Bounds in Simple Hexagonal Lattice and Classification of 11-stick Knots

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How to Classify Knots?

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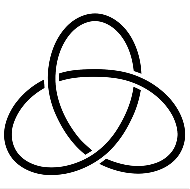
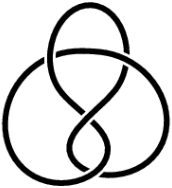
In particular, we say two knots are *equivalent* if there exists an ambient isotopy that transforms one to another.
However, it is sometimes hard to tell one knot from another...





Knot Invariants

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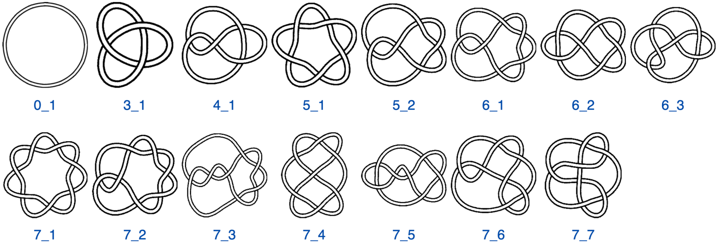
- Crossing Number
- Bridge Number
- ...

Definition

The *crossing number* of a knot type is the least number of crossings among all possible knots of this type.

Knot Types with Small Crossing Numbers

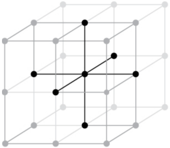
The crossing number gives us an idea of how simple/complex a knot really is.



Cubic Lattice

The *cubic lattice* is defined to be

$$\mathbb{L}^3 = (\mathbb{R} \times \mathbb{Z} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{Z} \times \mathbb{R}).$$



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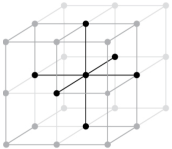

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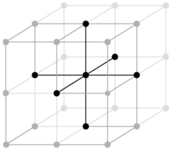

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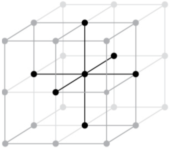


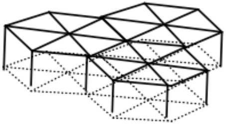
A *polygon* $\mathcal{P}$ in the cubic lattice is a continuous path consisting of line segments parallel to the x-, y-, and z-axes. A maximal line segment parallel to the x-axis is called an x*-stick*, and one can define y-stick and z-stick similarly. A *cubic lattice knot* is a non-intersecting polygon in the cubic lattice consisting of x-, y-, and z-sticks.

Simple Hexagonal Lattice

Let $x = \langle 1, 0, 0 \rangle$, $y = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, 0 \right\rangle$, and $z = \langle 0, 0, 1 \rangle$. The *simple hexagonal lattice* (sh-lattice) is defined to be the set of $\mathbb{Z}$-combinations of x, y, w, i.e.,

$$sh = \{ax + by + cw \mid a, b, c \in \mathbb{Z}\}.$$

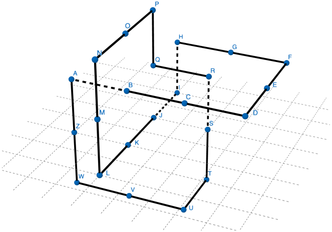
We define $z = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}, 0 \right\rangle$, i.e, $z = y - x$.



Mapping between Lattices

$$T : \mathbb{L}^3 \to \text{sh}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



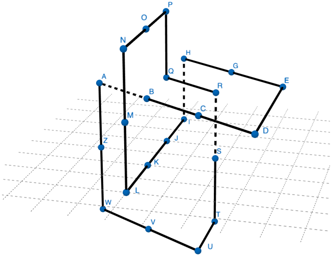


Figure: Effect of T on the Trefoil Knot

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Proposition

T is a well-defined linear transformation. Moreover, let $\mathcal{P}_L$ be a cubic lattice knot presentation and $\mathcal{P}_{sh}$ be its image over T, then T preserves

1. *the stick number of the lattice knot, i.e., $|\mathcal{P}_L| = |\mathcal{P}_{sh}|$.*
2. *the order and length of the sticks.*

Therefore, T preserves the overall structure and properties of lattice knots, only "squeezing" the knot a little.

Studying Knot Types

Definition

The *stick number of a knot type* $[K]$ is the least stick number among all knot conformations $\mathcal{P}$ of $[K]$ in a given lattice $\mathbb{A}$, i.e., $s_{\mathbb{A}}[K] = \min_{\mathcal{P} \in [K] \subset \mathbb{A}} |\mathcal{P}|$. We use $s_L[K]$ and $s_{\text{sh}}[K]$ to denote the stick number of $[K]$ with respect to $\mathbb{L}^3$ and sh, respectively.

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Question

Can we improve it to a strict bound, i.e., $s_{sh}[K] < s_L[K]$*?*

Proving the Strict Bound

Lemma

Project a polygon $\mathcal{P}$ in the cubic lattice down to the xy-plane. Suppose we have an x-stick named x and a y-stick named y of equal length, connected in the shape of an "L". If there are no z-sticks within the triangle with x and y as legs, then we can replace them with a z-stick in the sh-lattice after applying T.

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Proof.

By applying T on this triangle in $\mathbb{L}^3$, we obtain a triangle in the sh-lattice with no w-stick in the region. Therefore, we can replace the x- and y-stick with a z-stick. □

Proving the Strict Bound

- By moving the z-sticks from the lattice knot in $\mathbb{L}^3$ out of the triangular region, the theorem is trivial according to the lemma.
- Therefore, it suffices to apply a linear transformation on the y-coordinates of the z-sticks, which is possible because no other x- or y-sticks pass through the square region.

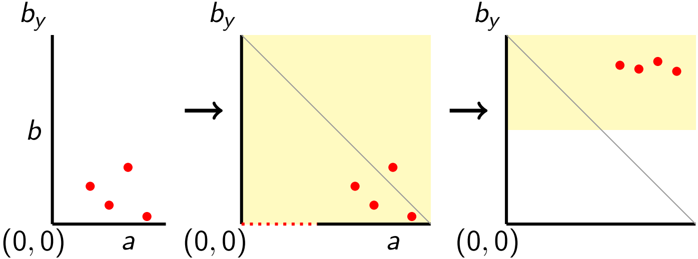

Figure: Illustration of Scaling

Previous Results on Classification

Classification of a few knots with small stick numbers has been known as follows:

	3_1	4_1	5_1	5_2
$\mathbb{L}^3$	12	14	16	16
sh	11	?	?	?

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	3_1	4_1	5_1	5_2
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We improve the classification by proving the following result:

Theorem

In the sh-lattice, the only non-trivial 11*-stick knots are* 3_1 *and* 4_1.

Stick Number of 4_1

Proposition

The stick number of a figure-eight knot in the sh-lattice is 11, *i.e.,* $s_{sh}(4_1) = 11$.

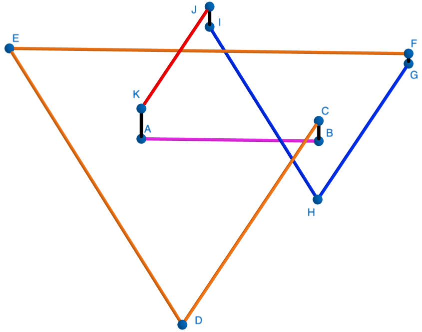

Figure: 4_1 knot in sh-lattice with 11 sticks

Number of w-sticks in a 11-stick Polygon

Lemma

An 11-stick polygon with five w-sticks has to be trivial.

Proof.

By studying the possible configuration of w-sticks in a lattice knot, we can determine the exact w-sticks in a knot, which is given by

$$w_{13}, w_{14}, w_{24}, w_{25}, w_{35}$$

where w_{ij} is a w-stick connecting w-level i and j. Therefore, one of the w-levels has two sticks and every other w-level has exactly one stick. Every possible configuration then turns out to be trivial. □

Corollary

A non-trivial irreducible 11-stick polygon $\mathcal{P}$ has exactly four w-sticks.

Determine the Stick Number of Each Type

Lemma

A non-trivial 11*-stick polygon has at least three x-sticks, at least two y-sticks, and at least one z-stick, up to permutation of stick types.*

Corollary

A non-trivial 11*-stick polygon must have either*

1. $(4, 2, 1)$*: four x-sticks, two y-sticks, and one z-stick, or*
2. $(3, 3, 1)$*: three x-sticks, three y-sticks and one z-stick, or*
3. $(3, 2, 2)$*: three x-sticks, two y-sticks and two z-sticks.*

Theorem

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Updated Classification

	3_1	4_1	5_1	5_2
$\mathbb{L}^3$	12	14	16	16
sh	11	?	?	?

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sh	11	11	$12 \sim 14$	$12 \sim 14$

Future Work

- Determine the stick number of 5_1 and 5_2 in sh-lattice.
- Determine the relationship between stick number and crossing number for knots with small stick numbers.
- For a polygon $\mathcal{P}$ of type $[K]$, construct upper and lower bounds on the number of w-sticks, both in terms of stick number $s_{\text{sh}}[K]$ and in terms of crossing number $c[K]$.