

MATH 595 (Group Cohomology) Notes

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1 AUG 21, 2023: INTRODUCTION

Group cohomology works over different settings of groups, like finite groups, profinite groups, and topological groups. The course will develop towards

- duality in $H^*(G, -)$, and
- focus on computations, e.g., spectral sequences.

We first establish some notations.

- Let G be a group. If G has a topology, that would also be part of the information of G .
- A (left) G -module is an abelian group M with an action map

$$\begin{aligned} G \times M &\rightarrow M \\ (g, m) &\mapsto g \cdot m = gm \end{aligned}$$

satisfying

- $1 \cdot m = m$,
- $(gh) \cdot m = g \cdot (hm)$,
- $g(m + m') = gm + gm'$.

Remark 1.1. If G is a finite group, then the associated (non-commutative) group ring $\mathbb{Z}[G] := \bigoplus_{g \in G} \mathbb{Z}e_g$, where the multiplication is determined by $e_g e_h = e_{gh}$. Therefore, a G -module is just a $\mathbb{Z}[G]$ -module.

Example 1.2. • Trivial module \mathbb{Z} , or any abelian group with the trivial action $g \cdot a = a$.

- C_2 , or any group with $f : G \twoheadrightarrow C_2$, then G with C_2 as a quotient gives the sign representation \mathbb{Z}_{sgn} , with $g \cdot (a) = (-1)^{\rho(g)} a$.
- $\mathbb{Z}[G]$ is a G -module via the left multiplication action, and/or the conjugation action.

Definition 1.3 (Fixed points/Invariants). The set of fixed points of M over G is $M^G = \{m \in M \mid gm = m \ \forall g \in G\}$.

Definition 1.4 (Orbits/Coinvariants). The set of orbits of M over G is $M_G = M/(gm - m)$.

Example 1.5. If $M = \mathbb{Z}_{\text{sgn}}$, then everything gets multiplied by -1 , so there are no fixed points. The orbits of M over G would be $\mathbb{Z}_{\text{sgn}}/(-2) \cong \mathbb{Z}/2\mathbb{Z}$.

Example 1.6. If $M = \mathbb{Z}[G]$, then the fixed points are $\mathbb{Z} \left\{ \sum_{g \in G} e_g \right\}$.

Thinking in a categorical setting, we have a trivial action function $\mathbb{Z}\text{-Mod} \rightarrow G\text{-Mod}$, sending $ga \mapsto a$ for all $g \in G$ and $a \in A$. This gives an exact functor from \mathbf{Ab} to $G\text{-Mod}$. Then this functor has a right adjoint $()^G : G\text{-Mod} \rightarrow \mathbf{Ab}$, and a left adjoint $()_G : \mathbf{Ab} \rightarrow G\text{-Mod}$. More specifically, M^G becomes the maximal trivial action submodule of M , namely $\text{Hom}_G(\mathbb{Z}, M)$; M_G becomes the largest quotient of M with trivial action, namely $\mathbb{Z} \otimes_{\mathbb{Z}[G]} M$. This simplifies to the tensor-hom adjunction in some sense. For a more detailed derivation of this, see Chapter 6.1 of Weibel.

Remark 1.7. In general, as in the category of G -sets, we have the orbit functor $X \mapsto X/G$ and the fixed point functor $X \mapsto X^G$. The orbit functor is left adjoint to the free G -set functor, and the fixed point functor is the right adjoint of the trivial G -set functor.

Remark 1.8. Read more about the setting in profinite groups with their topologies in Neukirch-Schmidt-Wingberg.

Definition 1.9 (Profinite Group). A profinite group of a collection of groups is $G = \varprojlim_i G_i$ as an inverse limit, where each G_i is a finite group of the form G/U_i for some open U_i . This gives a topology to the profinite group.

Remark 1.10. The groups rings $\mathbb{Z}[[G]] = \varprojlim_i \mathbb{Z}[G_i]$. For instance, let $G = \hat{\mathbb{Z}}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$, then $\mathbb{Z}_p[[G]] = \varprojlim_n \mathbb{Z}_p[\mathbb{Z}/p^n\mathbb{Z}]$, where each $\mathbb{Z}[\mathbb{Z}/p^n\mathbb{Z}] \cong \bigoplus_{i=0}^{n-1} \mathbb{Z}\{e_i\}$ where $e_i \cdot e_j = e_{ij}$. Therefore, $\mathbb{Z}_p[[G]]$ is now equivalent to $\varprojlim_n \mathbb{Z}_p[t]/(t^{p^n} - 1_e)$, and hence becomes a power series.

Remark 1.11. By a change of variables, this becomes $\varprojlim_n \mathbb{Z}_p[x]/(x^{p^n})$, but this only works in the finite group \mathbb{Z}_p case, and not in general for \mathbb{Z} .

Example 1.12. $\mathbb{Z}[C_n] \cong \mathbb{Z}\{e\} \oplus \mathbb{Z}\{g\} \oplus \mathbb{Z}\{g^2\} \oplus \cdots \oplus \mathbb{Z}\{g^{n-1}\} \cong \mathbb{Z}[g]/(g^n - 1_e)$.

2 AUG 23, 2023: COHOMOLOGY OF GROUPS

Definition 2.1. Let G be a group, then we have a diagram

$$EG : \cdots \rightrightarrows G \times G \rightrightarrows G$$

where the arrows are given by

$$EG^n = G^{n+1} \xrightarrow{d_i} G^n$$

for all $0 \leq i \leq n$. In the sense of simplicial sets, we have $d_i(g_0, \dots, g_n) = (g_0, \dots, \hat{g}_i, \dots, g_n)$.

Now let M be a G -module, then we define $X^n = X^n(G, M) = \text{Map}_{\text{Set}}(G^{n+1}, M)$. G now has an action on this set, given by

$$(g \circ f)(g_0, \dots, g_n) = gf(g^{-1}g_0, \dots, g^{-1}g_n).$$

The action on d^i 's are contravariant, namely we obtain $d_i^* : X_n \rightarrow X^{n+1}$ with an inherited structure. Note that M sits inside X^0 , therefore we have a complex $(*)$:

$$0 \longrightarrow M \xleftarrow{\partial_0} X^0 \xrightarrow{\partial_1} X^1 \xrightarrow{\partial_2} X^2 \xrightarrow{\partial_3} \cdots$$

Here ∂_0 includes M as the constant functions into X , namely $\partial_0(m) = f$ for $f(g) = m$, and so on. In general, for $n > 0$, we have

$$\partial_n = \sum_{i=0}^n (-1)^i d_i^*.$$

Lemma 2.2. The complex $(*) : M \rightarrow X^\cdot$ is an exact complex of G -modules, i.e., $\partial^2 = 0$ and $\ker(\partial_{n+1}) = \text{im}(\partial_n)$, and the ∂_i 's preserves the G -action. This is called the standard resolution of M as a G -module.

Proof. Exercise. □

Definition 2.3. The G -fixed points of the X^n 's are defined by $C^n(G, M) = (X^n(G, M))^G$, called the homogeneous n -cochains of G with coefficients in M . Because the complex preserves G -actions, then we obtain a complex of $C^n(G, M)$'s, given by

$$0 \longrightarrow C^0(G, M) \xrightarrow{\partial_0} C^1(G, M) \xrightarrow{\partial_1} \dots$$

Remark 2.4. To see what the induced mapping is, suppose $A \rightarrow B$ is a G -module map, then there is an induced map of fixed points $A^G \rightarrow B^G$ by the restriction. In particular, let $a \in A$ be fixed with $ga = a$ for all $g \in G$, then $f(a) = f(ga) = gf(a)$.

Remark 2.5. In the complex of Definition 2.3, $\partial^2 = 0$ as well, but in general this is not an exact sequence.

Definition 2.6 (Group Cohomology). The group cohomology of G with coefficients in M is the collection

$$\{H^n(G, M)\}_{n \geq 0},$$

where $H^n(G, M) := H^n(C^\bullet(G, M)) = \ker(\partial : C^n \rightarrow C^{n+1}) / \text{im}(\partial : C^{n-1} \rightarrow C^n)$. We usually use the notion of cocycles $Z^n(G, M) = \ker(\partial : C^n \rightarrow C^{n+1})$ and coboundaries $B^n(G, M) = \text{im}(\partial : C^{n-1} \rightarrow C^n)$.

Exercise 2.7. Show that $H^0(G, M)$ is isomorphic to M^G .

Definition 2.8. The inhomogeneous cochains $C_i^n(G, M)$ are given by

- $C_i^0 = M$, and
- for $n > 0$, $C_i^n = \text{Map}(G^n, M)$,

with coboundary maps $\partial^{n+1} : C_i^n \rightarrow C_i^{n+1}$, given by

- $\partial^1(m)(g) = gm - m$,
- $\partial^2(f)(g_1, g_2) = g_1 f(g_2) - f(g_1 g_2) + f(g_1)$, and so on, with
- $\partial^{n+1}(f)(g_1, \dots, g_{n+1}) = g_1 f(g_2, \dots, g_{n+1}) + \sum_{i=1}^n (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) + (-1)^{n+1} f(g_1, \dots, g_n)$.

This gives the inhomogeneous setting of this cochain.

Lemma 2.9. The maps

$$\begin{aligned} C^n(G, M) &\rightarrow C_i^n(G, M) \\ (\varphi : G^{n+1} \rightarrow M) &\mapsto (f : G^n \rightarrow M) \\ f(g_1, \dots, g_n) &:= \varphi(1, g_1, g_1 g_2, \dots, g_1 g_2 \cdots g_n) \end{aligned}$$

give a cochain homotopy equivalence $C^\bullet(G, M) \xrightarrow{\sim} C_i^\bullet(G, M)$, and hence this is a quasi-isomorphism.

Corollary 2.10. The cohomology $H^*(C_i^\bullet(G, M)) \cong H^*(G, M)$.

Remark 2.11. Any cohomology class can be represented by a normalized inhomogeneous cocycle $f : G^n \rightarrow M$, i.e., $f(g_1, \dots, g_n) = 0$ where $g_i = 1$ for some i .

Remark 2.12. Even for $G = C_2$, C_i^n or C^n get large as n grows.

Remark 2.13. • Using homological algebra, we can find other cochain complexes which computes group cohomology $H^*(G, M)$.

- We would also understand $H^*(G, M)$ as the failure of exactness of $()^G : G\text{-Mod} \rightarrow \mathbf{Ab}$. Therefore, when taking the fixed points, the exact sequence may not be mapped to another exact sequence. In particular, if we take an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of G -modules, the induced sequence

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G$$

do not give a surjection at $B^G \rightarrow C^G$. One needs to take higher cohomology to obtain a long exact sequence. Hence, $()^G : G\text{-Mod} \rightarrow \mathbf{Ab}$ is a left exact functor, but not necessarily right exact.

3 AUG 25, 2023: COHOMOLOGY OF GROUPS, CONTINUED

Example 3.1. Let G be C_2 , or any group with a surjection p onto C_2 , then it has an action on \mathbb{Z}_{sgn} given by $g \cdot a = (-1)^{p(g)}a$, therefore we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}_{\text{sgn}} \xrightarrow{\times 2} \mathbb{Z}_{\text{sgn}} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

and taking the fixed point functor we have

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

Remark 3.2. Higher homologies measure the failure of exactness.

Remark 3.3. The collection $\{H^n(G, -)\}_{n \in \mathbb{Z}}$ satisfies

- $H^n(G, -) = 0$ for $n < 0$;
- for short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $G\text{-Mod}$, we have a long exact sequence

$$0 \longrightarrow H^0(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, C) \xrightarrow{\delta} H^1(G, A) \longrightarrow \cdots$$

where δ is the connecting homomorphism.

- the connecting homomorphisms δ are natural, i.e., given a commuting diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \longrightarrow 0 \end{array}$$

the induced diagram

$$\begin{array}{ccc} H^n(G, C) & \xrightarrow{\delta} & H^{n+1}(G, A) \\ \downarrow & & \downarrow \\ H^n(G, C') & \xrightarrow{\delta} & H^{n+1}(G, A') \end{array}$$

also commutes, and $\{H^n(G, -)\}_{n \in \mathbb{Z}}$ is a cohomological δ -functor. Note that a δ -functor is additive, and usually occurs in abelian categories.

Definition 3.4 (δ -functor). A map of δ -functors $T^* \rightarrow F^*$ is a collection of natural transformations $T^n \rightarrow F^n$, commuting with the δ 's, i.e.,

$$\begin{array}{ccc} T^n & \longrightarrow & F^n \\ \delta_T \downarrow & & \downarrow \delta_F \\ T^{n+1} & \longrightarrow & F^{n+1} \end{array}$$

A δ -functor T^* is universal if, given any other δ -functor F^* , a map $T^* \rightarrow F^*$ is uniquely determined by $T^0 \rightarrow F^0$.

Proposition 3.5. $H^*(G, -) : G\text{-Mod} \rightarrow \mathbf{Ab}$ is a δ -functor.

Proof. We need to show:

- each $H^n(G, -)$ is a well-defined functor,
- the connecting homomorphisms δ 's gives a long exact sequence,
- the naturality of δ .

First, let $f : A \rightarrow B$ be in $G\text{-Mod}$, then $C^*(G, A) \rightarrow C^*(G, B)$ is equivalent to $\text{Map}(G^{*+1}, A)^G \rightarrow \text{Map}(G^{*+1}, B)^G$ by composition with f . One can show that this is equivariant, i.e., respects the G -action, so it is well-defined to take the fixed points, and thus commutes with ∂ 's.

Second, we need to apply the snake lemma. Given a short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, we claim:

Claim 3.6. $0 \rightarrow C^*(G, A) \rightarrow C^*(G, B) \rightarrow C^*(G, C) \rightarrow 0$ is a short exact sequence of cochain complexes, i.e., $C^*(G, -) : G\text{-Mod} \rightarrow \mathbf{coCh}$ is an exact functor.

Subproof. Exercise. ■

Now take the complex

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^n(G, A) & \longrightarrow & C^n(G, B) & \longrightarrow & C^n(G, C) \longrightarrow 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\ 0 & \longrightarrow & C^{n+1}(G, A) & \longrightarrow & C^{n+1}(G, B) & \longrightarrow & C^{n+1}(G, C) \longrightarrow 0 \end{array}$$

and quotient the boundaries everywhere (and thus lose the injectivity/surjectivity when applicable)

$$\begin{array}{ccccccc} C^n(G, A)/B^n(G, A) & \longrightarrow & C^n(G, B)/B^n(G, B) & \longrightarrow & C^n(G, C)/B^n(G, C) & \longrightarrow & 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\ 0 & \longrightarrow & Z^{n+1}(G, A) & \longrightarrow & Z^{n+1}(G, B) & \longrightarrow & Z^{n+1}(G, C) \end{array}$$

Taking the kernels and cokernels on ∂ 's, we obtain a complex

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & H^n(G, A) & \longrightarrow & H^n(G, B) & \longrightarrow & H^n(G, C) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & C^n(G, A)/B^n(G, A) & \longrightarrow & C^n(G, B)/B^n(G, B) & \longrightarrow & C^n(G, C)/B^n(G, C) \longrightarrow 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\ 0 & \longrightarrow & Z^{n+1}(G, A) & \longrightarrow & Z^{n+1}(G, B) & \longrightarrow & Z^{n+1}(G, C) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & H^{n+1}(G, A) & \longrightarrow & H^{n+1}(G, B) & \longrightarrow & H^{n+1}(G, C) \end{array}$$

By the snake lemma, we obtain the long exact sequence. □

Proposition 3.7. If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence such that $H^*(G, B) = 0$ for $* > 0$ (or at least $H^n(G, B) = 0 = H^{n+1}(G, B)$), then $\delta : H^n(G, C) \rightarrow H^{n+1}(G, A)$ is an isomorphism.

Definition 3.8 (Acyclic, Cohomologically Trivial). A G -module M is

- acyclic if $H^*(G, M) = 0$ for $* > 0$,
- cohomologically trivial if $H^*(H, M) = 0$ for $* > 0$ and any (closed) subgroup $H \subseteq G$.

Definition 3.9 (Induced Module). Given any G -module M , the induced module $\text{ind}_G(M) = \text{Map}(G, M) = X^0(G, M)$.

Example 3.10. M could have the trivial action.

Exercise 3.11. For any M , the induced module of M over G is isomorphic (under the G -action) to the induced module of module given by forgetful action over G .

Remark 3.12. • $\text{Ind}_G(-) : G\text{-Mod} \rightarrow G\text{-Mod}$ is exact.

- We say A is an induced module if $A \cong \text{Ind}_G(M)$ for some module M . If A is an induced G -module, then A is induced as an H -module for any subgroup $H \subseteq G$.

Lemma 3.13. Induced modules are cohomologically trivial.

Proof. There is an isomorphism

$$C^*(G, \text{Ind}_G(M)) \cong X^*(G, M).$$

□

Remark 3.14. We have an equivariant inclusion of fixed points

$$M \hookrightarrow \text{Ind}_G(M)$$

which is an embedding, and we take $Q \cong \text{Ind}_G(M)/M$, then this extends to a short exact sequence

$$0 \longrightarrow M \hookrightarrow \text{Ind}_G(M) \longrightarrow Q \longrightarrow 0$$

then $H^{n+1}(G, M) \cong H^n(G, Q)$. One say that $H^*(G, -)$ is effaceable. By Tohoku, an effaceable is universal.

4 AUG 28, 2023: FIRST COHOMOLOGY OF GROUPS

There are three ways to think about $H^1(G, M)$.

4.1 CROSSED HOMOMORPHISMS

Recall that $H^1(G, M) = Z^1_i(G, M)/B^1_i(G, M)$ as inhomogeneous cochains, where

- $Z^1_i(G, M) = \ker(\text{Map}(G, M) \rightarrow \text{Map}(G \times G, M))$ where the map sends $f \mapsto (g, h) \mapsto gf(h) - f(gh) + f(g)$. The kernel of this is exactly the maps f such that $f(gh) = gf(h) + f(g)$, and note that this is not a group homomorphism.
- $B^1_i(G, M) = \text{im}(M \rightarrow \text{Map}(G, M))$ given by $m \mapsto (g \mapsto gm - m)$, where the image is called a principal crossed homomorphism.

Exercise 4.1. $B^1_i(G, M) \cong M/M^G$ as an isomorphism of $\mathbb{Z}[G]$ -modules.

Remark 4.2. If the G -action is trivial, then $H^1(G, M) = \text{Hom}_{\text{Grp}}(G, M)$.

Corollary 4.3. If G is a finite group with trivial action, then $H^1(G, \mathbb{Z}) = 0$.

Theorem 4.4 (Hilbert's Theorem 90). Let L/K be a Galois extension with (finite or profinite) Galois group G , then $H^1(G, L^\times) = 0$.

Proof. Let $f : G \rightarrow L^\times$ be a crossed homomorphism. We know the addition is given by $f(gh) = gf(h) + f(g)$, and the multiplication is given by $f(gh) = (g \cdot f(h))f(g)$, where \cdot represents the group action. Now for any $l \in L^\times$, the multiplication with respect to l is given by $m_l = \sum_{h \in G} f(h)(h \cdot l)$. We can first choose l so that $m_l \neq 0$, since the Galois conjugates $h \cdot l$ over $l \in L$ are linearly independent. For $g \in G$, we have

$$\begin{aligned} g \cdot m_l &= \sum_{h \in G} (g \cdot f(h))(gh \cdot l) \\ &= \sum_{h \in G} \frac{f(gh)}{f(g)} (gh \cdot l) \\ &= \frac{1}{f(g)} \sum_{h \in G} f(gh)(gh \cdot l) \\ &= \frac{1}{f(g)} m_l. \end{aligned}$$

Therefore, $f(g) = \frac{m_l}{g \cdot m_l}$. For any crossed homomorphism, there exists $m \in L^\times$ such that $f(g) = \frac{gm}{m}$, so every crossed homomorphism is principal. □

Exercise 4.5. Let G acts over a commutative ring R , then $H^1(G, R^\times)$ classifies invariant R -modules with a compatible G -action.

4.2 NON-ABELIAN H^1 AND TORSORS

Let A be a group with G -action, so let the action $g \cdot a = {}^g a$. Hence, $g \cdot (ab) = {}^g a {}^g b$. Define the G -cocycles to be $f : G \rightarrow A$ such that $f(gh) = f(g) {}^g f(h)$. Two cocycles f and f' are said to be cohomologous as $f \sim f'$ if there exists $a \in A$ such that for all $g \in G$, $f'(g) = a^{-1} f(g) {}^g a$. This becomes an equivalence relation on the set of G -cocycles with coefficients in A , then $H^1(G, A)$ is the set of equivalence classes of G -cocycles. Now the first cohomology $H^1(G, A)$ has only a pointed set structure with distinguished point $f \equiv 1$, the constant function at 1.

Exercise 4.6. This definition is equivalent to the inhomogeneous cochain definition in the abelian case.

Definition 4.7. An A -torsor is a G -set X with action

$$\begin{aligned} X \times A &\rightarrow A \\ (x, a) &\mapsto xa \end{aligned}$$

that is free and transitive, i.e., for any $x, y \in X$, there exists a unique $a \in A$ such that $y = xa$. Moreover, the action $X \times A \rightarrow X$ respects the G -action, i.e., ${}^g(xa) = {}^g x {}^g a$.

Remark 4.8. • A is an A -torsor.

- An isomorphism of A -torsors is a bijection that respects the G - and A - action.
- If $A \subseteq B$ is a sub- G -group, then bA is an A -torsor.
- An A -torsor is a principal A -bundle on the classifying space BG .

Theorem 4.9. There is a canonical bijection of pointed sets

$$H^1(G, A) \cong \text{Torsor}(G, A)$$

Proof. • The backwards map $\lambda : \text{Torsor}(G, A) \rightarrow H^1(G, A)$ is defined as follows: for $x \in \text{Torsor}(G, A)$, we want to define a cocycle $f(X) : G \rightarrow A$. For arbitrary $x \in X$, note that for any $g \in G$, there exists a unique $f_x(g) \in A$ such that ${}^g x = x f_x(g)$ by the simple transitivity of the A -action on X . To see this is well-defined, if we have another $y \in X$, then $y = xb$ for some $b \in A$, then $f_y(g) = b^{-1} f_x(g) {}^g b$, so f_x and f_y are cohomologous and define the same class in $H^1(G, A)$, which is defined to be the image $\lambda(X)$.

- To define $\mu : H^1(G, A) \rightarrow \text{Torsor}(G, A)$, given a cocycle $f : G \rightarrow A$, let X_f be the group A , then the action of A on X_f is by multiplication on the right, and one can twist the G -action on it using cocycle $f : G \rightarrow A$ with ${}^g x = f(g)gx$, which defines an A -torsor. This is well-defined.

□

Remark 4.10. Suppose

$$1 \longrightarrow A \longrightarrow B \xrightarrow{p} C \longrightarrow 1$$

is a short exact sequence of G -groups, i.e., A is a sub- G -group and $C \cong B/A$, then there is a long exact sequence

$$1 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \xrightarrow{\delta} H^1(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, C)$$

where δ is given by $\delta(c) = p^{-1}(c)$. For the exactness in the sense of pointed sets to work, the kernel is the subset mapping to the distinguished element.

4.3 EXTENSION SPLITTING

Consider the a split extension

$$1 \longrightarrow A \longrightarrow E \xrightarrow{p} G \longrightarrow 1$$

That is, E is the direct product $A \times G$ with group action $(a, g)(a', g') = (a {}^g a', gg')$, and by definition E is the semidirect product $A \rtimes G$. Equivalently, there exists a section (as group homomorphism) $s : G \rightarrow E$.

There is an equivalence relation on the set of sections to the projection $p : E \rightarrow G$, where the sections $s, s' : G \rightarrow E$ are conjugates if there exists $a \in A$ such that $s'(g) = a^{-1} s(g) a$. We denote $\text{sec}(E \rightarrow G)$ to be the conjugacy class of sections of p . Note that the class of trivial section $s : g \mapsto (1, g) \in E$ is the distinguished element.

Proposition 4.11. The pointed set $H^1(G, A)$ is isomorphic to $\text{sec}(E \rightarrow G)$.

Proof. Take $\varphi \in \text{sec}(E \rightarrow G)$, then the composition $G \xrightarrow{\varphi} E \xrightarrow{\pi_1} A$, where π_1 is the set-theoretic projection to the first component, defines a cocycle $G \rightarrow A$. Conversely, given a cocycle $f : G \rightarrow A$, the section is given by $g \mapsto (f(g), g)$. \square

Exercise 4.12. Expand the proof above.

Exercise 4.13. Describe $\mathbb{Z} \rtimes C_2$ where C_2 acts on \mathbb{Z} by inversion. How many sections are there of $\mathbb{Z} \rtimes C_2 \rightarrow C_2$?

Exercise 4.14. How many sections are there to the projection $D_{2n} \rightarrow C_2$?

5 AUG 30, 2023: H^2 , ABELIAN EXTENSIONS, AND BRAUER GROUP

Suppose we have an abelian extension, that is, let A be abelian, the short exact sequence of group extensions

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1$$

is such that $E/i(A) \cong G$. Note that A can be regarded as a normal subgroup in E given this notation.

Note that two extensions are equivalent if there exists a group isomorphism $\varphi : E \rightarrow E'$ such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G \longrightarrow 1 \\ & & \parallel & & \downarrow \varphi & & \parallel \\ 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G \longrightarrow 1 \end{array}$$

commutes.

Consider the continuous functions

$$\varphi : G \times G \rightarrow A$$

such that $\varphi(g_1g_2, g_3) + \varphi(g_1, g_2) = \varphi(g_1, g_2g_3) + g_1\varphi(g_2, g_3)$. We know $H^2(G, M)$ is the quotient of all such functions over the coboundaries, i.e., the functions φ such that $\varphi(g_1, g_2) = f(g_1) - f(g_1g_2) + g_1f(g_2)$.

Now $E \cong A \times G$ can be considered as a bijection, so we pick a set-theoretic section $s : G \rightarrow E$ with $s(1) = 1$, and now every element in E is written as $as(g)$ uniquely for some $a \in A$ and $g \in G$, we have

$$s(g)a = s(g)as(g)^{-1}s(g) = {}^g as(g).$$

Note that s may not be a homomorphism, but we have $s(g)s(h) = f(g, h)s(gh)$ since $s(g)s(h)$ and $s(gh)$ are both lifts of gh .

As a consequence, we have

$$(s(g_1)s(g_2))s(g_3) = f(g_1, g_2)s(g_1g_2)s(g_3) = f(g_1, g_2)f(g_1g_2, g_3)s(g_1g_2g_3)$$

and

$$s(g_1)(s(g_2)s(g_3)) = s(g_1)f(g_2, g_3)s(g_2, g_3) = {}^{g_1}f(g_2, g_3)s(g_1)s(g_2g_3) = {}^{g_1}f(g_2, g_3)f(g_1, g_2g_3)s(g_1g_2g_3).$$

In additive notation, we have

$$f(g_1, g_2) + f(g_1g_2, g_3) = g_1f(g_2, g_3) + f(g_1, g_2g_3).$$

Therefore, f becomes an inhomogeneous 2-cocycle.

Proposition 5.1. The induced map $\lambda : \text{ext}(G, A) \rightarrow H^2(G, A)$ is a well-defined bijection between the set of equivalence classes of extensions and $H^2(G, A)$.

Example 5.2. The two elements in $H^2(C_2, \mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ are given by non-split extension of Q_8

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow Q_8 \longrightarrow C_2 \longrightarrow 1$$

and the identity element given by $D_8 \cong \mathbb{Z}/4\mathbb{Z} \rtimes C_2$

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow D_8 \longrightarrow C_2 \longrightarrow 1$$

where D_8 has the action of C_2 over $\mathbb{Z}/4\mathbb{Z}$.

Proposition 5.3. An associative finite-dimensional K -algebra A is a CSA if and only if one of the following equivalent conditions hold:

1. Base-changed to the separable closure \bar{K} of K via $\bar{K} \otimes_K A$, $A \cong M_n(\bar{K})$ for some integer $n \geq 1$.
2. there exists a finite Galois extension L/K such that base-changed to L via $L \otimes_K A$, A becomes isomorphic to a matrix algebra $M_n(L)$ for some integer $n \geq 1$.
3. $A \cong M_n(D)$ matrix algebra for some $m \geq 1$ and some finite division algebra D over K .

A CSA A over K is said to be split over L if the above holds, i.e., $A \otimes_K L \cong M_n(L)$. One can define an equivalence class on CSAs, such that $A \sim B$ if and only if $A \otimes_K M_n(K) \cong B \otimes_K M_m(K)$. Now the Brauer group of K is the abelian group of equivalence classes of CSAs over K equipped with tensor product.

Suppose L/K is an extension, then there exists a homomorphism of base-change of algebras $\text{Br}(K) \rightarrow \text{Br}(L)$. We say the kernel $\text{Br}(L | K)$ is the relative Brauer group of K -CSAs that split over L . The absolute Brauer group is $\text{Br}(\bar{K} | K) = \text{Br}(K)$, then

$$\text{Br}(K) = \bigcup_{L/K \text{ finite}} \text{Br}(L | K).$$

Now let L/K be a finite Galois extension with Galois group G , and we pick a normalized inhomogeneous 2-cycle $\varphi : G \times G \rightarrow L^\times$ as the representative of its class, and we can construct A_φ as a K -CSA, then $A_\varphi = \bigoplus_{g \in G} L e_g$ has dimension $|G|^2$, where e_g 's are the generators, with a multiplication operation $(l e_g)(m e_h) = l(g \cdot m) \varphi(g, h) e_{gh}$ which can be extended via distribution. A_φ is said to be the crossed product of L and G via φ .

Theorem 5.4. 1. A_φ is a split algebra over L .

2. If φ, φ' are two normalized inhomogeneous 2-cocycles, then $A_\varphi \sim A_{\varphi'}$ if and only if $\varphi \sim \varphi'$.
3. $A_{\varphi\varphi'} \sim A_\varphi \otimes_K A_{\varphi'}$.
4. Any K -CSA which is split over L is similar to a crossed product A_φ for some $\varphi : G \times G \rightarrow L^\times$.

Corollary 5.5. $H^2(G, L^\times)$ is isomorphic to $\text{Br}(L | K)$, and $H^2(\text{Gal}(\bar{K}/K), \bar{K}^\times)$ is isomorphic to $\text{Br}(K)$.

6 SEPT 1, 2023: COHOMOLOGY OF CYCLIC AND FREE GROUPS

Recall that we can compute $H^*(G, M)$ using any acyclic resolution of M . We want to describe $H^*(G, M)$ for specific G using nice resolutions.

We have

$$\dots \rightarrow G^3 \xrightarrow{\delta} G^2 \xrightarrow{\delta} G$$

and to obtain $X^*(G, M)$ we map out of the resolution and into M , so $\text{Map}(G, M) \cong \text{Hom}(\mathbb{Z}[G], M)$ as G -modules, and in general we obtain

$$\text{Map}(G^k, M) \cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]^{\otimes k}, M)$$

as \mathbb{Z} -modules.

We denote F^{st} to be the standard free resolution given by

$$\mathbb{Z}[G]^{\otimes k} \xrightarrow{d} \mathbb{Z}[G]^{\otimes(k-1)} \rightarrow \dots \rightarrow \mathbb{Z}[G]^{\otimes 2} \xrightarrow{d_1 - d_0} \mathbb{Z}[G]$$

To obtain $X^*(G, M)$, we can map this into M . Now the standard resolution becomes an augmentation of \mathbb{Z} that makes $X^*(G, M)$ exact, free, and acyclic. The kernel of $\mathbb{Z}[G] \rightarrow \mathbb{Z}$ is the augmentation ideal of G as of $\mathbb{Z}[G]$. Since this is a G -equivariant map, then the augmentation ideal is a G -submodule of $\mathbb{Z}[G]$, as a free abelian group generated by the set $\{(g-1) \mid 1 \neq g \in G\}$.

Lemma 6.1. If $P_* \rightarrow \mathbb{Z}$ is any free resolution of \mathbb{Z} as a G -module, then for a G -module M , we have $H^*(G, M) \cong H^*(\text{Hom}(P_*, M))^G$.

Proof. Since each P_i is free, then $\text{Hom}(P_i, M)$ is an acyclic module, so $M \rightarrow \text{Hom}(P_*, M)$ is an acyclic resolution of M . Now apply Proposition 2.28 in the notes. \square

Remark 6.2. $H^*(G, M) \cong \text{Ext}_{\mathbb{Z}[G]}^*(\mathbb{Z}, M)$ as universal δ -functors.

Now let C_n be the cyclic group of order n , generated by element g , then $\mathbb{Z}[C_n] \cong \mathbb{Z}[g]/(g^n - 1)$, so we have $0 = g^n - 1 = (g-1)N_g$ in $\mathbb{Z}[C_n]$ where N_g is the norm element $N_g = 1 + g + \cdots + g^{n-1}$, so we have a free resolution of \mathbb{Z} :

$$\cdots \longrightarrow \mathbb{Z}[C_n] \xrightarrow{1-g} \mathbb{Z}[C_n] \xrightarrow{N_g} \mathbb{Z}[C_n] \xrightarrow{1-g} \mathbb{Z}[C_n] \xrightarrow{\varepsilon} \mathbb{Z}$$

where augmentation ε sends g to 1. This allows us to compute the cohomology of any C_n -modules.

Proposition 6.3. Let M be an C_n -module, then

$$H^i(G, M) = \begin{cases} M^G, & i = 0 \\ \{m \in M \mid N_g m = 0\} / (1-g)M, & i > 0 \text{ odd} \\ M^G / N_g M, & i > 0 \text{ even} \end{cases}$$

Proof. Taking $\text{Hom}(P_*, M)^G$ gives

$$\cdots \longleftarrow M \xleftarrow{1-g} M \xleftarrow{N_g} M \xleftarrow{1-g} M \longleftarrow \cdots$$

\square

Remark 6.4. If M has trivial action, then

$$H^i(G, M) = \begin{cases} M, & i = 0 \\ M[n], & i > 0 \text{ odd} \\ M/n, & i > 0 \text{ even} \end{cases}$$

where $M[n]$ is the n -torsion in M .

Now if $T = \mathbb{Z}$ be with generator t , then $\mathbb{Z}[T]$ is isomorphic to the Laurent polynomials, so we have a resolution

$$0 \longrightarrow \mathbb{Z}[T] \xrightarrow{1-t} \mathbb{Z}[T] \longrightarrow \mathbb{Z}$$

since $(1-t)$ is not a zero-divisor of $\mathbb{Z}[T]$. Therefore, taking $\text{Hom}(P_*, M)^T$ gives

$$0 \longleftarrow M \xleftarrow{1-t} M$$

$$H^i(T, M) = \begin{cases} M^T, & i = 0 \\ M_T, & i = 1 \\ 0, & \text{otherwise} \end{cases}$$

Now let X be a set, and let G_X be the free group on X .

Proposition 6.5. The augmentation ideal I_X is a free $\mathbb{Z}[G_X]$ -module, generated by the set $\{(x-1) \mid x \in X\}$, and so the exact sequence

$$0 \longrightarrow I_X \longrightarrow \mathbb{Z}[G_X] \longrightarrow \mathbb{Z} \longrightarrow 0$$

is a free resolution of \mathbb{Z} as a G_X -module.

Proof. As \mathbb{Z} -bases of I_X , we have $\{(g-1) \mid g \in G_X\}$, but $\{h(x-1) \mid h \in G, x \in X\}$ is also a \mathbb{Z} -linear basis for I_X . \square

Remark 6.6. Groups are free if and only if they have cohomological dimension 1.