

# MATH 518 Notes

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**Definition 1.1.** Let  $M$  be a topological space. An *atlas* on  $M$  is a collection  $\{\varphi_\alpha : U_\alpha \rightarrow W_\alpha\}_{\alpha \in A}$  of homeomorphisms called *coordinate charts*, so that

1.  $\{U_\alpha\}_{\alpha \in A}$  is an open cover of  $M$ ,
2. for all  $\alpha \in A$ ,  $W_\alpha$  is an open subset of some  $\mathbb{R}^{n_\alpha}$ ,
3. for all  $\alpha, \beta \in A$ , the induced map  $\varphi_\beta \circ \varphi_\alpha^{-1}|_{U_\alpha \cap U_\beta}$  is  $C^\infty$ , i.e., smooth.

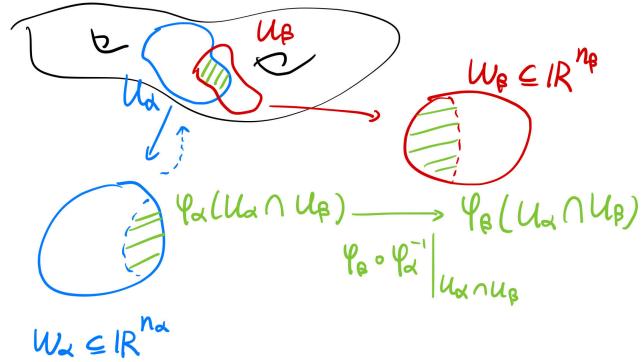


Figure 1: Atlas and Coordinate Chart

**Example 1.2.** Let  $M = \mathbb{R}^n$  be equipped with standard topology, and let  $A = \{*\}$ , so  $U_* = \mathbb{R}^n$  is the open cover of itself. Now the identity map

$$\begin{aligned}\varphi_* : U_* &\rightarrow \mathbb{R}^n \\ u &\mapsto u\end{aligned}$$

is an atlas on  $\mathbb{R}^n$ .

**Example 1.3.** Let  $M = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$  be equipped with subspace topology. Let  $U_\alpha = S^1 \setminus \{(1, 0)\}$  and  $U_\beta = S^1 \setminus \{(-1, 0)\}$ , and let  $A = \{\alpha, \beta\}$ . Let  $W_\alpha = (0, 2\pi)$  and  $W_\beta = (-\pi, \pi)$ . We define  $\varphi_\alpha^{-1}(\theta) = (\cos(\theta), \sin(\theta))$  and  $\varphi_\beta^{-1}(\theta) = (\cos(\theta), \sin(\theta))$ , then

$$(\varphi_\beta \circ \varphi_\alpha^{-1})(\theta) = \begin{cases} \theta, & 0 < \theta < \pi \\ \theta - 2\pi, & \pi < \theta < 2\pi \end{cases}$$

is smooth.

**Example 1.4.** Let  $X$  be a topological space with discrete topology, and let  $A = X$ , then  $\{\varphi_x : \{x\} \rightarrow \mathbb{R}^0\}_{x \in X}$  gives an atlas.

**Example 1.5.** Let  $V$  be a finite-dimensional real vector space of dimension  $n$ . Pick a basis  $\{v_1, \dots, v_n\}$  of  $V$ , then there is a linear bijection  $\varphi$  with inverse

$$\begin{aligned}\varphi^{-1} : \mathbb{R}^n &\rightarrow V \\ (x_1, \dots, x_n) &\mapsto \sum_{i=1}^n x_i v_i.\end{aligned}$$

The topology on  $V$  needs to make  $\varphi^{-1}$  a homeomorphism, and the obvious choice is just the collection of preimages, namely

$$\mathcal{T} = \{\varphi^{-1}(W) \mid W \subseteq \mathbb{R}^n \text{ open}\},$$

then  $\varphi : V \rightarrow \mathbb{R}^n$  becomes an atlas.

**Definition 1.6.** Two atlases  $\{\varphi_\alpha : U_\alpha \rightarrow W_\alpha\}_{\alpha \in A}$  and  $\{\psi_\beta : V_\beta \rightarrow O_\beta\}_{\beta \in B}$  on a topological space  $M$  are *equivalent* if for all  $\alpha \in A$  and  $\beta \in B$ ,

$$\psi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap V_\beta) \subseteq \mathbb{R}^{n_\alpha} \rightarrow \psi_\beta(U_\alpha \cap V_\beta) \subseteq \mathbb{R}^{n_\beta}$$

is always  $C^\infty$ , with  $C^\infty$ -inverses. Such continuous maps are called *diffeomorphisms*. Alternatively, the two atlases are equivalent if their union  $\{\varphi_\alpha\}_{\alpha \in A} \cup \{\psi_\beta\}_{\beta \in B}$  is always an atlas.

**Exercise 1.7.** Equivalence of atlases is an equivalence condition.

**Definition 1.8.** A (smooth) *manifold* is a topological space together with an equivalence class of atlases.

**Convention.** All manifolds are assumed to be smooth of  $C^\infty$ , but not necessarily *Hausdorff* and/or *second countable*.

**Example 1.9.** Continuing from [Example 1.5](#), now suppose  $\{w_1, \dots, w_n\}$  gives another basis of  $V$ , with

$$\begin{aligned}\psi^{-1} : \mathbb{R}^n &\rightarrow V \\ (y_1, \dots, y_n) &\mapsto \sum_{i=1}^n y_i w_i.\end{aligned}$$

This gives a change-of-basis matrix, so it is automatically  $C^\infty$  as a multiplication of invertible matrices. Therefore, the topology here does not depend on the chosen basis.

**Recall.** A topological space  $X$  is *Hausdorff* if for all distinct points  $x, y \in X$ , there exists open neighborhoods  $U \ni x$  and  $V \ni y$  such that  $U \cap V = \emptyset$ .

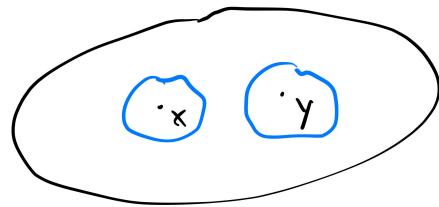


Figure 2: Hausdorff Condition

**Convention.** Via our definition ([Definition 1.8](#)), not all manifolds are Hausdorff.

**Example 1.10.** Let  $Y = \mathbb{R} \times \{0, 1\}$ , i.e., a space with two parallel lines, with a fixed topology. Define  $\sim$  to be the smallest equivalence relation on  $Y$  such that  $(x, 0) \sim (x, 1)$  for  $x \neq 0$ , and define  $X = Y / \sim$ .  $X$  is called the *line with two origins*, and it is second countable but not Hausdorff.

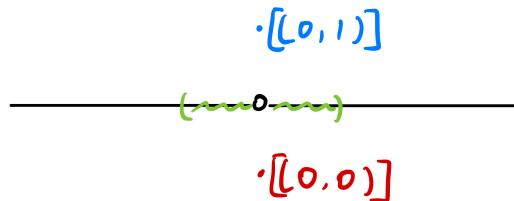


Figure 3: Line with Two Origins

**Example 1.11.** Take charts

$$\begin{aligned} \{\varphi : M = \mathbb{R} \rightarrow \mathbb{R}\} \\ x \mapsto x \end{aligned}$$

and

$$\begin{aligned} \{\psi : M = \mathbb{R} \rightarrow \mathbb{R}\} \\ x \mapsto x^3 \end{aligned}$$

on  $M = \mathbb{R}$ , then

$$\begin{aligned} \varphi \circ \psi^{-1} : \mathbb{R} \rightarrow \mathbb{R} \\ x \mapsto x^{\frac{1}{3}} \end{aligned}$$

is not  $C^\infty$ , so  $\varphi$  and  $\psi$  are two different charts, hence give two different manifolds.

**Definition 1.12.** A map  $F : M \rightarrow N$  between two manifolds is *smooth* if

1.  $F$  is continuous, and
2. for all charts  $\varphi : U \rightarrow \mathbb{R}^m$  on  $M$  and charts  $\psi : V \rightarrow \mathbb{R}^n$  on  $N$ ,  $\psi^{-1} \circ F \circ \varphi|_{\varphi(U \cap F^{-1}(V))}$  is  $C^\infty$ .

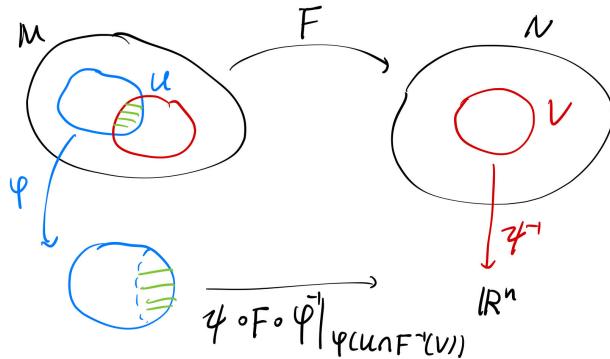


Figure 4: Smooth Map between Manifolds

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**Exercise 2.1.** 1.  $\text{id} : M \rightarrow M$  is smooth.

2. If  $f : M \rightarrow N$  and  $g : N \rightarrow Q$  are smooth maps between manifolds, then so is  $gf : M \rightarrow Q$ .

**Punchline.** The manifolds and the smooth maps between manifolds form a category.

**Recall.** A smooth map  $f : M \rightarrow N$  is called a *diffeomorphism*, as seen in [Definition 1.6](#), if it has a smooth inverse. This is the notion of an isomorphism in the category of manifolds.

**Warning.** 1. Following [Example 1.11](#),

$$\begin{aligned} f : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto x^3 \end{aligned}$$

has an inverse

$$\begin{aligned} f^{-1} : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto x^{\frac{1}{3}}, \end{aligned}$$

but  $f^{-1}$  is not differentiable at  $x = 0$ . Hence,  $f$  is not a diffeomorphism.

2. Take  $\mathbb{R}$  with discrete topology, then all singletons are open sets, then the map

$$\begin{aligned} f : \mathbb{R}_{\text{dis}} &\rightarrow \mathbb{R}_{\text{std}} \\ x &\mapsto x \end{aligned}$$

is a smooth bijection, but  $f^{-1}$  is not continuous.

**Example 2.2.** Consider  $M = (\mathbb{R}, \{\psi = \text{id} : \mathbb{R} \rightarrow \mathbb{R}\})$  and  $N = (\mathbb{R}, \{\psi : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^3\})$  as two manifolds on  $\mathbb{R}$  with standard topology. To see that they are equivalent, consider the homeomorphism

$$\begin{aligned} f : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto x^{\frac{1}{3}}, \end{aligned}$$

then  $(\psi \circ f \circ \varphi^{-1})(x) = \psi(f(x)) = (x^{\frac{1}{3}})^3 = x$ , so  $f$  is smooth, and  $(\psi \circ f \circ \varphi^{-1})^{-1} = \varphi \circ f^{-1} \circ \psi^{-1} = \text{id}$ , therefore  $f^{-1}$  is also smooth. Hence,  $f$  is a diffeomorphism.

We will now consider the real projective space  $\mathbb{R}P^{n-1}$  and the quotient map  $\pi : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}P^{n-1}$ .

**Definition 2.3.** Define a binary relation on  $\mathbb{R}^n \setminus \{0\}$  by  $v_1 \sim v_2$  if and only if there exists  $\lambda \neq 0$  such that  $v_1 = \lambda v_2$ . This is an equivalence relation, and we identify the equivalence class  $[v]$  of  $v \in \mathbb{R}^n \setminus \{0\}$  as a line  $\mathbb{R}v = \text{span}_{\mathbb{R}}\{v\}$  through  $v$ . Then we define the *real projective space*  $\mathbb{R}P^{n-1} = (\mathbb{R}^n \setminus \{0\}) / \sim$ .

The natural topology on  $\mathbb{R}P^{n-1}$  is the quotient topology, where  $\pi : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}P^{n-1}$  is surjective and continuous, so we define  $U \subseteq \mathbb{R}P^{n-1}$  to be open if and only if  $\pi^{-1}(U)$  is open in  $\mathbb{R}^n \setminus \{0\}$ .

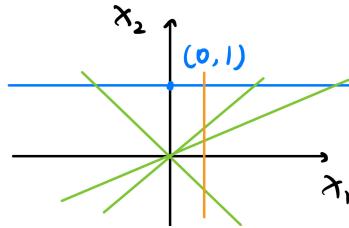


Figure 5: Stereographical Projection

**Claim 2.4.**  $\mathbb{R}P^{n-1}$  is a manifold.

*Proof.* Define

$$\begin{aligned} \varphi_i : U_i &\rightarrow \mathbb{R}^{n-1} \\ [v_1, \dots, v_n] &\mapsto \left( \frac{v_1}{v_i}, \dots, \frac{v_{i-1}}{v_i}, \frac{v_{i+1}}{v_i}, \dots, \frac{v_n}{v_i} \right), \end{aligned}$$

then

$$\begin{aligned}\varphi_i^{-1} : \mathbb{R}^{n-1} &\mapsto U_i \\ (x_1, \dots, x_{n-1}) &\mapsto [(x_1, \dots, x_{i-1}, 1, x_i, \dots, x_{n-1})],\end{aligned}$$

therefore

$$\begin{aligned}\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) &\rightarrow \varphi_j(U_i \cap U_j) \\ (x_1, \dots, x_{n-1}) &\mapsto \varphi_j([(x_1, \dots, x_{i-1}, 1, x_i, \dots, x_{n-1})]) \\ &= \begin{cases} \left( \frac{x_1}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots, \frac{x_{n-1}}{x_j} \right), & j < i \\ (x_1, \dots, x_{n-1}), & j = i \\ \left( \frac{x_1}{x_{j-1}}, \dots, \frac{1}{x_{j-1}}, \dots, \frac{x_{j-2}}{x_{j-1}}, \frac{x_j}{x_{j-1}}, \dots, \frac{x_{n-1}}{x_{j-1}} \right), & j > i \end{cases}\end{aligned}$$

Therefore, this is  $C^\infty$  as a rational map on  $\varphi_i(U_i \cap U_j)$ , and so this gives an atlas, hence  $\mathbb{R}P^{n-1}$  is a manifold.  $\square$

**Claim 2.5.**  $\pi : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}P^{n-1}$  is smooth.

*Proof.* Note that

$$\begin{aligned}\psi : \mathbb{R}^n \setminus \{0\} &\hookrightarrow \mathbb{R}^n \\ x &\mapsto x\end{aligned}$$

is an atlas on  $\mathbb{R}^n \setminus \{0\}$ , and

$$\begin{aligned}\varphi_i \circ \pi \circ \varphi_i^{-1} : \mathbb{R}^n \setminus \{0\} &\rightarrow \mathbb{R}^{n-1} \\ (v_1, \dots, v_n) &\mapsto \varphi_i([(v_1, \dots, v_n)]) \\ &= \left( \frac{v_1}{v_i}, \dots, \frac{v_{i-1}}{v_i}, \frac{v_{i+1}}{v_i}, \dots, \frac{v_n}{v_i} \right).\end{aligned}$$

This is  $C^\infty$  on  $\pi^{-1}(U_i) = \{(v_1, \dots, v_n) \mid v_i \neq 0\}$ , so  $\pi$  is smooth.  $\square$

**Definition 2.6.** A *smooth function* on a manifold  $M$  is a function  $f : M \rightarrow \mathbb{R}$  so that for any coordinate chart  $\varphi : U \rightarrow \varphi(U)$  open in  $\mathbb{R}^m$ , the function  $f \circ \varphi^{-1} : \varphi(U) \rightarrow \mathbb{R}$  is smooth.

**Remark 2.7.**  $f : M \rightarrow \mathbb{R}$  is smooth if and only if  $f : M \rightarrow (\mathbb{R}, \{\text{id} : \mathbb{R} \rightarrow \mathbb{R}\})$ , usually called the *standard manifold structure on  $\mathbb{R}$* , is smooth.

**Notation.** We denote  $C^\infty(M)$  to be the set of all smooth functions  $f : M \rightarrow \mathbb{R}$ .

**Remark 2.8.**  $C^\infty(M)$  is a smooth  $\mathbb{R}$ -vector space, that is, for all  $\lambda, \mu \in \mathbb{R}$  and  $f, g \in C^\infty(M)$ ,

- $(\lambda f + \mu g)(x) = \lambda f(x) + \mu g(x)$  for all  $x \in M$ ,
- $(f \cdot g)(x) = f(x)g(x)$  for all  $x \in M$ .

Therefore,  $C^\infty(M)$  becomes a (commutative, associative)  $\mathbb{R}$ -algebra.

**Fact.** Connecting manifolds have the notion of dimension. That is, the dimensions of open subsets induced by coordinate charts are the same.

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**Definition 3.1.** Let  $M$  be a manifold, then for every point  $q \in M$ , there exists a well-defined non-negative integer  $\dim_M(q)$ , so that for any coordinate chart  $\varphi : U \rightarrow \mathbb{R}^m$  for  $U \ni q$ , we have  $\dim_M(q) = m$  for some non-negative integer  $m$  that only depend on  $M$ . Consequently,  $\dim_M : M \rightarrow \mathbb{Z}^{\geq 0}$  is a locally constant function. This integer  $m$  is called the *dimension* of  $M$ .

*Proof.* Indeed, say  $\psi : V \rightarrow \mathbb{R}^n$  is another chart with  $U \cap V \ni q$ , then  $\psi \circ \varphi^{-1}|_{\varphi(U \cap V)} : \varphi(U \cap V) \subseteq \mathbb{R}^m \rightarrow \psi(U \cap V) \subseteq \mathbb{R}^n$  is a diffeomorphism, therefore the Jacobian  $D(\psi \circ \varphi^{-1})(\varphi(a)) : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a linear isomorphism, thus  $m = n$ .  $\square$

**Definition 3.2.** Suppose  $(M, \{\varphi_\alpha : U_\alpha \rightarrow \mathbb{R}^m\}_{\alpha \in A})$  and  $(N, \{\psi_\beta : V_\beta \rightarrow \mathbb{R}^n\}_{\beta \in B})$  are two manifolds. One can give a manifold structure to the product set  $M \times N$ , called the *product manifold*, as follows:

- give  $M \times N$  the product topology,
- let  $\{\varphi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \rightarrow \mathbb{R}^m \times \mathbb{R}^n\}_{(\alpha, \beta) \in A \times B}$  to be the atlas on  $M \times N$ . This is well-defined since the transition maps of  $\alpha, \alpha' \in A$  and  $\beta, \beta' \in B$  are over  $(U_\alpha \times V_\beta) \cap U_{\alpha'} \times V_{\beta'} = (U_\alpha \cap U_{\alpha'}) \times (V_\beta \cap V_{\beta'})$  with  $(\varphi_{\alpha'} \times \psi_{\beta'}) \circ (\varphi_\alpha \times \psi_\beta)^{-1} = (\varphi_{\alpha'} \circ \varphi_\alpha^{-1}, \psi_{\beta'} \circ \psi_\beta^{-1})$ . This is smooth since products of smooth maps are smooth.

**Punchline.** The product construction of manifolds gives the categorical product in the category of manifolds.

**Property.** 1. The projection maps

$$\begin{aligned} p_M : M \times N &\rightarrow M \\ (m, n) &\mapsto m \end{aligned}$$

and

$$\begin{aligned} p_N : M \times N &\rightarrow N \\ (m, n) &\mapsto n \end{aligned}$$

are  $C^\infty$ .

2. *Universal Property of Product:* for any manifold  $Q$  and smooth maps  $f_M : Q \rightarrow M$  and  $f_N : Q \rightarrow N$ , there exists a unique map

$$\begin{aligned} g : Q &\rightarrow M \times N \\ q &\mapsto (f(q), g(q)) \end{aligned}$$

such that  $p_M \circ g = f_M$ , and  $p_N \circ g = f_N$ .

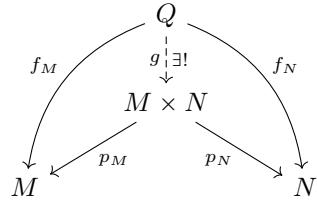


Figure 6: Universal Property of Product

**Recall.** • A topological space  $X$  is *second countable* if the topology has a countable basis: there exists a collection  $\mathcal{B} = \{B_i\}_{i \in \mathbb{N}}$  of open sets so that any open set of  $X$  is a union of some  $B_i$ 's.

- A cover  $\{U_\alpha\}_{\alpha \in A}$  of a topological space is *locally finite* if for all  $x \in X$ , there exists a neighborhood  $N$  of  $x$  such that  $N \cap U_\alpha = \emptyset$  for all but finitely many  $\alpha$ 's.

**Example 3.3.** Let  $X = \mathbb{R}$ , then

- $\{U_n = (-n, n)\}_{n \geq 0}$  is an open cover, but is not locally finite,
- $\{U_n = (n, n+2)\}_{n \in \mathbb{Z}}$  is a locally finite open cover of  $\mathbb{R}$ ,
- $\{U_n = (n, n+2]\}_{n \in \mathbb{Z}}$  is a locally finite cover of  $\mathbb{R}$ , but is not an open cover.

**Recall.** An (open) cover  $\{V_\beta\}_{\beta \in B}$  is a *refinement* of a cover  $\{U_\alpha\}_{\alpha \in A}$  if for all  $\beta$ , there exists  $\alpha = \alpha(\beta)$  such that  $V_\beta \subseteq U_{\alpha(\beta)}$ .

**Definition 3.4.** A Hausdorff topological space is *paracompact* if every open cover has a locally finite open refinement.

**Fact.** A connected Hausdorff manifold is paracompact if and only if it is second countable.

**Corollary 3.5.** A Hausdorff manifold is paracompact if and only if its connected components are second countable.

**Example 3.6.**  $\mathbb{R}$  with discrete topology is paracompact but not second countable.

**Convention.** Usually, we assume manifolds are paracompact, except when we need a non-Hausdorff manifold. This condition is required for the existence of *partition of unity* (i.e., constant function id).

**Recall.** If  $X$  is a space, and  $Y \subseteq X$  is a subset, then the *closure*  $\bar{Y}$  of  $Y$  is the smallest closed set containing  $Y$ .

**Definition 3.7.** Given a topological space  $X$  and a function  $f : X \rightarrow \mathbb{R}$ , the *support* of  $f$  over  $X$  is

$$\text{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}.$$

**Example 3.8.** The function

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

is  $C^\infty$ , with support  $\overline{(0, \infty)} = [0, \infty)$ .

**Definition 3.9.** Let  $M$  be a topological space and let  $\{U_\alpha\}_{\alpha \in A}$  be an open cover. A *partition of unity* subordinate to the cover is a collection of continuous functions  $\{\psi_\alpha : M \rightarrow [0, 1]\}_{\alpha \in A}$  such that

1.  $\text{supp}(\psi_\alpha) \subseteq U_\alpha$  for all  $\alpha \in A$ ,
2.  $\{\text{supp}(\psi_\alpha)\}_{\alpha \in A}$  is a locally finite closed cover of  $M$ ,
3.  $\sum_{\alpha \in A} \psi_\alpha(x) = 1$  for all  $x \in M$ .

**Remark 3.10.** For all  $x \in M$ , there exists  $\alpha_1, \dots, \alpha_n$  such that  $x \in \text{supp}(\psi_{\alpha_i})$ . Hence, for  $\alpha \neq \alpha_1, \dots, \alpha_n$ ,  $\psi_\alpha(x) = 0$ . Therefore, the summation in [Definition 3.9](#) is finite.

**Theorem 3.11.** Let  $M$  be a paracompact manifold with open cover  $\{U_\alpha\}_{\alpha \in A}$ , then there exists a partition of unity  $\{\psi_\alpha : U_\alpha \rightarrow [0, 1]\}_{\alpha \in A} \subseteq C^\infty(M)$  subordinate to the cover.

**Example 3.12.** Let  $M = \mathbb{R}$  and consider for  $n > 0$  the open sets  $\{U_n = (-n, n)\}_{n \in \mathbb{N}}$ . This is not locally finite at one point.

**Example 3.13.** Let  $M = \mathbb{R}^n$ , then for all  $x \in \mathbb{R}^n$  and for  $r > 0$ , we have  $B_r(x) = \{x' \in \mathbb{R}^n \mid \|x - x'\| < r\}$  and so  $\{B_r(x)\}_{r > 0, x \in \mathbb{R}^n}$  is an open cover, but this is not locally finite everywhere.

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We will start to talk about tangent vectors.

**Recall.** For any point  $q \in \mathbb{R}^n$  and any vector  $v \in \mathbb{R}^n$ , and any  $f \in C^\infty(\mathbb{R}^n)$ , the *directional derivative* of  $f$  at  $q$  in direction  $v$  with respect to  $f$  is

$$D_v f(q) = \frac{d}{dt}|_{t=0} f(q + tv).$$

This gives a map  $D_v(-)(q) : C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$  which is

- linear, and

- Leibniz rule holds, i.e.,

$$D_v(fg)(q) = D_v(f)(q) \cdot g(q) + f(q)D_v(g)(q).$$

In other words,  $D_v(-)(q) : C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$  is a derivation.

**Definition 4.1.** Let  $q$  be a point of a manifold  $M$ . A *tangent vector* to  $M$  at  $q$  is an  $\mathbb{R}$ -linear map  $v : C^\infty(M) \rightarrow \mathbb{R}$  such that for all  $f, g \in C^\infty(M)$ ,

$$v(fg) = v(f)g(q) + f(q)v(g).$$

**Remark 4.2.**  $v$  gives smooth vector fields over  $M$  an  $C^\infty(M)$ -module structure via evaluation.

**Lemma 4.3.** The set  $T_q M$  of all tangent vectors to  $M$  at  $q$  is an  $\mathbb{R}$ -vector space.

**Lemma 4.4.** Suppose  $c \in C^\infty(M)$  is a constant function, then for all  $q$  and all  $v \in T_q M$ ,  $v(c) = 0$ .

*Proof.* We have  $v(1) = v(1 \cdot 1) = 1(q)v(1) + v(1)1(q) = 2v(1)$ , so  $v(1) = 0$ . For a constant function  $c$ , we have

$$v(c) = v(c \cdot 1) = cv(1) = c(0) = 0.$$

□

**Lemma 4.5** (Hadamard). For any  $f \in C^\infty(\mathbb{R}^n)$ , there exists  $g_1, \dots, g_n \in C^\infty(\mathbb{R}^n)$  such that

- $f(x) = f(0) + \sum_{i=1}^n x_i g_i(x)$ , and
- $g_i(0) = \left( \frac{\partial}{\partial x_i} f \right)(0)$ .

*Proof.* We have

$$\begin{aligned} f(x) - f(0) &= \int_0^1 \frac{d}{dt}(f(tx))dt \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(tx) \cdot x_i dt \\ &= \sum_{i=1}^n x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx) dt \\ &= \sum_{i=1}^n x_i g_i(x). \end{aligned}$$

Therefore,  $g_i(0) = \int_0^1 \frac{\partial f}{\partial x_i}(t \cdot 0) dt = \frac{\partial f}{\partial x_i}(0)$ . □

**Remark 4.6.** For  $1 \leq i \leq n$ , we have canonical tangent vectors to  $\mathbb{R}^n$  at 0 given by

$$\begin{aligned} \frac{\partial}{\partial x_i}|_0 : C^\infty(\mathbb{R}^n) &\rightarrow \mathbb{R} \\ f &\mapsto \frac{\partial f}{\partial x_i}(0). \end{aligned}$$

**Lemma 4.7.**  $\left\{ \frac{\partial}{\partial x_1}|_0, \dots, \frac{\partial}{\partial x_n}|_0 \right\}$  is a basis of  $T_0 \mathbb{R}^n$ .

*Proof.* Suppose  $\sum c_i \frac{\partial}{\partial x_i}|_0 = 0$ , then

$$0 = \left( \sum_i c_i \frac{\partial}{\partial x_i}|_0 \right) (x_j) = \sum_i c_i \delta_{ij} = c_j.$$

Therefore,  $c_j = 0$  for all  $j$ , thus we have linear independence. For all  $v \in T_0 \mathbb{R}^n$ , i.e.,  $v : C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$  is a derivation, then  $v = \sum_i v(x_i) \frac{\partial}{\partial x_i}|_0$ . Let  $f \in C^\infty(\mathbb{R}^n)$ , then  $f(X) = f(0) + \sum x_i g_i(x)$ , thus

$$\begin{aligned} v(f) &= v(f(0)) + \sum_{i=1}^n v(x_i g_i(x)) \\ &= \sum_{i=1}^n v(x_i g_i(x)) \\ &= \sum_{i=1}^n (v(x_i) g_i(0) + x_i(0) v(g_i)) \\ &= \sum_{i=1}^n v(x_i) g_i(0) \\ &= \sum_{i=1}^n v(x_i) \frac{\partial f}{\partial x_i}(0). \end{aligned}$$

□

**Remark 4.8.** This shows  $\dim(T_0 \mathbb{R}^n) = n$  with the basis above.

Now let  $V$  be a finite-dimensional vector space with a basis  $e_1, \dots, e_n$ , then

$$\begin{aligned} \varphi : \mathbb{R}^n &\rightarrow V \\ (t_1, \dots, t_n) &\mapsto \sum_{i=1}^n t_i e_i \end{aligned}$$

is a linear bijection, with linear inverse

$$\begin{aligned} \psi : V &\rightarrow \mathbb{R}^n \\ v &\mapsto (\psi_1(v), \dots, \psi_n(v)) \end{aligned}$$

where  $\psi_i(v)$ 's are linear maps. To describe this with a basis, we have  $\psi(\sum_i a_i e_i) = (a_1, \dots, a_n)$ , i.e.,  $\psi_i(e_j) = \delta_{ij}$ .

**Claim 4.9.**  $\{\psi_1, \dots, \psi_n\}$  is a basis of  $V^* = \text{Hom}(V, \mathbb{R})$ , called the *dual basis* of  $\{e_1, \dots, e_n\}$ , denoted  $e_j^* = \psi_j$ .

*Proof.* Linear independence follows from  $e_j^*(e_i) = \delta_{ij}$ . Given  $\ell : V \rightarrow \mathbb{R}$  to be a linear map, then  $\ell = \sum \ell(e_i) e_i^*$  since  $\left(\sum_i \ell(e_i) e_i^*\right)(e_j) = \ell(e_j)$ . Given  $v \in T_0 \mathbb{R}^n$ ,  $v(f) = \sum a_i \left(\frac{\partial}{\partial x_i}|_0 f\right)$  for all  $f \in C^\infty(\mathbb{R}^n)$ . Note that  $\frac{\partial}{\partial x_i}|_0(x_j) = \delta_{ij}$ , so  $v(x_j) = \sum a_i \frac{\partial}{\partial x_i}|_0(x_j) = \sum_i a_i \delta_{ij} = a_j$ . Therefore, we have  $a_i = v(x_i)$  for all  $i$ , thus  $v(f) = \sum v(x_i) \left(\frac{\partial}{\partial x_i}|_0 f\right)$ . Thus, the dual basis to  $\frac{\partial}{\partial x_1}|_0, \dots, \frac{\partial}{\partial x_n}|_0$  is  $\{d(x_i)_0\}_{i=1}^n$  where  $(dx_i)_0(v) = v(x_i)$  for all  $i$ . Hence, we have  $v = \sum (dx_i)_0(v) \frac{\partial}{\partial x_i}|_0$ . □

**Remark 4.10.** Via a change of basis, this works at every point  $q$  on the local chart, so we can describe the tangent space on any point on a local chart.

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Let  $M$  be a manifold and  $x \in M$ . Recall that a tangent vector  $v : C^\infty(M) \rightarrow \mathbb{R}$  is a derivation, i.e., linear map, and the set of tangent vectors at  $q$  gives the tangent space.

**Example 5.1.** Let  $M = \mathbb{R}^n$ , and  $q = 0$ , then  $\left\{\frac{\partial}{\partial x_1}|_0, \dots, \frac{\partial}{\partial x_n}|_0\right\}$  is a basis of  $T_0 \mathbb{R}^n$ . Moreover, for all  $v \in T_0 \mathbb{R}^n$ ,  $v = \sum v(x_i) \frac{\partial}{\partial x_i}|_0$ , thus  $\{v \mapsto v(x_i)\}_{i=1}^n$  is the dual basis, with  $v(x_i) = (dx_i)_0(v)$  for all  $1 \leq i \leq n$ .

**Remark 5.2.** The proof used Hadamard's lemma ([Lemma 4.5](#)) and the fact that for all  $x \in \mathbb{R}^n$  and all  $t \in [0, 1]$ ,  $f(tx)$  is defined. Thus, the same argument should work for a version of Hadamard's lemma for star-shaped open subsets  $U \subseteq \mathbb{R}^n$ .

**Definition 5.3.** We say an open subset  $U \subseteq \mathbb{R}^n$  is a *star-shaped domain* if for all  $t \in [0, 1]$  and all  $x \in U$ ,  $tx \in U$ .

**Definition 5.4.** Let  $F : M \rightarrow N$  be a smooth map between two manifolds, and  $q \in M$  is a point, then

$$\begin{aligned} T_q F : T_q M &\rightarrow T_q N \\ v(f) &\mapsto v(f \circ F) \end{aligned}$$

via the pullback.

**Exercise 5.5.** Check that the definition makes sense, in particular:

- (i)  $(T_q F)(v)$  is a tangent vector to  $N$  of  $F(q)$ , and
- (ii)  $T_q F$  is a derivation.

**Remark 5.6.** (a) It is easy to deduce the *chain rule*. That is, given  $M \xrightarrow{F} N \xrightarrow{G} Q$  with  $q \in M$ , then  $T_q(G \circ F) = T_{F(q)}G \circ T_q F$  because for all  $f \in C^\infty(Q)$  and all  $v \in T_q M$ , we have

$$(T_q(G \circ F)(v))(f) = v(f \circ (G \circ F))$$

and

$$(T_{F(q)}G(T_q F(v))) = (T_q F)(v)(f \circ G) = v((f \circ G) \circ F).$$

- (b)  $T_q(\text{id}_M) = \text{id}_{T_q M}$ .

As a result, we know  $T$  is a functor from the category of pointed manifolds to the category of  $\mathbb{R}$ -vector spaces.

**Corollary 5.7.** If  $F : M \rightarrow N$  is a diffeomorphism, then for all  $q \in M$ ,  $T_q F : T_q M \rightarrow T_{F(q)}N$  is an isomorphism.

*Proof.* Since  $F$  is a diffeomorphism, then it has a smooth inverse  $G : N \rightarrow M$ , so

$$\text{id}_{T_q M} = T_q(\text{id}_M) = T_q(G \circ F) = T_{F(q)}G \circ T_q F$$

and

$$\text{id}_{T_{F(q)}N} = T_{F(q)}(\text{id}_N) = T_{F(q)}(F \circ G) = T_{F(q)}F \circ T_{F(q)}G.$$

□

We also need to show that  $\dim(T_q M) = \dim_q(M)$ , which is a result of [Lemma 5.8](#), whose proof will be postponed till next time.

**Lemma 5.8.** Let  $M$  be a manifold and  $q \in M$ , and let  $U$  be an open neighborhood of  $q$  in  $M$ , and let  $i : U \hookrightarrow M$  be an inclusion, then

$$\begin{aligned} I = T_q i : T_q U &\rightarrow T_q M \\ v(f) &\mapsto v(f|_U) \end{aligned}$$

is an isomorphism for all  $v \in T_q M$  and all  $U \subseteq M$ .

**Notation.** We denote  $r_1, \dots, r_n : \mathbb{R}^m \rightarrow \mathbb{R}$  to be the standard coordinates on  $\mathbb{R}^m$ .

Let  $M$  be a manifold,  $q_0 \in M$ , and  $\varphi : U \rightarrow \mathbb{R}^m$  is a coordinate chart with  $q_0 \in U$ . Now let  $x_i = r_i \circ \varphi$ , then  $\varphi(q) = (x_1(q), \dots, x_m(q))$ .

We may now assume that

- $\varphi(q_0) = 0$ , otherwise, we replace  $\varphi(q)$  by  $\varphi(q) := \varphi(q) - \varphi(q_0)$ , and
- $\varphi(U)$  is an open ball  $B_R(0) = \{r \in \mathbb{R}^m \mid \|r\| < R\}$  because there exists  $R > 0$  such that  $B_R(0) \subseteq \varphi(U)$ , and we can then replace  $U$  with  $\varphi^{-1}(B_R(0))$  and restrict the charts  $\varphi$  to  $\varphi|_{\varphi^{-1}(B_R(0))}$ .

We now define

$$\begin{aligned}\frac{\partial}{\partial x_j}|_{q_0} : C^\infty(U) &\rightarrow \mathbb{R} \\ f &\mapsto \frac{\partial}{\partial r_j}|_0(f \circ \varphi^{-1})\end{aligned}$$

**Claim 5.9.**  $\left\{ \frac{\partial}{\partial x_j}|_{q_0} \right\}_{j=1}^m$  is a basis of  $T_q M$  and for all  $v \in T_{q_0} M$ ,  $v = \sum v(x_j) \frac{\partial}{\partial x_j}|_{q_0}$ .

*Proof.* By Hadamard's lemma [Lemma 4.5](#) on  $B_R(0)$ , for all  $f \in C^\infty(U)$ , we have  $f \circ \varphi^{-1} \in C^\infty(B_R(0))$ , so there exists  $g_1, \dots, g_m \in C^\infty(B_R(0))$  such that  $(f \circ \varphi^{-1})(r) = f(\varphi^{-1}(0)) + \sum r_i g_i(r)$ . Therefore,  $f(q) = f(q_0) + \sum (r_i \circ \varphi)(q)(g_i \circ \varphi)(q)$ , hence  $f = f(q_0) + \sum x_i (g_i \circ \varphi)$ , and  $(g_i \circ \varphi)(q_0) = g_i(0) = \frac{\partial}{\partial r_i}|_0(f \circ \varphi^{-1}) = \frac{\partial}{\partial x_i}|_0(f)$ .

Hence, for all  $v \in T_{q_0}(U)$ , we know

$$\begin{aligned}v(f) &= v(f(q_0)) + v\left(\sum x_i \cdot (g_i \circ \varphi)\right) \\ &= \sum_i v(x_i)(g_i \circ \varphi)(q_0) \\ &= \sum v(x_i) \frac{\partial}{\partial x_i}|_{q_0}(f).\end{aligned}$$

□

**Remark 5.10.** 1. The linear functionals

$$\begin{aligned}(dx_i)_{q_0} : T_{q_0} U &\rightarrow \mathbb{R} \\ v &\mapsto v(x_i)\end{aligned}$$

is the basis of  $(T_{q_0} U)^*$  dual to  $\left\{ \frac{\partial}{\partial x_i}|_{q_0} \right\}$ .

2.  $(T_0 \varphi^{-1})\left(\frac{\partial}{\partial r_i}|_0\right) = \frac{\partial}{\partial x_i}|_{q_0}$  by definition. Since  $\left\{ \frac{\partial}{\partial x_i}|_0 \right\}_{i=1}^n$  is a basis of  $T_0(B_R(0))$ , then  $\left\{ \frac{\partial}{\partial x_i}|_{q_0} \right\}$  has to be a basis.

**Lemma 5.11.** Let  $M$  be a manifold and  $q \in M$  a point. Let  $U \ni q$  be an open neighborhood, and  $f \in C^\infty(M)$  such that  $f|_U = 0$ , then for all  $v \in T_q M$ , we have  $v(f) = 0$ .

*Proof.* We have shown the existence of a bump function  $\rho \in C^\infty(M)$  in homework 1, that is,  $0 \leq \rho(x) \leq 1$ ,  $\text{supp}(\rho) \subseteq U$  and  $\rho \equiv 1$  near  $q$ .

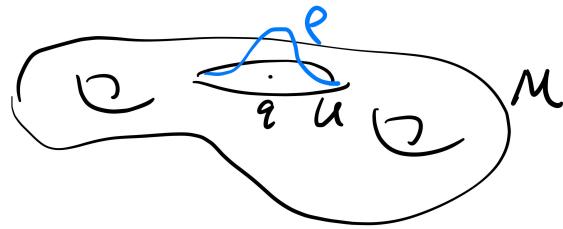


Figure 7: Bump Function

Therefore,  $\rho f \equiv 0$ , so  $v(f) = v(\rho f)(q) + \rho(q)v(f) = v(\rho f) = 0$ .

□

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**Recall.** Given a coordinate chart  $\varphi = (x_1, \dots, x_m) : U \rightarrow \mathbb{R}^m$ , and  $q \in U$  with  $f(q) = 0$ , we defined  $\left\{ \frac{\partial}{\partial x_i}|_q \right\}_{i=1}^m \subseteq T_q U$  by  $\frac{\partial}{\partial x_i}|_q f = \frac{\partial}{\partial r_i}(f \circ \varphi^{-1})|_{\varphi(q)}$  where  $\frac{\partial}{\partial r_i}$ 's are the standard partials on  $C^\infty(\mathbb{R}^m)$ . We know this is a basis with dual basis

$$(dx_i)_q : T_q M \rightarrow \mathbb{R}$$

$$v \mapsto v(x_i)$$

therefore  $v = \sum v(x_i) \frac{\partial}{\partial x_i}|_q$  for all  $v$ . Note that

$$C^\infty(M) \rightarrow C^\infty(U)$$

$$f \mapsto f|_U$$

is not surjective.

Also, we know  $v \in T_q M$  is local, if  $f, g \in C^\infty(M)$  agree on a neighborhood of  $q$ , then  $v(f) = v(g)$ .

Finally, given  $F : M \rightarrow N$ , this induces

$$T_q F : T_q M \rightarrow T_{F(q)} N$$

$$v \mapsto v(f \circ F).$$

**Lemma 6.1.** Given a manifold  $M$  and  $q \in M$ , open neighborhood  $q \in U \subseteq M$  and  $i : U \hookrightarrow M$  inclusion, then

$$I \equiv T_q i : T_q U \rightarrow T_q M$$

is an isomorphism with  $(I(v))(f) = v(f|_U)$  for all  $f \in C^\infty(M)$ .

*Proof.* Suppose  $v \in \ker(I)$ , then  $v(f|_U) = 0$  for all  $f \in C^\infty(M)$ . We want  $v(h) = 0$  for all  $h \in C^\infty(U)$ . We first choose bump function  $\rho : M \rightarrow [0, 1]$  that is  $C^\infty$ , and  $\rho \equiv 1$  near  $q$ , and suppose  $\text{supp}(\rho) \subseteq U$ , hence  $\rho|_{M \setminus U} \equiv 0$ . Then define  $\rho h \in C^\infty(M)$  via

$$\rho h(x) = \begin{cases} \rho(x)h(x), & x \in U \\ 0, & x \notin U \end{cases}$$

Now  $\rho h|_U \equiv h$  near  $q$ , i.e., identically 1. Therefore,  $v(h) = v(\rho h|_U) = 0$ , so  $v \equiv 0$ .

It remains to show that for all  $w \in T_q M$ , there exists  $v \in T_q U$  such that  $I(v) = w$ , i.e., for all  $f \in C^\infty(M)$ ,  $w(f) = v(f|_U)$ . Take the same  $\rho \in C^\infty(M, [0, 1])$  as above, define  $v(h) = w(\rho h)$  for all  $h \in C^\infty(M)$ , and we can check that

- $v \in T_q M$ , and
- for all  $f \in C^\infty(M)$ ,  $v(f|_U) = w(f)$ .

Note that  $v$  is  $\mathbb{R}$ -linear, and for all  $f, g \in C^\infty(W)$  we have  $v(fg) = w(\rho fg) = w(\rho^2 fg)$  since  $\rho fg = \rho^2 fg$  near  $q$ , then we have

$$\begin{aligned} v(fg) &= w(\rho^2 fg) \\ &= w((\rho f)(\rho g)) \\ &= v(\rho f) \cdot (\rho g)(g) + \rho(f)(q) \cdot v(\rho g) \\ &= v(f)g(q) + f(q)v(g). \end{aligned}$$

Finally, for all  $f \in C^\infty(M)$ , we have  $v(f|_U) = w(\rho f) = w(f)$  since  $\rho f = f$  near  $q$ .  $\square$

**Notation.** We now suppress the isomorphisms  $I : T_q U \rightarrow T_q M$ . In particular, given a chart  $\varphi = (x_1, \dots, x_m) : U \rightarrow \mathbb{R}^m$ , we view  $\left\{ \frac{\partial}{\partial x_i}|_q \right\}_{i=1}^m$  as a basis of  $T_q M$ .

**Lemma 6.2.** Let  $V$  be a finite-dimensional vector space with  $q \in V$ , then

$$\begin{aligned}\varphi : V &\rightarrow T_q V \\ v(f) &\mapsto \frac{d}{dt}|_0 f(q + tv)\end{aligned}$$

for all  $f \in C^\infty(V)$ , is an isomorphism.

*Proof.* One can see this is linear, so it suffices to show injectivity. We have

$$\ker(\varphi) = \{v \in V \mid \frac{d}{dt}|_0(q + tv) = 0 \ \forall f \in C^\infty(V)\}.$$

If  $0 \neq v \in \ker(\varphi)$ , then there exists  $\ell : V \rightarrow \mathbb{R}$  such that  $\ell(V) \neq 0$ , so

$$0 \neq \frac{d}{dt}|_0(\ell(q + tv)) = \frac{d}{dt}|_0(\ell(q) + t\ell(v)) = \ell(v).$$

□

**Definition 6.3.** A curve through a point  $q \in M$  on a manifold  $M$  is a  $C^\infty$ -map  $\gamma : (a, b) \rightarrow M$  with  $0 \in (a, b)$  such that  $\gamma(0) = q$ .

**Definition 6.4.** Given  $\gamma : (a, b) \rightarrow M$  with  $\gamma(0) = q$ , we define  $\dot{\gamma}(0) \in T_q M$  by  $\dot{\gamma}(0)f = \frac{d}{dt}|_0 f(\gamma(t)) = \frac{d}{dt}|_0(f \circ \gamma)$  for all  $f \in C^\infty(M)$ .

**Remark 6.5.**

$$\begin{aligned}t : (a, b) &\rightarrow \mathbb{R} \\ x &\mapsto x\end{aligned}$$

is a coordinate chart on  $(a, b)$ , where  $\frac{d}{dt}|_0 \in T_0(a, b)$  is a basis vector. Since  $\gamma$  is  $C^\infty$ ,

$$\begin{aligned}T_0\gamma : T_0(a, b) &\rightarrow T_{\gamma(0)}M \equiv T_q M \\ ((T_0\gamma)(\frac{d}{dt}|_0))f &= \frac{d}{dt}|_0(f \circ \gamma) = \dot{\gamma}(0),\end{aligned}$$

so  $\dot{\gamma}(0) = (T_0\gamma)(\frac{d}{dt}|_0)$ .

Let  $\mathcal{C} = \{\gamma : I \rightarrow M \mid \gamma(0) = q, I \text{ interval depending on } \gamma\}$ , then we have a map

$$\begin{aligned}\Phi : \mathcal{C} &\rightarrow T_q M \\ \gamma &\mapsto \dot{\gamma}(0)\end{aligned}$$

Note that  $\Phi$  is not injective. However, there is an equivalence relation  $\sim$  on  $\mathcal{C}$  defined by  $\gamma \sim \sigma$  if and only if  $\Phi(\gamma) = \Phi(\sigma)$ , so this gives an injection

$$\begin{aligned}\tilde{\Phi} : \mathcal{C}/\sim &\rightarrow T_q M \\ [\gamma] &\mapsto \dot{\gamma}(0).\end{aligned}$$

**Claim 6.6.**  $\tilde{\Phi}$  is onto.

*Proof.* Choose coordinates  $\varphi = (x_1, \dots, x_m) : U \rightarrow \mathbb{R}^m$  near  $q$  such that  $(x_1, \dots, x_m)(q) = 0$ . Now, for all  $v \in T_q M$ , we have  $v = \sum v(x_i) \frac{\partial}{\partial x_i}|_q$ . Consider  $\gamma(t) = \varphi^{-1}(tv(x_1), \dots, tv(x_m))$ , then  $\gamma(0) = \varphi^{-1}(0) = q$  and for any  $f \in C^\infty(M)$ , we have

$$\begin{aligned}\dot{\gamma}(0)f &= \frac{d}{dt}|_0(f \circ \varphi^{-1})(tv(x_1), \dots, tv(x_m)) \\ &= \sum \frac{\partial}{\partial r_i}(f \circ \varphi^{-1})|_0 \cdot v(x_i) \\ &= \sum v(x_i) \frac{\partial}{\partial x_i}|_q f \\ &= v(f).\end{aligned}$$

□

**Lemma 6.7.** For any smooth map  $F : M \rightarrow N$  between manifolds, for all  $q \in M$ , we have

$$T_q F(\dot{\gamma}(0)) = (F \circ \gamma)'(0).$$

*Proof.*

$$\begin{aligned} T_q F(\dot{\gamma}(0)) &= T_q F(T_0 \gamma \left( \frac{d}{dt}|_0 \right)) \\ &= T_0(F \circ \gamma) \left( \frac{d}{dt}|_0 \right) \\ &= (F \circ \gamma)'(0). \end{aligned}$$

□

**Example 6.8.** Let  $M = N = \mathbb{C}$  and  $F(z) = e^z$ . We claim that  $(T_z F)(v) = e^z v$ , which uses  $\mathbb{C} \cong T_w \mathbb{C}$  for all  $w \in \mathbb{C}$ . Indeed, since  $\frac{d}{dt}|_0 e^{tv} = v$ , then

$$\begin{aligned} (T_z F)(v) &= \frac{d}{dt}|_0 F(z + tv) \\ &= \frac{d}{dt}|_0 e^{z+tv} \\ &= \frac{d}{dt}|_0 (e^z e^{tv}) \\ &= e^z v. \end{aligned}$$

Note that  $T_z F$  is an isomorphism for all  $z$ , given by

$$\begin{array}{ccc} T_z \mathbb{C} & \xrightarrow{T_z F} & T_{F(z)} \mathbb{C} \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{C} & \xrightarrow[e^z \cdot -]{} & \mathbb{C} \end{array}$$

Also note that this is not a diffeomorphism, since the inverse is the complex logarithm function, which is only well-behaved on the principal branches.

7 SEPT 6, 2023

**Definition 7.1.** Given a manifold  $M$ ,  $q \in M$ , and  $f \in C^\infty(M)$ , we define the *exact differential* to be a linear map

$$\begin{aligned} df_q : T_q M &\rightarrow \mathbb{R} \\ v &\mapsto v(f) \end{aligned}$$

in  $\text{Hom}(T_q M, \mathbb{R}) =: T_q^* M$ , the cotangent space.

**Exercise 7.2.** •  $df_q$  is linear,

- $f \equiv g$  near  $q$ , then  $df_q = dg_q$ .

We have seen differentials before: given a coordinate chart  $\varphi = (x_1, \dots, x_m) : U \rightarrow \mathbb{R}^m$  is a coordinate chart, then  $\{(dx_i)_q\}_{i=1}^m$  is a basis of  $T_q^* M$  dual to  $\{\frac{\partial}{\partial x_i}|_q\}_{i=1}^m$ . Note that for all  $\eta \in T_q^* M \equiv (T_q M)^*$ , then  $\eta = \sum \eta \left( \frac{\partial}{\partial x_i}|_q \right) (dx_i)_q$ .

**Lemma 7.3.** Let  $M$  be a manifold,  $q \in M$ , and  $f \in C^\infty(M)$ , then the derivative

$$(T_q f)(v) = df_q(v) \frac{d}{dt}|_{f(q)}.$$

*Proof.* Note that  $\{dt_{f(q)}\}$  is a basis of  $T_{f(q)}^*\mathbb{R}$ , then

$$dt_{f(q)}(T_q f(v)) = (T_q f(v))t = v(t \circ f) = v(f) = df_q(v),$$

so  $(T_q f)(v) = df_q(v) \frac{d}{dt}|_{f(q)}$ .  $\square$

**Recall.** Let  $T : V \rightarrow W$  be a linear map, and let  $\{e_1, \dots, e_n\}$  be a basis of  $V$ , and let  $\{f_1, \dots, f_n\}$  be a basis of  $W$ , with dual basis  $\{f_1^*, \dots, f_n^*\}$  in  $W^*$ . Then let  $t_{ij} = f_i^*(Te_j)$ , then

$$T(e_j) = \sum_i f_i^*(Te_j) f_i = \sum_i t_{ij} f_i.$$

For all  $F : \mathbb{R}^m \rightarrow \mathbb{R}^n$ , consider the coordinates  $(x_1, \dots, x_m) : \mathbb{R}^m \rightarrow \mathbb{R}$  and  $(y_1, \dots, y_n) : \mathbb{R}^n \rightarrow \mathbb{R}$ , which gives coordinates  $\{(\frac{\partial}{\partial x_i}|_q)\}$  and  $\{(\frac{\partial}{\partial y_i}|_{F(q)})\}$ , respectively. With  $T = T_q F$ , we have

$$t_{ij} = (dy_i)_{F(q)}(T_q F(\frac{\partial}{\partial x_j}|_q)) = (T_q F(\frac{\partial}{\partial x_j}|_q))y_i = \frac{\partial}{\partial x_j}|_q(y_i \circ F).$$

If we denote  $F = (F_1, \dots, F_n)$  where  $F_i = y_i \circ F$  then this is just  $\frac{\partial F_i}{\partial x_j}(q)$ , so  $\left(\frac{\partial F_i}{\partial x_j}(q)\right)$  is the matrix of  $T_q F$ .

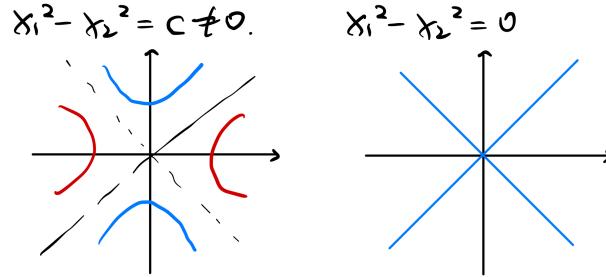
**Definition 7.4.** Let  $F : M \rightarrow N$  be a smooth map, we say  $c \in N$  is a *regular value* of  $F$  if either  $F^{-1}(c) = \emptyset$ , or for all  $q \in F^{-1}(c)$ ,  $T_q F : T_q M \rightarrow T_{F(q)}N = T_c N$  is onto.

We say  $c \in N$  is a *singular value* if it is not a regular value.

**Example 7.5.** Consider

$$\begin{aligned} F : \mathbb{R}^2 &\rightarrow \mathbb{R} \\ (x_1, x_2) &\mapsto x_1 - x_2^2 \end{aligned}$$

for all  $q = (x_1, x_2) \in \mathbb{R}^2$ , then  $T_q F$  is the matrix  $\left(\frac{\partial F}{\partial x_1}(q), \frac{\partial F}{\partial x_2}(q)\right) = (2x_1, 2x_2)$ . Hence,  $c \neq 0$  is a regular value, and  $c = 0$  is a singular value.



**Definition 7.6.** An *embedded submanifold* (of dimension  $k$ ) of a manifold  $M$  is a subspace  $Z \subseteq M$  such that for all  $q \in Z$  there exists a coordinate chart  $\varphi = (x_1, \dots, x_k, x_{k+1}, \dots, x_m) : U \rightarrow \mathbb{R}^m$  with  $\varphi(U \cap Z) = \{(r_1, \dots, r_m) \in \varphi(U) \mid r_k = \dots = r_m = 0\}$ .

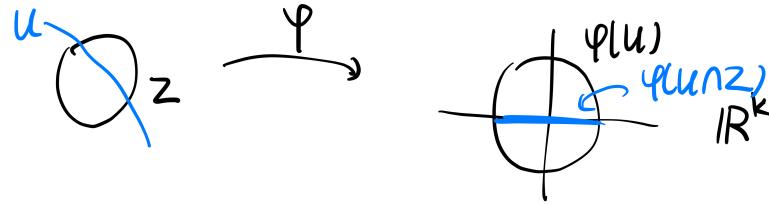


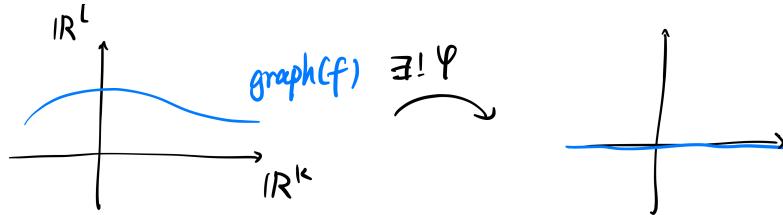
Figure 8: Embedded Submanifold

- Remark 7.7.**
- Any open subset  $U \subseteq M$  is an embedded submanifold.
  - Any singleton in  $M$  is an embedded submanifold.

**Example 7.8.** Let  $f : \mathbb{R}^k \rightarrow \mathbb{R}^l$  be  $C^\infty$ , then the graph of  $f$  is

$$\text{graph}(f) = \{(x, f(x)) \in \mathbb{R}^k \times \mathbb{R}^l \mid x \in \mathbb{R}^k\}$$

is an embedded submanifold of  $\mathbb{R}^k \times \mathbb{R}^l$ .



Here  $\varphi(x, y) = (x, y - f(x))$  is a coordinate chart of  $\mathbb{R}^k \times \mathbb{R}^l$  with inverse  $\varphi^{-1}(x, y') = (x, y' + f(x))$ .

**Theorem 7.9** (Regular Value Theorem). Let  $c \in N$  be a regular value of smooth function  $F : M \rightarrow N$ . If  $F^{-1}(c) = \emptyset$ , then for all  $q \in F^{-1}(c)$ ,  $T_q F : T_q M \rightarrow T_q N$  is onto, so  $F^{-1}(c)$  is an embedded submanifold of  $M$ . Moreover,  $T_q F^{-1}(c) = \ker(T_q F)$  and  $\dim(F^{-1}(c)) = \dim(M) - \dim(N)$ .

**Example 7.10.** Consider

$$\begin{aligned} F : \mathbb{R}^m &\rightarrow \mathbb{R} \\ x &\mapsto \sum x_i^2 = \|x\|^2 \end{aligned}$$

Now  $T_q F$  gives a local chart with  $(2x_1, \dots, 2x_m)$ . Any  $c \neq 0$  is a regular value. We have  $F^{-1}(c) = \{x \mid \|x\|^2 = c\}$  is the sphere of radius  $\sqrt{c}$  for  $c > 0$ . Moreover,  $F^{-1}(0) = \{0\}$ , an embedded submanifold, but  $\dim(\{0\}) \neq \dim(\mathbb{R}^m) - \dim(\mathbb{R})$ .

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**Recall.** A subset  $Z$  of a manifold  $M$  is an embedded submanifold (of dimension  $k$  and codimension  $m - k$  for  $m = \dim(M)$ ) if for all  $z \in Z$ , there exists a coordinate chart  $\varphi : U \rightarrow \mathbb{R}^m$  and  $z \in U$  which is adapted to  $Z$ , i.e.,  $\varphi(U \cap Z) = \varphi(U) \cap (\mathbb{R}^k \times \{0\})$ .

**Remark 8.1.**

- Submanifolds of codimension 0 are open subsets.

- Submanifolds of codimension  $m = \dim(M)$  are discrete sets of points.

We will proceed to prove [Theorem 7.9](#).

**Remark 8.2.** Once we proved  $F^{-1}(c)$  is embedded and  $\dim(F^{-1}(c)) = \dim(M) - \dim(N)$ , then the last statement follows. Indeed, given  $v \in T_q(F^{-1}(c))$ , there exists  $\gamma : (a, b) \rightarrow F^{-1}(c)$  such that  $\gamma(0) = q$ ,  $\gamma'(0) = v$ , and  $F(\gamma(t)) = c$  for all  $t$ . Therefore,

$$0 = \frac{d}{dt}|_0 F(\gamma(t)) = T_q F(\gamma'(0)) = T_q F v,$$

so  $v \in \ker(T_q F)$ , and so  $T_q F^{-1}(c) \subseteq \ker(T_q F)$ . By dimension argument, we have equality.

We will introduce inverse function theorem and implicit function theorem.

**Theorem 8.3** (Inverse Function Theorem). Let  $U \subseteq \mathbb{R}^n$  be open,  $f : U \rightarrow \mathbb{R}^n$  be  $C^\infty$  with  $q \in U$  such that  $T_q f = Df(q) : T_q U = \mathbb{R}^n \rightarrow \mathbb{R}^n = T_{F(q)} \mathbb{R}^n$  is an isomorphism. Then there exists an open neighborhood  $q \in V \subseteq U$  and  $f(q) \in W$  such that  $f : V \rightarrow W$  is a diffeomorphism.

**Notation.** Given  $F : \mathbb{R}^k \times \mathbb{R}^l \rightarrow \mathbb{R}^m$  for  $(a, b) \in \mathbb{R}^k \times \mathbb{R}^l$ , then we denote

- $\frac{\partial F}{\partial x}(a, b) = T_{(a,b)}F|_{\mathbb{R}^k \times \{0\}} = DF(a, b)|_{\mathbb{R}^k \times \{0\}}$ ,
- $\frac{\partial F}{\partial y}(a, b) = T_{(a,b)}F|_{\{0\} \times \mathbb{R}^l} = DF(a, b)|_{\{0\} \times \mathbb{R}^l}$ .

**Theorem 8.4** (Implicit Function Theorem). Let  $F : \mathbb{R}^k \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be  $C^\infty$ , let  $(a, b) \in \mathbb{R}^k \times \mathbb{R}^l$ . Suppose  $\frac{\partial F}{\partial y}(a, b) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an isomorphism, then there exists a neighborhood  $W \ni (a, b)$  and  $U \ni a$  in  $\mathbb{R}^k$ , as well as  $C^\infty$ -map  $g : U \rightarrow \mathbb{R}^n$  such that  $F^{-1}(c) \cap W = \text{graph}(g) \cap W$ .

**Remark 8.5.** inverse function theorem and implicit function theorem are equivalent.

*Proof.* Consider

$$\begin{aligned} H : \mathbb{R}^k \times \mathbb{R}^n &\rightarrow \mathbb{R}^k \times \mathbb{R}^n \\ (x, y) &\mapsto (x, F(x, y)) \end{aligned}$$

then  $H(a, b) = (a, F(a, b)) = (a, c)$ . The partials give

$$DH(a, b) = \begin{pmatrix} I & 0 \\ \frac{\partial F}{\partial x}(a, b) & \frac{\partial F}{\partial y}(a, b) \end{pmatrix}$$

As  $\frac{\partial F}{\partial y}(a, b)$  is invertible, so is  $DH(a, b)$ , so there exists neighborhoods  $(a, b) \in W \subseteq \mathbb{R}^n \times \mathbb{R}^k$  and  $a \in U \subseteq \mathbb{R}^k$ ,  $c \in V \subseteq \mathbb{R}^n$ , such that  $H : W \rightarrow U \times V$  is a diffeomorphism. Consider

$$\begin{aligned} G = H^{-1} : U \times V &\rightarrow W \subseteq \mathbb{R}^n \times \mathbb{R}^l \\ (u, v) &\mapsto (G_1(u, v), G_2(u, v)) \end{aligned}$$

therefore

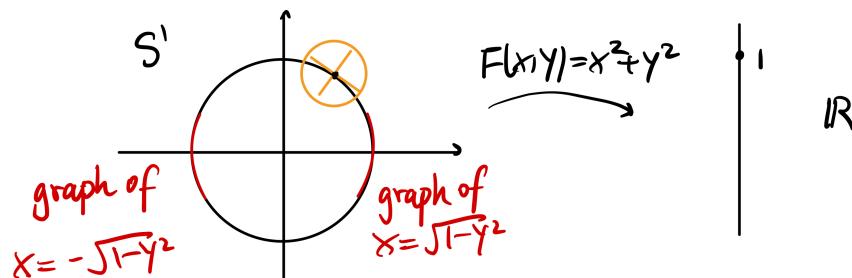
$$(u, v) = H(H^{-1}(u, v)) = H(G_1(u, v), G_2(u, v)) = (G_1(u, v), F(G_1(u, v), G_2(u, v)))$$

so  $G_1(u, v) = u$ , and  $v = F(u, G_2(u, v))$  for all  $u, v$ , hence  $c = F(u, G_2(u, c))$  for all  $u$ . Now let  $g(u) = G_2(u, c)$ , then  $F(u, g(u)) = c$  for all  $u$ . Hence,  $\text{graph}(g) \subseteq F^{-1}(c)$ .  $\square$

*Proof of Regular Value Theorem.* Let  $F : M \rightarrow N$ ,  $c \in N$ ,  $F^{-1}(c) \neq \emptyset$ . Now for all  $q \in F^{-1}(c)$ , then  $T_q F : T_q M \rightarrow T_q N$  is onto. Given  $q \in F^{-1}(c)$ , we want a chart  $T$  from a neighborhood of  $q$  to  $\mathbb{R}^m$ , adapted to  $F^{-1}(c)$ . Let  $\varphi : U \rightarrow \mathbb{R}^m$  and  $\psi : V \rightarrow \mathbb{R}^m$  be charts such that  $q \in U$ ,  $c \in V$ , then

$$\tilde{F} = \psi \circ F \circ \varphi^{-1}|_{\varphi(F^{-1}(V) \cap U)} : \varphi(F^{-1}(V) \cap U) \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^n$$

is  $C^\infty$ . Now  $\psi(c)$  is a regular value in  $\tilde{F}$ . Let  $r = \varphi(q)$ , then we have  $D\tilde{F}(r) : \mathbb{R}^m \rightarrow \mathbb{R}^n$ . Let  $X = \ker(D\tilde{F}(r))$  and  $Y$  be a complement in  $\mathbb{R}^m$ . So  $\mathbb{R}^m = X \otimes Y$  and  $D\tilde{F}(r)|_Y : Y \rightarrow \mathbb{R}^n$  is an isomorphism. Apply inverse function theorem to  $\tilde{F}$  from the intersection of  $X \times Y$  and the open subset to  $\mathbb{R}^n$ .



$\square$

**Example 8.6.** Let  $\text{Sym}^2(\mathbb{R}^n)$  be the  $n \times n$  symmetric real matrices, also known as  $\mathbb{R}^{\frac{n^2-n}{2}+n}$ . There is

$$\begin{aligned} F : \text{GL}(n, \mathbb{R}) &\rightarrow \text{Sym}^2(\mathbb{R}^n) \\ A &\mapsto A^T A \\ F^{-1}I &= \{A \in \text{GL}(n, \mathbb{R}) \mid A^T A = I\} \leftrightarrow I \end{aligned}$$

**Remark 8.7.** We have  $F = F \circ L_A$  for all  $A \in O(U)$ , then for all  $A$ , we have  $T_A F$  onto.

**Claim 8.8.** 1 is a regular value of  $F$ , so  $O(n)$  is an embedded submanifold of  $\text{GL}(n, \mathbb{R})$ .

*Proof.*

$$\begin{aligned} (T_I F)(v) &= \frac{d}{dt}|_0 (I + tv)^T (I + tv) \\ &= \frac{d}{dt}|_0 (I^2 + tv^T + tv + t^2 v^T v) \\ &= v^T + v \end{aligned}$$

and this is surjective since for all  $Y \in \text{Sym}^2(\mathbb{R})$ , we have  $Y = \frac{1}{2}(Y^T + Y)$ , so  $Y = (T_I F)(\frac{1}{2}Y)$ .  $\square$

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**Recall.** Let  $F : M \rightarrow N$  be  $C^\infty$ , let  $c \in N$  be a regular value such that  $F^{-1}(c) \neq \emptyset$ . (For all  $q \in F^{-1}(c)$ ,  $T_q F : T_q M \rightarrow T_q N$  is onto.) Then:

- i  $F^{-1}(c)$  is an embedded submanifold of  $M$ .
- ii  $\dim(M) = \dim(F^{-1}(c)) = \dim(N)$ .
- iii for all  $q \in F^{-1}(c)$ ,  $T_q F^{-1}(c) = \ker(T_q F)$ .

The proof uses inverse function theorem and/or implicit function theorem, and the key is to note that locally  $f^{-1}(c)$  is a graph.

Also,  $O(n) = \{A \in \text{GL}(n, \mathbb{R}) \mid A^T A = I\}$  is an embedded submanifold.

**Definition 9.1.** A *Lie group*  $G$  is a group and a manifold so that

- i the multiplication map

$$\begin{aligned} m : G \times G &\rightarrow G \\ (a, b) &\mapsto (a, b) \end{aligned}$$

is  $C^\infty$ .

- ii the inverse map

$$\begin{aligned} \text{inv} : G &\rightarrow G \\ g &\mapsto g^{-1} \end{aligned}$$

is  $C^\infty$ .

**Notation.**  $e_G = 1_G$  is the identity element.

**Example 9.2.**  $G = \mathbb{R}^n$  with  $m(v, w) = v + w$ , and  $\text{inv}(v) = -v$  gives a Lie group.

**Example 9.3.** Let  $G = \text{GL}(n, \mathbb{R})$  be with  $e_G = \text{diag}(1, \dots, 1) = I$ , with maps  $m(A, B) = AB$  and  $\text{inv}(A) = A^{-1}$ .

**Remark 9.4.** One can think of a Lie group  $G$  as four pieces of data:

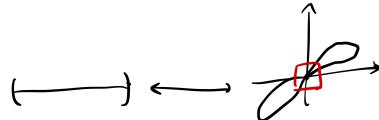
- manifold  $G$ ,
- map  $m : G \times G \rightarrow G$ ,
- map  $\text{inv} : G \rightarrow G$ ,
- $e_G \in G$ .

Note that a subgroup  $H$  of a Lie group  $G$  is not necessarily a Lie group. The sufficient condition would be  $H$  is an embedded submanifold of  $G$ , i.e.,

- $m|_{H \times H} : H \times H \rightarrow H$  are  $C^\infty$ ,
- $\text{inv}|_H : H \rightarrow H$

are  $C^\infty$ . Note  $m|_{H \times H} : H \times H \rightarrow G$  is  $C^\infty$  since  $i : H \hookrightarrow G$  is  $C^\infty$  and  $m|_{H \times H} = m(i \times i)$ .

**Example 9.5.** For example, think of the embedding



but at the origin the preimage is split into three pieces, because the inverse is not continuous, which does not embed into a submanifold.

**Lemma 9.6.** If  $i : Q \hookrightarrow M$  is an embedded submanifold, and  $f : N \rightarrow M$  is a smooth map such that  $f(N) \subseteq Q$ , then  $g : N \rightarrow Q$  with  $g(n) = f(n)$  is  $C^\infty$ .

$$\begin{array}{ccc} N & \xrightarrow{f} & M \\ & \searrow g & \downarrow i \\ & & Q \end{array}$$

*Proof.* Since  $Q \hookrightarrow M$  is embedded, for all  $q \in Q$ , there exists an adapted chart  $\varphi = (x_1, \dots, x_n, x_{k+1}, \dots, x_m) : U \rightarrow \mathbb{R}^m$  such that  $Q \cap U = \{x_k = \dots = x_n = 0\}$ . Consider  $\varphi \circ f|_{f^{-1}(U)} : f^{-1}(U) \rightarrow \mathbb{R}^m$ , then  $f(f^{-1}(U)) \subseteq Q \cap U$ .



Then  $\varphi \circ f|_{f^{-1}(U)} = \varphi(U \cap Q) = \{(r_1, \dots, r_k, r_{k+1}, \dots, r_m) \mid r_{k+1} = \dots = r_n = 0\}$ , so  $\varphi \circ f = (h_1, \dots, h_k, 0, \dots, 0)$  where  $h_1, \dots, h_k \in C^\infty(f^{-1}(U))$ . Therefore,  $\varphi|_{U \cap Q} g|_{f^{-1}(U)} = (h_1, \dots, h_k)$ .  $\square$

**Example 9.7.**  $O(n) \subseteq \text{GL}(n, \mathbb{R})$  is embedded, thus a Lie group.

**Example 9.8.**  $\text{SL}(n, \mathbb{R}) = \{A \in \text{GL}(n, \mathbb{R}) \mid \det(A) = 1\}$  is also a Lie group.

**Claim 9.9.**  $1 \in \mathbb{R}$  is a regular value of  $\det : \text{GL}(n, \mathbb{R}) \rightarrow \mathbb{R}$ .

*Proof.* The key fact is that  $T_I(\det) : \mathbb{R}^{n^2} \rightarrow \mathbb{R}$  is an  $(n \times n)$ -matrix given by  $A \mapsto \text{tr}(A)$ . Indeed, note that the trace is the differential of the determinant.  $\square$

**Definition 9.10.** A (real) *Lie algebra* is a (real) vector space  $\mathfrak{g}$  with an  $\mathbb{R}$ -bilinear map

$$\begin{aligned} [\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} &\rightarrow \mathfrak{g} \\ (X, Y) &\mapsto [X, Y] \end{aligned}$$

such that for all  $X, Y, Z \in \mathfrak{g}$ ,

- $[Y, X] = -[X, Y]$ ,
- $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$ .

**Example 9.11.** Let  $\mathfrak{g} = M_n(\mathbb{R})$ ,  $[X, Y] = XY - YX$  is the anti-commutator.

**Example 9.12.** Let  $M$  be a manifold,  $\mathfrak{g} = \text{Der}(C^\infty(M)) = \{X : C^\infty(M) \rightarrow C^\infty(M) \mid X(fg) = X(f) \cdot g + f \cdot X(g)\}$ . Therefore,  $\mathfrak{g}$  is a Lie algebra with the bracket  $[X, Y](f) = X(Y(f)) - Y(X(f))$  for all  $f \in C^\infty(M)$ . This is the Lie algebra of vector fields on  $M$ .

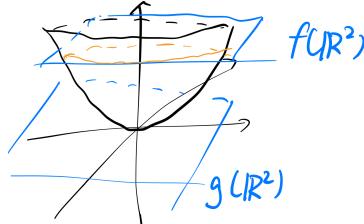
**Example 9.13.** Let  $\mathfrak{g} = \mathbb{R}^3$ , then  $[v, w] := v \times w$  is a Lie algebra with cross product.

We will see that for all Lie group  $G$ ,  $\mathfrak{g} = \text{Lie}(G) = T_e G$  is naturally a Lie algebra.

**Definition 9.14.** Let  $F : M \rightarrow N$  be a  $C^\infty$ -map,  $Z \subseteq N$  be an embedded submanifold. We say  $F$  is *transverse* to  $Z$ , denoted  $F \pitchfork Z$ , if for all  $x \in F^{-1}(Z)$ ,  $T_x F(T_x M) + T_{F(x)} Z = T_{F(x)} N$ .

**Example 9.15.** If  $Z = \{c\}$ , then  $F \pitchfork c$  if and only if for all  $q \in F^{-1}(c)$ ,  $(T_q F)(T_q N) + T_c c = T_c N$ , if and only if for all  $q \in F^{-1}(c)$ ,  $(T_q F)(T_q N) = T_c N$ , if and only if  $c$  is a regular value of  $F$ .

**Example 9.16.** Let  $M = \mathbb{R}^2$ ,  $N = \mathbb{R}^3$ ,  $Z = \{(x, y, z) \mid z = x^2 + y^2\}$ , with  $f(x, y) = (x, y, 1)$  and  $g(x, y) = (x, y, 0)$ , then  $f \pitchfork Z$  but  $g \nparallel Z$ .



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**Theorem 10.1.** Suppose  $f : M \rightarrow N$  is transverse to an embedded submanifold  $Z \subseteq N$ , then

- (i)  $f^{-1}(z)$  is an embedded submanifold of  $M$ .
- (ii) If  $f^{-1}(z) \neq \emptyset$ , then  $\dim(M) - \dim(f^{-1}(z)) = \dim(N) - \dim(Z)$ , i.e.,  $\text{codim}(f^{-1}(z)) = \text{codim}(Z)$ .

*Proof.* Fix  $z_0 \in Z$  with  $f^{-1}(z_0) \neq \emptyset$ , let  $\psi : V \rightarrow \mathbb{R}^n$  be a coordinate chart on  $N$ , adapted to  $Z$  such that  $\psi(V \cap Z) = \psi(V) \cap (\mathbb{R}^k \setminus \{0\})$ . Let  $\pi : \mathbb{R}^k \times \mathbb{R}^{n-k} \rightarrow \mathbb{R}^{n-k}$  be the canonical projection, then

$$(\pi \circ \psi)^{-1}(0) = \psi^{-1}(\pi^{-1}(0)) = \psi^{-1}(\psi(V) \cap (\mathbb{R}^k \times \{0\})) = Z \cap V,$$

therefore

$$(\pi \circ \psi \circ f)^{-1}(0) = f^{-1}(Z \cap V) = f^{-1}(Z) \cap f^{-1}(V).$$

**Claim 10.2.** 0 is a regular value of  $\pi \circ \psi \circ f|_{f^{-1}(V)}$ .

*Subproof.* Take arbitrary  $x \in (\pi \circ \psi \circ f)^{-1}(0) = f^{-1}(V) \cap f^{-1}(Z)$ , then  $T_x f(T_x M) + T_{f(x)} Z = T_{f(x)} N$ . Note that  $T_x M = T_x(f^{-1}(V))$ . Therefore,

$$\mathbb{R}^k \times \mathbb{R}^{n-k} = T_{f(x)} \psi(T_{f(x)} N) = T_{f(x)} \psi(T_x f(T_x f^{-1}(V))) + T_{f(x)} \psi(T_{f(x)} Z)$$

by applying  $T_{f(x)} \psi$  on both sides. Now apply  $T_{\psi(f(x))} \psi$  on both sides, then  $T_{f(x)} \psi(T_{f(x)} Z)$  vanishes, so we get

$$\begin{aligned} \mathbb{R}^{n-k} &= T_{\psi(f(x))} \pi(T_{f(x)} \psi(T_x f(T_x f^{-1}(V)))) \\ &= T_x(\pi \circ \psi \circ f)(T_x f^{-1}(V)). \end{aligned}$$

■

□

**Definition 10.3.** A  $C^\infty$ -map  $f : Q \rightarrow M$  is an *embedding* if

- (i)  $f(Q) \subseteq M$  is an embedded submanifold, and
- (ii)  $f : Q \rightarrow f(Q)$  is a diffeomorphism.

**Remark 10.4.** We know  $f : Q \rightarrow f(Q)$  is  $C^\infty$  since  $f(Q) \subseteq M$  is embedded and  $f : Q \rightarrow M$  is given by the composition of  $i : f(Q) \hookrightarrow M$  and  $f : Q \rightarrow f(Q)$ .

**Remark 10.5.** 1. Since  $f : Q \rightarrow f(Q)$  is a diffeomorphism, then it is a homeomorphism. Thus  $f : Q \rightarrow M$  is a topological embedding.

- 2. For all  $q \in Q$ , then  $T_q f : T_q Q \rightarrow T_{f(q)} M$  is injective, i.e.,  $T_q f(T_q Q) = T_{f(q)} f(Q)$ .

**Example 10.6** (Non-example). Let  $Q = \mathbb{R}$  with discrete topology, then  $Q$  is a paracompact but not second countable as a 0-dimensional manifold. Consider

$$\begin{aligned} f : Q &\rightarrow \mathbb{R}^2 \\ x &\mapsto (x, 0) \end{aligned}$$

be a  $C^\infty$ -map, then this is not an embedding.

**Example 10.7.** Let  $M$  be a manifold with  $f \in C^\infty(M)$ , then

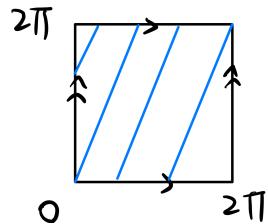
$$\begin{aligned} g : M &\rightarrow M \times \mathbb{R} \\ q &\mapsto (q, f(q)) \end{aligned}$$

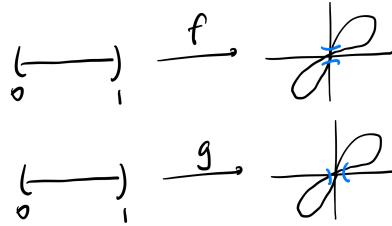
gives an embedding of  $M$  into  $R \times \mathbb{R}$ , as the graph of  $f$ .

**Definition 10.8.** A  $C^\infty$ -map  $f : Q \rightarrow M$  is an *immersion* if for all  $q \in Q$ ,  $T_q f : T_q Q \rightarrow T_{f(q)} M$  is injective.

**Example 10.9.** Consider

$$\begin{aligned} f : \mathbb{R} &\rightarrow S^1 \times S^1 \\ \theta &\mapsto (e^{i\theta}, e^{i\sqrt{2}\theta}) \end{aligned}$$





**Example 10.10.** Now  $g \circ f^{-1} : (0, 1) \rightarrow (0, 1)$  is not an embedding, as it is not continuous.

**Definition 10.11.** The *rank* of a  $C^\infty$ -map  $f : M \rightarrow N$  at a point  $q \in M$  is the rank of the linear map  $T_q f : T_q M \rightarrow T_{f(q)} N$ , i.e.,  $\text{rank}_q(f) = \dim(T_q f(T_q M))$ .

**Example 10.12.** If  $f : M \rightarrow N$  is an immersion, then  $\text{rank}_q(f) = \dim_q(M)$ .

**Remark 10.13.** Immersions are embeddings.

**Theorem 10.14 (Rank Theorem).** Let  $F : M \rightarrow N$  be a  $C^\infty$ -map of constant rank  $k$ . Then for all  $q \in M$ , there exists coordinates  $\varphi = (x_1, \dots, x_m) : U \rightarrow \mathbb{R}^m$  on  $M$  with  $q \in U$ , and  $\psi = (y_1, \dots, y_n) : V \rightarrow \mathbb{R}^n$  with  $F(q) \in V$  such that  $(\psi \circ F \circ \varphi^{-1})(r_1, \dots, r_m) = (r_1, \dots, r_k, 0, \dots, 0)$  for all  $r = (r_1, \dots, r_m) \in \varphi(F^{-1}(V) \cap U)$ .

**Notation.** Given a collection of sets  $\{S_\alpha\}_{\alpha \in A}$ ,  $\coprod_{\alpha \in A} S_\alpha$  is the disjoint union of the collection.

We will give the following construction of a tangent bundle.

**Remark 10.15.** Given a manifold  $M$ , we form a set  $TM = \coprod_{q \in M} T_q M$ . Given a chart  $\varphi = (x_1, \dots, x_n) : U \rightarrow \mathbb{R}^m$  on  $M$ , the corresponding candidate chart is  $\tilde{\varphi} : TU = \coprod_{q \in U} T_q M \rightarrow \varphi(U) \times \mathbb{R}^m$ . One can check that if  $\varphi : U \rightarrow \mathbb{R}^m$  and  $\psi : V \rightarrow \mathbb{R}^m$  are charts on  $M$  with  $U \cap V \neq \emptyset$ , then  $\tilde{\varphi} \tilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^m \rightarrow \psi(U \cap V) \times \mathbb{R}^m$  is  $C^\infty$ . Now we give  $TM$  the topology making  $\tilde{\varphi}$ 's homeomorphic onto their images, then  $\{\tilde{\varphi} : TU \rightarrow \varphi(U) \times \mathbb{R}^m\}$  will be an atlas on  $TM$ .

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**Definition 11.1.** A map  $f : M \rightarrow N$  is a *submersion* if for all  $p \in M$ , the differential  $T_p f : T_p M \rightarrow T_{f(p)} N$  is onto.

**Remark 11.2.** Every value over a submersion is regular.

**Recall.** For a manifold  $M$ , we defined the set  $TM = \coprod_{q \in M} T_q M = \bigcup (\{q\} \times T_q M)$ , which is called a tangent bundle, with additional structures. We will show that  $TM$  is a manifold, and

$$\begin{aligned}\pi : TM &\rightarrow M \\ (q, v) &\mapsto q\end{aligned}$$

is  $C^\infty$  and a submersion.

*Proof.* Let  $\varphi = (x_1, \dots, x_m) : U \rightarrow \mathbb{R}^m$  be a coordinate chart on  $M$ . For any  $q \in U$ , let  $\left\{ \frac{\partial}{\partial x_1} \Big|_q, \dots, \frac{\partial}{\partial x_m} \Big|_q \right\}$  be a basis of  $T_q M$ . The dual basis is  $\{(dx_1)_q, \dots, (dx_m)_q\}$ . For any  $v \in T_q M$ , we have  $v = \sum v(x_i) \frac{\partial}{\partial x_i} \Big|_q := \sum (dx_i)_q(v) \frac{\partial}{\partial x_i} \Big|_q$ , and

$$\begin{aligned}T_q M &\rightarrow \mathbb{R} \\ v &\mapsto ((dx_1)_q(v), \dots, (dx_m)_q(v))\end{aligned}$$

is a linear isomorphism. Define

$$\begin{aligned}\tilde{\varphi} : TU &= \coprod_{q \in M} T_q M \rightarrow \mathbb{R}^m \times \mathbb{R}^m \\ (q, v) &\mapsto (x_1(q), \dots, x_m(q), (dx_1)_q(v), \dots, (dx_m)_q(v)).\end{aligned}$$

Suppose  $\psi = (y_1, \dots, y_m) : V \rightarrow \mathbb{R}^m$  is another chart, we then have

$$\begin{aligned}\tilde{\psi} : TV &\rightarrow \mathbb{R}^m \times \mathbb{R}^m \\ (q, v) &\mapsto (y_1(q), \dots, y_m(q), (dy_1)_q(v), \dots, (dy_m)_q(v)).\end{aligned}$$

**Claim 11.3.** For any  $(r, w) \in \varphi(U \cap V) \times \mathbb{R}^m$ , we have

$$\begin{aligned}(\tilde{\psi} \circ \tilde{\varphi}^{-1})(r, w) &= ((\psi \circ \varphi^{-1})(r), \sum_j \frac{\partial y_1}{\partial x_j}(\varphi^{-1}(r))w_i, \dots, \sum_j \frac{\partial y_m}{\partial x_j}(\varphi^{-1}(r))w_i) \\ &= \left( (\psi \circ \varphi^{-1})(r), \left( \frac{\partial y_i}{\partial x_j}(\varphi^{-1}(r)) \right) \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} \right)\end{aligned}$$

*Subproof.*

**Recall.** If  $T : A \rightarrow B$  is a linear map, with  $\{e_1, \dots, e_n\}$  basis of  $A$ ,  $\{f_1, \dots, f_n\}$  is a basis of  $B$ , with dual basis  $\{f_1^*, \dots, f_n^*\}$ , then we set  $t_{ij} = f_u^*(Te_j)$ , i.e.,

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{(t_{ij})} & \mathbb{R}^n \\ (v_1, \dots, v_n) \mapsto \sum v_i e_i & \downarrow & \downarrow \\ A & \xrightarrow{T} & B \end{array}$$

In our case, we have  $A = B = T_q M$  with  $T = \text{id}$ , with basis  $\left\{ \frac{\partial}{\partial x_i} \Big|_q \right\}$  of  $A$ ,  $\{f_1, \dots, f_n\} = \left\{ \frac{\partial}{\partial y_1} \Big|_q, \dots, \frac{\partial}{\partial y_m} \Big|_q \right\}$  and dual basis  $\{f_1^*, \dots, f_m^*\} = \{(dy_1)_q, \dots, (dy_m)_q\}$ , then

$$\begin{aligned}t_{ij} &= (dy_i)_q \left( \frac{\partial}{\partial x_j} \Big|_q \right) \\ &= \frac{\partial}{\partial x_j} (y_i)(q) \\ &= \frac{\partial y_i}{\partial x_j}(\varphi^{-1}(q)).\end{aligned}$$

■

We define the topology on  $TM$  to be the topology generated by the sets of form  $\tilde{\varphi}^{-1}(W)$  where  $\varphi : U \rightarrow \mathbb{R}^m$  is a coordinate chart with open subset  $W \subseteq \mathbb{R}^m \times \mathbb{R}^m$ . Given an atlas  $\{\varphi_\alpha : U_\alpha \rightarrow \mathbb{R}^m\}$  on  $M$ , we get an induced atlas  $\{\tilde{\varphi}_\alpha : TU_\alpha \rightarrow \mathbb{R}^m \times \mathbb{R}^m\}$  on  $TM$ . One can check that the choice of an atlas on  $M$  does not matter. □

**Exercise 11.4.** • If  $M$  is Hausdorff, then so is  $TM$ .

- If  $M$  is second countable, then so is  $TM$ .

**Lemma 11.5.** The canonical projection  $\pi : TM \rightarrow M$  is  $C^\infty$  and is a submersion.

*Proof.* Let  $\varphi = (x_1, \dots, x_m) : U \rightarrow \mathbb{R}^m$  be a coordinate chart,  $\tilde{\varphi} : TU \rightarrow \mathbb{R}^m \times \mathbb{R}^m$  be the induced chart on  $TM$ , then

$$\begin{aligned} (\varphi \circ \pi \circ \tilde{\varphi}^{-1})(r, w) &= \varphi \circ \pi \left( \varphi^{-1}(r), \sum_i w_i \frac{\partial}{\partial x_i} \Big|_q \right) \\ &= \varphi(\varphi^{-1}(r)) \\ &= r. \end{aligned}$$

Moreover,

$$(T_{(r,w)}(\varphi \circ \pi \circ \tilde{\varphi}^{-1})) (v, w') = v$$

where  $(v, w') \in T_{(r,w)}(\varphi(U) \times \mathbb{R}^m) \cong \mathbb{R}^n \times \mathbb{R}^m$ . Therefore,  $T_{(q,v)}\pi : T_{(q,v)}TM \rightarrow T_q M$  is onto, hence a submersion.  $\square$

**Definition 11.6.** A (*algebraic*) *vector field* on a manifold  $M$  is a derivation  $v : C^\infty(M) \rightarrow C^\infty(M)$ , i.e.,  $v$  is  $\mathbb{R}$ -linear and  $v(fg) = v(f)g + fv(g)$  for all  $f, g \in C^\infty(M)$ .

**Definition 11.7.** A (*geometric*) *vector field* on a manifold  $M$  is a section of the tangent bundle  $TM$  of  $M$ , i.e.,  $X : M \rightarrow TM$  is  $C^\infty$  with  $\pi \circ X = \text{id}_M$ . Geometrically, this depicts tangent vectors over a point with directions in  $X(q)$ .

**Notation.**

- $\text{Der}(C^\infty(M))$  is the set of all derivations of  $C^\infty(M)$ .

- $\mathfrak{X}(M) = \Gamma(TM)$  is the set of sections of  $\pi : TM \rightarrow M$ .

**Proposition 11.8.** Given a section  $v : M \rightarrow TM$  in  $\mathfrak{X}(M)$ , we can try and define

$$\begin{aligned} D_v : C^\infty(M) &\rightarrow C^\infty(M) \\ (D_v(f))(q) &\mapsto v(q)f \end{aligned}$$

and this assignment  $v \mapsto D_v$  is a linear isomorphism.

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**Recall.**  $TM = \coprod_{q \in M} T_q M$  is a manifold. To show this, given chart  $\varphi = (x_1, \dots, x_m) : U \rightarrow \mathbb{R}^m$  on  $M$ , we set

$$\begin{aligned} \tilde{\varphi} = (x_1, \dots, x_m, dx_1, \dots, dx_m) : TU &\equiv \coprod_{q \in U} T_q M \rightarrow \mathbb{R}^m \rightarrow \mathbb{R}^m \\ (q, v) &\mapsto (\varphi(q), (dx_1)_q(v), \dots, (dx_m)_q(v)) \end{aligned}$$

with inverse

$$\tilde{\varphi}^{-1}(r, u) = (\varphi^{-1}(r), \sum_i u_i \frac{\partial}{\partial q_i} \Big|_{\varphi(r)}).$$

Also,

$$\begin{aligned} \pi : TM &\rightarrow M \\ (q, v) &\mapsto q \end{aligned}$$

is a  $C^\infty$ -submersion.

We defined vector fields in two ways,

- as sections of tangent bundle  $\pi : TM \rightarrow M$ , i.e., as  $C^\infty$ -maps  $X : M \rightarrow TM$  such that  $\pi X = \text{id}$ , i.e.,  $X(q) \in T_q M$ , and
- as derivations  $c : C^\infty(M) \rightarrow C^\infty(M)$ , i.e., as  $\mathbb{R}$ -linear maps such that  $c(fg) = fv(g) + fv(f)g$  for all  $f, g \in C^\infty(M)$ .

**Remark 12.1.** Both  $\Gamma(TM)$  and  $\mathfrak{X}(M)$  are  $\mathbb{R}$ -vector spaces, and  $C^\infty(M)$ -modules.

We now prove [Proposition 11.8](#).

*Proof.* Given  $v \in \Gamma(TM)$  and  $f \in C^\infty(M)$ , consider a function

$$\begin{aligned} D_v f : M &\rightarrow \mathbb{R} \\ (D_v(f))(q) &= v(q)f \end{aligned}$$

To go back, given  $X \in \text{Der}(C^\infty(M))$ , for any  $q \in M$ , we have  $\text{ev}_q : C^\infty(M) \rightarrow \mathbb{R}$ , and then  $\text{ev}_q \circ X : C^\infty(M) \rightarrow \mathbb{R}$  is a tangent vector. Define  $v_X(q) = \text{ev}_q \circ X$ , and we can check other requirements like  $C^\infty$  and so on.

**Claim 12.2.**  $D_v f$  is  $C^\infty$ .

*Subproof.* Given a chart  $\varphi = (x_1, \dots, x_m) : U \rightarrow \mathbb{R}^m$ , we have

$$\begin{aligned} \tilde{\varphi} : TU &\rightarrow \mathbb{R}^m \times \mathbb{R}^m \\ (q, v) &\mapsto (\varphi(q), dx_1(v), \dots, dx_m(v)) \end{aligned}$$

Since  $v$  is  $C^\infty$ , the map  $\tilde{\varphi} \circ v|_U : U \rightarrow \mathbb{R}^m \times \mathbb{R}^m$ , defined by  $(\tilde{\varphi} \circ v)(q) = (\varphi(q), (dx_1)_q(v(q)), \dots, (dx_m)_q(v(q)))$ , is  $C^\infty$ . Therefore, the assignment  $q \mapsto (dx_i)_q(v(q))$  are  $C^\infty$  on  $U$ . Hence,  $v = \sum v_i \frac{\partial}{\partial x_i}$  where  $v_i(q) = (dx_i)_q(v(q))$  for all  $i$ . So  $(D_v f)|_U = \left( \sum v_i \frac{\partial}{\partial x_i} \right) f = \sum v_i \frac{\partial f}{\partial x_i}$ . This concludes the proof.  $\blacksquare$

Also, for all  $f, g \in C^\infty(M)$  and all  $q$ , we have

$$\begin{aligned} (D_v(fg))(q) &= v(q)(fg) \\ &= (v(q)f)g(q) + f(q)(v(q)g) \\ &= ((D_v f)g + f(D_v g))(q). \end{aligned}$$

Recall that derivations are local, i.e., for  $X \in \text{Der}(C^\infty(M))$  and  $f \in C^\infty(M)$  and  $f|_U \equiv 0$ , then  $Xf|_U \equiv 0$ . As a consequence, for  $U \subseteq M$  open, define  $X|_U : C^\infty(U) \rightarrow C^\infty(U)$  such that  $(X|_U)(f|_U) = (Xf)|_U$  for all  $f \in C^\infty(M)$ . Now given a chart  $\varphi = (x_1, \dots, x_m) : U \rightarrow \mathbb{R}^m$ , we know  $x_i$ 's are in  $C^\infty(U)$ , then  $(X|_U)(x_i)$  is a smooth function on  $U$ . Therefore,

$$\begin{aligned} v_X|_U &= \sum (dx_i)(v_X) \frac{\partial}{\partial x_i} \\ &= \sum v_X X(x_i) \frac{\partial}{\partial x_i} \\ &= \sum X|_U(x_i) \frac{\partial}{\partial x_i}, \end{aligned}$$

and thus  $v_X|_U : U \rightarrow TU$  is  $C^\infty$ , and since  $U$  is arbitrary, then  $v_X \in \Gamma(TM)$ .  $\square$

**Recall.** For any  $X, Y \in \text{Der}(C^\infty(M))$ ,  $[X, Y] \in \text{Der}(C^\infty(M))$ . Therefore,  $\text{Der}(C^\infty(M))$  is a real Lie algebra with bracket  $(X, Y) \mapsto [X, Y]$ . Note that  $\text{Der}(C^\infty(M)) \subseteq \text{Hom}_{\mathbb{R}}(C^\infty(M), C^\infty(M))$ .

**Recall.** If  $(A, \circ)$  is a real associative algebra, then  $[a, b] := a \circ b - b \circ a$  gives  $A$  the structure of a Lie algebra, and  $\text{Der}(C^\infty(M)) \subseteq \text{Hom}_{\mathbb{R}}(C^\infty(M), C^\infty(M))$ .

Now given a  $C^\infty$ -map  $f : M \rightarrow N$  of manifolds, we get a map

$$\begin{aligned} Tf : TM &\rightarrow TN \\ (q, v) &\mapsto (f(q), T_q f v) \end{aligned}$$

**Exercise 12.3.**  $Tf$  is  $C^\infty$ .

**Remark 12.4.** Given  $f : M \rightarrow N$  and  $v \in \Gamma(TM)$ , we may not have a commutative diagram:

$$\begin{array}{ccc} TM & \xrightarrow{Tf} & TN \\ v \uparrow & & \uparrow ? \\ M & \xrightarrow{f} & N \end{array}$$

**Definition 12.5.** Let  $f : M \rightarrow N$  be a smooth map on manifolds, then  $v \in \Gamma(TM)$  and  $w \in \Gamma(TN)$  are  $f$ -related if we have a commutative diagram

$$\begin{array}{ccc} TM & \xrightarrow{Tf} & TN \\ v \uparrow & & \uparrow w \\ M & \xrightarrow{f} & N \end{array}$$

That is, for any  $q \in M$ ,  $w(f(q)) = (f(q), T_q f(v(q)))$ .

Equivalently, for  $f : M \rightarrow N$ , we say  $X \in \text{Der}(C^\infty(M))$  is  $f$ -related to  $Y \in \text{Der}(C^\infty(N))$  if for all  $h \in C^\infty(N)$ , we have  $Y(h) \circ f = X(h \circ f)$  in  $C^\infty(M)$ .

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**Recall.** Let  $M$  be a manifold, we have a bijection

$$\begin{aligned} \Gamma(TM) &\rightarrow \text{Der}(C^\infty(M)) \\ v &\mapsto D_v : (Dv f)(q) = v_q(f) \quad \forall f, q \end{aligned}$$

with inverse by assignment  $X \mapsto v_X$  where  $v_X(q)f = (Xf)(q)$ .

**Lemma 13.1.** Let  $f : M \rightarrow N$ , then  $v \in \Gamma(TM)$  is  $f$ -related to  $w \in \Gamma(TN)$  if and only if  $D_v \in \text{Der}(C^\infty(M))$  is  $f$ -related to  $D_w \in \text{Der}(C^\infty(N))$ .

*Proof.*  $v$  is  $f$ -related to  $w$  if and only if  $(T_q f)(v(q)) = w(f(q))$  for all  $q$ , if and only if  $((T_q f)(v(q)))h = (w(f(q)))h$  for all  $q$  and all  $h$ , if and only if  $(D_v(h \circ f))(q) = (D_w h)(f(q))$ , if and only if  $D_v(h \circ f) = D_w(h \circ f)$ .  $\square$

**Lemma 13.2.** Suppose  $f : M \rightarrow N$ , let  $X_1, X_2 \in \text{Der}(C^\infty(M))$ , and  $Y_1, Y_2 \in \text{Der}(C^\infty(N))$  such that  $X_i$  is  $f$ -related to  $Y_i$  for  $i = 1, 2$ , then  $[X_1, X_2]$  is  $f$ -related to  $[Y_1, Y_2]$ .

*Proof.* For any  $h \in C^\infty(N)$ ,  $X_i(h \circ f) = Y_i(h) \circ f$  for  $i = 1, 2$ . Therefore,

$$\begin{aligned} ([X_1, X_2])(h \circ f) &= X_1(X_2(h \circ f)) - X_2(X_1(h \circ f)) \\ &= X_1(Y_2(h) \circ f) - X_2(Y_1(h) \circ f) \\ &= Y_1(Y_2(h)) \circ f - Y_2(Y_1(h)) \circ f \\ &= ([Y_1, Y_2](h)) \circ f. \end{aligned}$$

$\square$

**Definition 13.3.** Let  $Q \subseteq M$  be an embedded submanifold. A vector field  $Y \in \Gamma(TM)$  is tangent to  $Q$  if for all  $q \in Q$ ,  $Y(q) \in T_q Q$ .

**Example 13.4.** If  $M = \mathbb{R}^2$ , let  $Q = \mathbb{R} \times \{0\}$ , then  $Y(x_1, x_2) = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$ , so  $Y(x, 0) = x_1 \frac{\partial}{\partial x_1} + 0 \in T_{(x, 0)} Q$ . Equivalently, we have  $i : Q \hookrightarrow M$  to be an inclusion, so  $Ti : TQ \hookrightarrow TM$  is an embedding since  $i$  is, as  $Y(q) \in T_q Q$  for all  $q \in Q$  indicates  $(Y \circ i)(Q) \subseteq TQ$ :

$$\begin{array}{ccc} Q & \xrightarrow{i} & M \\ Y \circ i \downarrow & & \downarrow Y \\ TQ & \xhookrightarrow[Ti]{} & TM \end{array}$$

Hence,  $Y \circ i : Q \rightarrow TQ$  is a vector field on  $Q$ , and  $Y \circ i$  is  $i$ -related to  $Y$ .

**Lemma 13.5.** Let  $Q \subseteq M$  be an embedded submanifold, let  $Y_1, Y_2 \in \Gamma(TM)$  which are tangent to  $Q$ , then  $[Y_1, Y_2]$  is tangent to  $Q$ .

*Proof.* Since  $Y_i|_Q$  is  $i$ -related to  $Y_i$ , then  $[Y_1, Y_2]|_Q$  is  $i$ -related to  $[Y_1, Y_2]$ .  $\square$

**Definition 13.6.** Let  $G$  be a Lie group, then we give  $T_e G$  the structure of a Lie algebra. A vector field  $X : G \rightarrow TG$  is *left-invariant* if for all  $a \in G$ ,  $TL_a(X(g)) = X(L_a g)$  for all  $g \in G$  and all  $a \in G$ , that is,  $X$  is  $L_a$ -related to  $X$  where  $L_a(g) = ag$  is the left translation.

Recall. •  $(La)^{-1} = L_{a^{-1}}$ .

- By [Lemma 13.2](#), if  $X$  and  $Y$  are left-invariant, then so is  $[X, Y]$ .

**Notation.** We denote  $\mathfrak{g} = \text{Lie}(G)$  to be the Lie algebra of the left-invariant vector fields.

**Lemma 13.7.** Let  $G$  be a Lie group, let  $\mathfrak{g}$  be the space of left-invariant vector fields, then the evaluation map

$$\begin{aligned}\text{ev}_e : \mathfrak{g} &\rightarrow T_e G \\ X &\mapsto X(e)\end{aligned}$$

is an  $\mathbb{R}$ -linear bijection. In particular, they have the same dimension.

*Proof.* Obviously  $\text{ev}_e$  is linear. If  $X(e) = 0$ , then for all  $a \in G$ ,  $X(a) = X(L_a e) = (TL_a)_e(X(e)) = 0$ , so  $\text{ev}_e$  is injective. Conversely, given  $v \in T_e G$ , define

$$\begin{aligned}\tilde{v} : G &\rightarrow TG \\ a &\mapsto (TL_a)_e v\end{aligned}$$

then  $\tilde{v}$  is left-invariant. We know

$$\begin{aligned}m : G \times G &\rightarrow G \\ (a, b) &\mapsto ab\end{aligned}$$

is  $C^\infty$ , so  $T_m : TG \times TG \rightarrow TG$  is  $C^\infty$ . Consider

$$\begin{aligned}f : G &\rightarrow TG \times TG \\ a &\mapsto ((a, 0), (e, v)).\end{aligned}$$

**Claim 13.8.**  $(T_m \circ f)(a) = (T_e L_a)(v)$ .

*Subproof.* Pick  $\gamma : I \rightarrow G$  such that  $\gamma(0) = e$  and  $\dot{\gamma}(0) = v$ , then

$$\begin{aligned}\sigma : I &\rightarrow G \times G \\ t &\mapsto (a, \gamma(t))\end{aligned}$$

is  $C^\infty$  where  $\sigma(0) = (a, e)$ , and  $\frac{d}{dt}|_0 (a, \gamma(t)) = (0, v) \in T_{(a,e)}(G \times G)$ . Now

$$\begin{aligned}T_m(f(a)) &= (T_m)_{(a,e)}(0, v) \\ &= \frac{d}{dt}|_0 m(\sigma(t)) \\ &= \frac{d}{dt}|_0 a\gamma(t) \\ &= \frac{d}{dt}|_0 L_a(\gamma(t)) \\ &= (T_e L_a)(\dot{\gamma}(0)) \\ &= (T_e L_a)(v) \\ &= \tilde{v}(a).\end{aligned}$$

■

□

Therefore, the left-invariant vector field  $\text{Lie}(G)$  is isomorphic to  $T_e G$  as  $\mathbb{R}$ -vector spaces.

**Definition 13.9.** Let  $X : M \rightarrow TM$  be a vector field. An *integral curve*  $\gamma : I \rightarrow M$  of  $X$  passing through  $q$  at  $t = 0$  is a  $C^\infty$ -map  $\gamma : I \rightarrow M$  such that  $\gamma(0) = q$  and  $\dot{\gamma}(t) = X(\gamma(t))$  for all  $t \in I$ . Here  $\dot{\gamma}(t) = (T_t \gamma) \left( \frac{d}{dt}|_t \right) \in T_{\gamma(t)} M$ . Equivalently,  $\dot{\gamma}(t)f = X(\gamma(t))f = \frac{d}{dt}|_t (f \circ \gamma)$  for all  $f \in C^\infty(M)$ .

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**Remark 14.1.** if  $\varphi = (x_1, \dots, x_m) : U \rightarrow \mathbb{R}^m$  is a coordinate chart and  $v$  is a vector field on  $U$ , so  $v = \sum v_i \frac{\partial}{\partial x_i}$  for  $v_1, \dots, v_m$  in  $C^\infty(U)$ . This is a section  $q \mapsto \sum v_i(q) \frac{\partial}{\partial x_i} \Big|_q \in \Gamma(TU)$  and for all  $f \in C^\infty(U)$ ,  $f \mapsto \sum v_i \frac{\partial f}{\partial x_i} \in C^\infty(U)$  which is a derivation.

**Recall.** An integral curve of  $X \in \Gamma(TM)$  is a curve  $\gamma : I \rightarrow M$  with  $\gamma(0) = q$  such that  $\frac{d\gamma}{dt} \Big|_t = X(\gamma(t))$ .

**Example 14.2.** Let  $M = U$  be open in  $\mathbb{R}^m$ , and  $X = \sum x_i \frac{\partial}{\partial x_i}$ . Let  $\gamma(t) = (\gamma_1(t), \dots, \gamma_m(t))$  for  $\gamma_i \in C^\infty(I)$ , then  $\frac{d\gamma}{dt} \Big|_t = \sum \gamma'_i(t) \frac{\partial}{\partial \gamma_i}$ . Therefore,  $\frac{d\gamma}{dt} = X(\gamma(t))$  amounts to  $\sum \gamma'_i(t) \frac{\partial}{\partial \gamma_i} = \sum x_i(\gamma(t)) \frac{\partial}{\partial \gamma_i}$ . Therefore,  $\gamma'_i(t) = x_i(\gamma_1(t), \dots, \gamma_m(t))$ .

Hence,  $\gamma$  is an integral curve of  $X$  if and only if  $\gamma$  solves such a system of equations with initial condition  $\gamma(0) = q$ .

**Theorem 14.3.** Let  $U \subseteq \mathbb{R}^m$  be open,  $X = (x_1, \dots, x_m) : U \rightarrow \mathbb{R}^m$  be  $C^\infty$ , then for all  $q_0 \in U$ , there exists an open neighborhood  $V$  of  $q_0$  in  $U$  and  $\varepsilon > 0$ , and a  $C^\infty$ -map  $\Phi : V \times (-\varepsilon, \varepsilon) \rightarrow U$  such that for all  $q \in V$ ,  $\gamma_q(t) := \Phi(q, t)$  solves  $\gamma'_i(t) = x_i(\gamma_1(t), \dots, \gamma_m(t))$  with initial condition  $\gamma_q(0) = q$ . Moreover, such mapping  $\Phi$  is unique.

*Proof.* Apply contraction mapping principle.  $\square$

**Example 14.4.** Say  $U = (-1, 1)$ , let

$$\begin{aligned} X : (-1, 1) &\rightarrow \mathbb{R} \\ x &\mapsto \frac{d}{dx} \end{aligned}$$

with  $X(q) = 1$  be the ODE, i.e.,  $\frac{dX}{dt} = 1$  with  $X(0) = q$ , then  $\Phi(q, t) = q + t$ . The domain of definition of  $\Phi$  is  $W = \{(q, t) \mid q \in (-1, 1), q + t \in (-1, 1)\}$ .

**Remark 14.5.** We need to keep track of the initial conditions. Say  $\gamma : (a, b) \rightarrow M$  is an integral curve of vector field  $X$  on  $M$  with  $\gamma(0) = q$ , then for all  $t_0 \in (a, b)$ , we know

$$\begin{aligned} \sigma : (a - t_0, b - t_0) &\rightarrow M \\ s &\mapsto \gamma(s + t_0) \end{aligned}$$

is also an integral curve. Therefore,  $\gamma$  and  $\sigma$  has the same image.

*Proof.*

$$\begin{aligned} \frac{d}{dt} \Big|_t \sigma &= \frac{d}{ds} \Big|_t \gamma(s + t_0) \\ &= \frac{d}{du} \Big|_{u=t+t_0} \gamma(u) \\ &= X(\gamma(t + t_0)) \\ &= X(\sigma(t)). \end{aligned}$$

$\square$

**Lemma 14.6.** Let  $X : M \rightarrow TM$  be a vector field,  $\varphi = (x_1, \dots, x_m) : U \rightarrow \mathbb{R}^m$  be a coordinate chart and  $X = \sum x_i \frac{\partial}{\partial x_i}$  where  $x_i \in C^\infty(U)$ , then  $\gamma : I \rightarrow U$  with  $\gamma(0) = q$  is an integral curve of  $X$  if and only if  $(x_1 \circ \gamma, \dots, x_m \circ \gamma) : I \rightarrow \mathbb{R}^m$  solves  $y'_i = Y_i(Y_1, \dots, y_m)$  with  $y_i(0) = x_i(\gamma(0))$ . Here  $Y_i = X_i \circ \varphi^{-1} \in C^\infty(\varphi^{-1}(U))$ .

*Proof.* We have  $\dot{\gamma}(t) = \sum dx_i(\dot{\gamma}(t)) \frac{\partial}{\partial x_i} = \sum (x_i \circ \gamma)'(t) \frac{\partial}{\partial x_i}$ . Therefore,  $\dot{\gamma}(t) = X(\gamma(t))$  if and only if  $(X_i \circ \gamma)' = X_i(\gamma(t)) = (X_i \circ \varphi^{-1})(\varphi(\gamma(t))) = Y_i(X_1 \circ \gamma(t), \dots, X_m \circ \gamma(t))$  for all  $i$ .  $\square$

**Corollary 14.7.** Let  $X : M \rightarrow TM$  be a vector field, then for all  $q \in M$ , there exists an integral curve  $\gamma : I \rightarrow M$  of  $X$  such that  $\gamma(0) = q$ . Moreover,  $\gamma$  depends smoothly on  $q$ , and is locally unique: for all integral curve  $\sigma : J \rightarrow M$  of  $X$  mapping  $0 \mapsto q$ , there exists  $\delta > 0$  such that  $(-\delta, \delta) \in I \cap J$  and  $\gamma|_{(-\delta, \delta)} = \sigma|_{(-\delta, \delta)}$ .

**Remark 14.8.** It may not be the case that  $\gamma|_{I \cap J} = \sigma|_{I \cap J}$ . This is true if  $M$  is Hausdorff.

**Example 14.9.** Consider line with two origins in [Example 1.10](#), with translations that agree before the origins.

**Lemma 14.10.** Suppose  $\gamma : I \rightarrow M$  and  $\sigma : J \rightarrow M$  are continuous curves, and  $M$  is Hausdorff, then the set  $Z = \{t \in I \cap J \mid \gamma(t) = \sigma(t)\}$  is closed in  $I \cap J$ .

*Proof.* Note that

$$\begin{aligned} (\gamma, \sigma) : I \cap J &\rightarrow M \times M \\ t &\mapsto (\gamma(t), \sigma(t)) \end{aligned}$$

is continuous, and  $Z = (\gamma, \sigma)^{-1}(\Delta_M)$ .  $\square$

**Lemma 14.11.** Let  $\gamma : I \rightarrow M$  and  $\sigma : J \rightarrow M$  be two integral curves of a vector field  $X$  on  $M$  with  $\sigma(0) = \gamma(0)$ , then  $W = \{t \in I \cap J \mid \gamma(t) = \sigma(t)\}$  is open in  $I \cap J$ .

*Proof.* Given  $t_0 \in W$ , then  $t_0 \in I \cap J$  and  $\sigma(t_0) = \gamma(t_0)$ , and we consider  $\tilde{\sigma}(t) := \sigma(t + t_0)$  and  $\tilde{\gamma}(t) = \gamma(t + t_0)$ , then  $\tilde{\sigma}(0) = \sigma(t_0) = \gamma(t_0) = \tilde{\gamma}(0)$ . Both  $\tilde{\gamma}$  and  $\tilde{\sigma}$  are integral curves of  $X$  with  $\tilde{\sigma}(0) = \tilde{\gamma}(0)$ , therefore by [Corollary 14.7](#), there exists  $\delta > 0$  such that  $\tilde{\sigma}|_{(-\delta, \delta)} = \tilde{\gamma}|_{(-\delta, \delta)}$ , then  $t_0 + (-\delta, \delta) = (t_0 - \delta, t_0 + \delta) \subseteq W$ .  $\square$

**Lemma 14.12.** Let  $M$  be a Hausdorff manifold,  $X \in \Gamma(TM)$ ,  $\gamma : I \rightarrow M$  and  $\sigma : J \rightarrow M$  be two integral curves with  $\gamma(0) = \sigma(0)$ , then  $\gamma|_{I \cap J} = \sigma|_{I \cap J}$ .

*Proof.* Since  $I$  and  $J$  are intervals, then  $I \cap J$  is connected. By [Lemma 14.11](#) and [Lemma 14.10](#),  $W = \{t \in I \cap J \mid \gamma(t) = \sigma(t)\}$  is clopen, thus  $W = I \cap J$ .  $\square$

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**Recall.** We introduced integral curves of vector fields, and in particular we introduced [Lemma 14.12](#).

**Corollary 15.1.** For any vector field  $X \in \Gamma(TM)$  and any  $q \in M$ , there exists a unique maximal integral curve  $\gamma_q : I_q \rightarrow M$  of  $X$  with  $\gamma_q(0) = q$ . Here *maximal* means that if  $\sigma : J \rightarrow M$  is another integral curve of  $X$  with  $\sigma(0) = q$ , then  $J \subseteq I_q$  and  $\sigma = \gamma_q|_J$ .

*Proof.* Consider the subset  $\Gamma \subseteq \mathbb{R} \times M$  defined as follows: let  $Y$  be the set of all integral curves  $\gamma$  of  $X$  with  $\gamma(0) = q$ , then define  $\Gamma = \bigcup_{\gamma \in Y} \text{graph}(\gamma)$ . By [Lemma 14.12](#),  $\Gamma$  is a graph of a smooth curve, which is the desired maximal integral curve  $\gamma_q$  of  $X$  with  $\gamma_q(0) = q$ .  $\square$

**Lemma 15.2.** Let  $f : M \rightarrow N$  be a map of manifolds, with  $X \in \Gamma(TM)$  and  $Y \in \Gamma(TY)$ , and  $Tf \circ X = Y \circ f$ , i.e.,  $X$  and  $Y$  are  $f$ -related, then for any integral curve  $\gamma$  of  $X$ ,  $f \circ \gamma$  is an integral curve of  $Y$ .

*Proof.* We have

$$\begin{aligned} \frac{d}{dt} (f \circ \gamma)|_t &= T_t(f \circ \gamma) \left( \frac{d}{dt} \right) \\ &= T_{\gamma(t)}f \left( T_t \gamma \left( \frac{d}{dt} \right) \right) \\ &= T_{\gamma(t)}f(X(\gamma(t))) \\ &= Y(f(\gamma(t))) \\ &= Y((f \circ \gamma)(t)). \end{aligned}$$

$\square$

**Example 15.3.** Let  $M = (-1, 1)$ ,  $N = \mathbb{R}$ ,  $f : (-1, 1) \hookrightarrow \mathbb{R}$  be the inclusion. Let  $X = \frac{d}{dt}$  and  $Y = \frac{d}{dt}$ , then

$$\begin{aligned}\gamma : (-1, 1) &\rightarrow M \\ t &\mapsto t\end{aligned}$$

is a maximal integral curve of  $X$  with  $\gamma(0) = 0$ . Note that it is not a maximal integral curve of  $Y$  because  $f \circ \gamma$  is not an integral curve of  $Y$  that is not maximal.

**Example 15.4.** Let  $M = \mathbb{R}^2$  and  $N = \mathbb{R}$ , then consider  $f(x, y) = x$  with  $X = \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$ , with  $Y(x) = \frac{d}{dx}$ , then  $\gamma_x(t) = x + t$  is the integral curve of  $Y$  with  $\gamma_x(0) = x$ . It is defined for all  $t \in \mathbb{R}$ .

To compute integral curves of  $X$ , we solve

$$\begin{cases} \dot{x} = 1, x(0) = x_0 \\ \dot{y} = y^2, y(0) = y_0, \end{cases}$$

then  $x(t) = x_0 + t$  and  $\frac{1}{y} \frac{dy}{dt} = 1$ , therefore

$$\int_0^t \frac{1}{y^2} \frac{dy}{dt} dt = \int_0^t dt$$

and so  $t = -\frac{1}{y} \Big|_0^t = \frac{1}{y_0} - \frac{1}{y(t)}$ , hence  $y(t) = \frac{y_0}{1-y_0 t}$ . Thus,  $t \in (-\infty, \frac{1}{y_0})$ . That is, the curve runs off to  $\infty$  in finite time.

**Definition 15.5.** Let  $X$  be a vector field on a (Hausdorff) manifold  $M$ , and let  $\gamma_q : I_q \rightarrow M$  be the unique maximal integral curve with  $\gamma_q(0) = q$ . Let  $W = \bigcup_{q \in M} \{q\} \times I_q \subseteq M \times \mathbb{R}$ , then the (*local*) flow of  $X$  is the map

$$\begin{aligned}\Phi : W &\rightarrow M \\ (q, t) &\mapsto \gamma_q(t)\end{aligned}$$

We say  $\Phi$  is a *global flow* if  $W = M \times \mathbb{R}$ , and in this case we say  $X$  is *complete*.

**Theorem 15.6.** Let  $\Phi : M \rightarrow M$  be a flow of a vector field, then

1.  $M \times \{0\} \subseteq W$ ,
2.  $W$  is open, and
3.  $\Phi$  is  $C^\infty$ .

*Proof.* See Lee. □

**Example 15.7.** Let  $X = y^2 \frac{d}{dy} \in \Gamma(\mathbb{R})$ , then  $W = \{(y, t) \in \mathbb{R} \times \mathbb{R} \mid t < \frac{1}{y} \text{ when } y > 0, t \text{ arbitrary when } y = 0, t > \frac{1}{y} \text{ if } y < 0\}$ . The flow is  $\Phi(y, t) = \frac{y}{1-yt}$ .

**Lemma 15.8.** Let  $\Phi : W \rightarrow M$  be a local flow of a vector field  $X$ , then  $\Phi(q, s+t) = \Phi(\Phi(q, s), t)$  whenever both sides are defined.

**Remark 15.9.** Note that if  $s = -t$ , then the left-hand side is defined, but the right-hand side is not.

*Proof.* Fix  $q$  and fix  $s$  such that  $(q, s) \in W$ . Consider  $\sigma(t) = \Phi(q, s+t) = \gamma_q(s+t)$ , and  $\tau(t) = \Phi(\Phi(q, s), t) = \gamma_{\Phi(q, s)}(t)$ , then  $\tau(0) = \Phi(q, s) = \gamma_q(s) = \sigma(0)$ . Both  $\sigma(t)$  and  $\tau(t)$  are integral curves, and that they agree at  $t = 0$ , then  $\sigma(t) = \tau(t)$  for all  $t$  in the intersection of their domains of definition. Therefore, the two equations agree whenever both sides are defined. □

**Definition 15.10.** An (*left*) *action* of a Lie group  $G$  on a manifold  $M$  is a  $C^\infty$ -map

$$\begin{aligned}G \times M &\rightarrow M \\ (g, q) &\mapsto g \cdot q\end{aligned}$$

such that

1.  $e \cdot q = q$  for all  $q$ , and
2.  $g_1 \cdot (g_2 \cdot q) = (g_1 g_2) \cdot q$ .

**Claim 15.11.** If  $X$  is complete, then its flow is an action of the Lie group  $(\mathbb{R}, +, \cdot)$ .

*Proof.* Define  $t \cdot q = \Phi(q, t)$ , then

$$\begin{aligned} t \cdot (s \cdot q) &= \Phi(\Phi(q, s), t) \\ &= \Phi(q, s + t) \\ &= (t + s) \cdot q \end{aligned}$$

and  $0 \cdot q = \Phi(q, 0) = q$ . □

**Remark 15.12.** If we have a group action, we determine the groupoid structure, and therefore we recover the groupoid version of the lemma.

**Remark 15.13.** For a Lie group  $G$ , the multiplication  $m : G \times G \rightarrow G$  is a left action of  $G$  on  $G$ , with  $e \cdot g = g$  and  $a \cdot (b \cdot g) = (a \cdot b) \cdot g$ .

**Remark 15.14.** For any manifold, there exists a group  $\text{Diff}(M) = \{f : M \rightarrow M \mid f \text{ is a diffeomorphism}\}$ , where the operation is function composition, and the identity is the identity map.

**Exercise 15.15.** An (left) action  $G \times M \rightarrow M$  of a Lie group  $G$  on a manifold  $M$  gives rise to a homomorphism

$$\begin{aligned} \rho : G &\rightarrow \text{Diff}(M) \\ (\rho(g))(q) &\mapsto g \cdot q \end{aligned}$$

In particular, the multiplication  $m : G \times G \rightarrow G$  gives rise to

$$\begin{aligned} L : G &\rightarrow \text{Diff}(G) \\ a &\mapsto L_a \end{aligned}$$

**Definition 15.16.** An *abstract local flow* on a manifold  $M$  is a  $C^\infty$ -map  $\psi : W \rightarrow M$ , where  $W$  is an open neighborhood of  $M \times \{0\}$  in  $M \times \mathbb{R}$ , so that  $\psi(q, 0) = q$  for all  $q \in M$  and  $\psi(q, s + t) = \psi(\psi(q, s), t)$  whenever both sides are defined.

We will show that any abstract local flow is part of a flow on a vector field.

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**Recall.** Given a vector field  $X$  on a manifold  $M$ , we define the flow to be  $\Phi : W \rightarrow \mathbb{R}$  for some open neighborhood of  $M \times \{0\}$  in  $M \times \mathbb{R}$ . The defining property of  $\Phi$  would be that for every  $q \in M$ ,  $W \cap (\{q\} \times \mathbb{R}) = \{q\} \times I_q$  and  $I_q \ni t \mapsto \Phi(q, t)$  is the maximal integral curve of  $X$ . We also proved that  $\Phi(q, t + s) = \Phi(\Phi(q, t), s)$  for all  $q, t, s$  such that both sides are defined.

We say the flow is a global flow if  $W = M \times \mathbb{R}$ , that is, for all  $q \in M$ , the maximal integral curve  $\gamma_q \in I_q \rightarrow M$  of  $X$  with  $\gamma_q(0) = q$  is defined for all  $t \in \mathbb{R}$ , i.e.,  $I_q = \mathbb{R}$ .

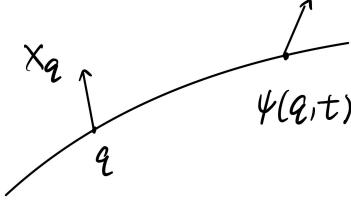
**Lemma 16.1.** Let  $M$  be a manifold,  $U \subseteq M \times \mathbb{R}$  be an open neighborhood of  $M \times \{0\}$  with  $U \cap (\{q\} \times \mathbb{R})$  connected for all  $q \in M$ , and  $\psi : U \rightarrow M$  a smooth map such that

1.  $\psi(q, 0) = q$  for all  $q$ , and
2.  $\psi(q, s + t) = \psi(\psi(q, s), t)$  whenever both sides are defined,

then there exists a vector field  $X$  on  $M$  such that for all  $q \in M$ , the assignment  $t \mapsto \psi(q, t)$  is an integral (but not necessarily maximal) curve of  $X$  with  $\psi(q, 0) = q$ .

*Proof.* For all  $q \in M$ , we define  $X(q) = \left. \frac{d}{dt} \right|_0 \psi(q, t)$ , then

$$\begin{aligned} \left. \frac{d}{dt} \right|_t \psi(q, t) &= \left. \frac{d}{dt} \right|_0 \psi(q, t + s) \\ &= \left. \frac{d}{ds} \right|_0 \psi(\psi(q, t), s) \\ &= X(\psi(q, t)). \end{aligned}$$



□

**Lemma 16.2.** Let  $\Phi : W \rightarrow M$  be a flow of a vector field  $X$  on a manifold  $M$ . Suppose there exists  $\varepsilon > 0$  such that  $M \times [-\varepsilon, \varepsilon] \subseteq W$ , then  $W = M \times \mathbb{R}$ , i.e., the vector field  $X$  is complete.

*Proof.* We want to show that for all  $q \in M$ ,  $I_q := \{t \in \mathbb{R} \mid (q, t) \in W\}$  is  $\mathbb{R}$ . Since  $I_q$  is connected, then it suffices to show that  $I_q$  is unbounded. By assumption,  $\varphi_\varepsilon(q) := \varphi(q, \varepsilon)$  and  $\varphi_{-\varepsilon}(q) := \varphi(q, -\varepsilon)$  are defined for all  $q \in M$ , since  $q = \varphi(q, 0) = \varphi(\varphi(q, \varepsilon), -\varepsilon) = \varphi(\varphi(q, -\varepsilon), \varepsilon)$ , therefore  $(\varphi_\varepsilon)^{-1}$  exists and is just  $\varphi_{-\varepsilon}$ .

Given  $q \in M$ , we consider  $\mu(t) = \varphi(q, t + \varepsilon) = \gamma_q(\varepsilon + t)$ , and it is easy to check that  $\mu'(t) = X(\mu(t))$ , therefore  $\mu$  is an integral curve of  $X$  with  $\mu(0) = \gamma_q(\varepsilon)$ . Since  $\gamma_q$  is defined on  $I_q$ , then  $\mu$  is defined for all  $t$  such that  $t + \varepsilon \in I_q$ , that is,  $t \in I_q - \varepsilon$ . Since  $\gamma_{\varphi_\varepsilon(q)} : I_{\varphi_\varepsilon(q)} \rightarrow M$  is a maximal integral curve of  $X$  such that  $\gamma_{\varphi_\varepsilon(q)}(0) = \Phi_\varepsilon(q) = \gamma_q(\varepsilon)$ , so  $I_q - \varepsilon \subseteq I_{\varphi_\varepsilon(q)}$ , and similarly  $I_q + \varepsilon \subseteq I_{\varphi_{-\varepsilon}(q)}$ , therefore  $I_{\varphi_\varepsilon(q)} + \varepsilon \subseteq I_{\varphi_{-\varepsilon}}(\varphi_\varepsilon(q)) = I_q$ . Therefore,  $I_q - \varepsilon = I_{\varphi_\varepsilon(q)}$ . By induction, we conclude that for all  $n > 0$ ,  $I_q - n\varepsilon = I_{(\varphi_\varepsilon)^n(q)}$ . Since  $0 \in I_{q'}$  for all  $q'$ , and  $0 \in I_q - n\varepsilon$ , so  $n\varepsilon \in I_q$  for all  $n \in \mathbb{N}$ . Similar argument shows that  $-n\varepsilon \in I_q$  for all  $n \in \mathbb{N}$ . That is,  $I_q$  is neither bounded above nor bounded below. □

**Definition 16.3.** The support of a vector field  $X \in \Gamma(TM)$  is  $\text{supp}(X) = \overline{\{q \in M \mid X(q) \neq 0\}}$ .

**Corollary 16.4.** Suppose  $X \in \Gamma(TM)$  has compact support, then  $X$  is complete: its flow exists for all time.

*Proof.* Note that  $X \equiv 0$  on  $M \setminus \text{supp}(X)$ , so for all  $q \in M \setminus \text{supp}(X)$ . Note that  $\gamma_q(t) = q$  is the maximal integral curve of  $X$ , which exists for all  $t$ , so  $(M \setminus \text{supp}(X)) \times \mathbb{R} \subseteq W$ , which is the domain of the flow  $\varphi$ . Since  $\text{supp}(X)$  is compact, then  $(\text{supp}(X) \times \{0\}) \subseteq W$  is compact. Since  $W$  is open, then by tube lemma, there exists  $\varepsilon > 0$  such that  $\text{supp}(X) \times (-2\varepsilon, 2\varepsilon) \subseteq W$ , hence  $\text{supp}(X) \times [-\varepsilon, \varepsilon] \subseteq W$ . Therefore,

$$(M \setminus \text{supp}(X)) \times [-\varepsilon, \varepsilon] \subseteq (M \setminus \text{supp}(X)) \times \mathbb{R} \subseteq W,$$

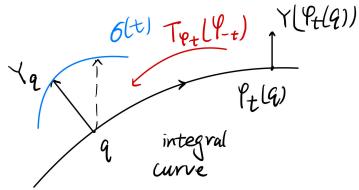
so  $M \times [-\varepsilon, \varepsilon] \subseteq W$ . Now apply Lemma 16.2. □

We will start talking about Lie derivatives. Let  $X, Y \in \Gamma(TM)$  be two vector fields. For simplicity we assume  $X$  and  $Y$  have global flow  $\varphi(q, t) = \varphi_t(q)$ , and  $\psi(q, t) = \psi_t(q)$ , respectively. (It suffices to have the flow maintained for small neighborhood of time.) Fix  $q \in M$ . Consider

$$\begin{aligned} \sigma : \mathbb{R} &\rightarrow T_q M \\ t &\mapsto (T_{\varphi_t(q)} \varphi_{-t})(Y(\varphi_t(q))) \end{aligned}$$

**Remark 16.5.** For any curve  $\gamma : \mathbb{R} \rightarrow M$ ,  $\dot{\gamma}(t) \in T_{\gamma(t)}(T_q M) = T_q M$  since  $\gamma_q M$  is a vector space. In particular,

$$\left. \frac{d\sigma}{dt} \right|_0 = \left. \frac{d}{dt} \right|_0 (T_{\varphi_t(q)} \varphi_{-t}(Y(\varphi_t(q)))) \in T_q M.$$



**Definition 16.6.** The *Lie derivative*  $L_X Y$  of  $Y$  with respect to  $X$  is defined by

$$(L_X Y)(q) = \left. \frac{d}{dt} \right|_0 T_{\varphi_t(q)} \varphi_{-t} (Y(\varphi_t(q))) = \lim_{t \rightarrow 0} \frac{1}{t} (T_{\varphi_t(q)} \varphi_{-t} (Y(\varphi_t(q))) - Y_q).$$

**Theorem 16.7.** For any two vector fields  $X, Y \in \Gamma(TM)$ ,  $L_X Y = [X, Y]$ .

To prove this, we will prove the following.

**Lemma 16.8.** Let  $M$  be a manifold and  $\gamma : I \rightarrow T_q M$  be a curve. Let  $f \in C^\infty(M)$ , then

$$\left. \frac{d}{dt} \right|_0 (\gamma(t)f) = \left( \left. \frac{d\gamma}{dt} \right|_0 \right) f.$$

*Proof.* Choose a chart  $(x_1, \dots, x_n) : U \rightarrow \mathbb{R}^m$  with  $q \in U$ , then  $\gamma(t) = \sum \gamma_i(t) \left. \frac{\partial}{\partial x_i} \right|_q$ , where each  $\gamma_i : I \rightarrow \mathbb{R}$  is  $C^\infty$ .

Now  $\left. \frac{d\gamma}{dt} \right|_0 = \sum \gamma'_i(0) \left. \frac{\partial}{\partial x_i} \right|_q$ . We also know that  $\gamma(t)f = \sum \gamma_i(t) \left. \frac{\partial f}{\partial x_i} \right|_q$ , therefore  $\left. \frac{d}{dt} \right|_0 \gamma(t) = \left. \frac{d}{dt} \right|_0 \left( \sum \gamma_i(t) \left. \frac{\partial f}{\partial x_i} \right|_q \right) = \sum \gamma'_i(0) \left. \frac{\partial f}{\partial x_i} \right|_q$  as well.  $\square$

**Lemma 16.9.** Let  $X$  and  $Y$  be two vector fields with flows  $\{\varphi_t\}$  and  $\{\psi_t\}$ , viewed as family of diffeomorphisms with  $\mathbb{R}$ -actions. For any  $f \in C^\infty(M)$ ,

$$(L_X Y)(q)f = \left. \frac{\partial^2}{\partial s \partial t} \right|_{(0,0)} (f \circ \varphi_{-t} \circ \psi_s \circ \varphi_t)(q).$$

*Proof.* We have

$$\begin{aligned} (L_X Y)(q)f &= \left( \left. \frac{d}{dt} \right|_0 T_{\varphi_{-t}} (Y(\varphi_t(q))) \right) f \\ &= \left. \frac{d}{dt} \right|_0 (T_{\varphi_{-t}} (Y(\varphi_t(q))f)) \\ &= \left. \frac{d}{dt} \right|_0 Y(\varphi_t(q))(f \circ \varphi_{-t}) \\ &= \left. \frac{d}{dt} \right|_0 \left. \frac{\partial}{\partial s} \right|_0 (f \circ \varphi_{-t})(\psi_s(\varphi_t(q))) \\ &= \left. \frac{\partial^2}{\partial t \partial s} \right|_{(0,0)} (f \circ \varphi_{-t} \circ \psi_s \circ \varphi_t)(q) \\ &= \left. \frac{\partial^2}{\partial s \partial t} \right|_{(0,0)} (f \circ \varphi_{-t} \circ \psi_s \circ \varphi_t)(q). \end{aligned}$$

$\square$