MATH 526 Notes

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August 28, 2023

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Let X be a topological space with basepoint $x_0 \in X$. We already know two invariants,

- the fundamental group $\pi_1(X, x_0)$, and
- the homology groups $H_n(X)$ for $n \geqslant 0$, which are abelian groups.

We will look at two more invariants,

- the cohomology groups $H^n(X)$ for $n \geqslant 0$, and
- the higher homotopy groups $\pi_n(X, x_0)$ for $n \geqslant 0$.

In particular, $\pi_*(X, x_0)$ is a very good invariant in the following sense:

Theorem 1.1 (Whitehead)**.** If $f : (X, x_0) \to (Y, y_0)$ is a map of CW-complexes, then f is a homotopy equivalence if and only if $\pi_*(f) : \pi_*(X, x_0) \to \pi_*(Y, y_0)$ is an isomorphism.

However, π_* is very hard to compute. On the other hand, $H^*(X)$ is relatively easy to compute, but this is not a complete invariant. For instance, $\mathbb{C}P^2$ and $S^2 \vee S^4$ have isomorphic cohomology groups, but they are not equivalent. $H^*(X)$ is closely related to $H_*(X)$, but $H^*(X)$ is a graded ring structure with cup product. It is contravariant in X, where $H_*(X)$ is covariant. The cup product is defined by the composition of induced diagonal map with an external product:

$$H^i(X) \times H^j(X) \xrightarrow{\times} H^{i+j}(X \times X) \xrightarrow{\Delta^*} H^{i+j}(X)$$

Other things we will talk about include:

- Natural transformations $H^i(-) \to H^j(-)$ encoded by Steenrod operations.
- $H^n(-)$ becomes a representable functor, i.e., $H^n(X) = [X, K(\mathbb{Z}, n)]$, where $K(\mathbb{Z}, n)$ is the Eilenberg-Maclane space, and the bracket indicates the homotopy classes of maps.
- Poincaré duality in $H^*(M)$ for compact manifold M, namely the cup product gives

$$H^i(M) \otimes H^{\dim(M)-i}(M) \xrightarrow{\smile} H^{\dim(M)}(M).$$

- Characteristic classes in $H^*(X)$ associated to vector bundles over X.

Recall for a topological space X, we obtain a collection of (singular) homology groups $H_n(X)$, with $H_*(X) = \bigoplus_{n \geqslant 0} H_n(X)$. The functoriality of morphisms says that $X \xrightarrow{f} Y \xrightarrow{g} Z$ induces $f_* g_* = (fg)_* : H_*(X) \xrightarrow{f_*} H_*(Y) \xrightarrow{g_*} H_*(Z)$. So

$$H_*(-) : \mathbf{Top} \to \mathbf{Ab}$$

is a well-defined functor. This factors into

$$\begin{tikzcd} \mathbf{Top} \arrow[rr, "H_*(-)"] \arrow[dr, "C_*(-)"'] & & \mathbf{Ab} \\ & \mathbf{Ch} \arrow[ur, "H_*(-)"'] & \end{tikzcd}$$

Here $C_*(-)$ is usually the singular chain, given by $\partial : C_n(X) \to C_{n-1}(X)$, where $C_n(X)$ is the free abelian group generated by $\operatorname{Hom}_{\mathbf{Top}}(\Delta^n, X) \cong \bigoplus_{\sigma:\Delta^n \to X} \mathbb{Z}\sigma$. $\Delta^n \subseteq \mathbb{R}^{n+1}$ is the set of tuples $(t_0, \ldots, t_n)$ such that the coordinates sum to 1. The boundary is $\partial\sigma = \sum_{0 \leqslant i \leqslant n} (-1)^i \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$.

We say $C_*(-)$ is homotopy invariant, i.e., if $f : X \to Y$ is a homotopy equivalence, then the induced map $C_*(X) \to C_*(Y)$ on chain complexes is a chain equivalence.

Remark 1.2. $C_*^{\Delta}(X)$ and $C_*^{\mathrm{CW}}(X)$ are both chain equivalent to $C_*(X)$.

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Here is a list of properties of $C_*(-) : \mathbf{Top} \to \mathbf{Ch}$:

- Functoriality: given a continuous map $f : X \to Y$, there is an induced map

$$\begin{aligned} f_* : C_*(X) &\to C_*(Y) \\ (\sigma : \Delta^n \to X) &\mapsto (f\sigma : \Delta^n \to Y) \end{aligned}$$

- Homotopy invariance: given $f, g : X \to Y$ such that $f \simeq g$, i.e., there is $H : X \times [0,1] \to Y$ such that $H|_0 = f$ and $H|_1 = g$, then $f_* \simeq g_*$ as a chain homotopy equivalence, i.e., there exists maps $h_n : C_n(X) \to C_{n+1}(Y)$ making a diagram

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(X) \arrow[r] \arrow[d, shift right, "g"'] \arrow[d, shift left, "f"] \arrow[dl, "h"] & C_n(X) \arrow[r] \arrow[d, shift right, "g"'] \arrow[d, shift left, "f"] \arrow[dl, "h"] & C_{n-1}(X) \arrow[r] \arrow[d, shift right, "g"'] \arrow[d, shift left, "f"] \arrow[dl, "h"] & \cdots \\ \cdots \arrow[r] & C_{n+1}(Y) \arrow[r] & C_n(Y) \arrow[r] & C_{n-1}(Y) \arrow[r] & \cdots \end{tikzcd}$$

such that $f - g = \partial h + h\partial$. Therefore $f_* = g_* : H_*(X) \to H_*(Y)$.

Remark 2.1. $f : A_* \to B_*$ is a chain equivalence if there exists $g : B_* \to A_*$ and $fg \simeq \mathrm{id}_B$ and $gf \simeq \mathrm{id}_A$, then $f_* : H_*(A_*) \to H_*(B_*)$ is an isomorphism, i.e., f is a quasi-isomorphism.

Example 2.2. The complexes $A : 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to 0$ and $B : 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0$ gives a quasi-isomorphism $f : A \to B$ in the canonical way, but this is not a chain equivalence, since the backwards map has to be zero.

- Additivity: $C_*(\coprod_{\alpha} X_\alpha) \cong \bigoplus_{\alpha} C_*(X_\alpha)$.
- Excision: given a pair (X, A) with $Z \subseteq A$ such that $\bar{Z} \subseteq \operatorname{int}(A)$, then we have $C_*(X \backslash Z, A \backslash Z) \cong C_*(X, A)$.
- Mayer-Vietoris: given $A, B \subseteq X$, with $X = \operatorname{int}(A) \cup \operatorname{int}(B)$, then we have a short exact sequence

$$0 \longrightarrow C_*(A \cap B) \longrightarrow C_*(A) \oplus C_*(B) * C_*(X) \longrightarrow 0$$

The cochain complex is obtained via inverting the indices and maps δ from a chain complex. This induces a cohomology $H^*(C^*) = \ker(\delta)/\operatorname{im}(\delta)$ as the quotient of cocycles over coboundaries. Now $f : A^* \to B^*$ is a quasi-isomorphism if $f^* : H^*(A^*) \to H^*(B^*)$ is an isomorphism. Similarly, one can define the cochain homotopy equivalence.

Example 2.3. If $C_* \in \mathbf{Ch}$, and $k \in \mathbf{Ab}$, then we can form cochain complex $C_k^* := \operatorname{Hom}(C_*, k)$, where $C_k^n = \operatorname{Hom}_{\mathbf{Ab}}(C_n, k) \xrightarrow{\delta} C_k^{n+1}$ by sending $f : C_n \to k$ to $f\partial : C_{n+1} \to C_n \to k$.

- $\mathrm{Hom}(-,k)$: **Ch** → **coCh** is a functor.
- The functor preserves quasi-isomorphisms between chain complexes of free abelian groups.

Definition 2.4. For $k \in$ **Ab**, the singular cochains with coefficients in k is

$$\begin{tikzcd} C^*(-,k) : \mathbf{Top} \arrow[rr] \arrow[dr, "C_*(-)"'] & & \mathbf{coCh} \\ & \mathbf{Ch} \arrow[ur, "\mathrm{Hom}(-,k)"'] & \end{tikzcd}$$

The cohomology of X with coefficients in k is defined by $H^*(X;k) = H^*(C^*X, k)$. We have the convention $C^*(X) = C^*(X, \mathbb{Z})$.

Alternatively, we take the opposite categories **Top*** and **Ch*** so that the functors are viewed as covariant.

The corresponding map $\delta : C^n(X;k) \to C^{n+1}(X;k)$ is given by δf that maps $\sigma \in C_{n+1}(X)$ to $(-1)^{n+1} f(\partial\sigma)$. Although the cochains are in general the dual of chains, the cohomology is not going to be the dual of the homology.

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Recall:

$$\begin{tikzcd} \mathbf{Top}^{\mathrm{op}} \arrow[r, "C_*"] \arrow[rrr, bend left, "H^*(-,k)"] & \mathbf{Ch}^{\mathrm{op}} \arrow[r, "\mathrm{Hom}(-,k)"] & \mathbf{coCh} \arrow[r, "H^*"] & \mathbf{GrAb} \end{tikzcd}$$

Properties of $H^*(-,k)$: **Top** → **GrAb**:

- Dimension:

  Claim 3.1. $H^i(\{*\}, k) = \begin{cases} 0, & i \neq 0 \\ k, & i = 0 \end{cases}$

  Proof. Note that each degree of cohomology is given the free abelian group generated by $\mathrm{Hom}(\Delta^n, \{*\})$, but the singleton set is the terminal object in the category of topological spaces, so there is always a unique generator, thus the chain complex is given by $\mathbb{Z}$'s on each degree $n \geqslant 0$.

  Now the generating map at degree n is $\sigma_n : \Delta^n \to \{*\}$, and see Homework 1 where we proved the homology. Now looking at $C^*(\{*\}, k)$, we have

  $$k \xrightarrow{0} k \xrightarrow{\cong} k \xrightarrow{0} k \longrightarrow \cdots$$

  and this gives the cohomology. □

- Homotopy: if $f \simeq g : X \to Y$, then $f^* = g^* : H^*(Y,k) \to H^*(X,k)$.

  Proof. We have $f_* = g_* : C_*X \to C_*Y$, and then $\mathrm{Hom}(f_*, k) \cong \mathrm{Hom}(g_*, k)$, so $H^*(-)$ is invariant under cochain homotopies. □

- Additivity: $H^*(\coprod_\alpha X_\alpha, k) \cong \prod_\alpha H^*(X_\alpha, k)$.

  Proof. We know that for chains there is $C_*(\coprod_\alpha X_\alpha) = \bigoplus_\alpha C_*(X_\alpha)$, so the cochain version says that $C^*(\coprod_\alpha, X_\alpha, k) \cong \mathrm{Hom}(\bigoplus_\alpha C_*(X_\alpha), k) \cong \prod_\alpha \mathrm{Hom}(C_*(X_\alpha), k) \cong \prod_\alpha C^*(X_\alpha)$ and H^* : **coCh** → **GrAb** commutes with the product. □

- Exactness: for a pair (X, A), there is a natural long exact sequence

$$\cdots \longrightarrow H^n(X, A; k) \longrightarrow H^n(X; k) \longrightarrow H^n(A; k) \longrightarrow \cdots$$

  Proof. We have a short exact sequence

$$0 \longrightarrow C_*A \longrightarrow C_*X \longrightarrow C_*(X, A) \longrightarrow 0$$

  where $C_*A \to C_*X$ is an inclusion of summands. Therefore, the quotient $C_*(X, A)$ is also a chain complex of free abelian groups. Therefore, taking the cochains also gives a short exact sequence. We then obtain a short exact sequence of cochain complexes

$$0 \longrightarrow C^*(X, A; k) \longrightarrow C^*(X; k) \longrightarrow C^*(A; k) \longrightarrow 0$$

  and can then apply cohomology functor. $\square$

- Excision: given a pair (X, A) and Z such that $\bar{Z} \subseteq \operatorname{int}(A)$, we have $H^*(X, A; k) \cong H^*(X \backslash Z, A \backslash Z; k)$.
- Mayer-Vietoris: given $A, B \subseteq X$ such that $\operatorname{int}(A) \cup \operatorname{int}(B) = X$, then we have a natural long exact sequence

$$\cdots \longrightarrow H^n(X; k) \longrightarrow H^n(A; k) \oplus H^n(B; k) \longrightarrow H^n(A \cap B; k) \longrightarrow \cdots$$

Definition 3.2. A functor $E^* : \mathbf{Top}^{\mathrm{op}} \to \mathbf{GrAb}$ is called a generalized cohomology theory if it satisfies the four middle property (except the dimension property and Mayer-Vietoris).

Remark 3.3. If E^* also satisfies the dimension property, then E^* is naturally isomorphic to the cohomology $H^*(-; k)$. There are also other generalized cohomology theories like K-theory, cobordism, etc.

The Mayer-Vietoris becomes a consequence of the first five properties.

We will now try to use homological algebra to relate $H_*(X) = H_*(CX)$ and $H^*(X; k) = H^*(\operatorname{Hom}(C_*X, k))$.

Definition 3.4. We say $C_*(X; k) \cong C_*(X) \otimes_{\mathbb{Z}} k$ and $H_*(X; k) \cong H_*(C_*X \otimes k)$ gives the singular homology of X with coefficients in k.

Lemma 3.5. $- \otimes k : \mathbf{Ab} \to \mathbf{Ab}$ is a right exact functor. $\operatorname{Hom}(-, k) : \mathbf{Ab}^{\mathrm{op}} \to \mathbf{Ab}$ is left exact.

Proof. Exercise. $\square$

Remark 3.6. The covariant hom functor is also left exact.

Remark 3.7. The left adjoint is right exact, the right adjoint is left exact. In particular, we have the hom-tensor adjunction

$$\operatorname{Hom}(A, \operatorname{Hom}(B, C)) \cong \operatorname{Hom}(A \otimes B, C).$$

Note that

$$\operatorname{Hom}(A, \operatorname{Hom}(B, C)) \cong \operatorname{Hom}(A \otimes B, C) \cong \operatorname{Hom}(B \otimes A, C) \cong \operatorname{Hom}(B, \operatorname{Hom}(A, C))$$

Example 3.8. Consider

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

Tensoring with $\mathbb{Z}/n\mathbb{Z}$, we do not have exactness.

Example 3.9.

$$0 \longrightarrow A \longrightarrow A \oplus C \longrightarrow C \longrightarrow 0$$

is always exact after tensoring $- \otimes k$ or applying the hom functor $\operatorname{Hom}(-, k)$.

Definition 3.10. A short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ is split if any of the following equivalence conditions hold:

(i) p has a section $s : C \to B$ such that $ps = 1$;

(ii) i has a retraction $r : B \to A$ such that $ri = 1$;

(iii) $B \cong A \oplus C$, i.e.,

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & B \arrow[r, "p"] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, hook, "i"] & A \oplus C \arrow[r, two heads, "p"] & C \arrow[r] & 0 \end{tikzcd}$$

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We will prove that (ii) implies (iii).

Suppose $b \in B$, then $b = (b - irb) + irb$, which is a decomposition of elements in $\ker(r)$ and in $\mathrm{im}(i)$, respectively. Also, $\ker(r) \cap \mathrm{im}(i) = 0$, therefore $B = \ker(r) \cap \mathrm{im}(i)$. Since i is an inclusion, then $\mathrm{im}(i) \cong A$. Now $p : B \to C$ factors through the projection onto $\ker(r)$ since $ri = 0$. By restricting p onto $\ker(r)$, we see p is also injective, thereby an isomorphism.

Lemma 4.1. If we have a split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, bend left, "i"] & B \arrow[l, bend left, "r"] \arrow[r, bend left, "p"] & C \arrow[l, bend left, "s"] \arrow[r] & 0 \end{tikzcd}$$

then $- \otimes k$ and $\mathrm{Hom}(-, k)$ preserves the split exactness, i.e.,

$$0 \longrightarrow A \otimes k \longrightarrow B \otimes k \longrightarrow C \otimes k \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Hom}(C, k) \longrightarrow \mathrm{Hom}(B, k) \longrightarrow \mathrm{Hom}(A, k) \longrightarrow 0$$

The point is tensors and homs preserve retracts.

Proof.
- $(r \otimes \mathrm{id}_k)(i \otimes \mathrm{id}_k) = ri \otimes \mathrm{id}_k = \mathrm{id}_{A \otimes k}$, so $i \otimes \mathrm{id}_k$ is split injective.
- Similarly, $\mathrm{Hom}(i, \mathrm{id})$ is split surjective.

□

Example 4.2. Given a surjection $B \to C \to 0$ such that C is free abelian, then there is always a section $s : C \to B$ making the exact sequence split. (That is, C is projective.) That is, if $0 \to A \to B \to C \to 0$ is an exact sequence where C is free, then the sequence is split exact.

Definition 4.3. Let $C \in \mathbf{Ab}$. A free resolution of C is a chain complex of free objects

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

and an augmentation $F_0 \to C$, so that

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0$$

is acyclic, i.e., exact everywhere.

Example 4.4.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow 0$$

is a free resolution of $\mathbb{Z}/n\mathbb{Z}$. So is

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

as well as

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\mathrm{id} \oplus (\times n)} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0$$

Lemma 4.5. Any $C \in \mathbf{Ab}$ admits a free resolution, and moreover, it admits a resolution of length 1¡ given by

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0$$

Proof. Choose a surjection $p : F_0 \to C$ from a free abelian group F_0 to C. Let $F_1 = \ker(p)$, then F_1 is free, so we are done. □

Lemma 4.6. Free resolutions are essentially unique, i.e., if $F_\cdot \to C$ and $F'_\cdot \to C$ are free resolutions, then there is a quasi-isomorphism $F_\cdot \xrightarrow{\sim} F'_\cdot$ which commutes with the augmentations to C.

Definition 4.7. Let $C \in \mathbf{Ab}$ and let $F_\cdot \to C$ be a free resolution, then we define the torsion groups to be $\mathrm{Tor}_n^{\mathbb{Z}}(C,k) = H_n(F_\cdot \otimes k)$, and the ext groups to be $\mathrm{Ext}_{\mathbb{Z}}^n(C,k) = H^n(\mathrm{Hom}_{\mathbb{Z}}(F_\cdot, k))$.

Remark 4.8.

- Tor and Ext are independent of the choice of resolutions.
- $\mathrm{Tor}_n^{\mathbb{Z}}$ and $\mathrm{Ext}_{\mathbb{Z}}^n$ are zero for $n > 1$.
- $\mathrm{Tor}_n^{\mathbb{Z}}(C,k) \cong \mathrm{Tor}_n^{\mathbb{Z}}(k,C)$.
- $\mathrm{Tor}_0^{\mathbb{Z}}(C,k) \cong C \otimes k$.
- $\mathrm{Ext}_{\mathbb{Z}}^0(C,k) \cong \mathrm{Hom}(C,k)$.

Example 4.9.

- If C is free, then $\mathrm{Tor}_1(C,k) = \mathrm{Ext}^1(C,k) = 0$.
- $\mathrm{Tor}_1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.
- $\mathrm{Tor}_1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = 0$.
- $\mathrm{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.
- $\mathrm{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.
- $\mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = 0$.

Proof. Look at

$$0 \longrightarrow F_1 = \mathbb{Z} \longrightarrow F_0 = \mathbb{Z} \longrightarrow C = \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

then $\mathrm{Tor}_*(\mathbb{Z}/p\mathbb{Z}, k) = H_*(F_1 \otimes k = k \xrightarrow{\times p} F_0 \otimes k = k) = \begin{cases} k[p], * = 0 \\ k/pk, * = 1 \end{cases}$. Here $k[p]$ denotes p-torsion subgroup of k. Moreover, $\mathrm{Ext}^*(\mathbb{Z}/p\mathbb{Z}, k) = H^*(\mathrm{Hom}(F_1,k) = k \xleftarrow{\times p} \mathrm{Hom}(F_0,k) = k) = \begin{cases} k[p], * = 0 \\ k/pk, * = 1 \end{cases}$. □