MATH 519 Notes

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Contents

1 Riemannian Geometry

1.1 Riemannian Metrics

Definition 1.1.1. A *Riemannian metric* on a manifold M is a family of inner products

$$\langle -,-\rangle_p : T_pM \times T_pM \to \mathbb{R}$$

which vary smoothly with the point p. We then say M is a *Riemannian manifold*.

Remark 1.1.2. Equivalently, we can think of it as a map

$$g : \mathfrak{X}(M) \to \mathfrak{X}(M) \to C^\infty(M)$$

of vectors field $\mathfrak{X}(M)$ on manifold M, defined by

$$g_p(x,y) = \langle x,y\rangle_p$$

and satisfying the properties that

a. it is C^∞-bilinear,

b. symmetric, i.e., $g(x,y) = g(y,x)$, and

c. positive-definite, i.e., $g_p(x,x) = 0$ if and only if $x = 0$.

Therefore, g is a symmetric tensor of type-$(2,0)$. In local charts (U, x^i), the tensor has local coordinates given by

$$\begin{aligned} g|_U &= \sum_{i,j} g_{ij}\, dx^i \otimes dx^j \\ &= \sum_{i,j} g_{ij}\, dx^i dx^j \text{ by symmetry} \end{aligned}$$

for $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$. Here by convention, we denote the symmetric product $dx^i dx^j = \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i)$.

Exercise 1.1.3. If (V, y^i) is another chart (such that $U \cap V \neq \varnothing$), then we have $g = \sum_{i,j} \bar{g}_{i,j}\, dy^i dy^j$. How are g_{ij}'s related to $\bar{g}_{ij}$?

Remark 1.1.4. The (symmetric) tensors of type-$(p,0)$ behave like differential forms. Therefore, say, given a $C^\infty(M)$-multilinear tensor of type-$(p,0)$ $T : \mathfrak{X}(M)^p \to C^\infty(M)$, and given $\Phi : N \to M$, then we have a pullback

$$\Phi^* T : \mathfrak{X}(N)^p \to C^\infty(N)$$

which is defined as per differential forms, i.e.,

$$(\Phi^* T)_x(x_1, \ldots, x_p) = T_{\Phi(x)}(d_x\Phi(x_1), \ldots, d_x\Phi(x_p)).$$

In local coordinates, we can represent $\Phi = (\Phi^1, \ldots, \Phi^p)$ and $T = \sum_{i_1,\ldots,i_p} T_{i_1,\ldots,i_p}\, dx^{i_1} \otimes \cdots \otimes dx^{i_p}$, then we have

$$(\Phi^* T) = \sum_{i_1,\ldots,i_p} (T_{i_1,\ldots,i_p} \circ \Phi)\, d\Phi^{i_1} \otimes \cdots \otimes d\Phi^{i_p}.$$

Example 1.1.5. Consider $M = \mathbb{R}^3$ in coordinates (x,y,z), and $T = z^2 dx \otimes dy + 2xy dz \otimes dx$ given by a 2-tensor. We have a map

$$\begin{aligned} \Phi : \mathbb{R}^2 &\to \mathbb{R}^3 \\ (u,v) &\mapsto (e^u v, v, uv), \end{aligned}$$

then to compute $\Phi^* T$, we assign $x = e^u$, $y = v$, and $z = uv$, then

$$\begin{aligned} \Phi^* T &= (uv)^2 de^u \otimes dv + 2e^u v d(uv) \otimes de^u \\ &= (uv)^2 e^u du \otimes dv + 2e^u v(vdu + udv) \otimes e^u du \\ &= 2ue^{2u} du \otimes du + u^2v^2 e^u du \otimes dv + 2e^{2u} vudv \otimes du. \end{aligned}$$

Definition 1.1.6. A (smooth) map $\Phi : (M_1, g_1) \to (M_2, g_2)$ between Riemannian manifolds is called an *isometric immersion* if it is an immersion and $\Phi^* g_2 = g_1$. In particular, if Φ is a (respectively, local) diffeomorphism, then we say Φ is a (respectively, local) isometry.

Example 1.1.7.

1. Consider $M = \mathbb{R}^n$ with local coordinates $(x^1, \ldots, x^n)$, the inner product structure on the tangent space gives the *(standard) distance function* $g_0 = \sum_{i=1}^{n} (dx^i)^2$ as the metric.

2. If $N \subseteq \mathbb{R}^n$ is a submanifold, then the inclusion gives an induced Riemannian metric $g_N = i^* g_0$ where $i : N \hookrightarrow \mathbb{R}^n$ is the inclusion.

($\star$) Consider $N = \mathbb{S}^2_R \subseteq \mathbb{R}^3$ be the 2-sphere of radius R with local coordinates (θ, φ) and (x, y, z), respectively. We note that $x = R\cos(\theta)\sin(\varphi)$, $y = R\sin(\theta)\sin(\varphi)$, and $z = R\cos(\varphi)$. We should think of these expressions as defining the inclusion map i from the 2-sphere to $\mathbb{R}^3$, thereby inducing

$$\begin{aligned} g_{\mathbb{S}^2_R} &= i^* g_0 \\ &= (dx)^2 + (dy)^2 + (dz)^2 \\ &= R^2(\sin^2(\varphi)(d\theta)^2 + (d\varphi)^2) \end{aligned}$$

3. Let G be a Lie group with Lie algebra $\mathfrak{g} = T_e G$. Picking an inner product $\langle -, - \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ on the vector space $\mathfrak{g}$, we induce a Riemannian structure on the Lie group G, namely the left translation

$$g_h(x, y) = \langle d_h L_{h^{-1}}(x), d_h L_{h^{-1}}(y) \rangle$$

for $h \in G$ and left translation L.[1] This is a left-invariant metric, i.e., $g_y(u, v) = g_{L_x(y)}(d_y L_x u, d_y L_x v)$: every left translation $L_h : G \to G$ is an isometry. Moreover, one can show that the Lie algebra on the Lie group must be of compact type.

We end the lecture with two important results.

Theorem 1.1.8. Every manifold admits a Riemannian metric.

Proof 1. Use Whitney embedding theorem and pullback g_0. □

Proof 2. Use partition of unity: a $\mathbb{C}$-combination of inner products is still an inner product, so we get to glue the local inner product structures together as a global one. □

Remark 1.1.9. We see that the first proof is better than the second one, in the sense that it works in general for any analytic manifold, while the second one only works for Riemannian manifolds.

Theorem 1.1.10 (Nash Embedding)**.** Every Riemannian manifold (M, g) admits an isometric embedding $i : (M, g) \hookrightarrow (\mathbb{R}^n, g_0)$ for some n.

End of Lecture 1

1.2 Geodesics

Definition 1.2.1. Let (M, g) be a Riemannian manifold.

- For any $v \in T_x M$, we have $||v||^2 = g(v, v)$. In particular, if $v \in \mathbb{R}^m$, then this is its norm $|v|^2 = \sum_{i=1}^{m} v_i^2$.

[1] Correspondingly, there is a Riemannian structure given by the right translation.

- Given a path $\gamma : [a, b] \to M$ that is piecewise smooth, then its *length* is given by $L(\gamma) = \int\limits_a^b ||\dot{\gamma}(t)||dt$, and its *energy* is given by $E(\gamma) = \frac{1}{2}\int\limits_a^b ||\dot{\gamma}(t)||^2 dt$.
- The *distance function* $d : M \times M \to \mathbb{R}$ is given by
$$d(p, q) = \min\{L(\gamma) : \gamma : [a, b] \to M \text{ piecewise smooth curve} : \gamma(a) = p, \gamma(b) = q\}.$$
In such cases, we usually assume M to be connected, or just deal with a connected component.

Proposition 1.2.2. The distance function is just the distance in the usual sense. That is, (M, d) is a metric space, and the topology it defined is the same as the one on M.

Proof. Let us first show that (M, d) is a metric space. Most of this is obvious, so the only part we need to show is that if $d(p, q) > 0$ whenever $p \neq q$. Fix a chart (U, φ) centered at $p \in M$, corresponding to $\varphi(U)$ on $\mathbb{R}^m$. Without loss of generality, we choose $q \notin U$. Choose $\varepsilon > 0$ such that $D_\varepsilon := \{v \in \mathbb{R}^m : |v| \leqslant \varepsilon\}$. If $\gamma(a) = p$ and $\gamma(b) = q$, then $L(\gamma) \geqslant L(\gamma \cap \varphi(D_\varepsilon))$. Therefore, it suffices to show that there exists $c > 0$ such that given a curve $\gamma' : [a, b] \to \varphi(D_\varepsilon)$ where $\gamma'(a) = p$ and $\gamma'(b) \in \varphi(\partial D_\varepsilon)$, then $L(\gamma') \geqslant \varepsilon$.

More specifically, let us write the chart as $\varphi = (x^1, \ldots, x^m)$ and $g = \sum\limits_{i,j} g_{ij}(x)dx^i dx^j$. Let us define
$$\lambda(x) = \min\{g_{ij}(x)v^i v^j : |v| = 1, v \in \mathbb{R}^m\},$$
but since D_ε is compact, then we have $\lambda(x) \geqslant \lambda_0 > 0$ for all $x \in D_\varepsilon$, therefore
$$||v|| = \sum_{i,j} g_{ij}(x)v^i v^j = \sum_{i,j} g_{ij}(x)\frac{v^i}{|v|}\frac{v^j}{|v|}|v|^2 \geqslant \lambda_0|v|^2$$
which is true for any tangent vector in the disk D_ε. We compute that the length on the chart
$$\varphi \circ \gamma'(t) = (\gamma'^1(t), \ldots, \gamma'^m(t)),$$
where we find
$$\dot{\gamma}'(t) = \sum_{i=1}^m \dot{\gamma}'^i(t) \left.\frac{\partial}{\partial x^i}\right|_{\gamma'(t)}.$$
Therefore, we calculate
$$\begin{aligned} L(\gamma') &= \int_a^b ||\dot{\gamma}'(t)||dt \\ &= \int_a^b (g_{ij}\dot{\gamma}'^i(t)\dot{\gamma}'^j(t))^{\frac{1}{2}} \\ &\geqslant \int_a^b \lambda_0|\dot{\gamma}'(t)|dt \\ &\geqslant \lambda_0\varepsilon \\ &= c. \end{aligned}$$
To check that the topologies agree, we just need to check this on any chart. In particular, on a chart, we have
$$\lambda_0|v|^2 \leqslant g_{ij}(x)v^i v^j \leqslant \mu_0|v|^2$$
which sandwiches the distance between the two points in the ball. This means that for $x = \varphi(p)$ and $y = \varphi(q)$, then the distance
$$\lambda_0|x - y| \leqslant d(p, q) \leqslant \mu_0|x - y|$$
which means they define the same open set in any chart. $\square$

Remark 1.2.3. Length is invariant under parametrization. That is, given $\gamma : [a, b] \to M$ and $\tau : [c, d] \to [a, b]$ such that $\tau(c) = a$ and $\tau(d) = b$, then $L(\gamma \circ \tau) = L(\gamma)$. This is given by the chain rule: we have

$$\dot{\overline{(\gamma \circ \tau)}}(t) = \dot{\gamma}(\tau(t))\dot{\tau}(t),$$

so taking the length gives

$$\begin{aligned} L(\gamma \circ \tau) &= \int_c^d ||\dot{\overline{(\gamma \circ \tau)}}||dt \\ &= \int_c^d ||\dot{\gamma}(\tau(t))|| \cdot ||\dot{\tau}(t)||dt \\ &= \int_a^b ||\dot{\gamma}(s)||ds \\ &= L(\gamma) \end{aligned}$$

where we define $s = \tau(t)$.

Remark 1.2.4. Energy is not invariant due to the quadratic in its formula.

However, length and energy are related as follows.

Theorem 1.2.5 (Length-energy Inequality)**.** If $\gamma : [a, b] \to M$, then

$$L(\gamma)^2 \leqslant 2(b - a)E(\gamma)$$

with equality holds if and only if the length of the tangent $||\dot{\gamma}(t)||$ is constant.

Proof. By Hölder's inequality, we have

$$\int_a^b |f(t)g(t)|dt \leqslant \left(\int_a^b |f|^2 dt\right)^{\frac{1}{2}} \left(\int_a^b |g|^2 dt\right)^{\frac{1}{2}}$$

where equality holds if and only if there exists $\lambda, \mu \in \mathbb{R}$ such that $\lambda f^2 = \mu g^2$. Therefore, taking the length gives

$$\begin{aligned} L(\gamma)^2 &= \int_a^b ||\dot{\gamma}||dt \\ &\leqslant 2\left(\int_a^b 1dt\right) \cdot \frac{1}{2}\left(\int_a^b ||\dot{\gamma}||^2 dt\right) \\ &= 2(b - a)E(\gamma), \end{aligned}$$

and equality holds if and only if the length of the tangent is constant. □

Definition 1.2.6. A *geodesic* is a curve that minimizes energy.

Remark 1.2.7. To get around the fact that energy is not invariant under parametrization, we will define

$$P(p, q) = \{\gamma : [0, 1] \to M : \gamma(0) = p, \gamma(1) = q\},$$

then we can rewrite energy as

$$\begin{aligned} E : P(p, q) &\to \mathbb{R} \\ \gamma &\mapsto \frac{1}{2}\int_0^1 ||\dot{\gamma}(t)||^2 dt. \end{aligned}$$

Remark 1.2.8. If we try to minimize the length, since the length is invariant under parametrization, we can just restrict to the curves parametrized by the arc length, which means the norm of the derivative $||\dot{\gamma}(t)||$ is 1, but in that case this is equivalent as minimizing the energy function by Theorem 1.2.5. Conversely, we will show that the geodesic has "constant" norm of derivative, so again by Theorem 1.2.5, minimizing the length function and minimizing the energy function are the same thing.

Fix a curve $\gamma_0 : [0, 1] \to M$, and take a piecewise smooth variation of γ_0. For simplicity, we may assume γ_0 is smooth, so that we only require a *smooth variation* of γ_0, which is a smooth curve $\gamma : (-\delta, \delta) \times [0, 1]$ such that $\gamma(0, t) = \gamma_0(t)$, and we define $\gamma_\varepsilon(t) = \gamma(\varepsilon, t)$. If γ_0 minimizes E, then we will see that

$$0 = \frac{d}{d\varepsilon} E(\gamma_\varepsilon)\bigg|_{\varepsilon=0} = \frac{1}{2}\frac{d}{d\varepsilon}\int_0^1 ||\gamma_\varepsilon||^2 dt\bigg|_{\varepsilon=0}. \tag{1.2.9}$$

We will denote $||\gamma_\varepsilon||^2$ by $\mathcal{L}(\gamma_\varepsilon)$, given by the function

$$\begin{aligned} \mathcal{L} : TM &\to \mathbb{R} \\ v &\mapsto g(v, v) = ||v||^2 \end{aligned}$$

known as the *Lagrangian*. In particular, Equation (1.2.9) is equivalent to the Euler-Lagrange equations for $\mathcal{L}$.

To do this in global coordinates, we will require Čech spaces. Instead, we will do this in local coordinates (U, φ), where we define

$$\mathcal{L}(x, v) := g_{ij}(x)v^i v^j.$$

End of Lecture 2

We now summarize the setting. For a Riemannian manifold (X, g), where we take

$$X = \{\gamma : [0, 1] \to M \mid \gamma(0) = p, \gamma(1) = q\}$$

and we have an energy function

$$\begin{aligned} E : X &\to \mathbb{R} \\ \gamma &\mapsto \frac{1}{2}\int_0^1 ||\dot{\gamma}(t)||^2 dt. \end{aligned}$$

We are now interested in the critical points of this function, which is the interest in studying calculus of variations. In a more general setting, consider a function $\mathcal{L} : TM \to \mathbb{R}$ on the tangent bundle of the manifold, where we should think of as $\mathcal{L}(v) = \frac{1}{2}g(v, v)$ in our case. Now consider the function

$$\begin{aligned} \mathcal{F} : X &\to \mathbb{R} \\ \gamma &\mapsto \frac{1}{2}\mathcal{L}(\gamma(t), \dot{\gamma}(t))dt. \end{aligned}$$

Let us fix a chart $U \subseteq M$ so we can run a local argument (assuming $p, q \in U$), that is, assuming $\gamma_0 : [0, 1] \to U$. Now take $\gamma(\varepsilon, t) = \gamma_\varepsilon : [0, 1] \to U$ for

$$\gamma : (-\delta, \delta) \times [0, 1] \to \mathbb{R}$$

for some small $\delta > 0$. Therefore $\gamma_0 \in X$ is a point, and γ_ε is now a curve on X, which defines a function $\mathcal{F}$ into $\mathbb{R}$. We are therefore interested in finding curves so that

$$\frac{d}{d\varepsilon}\mathcal{F}(\gamma_\varepsilon)\bigg|_{\varepsilon=0} = 0.$$

This is just asking

$$
\begin{aligned}
&\frac{d}{d\varepsilon}\int_0^1 \mathcal{L}(\gamma_\varepsilon(t),\dot{\gamma}_\varepsilon(t))dt\Big|_{\varepsilon=0} = \int_0^1\left(\sum_{i=1}^m \frac{\partial\mathcal{L}}{\partial x^i}(\gamma_\varepsilon(t))\frac{d\gamma_\varepsilon^i}{d\varepsilon}+\frac{\partial\mathcal{L}}{\partial v^i}(\gamma_\varepsilon(t),\dot{\gamma}_\varepsilon(t))\frac{d}{d\varepsilon}\dot{\gamma}_\varepsilon^i(t)\Big|_{\varepsilon=0}\right)\\
&=\int_0^1\sum_{i=1}^m\left(\frac{\partial\mathcal{L}}{\partial x^i}(\gamma_0(t),\dot{\gamma}_0(t))\frac{d\gamma_\varepsilon^i}{d\varepsilon}\Big|_{\varepsilon=0}-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial v^i}(\gamma_0,\dot{\gamma}_0(t))\frac{d\gamma_\varepsilon^i}{d\varepsilon}\Big|_{\varepsilon=0}+\frac{d}{dt}\left(\frac{d\mathcal{L}}{dv^i}(\gamma_0(t),\dot{\gamma}_0(t))\frac{d}{d\varepsilon}\gamma_\varepsilon^i(t)\Big|_{\varepsilon=0}\right)\right)\\
&=\int_0^1\sum_{i=1}^m\left(\frac{\partial\mathcal{L}}{\partial x^i}(\gamma_0(t),\dot{\gamma}_0(t))-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial v^i}(\gamma_0(t),\dot{\gamma}_0(t))\right)\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}\Big|_{\varepsilon=0}dt+\sum_{i=1}^m\left(\frac{\partial\mathcal{L}}{\partial v^i}(\gamma_0(t),\dot{\gamma}_0(t))\ \frac{d}{d\varepsilon}\gamma_\varepsilon^i(t)\Big|_{\varepsilon=0}\Big|_{t=0}^{t=1}\right)\\
&=\int_0^1\sum_{i=1}^m\left(\frac{\partial\mathcal{L}}{\partial x^i}(\gamma_0(t),\dot{\gamma}_0(t))-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial v^i}(\gamma_0(t),\dot{\gamma}_0(t))\right)\frac{d\gamma_\varepsilon(t)}{d\varepsilon}\Big|_{\varepsilon=0}dt
\end{aligned}
$$

under enough smoothness conditions. But this expression is zero for any γ_ε, therefore this implies that

$$\int_0^1\sum_{i=1}^m\left(\frac{\partial\mathcal{L}}{\partial x^i}(\gamma_0(t),\dot{\gamma}_0(t))-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial v^i}(\gamma_0(t),\dot{\gamma}_0(t))\right)\frac{d\gamma_\varepsilon(t)}{d\varepsilon}\Big|_{\varepsilon=0}dt=0 \tag{1.2.10}$$

for any $t\in[0,1]$ and $i=1,\dots,m$. This is known as the *Euler-Lagrange equation*.

With $\mathcal{L}(x,v)=\sum_{i,j}g_{ij}(x)v^iv^j$, we have the Euler-Lagrange Equation local charts (U,φ) given by

$$\ddot{\gamma}_0^i(T)+\sum_{j,k=1}^m\Gamma_{jk}^i(\gamma_0(t))\dot{\gamma}^j(t)\dot{\gamma}^k(t)=0 \tag{1.2.11}$$

for $i=1,\dots,m$, where we define *Christoffel symbols*

$$\Gamma_{jk}^i=\frac{1}{2}\sum_{\ell=1}^m g^{i\ell}\left(\frac{\partial g_{j\ell}}{\partial x^k}+\frac{\partial g_{k\ell}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^\ell}\right)$$

where (g^{ij}) is the inverse of (g_{ij}) given as a matrix. In particular, the Christoffel symbols are not tensors.

Definition 1.2.12 (Einstein Convention). A (p,q)-tensor $T\in\Gamma(\bigotimes^p T^*M\bigotimes^q TM)$ can be described as a C^∞-multilinear function

$$T:(\mathfrak{X}^1(M))^p\times(\Omega^1(M))^q\to C^\infty(M)$$

and in particular we can write

$$T=T_{i_1,\dots,i_p}^{j_1,\dots,j_q}(x)dx^{i_1}\otimes\cdots dx^{i_p}\otimes\frac{\partial}{\partial x^{j_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{j_q}}$$

in local charts. This avoids writing over the summation $\sum\limits_{\substack{i_1,\dots,i_p\\ j_1,\dots,j_q}}$ and we can just denote $g=g_{ij}dx^idx^j$.

Exercise 1.2.13. Show that any solution γ_0 of Equation (1.2.11) has $||\dot{\gamma}(t)||$ constant.

1.3 Connections

Definition 1.3.1 (Affine Connection). An affine connection on a manifold M is a $\mathbb{R}$-bilinear map

$$
\begin{aligned}
\nabla:\mathfrak{X}(M)\times\mathfrak{X}(M)&\to\mathfrak{X}(M)\\
(X,Y)&\mapsto\nabla_XY
\end{aligned}
$$

satisfying

i. it is C^∞-bilinear in the first entry: $\nabla_{fX}Yy = f\nabla_X Y$ for any $f \in C^\infty(M)$;

ii. $\nabla_X(fY) = f\nabla_X Y + X(f)Y$ for any $f \in C^\infty(M)$.

Remark 1.3.2. There are other ways of defining connections. For instance, we can say it is a linear operator

$$d^\nabla : \Omega^*(M, TM) \to \Omega^{*+1}(M, TM)$$

satisfying $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega|}\omega \wedge d^\nabla \eta$ for $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M, TM)$.

Example 1.3.3.

1. Set $M = \mathbb{R}^n$ and $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$, then

$$\begin{aligned}\nabla_X Y &= X^i \nabla_{\frac{\partial}{\partial x^i}} \left(Y^j \frac{\partial}{\partial x^j}\right) \\ &= X^i Y^j \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} + X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j}.\end{aligned}$$

We can now set

$$\Gamma^k_{ij} \frac{\partial}{\partial x^k} = \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}$$

to be arbitrary functions so that we get a connection. For instance, we can set them to be zeros, which gives a canonical connection in $\mathbb{R}^n$, namely the *flat connection* or the trivial connection in $\mathbb{R}^n$ with $\Gamma^k_{ij} \equiv 0$. That is,

$$\nabla_X Y = X(Y^j) \frac{\partial}{\partial x^j}.$$

2. Let $M = G$ be a Lie group with a Lie algebra $\mathfrak{g} = T_e G$, and fix a basis $\{e_1, \dots, e_n\}$ for $\mathfrak{g}$, with left-invariant vector fields $\{E_1, \dots, E_n\} \subseteq \mathfrak{X}(M)$. This gives a basis, so any vector fields $X, Y \in \mathfrak{X}(M)$ can be written as $X = X^i E_i$ and $Y = Y^j E_j$. Now we get

$$\nabla_X Y = X^i Y^j \nabla_{E^i} E^j + X(Y^i) E_j$$

just as in the previous example. If we set this to be arbitrary, we may get any connection. In particular, for $\nabla_{E_j} E_j = c[E_i, E_j]$ to be a Lie bracket multiplied by some fixed constant c.

We now know a connection always exists for an arbitrary manifold: the first example tells us that the connections exist locally, so it is just a question of how we glue connections together.

Proposition 1.3.4. Every manifold M has a connection. The space of connections is an affine space modeled on the vector space of $(2, 1)$-tensors.

Proof. Given a chart, we apply the first example in Example 1.3.3. So take a cover $C = \{(U_i, \varphi_i)\}$ of M by charts, and choose a connection ∇^i on each chart U_i. Now take a partition of unity $\{\rho_i\}$ subordinated to C, then we can define a global connection

$$\nabla_X Y = \sum_i \rho_i \nabla^i_{X|_{U_i}} Y|_{U_i}.$$

To prove the second statement, given two connections ∇^1 and ∇^2, we note that $T(x, y) := \nabla^1_X Y - \nabla^2_X Y$ is C^∞-linear, so this defines a C^∞-linear map

$$T : \mathfrak{X}^1(M) \times \mathfrak{X}^1(M) \to \mathfrak{X}^1(M)$$

which defines a $(2, 1)$-tensor. $\square$

End of Lecture 3

Let ∇ be the connection. For a chart (U, x^i), we observed that

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k}$$

characterized the vector fields.

Remark 1.3.5. Γ_{ij}^k's are not components of a tensor field. Note that the assignment $X \mapsto \nabla_X Y$ is $C^\infty(M)$-linear for fixed Y, but $Y \mapsto \nabla_X Y$ is not $C^\infty(M)$-linear.

Definition 1.3.6. The *torsion* of ∇ is the map

$$\begin{aligned} T^\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ (X, Y) &\mapsto \nabla_X Y - \nabla_Y X - [X, Y] \end{aligned}$$

Remark 1.3.7. This is a $(2, 1)$-tensor: we can define

$$\begin{aligned} \tilde{T}^\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Omega^1(M) &\to C^\infty(M) \\ (X, Y, \alpha) &\mapsto \langle T^\nabla(X, Y), \alpha \rangle \end{aligned}$$

which is $C^\infty(M)$-linear in each entry. Indeed,

- $\alpha \mapsto \tilde{T}^\nabla(X, Y, \alpha)$ is $C^\infty(M)$-linear,
- $\tilde{T}^\nabla(X, Y, \alpha) = -\tilde{T}^\nabla(Y, X, \alpha)$,
- and

$$\begin{aligned} T^\nabla(fX, Y, \alpha) &= \nabla_{fX} Y - \nabla_Y (fX) - [fX, Y] \\ &= f\nabla_X Y - f\nabla_Y X - Y(f)X - f[X, Y] + Y(f)X \\ &= fT^\nabla(X, Y). \end{aligned}$$

Therefore, in a local chart (U, x^i), we get

$$T^\nabla\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = T_{ij}^k \frac{\partial}{\partial x^k}$$

and therefore

$$T^\nabla = T_{ij}^k dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k} = \frac{1}{2} T_{ij}^k dx^i \wedge dx^j \otimes \frac{\partial}{\partial x^k}$$

In particular, T_{ij}^k's are symmetric in i and j. In terms of Christoffel symbols, we write

$$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k.$$

Definition 1.3.8. A connection ∇ is called *symmetric* or *torsion-free* if the torsion T^∇ vanishes.

Remark 1.3.9. In a local chart (U, x^i), ∇ is torsion-free if and only if $\Gamma_{ij}^k = \Gamma_{ji}^k$ for all i, j.

Example 1.3.10.

1. $\mathbb{R}^n$ with ∇ determined by
$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$$
which is torsion-free.

2. For a Lie group G with ∇ determined by $\nabla_X Y = c[X, Y]$ for any $X, Y \in \mathfrak{X}_{\text{left-invariant}}(G)$. This connection has torsion:
$$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 2c[X, Y] - [X, Y] = (2c - 1)[X, Y].$$
Therefore, ∇ is torsion-free if either

- $\mathfrak{g}$ is abelian, i.e., $[X, Y] = 0$ for all $X, Y \in \mathfrak{X}_{\text{left-invariant}}(G) \simeq \mathfrak{g}$, or
- $\mathfrak{g}$ is arbitrary but $c = \frac{1}{2}$.

Remark 1.3.11. Given a connection, we can differentiate any tensor fields along a vector field.

- For a 1-form $\alpha \in \Omega^1(M)$, we construct a new 1-form
$$\nabla_X \alpha \in \Omega^1(M)$$
given by $(\nabla_X \alpha)(Y) = X(\alpha(Y)) - \alpha(\nabla_X Y)$. This is equivalent to the property
$$X(\langle \alpha, Y\rangle) = \langle \nabla_X \alpha, Y\rangle + \langle \alpha, \nabla_X Y\rangle$$
for $\langle \alpha, Y\rangle = \alpha(Y)$.
- In general, given any (p, q)-tensor, we think of it as a map
$$T : (\mathfrak{X}(M))^p \times (\Omega^1(M))^q \to C^\infty(M),$$
we define a (p, q)-tensor
$$\begin{aligned}&\nabla_X T : (\mathfrak{X}(M))^p \times (\Omega^1(M))^q \to C^\infty(M)\\ &X(T(Y_1, \dots, Y_p, \alpha_1, \dots, \alpha_q)) = (\nabla_X T)(Y_1, \dots, Y_p, \alpha_1, \alpha_q) + \sum_i T(Y_1, \dots, \nabla_X Y_i, \dots, Y_p, \alpha_1, \dots, \alpha_q)\\ &\qquad\qquad + \sum_i T(Y_1, \dots, Y_p, \alpha_1, \dots, \nabla_X \alpha_i, \dots, \alpha_q)\end{aligned}$$
where we think of $T \in C^\infty(M)$ so we get to apply X on T since $\mathfrak{X}(M)$ is the set of derivations $X : C^\infty(M) \to C^\infty(M)$.
In the notation that
$$T = T^{j_1,\dots,j_q}_{i_1,\dots,i_p} \alpha^{i_1} \otimes \cdots \otimes \alpha^{i_p} \otimes X_{j_1} \otimes \cdots \otimes X_{j_q},$$
then we can also rewrite $\nabla_X T$ in this form as well.

Definition 1.3.12. A connection ∇ is *compatible* with a Riemannian metric g if $\nabla_X g = 0$ for all $X \in \mathfrak{X}(M)$. We also just write $\nabla g = 0$.

Remark 1.3.13. Explicitly, $\nabla g = 0$ is equivalent to the statement that $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ for any $X, Y, Z \in \mathfrak{X}(M)$.

Exercise 1.3.14. Show that if ∇ is compatible with g and $\nabla_X Y = 0$, then $||Y||$ is constant along the orbit, i.e., the integral curves of the vector field X, that is, $X(||Y||^2) = 0$.

Theorem 1.3.15. Given a Riemannian manifold (M, g), there exists a unique torsion-free connection compatible with the Riemannian metric g.

Definition 1.3.16. The connection specified in Theorem 1.3.15 is called the *Levi-Civita connection* of (M, g).

Remark 1.3.17. Not all torsion-free connections the Levi-Civita connection of some Riemannian manifold.

Proof. Assuming ∇ satisfies $\nabla g = 0$ and $T^\nabla = 0$, we see that

- $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$,
- $Y(g(Z, X)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X)$, and
- $Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$,

therefore

$$\begin{aligned}X(g(Y,Z))+Y(g(Z,X))-Z(g(X,Y)) &= g(\nabla_X Y+\nabla_Y X,Z)+g(\nabla_X Z-\nabla_Z X,Y)+g(\nabla_Y Z-\nabla_Z Y,X)\\ &= g(2\nabla_X Y+[Y,X],Z)+g([X,Z],Y)+g([Y,Z],X),\end{aligned}$$

therefore

$$\begin{aligned}g(\nabla_X Y,Z)=\frac{1}{2}(&X(g(Y,Z))+Y(g(Z,X))-Z(g(X,Y))\\ &-g([Y,X],Z)-g([X,Z],Y)-g([Y,Z],X)).\end{aligned}$$

One can then check that this is the torsion-free connection we need: in particular, show that X and Y in $\nabla_X Y$ satisfies the properties of a connection. $\square$

End of Lecture 4

Remark 1.3.18. On a local chart (U,x^i), we see that

$$\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}=\Gamma_{ij}^k\frac{\partial}{\partial x^k},$$

by writing $X=\frac{\partial}{\partial x^i}$, $Y=\frac{\partial}{\partial x^j}$, and $Z=\frac{\partial}{\partial x^k}$, we have

$$\Gamma_{ij}^k g_{\ell k}=\frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i}+\frac{\partial g_{ki}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k}\right).$$

Example 1.3.19.

1. Consider $\mathbb{R}^n$ with $g_0=\sum_{i=1}^m (dx^i)^2$, the Levi-Civita connection is the flat connection given by $\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}=0$.

2. Let G be a Lie group, we have the torsion-free connection

$$\nabla_X Y=\frac{1}{2}[X,Y]$$

for any $X,Y\in\mathfrak{X}_{\text{left-invariant}}(G)$.

For any connection ∇, we know

$$\nabla_{fX}Y=f\nabla_X Y.$$

Therefore, suppose X_1 and X_2 agree at a point x, i.e., $X_1|_x=X_2|_x$, then

$$(\nabla_{X_1}Y)|_x=(\nabla_{X_2}Y)|_x.$$

Therefore, for any tangent vector $v\in T_xM$ and any tangent field Y defined in a neighborhood of x, $\nabla_v Y\in T_xM$ is a well-defined tangent vector of M at x.

Definition 1.3.20. Let $\gamma:[a,b]\to M$ be a path and $V:[a,b]\to TM$ be a vector field along γ, i.e., $V(t)\in T_{\gamma(t)}M$ for all $t\in[a,b]$, or

$$\begin{tikzcd} & TM \arrow[d, "\pi"] \\ {[a,b]} \arrow[ur, "V"] \arrow[r, "\gamma"'] & M \end{tikzcd}$$

then the *covariant derivative* of V along γ is the vector field $D_\gamma V$ along γ given by

$$(D_\gamma V)(t)=\nabla_{\dot\gamma(t)}\tilde V_t+\left.\frac{d}{dt}\tilde V_t\right|_{\gamma(t)},$$

where vector field $\tilde V_t\in\mathfrak{X}(M)$ is any time-dependent extension of V, i.e., it is smooth in both variables, such that $\tilde V_t(\gamma(t))=\tilde V(\gamma(t),t):=V(t)$.

Remark 1.3.21. In general, one needs time-dependent extensions since curve may intersect itself. That is, if there is a self-intersecting curve, the tangent vector at the intersection point may change, depending on the time variable.

Remark 1.3.22. The definition is independent of the choice of extension. We just need to check this in a local chart (U, x^i). Consider $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$, and vector field $V(t) = V^i(t)\ \frac{\partial}{\partial x^i}\big|_{\gamma(t)}$, and let $\dot{\gamma}(t) = \dot{\gamma}^i(t)\ \frac{\partial}{\partial x^i}\big|_{\gamma(t)}$. Given an extension

$$\tilde{V}_t = \tilde{V}^i(x,t)\frac{\partial}{\partial x^i},$$

with $\tilde{V}^i(\gamma(t),t) = V^i(t)$, we apply the formula and get

$$\begin{aligned}
(D_\gamma V)(t) &= \nabla_{\dot{\gamma}^i(t)\ \frac{\partial}{\partial x^i}|_{\gamma(t)}}\left(\tilde{V}^j(x,t)\frac{\partial}{\partial x^j}\right) + \frac{d}{dt}\left(\tilde{V}^i(x,t)\frac{\partial}{\partial x^i}\right)\Bigg|_{\gamma(t)} \\
&= \dot{\gamma}^i(t)\frac{\partial \tilde{V}^j(x,t)}{\partial x^i}\frac{\partial}{\partial x^j}\Bigg|_{\gamma(t)} + \dot{\gamma}^i(t)\tilde{V}^j(\gamma(t),t)\Gamma^k_{ij}(\gamma(t))\frac{\partial}{\partial x^k}\Bigg|_{\gamma(t)} + \frac{\partial \tilde{V}^j}{\partial t}(\gamma(t),t)\frac{\partial}{\partial x^j}\Bigg|_{\gamma(t)} \\
&= \frac{d}{dt}\left(\tilde{V}^j(\gamma(t),t)\right)\frac{\partial}{\partial x^j}\Bigg|_{\gamma(t)} + \dot{\gamma}^i(t)\tilde{V}^j(\gamma(t),t)\Gamma^k_{ij}(\gamma(t))\frac{\partial}{\partial x^k}\Bigg|_{\gamma(t)} \quad \text{by chain rule} \\
&= \left(\dot{\tilde{V}}^k(t) + \dot{\gamma}^i(t)V^j(t)\Gamma^k_{ij}(\gamma(t))\right)\frac{\partial}{\partial x^k}\Bigg|_{\gamma(t)} \qquad (1.3.23)
\end{aligned}$$

which is independent of the choice of extension.

Definition 1.3.24. Given a path $\gamma : [a,b] \to M$,

1. a vector field V along γ is *parallel* if $D_\gamma V(t) = 0$ for any $t \in [a,b]$,
2. γ is a *geodesic* if $\dot{\gamma}(t)$ is parallel along γ, that is, $(D_\gamma\dot{\gamma})(t) = 0$ for any $t \in [a,b]$.

Remark 1.3.25. Equation (1.3.23) now explains what a parallel vector field and a geodesic would be.

- A vector field V along γ is parallel if it satisfies the equation

$$\dot{V}^k(t) + \Gamma^k_{ij}(\gamma(t))\dot{\gamma}^i(t)V^j(t) = 0 \qquad (1.3.26)$$

  for any $t \in [a,b]$ and $k = 1, \ldots, m$.
- A geodesic satisfies the equation

$$\ddot{\gamma}^k(t) + \Gamma^k_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t) = 0 \qquad (1.3.27)$$

  for any $t \in [a,b]$. Note that we can also rewrite this a system of first-order differential equations given by

$$\begin{cases} \dot{\gamma}^k &= v^k(t) \\ \dot{v}^k(t) &= -\Gamma^k_{ij}(\gamma(t))v^iv^j \end{cases} \qquad (1.3.28)$$

  interpreted from manifold to tangent bundle.

Proposition 1.3.29. Given a connection ∇ and path $\gamma : [a,b] \to M$, for any tangent vector $v_0 \in T_{\gamma(a)}M$, there exists a unique parallel vector field V along γ such that $V(a) = v_0$.

Remark 1.3.30. Because Equation (1.3.26) is a first-order linear ordinary differential equation, then given a tangent vector at the beginning of the path, we can "parallel transport" it along the path, and get a tangent vector at the end of the path, which then gives a vector field.

Definition 1.3.31. The *parallel transport* along a path $\gamma : [a,b] \to M$ is

$$\begin{aligned} \tau_\gamma : T_{\gamma(a)}M &\to T_{\gamma(b)}M \\ v_0 &\mapsto V(b) \end{aligned}$$

where $V(t)$ is given by Proposition 1.3.29.

Remark 1.3.32. We see that the parallel transport is a linear isomorphism, due to the following properties of covariant derivative $D_\gamma V$ of V along γ.

- D is linear: $D_\gamma(V_1 + V_2) = D_\gamma V_1 + D_\gamma V_2$.
- D satisfies the Leibniz rule: $D_\gamma(fV) = fD_\gamma(V) + \langle df, \dot{\gamma}\rangle V$.

As opposed to Equation (1.3.26), Equation (1.3.27) is a second-order non-linear ordinary differential equation, which means the solution tends to appear only in small intervals of time. This draws the following result.

Proposition 1.3.33. Given ∇ on M and $v_0 \in T_{x_0}(M)$, there exists a unique maximal geodesic $\gamma : I \to M$ such that $\gamma(0) = x_0$ and $\dot{\gamma}(0) = v_0$, where I is an open interval containing 0.

Example 1.3.34. Consider $\mathbb{R}^n$ with flat connection ∇, a path γ is a geodesic if and only if $\ddot{\gamma}^i(t) = 0$ for all i, therefore the geodesics are straight lines.

Definition 1.3.35. A connection is *complete* if geodesics exist for all time.

Definition 1.3.36. A *geodesic* in a Riemannian manifold (M, g) is a geodesic for Levi-Civita connection.

Remark 1.3.37. Arbitrary connections on compact manifolds will not be complete. However, we will see that this will happen for Riemannian manifolds.

The following definition is motivated by Equation (1.3.28).

Definition 1.3.38. On (M, ∇), the *spray* $X^\nabla \in \mathfrak{X}(TM)$ in local coordinates (x^i, v^j) is given by

$$X^\nabla(x, v) = X^\nabla\big|_{x,v} = v^i \frac{\partial}{\partial x^i} - \Gamma^k_{ij} v^i v^j \frac{\partial}{\partial v^k}.$$

The flow of the spray is called the *geodesic flow*.

Remark 1.3.39. Let $p : TM \to M$ be the projection and $m_t(v) = tv$ be the multiplication. The spray as a vector field is the unique one satisfying

- $d_v p(X^\nabla_v) = v$, and
- $(m_t)_*(X^\nabla) = \frac{1}{t} X^\nabla$ for all $t \in \mathbb{R}_+$.

End of Lecture 5

Exercise 1.3.40. Show that

a. X^∇ is independent of the choice of local charts;

b. X^∇ satisfies two properties:
   - Given the projection of tangent bundle $p : TM \to M$, we have $d_v p(X^\nabla) = v$;
   - Given the multiplication $m_t(v) = tv$, we have $(m_t)_* X^\nabla = \frac{1}{t} X^\nabla$ for all $t > 0$;

c. any vector field $X \in \mathfrak{X}(TM)$ that satisfies part a. and b. is the spray of a connection ∇.

Remark 1.3.41. Note that Equation (1.3.27) or X^∇ only depend on the symmetric part of Γ^k_{ij}:

$$\Gamma^k_{ij}(x) v^i v^j = \frac{1}{2}\left(\Gamma^k_{ij}(x) + \Gamma^k_{ji}(x)\right) v^i v^j.$$

(Here we implicitly assume there is a summation going on, as it usually happens in Einstein notation.) Therefore, geodesics do not give a complete characterization for the torsion.

Proposition 1.3.42.

i. Given any connection ∇, there exists a unique connection $\bar{\nabla}$ that has the same geodesic as ∇, but is torsion-free, i.e., $T^{\bar{\nabla}} = 0$.

ii. Two connections ∇_1 and ∇_2 with the same geodesics and torsions coincide.

Proof. Given ∇, we can define a *dual connection* ∇^* by

$$\nabla^*_X Y = \nabla_Y X + [X, Y].$$

Indeed,

$$\begin{aligned}\nabla^*_{fX} Y &= \nabla_Y(fX) + [fX, Y] \\ &= f\nabla_Y X + Y(f)X + f[X, Y] - Y(f)X \\ &= f\nabla^*_X Y,\end{aligned}$$

and

$$\begin{aligned}\nabla^*_X(fY) &= \nabla_{fY} X + [X, fY] \\ &= f\nabla_Y X + f[X, Y] + X(f)Y \\ &= f\nabla^*_X Y + X(f)Y,\end{aligned}$$

so taking combinations give a connection $\bar{\nabla} = \frac{1}{2}(\nabla_X Y + \nabla^*_X Y)$, such that

$$T^{\bar{\nabla}}(X, Y) = 0.$$

□

Remark 1.3.43. Let $v \in T_x M$ and $\gamma_v : [0, b) \to M$ be the geodesic with $\dot{\gamma}(0) = v$. Take $\lambda > 0$, we have a parametrization $t \mapsto \gamma_{\lambda v}(t) \equiv \gamma(t)$, and

$$\begin{cases} (D_\gamma \dot{\gamma})(t) & = \lambda^2 (D_{\gamma_v} \dot{\gamma}_v)(\lambda t) = 0 \\ \dot{\gamma}(0) & = \lambda v \end{cases}$$

therefore $\gamma : \left[0, \frac{b}{\lambda}\right) \to M$ is a geodesic with $\dot{\gamma}(0) = \lambda v$, and in particular $\gamma = \gamma_{\lambda v}$. Therefore, if we choose v sufficiently small, we can choose γ so that the geodesic exists for $t \in [0, 1]$.

Definition 1.3.44. The *exponential map* is defined as

$$\begin{aligned}\mathrm{Exp}^\nabla : V &\to M \\ v &\mapsto \gamma_v(1)\end{aligned}$$

which exists in a neighborhood $0_M \subseteq V \subseteq TM$ containing the zero section 0_M. We denote Exp^M_x to be the map

$$\mathrm{Exp}^M_x : V \cap T_x M \to M$$

for $x \in M$.

Remark 1.3.45. The exponential map $\mathrm{Exp}^\nabla(t\cdot)$[2] cannot be a flow of a vector field, but if we take the flow of the geodesic spray, the diagram

$$\begin{tikzcd} TM \supseteq V \arrow[r, "\varphi^t_{X^\nabla}"] \arrow[dr, "\mathrm{Exp}^\nabla(t\cdot)"'] & TM \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

commutes.

[2]We write "$t\cdot$" to represent an one-parameter group of diffeomorphisms.

Theorem 1.3.46. Given $x \in M$, there exists an open neighborhood $0_x \in V \subseteq T_xM$ of x and an open neighborhood $x \in U \subseteq M$ such that $\operatorname{Exp}_x^\nabla : V \to U$ is a diffeomorphism.

Proof. We need to check the differential of exponential map around 0_x is zero, then we have such a construction. That is, we need to check that

$$d_{0_x} \operatorname{Exp}_x^\nabla : T_{0_x}(T_xM) \simeq T_xM \to T_xM$$

is a linear isomorphism, which as we will see, is actually the identity map. Indeed,

$$\begin{aligned} d_{0_x} \operatorname{Exp}_x^\nabla(v) &= \frac{d}{dt} \operatorname{Exp}_x^\nabla(tv)\Big|_{t=0} \\ &= \frac{d}{dt}\gamma_{tv}(1)\Big|_{t=0} \\ &= \frac{d}{dt}\gamma_v(t)\Big|_{t=0} \\ &= v. \end{aligned}$$

□

Definition 1.3.47. The local coordinates in the chart given by Theorem 1.3.46 are called the *normal coordinates* centered at $x \in M$, i.e.,

$$U \xrightarrow{\left(\operatorname{Exp}_x^\nabla\right)^{-1}} V \subseteq T_xM \xrightarrow{\simeq} \mathbb{R}^m$$

where we choose a basis $\{e_1, \dots, e_m\}$ for T_xM to get the isomorphism.

Remark 1.3.48. In normal coordinates centered at $x \in M$,

- geodesics through x correspond to straight lines,
- geodesics through $y \neq x$ are not, in general, straight lines.

1.4 Geodesics in Riemannian Geometry

Recall that geodesics for (M, g) are just the geodesics for the Levi-Civita connection ∇.

Lemma 1.4.1. Geodesics have constant velocity.

Proof. Let $\gamma : [a, b] \to M$ be a geodesic, then

- the derivative

$$\begin{aligned} \frac{d}{dt}||\dot\gamma(t)||^2 &= \frac{d}{dt} g(\dot\gamma(t), \dot\gamma(t)) \\ &= g(D_\gamma\dot\gamma(t), \dot\gamma(t)) + g(\dot\gamma(t), D_\gamma\dot\gamma(t)) \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

Note $\nabla g = 0$, therefore

$$\begin{aligned} g(Y, \nabla_X Z) &= X(g(Y, Z)) - g(\nabla_X Y, Z) \\ &= \nabla_X g(Y, Z) \\ &= 0. \end{aligned}$$

□

Definition 1.4.2. For a Riemannian manifold (M, g), let ∇ be the Levi-Civita connection, then for $x \in U \subseteq M$ and $0_x \in V \subseteq T_xM$, we have a diagram

$$\begin{tikzcd} U \arrow[r, "(\mathrm{Exp}_x^\nabla)^{-1}"] & V \arrow[r, hook] \arrow[d, hook] & \mathbb{R}^n \arrow[dl, "\simeq"] \\ & T_xM & \end{tikzcd}$$

after choosing orthonormal basis $\{e_1, \ldots, e_n\}$. The coordinates given by this diagram is called the *metric normal coordinates*. Given such g_x, we build up a local chart (U, x^i).

Remark 1.4.3. In this chart, writing $g = g_{ij}(x)dx^idx^j$ gives $g_{ij}(0) = \delta_{ij}$. At the origin, we have the Euclidean metric, but that is not true outside the origin. Instead, we get

$$\begin{aligned} g &= g_{ij}(x)dx^idx^j \\ &= \sum_{i=1}^m (dx^i)^2 + O(2) \end{aligned}$$

is of second-order in x. Indeed, the geodesics through $x = 0$ are assigned as $t \mapsto vt$ for $v \in \mathbb{R}^n$, therefore $\Gamma_{ij}^k(0) = 0$ and $\frac{\partial g_{ij}}{\partial x^k}(0) = 0$.

End of Lecture 6

Remark 1.4.4. If $\gamma : [a, b] \to M$ is a geodesic, then we know that $||\dot{\gamma}(t)||$ is constant. Now suppose we have

$$\begin{aligned} s : [a, b] &\to [0, L(\gamma)] \\ t &\mapsto \int_0^t ||\dot{\gamma}(t)||dt, \end{aligned}$$

then we get to write

$$s(t) = \frac{L(\gamma)}{b-a}(t-a)$$

is an affine function. Therefore, such repraparametrization $\gamma = \gamma(s)$ is still a geodesic.

In general, if we choose an arbitrary reparametrization $\tau : [0, d] \to [a, b]$, then $\gamma \circ \tau$ is not a geodesic. For instance, we can take $||\dot{\overline{\gamma \circ \tau}}|| = ||\dot{\gamma}(\tau(t))|| \cdot |\tau'|$ but this may not be constant.

We saw last time the notion of normal neighborhood U for (M, g) centered at $x_0 \in M$. This is given by

$$\begin{tikzcd} \mathbb{R}^n \simeq T_{x_0}M \supseteq V \arrow[r, "\mathrm{Exp}_{x_0}"] & U \subseteq M \arrow[l, bend left, "\varphi"] \end{tikzcd}$$

landing back in $\mathbb{R}^n$ after fixing an orthonormal basis for $T_{x_0}M$. We take up the following conventions.

- The *normal sphere* is denoted $S_\varepsilon(x_0) = \{x \in U : |\varphi(x)| = \varepsilon\}$.
- The *normal ball* is denoted $B_\varepsilon(x_0) = \{x \in U : |\varphi(x) < \varepsilon|\}$.

These notions don't depend on choices. In a normal chart, the metric is given by

$$\begin{aligned} g|_U &= g_{ij}(x)dx^idx^j \\ &= g_{ij}(0)dx^idx^j + \frac{\partial g_{ij}(0)}{\partial x^k}x^kdx^idx^j + \cdots \\ &= \sum_{i=1}^n (dx^i)^2 + O(2). \end{aligned}$$

In *"spherical" normal coordinates*, we write

$$\begin{cases} x^1 &= r\sin(\varphi^1) \\ x^2 &= r\cos(\varphi^1)\sin(\varphi^2) \\ \vdots & \\ x^{n-1} &= r\cos(\varphi^1)\cos(\varphi^2)\cdots\cos(\varphi^{n-2})\sin(\varphi^{n-1}) \\ x^n &= r\cos(\varphi^1)\cos(\varphi^2)\cdots\cos(\varphi^{n-1}) \end{cases}$$

for $r \in [0,\infty)$, $\varphi^1,\dots,\varphi^{n-2} \in [0,\pi]$, and $\varphi^{n-1} \in [0,2\pi]$.

Proposition 1.4.5. In spherical normal coordinates,

$$g = (dr)^2 + g_{ij}(r,\varphi^1,\dots,\varphi^{n-1})d\varphi^i d\varphi^j$$

where $g_{ij}(0,\varphi^1,\dots,\varphi^{n-1}) = 0$.

Proof. We have

$$g = g_{rr}(dr)^2 + g_{ri}drd\varphi^i + g_{ij}d\varphi^i d\varphi^J$$

where $g_{rr}(r,\varphi) = g\left(\frac{\partial}{\partial r}\right)$. Recall that the assignment $\gamma : t \mapsto tv$ is geodesic, and $\frac{\partial}{\partial r}\big|_v$ are the derivatives $\dot{\gamma}(t)$, and in particular $\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r} = 0$, then by computation we get

$$\begin{aligned} \frac{\partial}{\partial r}g_{rr} &= \frac{\partial}{\partial r}\left(g\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right)\right) \\ &= 2g\left(\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right) \\ &= 0. \end{aligned}$$

Therefore, g_{rr} is constant along the ray, thus $g_{rr}(0,\varphi) = 1$ since $g(0) = \sum_{i=1}^{m}(dx^i)^2$, which means $g_{rr}(r,\varphi) = 1$.

We now observe that the vector fields commute, i.e.,

$$\left[\frac{\partial}{\partial r},\frac{\partial}{\partial \varphi^i}\right] = 0,$$

and since $T^\nabla = 0$, thus

$$\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial \varphi} = \nabla_{\frac{\partial}{\partial \varphi}}\frac{\partial}{\partial r}.$$

By definition, we have

$$g_{ri} = g_r\left(\frac{\partial}{\partial r},\frac{\partial}{\partial \varphi^i}\right),$$

thus

$$\begin{aligned} \frac{\partial}{\partial r}g_{ri} &= g\left(\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r},\frac{\partial}{\partial \varphi^i}\right) + g\left(\frac{\partial}{\partial r},\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial \varphi^i}\right) \\ &= 0 + \frac{1}{2}\frac{\partial}{\partial \varphi^i}g\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right) \\ &= 0. \end{aligned}$$

This implies that g_{ri} is constant along the ray, hence

$$g_{ri}(r,\varphi) = g_{ri}(0,\varphi) = 0.$$

□

Corollary 1.4.6. If $\gamma : [0,1] \to M$ is any curve such that $\gamma(0) = x_0$ and $\gamma(1) \in S_x(x_0)$, then

$$L(\gamma) \geqslant \varepsilon$$

and equality holds if and only if γ is a reparametrization of a geodesic (but not necessarily one itself).

Remark 1.4.7. When points are close enough, i.e., contained in the normal neighborhood, the geodesics minimize the length.

Proof. We may assume that

- $\gamma(t) \neq x_0$ for all $t \in [0,1]$ by reparametrization, since this would not affect the length,
- the curve $\gamma(t) \subseteq U$ for some normal neighborhood U of x_0.

Using sphereical normal coordinates, we get

$$\begin{aligned}
L(\gamma) &= \int_0^1 ||\dot{\gamma}(t)|| dt \\
&= \int_0^1 \left(\dot{\gamma}^{r^2}(t) + g_{ij}(\dot{\gamma}^{\varphi_i}(t))\dot{\gamma}^{\varphi_j}(t)\right)^2 dt \\
&\geqslant \int_0^1 |\dot{\gamma}^r(t)| dt \\
&= \gamma^r(1) - \gamma^r(0) \\
&= \varepsilon.
\end{aligned}$$

In particular, the equality holds if and only if

$$\begin{cases} g_{ij}\dot{\gamma}^{\varphi_i}\dot{\gamma}^{\varphi_j} & = 0 \\ \dot{\gamma}^r(t) & > 0 \end{cases}$$

which gives

$$\begin{cases} \dot{\gamma}^{\varphi_i}(t) & = 0 \\ \dot{\gamma}^r(t) & > 0 \end{cases}$$

In this case, $\gamma^{\varphi_i}(t) = \varphi^i(0)$ is constant, then we have

$$\gamma(t) = \exp\left(\left(\gamma^r(t), \varphi^1(0), \ldots, \varphi^{n-1}(0)\right)\right)$$

given by the exponential map acting on a reparametrization of $t \mapsto \dot{\gamma}(0)t$. $\square$

What can we say when the sphere is huge?

Theorem 1.4.8. Suppose $\gamma : [a,b] \to M$ is a smooth curve such that $\gamma(0) = x$ and $\gamma(b) = y$, and for every piecewise smooth curve $\eta : [c,d] \to M$ with $\eta(c) = x$ and $\eta(d) = y$, one has

$$L(\eta) \geqslant L(\gamma),$$

then γ is a reparametrized geodesic.

Proof. If γ is contained in a normal neighborhood, we may apply Corollary 1.4.6. Otherwise, the intersection of γ with any normal neighborhood U satisfies the assumption of Theorem 1.4.8 for any values of parameter t such that for any $t \in [c,d]$, we have $\gamma(t) \in U$. We may then apply the local case again. $\square$

Question. Here is a rather open question. In a Riemannian manifold (M, g) with $(x, y) \in M$ fixed, are there geodesics connecting x and y? If yes, how many? In this case geodesics mean either unparametrized geodesics or ones up to reparametrization.

End of Lecture 7

Remark 1.4.9. The proof of Theorem 1.4.8 last time actually requires more than just having normal neighborhoods. What we need is a notion of totally normal neighborhoods.

Definition 1.4.10. A *totally normal neighborhood* $U \subseteq M$ is one such that for any $x \in U$, $U \subseteq B_\varepsilon(x)$ for some $\varepsilon > 0$.

Proposition 1.4.11. Totally normal neighborhoods always exists.

Proof. We define the geodesic flow

$$\begin{aligned} \Phi : \mathbb{R} \times TM \supseteq D &\to \mathbb{R} \times TM \\ (t, v) &\mapsto (t, \varphi^t_{X^\nabla}(v)) \end{aligned}$$

For any $x_0 \in M$, we have $(0, 0_{x_0}) \in D$, and Φ is a diffeomorphism on some open V containing this point. Therefore, there exists $0_{x_0} \in \bar{V} \subseteq T_{x_0}M$ and $\varepsilon > 0$ such that $[0, \varepsilon] \times \bar{v} \subseteq V$, which means that $U = \exp_{x_0}(V)$ is a totally normal neighborhood. $\square$

Corollary 1.4.12. Geodesics contained in a totally normal neighborhood are length-minimizing.

Example 1.4.13. Given two points, is there a geodesic connecting them? How many are there precisely?

1. Suppose $M = \mathbb{R}^n$ with standard Euclidean metric g_0. The geodesics are the straight lines, therefore any two points are connected by a unique geodesic.
2. Suppose $M = \mathbb{R}^n \backslash \{0\}$ with induced metric $g = g_0|_M$. Note that the points x and $-x$ are not connected by a geodesic.
3. Suppose $M = \mathbb{S}^n \subseteq \mathbb{R}^{n+1} \backslash \{0\}$ with induced metric $g = g_0|_{\mathbb{S}^n}$. In this case, the geodesics are maximal circles. To see why,
   - take $v \in T_x\mathbb{S}^n$ and let $\gamma_v(t)$ be the geodesic with $\dot{\gamma}_v(0) = v$;
   - isometries take geodesics to geodesics;
   - let H be the 2-plane containing x and v, then set $r : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ to be the reflection on H, which is an isometry on $\mathbb{R}^{n+1}$ with g_0;
   - in particular, $r = r|_{\mathbb{S}^n} : \mathbb{S}^n \to \mathbb{S}^n$ is still an isometry, and in particular $\gamma(t) = r \circ \gamma_v(t)$ is a geodesic;
   - but note that γ satisfies

     $$\begin{cases} \gamma(0) &= x \\ \dot{\gamma}(0) &= dr \circ \dot{\gamma}_v(0) = v \end{cases}$$

     which means that $\gamma = \gamma_v$.

   Therefore, for any $x, y \in \mathbb{S}^n$ that are not antipodal, i.e., $y \neq -x$, there are two geodesics containing x and y. In the case where $y = -x$, there are infinitely many geodesics connecting x and y.
4. Suppose $M = \mathbb{T}^n$ and let $g = (d\theta^1)^2 + \cdots + (d\theta^m)^2$, then any two points are connected by infinitely many geodesics.

Definition 1.4.14. A geodesically-complete Riemann manifold is a Riemannian manifold (M, g) such that every maximal geodesic $\gamma : I \to M$ has $I = \mathbb{R}$.

Theorem 1.4.15 (Hopf-Rinow)**.** Given a Riemannian manifold (M, g), the following are equivalent:

i. (M, g) is geodesic-complete;

ii. (M, d) is a complete metric space, where d is the metric induced by length;

iii. there exists a point $p \in M$ such that the exponential map $\exp_x$ has domain the entire tangent space $T_x M$.

Moreover, if any of the conditions above holds, then for any $x, y \in M$, there exists a geodesic connecting x and y, with $d(x, y) = L(\gamma)$.

Remark 1.4.16. By completeness, the last condition is actually true for any point $p \in M$.

Corollary 1.4.17. Every compact Riemannian manifold is geodesically-complete.

Proof. Any compact metric space is complete. □

Corollary 1.4.18. A closed embedded submanifold N of a geodesically-complete Riemannian manifold (M, g) can be upgraded to a geodesically-complete Riemannian manifold $(N, g|_N)$.

Proof. If $\gamma : [a, b] \to M$ is a smooth curve with $\gamma(t) \in N$ for all $t \in [a, b]$, then since N is embedded, γ must also be a smooth curve in N. Therefore, for any $x, y \in N$, $d_M(x, y) \leqslant d_N(x, y)$, so a Cauchy sequence in N must be a Cauchy sequence in M. In particular, it has a subsequence that converges in M. Since N is closed and embedded, it convergences in N. Therefore, (N, d_N) is a complete metric space as well. □

Remark 1.4.19. We still keep the implicit assumption that Riemannian manifolds are connected: otherwise we may not have paths connecting two points.

We will now prove Theorem 1.4.15.

- The proof of i. implying iii. is obvious: this follows from the definition of the exponential map.
- To prove iii. implies ii., we require the following lemma.

  Lemma 1.4.20. If condition iii. of Theorem 1.4.15 holds, then for each $x, y \in M$, there exists a geodesic γ connecting x and y such that $L(\gamma) = d(x, y)$.

  To prove ii., we will show that having $K \subseteq M$ bounded and closed implying K is compact. Since K is bounded, then we have $K \subseteq B_R(x) = \{y \in M : d(x, y) \leqslant R\}$ for some R. By Lemma 1.4.20, for any $y \in K$, there exists a geodesic $\gamma : [a, b] \to M$ such that $\gamma(a) = x$ and $\gamma(b) = y$, with $L(\gamma) = d(x, y)$. Therefore, $K \subseteq \exp_x(\{v \in T_x M : ||v|| \leqslant R\})$, because the domain of $\exp_x$ is the entire tangent space. Note that $\{v \in T_x M : ||v|| \leqslant R\}$ is compact, and since $\exp_x$ is continuous, then $\exp_x(\{v \in T_x M : ||v|| \leqslant R\})$ is compact as well. Being a closed subset of a compact set, we note that K is compact as well.
- To prove that ii. implies i., let $\gamma : [a, b) \to M$ be a geodesic. Assume for now that $b < \infty$, then we have an increasing sequence $\{t_n\}_{n \geqslant 1}$ converging to b. The geodesic γ has the property $||\dot{\gamma}(t)|| = c$, so by reparametrization $s = \frac{t}{c}$, we assume $||\dot{\gamma}(s)|| = 1$. Therefore, we have

$$\begin{aligned} d(\gamma(t_n), \gamma(t_{n+1})) &\leqslant L(\gamma|_{[t_n, t_{n+1}]}) \\ &= \int_{t_n}^{t_{n+1}} ||\dot{\gamma}(t)|| dt \\ &= t_{n-1} - t_n \\ &\xrightarrow[n \to \infty]{} 0 \end{aligned}$$

  Since the sphere of radius 1 is compact, and $||\dot{\gamma}(t_m)|| = 1$, then there exists a converging subsequence $\dot{\gamma}(t_{n_k}) \to v$. In particular, $(\gamma(t)_{n_k}, \dot{\gamma}(t_{n_k}))$ converges, therefore $(\gamma(t), \dot{\gamma}(t))$ is an integral curve X^∇ that is bounded as $t \to b$, hence $(\gamma(t), \dot{\gamma}(t))$ exists in the interval $[a, b + \varepsilon)$ for some ε. But that means (a, b) is not maximal, contradiction. Therefore, $b = \infty$. Similar proof shows that $a = -\infty$.

End of Lecture 8

We omitted the proof of Lemma 1.4.20 in class, but we record it here for completeness.

Supplement. *Let $\rho = d(x, y)$. Choose $0 < \varepsilon < \rho$ such that $S_\varepsilon(x)$ is a normal sphere. This is compact, therefore there exists $x_0 \in S_\varepsilon(x)$ such that*

$$d(x_0, y) = \min\{d(z, y) : z \in S_\varepsilon(x_0)\}.$$

By the definition of a normal sphere, there exists some $v \in T_xM$ such that $||v|| = 1$ and $\exp_x(\varepsilon v) = x_0$. We claim that $y = \exp_x(\rho v)$, therefore $\gamma(t) = \exp_x(tv)$ is the desired geodesic. To prove this, let

$$A = \{t \in [0, \rho] : d(\exp_x(tv), y) = \rho - t\} \subseteq \mathbb{R}.$$

Since the assignment $f : t \mapsto d(\exp_x(tv), y) + t$ is continuous, then $A = f^{-1}(\rho)$ is closed, bounded, and non-empty since $0 \in A$. Therefore, A has a maximum. If we can show that any $t_0 \in [0, \rho)$ is not a maximum, then ρ must be a maximum. If that is the case, then $\rho \in A$, thus $d(\exp_x(\rho v), y) = 0$, hence $\exp_x(\rho v) = y$.

We may assume, towards contradiction, that $t_0 = \max(A) \in [0, \rho)$, and set $y_0 = \exp(t_0 v)$, then choose $\delta \in (0, \rho - t_0)$ such that $S_\delta(y_0)$ is a normal sphere. Let z_0 be such that $d(z_0, y) = \min\{d(z, y) : z \in S_\delta(y_0)\}$. It suffices to show that $z_0 = \exp_x((t_0 + \delta)v)$, and $d(z_0, y) = \rho - (t_0 + \delta)$, then $t_0 \neq \max(A)$. We note that

$$\begin{aligned} \rho - t_0 &= d(y_0, y) \\ &= \delta + \min_{z \in S_\delta(y_0)} d(z, y) \\ &= \delta + d(z_0, y), \end{aligned}$$

therefore

$$\begin{aligned} d(x, z_0) &\geqslant d(x, y) - d(z_0, y) \\ &= \rho - d(z_0, y) \\ &= t_0 + \delta. \end{aligned}$$

Set $z_0 = \exp_{y_0}(\delta w)$ for some w, then the curve

$$\exp_{x, 0<t<t_0}(tv) \cup \exp_{x, 0<t<\delta}(tw)$$

has length $t_0 + \delta$. In particular, this must be a reparametrized geodesic, therefore $\gamma(t) = \exp_x(tv)$ is a geodesic through z_0, so $\exp_x((t_0 + \delta)v) = z_0$, as desired.

Remark 1.4.21.

1. How do we count geodesics? We can show that in a complete Riemannian manifold (M, g), in every path-homotopy class, there exists a geodesic that minimizes the length among all curves in the class.
2. Every manifold admits a geodesically-complete metric. In fact, given any metric g, there exists a geodesically-complete metric $g' = fg$ for some non-negative function $0 < f \in C^\infty$. We say that g' is *conformal* to g, and f is a *conformal factor*. That is, in every conformal class, there exists a geodesically-complete metric.
3. Instead of curves, what about other submanifolds, e.g., minimizing surfaces? This is more of the current research.

1.5 Curvature

We saw that on normal charts, a metric g starts with a Euclidean metric with error term has of order at least 2. How do we measure this higher-order term, i.e., the failure of g being locally equivalent to Euclidean metric? More generally, how can we measure the failure of the connection ∇ being locally equivalent to the standard connection with $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = 0$?

Definition 1.5.1. The *curvature* of a connection ∇ is

$$R^\nabla(X, Y)(Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for $X, Y, Z \in \mathfrak{X}(M)$.

Remark 1.5.2.

- This is a $(3,1)$-tensor, i.e., 3-covariant, 1-contravariant tensor, given by
$$\begin{aligned}\mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M)\times\Omega^1(M) &\to C^\infty(M)\\ (X,Y,Z,\alpha) &\mapsto \langle R^\nabla(X,Y)Z,\alpha\rangle\end{aligned}$$
and is $C^\infty(M)$-linear in each entry.
- There is an assignment
$$\begin{aligned}TM &\to TM\\ (X,Y) &\mapsto R^\nabla(X,Y)\end{aligned}$$
such that $R^\nabla(X,Y)=-R^\nabla(Y,X)$ for $R^\nabla\in\Omega^2(M,\mathrm{End}(TM))$.
- For Euclidean connection ∇, $R^\nabla\equiv 0$.

Theorem 1.5.3 (Bianchi's Identity). If $T^\nabla=0$, then the cyclic permutations of (X,Y,Z)
$$R^\nabla(X,Y)Z+\mathrm{cycPerm}(X,Y,Z):=R^\nabla(X,Y)Z+R^\nabla(Y,Z)X+R^\nabla(Z,X)Y=0.$$

Proof. Note that $T^\nabla=0$ if and only if $\nabla_X Y-\nabla_Y X=[X,Y]$, therefore we may compute
$$R^\nabla(X,Y)Z+\mathrm{cycPerm}(X,Y,Z)=[X,[Y,Z]]+\mathrm{cycPerm}(X,Y,Z)=0$$
by Jacobi identity. □

In a local chart (U,x^i), we have
$$R^\nabla\big|_U=R^\ell_{ijk}(x)dx^i\otimes dx^j\otimes dx^k\otimes\frac{\partial}{\partial x^\ell}$$
where $R^\ell_{ijk}=\left\langle R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k},dx^\ell\right\rangle$ with the property that

- $R^\ell_{ijk}=-R^\ell_{jik}$, and
- $R^\ell_{ijk}+R^\ell_{jki}+R^\ell_{kij}=0$.

We now give a geometric interpretation of the curvature, using covariant derivative along paths. Denote
$$\gamma:[0,1]\times[0,1]\to M$$
parametrized by (t,ε), then we get two families of disjoint curves intersecting each other that parametrizes the surface, given by
$$\begin{aligned}\gamma_\varepsilon:[0,1]&\to M\\ t&\mapsto\gamma_\varepsilon(t)\end{aligned}$$
and
$$\begin{aligned}\gamma_t:[0,1]&\to M\\ \varepsilon&\mapsto\gamma_t(\varepsilon)\end{aligned}$$
This allows us to define a vector field along γ
$$c:[0,1]\times[0,1]\to TM$$
where $c(t,\varepsilon)\in T_{\gamma(t,\varepsilon)}M$.

Proposition 1.5.4. For a connection ∇, the covariant derivative

$$T^{\nabla}(\dot{\gamma}_{\varepsilon}, \dot{\gamma}_t) = D_{\gamma_\varepsilon}\dot{\gamma}_t - D_{\gamma_t}\dot{\gamma}_\varepsilon,$$

and

$$R^{\nabla}(\dot{\gamma}_\varepsilon, \dot{\gamma}_t)c = D_{\gamma_\varepsilon} D_{\gamma_t} c - D_{\gamma_t} D_{\gamma_\varepsilon} c.$$

Remark 1.5.5. Here $\dot{\gamma}_t$ is a vector field along γ_ε so we get to derive it, and the other way around.

We will postpone the proof of Proposition 1.5.4 because it will be a mess: taking a time-dependent parametrization will introduce a third variable, c.f., [Spi70]. Instead, after learning about pullback connections of vector bundles, we will come back to this.

Corollary 1.5.6. If ∇ is flat, i.e., $R^{\nabla} \equiv 0$, then parallel transport is invariant under path-homotopy. That is, if $\gamma_0 \sim \gamma_1$ is a path-homotopy, then $\tau_{x_0} = \tau_{x_1}$.

Proof. Let us take $\gamma : [0,1] \times [0,1] \to M$ be a path-homotopy between γ_0 and γ_1. We will assume that γ is C^∞: any C^0 path-homotopy can then by approximated by the smooth homotopies. We now have

$$\begin{cases} \gamma_0(t) = \gamma(t,0), & \gamma(0,\varepsilon) = x_0 \\ \gamma_1(t) = \gamma(t,1), & \gamma(1,\varepsilon) = x_0 \end{cases}.$$

Given a tangent vector $v_0 \in T_{x_0}M$, we define

$$\begin{aligned} c : [0,1] \times [0,1] &\to TM \\ (t,\varepsilon) &\mapsto \tau^t_{\gamma_\varepsilon}(v_0), \end{aligned}$$

which is equivalent to saying $D_{\gamma_\varepsilon} c = 0$. Since $c(0,\varepsilon) = v_0$, then

$$D_{\gamma_0(\varepsilon) = \gamma_{t=0}} c = 0,$$

and we want to show that $c(1,\varepsilon)$ is constant, i.e., $D_{\gamma_{t=1}} c = 0$. Because $R^{\nabla} = 0$, then

$$D_{\gamma_\varepsilon} D_{\gamma_t} c = D_{\gamma_t} D_{\gamma_\varepsilon} c = 0.$$

In particular, $D_{\gamma_{t=1}} c = 0$. □

Example 1.5.7. We can give an example where Corollary 1.5.6 fails if we remove the assumption of path-homotopy invariance. Take $M = \mathbb{T}^2$ and connection ∇ with

$$\begin{cases} \nabla_{\frac{\partial}{\partial x^1}} dx^1 = \nabla_{\frac{\partial}{\partial x^2}} dx^1 = 0 \\ \nabla_{\frac{\partial}{\partial x^1}} dx^2 = dx^2, \nabla_{\frac{\partial}{\partial x^2}} dx^2 = dx^2 = dx^1. \end{cases}$$

and consider the monodromy $\ell^{\nabla} : \pi_1(M, x_0) \to \mathrm{GL}(T_{x_0}M)$.

End of Lecture 9

Note that the curvature for connection ∇ is defined by

$$R^{\nabla}(X,X) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Definition 1.5.8. For a Riemannian manifold (M,g), the *Riemannian curvature tensor* is the 4-covariant tensor

$$R(x,y,z,w) = g(R^{\nabla}(X,Y)(Z),W).$$

In local coordinates (U, x^i), we can write

$$R^\nabla = R^\ell_{ijk} dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^\ell}$$

for

$$R^\ell_{ijk} = \left\langle R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}\right\rangle$$

and

$$R = R_{ijk\ell} dx^i \otimes dx^j \otimes dx^k \otimes dx^\ell$$

for

$$R_{ijk\ell} = R\left(\partial_{x^i}, \partial_{x^j}, \partial_{x^k}, \partial_{x^\ell}\right) = g(R(\partial_{x^i}, \partial_{x^j})\partial_{x^k}, \partial_{x^\ell}) = g_{\ell m} R^m_{ijk}$$

where we write $\partial_{x^j} = \frac{\partial}{\partial x^j}$.

We have the following symmetries of R.

Proposition 1.5.9.

i. Bianchi's Identity: $R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0$.

ii. $R(X, Y, Z, W) = -R(Y, X, Z, W)$.

iii. $R(X, Y, Z, W) = -R(X, Y, W, Z)$.

iv. $R(X, Y, Z, W) = R(Z, W, X, Y)$.

Proof. We have already seen that the first two are true. We will prove iii. and iv.

iii. It is enough to show that $R(X, Y, Z, Z) = 0$ by polarity: if this holds, then

$$\begin{aligned} 0 &= R(X, Y, Z + W, Z + W) \\ &= R(X, Y, Z, Z) + R(X, Y, Z, W) + R(X, Y, W, Z) + R(X, Y, W, W) \\ &= 0 + R(X, Y, Z, W) + R(X, Y, W, Z) + 0 \\ &= R(X, Y, Z, W) + R(X, Y, W, Z). \end{aligned}$$

Since $\nabla g = 0$, then

$$\begin{cases} X(g(\nabla_Y Z, Z)) &= g(\nabla_X \nabla_Y Z, Z) + g(\nabla_Y Z, \nabla_X Z) \\ [X, Y](g(Z, Z)) &= 2g(\nabla_{[X,Y]} Z, Z). \end{cases}$$

Therefore,

$$\begin{aligned} R(X, Y, Z, Z) &= g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, Z) \\ &= X(g(\nabla_Y Z, Z)) - Y(g(\nabla_X Z, Z)) - \frac{1}{2}[X, Y]g(Z, Z) \\ &= \frac{1}{2}X(Y(g(Z, Z))) - \frac{1}{2}Y(X(g(Z, Z))) - \frac{1}{2}[X, Y](g(Z, Z)) \\ &= 0. \end{aligned}$$

iv. Apply Bianchi's identity (with appropriate signs) four times to ii. and iii.

□

Remark 1.5.10. We also have a point of view that characterize the curvature as an operator. By ii. and iii. of Proposition 1.5.9, we have

$$\tilde{\rho} : \Lambda^2 T_p M \otimes \Lambda^2 T_p M \to \mathbb{R}$$

$$(X \wedge Y, Z \wedge W) \mapsto R(X, Y, Z, W)$$

By iv., $\tilde{\rho}$ is a symmetric bilinear form on the vector space $\Lambda^2 T_pM$. Now g_p induces inner product on $\Lambda^2 T_pM$:

$$g_p(X \wedge Y, Z \wedge W) := \det \begin{pmatrix} g_p(X, Z) & g_p(X, W) \\ g_p(Y, Z) & g_p(Y, W) \end{pmatrix}$$

Therefore, we have the *curvature operator* $\rho : \Lambda^2 TM \to \Lambda^2 TM$ defined by

$$\Lambda^2 T_pM \xrightarrow{\tilde{\rho}} (\Lambda^2 T_pM)^* \xrightarrow{g_p} \Lambda^2 T_pM$$

where we use the metric to identify the dual with the vector space.

A different incarnation of curvature would be the sectional curvature.

Definition 1.5.11. The *sectional curvature* of a 2-plane generated by $v, w, \in T_pM$ is given by

$$K_p(v \wedge w) := \frac{R(v, w, w, v)}{||v \wedge w||^2} = \frac{R(v, w, w, v)}{g(v, v)g(w, w) - g(v, w)^2}.$$

Remark 1.5.12. This is not a linear map in v and w, but it associates a 2-plane with a real number. The 2-planes give a Grassmannian in the tangent bundle, therefore we can think of this as $K : \mathrm{Gr}_2(TM) \to \mathbb{R}$.

Proposition 1.5.13. The sectional curvature completely determines the Riemannian curvature tensor.

Proof. This is proven from the following observations.

- If R_1 and R_2 are tensors satisfying all the symmetries in Proposition 1.5.9, then so does their difference $R_1 - R_2$.
- If R satisfies Proposition 1.5.9 and
$$R(X, Y, Y, X) = 0$$
for all vector fields X, Y, then $R = 0$.

The first observation is obvious. We will prove the second observation using polarity. We have

$$\begin{aligned} 0 &= R(X + Z, Y, Y, X + Z) \\ &= R(X, Y, Y, X) + R(X, Y, Y, Z) + R(Z, Y, Y, X) + R(Z, Y, Y, Z) \\ &= 0 + R(X, Y, Y, Z) + R(Z, Y, Y, X) + 0 \\ &= R(X, Y, Y, Z) + R(Z, Y, Y, X) \\ &= 2R(X, Y, Y, Z), \end{aligned}$$

thus $R(X, Y, Y, Z) = 0$. We then take

$$\begin{aligned} 0 &= R(X, Y + Z, Y + Z, W) \\ &= R(X, Y, Y, W) + R(X, Y, Z, W) + R(X, Z, Y, W) + R(X, Z, Z, W) \\ &= 0 + R(X, Y, Z, W) + R(X, Z, Y, W) + 0 \\ &= R(X, Y, Z, W) + R(X, Z, Y, W), \end{aligned}$$

hence $R(X, Y, Z, W) = R(Z, X, Y, W)$. By Bianchi's identity,

$$R(X, Y, Z, W) = R(Y, Z, X, W) = R(Z, X, Y, W) = 0.$$

$\square$

Remark 1.5.14. If $\dim(M) = 2$, then there is only one 2-plane, hence the Grassmannian is canonically $\mathrm{Gr}_2(TM) \simeq M$, and the sectional curvature becomes a function $K : M \to \mathbb{R}$, known as the *Gaussian curvature*. Again, this function completely determines the Riemannian curvature tensor in the case of $\dim(M) = 2$, then

$$R(X, Y, Z, W) = -K(g(X, Z)g(Y, W) - g(X, W)g(Y, Z)) \tag{1.5.15}$$

since we compute the Riemannian curvature tensor of $g(X, Z)g(Y, W) - g(X, W)g(Y, Z)$ to be -1. This ensures the 2-sphere has a positive sectional curvature.

Definition 1.5.16. A Riemannian manifold (M, g) is *isotropic* at a point $p \in M$ if the sectional curvature K_p at the point p is constant. We say (M, g) has *constant curvature* if it is isotropic and sectional curvature does not depend on p.

Remark 1.5.17. For any isotropic (M, g), the Riemannian curvature tensor is given by Equation (1.5.15).

Exercise 1.5.18. If $\dim(M) \geqslant 3$, then (M, g) being isotropic (at every point) implies constant curvature.

Example 1.5.19.

1. $\mathbb{R}^n$ with the flat metric $g_0 = \sum_{i=1}^{n} (dx^i)^2$ has $R \equiv 0$, therefore it has constant curvature 0.

2. Consider the n-sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\} \hookrightarrow \mathbb{R}^{n+1}$ with $g = g_0|_{\mathbb{S}^n}$. The orthonormal group $\mathrm{SO}(n+1)$ acts on $\mathbb{R}^{n+1}$ by isometries, but the action also preserves the sphere: $\mathrm{SO}(n+1)$ acts on $\mathbb{S}^n$ by isometries. For instance, fix the north pole $p = (0, \dots, 0, 1) \in \mathbb{S}^n$, then the isometry group or stabilizer group is given by
$$\mathrm{SO}(n+1)_p = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in \mathrm{SO}(n) \right\}.$$
Therefore, $\mathrm{SO}(n+1)_p \simeq \mathrm{SO}(n)$ gives an action on $T_p\mathbb{S}^n$ preserving the inner product g_p at the point p. As a vector space, $T_p\mathbb{S}^n$ is just $\mathbb{R}^n$, then the action acts transitively on 2-planes of $T_p\mathbb{S}^n$. In particular, K_p is constant (which can be calculated to be 1) at every point $p \in M$, which means $\mathbb{S}^n$ is of constant curvature 1.

   More generally, taking a sphere of radius R gives radius $\frac{1}{R}$ for $\mathbb{S}^n_R$.

We see that a submanifold with restricted metric can have different curvature from the manifold.

End of Lecture 10

Remark 1.5.20. If $\dim(M) \geqslant 3$, then $K(P)$ is the Gaussian curvature of $\exp(P)$, where P is the span by vectors.

Example 1.5.21.

1. Consider the hyperbolic space $\mathbb{H}^n = \{x \in \mathbb{R}^{n+1} : (x, x) = -1\}$ with Minkowski bilinear form $(v, w) = -v^0w^0 + \sum_{i=1}^{n} v^iw^i$. This is not inner product: it is not positive-definite. To get a bilinear form, we note there is an inclusion $i : \mathbb{H}^n \hookrightarrow \mathbb{R}^{n+1}$, where on $\mathbb{R}^{n+1}$ we take the metric
$$g = -(dx^0)^2 + \sum_{i=1}^{m} (dx^i)^2$$
as a symmetric bilinear tensor, then we pullback along i to get a Riemannian metric $(\mathbb{H}^n, i^*g)$.

   Exercise 1.5.22. Check that $T_x\mathbb{H}^n = \{v \in \mathbb{R}^{n+1} : (x, v) = 0\}$, and $g|_{T_x\mathbb{H}^n}$ is actually positive-definite, which gives a Riemannian metric i^*g.

   The linear transformations $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, such that $(Ax, Ay) = (x, y)$ for all x, y with $\det(A) = 1$, is denoted by $\mathrm{SO}(n, 1)$. We denote $G := \mathrm{SO}(n, 1)^\circ$ to be its connected component of the identity. This admits a (transitive) $\mathrm{SO}(n, 1)^\circ$-action on $\mathbb{H}^n$ by isometries of the hyperbolic space. Fixing a point $x \in \mathbb{H}^n$, we note that G_x is the isotropy group acting on $(T_x\mathbb{H}^n, g_x)$ by isometries. This action is still transitive on 2-planes, therefore it has constant sectional curvature.

   Remark 1.5.23. Applying the stereographic projection, we get a disk
$$D = \{x \in \mathbb{R}^n : |x| < 1\}$$
with a metric
$$g_D = \frac{4((dx^1)^2 + \cdots + (dx^n)^2)}{(1 - ||x||^2)^2}.$$
The isometry group is still the group G denoted above, just with a different notion of action on the disk model.

2. Suppose G is a Lie group with bi-invariant metric, i.e., a metric that is both left-translation invariant and right-translation invariant, then we have a Levi-Civita connection

$$\nabla_X Y = \frac{1}{2}[X,Y]$$

for left-invariant vectors fields $X, Y \in \mathfrak{X}_{\text{left-invariant}}(G) \simeq \mathfrak{g}$. One can compute $R^\nabla(X,Y)Z = \frac{1}{4}[Z,[X,Y]]$ in terms of left-invariant vector fields. Since the metric is bi-invariant, then

$$g([Z,X],Y) + g(X,[Z,Y]) = 0$$

and therefore the Riemannian curvature tensor is

$$R(X,Y,Z,W) = g(R^\nabla(X,Y)Z,W) = -\frac{1}{4}g([X,Y],[Z,W]).$$

We may then calculate the sectional curvature to be non-negative since

$$R(X,Y,Y,X) = \frac{1}{4}||[X,Y]||^2.$$

The sectional curvature may not be constant. For abelian Lie group, this is indeed constant.

Definition 1.5.24. We define the *Ricci curvature* $\mathrm{Ric}(Y,Z)$ to be the trace of Riemannian curvature using the metric g. That is, if we consider the assignment

$$\begin{aligned} TM &\to TM \\ X &\mapsto R^\nabla(X,Y)Z \end{aligned}$$

this is a linear transformation. In particular, every linear map has a trace, therefore the precise definition would be

$$\mathrm{Ric}(Y,Z) = \mathrm{tr}(X \mapsto R^\nabla(X,Y)Z).$$

This is a symmetric covariant tensor.

Remark 1.5.25. Being a symmetric covariant tensor,

1. Ric is completely determined by the quadratic $Q(x) = \mathrm{Ric}(X,X)$ by the polarity argument;
2. in dimension 2 (or in the isotropic case: $\mathrm{Ric}(X,Y) = K(n-1)g(X,Y)$ for $n = \dim(M)$), we have

$$\mathrm{Ric}(X,Y) = Kg(X,Y).$$

following from Equation (1.5.15);

3. one can show that $Q(x) = \mathrm{Ric}(X,X)$ is the average of sectional curvature $K(P)$ for $X \subseteq P \subseteq T_xM$. Therefore, Ricci curvature does not determine the curvature tensor or the sectional curvature in general (i.e., for $\dim(M) \geqslant 3$, but this in fact still holds for $\dim(M) = 3$).
4. By definition, the Ricci curvature only depends on the connection, so it is defined more generally than the Riemannian manifolds.

In the remark above, we notice that the Ricci curvature is proportional to the metric.

Definition 1.5.26. An *Einstein metric* g is one such that

$$\mathrm{Ric} = cg$$

where c is the cosmological constant.

Remark 1.5.27. If M has constant sectional curvature, then $c = K(n-1)$ where $n = \dim(M)$. Therefore, constant sectional curvature implies Einstein metric, but not the other way around, e.g., *Fubini-Study metric* in $\mathbb{CP}^n$, c.f., Example 1.6.4 and Remark 1.6.5.

Given a bilinear form $\mathrm{Ric} : T_xM \times T_xM \to \mathbb{R}$, we get a mapping

$$\begin{aligned} T_xM &\to T_x^*M \\ v &\mapsto \mathrm{Ric}_x(v, -). \end{aligned}$$

This defines a mapping L_x via

$$T_xM \longrightarrow T_x^*M \xrightarrow{g} T_xM$$

Definition 1.5.28. The *scalar curvature* of (M, g) is defined by to be

$$\begin{aligned} S : M &\to \mathbb{R} \\ x &\mapsto \mathrm{tr}_g(\mathrm{Ric}_x) = \mathrm{tr}(L_x). \end{aligned}$$

In terms of local coordinates, we may express

$$\mathrm{Ric} = R_{ij} dx^i \otimes dx^j$$

where

$$R_{ij}(x) = R^\ell_{\ell ij}(x) = g^{\ell m} R_{\ell m ij},$$

and we take the convention

$$dx^i dx^j = \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i),$$

therefore $g^{\ell m}$ is just the inverse of $g = g_{ij} dx^i dx^j$. With this, we can now write

$$S(x) = g^{ij} R_{ij}$$

for the scalar curvature.

End of Lecture 11

Remark 1.5.29. Let us recall the following notions of curvature we have seen so far.

- The Riemann curvature tensor R, corresponding to the sectional curvature K.
- The Ricci tensor Ric, given by $\mathrm{Ric}(Y, Z) = \mathrm{tr}(X \mapsto R^\nabla(X, Y)Z)$.
- The scalar tensor $S = \mathrm{Tr}_g \mathrm{Ric}$.

The natural question being, why these tensors specifically? This is because the curvature tensor R has symmetries $S^2(\Lambda^2 V)$ that encode all of them except Bianchi's identity, and to encode this identity, we have a map

$$\begin{aligned} L : S^2(\Lambda^2 V) &\to \Lambda^4 V \\ L(R)(X, Y, Z, W) &= R(X, Y, Z, W) + \mathrm{cycPerm}(X, Y, Z) \end{aligned}$$

This motivates us to examine $\ker(L)$. The $O(m)$-action on $V = (T_xM, g_x)$ gives an action on $S^2(\Lambda^2 V)$, which in turn lifts into an action on curvature tensor $R \in \ker(L)$. We may then decompose the $O(m)$-action on $\ker(L)$ into irreducible subspaces

$$\ker(L) = V_0 \oplus V_1 \oplus V_2.$$

The corresponding decomposition of R is the following:

$$\begin{aligned} R(X, Y, Z) = &- \frac{S}{m(m-1)} \left(g(X, Y)g(Y, W) - g(X, W)g(Y, Z)\right) \\ &- \frac{1}{m-2} \left(\mathrm{Ric}_0(X, Z)g(Y, W) + \mathrm{Ric}_0(Y, W)g(X, Z) - \mathrm{Ric}_0(X, W)g(Y, Z) - \mathrm{Ric}_0(Y, Z)g(X, W)\right) \\ &+ W(X, Y, Z, W) \end{aligned}$$

where $\mathrm{Ric}_0(X, Y) = \mathrm{Ric}(X, Y) - \frac{1}{m} S g(X, Y)$ is the *traceless Ricci tensor*. We see that the three components corresponding to V_0, V_1, and V_2, and they are called the scalar curvature component, traceless Ricci component, and the Weyl tensor.

Remark 1.5.30.

- When the scalar curvature component is 0, R is called scalar flat.
- When the traceless Ricci component is 0, R is called Einstein.
- When both the traceless Ricci component and the Weyl tensor is 0, R is said to be isotropic.

1.6 Quotients and Isometry Groups

Given a Riemannian manifold (M, g) and a surjective submersion $\Phi : M \to N$, then M is like a quotient of N by some smooth equivalence relation. How do we build up an induced metric on N using the quotient and g? The general answer would be no, but we would like to understand when we can get one.

Suppose q is a point in N, with tangent space T_qN, then via $d_p\Phi$, it corresponds to $\Phi^{-1}(q)$ upstairs, where p is a point upon it. Other than the tangent space T_pM, we can look at the orthogonal complement $H_p = (\ker(d_p\Phi))^\perp$ as well, which gives an isomorphism

$$d_p\Phi : H_p = (\ker(d_p\Phi))^\perp \simeq T_qN,$$

so we would like to build up the metric on T_qN. The issue being, there are multiple points in the fiber. This motivates the following definition.

Definition 1.6.1. A Riemannian submersion is a submersion

$$\Phi : (M, g) \to (N, \bar{g})$$

such that

$$\bar{g}_{\Phi(p)}(d_p\Phi(v), d_p\Phi(w)) = g_p(v, w)$$

for all $v, w \in (\ker(d_p\Phi))^\perp$. This definition does not depend on $\bar{g}$: it is completely determined by the structure on g.

Remark 1.6.2. This is not just a pullback. This definition is talking about the inverse of the metric, i.e., given on the cotangent space. That is, given the corresponding injective map

$$(d_p\Phi)^* : T^*_{\Phi(p)}N \to T^*_pN$$

with metric g_p^{-1} downstairs, and the theorem says the corresponding metric matches.

Here is one way of getting a Riemannian submersion.

Theorem 1.6.3. Let G be a Lie group that acts on (M, g) properly and freely, and by isometries. By the assumption, the orbit space is a manifold, then the map to the orbit space $\pi : M \to M/G$ is a Riemannian submersion for a unique Riemannian metric $\bar{g}$ on M/G.

Proof. Set $N = M/G$, then the fibers of $q \in M/G$ are exactly given by the orbit of the action, i.e.,

$$\pi^{-1}(q) = O.$$

Fix $k \in G$, then

$$\begin{aligned} \Psi_k : M &\to M \\ x &\mapsto k \cdot x \end{aligned}$$

In particular, it sends the orbit into itself, i.e., $\Psi_k(O) \subseteq O$, and moreover, $d_p\Psi_k : T_pM \to T_{k\cdot p}M$ is an isometry. Note that the tangent space $T_pO = \ker(d_p\pi)$ by definition of the orbit space, then $d_p\Psi_k(T_pO) = T_{k\cdot p}O$, thus it must map the orthogonal to the orthogonal:

$$d_p\Psi_k((T_pO)^\perp) = (T_{k\cdot p}O)^\perp,$$

and in particular it maps the metric restricted to the orthogonal to the metric restricted to the orthogonal, i.e., preserves restriction $g|_{TO}$ for every $k \in G$. Therefore, π is a Riemannian submersion. $\square$

Example 1.6.4.

1. Let G be a Lie group with a right-invariant Riemannian metric[3], and let $H \subseteq G$ be a closed subgroup. Now H acts on G via right translations $h \cdot k = kh^{-1}$. Therefore, this action is given by isometries, because G is right-invariant. By Theorem 1.6.3, there exists a unique metric $\bar{g}$ on G/H such that $\pi : (G, g) \to (G/H, \bar{g})$ is a Riemannian submersion.

2. We now apply this to construct a metric on $M = \mathbb{CP}^n$. This is the quotient $\mathbb{C}^{n+1}/\mathbb{C}^*$, given by lines on $\mathbb{C}^{n+1}$ with identification by non-zero multiplication. Since $\mathbb{C}^{n+1} \simeq \mathbb{R}^{2n+2}$, we equip it with a Euclidean metric g_0. The issue being, $\mathbb{C}^* \simeq \mathbb{R}_+ \times \mathbb{S}^1$ is given by dilations and rotations, but the dilations on $\mathbb{R}^{2n+2}$ will not be isometries, i.e., does not preserve the inner products, thus g_0 is not dilation invariant. Instead, we think of $\mathbb{CP}^n$ as a sphere, then it is only given by a $\mathbb{S}^1$-action, i.e., $\mathbb{CP}^n \simeq \mathbb{S}^{2n+1}/\mathbb{S}^1$. In this case, $(\mathbb{S}^{2n+1}, g_{\mathbb{S}^{2n+1}} := g_0|_{\mathbb{S}^{2n+1}})$ has an $\mathbb{S}^1$-action by isometries, which gives the *Fubini-Study metric* $(\mathbb{CP}^n, g_{\mathbb{CP}^n})$.

Remark 1.6.5. Here are some properties of the Fubini-Study metric.

1. This is an Einstein metric: $\mathrm{Ric} = (2n+1)g_{\mathbb{CP}^n}$. This implies having constant scalar curvature S.

2. However, the sectional curvature is not constant (when $n > 1$):

$$K(P) = 1 + 3g_{\mathbb{CP}^n}(X, JY)^2$$

where $\{X, Y\}$ is an orthonormal basis of the plane P, and J is the complex structure given at every point, i.e., $J_p : T_pM \to T_pM$ is such that $J_p^2 = -I$. For $n = 1$, the curvature is constant.

Note that all the structures we are considering here are over $\mathbb{R}$. The analog of sectional curvature over $\mathbb{C}$ is the holomorphic sectional curvature, and in which case we take $Y = JX$ on the complex line P, and in that case we have constant holomorphic sectional curvature 4. In particular, we verify that statement of Remark 1.5.27: Einstein metric does not necessarily have constant sectional curvature.

Lemma 1.6.6. Let (M, g) be a Riemannian manifold and $X \in \mathfrak{X}(M)$, then the following are equivalent:

1. $\varphi_X^t : M \to M$ is a local isometry;
2. $L_X g = 0$;
3. $g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$ for all $Y, Z \in \mathfrak{X}(M)$.

Definition 1.6.7. A vector field X satisfying any condition in Lemma 1.6.6 is called a *Killing vector field* or an *infinitesimal isometry* of (M, g). We denote $\mathfrak{X}(M, g) \subseteq \mathfrak{X}(M)$ to be the linear subspace of Killing vector fields.

End of Lecture 12

Proof.

1. $\iff$ 2.: Note that

$$\begin{aligned}
(\varphi_X^t)^* g = g &\iff \frac{d}{dt}(\varphi_X^t)^* g = 0 \\
&\iff (\varphi_X^t)^*(L_X g) = 0 \\
&\iff L_X g = 0.
\end{aligned}$$

1. $\iff$ 3.: We have

$$\begin{aligned}
(L_X g)(Y, Z) = 0 &\iff X(g(Y, Z)) = g(L_X Y, Z) + g(Y, L_X Z) \\
&\iff X(g(Y, Z)) = g([X, Y], Z) + g(Y, [X, Z]) \\
&\qquad\qquad\qquad\quad = g(\nabla_X Y - \nabla_Y X, Z) + g(Y, \nabla_X Z - \nabla_Z X) \\
&\iff X(g(Y, Z)) - g(\nabla_X Y - Z) - g(Y, \nabla_X Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X) \\
&\iff g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0
\end{aligned}$$

where the last equivalence holds since $(\nabla_X g)(Y, Z) = 0$.

∎

[3] A Lie group always has a right-invariant structure. We can also construct the left-invariant structure instead, but not a bi-invariant one, c.f., Homework 1.

Remark 1.6.8. $\mathfrak{X}(M,g)$ is a Lie subalgebra of $\mathfrak{X}(M)$:

$$L_{[X,Y]} = L_X L_Y - L_Y L_X.$$

Corollary 1.6.9. Let G be a connected Lie group, then a G-action on (M,g) is by isometries if and only if the *infinitesimal action*

$$\rho : \mathfrak{g} \to \mathfrak{X}(M)$$

takes values in $\mathfrak{X}(M,g)$.

Remark 1.6.10. Given a point on the corresponding Lie algebra, the infinitesimal action is really defined as in the proof below: we take a one-parameter family of Lie group, therefore taking the derivative at 0 we get the infinitesimal action which gives the tangent space at the point: in this case we retrieve the vector field in $\mathfrak{X}(M)$.

Proof. G is generated by elements of the form $\exp(X)$ with $X \in \mathfrak{g}$. Since $\rho(X)$ is the infinitesimal action

$$\rho(X)|_X = \frac{d}{dt}\exp(-tx)\cdot x|_{t=0} = \frac{d}{dt}\,\varphi^t_{\rho(X)}(X)\Big|_{t=0},$$

then we can apply Lemma 1.6.6. □

Definition 1.6.11. We define the *group of isometries* to be

$$I(M,g) \equiv \{\varphi : (M,g) \to (M,g) \mid \varphi^* g = g\}.$$

This has a natural topology generated by open sets

$$V(K,U) = \{\varphi \in I(M,g) : \varphi(K) \subseteq U\}$$

for compact K and open U, which is the *compact-open topology*.

Remark 1.6.12. It is not hard to show that

$$\begin{aligned} I(M,g) \times I(M,g) &\to I(M,g) \\ (\varphi,\psi) &\mapsto \varphi \circ \psi \end{aligned}$$

and

$$\begin{aligned} I(M,g) &\to I(M,g) \\ \varphi &\mapsto \varphi^{-1} \end{aligned}$$

are continuous under the given topology. Therefore, the group of isometries is a topological group. Moreover, one can show that this is a finite-dimensional Lie group.

Theorem 1.6.13 (Myers-Steenrod)**.** For Riemannian manifold (M,g), the group $I(M,g)$ is a finite-dimensional Lie group, and the $I(M,g)$-action on M is a proper action. Moreover, if (M,g) is complete, then the corresponding Lie algebra of $I(M,g)$ is the Killing vector field $\mathfrak{X}(M,g)$.

Remark 1.6.14.

- Given a fixed point x, since the action is proper, then the isotropy group $I(M,g)_x$ at x is a compact Lie group.
- If (M,g) is not complete, then in general the corresponding Lie algebra is strictly contained in $\mathfrak{X}(M,g)$.
- if (M,g) is compact, then it is complete and $I(M,g)$ is a compact Lie group.

Theorem 1.6.15. Let G be a Lie group and G acts on M properly and *effectively*, i.e., given G-action on X, the kernel of the given map $G \to \Sigma(X)$ is trivial, then there exists a Riemannian metric g such that the action is by isometries. Therefore, G can be identified with a subgroup of $I(M,g)$ for some g.

1.7 Cartan's Structure Equations

Definition 1.7.1. A *local frame* in a manifold M over an open set $U \subseteq M$ is a family of vector fields $\{X_1, \dots, X_n\} \subseteq \mathfrak{X}(U)$ such that for any $x \in U$,

$$\{X_1|_x, \dots, X_n|_x\}$$

is a basis for T_xM.

Dually, a *local coframe* in a manifold M over an open set $U \subseteq M$ is a family of vector fields $\{\omega^1, \dots, \omega^n\} \subseteq \mathfrak{X}(U)$ such that for any $x \in U$,

$$\{\omega^1|_x, \dots, \omega^n|_x\}$$

is a basis for T_x^*M.

Remark 1.7.2.

- Any frame determines a coframe by duality, and vice versa, by

$$\omega^i(X_j) = \delta^i_j$$

  given by the Kronecker δ-function.

- Local frames and coframes always exist: given a chart (U, φ_i), we get the vector fields associated to the chart

$$\left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$$

  and the dual components

$$\{dx^1, \dots, dx^n\}.$$

  But globally they may not exist: for instance, for $M = \mathbb{S}^2$, the global vector fields do not exist, c.f., hairy ball theorem.

- For any local frame, the Lie brackets satisfy

$$[X_i, X_j] = c^k_{ij} X_k$$

  for some $c^k_{ij} \in C^\infty(U)$. Dually, the de Rham differentials satisfy

$$d\omega^k = -\frac{1}{2} c^k_{ij} \omega^i \wedge \omega^j$$

  for local coframes.

Given a connection ∇, fixing a local frame (with dual coframe) gives

$$\nabla_{X_i} X_j = \Gamma^k_{ij} X_k$$

for some functions $\Gamma^k_{ij} \in C^\infty(U)$. In local charts, this gives the definition of the Christoffel symbols. It is worth noting that the frames may not commute in general, therefore we do not always have to take the usual local frame/coframe as given in Remark 1.7.2. More precisely, if $[X_i, X_j] \neq 0$, then if $T^\nabla = 0$, we have $\Gamma^k_{ij} \neq \Gamma^k_{ji}$.

Now consider the 1-forms

$$\omega^k_j = \Gamma^k_{ij} \omega^i \in \Omega^1(U)$$

on U, so in terms of matrices, we have

$$[\omega^k_j] \in \Omega^1(U, \mathfrak{gl}_m(\mathbb{R}))$$

as a *connection* 1*-form*. Therefore, we always have

$$\nabla_Z X_j = \omega^k_j(Z) X_k.$$

This encodes the information of the vector field locally in terms of frames. Now we can characterize the torsion and curvature in a similar way.

$$T^\nabla(X_i, X_j) = T^k_{ij} X_k$$

for $T_{ij}^k \in C^\infty(U)$. We can then define

$$\theta^k = \frac{1}{2} T_{ij}^k \omega^i \wedge \omega^j$$

so θ is a family of vector-valued 2-forms on U, inducing the *torsion 2-form* $[\theta^k] \in \Omega^2(U, \mathbb{R})$. Moreover,

$$T^\nabla(X, Y) = \theta^k(X, Y) X_k.$$

End of Lecture 13

Recall that we have the following setting.

- Given (M, ∇), we fix a local frame $\{X_1, \dots, X_m\}$ with dual coframe $\{\omega^1, \dots, \omega^m\}$ over $U \subseteq M$.
- $\nabla_{X_i} X_j = \Gamma_{ij}^k X_k$ for $\Gamma_{ij}^k \in C^\infty(U)$. Set $\omega_j^k = \Gamma_{ij}^k \omega^i$, then $[\omega_j^k] \in \Omega^1(U; \mathfrak{gl}_m)$ is known as the connection 1-form. Therefore, $\nabla_Y X_j = \omega_j^k(Y) X_k$ by definition.
- Set $T(X_i, X_j) = T_{ij}^k X_k$, then the torsion 2-forms $[\theta^k] \in \Omega^2(U, \mathbb{R}^n)$ are defined by $\theta^k = \frac{1}{2} T_{ij}^k \omega^i \wedge \omega^j$. Therefore $T(Y, Z) = \theta^k(Y, Z) X_k$.

We can now write $R^\nabla(X_i, X_j) X_k = R_{ijk}^\ell X_\ell$, then we define the *curvature 2-form* to be $[\Omega_k^\ell] \in \Omega^2(U, \mathfrak{gl}_m)$, defined as

$$\Omega_k^\ell = \frac{1}{2} R_{ijk}^\ell \omega^i \wedge \omega^j.$$

Proposition 1.7.3. *Cartan's structural equations* are then defined by

$$d\omega^i = -\omega_j^i \wedge \omega^j + \theta^i \tag{1.7.4}$$

and

$$d\omega_j^i = -\omega_k^i \wedge \omega_j^k + \Omega_j^i \tag{1.7.5}$$

In particular, this gives

$$\begin{cases} d\omega &= -[\omega_j^i] \wedge \omega + \theta \\ d[\omega_j^i] &= [\omega_j^i] \wedge [\omega_j^I] + \Omega \end{cases}.$$

Proof. We have

$$\begin{aligned} d\omega^i(X_r, X_s) &= X_r(\omega^i(X_s)) - X_s(\omega^i(X_r)) - \omega^i([X_r, X_s]) \\ &= \omega^i(T(X_r, X_s) - \nabla_{X_r} X_s + \nabla_{X_s} X_r) \\ &= \omega^i(\theta^k(X_r, X_s) X_k - \omega_r^k(X_s) X_k + \omega_s^k(X_r) X_k) \\ &= \theta^i(X_r, X_s) - \omega_r^k(X_s) + \omega_s^k(X_r) \\ &= \theta^i(X_r, X_s) - \omega_j^i(X_s) \omega^j(X_k) + \omega_j^i(X_r) \omega^j(X_s) \\ &= (\theta^i - \omega_j^i \wedge \omega^j)(X_s, X_r). \end{aligned}$$

Moreover,

$$\begin{aligned} \nabla_{X_r} \nabla_{X_s} X_j &= \nabla_{X_r}(\omega_j^i(X_s) X_i) \\ &= X_r(\omega_j^i(X_s)) X_i + \omega_j^k(X_s) \omega_k^i(X_r) X_i, \end{aligned}$$

but

$$\nabla_{X_s} \nabla_{X_r} X_j = X_s(\omega_j^i(X_r)) X_i + \omega_j^k(X_r) \omega_k^i(X_s) X_i,$$

and

$$\nabla_{[X_s,X_r]}X_j = -\omega^i_j([X_s,X_r])X_i.$$

Therefore,

$$\begin{aligned}\nabla_{X_r}\nabla_{X_s}X_j - \nabla_{X_s}\nabla_{X_r}X_j - \nabla_{[X_s,X_r]}X_j &= (X_r(\omega^i_j(X_s))X_i - X_s(\omega^i_j(X_r))X_i - \omega^i_j([X_s,X_r])X_i)\\ &\quad + (\omega^k_j(X_s)\omega^i_k(X_r)X_i - \omega^k_j(X_r)\omega^i_k(X_s)X_i)\\ &= d\omega^i_j(X_r,X_s)X_i + \omega^k_j \wedge \omega^i_k(X_s,X_r)X_i\\ &= R(X_r,X_s)X_j\\ &= \Omega^i_j(X_r,X_s)X_i.\end{aligned}$$

□

Remark 1.7.6. For Levi-Civita connection ∇ of (M,g), Equation (1.7.4) corresponds to

$$d\omega^i = -\omega^i_j \wedge \omega^j, \tag{1.7.7}$$

and Equation (1.7.5) corresponds to

$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j + \Omega^i_j. \tag{1.7.8}$$

Moreover, the metric corresponds to

$$dg_{ij} = g_{ik}\omega^k_j + g_{kj}\omega^k_i \tag{1.7.9}$$

where $g_{ij} = g(X_i,X_j)$. But on a metric, we get to take an orthonormal frame instead of an ordinary one, and in such cases g_{ij} is constant, therefore Equation (1.7.9) becomes

$$\omega^i_j + \omega^j_i = 0.$$

In particular, this means $[\omega^i_j] \in \Omega^1(U,\mathfrak{so}_m)$, where $\mathfrak{so}_m = \{A \in \mathfrak{gl}_m : A + A^t = 0\}$, that is, the skew-symmetric matrices in the orthogonal group. Moreover, in that case Equation (1.7.8) becomes

$$\Omega^i_j + \Omega^j_i = 0,$$

therefore $[\Omega^i_j] \in \Omega^2(U,\mathfrak{so}_m)$.

Proof. We will now prove that Equation (1.7.9) holds. This is given by

$$\begin{aligned}dg_{ij}(X_r) &= X_r(g_{ij})\\ &= X_r(g(X_i,X_j))\\ &= g(\nabla_{X_r}X_i,X_j) + g(X_i,\nabla_{X_r}X_j)\\ &= g(\omega^k_i(X_r)X_k,X_j) + g(X_i,\omega^k_j(X_r)X_k)\\ &= g_{kj}\omega^k_i(X_r) + g_{ik}\omega^k_j(X_r).\end{aligned}$$

□

What happens if we are dealing with a surface with a metric?

Corollary 1.7.10. If (M,g) is 2-dimensional, then for any orthonormal coframe, we get

$$\begin{cases} d\omega^1 &= -\Omega^1_2 \wedge \omega^2\\ d\omega^2 &= \omega^1_2 \wedge \omega^1\\ d\omega^1_2 &= \Omega^1_2 \end{cases}$$

where

$$[\omega^i_j] = \begin{pmatrix} 0 & \omega^1_2\\ -\omega^1_2 & 0\end{pmatrix}$$

and

$$[\Omega^i_j] = \begin{pmatrix} 0 & \Omega^1_2 \\ -\Omega^1_2 & 0 \end{pmatrix}.$$

Because this is a coframe, then the 2-form can be written as a function in terms of $\omega^1 \wedge \omega^2$ as the unique 2-form, i.e., $d\omega^1_2 = \Omega^1_2 = K\omega^1 \wedge \omega^2$, where K is the Gaussian curvature, also known as the sectional curvature. In particular, if $\{X_i\}$ is orthonormal, then

$$K = R(X_1, X_2, X_2, X_1),$$

so

$$\Omega^1_2 = K\omega^1 \wedge \omega^2.$$

In this case,

$$R(X, Y, Z, W) = K(g(X, Z)g(Y, W) - g(X, W)g(Y, Z)).$$

Remark 1.7.11. In the case where $g_{\mathbb{S}^2_R} = R^2(\sin^2\varphi(d\theta)^2 + (d\varphi)^2)$, then

$$\begin{cases} \omega^1 & = R\sin\varphi d\theta \\ \omega^2 & = Rd\varphi \end{cases}.$$

In the orthonormal coframe, $\omega^1_2 = \cos\varphi d\varphi$, therefore $d\omega^1_2 = \frac{1}{R^2}\omega^1 \wedge \omega^2$, and therefore $K = \frac{1}{R^2}$.

End of Lecture 14

Recall:

- let (M, g) be a Riemannian manifold, then consider the frame (and corresponding coframe) $\{X_1, \ldots, X_n\}$ and $\{\theta^1, \ldots, \theta^n\}$ for $\theta \in \Omega^1(U, \mathbb{R}^n)$ that is orthogonal over $U \subseteq M$, then we have
- the connective 1-form $\omega = [\omega^i_j] \in \Omega^1(U, \mathfrak{so}_m)$, and
- the curvature 1-form $\Omega = [\Omega^i_j] \in \Omega^2(U, \mathfrak{so}_n)$.
- We then saw that the structural equations hold:

$$\begin{cases} d\theta^i & = -\omega^i_j \wedge \theta^j \\ d\omega^i_j & = \omega^i_k \wedge \omega^k_j + \Omega^i_j \end{cases}$$

  or correspondingly,

$$\begin{cases} d\theta & = -\omega \wedge \theta \\ d\omega & = -\omega \wedge \omega + \Omega \end{cases}$$

We have not discussed the corresponding *Bianchi's identity*.

Proposition 1.7.12. There are two Bianchi's identities,

- the first Bianchi's identity: $\Omega \wedge \theta = 0$;
- the second Bianchi's identity: $d\Omega = \Omega \wedge \omega - \omega \wedge \Omega$.

Proof. By differentiating the first structural equation, we get

$$\begin{aligned} 0 &= d^2\theta \\ &= -d\omega \wedge \theta + \omega \wedge d\theta \\ &= \omega \wedge \omega \wedge \theta - \Omega \wedge \theta - \omega \wedge \omega \wedge \theta \\ &= -\Omega \wedge \theta \end{aligned}$$

by applying the two structural equations. Similarly, differentiating the second structural equation gives

$$\begin{aligned} 0 &= d^2\omega \\ &= -d\omega \wedge \omega + \omega \wedge d\omega + d\Omega \\ &= \omega \wedge \omega \wedge \omega - \Omega \wedge \omega - \omega \wedge \omega \wedge \omega + \omega \wedge \Omega + d\Omega \\ &= d\Omega + \omega \wedge \Omega - \Omega \wedge \omega. \end{aligned}$$

□

Exercise 1.7.13. Check that the first Binachi's identity is equivalent to Theorem 1.5.3:

$$R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0.$$

Remark 1.7.14. One can actually check that these are identities in the global sense.

Corollary 1.7.15. If (M, g) is isotropic, then

$$\Omega_i^j = K\theta^i \wedge \theta^j$$

where K is the sectional curvature.

Proof. This is a rephrasing of curvature for isotropic manifolds. Note that

$$R^\nabla(X, Y)Z = \Omega_k^\ell(X, Y)\theta^k(Z)X_\ell.$$

For orthonormal frames, we may compute

$$R^\nabla(X_i, X_j)X_k = \Omega_k^\ell(X_i, X_j)X_\ell,$$

so

$$R(X_i, X_j, X_k, X_\ell) = \Omega_k^\ell(X_i, X_j)$$

by contraction, which is really just a Kronecker delta function depending on choices of i and j. For isotropic Riemannian manifold (M, g), we know the curvature is given by

$$R(X, Y, Z, W) = -K(g(X, Z)g(Y, W) - g(X, W)g(Y, Z),$$

so for orthonormal frames, we get

$$R(X_i, X_j, X_k, X_\ell) = -K(\delta_i^k\delta_j^\ell - \delta_j^\ell\delta_j^k) = \Omega_k^\ell(X_i, X_j),$$

which is equivalent to saying

$$\Omega_k^\ell = K\omega^\ell \wedge \omega^k.$$

□

Exercise 1.7.16. Check that if dimension is at least 3, then K must be constant.

Example 1.7.17. Consider

$$i : \mathbb{S}_R^2 = \{(x, y, z) : x^2 + y^2 + z^2 = R^2\} \hookrightarrow \mathbb{R}^3$$

For spherical coordinates, we get

$$g_{\mathbb{S}^2} = i^*g_0 = R^2(\sin^2\varphi(d\theta)^2 + (d\varphi)^2)$$

in the usual spherical coordinate system (R, φ, θ). We then get

- $X_1 = \frac{1}{R\sin\varphi}\frac{\partial}{\partial\theta}$, $X_2 = \frac{1}{R}\frac{\partial}{\partial\varphi}$;
- $\theta^1 = R\sin\varphi d\theta$, and $\theta^2 = Rd\varphi$

as frames and coframes. Since we are in dimension 2, then the manifold is isotropic, and

$$[\omega_j^i] = \begin{bmatrix} 0 & \omega_2^1 \\ -\omega_2^1 & 0 \end{bmatrix}$$

and

$$\Omega_2^1 = \begin{bmatrix} 0 & \Omega_2^1 \\ \Omega_2^1 & 0 \end{bmatrix}$$

and by the structural equations, we have

$$\begin{aligned} d\theta^1 &= -\omega_2^1 \wedge \theta^2 \\ d\theta^2 &= \omega_2^1 \wedge \theta^1 \end{aligned}$$

By directly differentiating the coframes, we get

$$\begin{aligned} d\theta^1 &= -R\cos\varphi d\theta \wedge d\varphi \\ d\theta^2 &= 0, \end{aligned}$$

which forces $\omega_2^1 = R\cos\varphi d\theta$, $\theta^2 = d\varphi$, and $\omega_2^1 \wedge \theta^1 = 0$. Moreover, we have $d\omega_2^1 = \Omega_2^1$, therefore by comparing with the differentiation, we get

$$d\omega_2^1 = R\sin\varphi d\theta \wedge d\varphi = \Omega_2^1$$

Because we are in dimension 2, then the manifold is isotropic, so by Corollary 1.7.15,

$$\Omega_2^1 = K\theta^1 \wedge \theta^2,$$

hence $K = \frac{1}{R^2}$. This gives

$$d\omega_2^1 = \frac{1}{R^2} R\sin\varphi d\theta \wedge Rd\varphi,$$

hence

$$\theta^1 = R\sin\varphi d\theta$$

and

$$\theta^2 = Rd\varphi.$$

Given a Riemannian manifold (M, g) with constant curvature. Let $x \in M$ be a point, then there is an exponential map as follows: for $0_x \in V \subseteq T_xM$, there is some open subset $x \in U \subseteq M$ such that

$$\exp_x : V \xrightarrow{\simeq} U.$$

Pick a basis $\{e_1, \dots, e_n\}$ for T_xM, we can do parallel transport for straight lines in V, and since we can reach any point by the exponential map, we then get a frame $\{X_1, \dots, X_n\}$ over U. We can use this frame to write down the structural equations, but because the curvature is constant, we can write down a much more simplified version of structural equations in this frame. Eventually, the exponential map $\exp_x$ is a local isometry between constant-curvature metrics in $\mathbb{R}^n$ (respectively, $\mathbb{S}_K^n$, H_K^n). After even a bit more work, we have the following global conclusions.

Theorem 1.7.18 (Killing-Hopf)**.** Let (M, g) be a complete Riemannian manifold of constant curvature, then

i. if M is simply connected, then there exists an isometry between M and
   - $(\mathbb{R}^n, g_0)$, if $K = 0$;
   - $(\mathbb{S}_K^n, g_{\mathbb{S}^n})$, if $K > 0$;
   - $(\mathbb{H}_K^n, g_{\mathbb{H}_K^n})$, if $K < 0$.

   This gives a precise classification.

   In the case where M is not simply connected, we recover the classification by quotients: recall that the action of fundamental group on the manifold is induced by the deck transformations on the universal covering, via concatenation of paths, then

ii. if $\pi_1(M) = \Gamma$, then M is isometric to a quotient of the form $\tilde{M}/\Gamma$, with Γ acting freely and properly on $\tilde{M}$ by isometries, and where $\tilde{M}$ is one of the constant-curvature model spaces mentioned above.

We now discuss the change of frames and coframes. Consider two frames $(U, X_1, \dots, X_m)$ and $(\bar{U}, \bar{X}_1, \dots, \bar{X}_m)$, with two dual coframes $(U, \theta^1, \dots, \theta^m)$ and $(\bar{U}, \bar{\theta}^1, \dots, \bar{\theta}^m)$, such that $U \cap \bar{U} \neq \varnothing$. Therefore,

$$\begin{cases} \bar{X}_i &= X_k A_i^k \\ \bar{\theta}^i &= A_k^i \theta^k \end{cases},$$

where $A = [A_i^k] : U \cap \bar{U} \to O(m)$. Now further assuming the frames/coframes are orthogonal, then we have $AA^T = A^T A = I$, so we write

$$\begin{cases} \bar{X} &= XA \\ \bar{\theta} &= A^T\theta \end{cases}$$

compactly.

Proposition 1.7.19.

- $\bar{\omega} = A^T \omega A + A^T dA$ (where $\bar{\omega}_j^i = A_k^i \omega_\ell^k A_j^\ell + A_k^i dA_j^k$);
- $\bar{\Omega} = A^T \Omega A$.

Proof. Since $A^T A = I$, then $\theta = A\bar{\theta}$, and by differentiation,

$$(dA)^T A + A^T dA = 0.$$

We have

$$\begin{aligned} d\bar{\theta} &= (dA)^T \wedge \theta + A^T d\theta \\ &= (dA)^T \wedge A\bar{\theta} - A^T(\omega \wedge \theta) \\ &= (dA)^T \wedge \bar{\theta} - A^T(\omega \wedge A\bar{\theta}) \\ &= -(A^T dA + \bar{A}\omega A) \wedge \bar{\theta} \end{aligned}$$

by Proposition 1.7.3. Therefore $\bar{\omega} = A^T dA + \bar{A}\omega A$. The second equality can be done similarly, but in a more involved manner. see notes □

End of Lecture 15

Recall that, given orthonormal frames $\{X_i\}$ and $\{\bar{X}_i\}$, which gives rise to coframes $\{\theta^i\}$ and $\{\bar{\theta}^i\}$, then they are related by $\bar{\theta} = A\theta$ for $A = (a_j^i) \in O(m)$. In turn, we have connection 1-form $\bar{\omega} = A^T \omega A + A^T dA$ and curvature 2-form $\bar{\Omega} = A^T \Omega A$. In the case of dimension 2, if $\{\bar{\theta}\}$ and $\{\theta\}$ have the same orientation, then we can write $A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ for some $\varphi : U \cap \bar{U} \to \mathbb{S}^1$. Therefore, we can write $\omega = \begin{pmatrix} 0 & \omega_2^1 \\ -\omega_2^1 & 0 \end{pmatrix}$ and $\Omega = \begin{pmatrix} 0 & \Omega_2^1 \\ -\Omega_2^1 & 0 \end{pmatrix}$. In the view that $A \in C^\infty(U, O(m))$ and therefore $dA \in \Omega^1(U, \Omega(m))$, we have $\bar{\omega}_2^1 = \omega_2^1 - d\varphi$ and $\bar{\Omega}_2^1 = \Omega_2^1$ and

$$A^T dA = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} -\sin\varphi d\varphi & -\cos\varphi d\varphi \\ \cos\varphi d\varphi & -\sin\varphi d\varphi \end{pmatrix} = \begin{pmatrix} 0 & -d\varphi \\ d\varphi & 0 \end{pmatrix}.$$

1.8 Gauss-Bonnet Theorem

Theorem 1.8.1 (Gauss-Bonnet)**.** Let (M, g) be an compact (i.e., without boundary) oriented Riemannian 2-manifold, then

$$\int_M K_g V_g = 2\pi\chi(M)$$

where K_g is the Gaussian curvature of g, V_g is the Riemannian volume form of (M, g), and $\chi(M)$ is the Euler characteristic of M.

Remark 1.8.2. This is a result that connects geometry with topology.

1. For any manifold M, $\chi(M) = \sum_{i=0}^{m}(-1)^i\beta_i$ where $\beta_i = \dim(H^i(M))$ is the Betti number. In particular, for an oriented surface, we recover the Riemann-Roch theorem $\chi(M) = 2 - 2g$, where g is the genus of M. For instance, $\chi(\mathbb{S}^2) = 2$, $\chi(\mathbb{T}^2) = 0$, and $\chi(M) = -2$ for a manifold with 2 punctures.
2. This result generalizes as follows. For any even-dimensional manifold M, Chern proved that

   $$\int_M P(R)V_g = C_m\chi(M)$$

   where $P(R)$ is a polynomial in terms of the curvature R of g, and C_m is a constant that only depends on $\dim(M) = m$.
3. There is a version of Theorem 1.8.1 for compact oriented 2-manifolds with boundaries:

   $$\int_M K_g V_g + \int_{\partial M} k_g = 2\pi\chi(M)$$

   where the geodesic curvature k_g on ∂M coincides with the covariant derivative $D_{\gamma_i}\dot{\gamma}_i(t)$ for $\partial M = \bigcup\{\gamma_i\}$.

Corollary 1.8.3. $\mathbb{S}^2$ and $\mathbb{T}^2$ do not admit a metric with negative Gaussian curvature.

Proof. If such g exists, then

$$\int_M K_g dV_g < 0$$

which is impossible since $\chi(\mathbb{S}^2), \chi(\mathbb{T}^2) \geqslant 0$. □

We take a detour into Riemannian volume forms.

Lemma 1.8.4. Let (M, g) be an oriented Riemannian manifold, then there exists a unique volume form V_g such that for any positive, orthonormal frame $\{X_1, \dots, X_n\}$: $V_g(X_1, \dots, X_n) = 1$.

Proof. Suppose $\{X_i\}$ and $\{\bar{X}_i\}$ are both positively-oriented orthonormal frames, where $\bar{X} = XA$ with $A : U \cap \bar{U} \to \mathrm{SO}(n)$, then we can pick some volume form $\mu \in \Omega^n(M)$ defining the given orientation, and calculation of this volume form on the given frame shows that

$$\mu(\bar{X}_1, \dots, \bar{X}_m) = \det(A)\mu(X_1, \dots, X_m) = \mu(X_1, \dots, X_m) = c,$$

so set

$$V_g = \frac{1}{c}\mu \in \Omega^m(M).$$

□

Remark 1.8.5.

- If the manifold is not oriented, then there is no longer a volume form, but we may recover the notion of density.

- If $\{\theta^i\}$ is a positively-oriented orthonormal coframe, then $V_g = \theta^1 \wedge \cdots \wedge \theta^m$.
- If (U, x^i) is a positively-orientated chart, i.e., mapping the orientation of the chart to the standard orientation of $\mathbb{R}^n$, then $g = g_{ij} dx^i dx^j$, hence $V_g = \deg(g_{ij})^{\frac{1}{2}} dx^1 \wedge \cdots \wedge dx^n$.

Proof of Theorem 1.8.1. This makes use of Theorem 1.8.6 which we will prove later on in the course.

Theorem 1.8.6 (Poincaré-Hopf)**.** Suppose $X \in \mathfrak{X}(M)$ has finite number of zeros, say $\{p_1, \dots, p_N\}$, then

$$\chi(M) = \sum_{i=1}^{N} \operatorname{ind}_{p_i}(X).$$

The index is a notion of rotation of vector field around each zero. To compute the index at a zero $p \in M$, we choose a chart (U, x^i) of p, pick a ball $B_\varepsilon(p)$. We can then look at the Gauss map

$$\begin{aligned} G : \partial B_\varepsilon(p) &\to \mathbb{S}^{m-1} \\ x &\mapsto \frac{X(x)}{||X(x)||} \end{aligned}$$

and define $\operatorname{ind}_p(X) = \deg(G)$, where the degree is the unique integer such that

$$\int_{\partial B_\varepsilon} G^* \alpha = \deg(G) \int_{\mathbb{S}^{m-1}} \alpha$$

for closed form $\alpha \in \Omega^{m-1}(\mathbb{S}^{m-1})$.

To prove Theorem 1.8.1, we choose $X \in \mathfrak{X}(M)$ with zeros $\{p_1, \dots, p_N\}$. On $M \backslash \{p_1, \dots, p_N\}$, there is a positively-oriented orthonormal frame $\{X_1 = \frac{X}{||X||}, X_2\}$ with dual (positively-oriented orthonormal) coframe $\{\theta^1, \theta^2\}$. We choose some small enough balls $B_{\varepsilon_i}(p_i)$'s for all i, so that it contains only zero p_i, and is contained in some chart $(U_i i, \varphi_i)$. Therefore,

$$\begin{aligned} \int_{M \backslash \bigcup_i B_{\varepsilon_i}(p_i)} K_g V_g &= \int_{M \backslash \bigcup_i B_{\varepsilon_i}(p_i)} K_g \theta^1 \wedge \theta^2 \\ &= \int_{M \backslash \bigcup_i B_{\varepsilon_i}(p_i)} d\omega_2^1 \\ &= \int_{\partial(M \backslash \bigcup_i B_{\varepsilon_i}(p_i))} \omega_2^1 \text{ by Stokes' theorem} \\ &= \sum_{i=1}^{N} \int_{\partial B_{\varepsilon_i}(p_i)} \omega_1^2. \end{aligned}$$

It then suffices to show that taking $\varepsilon_i \to 0$ for arbitrary i gives

$$\lim_{\varepsilon_i \to 0} \int_{\partial B_{\varepsilon_i}(p_i)} \omega_1^2 = 2\pi \operatorname{ind}_{p_i}(X).$$

Now choose frame $\{\bar{X}_1, \bar{X}_2\}$ on each U_i that is positively-oriented and orthonormal, then we have $\bar{X} = XA$ for

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin(\theta) & \cos\theta \end{pmatrix}$$

with $\theta : U_i \backslash \{p_i\} \to \mathbb{S}^1$ as the angle between $\bar{X}_i$ and X_i, therefore

$$\theta|_{\partial B_{\varepsilon_i}(p_i)} = G$$

with $G(x) = \frac{x}{||x||}$. At the start of the lecture, we saw $\bar{\omega}_1^2 = \omega_1^2 - d\theta$, then

$$\bar{\omega}_1^2|_{\partial B_{\varepsilon_i}(p_i)} = \omega_1^2|_{\partial B_{\varepsilon_i}(p_i)} - G^* d\theta$$

where $d\theta$ is the standard angle function on $\mathbb{S}^1$. This shows us that

$$\begin{aligned}
\int\limits_{\partial B_{\varepsilon_i}(p_i)} \omega_1^2 &= \int\limits_{\partial B_{\varepsilon_i}(p_i)} G^* d\theta - \int\limits_{\partial B_{\varepsilon_i}(p_i)} \bar{\omega}_1^2 \\
&\xrightarrow[\varepsilon_i \to 0]{} \int\limits_{\partial B_{\varepsilon_i}(p_i)} G^* d\theta \\
&= \deg(G) \int\limits_{\mathbb{S}^1} d\theta \\
&= 2\pi \operatorname{ind}_{p_i}(X),
\end{aligned}$$

where the second term vanishes whenever $\varepsilon_i \to 0$ just like integrating a smooth function. ☐

End of Lecture 16

1.9 Hodge Decomposition

Let $(V, \langle -, - \rangle)$ be an Euclidean vector space with an inner product, which defines an inner product

$$\langle -, - \rangle : \Lambda^k V \times \Lambda^k V \to \mathbb{R}$$

that is uniquely determined by

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \det(\langle v_i, w_j \rangle).$$

If $\{e_1, \ldots, e_n\}$ is an orthonormal basis, then we get an orthonormal basis on $\Lambda^k V$ using the set

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} : i_1 < \cdots < i_k\}.$$

Lemma 1.9.1. If $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ are orthonormal bases that define the same orientation, then $e_1 \wedge \cdots \wedge e_n = f_1 \wedge \cdots \wedge f_n$.

Proof. We have seen a similar proof last time: write $f_i = \sum\limits_j a_i^j e_j$, then $A = (a_i^j) \in \mathrm{SO}(n)$, i.e., it has determinant 1, since they define the same orientation, therefore

$$f_1 \wedge \cdots \wedge f_n = \det(A) e_1 \wedge \cdots \wedge e_n = e_1 \wedge \cdots \wedge e_n.$$

☐

Fix some orientation V, given by $\mu = e_1 \wedge \cdots \wedge e_n$ as a notion of unit n-vector, where $\{e_i\}$'s give a positively-oriented orthonormal basis.

Proposition 1.9.2. There is a unique linear map $* : \Lambda^k V \to \Lambda^{n-k} V$ such that

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \mu \tag{1.9.3}$$

for any $\alpha, \beta \in \Lambda^k V$.

Proof. If Equation (1.9.3) holds, then if $\{e_i\}$ is a positively-oriented orthonormal basis, then we find that

$$*(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \pm e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}, \tag{1.9.4}$$

where $\{e_1, \ldots, e_{i_k}, e_{j_1 \ldots, e_{j_{n-k}}}\}$ is basis, and the sign $\pm$ is determined by whether this basis is positively- or negatively-oriented. Therefore, $*$ is unique if it exists. But Equation (1.9.4) defines $*$ on a basis. ☐

Remark 1.9.5. The operator in Proposition 1.9.2 satisfies the following properties.

1. $*1 = e_1 \wedge \cdots \wedge e_n$.
2. $*$ satisfies Equation (1.9.4).
3. $*(Av_1 \wedge \cdots \wedge Av_k) = \det(A) * (v_1 \wedge \cdots \wedge v_k)$.
4. $*(\alpha \wedge *\beta) = \langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = *(\beta \wedge *\alpha)$.
5. $** = (-1)^{k(n-k)}$ defines an operator $\Lambda^k V \to \Lambda^k V$.

Assuming $\{e_1, \ldots, e_{i_k}, e_{j_1}, \ldots, e_{j_{n-k}}\}$ is positively-oriented, then we know Equation (1.9.4) holds, therefore

$$**(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \pm e_{i_1} \wedge \cdots \wedge e_{i_k}$$

where the sign $\pm$ depends on the number of sign changes required to reach $\{e_{j_1}, \ldots, e_{j_{n-k}}, e_{i_1}, \ldots, e_{i_j}\}$, i.e., $(-1)^{k(n-k)}$.

Remark 1.9.6. If $\{v_1, \ldots, v_n\}$ is any positively-oriented basis (that is not assumed to be orthonormal), then

$$*1 = \frac{1}{\det(\langle v_i, v_j \rangle)} v_1 \wedge \cdots \wedge v_n.$$

Suppose (M, g) is a Riemannian manifold with a fixed choice of orientation. For any point $x \in M$, there is a notion of inner product g_x on $T_* M$, so there is an identification $T_x M \simeq T_x^* M$ of vector spaces given by $v \mapsto g_x(v, -)$, which therefore transforms the inner product into $T_x^* M$, now denoted g_x^*. In local charts, if we write

$$g = g_{ij} dx^i dx^j,$$

then $g_x = (g_{ij}(x))$ and $g_x^* = (g_{ij})^{-1} = (g^{ij})$.

Performing the operator $*$ on each cotangent space, we get an operator

$$* : \Omega^k(M) \to \Omega^k(M).$$

Definition 1.9.7. The operator $* : \Omega^k(M) \to \Omega^k(M)$ defined in called the *Hodge star operator*.

If $\{\theta^i\}$ is a positively-oriented orthonormal coframe, then

$$*(\theta^{i_1} \wedge \cdots \wedge \theta^{i_k}) = \pm \theta^{j_1} \wedge \cdots \wedge \theta^{j_{n-k}},$$

where the choice of sign follows from the previous choices. In particular,

$$*1 = \theta^1 \wedge \cdots \wedge \theta^n = V_g$$

is the Riemannian volume form. More particularly, if M is a compact manifold, then the *volume* of M is defined by

$$\mathrm{Vol}(M) = \int_M *1.$$

Definition 1.9.8. We define L^2*-inner product* on the differential k-forms $\Omega_c^k(M)$ with compact support as

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle V_g = \int_M (\alpha \wedge *\beta).$$

We now assume M is compact, i.e., $\Omega_c^k(M) = \Omega^k(M)$.

Proposition 1.9.9. Given a oriented Riemannian manifold (M, g), the de Rham differential $d : \Omega^k(M) \to \Omega^{k+1}(M)$ has a formal adjoint $d^* : \Omega^{k+1}(M) \to \Omega^k(M)$, i.e., $(d\alpha, \beta) = (\alpha, d^*\beta)$ for all $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{k+1}(M)$, called the *codifferential*, defined by

$$d^*\beta = (-1)^{nk} * d * \beta.$$

Proof. We have

$$\begin{aligned} d(\alpha \wedge *\beta) &= (d\alpha) \wedge *\beta + (-1)^k \alpha \wedge d*\beta \\ &= (d\alpha) \wedge *\beta + (-1)^k \alpha \wedge (-1)^{k(n-k)} **d*\beta \\ &= \langle d\alpha, \beta \rangle + (-1)^{kn} \langle \alpha, d^*\beta \rangle . \end{aligned}$$

By Stokes' theorem,

$$\begin{aligned} 0 &= \int_M d(\alpha \wedge *\beta) \\ &= (d\alpha, \beta) + (-1)^{kn} (\alpha, d^*\beta). \end{aligned}$$

□

Definition 1.9.10. The *Laplace-Beltrami operator* is

$$\Delta = dd^* + d^*d : \Omega^k(M) \to \Omega^k(M).$$

Proposition 1.9.11. Δ satisfies the following properties.

i. Δ is formally self-adjoint, i.e., $(\Delta\alpha, \beta) = (\alpha, \Delta\beta)$ for all $\alpha, \beta \in \Omega^*(M)$.

ii. $\Delta\alpha = 0$ if and only if $d\alpha = d^*\alpha = 0$.

iii. $\Delta * = *\Delta$.

Proof. i. We have

$$\begin{aligned} (\Delta\alpha, \beta) &= (dd^*\alpha, \beta) + (d^*d\alpha, \beta) \\ &= (d^*\alpha, d^*\beta) + (d\alpha, d\beta) \\ &= (\alpha, \Delta\beta). \end{aligned}$$

ii. If $d^*\alpha = d\alpha = 0$, then $\Delta\alpha = 0$. Conversely, if $\Delta\alpha = 0$, then $||d^*\alpha||^2 + ||d\alpha||^2 = (\Delta\alpha, \alpha) = 0$, therefore $d^*\alpha = d\alpha = 0$.

□

Definition 1.9.12. The harmonic k-forms are defined by $\mathcal{H}^k(M) = \{\alpha \in \Omega^k(M) : \Delta\alpha = 0\}$.

Remark 1.9.13. From the definition, the harmonic functions (still under the assumption that M is compact) are the ones that are constant on each connected component of the manifold. Therefore, $\mathcal{H}^0(M)$ is the vector space of dimension the number of connected components.

Let us now express the Laplace-Beltrami operator in local coordinates.

Example 1.9.14. Let $M = \mathbb{R}^n$ and $g_0 = \sum (dx^i)^2$ be the flat metric, under the usual orientation, then we have $df = \frac{\partial f}{\partial x^i} dx^i$ using a basis $\{dx^1, \ldots, dx^n\}$. Therefore,

$$*df = \sum_{i=1}^n (-1)^i \frac{\partial f}{\partial x^i} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

and

$$*dx^i = (-1)^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n,$$

therefore

$$\Delta f = dd^*f + d^*df$$

$$
\begin{aligned}
&= *d\left(\sum_{i=1}^{n}(-1)^{i}\frac{\partial f}{\partial x^{i}}dx^{1}\cdots\widehat{dx^{i}}\cdots dx^{n}\right)\\
&= -\sum_{i=1}^{n}\frac{\partial^{2} f}{\partial (x^{i})^{2}}
\end{aligned}
$$

which is the negative of the usual Laplacian, since $dd^* f = 0$. Similarly,

$$\Delta\omega = \sum_{j=1}^{n}\frac{\partial^{2}\omega_{i_1\cdots i_k}}{\partial (x^{j})^{2}}dx^{i_1}\wedge\cdots\wedge dx^{i_k}.$$

Note that this does not use the compactness assumption, because this does not involve the L^2-inner product defined above.

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Recall that

- we defined the Hodge star operator
$$*:\Omega^k(M)\to\Omega^{n-k}(M).$$
On orthogonal positively-oriented coframe, this gives
$$*(\theta^{i_1}\wedge\cdots\wedge\theta^{i_k}) = \pm\theta^{j_1}\wedge\cdots\wedge\theta^{j_{n-k}}$$
where the sign depends on whether the set $\{\theta^{i_1},\dots,\theta^{i_k},\theta^{j_1},\dots,\theta^{j_{n-k}}\}$ is oriented;
- the L^2-inner product is defined by
$$(\alpha,\beta) = \int_M \alpha\wedge *\beta;$$
- and we defined the codifferential to be
$$d^*:\Omega^k(M)\to\Omega^{k-1}(M)$$
to be $d^* = -(-1)^{m(k+1)}*d*$, which is the formal adjoint of de Rham differential d, i.e., $(d^*\alpha,\beta) = (\alpha, d\beta)$;
- we defined the Laplace-Beltrami operator to be
$$\Delta = dd^* + d^*d : \Omega^k(M)\to\Omega^k(M)$$
which is self-adjoint;
- we define the harmonic k-forms to be the set
$$\mathcal{H}^k(M) = \{\alpha\in\Omega^k(M) : \Delta\alpha = 0\}.$$

Exercise 1.9.15. In a local chart (U, x^i),

$$\Delta f = -\frac{1}{(\det(g))^{\frac{1}{2}}}\frac{\partial}{\partial x^{i}}\left((\det(g))^{\frac{1}{2}}g^{ij}\frac{\partial f}{\partial x^{j}}\right)$$

for $g = g_{ij}dx^i dx^j$ and $(g^{ij}) = (g_{ij})^{-1}$.

Theorem 1.9.16 (Hodge Decomposition)**.** There is an orthogonal decomposition

$$
\begin{aligned}
\Omega^k(M) &= \Delta(\Omega^k(M))\oplus\mathcal{H}^k(M)\\
&= d(d^*\Omega^k(M))\oplus d^*(d\Omega^k(M))\oplus\mathcal{H}^k(M).
\end{aligned}
$$

In particular, $\Delta\omega = \alpha$ has solutions if and only if $\alpha\in\mathcal{H}^k(M)^\perp$.

We first list a few consequences of Theorem 1.9.16.

Definition 1.9.17. Let $H : \Omega^k(M) \to \mathcal{H}^k(M)$ be the orthogonal projection. The *Green operator* is a linear operator defined by

$$\begin{aligned} G : \Omega^k(M) &\to \mathcal{H}^k(M)^\perp \\ \alpha &\mapsto \omega, \end{aligned}$$

where ω is the unique solution of the equation $\Delta\omega = \alpha - H(\alpha)$.

Lemma 1.9.18. G commutes with any linear operator $T : \Omega^*(M) \to \Omega^*(M)$ that commutes with Δ. In particular, G commutes with differential d, codifferential d^*, and Δ itself.

Proof. Assume that $T\Delta = \Delta T$, then $T(\mathcal{H}^k(M)) \subseteq \mathcal{H}^k(M)$, and since $\mathcal{H}(M)^\perp = \operatorname{im}(\Delta)$, therefore $T(\mathcal{H}^k(M)^\perp) \subseteq \mathcal{H}^k(M)^\perp$. By the description of the Green operator, we can write

$$G = \left(\Delta|_{\mathcal{H}(M)^\perp}\right)^{-1} \circ \mathrm{pr}_{\mathcal{H}^k(M)^\perp}.$$

This gives $G \circ T = T \circ G$. ☐

Corollary 1.9.19. The de Rham cohomology $H^*(M)$ of a manifold M is finite-dimensional, and every class in $H^k(M)$ has a unique harmonic representative.

Proof. Given $\alpha \in \Omega^k(M)$, then

$$\begin{aligned} \alpha &= \Delta G(\alpha) + H(\alpha) \text{ by Theorem 1.9.16} \\ &= dd^*G(\alpha) + d^*dG(\alpha) + H(\alpha) \\ &= dd^*G(\alpha) + d^*G(d\alpha) + H(\alpha) \text{ by Lemma 1.9.18}. \end{aligned}$$

In particular, if $d\alpha = 0$, then $\alpha = d(G(d^*\alpha)) + H(\alpha)$, so $[\alpha] = [H(\alpha)]$. One should now check that the harmonic forms are well-defined in representatives: given $[\alpha_1] = [\alpha_2]$, we should have $H(\alpha_1) = H(\alpha_2)$. Assume that $[\alpha_1] = [\alpha_2]$ and $\Delta\alpha_1 = \Delta\alpha_2 = 0$, then it suffices to show that $\alpha_1 = \alpha_2$. We see that $\alpha_1 - \alpha_2 = d\beta$ is exact, so

$$(\alpha_1 - \alpha_2, d\beta) = (d^*(\alpha_1 - \alpha_2), \beta),$$

but having $\Delta\alpha_1 = \Delta\alpha_2 = 0$, it is equivalent to saying that $d(\alpha_1 - \alpha_2) = d^*(\alpha_1 - \alpha_2) = 0$, therefore

$$(\alpha_1 - \alpha_2, d\beta) = (d^*(\alpha_1 - \alpha_2), \beta) = 0.$$

Now $||\alpha_1 - \alpha_2||^2 = (\alpha_1 - \alpha_2, \alpha_1 - \alpha_2) = (\alpha_1 - \alpha_2, d\beta) = 0$, therefore $\alpha_1 = \alpha_2$. ☐

Lemma 1.9.20. In Theorem 1.9.16, the first decomposition implies the second decomposition.

Proof. Say $\Delta\alpha = 0$, or equivalently $d\alpha = d^*\alpha = 0$, then

$$\begin{aligned} (\alpha, d\beta) &= (d^*\alpha, \beta) = 0 \\ (\alpha, d^*\beta) &= (d\alpha, \beta) = 0 \end{aligned}$$

for any $\beta \in \Omega^k(M)$. Therefore, the harmonic forms is orthogonal to images of d and d^*. Finally,

$$(d\beta_1, d^*\beta_2) = (d^2\beta, d\beta) = 0.$$

This shows that all three factors in the second decomposition are pairwise orthogonal. ☐

It then remains to show the first decomposition. Suppose $(V, (\cdot,\cdot))$ is a Euclidean vector space, then for any $v \in V$, we can look at the functionals

$$\begin{aligned} \ell_v : V &\to \mathbb{R} \\ w &\mapsto (v, w) \end{aligned}$$

such that $|\ell_v(w)| = |(v, w)| \leqslant ||v|| \cdot ||w|| = c||w||$ for some constant c, therefore ℓ_v is a bounded linear function. If $\dim(V) < \infty$, then

- any functional $\ell : V \to \mathbb{R}$ is bounded, and in fact
- any functional $\ell : V \to \mathbb{R}$ is of the form $\ell(w) = (v, w)$ for some $v \in V$.

However, if $\dim(V) = \infty$, both properties may fail. The space of differential forms is one such space, therefore causing us problems. Regardless, we have

Theorem 1.9.21 (Riesz Representation Theorem)**.** If $(V, (\cdot,\cdot))$ is a Hilbert space, and $\ell : V \to \mathbb{R}$ is a bounded linear functional, then $\ell(w) = (v, w)$ for some unique $v \in W$.

We may want to apply this theorem, but the issue being, $\Omega^*(M)$ is not a Hilbert space, since it is not complete. To take the completion, another issue occurs: the notion of completion is then not unique. We want to find the right notion of completion $(W, (\cdot,\cdot))$ with $V \subseteq W$ and $\bar{V} = W$, which is given by

$$W = \{\alpha : \alpha, d\alpha, d^*\alpha \in L^2\},$$

whatever this means. The correct way of doing this is using the notion of a Sobolev space, but we digress. After completion, we look at the solutions $w \in W$ such that $\Delta\omega = \alpha$. Assuming that a solution exists, then

$$(\Delta\omega, \varphi) = (\alpha, \varphi)$$

for any $\varphi \in \Omega^k(M))$. To define this, we note that Δ is still self-adjoint after the completion, therefore this is equivalent to

$$(\omega, \Delta\varphi) = (\alpha, \varphi)$$

for any $\varphi \in \Omega^k(M))$. This is really the definition of α above, i.e., in the weak sense. The point being, the solutions ω of $\Delta\omega = \alpha$ are exactly the linear functionals

$$\ell_w : \Omega^k(M) \to \mathbb{R}$$

such that $\ell_\omega(\Delta\varphi) = (\alpha, \varphi)$. These are known as weak solutions, i.e., a solution in W by Theorem 1.9.21.

Definition 1.9.22. A *weak solution* of $\Delta\omega = \alpha$ is a bounded linear functional $\ell_w : \Omega^k(M) \to \mathbb{R}$ such that $\ell_\omega(\Delta\varphi) = (\alpha, \varphi)$.

Remark 1.9.23. ℓ_w should then be thought of as a function on W by Theorem 1.9.21, i.e., taking a completion on k-forms.

Any soluition now gives rise to a weak solution. We still need to connect weak solutions back to the regular solutions.

Theorem 1.9.24 (Regularity)**.** Given $\alpha \in \Omega^k(M)$ and weak solution $\ell_w : W \to \mathbb{R}$, then there exists $\omega \in \Omega^k(M)$ such that

$$\ell_\omega(\varphi) = (\omega, \varphi)$$

for all $\varphi \in \Omega^k(M)$.

Theorem 1.9.25. If $\{\alpha_n\} \subseteq \Omega^k(M)$ is a sequence of smooth functions that is bounded, and whose Laplacian is also bounded, i.e., $||\alpha_n|| \leqslant C$ and $||\Delta\alpha_n|| \leqslant C$ for some C for all $n \in \mathbb{N}$, then there exists a Cauchy subsequence $\{\alpha_{n_k}\}$.

End of Lecture 18

Proof of Theorem 1.9.16. We first show that $\mathcal{H}^k(M)$ is finite-dimensional. Assume not, then let $\{\alpha_n\} \subseteq \mathcal{H}^k(M)$ be such that $||\alpha_n|| = 1$ and $(\alpha_n, \alpha_n) = 0$ for all $n \neq m$, then this sequence has no Cauchy subsequences, which contradicts Theorem 1.9.25.

Given Lemma 1.9.20, it suffices to prove the first decomposition of Theorem 1.9.16. Fix orthonormal basis $\{\omega_1, \ldots, \omega_N\}$ for $\mathcal{H}^k(M)$. For any $\alpha \in \Omega^k$, we can write

$$\alpha = \beta + \sum_{i=1}^{N} (\alpha, \omega_i)\omega_i,$$

therefore

$$\Omega^k(M) = \mathcal{H}^k(M)^\perp \oplus \mathcal{H}^k(M).$$

It remains to show $\Delta(\Omega^k(M)) = \mathcal{H}^k(M)^\perp$. One direction is easy: to show $\Delta(\Omega^k(M)) \subseteq \mathcal{H}^k(M)^\perp$, note that for any $\varphi \in \mathcal{H}^k(M)$, we get

$$(\Delta\omega, \varphi) = (\omega, \Delta\varphi) = 0.$$

To show the other inclusion $\mathcal{H}^k(M)^\perp \subseteq \Delta(\Omega^k(M))$, we need the following lemma, stating that the inverse of Laplacian is continuous, assuming such inverse exists.

Lemma 1.9.26. There exists some $c > 0$ such that $||\varphi|| \leqslant c||\Delta\varphi||$ for all $\varphi \in \mathcal{H}^k(M)^\perp$.

Let $\alpha \in \mathcal{H}^k(M)^\perp$, we define

$$\begin{aligned} \ell : \Delta(\Omega^k(M)) &\to \mathbb{R} \\ \Delta\varphi &\mapsto (\varphi, \alpha) \end{aligned}$$

We first show that this is well-defined. Suppose $\Delta\varphi_1 = \Delta\varphi_2$, then $\Delta(\varphi_1 - \varphi_2) = 0$, hence $\varphi_1 - \varphi_2 \in \mathcal{H}^k(M)$, thus $(\varphi_1 - \varphi_2, \alpha) = 0$. Now we check that ℓ is bounded. By Lemma 1.9.26, we have

$$\begin{aligned} |\ell(\Delta\varphi)| &= |\ell(\Delta(\varphi - H(\varphi))| \\ &= (\varphi - H(\varphi), \alpha) \\ &\leqslant ||\alpha|| \cdot ||\varphi - H(\varphi)|| \\ &\leqslant c||\alpha|| \cdot ||\Delta(\varphi - H(\varphi))|| \\ &= c||\alpha|| \cdot ||\Delta\varphi||. \end{aligned}$$

By Hahn–Banach theorem, we know that bounded operator in the closed subspace can be extended to a bounded operator on the entire space, therefore there exists an extension $\ell : W \to \mathbb{R}$ which is bounded. Hence, ℓ is a weak solution of the equation. Finally, by Theorem 1.9.24, $\ell(\varphi) = (\omega, \varphi)$ with $\omega \in \Omega^k$, therefore $\Delta\omega = \alpha$. This proves the inclusion. Finally, we give a proof of Lemma 1.9.26.

Proof of Lemma 1.9.26. Suppose not, then there exists a sequence $\{\alpha_n\} \subseteq \mathcal{H}^k(M)^\perp$ such that the norm is constant, i.e., we may assume $||\alpha_n|| = 1$, and $||\Delta\alpha_n|| \to 0$ as $n \to \infty$. By Theorem 1.9.25, it has a Cauchy subsequence $\alpha_{n_k} \subseteq \mathcal{H}^k(M)^\perp$. That is, for any $\varphi \in \Omega^k(M)$, $(\alpha_{n_k}, \varphi) \in \mathbb{R}$ is Cauchy, hence $\lim_k \to \infty(\alpha_{n_k}, \varphi)$ exists. Now we define a linear operator

$$\begin{aligned} \ell : \Omega^k(M) &\to \mathbb{R} \\ \varphi &\mapsto \lim_{k\to\infty} (\alpha_{n_k}, \varphi) \end{aligned}$$

We claim that ℓ is bounded. Indeed,

$$\begin{aligned} |\ell(\varphi)| &= |\lim_{k\to\infty} (\alpha_{n_k}, \varphi)| \\ &= \lim_{k\to\infty} |(\alpha_{n_k}, \varphi)| \\ &\leqslant \lim_{k\to\infty} ||\alpha_{n_k}|| \cdot ||\varphi|| \\ &= ||\varphi||. \end{aligned}$$

Moreover, we check that ℓ is a weak solution. Indeed,

$$\begin{aligned} \ell(\Delta\varphi) &= \lim_{k\to\infty} (\alpha_{n_k}, \Delta\varphi) \\ &= \lim_{k\to\infty} (\Delta\alpha_{n_k}, \varphi) \\ &= 0. \end{aligned}$$

By Theorem 1.9.24, we can write $\ell(\varphi) = (\omega, \varphi)$ for some smooth form $\omega \in \Omega^k$, such that $\Delta\omega = 0$. Therefore, $\alpha_{n_k} \to \omega$, so $||\omega|| = 1$ and $\omega \in \mathcal{H}(M)^\perp$. However, since $\Delta\omega = 0$, we note $\omega \in \mathcal{H}^k(M)$, which is a contradiction. ■

□

Remark 1.9.27. There is also a complex version of Theorem 1.9.16, which involves $\bar{\partial}$. For instance, c.f., [GH14].

2 Bundle Theory

2.1 Vector Bundles

Definition 2.1.1. For a map $\pi : E \to M$, a *trivializing chart* of dimension/rank r is a chart (U, ϕ) where

- $U \subseteq M$ is open, and
- $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ is a diffeomorphism,

such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^r \arrow[dl, "\mathrm{pr}_U"] \\ & U & \end{tikzcd}$$

commutes.

Notation. Fix $p \in M$,

- we denote $E_p = \pi^{-1}(p)$ to be the fiber over p;
- we denote ϕ^p to be the diffeomorphism given by the composition

$$E_p \xrightarrow{\phi} \{p\} \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$$

Therefore, under this notation, E_p is a vector space. Unpacking all of this, we note that ϕ^p and the projection determines ϕ itself, via

$$\phi(v) = (\pi(v), \phi^{\pi(v)}(v)).$$

Definition 2.1.2. An *atlas* of trivializing charts for $\pi : E \to M$ is a collection of trivializing charts $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ such that

i. $\{U_\alpha\}_{\alpha \in I}$ is an open cover of M, and

ii. given any $\alpha \neq \beta$, for any $p \in U_\alpha \cap U_\beta$, we have a linear isomorphism

$$\phi_\alpha^p \circ (\phi_\beta^p)^{-1} : \mathbb{R}^r \to \mathbb{R}^r$$

demonstrating compatibility.

Remark 2.1.3. From the definition, it is clear that π is a surjective submersion.

Definition 2.1.4. A *vector bundle* $\xi = (E, \pi, M)$ is a map $\pi : E \to M$ together with a maximal atlas $\mathcal{C}$ of trivializing charts.

Remark 2.1.5.

- By a maximal atlas, we mean that if (U, ϕ) is any trivializing chart such that for any $p \in U_\alpha \cap U$, $\phi_\alpha^p \circ (\phi^p)^{-1}$ and $\phi^p \circ (\phi_\alpha^p)^{-1}$ are linear isomorphisms, then $(U, \phi) \in \mathcal{C}$.
- Any atlas is contained in a unique maximal atlas, therefore determining a unique vector bundle. Therefore, to define a vector bundle, it suffices to give an atlas.
- In the case of complex vector bundles, we should change all instances of $\mathbb{R}^r$ to $\mathbb{C}^r$, and ($\mathbb{R}$-)linear isomorphisms are now complex linear isomorphisms. However, the manifold is still a real manifold. This is different from holomorphic vector bundles, which are complex vector bundles over complex manifolds.

End of Lecture 19

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