

Motivic Cohomology Notes

Jiantong Liu

February 28, 2024

These notes were taken from a **Motivic Cohomology course** taught by Professor N. Yang in Spring 2024 at BIMSA. Any mistakes and inaccuracies would be my own. References for this course include [MVW06] and [Ros96].

0 INTRODUCTION

Let $X \in \mathbf{Sm}/k$ be a smooth separated scheme over a field k . The study of motivic cohomology started with the hope that

Conjecture 0.1 (Beilinson and Lichtenbaum, 1982-1987). There exists certain complexes $\mathbb{Z}(n)$ for $n \in \mathbb{N}$ of sheaves in Zariski topology on \mathbf{Sm}/k such that

1. $\mathbb{Z}(0)$ is (quasi-isomorphic to) the constant sheaf \mathbb{Z} , i.e., the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$

concentrated at degree 0;

2. $\mathbb{Z}(1)$ is the complex $\mathcal{O}^*[-1]$, i.e., the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{O}^* \longrightarrow 0 \longrightarrow \cdots$$

concentrated at degree 1;

3. for every field F/k , the hypercohomology¹

$$\mathbb{H}_{\mathrm{Zar}}^n(F, \mathbb{Z}(n)) = H^n(\mathbb{Z}(n)(\mathrm{Spec}(F))) = K_n^M(F),$$

where $K_n^M(F)$ is the n th Milnor K-theory of a field F , given by the quotient of the tensor algebra $T(F^*)/\{x \otimes (1-x) : x \in F^*\}$ over \mathbb{Z} ; (lecture 5 of [MVW06], page 29)

Example 0.2.

- a. $K_0^M(F) = K_0(F) = \mathbb{Z}$;
 - b. $K_1^M(F) = K_1(F) = F^\times$;
 - c. $K_2^M(F) = K_2(F)$.
4. $\mathbb{H}_{\mathrm{Zar}}^{2n}(X, \mathbb{Z}(n)) = \mathrm{CH}^n(X)$ (lecture 17 of [MVW06], page 135), where the n th classical Chow group $\mathrm{CH}^n(X)$ is the free group given by

$$\mathrm{CH}^n(X) = \mathbb{Z}\{\text{cycles of codimension } n\}/\text{rational equivalences};$$

¹Here we use the convention that the (hyper)cohomology of F should be interpreted as of $\mathrm{Spec}(F)$, the corresponding space.

5. there is a natural Atiyah–Hirzebruch spectral sequence

$$E_2^{p,q} = \mathbb{H}_{\text{Zar}}^p(X, \mathbb{Z}(q)) \Rightarrow K_{2q-p}(X).$$

Moreover, tensoring with \mathbb{Q} , the spectral sequence degenerates and one has

$$\mathbb{H}_{\text{Zar}}^i(X, \mathbb{Z}(n))_{\mathbb{Q}} = \text{gr}_{\gamma}^n(K_{2n-i}(X)_{\mathbb{Q}})$$

where gr_{γ}^n 's are the quotients (graded pieces) of γ -filtration. ([Lev94]; [Lev99], Theorem 11.7)

Remark 0.3. Such choices of complexes $\mathbb{Z}(q)$ exists, and is called the motivic complex. For a clear definition of these complexes, see Lecture 3 of [MVW06]. Moreover, by convention $\mathbb{Z}(q) = 0$ for $q < 0$.

Definition 0.4. The motivic cohomology of X is defined by $H^{p,q}(X, \mathbb{Z}) = \mathbb{H}_{\text{Zar}}^p(X, \mathbb{Z}(q))$, the hypercohomology of the motivic complexes with respect to the Zariski topology.

Remark 0.5. In general, a motivic cohomology with coefficient in an abelian group A is a family of contravariant functors $H^{p,q}(-, A) : \text{Sm}/k \rightarrow \text{Ab}$.

Remark 0.6. The motivic cohomology of X satisfies the cancellation property: set $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$, then

$$H^{p,q}(X \times \mathbb{G}_m, \mathbb{Z}) = H^{p,q}(X, \mathbb{Z}) \oplus H^{p-1, q-1}(X, \mathbb{Z}).$$

Remark 0.7. It turns out that the group remains unchanged if we replace the Zariski topology by Nisnevich topology.² If one uses étale topology instead, we retrieve Lichtenbaum motivic cohomology $H_L^{p,q}(X, \mathbb{Z})$. If $\text{char}(k) \nmid n$, it admits the comparison

$$H_L^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) = H_{\text{étale}}(X, \mathbb{Z}/n\mathbb{Z}(q)),$$

where $\mathbb{Z}/n\mathbb{Z}(q)$ is the q -twist $\mu_n^{\otimes q}$.

We may compare Lichtenbaum motivic cohomology with motivic cohomology by the following theorem, formerly known as Beilinson-Lichtenbaum Conjecture³:

Theorem 0.8 ([Voe11]). The natural map

$$H^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) \rightarrow H_L^{p,q}(X, \mathbb{Z}/n\mathbb{Z})$$

is an isomorphism if $p \leq q$, is a monomorphism if $p = q + 1$, and gives a spectral sequence for any pair of p, q .

Corollary 0.9. For $p \leq q$, we have

$$H^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) = H_{\text{étale}}^p(X, \mathbb{Z}/n\mathbb{Z}(q)).$$

In particular, for $X = \text{Spec}(k)$ as a point, this is the theorem formerly known as Milnor conjecture:

Corollary 0.10 ([Voe97], [Voe03a], [Voe03b]).

- $H^{p,p}(k, \mathbb{Z}/n\mathbb{Z}) = K_p^M(k)/n = H_{\text{étale}}^p(X, \mathbb{Z}/n\mathbb{Z}(p))$ as the Galois cohomology;
- in general,

$$H^{p,q}(k, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} 0, & p > q \\ H^{p,p}(k, \mathbb{Z}/n\mathbb{Z}) \cdot \tau^{q-p}, & p < q \end{cases}$$

where $\tau \in \mu_n(k) = H^{0,1}(k, \mathbb{Z})$ is a primitive n th root of unity.

Remark 0.11. Unlike finite coefficients, $H^{p,q}(k, \mathbb{Z})$ is quite hard to compute for small $p < q$; for $p \geq q$, this is 0.

A current long-standing conjecture is

²Recall that the Nisnevich topology is a Grothendieck topology on the category of schemes that is finer than the Zariski topology but coarser than the étale topology.

³This is also known as the norm residue isomorphism theorem, or (formerly) Bloch-Kato conjecture.

Conjecture 0.12 (Beilinson-Soulé Vanishing Conjecture, [Lev93]). $H^{p,q}(k, \mathbb{Z}) = 0$ if $p < 0$.

Remark 0.13. Here are a few known cases:

- for $\text{char}(k) = 0$, this is known for number fields ([Bor74]), function fields of genus 0 ([Dég08]), curves over number fields, and their inductive limits (more precise references required); ([DG05])
- for $\text{char}(k) > 0$, this is known for finite fields ([Qui72]) and global fields ([Har77]).

Remark 0.14. The motivic cohomology could be realized in a tensor triangulated category, namely the category of effective motives $DM(k)$. For any pair of p, q , we can find an Eilenberg-MacLane space and a corresponding representable functor so that

$$H^{p,q}(X, \mathbb{Z}) = \text{Hom}_{DM}(\mathbb{Z}(X), \mathbb{Z}(q)[p])$$

where $\mathbb{Z}(X)$ is the motive of X and $\mathbb{Z}(q)[p] = \mathbb{G}_m^q[p - q]$.⁴

Remark 0.15. Dually, we can define the motivic homology by

$$H_{p,q}(X, \mathbb{Z}) = \text{Hom}_{DM}(\mathbb{Z}(q)[p], \mathbb{Z}(X)).$$

Remark 0.16 ([MVW06] Properties 14.5, page 110). By taking the hom functor from the aspect of motives, we can derive theorems for all (co)homologies which can be represented in DM . The main derives are the following:

1. If $E \rightarrow X$ is an \mathbb{A}^n -bundle, then motives $\mathbb{Z}(E) = \mathbb{Z}(X)$ in DM .
2. If $\{U, V\}$ is a Zariski open covering of X , we have a Mayer-Vietoris sequence

$$\mathbb{Z}(U \cap V) \longrightarrow \mathbb{Z}(U) \oplus \mathbb{Z}(V) \longrightarrow \mathbb{Z}(X) \longrightarrow \mathbb{Z}(U \cap V)[1]$$

in the form of a distinguished triangle in DM .

3. If $Y \subseteq X$ is a closed embedding of codimension c in Sm/k , then we have a Gysin triangle

$$\mathbb{Z}(X \setminus Y) \longrightarrow \mathbb{Z}(X) \longrightarrow \mathbb{Z}(Y)(c)[2c] \longrightarrow \mathbb{Z}(X \setminus Y)[1]$$

which is a distinguished triangle where $\mathbb{Z}(Y)(c)[2c] := \mathbb{Z}(Y) \otimes \mathbb{Z}(c)[2c]$.

4. For any vector bundle of rank n on X , we have the projective bundle formula

$$\mathbb{Z}(\mathbb{P}(E)) = \bigoplus_{i=0}^n \mathbb{Z}(X)(i)[2i]$$

which defines the Chern class of E .

5. Let X be a proper smooth scheme and let d_X be its dimension, then $\mathbb{Z}(X)$ has a strong dual $\mathbb{Z}(X)(-d_X)[-2d_X]$ in DM by stabilization. This gives a Poincaré duality⁵

$$H^{p,q}(X, \mathbb{Z}) \cong H_{2d_X - p, d_X - q}(X, \mathbb{Z})$$

1 INTERSECTION THEORY

Definition 1.1. Let X be a scheme of finite type over k . We define the i th cycle on the scheme X to be a free abelian group

$$Z_i(X) = \bigoplus_{\substack{\text{irreducible closed } c \subseteq X \\ \text{with } \dim(c)=i}} \mathbb{Z} \cdot c$$

and set $Z(X) = \bigoplus_i Z_i(X)$. Define a set $K_i(X)$ to be the set of coherent sheaves \mathcal{F} on X with $\dim(\text{supp}(\mathcal{F})) \leq i$.⁶

⁴Again, this notation goes back to the concise definition of the motivic complexes: see Lecture 3 from [MVW06] as well as the concept of presheaves with transfers.

⁵We can use cohomology with compact support for this.

⁶Despite the notation, this has nothing to do with a K-theory.

Remark 1.2. Let (A, \mathfrak{m}) be a Noetherian local ring and M be an A -module, then by the dimension theorem, we know $\dim(M) = d(M) = \dim(\text{supp}(M))$, where $d(M)$ is the degree of the Hilbert-Samuel polynomial $P_{\mathfrak{m}}(M, n)$.

Definition 1.3. Let $X \in \text{Sm}/k$ and let $U, V \subseteq X$ be irreducible and closed. Suppose $W \subseteq U \cap V$ is a irreducible and closed component. If $\dim(W) = \dim(U) + \dim(V) - \dim(X)$, i.e., $\text{codim}(W) = \text{codim}(U) + \text{codim}(V)$, we say that U and V intersect properly at W .

Remark 1.4. This condition is weaker than saying they intersect transversely: we do not require information about tangent spaces.

Theorem 1.5. Let $A \supseteq k$ be a Noetherian regular ring, M, N be finitely-generated A -modules, and suppose $\ell(M \otimes_A N) < \infty$, then

1. $\ell(\text{Tor}_i^A(M, N)) < \infty$ for all $i \geq 0$;
2. the Euler-Poincaré characteristic $\chi(M, N) := \sum_{i=0}^{\dim(A)} (-1)^i \ell(\text{Tor}_i^A(M, N)) \geq 0$;
3. by Remark 1.2, we have $\dim(M) + \dim(N) \leq \dim(A)$;
4. in particular, we have $\dim(M) + \dim(N) < \dim(A)$ if and only if $\chi(M, N) = 0$.

Proof. See [Ser12], page 106. □

Remark 1.6. Part 3. from Theorem 1.5 implies that $\dim(W) \geq \dim(U) + \dim(V) - \dim(X)$, i.e., $\text{codim}(W) \leq \text{codim}(U) + \text{codim}(V)$ in the notation of Definition 1.3.

Definition 1.7. Let X, U, V, W be as in Definition 1.3, then we define the intersection multiplicity $m_W(U, V)$ of U and V at W by

$$m_W(U, V) = \chi^{\mathcal{O}_{X,W}}(\mathcal{O}_{X,W}/P_U, \mathcal{O}_{X,W}/P_V)$$

where P_U and P_V are prime ideals defining U and V , respectively.

Remark 1.8. By Theorem 1.5, we know $m_W(U, V) \geq 0$, and $m_W(U, V) = 0$ if and only if U and V do not intersect properly at W .

Definition 1.9. Let $X \in \text{Sm}/k$, and let $U \in Z_a(X)$ and $V \in Z_b(X)$. If U and V intersect properly at every component, then we define

$$U \cdot V = \sum_{\substack{W \subseteq U \cap V \\ \dim(W) = a+b-d_X}} m_W(U, V) \cdot W \in Z_{a+b-d_X}(X).$$

Example 1.10. Let X be a smooth projective surface, and let C and D be divisors on X . For any point $x \in C \cap D$, locally we think of $C = \{f = 0\}$ and $D = \{g = 0\}$ around x , then $m_x(C, D) = \ell_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(f, g))$.

Definition 1.11. Suppose X is a scheme of finite type over k , and $\mathcal{F} \in K_n(X)$ is a coherent sheaf, then we define $Z_a(\mathcal{F}) = \sum_{\dim(\bar{\eta})=a} (\mathcal{O}_{X,\eta}(\mathcal{F}_{\bar{\eta}}) \cdot \bar{\eta}) \in Z_a(X)$.

Therefore, we define the cycle of \mathcal{F} as an element of the cycle of X .

Definition 1.12 ([Har13], Exercise III.6.9). Every coherent sheaf \mathcal{F} on $X \in \text{Sm}/k$ has a resolution

$$0 \longrightarrow E_k \longrightarrow E_{k-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

where E_i 's are locally free of finite rank. Therefore, for any coherent sheaf \mathcal{G} , we can define the Tor functor of coherent sheaves by

$$\text{Tor}_i(\mathcal{F}, \mathcal{G}) = H_i(E_* \otimes_F \mathcal{G}).$$

Proposition 1.13. Let $X \in \text{Sm}/k$. Suppose $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(X)$ intersect properly, then

$$Z_a(\mathcal{F}) \cdot Z_b(\mathcal{G}) = \sum_{i=0}^{d_X} (-1)^i \cdot Z_{a+b-d_X}(\text{Tor}_i(\mathcal{F}, \mathcal{G})).$$

Proof. We only have to do it locally, so we can assume X to be affine, and count the coefficients of $\bar{\xi}$ where $\dim(\xi) = a + b - d_X$. It suffices to show that the stalks at ξ satisfies

$$\chi(F_\xi, G_\xi) = \sum_{\substack{\dim(\bar{\lambda})=a \\ \dim(\bar{\eta})=b \\ \xi \in \bar{\lambda} \cap \bar{\eta}}} \ell(\mathcal{F}_\lambda) \cdot \ell(G_\eta) \cdot m_{\bar{\xi}}(\bar{\lambda}, \bar{\eta}).$$

Because our ring is Noetherian, then \mathcal{F} admits a filtration

$$0 = M_0 \subseteq \cdots \subseteq M_d = \mathcal{F}$$

such that $M_i/M_{i-1} \cong \mathcal{O}_X/\mathcal{I}$ is coherent for prime ideal \mathcal{I} . By the additivity of both sides of the isomorphism, we may assume $\mathcal{F} = \mathcal{O}_X/\mathfrak{p}$ with dimension at most a , where $\mathfrak{p} \sim \lambda \in X$. Similarly, we may assume $\mathcal{G} = \mathcal{O}_X/\mathfrak{q}$ with dimension at most b , where $\mathfrak{q} \sim \eta \in X$. Moreover, set $\xi \in \bar{\lambda} \cap \bar{\eta}$. By definition, we now have $\chi(\mathcal{F}_\xi, \mathcal{G}_\xi) = m_{\bar{\xi}}(\bar{\lambda}, \bar{\eta})$.

- If $\dim(\bar{\lambda}) = a$ and $\dim(\bar{\eta}) = b$, then the equality follows from the fact that $\ell(\mathcal{F}_\lambda) = \ell(\mathcal{G}_\eta) = 1$.
- If not, then either $\dim(\bar{\lambda}) < a$ or $\dim(\bar{\eta}) < b$, then $\bar{\lambda}$ and $\bar{\eta}$ do not intersect properly at $\bar{\xi}$, so both the left-hand side and the right-hand side become 0.

□

Proposition 1.14. The intersection product is commutative.

Proof. This is obvious since the Tor functor is commutative.

□

Proposition 1.15. The intersection product is associative.

REFERENCES

- [Bor74] Armand Borel. Stable real cohomology of arithmetic groups. In *Annales scientifiques de l'École Normale Supérieure*, volume 7, pages 235–272, 1974.
- [Dég08] Frédéric Déglise. Motifs génériques. *Rendiconti del Seminario Matematico della Università di Padova*, 119:173–244, 2008.
- [DG05] Pierre Deligne and Alexander B Goncharov. Groupes fondamentaux motiviques de Tate mixte. In *Annales scientifiques de l'École Normale Supérieure*, volume 38, pages 1–56. Elsevier, 2005.
- [Har77] Günter Harder. Die Kohomologie s-arithmetischer Gruppen über Funktionenkörpern. *Inventiones mathematicae*, 42:135–175, 1977.
- [Har13] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer Science & Business Media, 2013.
- [Lev93] Marc Levine. Tate motives and the vanishing conjectures for algebraic K-theory. *Algebraic K-theory and algebraic topology*, pages 167–188, 1993.
- [Lev94] Marc Levine. Bloch's higher Chow groups revisited. *Astérisque*, 226(10):235–320, 1994.
- [Lev99] Marc Levine. K-theory and motivic cohomology of schemes. *preprint*, 166:167, 1999.
- [MVW06] Carlo Mazza, Vladimir Voevodsky, and Charles A Weibel. *Lecture notes on motivic cohomology*, volume 2. American Mathematical Soc., 2006.
- [Qui72] Daniel Quillen. On the cohomology and K-theory of the general linear groups over a finite field. *Annals of Mathematics*, 96(3):552–586, 1972.
- [Ros96] Markus Rost. Chow groups with coefficients. *Documenta Mathematica*, 1:319–393, 1996.
- [Ser12] Jean-Pierre Serre. *Local algebra*. Springer Science & Business Media, 2012.

- [Voe97] Vladimir Voevodsky. The milnor conjecture, 1997.
- [Voe03a] Vladimir Voevodsky. Motivic cohomology with $\mathbf{z}/2$ -coefficients. *Publications Mathématiques de l'IHÉS*, 98:59–104, 2003.
- [Voe03b] Vladimir Voevodsky. Reduced power operations in motivic cohomology. *Publications Mathématiques de l'IHÉS*, 98:1–57, 2003.
- [Voe11] Vladimir Voevodsky. On motivic cohomology with \mathbf{z}/l -coefficients. *Annals of mathematics*, pages 401–438, 2011.