MATH 214A Notes

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1 Lecture 1

Algebraic geometry is about shapes defined by polynomial equations. One may realize it is especially easier to understand algebraic sets over $\mathbb{C}$.

Example 1.1. $\{(x,y) \in \mathbb{C}^2 : x^2 + y^2 = 1\} \cong \mathbb{C}\backslash\{0\}$.

Algebraic geometry studies algebraic curves over $\mathbb{C}$, i.e., structure of dimension 1. Because the field $\mathbb{C}$ is algebraically closed, then every polynomial $f \in \mathbb{C}[x]$ can be factored into degree 1 polynomials, i.e., $f(x) = a(x-b_1)\cdots(x-b_n)$ for some $a \in \mathbb{C}$, $n \geq 0$, and $b_1, \ldots, b_n \in \mathbb{C}$. This would not happen over $\mathbb{R}$, for instance.

Algebraic geometry looks at equations with more variables, in general.

Example 1.2. Consider $\{x \in \mathbb{R} : x^3 + ax^2 + bx + c = 0\}$ for some $a, b, c \in \mathbb{R}$. Typically, the equation has 1 root or 3 roots, depending on the shape of the diagram. However, if we substitute $\mathbb{R}$ with $\mathbb{C}$, then we essentially always have 3 roots in this equation, even though sometimes there exists a double root.

To classify algebraic varieties, one key step for varieties over $\mathbb{C}$ is to look at them just as topological spaces.

Example 1.3. Consider $\{(x,y) \in \mathbb{C}^2 : x^d + y^d = 1\}$. This is a complex curve homeomorphic to a real 2-manifold of genus g minus a finite set. In this case, we have $g = \frac{(d-1)(d-2)}{2}$.

Theorem 1.4 (Faltings)**.** If an algebraic curve X over $\mathbb{Q}$ has genus $g \geq 2$, then the set of rational points $X(\mathbb{Q})$ is finite.

In some sense, complexity in algebra and topology are related.

Sometimes people also look at the connection between algebraic geometry and number theory.

Example 1.5. What is $\{(x, y, z) \in \mathbb{Z}^3 : x^5 + y^5 = z^5\}$? The only solution is $(0, 0, 0)$. Note that this set is equivalent to $\{(x, y) \in \mathbb{Q}^2 : x^5 + y^5 = 1\}$.

Number theory allows us to study numbers in finite fields. We can define numbers like the genus and topology even in finite characteristics.

Definition 1.6 (Affine Space)**.** Let k be an algebraically closed field. The *affine n-space over k* is

$$\mathbb{A}_k^n = k^n = \{(a_1, \dots, a_n) : a_1, \dots, a_n \in k\}.$$

Let $R = k[x_1, \dots, x_n]$. An element $f \in R$ determines a function $\mathbb{A}_k^n \to k$. For an element $f \in R$, its *zero set* is $\{f = 0\} \subseteq \mathbb{A}_k^n$, often defined by

$$Z(f) = \{f = 0\} := \{(a_1, \dots, a_n) \in \mathbb{A}_k^n : f(a_1, \dots, a_n) = 0\}.$$

Similarly, for a set T, its zero set is

$$Z(T) = \{a \in \mathbb{A}_k^n : f(a) = 0 \ \forall f \in T\}.$$

An *affine algebraic set* over k is a subset of $\mathbb{A}_k^n$ for some $n \geq 0$ of the form $Z(T)$ for some subset $T \subseteq R = k[x_1, \dots, x_n]$.

Remark 1.7. Given a subset $T \subseteq R$, let $I \subseteq R$ be the ideal generated by T, then $Z(T) = Z(I)$.

Example 1.8. What is the algebraic set of the affine line $\mathbb{A}_k^1$? We want to find all subsets of $\mathbb{A}_k^1 \cong k$ defined by some ideal $I \subseteq k[x]$. If $I = \{0\}$, then $Z(I) = \mathbb{A}_k^1$. If not, then pick $f \neq 0$ in I, then $Z(I) \subseteq Z(f)$, and $f = a(x - b_1) \cdots (x - b_n)$, so $Z(f) = \{b_1, \dots, b_n\}$.

We conclude that an affine set in $\mathbb{A}_k^1$ is either all of $\mathbb{A}_k^1$ or a finite set of points.

2 Lecture 2

Definition 2.1 (Zariski Topology)**.** Let k be an algebraically closed field and let $n \geq 0$. The *Zariski Topology* on $\mathbb{A}_k^n \cong k^n$ is defined by closed sets, which is defined as follows: a subset $S \subseteq \mathbb{A}_k^n$ is closed if and only if it is of the form $S = Z(I)$ for some ideal $I \subseteq R$ where $R = k[x_1, \dots, x_n]$.

Example 2.2. The twisted cubic curve in $\mathbb{A}^3_k$ is defined as

$$\{(\mathcal{A}, \mathcal{A}^2, \mathcal{A}^3) : \mathcal{A} \in k\} \subseteq \mathbb{A}^3_k.$$

This is Zariski-closed in $\mathbb{A}^3_k$ since

$$S = \{y = x^2, z = x^3\} \subseteq \mathbb{A}^3_k$$

is equivalent to $Z(\{y - x^2, z - x^3\}$, which is just $Z(I)$ where $I \subseteq k[x, y, z]$ is just the ideal $(y - x^2, z - x^3)$.

Remark 2.3. If $k = \mathbb{C}$, then we also have the classical topology on $\mathbb{A}^n_{\mathbb{C}} = \mathbb{C}^n$, based on the usual metric on $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

It is easy to see that Zariski-closed in $\mathbb{A}^n_{\mathbb{C}}$ implies closure in the classical topology. The converse is obviously not true, for example consider the closed balls in $\mathbb{C}^3$.

Lemma 2.4. The Zariski topology in $\mathbb{A}^n_k$ is a well-defined topology.

Proof. By definition, a topological space is a set with a colletion of subsets called "the open subsets of X", such that

1. $\varnothing$ and X are open in X,
2. union of any collection of open sets is open,
3. intersection of finitely many open sets is open.

Equivalently, the closed subsets of X satisfy

1. $\varnothing$ and X are closed in X,
2. intersection of any collection of closed sets is closed,
3. union of finitely many closed sets is closed.

Indeed,

1. $\mathbb{A}^n_k = Z(0)$ and $\varnothing = Z(R)$.
2. Given a collection S_α of closed subsets of $X = \mathbb{A}^n_k$ where $\alpha \in I$ set, which could be infinite, the intersection of the collection is just the union of the zero sets.

   By definition, for each $\alpha \in I$, we can choose an ideal $I_\alpha \subseteq R$ with $S_\alpha = Z(I_\alpha) \subseteq \mathbb{A}^n_k$.

   Define $I = \sum\limits_{\alpha \in I} I_\alpha \subseteq R$ (i.e., the set of all possible finite sums), then $Z(I) = \bigcap\limits_{\alpha \in I} Z(I_\alpha) = \bigcap\limits_{\alpha \in I} S_\alpha$, so it is closed.

3. Given closed sets $S, T \subseteq \mathbb{A}_k^n$, we want to show that $S \cup T$ is closed. By definition, choose I and J such that $S = Z(I)$ and $T = Z(J)$. Take $K = I \cap J$ or $J = IJ$ (i.e., finite sum of elements ab with $a \in I$ and $b \in J$), then it suffices to show that $Z(I \cap J) = Z(IJ) = Z(I) \cup (J)$.

   Example 2.5. Note that the two structures may not be equivalent. Let $R = k[x]$ and let $I = J = (x)$. Now $Z(I) = Z(J) = \{0\}$, then $I \cap J = (x)$, but $IJ = (x^2)$.

   Remark 2.6. Essentially, if $I = (f_1, \dots, f_r)$ and $J = (g_1, \dots, g_s)$, then $IJ = (f_i g_j : \forall i, j)$.

   However, things look better if we look at their radicals.

   Exercise 2.7. Show that for any commutative R and ideals I and J, the radicals satisfy $\operatorname{rad}(I \cap J) = \operatorname{rad}(IJ)$.

   To finish the proof, we show that $Z(IJ) = Z(I) \cup Z(J)$. Indeed, we have $IJ \subseteq I$ and $IJ \subseteq J$, so $Z(IJ) \supseteq Z(I)$ and $Z(IJ) \supseteq Z(J)$, so $Z(I) \cup Z(J) \subseteq Z(IJ)$.

   Conversely, we want to show $Z(IJ) \subseteq Z(I) \cup Z(J) \subseteq \mathbb{A}_k^n$.

   Let $a = (a_1, \dots, a_n) \in k^n$ be a point in $Z(IJ)$. Suppose $a \notin Z(I)$ and $a \notin Z(J)$, so there exists $f \in I$ such that $f(a) \neq 0$, and there exists $g \in J$ such that $g(a) \neq 0$, then $(fg)(a) = f(a)g(a) = 0$, but $fg \in IJ$, $(fg)(a) \neq 0$, contradiction.

□

Remark 2.8. Note that $\mathbb{A}_k^n$ is not Hausdorff for $n > 1$. In fact, the intersection of any two non-empty open subsets is non-empty.

For $\mathbb{A}_k^1$, an open subset of $\mathbb{A}_k^1$ is either $\varnothing$ or a $\mathbb{A}_k^1$-finite set. Note that k is infinite since it is algebraically closed, so the intersection of two intervals on $\mathbb{A}_k^1$ (with finitely many isolated points excluded) should not be empty.

Definition 2.9 (Connected, Irreducible)**.** A topological space X is *connected* if $X \neq \varnothing$, and you cannot write X as the disjoint union of two non-empty closed subsets.

A topological space X is *irreducible* if $X \neq \varnothing$, and you cannot write X as the union of two proper closed subsets.

Example 2.10. For example, the set defined by two parallel lines is not connected; the set defined by the union of a circle and a line passing through the circle is connected, but not irreducible.

Remark 2.11. A Hausdorff space with at least 2 points is never irreducible.

Example 2.12. $[0,1]$ is not irreducible since $[0,1] = [0,\frac{1}{2}] \cup [\frac{1}{2},1]$, but $\mathbb{A}_k^n$ is irreducible.

Theorem 2.13 (Hilbert's Nullstellensatz)**.** For an algebraically closed field k and $n \geq 0$, there is a one-to-one correspondence between radical ideals in $R = k[x_1, \ldots, x_n]$ and the Zariski closed subsets of $\mathbb{A}_k^n$. More precisely, this correspondence is given by the mapping $I \mapsto Z(I)$ for radical ideals I and the mapping $S \mapsto I(S) = \{f \in R : f(a) = 0\ \forall a \in S\}$ for closed subset $S \subseteq \mathbb{A}_k^n$.

Definition 2.14 (Reduced Ring, Radical Ideal)**.** A commutative ring R is *reduced* if every nilpotent element is 0, i.e., if $a \in R$ such that $a^m = 0$ for some $m > 0$, then $a = 0$.

An ideal I in a commutative ring R is *radical* if the ring R/I is radical. In particular, $I \subseteq R$ is radical if and only if for any $a \in R$ with $a^m \in I$ for some $m > 0$, we know $a \in I$.

For any ideal I, $\mathrm{rad}(I) = \{a \in R : a^m \in I \text{ for some } m > 0\}$.

Lemma 2.15. An affine algebraic set $X \subseteq \mathbb{A}_k^n$ is irreducible if and only if $I(Y) \subseteq R$ is prime.

Proof. ($\Longrightarrow$): Let $Y \subseteq \mathbb{A}_k^n$ be an irreducible algebraic set.

We define the subspace topology on Y as follows: a subset of Y is closed in Y if it is the intersection of some closed subset (of X) and Y.

Therefore, since $Y \neq \varnothing$, so $I(Y) \neq R$ as $1 \in R$ is not in $I(Y)$.

Suppose $f, g \in R$ with $fg \in I(Y)$. We want to show that f or g is in $I(Y)$. Since $fg \in I(Y)$, $Y = (Y \cap \{f = 0\}) \cup (Y \cap \{g = 0\})$ is the union of two closed sets in Y. Therefore, since Y is irreducible, then either $Y = Y \cap \{f = 0\}$, or $Y = Y \cap \{g = 0\}$. That is, $f \in I(Y)$ or $g \in I(Y)$, as desired.

($\Longleftarrow$): Given an affine algebraic set $X \subseteq \mathbb{A}_k^n$ such that the ideal $I(X) \subseteq R$ is prime. That means $1 \notin I(X)$, and, if $f, g \in R$ such that $fg \in I(X)$, then $f \in I(X)$ or $g \in I(X)$. Note that if $X = \varnothing$, then $I(X)$ would be R, which is not prime. Therefore, $X \neq \varnothing$. Suppose $X = S_1 \cup S_2$ for closed subsets $S_1, S_2 \subsetneq X$. We pick $p \in S_2 \backslash S_1$ and $q \in S_1 \backslash S_2$. Since S_1 and S_2 are closed in $\mathbb{A}_k^n$, there is a polynomial $f \in I(S_1)$ and $f(q) \neq 0 \in k$. Similarly, there is a polynomial $g \in I(S_2)$ but with $g(p) \neq 0$. Then $fg \in I(X)$. Since $I(X)$ is prime, $f \in I(X)$ or $g \in I(X)$, contradiction. $\square$

3 Lecture 3

Remark 3.1. For any subset $X \subseteq \mathbb{A}_k^n$, $I(X) \subseteq R$ is radical.

Proof. If $f \in R$ has $f^m \in I(X)$ for some $m > 0$, then $f \in I(X)$. Therefore, at any $p \in X$, $f(p)^m = 0 \in k$. Hence, $f(p) = 0 \in k$. $\square$

Remark 3.2. $Z(I) = Z(\mathrm{rad}(I))$ for ideal $I \subseteq R = k[x_1, \dots, x_n]$.

Example 3.3. Affine n-space $\mathbb{A}_k^n$ is irreducible.

Proof. Think of $\mathbb{A}_k^n$ as a closed set in itself, then $I(\mathbb{A}_k^n) = 0$, and so $\mathbb{A}_k^n$ is irreducible if and only if $0 \subseteq k[x_1, \dots, x_n]$ is prime, if and only if $k[x_1, \dots, x_n]$ is a domain. $\square$

Remark 3.4. For any irreducible topological space, the intersection of any two non-empty open subsets is non-empty. (So this holds in $\mathbb{A}_k$ per se.)

Definition 3.5 (Affine Variety)**.** An *affine variety* over k is an irreducible affine algebraic set in some $\mathbb{A}_k^n$.

Definition 3.6 (Irreducible)**.** Let R be a domain. Any element $f \in R$ is *irreducible* if $f \neq 0$ and for any $g, h \in R$ such that $f = gh$, either g or h must be a unit.

Remark 3.7. This concept is useless unless R is a UFD, where R admits a unique factorization.

Proposition 3.8. If R is a UFD, and $f \in R$ is irreducible, then (f) is a prime ideal. In particular, for any field k, the polynomial ring $k[x_1, \dots, x_n]$ is a UFD.

We now have the notion of an irreducible polynomial $f \in k[x_1, \dots, x_n]$ over k. In particular, the units in the polynomial ring $k[x_1, \dots, x_n]$ is just k^*, i.e., the units in k.

Remark 3.9. The proposition implies that for any irreducible polynomial f over a field k, the ideal $(f) \subseteq R$ is prime.

Corollary 3.10. For an irreducible polynomial $f \in k[x_1, \dots, x_n]$ over an algebraically closed field k, $\{f = 0\} \subseteq \mathbb{A}_k^n$ is an affine variety over k. This is called an *irreducible hypersurface* in $\mathbb{A}_k^n$.

For $n = 1$, an irreducible polynomial in $k[x]$ (with k algebraically closed) is of the form $c(x - a)$ for $a, c \in k$.

Recall the following exercise in homework:

Exercise 3.11. Let $g \in k[x_1, \dots, x_{n-1}]$. Then $x_n^2 - g(x_1, \dots, x_{n-1})$ is irreducible over k if and only if g is not a square in $k[x_1, \dots, x_{n-1}]$.

For example, $x^2 - y^{17}$ is irreducible over $\mathbb{C}$, i.e., $\{x^2 = y^{17}\} \subseteq \mathbb{A}_{\mathbb{C}}^2$ is a variety.

Example 3.12. Over $\mathbb{R}$, x^2+y^2 is irreducible since $-y^2$ is not a square in $\mathbb{R}[y]$. Geometrically, we see that the set $\{(x,y) \in \mathbb{R}^2 : x^2+y^2=0\} = \{(0,0)\}$.

Over $\mathbb{C}$, as $x^2+y^2=(x+iy)(x-iy)$, then geometrically we see $\{(x,y) \in \mathbb{C}^2 : x^2+y^2 = 0\} = \{(x=iy)\} \cup \{(x=-iy)\}$.

Note that for $n \geq 3$, $x_1^2 + \cdot + x_n^2$ is irreducible over $\mathbb{C}$.

Definition 3.13 (Coordinate Ring)**.** For an affine algebraic set $X \subseteq \mathbb{A}_k^n$, the *coordinate ring* of X (or *ring of regular functions* on X) is $\mathcal{O}(X) := k[x_1, \ldots, x_n]/I(X)$. This is isomorphic to the image of mapping from $k[x_1, \ldots, x_n]$ to the ring of all functions $X \to k$.

Example 3.14. Consider $X = \{x^2 = y^3\} \subseteq \mathbb{A}_{\mathbb{C}}^2$. Then $x^5 - 7y$ is a regular function on X, and is equal to $x^5 - 7x + 8(x^2 - y^3)$ on X.

Remark 3.15. For an affine algebraic set X, $\mathcal{O}(X)$ is a finitely-generated commutative k-algebra. Also, for an affine variety $X \subseteq \mathbb{A}_k^n$, $\mathcal{O}(X)$ is a domain as well.

Conversely, for any finitely-generated commutative k-algebra R (which is a domain), $R \cong \mathcal{O}(X)$ for some affine variety $X \subseteq \mathbb{A}_k^n$ for some $n \geq 0$. Similar classification holds for general affine algebraic sets.

Proof. Let R be a finitely-generated k-algebra which is a domain, then $R = k[x_1, \ldots, x_n]/I$ for some $n \geq 0$ and some ideal I. Since R is a domain, I is prime. So $Z(I) \subseteq \mathbb{A}_k^n$ is an affine variety X.

We want to show that $R \cong \mathcal{O}(X)$ as k-algebras. Here $\mathcal{O}(X) \cong k[x_1, \ldots, x_n]/I(X)$, where we can denote $I(X) = I(Z(I))$. By Nullstellensatz, $I(Z(I))$ is just I if it is radical. Now since I is prime, then it is radical indeed, and we are done. $\square$

Example 3.16. $\mathbb{A}_k^1$ and $X = \{y = x^2\} \subseteq \mathbb{A}_k^2$ have isomorphic coordinate rings (as k-algebras).

Proof. One would realize that $\mathcal{O}(\mathbb{A}_k^1) = k[x]$ and $\mathcal{O}(X) = k[x,y]/I(X)$. Note that $y - x^2$ is irreducible, so $(y - x^2) \subseteq k[x,y]$ is prime, then $I(X) = I(Z(y-x^2)) = (y - x^2)$. Therefore, $\mathcal{O}(X) = k[x,y]/I(X) \cong k[x,y]/(y-x^2) \cong k[x]$.

Geometrically, the two structures are just a horizontal line and a quadratic curve, respectively. The isomorphic is given by the projection of the quadratic curve onto the horizontal axis. $\square$

4 Lecture 4

Definition 4.1 (Noetherian)**.** A topological space X is *Noetherian* if every descending sequence of closed subsets $X \supset Y_1 \supset Y_2 \supset \cdots$, there is some $N \in \mathbb{Z}^+$ such that $Y_N = Y_{N+1} =$

$\cdots$. This is essentially a DCC on X.

Remark 4.2. Note that $\mathbb{R}$ and $[0,1]$ are not Noetherian with the classical topology.

Lemma 4.3. Every affine algebraic set over an algebraically closed field k is Noetherian (as a topological space).

Proof. We are given a closed subset $X \subseteq \mathbb{A}_k^n$ for some $n \geq 0$. Here $\mathcal{O}(X)$ is a finitely-generated (commutative) k-algebra (and a reduced ring). By the Nullstellensatz, we have a one-to-one correspondence between closed subsets of X and radical ideals of $\mathcal{O}(X)$. To see this, we know a one-to-one correspondence between closed subsets of $\mathbb{A}_k^n$ and radical ideals in $k(x_1, \ldots, x_n)$, then $\mathcal{O}(X) = k[x_1, \ldots, x_n]/I(X)$. By Hilbert's basis theorem, $\mathcal{O}(X)$ is a Noetherian ring, i.e., every ideal in $\mathcal{O}(X)$ is finitely-generated as an ideal, or equivalently, the ACC condition. Therefore, every decreasing sequence of closed subsets of X terminates, i.e., X is Noetherian as a topological space. $\square$

Theorem 4.4. Every Noetherian topological space X can be written as a finite union of irreducible closed subsets, i.e., $X = Y_1 \cup \cdots \cup Y_n$ for some $n \geq 0$ and irreducible closed subsets Y_i of X.

Moreover, if we also require that Y_i is not contained in Y_j for all $i \neq j$, then this decomposition is unique up to reordering.

Remark 4.5. We call the Y_i's (with all the conditions above) the *irreducible component* of X.

Definition 4.6 (Dimension)**.** The *dimension* of a topological space X is $\dim(X) = \sup\{n \geq 0 :$ there is a chain of length n of irreducible closed subsets of $X,\ Y_0 \subsetneq Y_1 \subsetneq Y_2 \subsetneq \cdots \subsetneq Y_n\}$.

Exercise 4.7. Show that $\dim(\mathbb{R}^3) = 0$ for $\mathbb{R}^3$ with the classical topological space.

Example 4.8. $\dim(\mathbb{A}_k^1) = 1$ with the Zariski topology. Recall that any closed set on this space is either itself or a set of finitely many points. Therefore, the largest chain of irreducible closed subsets has length $\{a\} \subsetneq \mathbb{A}_k^1$ for any $a \in k$.

Definition 4.9 (Krull Dimension)**.** The *(Krull) dimension* of a commutative ring R is $\sup\{n \geq 0 :$ there is a chain of length n of prime ideals in $R : \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n\}$.

Lemma 4.10. Let X be an affine algebraic set over k. Then $\dim(X) = \dim(\mathcal{O}(X))$, i.e., the dimension of the topological space equals the (Krull) dimension of the ring.

Proof. We have a one-to-one correspondence between prime ideals in $\mathcal{O}(X)$ and irreducible closed subsets of X (containing whatever $I(X)$ we quotient out), reversing the directions of the inclusions. $\square$

Definition 4.11 (transcendence degree)**.** Let $k \subseteq E$ be a field extension (not necessarily finite, or even algebraic). There is a set I and a set of elements $x_i \in E$ for $i \in I$ such that $k \subseteq k(x_i : i \in I) \subseteq$, where $k(x_i : i \in I) = \mathrm{Frac}(k[x_i : i \in I])$ is the rational function field on a set of variables, such that E is algebraic over $k(x_i : i \in I)$. The *transcendence degree* of E over k is the cardinality $|I|$. This is well-defined.

Theorem 4.12. Let k be any field and let A be a domain which is also a finitely-generated (commutative) k-algebra. Then $\dim(A)$ is the transcendence degree of $\mathrm{Frac}(A)/k$, i.e., $\dim(A) = \operatorname{tr}\deg(\mathrm{Frac}(A)/k)$.

Corollary 4.13. For any $n \geq 0$ and algebraically closed field k, $\dim(\mathbb{A}_k^n) = n$.

Proof. We have $\dim(\mathbb{A}_k^n) = \dim(k[x_1, \ldots, x_n]) = \dim(\mathcal{O}(\mathbb{A}_k^n)) = \operatorname{tr}\deg(k(x_1, \ldots, x_n)/k) = n$. $\square$

Proposition 4.14 (Krull's Principal Ideal Theorem)**.** Let A be a Noetherian ring, and let $f \in A$ be an element which is neither zero divisor nor a unit, then every minimal prime ideal $\mathfrak{p}$ containing f has height 1.

Corollary 4.15. A variety in $\mathbb{A}_k^n$ has dimension $n-1$ if and only if it is the zero set $Z(f)$ of a single non-constant irreducible polynomial in $A = k[x_1, \ldots, x_n]$.

Proof. See Hartshorne Section I.1 Proposition 1.13. $\square$

In the classical topology, $\mathbb{C}P^n$ is a compact complex manifold, containing $\mathbb{C}^n$ as an open subset; note that $\mathbb{C}^n$ is not compact for $n \geq 1$.

Example 4.16. The 2-sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is compact in the classical topology in $\mathbb{R}^3$. However, $S^2_{\mathbb{C}} = \{(x, y, z) \in \mathbb{A}^3_{\mathbb{C}} : x^2 + y^2 + z^2 = 1\}$ is not compact in the classical topology in $\mathbb{C}^3$.

Indeed, consider the function z desecending in $\mathbb{C}$. So we have an unbounded compact function on $S^2_{\mathbb{C}}$ with values decreasing in $\mathbb{C}$, so $S^2_{\mathbb{C}}$ is not compact.

Definition 4.17 (Projective Space)**.** For $n \geq 0$ and k algebraically closed, the *projective n-space over k P_k^n* is the set of one-dimensional k-linear subspaces of the k-vector space k^{n+1}.

Example 4.18. P_k^0 is just a point.

Definition 4.19 (Homogeneous Coordinates)**.** For $a_0, \ldots, a_n \in k$, not all zeros, we write $[a_0, \ldots, a_n] \in P_k^n$ to mean the line $k(a_0, \ldots, a_n) \subseteq k^{n+1}$.

Remark 4.20. Note that $[0, \ldots, 0]$ is not defined in P_k^n.

Clearly, $[a_0, \ldots, a_n] = [b_0, \ldots, b_n]$ if and only if there exists $c \in k^*$ such that $b_i = ca_i$ for all $0 \leq i \leq n$.

Example 4.21. We can define a bijection $P_k^1 \cong \mathbb{A}_k^1 \cup \{\infty\}$ by the following correspondence: every point in P_k^1, $[a_0, a_1]$ with coordinates not both 0, is either equal to $[0, 1]$ or to $[1, b]$ for some $b \in k$, and that is a unique way of writing the point.

Remark 4.22. By adding a point of infinity, we make sure parallel lines intersect at infinity.

5 Lecture 5

Remark 5.1. In fact, we can make a generalization: $P_k^1 := \mathbb{A}_k^1 \cup \{\infty\}$. Let k be an algebraically closed field and let $n \geq 0$, let $0 \leq i \leq n$, then $[x_0, \ldots, x_n] \in P^n(k)$. Note that there exists a bijective correspondence between $\{x_i \neq 0\}$ ($\subseteq P_k^n$) and $\mathbb{A}_k^n$, via $[x_0, \ldots, x_i, \ldots, x_n] \mapsto (\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i})$. Clearly P_k^n is covered by these $n+1$ "coordinate charts", as in $P_k^1 \cong (P_k^1 \backslash \{\infty\}) \cup (P_k^1 \backslash \{0\}) \cong \mathbb{A}_k^1 \cup \mathbb{A}_k^1$.

We can also see that $P_k^2 = \{x_0 \neq 0\} \cup P_k^1 \cong \mathbb{A}_k^2 \cup P_k^1 = \mathbb{A}_k^2 \cup \mathbb{A}_k^1 \cup \{*\}$, where $* = [0, x_1, x_2] \in P_k^2$.

Definition 5.2 (Homogeneous Polynomial)**.** A polynomial $f \in k[x_0, \ldots, x_n]$ is *homogeneous* of degree $d \geq 0$ if $f = \sum\limits_{\text{finite sum}} a_{i_0, \ldots, i_n} x_0^{i_0} \ldots x_i^{i_n}$ with $a_I \in k$ and $i_0 + \ldots + i_n = d$.

Remark 5.3. Note that a polynomial f (homogeneous or not) does not give a well-defined function $f : P^n(k) \to k$: for a point $[b_0, \ldots, b_n] \in P^n(k)$, if there is another point in the same class (off by a scaling), the polynomial then produces a different value.

But, if f is homogeneous of degree d, then $f(ca_0, \ldots, ca_n) = c^d f(a_0, \ldots, a_n)$ for any $c \in k$.

Therefore, the zero set of a homogeneous polynomial f is a well-defined subset of P_k^n, $Z(f) = \{f = 0\} \subseteq P_k^n$, called a *hypersurface* in P_k^n.

Definition 5.4 (Projective Algebraic Set)**.** A *projective algebraic set* over k is a subset $X \subseteq P_k^n$ (for some $n \geq 0$) that equal to $Z(T) := \bigcap\limits_{f \in T} Z(f)$ for some set T of homogeneous polynomials in $k[x_0, \ldots, x_n]$.

Remark 5.5. We will see later that this set T is defined as $T = Z(I)$ for a homogeneous ideal in $k[x_0, \ldots, x_n]$.

Definition 5.6 (Zariski Topology)**.** The *Zariski topology on* P_k^n (for $n \geq 0$) is the topology whose closed subsets are the projective algebraic sets in P_k^n.

Remark 5.7. This is a topology.

There is a correspondence $\mathbb{A}_k^{n+1} \backslash \{0\} \to P^n$ given by sending $(x_0, \ldots, x_n)$ to $[x_0, \ldots, x_n]$.

Definition 5.8 (Cone)**.** A *cone* in $\mathbb{A}_k^{n+1}$ is a closed subset that is a union of lines through 0.

Remark 5.9. The zero set of a homogeneous polynomial in $\mathbb{A}_k^{n+1}$ is a cone.

Definition 5.10 (Graded Ring)**.** A *graded ring* is a (commutative ring) $R = \bigoplus_{i \geq 0} R_i$ such that $R_i R_j \subseteq R_{i+j}$ for all i, j.

Example 5.11. $k[x_0, \ldots, x_n]$ is graded with $|x_i| = 1$ for each i.

Definition 5.12 (Homogeneous Ideal)**.** An ideal I in a graded ring R is *homogeneous* if

$$I = \sum_{d \geq 0} (I \cap R_d).$$

In particular, this implies that

$$I = \bigoplus_{d \geq 0} (I \cap R_d).$$

Definition 5.13 (Zero Set)**.** For a homogeneous ideal $I \subseteq k[x_0, \ldots, x_n]$, its zero set in P_k^n is $Z(I) = \bigcap_{f \in I \text{ homogeneous}} Z(f)$.

Remark 5.14. If $I = (g_1, \ldots, g_r)$ with $g_1, \ldots, g_r$ homogeneous, then $Z(I) = Z(g_1) \cap \cdots \cap Z(g_r)$.

Definition 5.15 (Projective Algebraic Variety)**.** A *projective algebraic variety* is an irreducible projective algebraic set $X \subseteq P_k^n$ for some $n \geq 0$.

Remark 5.16. A projective algebraic set over k is a Noetherian topological space. So it is a finite union of its irreducible components.

Remark 5.17. Given an affine algebraic set $X \subseteq \mathbb{A}_k^n$, we can think of $\mathbb{A}_k^n$ as an open subset of P_k^n, and therefore produces a bijective correspondence between $\{x_0 \neq 0\} (\subseteq CP^n) \Leftrightarrow \mathbb{A}_k^n$.

Note that

1. The bijection above is a homeomorphis.

2. $\{x_0 \neq 0\} \subseteq P_k^n$ is open.

We can then consider its *projective closure*, i.e., its closure in P_k^n.

Remark 5.18. How would we usually calculate that closure?

Given as set of polynomials with $X = \{f(x_1, \dots, x_n) = 0, \dots\} \subseteq \mathbb{A}_k^n$, then say that f_i has degree at most d, then we can write down an "associated" homogeneous polynomial $g_i(x_1, \dots, x_n)$ with degree d by $x_1^{i_1} \dots x_n^{i_n} \mapsto x_0^{d-i_1-\dots-i_n} x_1^{i_1} \dots x_n^{i_n}$.

The correspondence is now given by

$$[1, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n] \in P_k^n \iff (x_1, \dots, x_n) \in \mathbb{A}_k^n.$$

Therefore,

$$\{g_1 = 0, \dots, g_r = 0\}(\subseteq P_k^n) \cap \{x_0 \neq 0\}(\cong \mathbb{A}_k^n) = \{f_1 = 0, \dots, f_r = 0\} \subseteq \mathbb{A}_k^n.$$

The subtlety is that the set on the left might be bigger than the precise closure in P_k^n of the set in the right. (That is, the calculation from right to left may not be well-defined.)

Definition 5.19 (Regular Function)**.** Let X be an affine algebraic set over algebraically closed field k. (That is, $X \subseteq \mathbb{A}_k^n$ is closed.) Let $U \subseteq X$ be an open subset, then a function $f : U \to K$ is called *regular* if for every $x \in U$ there exists an open neighborhood $x \in V \subseteq U$ on which we can write $f = \frac{g}{h}$ where g and h are polynomials in $k[x_1, \dots, x_n]$ such that $h \neq 0$ at all points of V.

Remark 5.20. This is a locally defined class of functions. That is, the expression may not be the same in different neighborhoods.

Example 5.21. $\frac{1}{x}$ is a regular function on $\mathbb{A}_k^1 \backslash \{0\}$. In fact, as we will see, the ring of all regular functions $\mathcal{O}(\mathbb{A}_k^1 \backslash \{0\}) \cong k[x][\frac{1}{x}]$, i.e., the ring of Laurent polynomials.

Remark 5.22. Note that for a function to be regular on the entire affine variety, this is equivalent to the following: a function is *regular* on the entire affine variety if it can globally be written as a polynomial.

Therefore, it is not so interesting to define regularity on an affine algebraic set with the same definition: one can just take the definition on the entire affine variety and restrict its domain. Our alternative definition essentially looks for the localization on open subsets.

6 Lecture 6

Definition 6.1 (Quasi-affine Algebraic Set)**.** A *quasi-affine algebraic set* over k an algebraically closed field is an open subset of an affine algebraic (closed) set $X \subseteq \mathbb{A}_k^n$. That is, $X \cap U$ where U is open in $\mathbb{A}_k^n$, i.e., $X - Y$ where Y is closed in $\mathbb{A}_k^n$, i.e., $X - Y$ where Y is a closed in X. This describes the idea of "a solution set minus another solution set".

Lemma 6.2. A regular function $f : U \to k$ on a quasi-affine algebraic set U is continuous as a mapping $f : U \to \mathbb{A}^1_k$ (with the Zariski topology).

Proof. We have to show that for every closed $S \subseteq \mathbb{A}^k_1$, $f^{-1}(U)$ is closed in U. By our knowledge of the closed subset of $\mathbb{A}^1_k$, it suffices to prove this for $S = \{a\}$ for some $a \in k$. By assumption, U is covered by open set $V \subseteq U$, on which $f = \frac{g}{h}$ with $g, h \in x[k_1 \ldots, x_n]$ with $h\mid_V \neq 0$ everywhere on V.

Lemma 6.3. For a topological space X with an open covering by open V_α, a subset S is closed in X if and only if $S \cap V_\alpha$ is closed in V_α for all α, and likewise for open subsets.

Subproof. Left as an exercise. ■

So it suffices to show that $f^{-1}(a) \cap V$, for each open $V \subseteq U$ as above. Now $f^{-1}(a) \cap V = \{x \in V : f(x) = a\} = \{x \in V : \frac{g(x)}{h(x)} = a\} = \{x \in V : g(x) - ah(x) = 0\}$, but this is a polynomial function on $\mathbb{A}^n_k$, restricted to V, and therefore this is a closed subset of V. □

Definition 6.4 (Quasi-projective Algebraic Set)**.** A *quasi-projective algebraic set* V over k is an open subset V of some projective algebraic set $X \subseteq P^n_k$ for some $n \geq 0$.

Remark 6.5. A quasi-affine algebraic set can be viewed as a quasi-projective algebraic set in P^n_k by the inclusion $\mathbb{A}^n_k \subseteq P^n_k$ as $\mathbb{A}^n_k = \{x_i \neq 0\} \subseteq P^n_k$ for any $0 \leq i \leq n$.

Definition 6.6 (Morphism of Quasi-projective Algebraic Set)**.** Let X and Y be quasi-projective algebraic sets over k. A *morphism* $f : X \to Y$ is a continuous function such that for every open $U \subseteq Y$ and every regular function g on U, the composition $g \circ f : f^{-1}(U) \to k$ is a regular function open in X.

Definition 6.7 (Regular functions on Quasi-projective Algebraic Set)**.** Let U be a quasi-projective algebraic set over k. A function $f : U \to k$ is *regular* if and only if for every point $x \in U$, there is an open $x \in V \subseteq U$ and $g, h \in k[x_0, \ldots, x_n]$ homogeneous of the same degree d such that

1. $h \neq 0$ at every point of V, and

2. $f = \frac{g}{h}$ on V.

Remark 6.8. Note that for homogeneous polynomial g, h of the same degree d,

$$\frac{g(ca_0, \ldots, ca_n)}{h(ca_0, \ldots, ca_n)} = \frac{c^d g(a_0, \ldots, a_n)}{c^d h(a_0, \ldots, a_n)} = \frac{g(a_0, \ldots, a_n)}{h(a_0, \ldots, a_n)}.$$

Remark 6.9. In defining a morphism, it is not enough to take $U = Y$ in the definition.

Example 6.10. The ring of regular functions on P_k^1 is just k, i.e., the constant functions.

Remark 6.11. Note that $P_k^1\backslash\{\infty\} \cong P_k^1\backslash\{0\} \cong \mathbb{A}_k^1$.

Proof Sketch. We will see that $\mathcal{O}(\mathbb{A}_k^1) = k[x]$, even by our new definition. So a regular function$f : P_k^1 \to k$ would restrict to regular functions on $V_0 = \{x_0 \neq 0\} \cong \mathbb{A}_k^1$ but also in $V_1 = \{x_1 \neq 0\} \cong \mathbb{A}_k^1$, and as $[x_0, x_1] \in P_k^1$, therefore f would be in $k[x]$ and also $k[y]$. But $[1, a] = [\frac{1}{a}, 1]$, so f is both a polynomial in x and in $\frac{1}{x}$, which forces f to be a constant. □

Example 6.12. For a quasi-projective algebraic set X, a morphism $f : X \to \mathbb{A}_k^n$ is of the form $f(x) = (f_1(x), \ldots, f_n(x))$ where $f_1, \ldots, f_n$ are regular functions on X, and the converse is true.

Corollary 6.13. If X is a quasi-projective variety (meaning that it is irreducible), and f is a regular function on X that is not identically zero, then every irreducible component of the closed subset $\{f = 0\} \subseteq X$ has dimension $\dim(X) - 1$.

Proof. This is a corollary of Krull's Principal Ideal Theorem. □

Theorem 6.14. Let $X \subseteq \mathbb{A}_k^n$ be a closed subset (i.e. an affine algebraic set), then the definition of the ring $\mathcal{O}(X)$ of regular functions agrees with our old definition $k[x_1, \ldots, x_n]/I(X)$.

Proof.

Definition 6.15. For an affine algebraic set $X \subseteq \mathbb{A}_k^n$, a *standard open subset* of X is a subset of the form $\{g \neq 0\} \subseteq X$, where $g \in k[x_1, \ldots, x_n]$.

Lemma 6.16. The standard open subsets of X form a basis for the topology of X, for X an affine algebraic set.

Subproof. We have to show that every open subset of X is a union of standard ones. By definition, an open set $U \subseteq X$ is $X - \{g_1 = 0, \ldots, g_r = 0\}$ for some $g_1, \ldots, g_r \in k[x_1, \ldots, x_n]$, and this is just the set $\bigcup\limits_{1 \leq i \leq r} \{g_i \neq 0\}$. ■

Write $\mathcal{O}(X)$ for our new descriptions of regular functions. Clearly there is a homomorphism of k-algebras

$$\varphi : k[x_1, \ldots, x_n]/I(X) \to \mathcal{O}(X),$$

and clearly φ is injective. We now show that it is surjective. Let $f \in \mathcal{O}(X)$, we know we can cover X by open sets $U_\alpha \subseteq X$ on which $f = \frac{g_\alpha}{h_\alpha}$ with g_α, h_α as polynomials in $k[x_1, \ldots, x_n]$, and $h_\alpha \neq 0$ everywhere on U_α. By Lemma 6.16, we can assume that each U_α is a standard open subset in X, i.e., $U_\alpha = \{k_\alpha \neq 0\} \subseteq X$ for some $k_\alpha \in k[x_1, \ldots, x_n]$. Note that on U_α,

$$f = \frac{g_\alpha}{h_\alpha} = \frac{g_\alpha k_\alpha}{h_\alpha k_\alpha},$$

and this is still well-defined. Note that $\{k_\alpha \neq 0\} = \{h_\alpha k_\alpha \neq 0\} \subseteq X$. Therefore, we can replace h_α and k_α by $h_\alpha k_\alpha$ in our discussion. We now have polynomials g_α and h_α such that

$$X = \bigcup_\alpha \{h_\alpha \neq 0\}$$

and, on $\{h_\alpha \neq 0\}$, $f = \frac{g_\alpha}{h_\alpha}$. Note that $h_\alpha^2 \cdot f = g_\alpha h_\alpha$ on $\{h_\alpha \neq 0\} \subseteq X$, and also on $\{h_\alpha = 0\} \subseteq X$. Therefore, the equation is true on all of X.

Because $X = \bigcup_\alpha \{h_\alpha \neq 0\}$, we have $Z(h_\alpha^2 : \alpha \in \zeta) \subseteq X$ as the empty set $\varnothing$. By the Nullstellensatz, let $I = (h_\alpha : \alpha \in \zeta) \subseteq k[x_1, \ldots, x_n]/I(X) = R/I(X)$, then it has $\operatorname{rad}(I) = R$. In particular, $I = R$. Therefore, 1 can be expressed as some finite sum of the forms $r_\alpha h_\alpha^2$ for some $r_\alpha \in R$. Hence, on all of X, $1 \cdot f = (\sum r_\alpha h_\alpha^2) \cdot f = \sum r_\alpha h_\alpha^2 f = \sum r_\alpha g_\alpha h_\alpha \in R = k[x_1, \ldots, x_n]/I(X)$. $\square$

7 Lecture 7

Lemma 7.1. Let X be a quasi-projective algebraic set over k algebraically closed. $\mathcal{O}(X)$ is a ring, in fact a commutative reduced k-algebra.

Proof. The main point is to show that the sum and product of regular functions are still regular. Call our set U, then given functions $f_1, f_2 : U \to k$ that locally are of the form $\frac{g}{h}$ with $g, h \in k[x_1, \ldots, x_n]$, both homogeneous of same degree d, with $h \neq 0$ of the given point p. Then say $f_1 = \frac{g_1}{h_1}$ near p and $f_2 = \frac{g_2}{h_2}$ near p. Obviously, $f_1 f_2 = \frac{g_1 g_2}{h_1 h_2}$ where the numerator and the denominator are homogeneous of the same degree, and the denominator is still non-zero at this point. The sum is similar: $\frac{g_1}{h_1} + \frac{g_2}{h_2} = \frac{g_1 h_2 + h_1 g_2}{h_1 h_2}$, and therefore we have the same argument. $\square$

Lemma 7.2. For a quasi-projective algebraic set X over k, a morphism $f : X \to \mathbb{A}_k^n$ is equivalent to a list of n regular functions $f_1, \ldots, f_n$ on X.

Proof. Clearly, a function $U \to \mathbb{A}_k^n = k^n$ is equivalent to a list of n functions $U \to k$, i.e., $f(x) = (f_1(x), \ldots, f_n(x))$. If f is a morphism, then the pullbacks of the n regular functions, $x_1, \ldots, x_n \in \mathcal{O}(\mathbb{A}_k^n) = k[x_1, \ldots, x_n]$, so $f_1, \ldots, f_n$ are regular functions on X.

Conversely, suppose $f_1, \ldots, f_n$ are regular functions on $X = U$. To show that $f(x) = (f(x_1), \ldots, f(x_n))$ is a morphism $U \to \mathbb{A}_k^n$ over k, let $V \subseteq \mathbb{A}_k^n$ be open and let $g \in \mathcal{O}(U)$.

(One can check that f is indeed continuous.) To show that $i(g) = g \circ f$ is regular on $f^{-1}(V)$, here g can be written locally as $\frac{h}{k}$, with h, k polynomials near each point $p \in U$ with $k(p) \neq 0$. We want to show that $\frac{h(f_1,\ldots,f_n)}{k(f_1,\ldots,f_n)}$ is regular on $f^{-1}(V)$, so one has to write this as a ratio of homogeneous polynomials of the same degree, using that each function is of that form (near p). $\square$

Remark 7.3. For a quasi-affine algebraic set $Y \subseteq \mathbb{A}_k^n$ and X a quasi-projective algebraic set X over k, a morphism $f : X \to Y$ is equal to n regular functions $f_1, \ldots, f_n \in \mathcal{O}(X)$ such that $(f_1(x), \ldots, f_n(x)) \in Y$ for every $x \in X$.

Remark 7.4. The morphisms of quasi-projective algebraic sets over k form a category.

Definition 7.5 (Isomorphism)**.** An *isomorphism* $f : X \to Y$ of quasi-projective algebraic set over k is a morphism $f : X \to Y$ that has a two-sided inverse.

Example 7.6. $X = \mathbb{A}_k^1 \backslash \{0\} \cong \{xy = 1\} \subseteq \mathbb{A}_k^2 = Y$. Note that X is quasi-affine and Y is affine.

Proof. Use the morphism $Y \to X$ by $(x, y) \mapsto x$ and $X \to Y$ by $x \mapsto (x, x^{-1})$, and this is well-defined since $x^{-1} \in \mathcal{O}(\mathbb{A}_k^1 \backslash \{0\})$. $\square$

Remark 7.7. Sometimes we say that a quasi-projective algebraic set is affine if it is isomorphic to an affine algebraic set, i.e., a closed subset of some $\mathbb{A}_k^n$.

Example 7.8. The hypersurface $\{x_n = f(x_1, \ldots, x_{n-1})\} \subseteq \mathbb{A}_k^n$ is isomorphic to $\mathbb{A}_k^{n-1}$, where f is any polynomial in $k[x_1, \ldots, x_{n-1}]$.

Example 7.9. Let $X \subseteq \mathbb{A}_k^n$ be an affine algebraic set over k (i.e., a closed subset of $\mathbb{A}_k^n$). Let $g \in \mathcal{O}(X)$, then the standard open subset $\{g \neq 0\}$ is affine, in fact it is isomorphic to $\{(x_1, \ldots, x_n, x_{n+1}) : x_{n+1} g(x_1, \ldots, x_n) = 1\} \subseteq \mathbb{A}_k^{n+1}$.

Proof. Map $U = \{g \neq 0\} \subseteq X$ by $(a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n, g(a_1, \ldots, a_n)^{-1}) \in Y$, then this is a morphism. The inverse morphism is given by $(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n) \in U = \{g \neq 0\}$. $\square$

Example 7.10. $\mathbb{A}_k^2 \backslash \{0\} = \{x_1 = 0\} \cup \{x_2 = 0\}$ is a quasi-affine algebraic set which is not affine.

Corollary 7.11. Let $X \subseteq \mathbb{A}_k^n$ be an affine algebraic set (i.e., closed in $\mathbb{A}_k^n$), and let $g \in \mathcal{O}(X)$, then $\mathcal{O}(\{g \neq 0\}) = \mathcal{O}(X)[\frac{1}{g}]$.

Proof. A morphism $f : X \to Y$ of quasi-projective algebraic sets induces a k-algebraic homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. Therefore, an isomorphism $f : X \to Y$ induces an isomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ of k-algebras. Therefore,

$$\begin{aligned}\mathcal{O}(\{g \neq 0\}) &= \mathcal{O}(\{x_{n+1}g(x_1, \ldots, x_n) = 1\} \subseteq \mathbb{A}_k^{n+1}) \\ &= k[x_1, \ldots, x_{n+1}]/(f_1(x_1, \ldots, x_n), \ldots, f_r(x_1, \ldots, x_n), x_{n+1}g(x_1, \ldots, x_n) - 1) \\ &= \mathcal{O}(X)[x_{n+1}]/(x_{n+1}g(x_1, \ldots, x_n) - 1) \\ &\cong \mathcal{O}(X)[\frac{1}{g}].\end{aligned}$$

$\square$

Theorem 7.12. The correspondence mentioned in the proof above can be formalized. Let $f : X \to Y$ be a morphism of quasi-projective algebraic sets over an algebraically closed field k. f determines a pullback homomorphism of k-algebras $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. Moreover, if Y is affine (i.e., isomorphic to a closed subset of some $\mathbb{A}_k^n$), then this construction gives a one-to-one correspondence between morphisms $X \to Y$ and k-algebra homomorphisms $\mathcal{O}(Y) \to \mathcal{O}(X)$. It follows that if both X and Y are affine, then X and Y are isomorphic if and only if the k-algebras $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are isomorphic.

8 Lecture 8

Lemma 8.1. Let $X \subseteq \mathbb{A}_k^{n+1}$ be a cone (that is, X is closed and is a union of lines through 0), then the ideal $I(X) \subseteq k[x_0, \ldots, x_n]$ is homogeneous.

Proof. We have to say: for any $f \in I(X)$, if we write $f = f_0 + \ldots + f_d$ with f_i homogeneous of degree i, then f_i should be in $I(X)$.

Let $(a_0, \ldots, a_n)$ be a point in X, then we know that (because X is a cone and $f \in I(X)$) $f(ca_0, \ldots, ca_n) = 0$ for all $c \in k$. In particular, $f_0(a_0, \ldots, a_n) + cf_1(a_0, \ldots, a_n) + \cdots + c^d f_d(a_0, \ldots, a_n)$. Note that every term is in k, but as polynomial in c, this polynomial $g(c) \in k[c]$ such that $g(c) = 0$ for all $c \in k$. Hence, all its coefficients are 0.

Since k is algebraically closed, it is infinite. So $g = 0 \in k[c]$, that is, $f_i(a_0, \ldots, a_n) = 0$ for each $0 \leq i \leq d$. Since $(a_0, \ldots, a_n) \in X$ are arbitrary, $f_i \in I(X)$, so the ideal $I(X)$ is homogeneous. $\square$

Remark 8.2. Note that the zero set in P^n of the ideal $(x_0, \ldots, x_n)$ in $k[x_0, \ldots, x_n]$ since $[0, \ldots, 0]$ is not a point in P^n. We get a one-to-one correspondence between homogenous prime ideals that are not $(x_0, \ldots, x_n)$ (called the *irrelevant ideal*), and irreducible closed subsets of P_k^n.

Definition 8.3 (Local Ring)**.** Let X be a quasi-projective algebraic set over k algebraically closed. Then for a point $p \in X$, the *local ring* of X at p is

1. an equivalence class of pairs (U, f) with open $p \in U \subseteq X$ and $f \in \mathcal{O}(U)$, with $(U, f) \sim (V, g)$ if there is an open neighborhood $p \in W \subseteq U \cap V$ such that $f\mid_W = g\mid_W$. (That is, an element of $\mathcal{O}_{X,p}$ is a germ of regular functions at p.)

2. The direct limit $\varinjlim_{p \in U \subseteq X} \mathcal{O}(U)$, i.e., with $p \in U \subseteq V \subseteq X$, there is a restriction map $\mathcal{O}(V) \to \mathcal{O}(U)$.

Lemma 8.4. $\mathcal{O}_{X,p}$ is a local ring.

Proof. That is, we want to show that $\mathcal{O}_{X,p}$ has exactly one maximal ideal. Equivalently, $\mathcal{O}_{X,p}$ has a maximal ideal $\mathfrak{m}$ such that for all $f \in \mathcal{O}_{X,p} \backslash \mathfrak{m}_{X,p}$, then $f \in \mathcal{O}_{X,p}^*$. Let $\mathfrak{m} = \ker(\mathcal{O}_{X,p} \to k)$, i.e., the kernel of the evaluation at p. One can see this is surjective (using constant functions), then let $f \in \mathcal{O}_{X,p} \backslash \mathfrak{m}$, then we can view $f \in \mathcal{O}(U)$ for some open set $p \in U \subseteq X$. Then $\{f \neq 0\} \subseteq U$ is an open subset of X containing p, so $\frac{1}{f} \in \mathcal{O}(V)$, hence $\frac{1}{f} \in \mathcal{O}_{X,p}$. □

Lemma 8.5. Let X be an affine algebraic set over k, then for a point $p \in X$ with $\mathfrak{m} = \ker(\mathcal{O}_{X,p} \to k)$ as the evaluation map at p, then $\mathcal{O}_{X,p} = \mathcal{O}(X)_{\mathfrak{m}}$ as the localization.

Proof. For a commutative ring R and prime ideal $\mathfrak{p} \subseteq R$, an element of the localization $R_{\mathfrak{p}}$ can be written as $\frac{a}{b}$ with $a \in R$ and $b \in R \backslash \mathfrak{p}$. So an element of $\mathcal{O}(X)_{\mathfrak{m}}$ is a fraction $\frac{a}{b}$ with $a \in \mathcal{O}(X)$ and $b \in \mathcal{O}(X)$ with $b(p) \neq 0$. Therefore $\frac{a}{b} \in \mathcal{O}(\{b \neq 0\})$ hence is contained in $\mathcal{O}_{X,p}$. □

Remark 8.6. Recall that $\mathcal{O}(\{g \neq 0\}) = \mathcal{O}[x][\frac{1}{g}]$.

Remark 8.7. An isomorphism $f : X \to Y$ of quasi-projective algebraic sets over k induces an isomorphism of local rings $\mathcal{O}_{Y,f(p)} \cong \mathcal{O}_{X,p}$.

Definition 8.8 (Dimension Near a Point)**.** Let $X \subseteq \mathbb{A}_k^n$ be a closed subset, write $I(X) = (f_1, \dots, f_r) \in k[x_1, \dots, x_n]$, and let $p \in X$. Let m be the dimension of X near p, i.e., the dimension of U for all small enough open neighborhoods of p.

Remark 8.9. If X is irreducible, then it has the same dimension near every point. Note that we can define derivatives of polynomials manually:

$$\frac{\partial}{\partial x_j}(x_1^{i_1}, \dots, x_n^{i_n}) = i_j x_1^{i_1} \dots x_j^{i_j - 1} \dots x_n^{i_n}$$

Note that we have a unique ring homomorphism $\mathbb{Z} \to k$, and can be viewed as a polynomial in $k[x_1, \ldots, x_n]$.

We have

$$\frac{\partial}{\partial x}(fg) = f\frac{\partial g}{\partial x} + \frac{\partial f}{\partial x}g$$

and etc.

Remark 8.10. If k has characteristic $p > 0$, then $p = 0 \in k$, so $\frac{\partial}{\partial x}(x^p) = px^{p-1} = 0 \in k[x]$. We now get a $n \times r$ matrix in k, of the form $\left(\frac{\partial f_i}{\partial x_j} \mid_p\right)$, and therefore a map $A^n \to A^r$.

Definition 8.11 (Smooth)**.** $X \subseteq \mathbb{A}^n_k$ is *smooth* over k at $p \in X(k)$ if the matrix $D_p = \left(\frac{\partial f_i}{\partial x_j} \mid_p\right)$ has rank $n - m$ where m is the dimension of X near p.

Definition 8.12 (Zariski Tangent Space)**.** The *Zariski tangent space* is defined to be $T_{X,p} = \ker(D_p : k^n \to k^r)$. The smoothness of X at p means that (X, p) has dimension $\dim(X)$ near p. Note that we always have a $\geq$ relation.

Example 8.13. Let $X = \{xy = 0\} \subseteq \mathbb{A}^2_k$. Where is X smooth? Let $(a, b) \in X(k)$, then the matrix $D_p = \left(\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y}\right) \mid_{(a,b)} = (y\ x) \mid_{a,b} = (b\ a) \in M_{1\times 2}(k)$. Therefore, X is smooth if and only if this matrix has rank 1 (note that it always has rank at most 1), if and only if $a \neq 0$ or $b \neq 0$.

Thus, X is smooth (of dimension 1) everywhere except $(0, 0)$.

Example 8.14. Where is the curve $X = \{xy = 1\} \subseteq \mathbb{A}^2_K$ smooth?

The matrix of derivatives is (write $f = xy - 1$) $(y\ x)$, and so X is smooth at (x, y) if and only if $(x, y) \neq (0, 0)$. But $(0, 0)$ is not on the curve, so X is smooth everywhere.

9 Lecture 9

Remark 9.1.

1. Smoothness does not depend on the choice of generators $g_1, \ldots, g_r$.
2. This "commutes with localization".
3. Smoothness is preserved by isomorphisms.

Example 9.2 (Zariski Tangent Space)**.** Consider $X = \{xy = 0\} \subseteq \mathbb{A}^2_k$, then at every point $x \in X$, we define a vector space $T_pX \subseteq k^n$ for $X \subseteq \mathbb{A}^n_k$. The tangent space is two-dimensional at the origin, and is one-dimensional everywhere else.

Definition 9.3 (Presheaf)**.** Let X be a topological space. A *presheaf* of Abelian groups on X is an Abelian group $A(U)$ for every open set $U \subseteq X$, together with restriction homomorphisms $r^V_U : A(V) \to A(U)$ for every open $U \subseteq V \subseteq X$, such that

- $r_U^U = 1_{A(U)}$ for every $U \subseteq X$,
- $r_U^W = r_U^V r_V^W$ for open $U \subseteq V \subseteq W \subseteq X$ as homomorphism $A(W) \to A(U)$.

Example 9.4. Let X be a topological space. Let $C(U)$ be the presheaf of continuous $\mathbb{R}$-valued functions.

Example 9.5. Let X be C^∞-manifold, then we have the presheaf of C^∞ (smooth) $\mathbb{R}$-valued functions.

Example 9.6. Let X be a complex manifold. We have the presheaf $\mathcal{O}_{an}$ of $\mathbb{C}$-analytic functions (on open subsets of X). For instance, if $X = \mathbb{C}P^1$, then $\mathcal{O}_{an}(X) = \mathbb{C}$.

Example 9.7. Let X be a quasi-projective algebraic set over k algebraically closed, then we have the presheaf $\mathcal{O}_X$ of regular functions.

Remark 9.8. We may call $A(U)$ the Abelian group of section of A on U.

Remark 9.9. Let X be a topological space. Define a category $\mathbf{Top}(X)$ with objects the open subsets of X, and $\mathbf{Hom}_{\mathbf{Top}(X)}(U, V) = \begin{cases} *, & \text{if } U \subseteq V \\ \varnothing, & \text{if } U \nsubseteq V \end{cases}$. A presheaf of Abelian groups on X is exactly a contravariant functor $\mathbf{Top}(X) \to \mathbf{Ab}$.

Definition 9.10 (Sheaf)**.** A *sheaf* of Abelian groups on a topological space X is a presheaf A of Abelian groups such that

- for every open set $U \subseteq X$ and every open cover $\{U_\alpha\}_{\alpha \in I}$ of U if $a, b \in A(U)$ such that $a \mid_{U_\alpha} = b \mid_{U_\alpha}$ for every $\alpha \in I$, then $a = b \in A(U)$,
- for every open set $U \subseteq X$ and every open cover $\{U_\alpha\}_{\alpha \in I}$ of U, for any collection of $a_\alpha \in A(U_\alpha)$ for all $\alpha \in I$, if $a_\alpha \mid_{U_\alpha \cap U_\beta} = a_\beta \mid_{U_\alpha \cap U_\beta}$ for all $\alpha, \beta \in I$, then there is an $a \in A(U)$ such that $a \mid_{U_\alpha} = a_\alpha$ for all $\alpha \in I$.

Remark 9.11. If A is a sheaf, then the $a \in A(U)$ described in the second property is unique, given by the first property.

Example 9.12. The presheaves described above are sheaves.

Remark 9.13. If A is a sheaf, then $A(\varnothing) = 0$ is the trivial Abelian group.

Proof. Take $U = \varnothing$, notice that U is covered by no open subsets. $\square$

Example 9.14. Let A be an Abelian group and X be a topological space. The constant presheaf T_A on X is defined by $T_A(U) = A$ for every open $U \subseteq X$. This is not a sheaf if $A \neq 0$, since $T_A(\varnothing) = A$, not 0.

Example 9.15. Let A be an Abelian group on a space X. Define a presheaf S_A on X by $S_A(U) = \begin{cases} 0, & \text{if } V = \varnothing \\ A, & \text{otherwise} \end{cases}$. This is not a sheaf, for many spaces X, e.g., $X = \mathbb{R}$ with classical topology. Take the real line $\mathbb{R}$, and two disjoint open subsets U_1 and U_2, then let $U = U_1 \cup U_2 \subseteq \mathbb{R}$. Now $7 \in S_{\mathbb{Z}}(U_1)$ and $8 \in S_{\mathbb{Z}}(U_2)$, then the sections agree on the intersection, but there is not $a \in S_{\mathbb{Z}}(U_1 \cup U_2) = \mathbb{Z}$ that restricts to both 7 and 8.

Example 9.16. For a topological space X and Abelian group A, the sheaf A_X of locally constant A-valued functions on X is $A_X(U)$, the set of functions $f : U \to A$ for $U \subseteq X$ open that are locally constant, i.e., for every $p \in U$, there exists $p \in V \subseteq U$ such that $f\mid_V$ is constant.

Definition 9.17 (Stalk)**.** Let A be a presheaf on a space X. The *stalk* of A at a point $p \in X$ is $A_p = \varinjlim_{p \in U \subseteq X} A(U)$ for any open U of X containing p. That is, an element A_p is a germ of section of A at p.

Example 9.18. For a quasi-projective algebraic set X over k, the stalk $\mathcal{O}_{X,p}$ is exactly the local ring of X at p.

Definition 9.19 (Homomorphism of Presheaves)**.** A *homomorphism of presheaves* of Abelian groups A and B on a space X is a natural transformation $A \to B$ (as contravariant functors on $\mathbf{Top}(X)$): for every open $U \subseteq X$ we are given a homomorphism $f_U : A(U) \to B(U)$ of Abelian groups such that for every open inclusion $U \subseteq V$, the diagram

$$\begin{tikzcd} A(V) \arrow[r] \arrow[d] & B(V) \arrow[d] \\ A(U) \arrow[r] & B(U) \end{tikzcd}$$

commutes.

10 Lecture 10

Algebraic geometry classifies closed subsets of $\mathbb{A}_k^n$, so it is the study of affine algebraic sets up to isomorphisms. As we mentioned, we have a correspondence between coordinate rings and the commutative k algebras (finitely-generated over k and reduced), but the latter is hard to classify. The main technique we use is to switch from affine algebraic geometry to projective algebraic geometry. In projective algebraic geometry, we get invariants as cohomology groups of sheaves, which is a measurement of difference between local and global behaviors.

Definition 10.1 (Homomorphism of Sheaves)**.** A *homomorphism* $f : A \to B$ *of sheaves of Abelian groups* over X is the same thing. That is, $\mathbf{Sh}(X)$ is a full subcategory of $\mathbf{PreSh}(X)$.

Remark 10.2. A map $f : A \to B$ of presheaves on a space X determines a homomorphism of Abelian groups $f_p : A_p \to B_p$ for every point $p \in X$. This well-defined mapping is given by $s \in A(U) \mapsto f(s) \in B(U)$, and thus is mapped to a germ $f(s)_p \in B_p$.

Proposition 10.3. Let $f : A \to B$ be a homomorphism of sheaves on X. Then $f_p : A_p \to B_p$ is an isomorphism for every $p \in X$ if and only if $f : A \to B$ is an isomorphism.

Remark 10.4. This is not true for presheaves.

Example 10.5. Let T be the constant presheaf on a space X associated to $\mathbb{Z}$. That is, $T(U) = \mathbb{Z}$ for every open $U \subseteq X$. Then there is a natural map $T \to \mathbb{Z}_X$ of presheaves where $\mathbb{Z}_X$ is the sheaf of local of locally constant $\mathbb{Z}$-valued functions.

For instance, if $X = \mathbb{R}$, f is not an isomorphism, but f induces an isomorphism on stalks. Both presheaves have stalk at every point as $\mathbb{Z}$.

Proof. It is clear that if f is an isomorphism, then $f_p : A_p \to B_p$ is an isomorphism for every $p \in X$.

Conversely, let $f : A \to B$ be a homomorphism of sheaves on X, with isomorphism of Abelian groups $f_p : A_p \to B_p$ at every point $p \in X$. We have to show that for every $U \subseteq X$, the homomorphism of Abelian groups $f_U : A(U) \to B(U)$ must be an isomorphism. First, we show that f_U is injective. Let $s \in A(U)$ be such that $f_U(s) = 0 \in B(U)$. We have a commutative diagram

$$\begin{array}{ccc} A(U) & \xrightarrow{f_U} & B(U) \\ \downarrow & & \downarrow \\ A_p & \xrightarrow[f_p]{} & B_p \end{array}$$

so s mapping from both ways goes to 0 since $f_U(s) = 0$, but f_p is an isomorphism here, so the germ of s at every point $p \in U$ is 0. Therefore, for every point $p \in U$ we can choose an open set $p \in U_p \subseteq U$ such that $s\,|_{U_p}= 0 \in A(U_p)$. By the definition of sheaves, it follows that $s = 0 \in A(U)$, hence injective. To show that f_U is surjective, let U be an open subset of X and let $t \in B(U)$. We want to find $s \in A(U)$ with $f(s) = t$. For each point $p \in U$, the germ $t_p \in B_p$ is the image of a unique element $s_p \in A_p$. That is, for each $p \in U$, there is an open set $p \in U_p \subseteq U$ and a section $w_p \in A(U_p)$ such that the germ of w_p at p is $s_p \in A_p$. It is not necessarily true that $w_p |_{U_p \cap U_q} = w_q |_{U_p \cap U_q} \in A(U_p \cap U_q)$. However, we know that $f(w_p) \in B(U_p)$ has germ at p equal to t_p, the germ of $t \in B(U)$ at p, which is the same thing as the germ of $t |_{U_p} \in B(U_p)$. Thus, there is an open neighborhood $p \in V_p \subseteq U_p \subseteq U$ such that $f(w_p) |_{V_p} = t |_{V_p} \in B(V_p)$. Clearly V_p's form an open cover of U since $p \in V_p$.

Claim 10.6. For every $p, q \in U$, $w_p |_{V_p}$ agrees with $w_q |_{V_q}$ in $A(V_p \cap V_q)$.

Subproof. We know that both $w_p |_{V_p \cap V_q}$ and $w_q |_{V_p \cap V_q}$ map to $t |_{V_p \cap V_q}$. In particular, $f(w_p) |_{V_p \cap V_q}$ has the same germ at every point of $V_p \cap V_q$ as $f(w_q) |_{V_p \cap V_q}$. By our assumption (that $f_x : A_x \to B_x$ is isomorphic for all $x \in X$), it follows that $w_p |_{V_p \cap V_q}$ and $w_q |_{V_p \cap V_q}$ have the same germ at every point in A_x for $x \in V_p \cap V_q$. By the proof of injectivity, we know that for a sheaf A on a space X, $A(U) \to \prod_{p \in U} A_p$ is injective. Therefore, $w_p |_{V_p \cap V_q} = w_q |_{V_p \cap V_q}$. ■

Since A is a sheaf, it follows that there is a unique section $s \in A(U)$ such that $s |_{V_p} = w_p |_{V_p}$ for every $p \in U$. We want to show that $f(s) = t \in B(U)$. Indeed, we know by construction that the sections in $B(U)$ have the same germ at every point in U, so since $A(U) \to \prod_{p \in U} A_p$ is injective, $f(s) = t \in B(U)$ as desired. □

Definition 10.7 (Kernel of Sheaves)**.** Let $f : A \to B$ be a homomorphism of sheaves (of Abelian groups) on a topological space X. The *kernel* of f, denoted $\ker(f)$, is the sheaf $(\ker(f))(U) = \ker(f : A(U) \to B(U))$ for $U \subseteq X$ open.

Definition 10.8 (Image of Sheaves)**.** Let $f : A \to B$ be a homomorphism of sheaves (of Abelian groups) on a topological space X. The *image* of f, denoted $\operatorname{im}(f)$, is defined by $(\operatorname{im}(f))(U) = \operatorname{im}(f : A(U) \to B(U))$ for $U \subseteq X$ open. Note that this only a presheaf in general.

11 Lecture 11

Example 11.1. Let $X = S^1$ for $U \subseteq X$ open. Let A be the sheaf of continuous $\mathbb{C}$-valued functions on S^1. Let B be the sheaf of $\mathbb{C}*$-valued continuous functions on S^1. The structure on $B(U)$ is $(fg)(x) = f(x)g(x) \in \mathbb{C}^*$ for $x \in S^1$.

We have a homomorphism of sheaves $\exp : A \to B$ given by $\exp(f)(x) = e^{f(x)} \in B(U)$ for $f \in A(U)$ and $x \in U$ where $U \subseteq S^1$ open. This is a homomorphism since $e^{f+g} = e^f \cdot e^g$.

We claim that $\operatorname{im}(\exp)$ is a presheaf that is not a sheaf. Consider the section $z \in B(S^1)$, the obvious inclusion $S^1 \hookrightarrow \mathbb{C}^*$, then $z \notin (\operatorname{im}(\exp))(S^1)$: $z = e^{f(z)}$ for some continuous $\mathbb{C}$-valued function on S^1 corresponds to $f(z) = \log(z) + \mathbb{Z} \cdot 2\pi i$, which is not possible. But if

$U = S^1\backslash\{1\}$ and $V = S^1\backslash\{-1\}$, then $z\mid_U$ and $z\mid_V$ are in $(\mathrm{im}(f))(U)$ and $(\mathrm{im}(f))(V)$. These sections clearly agrees on $U \cap V$, but they cannot be glued to an element of $(\mathrm{im}(f))(S^1)$, so this is not a sheaf.

Definition 11.2 (Skyscraper Sheaf)**.** For a topological space X and a point $p \in X$ and an Abelian group A, the *skyscraper sheaf* A_p on X is defined by $A_p(U) = \begin{cases} A, & \text{if } p \in U \\ 0, & \text{if } p \notin U \end{cases}$ for $U \subseteq X$ open.

Example 11.3. Let k be an algebraically closed field $X = P^1_k$ (with Zariski topology), consider the sheaves $A = \mathcal{O}_X$ and $B = k_0 \oplus k_\infty$. The direct sum of two sheaves A and B is defined by $(A \oplus B)(U) = A(U) \oplus B(U)$. There is an inclusion homomorphism of sheaves $A \to B$ by evaluation at 0 and ∞, note that $0 = [1,0] \in P^1_k$ and $\infty = [0,1] \in P^1_k$. Define $f = \exp : A \to B$. We claim that $\mathrm{im}(f)$ is a presheaf but not a sheaf on $P^1_k \cong \mathbb{A}^1_k$. Let $U = P^1_k\backslash\{0\}$ and $V = P^1_k\backslash\{\infty\}$. Consider the sections $0 \in (\mathrm{im}(f))(U) = k(U)$ and $1 \in (\mathrm{im}(f))(V) = k(V)$. Therefore, these sections agree on $(\mathrm{im})(U \cap V)$, but these sections cannot be glued to an element of $(\mathrm{im}(f))(P^1_k) = \mathrm{im}(A(P^1_k) \to B(P^1_k)) = \mathrm{im}(k \to k \oplus k) \cong k$.

Theorem 11.4. Let A be a presehaf of Abelian groups on a space. Then there is a sheaf A^+, the *sheafification* of A with a map $A \to A^+$ which is universal for maps of A to sheave. That is, for every sheaf B with a map $A \to B$, there is a unique map $A^+ \to B$ making the diagram commutes:

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \\ & A^+ \arrow[ur, dashed] & \end{tikzcd}$$

Proof Sketch. For an open $U \subseteq X$, define

$$A^+(U) = \left\{ \prod_{p \in U} S_p \text{ with } s_p \in A_p \right\} \subseteq \prod_{o \in U} A_p$$

for all $p \in U$, such that for all $p \in U$ there exists $p \in V_p \in U$ and $t \in A(V_p)$ such that s_p is the germ t_p for all $p \in V_p$. This is a sheaf and has the universal property. □

Definition 11.5 (Image Sheaf)**.** For a map $f : A \to B$ of sheaves on X, let the *image sheaf* $\mathrm{im}(f)$ be the sheafification of the presheaf $U \to \mathrm{im}(A(U) \xrightarrow{f} B(U))$.

Definition 11.6 (Injective, Surjective)**.** A map $f : A \to B$ of sheaves is *injective* if $f : A(U) \to B(U)$ is injective for every open set $U \subseteq X$.

A map $f : A \to B$ of sheaves is *surjective* if the image sheaf $\mathrm{im}(f) \subseteq B$ is equal to B. Thus, we do not require that $f : A(U) \to B(U)$ to be surjective.

Equivalently, a map of sheaves $f : A \to B$ is surjective if and only if for every open $U \subseteq X$ and every $A \in B(U)$, there is a covering $\{U_\alpha\}_{\alpha \in I}$ of U such that $A\mid_{U_\alpha}$ is the image of f over some section in $A(U_\alpha)$.

Proposition 11.7. Let $f : A \to B$ be a map of sheaves on a space X, then f is injective (respectively, surjective, isomorphic) if and only if for every $p \in X$, $f_p : A_p \to B_p$ is injective (respectively, surjective, isomorphic).

Remark 11.8. The isomorphism of sheaves is just a bijective map on sheaves.

Definition 11.9 (Cokernel of Sheaves)**.** For a map of sheaves $f : A \to B$ of Abelian groups on a space X, the *cokernel sheaf* $\mathrm{coker}(f)$ is the sheafification of the presheaf $U \mapsto \mathrm{coker}(f : A(U) \to B(U))$, where the cokernel here is defined by $B(U)/f(A(U))$.

Remark 11.10. With these definitions, the category of sheaves of Abelian groups on a topological space X is an Abelian category.

Definition 11.11 (Direct Image)**.** Let $f : X \to Y$ be a continuous map of topological spaces. Let E be a sheaf on X. The *direct image sheaf* f_*E on Y is the sheaf $(f_*E)(U) = E(f^{-1}(U))$ for open $U \subseteq Y$.

Example 11.12. If $f : * \to X$ is a map, then an Abelian group A gives a sheaf on a point, and $(f_*)(A)$ is the skyscraper sheaf A_p.

Example 11.13. If Y is a closed subset of an algebraic set X over k, and $i : Y \to X$ is the inclusion, then $i_*(\mathcal{O}_Y)$ is a sheaf on X.

Definition 11.14 (Inverse Image)**.** For a continuous map $f : X \to Y$ of topological spaces, let E be a sheaf on Y. The *inverse image sheaf* $f^{-1}(E)$ on X is the sheafification of the presheaf $U \mapsto \varinjlim_{V} E(V)$ where V runs over all open subsets of Y that contains $f(U)$.

Example 11.15. Let $i : * \hookrightarrow X$ with image $p \in X$. For a sheaf E on X, the inverse image sheaf $f^{-1}(E)$ is the Abelian group E_p, the stalk at p.

12 Lecture 12

Definition 12.1 (Inverse Image)**.** Let $f : X \to Y$ be a continuous map and E a sheaf on Y. We define the inverse image sheaf $f^{-1}(E)$ on X as the sheafification of the presheaf $U \subseteq X \mapsto \varinjlim_{V \subseteq Y} E(V)$ for V open such that $f(U) \subseteq V$.

Remark 12.2. This is the left adjoint to f_*, that is,

$$\mathbf{Hom}_{\mathbf{Sh}(X)}(f^{-1}(E), F) \cong \mathbf{Hom}_{\mathbf{Sh}(Y)}(E, f_*F).$$

Remark 12.3. The sheafification in the definition of $f^{-1}(E)$ cannot be omitted. Take the map $f : X \to *$ for a topological space X. Take the sheaf $\mathbb{Z}_*$ on the point, the presheaf above is sent from $U \subseteq X$ open to $\mathbb{Z}$ if $U \neq \varnothing$, and to 0 if $U = \varnothing$.

As we have seen, this is not a sheaf. For instance,take $X = \mathbb{R}$ with classical topology. With sheafification, $f^{-1}(\mathbb{Z}^() = \mathbb{Z}_X$, the sheaf of locally constant $\mathbb{Z}$-valued functions.

Remark 12.4 (Motivating Scheme)**.** For an algebraically closed field k, there is an equivalence of categories (with reversed orderings) between affine algebraic sets over k and reduced commutative finitely-generated k-algebras.

Given an affine algebraic set X, we send it to $\mathcal{O}(X) = k[x_1, \ldots, x_n]/I(X)$. This is contravariant, as it sends a morphism $X \to Y$ to the k-algebra homomorphism $\mathcal{O}(Y) \to \mathcal{O}(X)$.

Given a k-algebra A, choose a presentation of A as $A \cong k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ for some elements f_i's, then we send it to $\{f_1 = 0, \ldots, f_r = 0\} \subseteq \mathbb{A}^n_k$.

We now want to find a similar correspondence for all commutative rings. For example, the local ring of an algebraic set at a point is usually not a finitely-generated algebra, e.g., $\mathcal{O}_{\mathbb{A}^1_k,0} \cong k[x]_{(x)} = k[x, \frac{1}{x-a}$ for all $a \in k^*]$.

By the Nullstellensatz, in the case where k is algebraically closed, let $X = \mathbf{Max}(\mathcal{O}(X))$, the set of maximal ideals in $\mathcal{O}(X)$. For instance, the maximal ideals in $k[x_1, \ldots, x_n]$ are given by elements $(a_1, \ldots, a_n) \in k^n$, $I = (x_1 - a_1, \ldots, x_n - a_n) \subseteq k[x_1, \ldots, x_n]$. However, the right choice would be to send $A \mapsto \operatorname{Spec}(A)$, the set of prime ideals in A.

Remark 12.5. For a homomorphism $f : A \to B$ of commutative rings, there is a natural map $f^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ by sending a given prime $\mathfrak{p} \subseteq B$ to $f^{-1}(\mathfrak{p}) \subseteq A$, which is prime in A. Note that $A/f^{-1}(\mathfrak{p}) \subseteq B/\mathfrak{p}$.

If $\mathfrak{p}$ is maximal, then $f^{-1}(\mathfrak{p})$ need not be maximal. For example, take the ring homomorphism $\mathbb{Z} \to \mathbb{Q}$, then $(0) \subseteq \mathbb{Q}$ is maximal, but $f^{-1}(0) = 0 \subseteq \mathbb{Z}$ is prime but not maximal.

Definition 12.6 (Spectrum)**.** For a commutative ring A, its *spectrum* $\operatorname{Spec}(A)$ is the set of prime ideals in A. For an ideal $I \subseteq A$, define its zero set $Z(I) = \{\mathfrak{p} \in \operatorname{Spec}(A) : I \subseteq \mathfrak{p}\}$. For a commutative ring A, a closed subset of $\operatorname{Spec}(A)$ is a subset of the form $Z(I)$ for some ideal $I \subseteq A$.

Remark 12.7 (Why is this the right construction?)**.** Given an element $f \in A$, we can think of f as a function whose values "near a point $\mathfrak{p}p \in \operatorname{Spec}(A)''$ is $f \in A_\mathfrak{p}$ (with a ring homomorphism $A \to A_{\mathfrak{p}}$) and values "at the point $\mathfrak{p}$ is $f \in A/\mathfrak{p}$, which is a domain, or we can think of it as $f \in \operatorname{Frac}(A/\mathfrak{p})$, a field.

Therefore, $Z(I) = \bigcap\limits_{f \in I} Z(f)$, where $Z(f) = \{\mathfrak{p} \in \operatorname{Spec}(A) : f = 0 \in A/\mathfrak{p}\} = \{\mathfrak{p} \in \operatorname{Spec}(A) : f \in \mathfrak{p}\}$.

Proposition 12.8. For every commutative ring A, $\operatorname{Spec}(A)$ is a topological space.

Proof. We have to show:

- $\varnothing$ and $\operatorname{Spec}(A)$ are closed in $\operatorname{Spec}(A)$,
- the union of two closed subsets in $\operatorname{Spec}(A)$ is closed,
- the intersection of any collection of closed subsets is closed.

1. $Z((1)) = Z(A) = \{\mathfrak{p} \in \operatorname{Spec}(A)(1) \subseteq \mathfrak{p}\} = \varnothing$ because a prime ideal does not contain 1; and $Z((0)) = \{\mathfrak{p} : 0 \in \mathfrak{p}\} = \operatorname{Spec}(A)$, so those are closed.

2. Given closed subsets $Z(I)$ and $Z(J)$ in $\operatorname{Spec}(A)$ for ideals I and J, we want to show that $Z(I) \cup Z(J) = Z(K)$ for some ideal $K \subseteq A$. We could either take $K = IJ$ or $K = I \cap J$. Let us use $K = IJ$. That is, we want to show a prime ideal $\mathfrak{p} \subseteq A$ satisfies $I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$ if and only if $IJ \subseteq \mathfrak{p}$.

   We have $IJ \subseteq I$ and $IJ \subseteq J$, so $(\Rightarrow)$ is clear. Conversely, suppose that $IJ \subseteq \mathfrak{p}$ for some prime $\mathfrak{p} \subseteq A$, and suppose that $I, J \not\subseteq \mathfrak{p}$, then there are elements $f \in I \backslash \mathfrak{p}$ and $g \in J \backslash \mathfrak{p}$, then $fg \notin \mathfrak{p}$, but $fg \in IJ \subseteq \mathfrak{p}$, contradiction, so $Z(I) \cup Z(J) = Z(IJ)$.

3. $\bigcap\limits_{\alpha \in S} Z(I_\alpha) = Z(\sum\limits_{\alpha \in S} I_\alpha)$.

$\square$

Definition 12.9 (Sheaf of Regular Functions of Spectrum)**.** Let A be a commutative ring. Then the *sheaf of regular functions* on the topological space $X = \operatorname{Spec}(A)$ is defined by: for an open subset $U \subseteq X$, $\mathcal{O}_X(U) = \{S = (s_p : p \in U)\}$ where $s_p \in A_p$ (the localization) such that U is covered by open subsets $V \subseteq U$, on which s can be written as $\frac{f}{g}$ for some $f, g \in A$ such that "$g \neq 0$ of every point of V", that is, $g \notin p$ for every point $p \in V$.

Remark 12.10. It is easy to see that $\mathcal{O}_X(U)$ is a commutative ring for each open $U \subseteq X = \operatorname{Spec}(A)$. It is also easy to verify that $\mathcal{O}_X$ is a sheaf of commutative rings on $X = \operatorname{Spec}(A)$.

Definition 12.11 (Standard Open Subset)**.** A *standard open subset* of $\operatorname{Spec}(A)$ for any commutative ring A is a subset of the form $\{f \neq 0\} = \operatorname{Spec}(A) \backslash Z((f)) \subseteq \operatorname{Spec}(A)$ for some $f \in A$.

Remark 12.12. It is easy to verify that the standard open sets form a basis for the topologies of $\mathrm{Spec}(A)$.

13 Lecture 13

Note that from now on a a ring will always be commutative, unless stated otherwise.

Definition 13.1 (Ringed Space)**.** A *ringed space* X is a topological space with a sheaf of commutative rings.

Example 13.2.

1. A quasi-projective algebraic set over algebraically closed field k is a ringed space.
2. For every ring A, $\mathrm{Spec}(A)$ is a ringed space.

Definition 13.3 (Affine Scheme)**.** An *affine scheme* is a ringed space that is isomorphic to $\mathrm{Spec}(A)$ as a ringed space, for some ring A.

Lemma 13.4. Let A be a ring. The topological space $\mathrm{Spec}(A)$ is quasi-compact.

Proof. That is, the topological space X is compact (but not necessarily Hausdorff). That is, for any open cover $\{U_\alpha\}_{\alpha\in I}$ of $X = \mathrm{Spec}(A)$, here is a finite subset $J \subseteq I$ such that $X = \bigcup\limits_{\alpha\in J} U_\alpha$.

We can choose ideal $I_\alpha \subseteq A$ for $\alpha \in I$ such that $U_\alpha = X\backslash Z(I_\alpha)$, so $X = \bigcup\limits_{\alpha\in I} U_\alpha$, then $\bigcap\limits_{\alpha\in I} Z(I_\alpha) = \varnothing \subseteq X$, so $Z(\sum\limits_{\alpha\in I} I_\alpha = \varnothing$. Recall that $Z(I) \subseteq \mathrm{Spec}(A)$ means $\{\mathfrak{p} \in \mathrm{Spec}(A) : I \subseteq \mathfrak{p}\}$, then $Z(I) = \varnothing$ if and only if I is not contained in any prime ideal, but every ideal $I \subsetneq A$ is contained in some maximal ideal, so $1 \in \sum\limits_{\alpha\in I} I_\alpha$, and so there exists a finite subset $J \subseteq I$ with $1 \in \sum\limits_{\alpha\in J} I_\alpha$, thus $\bigcap\limits_{\alpha\in J} Z(I_\alpha) = \varnothing$, that is, $\bigcup\limits_{\alpha\in J} U_\alpha = X$. □

Theorem 13.5. Let A be a ring, and let $X = \mathrm{Spec}(A)$ be a ringed space.

1. There is a natural isomorphism $A \cong \mathcal{O}(X)$.
2. For any element $g \in A$, there is a natural isomorphism $A\left[\frac{1}{g}\right] \cong \mathcal{O}(\{g \neq 0\})$, where $\{g \neq 0\}$ is called the standard open subset of $X = \mathrm{Spec}(A)$.
3. For every $p \in \mathrm{Spec}(A)$, the stalk $\mathcal{O}_{X,p} \cong A_p$, the localization of A at the prime ideal p.

Example 13.6. Let F be a field, e.g., $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for p a prime number, or $\mathbb{Q}$, or $\mathbb{R}$, or $\mathbb{C}$. Then $\mathrm{Spec}(F)$ is a point, corresponding to the prime ideal $0 \subseteq F$. We have $\mathcal{O}(\varnothing) = 0$ and $\mathcal{O}(*) = F$.

Lemma 13.7. For any ring A, there are a natural bijective correspondences between:

1. the closed points in $\mathrm{Spec}(A)$ and the maximal ideals in A.
2. the points in $\mathrm{Spec}(A)$ and the prime ideals in A.
3. the closed subsets of $\mathrm{Spec}(A)$ and the radical ideals in A.

Proof. (2) is clear. For (1), what is the closure of a prime ideal $p \in \mathrm{Spec}(A) = X$? It is the closed subset $Z(p) \subseteq X$. By definition, this is the set of primes q containing p. Clearly $Z(p)$ is a closed subset of X that contains the point p. If $Z(I)$ is some other closed subset containing p, then $I \subseteq p$, so $Z(p) \subseteq Z(I)$. Therefore, $Z(p)$ is the closure of the point p. So a point $p \in \mathrm{Spec}(A)$ is closed if and only if $Z(p) = \{p\}$ if and only if the only prime ideal containing p is p, i.e., p is a maximal ideal.

To prove (3), recall that for any commutative ring A, $\mathrm{nilrad}(A) = \{x \in A : x^n = 0 \text{ for some } n > 0\}$, also known as the intersection of all prime ideals in A. Therefore, for any ideal $I \subseteq A$, then $\mathrm{rad}(I)$ is the intersection of all prime ideals in A containing I. So for an ideal I, knowing $Z(I) = \{p \in \mathrm{Spec}(A) : I \subseteq p\}$ is equivalent to knowing the intersection of all primes containing I, i.e., knowing $\mathrm{rad}(I)$. □

Example 13.8 (What is $\mathrm{Spec}(\mathbb{Z})$?)**.** The prime ideals in $\mathbb{Z}$ are the maximal ideals (2), (3), (5), and so on, and the zero ideal $(0) \subseteq \mathbb{Z}$. Geometrically speaking, the points (2), (3), (5), and so on are closed, but the closure of the point (0) is $Z((0)) = \mathrm{Spec}(\mathbb{Z})$.

Definition 13.9 (Generic Point)**.** For a topological space X, a *generic point* of X is a point whose closure is X.

Remark 13.10. For every domain A, $\mathrm{Spec}(A)$ has a generic point, namely the prime ideal $(0) \subseteq A$.

The closed subsets of $\mathrm{Spec}(\mathbb{Z})$ are the subsets of the form $Z(I)$ for some ideal I. Since $\mathbb{Z}$ is a PID, $I = (a)$ for some $a \in \mathbb{Z}$. Therefore, every closed subset $\mathrm{Spec}(\mathbb{Z})$ is of the form $\{a = 0\}$ for some $a \in \mathbb{Z}$.

Example 13.11. $\{15 = 0\} \subseteq \mathrm{Spec}(\mathbb{Z})$ is $\{(3), (5)\}$, and $\{0 = 0\}$ is all of $\mathrm{Spec}(\mathbb{Z})$. Therefore, every closed subset of $\mathrm{Spec}(\mathbb{Z})$ is either $\mathrm{Spec}(\mathbb{Z})$ or a finite set of closed points. And we have, for example, $\mathcal{O}(\{15 \neq 0\} = \mathbb{Z}\left[\frac{1}{15}\right]$. So, for instance, $7, \frac{2}{3}, \frac{8}{15}$, are all regular functions on $\{15 \neq 0\}$.

Example 13.12 (What is $\mathrm{Spec}(\mathbb{Z}/6\mathbb{Z})$?)**.** We can use that $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} = \{(a, b) : a \in \mathbb{Z}/2\mathbb{Z}, b \in \mathbb{Z}/3\mathbb{Z}\}$. Therefore, $\mathrm{Spec}(\mathbb{Z}/6\mathbb{Z}) \cong \mathrm{Spec}(\mathbb{Z}/2\mathbb{Z}) \coprod \mathrm{Spec}(\mathbb{Z}/3\mathbb{Z})$, therefore given by two closed points (2) and (3). Therefore, not every affine scheme has a generic point.

Example 13.13. Affine n-space over a Commutative Ring R, $\mathbb{A}^n_R$, means $\mathrm{Spec}(R[x_1, \dots, x_n])$.

Example 13.14. Let k be an algebraically closed field. What is $\mathbb{A}^1_k$ in this new sense? It is $\mathrm{Spec}(k[x])$. The prime ideals are the maximal ideals $(x - a)$ for $a \in k$, and the prime (but not maximal) ideal (0). Therefore, $\mathbb{A}^1_k = k \cup \{*\}$, where $*$ denotes the generic point.

14 Lecture 14

Definition 14.1 (Discrete Valuation)**.** A *discrete valuation* v on a field F is a surjective function $v : F \to \mathbb{Z} \cup \{\infty\}$ such that

1. For $a \in F$, $v(a) = \infty$ if and only if $a = 0 \in F$.

2. $v(ab) = v(a) + v(b)$ for all $a, b \in F$.

3. $v(a + b) \geq \min(v(a), v(b))$ for all $a, b \in F$.

Definition 14.2 (Discrete Valuation Ring)**.** A *discrete valuation ring* (DVR) is the subring $\{a \in f : v(a) \geq 0\}$ of ra discretely valued field (F, v).

Example 14.3. For a field k, get a valuation on the field $k(x)$ by $v(x^a \cdot \frac{p}{q}) = a$, if $a \in \mathbb{Z}$ and $p, q \in k[x]$ not multiples of x, so this valuation measure the order of vanishing of $f \in k(x)$ at $0 \in \mathbb{A}^1_k$. Therefore, it is negative if f has a pole at $0 \in \mathbb{A}^1_k$. The associated DVR is $\{f \in k(x) : v(f) \geq 0\} = k[x]_{(x)}$, the localization at this prime ideal.

Example 14.4. Get the p-adic valuation on $\mathbb{Q}$ for a prime number p by $v(p^n \frac{u}{v}) = a$ if $u, v \in \mathbb{Z}\backslash(p)$. The associated DVR is the local ring $\mathbb{Z}_{(p)}$.

Remark 14.5 (What is $\mathrm{Spec}(A)$ for a DVR A?)**.** The ideal in a DVR A are just $\{0\}$ and $J_a = \{f \in R : v(f) \geq a\}$ for $a \in \mathbb{N}$. The only prime ideals are (0) and $J_1 = \{f \in R : v(f) \geq 1\} = (q)$ for some $q \in R$ with $v(q) = 1$, which gives a maximal ideal. Therefore, $\mathrm{Spec}(A)$ is given a closed point J_1 and a generic point (0). The open subset of $\mathrm{Spec}(A)$ are $\varnothing$, the generic point $\{g \neq 0\}$, and $\mathrm{Spec}(A)$, so the ring of regular functions on these open subsets are 0, and $A[\frac{1}{g}] = \mathrm{Frac}(A)$, and A.

Remark 14.6. Recall that for a commutative ring R, the affine n-space over R is the affine scheme $\mathrm{Spec}(R[x_1, \dots, x_n])$. For k algebraically closed, the scheme $\mathbb{A}^1_k$ is $k \cup \{*\}$ where $*$ is the generic point. What about $\mathbb{A}^2_k$? The points of the scheme $\mathbb{A}^2_k$ are the prime ideals in $k[x, y]$. By the Nullstellensatz, the subset of closed points is k^2 (by $(a, b) \in k^2$, get the maximal ideal $(x - a, y - b) \subseteq k[x, y] = R$). The subspace topology on k^2 is the Zariski topology. The other point of the scheme $\mathbb{A}^2_k$ are in one-to-one correspondence with the irreducible closed subset of dimension at least 1 in k^2.

Remark 14.7. There is a one-to-one correspondence between open subset of the scheme $\mathbb{A}_k^2$ and open subsets of k^2, given by $U \mapsto U \cap k^2$. Moreover, the ring of regular functions on open $U \subseteq \mathbb{A}_k^2$ is the same as the regular functions on $U \cap k^2$.

Example 14.8. $\text{Spec}(0) = \varnothing$ because $R \subseteq R$ is not prime ideal, because a domain is defined to have $1 \neq 0$. And if R is a non-zero ring, then $\text{Spec}(R) \neq \varnothing$. Also, if R is a ring with $\text{Spec}(R) = \varnothing$, then $R \cong \mathcal{O}(\text{Spec}(R)) = 0$.

Example 14.9 (What is the scheme $\mathbb{A}_{\mathbb{R}}^1$?)**.** A point of $\mathbb{A}_{\mathbb{R}}^1$ is a prime ideal in $\mathbb{R}[x]$. For any field, this is a PID, the prime ideal are (0) and the maximal ideals, which are of the form (g) for an irreducible polynomial $g \in R[x]$. For an irreducible polynomial $g(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ over $\mathbb{R}$, we have a complete factorization over $\mathbb{C}$ given by $g(x) = (x - c_1) \cdots (x - c_n)$ for some $c_1, \ldots, c_n \in \mathbb{C}$. Note that multiplying the complex conjugations maintain the coefficients as reals. Therefore, the irreducible real polynomials in one variable are $x - b$ for $b \in \mathbb{R}$, and $(x - c)(x - \bar{c})$ for $c \in \mathbb{C} \backslash \mathbb{R}$. Therefore, as a set $\mathbb{A}_{\mathbb{R}}^1$ is just the quotient of $\mathbb{C}$ over the action of complex conjugation, and the single generic point.

Note that $I = (x^2 + 1) \subseteq \mathbb{R}[x]$ has $Z(I) \subseteq R$ empty, which is the same as $Z((1))$, but $\text{rad}(x^2 + 1) \neq \text{rad}(1)$.

Definition 14.10 (Ringed Space)**.** A *ringed space* is a topological space X with a sheaf of commutative rings $\mathcal{O}_X$.

Definition 14.11 (Locally Ringed Space)**.** A *locally ringed space* is a ringed space such that for every $p \in X$, the stalk $\mathcal{O}_{X,p}$ is a local ring.

Example 14.12. For a commutative ring A, $X = \text{Spec}(A)$ is a locally ringed space because $\mathcal{O}_{X,p} = A_p$, the localization of A at p.

Definition 14.13 (Affine Scheme)**.** An *affine scheme* X is a locally ringed space that is isomorphic (as a locally ringed space) to $\text{Spec}(A)$ for some commutative ring A. A *scheme* is a locally ringed space that has an open cover by affine schemes (as locally ringed space).

Example 14.14. Every open subset of a scheme is a scheme.

Proof Sketch. Let X be a scheme, and $V \subseteq X$ open. We are given $X = \bigcup_{\alpha \in I} U_\alpha$, with $U_\alpha \cong \text{Spec}(R_\alpha)$ then U is a locally ringed space with U_α's. Then $V = \bigcup_{\alpha \in I} (V \cap U_\alpha)$, but $V \cap U_\alpha$ need not be an affine scheme, but it is an open subset of an affine scheme.

Note that every open subset of $U_\alpha = \text{Spec}(R_\alpha)$ is a union of some standard open subsets $\{g_\beta \neq 0\} \subseteq U_\alpha$ for $g_\beta \in R_\alpha$. Therefore, V is a union of affine scheme, namely the $\{g_\beta \neq 0\}$ are of the form $\text{Spec}(R_\alpha[\frac{1}{g}]_\beta)$, so it is a scheme. $\square$

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