhood of $(0,0) \in N$ to an open set in $\mathbb{H}^2$.

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Theorem 31.1. Given a manifold with boundary M, there exists a manifold $\tilde{M}$ containing M and the property that for all $q \in M$, there exists coordinate chart $\varphi : U \to \mathbb{R}^m$ on $\tilde{M}$ so that $\varphi(U \cap M) = \varphi(U) \cap \mathbb{H}^m$.

Proof. Omitted: required vector fields and flows on manifolds with boundaries. □

Definition 31.2. A *regular domain* D in a manifold M is a subset $D \subseteq M$ so that for all $q \in D$ there exists a chart $\varphi : U \to \mathbb{R}^m$ on M with $\varphi(U \cap D) = \mathbb{H}^m \cap \varphi(U) \cap \{r_1 \leqslant 0\}$. We call such charts φ *adapted to* D.

Example 31.3. $\{w \in \mathbb{R}^m \mid \sum_{i=1}^m x_i^2 \leqslant 1\}$ is a regular domain in $\mathbb{R}^m$, but $[0,1]^2 \subseteq \mathbb{R}^2$ is not a regular domain.

It is not hard to prove:

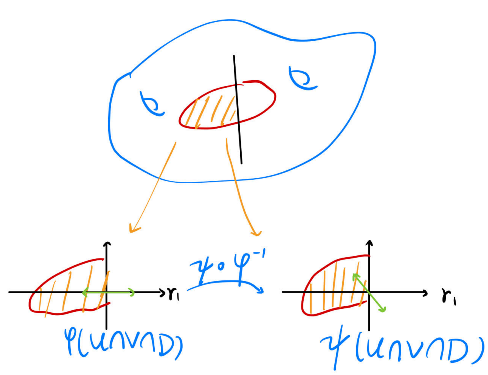
- Any regular domain is a manifold with boundary.
- If $D \subseteq M$ is a regular domain, define $\partial D = \{q \in D \mid \forall \text{open neighborhood } U \ni q, U \cap (M \setminus D) \neq \varnothing\}$, then ∂D is a codimension-1 embedded submanifold of M: this is because for a chart $\varphi : U \to \mathbb{R}^m$ adapted to D, we have $\varphi(U \cap \partial D) = \{r \in \varphi(U) \subseteq \mathbb{R}^m \mid r_1 = 0\}$.

Lemma 31.4. Let $D \subseteq M$ be a regular domain, let $\varphi : U \to \mathbb{R}^m$ and $\psi : V \to \mathbb{R}^m$ be two charts adapted to D, such that $U \cap V \cap \partial D \neq \varnothing$, then for any point $q \in U \cap V \cap \partial D$, we have

$$D(\psi \circ \varphi^{-1})(\varphi(q)) = \begin{pmatrix} a & 0 & \cdots & 0 \\ * & & & \\ \vdots & & D(\psi\circ\varphi^{-1})\big|_{\{0\}\times\mathbb{R}^{m-1}} & \\ * & & & \end{pmatrix}$$

with $a > 0$.

Proof. This is illustrated in the following picture.



□

Corollary 31.5. Let $D \subseteq M$ be a regular domain, then there exists a vector field $\vec{n}$ defined in an open neighborhood of ∂D such that for all $q \in \partial D$, $\vec{n}(q) \neq 0$ and points out of D.

Proof. Cover ∂D by the domains $\{U_\alpha\}$ of adapted coordinate charts $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^m\}$. Let $U_0 = M \backslash D$, then choose a partition of unity $\{\rho_0\} \cup \{\rho_\alpha\}_{\alpha\in A}$ such that $\operatorname{supp}(\rho_0) \subseteq M\backslash D$ and $\operatorname{supp}(\rho_\alpha) \subseteq U_\alpha$ for all α. Let $W = \bigcup\limits_{\alpha\in A} U_\alpha$. For all α, $\varphi_\alpha = (x_1^{(\alpha)}, \dots, x_m^{(\alpha)}) : U_\alpha \to \mathbb{R}^m$. Let $\vec{n} = \sum \rho_\alpha \frac{\partial}{\partial x_i^{(\alpha)}}$.

Claim 31.6. $\vec{n}$ points outwards of D.

Subproof. Given $q \in \partial D \cap U_\alpha \cap U_\beta$, then

$$\left.\frac{\partial}{\partial x_1^{(\alpha)}}\right|_q = a_1 \left.\frac{\partial}{\partial x_1^{(\alpha)}}\right|_q + P\left(\left.\frac{\partial}{\partial x_1^{(\alpha)}}\right|_q, i > 1\right)$$

where P is a polynomial and $a_1 > 0$ by Lemma 31.4. ■

Given $q \in \partial D$, choose α_0 such that $\rho_{\alpha_0}(q) > 0$, then $\vec{n}(q) = \sum\limits_\alpha \rho_\alpha(q) \left.\frac{\partial}{\partial x_1^{(\alpha_0)}}\right|_q$, but this is of the form $c \left.\frac{\partial}{\partial x_1^{(\alpha_0)}}\right|_q + \cdots$ where $c > 0$. □

Example 31.7. Let $M = \mathbb{R}$ and $D = [0, 1]$, we have $\vec{n} = \frac{d}{dx}$ near 1 and $\vec{n} = -\frac{d}{dx}$ near 0.
Let $M = \mathbb{R}^2$, $D = \{(x_1, x_2) \mid x_1^2 + x_2^2 \leqslant 1\}$, then $\vec{n} = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$.

Lemma 31.8. Let M be an orientable manifold, let $D \subseteq M$ be a regular domain, let $\mu \in \Omega^{\text{top}}(M)$ be a volume form, and let $\vec{n} = W \to TM|_W$ be the outward normal vector field. Then $\nu = (\iota(\vec{n})\mu)|_{\partial D}$ is a volume form on ∂D.

Example 31.9. Let $\mu = dx$, if we contract the vector field using the first example, then $\iota(\vec{n})dx$ is 1 near 1 and is 0 near 0.
Let $\mu = dx_1 \wedge dx_2$, then $\iota(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}) dx_1 \wedge dx_2 = x_1 dx_2 + x_1 dx_2 - x_2 dx_1$, restricting this to the boundary S^1 gives $d\theta$.

Proof. We compute in an adapted chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$, then $\mu|_U = f dx_1 \wedge \cdots \wedge dx_n$ where $f \neq 0$. Assuming we do not have to shrink it any further, we have $\vec{n}|_U = n_1 \frac{\partial}{\partial x_1} + \cdots + n_m \frac{\partial}{\partial x_m}$ with $n_1 > 0$. By contraction, we have $\partial D \cap U = \{x_1 = 0\}$, so $dx_1|_{\partial D \cap U} = 0$, therefore

$$\begin{aligned}
\iota(\vec{n})\mu|_{\partial D} &= (n_1 f_1 dx_2 \wedge \cdots \wedge dx_m + \cdots)|_{\{x_1=0\}} \\
&= (n_1 f_2|_{\{x_1=0\}}) dx_2 \wedge \cdots \wedge dx_m \\
&\neq 0
\end{aligned}$$

since the omitted terms only involved dx_1. □

Definition 31.10. $\iota(\vec{n})\mu|_{\partial D}$ is the *orientation induced on ∂D by μ*.

Theorem 31.11 (Stokes). Let $D \subseteq M$ be a regular domain, $\omega \in \Omega_c^{\dim(M)-1}(M)$, then

$$\int_{\partial D} \omega|_{\partial D} = \int_D d\omega := \int_{D\backslash\partial D} d\omega$$

where ∂D is given the induced orientation.

Example 31.12. Let $M = \mathbb{R}$, $D = [0, 1]$, then $\omega = f \in C^\infty(\mathbb{R}) = \Omega_c^{1-1}(\mathbb{R})$ then

$$\int_{[0,1]} df = \int_{\partial[0,1]} f = \int_{\{0,1\}} f = f(1) - f(0).$$

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Proof. We first prove the special case where $M = \mathbb{R}^m$, $D = \mathbb{H}^m = \{x \in \mathbb{R}^m \mid x_1 \leqslant 0\}$, and $\mu = dx_1 \wedge \cdots \wedge dx_m \in \Omega^{\text{top}}(\mathbb{R}^m)$. Note that $\partial D = \{x_1 = 0\} = \{0\} \times \mathbb{R}^{m-1}$ and $\vec{n} = \frac{\partial}{\partial x_1}$, now

$$\iota(\vec{v})\mu|_{\partial D} = dx_2 \wedge \cdots \wedge dx_n|_{\{0\}\times\mathbb{R}^{m-1}}.$$

Given $\omega \in \Omega_c^{m-1}(\mathbb{R}^m)$, then

$$\omega = \sum_{j=1}^{m} (-1)^{j-1} f_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_m$$

for some $f_1, \ldots, f_m \in C_c^\infty(\mathbb{R}^m)$. There exists $R > 0$ such that $\operatorname{supp}(f_j) \subseteq (-R, R)^m$ for all j. Therefore

$$f_j(x_1, \ldots, x_{j-1}, -R, x_{j+1}, \ldots, x_m) = 0 = f_j(x_1, \ldots, x_{j-1}, R, x_{j+1}, \ldots, x_m)$$

for all j. Therefore

$$\begin{aligned} d\omega &= \sum (-1)^{j-1} \frac{\partial f_j}{\partial x_j} dx_{\wedge} dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_m \\ &= \left(\sum_j \frac{\partial f}{\partial x_j} \right) dx_1 \wedge \cdots \wedge dx_m, \end{aligned}$$

and so

$$\begin{aligned} \int_D d\omega &= \int_{\{x_1 \leqslant 0\}} \left(\sum \frac{\partial f}{\partial x_j} \right) dx_1 \cdots dx_j \\ &= \int_{\mathbb{R}^{m-1}} \left(\int_{-R}^{0} \frac{\partial f}{\partial x_1} dx_1 \right) dx_2 \cdots dx_m + \sum_{j>1} \int_{[-R,R]^{m-1} \cap \{x_1 \leqslant 0\}} \left(\int_{-R}^{R} \frac{\partial f}{\partial x_j} dx_j \right) dx_1 \cdots \widehat{dx_j} \cdots dx_m \\ &= \int_{\mathbb{R}^{m-1}} (f_1(0, x_2, \ldots, x_m) - f_1(-R, x_2, \ldots, x_m)) dx_2 \cdots dx_m \\ &= \int_{\mathbb{R}^{m-1}} f_1(0, x_2, \ldots, x_m) dx_2 \cdots dx_m \\ &= \int_{\partial D} (f_1 dx_2 \wedge \cdots \wedge dx_m)|_{\partial D}. \end{aligned}$$

Similarly, note that $dx_1|_{\{0\}\times\mathbb{R}^{m-1}} = 0$, and so $dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_m\Big|_{\{0\}\times\mathbb{R}^{m-1}} = 0$, therefore

$$\int_{\partial D} \omega|_{\partial D} = \int_{\partial D} (f_1 dx_2 \wedge \cdots \wedge dx_m)|_{\partial D}$$

as well.

We now prove the general case. Fix a volume form μ on M, let $\vec{n}$ be in a neighborhood of ∂D and $\iota(\vec{n})\mu|_{\partial D}$ induced orientation. Since $\operatorname{supp}(\omega)$ is compact, then there exists finitely many charts $\{\varphi_j : U_j \to \mathbb{R}^m\}_{j=1}^N$ on M such that

1. $\bigcup_{j=1}^{N} U_j \supseteq \operatorname{supp}(\omega)$,
2. U_j's are connected,
3. $\varphi_j : U_j \to \mathbb{R}^m$ are adapted to D.

We now assign to each φ_j a sign $\pm$, where $+$ means φ_j preserves orientation, i.e., $\varphi_j^*(dr_1 \wedge \cdots \wedge dr_m) = f_j \cdot \mu|_U$ and $f_j > 0$, and $-$ means otherwise. Let $U_0 = M \setminus \operatorname{supp}(\omega)$, and let $\{\rho_1, \ldots, \rho_m\}$ be a partition of unity with $\operatorname{supp}(\rho_j) \subseteq U_j$ for $j = 0, \ldots, m$. Note $\rho_0|_{\operatorname{supp}(\omega)} = 0$, so $\sum\limits_{j=1}^N \rho_j|_{\operatorname{supp}(\omega)} = 1$. Therefore $\omega = \sum\limits_{j=1}^N \rho_j\omega$, and $d\omega = \sum\limits_{j=1}^N d(\rho_j\omega)$. Note that $\operatorname{supp}(\rho_j\omega)$ and $\operatorname{supp}(d(\rho_j\omega)) \subseteq U_j$, now

$$\begin{aligned}\int_M d(\rho_j\omega) &= \int_{U_j} d(\rho_j\omega) \\ &= \operatorname{sgn}(\varphi_j) \cdot \int_{\varphi_j(U_j)} (\varphi_j^{-1})^* d(\rho_j\omega).\end{aligned}$$

and

$$\begin{aligned}\int_D d\omega &= \sum_{j=1}^N \int_{D \cap U_j} d(\rho_j\omega) \\ &= \sum_{j=1}^N \operatorname{sgn}(\varphi) \int_{\varphi_(U_j \cap D)} (\varphi_j^{-1})^* d(\rho_j\omega) \\ &= \sum_{j=1}^N \operatorname{sgn}(\varphi) \int_{\varphi_j(U_j) \cap \{x_j \leqslant 0\}} (\varphi_j^{-1})^* d(\rho_j\omega) \\ &= \sum \operatorname{sgn}(\varphi_j) \cdot \int_{\varphi_j(U_j) \cap \{x_1 \leqslant 0\}} d((\varphi_j^{-1})^* \rho_j\omega) \\ &= \sum \operatorname{sgn}(\varphi_j) \int_{\varphi_j(U_j) \cap \{x_1 \leqslant 0\}} (\varphi_j^{-1})^* \rho_j\omega\big|_{\{x_1=0\}} \\ &= \sum \operatorname{sgn}(\varphi_j) \int_{\varphi_j(U_j \cap \partial D)} (\varphi_j^{-1})^* \rho_j\omega\big|_{\{x_1=0\}} \\ &= \sum \int_{U_j \cap \partial D} (\rho_j\omega)|_{\partial D \cap U_j} \\ &= \int_{\partial D} \sum \rho_j\omega|_{\partial D} \\ &= \int_{\partial D} \omega|_{\partial D}.\end{aligned}$$

□

Definition 32.1. Let M be an oriented manifold, let $\mu \in \Omega^{\text{top}}(M)$ be a volume form. For any vector field X, we have $L_X\mu = \operatorname{div}_\mu(X) \cdot \mu$ for $\operatorname{div}_\mu(X) \in C^\infty(M)$, the *divergence* of X with respect to μ.

Example 32.2. Let $M = \mathbb{R}^3$, $\mu = dx \wedge dy \wedge dz$, then let $X = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} = \frac{\partial}{\partial z}$, then

$$\begin{aligned}L_X\mu &= d(\iota(X)dx \wedge dy \wedge dz) \\ &= d(fdy \wedge dz - gdx \wedge dz + hdx \wedge dy) \\ &= \frac{\partial f}{\partial x}dx \wedge dy \wedge dz - \frac{\partial g}{\partial y}dy \wedge dx \wedge dz + \frac{\partial h}{\partial z}dz \wedge dx \wedge dy \\ &= \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right) dx \wedge dy \wedge dz.\end{aligned}$$

Therefore, the divergence is $\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}+\frac{\partial h}{\partial z}$.

Theorem 32.3 (Divergence)**.** Let M be an orientable manifold and $\mu \in \Omega^{\text{top}}(M)$ volume form, let $X \in \mathfrak{X}(M)$ to be a vector field. Let $D \subseteq M$ be a compact regular domain, then

$$\int_D (\operatorname{div}_\mu(X)\mu = \int_{\partial D} \iota(X)\mu.$$

Proof. We have

$$\int_D \operatorname{div}_\mu(X)\mu = \int_D (L_X \mu) = \int_D d(\iota(X)\mu)$$

by Cartan's formula, then by Stokes' theorem, this is $\int_{\partial D} \iota(X)\mu$. $\square$

33 Nov 8, 2023

Definition 33.1. A *vector bundle* over a manifold M with typical fiber a real finite-dimensional vector space V is a surjective C^∞-map $\pi : E \to M$ such that

1. for all $q \in M$, $\pi^{-1}(q) =: E_q$ is a vector space isomorphic to V;
2. for all $q \in M$, there exists an open neighborhood U of q and a diffeomorphism

$$\begin{aligned}\varphi : \pi^{-1}(U) &\to U \times V\\ \pi^{-1}(q') &\mapsto \{q'\} \times V\end{aligned}$$

for all $q' \in U$, i.e., the commutative diagram

$$\begin{tikzcd} E \supseteq \pi^{-1}(U) \arrow[rr] \arrow[dr, "\pi"'] & & U \times V \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

commutes, and $\varphi|_{E_{q'}} E_{q'} \to \{q'\} \times V$ is an isomorphism.

We say E is the *total space* of the vector bundle $\pi : E \to M$, M is the *base space*, and the maps $\varphi : \pi^{-1}(U) \to U \times V$ are *local trivializations*.

Notation. We denote the vector bundle by (V, E, M), or $\pi : E \to M$, or $V \hookrightarrow E \xrightarrow{\pi} M$, or just E.

Example 33.2. For any manifold M and any finite-dimensional vector space V,

$$\begin{aligned}\pi_M : M \times V &\to M\\ (q, v) &\mapsto q\end{aligned}$$

for all $(q, v) \in M \times V$ is a vector bundle, called the product bundle or trivial bundle.

Example 33.3. For any manifold M, the tangent bundle given by $\pi : TM \to M$ is a vector bundle, with typical fiber $\mathbb{R}^m$ where $m = \dim(M)$. To see that this is a local trivialization, let $\psi : U \to \mathbb{R}^m$ be a coordinate chart on M, then

$$\begin{aligned}\varphi : TU &\to U \times \mathbb{R}^m\\ (q, v) &\mapsto (q, (T_q\psi)(v))\end{aligned}$$

Example 33.4. For any $k \geqslant 0$, any manifold M, the exterior power

$$\bigwedge^k (T^*M) \to M$$

is a vector bundle over M with typical fiber $\bigwedge^k((\mathbb{R}^m)^*)$. For any chart $\psi : U \to \mathbb{R}^m$, we get

$$\begin{aligned}\varphi : \bigwedge^k (T^*U) &\to U \times \bigwedge^k((\mathbb{R}^m)^*) \\ (q, \eta) &\mapsto (q, \bigwedge^k((T_q\psi)^{-1})^*\eta).\end{aligned}$$

Remark 33.5. Let $\pi : E \to M$ be a vector bundle and $W \subseteq M$ be open. We define $E|_W = \pi^{-1}(W)$.

Exercise 33.6. The restriction of E to W, $\pi : E|_W \to W$ is a vector bundle over W.

Definition 33.7. Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be two vector bundles, then *a map of vector bundles* is a smooth map $f : E \to F$ such that

1. for all $q \in M$, $f(E_q) \subseteq F_q$, i.e.,

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & M & \end{tikzcd}$$

   commutes;

2. $f|_{E_q} : E_q \to F_q$ is linear.

Exercise 33.8. Fix a manifold M, then the collection of vector bundles over M and their maps form a category.

Definition 33.9. A vector bundle map $f : E \to F$, i.e., $E \to M$ and $F \to M$ are two vector bundles, is an *isomorphism* if there exists a vector bundle map $g : F \to E$ such that $gf = \mathrm{id}_E$ and $fd = \mathrm{id}_F$.

Definition 33.10. A vector bundle $\pi : E \to M$ is a *trivial bundle* if it is isomorphic to a product bundle.

Example 33.11. Let G be a Lie group, then $TG \to G$ is trivial. To see this, we write down the map

$$\begin{aligned} f : TG &\to G \times \mathfrak{g} \\ (g, v) &\mapsto (g, T_g L_{g^{-1}} v) \end{aligned}$$

or

$$\begin{aligned} f : TG &\to G \times \mathfrak{g} \\ (g, v) &\mapsto (g, T_g R_{g^{-1}} v) \end{aligned}$$

Remark 33.12. $TS^2 \to S^2$ is not trivial, c.f., the Hairy ball theorem.

Remark 33.13. By definition, for any vector bundle $E \to M$, for all $q \in M$, there exists an open neighborhood U of q such that $E|_U$ is trivial.

Exercise 33.14. For any vector bundle $\pi : E \to M$, π is a submersion. *Hint*: note that this is a local statement, and note that this is true for product bundles.

Definition 33.15. A *section* of a vector bundle $\pi : E \to M$ is a C^∞-map $s : M \to E$ such that $\pi(s(q)) = q$, i.e., $\pi \circ s = \mathrm{id}_M$, i.e., $s(q) \in E_q$ for all q.

Notation. We denote $\Gamma(E) = \Gamma(E; M)$ to be the set of all sections of $\pi : E \to M$.

Example 33.16.
- $\Gamma(TM)$ is the set of vector fields,

- $\Gamma(\bigwedge^k(T^*M))\Omega^k(M)$ is the set of differential k-forms;
- $\Gamma(M \times V \xrightarrow{\pi_M} M = \{(s_1, s_2) : M \to M \times V \mid s_1(q) = q\ \forall q\} = C^\infty(M, V)$ is the set of V-valued C^∞-functions.

Lemma 33.17. The set of sections $\Gamma(E)$ of a vector bundle $\pi : E \to M$ is a projective $C^\infty(M)$-module.

Proof. We will prove that this is a module. For example, given $s_1, s_2 \in \Gamma(E)$, we define $(s_1 + s_2)(q) = s_1(q) + s_2(q)$; given $f \in C^\infty(M)$, we define $f \cdot s_1)(q) = f(q)s_1(q)$ for all q. We need to check $s_1 + s_2$, fs_1 are both in C^∞. If $E \to M$, then it suffices to check on $\pi_M : M \to V$ gives $\Gamma(E) = C^|infty(M, V)$ and $s_1 + s_2 : M \to V$ is the composition

$$M \xrightarrow{(s_1,s_2)} V \times V \xrightarrow{+} V$$

Similarly, we have fs as the composition

$$M \xrightarrow{(f,s)} \mathbb{R} \times V \xrightarrow{\cdot} V$$

by scalar multiplication and the function pair. The general case follows since any vector bundle locally "is" a product bundle. ☐

Definition 33.18. A *local section* of $\pi : E \to M$ is a section of $E|_W \to W$ for some open set $W \subseteq M$.

Remark 33.19. Given a vector bundle $E \to M$, let $\{U_\alpha\}_{\alpha \in A}$ to be an open cover, then let $\{s_\alpha \in \Gamma(E|_{U_\alpha})\}_{\alpha \in A}$ be the corresponding local sections. Choose a partition of unity $\{\rho_\alpha\}_{\alpha \in A}$ subordinate to this cover, then $s = \sum\limits_{\alpha \in A} \rho_\alpha s_\alpha$ gives a partition of unity, given that the zero section would be smooth so that the set would be non-empty.

34 Nov 10, 2023

Definition 34.1. The *rank* of a vector bundle $\pi : E \to M$ is the dimension of the fiber E_q for $q \in M$.

Lemma 34.2. Let $f : M \times \mathbb{R}^k \to M \times \mathbb{R}^k$ be an isomorphism of product bundles over M via $\pi_M : M \times \mathbb{R}^k \to M$, then there exists a smooth map $g : M \to \mathrm{GL}(k, \mathbb{R})$ so that $f(q, v) = (q, g(q)v)$ for all $(q, v) \in M \times V$.

Proof. Since $\pi_M(f(q, v)) = q$, $f(q, v) = (q, \varphi(q, v))$ for some C^∞-map $\varphi : M \times \mathbb{R}^k \to \mathbb{R}^k$, so in particular for all $v \in V$ we know the assignment $M \ni q \mapsto \varphi(q, v) \in \mathbb{R}^k$ is C^∞. Let $\{e_1, \ldots, e_k\}$ be the standard basis of $\mathbb{R}^k$, then the functions

$$\begin{aligned} a_j : M &\to \mathbb{R}^k \\ q &\mapsto \varphi(q, e_j) \end{aligned}$$

are C^∞, and if we write $a_j(q) = \begin{pmatrix} a_{1j}(q) \\ \vdots \\ a_{kj}(q) \end{pmatrix}$ with $a_{ij} \in C^\infty(M; \mathbb{R})$, then for all $q \in M$ we have the linear map

$$\begin{aligned} \mathbb{R}^k &\to \mathbb{R}^k \\ v &\mapsto \varphi(q, v) \end{aligned}$$

therefore $\varphi(q, v) = (a_{ij}(q)) \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}$ for all q, v, therefore $g(q) = (a_{ij}(q)) \in \mathrm{GL}(k, \mathbb{R})$. ☐

Lemma 34.3. Let $\pi : E \to M$ be a vector bundle of rank k, and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M such that $E|_{U_\alpha}$ is trivial for all α. Let $U_{\alpha\beta} = U_\alpha \cap U_\beta$, and $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$, then there exists a family $\{\varphi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(k, \mathbb{R})\}_{(\alpha,\beta) \in A^2}$ of C^∞-maps so that

1. $\varphi_{\alpha\alpha}(q) = \mathrm{id}$ for all $q \in U_{\alpha\alpha}$,

2. $\varphi_{\alpha\beta}(q) \circ \varphi_{\beta\alpha}9q) = \mathrm{id}$ for all $q \in U_{\alpha\beta}$, and
3. $\varphi_{\alpha\beta}(q) \circ \varphi_{\beta\gamma}(q) = \varphi_{\alpha\gamma}(q)$ for all $q \in U_{\alpha\beta\gamma}$.

Proof. Since $E|_{U_\alpha}$ is trivial, then there exists isomorphisms $\varphi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^k$, then for any $\alpha, \beta \in A$ we consider

$$\begin{aligned} \varphi_\alpha \circ \varphi_\beta^{-1} : U_{\alpha\beta} \times \mathbb{R}^k &\to U_{\alpha\beta} \times \mathbb{R}^k \\ (q, v) &\mapsto (q, \varphi_{\alpha\beta}(q)v) \end{aligned}$$

for some C^∞-map $\varphi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(\mathbb{R}^k)$ by Lemma 34.2. Then $(q, v) = (\varphi_\alpha \circ \varphi_\alpha^{-1})(q, v) = (q, \varphi_{\alpha\alpha}(q)v)$. Therefore, $\varphi_{\alpha\alpha}(q) = \mathrm{id} \in \mathrm{GL}(k, \mathbb{R})$.

We also have $(q, v) = (\varphi_\alpha \circ \varphi_\beta^{-1} \circ \varphi_\beta \circ \varphi_\alpha^{-1})(q, v) = (q, \varphi_{\alpha\beta}(q)(\varphi_{\beta\alpha}(q)v))$. Similarly, the last one holds. □

Definition 34.4. Given a manifold M and an over cover $\{U_\alpha\}_{\alpha \in A}$, a family of maps $\{\varphi_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}(k, \mathbb{R})\}_{\alpha,\beta \in A}$ satisfying these properties in Lemma 34.3 is called a *Čech cocycle*.

Two cocycles $\{\varphi_{\alpha\beta} : U_\alpha \to \mathrm{GL}(k, \mathbb{R})\}_{\alpha,\beta}$ and $\{\psi_{\alpha\beta} : U_\alpha \to \mathrm{GL}(k, \mathbb{R})\}_{\alpha,\beta}$ are *isomorphic*, i.e., differ by a coboundary, if there exists a family of C^∞-maps $\{f_\alpha : U_\alpha \to \mathrm{GL}(k, \mathbb{R})\}_{\alpha \in A}$ such that $\psi_{\alpha\beta}(q) = f_\alpha(q)\varphi_{\alpha\beta}(q)f_\beta(q)^{-1}$ for all $q \in U_{\alpha\beta}$.

Remark 34.5. There is an equivalence of categories between the category of vector bundles and the category of Čech cocycles.

Our goal is to define operations on vector bundles. For example, for vector bundle $E \to M$ we would like to have

- the dual bundle $E^* \to M$ with $(E^*)_q = \mathrm{Hom}(E_q, \mathbb{R})$ for all q;
- when $E = TM$, we have $E^* = T^*M$;
- the kth tensor power $E^{\otimes k} \to M$ with fibers $(E^{\otimes k})_q := (E_q)^{\otimes k}$;
- the kth exterior power $\bigwedge^k(E) \to M$ with $(\bigwedge^k E)_q = \bigwedge^k(E_q)$ for all q, so that $\bigwedge^k(T^*M)$ is the right bundle;
- given two vector bundles $E \to M$ and $F \to M$, we would want to define the direct sum as $E \otimes F \to M$ with tensor product $E \otimes F \to M$, and the hom bundle $\mathrm{Hom}(E, F)$ with $(\mathrm{Hom}(E, F))_q = \mathrm{Hom}(E_q, F_q)$. Also, it would be nice to know that there is an isomorphism

$$\mathrm{Hom}(E, F) \cong E^* \otimes F$$

Definition 34.6. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories, their *product* $\mathscr{C} \times \mathscr{D}$ is a category with objects $\mathrm{Ob}(\mathscr{C} \times \mathscr{D}) = \mathrm{Ob}(\mathscr{C}) \times \mathrm{Ob}(\mathscr{D})$, and $\mathrm{Mor}(\mathscr{C} \times \mathscr{D}) = \mathrm{Mor}(\mathscr{C}) \times \mathrm{Mor}(\mathscr{D})$, with entrywise composition.

Exercise 34.7. Let $\mathscr{C}$ and $\mathscr{D}$ both be the poset category of two objects, the product category has four objects and nine morphisms.

Definition 34.8. Given a category $\mathscr{C}$, the *core* of the category is the wide subcategory with same objects but only the isomorphisms of $\mathscr{C}$ as morphisms.

Example 34.9. The core of the poset category of two objects is a discrete category of two objects.

The core of the category of vector spaces has all vector spaces as objects and linear isomorphisms as morphisms. Note that for any object V in the core of this category, the hom set on V, $\mathrm{Hom}_{\mathrm{Core}(\mathbf{Vect})}(V, V) = \mathrm{GL}(V)$, which is a Lie group.

Definition 34.10. A functor $F : \mathrm{Core}(\mathbf{Vect}))^n \to \mathrm{Core}(\mathbf{Vect})$ is *smooth* or of C^∞ if for all $(V_1, \ldots, V_n) \in \mathrm{Core}(\mathbf{Vect}))^n$, the functor $F : \mathrm{Hom}((V_1, \ldots, V_n), (V_1, \ldots, V_n)) \to \mathrm{Hom}(F(V_1, \ldots, V_n), F(V_1, \ldots, V_n))$ can be interpreted as a functor from product $\mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_n)$ of Lie groups to $\mathrm{GL}(F(V_1, \ldots, V_n))$, which is C^∞.

Example 34.11.

$$\begin{aligned} F : \mathrm{Core}(\mathbf{Vect}) &\to \mathrm{Core}(\mathbf{Vect}) \\ (T : V \to W) &\mapsto ((T^{-1})^* : V^* \to W^*) \end{aligned}$$

is a smooth functor.

Example 34.12.

$$\begin{aligned} F : \mathrm{Core}(\mathbf{Vect})^2 &\to \mathrm{Core}(\mathbf{Vect}) \\ (T_1 : V_1 \to W_1, T_2 : V_2 \to W_2) &\mapsto (T_1 \oplus T_2 : V_1 \oplus V_2 \to W_1 \oplus W_2) \end{aligned}$$

is a smooth functor as well, since

$$\begin{aligned} \mathrm{GL}(V) \times \mathrm{GL}(W) &\to \mathrm{GL}(V \otimes W) \\ (T_1, T_2) &\mapsto \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \end{aligned}$$

is smooth.

35 Nov 13, 2023

Remark 35.1. Given a vector bundle $\pi : E \to M$, a section $s : M \to E$ of π is C^∞ if and only if for all open subsets $U \subseteq M$, $s|_U$ is C^∞, if and only if for any collection $\{\varphi_\alpha : E|_{U_\alpha} \to U_\alpha \to V\}$ of local trivializations with $\bigcup\limits_\alpha U_\alpha = M$, the composition

$$U_\alpha \xrightarrow{s|_{U_\alpha}} E|_{U_\alpha} \xrightarrow{\varphi_\alpha} U_\alpha \times V \xrightarrow{\pi_V} V$$

Theorem 35.2. Given a smooth functor $F : \mathrm{Core}(\mathbf{Vect})^n \to \mathrm{Core}(\mathbf{Vect})$, then for any manifold M and any vector bundles $\pi_i : E_i \to M$, there exists a vector bundle $\mathbb{F}(E_1, \ldots, E_n) \to M$ with fibers $(\mathbb{F}(E_1, \ldots, E_n))_q = F((E_1)_q, \ldots, (E_n)_q)$.

To prove this, we need

Lemma 35.3. Let N be a set and let $\{O_\alpha\}_{\alpha \in A}$ be a cover of N, i.e., $O_\alpha \subseteq N$ for all α and $\bigcup\limits_{\alpha \in A} O_\alpha = N$. Suppose there exists a collection $\{W_\alpha\}_{\alpha \in A}$ of manifolds and bijections $f_\alpha : O_\alpha \to W_\alpha$ such that

1. for all α, β, denote $O_{\alpha\beta} = O_\alpha \cap O_\beta$, then $f_\alpha(O_{\alpha\beta}) \subseteq W_\alpha$ are open, and such that
2. $f_\alpha \circ f_\beta^{-1} : f_\beta(O_{\alpha\beta}) \to f_\alpha(O_{\alpha\beta})$ for all α, β are smooth,

then N has a topology so that f_α's are homeomorphisms, and a manifold structure such that $f_\alpha : O_\alpha \to W_\alpha$ are diffeomorphisms.

Proof. Similar to the proof we did before. □

Proof of Theorem 35.2. We will prove the case for $n = 2$. Suppose we have a smooth functor $F : \mathrm{Core}(\mathbf{Vect}) \times \mathrm{Core}(\mathbf{Vect}) \to \mathrm{Core}(\mathbf{Vect})$ and vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$, then we may assume there exists an open cover $\{U_\alpha\}_{\alpha \in A}$ of M such that $E_1|_{U_\alpha}$ and $E_2|_{U_\alpha}$ are all trivial. Let $\{\varphi_\alpha^{(i)} : E_i|_{U_\alpha} \to U_\alpha \times V_i\}_{\alpha \in A}$ be a choice of trivializations. Let $\mathbb{F}(E_1, E_2) = \coprod\limits_{q \in M} F((E_1)_q, (E_2)_q) = \bigcup\limits_{q \in M} \{q\} \times F((E_1)_q, (E_2)_q)$. We then have a map

$$\begin{aligned} \pi : \mathbb{F}(E_1, E_2) &\to M \\ (q, w) &\mapsto q \end{aligned}$$

for all $q \in M$ and $w \in F((E_1)_q, (E_2)_q)$, where we consider $\mathbb{F}(E_1, E_) = \bigcup\limits_{\alpha \in A} \mathbb{F}(E_1, E_2)|_{U_\alpha} \equiv \bigcup\limits_{\alpha \in A} \pi^{-1}(U_\alpha)$ for all α and all $q \in U_\alpha$ and for $i = 1, 2$. By functoriality of F, we have isomorphisms

$$F\left(\varphi_\alpha^{(1)}\Big|_{(E_1)_q}, \varphi_\alpha^{(2)}\Big|_{(E_2)_q}\right) : F((E_1)_q, (E_2)_q) \to F(V_1, V_2).$$

Define

$$\varphi_\alpha : \mathbb{F}(E_1, E_2)|_{U_\alpha} \to U_\alpha \times F(V_1, V_2)$$

$$(q,w)\mapsto (q, F(\varphi_\alpha^{(1)}\big|_{(E_1)_q}, \varphi_\alpha^{(2)}\big|_{(E_2)_q})w)$$

Recall that we have smooth maps

$$\begin{aligned}\varphi_{\alpha\beta}^{(i)} &: U_{\alpha\beta}\to \mathrm{GL}(V_i)\\ q &\mapsto \varphi_\alpha^{(i)}\big|_{(E_i)_q}\circ(\varphi_\beta^{(i)}\big|_{(E_i)_q})^{-1}\end{aligned}$$

and then

$$(\varphi_\alpha^{(i)}\circ(\varphi_\beta^{(i)})^{-1})(q,v)=(q,\varphi_{\alpha\beta}^{(i)}(q)v).$$

Therefore we have

$$\begin{aligned}F(\varphi_\alpha^{(1)}\big|_{(E_1)_q},\varphi_\alpha^{(2)}\big|_{(E_2)_q})\circ F(\varphi_\beta^{(1)}\big|_{(E_1)_q},\varphi_\beta^{(2)}\big|_{(E_2)_q})^{-1} &= F(\varphi_\alpha^{(1)}\big|_{(E_1)_q}\circ(\varphi_\beta^{(1)}\big|_{(E_1)_q})^{-1},\varphi_\alpha^{(2)}\big|_{(E_2)_q}\circ(\varphi_\beta^{(2)}\big|_{(E_2)_q})^{-1})\\ &= F(\varphi_{\alpha\beta}^{(1)}(q),\varphi_{\alpha\beta}^{(2)}(q))\end{aligned}$$

Now since F is C^∞, then

$$\begin{aligned}\varphi_{\alpha\beta} &: U_{\alpha\beta}\to \mathrm{GL}(F(V_1,V_2))\\ q &\mapsto (\varphi_{\alpha\beta}^{(1)}(q),\varphi_{\alpha\beta}^{(2)}(q))\end{aligned}$$

is also C^∞, which is the composition

$$U_{\alpha\beta}\xrightarrow{(\varphi_{\alpha\beta}^{(1)},\varphi_{\alpha\beta}^{(2)})}\mathrm{GL}(V_1)\times\mathrm{GL}(V_2)\xrightarrow{F}\mathrm{GL}(F(V_1,V_2)).$$

It follows that

$$\begin{aligned}\varphi_\alpha\circ\varphi_\beta^{-1} &: U_{\alpha\beta}\times F(V_1,V_2)\to U_{\alpha\beta}\times F(V_1,V_2)\\ (q,w) &\mapsto (q, F\circ(\varphi_{\alpha\beta}^{(1)}(q),\varphi_{\alpha\beta}^{(2)}(q))(w))\end{aligned}$$

is C^∞. By Lemma 35.3, we get $\mathbb{F}(E_1,E_2)$ is a manifold, and for all α, $\varphi_\alpha:\mathbb{F}(E_1,E_2)|_{U_\alpha}\to U_\alpha\times F(V_1,V_2)$ are smooth diffeomorphisms. Fiberwise, they are linear isomorphisms. Consequently, $\pi:\mathbb{F}(E_1,E_2)\to M$ is a vector bundle with typical fiber $F(V_1,V_2)$. □

Remark 35.4. Recall that we can think of a section ω of $\bigwedge^k T^*M\to M$ as assigning for all $q\in M$ a k-linear alternating map $\omega_q:(T_qM)^k\to\mathbb{R}$, and the fact that ω is C^∞ translates into in each coordinate chart $(x_1,\dots,x_m):U\to\mathbb{R}^m$ $\omega=\sum\limits_{|I|=k}a_I dx_I$ and $a_I\in C^\infty(U,\mathbb{R})$. Similarly, a section σ of $(T^*M)^{\otimes 2}\to M$ assigns to each $q\in M$ a bilinear map $\sigma_q:T_qM\times T_qM\to\mathbb{R}$, and for every chart $(x_1,\dots,x_m)$, $\sigma|_U=\sum a_{ij}dx_i\otimes dx_j$ for all $a_{ij}\in C^\infty(U,\mathbb{R})$.

Definition 35.5. A *Riemannian metric* on a manifold M is a section g of $(T^*M)^{\otimes 2}\to M$ so that for all $q\in M$, $g_q: T_qM\times T_qM\to\mathbb{R}$ is symmetric, i.e., $g_q(v,w)=g_q(w,v)$ for all v,w, and positive definite, i.e., $g_q(v,v)\geqslant 0$ for all v, and $g_q(v,v)=0$ if and only if $v=0$.

Exercise 35.6. $g\in\Gamma((T^*M)^{\otimes 2})$ is a Riemannian metric if and only if for every coordinate chart $(x_1,\dots,x_m):U\to\mathbb{R}$, $g=\sum g_{ij}dx_i\otimes dx_j$, and for all $q\in U$, the matrix $(g_{ij}(q))$ is symmetric and positive definite.

Exercise 35.7. Any manifold admits a Riemannian metric.