MATH 518 Notes

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Definition 1.1. Let M be a topological space. An *atlas* on M is a collection $\{\varphi_\alpha : U_\alpha \to W_\alpha\}_{\alpha \in A}$ of homeomorphisms called *coordinate charts*, so that

1. $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M,
2. for all $\alpha \in A$, W_α is an open subset of some $\mathbb{R}^{n_\alpha}$,
3. for all $\alpha, \beta \in A$, the induced map $\varphi_\beta \circ \varphi_\alpha^{-1}|_{U_\alpha \cap U_\beta}$ is C^∞, i.e., smooth.

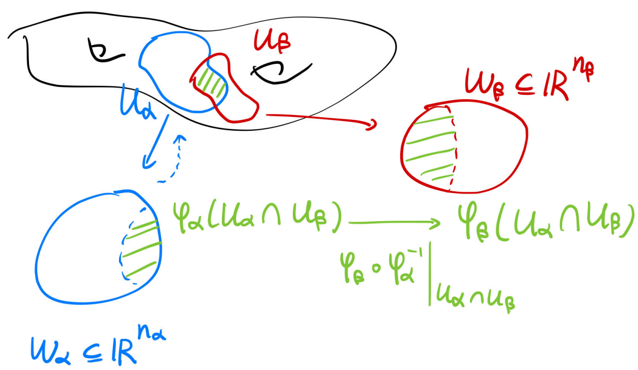

Figure 1: Atlas and Coordinate Chart

Example 1.2. Let $M = \mathbb{R}^n$ be equipped with standard topology, and let $A = \{*\}$, so $U_* = \mathbb{R}^n$ is the open cover of itself. Now the identity map

$$\begin{aligned} \varphi_* : U_* &\to \mathbb{R}^n \\ u &\mapsto u \end{aligned}$$

is an atlas on $\mathbb{R}^n$.

Example 1.3. Let $M = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be equipped with subspace topology. Let $U_\alpha = S^1 \backslash \{(1, 0)\}$ and $U_\beta = S^1 \backslash \{(-1, 0)\}$, and let $A = \{\alpha, \beta\}$. Let $W_\alpha = (0, 2\pi)$ and $W_\beta = (-\pi, \pi)$. We define $\varphi_\alpha^{-1}(\theta) = (\cos(\theta), \sin(\theta))$ and $\varphi_\beta^{-1}(\theta) = (\cos(\theta), \sin(\theta))$, then

$$(\varphi_\beta \circ \varphi_\alpha^{-1})(\theta) = \begin{cases} \theta, 0 < \theta < \pi \\ \theta - 2\pi, \pi < \theta < 2\pi \end{cases}$$

is smooth.

Example 1.4. Let X be a topological space with discrete topology, and let $A = X$, then $\{\varphi_x : \{x\} \to \mathbb{R}^0\}_{x \in X}$ gives an atlas.

Example 1.5. Let V be a finite-dimensional real vector space of dimension n. Pick a basis $\{v_1, \ldots, v_n\}$ of V, then there is a linear bijection φ with inverse

$$\begin{aligned}\varphi^{-1} : \mathbb{R}^n &\to V\\ (x_1, \ldots, x_n) &\mapsto \sum_{i=1}^n x_i v_i.\end{aligned}$$

The topology on V needs to make φ^{-1} a homeomorphism, and the obvious choice is just the collection of preimages, namely

$$\mathcal{T} = \{\varphi^{-1}(W) \mid W \subseteq \mathbb{R}^n \text{ open}\},$$

then $\varphi : V \to \mathbb{R}^n$ becomes an atlas.

Definition 1.6. Two atlases $\{\varphi_\alpha : U_\alpha \to W_\alpha\}_{\alpha \in A}$ and $\{\psi_\beta : V_\beta \to O_\beta\}_{\beta \in B}$ on a topological space M are *equivalent* if for all $\alpha \in A$ and $\beta \in B$,

$$\psi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap V_\beta) \subseteq \mathbb{R}^{n_\alpha} \to \psi_\beta(U_\alpha \cap V_\beta) \subseteq \mathbb{R}^{n_\beta}$$

is always C^∞, with C^∞-inverses. Such continuous maps are called *diffeomorphisms*. Alternatively, the two atlases are equivalent if their union $\{\varphi_\alpha\}_{\alpha \in A} \cup \{\psi_\beta\}_{\beta \in B}$ is always an atlas.

Exercise 1.7. Equivalence of atlases is an equivalence condition.

Definition 1.8. A (smooth) *manifold* is a topological space together with an equivalence class of atlases.

Convention. All manifolds are assumed to be smooth of C^∞, but not necessarily *Haudorff* and/or *second countable*.

Example 1.9. Continuing from Example 1.5, now suppose $\{w_1, \ldots, w_n\}$ gives another basis of V, with

$$\begin{aligned}\psi^{-1} : \mathbb{R}^n &\to V\\ (y_1, \ldots, y_n) &\mapsto \sum_{i=1}^n y_i w_i.\end{aligned}$$

This gives a change-of-basis matrix, so it is automatically C^∞ as a multiplication of invertible matrices. Therefore, the topology here does not depend on the chosen basis.

Recall. A topological space X is *Hausdorff* if for all distinct points $x, y \in X$, there exists open neighborhoods $U \ni x$ and $V \ni y$ such that $U \cap V = \varnothing$.

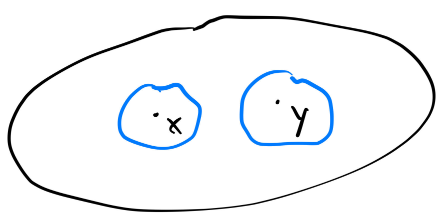

Figure 2: Hausdorff Condition

Convention. Via our definition (Definition 1.8), not all manifolds are Hausdorff.

Example 1.10. Let $Y = \mathbb{R} \times \{0, 1\}$, i.e., a space with two parallel lines, with a fixed topology. Define $\sim$ to be the smallest equivalence relation on Y such that $(x, 0) \sim (x, 1)$ for $x \neq 0$, and define $X = Y/\sim$. X is called the *line with two origins*, and it is second countable but not Hausdorff.

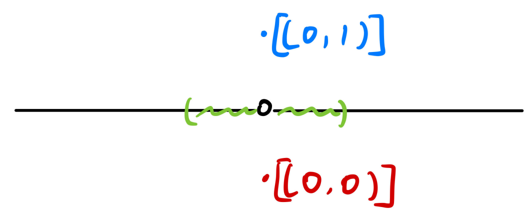

Figure 3: Line with Two Origins

Example 1.11. Take charts

$$\begin{aligned}\{\varphi : M = \mathbb{R} &\to \mathbb{R}\} \\ x &\mapsto x\end{aligned}$$

and

$$\begin{aligned}\{\psi : M = \mathbb{R} &\to \mathbb{R}\} \\ x &\mapsto x^3\end{aligned}$$

on $M = \mathbb{R}$, then

$$\begin{aligned}\varphi \circ \psi^{-1} : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^{\frac{1}{3}}\end{aligned}$$

is not C^∞, so φ and ψ are two different charts, hence give two different manifolds.

Definition 1.12. A map $F : M \to N$ between two manifolds is *smooth* if

1. F is continuous, and
2. for all charts $\varphi : U \to \mathbb{R}^m$ on M and charts $\psi : V \to \mathbb{R}^n$ on N, $\psi^{-1} \circ F \circ \varphi|_{\varphi(U \cap F^{-1}(V))}$ is C^∞.

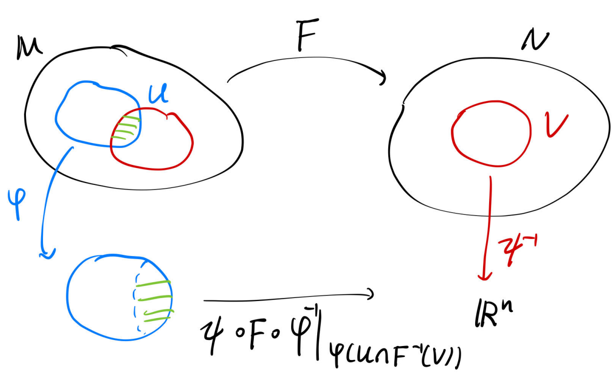

Figure 4: Smooth Map between Manifolds

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Exercise 2.1. 1. $\mathrm{id} : M \to M$ is smooth.

2. If $f : M \to N$ and $g : N \to Q$ are smooth maps between manifolds, then so is $gf : M \to Q$.

Punchline. The manifolds and the smooth maps between manifolds form a category.

Recall. A smooth map $f : M \to N$ is called a *diffeomorphism*, as seen in Definition 1.6, if it has a smooth inverse. This is the notion of an isomorphism in the category of manifolds.

Warning.

1. Following Example 1.11,
$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^3 \end{aligned}$$
has an inverse
$$\begin{aligned} f^{-1} : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^{\frac{1}{3}}, \end{aligned}$$
but f^{-1} is not differentiable at $x = 0$. Hence, f is not a diffeomorphism.
2. Take $\mathbb{R}$ with discrete topology, then all singletons are open sets, then the map
$$\begin{aligned} f : \mathbb{R}_{\text{dis}} &\to \mathbb{R}_{\text{std}} \\ x &\mapsto x \end{aligned}$$
is a smooth bijection, but f^{-1} is not continuous.

Example 2.2. Consider $M = (\mathbb{R}, \{\psi = \mathrm{id} : \mathbb{R} \to \mathbb{R}\})$ and $N = (\mathbb{R}, \{\psi : \mathbb{R} \to \mathbb{R}, x \mapsto x^3\})$ as two manifolds on $\mathbb{R}$ with standard topology. To see that they are equivalent, consider the homeomorphism
$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^{\frac{1}{3}}, \end{aligned}$$
then $(\psi \circ f \circ \varphi^{-1})(x) = \psi(f(x)) = (x^{\frac{1}{3}})^3 = x$, so f is smooth, and $(\psi \circ f \circ \varphi^{-1})^{-1} = \varphi \circ f^{-1} \circ \psi^{-1} = \mathrm{id}$, therefore f^{-1} is also smooth. Hence, f is a diffeomorphism.

We will now consider the real projective space $\mathbb{R}P^{n-1}$ and the quotient map $\pi : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}P^{n-1}$.

Definition 2.3. Define a binary relation on $\mathbb{R}^n \backslash \{0\}$ by $v_1 \sim v_2$ if and only if there exists $\lambda \neq 0$ such that $v_1 = \lambda v_2$. This is an equivalence relation, and we identify the equivalence class $[v]$ of $v \in \mathbb{R}^n \backslash \{0\}$ as a line $\mathbb{R}v = \mathrm{span}_{\mathbb{R}}\{v\}$ through v. Then we define the *real projective space* $\mathbb{R}P^{n-1} = (\mathbb{R}^n \backslash \{0\}) / \sim$.

The natural topology on $\mathbb{R}P^{n-1}$ is the quotient topology, where $\pi : \mathbb{R}^n \backslash \{0\} \twoheadrightarrow \mathbb{R}P^{n-1}$ is surjective and continuous, so we define $U \subseteq \mathbb{R}P^{n-1}$ to be open if and only if $\pi^{-1}(U)$ is open in $\mathbb{R}^n \backslash \{0\}$.

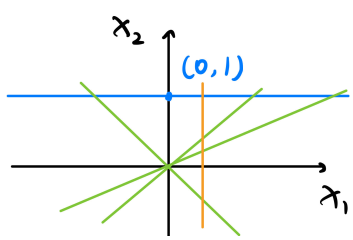

Figure 5: Stereographical Projection

Claim 2.4. $\mathbb{R}P^{n-1}$ is a manifold.

Proof. Define
$$\begin{aligned} \varphi_i : U_i &\to \mathbb{R}^{n-1} \\ [v_1, \ldots, v_n] &\mapsto \left(\frac{v_1}{v_i}, \ldots, \frac{v_{i-1}}{v_i}, \frac{v_{i+1}}{v_i}, \ldots, \frac{v_n}{v_i}\right), \end{aligned}$$

then

$$\begin{aligned}\varphi_i^{-1} : \mathbb{R}^{n-1} &\mapsto U_i \\ (x_1, \ldots, x_{n-1}) &\mapsto [(x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_{n-1})],\end{aligned}$$

therefore

$$\begin{aligned}\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) &\to \varphi_j(U_i \cap U_j) \\ (x_1, \ldots, x_{n-1}) &\mapsto \varphi_j([(x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_{n-1})]) \\ &= \begin{cases} \left(\frac{x_1}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_{n-1}}{x_j}\right), & j < i \\ (x_1, \ldots, x_{n-1}), & j = i \\ \left(\frac{x_1}{x_{j-1}}, \ldots, \frac{1}{x_{j-1}}, \ldots, \frac{x_{j-2}}{x_{j-1}}, \frac{x_j}{x_{j-1}}, \ldots, \frac{x_{n-1}}{x_{j-1}}\right), & j > i \end{cases}\end{aligned}$$

Therefore, this is C^∞ as a rational map on $\varphi_i(U_i \cap U_j)$, and so this gives an atlas, hence $\mathbb{R}P^{n-1}$ is a manifold. □

Claim 2.5. $\pi : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}P^{n-1}$ is smooth.

Proof. Note that

$$\begin{aligned}\psi : \mathbb{R}^n \backslash \{0\} &\hookrightarrow \mathbb{R}^n \\ x &\mapsto x\end{aligned}$$

is an atlas on $\mathbb{R}^n \backslash \{0\}$, and

$$\begin{aligned}\varphi_i \circ \pi \circ \varphi_i^{-1} : \mathbb{R}^n \backslash \{0\} &\to \mathbb{R}^{n-1} \\ (v_1, \ldots, v_n) &\mapsto \varphi_i([(v_1, \ldots, v_n)]) \\ &= \left(\frac{v_1}{v_i}, \ldots, \frac{v_{i-1}}{v_i}, \frac{v_{i+1}}{v_i}, \ldots, \frac{v_n}{v_i}\right).\end{aligned}$$

This is C^∞ on $\pi^{-1}(U_i) = \{(v_1, \ldots, v_n) \mid v_i \neq 0\}$, so π is smooth. □

Definition 2.6. A *smooth function* on a manifold M is a function $f : M \to \mathbb{R}$ so that for any coordinate chart $\varphi : U \to \varphi(U)$ open in $\mathbb{R}^m$, the function $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is smooth.

Remark 2.7. $f : M \to \mathbb{R}$ is smooth if and only if $f : M \to (\mathbb{R}, \{\mathrm{id} : \mathbb{R} \to \mathbb{R}\})$, usually called the *standard manifold structure on* $\mathbb{R}$, is smooth.

Notation. We denote $C^\infty(M)$ to be the set of all smooth functions $f : M \to \mathbb{R}$.

Remark 2.8. $C^\infty(M)$ is a smooth $\mathbb{R}$-vector space, that is, for all $\lambda, \mu \in \mathbb{R}$ and $f, g \in C^\infty(M)$,

- $(\lambda f + \mu g)(x) = \lambda f(x) + \mu g(x)$ for all $x \in M$,
- $(f \cdot g)(x) = f(x)g(x)$ for all $x \in M$.

Therefore, $C^\infty(M)$ becomes a (commutative, associative) $\mathbb{R}$-algebra.

Fact. Connecting manifolds have the notion of dimension. That is, the dimensions of open subsets induced by coordinate charts are the same.

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Definition 3.1. Let M be a manifold, then for every point $q \in M$, there exists a well-defined non-negative integer $\dim_M(q)$, so that for any coordinate chart $\varphi : U \to \mathbb{R}^m$ for $U \ni q$, we have $\dim_M(q) = m$ for some non-negative integer m that only depend on M. Consequently, $\dim_M : M \to \mathbb{Z}^{\geqslant 0}$ is a locally constant function. This integer m is called the *dimension* of M.

Proof. Indeed, say $\psi : V \to \mathbb{R}^n$ is another chart with $U \cap V \ni q$, then $\psi \circ \varphi^{-1}|_{\varphi(U \cap V)} : \varphi(U \cap V) \subseteq \mathbb{R}^m \to \psi(U \cap V) \subseteq \mathbb{R}^n$ is a diffeomorphism, therefore the Jacobian $D(\psi \circ \varphi^{-1})(\varphi(a)) : \mathbb{R}^m \to \mathbb{R}^n$ is a linear isomorphism, thus $m = n$. $\square$

Definition 3.2. Suppose $(M, \{\varphi_\alpha : U_\alpha \to \mathbb{R}^m\}_{\alpha \in A})$ and $(N, \{\psi_\alpha : V_\beta \to \mathbb{R}^n\}_{\beta \in B})$ are two manifolds. One can give a manifold structure to the product set $M \times N$, called the *product manifold*, as follows:

- give $M \times N$ the product topology,
- let $\{\varphi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n\}_{(\alpha,\beta) \in A \times B}$ to be the atlas on $M \times N$. This is well-defined since the transition maps of $\alpha, \alpha' \in A$ and $\beta, \beta' \in B$ are over $(U_\alpha \times V_\beta) \cap U_{\alpha'} \times V_{\beta'} = (U_\alpha \cap U_{\alpha'}) \times (V_\beta \cap V_{\beta'})$ with $(\varphi_{\alpha'} \times \psi_{\beta'}) \circ (\varphi_\alpha \times \psi_\beta)^{-1} = (\varphi_{\alpha'} \circ \varphi_\alpha^{-1}, \psi_{\beta'} \circ \psi_\beta^{-1})$. This is smooth since products of smooth maps are smooth.

Punchline. The product construction of manifolds gives the categorical product in the category of manifolds.

Property.

1. The projection maps

$$\begin{aligned} p_M : M \times N &\to M \\ (m, n) &\mapsto m \end{aligned}$$

and

$$\begin{aligned} p_N : M \times N &\to N \\ (m, n) &\mapsto n \end{aligned}$$

are C^∞.

2. *Universal Property of Product*: for any manifold Q and smooth maps $f_M : Q \to M$ and $f_N : Q \to N$, there exists a unique map

$$\begin{aligned} g : Q &\to M \times N \\ q &\mapsto (f(q), g(q)) \end{aligned}$$

such that $p_M \circ g = f_M$, and $p_N \circ g = f_N$.

$$\begin{tikzcd}
& Q \arrow[ddl, bend right, "f_M"'] \arrow[d, dashed, "g"', "\exists!"] \arrow[ddr, bend left, "f_N"] & \\
& M \times N \arrow[dl, "p_M"] \arrow[dr, "p_N"'] & \\
M & & N
\end{tikzcd}$$

Figure 6: Universal Property of Product

Recall.

- A topological space X is *second countable* if the topology has a countable basis: there exists a collection $\mathcal{B} = \{B_i\}_{i \in \mathbb{N}}$ of open sets so that any open set of X is a union of some B_i's.
- A cover $\{U_\alpha\}_{\alpha \in A}$ of a topological space is *locally finite* if for all $x \in X$, there exists a neighborhood N of X such that $N \cap U_\alpha = \varnothing$ for all but finitely many α's.

Example 3.3. Let $X = \mathbb{R}$, then

- $\{U_n = (-n, n)\}_{n \geqslant 0}$ is an open cover, but is not locally finite,
- $\{U_n = (n, n+2)\}_{n \in \mathbb{Z}}$ is a locally finite open cover of $\mathbb{R}$,
- $\{U_n = (n, n+2]\}_{n \in \mathbb{Z}}$ is a locally finite cover of $\mathbb{R}$, but is not an open cover.

Recall. An (open) cover $\{V_\beta\}_{\beta\in B}$ is a *refinement* of a cover $\{U_\alpha\}_{\alpha\in A}$ if for all β, there exists $\alpha = \alpha(\beta)$ such that $V_\beta \subseteq U_{\alpha(\beta)}$.

Definition 3.4. A Hausdorff topological space is *paracompact* if every open cover has a locally finite open refinement.

Fact. A connected Hausdorff manifold is paracompact if and only if it is second countable.

Corollary 3.5. A Haudorff manifold is paracompact if and only if its connected components are second countable.

Example 3.6. $\mathbb{R}$ with discrete topology is paracompact but not second countable.

Convention. Usually, we assume manifolds are paracompact, except when we need a non-Haudorff manifold. This condition is required for the existence of *partition of unity* (i.e., constant function id).

Recall. If X is a space, and $Y \subseteq X$ is a subset, then the *closure* $\bar{Y}$ of Y is the smallest closed set containing Y.

Definition 3.7. Given a topological space X and a function $f : X \to \mathbb{R}$, the *support* of f over X is

$$\operatorname{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}.$$

Example 3.8. The function

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leqslant 0 \end{cases}$$

is C^∞, with support $\overline{(0,\infty)} = [0,\infty)$.

Definition 3.9. Let M be a topological space and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover. A *partition of unity* subordinate to the cover is a collection of continuous functions $\{\psi_\alpha : M \to [0,1]\}_{\alpha\in A}$ such that

1. $\operatorname{supp}(\psi_\alpha) \subseteq U_\alpha$ for all $\alpha \in A$,
2. $\{\operatorname{supp}(\psi_\alpha)\}_{\alpha\in A}$ is a locally finite closed cover of M,
3. $\sum\limits_{\alpha\in A} \psi_\alpha(x) = 1$ for all $x \in M$.

Remark 3.10. For all $x \in M$, there exists $\alpha_1, \ldots, \alpha_n$ such that $x \in \operatorname{supp}(\psi_{\alpha_i})$. Hence, for $\alpha \neq \alpha_1, \ldots, \alpha_n$, $\psi_\alpha(x) = 0$. Therefore, the summation in Definition 3.9 is finite.

Theorem 3.11. Let M be a paracompact manifold with open cover $\{U_\alpha\}_{\alpha\in A}$, then there exists a partition of unity $\{\psi_\alpha : U_\alpha \to [0,1]\}_{\alpha\in A} \subseteq C^\infty(M)$ subordinate to the cover.

Example 3.12. Let $M = \mathbb{R}$ and consider for $n > 0$ the open sets $\{U_n = (-n,n)\}_{n\in\mathbb{N}}$. This is not locally finite at one point.

Example 3.13. Let $M = \mathbb{R}^n$, then for all $x \in \mathbb{R}^n$ and for $r > 0$, we have $B_r(x) = \{x' \in \mathbb{R}^n \mid ||x - x'|| < r\}$ and so $\{B_r(x)\}_{r>0, x\in\mathbb{R}^n}$ is an open cover, but this is not locally finite everywhere.

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We will start to talk about tangent vectors.

Recall. For any point $q \in \mathbb{R}^n$ and any vector $v \in \mathbb{R}^n$, and any $f \in C^\infty(\mathbb{R}^n)$, the *directional derivative* of q in direction v with respect to f is

$$D_v f(q) = \frac{d}{dt}|_0 f(q + tv).$$

This gives a map $D_v(-)(q) : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ which is

- linear, and

- Leibniz rule holds, i.e.,
$$D_v(fg)(q) = D_v(f)(q) \cdot g(q) + f(q)D_v(g)(q).$$

In other words, $D_v(-)(q) : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation.

Definition 4.1. Let q be a point of a manifold M. A *tangent vector* to M at q is an $\mathbb{R}$-linear map $v : C^\infty(M) \to \mathbb{R}$ such that for all $f, g \in C^\infty(M)$,
$$v(fg) = v(f)g(q) + f(q)v(g).$$

Remark 4.2. v gives smooth vector fields over M an $C^\infty(M)$-module structure via evaluation.

Lemma 4.3. The set T_qM of all tangent vectors to M at q is an $\mathbb{R}$-vector space.

Lemma 4.4. Suppose $c \in C^\infty(M)$ is a constant function, then for all q and all $v \in T_qM$, $v(c) = 0$.

Proof. We have $v(1) = v(1 \cdot 1) = 1(q)v(1) + v(1)1(q) = 2v(1)$, so $v(1) = 0$. For a constant function c, we have
$$v(c) = v(c \cdot 1) = cv(1) = c(0) = 0.$$
□

Lemma 4.5 (Hadamard)**.** For any $f \in C^\infty(\mathbb{R}^n)$, there exists $g_1, \ldots, g_n \in C^\infty(\mathbb{R}^n)$ such that

- $f(x) = f(0) + \sum\limits_{i=1}^n x_i g_i(x)$, and
- $g_i(0) = \left(\frac{\partial}{\partial x_i} f\right)(0)$.

Proof. We have
$$\begin{aligned} f(x) - f(0) &= \int_0^1 \frac{d}{dt}(f(tx))dt \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(tx) \cdot x_i dt \\ &= \sum_{i=1}^n x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt \\ &= \sum_{i=1}^n x_i g_i(x). \end{aligned}$$

Therefore, $g_i(0) = \int_0^1 \frac{\partial f}{\partial x_i}(t \cdot 0)dt = \frac{\partial f}{\partial x_i}(0)$. □

Remark 4.6. For $1 \leqslant i \leqslant n$, we have canonical tangent vectors to $\mathbb{R}^n$ at 0 given by
$$\begin{aligned} \frac{\partial}{\partial x_i}|_0 : C^\infty(\mathbb{R}^n) &\to \mathbb{R} \\ f &\mapsto \frac{\partial f}{\partial x_i}(0). \end{aligned}$$

Lemma 4.7. $\left\{\frac{\partial}{\partial x_1}|_0, \ldots, \frac{\partial}{\partial x_n}|_0\right\}$ is a basis of $T_0\mathbb{R}^n$.

Proof. Suppose $\sum c_i \frac{\partial}{\partial x_i}|_0 = 0$, then
$$0 = \left(\sum_i c_i \frac{\partial}{\partial x_i}|_0\right)(x_j) = \sum_i c_i \delta_{ij} = c_j.$$

Therefore, $c_j = 0$ for all j, thus we have linear independence. For all $v \in T_0\mathbb{R}^n$, i.e., $v : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation, then $v = \sum\limits_i v(x_i)\frac{\partial}{\partial x_i}|_0$. Let $f \in C^\infty(\mathbb{R}^n)$, then $f(X) = f(0) + \sum x_i g_i(x)$, thus

$$\begin{aligned} v(f) &= v(f(0)) + \sum_{i=1}^n v(x_i g_i(x)) \\ &= \sum_{i=1}^n v(x_i g_i(x)) \\ &= \sum_{i=1}^n (v(x_i)g_i(0) + x_i(0)v(g_i)) \\ &= \sum_{i=1}^n v(x_i)g_i(0) \\ &= \sum_{i=1}^n v(x_i)\frac{\partial f}{\partial x_i}(0). \end{aligned}$$

□

Remark 4.8. This shows $\dim(T_0\mathbb{R}^n) = n$ with the basis above.

Now let V be a finite-dimensional vector space with a basis $e_1, \dots, e_n$, then

$$\begin{aligned} \varphi : \mathbb{R}^n &\to V \\ (t_1, \dots, t_n) &\mapsto \sum_{i=1}^n t_i e_i \end{aligned}$$

is a linear bijection, with linear inverse

$$\begin{aligned} \psi : V &\to \mathbb{R}^n \\ v &\mapsto (\psi_1(v), \dots, \psi_n(v)) \end{aligned}$$

where $\psi_i(v)$'s are linear maps. To describe this with a basis, we have $\psi(\sum\limits_i a_i e_i) = (a_1, \dots, a_n)$, i.e., $\psi_i(e_j) = \delta_{ij}$.

Claim 4.9. $\{\psi_1, \dots, \psi_n\}$ is a basis of $V^* = \operatorname{Hom}(V, \mathbb{R})$, called the *dual basis* of $\{e_1, \dots, e_n\}$, denoted $e_j^* = \psi_j$.

Proof. Linear independence follows from $e_j^*(e_i) = \delta_{ij}$. Given $\ell : V \to \mathbb{R}$ to be a linear map, then $\ell = \sum \ell(e_i)e_i^*$ since $\left(\sum\limits_i \ell(e_i)e_i^*\right)(e_j) = \ell(e_j)$. Given $v \in T_0\mathbb{R}^n$, $v(f) = \sum a_i \left(\frac{\partial}{\partial x_i}|_0 f\right)$ for all $f \in C^\infty(\mathbb{R}^n)$. Note that $\frac{\partial}{\partial x_i}|_0(x_j) = \delta_{ij}$, so $v(x_j) = \sum a_i \frac{\partial}{\partial x_i}|_0(x_j) = \sum\limits_i a_i\delta_{ij} = a_j$. Therefore, we have $a_i = v(x_i)$ for all i, thus $v(f) = \sum v(x_i)\left(\frac{\partial}{\partial x_i}|_0 f\right)$. Thus, the dual basis to $\frac{\partial}{\partial x_1}|_0, \dots, \frac{\partial}{\partial x_n}|_0$ is $\{d(x_i)_0\}_{i=1}^n$ where $(dx_i)_0(v) = v(x_i)$ for all i. Hence, we have $v = \sum (dx_i)_0(v)\frac{\partial}{\partial x_i}|_0$. □

Remark 4.10. Via a change of basis, this works at every point q on the local chart, so we can describe the tangent space on any point on a local chart.

5 Aug 30, 2023

Let M be a manifold and $x \in M$. Recall that a tangent vector $v : C^\infty(M) \to \mathbb{R}$ is a derivation, i.e., linear map, and the set of tangent vectors at q gives the tangent space.

Example 5.1. Let $M = \mathbb{R}^n$, and $q = 0$, then $\left\{\frac{\partial}{\partial x_1}|_0, \dots, \frac{\partial}{\partial x_n}|_0\right\}$ is a basis of $T_0\mathbb{R}^n$. Moreover, for all $v \in T_0\mathbb{R}^n$, $v = \sum v(x_i)\frac{\partial}{\partial x_i}|_0$, thus $\{v \mapsto v(x_i)\}_{i=1}^n$ is the dual basis, with $v(x_i) = (dx_i)_0(v)$ for all $1 \leqslant i \leqslant n$.

Remark 5.2. The proof used Hadamard's lemma (Lemma 4.5) and the fact that for all $x \in \mathbb{R}^n$ and all $t \in [0,1]$, $f(tx)$ is defined. Thus, the same argument should work for a version of Hadamard's lemma for star-shaped open subsets $U \subseteq \mathbb{R}^n$.

Definition 5.3. We say an open subset $U \subseteq \mathbb{R}^n$ is a *star-shaped domain* if for all $t \in [0,1]$ and all $x \in U$, $tx \in U$.

Definition 5.4. Let $F : M \to N$ be a smooth map between two manifolds, and $q \in M$ is a point, then

$$\begin{aligned} T_qF : T_qM &\to T_qN \\ v(f) &\mapsto v(f \circ F) \end{aligned}$$

via the pullback.

Exercise 5.5. Check that the definition makes sense, in particular:

(i) $(T_qF)(v)$ is a tangent vector to N of $F(q)$, and

(ii) T_qF is a derivation.

Remark 5.6. (a) It is easy to deduce the *chain rule*. That is, given $M \xrightarrow{F} N \xrightarrow{G} Q$ with $q \in M$, then $T_q(G \circ F) = T_{F(q)}G \circ T_qF$ because for all $f \in C^\infty(Q)$ and all $v \in T_qM$, we have

$$(T_q(G \circ F)(v))(f) = v(f \circ (G \circ F))$$

and

$$(T_{F(q)}G(T_qF(v))) = (T_qF)(v)(f \circ G) = v((f \circ G) \circ F).$$

(b) $T_q(\mathrm{id}_M) = \mathrm{id}_{T_qM}$.

As a result, we know T is a functor from the category of pointed manifolds to the category of $\mathbb{R}$-vector spaces.

Corollary 5.7. If $F : M \to N$ is a diffeomorphism, then for all $q \in M$, $T_qF : T_qM \to T_{F(q)}N$ is an isomorphism.

Proof. Since F is a diffeomorphism, then it has a smooth inverse $G : N \to M$, so

$$\mathrm{id}_{T_qM} = T_q(\mathrm{id}_M) = T_q(G \circ F) = T_{F(q)}G \circ T_qF$$

and

$$\mathrm{id}_{T_{F(q)}N} = T_{F(q)}(\mathrm{id}_N) = T_{F(q)}(F \circ G) = T_{F(q)}F \circ T_{F(q)}G.$$

□

We also need to show that $\dim(T_qM) = \dim_q(M)$, which is a result of Lemma 5.8, whose proof will be postponed till next time.

Lemma 5.8. Let M be a manifold and $q \in M$, and let U be an open neighborhood of q in M, and let $i : U \hookrightarrow M$ be an inclusion, then

$$\begin{aligned} I = T_qi : T_qU &\to T_qM \\ v(f) &\mapsto v(f|_U) \end{aligned}$$

is an isomorphism for all $v \in T_qM$ and all $U \subseteq M$.

Notation. We denote $r_1, \ldots, r_n : \mathbb{R}^m \to \mathbb{R}$ to be the standard coordinates on $\mathbb{R}^m$.

Let M be a manifold, $q_0 \in M$, and $\varphi : U \to \mathbb{R}^m$ is a coordinate chart with $q_0 \in U$. Now let $x_i = r_i \circ \varphi$, then $\varphi(q) = (x_1(q), \ldots, x_m(q))$.

We may now assume that

- $\varphi(q_0) = 0$, otherwise, we replace $\varphi(q)$ by $\varphi(q) := \varphi(q) - \varphi(q_0)$, and
- $\varphi(U)$ is an open ball $B_R(0) = \{r \in \mathbb{R}^m \mid ||r|| < R\}$ because there exists $R > 0$ such that $B_R(0) \subseteq \varphi(U)$, and we can then replace U with $\varphi^{-1}(B_R(0))$ and restrict the charts φ to $\varphi|_{\varphi^{-1}(B_R(0))}$.

We now define

$$\begin{aligned}\frac{\partial}{\partial x_j}|_{q_0}: C^\infty(U) &\to \mathbb{R}\\ f &\mapsto \frac{\partial}{\partial r_j}|_0(f\circ\varphi^{-1})\end{aligned}$$

Claim 5.9. $\left\{\frac{\partial}{\partial x_j}|_{q_0}\right\}_{j=1}^m$ is a basis of T_qM and for all $v\in T_{q_0}M$, $v=\sum v(x_j)\frac{\partial}{\partial x_j}|_{q_0}$.

Proof. By Hadamard's lemma Lemma 4.5 on $B_R(0)$, for all $f\in C^\infty(U)$, we have $f\circ\varphi^{-1}\in C^\infty(B_R(0))$, so there exists $g_1,\dots,g_m\in C^\infty(B_R(0))$ such that $(f\circ\varphi^{-1})(r)=f(\varphi^{-1}(0))+\sum r_ig_i(r)$. Therefore, $f(q)=f(q_0)+\sum(r_i\circ\varphi)(q)(g_i\circ\varphi)(q)$, hence $f=f(q_0)+\sum x_i(g_i\circ\varphi)$, and $(g_i\circ\varphi)(q_0)=g_i(0)=\frac{\partial}{\partial r_i}|_0(f\circ\varphi^{-1})=\frac{\partial}{\partial x_i}|_0(f)$.

Hence, for all $v\in T_{q_0}(U)$, we know

$$\begin{aligned}v(f)&=v(f(q_0))+v\left(\sum x_i\cdot(g_i\circ\varphi)\right)\\&=\sum_i v(x_i)(g_i\circ\varphi)(q_0)\\&=\sum v(x_i)\frac{\partial}{\partial x_i}|_{q_0}(f).\end{aligned}$$

□

Remark 5.10.

1. The linear functionals

$$\begin{aligned}(dx_i)_{q_0}: T_{q_0}U &\to \mathbb{R}\\ v&\mapsto v(x_i)\end{aligned}$$

is the basis of $(T_{q_0}U)^*$ dual to $\left\{\frac{\partial}{\partial x_i}|_{q_0}\right\}$.

2. $(T_0\varphi^{-1})\left(\frac{\partial}{\partial r_i}|_0\right)=\frac{\partial}{\partial x_i}|_{q_0}$ by definition. Since $\left\{\frac{\partial}{\partial x_i}|_0\right\}_{i=1}^n$ is a basis of $T_0(B_R(0))$, then $\left\{\frac{\partial}{\partial x_i}|_{q_0}\right\}$ has to be a basis.

Lemma 5.11. Let M be a manifold and $q\in M$ a point. Let $U\ni q$ be anopen neighborhood, and $f\in C^\infty(M)$ such that $f|_U=0$, then for all $v\in T_qM$, we have $v(f)=0$.

Proof. We have shown the existence of a bump function $\rho\in C^\infty(M)$ in homework 1, that is, $0\leqslant\rho(x)\leqslant 1$, $\operatorname{supp}(\rho)\subseteq U$ and $\rho\equiv 1$ near q.

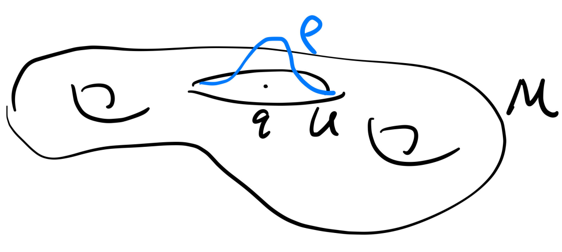

Figure 7: Bump Function

Therefore, $\rho f\equiv 0$, so $v(f)=v(\rho)f(q)+\rho(q)v(f)=v(\rho f)=0$. □

6 Sept 1, 2023

Recall. Given a coordinate chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$, and $q \in U$ with $f(q) = 0$, we defined $\left\{ \frac{\partial}{\partial x_i}|_q \right\}_{i=1}^m \subseteq T_qU$ by $\frac{\partial}{\partial x_i}|_q f = \frac{\partial}{\partial r_i}(f \circ \varphi^{-1})|_{\varphi(q)}$ where $\frac{\partial}{\partial r_i}$'s are the standard partials on $C^\infty(\mathbb{R}^m)$.

We know this is a basis with dual basis

$$\begin{aligned} (dx_i)_q : T_qM &\to \mathbb{R} \\ v &\mapsto v(x_i) \end{aligned}$$

therefore $v = \sum v(x_i) \frac{\partial}{\partial x_i}|_q$ for all v. Note that

$$\begin{aligned} C^\infty(M) &\to C^\infty(U) \\ f &\mapsto f|_U \end{aligned}$$

is not surjective.

Also, we know $v \in T_qM$ is local, if $f, g \in C^\infty(M)$ agree on a neighborhood of q, then $v(f) = v(g)$.

Finally, given $F : M \to N$, this induces

$$\begin{aligned} T_qF : T_qM &\to T_{F(q)}N \\ v &\mapsto v(f \circ F). \end{aligned}$$

Lemma 6.1. Given a manifold M and $q \in M$, open neighborhood $q \in U \subseteq M$ and $i : U \hookrightarrow M$ inclusion, then

$$I \equiv T_qi : T_qU \to T_qM$$

is an isomorphism with $(I(v))(f) = v(f|_U)$ for all $f \in C^\infty(M)$.

Proof. Suppose $v \in \ker(I)$, then $v(f|_U) = 0$ for all $f \in C^\infty(M)$. We want $v(h) = 0$ for all $h \in C^\infty(U)$. We first choose bump function $\rho : M \to [0, 1]$ that is C^∞, and $\rho \equiv 1$ near q, and suppose $\operatorname{supp}(\rho) \subseteq U$, hence $\rho|_{M \setminus U} \equiv 0$. Then define $\rho h \in C^\infty(M)$ via

$$\rho h(x) = \begin{cases} \rho(x)h(x), & x \in U \\ 0, & x \notin U \end{cases}$$

Now $\rho h|_U \equiv h$ near q, i.e., identically 1. Therefore, $v(h) = v(\rho h|_U) = 0$, so $v \equiv 0$.

It remains to show that for all $w \in T_qM$, there exists $v \in T_qU$ such that $I(v) = w$, i.e., for all $f \in C^\infty(M)$, $w(f) = v(f|_U)$. Take the same $\rho \in C^\infty(M, [0.1])$ as above, define $v(h) = w(\rho h)$ for all $h \in C^\infty(M)$, and we can check that

- $v \in T_qM$, and
- for all $f \in C^\infty(M)$, $v(f|_U) = w(f)$.

Note that v is $\mathbb{R}$-linear, and for all $f, g \in C^\infty(W)$ we have $v(fg) = w(\rho fg) = w(\rho^2 fg)$ since $\rho fg = \rho^2 fg$ near q, then we have

$$\begin{aligned} v(fg) &= w(\rho^2 fg) \\ &= w((\rho f)(\rho g)) \\ &= v(\rho f) \cdot (\rho g)(g) + \rho(f)(q) \cdot v(\rho g) \\ &= v(f)g(q) + f(q)v(g). \end{aligned}$$

Finally, for all $f \in C^\infty(M)$, we have $v(f|_U) = w(\rho f) = w(f)$ since $\rho f = f$ near q. $\square$

Notation. We now suppress the isomorphisms $I : T_qU \to T_qM$. In particular, given a chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$, we view $\left\{ \frac{\partial}{\partial x_i}|_q \right\}_{i=1}^m$ as a basis of T_qM.

Lemma 6.2. Let V be a finite-dimensional vector space with $q \in V$, then

$$\begin{aligned}\varphi : V &\to T_qV\\ v(f) &\mapsto \frac{d}{dt}|_0 f(q+tv)\end{aligned}$$

for all $f \in C^\infty(V)$, is an isomorphism.

Proof. One can see this is linear, so it suffices to show injectivity. We have

$$\ker(\varphi) = \{v \in V \mid \frac{d}{dt}|_0(q+tv) = 0\ \forall f \in C^\infty(V)\}.$$

If $0 \neq v \in \ker(\varphi)$, then there exists $\ell : V \to \mathbb{R}$ such that $\ell(V) \neq 0$, so

$$0 \neq \frac{d}{dt}|_0(\ell(q+tv)) = \frac{d}{dt}|_0(\ell(q) + t\ell(v)) = \ell(v).$$

□

Definition 6.3. A *curve* through a point $q \in M$ on a manifold M is a C^∞-map $\gamma : (a,b) \to M$ with $0 \in (a,b)$ such that $\gamma(0) = q$.

Definition 6.4. Given $\gamma : (a,b) \to M$ with $\gamma(0) = q$, we define $\dot{\gamma}(0) \in T_qM$ by $\dot{\gamma}(0)f = \frac{d}{dt}|_0 f(\gamma(t)) = \frac{d}{dt}|_0(f \circ \gamma)$ for all $f \in C^\infty(M)$.

Remark 6.5.

$$\begin{aligned}t : (a,b) &\to \mathbb{R}\\ x &\mapsto x\end{aligned}$$

is a coordinate chart on (a,b), where $\frac{d}{dt}|_0 \in T_0(a,b)$ is a basis vector. Since γ is C^∞,

$$\begin{aligned}T_0\gamma : T_0(a,b) &\to T_{\gamma(0)}M \equiv T_qM\\ ((T_0\gamma)(\frac{d}{dt}|_0))f &= \frac{d}{dt}|_0(f \circ \gamma) = \dot{\gamma}(0),\end{aligned}$$

so $\dot{\gamma}(0) = (T_0\gamma)\left(\frac{d}{dt}|_0\right)$.

Let $\mathscr{C} = \{\gamma : I \to M \mid \gamma(0) = q, I \text{ interval depending on } \gamma\}$, then we have a map

$$\begin{aligned}\Phi : \mathscr{C} &\to T_qM\\ \gamma &\mapsto \dot{\gamma}(0)\end{aligned}$$

Note that Φ is not injective. However, there is an equivalence relation $\sim$ on $\mathscr{C}$ defined by $\gamma \sim \sigma$ if and only if $\Phi(\gamma) = \Phi(\sigma)$, so this gives an injection

$$\begin{aligned}\tilde{\Phi} : \mathscr{C}/\sim &\to T_qM\\ [\gamma] &\mapsto \dot{\gamma}(0).\end{aligned}$$

Claim 6.6. $\tilde{\Phi}$ is onto.

Proof. Choose coordinates $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ near q such that $(x_1, \ldots, x_m)(q) = 0$. Now, for all $v \in T_qM$, we have $v = \sum v(x_i)\frac{\partial}{\partial x_i}|_q$. Consider $\gamma(t) = \varphi^{-1}(tv(x_1), \ldots, tv(x_m))$, then $\gamma(0) = \varphi^{-1}(0) = q$ and for any $f \in C^\infty(M)$, we have

$$\begin{aligned}\dot{\gamma}(0)f &= \frac{d}{dt}|_0(f \circ \varphi^{-1})(tv(x_1), \ldots, tv(x_m))\\ &= \sum \frac{\partial}{\partial r_i}(f \circ \varphi^{-1})|_0 \cdot v(x_i)\\ &= \sum v(x_i)\frac{\partial}{\partial x_i}|_q f\\ &= v(f).\end{aligned}$$

□

Lemma 6.7. For any smooth map $F : M \to N$ between manifolds, for all $q \in M$, we have

$$T_qF(\dot{\gamma}(0)) = (F \circ \gamma)^{\cdot}(0).$$

Proof.

$$\begin{aligned} T_qF(\dot{\gamma}(0)) &= T_qF(T_0\gamma\left(\frac{d}{dt}|_0\right)) \\ &= T_0(F \circ \gamma)\left(\frac{d}{dt}|_0\right) \\ &= (F \circ \gamma)^{\cdot}(0). \end{aligned}$$

□

Example 6.8. Let $M = N = \mathbb{C}$ and $F(z) = e^z$. We claim that $(T_zF)(v) = e^z v$, which uses $\mathbb{C} \cong T_w\mathbb{C}$ for all $w \in \mathbb{C}$. Indeed, since $\frac{d}{dt}|_0 e^{tv} = v$, then

$$\begin{aligned} (T_zF)(v) &= \frac{d}{dt}|_0 F(z + tv) \\ &= \frac{d}{dt}|_0 e^{z+tv} \\ &= \frac{d}{dt}|_0 (e^z e^{tv}) \\ &= e^z v. \end{aligned}$$

Note that T_zF is an isomorphism for all z, given by

$$\begin{tikzcd} T_z\mathbb{C} \arrow[r, "T_zF"] \arrow[d, "\cong"] & T_{F(z)}\mathbb{C} \arrow[d, "\cong"] \\ \mathbb{C} \arrow[r, "e^z \cdot -"'] & \mathbb{C} \end{tikzcd}$$

Also note that this is not a diffeomorphism, since the inverse is the complex logarithm function, which is only well-behaved on the principal branches.