

Motivic Cohomology Notes

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0 INTRODUCTION

Let $X \in \mathbf{Sm}/k$ be a smooth separated scheme over a field k . The study of motivic cohomology started with the hope that

Conjecture 0.1 (Beilinson and Lichtenbaum, 1982-1987). There exists some complexes $\mathbb{Z}(n)$ for $n \in \mathbb{N}$ of sheaves in Zariski topology on \mathbf{Sm}/k such that

1. $\mathbb{Z}(0)$ is (quasi-isomorphic to) the constant sheaf \mathbb{Z} , i.e., the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$

concentrated at degree 0;

2. $\mathbb{Z}(1)$ is the complex $\mathcal{O}^*[-1]$, i.e., the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{O}^* \longrightarrow 0 \longrightarrow \cdots$$

concentrated at degree 1;

3. for every field F/k , the hypercohomology over Zariski topology satisfies¹

$$\mathbb{H}_{\mathrm{Zar}}^n(F, \mathbb{Z}(n)) = H^n(\mathbb{Z}(n)(\mathrm{Spec}(F))) = K_n^M(F),$$

where $K_n^M(F)$ is the n th Milnor K-theory of a field F , given by the quotient of the tensor algebra $T(F^*)/\{x \otimes (1-x) : x \in F^*\}$ over \mathbb{Z} ; (lecture 5 of [\[MVW06\]](#), page 29)

Example 0.2.

- a. $K_0^M(F) = K_0(F) = \mathbb{Z}$;
 - b. $K_1^M(F) = K_1(F) = F^\times$;
 - c. $K_2^M(F) = K_2(F)$.
4. $\mathbb{H}_{\mathrm{Zar}}^{2n}(X, \mathbb{Z}(n)) = \mathrm{CH}^n(X)$ (lecture 17 of [\[MVW06\]](#), page 135), where the n th classical Chow group $\mathrm{CH}^n(X)$ is the free group given by

$$\mathrm{CH}^n(X) = \mathbb{Z}\{\text{cycles of codimension } n\}/\text{rational equivalences};$$

¹Here we use the convention that the (hyper)cohomology of F should be interpreted as of $\mathrm{Spec}(F)$, the corresponding space.

5. there is a natural Atiyah–Hirzebruch spectral sequence

$$E_2^{p,q} = \mathbb{H}_{\text{Zar}}^p(X, \mathbb{Z}(q)) \Rightarrow K_{2q-p}(X).$$

Moreover, tensoring with \mathbb{Q} , the spectral sequence degenerates and one has

$$\mathbb{H}_{\text{Zar}}^i(X, \mathbb{Z}(n))_{\mathbb{Q}} = \text{gr}_{\gamma}^n(K_{2n-i}(X)_{\mathbb{Q}})$$

where gr_{γ}^n 's are the quotients (graded pieces) of γ -filtration. ([Lev94]; [Lev99], Theorem 11.7)

Remark 0.3. Such choice of complexes $\mathbb{Z}(q)$ exists, and is called the motivic complex. For a clear definition of these complexes, see Lecture 3 of [MVW06]. Moreover, by convention $\mathbb{Z}(q) = 0$ for $q < 0$.

Definition 0.4. The motivic cohomology of X is defined by $H^{p,q}(X, \mathbb{Z}) = \mathbb{H}_{\text{Zar}}^p(X, \mathbb{Z}(q))$, the hypercohomology of the motivic complexes with respect to the Zariski topology.

Remark 0.5. In general, a motivic cohomology with coefficient in an abelian group A is a family of contravariant functors $H^{p,q}(-, A) : \text{Sm}/k \rightarrow \text{Ab}$.

Remark 0.6. The motivic cohomology of X satisfies the cancellation property: set $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$, then

$$H^{p,q}(X \times \mathbb{G}_m, \mathbb{Z}) = H^{p,q}(X, \mathbb{Z}) \oplus H^{p-1, q-1}(X, \mathbb{Z}).$$

Remark 0.7. It turns out that the group remains unchanged if we replace the Zariski topology by Nisnevich topology.² If one uses étale topology instead, we retrieve Lichtenbaum motivic cohomology $H_L^{p,q}(X, \mathbb{Z})$. If $\text{char}(k) \nmid n$, it admits the comparison

$$H_L^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) = H_{\text{étale}}^p(X, \mathbb{Z}/n\mathbb{Z}(q)),$$

where $\mathbb{Z}/n\mathbb{Z}(q)$ is the q -twist $\mu_n^{\otimes q}$.

We may compare Lichtenbaum motivic cohomology with motivic cohomology by the following theorem, formerly known as Beilinson-Lichtenbaum Conjecture³:

Theorem 0.8 ([Voe11]). The natural map

$$H^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) \rightarrow H_L^{p,q}(X, \mathbb{Z}/n\mathbb{Z})$$

is an isomorphism if $p \leq q$, is a monomorphism if $p = q + 1$, and gives a spectral sequence for any pair of p, q .

Corollary 0.9. For $p \leq q$, we have

$$H^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) = H_{\text{étale}}^p(X, \mathbb{Z}/n\mathbb{Z}(q)).$$

In particular, for $X = \text{Spec}(k)$ as a point, this is the theorem formerly known as Milnor conjecture:

Corollary 0.10 ([Voe97], [Voe03a], [Voe03b]).

- $H^{p,p}(k, \mathbb{Z}/n\mathbb{Z}) = K_p^M(k)/n = H_{\text{étale}}^p(X, \mathbb{Z}/n\mathbb{Z}(p))$ as the Galois cohomology;
- in general,

$$H^{p,q}(k, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} 0, & p > q \\ H^{p,p}(k, \mathbb{Z}/n\mathbb{Z}) \cdot \tau^{q-p}, & p \leq q \end{cases}$$

where $\tau \in \mu_n(k) = H^{0,1}(k, \mathbb{Z})$ is a primitive n th root of unity.

Remark 0.11. Unlike the case with finite coefficients, $H^{p,q}(k, \mathbb{Z})$ is quite hard to compute for small $p < q$; for $p \geq q$, this is 0.

²Recall that the Nisnevich topology is a Grothendieck topology on the category of schemes that is finer than the Zariski topology but coarser than the étale topology.

³This is also known as the norm residue isomorphism theorem, or (formerly) Bloch-Kato conjecture.

A current long-standing conjecture is

Conjecture 0.12 (Beilinson-Soulé Vanishing Conjecture, [Lev93]). $H^{p,q}(k, \mathbb{Z}) = 0$ if $p < 0$.

Remark 0.13. Here are a few known cases:

- for $\text{char}(k) = 0$, this is known for number fields ([Bor74]), function fields of genus 0 ([Dég08]), curves over number fields, and their inductive limits (more precise references required); ([DG05])
- for $\text{char}(k) > 0$, this is known for finite fields ([Qui72]) and global fields ([Har77]).

Remark 0.14. The motivic cohomology could be realized in a tensor triangulated category, namely the category of effective motives $\text{DM}(k)$. For any pair of p, q , we can find an Eilenberg-MacLane space and a corresponding representable functor so that

$$H^{p,q}(X, \mathbb{Z}) = \text{Hom}_{DM}(\mathbb{Z}(X), \mathbb{Z}(q)[p])$$

where $\mathbb{Z}(X)$ is the motive of X and $\mathbb{Z}(q)[p] = \mathbb{G}_m^{\wedge q}[p - q]$.⁴

Remark 0.15. Dually, we can define the motivic homology by

$$H_{p,q}(X, \mathbb{Z}) = \text{Hom}_{DM}(\mathbb{Z}(q)[p], \mathbb{Z}(X)).$$

Remark 0.16 ([MVW06] Properties 14.5, page 110). By taking the hom functor from the aspect of motives, we can derive theorems for all (co)homologies which can be represented in DM . The main derives are the following:

1. If $E \rightarrow X$ is an \mathbb{A}^n -bundle, then motives $\mathbb{Z}(E) = \mathbb{Z}(X)$ in DM .
2. If $\{U, V\}$ is a Zariski open covering of X , we have a Mayer-Vietoris sequence

$$\mathbb{Z}(U \cap V) \longrightarrow \mathbb{Z}(U) \oplus \mathbb{Z}(V) \longrightarrow \mathbb{Z}(X) \longrightarrow \mathbb{Z}(U \cap V)[1]$$

in the form of a distinguished triangle in DM .

3. If $Y \subseteq X$ is a closed embedding of codimension c in Sm/k , then we have a Gysin triangle

$$\mathbb{Z}(X \setminus Y) \longrightarrow \mathbb{Z}(X) \longrightarrow \mathbb{Z}(Y)(c)[2c] \longrightarrow \mathbb{Z}(X \setminus Y)[1]$$

which is a distinguished triangle where $\mathbb{Z}(Y)(c)[2c] := \mathbb{Z}(Y) \otimes \mathbb{Z}(c)[2c]$.

4. For any vector bundle of rank n on X , we have the projective bundle formula

$$\mathbb{Z}(\mathbb{P}(E)) = \bigoplus_{i=0}^n \mathbb{Z}(X)(i)[2i]$$

which defines the Chern class of E .

5. Let X be a proper smooth scheme and let d_X be its dimension, then $\mathbb{Z}(X)$ has a strong dual $\mathbb{Z}(X)(-d_X)[-2d_X]$ in DM by stabilization. This gives a Poincaré duality⁵

$$H^{p,q}(X, \mathbb{Z}) \cong H_{2d_X - p, d_X - q}(X, \mathbb{Z}).$$

⁴Again, this notation goes back to the concise definition of the motivic complexes: see Lecture 3 from [MVW06] as well as the concept of presheaves with transfers.

⁵We can use cohomology with compact support for this.

1 INTERSECTION THEORY

1.1 CYCLES OF SCHEME

Definition 1.1. Let X be a scheme of finite type over k . We define the i th cycle on the scheme X to be a free abelian group

$$Z_i(X) = \bigoplus_{\substack{\text{irreducible closed } c \subseteq X \\ \text{with } \dim(c)=i}} \mathbb{Z} \cdot c$$

and set $Z(X) = \bigoplus_i Z_i(X)$. Define a set $K_i(X)$ to be the set of coherent sheaves \mathcal{F} on X with $\dim(\text{supp}(\mathcal{F})) \leq i$.⁶

Remark 1.2. Let (A, \mathfrak{m}) be a Noetherian local ring and M be an A -module, then by the dimension theorem, we know $\dim(M) = d(M) = \dim(\text{supp}(M))$, where $d(M)$ is the degree of the Hilbert-Samuel polynomial $P_{\mathfrak{m}}(M, n)$.

Definition 1.3. Let $X \in \text{Sm}/k$ and let $U, V \subseteq X$ be irreducible and closed. Suppose $W \subseteq U \cap V$ is a irreducible and closed component. If $\dim(W) = \dim(U) + \dim(V) - \dim(X)$, i.e., $\text{codim}(W) = \text{codim}(U) + \text{codim}(V)$, we say that U and V intersect properly at W .

Remark 1.4. This condition is weaker than saying they intersect transversely: we do not require information about tangent spaces.

Theorem 1.5. Let $A \supseteq k$ be a Noetherian regular ring, M, N be finitely-generated A -modules, and suppose $\ell(M \otimes_A N) < \infty$, then

1. $\ell(\text{Tor}_i^A(M, N)) < \infty$ for all $i \geq 0$;
2. the Euler-Poincaré characteristic $\chi(M, N) := \sum_{i=0}^{\dim(A)} (-1)^i \ell(\text{Tor}_i^A(M, N)) \geq 0$;
3. by Remark 1.2, we have $\dim(M) + \dim(N) \leq \dim(A)$;
4. in particular, we have $\dim(M) + \dim(N) < \dim(A)$ if and only if $\chi(M, N) = 0$.

Proof. See [Ser12], page 106. □

Remark 1.6. Part 3. from Theorem 1.5 implies that $\dim(W) \geq \dim(U) + \dim(V) - \dim(X)$, i.e., $\text{codim}(W) \leq \text{codim}(U) + \text{codim}(V)$ in the notation of Definition 1.3.

Definition 1.7. Let X, U, V, W be as in Definition 1.3, then we define the intersection multiplicity $m_W(U, V)$ of U and V at W by

$$m_W(U, V) = \chi^{\mathcal{O}_{X,W}}(\mathcal{O}_{X,W}/P_U, \mathcal{O}_{X,W}/P_V)$$

where P_U and P_V are prime ideals defining U and V , respectively.

Remark 1.8. By Theorem 1.5, we know $m_W(U, V) \geq 0$, and $m_W(U, V) = 0$ if and only if U and V do not intersect properly at W .

1.2 INTERSECTION PRODUCT AND CROSS PRODUCT

Definition 1.9. Let $X \in \text{Sm}/k$, and let $U \in Z_a(X)$ and $V \in Z_b(X)$. If U and V intersect properly at every component, then we define the intersection product to be the cycle

$$U \cdot V = \sum_{\substack{W \subseteq U \cap V \\ \dim(W)=a+b-d_X}} m_W(U, V) \cdot W \in Z_{a+b-d_X}(X).$$

Example 1.10. Let X be a smooth projective surface, and let C and D be divisors on X . For any point $x \in C \cap D$, locally we think of $C = \{f = 0\}$ and $D = \{g = 0\}$ around x , then $m_x(C, D) = \ell_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(f, g))$.

⁶Despite the notation, this has nothing to do with a K-theory.

Definition 1.11. Suppose X is a scheme of finite type over k , and $\mathcal{F} \in K_n(X)$ is a coherent sheaf, then we define $Z_a(\mathcal{F}) = \sum_{\dim(\bar{\eta})=a} (\mathcal{O}_{X,\eta}(\mathcal{F}_\eta) \cdot \bar{\eta}) \in Z_a(X)$.

Therefore, we define the cycle of \mathcal{F} as an element of the cycle of X .

Definition 1.12 ([Har13], Exercise III.6.9). Every coherent sheaf \mathcal{F} on $X \in \text{Sm}/k$ has a resolution

$$0 \longrightarrow E_k \longrightarrow E_{k-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

where E_i 's are locally free of finite rank. Therefore, for any coherent sheaf \mathcal{G} , we can define the Tor functor⁷ of coherent sheaves by

$$\text{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = H_i(E_* \otimes_{\mathcal{O}_X} \mathcal{G}).$$

Proposition 1.13. Let $X \in \text{Sm}/k$. Suppose $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(X)$ intersect properly, then

$$Z_a(\mathcal{F}) \cdot Z_b(\mathcal{G}) = \sum_{i=0}^{d_X} (-1)^i \cdot Z_{a+b-d_X}(\text{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})).$$

Proof. We only have to do it locally, so we can assume X to be affine, and count the coefficients of $\bar{\xi}$ where $\dim(\xi) = a + b - d_X$. It suffices to show that the stalks at ξ satisfies

$$\chi(F_\xi, G_\xi) = \sum_{\substack{\dim(\bar{\lambda})=a \\ \dim(\bar{\eta})=b \\ \xi \in \bar{\lambda} \cap \bar{\eta}}} \ell(\mathcal{F}_\lambda) \cdot \ell(\mathcal{G}_\eta) \cdot m_{\bar{\xi}}(\bar{\lambda}, \bar{\eta}).$$

Because our ring is Noetherian, then \mathcal{F} admits a filtration

$$0 = M_0 \subseteq \cdots \subseteq M_d = \mathcal{F}$$

such that $M_i/M_{i-1} \cong \mathcal{O}_X/\mathcal{I}$ is coherent for prime ideal \mathcal{I} . By the additivity of both sides of the isomorphism, we may assume $\mathcal{F} = \mathcal{O}_X/\mathfrak{p}$ with dimension at most a , where $\mathfrak{p} \sim \lambda \in X$. Similarly, we may assume $\mathcal{G} = \mathcal{O}_X/\mathfrak{q}$ with dimension at most b , where $\mathfrak{q} \sim \eta \in X$. Moreover, set $\xi \in \bar{\lambda} \cap \bar{\eta}$. By definition, we now have $\chi(\mathcal{F}_\xi, \mathcal{G}_\xi) = m_{\bar{\xi}}(\bar{\lambda}, \bar{\eta})$.

- If $\dim(\bar{\lambda}) = a$ and $\dim(\bar{\eta}) = b$, then the equality follows from the fact that $\ell(\mathcal{F}_\lambda) = \ell(\mathcal{G}_\eta) = 1$.
- If not, then either $\dim(\bar{\lambda}) < a$ or $\dim(\bar{\eta}) < b$, then $\bar{\lambda}$ and $\bar{\eta}$ do not intersect properly at $\bar{\xi}$, so both the left-hand side and the right-hand side become 0.

□

Proposition 1.14. The intersection product is commutative.

Proof. This is obvious since the Tor functor is commutative.

□

Proposition 1.15. The intersection product is associative.

Proof. Suppose we pick $\mathcal{F} \in K_a(X)$, $\mathcal{G} \in K_b(X)$, and $\mathcal{H} \in K_c(X)$ with support dimension at most a, b, c , respectively, and they intersect properly. Let L_* and M_* be free resolutions of \mathcal{F} and \mathcal{H} , respectively. Define a double complex $N_{ij} = L_i \otimes \mathcal{G} \otimes M_j$, then the associativity of tensor product allows us to calculate triple Tor

$$H_i(L_i \otimes H_j(\mathcal{G}) \otimes M_j) \cong \text{Tor}_i(\mathcal{F}, \mathcal{G}, \mathcal{H}) \cong H_i(H_j(L_i \otimes \mathcal{G}) \otimes M_j)$$

as the homology of two (tensor) double complexes. We obtain two spectral sequences

$${}^I E_{p,q}^2 = \text{Tor}_p(\mathcal{F}, \text{Tor}_q(\mathcal{G}, \mathcal{H})) \Rightarrow \text{Tor}_{p+q}(\mathcal{F}, \mathcal{G}, \mathcal{H})$$

⁷Since we are working over sheaves of \mathcal{O}_X -modules, using the same argument on the level of modules shows that the Tor functor is independent from the choice of resolution.

$${}^{II}E_{p,q}^2 = \mathrm{Tor}_p(\mathrm{Tor}_q(\mathcal{F}, \mathcal{G}), \mathcal{H}) \Rightarrow \mathrm{Tor}_{p+q}(\mathcal{F}, \mathcal{G}, \mathcal{H}).$$

Recall Euler-Poincaré characteristic is invariant with respect to taking spectral sequence (*), then

$$\begin{aligned} Z_a(\mathcal{F}) \cdot ((Z_b \mathcal{G}) \cdot Z_c(\mathcal{H})) &= Z_a(\mathcal{F}) \cdot \sum_q (-1)^q Z_{b+c-d_X}(\mathrm{Tor}_q(\mathcal{G}, \mathcal{H})) \text{ by Proposition 1.13} \\ &= \sum_{p,q} (-1)^{p+q} Z_{a+b+c-2d_X}({}^I E_{p,q}^2) \text{ by Proposition 1.13} \\ &= \sum_i (-1)^i Z_{a+b+c-2d_X}(\mathrm{Tor}_i(\mathcal{F}, \mathcal{G}, \mathcal{H})) \text{ by (*)} \\ &= \sum_{p,q} (-1)^{p+q} Z_{a+b+c-2d_X}({}^{II} E_{p,q}^2) \text{ by (*)} \\ &= \sum_p Z_{a+b-d_X}(\mathrm{Tor}_p(\mathcal{F}, \mathcal{G})) \cdot Z_c(\mathcal{H}) \text{ by Proposition 1.13} \\ &= (Z_a(\mathcal{F}) \cdot Z_b(\mathcal{G})) \cdot Z_c(\mathcal{H}) \text{ by Proposition 1.13.} \end{aligned}$$

□

Definition 1.16. Suppose $X_1, X_2 \in \mathrm{Sm}/k$, with $\mathcal{F}_1 \in K_{a_1}(X_1)$ and $\mathcal{F}_2 \in K_{a_2}(X_2)$. We define the cross product of cycles to be

$$Z_{a_1}(\mathcal{F}_1) \times Z_{a_2}(\mathcal{F}_2) = Z_{a_1+d_{X_2}}(p_1^* \mathcal{F}_1) \cdot Z_{a_2+d_{X_1}}(p_2^* \mathcal{F}_2),$$

where $p_i : X_1 \times X_2 \rightarrow X_i$ is the projection for $i = 1, 2$.

Exercise 1.17. One should check that this is well-defined.

Remark 1.18. Suppose $X_1, X_2 \in \mathrm{Sm}/k$, with $\mathcal{F}_1 \in K_{a_1}(X_1)$, $\mathcal{F}_2 \in K_{b_1}(X_1)$, $\mathcal{G}_1 \in K_{a_2}(X_2)$ and $\mathcal{G}_2 \in K_{b_2}(X_2)$. Suppose $Z_{a_1}(\mathcal{F}_1) \cdot Z_{a_2}(\mathcal{G}_1)$ and $Z_{b_1}(\mathcal{F}_2) \cdot Z_{b_2}(\mathcal{G}_2)$ are defined, then

- $Z_{a_1}(\mathcal{F}_1) \times Z_{a_2}(\mathcal{G}_1)$ and $Z_{b_1}(\mathcal{F}_2) \times Z_{b_2}(\mathcal{G}_2)$ intersect properly on $X_1 \times X_2$, and
- $(Z_{a_1}(\mathcal{F}_1) \times Z_{a_2}(\mathcal{G}_1)) \cdot (Z_{b_1}(\mathcal{F}_2) \times Z_{b_2}(\mathcal{G}_2)) = (Z_{a_1}(\mathcal{F}_1) \cdot Z_{b_1}(\mathcal{F}_2)) \times (Z_{a_2}(\mathcal{G}_1) \cdot Z_{b_2}(\mathcal{G}_2)).$

1.3 PUSHOUT AND PULLBACK

Definition 1.19. Suppose X, Y are schemes of finite type over k , and let $f : X \rightarrow Y$ be a proper map. For every irreducible closed subset $c \subseteq X$ of dimension a , we define the direct image to be

$$f_* c = \begin{cases} [k(c) : k(f(c))] \cdot f(c) \in Z_a(Y), & \dim(f(c)) = a \\ 0, & \dim(f(c)) < a \end{cases}$$

to be the direct image of c under f .

Lemma 1.20. Suppose X and Y are schemes of finite type over k of the same dimension n , and that $f : X \rightarrow Y$ is proper, then there exists an open subset $U \subseteq Y$ such that $\dim(Y \setminus U) < n$ and $f : f^{-1}(U) \rightarrow U$ is a finite morphism.

Proof. Suppose $\xi \in Y$ has $\dim(\bar{\xi}) = n$. We can find $U \ni \xi$ such that $f|_U$ has finite fibers by Exercise II.3.7 from [Har13]. By Exercise III.11.2 in [Har13], such f is finite. □

Proposition 1.21. Let $f : X \rightarrow Y$ be a proper morphism between schemes over k of finite type, and let $\mathcal{F} \in K_a(X)$, then

1. $f_* \mathcal{F} \in K_a(Y)$ and the right derived $R^i f_* \mathcal{F} \in K_{a-1}(Y)$ for $i > 0$.
2. $f_* Z_a(\mathcal{F}) = Z_a(f_* \mathcal{F})$.

Proof. 1. By Theorem III.8.8 from [Har13], $R^i f_* \mathcal{F}$ is coherent for all $i \geq 0$. We have $\text{supp}(R^i f_* \mathcal{F}) \subseteq \text{supp}(\mathcal{F})$. If f is finite, then f_* is exact, so $R^i f_* \mathcal{F} = 0$ for $i > 0$. For general cases, we may assume $\dim(f(\text{supp}(\mathcal{F}))) = a$ and set $W = \text{supp}(\mathcal{F})$. We have a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{h} & f(W) \\ i \downarrow & & \downarrow j \\ X & \xrightarrow{f} & Y \end{array}$$

where h is also proper. By Lemma 1.20, there exists $V \subseteq f(W)$ such that $\dim(f(W) \setminus V) < a$ and $h|_V$ is finite. Let \mathcal{I} be the ideal sheaf of W , then $\mathcal{I}^s \mathcal{F} / \mathcal{I}^{s+1} \mathcal{F} = i_* i^* \mathcal{I}^s \mathcal{F} / \mathcal{I}^{s+1} \mathcal{F}$. By the long exact sequence, it suffices to prove the case for $\mathcal{F} = i_* \mathcal{G}$. Then

$$(R^k f_*) i_* \mathcal{G} = R^k (fi)_* \mathcal{G} = j_* R^k h_* \mathcal{G}.$$

It suffices to consider h , but

$$(R^k h_* \mathcal{G})_V = R^k h_*(\mathcal{G}|_{f^{-1}(V)}) = 0$$

for $k > 0$, so $\text{supp}(R^k h_* \mathcal{G}) \subseteq f(W) \setminus V$ if $k > 0$.

2. If f is finite, let us write down the coefficients of ξ of dimension a on both sides, namely

$$\ell((f_* \mathcal{F})_\xi) = \sum_{\substack{\eta \in f^{-1}(\xi) \\ \dim(\eta) = a}} \ell(F_\eta) \cdot [k(\bar{\eta}) : k(\overline{f(\eta)})].$$

By additivity, one reduces to the case when X is affine and $F = \mathcal{O}_X / \mathfrak{p}$. For the general case, use Lemma 1.20, and the case where f is finite. \square

Definition 1.22. Suppose $f : X \rightarrow Y$ where $Y \in \text{Sm}/k$ and X is closed in $Z \in \text{Sm}/k$. Define $j : X \rightarrow Z \times Y$ to be the graph map. For any $C \in Z_a(X)$ and $D \in Z_b(Y)$ such that C and $f^{-1}(D)$ intersect properly, define the intersection cycle to be

$$C \cdot_f D = j_*^{-1}(j(C) \cdot (Z \times D)) \in Z_{a+b-d_Y}(X)$$

In particular, $f^*(D) = X \cdot_f D$ for $C = X$.

Proposition 1.23. Using the notation above, for $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(Y)$, if \mathcal{F} and $f^* \mathcal{G}$ intersect properly, we have

$$Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G}) = \sum_{i=0}^{d_Y} (-1)^i Z_{a+b-d_Y}(L_i(\mathcal{F} \otimes f^* \mathcal{G}))$$

Proof. Denote $p_2 : Z \times Y \rightarrow Y$ to be the projection onto the second coordinate. By linearity, $Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G}) = j_*^{-1}(Z_a(j_* \mathcal{F}) \cdot Z_{b+d_Z}(p_2^* \mathcal{G}))$ for $j : X \rightarrow Z \times Y$. Suppose $L_* \rightarrow \mathcal{G}$ is the locally free resolution of \mathcal{G} . Note that for all $i \geq 0$, we have

$$j^*(j_* \mathcal{F} \otimes p_2^* L_i) = F \otimes f^* L_i,$$

which induces an isomorphism

$$j_* \mathcal{F} \otimes p_2^* L_i = j_*(\mathcal{F} \otimes f^* L_i).$$

Hence $\text{Tor}_i^{\mathcal{O}_{Z \times Y}}(j_* \mathcal{F}, p_2^* \mathcal{G}) = j_* L_i(F \otimes f^* \mathcal{G})$. So

$$j_*^{-1} Z_{a+b-d_Y}(\text{Tor}_i^{\mathcal{O}_{Z \times Y}}(j_* \mathcal{F}, p_2^* \mathcal{G})) = Z_{a+b-\dim(Y)}(L_i(F \otimes f^* \mathcal{G})).$$

Therefore the statement follows. \square

Proposition 1.24. Let $X \in \text{Sm}/k$, $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(X)$ such that \mathcal{F} and \mathcal{G} intersect properly. Let $\Delta : X \rightarrow X \times X$ be the diagonal map, then

$$\Delta^*(Z_a(\mathcal{F}) \times Z_b(\mathcal{G})) = Z_a(\mathcal{F}) \cdot Z_b(\mathcal{G}).$$

Proof. See page 115 of [Ser12]. □

Proposition 1.25. f^* is compatible with intersection product, and $f^*g^* = (gf)^*$.

Proof. See page 119 of [Ser12]. □

Lemma 1.26. Let \mathcal{A} be an abelian category with enough projectives (respectively, injectives) and F be a right (respectively, left) exact functor from \mathcal{A} . Suppose C is chain complex in \mathcal{A} , then there exists a double complex $M_{*,*}$ in \mathcal{A} such that

$${}^I E_{p,q}^2 = L_p F H_q(C) \quad (\text{respectively, } R^{-p} F(H_q(C))).$$

Proof. To do this when F is right exact, use the Cartan-Eilenberg resolution⁸ $C_* \rightarrow C$ and consider the double complex FC_* . □

Proposition 1.27. Suppose $f : X \rightarrow Y$ is in Sm/k , suppose $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(Y)$, then

$$Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G}) = Z_a(\mathcal{F}) \cdot f^* Z_b(\mathcal{G})$$

if both sides are defined.

Proof. We may assume X is affine. Let $L_* \rightarrow \mathcal{G}$ be a free resolution and apply Lemma 1.26 to f^*L_* and $F \otimes -$, then we find a double complex such that

$$\begin{aligned} {}^I E_{p,q}^2 &= \text{Tor}_p(\mathcal{F}, L_q f^* \mathcal{G}) \\ {}^{II} E_{p,q}^2 &= L_p(F \otimes f^* \mathcal{G}). \end{aligned}$$

□

Proposition 1.28. Let $X \subseteq Z$ and $Y, Z \in \text{Sm}/k$ and $f : X \rightarrow Y$ be proper. Suppose $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(Y)$, and suppose \mathcal{F} and $f^* \mathcal{G}$ intersect properly, then

$$f_*(Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G})) = (f_* Z_a(\mathcal{F})) \cdot Z_b(\mathcal{G}).$$

Proof. Pick $L_* \rightarrow \mathcal{G}$ to be a resolution and apply Lemma 1.26 to $F \otimes f^* L_*$ and f_* , then we have a double complex $M_{*,*}$ such that

$${}^I E_{p,q}^2 = R^{-p} f_* L_q(F \otimes f^* \mathcal{G}).$$

On the other hand, $H_q(M_{*,n}) = R^{-q} f_*(F \otimes f^* L_n) = (R^{-q} f_* \mathcal{F}) \otimes L_n$, therefore

$${}^{II} E_{p,q}^2 = \text{Tor}_p(R^{-q} f_* \mathcal{F}, \mathcal{G}).$$

□

Corollary 1.29. Under the same hypothesis as Proposition 1.28, we have

$$f_*(Z_a(\mathcal{F}) \cdot f^*(Z_b(\mathcal{G}))) = f_*(Z_a(\mathcal{F})) \cdot Z_b(\mathcal{G}).$$

⁸See Proposition 11 on page 210 of [GM13].

2 SHEAVES WITH TRANSFERS

We fix a base scheme $S \in \mathbf{Sm}/k$.

Definition 2.1. Let $X, Y \in \mathbf{Sm}/S$, then we define the group of finite correspondences

$$\mathrm{Cor}_S(X, Y) = \mathbb{Z}\{\text{irreducible closed } C \subseteq X \times_S Y \mid C \rightarrow X \text{ finite, } \dim(C) = \dim(X)\}$$

to be the free abelian group generated by elementary correspondences from X to Y .

Example 2.2. For any $f : X \rightarrow Y$, the graph $\Gamma_f = (x, f(x)) \subseteq X \times_S Y$ is a finite correspondence from $X \rightarrow Y$.

Example 2.3. If $f : X \rightarrow Y$ is finite and $\dim(X) = \dim(Y)$, then the graph Γ_f is also a finite correspondence from $Y \rightarrow X$.

Definition 2.4. Define an additive category Cor_S whose objects are the same as \mathbf{Sm}/S , and the hom sets defined as $\mathrm{Hom}_{\mathrm{Cor}_S}(X, Y) = \mathrm{Cor}_S(X, Y)$ as in [Definition 2.1](#). The contravariant additive functors

$$F : \mathrm{Cor}_S^{\mathrm{op}} \rightarrow \mathbf{Ab}$$

are called presheaves with transfers. The corresponding category is denoted by $\mathrm{PSh}(S) = \mathrm{PSh}(\mathrm{Cor}_S)$, which is abelian with enough injectives and projectives. We have a functor $\gamma : \mathbf{Sm}/S \rightarrow \mathrm{Cor}_S$ by [Example 2.3](#).

Example 2.5. Every $X \in \mathbf{Sm}/S$ gives an element $\mathbb{Z}(X) \in \mathrm{PSh}(S)$ defined by $\mathbb{Z}(X)(Y) = \mathrm{Cor}_S(Y, X)$. Therefore, we say $\mathbb{Z}(X)$ is the presheaf with transfers represented by X . By Yoneda Lemma we know there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{PSh}(S)}(\mathbb{Z}(X), F) \cong F(X).$$

Moreover, representable functors give embeddings of \mathbf{Sm}/S and Cor_S into $\mathrm{PSh}(S)$ via

$$\begin{aligned} \mathbf{Sm}/S &\xrightarrow{\gamma} \mathrm{Cor}_S \longrightarrow \mathrm{PSh}(S) \\ X &\longmapsto X \longmapsto \mathbb{Z}(X) \end{aligned}$$

In particular, $\mathbb{Z}(S) = \mathbb{Z}$.

Example 2.6. The presheaves \mathcal{O} and \mathcal{O}^* are in $\mathrm{PSh}(S)$. For any $C \in \mathrm{Cor}_S(X, Y)$ and $f \in \mathcal{O}(Y)$ (respectively, $\mathcal{O}^*(Y)$), we have a diagram

$$\begin{array}{ccc} C & \xrightarrow{i} & X \times_S Y \xrightarrow{p_2} Y \\ & & \downarrow p_1 \\ & & X \end{array}$$

and can define $\mathcal{O}(C)(f) = \mathrm{Tr}_{C/X}((p_2 \circ i)^*(f))$ (respectively, $\mathcal{O}^*(C)(f) = \mathrm{N}_{C/X}((p_2 \circ i)^*(f))$).

Definition 2.7. Let us describe the composition in Cor_S . Suppose $f \in \mathrm{Cor}_S(X, Y)$ and $g \in \mathrm{Cor}_S(Y, Z)$, then from the diagram

$$\begin{array}{ccc} & X \times_S Z & \\ p_{13} \uparrow & & \\ X \times_S Y \times_S Z & \xrightarrow{p_{23}} & Y \times_S Z \\ \downarrow p_{12} & & \\ & X \times_S Y & \end{array}$$

we define the composition $g \circ f = p_{13*}(p_{23}^*(g)p_{12}^*(f))$.

Exercise 2.8. One should check that all intersections are proper.

Proposition 2.9. The composition law is associative.

Proof. Suppose

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

are morphisms in Cor_S , then we have two Cartesian squares

$$\begin{array}{ccc} X \times_S Y \times_S Z \times_S W & \longrightarrow & X \times_S Z \times_S W \\ \downarrow & & \downarrow \\ X \times_S Y \times_S Z & \longrightarrow & X \times_S Z \end{array}$$

and

$$\begin{array}{ccc} X \times_S Y \times_S Z \times_S W & \longrightarrow & X \times_S Y \times_S W \\ \downarrow & & \downarrow \\ Y \times_S Z \times_S W & \longrightarrow & Y \times_S W \end{array}$$

Now using the base-change formula, we know

$$\begin{aligned} h \circ (g \circ f) &= p_{XW}^{XZW} (p_{ZW}^{XZW*} (h) p_{XZ}^{XZW*} p_{XZ}^{XYZ} (p_{YZ}^{XYZ*} (g) p_{XY}^{XYZ*} (f))) \\ &= p_{XW}^{XZW} (p_{ZW}^{XZW*} (h) p_{XZW}^{XYZ} p_{XYZ}^{XYZ*} (p_{YZ}^{XYZ*} (g) p_{XY}^{XYZ*} (f))) \\ &= p_{XW}^{XZW} (p_{ZW}^{XZW*} (h) p_{XZW}^{XYZ} (p_{YZ}^{XYZ*} (g) p_{XY}^{XYZ*} (f))) \\ &= p_{XW}^{XZY} p_{XZW}^{XYZ} (p_{ZW}^{XZY*} (h) p_{YZ}^{XYZ*} (g) p_{XY}^{XYZ*} (f)) \\ &= p_{XW}^{XZY} p_{XZW}^{XYZ} (p_{ZW}^{XZY*} (h) p_{YZ}^{XYZ*} (g) p_{XY}^{XYZ*} (f)) \\ &= p_{XW}^{XZY} p_{XZW}^{XYZ} (p_{ZW}^{XZY*} (h) p_{YZ}^{XYZ*} (g) p_{XY}^{XYZ*} (f)) \\ &= p_{XW}^{XZY} p_{XZW}^{XYZ} (p_{ZW}^{XZY*} (h) p_{YZ}^{XYZ*} (g) p_{XY}^{XYZ*} (f)) \\ &= p_{XW}^{XZY} p_{XZW}^{XYZ} (p_{ZW}^{XZY*} (h) p_{YZ}^{XYZ*} (g) p_{XY}^{XYZ*} (f)) \\ &= p_{XW}^{XZY} p_{XZW}^{XYZ} (p_{ZW}^{XZY*} (h) p_{YZ}^{XYZ*} (g) p_{XY}^{XYZ*} (f)) \\ &= p_{XW}^{XZY} p_{XZW}^{XYZ} (p_{ZW}^{XZY*} (h) p_{YZ}^{XYZ*} (g) p_{XY}^{XYZ*} (f)) \\ &= p_{XW}^{XZY} p_{XZW}^{XYZ} (p_{ZW}^{XZY*} (h) p_{YZ}^{XYZ*} (g) p_{XY}^{XYZ*} (f)) \\ &= h \circ g \circ f. \end{aligned}$$

□

Theorem 2.10. We have $\mathcal{O}(g \circ f) = \mathcal{O}(f) \circ \mathcal{O}(g)$ and $\mathcal{O}^*(g \circ f) = \mathcal{O}^*(f) \circ \mathcal{O}^*(g)$.

Proof. We sketch the proof for \mathcal{O} . Pick $X \in \text{Sm}/k$. For every $a \in \mathbb{N}$, define $\mu_a(x) = \bigoplus_{\dim(V)=a} K(V)$. Therefore, we have a pairing

$$\begin{aligned} \mathcal{O}(X) \times Z_a(X) &\rightarrow \mu_a(X) \\ (S, V) &\mapsto S|_V \end{aligned}$$

by restricting the regular function on the closed subset. For any map $f : X \rightarrow Y$ where X contains irreducible and closed C , suppose C is finite over Y and $s \in K(C)$, then we define $f_*(s) = \text{Tr}_{K(C)/K(f(C))}(s)$.⁹ Therefore, for any finite correspondence $C \in \text{Cor}(Y, X)$ and $s \in \mathcal{O}(X)$, we have

$$\begin{array}{ccc} C & \hookrightarrow & X \times Y \xrightarrow{p_2} Y \\ & & \downarrow p_1 \\ & & X \end{array}$$

and thus $\mathcal{O}(C)(s) = p_{2*}(p_1^*(s)|_C)$.

Now suppose we have closed subsets $C \subseteq X$ and $D \subseteq Y$, with

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow & \nearrow \text{finite} & \\ C & & D \end{array}$$

⁹Here $f(C)$ is closed since f is finite.

and that C and $f^{-1}(D)$ intersect properly, then one can show that

$$f_*(s|_C)|_D = f_*(s|_{C \cdot_f D})$$

by Tor formula. Moreover, for diagrams like

$$\begin{array}{ccc} X \times_S Y \times_S Z & \xrightarrow{p_{23}} & YZ \\ p_{12} \downarrow & & \downarrow p_1 \\ X \times_S Y & \xrightarrow{p_2} & Y \end{array}$$

where $C \subseteq YZ$ and C is finite over Y , then one can show that for all $s \in \mathcal{O}(Y \times_S Z)$ and C finite over Y , we have

$$p_2^* p_{1*}(s|_C) = p_{12*}(p_{23}^*(s)|_{p_{23}^*(C)}).$$

We finish the proof by working with formal calculation. \square

Definition 2.11. Suppose $\mathcal{F}_1, \mathcal{F}_2, \mathcal{G} \in \text{PSh}(S)$ be presheaves with transfers. A bilinear function $\varphi : \mathcal{F}_1 \times \mathcal{F}_2 \rightarrow \mathcal{G}$ is a collection of bilinear maps

$$\varphi_{x_1, x_2} : \mathcal{F}_1(x_1) \times \mathcal{F}_2(x_2) \rightarrow \mathcal{G}(x_1 \times_S x_2)$$

for every $x_1, x_2 \in \text{Sm}/S$ any any morphisms $f_i \in \text{Cor}_S(x_i, x'_i)$ for $i = 1, 2$, such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}_1(x'_1) \times \mathcal{F}_2(x_2) & \xrightarrow{\varphi_{x'_1, x_2}} & \mathcal{G}(x'_1 \times_S x_2) \\ \mathcal{F}_1(f_1) \times \text{id} \downarrow & & \downarrow (f_1 \times \text{id}) \\ \mathcal{F}_1(x_1) \times \mathcal{F}_2(x_2) & \xrightarrow{\varphi_{x_1, x_2}} & \mathcal{G}(x_1 \times_S x_2) \end{array}$$

for f_1 and similarly there is a diagram that commutes for f_2 .

Definition 2.12. Define the tensor product $\mathcal{F}_1 \times \mathcal{F}_2$ to be the presheaf such that for every \mathcal{G} , the hom set $\text{Hom}(\mathcal{F}_1 \times \mathcal{F}_2, \mathcal{G})$ is the same as the collection of bilinear functions $\mathcal{F}_1 \times \mathcal{F}_2 \rightarrow \mathcal{G}$.

Proposition 2.13. The tensor product $\mathcal{F}_1 \otimes \mathcal{F}_2$ exists.

Proof. For every $Z \in \text{Sm}/S$, define

$$(\mathcal{F}_1 \otimes \mathcal{F}_2)(Z) = \bigoplus_{X, Y \in \text{Sm}/S} \mathcal{F}_1(X) \otimes_Z \mathcal{F}_2(Y) \otimes_Z \text{Cor}_S(Z, X \times_S Y) / \sim$$

where \sim is the subgroup generated by the relations $\varphi \otimes \psi(f \times \text{id}_Y) \circ h = f^*(\varphi) \otimes \psi \otimes h$ where $f \in \text{Cor}_S(X', X)$, $\varphi \in \mathcal{F}_1(X)$, $\psi \in \mathcal{F}_2(Y)$, $h \in \text{Cor}_S(Z, X' \times Y)$, and the relations $\varphi \otimes \psi \otimes (\text{id}_X \times g) \circ h = \varphi \otimes g^*(\psi) \otimes h$ where $g \in \text{Cor}_S(Y', Y)$, $\varphi \in \mathcal{F}_1(X)$, $\psi \in \mathcal{F}_2(Y)$, $h \in \text{Cor}_S(Z, X \times Y')$. \square

Definition 2.14. A pointed presheaf (\mathcal{F}, x) is a split injective map given by the constant presheaf $x : \mathbb{Z} \rightarrow \mathcal{F}$ for some $\mathcal{F} \in \text{PSh}(S)$. We set $\mathcal{F}^{\wedge 1} = \mathcal{F}/x$. For any two pointed presheaves (\mathcal{F}_1, x_1) and (\mathcal{F}_2, x_2) , define $\mathcal{F}_1 \wedge \mathcal{F}_2 = (\mathcal{F}_1 \otimes \mathcal{F}_2) / ((\mathcal{F}_1 \otimes x_2) \oplus (x_1 \otimes \mathcal{F}_2))$. This allows us to define $\mathcal{F}^{\wedge n}$ inductively.

Proposition 2.15.

- $\mathbb{Z}(X) \otimes \mathbb{Z}(Y) = \mathbb{Z}(X \times Y)$;
- $\mathcal{F}^{\wedge 1} \otimes \mathcal{G}^{\wedge 1} = \mathcal{F} \wedge \mathcal{G}$.

Definition 2.16. For any $\mathcal{F} \in \text{PSh}(S)$ and $X \in \text{Sm}/S$, define $\mathcal{F}^\times \in \text{PSh}(S)$ by $\mathcal{F}^\times(Y) = \mathcal{F}(X \times_S Y)$. For any $\mathcal{F}, \mathcal{G} \in \text{PSh}(S)$, define $\underline{\text{Hom}}(\mathcal{F}, \mathcal{G}) \in \text{PSh}(S)$ by $\underline{\text{Hom}}(\mathcal{F}, \mathcal{G})(x) = \text{Hom}(\mathcal{F}, \mathcal{G}^\times)$.

Proposition 2.17. We have a tensor-hom adjunction

$$\text{Hom}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \cong \text{Hom}(\mathcal{F}, \underline{\text{Hom}}(\mathcal{G}, \mathcal{H})).$$

Let us give a brief introduction to Nisnevich topology, c.f., section 3 and 4 from Chapter I of [Mil80].

Definition 2.18. Suppose $f : Y \rightarrow X$ is a morphism between schemes that are locally of finite type.

1. It is called unramified if for all $y \in Y$, the maximal ideals satisfy $\mathfrak{m}_{f(y)}\mathcal{O}_{Y,y} = \mathfrak{m}_y$, and $k(y)/k(f(y))$ is a finite separable field extension of function fields.
2. It is called étale if it is both flat and unramified.
3. It is called Nisnevich if for all $x \in X$, there is some $y \in Y$ such that $f(y) = x$, $k(y) = k(x)$, and f is étale.

Definition 2.19. A morphism $f : Y \rightarrow X$ is called a Nisnevich covering if f is Nisnevich and surjective.

Definition 2.20. Suppose $\mathcal{F} \in \text{PSh}(S)$. We say that it is a Nisnevich sheaf with transfers if for any $X \in \text{Sm}/S$ and Nisnevich covering $\pi : Y \rightarrow X$, the sequences

$$\begin{aligned} 0 \longrightarrow \mathcal{F}(X) &\xrightarrow{\pi^*} \mathcal{F}(Y) \xrightarrow{p_1^* - p_2^*} \mathcal{F}(Y \times_X Y) \xrightarrow{p_1} Y \\ 0 \longrightarrow \mathcal{F}(X) &\xrightarrow{\pi^*} \mathcal{F}(Y) \xrightarrow{p_1^* - p_2^*} \mathcal{F}(Y \times_X Y) \xrightarrow{p_2} Y \end{aligned}$$

are exact. The category of Nisnevich sheaves with transfers is denoted by $\text{Sh}(S)$.

Definition 2.21. A local ring is called Henselian if for any monic polynomial $f \in A[t]$ such that its image \bar{f} in the residue field satisfies $\bar{f} = g_0 h_0$ in $k(A)[T]$ where g_0, h_0 are monic and relatively prime, there are monic $g, h \in A[T]$ such that $\bar{g} = g_0, \bar{h} = h_0$ in the residue fields, and $f = gh$.

Example 2.22. Complete local rings are Henselian.

Theorem 2.23 ([Mil80], Theorem I.4.2.). Let A be a local ring, $X = \text{Spec}(A)$, and $x \in X$ be the closed point, then the following are equivalent:

1. A is Henselian;
2. any finite A -algebra B is a direct product of local rings $B \cong \prod_{i \in I} B_i$, where each B_i is of the form $B_{\mathfrak{m}_i}$ for some maximal ideal \mathfrak{m}_i of B ;
3. if $f : Y \rightarrow X$ has finite fibers and is separated, then $Y = \coprod_{i=0}^n Y_i$ where $X \notin f(Y_0)$, and for $i \geq 1$, Y_i is finite over X and is the spectrum of a local ring;
4. if $f : Y \rightarrow X$ is étale and there exists $y \in Y$ such that $f(y) = x$ and $k(y) = k(x)$, then f has a section $s : X \rightarrow Y$ such that $f \circ s = \text{id}_X$.

Now let A be a Noetherian ring and $\mathfrak{p} \in \text{Spec}(A)$. Consider the set I whose elements are pairs (B, \mathfrak{q}) , where B is a connected étale A -algebra, $\mathfrak{q} \in \text{Spec}(B)$, $\mathfrak{q} \cap A = \mathfrak{p}$, i.e., \mathfrak{q} lies over \mathfrak{p} , and $k(\mathfrak{p}) = k(\mathfrak{q})$. We say that $(B_1, \mathfrak{q}_1) \leq (B_2, \mathfrak{q}_2)$ if there is an A -morphism $f : B_1 \rightarrow B_2$ such that $f^{-1}(\mathfrak{q}_2) = \mathfrak{q}_1$. This gives a poset structure.

Proposition 2.24. The set I is a directed set and the ring $\varinjlim_{(B, \mathfrak{q})} B = A_{\mathfrak{p}}^h$, i.e., the Henselization of $A_{\mathfrak{p}}$, is Henselian and admits the following universal property: for any Henselian A -algebra C such that $\mathfrak{m}_C \cap A = \mathfrak{p}$, there is a unique morphism $\varphi : A_{\mathfrak{p}}^h \rightarrow C$ (as a local homomorphism) such that the diagram

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & \nearrow \exists! \varphi & \\ A_{\mathfrak{p}}^h & & \end{array}$$

Proof. This makes use of Lemma I.4.8 from [Mil80]. □

Let φ_X be the smallest Nisnevich site on X . Suppose X is Noetherian, pick $x \in X$, and $\mathcal{F} \in \text{PSh}(\varphi_X)$. We write $\mathcal{F}_x = \mathcal{F}(\mathcal{O}_{X,x}^h) = \varinjlim_{(V,u)} \mathcal{F}(V)$ as the stalk of \mathcal{F} at x , taking all the pairs (V, u) with étale morphism

$$\begin{aligned} V &\rightarrow X \\ u &\mapsto x \end{aligned}$$

with $k(u) = k(x)$.

Proposition 2.25. Let

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

be a complex in $\text{Sh}(\varphi_X)$. The following are equivalent:

1. the complex is exact;
2. for every $x \in X$, the complex

$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{H}_x \longrightarrow 0$$

is exact.

Proof. This mimics the idea in the usual sheaf theory with Zariski topology. To do so, we need to construct a sheafification in the sense of Nisnevich, explained as follows: suppose $\mathcal{F} \in \text{PSh}(\varphi_X)$, define \mathcal{F}^+ as the following: for every Nisnevich covering $\{V_i\}$ of U , define

$$\mathcal{F}(U) = \{(s_i) \in \prod_i \mathcal{F}(V_i) : s_i|_{V_i \times_X V_j} = s_j|_{V_i \times_S V_j}\}.$$

Now let $\mathcal{F}^+(U) = \varinjlim_{V \supseteq U} \mathcal{F}(V)$, then \mathcal{F}^{++} is a Nisnevich sheaf with the same stalks as \mathcal{F} , with a map $\mathcal{F} \rightarrow \mathcal{F}^{++}$. \square

For any Noetherian scheme X with $\dim(X) < \infty$, we define the cochain to be

$$C^p(X) = \{Y \subseteq X \mid \text{codim}(Y) \geq p\}.$$

For $\mathcal{F} \in \text{Sh}(\varphi_X)$. For any $Z \subseteq W$ where $Z \in C^{p+1}(X)$ and $W \in C^p(X)$, we have a long exact sequence

$$\cdots \longrightarrow H_Z^i(X, \mathcal{F}) \longrightarrow H_W^i(X, \mathcal{F}) \longrightarrow H_{W \setminus Z}^i(X \setminus Z, \mathcal{F}) \longrightarrow H_Z^{i+1}(X, \mathcal{F}) \longrightarrow \cdots$$

with supports specified as subscripts, using

$$0 \longrightarrow \mathcal{F}_Z(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}(X \setminus Z) \longrightarrow H_Z^{i+1}(X, \mathcal{F}) \longrightarrow \cdots$$

which gives a right derived functor $H_Z^i = R^i \Gamma_Z(X, -)$. Now define $H^i(C^p(X), \mathcal{F}) = \varinjlim_{Z \in C^p(X)} H_Z^i(X, \mathcal{F})$, then

$$H^i(C^p(X)/C^{p+1}(X), \mathcal{F}) = \varinjlim_{\substack{Z \subseteq W \\ W \in C^p, Z \in C^{p+1}}} H_{W \setminus Z}^i(X \setminus Z, \mathcal{F}).$$

Taking limit with respect to pairs $Z \subseteq W$ where $W \in C^p(X)$ and $Z \in C^{p+1}(X)$, we get a long exact sequence

$$\cdots \rightarrow H^i(C^{p+1}(X), \mathcal{F}) \rightarrow H^i(C^p(X), \mathcal{F}) \rightarrow H^i(C^p(X)/C^{p+1}(X), \mathcal{F}) \rightarrow H^{i+1}(C^{p+1}(X), \mathcal{F}) \rightarrow \cdots$$

Set $F^p H^i(X, \mathcal{F}) = \text{im}(H^i(C^p(X), \mathcal{F}) \rightarrow H^i(X, \mathcal{F}))$, then we obtain the Coniveau spectral sequence

$$E_1^{p,q} = H^{p+q}(C^p(X)/C^{p+1}(X), \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

Definition 2.26. Suppose $x \in X$. Define the local cohomology

$$H_x^i(X, \mathcal{F}) = \varinjlim_{\text{open } x \in V \subseteq X} H_{\bar{x} \cap V}^i(V, \mathcal{F}).$$

This allows us to calculate $E_1^{p,q}$ as

$$E_1^{p,q} = \bigoplus_{\text{codim}(\bar{x})=p} H_x^{p+q}(X, \mathcal{F}).$$

Proposition 2.27 (Excision). Suppose $\varphi : Y \rightarrow X$ is a étale morphism of sheaves, and suppose $Z \subseteq X$ is a closed subset such that $\varphi^{-1}(Z) = Z$. For any $\mathcal{F} \in \text{Sh}(\varphi_X)$, we have

$$H_Z^i(Y, \varphi^* \mathcal{F}) = H_Z^i(X, \mathcal{F}).$$

Proof. The morphism

$$Y \coprod (X \setminus Z) \rightarrow X$$

is a Nisnevich covering, but by the (Nisnevich) sheaf condition on \mathcal{F} , we have a Cartesian square

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(Y) \\ \downarrow & & \downarrow \\ \mathcal{F}(X \setminus Z) & \longrightarrow & \mathcal{F}(Y \setminus Z) \end{array}$$

which shows the result for $i = 0$. The map φ^* is exact and has a left adjoint $\varphi_!$, namely the extension by zero. In particular, φ^* preserves injective objects, and the statement for $i > 0$ follows. \square

Corollary 2.28. The local cohomology agrees with the supported cohomology, i.e.,

$$H_X^i(X, \mathcal{F}) \cong H_x^i(\text{Spec}(\mathcal{O}_{X,x}^h), \mathcal{F}_x).$$

Theorem 2.29. For all $n > \dim(X)$, $H^n(X, \mathcal{F}) = 0$.

Proof. We proceed by induction on $\dim(X)$. If $\dim(X) = 0$, then X is a disjoint union of spectra of Henselian rings, so the statement holds. Now suppose the statement is true for any scheme Y such that $\dim(Y) < \dim(X)$, then the sequence

$$H^i(\text{Spec}(\mathcal{O}_{X,x}^h), \mathcal{F}_x) \rightarrow H^i(\text{Spec}(\mathcal{O}_{X,x}^h \setminus \{x\}), \mathcal{F}_x) \rightarrow H_x^{i+1}(\text{Spec}(\mathcal{O}_{X,x}^h), \mathcal{F}_x) \rightarrow H^{i+1}(\text{Spec}(\mathcal{O}_{X,x}^h), \mathcal{F}_x)$$

has first term and last term as zero, therefore the two terms in the middle agree for $i > 0$, i.e.,

$$H^i(\text{Spec}(\mathcal{O}_{X,x}^h \setminus \{x\}), \mathcal{F}_x) \cong H_x^{i+1}(\text{Spec}(\mathcal{O}_{X,x}^h), \mathcal{F}_x)$$

for $i > 0$. By induction, $H^{n-1}(\text{Spec}(\mathcal{O}_{X,x}^h \setminus \{x\}), \mathcal{F}_x) = 0$ if $n > \dim(\bar{x})$,¹⁰ then

$$E_1^{p,q} \cong \bigoplus_{\text{codim}(\bar{x})=p} H_x^{p+q}(\text{Spec}(\mathcal{O}_{X,x}^h), \mathcal{F}_x) = 0$$

when $p + q > \dim(X)$. Therefore, $H^n(X, \mathcal{F}) = 0$ for $n > \dim(X)$. \square

Theorem 2.30. Let $X, U \in \text{Sm}/S$ and $p : U \rightarrow X$ be a Nisnevich covering. Denote the n -fold product $A \times_B A \times_B \cdots \times_B A$ by A_B^n , then the Čech complex of sheaves (associated to the complex over Sm/S)

$$\check{C}(U/X) \longrightarrow \cdots \longrightarrow \mathbb{Z}(U_X^n) \xrightarrow{d_n} \cdots \longrightarrow \mathbb{Z}(U \times_X U) \xrightarrow{d_2} \mathbb{Z}(U) \longrightarrow \mathbb{Z}(X) \longrightarrow 0$$

is exact, where $d_n = \sum_i (-1)^i \mathbb{Z}(p_i)$ for i th omission map $p_i : U_X^n \rightarrow U_X^{n-1}$.

¹⁰Note that removing the closure of the point reduces the length by 1.

Proof. It suffices to show exactness stalkwise, so to do things locally, we suppose $Y = \text{Spec}(A)$ where A is Henselian, regular and local, and $a \in \text{Cor}_S(Y, U_X^n)$ such that $d_n(A) = 0$. Define $T = \text{supp}(a)$ and $R = T \times_{X \times Y} (U \times Y)$. Since T is a finite correspondence, so by definition it is finite over Y , but Y is Henselian, so T is the spectrum of a disjoint union of Henselian rings by [Theorem 2.23](#). Now R is a Nisnevich covering over T , so the map $R \rightarrow T$ admits a section $s : T \rightarrow R$, where s is both an open immersion and a closed immersion. This gives a diagram of Cartesian squares

$$\begin{array}{ccc} R_T^n & \longrightarrow & (U \times Y)_{X \times Y}^n \times_{X \times Y} ((U \times Y) \setminus (R \setminus T)) \\ \text{id}^n \times s \downarrow & & \downarrow j_{n+1} \\ R_T^{n+1} & \longrightarrow & (U \times Y)_{X \times Y}^{n+1} \\ p_{n+1} \downarrow & & \downarrow p_{n+1} \\ R_T^n & \longrightarrow & (U \times Y)_{X \times Y}^n \end{array}$$

where j_{n+1} is a closed immersion. But note that the composition of the left column is just identity, so we define

$$b = (j_{n+1*}(p_{n+1} \circ j_{n+1}^*)(a)) \in \text{Cor}_S(Y, U_X^{n+1}).$$

By intersection theory, one can check that $d_{n+1}(b) = a$. □

Theorem 2.31. There is a unique sheafification function $a : \text{PSh}(S) \rightarrow \text{Sh}(S)$ such that the following diagram commutes:

$$\begin{array}{ccc} \text{PSh}(S) & \xrightarrow{a} & \text{Sh}(S) \\ \downarrow & & \downarrow \\ \text{PSh}(\text{Sm}/S) & \xrightarrow{+} & \text{Sh}(\text{Sm}(S)) \end{array}$$

Proof. Take $\mathcal{F}_1, \mathcal{F}_2 \in \text{Sh}(S)$. We first prove uniqueness. Suppose $\mathcal{F}_1|_{\text{Sm}/S} = \mathcal{F}_2|_{\text{Sm}/S} = (\mathcal{F}|_{\text{Sm}/S})^+$, set $s \in \mathcal{F}_1(Y) = \mathcal{F}_2(Y)$ and $T \in \text{Cor}_S(X, Y)$ where X is Henselian, then there is a Nisnevich covering $p : U \rightarrow Y$ such that $s|_U = t^+$ where $t \in \mathcal{F}(U)$. Consider the Cartesian square

$$\begin{array}{ccc} T_U & \longrightarrow & X \times U \\ \downarrow & & \downarrow \\ T & \longrightarrow & X \times Y \end{array}$$

then since T is irreducible so T is the spectrum of some Henselian ring, which gives a section s of the map $T_U \rightarrow T$. Denote $D = \text{im}(s)$, then $D \in \text{Cor}_S(X, U)$. Therefore $p \circ D = T$, so we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}_1(X) & \xlongequal{\quad} & \mathcal{F}_2(X) \\ \mathcal{F}_1(D) \uparrow & & \uparrow \mathcal{F}_2(D) \\ \mathcal{F}_1(U) & \xlongequal{\quad} & \mathcal{F}_2(U) \\ \mathcal{F}_1(p) \uparrow & & \uparrow \mathcal{F}_2(p) \\ \mathcal{F}_1(Y) & \xlongequal{\quad} & \mathcal{F}_2(Y) \end{array} \quad \begin{array}{c} \mathcal{F}_1(T) \curvearrowright \\ \mathcal{F}_2(T) \curvearrowleft \end{array}$$

In particular, $\mathcal{F}_1 = \mathcal{F}_2$, so we have uniqueness. To prove existence, we make $(\mathcal{F}|_{\text{Sm}/S})^+$ a sheaf with transfers. Suppose $y \in (\mathcal{F}|_{\text{Sm}/S})^+(Y)$, and $y|_U = Z^+$, where $p : U \rightarrow Y$ is a Nisnevich covering and $Z \in \mathcal{F}(U)$ (and so Z^+ is the image of Z over sheafification). By shrinking U , we allow Z to agree on the intersection, i.e., we may assume that Z is mapped to 0 in $\mathcal{F}(U \times_Y U)$. This gives a sequence

$$0 \longrightarrow \text{Hom}(\mathbb{Z}(Y), (\mathcal{F}|_{\text{Sm}/S})^+) \longrightarrow \text{Hom}(\mathbb{Z}(0), (\mathcal{F}|_{\text{Sm}/S})^+) \longrightarrow \text{Hom}(\mathbb{Z}(U \times_X U), (\mathcal{F}|_{\text{Sm}/S})^+)$$

which is exact by [Theorem 2.30](#). We know that $p^*(Z) = 0$, so there exists $[y] : \mathbb{Z}(Y) \rightarrow (\mathcal{F}|_{\text{Sm}/S})^+$ such that $[y]|_U = y|_U$. Take $f \in \text{Cor}_S(X, Y)$, then by Yoneda lemma we know the composition

$$\mathbb{Z}(X) \xrightarrow{f} \mathbb{Z}(Y) \xrightarrow{[y]} (\mathcal{F}|_{\text{Sm}/S})^+$$

of Nisnevich sheaves produces the transfer of y with respect to f . \square

Remark 2.32. The category $\text{Sh}(S)$ is an abelian category, then the statement in [Proposition 2.25](#) holds for $\text{Sh}(S)$.

Proposition 2.33. Suppose $X \in \text{Sm}/S$ and $\{U_1, U_2\}$ is a Zariski covering of X , then we have an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathbb{Z}(U_1 \cap U_2) &\longrightarrow \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2) \longrightarrow \mathbb{Z}(X) \longrightarrow 0 \\ s &\longmapsto (s|_{U_1}, -s|_{U_2}) \\ (s_1, s_2) &\longmapsto s_1 + s_2 \end{aligned}$$

Proof. Note that $U_1 \coprod U_2$ is a Nisnevich covering of X . Applying the Čech complex of X in [Theorem 2.30](#), we obtain an exact sequence

$$\mathbb{Z}(U_1) \oplus \mathbb{Z}(U_1 \cap U_2)^{\oplus 2} \oplus \mathbb{Z}(U_2) \xrightarrow{d} \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2) \xrightarrow{+} \mathbb{Z}(X) \longrightarrow 0$$

where $d(x, y, a, b) = (a - y, y - a)$. \square

Definition 2.34. Define Sim to be the category of simplicial sets $[n] = \{0, \dots, n\}$ for $n \in \mathbb{N}$, where $\text{Hom}_{\text{Sim}}([n], [m])$ is the set of non-decreasing simplicial maps $[n] \rightarrow [m]$.

For any category \mathcal{C} , we define a simplicial (respectively, cosimplicial) object in \mathcal{C} to be a functor $\text{Sim}^{\text{op}} \rightarrow \mathcal{C}$ (respectively, $\text{Sim} \rightarrow \mathcal{C}$).

REFERENCES

- [Bor74] Armand Borel. Stable real cohomology of arithmetic groups. In *Annales scientifiques de l'École Normale Supérieure*, volume 7, pages 235–272, 1974.
- [Dég08] Frédéric Déglise. Motifs génériques. *Rendiconti del Seminario Matematico della Università di Padova*, 119:173–244, 2008.
- [DG05] Pierre Deligne and Alexander B Goncharov. Groupes fondamentaux motiviques de Tate mixte. In *Annales scientifiques de l'École Normale Supérieure*, volume 38, pages 1–56. Elsevier, 2005.
- [GM13] Sergei I Gelfand and Yuri I Manin. *Methods of homological algebra*. Springer Science & Business Media, 2013.
- [Har77] Günter Harder. Die Kohomologie s-arithmetischer Gruppen über Funktionenkörpern. *Inventiones mathematicae*, 42:135–175, 1977.
- [Har13] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer Science & Business Media, 2013.
- [Lev93] Marc Levine. Tate motives and the vanishing conjectures for algebraic K-theory. *Algebraic K-theory and algebraic topology*, pages 167–188, 1993.
- [Lev94] Marc Levine. Bloch's higher chow groups revisited. *Astérisque*, 226(10):235–320, 1994.
- [Lev99] Marc Levine. K-theory and motivic cohomology of schemes. *preprint*, 166:167, 1999.
- [Mil80] James S Milne. *Etale cohomology (PMS-33)*. Princeton university press, 1980.
- [MVW06] Carlo Mazza, Vladimir Voevodsky, and Charles A Weibel. *Lecture notes on motivic cohomology*, volume 2. American Mathematical Soc., 2006.

- [Qui72] Daniel Quillen. On the cohomology and k-theory of the general linear groups over a finite field. *Annals of Mathematics*, 96(3):552–586, 1972.
- [Ros96] Markus Rost. Chow groups with coefficients. *Documenta Mathematica*, 1:319–393, 1996.
- [Ser12] Jean-Pierre Serre. *Local algebra*. Springer Science & Business Media, 2012.
- [Voe97] Vladimir Voevodsky. The milnor conjecture, 1997.
- [Voe03a] Vladimir Voevodsky. Motivic cohomology with $\mathbf{Z}/2$ -coefficients. *Publications Mathématiques de l’IHÉS*, 98:59–104, 2003.
- [Voe03b] Vladimir Voevodsky. Reduced power operations in motivic cohomology. *Publications Mathématiques de l’IHÉS*, 98:1–57, 2003.
- [Voe11] Vladimir Voevodsky. On motivic cohomology with \mathbf{Z}/l -coefficients. *Annals of mathematics*, pages 401–438, 2011.