MATH 595 (Open Problems in Geometric Group Theory) Notes

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This course, differs from MATH 503 (geometric group theory), will focus on the open problems in geometric group theory. This is a study related to group theory, topology, and some algebraic topology (cell complexes, universal covering, fundamental group).

Definition 1.1. A group action G on X is free if the stabilizer of any point of X is trivial, i.e., given $x \in X$, if $g \in G$ fixes x, then $g = e$.

Definition 1.2. Let G act on X. The quotient by this action is the set of orbits $G\backslash X := \{Gx \mid x \in X\}$.

Example 1.3. Consider the unit circle S^1, it has a universal covering $\mathbb{R}$, which gives a projection down to S^1 by mapping each interval to the unit circle. The action of the real line over S^1 should preserve integer points, therefore this is equivalent to the action of $\mathbb{Z}$ over the real line $\mathbb{R}$, given by $(n, x) \mapsto n + x$, which does not fix any point. This is a free action, so this becomes a covering map. This becomes an action of topological and metric space.

Example 1.4. Consider the torus $T^2 \cong S^1 \times S^1$, the tangent bundle of the manifold structure of T^2 is known via the differentiable structure, without embedding into $\mathbb{R}^3$.

Alternatively, we can think of T^2 via the universal covering $\mathbb{R}^2$, which is exactly the local tangent plane at each point of the torus. Therefore, at every point of $\mathbb{R}^2$, there is a local tangent plane structure.

The action on $\mathbb{R}^2$ is given by $G = \mathbb{Z} \times \mathbb{Z}$. For $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ and $(x, y) \in \mathbb{R}^2$, we define $(a, b) \cdot (x, y) = (a + x, b + y)$. This defines the free action as well.

The point being, there are many topological structures on T^2, but we can always lift each of them to the universal cover $\mathbb{R}^2$. Therefore, studying of the topological structures becomes a study of group actions on the universal covering, as a type of invariants like homology, cohomology, etc.

Exercise 1.5. If X is a metric space, give the definition of a "natural" of a (pseudo-)metric on $G\backslash X$.