MATH 212B Notes

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1 Examples of Tensor-triangulated Categories

[1]We aim to discuss examples of tensor-triangulated categories as an entryway into the theory of tensor-triangulated geometry. These examples often involve two categories, a small (compact) category $\mathcal{K}$ and a large (triangulated) category $\mathcal{T}$.

1.1 Examples in Commutative Algebra and Algebraic Geometry

Definition 1.1. An element $x \in \mathcal{T}$ is *compact* if $\mathbf{Hom}_{\mathcal{T}}(x, -)$ commutes with coproducts.[2]

Example 1.2. Let A be a commutative ring. The large category is $\mathcal{T} = D(A\text{-}\mathbf{Mod})$, the derived category of A-modules. Note that this is the derived category of an Abelian (and Grothendieck) category, made up of complexes of A-modules, with quasi-isomorphisms inverted. $\mathcal{K}$ is the subcategory consisting the compact elements of $\mathcal{T}$[3], i.e., $\mathcal{T}^c$, which happens to be $D_{perf}(A)$, the derived category of perfect complexes of A, which is just $K_b(A\text{-}\mathbf{proj})$, the bounded complexes of finitely-generated projective A-modules. Therefore, on each degree of the complex we have finitely generated projective modules, and far enough on the left (and the right) there are zero terms. The maps in this complex are up to homotopy simply because quasi-isomorphisms between such complexes have to be homotopy-equivalent. $\mathcal{K}$ is now a triangulated category.

Remark 1.3. Note that the construction above does not require commutativity. What requires this property is the construction of the symmetric monoidal tensor product.

The category has a tensor product $\otimes$ induced from the tensor product of A, i.e., $- \otimes_A^L -$, given by the left derived functor of the derived category.

We can now generalize this example in algebraic geometry.

[1]This lecture coincides to Professor Paul Balmer's Talk. It is also based on his notes.

[2]We usually assume that $\mathcal{T}$ contains all coproducts.

[3]A theorem due to Amnon Neeman shows that this construction coincides with the collection of compact elements.

Example 1.4. Let X be a quasi-compact and quasi-separated scheme, i.e., the underlying space $|X|$ has a quasi-compact open basis. For example, let $X = \mathbf{Spec}(A)$ be the spectrum of a commutative ring. Denote $\mathcal{T} = D(X)$, (actually) the derived category of complexes of $\mathcal{O}_X$-modules with quasi-coherent homology. $\mathcal{K}$ is still the compact subcategory of $\mathcal{T}$, equivalent to $D_{perf}(X)$, those that are in $D_{perf}(A)$ for every affine $\mathbf{Spec}(A)$.

The triangular structures on $\mathcal{T}$ are really the traces that survived from the exact sequences of modules, and the tensor product is exact in each variable, therefore tensoring a fixed object preserves exact triangles.

These considerations of larger categories go hand-in-hand with the modern development of algebraic geometry like K-theory or homological algebra. One of the early motivations (other than the ones in homological algebra) was the pushforward. When we look at a vector bundle, we have things working nicely on the closed subschemes or on given schemes. We try pushing it to another scheme, like in the following example:

Example 1.5. Consider $i : \mathbf{Spec}(\mathbb{Z}/p\mathbb{Z}) \hookrightarrow \mathbf{Spec}(\mathbb{Z})$. Let $A = \mathbb{Z}$ and $k = \mathbb{Z}/p\mathbb{Z}$, then this is associated with the quotient $A \twoheadrightarrow k$. If V is a finite-dimensional k-vector space, we can view it as an A-module i_*V. A acts on V by projecting onto k and acts correspondingly. The k-dual of V has k-dimension $\dim_k(V^*) = \dim_k(V)$. But if we look the homomorphisms over A instead, we have $\mathbf{Hom}_A(i_*V, A) = 0$ because the module i_*V is killed by p since it is torsion, so every element lands in elements killed by p in A, but there is no such element. Therefore, the information about the dual gets lost.

We can look at an even easier example.

Example 1.6. If we take $V = \mathbb{Z}/p\mathbb{Z}$ itself, then i_*V is $\mathbb{Z}/p\mathbb{Z}$ as an A-module, but in the derived category of A, this is equivalent (quasi-isomorphic) to the complex

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to 0$$

This is a perfect complex, i.e., contained in $D_{perf}(A)$. If we try to dualize this perfect complex, we have $(i_*V)^*$ to dualize on every degree, but because it is contravariant, we have the complex

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to 0$$

Note that the two complexes has different degree 0's: if the first complex has degree 0 to be $\mathbb{Z}$ on the right, then the second complex has degree 0 to be $\mathbb{Z}$ on the left. In other words, it shifted by one. We can denote the dual to be $i_*V[-1]$.

Remark 1.7. Example 1.6 works for all finitely generated $V \in D_{perf}(k)$.

Example 1.8. Take $A = k[X_1, \dots, X_n]$ and $k = A/\langle X_1, \dots, X_n \rangle$. We then get $(i_*V)^* = (i_*V)[-n]$, i.e., shifted by $-n$.

Remark 1.9. In fact, a more precise way of writing the isomorphisms in the examples above is $(i_*V)^* \cong i_*(V^*)[-1]$ and $(i_*V)^* \cong (i_*(V^*))[-n]$, because the functor is contravariant.

This is interesting because the value n is the difference between dimensions of the two schemes we are looking at. For example, the first one is the difference between the codimensions of $\mathbf{Spec}(\mathbb{Z}/p\mathbb{Z})$ (which is 0) and $\mathbf{Spec}(\mathbb{Z})$ (which is 1).

We can make the following observations.

Remark 1.10.
- There are phenomena that make sense on derived (triangulated) categories but not on the level of modules (c.f. Example 1.5, where we lost information about the dual as a module).
- Some geometric information appears in the derived category $D(X)$, e.g., the relative dimension as seen in Remark 1.9.

Another classical example comes from K-theory. K-theory was born from Grothendieck's theory on Grothendieck–Riemann–Roch theorem (formalized by Borel–Serre in 1958), where he also looked at f_* for vector bundles.

Example 1.11.
- For example, let us look at a vector bundle over X and a (smooth enough) map $f : X \to Y$. We push the vector bundle down and get a perfect complex (which may not be a vector bundle anymore) over Y, then we look at the alternate sum of elements of this complex (resolution).
- Another example comes from the Thomason-Trobaugh paper in 1990s, where they developed the higher algebraic K-theory of schemes in algebraic geometry. This goes hand-in-hand with the development of perfect complexes with more theoretical information, i.e., under localizations.

Neeman concluded in the early 1990s that we could not expect certain K-theories to factor via homotopy categories because there are certain functors in these categories with sections, but no sections in those K-theories.

Very recently, Muro and Raptis give a big reconciliation on the K-theory of derivators.

Following the observations above, one can now ask: how much geometry of X survives in $D(X)$ or $D_{perf}(X)$? Note that the work *duality between* $D(X)$ *and* $D(\hat{X})$ by Mukai in 1981 shows that there are non-isomorphic schemes X and X' (in particular, Abelian varieties and their duals) such that $D(X)$ and $D(X')$ are equivalent as triangulated categories. However, this construction was not $\otimes$-compatible. Thomason (1997) highlighted the importance of $\otimes$ when classifying the triangulated subcategories of the derived category of perfect complexes, as he classified the tensor ideals of $D^{perf}(X)$. This is a very important precursor of tensor triangular geometry. An important corollary is the following:

Theorem 1.12. If $D(X) \cong D(X')$ as tensor triangulated categories (i.e., preserving the tensor), then the schemes are isomorphic, i.e., $X \cong X'$. Alternatively, the same result holds if $D_{perf}(X) \cong D_{perf}(X')$.

We now go over a few non-geometric examples.

1.2 Examples in Modular Representation Theory

Let G be a finite group and k be a field of positive characteristic ($p > 0$). In particular, we look at the case where $p \mid |G|$. Recall

Theorem 1.13 (Maschke)**.** If $p = 0$ or $p \nmid |G|$, then kG is semisimple. In particular, all modules are projective and injective, and the finitely generated ones decompose uniquely as a sum of irreducible (or simple) ones (according to Krull-Schmidt).

Therefore, the theory studies the case when kG is not semisimple. That is to say, there are non-projective modules. We look at the category of kG-modules and mod out the projective ones. Therefore, objects are still kG-modules, but if a map differs from another map by factoring via a projective, then it is zero, i.e., $f \sim 0 : M \to M'$ if there exists a projective kG-module P and maps such that the diagram

$$\begin{tikzcd} M \arrow[rr] \arrow[dr, dashed] & & M' \\ & P \arrow[ur, dashed] & \end{tikzcd}$$

commutes. In particular, the identity of all projective modules will factor by itself, and therefore become zero. Hence, all projective modules disappear and give the idea of an additive quotient. That is to say, the quotient category kG-**Mod**/kG-**Proj** is an additive category, i.e., receiving kG-modules and all projective modules become zero. Amusingly, the quotient category is a triangulated category $\mathcal{T}$. Again, the compact portion $\mathcal{K}$ of this quotient category is actually the finitely-generated ones, i.e., kG-**mod**/kG-**Proj**, where kG-**mod** is the category of finitely-generated kG-modules. The tensor product $\otimes$ is given over the field, i.e., as $\otimes_k$, with diagonal G-action. This means that $g \cdot (m_1 \otimes m_2) = (gm_1) \otimes (gm_2)$ in $M_1 \otimes_k M_2$. This tensor product is nice because it allows us to pass into the quotient. Therefore, these quotients have a tensor structure and are tensor triangulated categories, with the tensor compatible with the triangulation. Denoting $\mathrm{stab}(kG) = \mathcal{K}$ and $\mathrm{Stab}(kG) = \mathcal{T}$, this stable module category is the measure of modularity, i.e, how non-semisimple kG is. Note that we can have $\mathrm{Stab}(kG) = 0$ if $p \nmid |G|$. In fact, the restriction $\mathrm{Res}^G_H : \mathrm{Stab}(kG) \to \mathrm{Stab}(kH)$ can be an equivalence if $H \cap H^g$, i.e., intersecting with the conjugate, has order relatively prime to p for all $g \in G \backslash H$. In some sense, the modular representation theory of G and H are the same. For example, this happens when $p = 2$ and $G = S_3$ with $H = C_2$.

By Krull-Schmidt, every finitely generated kG-module can be decomposed in an essentially unique way as a sum of indecomposables (even in modular case). Therefore, we can apply the same idea to $\mathrm{stab}(kG)$. In some sense, knowing the decomposition of modules in there is the same as studying non-projectives in the indecomposables. If we look at the quotient $\otimes$-functor kG-**mod** $\twoheadrightarrow \mathrm{stab}(kG)$, (even if it is from an Abelian category to a triangulated category), if M is such that $M \otimes -$ is an equivalence on $\mathrm{stab}(kG)$, then if N is indecomposable in the stable category $\mathrm{stab}(kG)$, then so are $M^{\otimes n} \otimes N$ for all $n \in \mathbb{Z}$. Therefore, the invertible (as an equivalence) elements in kG-**mod** are mapped to the invertible elements in $\mathrm{stab}(kG)$. We see that $\otimes$-invertible in kG-**mod** is exactly saying that $\dim_k(M) = 1$. But there are more invertible elements in $\mathrm{stab}(kG)$, which are called endotrivial and crucial in modular representation theory.

1.3 Stable Homotopy Theory

Consider $\mathcal{T} = SH$, the stable homotopy category, also known as the homotopy category of **Top**-spectra. We can start with topological spaces and ask whether we can study them up to homotopy. This is possible for pointed spaces, as we can just suspend them. Therefore, in general, we consider "spaces" up to homotopy with the suspension $\sum = S^1 \wedge -$ (essentially the smash product) inverted. The compact portion $\mathcal{K} = SH^{fin}$ is classified by looking at finite CW-complexes and attaching finitely many disks to finitely many points[4], then we can look at the homotopy and stabilizes.

The motivation is that studying spaces (even up to homotopy) is too hard. Working stably, we can look at the spheres and their suspensions, where the homomorphisms (of the stable homotopy group of spheres) $\pi_i^{st} = \mathbf{Hom}_{SH}(S^i, S^0)$ are hard but interesting to study.

One can also look at Chromatic theory, which motivates all of this, and provides overall organization to the finite spectrum SH^{fin}. This helps us to study the homotopy groups, and even the tensor triangular categories and stable homotopy theory, and therefore we see it has the same role as $\mathbb{Z}$ in commutative algebra. To make this idea precise, we would need more structure, but we can just look at the tensor triangular categories with certain enrichment.

The category SH has its significance because of Brown Representability Theorem. This theorem had been generalized by Neeman on such categories (see theorem 4.1 from his work in 1996).

Remark 1.14. There are relations in equivariant versions of the same categories. Let $\mathcal{T} = SH(G)$ and $\mathcal{K} = SH(G)^c$. Although it should be similar to what we have seen before, there is a bit of subtlety in what we mean by stabilization: one is not stabilizing with respect to smashing with spheres, but with the ones that have a G-action in general. This construction helps us look at actions like restriction and induction.

1.4 Motivic Theory

Let S be a base scheme, e.g., the spectrum of the ground field $\mathbf{Spec}(k)$. Note that we sometimes want it to be a perfect field. We want to do similar things, but to study smooth schemes over S, and their homological properties. In particular, we want to make $\mathbb{A}^1 \times X \cong X$, so we can look at an algebraic form of homotopy, i.e., $\mathbf{Spec}(\mathbb{Z})(t)$ for some variable t, instead of the traditional $[0, 1]$. To do this, we have an algebraic theory called derived category of motives, with $\mathcal{K} = DM^{gm}(S) \subseteq DM(S) = \mathcal{T}$, and there is a topological theory where $\mathcal{K} = SH(S)^c \subseteq SH(S) = \mathcal{T}$. The first one is called the derived category of motives by Voevodsky, and the second one is the motivic stable homotopy category.

In both cases, each of those categories

- contains an object $[X]$ for every smooth scheme X over S, in a way that the motives satisfy $[\mathbb{A}^1 \times X] \cong [X]$.
- algebraic "coefficients" in complexes, and topological "coefficients" in spectra (spaces).

Remark 1.15. In some sense, our example in motivic theory has the same role as our example of stable homotopy theory in the algebraic geometry examples.

[4]This is known as the Spanier-Whitehead stable homotopy category of finite pointed CW-complexes.

1.5 More Examples

- KK-theory of C^*-algebras.
- Homological mirror symmetry.

2 Pre-triangulated Categories

Definition 2.1 (Suspended Category)**.** A suspended (or stable) category is a pair (K, Σ) where K is an additive category and $\Sigma : K \xrightarrow{\cong} K$ is an equivalence.

Example 2.2. Let $\mathcal{A}$ be an additive category. We can consider $\mathbf{Ch}(\mathcal{A})$ whose objects are complexes:

$$\cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots$$

such that $d^2 = 0$, i.e., $d_n \circ d_{n+1} = 0$, and with morphisms from $A. \to B.$ which are collection of $f_n : A_n \to B_n$ for all $n \in \mathbb{Z}$, and such that $df = fd$, i.e., a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r, "d"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_n \arrow[r, "d"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

The suspension is (almost) shifting the degree (with a sign difference). Indeed, it is suspended with $(\Sigma A)_n = A_{n-1}$ and $d_n^{\Sigma A} = -d_{n-1}^A$.

Remark 2.3. $f \sim g : A. \to B.$ are homotopic if there exists $\varepsilon_n : A_n \to B_{n+1}$ for all $n \in \mathbb{Z}$, such that $f - g = d\varepsilon + \varepsilon d$.

This notion appears when we discuss the uniqueness of resolutions, maps lifted up to homotopy, which are themselves unique up to homotopy.

Alternatively, we can define $f \sim g$ to be $f - g \in \mathcal{I} = \{h \sim 0\}$ in an additive construction, to be an ideal.

We define the category $\mathbf{K}(\mathcal{A}) = \mathbf{Ch}(\mathcal{A})/\sim$ with the same objects (chain complexes), and morphisms are up to homotopy, i.e., we get $\mathbf{Hom}_{\mathbf{K}(\mathcal{A})}(A, B) = \mathbf{Hom}_{\mathbf{Ch}(\mathcal{A})}(A, B)/\mathcal{I}(A, B)$.

Definition 2.4 (Triangle)**.** A triangle Δ is a diagram in K of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

or alternatively,

$$\begin{tikzcd}
 & C \arrow[dl, "h"'] & \\
A \arrow[rr, "f"'] & & B \arrow[ul, "g"']
\end{tikzcd}$$

A morphism of triangles is of the form $(u,v,w) : \Delta \to \Delta'$, denoted

$$\begin{tikzcd}
\Delta : \arrow[d, "{(u,v,w)}"] & A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[r, "g"] \arrow[d, "v"] & C \arrow[r, "h"] \arrow[d, "w"] & \Sigma A \arrow[d, "\Sigma u"] \\
\Delta' : & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C \arrow[r, "h'"] & \Sigma A
\end{tikzcd}$$

An isomorphism of triangles (u, v, w) is just a morphism where u, v, w are all triangles.

Remark 2.5. A pre-triangulated category is a structure on the categories (in the given models). If you take the category of chain complexes from an Abelian category, adjoin the inversions of quasi-isomorphisms, we do not get an Abelian category anymore, but a pre-triangulated category.

Definition 2.6 (Pre-triangulated Category)**.** A pre-triangulated category is a suspended category (K, Σ) together with a chosen (distinguished) class of triangles, called exact triangles, satisfying some axioms.

1. Book-keeping Axiom:
   - Exact triangles should be replete: if $\Delta \cong \Delta'$ and Δ is exact, then Δ' is exact.
   - For every object A, the triangle

     $$A \xrightarrow{\mathbf{id}} A \longrightarrow 0 \longrightarrow \Sigma A$$

     is exact.
   - Rotation Axiom: Note that the triangle is essentially a long exact sequence

     $$\cdots \longrightarrow \Sigma^{-1}C \xrightarrow{\Sigma^{-1}h} A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A \longrightarrow \Sigma B \longrightarrow \cdots$$

     and so the base (the first morphism in the triangle) does not really matter. The axiom says that

     $$\begin{tikzcd}
     & C \arrow[dl, "h"'] & \\
     A \arrow[rr, "f"'] & & B \arrow[ul, "g"']
     \end{tikzcd}$$

     is exact if and only if

     $$\begin{tikzcd}
     & \Sigma A \arrow[dl, "-\Sigma f"'] & \\
     B \arrow[rr, "g"'] & & C \arrow[ul, "h"']
     \end{tikzcd}$$

     is exact.
     Observe that by the replete axiom, the triangle above is essentially made by changing two signs, e.g., changing f and g to $-f$ and $-g$, which is equivalent to the triangle above. The replete axiom says that we can only change an even number of signs.

2. Existence Axiom: Every morphism $f : A \to B$ can be completed in an exact triangle:

$$\begin{tikzcd} & C \arrow[dl, "h"'] & \\ A \arrow[rr, "f"'] & & B \arrow[ul, "g"'] \end{tikzcd}$$

   However, the axiom does not guarantee it to be unique.

3. Morphism Axiom: Given exact rows

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[r, "g"] \arrow[d, "v"] & C \arrow[r, "h"] & \Sigma A \arrow[d, "\Sigma u"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C \arrow[r, "h'"] & \Sigma A \end{tikzcd}$$

   and u, v such that $f'u = vf$, then there exists a morphism $w : C \to C'$ such that (u, v, w) is a morphism of triangles, i.e., w completes the commutative diagram. (Again, no uniqueness.)

Remark 2.7. For any $f : A \to B$ we have a morphism of triangles

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[r, "\mathbf{id}"] \arrow[d, "f"] & A \arrow[r] \arrow[d] & 0 \arrow[d] \\ B \arrow[r] & B \arrow[r] & 0 \arrow[r] & \Sigma B \end{tikzcd}$$

Remark 2.8. We "conjugate" by rotation, that is, if we pre-compose a morphism with a rotation, and post-compose the inverse of the rotation, then we get a conjugated version of the original morphism. For instance, if we have

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[r, "g"] & C \arrow[r, "h"] \arrow[d, "w"] & \Sigma A \arrow[d, "\Sigma u"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C \arrow[r, "h'"] & \Sigma A \end{tikzcd}$$

then we can rotate the figure one step to the right, then by the morphism axiom, we get the desired morphism v, then we can rotate it back to the original form, saying there exists such morphism $v : B \to B'$ that makes this a morphism of triangles.

Example 2.9. $\mathbf{K}(\mathcal{A})$ for $\mathcal{A}$ additive is pre-triangulated. Let Σ be the shifting, then the exact triangles are those isomorphic (i.e., hotomopy equivalent) to the following: Suppose we have a morphism of complexes $f : A. \to B.$, that is,

$$\begin{tikzcd} \cdots \arrow[r] & A_n \arrow[r, "d"] \arrow[d, "f"] & A_{n-1} \arrow[r] \arrow[d, "f"] & \cdots \\ \cdots \arrow[r] & B_n \arrow[r, "d"] & B_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

and we want to build the composition, i.e., with $g : B. \to C.$, $h : C. \to D.$, and so on, we can construct C as the cone of f, denoted cone(f), and then $D = \Sigma A$. To construct them, we obviously have D as the suspension by 1 with sign of differential changed, i.e.,

$$
\begin{tikzcd}
A. : \arrow[d, "f"] & \cdots \arrow[r] & A_n \arrow[r, "d"] \arrow[d, "f"] & A_{n-1} \arrow[r] \arrow[d, "f"] & \cdots \\
B. : \arrow[d, "g"] & \cdots \arrow[r] & B_n \arrow[r, "d"] & B_{n-1} \arrow[r] & \cdots \\
\mathrm{cone}(f) : \arrow[d, "h"] & & & & \\
\Sigma A : & \cdots \arrow[r, "-d"] & A_{n-1} \arrow[r, "-d"] & A_{n-2} \arrow[r] & \cdots
\end{tikzcd}
$$

To make it additive, we do not have much choice on the cone. The easiest way to get the composition as 0 is

$$
\begin{tikzcd}
A. : \arrow[d, "f"] & \cdots \arrow[r] & A_n \arrow[r, "d"] \arrow[d, "f"] & A_{n-1} \arrow[r] \arrow[d, "f"] & \cdots \\
B. : \arrow[d, "g"] & \cdots \arrow[r] & B_n \arrow[r, "d"] \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & B_{n-1} \arrow[r] \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & \cdots \\
\mathrm{cone}(f) : \arrow[d, "h"] & \cdots \arrow[r] & A_{n-1} \oplus B_n \arrow[r, "\begin{pmatrix} -d & 0 \\ f & d \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & A_{n-2} \oplus B_{n-1} \arrow[r] \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & \cdots \\
\Sigma A : & \cdots \arrow[r, "-d"] & A_{n-1} \arrow[r, "-d"] & A_{n-2} \arrow[r] & \cdots
\end{tikzcd}
$$

Considering the composite from A to cone(f), the morphism is given by $\begin{pmatrix} 0 \\ f \end{pmatrix}$. Then the natural homotopy from A_{n-1} to $A_{n-1} \oplus B_n$ is given by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Therefore, the composition is just homotopic to 0, thus showing the sequence is exact vertically. Horizontally, we have

$$\begin{pmatrix} -d & 0 \\ f & d \end{pmatrix} \begin{pmatrix} -d & 0 \\ f & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

so the entire diagram is a complex indeed.

Definition 2.10 (Weak Kernel/Cokernel)**.** Recall that a kernel is the universal map into the source such that the composition is zero, and a cokernel is the universal map out of the target such that the composition is zero.

A weak kernel (respectively, cokernel) is just a kernel (respectively, cokernel) without the universal property, i.e., existence of factorization without the uniqueness. For instance, for the weak kernel triangle of A, B, C, for every map $t : B \to T$, there exists $\bar{t}$ so that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "0"] \arrow[ddrr, bend right, "0"'] & B \arrow[d, "g"] \arrow[ddr, bend left, "\forall t"] & \\
& C \arrow[dr, dashed, "\exists \bar{t}"] & \\
& & T
\end{tikzcd}$$

Proposition 2.11. Let K be a pre-triangulated category and consider an exact triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

(a) $g \circ f = 0$ and $h \circ g = 0$, and $(\Sigma f) \circ h = 0$, and so on (in the long exact triangle sequence).

(b) g is a weak cokernel of f, and $\Sigma^{-1} h$ is a weak kernel of f, and so on.

Proof. Left as an exercise. Hit the morphism axiom on the given triangle, and all triangles of the form

$$X \xrightarrow{\mathbf{id}} X \longrightarrow 0 \longrightarrow \Sigma X$$

for $X \in \{A, B, C\}$ (in part (a), and $X = T$ in part (b)) and rotations. For instance, we look at the two exact rows below:

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d] & C \arrow[r, "h"] \arrow[d] & \Sigma \\
0 \arrow[r] & C \arrow[r, "1"] & C \arrow[r] & 0
\end{tikzcd}$$

We can then take the identity map from C to C, and the map $g : B \to C$, and we can take rest of the morphisms to be the zero morphisms. $\square$

Corollary 2.12. For K pre-triangulated and $X \in K$, the functors

$$\mathbf{Hom}_K(X, -) : K \to \mathbf{Ab}$$

and

$$\mathbf{Hom}_K(-, X) : K^{op} \to \mathbf{Ab}$$

map triangles (written as long exact sequences) to long exact sequences. That is, if

$$\begin{tikzcd}
& C \arrow[dl, "h"'] & \\
A \arrow[rr, "f"'] & & B \arrow[ul, "g"']
\end{tikzcd}$$

is exact, then

$$\cdots \xrightarrow{(\Sigma^{-1}h)_*} \mathbf{Hom}(X, A) \xrightarrow{f_*} \mathbf{Hom}(X, B) \xrightarrow{g_*} \mathbf{Hom}(X, C) \xrightarrow{h_*} \mathbf{Hom}(X, \Sigma A) \longrightarrow \cdots$$

is exact.

Remark 2.13. This is the source to almost all long exact sequences we found in homological algebra. It is powerful to see that pre-triangulated categories can give us long exact sequences, and therefore give us spectral sequences in the right situation, and so on.

Remark 2.14. Using the above, Yoneda Lemma, and the Five Lemma in **Ab**, it is easy to see that for a morphism $(u, v, w) : \Delta \to \Delta'$ of exact triangles, if u and v are isomorphisms, so is w.

Lemma 2.15. Let Δ, Δ', and Δ'' be exact triangles in a pre-triangulated category, and two morphisms $(u, v, 0) : \Delta \to \Delta'$ and $(0, v', w') : \Delta' \to \Delta''$, then $v' \circ v = 0$.

Proof. Consider

$$\begin{tikzcd}
\Delta : & A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[r, "g"] \arrow[d, "v"] \arrow[dl, dashed, "\exists\tilde{v}"] & C \arrow[r, "h"] \arrow[d, "0"] & \Sigma A \arrow[d, "\Sigma u"] \\
\Delta' : & A' \arrow[r, "f'"] \arrow[d, "0"] & B' \arrow[r, "g'"] \arrow[d, "v'"] & C' \arrow[r, "h'"] \arrow[d, "w'"] \arrow[dl, dashed, "\exists\bar{v}'"] & \Sigma A \arrow[d, "0"] \\
\Delta'' : & A'' \arrow[r, "f''"] & B'' \arrow[r, "g''"] & C'' \arrow[r, "h''"] & \Sigma A''
\end{tikzcd}$$

Note that $g' \circ v = 0$, and therefore v lands in the kernel of g', but we only have a weak kernel, so there exists $\tilde{v}'' : B \to A'$ such that $f' \circ \tilde{v} = v$. Similarly, we have $v' \circ f' = 0$, and with the same reasoning (on the weak cokernel) shows that there exists $\bar{v}'$ such that $\bar{v}' \circ g' = v'$. We compute $v' \circ v = (\bar{v}' \circ g') \circ (f' \circ \tilde{v}) = 0$ because $g' \circ f' = 0$. □

Corollary 2.16. If $(0, 0, w) : \Delta \to \Delta$ is an endomorphism of an exact triangle, then (by rotating to $(0, w, 0)$ and $(0, w, 0)$) $w^2 = 0$.

Corollary 2.17. If $(\mathbf{id}, \mathbf{id}, w) : \Delta \to \Delta$ is an endomoprhism of an exact triangle, then $w = \mathbf{id} + x$ such that $x^2 = 0$. (This can be done by taking the difference of this morphism and $(\mathbf{id}, \mathbf{id}, \mathbf{id})$.) In particular, w is an automorphism.

Corollary 2.18. If $(u, v, w) : \Delta \to \Delta'$ is a morphism of exact triangles, and two of them are isomorphisms, then so is the third.

Proof. Without loss of generality, say u, v are isomorphisms. Then we can extend the morphism back to Δ (by taking some w'), using the morphism axiom:

$$\Delta \xrightarrow{(u,v,w)} \Delta' \xrightarrow{(\mu^{-1},v^{-1},w')} \Delta$$

Then the composition is $(\mathbf{id}, \mathbf{id}, w' \circ w)$. By the corollary, $w' \circ w$ is an isomorphism, and similarly, $w \circ w'$ is an isomorphism. Hence, w is an isomorphism. □

Remark 2.19. Looking at the existence axiom again: given $f : A \to B$, suppose

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, equal] & C \arrow[r, "h"] \arrow[d, dashed, "\exists w"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "f"] & B \arrow[r, "g'"] & C' \arrow[r, "h'"] & \Sigma A
\end{tikzcd}$$

are two exact triangles extending f, then there exists w by the morphism axiom, and w is an isomorphism by the above.

Therefore, the triple $(C, g, h) \cong (C', g', h')$. This is usually referred to as the cone of f. In notation, we write $C = \text{cone}(f)$, which is only up to isomorphism. The map $g : B \to C$ is called the homotopy cofiber of f, and $\Sigma^{-1}h : \Sigma^{-1}C \to A$ is called the homotopy fiber of f.

Proposition 2.20. *A morphism $f : A \to B$ is an isomorphism if and only if $\text{cone}(f) \cong 0$.*

Proof. By replete axiom, if f is an isomorphism, we have two exact triangles

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & 0 \arrow[r, "h"] & \Sigma A \\ A \arrow[u, equal] \arrow[r, "\mathbf{id}"] & A \arrow[u, "f"] \arrow[r, "g'"] & 0 \arrow[u, "\cong"] \arrow[r, "h'"] & \Sigma A \arrow[u, equal] \end{tikzcd}$$

and forces the cone to be 0. Conversely, we use the 2-out-of-3 and compare the exact sequences

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] & 0 \arrow[r, "h"] & \Sigma A \\ A \arrow[u, equal] \arrow[r, "\mathbf{id}"] & A \arrow[u, "f"] \arrow[r, "g'"] & 0 \arrow[u, equal] \arrow[r, "h'"] & \Sigma A \arrow[u, equal] \end{tikzcd}$$

then f is an isomorphism. □

We now make a few remarks comparing exact triangles and exact sequences.

Exercise 2.21. Given two triangles Δ and Δ', then $\Delta \oplus \Delta'$ is exact if and only if Δ and Δ' are exact.

Exercise 2.22. For every A, B, the following is exact:

$$A \xrightarrow{0} B \longrightarrow ? \longrightarrow \Sigma A$$

if and only if $? = B \oplus \Sigma A$.

Proposition 2.23. *Let $\Delta : A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ be exact. Then the following are equivalent:*

1. $f = 0$,
2. g is a split monomorphism,
3. h is a split epimorphism,
4. g is a monomorphism,
5. h is an epimorphism.

Proof. Obviously (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5). Note that (4) $\Rightarrow$ (1) since $g \circ f = 0 = g \circ 0$ and so $f = 0$. Similarly, (5) $\Rightarrow$ (1). Finally, (1) implies (2) and (3) because for any object (on the third slot) that makes the bottom row an exact triangle, it must be of the form $C \cong B \oplus \Sigma A$

$$\begin{tikzcd}
A \arrow[r, "0"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, equal] & C \arrow[r, "h"] \arrow[d, "\cong"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "0"] & B \arrow[r] & B \oplus \Sigma A \arrow[r] & \Sigma A
\end{tikzcd}$$

by the uniqueness of the triangle of the zero morphism. However, that means the bottom row splits. $\square$

Remark 2.24. There are no interesting monomorphisms or epimorphisms in pre-triangulated categories.

Example 2.25. In $\mathbf{K}(\mathbb{Z}\textbf{-Mod})$, the map

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "\cdot 2"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

is not a monomorphism.

Remark 2.26. A pre-triangulated category can only be Abelian (exact) if it is Abelian semisimple, in which case every exact triangle is just a direct sum of the trivial ones (and their rotations):

$$A \xrightarrow{\mathbf{id}} A \longrightarrow 0 \longrightarrow \Sigma A$$

$$0 \longrightarrow B \xrightarrow{\mathbf{id}} B \longrightarrow 0$$

$$\Sigma^{-1} C \longrightarrow 0 \longrightarrow C \xrightarrow{\mathbf{id}} C$$

Remark 2.27. Suppose K is a pre-triangulated category (or just additive) such that its arrow category $\mathbf{Arr}(K)$ is pre-triangulated. Then $K = 0$!

Proof. Pick $A \in K$, look at the identity morphism on A as an object in the arrow category, look at the morphism from this object to the zero morphism from A to 0, then that morphism is an epimorphism in the arrow category for obvious reasons.

$$\begin{tikzcd}
A \arrow[r, "\mathbf{id}"] \arrow[d, "\mathbf{id}"] & A \arrow[d] \\
A \arrow[r] \arrow[u, bend left, dashed, "u"] & 0
\end{tikzcd}$$

By the proposition, it must be split, so there exists some splitting (morphism of arrow category) backwards, but the backwards map u should be 0 because $\mathbf{id} \circ u = \mathbf{id}$ and $\mathbf{id} \circ u = 0$ as a backwards commutative square. In particular, this says that the object $A = 0$. $\square$

3 Verdier Octahedron Axiom and Triangulated Category

So far, in a pre-triangulated category, for every morphism $f : A \to B$, we have an object $\mathrm{cone}(f)$ (unique up to isomorphism) that measures f "homologically". For example, it provides a weak (co)kernel, vanishes if and only if $f = 0$, etc. A natural question to ask is: what about compositions? That is, if we have two composable morphisms

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$$

would there be a relation between the cones of f_1, f_2, and $f_2 \circ f_1$. This is what the Verdier Octahedron Axiom tells us about. In some sense, this should be called a composition axiom.

Definition 3.1. A Verdier triangulated category is a pre-triangulated category (K, Σ, Δ) such that for any composable morphisms

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$$

there exists a Verdier octahedron:

$$\begin{tikzcd}
A_1 \arrow[rr, "f_1"] \arrow[rrrr, bend left, "f_3"] & & A_2 \arrow[rr, "f_2"] \arrow[dl, "g_1"] & & A_3 \arrow[dl, "g_2"] \arrow[ddll, bend left, "g_3"] \\
& C_1 \arrow[ul, "h_1"] \arrow[dr, "f_4"] & & C_2 \arrow[ul, "h_2"] \arrow[ll, "(\Sigma g_1)\circ h_2"] & \\
& & C_3 \arrow[ur, "g_4"] \arrow[uull, bend left, "h_3"] & &
\end{tikzcd}$$

where $A_1A_2A_3, A_1C_1C_3, A_3C_2C_3$ and $A_2C_1C_2$ commutes, and $A_1A_2C_1$, $A_2A_3C_2$ and $C_1C_2C_3$ are exact. Moreover, $f_4 \circ g_1 = g_3 \circ f_2 : A_2 \to C_3$, and $\Sigma f_1 \circ h_3 = h_2 \circ g_4 : C_3 \to \Sigma A_2$.

Remark 3.2. In other words, the first part of the commutativity says that there exists an exact triangle

$$\mathrm{cone}(f_1) \xrightarrow{f_4} \mathrm{cone}(f_2 \circ f_1) \xrightarrow{g_4} \mathrm{cone}(f_2) \xrightarrow{\Sigma g_1 \circ h_2} \Sigma \mathrm{cone}(f_1)$$

also known as the bottom triangle $C_1C_2C_3$, that is compatible with the rest of the structures. The second part of the commutativity says that we have the following diagram:

$$\begin{tikzcd}
& & A_3 \arrow[dl, "g_2"] \arrow[dr, "g_3"] & \\
A_2 \arrow[urr, "f_2"] \arrow[ddr, "g_1"] & C_2 \arrow[l, "h_2"] \arrow[dd, "h_4"] & & C_3 \arrow[ll, dashed, "g_4"] \arrow[dl, "h_3"] \\
& & A_1 \arrow[uu, bend right, "f_3"] \arrow[ull, "f_1"] & \\
& C_1 \arrow[ur, "h_1"] \arrow[uurr, dashed, bend right, "f_4"] & &
\end{tikzcd}$$

Remark 3.3. Alternatively, we can think of the diagram as in the following form:

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "f_1"] \arrow[d] & A_2 \arrow[r, "f_2"] \arrow[d, "g_1"] & A_3 \arrow[r] \arrow[d] & 0 \arrow[d] \\
& 0 \arrow[r] & C_1 = A_{12} \arrow[r, "f_4"] \arrow[d] & C_3 = A_{13} \arrow[r, "h_3"] \arrow[d, "g_4"] & \Sigma A_1 \arrow[d, "\Sigma f_1"] \\
& & 0 \arrow[r] & C_2 = A_{23} \arrow[r, "h_2"] \arrow[d] & \Sigma A_2 \arrow[d, "\Sigma f_2"] \\
& & & 0 \arrow[r] & \Sigma A_3 \arrow[d] \\
& & & & 0
\end{tikzcd}$$

Problem 3.4 (Take-home Problem 1)**.** Suppose K is (Verdier) triangulated. Let

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
A' \arrow[r] & B'
\end{tikzcd}$$

be a commutative square. Then there exists an extended diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d] & C \arrow[r, "h"] \arrow[d] & \Sigma A \arrow[d] \\
A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & \Sigma A' \arrow[d] \\
A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & \Sigma A'' \arrow[d] \\
\Sigma A \arrow[r] & \Sigma B \arrow[r] & \Sigma C \arrow[r] & \Sigma^2 A
\end{tikzcd}$$

with exact rows and columns (that is, first three of each, with the last one being the suspension of the first), and all squares commute, except the bottom-right that anti-commutes.

Hint: Use 3 Octahedra.