Power Operations and Global Algebra

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Background. These are notes taken from Professor Nathaniel Stapleton's minicourse at University of Illinois in Fall 2024. Any mistakes and inaccuracies would be my own.

1 November 13, 2024

Background. In chromatic homotopy theory, we have a notion of height that measures complexity. In the case of height 1, we have a completion of complex K-theory

$$K \to K_p^\wedge = E_1$$

which then builds up to higher heights with E_2, E_3, and so on. When goes on to the height level of ∞, we have a map $\mathbb{S} \to H\mathbb{F}_p$, as the sphere spectrum also maps to each chromatic level. When valued in finite groups, this gives rise to objects in global algebra, which are the representative ring functor. This corresponds to the Burnside ring functor in terms of K-theory and E-cohomology of classifying spaces in terms of the spectrum $\{E_i\}_{i \geqslant 1}$.

1.1 The Complex Representation Ring

Definition 1.1. A G-representation is a finite-dimensional $\mathbb{C}$-vector space equipped with an action of G. A map $f : V \to W$ of G-representations is an equivariant linear map: $g \cdot f(v) = f \cdot g(v)$ for $g \in G$ and $v \in V$.

Given two G-representations V and W, we may build G-representations $V \oplus W$ and $V \otimes W$ with respect to the G-diagonal action.

Definition 1.2. Let $[V]$ be the isomorphism class of G-representation V. We may define addition and multiplication of G-representations V and W as

$$[V] + [W] = [V + W] \qquad [V][W] = [V \otimes W].$$

This gives rise to a symmetric monoidal structure, only lacking the additive inverses.

Taking the Grothendieck construction, we may fill in the additive inverses. Let $\mathrm{RU}(G)$ be the Grothendieck ring of the isomorphism class of G-representations under addition and multiplication above.

Lemma 1.3 (Schur)**.** If V and W are irreducible G-representations, i.e., no non-trivial G-subrepresentations, then

1. if $V \ncong W$ as G-representations, and $f : V \to W$ is a map of G-representations, then $f \equiv 0$;
2. if $V \cong W$, then any map $f : V \to V$ of G-representations must be defined by multiplication by a scalar.

Fact 1.4. Since every G-representation is a sum of irreducible G-representations in a unique way, then $\mathrm{RU}(G)$ is (additively) a free $\mathbb{Z}$-module with canonical basis given by the set of isomorphism classes of irreducible G-representations.

Therefore, $\mathrm{RU}(G)$ is quite simple with respect to the additive structure. However, it takes more effort to understand the ring multiplicatively.

Example 1.5. Let e be the trivial group, then the isomorphism classes are given by $\mathbb{N}$, so taking the Grothendieck completion gives $\mathrm{RU}(e) \cong \mathbb{Z}$.

Example 1.6. Assume A is an abelian group and V is an irreducible A-representation. For $a \in A$, the action map $a : V \to V$ is a map of A-representations. Since V is irreducible, then by Lemma 1.3, we know the map a is described by $av = cv$ for some $c \in \mathbb{C}$. Therefore, the subspace $\langle v \rangle$ is a subrepresentation of V, hence $V = \langle v \rangle$. That is, $\dim(V) = 1$.

Example 1.7. Consider $A = C_n \subseteq S^1 \subseteq \mathbb{C}$, then A inherits an $\mathbb{C}$-action. In particular, the action $\rho : C_n \times \mathbb{C} \to \mathbb{C}$ is such that $\rho^{\otimes n} = \mathrm{triv}$ and the tensor powers give n irreducible representations. Therefore, $\mathrm{RU}(C_n) \cong \mathbb{Z}[x]/(x^n - 1)$ where $x = [\rho]$.

Remark. The spectrum $\mathrm{Spec}(\mathbb{Z}[x]/(x^n - 1)) \cong \mathbb{G}_m[n]$ is the n-torsion of the multiplicative group.

Example 1.8. Consider the free $\mathbb{C}$-vector space $\mathbb{C}\{C_n\}$ based on the cyclic group C_n has a C_n-action. This is then called the regular representation. Since it can be written as a sum of irreducible representations, then one can show that

$$\mathbb{C}\{C_n\} \cong \bigoplus_{i=0}^{n-1} \rho^{\otimes i}.$$

Alternatively,

$$[\mathbb{C}\{C_n\}] = 1 + x + x^2 + \cdots + x^{n-1}$$

in the context of representation ring.

It is now natural to ask: how do representation rings interact as the group varies?

1.2 Restrictions and Transfers

Let $f : H \to G$ be a map of groups, then

- there is a (contravariant) restriction map $\mathrm{Res}_f : \mathrm{RU}(G) \to \mathrm{RU}(H)$: given G-representation V, we can send this to $H \xrightarrow{f} G$ acting on V, so thinking of V as an H-representation. In particular, the restriction map above is a ring map;
- we can also define a (covariant) transfer map $\mathrm{Tr}_f : \mathrm{RU}(H) \to \mathrm{RU}(G)$: given H-representation V, we may notice that it is the same thing as a $\mathbb{C}[H]$-module over the group ring, then by base-change, we consider it as $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ as a G-representation. This map is not a ring map: it is additive but not multiplicative in general.

Example 1.9. Consider the trivial map $i : e \to G$, then this corresponds to a restriction map

$$\begin{aligned} \mathrm{Res}_i : \mathrm{RU}(G) &\to \mathbb{Z} \\ V &\mapsto \dim(V) \end{aligned}$$

that describes the dimension, and a transfer map

$$\begin{aligned} \mathrm{Tr}_i : \mathbb{Z} &\to \mathrm{RU}(G) \\ \mathbb{C} &\mapsto \mathbb{C}[G] \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}[G] \end{aligned}$$

as the regular representation.

The restriction and transfer map interacts via the Frobenius reciprocity and a double coset formula.

Theorem 1.10 (Frobenius Reciprocity). Given $x \in \mathrm{RU}(G)$ and $y \in \mathrm{RU}(H)$, then $\mathrm{Tr}_f(\mathrm{Res}_f(x)y) = x\,\mathrm{Tr}_f(y)$. That is, the transfer map is a map of $\mathrm{RU}(G)$-modules for a module structure on $\mathrm{RU}(H)$ given by restriction along f.

Theorem 1.11 (Double Coset Formula). Given subgroups $H, K \subseteq G$, then

$$\mathrm{Res}^G_K \mathrm{Tr}^G_H = \sum_{[g] \in K\backslash G/H} \mathrm{Tr}^K_{K \cap H^{g^{-1}}} c_g \mathrm{Res}^{K^g \cap H}_H$$

where c_g is a conjugation action.

Example 1.12. Suppose $k \mid n$ and consider $f : C_k \to C_n$, then

$$\begin{aligned}\mathrm{Res}_f : \mathrm{RU}(C_n) \cong \mathbb{Z}[x]/(x^n - 1) &\to \mathrm{RU}(C_k) \cong \mathbb{Z}[x]/(x^k - 1)\\ x &\mapsto x\end{aligned}$$

is a surjection, and

$$\begin{aligned}\mathrm{Tr}_f : \mathrm{RU}(C_k) &\to \mathrm{RU}(C_n)\\ \mathbb{1} = [\mathbb{C}] &\mapsto [\mathbb{C}[C_n] \otimes_{\mathbb{C}[C_k]} \mathbb{C}] \cong [\mathbb{C}[C_n/C_k]]\end{aligned}$$

Since the restriction map is surjective and the transfer map is a map of modules, then the module structure implies that the transfer map is completely determined by the mapping of $\mathbb{1}$.

1.3 Character Theory

Let G/conj be the set of conjugacy classes of G. Let $\mathbb{Q}(\mu_\infty)$ be $\mathbb{Q}$ adjoining all roots of unity. Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be some G-representation, then the trace $\mathrm{Tr}(\rho(g))$ is a sum of roots of unity.

Remark. To see this, we note that every representation $\mathrm{GL}_n(\mathbb{C})$ can be conjugated to some representation of the unitary group, which can then be diagonalized. But G has finite order, so the elements on the diagonal has to be some roots of unity. Alternatively, apply Jordan canonical form.

Furthermore, the trace function satisfies $\mathrm{Tr}(\rho(hgh^{-1})) = \mathrm{Tr}(\rho(g))$. So this process gives a map

$$\chi : \mathrm{RU}(G) \to \mathrm{Cl}(G, \mathbb{Q}(\mu_\infty)) = \mathrm{Fun}(G/\operatorname{conj}, \mathbb{Q}(\mu_\infty))$$

into the class functions.

Fact 1.13. χ is an injective ring map: we win by sending a complicated (multiplicative) structure into a much simpler structure, since the ring structure is defined pointwise. Moreover, the base-change

$$\mathbb{Q}(\mu_\infty) \otimes_{\mathbb{Z}} \mathrm{RU}(G) \to \mathrm{Cl}(G, \mathbb{Q}(\mu_\infty))$$

is an isomorphism. Even more: $\mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \cong \hat{\mathbb{Z}}^* := \varinjlim_n (\mathbb{Z}/n\mathbb{Z})^*$.

Fact 1.14. Here $\mathrm{Aut}(\hat{\mathbb{Z}}) \cong \hat{\mathbb{Z}}^*$ acts on $G/\operatorname{conj} \cong \mathrm{Hom}_{\mathrm{cts}}(\hat{\mathbb{Z}}, G)/\operatorname{conj}$ naturally.

Combining the two actions, we have an isomorphism

$$\mathbb{Q} \otimes \mathrm{RU}(G) \cong \mathrm{Cl}(G, \mathbb{Q}(\mu_\infty))^{\hat{\mathbb{Z}}^\times}.$$

Example 1.15. Let $G = \Sigma_m$, then we have a map

$$\mathrm{RU}(\Sigma_m) \to \mathrm{Cl}(\Sigma_m, \mathbb{Q}(\mu_\infty))^{\hat{\mathbb{Z}}^\times}.$$

A conjugacy class $[\sigma]$ of Σ_m is determined completely by the cycle decomposition: given $\ell \in \hat{\mathbb{Z}}^*$ and $[\sigma] \in \Sigma_m/\operatorname{conj}$, we view $\ell \in (\mathbb{Z}/m!\mathbb{Z})^*$ and send $[\sigma]$ to $[\sigma^\ell]$ via ℓ. In particular, $[\sigma] = [\sigma^\ell]$ have the same cycle decomposition. Therefore, the action of $\hat{\mathbb{Z}}^*$ on conjugacy classes must be trivial. Hence, the given map tells us that

$$\mathrm{Cl}(\Sigma_m, \mathbb{Q}(\mu_\infty))^{\hat{\mathbb{Z}}^\times} \cong \mathrm{Cl}(\Sigma_m, \mathbb{Q}).$$

Comparing this with $\mathrm{Cl}(\Sigma_m, \mathbb{Z})$, we notice that the trace map ensures the fractions of integers never appear in the image, therefore this map factors into $\mathrm{Cl}(\Sigma_m, \mathbb{Z})$.

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Background. Recall that given a finite group G, we built a complex representation ring $\mathrm{RU}(G)$. Given a group homomorphism $f : H \to G$, we had a map of rings $\mathrm{Res}_f : \mathrm{RU}(G) \to \mathrm{RU}(H)$ and a map of abelian groups $\mathrm{Tr}_f : \mathrm{RU}(H) \to \mathrm{RU}(G)$. This transfer map satisfies Frobenius reciprocity, so it is a map of modules. Therefore, the image of the transfer is an ideal in $\mathrm{RU}(G)$.

2.1 Characters of Restriction and Transfer

Given a group homomorphism $\varphi : H \to G$, on the level of class functions, we have a restriction

$$\varphi_{H/G} : H/\operatorname{conj} \to G/\operatorname{conj}$$

with a commutative square

$$\begin{tikzcd} \mathrm{RU}(G) \arrow[r, "\mathrm{Res}_\varphi"] \arrow[d, "\chi"'] & \mathrm{RU}(H) \arrow[d, "\chi"] \\ \mathrm{Cl}(G, \mathbb{Q}(\mu_\infty)) \arrow[r, dashed, "\mathrm{Res}_{\varphi/\operatorname{conj}}"'] & \mathrm{Cl}(H, \mathbb{Q}(\mu_\infty)) \end{tikzcd}$$

This induced map can be constructed via base-change to $\mathbb{Q}(\mu_\infty)$, and then make suitable identifications.

Exercise 2.1. Show that this square actually commutes.

To ask the same thing for the transfer map is a bit difficult: suppose $H \subseteq G$, then we do have a transfer map

$$\begin{aligned} \mathrm{Tr}_H^G : \mathrm{Cl}(H, \mathbb{Q}(\mu_\infty)) &\to \mathrm{Cl}(G, \mathbb{Q}(\mu_\infty)) \\ \mathrm{Tr}_H^G(f)([g]) &= \frac{1}{|H|} \sum_{\substack{\ell \in G: \\ \ell g \ell^{-1} \in H}} f([\ell g \ell^{-1}]) \\ &= \sum_{\substack{\ell H \in G/H \\ \ell g \ell^{-1} \in H}} f([\ell g \ell^{-1}]) \text{ via Theorem 1.11} \end{aligned}$$

on the level of class functions, through one of these equivalent definitions.

Now suppose $\varphi : H \twoheadrightarrow G$ is a surjection, then the transfer map can be defined by

$$\mathrm{Tr}_\varphi(f)([g]) = \frac{1}{|\ker(\varphi)|} \sum_{h \in \varphi^{-1}(g)} f([h]).$$

Theorem 2.2 (Künneth Isomorphism)**.** Given finite groups G and H, then

$$\mathrm{RU}(G \times H) \cong \mathrm{RU}(G) \otimes_{\mathbb{Z}} \mathrm{RU}(H).$$

Let V be a G-representation, then we may map it to $V^{\otimes m}$. How do we retain the corresponding G-action? One way to do this is a coordinatewise action by $G^{\times m}$. However, there is also an action by the symmetric group where we permute the factors. Therefore, there is an action of the wreath product $G \wr \Sigma_m := G^{\times m} \rtimes \Sigma_m$ on $V^{\otimes m}$.

Definition 2.3. The mth power operation $\mathbb{P}^m$ is defined by $\mathbb{P}_m([V]) = [V^{\otimes m}]$ with the wreath product action above.

Fact 2.4.

- $\mathbb{P}^m$ is multiplicative: $(V \otimes W)^{\otimes m} \cong V^{\otimes m} \otimes W^{\otimes m}$.
- As vector spaces, we have $(V \oplus W)^{\otimes m} \cong \bigoplus_{i+j=m} \binom{m}{i} V^{\otimes i} \otimes W^{\otimes j}$. But what happens if we think of them as $(G \wr \Sigma_m)$-representations? This requires an $(G \wr \Sigma_m)$-action on each summand, which is not usually available.

The failure of additivity is in fact controlled by the transfer map:

$$\mathbb{P}_m([V] + [W]) = \mathbb{P}_m([V]) + \sum_{\substack{i+j=m \\ i,j>0}} \mathrm{Tr}_{G\wr\Sigma_i \times G\wr\Sigma_j}^{G\wr\Sigma_m} (\mathbb{P}_i([V])) \boxtimes \mathbb{P}_j([W])) + \mathbb{P}_m([W]). \tag{2.5}$$

In the boundary cases, i.e., $j = m$ and $j = 0$, the formula is intuitive: the interesting case is when j is between them. The idea being, let $\underline{m} = \underline{i} \sqcup \underline{j}$, then there is a map

$$V^{\otimes \underline{i}} \otimes W^{\otimes \underline{j}} \to \bigoplus_{\substack{x \subseteq \underline{m} \\ |x| = i}} V^{\otimes x} \otimes W^{\otimes (\underline{m} \setminus x)}.$$

In particular, this induces an inclusion

$$\mathbb{C}[G \wr \Sigma_m] \otimes_{\mathbb{C}[G \wr \Sigma_i \times G \wr \Sigma_j]} V^{\otimes \underline{i}} \otimes W^{\otimes \underline{j}} \hookrightarrow \bigoplus_{\substack{x \subseteq \underline{m} \\ |x| = i}} V^{\otimes x} \otimes W^{\otimes (\underline{m} \setminus x)}$$

which is $(G \wr \Sigma_i \times G \wr \Sigma_j)$-equivariant, which is then an isomorphism.

All of these power operation seems to only work on a given group G: it is not additive. However, there is really a way we can work it out on the representation ring.

2.2 From G-representations to $\mathrm{RU}(G)$

- Note that $\mathbb{P}_0([V]) = 1$, therefore $\mathbb{P}_0(-[V]) = 1$.
- $\mathbb{P}_1([V]) = [V]$, so $\mathbb{P}_1(-[V]) = -[V]$.
- By induction, we may define $\mathbb{P}_m$ on $\mathrm{RU}(G)$. For instance, by the formula of power operations and transfer, we have

  $$\mathbb{P}_2([V] + (-[V])) = \mathbb{P}_2([V]) + \mathrm{Tr}_{1,1}^2([V] \boxtimes (-[V])) + \mathbb{P}_2(-[V])$$

  and by pulling the negative sign out, we get

  $$\mathbb{P}_2(-[V]) = \mathrm{Tr}_{1,1}^2([V] \boxtimes [V]) - \mathbb{P}_2([V]).$$

  Exercise 2.6. $\mathbb{P}_2(-1) = 1 + x - 1 = x$.

  In general, we define a map $\mathbb{P}_2 : \mathbb{Z} \to \mathrm{RU}(\Sigma_2) = \mathrm{RU}(C_2)$.

Let us examine some properties of $\mathbb{P}_m$'s.

Remark.

- $\mathbb{P}_0(x) = 1$, $\mathbb{P}_1 = \mathrm{id}$, and $\mathbb{P}_m(1) = 1$.
- $\mathbb{P}_m(x + y)$ is controlled by binomial expansions, as seen above in Equation (2.5).
- $\mathrm{Res}_{G \wr \Sigma_i \times G \wr \Sigma_j}^{G \wr \Sigma_m} \mathbb{P}_m = \mathbb{P}_i \boxtimes \mathbb{P}_j$.

Let $I_{\mathrm{tr}} \subseteq \mathrm{RU}(G \wr \Sigma_m)$ be

$$\mathrm{im}\left(\bigoplus_{\substack{i+j=m \\ i,j>0}} \mathrm{Tr}_{G \wr \Sigma_i \times G \wr \Sigma_j}^{G \wr \Sigma_m} \right).$$

This gives a ring map

$$\mathbb{P}_m / I_{\mathrm{tr}} : \mathrm{RU}(G) \to \mathrm{RU}(G \wr \Sigma_m)/I_{\mathrm{tr}}$$

which is additive. In fact, whatever additive operations we build on the level of representation rings must factor through this map.

2.3 Interaction of Power Operations with Character Map

Definition 2.7. An (unordered) partition of a natural number m, denoted $\lambda \vdash m$, is a function $\lambda : \mathbb{N}_{>0} \to \mathbb{N}$ such that $\sum_i \lambda_i \cdot i = m$.

A partition $\lambda \vdash m$ of m decorated by G/conj is a function $\lambda : \mathbb{N}_{>0} \times G/\operatorname{conj} \to \mathbb{N}$ such that $\sum_{i,[g]} \lambda_{i,[g]} \cdot i = m$.

In particular, there is a canonical bijection

$$(G \wr \Sigma_m)/\operatorname{conj} \cong \operatorname{Parts}(m, G/\operatorname{conj})$$

where the right-hand side gives partitions of m decorated by G/conj. The idea being, if $\sigma(1\ \cdots\ m)$, then there is a conjugation $(g_1, \cdots, g_m, \sigma) \sim (g_1 g_2 \cdots g_m, e, \cdots, e, \sigma)$. The partition we get from this element has the property that $\lambda_{m,[g_1 \cdots g_m]} = 1$.

Proposition 2.8. We have a commtuative diagram

$$\begin{tikzcd}
\mathrm{RU}(G) \arrow[r, "\mathbb{P}_m"] \arrow[d, "\chi"'] & \mathrm{RU}(G \wr \Sigma_m) \arrow[d, "\lambda"] \\
\mathrm{Cl}(G, \mathbb{Q}(\mu_\infty) \arrow[r, dashed, "\mathbb{P}_m"'] & \mathrm{Cl}(G \wr \Sigma_m, \mathbb{Q}(\mu_\infty))
\end{tikzcd}$$

Since the operation is not additive, we cannot just base-change by $\mathbb{Q}(\mu_\infty)$ and find the bottom map $\mathbb{P}_m$. Regardless, such map still exists, which is defined by the formula

$$\mathbb{P}_m(f)(\lambda) = \prod_{i,[g]} f([g])^{\lambda_{i,[g]}}.$$

Proof. Proceed by induction. □

2.4 Symmetric Powers and Adams Operations from Power Operations

There is a diagonal map

$$\Delta : G \times \Sigma_m \to G \wr \Sigma_m,$$

which is induced by the diagonal map $G \to G^{\times m}$. On conjugacy classes, this gives an assignment

$$([g], \tau \vdash m) \mapsto \lambda_{i,[h]} = \begin{cases} \tau_i, & \text{if } [h] = [g^i] \\ 0, & \text{otherwise} \end{cases}$$

We have a diagram

$$\begin{tikzcd}
 & & \mathrm{RU}(G) & & \\
\mathrm{RU}(G) \arrow[r, "\mathbb{P}_m"] \arrow[urr, bend left, "\beta_m"] \arrow[ddrrr, bend right, "\psi_m"'] & \mathrm{RU}(G \wr \Sigma_m) \arrow[r, "\Delta^*"] & \mathrm{RU}(G \times \Sigma_m) \arrow[r, "\cong"] \arrow[u, "\mathrm{Tr}^G_{G \times \Sigma_m}"'] & \mathrm{RU}(G) \otimes \mathrm{RU}(\Sigma_m) \arrow[d, "\mathrm{id} \otimes \chi_{(1\ \cdots\ m)}"] \\
 & & & \mathrm{RU}(G) \otimes \mathbb{Z} \arrow[d, "\cong"] \\
 & & & \mathrm{RU}(G)
\end{tikzcd}$$

where

- β_m is the symmetric power operation, defined by $\beta_m([V]) = [V^{\otimes m}/\Sigma_m]$;

- ψ_m is the Adams operation. In fact, this is additive and therefore is a ring map.

Remark. Within the diagram above, we have a factorization

$$\begin{tikzcd} \mathrm{RU}(\Sigma_m) \arrow[rr, "\chi_{(1\ \cdots\ m)}"] \arrow[dr] & & \mathbb{Z} \\ & \mathrm{RU}(\Sigma_m)/I_{\mathrm{tr}} \arrow[ur, "\cong"'] & \end{tikzcd}$$

and therefore $\mathrm{RU}(G \times \Sigma_m)/I_{\mathrm{tr}} \cong \mathrm{RU}(G) \otimes (\mathrm{RU}(\Sigma_m)/I_{\mathrm{tr}})$.

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Going back to the diagram last time, we give formulas for each one of them on the level of class functions.

- Recall that $\Delta^*\mathbb{P}_m(f)([g], \tau \vdash m) = \prod_i f([g^i])^{\tau_i}$.
- We can read off the Adams operations from cycles using the assignment above, as $\psi_m(f)([g]) = f([g^m])$.
- The symmetric power operation is defined using the transfer along surjection, via

$$\beta_m(f) = \frac{1}{m!}\sum_{\tau \vdash m} \frac{m!}{\prod_i (i!)^{\tau_i}} \prod_i f([g^i])^{\tau_i} = \sum_{\tau \vdash m} \prod_i \left(\frac{f([g^i])}{i!}\right)^{\tau_i}.$$

  Note that the summation counts the size of conjugacy classes over τ.

Comparing terms of $\prod_i f([g^i])^{\tau_i}$ and $\sum_{\tau\vdash m}\prod_i \left(\frac{f([g^i])}{i!}\right)^{\tau_i}$, we note that

$$\sum_{m \geqslant 0} \beta_m t^m = \exp\left(\sum_{k>0} \frac{\psi_k}{k} t^k\right).$$

3.1 The Burnside Ring

Instead of on G-representations, we can do the same thing on finite G-sets using the Burnside ring.

Definition 3.1. A G-set is a finite set X with a G-action.

Given G-sets X and Y, we have disjoint union $X \sqcup Y$, and the product $X \times Y$, which has diagonal G-action. Using these notions, we may define addition and multiplications on G-setes.

We define $A(G)$ to be the Grothendieck ring of isomorphism classes of G-sets under addition and multiplication.

Example 3.2. $A(e) = \mathbb{Z}$.

Just like in the case of representation rings, we understand the Grothendieck ring using building blocks called the transitive G-sets. Two G-sets H and K are isomorphic, i.e., $G/H \cong G/K$, if and only if H is conjugate to K in G.

Remark. Additively, $A(G)$ is a free-abelian group on the isomorphism classes of transitive G-sets. We can write down the formula that says

$$A(G) \cong \bigoplus_{[H]} \mathbb{Z}\{[G/H]\}.$$

Let $\mathrm{Conj}(G) = \mathrm{Sub}(G)/\mathrm{conj}$, the conjugacy classes of subgroups of G.

To understand multiplication, we need to give a formula for the product of G-sets. This can be done using Theorem 1.11:

$$(G/H) \times (G/K) \cong \coprod_{[g]\in H\backslash G/K} G/(H^g \wedge K)$$

as G-sets. Even though we can write down the explicit formula, unless G is small, it is hard to figure out the answer using the double coset formula. Instead, we will embed this in a different ring.

First note that we have a linearization map

$$\begin{aligned} L : A(G) &\to \mathrm{RU}(G) \\ [X] &\mapsto [\mathbb{C}\{X\}] \end{aligned}$$

from a G-set to the free $\mathbb{C}$-vector space, along with an induced G-action. This is in fact a ring map. Moreover, given a group homomorphism $\varphi : H \to G$, we get a restriction map

$$\mathrm{Res}_\varphi : A(G) \to A(H)$$

and a transfer map

$$\begin{aligned} \mathrm{Tr}_\varphi : A(H) &\to A(G) \\ H \curvearrowright X &\mapsto G^\varphi \times_H X \end{aligned}$$

This is completely analogous to the G-representation map. In fact, through L, restriction and transfer maps are compatible across $A(G)$ and $\mathrm{RU}(G)$. Moreover, these maps satisfy Frobenius reciprocity and double coset formula as well.

Remark. It is hard to study L: sometimes L is injective or surjective, sometimes neither.

3.2 Character Theory

Let us study the character theory in this case. Let the Ghost ring $\mathrm{Marks}(G, \mathbb{Z}) := \mathrm{Fun}(\mathrm{Conj}(G), \mathbb{Z})$ be the collection of functions $\mathrm{Conj}(G) \to \mathbb{Z}$. The character map is defined by

$$\begin{aligned} \chi : A(G) &\to \mathrm{Marks}(G, \mathbb{Z}) \\ G \curvearrowright X &\mapsto \chi([X])([H]) = |X^H| \end{aligned}$$

Remark. Historically, the number of fixed points we get from the different G-sets is denoted by Marks.

Remark. The Burnside ring is at height ∞, so this is parametrized by on the level of all subgroups of elements, instead of the conjugacy classes.

Fact 3.3. This is an injective ring map and a rational isomorphism.

How does this relate to the character map of G-representations, given the linearization map? The answer would be in the best way possible:

$$\begin{tikzcd} A(G) \arrow[r, "L"] \arrow[d, "\chi"'] & \mathrm{RU}(G) \arrow[d, "\chi"] \\ \mathrm{Marks}(G, \mathbb{Z}) \arrow[r, dashed, "L"'] & \mathrm{Cl}(G, \mathbb{Q}(\mu_\infty)) \end{tikzcd}$$

where we define $L(f)([g]) = f([\langle g\rangle])$, i.e., we define it via the conjugacy class of the subgroup generated by g.

Remark. The Burnside ring does not satisfy a Künneth isomorphism formula: $A(G \times H) \ncong A(G) \otimes A(H)$ in general.

3.3 Power Operations

Again, there are power operations

$$\begin{aligned} \mathbb{P}_m : A(G) &\to A(G \wr \Sigma_m) \\ G \curvearrowright X &\mapsto (G \wr \Sigma_m) \curvearrowright X^{\times m} \end{aligned}$$

Something special happens: the power operations has the combinatorial property, having something to do with the partitions again. Let $\operatorname{Parts}(m, \operatorname{Conj}(G))$ be the set of integer partitions of m decorated by the set of conjugacy classes of subgroups of G, with

$$\lambda : \mathbb{N}_{>0} \times \operatorname{Conj}(G) \to \mathbb{N}$$

such that $\sum_{i,[H]} \lambda_{i,[H]} \cdot i = m$. One concern we do have is that the Burnside ring $A(G \wr \Sigma_m)$ is too big, so we want to reduce the size of the group.

Given a partition λ, we can write down a summation of wreath products

$$\prod_{i,[H]} (H \wr \Sigma_i)^{\times \lambda_{i,[H]}} \subseteq G \wr \Sigma_m.$$

We can just let $\mathring{A}(G, m) \subseteq A(G \wr \Sigma_m)$ to be the subgroup generated by $(G \wr \Sigma_m)$-sets of the form $(G \wr \Sigma_m)/\prod(H \wr \Sigma_i)^{\times \lambda_{i,[H]}}$. This is much smaller than $A(G \wr \Sigma_m)$.

Fact 3.4.

- $\mathring{A}(G, m)$ is a subring.
- $\mathring{A}(G, m)$ is a Burnside ring, given by the submissive $(G \wr \Sigma_m)$-sets: $(G \wr \Sigma_m) \curvearrowright X \hookrightarrow Y^{\times m}$ for $G \curvearrowright Y$.
- Elements in the image of power operations $\mathbb{P}_m$ are submissive by definition, so $\operatorname{im}(\mathbb{P}_m) \subseteq \mathring{A}(G, m)$.

Fact 3.5. We have a commutative diagram

$$\begin{tikzcd} A(G) \arrow[r, "\mathbb{P}_m"] \arrow[d, "\chi"'] & \hat{A}(G, m) \arrow[d, "\chi"] \\ \operatorname{Marks}(G, \mathbb{Z}) \arrow[r, dashed, "\mathbb{P}_m"'] & \operatorname{Fun}(\operatorname{Parts}(m, \operatorname{Conj}(G)), \mathbb{Z}) \end{tikzcd}$$

defined by $\mathbb{P}_m(f)(\lambda) = \prod_{i,[H]} f([H])^{\lambda_{i,[H]}}$.

Note that the right-hand side is still a rational isomorphism.

3.4 Morava E-theory

Let $n > 0$ and p be a prime, then we have a cohomology theory $E = E_{n,p}$. Given a group Be, we get

$$E(Be) = \mathbb{Z}_p[[u_1, \ldots, u_{n-1}]].$$

In particular, $E(BG)$ is a complete local ring, and often times a free $E(Be)$-module. However, unlike the other two cases, it has no canonical basis.

There are also restriction and transfer maps along all homomorphisms $\varphi : H \to G$. Analogously, it has a character theory called the Hopkins-Kuhn-Ravenel (HKR) character theory. The concept that plays the role of conjugacy classes is a map

$$\mathbb{Z}_p^{\times n} \to G$$

which is an n-tuple of pairwise commuting p-power order elements in G. Let

$$\operatorname{Cl}_{n,p}(G, R) = \operatorname{Fun}(\operatorname{Hom}_{\text{cts}}(\mathbb{Z}_p^{\times}, G)/\operatorname{conj}, R).$$

HKR constructed a $\mathbb{Q}$-algebra C_0 equipped with ring maps $E(B(\mathbb{Z}/p^k\mathbb{Z})^{\times n}) \to C_0$ for all k, as a choice of permutation of unity. They also defined a character map

$$\chi : E(BG) \to \mathrm{Cl}_{n,p}(G, C_0)$$

which is a ring map such that $C_0 \otimes_{E(Be)} E(BG) \cong \mathrm{Cl}_{n,p}(G, C_0)$. There is also a $\mathrm{GL}_n(\mathbb{Z}_p)$-action, giving an isomorphism

$$\mathbb{Q} \otimes E(BG) \to \mathrm{Cl}_{n,p}(G, C_0)^{\mathrm{GL}_n(\mathbb{Z}_p)}.$$

This E-theory also has corresponding power operations. We define

$$\mathbb{P}_m : E(BG) \to E(BG \wr \Sigma_m).$$

Since E-theory is constructed out of arithmetic geometry using formal groups, we have a fundamental result that allows us to grasp the power operations.

Theorem 3.6 (Ando-Hopkins-Strickland)**.** The map of commutative rings

$$E(BA) \to E(BA \times B\Sigma_m)/I_{\mathrm{tr}}$$

can be understood in terms of formal algebraic geometry.

This allows us to give a formula of power operations on class functions

$$\mathbb{P}_m : \mathrm{Cl}_{n,p}(G, C_0) \to \mathrm{Cl}_{n,p}(G \wr \Sigma_m, C_0)$$

that is compatible with the power operations in the mentioned setting. That is, we have

$$\begin{tikzcd}
 & & E(BG) & \\
E(BG) \arrow[urr, bend left, "\beta_m"] \arrow[drrr, bend right, "T_{p^k}"'] \arrow[r] & E(G \wr \Sigma_m) \arrow[r] & E(BG \times B\Sigma_m) \arrow[u, "\mathrm{Tr}^G_{G\times\Sigma_m}"'] \arrow[r, "\text{Künneth from } \Sigma_m", "\cong"'] & E(BG) \otimes_{E(Be)} E(B\Sigma_m) \arrow[d] \\
 & & & E(BG) \otimes_{E(Be)} E(B\Sigma_m)/I_{\mathrm{tr}} \arrow[d, "m=p^k"', "\text{adding ring maps together}"] \\
 & & & E(BG)
\end{tikzcd}$$

where the Hecke operator T_{p^k} is additive but not multiplicative.

Remark. When we do not assign $m = p^k$, then the diagram is trivial, i.e., the composition is zero.

By a result of Ganter, we have

$$\sum_{m \geqslant 0} \beta_m t^m = \exp\left(\sum_{k \geqslant 0} \frac{T_{p^k}}{p^k} t^{p^k}\right).$$

Remark. We have seen one version of this formula in symmetric power operation on $\mathrm{RU}(G)$, so there is the analogy. Similarly, there is a formula for Burnside ring just like this, however, that particular version is not derived from a diagram like the other two.