MATH 518 Notes

Jiantong Liu

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Definition 1.1. Let M be a topological space. An *atlas* on M is a collection $\{\varphi_\alpha : U_\alpha \to W_\alpha\}_{\alpha \in A}$ of homeomorphisms called *coordinate charts*, so that

1. $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M,
2. for all $\alpha \in A$, W_α is an open subset of some $\mathbb{R}^{n_\alpha}$,
3. for all $\alpha, \beta \in A$, the induced map $\varphi_\beta \circ \varphi_\alpha^{-1}|_{U_\alpha \cap U_\beta}$ is C^∞, i.e., smooth.

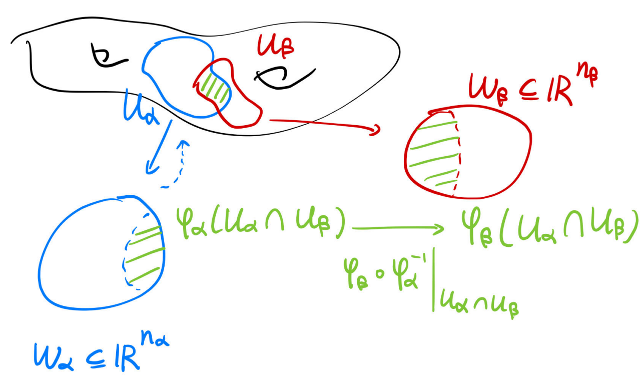

Figure 1: Atlas and Coordinate Chart

Example 1.2. Let $M = \mathbb{R}^n$ be equipped with standard topology, and let $A = \{*\}$, so $U_* = \mathbb{R}^n$ is the open cover of itself. Now the identity map

$$\begin{aligned} \varphi_* : U_* &\to \mathbb{R}^n \\ u &\mapsto u \end{aligned}$$

is an atlas on $\mathbb{R}^n$.

Example 1.3. Let $M = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be equipped with subspace topology. Let $U_\alpha = S^1 \backslash \{(1, 0)\}$ and $U_\beta = S^1 \backslash \{(-1, 0)\}$, and let $A = \{\alpha, \beta\}$. Let $W_\alpha = (0, 2\pi)$ and $W_\beta = (-\pi, \pi)$. We define $\varphi_\alpha^{-1}(\theta) = (\cos(\theta), \sin(\theta))$ and $\varphi_\beta^{-1}(\theta) = (\cos(\theta), \sin(\theta))$, then

$$(\varphi_\beta \circ \varphi_\alpha^{-1})(\theta) = \begin{cases} \theta, 0 < \theta < \pi \\ \theta - 2\pi, \pi < \theta < 2\pi \end{cases}$$

is smooth.

Example 1.4. Let X be a topological space with discrete topology, and let $A = X$, then $\{\varphi_x : \{x\} \to \mathbb{R}^0\}_{x\in X}$ gives an atlas.

Example 1.5. Let V be a finite-dimensional real vector space of dimension n. Pick a basis $\{v_1, \dots, v_n\}$ of V, then there is a linear bijection φ with inverse

$$\begin{aligned} \varphi^{-1} : \mathbb{R}^n &\to V \\ (x_1, \dots, x_n) &\mapsto \sum_{i=1}^n x_i v_i. \end{aligned}$$

The topology on V needs to make φ^{-1} a homeomorphism, and the obvious choice is just the collection of preimages, namely

$$\mathcal{T} = \{\varphi^{-1}(W) \mid W \subseteq \mathbb{R}^n \text{ open}\},$$

then $\varphi : V \to \mathbb{R}^n$ becomes an atlas.

Definition 1.6. Two atlases $\{\varphi_\alpha : U_\alpha \to W_\alpha\}_{\alpha\in A}$ and $\{\psi_\beta : V_\beta \to O_\beta\}_{\beta\in B}$ on a topological space M are *equivalent* if for all $\alpha \in A$ and $\beta \in B$,

$$\psi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap V_\beta) \subseteq \mathbb{R}^{n_\alpha} \to \psi_\beta(U_\alpha \cap V_\beta) \subseteq \mathbb{R}^{n_\beta}$$

is always C^∞, with C^∞-inverses. Such continuous maps are called *diffeomorphisms*. Alternatively, the two atlases are equivalent if their union $\{\varphi_\alpha\}_{\alpha\in A} \cup \{\psi_\beta\}_{\beta\in B}$ is always an atlas.

Exercise 1.7. Equivalence of atlases is an equivalence condition.

Definition 1.8. A (smooth) *manifold* is a topological space together with an equivalence class of atlases.

Convention. All manifolds are assumed to be smooth of C^∞, but not necessarily *Haudorff* and/or *second countable*.

Example 1.9. Continuing from Example 1.5, now suppose $\{w_1, \dots, w_n\}$ gives another basis of V, with

$$\begin{aligned} \psi^{-1} : \mathbb{R}^n &\to V \\ (y_1, \dots, y_n) &\mapsto \sum_{i=1}^n y_i w_i. \end{aligned}$$

This gives a change-of-basis matrix, so it is automatically C^∞ as a multiplication of invertible matrices. Therefore, the topology here does not depend on the chosen basis.

Recall. A topological space X is *Hausdorff* if for all distinct points $x, y \in X$, there exists open neighborhoods $U \ni x$ and $V \ni y$ such that $U \cap V = \varnothing$.

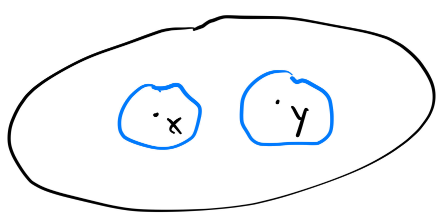

Figure 2: Hausdorff Condition

Convention. Via our definition (Definition 1.8), not all manifolds are Hausdorff.

Example 1.10. Let $Y = \mathbb{R} \times \{0, 1\}$, i.e., a space with two parallel lines, with a fixed topology. Define $\sim$ to be the smallest equivalence relation on Y such that $(x, 0) \sim (x, 1)$ for $x \neq 0$, and define $X = Y/\sim$. X is called the *line with two origins*, and it is second countable but not Hausdorff.

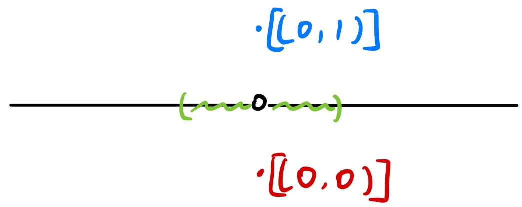

Figure 3: Line with Two Origins

Example 1.11. Take charts

$$\{\varphi : M = \mathbb{R} \to \mathbb{R}\}$$
$$x \mapsto x$$

and

$$\{\psi : M = \mathbb{R} \to \mathbb{R}\}$$
$$x \mapsto x^3$$

on $M = \mathbb{R}$, then

$$\varphi \circ \psi^{-1} : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^{\frac{1}{3}}$$

is not C^∞, so φ and ψ are two different charts, hence give two different manifolds.

Definition 1.12. A map $F : M \to N$ between two manifolds is *smooth* if

1. F is continuous, and
2. for all charts $\varphi : U \to \mathbb{R}^m$ on M and charts $\psi : V \to \mathbb{R}^n$ on N, $\psi^{-1} \circ F \circ \varphi|_{\varphi(U \cap F^{-1}(V))}$ is C^∞.

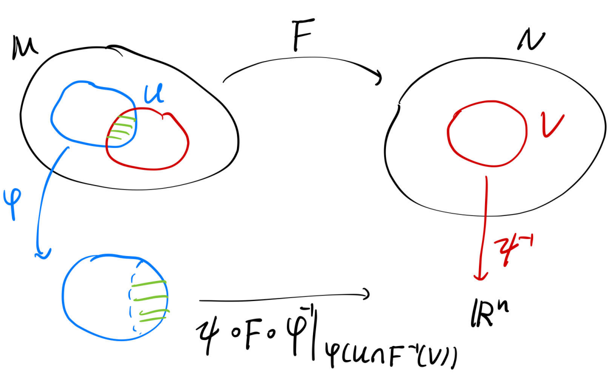

Figure 4: Smooth Map between Manifolds

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Exercise 2.1.

1. $\mathrm{id} : M \to M$ is smooth.
2. If $f : M \to N$ and $g : N \to Q$ are smooth maps between manifolds, then so is $gf : M \to Q$.

Punchline. The manifolds and the smooth maps between manifolds form a category.

Recall. A smooth map $f : M \to N$ is called a *diffeomorphism*, as seen in Definition 1.6, if it has a smooth inverse. This is the notion of an isomorphism in the category of manifolds.

Warning.

1. Following Example 1.11,
$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^3 \end{aligned}$$
has an inverse
$$\begin{aligned} f^{-1} : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^{\frac{1}{3}}, \end{aligned}$$
but f^{-1} is not differentiable at $x = 0$. Hence, f is not a diffeomorphism.

2. Take $\mathbb{R}$ with discrete topology, then all singletons are open sets, then the map
$$\begin{aligned} f : \mathbb{R}_{\text{dis}} &\to \mathbb{R}_{\text{std}} \\ x &\mapsto x \end{aligned}$$
is a smooth bijection, but f^{-1} is not continuous.

Example 2.2. Consider $M = (\mathbb{R}, \{\psi = \mathrm{id} : \mathbb{R} \to \mathbb{R}\})$ and $N = (\mathbb{R}, \{\psi : \mathbb{R} \to \mathbb{R}, x \mapsto x^3\})$ as two manifolds on $\mathbb{R}$ with standard topology. To see that they are equivalent, consider the homeomorphism
$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^{\frac{1}{3}}, \end{aligned}$$
then $(\psi \circ f \circ \varphi^{-1})(x) = \psi(f(x)) = (x^{\frac{1}{3}})^3 = x$, so f is smooth, and $(\psi \circ f \circ \varphi^{-1})^{-1} = \varphi \circ f^{-1} \circ \psi^{-1} = \mathrm{id}$, therefore f^{-1} is also smooth. Hence, f is a diffeomorphism.

We will now consider the real projective space $\mathbb{R}P^{n-1}$ and the quotient map $\pi : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}P^{n-1}$.

Definition 2.3. Define a binary relation on $\mathbb{R}^n \backslash \{0\}$ by $v_1 \sim v_2$ if and only if there exists $\lambda \neq 0$ such that $v_1 = \lambda v_2$. This is an equivalence relation, and we identify the equivalence class $[v]$ of $v \in \mathbb{R}^n \backslash \{0\}$ as a line $\mathbb{R}v = \mathrm{span}_{\mathbb{R}}\{v\}$ through v. Then we define the *real projective space* $\mathbb{R}P^{n-1} = (\mathbb{R}^n \backslash \{0\})/\sim$.

The natural topology on $\mathbb{R}P^{n-1}$ is the quotient topology, where $\pi : \mathbb{R}^n \backslash \{0\} \twoheadrightarrow \mathbb{R}P^{n-1}$ is surjective and continuous, so we define $U \subseteq \mathbb{R}P^{n-1}$ to be open if and only if $\pi^{-1}(U)$ is open in $\mathbb{R}^n \backslash \{0\}$.

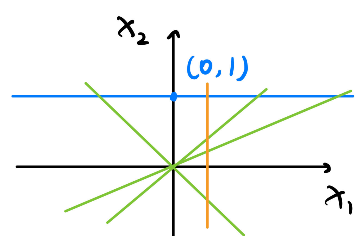

Figure 5: Stereographical Projection

Claim 2.4. $\mathbb{R}P^{n-1}$ is a manifold.

Proof. Define
$$\begin{aligned} \varphi_i : U_i &\to \mathbb{R}^{n-1} \\ [v_1, \ldots, v_n] &\mapsto \left(\frac{v_1}{v_i}, \ldots, \frac{v_{i-1}}{v_i}, \frac{v_{i+1}}{v_i}, \ldots, \frac{v_n}{v_i}\right), \end{aligned}$$

then

$$\begin{aligned}\varphi_i^{-1} : \mathbb{R}^{n-1} &\mapsto U_i \\ (x_1, \ldots, x_{n-1}) &\mapsto [(x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_{n-1})],\end{aligned}$$

therefore

$$\begin{aligned}\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) &\to \varphi_j(U_i \cap U_j) \\ (x_1, \ldots, x_{n-1}) &\mapsto \varphi_j([(x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_{n-1})]) \\ &= \begin{cases} \left(\frac{x_1}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_{n-1}}{x_j}\right), & j < i \\ (x_1, \ldots, x_{n-1}), & j = i \\ \left(\frac{x_1}{x_{j-1}}, \ldots, \frac{1}{x_{j-1}}, \ldots, \frac{x_{j-2}}{x_{j-1}}, \frac{x_j}{x_{j-1}}, \ldots, \frac{x_{n-1}}{x_{j-1}}\right), & j > i \end{cases}\end{aligned}$$

Therefore, this is C^∞ as a rational map on $\varphi_i(U_i \cap U_j)$, and so this gives an atlas, hence $\mathbb{R}P^{n-1}$ is a manifold. □

Claim 2.5. $\pi : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}P^{n-1}$ is smooth.

Proof. Note that

$$\begin{aligned}\psi : \mathbb{R}^n \backslash \{0\} &\hookrightarrow \mathbb{R}^n \\ x &\mapsto x\end{aligned}$$

is an atlas on $\mathbb{R}^n \backslash \{0\}$, and

$$\begin{aligned}\varphi_i \circ \pi \circ \varphi_i^{-1} : \mathbb{R}^n \backslash \{0\} &\to \mathbb{R}^{n-1} \\ (v_1, \ldots, v_n) &\mapsto \varphi_i([(v_1, \ldots, v_n)]) \\ &= \left(\frac{v_1}{v_i}, \ldots, \frac{v_{i-1}}{v_i}, \frac{v_{i+1}}{v_i}, \ldots, \frac{v_n}{v_i}\right).\end{aligned}$$

This is C^∞ on $\pi^{-1}(U_i) = \{(v_1, \ldots, v_n) \mid v_i \neq 0\}$, so π is smooth. □

Definition 2.6. A *smooth function* on a manifold M is a function $f : M \to \mathbb{R}$ so that for any coordinate chart $\varphi : U \to \varphi(U)$ open in $\mathbb{R}^m$, the function $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is smooth.

Remark 2.7. $f : M \to \mathbb{R}$ is smooth if and only if $f : M \to (\mathbb{R}, \{\mathrm{id} : \mathbb{R} \to \mathbb{R}\})$, usually called the *standard manifold structure on* $\mathbb{R}$, is smooth.

Notation. We denote $C^\infty(M)$ to be the set of all smooth functions $f : M \to \mathbb{R}$.

Remark 2.8. $C^\infty(M)$ is a smooth $\mathbb{R}$-vector space, that is, for all $\lambda, \mu \in \mathbb{R}$ and $f, g \in C^\infty(M)$,

- $(\lambda f + \mu g)(x) = \lambda f(x) + \mu g(x)$ for all $x \in M$,
- $(f \cdot g)(x) = f(x)g(x)$ for all $x \in M$.

Therefore, $C^\infty(M)$ becomes a (commutative, associative) $\mathbb{R}$-algebra.

Fact. Connecting manifolds have the notion of dimension. That is, the dimensions of open subsets induced by coordinate charts are the same.

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Definition 3.1. Let M be a manifold, then for every point $q \in M$, there exists a well-defined non-negative integer $\dim_M(q)$, so that for any coordinate chart $\varphi : U \to \mathbb{R}^m$ for $U \ni q$, we have $\dim_M(q) = m$ for some non-negative integer m that only depend on M. Consequently, $\dim_M : M \to \mathbb{Z}^{\geqslant 0}$ is a locally constant function. This integer m is called the *dimension* of M.

Proof. Indeed, say $\psi : V \to \mathbb{R}^n$ is another chart with $U \cap V \ni q$, then $\psi \circ \varphi^{-1}|_{\varphi(U \cap V)} : \varphi(U \cap V) \subseteq \mathbb{R}^m \to \psi(U \cap V) \subseteq \mathbb{R}^n$ is a diffeomorphism, therefore the Jacobian $D(\psi \circ \varphi^{-1})(\varphi(a)) : \mathbb{R}^m \to \mathbb{R}^n$ is a linear isomorphism, thus $m = n$. $\square$

Definition 3.2. Suppose $(M, \{\varphi_\alpha : U_\alpha \to \mathbb{R}^m\}_{\alpha \in A})$ and $(N, \{\psi_\alpha : V_\beta \to \mathbb{R}^n\}_{\beta \in B})$ are two manifolds. One can give a manifold structure to the product set $M \times N$, called the *product manifold*, as follows:

- give $M \times N$ the product topology,
- let $\{\varphi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n\}_{(\alpha,\beta) \in A \times B}$ to be the atlas on $M \times N$. This is well-defined since the transition maps of $\alpha, \alpha' \in A$ and $\beta, \beta' \in B$ are over $(U_\alpha \times V_\beta) \cap U_{\alpha'} \times V_{\beta'} = (U_\alpha \cap U_{\alpha'}) \times (V_\beta \cap V_{\beta'})$ with $(\varphi_{\alpha'} \times \psi_{\beta'}) \circ (\varphi_\alpha \times \psi_\beta)^{-1} = (\varphi_{\alpha'} \circ \varphi_\alpha^{-1}, \psi_{\beta'} \circ \psi_\beta^{-1})$. This is smooth since products of smooth maps are smooth.

Punchline. The product construction of manifolds gives the categorical product in the category of manifolds.

Property.

1. The projection maps
$$\begin{aligned} p_M : M \times N &\to M \\ (m, n) &\mapsto m \end{aligned}$$
and
$$\begin{aligned} p_N : M \times N &\to N \\ (m, n) &\mapsto n \end{aligned}$$
are C^∞.

2. *Universal Property of Product*: for any manifold Q and smooth maps $f_M : Q \to M$ and $f_N : Q \to N$, there exists a unique map
$$\begin{aligned} g : Q &\to M \times N \\ q &\mapsto (f(q), g(q)) \end{aligned}$$
such that $p_M \circ g = f_M$, and $p_N \circ g = f_N$.

$$\begin{tikzcd}
& Q \arrow[ddl, bend right, "f_M"'] \arrow[d, dashed, "g"', "\exists!"] \arrow[ddr, bend left, "f_N"] & \\
& M \times N \arrow[dl, "p_M"] \arrow[dr, "p_N"'] & \\
M & & N
\end{tikzcd}$$

Figure 6: Universal Property of Product

Recall.

- A topological space X is *second countable* if the topology has a countable basis: there exists a collection $\mathcal{B} = \{B_i\}_{i \in \mathbb{N}}$ of open sets so that any open set of X is a union of some B_i's.
- A cover $\{U_\alpha\}_{\alpha \in A}$ of a topological space is *locally finite* if for all $x \in X$, there exists a neighborhood N of X such that $N \cap U_\alpha = \varnothing$ for all but finitely many α's.

Example 3.3. Let $X = \mathbb{R}$, then

- $\{U_n = (-n, n)\}_{n \geqslant 0}$ is an open cover, but is not locally finite,
- $\{U_n = (n, n+2)\}_{n \in \mathbb{Z}}$ is a locally finite open cover of $\mathbb{R}$,
- $\{U_n = (n, n+2]\}_{n \in \mathbb{Z}}$ is a locally finite cover of $\mathbb{R}$, but is not an open cover.

Recall. An (open) cover $\{V_\beta\}_{\beta\in B}$ is a *refinement* of a cover $\{U_\alpha\}_{\alpha\in A}$ if for all β, there exists $\alpha = \alpha(\beta)$ such that $V_\beta \subseteq U_{\alpha(\beta)}$.

Definition 3.4. A Hausdorff topological space is *paracompact* if every open cover has a locally finite open refinement.

Fact. A connected Hausdorff manifold is paracompact if and only if it is second countable.

Corollary 3.5. A Haudorff manifold is paracompact if and only if its connected components are second countable.

Example 3.6. $\mathbb{R}$ with discrete topology is paracompact but not second countable.

Convention. Usually, we assume manifolds are paracompact, except when we need a non-Haudorff manifold. This condition is required for the existence of *partition of unity* (i.e., constant function id).

Recall. If X is a space, and $Y \subseteq X$ is a subset, then the *closure* $\bar{Y}$ of Y is the smallest closed set containing Y.

Definition 3.7. Given a topological space X and a function $f : X \to \mathbb{R}$, the *support* of f over X is

$$\operatorname{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}.$$

Example 3.8. The function

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leqslant 0 \end{cases}$$

is C^∞, with support $\overline{(0,\infty)} = [0,\infty)$.

Definition 3.9. Let M be a topological space and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover. A *partition of unity* subordinate to the cover is a collection of continuous functions $\{\psi_\alpha : M \to [0,1]\}_{\alpha\in A}$ such that

1. $\operatorname{supp}(\psi_\alpha) \subseteq U_\alpha$ for all $\alpha \in A$,
2. $\{\operatorname{supp}(\psi_\alpha)\}_{\alpha\in A}$ is a locally finite closed cover of M,
3. $\sum\limits_{\alpha\in A} \psi_\alpha(x) = 1$ for all $x \in M$.

Remark 3.10. For all $x \in M$, there exists $\alpha_1, \dots, \alpha_n$ such that $x \in \operatorname{supp}(\psi_{\alpha_i})$. Hence, for $\alpha \neq \alpha_1, \dots, \alpha_n$, $\psi_\alpha(x) = 0$. Therefore, the summation in Definition 3.9 is finite.

Theorem 3.11. Let M be a paracompact manifold with open cover $\{U_\alpha\}_{\alpha\in A}$, then there exists a partition of unity $\{\psi_\alpha : U_\alpha \to [0,1]\}_{\alpha\in A} \subseteq C^\infty(M)$ subordinate to the cover.

Example 3.12. Let $M = \mathbb{R}$ and consider for $n > 0$ the open sets $\{U_n = (-n,n)\}_{n\in\mathbb{N}}$. This is not locally finite at one point.

Example 3.13. Let $M = \mathbb{R}^n$, then for all $x \in \mathbb{R}^n$ and for $r > 0$, we have $B_r(x) = \{x' \in \mathbb{R}^n \mid ||x - x'|| < r\}$ and so $\{B_r(x)\}_{r>0, x\in\mathbb{R}^n}$ is an open cover, but this is not locally finite everywhere.

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We will start to talk about tangent vectors.

Recall. For any point $q \in \mathbb{R}^n$ and any vector $v \in \mathbb{R}^n$, and any $f \in C^\infty(\mathbb{R}^n)$, the *directional derivative* of q in direction v with respect to f is

$$D_v f(q) = \frac{d}{dt}|_0 f(q + tv).$$

This gives a map $D_v(-)(q) : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ which is

- linear, and

- Leibniz rule holds, i.e.,
$$D_v(fg)(q) = D_v(f)(q) \cdot g(q) + f(q)D_v(g)(q).$$

In other words, $D_v(-)(q) : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation.

Definition 4.1. Let q be a point of a manifold M. A *tangent vector* to M at q is an $\mathbb{R}$-linear map $v : C^\infty(M) \to \mathbb{R}$ such that for all $f, g \in C^\infty(M)$,
$$v(fg) = v(f)g(q) + f(q)v(g).$$

Remark 4.2. v gives smooth vector fields over M an $C^\infty(M)$-module structure via evaluation.

Lemma 4.3. The set T_qM of all tangent vectors to M at q is an $\mathbb{R}$-vector space.

Lemma 4.4. Suppose $c \in C^\infty(M)$ is a constant function, then for all q and all $v \in T_qM$, $v(c) = 0$.

Proof. We have $v(1) = v(1 \cdot 1) = 1(q)v(1) + v(1)1(q) = 2v(1)$, so $v(1) = 0$. For a constant function c, we have
$$v(c) = v(c \cdot 1) = cv(1) = c(0) = 0.$$
□

Lemma 4.5 (Hadamard)**.** For any $f \in C^\infty(\mathbb{R}^n)$, there exists $g_1, \ldots, g_n \in C^\infty(\mathbb{R}^n)$ such that

- $f(x) = f(0) + \sum_{i=1}^n x_i g_i(x)$, and
- $g_i(0) = \left(\frac{\partial}{\partial x_i} f\right)(0)$.

Proof. We have
$$\begin{aligned} f(x) - f(0) &= \int_0^1 \frac{d}{dt}(f(tx))dt \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(tx) \cdot x_i dt \\ &= \sum_{i=1}^n x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt \\ &= \sum_{i=1}^n x_i g_i(x). \end{aligned}$$

Therefore, $g_i(0) = \int_0^1 \frac{\partial f}{\partial x_i}(t \cdot 0)dt = \frac{\partial f}{\partial x_i}(0)$. □

Remark 4.6. For $1 \leqslant i \leqslant n$, we have canonical tangent vectors to $\mathbb{R}^n$ at 0 given by
$$\begin{aligned} \frac{\partial}{\partial x_i}|_0 : C^\infty(\mathbb{R}^n) &\to \mathbb{R} \\ f &\mapsto \frac{\partial f}{\partial x_i}(0). \end{aligned}$$

Lemma 4.7. $\left\{\frac{\partial}{\partial x_1}|_0, \ldots, \frac{\partial}{\partial x_n}|_0\right\}$ is a basis of $T_0\mathbb{R}^n$.

Proof. Suppose $\sum c_i \frac{\partial}{\partial x_i}|_0 = 0$, then
$$0 = \left(\sum_i c_i \frac{\partial}{\partial x_i}|_0\right)(x_j) = \sum_i c_i \delta_{ij} = c_j.$$

Therefore, $c_j = 0$ for all j, thus we have linear independence. For all $v \in T_0\mathbb{R}^n$, i.e., $v : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation, then $v = \sum\limits_i v(x_i)\frac{\partial}{\partial x_i}|_0$. Let $f \in C^\infty(\mathbb{R}^n)$, then $f(X) = f(0) + \sum x_i g_i(x)$, thus

$$\begin{aligned}
v(f) &= v(f(0)) + \sum_{i=1}^{n} v(x_i g_i(x)) \\
&= \sum_{i=1}^{n} v(x_i g_i(x)) \\
&= \sum_{i=1}^{n} (v(x_i)g_i(0) + x_i(0)v(g_i)) \\
&= \sum_{i=1}^{n} v(x_i)g_i(0) \\
&= \sum_{i=1}^{n} v(x_i)\frac{\partial f}{\partial x_i}(0).
\end{aligned}$$

□

Remark 4.8. This shows $\dim(T_0\mathbb{R}^n) = n$ with the basis above.

Now let V be a finite-dimensional vector space with a basis $e_1, \dots, e_n$, then

$$\begin{aligned}
\varphi : \mathbb{R}^n &\to V \\
(t_1, \dots, t_n) &\mapsto \sum_{i=1}^{n} t_i e_i
\end{aligned}$$

is a linear bijection, with linear inverse

$$\begin{aligned}
\psi : V &\to \mathbb{R}^n \\
v &\mapsto (\psi_1(v), \dots, \psi_n(v))
\end{aligned}$$

where $\psi_i(v)$'s are linear maps. To describe this with a basis, we have $\psi(\sum\limits_i a_i e_i) = (a_1, \dots, a_n)$, i.e., $\psi_i(e_j) = \delta_{ij}$.

Claim 4.9. $\{\psi_1, \dots, \psi_n\}$ is a basis of $V^* = \mathrm{Hom}(V, \mathbb{R})$, called the *dual basis* of $\{e_1, \dots, e_n\}$, denoted $e_j^* = \psi_j$.

Proof. Linear independence follows from $e_j^*(e_i) = \delta_{ij}$. Given $\ell : V \to \mathbb{R}$ to be a linear map, then $\ell = \sum \ell(e_i)e_i^*$ since $\left(\sum\limits_i \ell(e_i)e_i^*\right)(e_j) = \ell(e_j)$. Given $v \in T_0\mathbb{R}^n$, $v(f) = \sum a_i \left(\frac{\partial}{\partial x_i}|_0 f\right)$ for all $f \in C^\infty(\mathbb{R}^n)$. Note that $\frac{\partial}{\partial x_i}|_0(x_j) = \delta_{ij}$, so $v(x_j) = \sum a_i \frac{\partial}{\partial x_i}|_0(x_j) = \sum\limits_i a_i\delta_{ij} = a_j$. Therefore, we have $a_i = v(x_i)$ for all i, thus $v(f) = \sum v(x_i)\left(\frac{\partial}{\partial x_i}|_0 f\right)$. Thus, the dual basis to $\frac{\partial}{\partial x_1}|_0, \dots, \frac{\partial}{\partial x_n}|_0$ is $\{d(x_i)_0\}_{i=1}^n$ where $(dx_i)_0(v) = v(x_i)$ for all i. Hence, we have $v = \sum (dx_i)_0(v)\frac{\partial}{\partial x_i}|_0$. □

Remark 4.10. Via a change of basis, this works at every point q on the local chart, so we can describe the tangent space on any point on a local chart.

5 Aug 30, 2023

Let M be a manifold and $x \in M$. Recall that a tangent vector $v : C^\infty(M) \to \mathbb{R}$ is a derivation, i.e., linear map, and the set of tangent vectors at q gives the tangent space.

Example 5.1. Let $M = \mathbb{R}^n$, and $q = 0$, then $\left\{\frac{\partial}{\partial x_1}|_0, \dots, \frac{\partial}{\partial x_n}|_0\right\}$ is a basis of $T_0\mathbb{R}^n$. Moreover, for all $v \in T_0\mathbb{R}^n$, $v = \sum v(x_i)\frac{\partial}{\partial x_i}|_0$, thus $\{v \mapsto v(x_i)\}_{i=1}^n$ is the dual basis, with $v(x_i) = (dx_i)_0(v)$ for all $1 \leqslant i \leqslant n$.

Remark 5.2. The proof used Hadamard's lemma (Lemma 4.5) and the fact that for all $x \in \mathbb{R}^n$ and all $t \in [0,1]$, $f(tx)$ is defined. Thus, the same argument should work for a version of Hadamard's lemma for star-shaped open subsets $U \subseteq \mathbb{R}^n$.

Definition 5.3. We say an open subset $U \subseteq \mathbb{R}^n$ is a *star-shaped domain* if for all $t \in [0,1]$ and all $x \in U$, $tx \in U$.

Definition 5.4. Let $F : M \to N$ be a smooth map between two manifolds, and $q \in M$ is a point, then

$$\begin{aligned} T_qF : T_qM &\to T_qN \\ v(f) &\mapsto v(f \circ F) \end{aligned}$$

via the pullback.

Exercise 5.5. Check that the definition makes sense, in particular:

(i) $(T_qF)(v)$ is a tangent vector to N of $F(q)$, and

(ii) T_qF is a derivation.

Remark 5.6. (a) It is easy to deduce the *chain rule*. That is, given $M \xrightarrow{F} N \xrightarrow{G} Q$ with $q \in M$, then $T_q(G \circ F) = T_{F(q)}G \circ T_qF$ because for all $f \in C^\infty(Q)$ and all $v \in T_qM$, we have

$$(T_q(G \circ F)(v))(f) = v(f \circ (G \circ F))$$

and

$$(T_{F(q)}G(T_qF(v))) = (T_qF)(v)(f \circ G) = v((f \circ G) \circ F).$$

(b) $T_q(\mathrm{id}_M) = \mathrm{id}_{T_qM}$.

As a result, we know T is a functor from the category of pointed manifolds to the category of $\mathbb{R}$-vector spaces.

Corollary 5.7. If $F : M \to N$ is a diffeomorphism, then for all $q \in M$, $T_qF : T_qM \to T_{F(q)}N$ is an isomorphism.

Proof. Since F is a diffeomorphism, then it has a smooth inverse $G : N \to M$, so

$$\mathrm{id}_{T_qM} = T_q(\mathrm{id}_M) = T_q(G \circ F) = T_{F(q)}G \circ T_qF$$

and

$$\mathrm{id}_{T_{F(q)}N} = T_{F(q)}(\mathrm{id}_N) = T_{F(q)}(F \circ G) = T_{F(q)}F \circ T_{F(q)}G.$$

□

We also need to show that $\dim(T_qM) = \dim_q(M)$, which is a result of Lemma 5.8, whose proof will be postponed till next time.

Lemma 5.8. Let M be a manifold and $q \in M$, and let U be an open neighborhood of q in M, and let $i : U \hookrightarrow M$ be an inclusion, then

$$\begin{aligned} I = T_qi : T_qU &\to T_qM \\ v(f) &\mapsto v(f|_U) \end{aligned}$$

is an isomorphism for all $v \in T_qM$ and all $U \subseteq M$.

Notation. We denote $r_1, \ldots, r_n : \mathbb{R}^m \to \mathbb{R}$ to be the standard coordinates on $\mathbb{R}^m$.

Let M be a manifold, $q_0 \in M$, and $\varphi : U \to \mathbb{R}^m$ is a coordinate chart with $q_0 \in U$. Now let $x_i = r_i \circ \varphi$, then $\varphi(q) = (x_1(q), \ldots, x_m(q))$.

We may now assume that

- $\varphi(q_0) = 0$, otherwise, we replace $\varphi(q)$ by $\varphi(q) := \varphi(q) - \varphi(q_0)$, and
- $\varphi(U)$ is an open ball $B_R(0) = \{r \in \mathbb{R}^m \mid ||r|| < R\}$ because there exists $R > 0$ such that $B_R(0) \subseteq \varphi(U)$, and we can then replace U with $\varphi^{-1}(B_R(0))$ and restrict the charts φ to $\varphi|_{\varphi^{-1}(B_R(0))}$.

We now define

$$\begin{aligned}\frac{\partial}{\partial x_j}|_{q_0} : C^\infty(U) &\to \mathbb{R}\\ f &\mapsto \frac{\partial}{\partial r_j}|_0(f\circ\varphi^{-1})\end{aligned}$$

Claim 5.9. $\left\{\frac{\partial}{\partial x_j}|_{q_0}\right\}_{j=1}^m$ is a basis of T_qM and for all $v \in T_{q_0}M$, $v = \sum v(x_j)\frac{\partial}{\partial x_j}|_{q_0}$.

Proof. By Hadamard's lemma Lemma 4.5 on $B_R(0)$, for all $f \in C^\infty(U)$, we have $f\circ\varphi^{-1} \in C^\infty(B_R(0))$, so there exists $g_1,\dots,g_m \in C^\infty(B_R(0))$ such that $(f\circ\varphi^{-1})(r) = f(\varphi^{-1}(0)) + \sum r_ig_i(r)$. Therefore, $f(q) = f(q_0) + \sum(r_i\circ\varphi)(q)(g_i\circ\varphi)(q)$, hence $f = f(q_0) + \sum x_i(g_i\circ\varphi)$, and $(g_i\circ\varphi)(q_0) = g_i(0) = \frac{\partial}{\partial r_i}|_0(f\circ\varphi^{-1}) = \frac{\partial}{\partial x_i}|_0(f)$.

Hence, for all $v \in T_{q_0}(U)$, we know

$$\begin{aligned}v(f) &= v(f(q_0)) + v\left(\sum x_i\cdot(g_i\circ\varphi)\right)\\ &= \sum_i v(x_i)(g_i\circ\varphi)(q_0)\\ &= \sum v(x_i)\frac{\partial}{\partial x_i}|_{q_0}(f).\end{aligned}$$

□

Remark 5.10.

1. The linear functionals

$$\begin{aligned}(dx_i)_{q_0} : T_{q_0}U &\to \mathbb{R}\\ v &\mapsto v(x_i)\end{aligned}$$

is the basis of $(T_{q_0}U)^*$ dual to $\left\{\frac{\partial}{\partial x_i}|_{q_0}\right\}$.

2. $(T_0\varphi^{-1})\left(\frac{\partial}{\partial r_i}|_0\right) = \frac{\partial}{\partial x_i}|_{q_0}$ by definition. Since $\left\{\frac{\partial}{\partial x_i}|_0\right\}_{i=1}^n$ is a basis of $T_0(B_R(0))$, then $\left\{\frac{\partial}{\partial x_i}|_{q_0}\right\}$ has to be a basis.

Lemma 5.11. Let M be a manifold and $q \in M$ a point. Let $U \ni q$ be anopen neighborhood, and $f \in C^\infty(M)$ such that $f|_U = 0$, then for all $v \in T_qM$, we have $v(f) = 0$.

Proof. We have shown the existence of a bump function $\rho \in C^\infty(M)$ in homework 1, that is, $0 \leqslant \rho(x) \leqslant 1$, $\text{supp}(\rho) \subseteq U$ and $\rho \equiv 1$ near q.

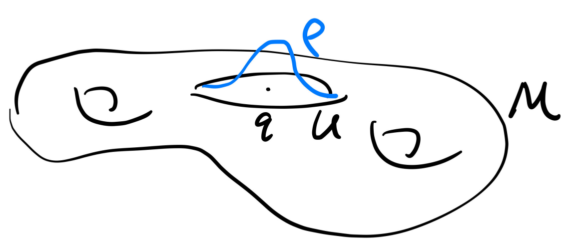


Figure 7: Bump Function

Therefore, $\rho f \equiv 0$, so $v(f) = v(\rho)f(q) + \rho(q)v(f) = v(\rho f) = 0$. □

6 Sept 1, 2023

Recall. Given a coordinate chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$, and $q \in U$ with $f(q) = 0$, we defined $\left\{ \frac{\partial}{\partial x_i}|_q \right\}_{i=1}^m \subseteq T_qU$ by $\frac{\partial}{\partial x_i}|_q f = \frac{\partial}{\partial r_i}(f \circ \varphi^{-1})|_{\varphi(q)}$ where $\frac{\partial}{\partial r_i}$'s are the standard partials on $C^\infty(\mathbb{R}^m)$.

We know this is a basis with dual basis

$$\begin{aligned} (dx_i)_q : T_qM &\to \mathbb{R} \\ v &\mapsto v(x_i) \end{aligned}$$

therefore $v = \sum v(x_i)\frac{\partial}{\partial x_i}|_q$ for all v. Note that

$$\begin{aligned} C^\infty(M) &\to C^\infty(U) \\ f &\mapsto f|_U \end{aligned}$$

is not surjective.

Also, we know $v \in T_qM$ is local, if $f, g \in C^\infty(M)$ agree on a neighborhood of q, then $v(f) = v(g)$.

Finally, given $F : M \to N$, this induces

$$\begin{aligned} T_qF : T_qM &\to T_{F(q)}N \\ v &\mapsto v(f \circ F). \end{aligned}$$

Lemma 6.1. Given a manifold M and $q \in M$, open neighborhood $q \in U \subseteq M$ and $i : U \hookrightarrow M$ inclusion, then

$$I \equiv T_qi : T_qU \to T_qM$$

is an isomorphism with $(I(v))(f) = v(f|_U)$ for all $f \in C^\infty(M)$.

Proof. Suppose $v \in \ker(I)$, then $v(f|_U) = 0$ for all $f \in C^\infty(M)$. We want $v(h) = 0$ for all $h \in C^\infty(U)$. We first choose bump function $\rho : M \to [0, 1]$ that is C^∞, and $\rho \equiv 1$ near q, and suppose $\operatorname{supp}(\rho) \subseteq U$, hence $\rho|_{M \setminus U} \equiv 0$. Then define $\rho h \in C^\infty(M)$ via

$$\rho h(x) = \begin{cases} \rho(x)h(x), & x \in U \\ 0, & x \notin U \end{cases}$$

Now $\rho h|_U \equiv h$ near q, i.e., identically 1. Therefore, $v(h) = v(\rho h|_U) = 0$, so $v \equiv 0$.

It remains to show that for all $w \in T_qM$, there exists $v \in T_qU$ such that $I(v) = w$, i.e., for all $f \in C^\infty(M)$, $w(f) = v(f|_U)$. Take the same $\rho \in C^\infty(M, [0.1])$ as above, define $v(h) = w(\rho h)$ for all $h \in C^\infty(M)$, and we can check that

- $v \in T_qM$, and
- for all $f \in C^\infty(M)$, $v(f|_U) = w(f)$.

Note that v is $\mathbb{R}$-linear, and for all $f, g \in C^\infty(W)$ we have $v(fg) = w(\rho fg) = w(\rho^2 fg)$ since $\rho fg = \rho^2 fg$ near q, then we have

$$\begin{aligned} v(fg) &= w(\rho^2 fg) \\ &= w((\rho f)(\rho g)) \\ &= v(\rho f) \cdot (\rho g)(g) + \rho(f)(q) \cdot v(\rho g) \\ &= v(f)g(q) + f(q)v(g). \end{aligned}$$

Finally, for all $f \in C^\infty(M)$, we have $v(f|_U) = w(\rho f) = w(f)$ since $\rho f = f$ near q. □

Notation. We now suppress the isomorphisms $I : T_qU \to T_qM$. In particular, given a chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$, we view $\left\{ \frac{\partial}{\partial x_i}|_q \right\}_{i=1}^m$ as a basis of T_qM.

Lemma 6.2. Let V be a finite-dimensional vector space with $q \in V$, then

$$\begin{aligned} \varphi : V &\to T_q V \\ v(f) &\mapsto \frac{d}{dt}|_0 f(q+tv) \end{aligned}$$

for all $f \in C^\infty(V)$, is an isomorphism.

Proof. One can see this is linear, so it suffices to show injectivity. We have

$$\ker(\varphi) = \{v \in V \mid \frac{d}{dt}|_0(q+tv) = 0 \,\forall f \in C^\infty(V)\}.$$

If $0 \neq v \in \ker(\varphi)$, then there exists $\ell : V \to \mathbb{R}$ such that $\ell(V) \neq 0$, so

$$0 \neq \frac{d}{dt}|_0(\ell(q+tv)) = \frac{d}{dt}|_0(\ell(q) + t\ell(v)) = \ell(v).$$

□

Definition 6.3. A *curve* through a point $q \in M$ on a manifold M is a C^∞-map $\gamma : (a,b) \to M$ with $0 \in (a,b)$ such that $\gamma(0) = q$.

Definition 6.4. Given $\gamma : (a,b) \to M$ with $\gamma(0) = q$, we define $\dot\gamma(0) \in T_q M$ by $\dot\gamma(0)f = \frac{d}{dt}|_0 f(\gamma(t)) = \frac{d}{dt}|_0(f \circ \gamma)$ for all $f \in C^\infty(M)$.

Remark 6.5.

$$\begin{aligned} t : (a,b) &\to \mathbb{R} \\ x &\mapsto x \end{aligned}$$

is a coordinate chart on (a,b), where $\frac{d}{dt}|_0 \in T_0(a,b)$ is a basis vector. Since γ is C^∞,

$$\begin{aligned} T_0\gamma : T_0(a,b) &\to T_{\gamma(0)}M \equiv T_q M \\ ((T_0\gamma)(\frac{d}{dt}|_0))f &= \frac{d}{dt}|_0(f \circ \gamma) = \dot\gamma(0), \end{aligned}$$

so $\dot\gamma(0) = (T_0\gamma)\left(\frac{d}{dt}|_0\right)$.

Let $\mathscr{C} = \{\gamma : I \to M \mid \gamma(0) = q, I \text{ interval depending on } \gamma\}$, then we have a map

$$\begin{aligned} \Phi : \mathscr{C} &\to T_q M \\ \gamma &\mapsto \dot\gamma(0) \end{aligned}$$

Note that Φ is not injective. However, there is an equivalence relation $\sim$ on $\mathscr{C}$ defined by $\gamma \sim \sigma$ if and only if $\Phi(\gamma) = \Phi(\sigma)$, so this gives an injection

$$\begin{aligned} \tilde\Phi : \mathscr{C}/\sim &\to T_q M \\ [\gamma] &\mapsto \dot\gamma(0). \end{aligned}$$

Claim 6.6. $\tilde\Phi$ is onto.

Proof. Choose coordinates $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ near q such that $(x_1, \ldots, x_m)(q) = 0$. Now, for all $v \in T_q M$, we have $v = \sum v(x_i) \frac{\partial}{\partial x_i}|_q$. Consider $\gamma(t) = \varphi^{-1}(tv(x_1), \ldots, tv(x_m))$, then $\gamma(0) = \varphi^{-1}(0) = q$ and for any $f \in C^\infty(M)$, we have

$$\begin{aligned} \dot\gamma(0)f &= \frac{d}{dt}|_0(f \circ \varphi^{-1})(tv(x_1), \ldots, tv(x_m)) \\ &= \sum \frac{\partial}{\partial r_i}(f \circ \varphi^{-1})|_0 \cdot v(x_i) \\ &= \sum v(x_i) \frac{\partial}{\partial x_i}|_q f \\ &= v(f). \end{aligned}$$

□

Lemma 6.7. For any smooth map $F : M \to N$ between manifolds, for all $q \in M$, we have

$$T_qF(\dot{\gamma}(0)) = (F \circ \gamma)^{\cdot}(0).$$

Proof.

$$\begin{aligned} T_qF(\dot{\gamma}(0)) &= T_qF(T_0\gamma\left(\frac{d}{dt}|_0\right)) \\ &= T_0(F \circ \gamma)\left(\frac{d}{dt}|_0\right) \\ &= (F \circ \gamma)^{\cdot}(0). \end{aligned}$$

□

Example 6.8. Let $M = N = \mathbb{C}$ and $F(z) = e^z$. We claim that $(T_zF)(v) = e^z v$, which uses $\mathbb{C} \cong T_w\mathbb{C}$ for all $w \in \mathbb{C}$. Indeed, since $\frac{d}{dt}|_0 e^{tv} = v$, then

$$\begin{aligned} (T_zF)(v) &= \frac{d}{dt}|_0 F(z + tv) \\ &= \frac{d}{dt}|_0 e^{z+tv} \\ &= \frac{d}{dt}|_0 (e^z e^{tv}) \\ &= e^z v. \end{aligned}$$

Note that T_zF is an isomorphism for all z, given by

$$\begin{tikzcd} T_z\mathbb{C} \arrow[r, "T_zF"] \arrow[d, "\cong"] & T_{F(z)}\mathbb{C} \arrow[d, "\cong"] \\ \mathbb{C} \arrow[r, "e^z \cdot -"'] & \mathbb{C} \end{tikzcd}$$

Also note that this is not a diffeomorphism, since the inverse is the complex logarithm function, which is only well-behaved on the principal branches.

7 Sept 6, 2023

Definition 7.1. Given a manifold M, $q \in M$, and $f \in C^\infty(M)$, we define the *exact differential* to be a linear map

$$\begin{aligned} df_q : T_qM &\to \mathbb{R} \\ v &\mapsto v(f) \end{aligned}$$

in $\operatorname{Hom}(T_qM, \mathbb{R}) =: T_q^*M$, the cotangent space.

Exercise 7.2.

- df_q is linear,
- $f \equiv g$ near q, then $df_q = dg_q$.

We have seen differentials before: given a coordinate chart $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ is a coordinate chart, then $\{(dx_i)_q\}_{i=1}^m$ is a basis of T_q^*M dual to $\{\frac{\partial}{\partial x_i}|_q\}_{i=1}^m$. Note that for all $\eta \in T_q^*M \equiv (T_qM)^*$, then $\eta = \sum \eta\left(\frac{\partial}{\partial x_i}|_q\right)(dx_i)_q$.

Lemma 7.3. Let M be a manifold, $q \in M$, and $f \in C^\infty(M)$, then the derivative

$$(T_qf)(v) = df_q(v)\frac{d}{dt}|_{f(q)}.$$

Proof. Note that $\{dt_{f(q)}\}$ is a basis of $T^*_{f(q)}\mathbb{R}$, then

$$dt_{f(q)}(T_q f(v)) = (T_q f(v))t = v(t \circ f) = v(f) = df_q(v),$$

so $(T_q f)(v) = df_q(v)\frac{d}{dt}|_{f(q)}$. ☐

Recall. Let $T : V \to W$ be a linear map, and let $\{e_1, \ldots, e_n\}$ be a basis of V, and let $\{f_1, \ldots, f_n\}$ be a basis of W, with dual basis $\{f_1^*, \ldots, f_n^*\}$ in W^*. Then let $t_{ij} = f_i^*(Te_j)$, then

$$T(e_j) = \sum_i f_i^*(Te_j)f_i = \sum_i t_{ij} f_i.$$

For all $F : \mathbb{R}^m \to \mathbb{R}^n$, consider the coordinates $(x_1, \ldots, x_m) : \mathbb{R}^m \to \mathbb{R}$ and $(y_1, \ldots, y_n) : \mathbb{R}^n \to \mathbb{R}$, which gives coordinates $\{(\frac{\partial}{\partial x_i}|_q)\}$ and $\{(\frac{\partial}{\partial y_i}|_{F(q)})\}$, respectively. With $T = T_q F$, we have

$$t_{ij} = (dy_i)_{F(q)}(T_q F(\frac{\partial}{\partial x_j}|_q)) = (T_q F(\frac{\partial}{\partial x_j}|_q))y_i = \frac{\partial}{\partial x_j}|_q(y_i \circ F).$$

If we denote $F = (F_1, \ldots, F_n)$ where $F_i = y_i \circ F$ then this is just $\frac{\partial F_i}{\partial x_j}(q)$, so $\left(\frac{\partial F_i}{\partial x_j}(q)\right)$ is the matrix of $T_q F$.

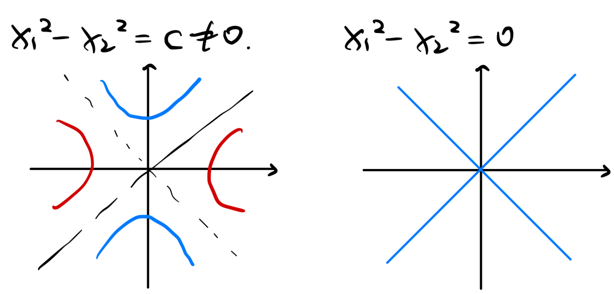
Definition 7.4. Let $F : M \to N$ be a smooth map, we say $c \in N$ is a *regular value* of F if either $F^{-1}(c) = \varnothing$, or for all $q \in F^{-1}(c)$, $T_q F : T_q M \to T_{F(q)}N = T_c N$ is onto.

We say $c \in N$ is a *singular value* if it is not a regular value.

Example 7.5. Consider

$$F : \mathbb{R}^2 \to \mathbb{R}$$
$$(x_1, x_2) \mapsto x_1 - x_2^2$$

for all $q = (x_1, x_2) \in \mathbb{R}^2$, then $T_q F$ is the matrix $\left(\frac{\partial F}{\partial x_1}(q), \frac{\partial F}{\partial x_1}(q)\right) = (2x_1, 2x_2)$. Hence, $c \neq 0$ is a regular value, and $c = 0$ is a singular value.



Definition 7.6. An *embedded submanifold* (of dimension k) of a manifold M is a subspace $Z \subseteq M$ such that for all $q \in Z$ there exists a coordinate chart $\varphi = (x_1, \ldots, x_k, x_{k+1}, \ldots, x_m) : U \to \mathbb{R}^m$ with $\varphi(U \cap Z) = \{(r_1, \ldots, r_m) \in \varphi(U) \mid r_k = \cdots = r_m = 0\}$.

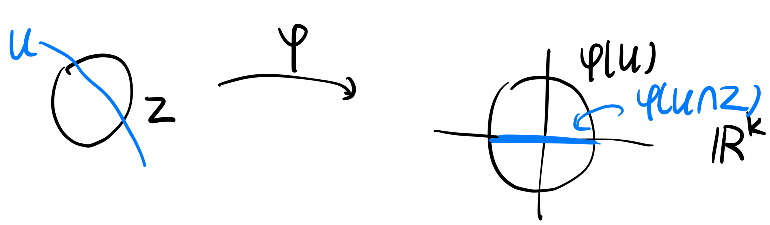


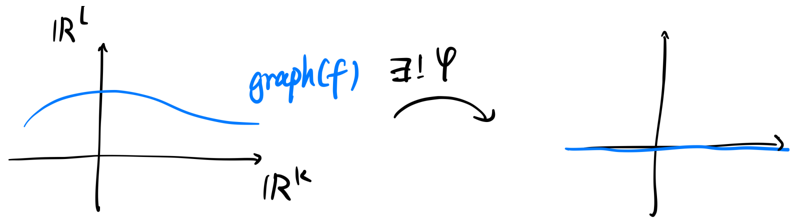
Figure 8: Embedded Submanifold

Remark 7.7.
- Any open subset $U \subseteq M$ is an embedded submanifold.
- Any singleton in M is an embedded submanifold.

Example 7.8. Let $f : \mathbb{R}^k \to \mathbb{R}^l$ be C^∞, then the graph of f is

$$\operatorname{graph}(f) = \{(x, f(x) \in \mathbb{R}^k \times \mathbb{R}^l \mid x \in \mathbb{R}^k\}$$

is an embedded submanifold of $\mathbb{R}^k \times \mathbb{R}^l$.



Here $\varphi(x, y) = (x, y - f(x))$ is a coordinate chart of $\mathbb{R}^k \times \mathbb{R}^l$ with inverse $\varphi^{-1}(x, y') = (x, y' + f(x))$.

Theorem 7.9 (Regular Value Theorem)**.** Let $c \in N$ be a regular value of smooth function $F : M \to N$. If $F^{-1}(c) = \varnothing$, then for all $q \in F^{-1}(c)$, $T_qF : T_qM \to T_qN$ is onto, so $F^{-1}(c)$ is an embedded submanifold of M. Moreover, $T_qF^{-1}(c) = \ker(T_qF)$ and $\dim(F^{-1}(c)) = \dim(M) - \dim(N)$.

Example 7.10. Consider

$$\begin{aligned} F : \mathbb{R}^m &\to \mathbb{R} \\ x &\mapsto \sum x_i^2 = ||x||^2 \end{aligned}$$

Now T_qF gives a local chart with $(2x_1, \ldots, 2x_m)$. Any $c \neq 0$ is a regular value. We have $F^{-1}(c) = \{x \mid ||x||^2 = c\}$ is the sphere of radius $\sqrt{c}$ for $c > 0$. Moreover, $F^{-1}(0) = \{0\}$, an embedded submanifold, but $\dim(\{0\}) \neq \dim(\mathbb{R}^m) - \dim(\mathbb{R})$.

8 Sept 8, 2023

Recall. A subset Z of a manifold M is an embedded submanifold (of dimension k and codimension $m - k$ for $m = \dim(M)$) if for all $z \in Z$, there exists a coordinate chart $\varphi : U \to \mathbb{R}^m$ and $z \in U$ which is adapted to Z, i.e., $\varphi(U \cap Z) = \varphi(U) \cap (\mathbb{R}^k \times \{0\})$.

Remark 8.1.
- Submanifolds of codimension 0 are open subsets.
- Submanifolds of codimension $m = \dim(M)$ are discrete sets of points.

We will proceed to prove Theorem 7.9.

Remark 8.2. Once we proved $F^{-1}(c)$ is embedded and $\dim(F^{-1}(c)) = \dim(M) - \dim(N)$, then the last statement follows. Indeed, given $v \in T_q(F^{-1}(c))$, there exists $\gamma : (a, b) \to F^{-1}(c)$ such that $\gamma(0) = q, \gamma'(0) = v$, and $F(\gamma(t)) = c$ for all t. Therefore,

$$0 = \frac{d}{dt}|_0 F(\gamma(t)) = T_qF(\gamma'(0)) = T_qFv,$$

so $v \in \ker(T_qF)$, and so $T_qF^{-1}(c) \subseteq \ker(T_qF)$. By dimension argument, we have equality.

We will introduce inverse function theorem and implicit function theorem.

Theorem 8.3 (Inverse Function Theorem)**.** Let $U \subseteq \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^n$ be C^∞ with $q \in U$ such that $T_qf = Df(q) : T_qU = \mathbb{R}^n \to \mathbb{R}^n = T_{F(q)}\mathbb{R}^n$ is an isomorphism. Then there exists an open neighborhood $q \in V \subseteq U$ and $f(q) \in W$ such that $f : V \to W$ is a diffeomorphism.

Notation. Given $F : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^m$ for $(a, b) \in \mathbb{R}^k \times \mathbb{R}^l$, then we denote

- $\frac{\partial F}{\partial x}(a, b) = T_{(a,b)}F|_{\mathbb{R}^k \times \{0\}} = DF(a, b)|_{\mathbb{R}^k \times \{0\}}$,
- $\frac{\partial F}{\partial y}(a, b) = T_{(a,b)}F|_{\{0\} \times \mathbb{R}^l} = DF(a, b)|_{\{0\} \times \mathbb{R}^l}$.

Theorem 8.4 (Implicit Function Theorem)**.** Let $F : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ be C^∞, let $(a, b) \in \mathbb{R}^k \times \mathbb{R}^l$. Suppose $\frac{\partial F}{\partial y}(a, b) : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, then there exists a neighborhood $W \ni (a, b)$ and $U \ni a$ in $\mathbb{R}^k$, as well as C^∞-map $g : U \to \mathbb{R}^n$ such that $F^{-1}(c) \cap W = \text{graph}(g) \cap W$.

Remark 8.5. inverse function theorem and implicit function theorem are equivalent.

Proof. Consider

$$\begin{aligned} H : \mathbb{R}^k \times \mathbb{R}^n &\to \mathbb{R}^k \times \mathbb{R}^n \\ (x, y) &\mapsto (x, F(x, y)) \end{aligned}$$

then $H(a, b) = (a, F(a, b)) = (a, c)$. The partials give

$$DH(a, b) = \begin{pmatrix} I & 0 \\ \frac{\partial F}{\partial x}(a, b) & \frac{\partial F}{\partial y}(a, b) \end{pmatrix}$$

As $\frac{\partial F}{\partial y}(a, b)$ is invertible, so is $DH(a, b)$, so there exists neighborhoods $(a, b) \in W \subseteq \mathbb{R}^n \times \mathbb{R}^k$ and $a \in U \subseteq \mathbb{R}^k$, $c \in V \subseteq \mathbb{R}^n$, such that $H : W \to U \times V$ is a diffeomorphism. Consider

$$\begin{aligned} G = H^{-1} : U \times V &\to W \subseteq \mathbb{R}^n \times \mathbb{R}^l \\ (u, v) &\mapsto (G_1(u, v), G_2(u, v)) \end{aligned}$$

therefore

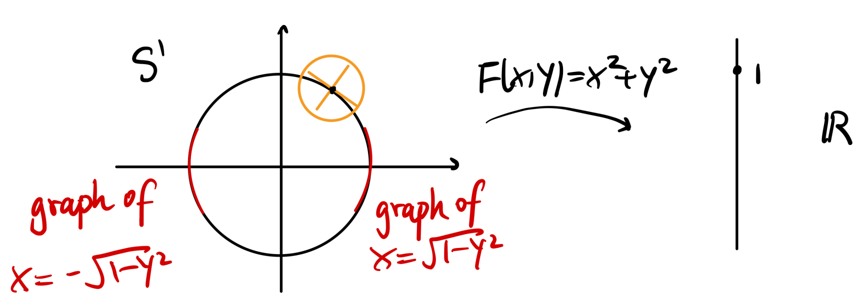
$$(u, v) = H(H^{-1}(u, v)) = H(G_1(u, v), G_2(u, v)) = (G_1(u, v), F(G_1(u, v), G_2(u, v))$$

so $G_1(u, v) = u$, and $v = F(u, G_2(u, v))$ for all u, v, hence $c = F(u, G_2(u, c))$ for all u. Now let $g(u) = G_2(u, c)$, then $F(u, g(u)) = c$ for all u. Hence, $\text{graph}(g) \subseteq F^{-1}(c)$. □

Proof of Regular Value Theorem. Let $F : M \to N$, $c \in N$, $F^{-1}(c) \neq \varnothing$. Now for all $q \in F^{-1}(c)$, then $T_qF : T_qM \to T_qN$ is onto. Given $q \in F^{-1}(c)$, we want a chart T from a neighborhood of q to $\mathbb{R}^m$, adapted to $F^{-1}(c)$. Let $\varphi : U \to \mathbb{R}^m$ and $\psi : V \to \mathbb{R}^m$ be charts such that $q \in U$, $c \in V$, then

$$\tilde{F} = \psi \circ F \circ \varphi^{-1}|_{\varphi(F^{-1}(V) \cap U} : \varphi(F^{-1}(V) \cap U) \subseteq \mathbb{R}^m \to \mathbb{R}^n$$

is C^∞. Now $\psi(c)$ is a regular value in $\tilde{F}$, Let $r = \varphi(q)$, then we have $D\tilde{F}(r) : \mathbb{R}^m \twoheadrightarrow \mathbb{R}^n$. Let $X = \ker(D\tilde{F}(r))$ and Y be a complement in $\mathbb{R}^m$. So $\mathbb{R}^m = X \otimes Y$ and $D\tilde{F}(r)|_Y : Y \to \mathbb{R}^n$ is an isomorphism. Apply inverse function theorem to $\tilde{F}$ from the intersection of $X \times Y$ and the open subset to $\mathbb{R}^n$.



□

Example 8.6. Let $\mathrm{Sym}^2(\mathbb{R}^n)$ be the $n \times n$ symmetric real matrices, also known as $\mathbb{R}^{\frac{n^2-n}{2}+n}$. There is

$$\begin{aligned} F : \mathrm{GL}_(n, \mathbb{R}) &\to \mathrm{Sym}^2(\mathbb{R}^n) \\ A &\mapsto A^T A \\ F^{-1}I = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T A = I\} &\leftarrow I \end{aligned}$$

Remark 8.7. We have $F = F \circ L_A$ for all $A \in O(U)$, then for all A, we have $T_A F$ onto.

Claim 8.8. 1 is a regular value of F, so $O(n)$ is an embedded submanifold of $\mathrm{GL}(n, \mathbb{R})$.

Proof.

$$\begin{aligned} (T_I F)(v) &= \frac{d}{dt}|_0 (I + tv)^T (I + tv) \\ &= \frac{d}{dt}|_0 (I^2 + tv^T + tv + t^2 v^T v) \\ &= v^T + v \end{aligned}$$

and this is surjective since for all $Y \in \mathrm{Sym}^2(\mathbb{R})$, we have $Y = \frac{1}{2}(Y^T + Y)$, so $Y = (T_I F)(\frac{1}{2}Y)$. □

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Recall. Let $F : M \to N$ be C^∞, let $c \in N$ be a regular value such that $F^{-1}(c) \neq \varnothing$. (For all $q \in F^{-1}(c)$, $T_q F : T_q M \to T_q N$ is onto.) Then:

i $F^{-1}(c)$ is an embedded submanifold of M.

ii $\dim(M) = \dim(F^{-1}(c)) = \dim(N)$.

iii for all $q \in F^{-1}(c)$, $T_q F^{-1}(c) = \ker(T_q F)$.

The proof uses inverse function theorem and/or implicit function theorem, and the key is to note that locally $f^{-1}(c)$ is a graph.

Also, $O(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T A = I\}$ is an embedded submanifold.

Definition 9.1. A *Lie group* G is a group and a manifold so that

i the multiplication map

$$\begin{aligned} m : G \times G &\to G \\ (a, b) &\mapsto (a, b) \end{aligned}$$

is C^∞.

ii the inverse map

$$\begin{aligned} \mathrm{inv} : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

is C^∞.

Notation. $e_G = 1_G$ is the identity element.

Example 9.2. $G = \mathbb{R}^n$ with $m(v, w) = v + w$, and $\mathrm{inv}(v) = -v$ gives a Lie group.

Example 9.3. Let $G = \mathrm{GL}(n, \mathbb{R})$ be with $e_G = \mathrm{diag}(1, \ldots, 1) = I$, with maps $m(A, B) = AB$ and $\mathrm{inv}(A) = A^{-1}$.

Remark 9.4. One can think of a Lie group G as four pieces of data:

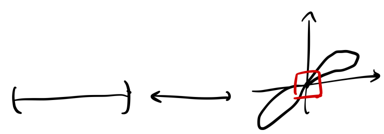
- manifold G,
- map $m : G \times G \to G$,
- map $\mathrm{inv} : G \to G$,
- $e_G \in G$.

Note that a subgroup H of a Lie group G is not necessarily a Lie group. The sufficient condition would be H is an embedded submanifold of G, i.e.,

- $m|_{H\times H} : H \times H \to H$ are C^∞,
- $\mathrm{inv}|_H : H \to H$

are C^∞. Note $m|_{H\times H} : H \times H \to G$ is C^∞ since $i : H \hookrightarrow G$ is C^∞ and $m|_{H\times H} = m(i \times i)$.

Example 9.5. For example, think of the embedding



but at the origin the preimage is split into three pieces, because the inverse is not continuous, which does not embed into a submanifold.

Lemma 9.6. If $i : Q \hookrightarrow M$ is an embedded submanifold, and $f : N \to M$ is a smooth map such that $f(N) \subseteq Q$, then $g : N \to Q$ with $g(n) = f(n)$ is C^∞.

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[dr, dashed, "g"'] & M \\ & Q \arrow[u, hook, "i"'] \end{tikzcd}$$

Proof. Since $Q \hookrightarrow M$ is embedded, for all $q \in Q$, there exists an adapted chart $\varphi = (x_1, \dots, x_n, x_{k+1}, \dots, x_m) : U \to \mathbb{R}^m$ such that $Q \cap U = \{x_k = \cdots = x_n = 0\}$. Consider $\varphi \circ f|_{f^{-1}(U)} : f^{-1}(U) \to \mathbb{R}^m$, then $f(f^{-1}(U)) \subseteq Q \cap U$.

Then $\varphi \circ f|_{f^{-1}(U)} = \varphi(U \cap Q) = \{(r_1, \dots, r_k, r_{k+1}, \dots, r_m) \mid r_{k+1} = \cdots = r_n = 0\}$, so $\varphi \circ f = (h_1, \dots, h_k, 0, \dots, 0)$ where $h_1, \dots, h_k \in C^\infty(f^{-1}(U))$. Therefore, $\varphi|_{U\cap Q}\, g|_{f^{-1}(U)} = (h_1, \dots, h_k)$. $\square$

Example 9.7. $O(n) \subseteq \mathrm{GL}(n, \mathbb{R})$ is embedded, thus a Lie group.

Example 9.8. $\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid \det(A) = 1\}$ is also a Lie group.

Claim 9.9. $1 \in \mathbb{R}$ is a regular value of $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$.

Proof. The key fact is that $T_I(\det) : \mathbb{R}^{n^2} \to \mathbb{R}$ is an $(n \times n)$-matrix given by $A \mapsto \mathrm{tr}(A)$. Indeed, note that the trace is the differential of the determinant. $\square$

Definition 9.10. A (real) *Lie algebra* is a (real) vector space $\mathfrak{g}$ with an $\mathbb{R}$-bilinear map

$$\begin{aligned}[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}&\to\mathfrak{g}\\(X,Y)&\mapsto[X,Y]\end{aligned}$$

such that for all $X,Y,Z\in\mathfrak{g}$,

- $[Y,X]=-[X,Y]$,
- $[X,[Y,Z]]=[[X,Y],Z]+[Y,[X,Z]]$.

Example 9.11. Let $\mathfrak{g}=M_n(\mathbb{R})$, $[X,Y]=XY-YX$ is the anti-commutator.

Example 9.12. Let M be a manifold, $\mathfrak{g}=\mathrm{Der}(C^\infty(M))=\{X:C^\infty(M)\to C^\infty(M)\mid X(fg)=X(f)\cdot g+f\cdot X(g)\}$. Therefore, $\mathfrak{g}$ is a Lie algebra with the bracket $[X,Y](f)=X(Y(f))-Y(X(f))$ for all $f\in C^\infty(M)$. This is the Lie algebra of vector fields on M.

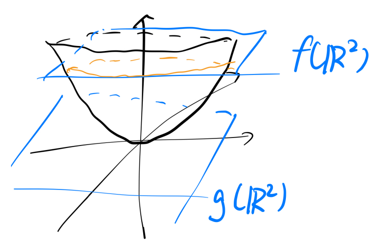
Example 9.13. Let $\mathfrak{g}=\mathbb{R}^3$, then $[v,w]:=v\times w$ is a Lie algebra with cross product.

We will see that for all Lie group G, $\mathfrak{g}=\mathrm{Lie}(G)=T_eG$ is naturally a Lie algebra.

Definition 9.14. Let $F:M\to N$ be a C^∞-map, $Z\subseteq N$ be an embedded submanifold. We say F is *transverse* to Z, denoted $F\pitchfork Z$, if for all $x\in F^{-1}(Z)$, $T_xF(T_xM)+T_{F(x)}Z=T_{F(x)}N$.

Example 9.15. If $Z=\{c\}$, then $F\pitchfork c$ if and only if for all $q\in F^{-1}(c)$, $(T_xF)(T_xN)+T_cc=T_cN$, if and only if for all $q\in F^{-1}(c)$, $(T_xF)(T_xN)=T_cN$, if and only if c is a regular value of F.

Example 9.16. Let $M=\mathbb{R}^2$, $N=\mathbb{R}^3$, $Z=\{(x,y,z)\mid z=x^2+y^2\}$, with $f(x,y)=(x,y,1)$ and $g(x,y)=(x,y,0)$, then $f\pitchfork Z$ but $g\not\pitchfork Z$.



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Theorem 10.1. Suppose $f:M\to N$ is transverse to an embedded submanifold $Z\subseteq N$, then

(i) $f^{-1}(z)$ is an embedded submanifold of M.

(ii) If $f^{-1}(z)\neq\varnothing$, then $\dim(M)-\dim(f^{-1}(z))=\dim(N)-\dim(Z)$, i.e., $\mathrm{codim}(f^{-1}(Z))=\mathrm{codim}(Z)$.

Proof. Fix $z_0\in Z$ with $f^{-1}(z_0)\neq\varnothing$, let $\psi:V\to\mathbb{R}^n$ be a coordinate chart on N, adapted to Z such that $\psi(V\cap Z)=\psi(V)\cap(\mathbb{R}^k\backslash\{0\})$. Let $\pi:\mathbb{R}^k\times\mathbb{R}^{n-k}\to\mathbb{R}^{n-k}$ be the canonical projection, then

$$(\pi\circ\psi)^{-1}(0)=\psi^{-1}(\pi^{-1}(0))=\psi^{-1}(\psi(V)\cap(\mathbb{R}^k\times\{0\}))=Z\cap V,$$

therefore

$$(\pi\circ\psi\circ f)^{-1}(0)=f^{-1}(Z\cap V)=f^{-1}(Z)\cap f^{-1}(V).$$

Claim 10.2. 0 is a regular value of $\pi\circ\psi\circ f|_{f^{-1}}(V)$.

Subproof. Take arbitrary $x \in (\pi \circ \psi \circ f)^{-1}(0) = f^{-1}(V) \cap f^{-1}(Z)$, then $T_x f(T_x M) + T_{f(x)} Z = T_{f(x)} N$. Note that $T_x M = T_x(f^{-1}(V))$. Therefore,

$$\mathbb{R}^k \times \mathbb{R}^{n-k} = T_{f(x)}\psi(T_{f(x)}N) = T_{f(x)}\psi(T_x f(T_x f^{-1}(V))) + T_{f(x)}\psi(T_{f(x)}Z)$$

by applying $T_{f(x)}\psi$ on both sides. Now apply $T_{\psi(f(x))}\psi$ on both sides, then $T_{f(x)}\psi(T_{f(x)}Z)$ vanishes, so we get

$$\begin{aligned}\mathbb{R}^{n-k} &= T_{\psi(f(x))}\pi(T_{f(x)}\psi(T_x f(T_x f^{-1}(V)))) \\ &= T_x(\pi \circ \psi \circ f)(T_x f^{-1}(V)).\end{aligned}$$

■

□

Definition 10.3. A C^∞-map $f : Q \to M$ is an *embedding* if

(i) $f(Q) \subseteq M$ is an embedded submanifold, and

(ii) $f : Q \to f(Q)$ is a diffeomorphism.

Remark 10.4. We know $f : Q \to f(Q)$ is C^∞ since $f(Q) \subseteq M$ is embedded and $f : Q \to M$ is given by the composition of $i : f(Q) \hookrightarrow M$ and $f : Q \to f(Q)$.

Remark 10.5.
1. Since $f : Q \to f(Q)$ is a diffeomorphism, then it is a homeomorphism. Thus $f : Q \to M$ is a topological embedding.
2. For all $q \in Q$, then $T_q f : T_q Q \to T_{f(q)} M$ is injective, i.e., $T_q f(T_q Q) = T_{f(q)} f(Q)$.

Example 10.6 (Non-example)**.** Let $Q = \mathbb{R}$ with discrete topology, then Q is a paracompact but not second countable as a 0-dimensional manifold. Consider

$$\begin{aligned} f : Q &\to \mathbb{R}^2 \\ x &\mapsto (x, 0) \end{aligned}$$

be a C^∞-map, then this is not an embedding.

Example 10.7. Let M be a manifold with $f \in C^\infty(M)$, then

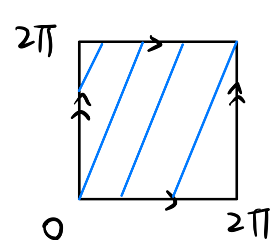
$$\begin{aligned} g : M &\to M \times \mathbb{R} \\ q &\mapsto (q, f(q)) \end{aligned}$$

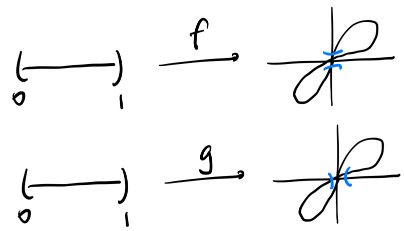
gives an embedding of M into $R \times \mathbb{R}$, as the graph of f.

Definition 10.8. A C^∞-map $f : Q \to M$ is an *immersion* if for all $q \in Q$, $T_q f : T_q Q \to T_{f(q)} M$ is injective.

Example 10.9. Consider

$$\begin{aligned} f : \mathbb{R} &\to S^1 \times S^1 \\ \theta &\mapsto (e^{i\theta}, e^{i\sqrt{2}\theta}) \end{aligned}$$





Example 10.10. Now $g \circ f^{-1} : (0, 1) \to (0, 1)$ is not an embedding, as it is not continuous.

Definition 10.11. The *rank* of a C^∞-map $f : M \to N$ at a point $q \in M$ is the rank of the linear map $T_q f : T_q M \to T_{f(q)}N$, i.e., $\operatorname{rank}_q(f) = \dim(T_q f(T_q M))$.

Example 10.12. If $f : M \to N$ is an immersion, then $\operatorname{rank}_q(f) = \dim_q(M)$.

Remark 10.13. Immersions are embeddings.

Theorem 10.14 (Rank Theorem)**.** Let $F : M \to N$ be a C^∞-map of constant rank k. Then for all $q \in M$, there exists coordinates $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$ on M with $q \in U$, and $\psi = (y_1, \dots, y_n) : V \to \mathbb{R}^n$ with $F(q) \in V$ such that $(\psi \circ F \circ \varphi^{-1})(r_1, \dots, r_m) = (r_1, \dots, r_k, 0, \dots, 0)$ for all $r = (r_1, \dots, r_m) \in \varphi(F^{-1}(V) \cap U)$.

Notation. Given a collection of sets $\{S_\alpha\}_{\alpha\in A}$, $\coprod\limits_{\alpha\in A} S_\alpha$ is the disjoint union of the collection.

We will give the following construction of a tangent bundle.

Remark 10.15. Given a manifold M, we form a set $TM = \coprod\limits_{q\in M} T_q M$. Given a chart $\varphi = (x_1, \dots, x_n) : U \to \mathbb{R}^m$ on M, the corresponding candidate chart is $\tilde{\varphi} : TU = \coprod\limits_{q\in U} T_q M \to \varphi(U) \times \mathbb{R}^m$. One can check that if $\varphi : U \to \mathbb{R}^m$ and $\psi : V \to \mathbb{R}^m$ are charts on M with $U \cap V \neq \varnothing$, then $\tilde{\psi}\tilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^m \to \psi(U \cap V) \times \mathbb{R}^m$ is C^∞. Now we give TM the topology making $\tilde{\varphi}$'s homeomorphic onto their images, then $\{\tilde{\varphi} : TU \to \varphi(U) \times \mathbb{R}^m\}$ will be an atlas on TM.

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Definition 11.1. A map $f : M \to N$ is a *submersion* if for all $p \in M$, the differential $T_q f : T_q M \to T_{f(q)}N$ is onto.

Remark 11.2. Every value over a submersion is regular.

Recall. For a manifold M, we defined the set $TM = \coprod\limits_{q\in M} T_q M = \bigcup(\{q\} \times T_q M)$, which is a called a tangent bundle, with additional structures. We will show that TM is a manifold, and

$$\begin{aligned} \pi : TM &\to M \\ (q, v) &\mapsto q \end{aligned}$$

is C^∞ and a submersion.

Proof. Let $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$ be a coordinate chart on M. For any $q \in U$, let $\left\{ \frac{\partial}{\partial x_1}\Big|_q, \dots, \frac{\partial}{\partial x_m}\Big|_q \right\}$ be a basis of $T_q M$. The dual basis is $\{(dx_1)_q, \dots, (dx_m)_q\}$. For any $v \in T_q M$, we have $v = \sum v(x_i) \frac{\partial}{\partial x_i}\Big|_q := \sum (dx_i)_q(v) \frac{\partial}{\partial x_i}\Big|_q$, and

$$\begin{aligned} T_q M &\to \mathbb{R} \\ v &\mapsto ((dx_1)_q(v), \dots, (dx_m)_q(v)) \end{aligned}$$

is a linear isomorphism. Define

$$\begin{aligned}\tilde{\varphi}: TU = \coprod_{q\in M} T_qM &\to \mathbb{R}^m \times \mathbb{R}^m \\ (q,v) &\mapsto (x_1(q),\dots,x_m(q),(dx_1)_q(v),\dots,(dx_m)_q(v)).\end{aligned}$$

Suppose $\psi = (y_1,\dots,y_m): V \to \mathbb{R}^m$ is another chart, we then have

$$\begin{aligned}\tilde{\psi}: TV &\to \mathbb{R}^m \times \mathbb{R}^m \\ (q,v) &\mapsto (y_1(q),\dots,y_m(q),(dy_1)_q(v),\dots,(dy_m)_q(v)).\end{aligned}$$

Claim 11.3. For any $(r,w) \in \varphi(U\cap V)\times\mathbb{R}^m$, we have

$$\begin{aligned}(\tilde{\psi}\circ\tilde{\varphi}^{-1})(r,w) &= ((\psi\circ\varphi^{-1})(r), \sum_j \frac{\partial y_1}{\partial x_j}(\varphi^{-1}(r))w_i,\dots,\sum_j \frac{\partial y_m}{\partial x_j}(\varphi^{-1}(r))w_i) \\ &= \left((\psi\circ\varphi^{-1})(r), \left(\frac{\partial y_i}{\partial x_j}(\varphi^{-1}(r))\right)\begin{pmatrix} w_1 \\ \vdots \\ w_m\end{pmatrix}\right)\end{aligned}$$

Subproof.

Recall. If $T: A \to B$ is a linear map, with $\{e_1,\dots,e_n\}$ basis of A, $\{f_1,\dots,f_n\}$ is a basis of B, with dual basis $\{f_1^*,\dots,f_n^*\}$, then we set $t_{ij} = f_u^*(Te_j)$, i.e.,

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "(t_{ij})"] \arrow[d, "(v_1,\dots,v_n)\mapsto\sum v_ie_i"'] & \mathbb{R}^n \arrow[d] \\ A \arrow[r, "T"'] & B \end{tikzcd}$$

In our case, we have $A = B = T_qM$ with $T = \mathrm{id}$, with basis $\left\{\frac{\partial}{\partial x_i}\Big|_q\right\}$ of A, $\{f_1,\dots,f_n\} = \left\{\frac{\partial}{\partial y_1}\Big|_q,\dots,\frac{\partial}{\partial y_m}\Big|_q\right\}$ and dual basis $\{f_1^*,\dots,f_m^*\} = \{(dy_1)_q,\dots,(dy_m)_q\}$, then

$$\begin{aligned} t_{ij} &= (dy_i)_q\left(\frac{\partial}{\partial x_j}\Big|_q\right) \\ &= \frac{\partial}{\partial x_j}(y_i)(q) \\ &= \frac{\partial y_i}{\partial x_j}(\varphi^{-1}(\gamma)).\end{aligned}$$

■

We define the topology on TM to be the topology generated by the sets of form $\tilde{\varphi}^{-1}(W)$ where $\varphi: U \to \mathbb{R}^m$ is a coordinate chart with open subset $W \subseteq \mathbb{R}^m\times\mathbb{R}^m$. Given an atlas $\{\varphi_\alpha: U_\alpha \to \mathbb{R}^m\}$ on M, we get an induced atlas $\{\tilde{\varphi}_\alpha: TU_\alpha \to \mathbb{R}^m\times\mathbb{R}^m\}$ on TM. One can check that the choice of an atlas on M does not matter. □

Exercise 11.4.

- If M is Hausdorff, then so is TM.
- If M is second countable, then so is TM.

Lemma 11.5. The canonical projection $\pi: TM \to M$ is C^∞ and is a submersion.

Proof. Let $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$ be a coordinate chart, $\tilde{\varphi} : TU \to \mathbb{R}^m \times \mathbb{R}^m$ be the induced chart on TM, then

$$\begin{aligned}(\varphi \circ \pi \circ \tilde{\varphi}^{-1})(r, w) &= \varphi \circ \pi \left(\varphi^{-1}(r), \sum_i w_i \left.\frac{\partial}{\partial x_i}\right|_q\right) \\ &= \varphi(\varphi^{-1}(r)) \\ &= r.\end{aligned}$$

Moreover,

$$\left(T_{(r,w)}(\varphi \circ \pi \circ \tilde{\varphi}^{-1})\right)(v, w') = v$$

where $(v, w') \in T_{(r,w)}(\varphi(U) \times \mathbb{R}^m) \cong \mathbb{R}^n \times \mathbb{R}^m$. Therefore, $T_{(q,v)}\pi : T_{(q,v)}TM \to T_qM$ is onto, hence a submersion. $\square$

Definition 11.6. A *(algebraic) vector field* on a manifold M is a derivation $v : C^\infty(M) \to C^\infty(M)$, i.e., v is $\mathbb{R}$-linear and $v(fg) = v(f)g + fv(g)$ for all $f, g \in C^\infty(M)$.

Definition 11.7. A *(geometric) vector field* on a manifold M is a section of the tangent bundle TM of M, i.e., $X : M \to TM$ is C^∞ with $\pi \circ X = \mathrm{id}_M$. Geometrically, this depicts tangent vectors over a point with directions in $X(q)$.

Notation.

- $\mathrm{Der}(C^\infty(M))$ is the set of all derivations of $C^\infty(M)$.
- $\mathfrak{X}(M) = \Gamma(TM)$ is the set of sections of $\pi : TM \to M$.

Proposition 11.8. Given a section $v : M \to TM$ in $\mathfrak{X}(M)$, we can try and define

$$\begin{aligned} D_v : C^\infty(M) &\to C^\infty(M) \\ (D_v(f))(q) &\mapsto v(q)f \end{aligned}$$

and this assignment $v \mapsto D_v$ is a linear isomorphism.

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Recall. $TM = \coprod_{q \in M} T_qM$ is a manifold. To show this, given chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$ on M, we set

$$\begin{aligned} \tilde{\varphi} = (x_1, \dots, x_m, dx_1, \dots, dx_m) : TU \equiv \coprod_{q \in U} T_qM &\to \mathbb{R}^m \to \mathbb{R}^m \\ (q, v) &\mapsto (\varphi(q), (dx_1)_q(v), \dots, (dx_m)_q(v)) \end{aligned}$$

with inverse

$$\tilde{\varphi}^{-1}(r, u) = (\varphi^{-1}(r), \sum w_i \left.\frac{\partial}{\partial q_i}\right|_{\varphi(r)}.$$

Also,

$$\begin{aligned} \pi : TM &\to M \\ (q, v) &\mapsto q \end{aligned}$$

is a C^∞-submersion.

We defined vector fields in two ways,

- as sections of tangent bundle $\pi : TM \to M$, i.e., as C^∞-maps $X : M \to TM$ such that $\pi X = \mathrm{id}$, i.e., $X(q) \in T_qM$, and
- as derivations $c : C^\infty(M) \to C^\infty(M)$, i.e., as $\mathbb{R}$-linear maps such that $v(fg) = fv(g) + v(f)g$ for all $f, g \in C^\infty(M)$.

Remark 12.1. Both $\Gamma(TM)$ and $\mathfrak{X}(M)$ are $\mathbb{R}$-vector spaces, and $C^\infty(M)$-modules.

We now prove Proposition 11.8.

Proof. Given $v \in \Gamma(TM)$ and $f \in C^\infty(M)$, consider a function

$$\begin{aligned} D_v f : M &\to \mathbb{R} \\ (D_v(f))(q) &= v(q)f \end{aligned}$$

To go back, given $X \in \mathrm{Der}(C^\infty(M))$, for any $q \in M$, we have $\mathrm{ev}_q : C^\infty(M) \to \mathbb{R}$, and then $\mathrm{ev}_q \circ X : C^\infty(M) \to \mathbb{R}$ is a tangent vector. Define $v_X(q) = \mathrm{ev} \circ X$, and we can check other requirements like C^∞ and so on.

Claim 12.2. $D_v f$ is C^∞.

Subproof. Given a chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$, we have

$$\begin{aligned} \tilde{\varphi} : TU &\to \mathbb{R}^m \times \mathbb{R}^m \\ (q, v) &\mapsto (\varphi(q), dx_1(v), \dots, dx_m(v)) \end{aligned}$$

Since v is C^∞, the map $\tilde{\varphi} \circ v|_U : U \to \mathbb{R}^m \times \mathbb{R}^m$, defined by $(\tilde{\varphi} \circ v)(q) = (\varphi(q), (dx_1)_q(v(q)), \dots, (dx_m)_q(v(q)))$, is C^∞. Therefore, the assignment $q \mapsto (dx_i)_q(v(q))$ are C^∞ on U. Hence, $v = \sum v_i \frac{\partial}{\partial x_i}$ where $v_i(q) = (dx_i)_q(v(q))$ for all i. So $(D_v f)|_U = \left(\sum v_i \frac{\partial}{\partial x_i}\right) f = \sum v_i \frac{\partial f}{\partial x_i}$. This concludes the proof. ■

Also, for all $f, g \in C^\infty(M)$ and all q, we have

$$\begin{aligned} (D_v(fg))(q) &= v(q)(fg) \\ &= (v(q)f)g(q) + f(q)(v(q)g) \\ &= ((D_v f)g + f(D_v g))(q). \end{aligned}$$

Recall that derivations are local, i.e., for $X \in \mathrm{Der}(C^\infty(M))$ and $f \in C^\infty(M)$ and $f|_U \equiv 0$, then $Xf|_U \equiv 0$. As a consequence, for $U \subseteq M$ open, define $X|_U : C^\infty(U) \to C^\infty(U)$ such that $(X|_U)(f|_U) = (Xf)|_U$ for all $f \in C^\infty(M)$. Now given a chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$, we know x_i's are in $C^\infty(U)$, then $(X|_U)(x_i)$ is a smooth function on U. Therefore,

$$\begin{aligned} v_X|_U &= \sum (dx_i)(v_X) \frac{\partial}{\partial x_i} \\ &= \sum v_X X(x_i) \frac{\partial}{\partial x_i} \\ &= \sum X|_U (x_i) \frac{\partial}{\partial x_i}, \end{aligned}$$

and thus $v_X|_U : U \to TU$ is C^∞, and since U is arbitrary, then $v_X \in \Gamma(TM)$. □

Recall. For any $X, Y \in \mathrm{Der}(C^\infty(M))$, $[X, Y] \in \mathrm{Der}(C^\infty(M))$. Therefore, $\mathrm{Der}(C^\infty(M))$ is a real Lie algebra with bracket $(X, Y) \mapsto [X, Y]$. Note that $\mathrm{Der}(C^\infty(M)) \subseteq \mathrm{Hom}_{\mathbb{R}}(C^\infty(M), C^\infty(M))$.

Recall. If $(A, \circ)$ is a real associative algebra, then $[a, b] := a \circ b - b \circ a$ gives A the structure of a Lie algebra, and $\mathrm{Der}(C^\infty(M)) \subseteq \mathrm{Hom}_{\mathbb{R}}(C^\infty(M), C^\infty(M))$.

Now given a C^∞-map $f : M \to N$ of manifolds, we get a map

$$\begin{aligned} Tf : TM &\to TN \\ (q, v) &\mapsto (f(q), T_q f v) \end{aligned}$$

Exercise 12.3. Tf is C^∞.

Remark 12.4. Given $f : M \to N$ and $v \in \Gamma(TM)$, we may not have a commutative diagram:

$$\begin{tikzcd} TM \arrow[r, "Tf"] & TN \\ M \arrow[u, "v"] \arrow[r, "f"'] & N \arrow[u, dashed, "?"'] \end{tikzcd}$$

Definition 12.5. Let $f : M \to N$ be a smooth map on manifolds, then $v \in \Gamma(TM)$ and $w \in \Gamma(TN)$ are *f-related* if we have a commutative diagram

$$\begin{tikzcd} TM \arrow[r, "Tf"] & TN \\ M \arrow[u, "v"] \arrow[r, "f"'] & N \arrow[u, "w"'] \end{tikzcd}$$

That is, for any $q \in M$, $w(f(q)) = (f(q), T_q f(v(q)))$.

Equivalently, for $f : M \to N$, we say $X \in \operatorname{Der}(C^\infty(M))$ is *f-related* to $Y \in \operatorname{Der}(C^\infty(N))$ if for all $h \in C^\infty(N)$, we have $Y(h) \circ f = X(h \circ f)$ in $C^\infty(M)$.

13 Sept 20, 2023

Recall. Let M be a manifold, we have a bijection

$$\begin{aligned} \Gamma(TM) &\to \operatorname{Der}(C^\infty(M)) \\ v &\mapsto D_v : (Dvf)(q) = v_q(f)\ \forall f, q \end{aligned}$$

with inverse by assignment $X \mapsto v_X$ where $v_X(q)f = (Xf)(q)$.

Lemma 13.1. Let $f : M \to N$, then $v \in \Gamma(TM)$ is f-related to $w \in \Gamma(TN)$ if and only if $D_v \in \operatorname{Der}(C^\infty(M))$ is f-related to $D_w \in \operatorname{Der}(C^\infty(N))$.

Proof. v is f-related to w if and only if $(T_q f)(v(q)) = w(f(q))$ for all q, if and only if $((T_q f)(v(q)))h = (w(f(q)))h$ for all q and all h, if and only if $(D_v(h \circ f))(q) = (D_w h)(f(q))$, if and only if $D_v(h \circ f) = D_w(h \circ f)$. ☐

Lemma 13.2. Suppose $f : M \to N$, let $X_1, X_2 \in \operatorname{Der}(C^\infty(M))$, and $Y_1, Y_2 \in \operatorname{Der}(C^\infty(N))$ such that X_i is f-related to Y_i for $i = 1, 2$, then $[X_1, X_2]$ is f-related to $[Y_1, Y_2]$.

Proof. For any $h \in C^\infty(N)$, $X_i(h \circ f) = Y_i(h) \circ f$ for $i = 1, 2$. Therefore,

$$\begin{aligned} ([X_1, X_2])(h \circ f) &= X_1(X_2(h \circ f)) - X_2(X_1(h \circ f)) \\ &= X_1(Y_2(h) \circ f) - X_2(Y_1(h) \circ f) \\ &= Y_1(Y_2(h)) \circ f - Y_2(Y_1(h)) \circ f \\ &= ([Y_1, Y_2](h)) \circ f. \end{aligned}$$

☐

Definition 13.3. Let $Q \subseteq M$ be an embedded submanifold. A vector field $Y \in \Gamma(TM)$ is *tangent* to Q if for all $q \in Q$, $Y(q) \in T_q Q$.

Example 13.4. If $M = \mathbb{R}^2$, let $Q = \mathbb{R} \times \{0\}$, then $Y(x_1, x_2) = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$, so $Y(x, 0) = x_1 \frac{\partial}{\partial x_1} + 0 \in T_{(x,0)} Q$. Equivalently, we have $i : Q \hookrightarrow M$ to be an inclusion, so $Ti : TQ \hookrightarrow TM$ is an embedding since i is, as $Y(q) \in T_q Q$ for all $q \in Q$ indicates $(Y \circ i)(Q) \subseteq TQ$:

$$\begin{tikzcd} Q \arrow[r, "i"] \arrow[d, dashed, "Y \circ i"'] & M \arrow[d, "Y"] \\ TQ \arrow[r, hook, "Ti"'] & TM \end{tikzcd}$$

Hence, $Y \circ i : Q \to TQ$ is a vector field on Q, and $Y \circ i$ is i-related to Y.

Lemma 13.5. Let $Q \subseteq M$ be an embedded submanifold, let $Y_1, Y_2 \in \Gamma(TM)$ which are tangent to Q, then $[Y_1, Y_2]$ is tangent to Q.

Proof. Since $Y_i|_Q$ is i-related to Y_i, then $[Y_1, Y_2]|_Q$ is i-related to $[Y_1, Y_2]$. ☐

Definition 13.6. Let G be a Lie group, then we give T_eG the structure of a Lie algebra. A vector field $X : G \to TG$ is *left-invariant* if for all $a \in G$, $TL_a(X(g)) = X(L_a g)$ for all $g \in G$ and all $a \in G$, that is, X is L_a-related to X where $L_a(g) = ag$ is the left translation.

Recall.

- $(La)^{-1} = L_{a^{-1}}$.
- By Lemma 13.2, if X and Y are left-invariant, then so is $[X, Y]$.

Notation. We denote $\mathfrak{g} = \mathrm{Lie}(G)$ to be the Lie algebra of the left-invariant vector fields.

Lemma 13.7. Let G be a Lie group, let $\mathfrak{g}$ be the space of left-invariant vector fields, then the evaluation map

$$\begin{aligned} \mathrm{ev}_e : \mathfrak{g} &\to T_eG \\ X &\mapsto X(e) \end{aligned}$$

is an $\mathbb{R}$-linear bijection. In particular, they have the same dimension.

Proof. Obviously ev_e is linear. If $X(e) = 0$, then for all $a \in G$, $X(a) = X(L_a e) = (TL_a)_e(X(e)) = 0$, so ev_e is injective. Conversely, given $v \in T_eG$, define

$$\begin{aligned} \tilde{v} : G &\to TG \\ a &\mapsto (TL_a)_e v \end{aligned}$$

then $\tilde{v}$ is left-invariant. We know

$$\begin{aligned} m : G \times G &\to G \\ (a, b) &\mapsto ab \end{aligned}$$

is C^∞, so $T_m : TG \times TG \to TG$ is C^∞. Consider

$$\begin{aligned} f : G &\to TG \times TG \\ a &\mapsto ((a, 0), (e, v)). \end{aligned}$$

Claim 13.8. $(T_m \circ f)(a) = (T_e L_a)(v)$.

Subproof. Pick $\gamma : I \to G$ such that $\gamma(0) = e$ and $\dot{\gamma}(0) = v$, then

$$\begin{aligned} \sigma : I &\to G \times G \\ t &\mapsto (a, \gamma(t)) \end{aligned}$$

is C^∞ where $\sigma(0) = (a, e)$, and $\frac{d}{dt}\big|_0 (a, \gamma(t)) = (0, v) \in T_{(a,e)}(G \times G)$. Now

$$\begin{aligned} T_m(f(a)) &= (T_m)_{(a,e)}(0, v) \\ &= \frac{d}{dt}\bigg|_0 m(\sigma(t)) \\ &= \frac{d}{dt}\bigg|_0 a\gamma(t) \\ &= \frac{d}{dt}\bigg|_0 L_a(\gamma(t)) \\ &= (T_e L_a)(\dot{\gamma}(0)) \\ &= (T_e L_a)(v) \\ &= \tilde{v}(a). \end{aligned}$$

■

□

Therefore, the left-invariant vector field $\mathrm{Lie}(G)$ is isomorphic to T_eG as $\mathbb{R}$-vector spaces.

Definition 13.9. Let $X : M \to TM$ be a vector field. An *integral curve* $\gamma : I \to M$ of X passing through q at $t = 0$ is a C^∞-map $\gamma : I \to M$ such that $\gamma(0) = q$ and $\dot{\gamma}(t) = X(\gamma(t))$ for all $t \in I$. Here $\dot{\gamma}(t) = (T_t\gamma)\left(\frac{d}{dt}\big|_t\right) \in T_{\gamma(t)}M$. Equivalently, $\dot{\gamma}(t)f = X(\gamma(t))f = \frac{d}{dt}\big|_t (f \circ \gamma)$ for all $f \in C^\infty(M)$.

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Remark 14.1. if $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ is a coordinate chart and v is a vector field on U, so $v = \sum v_i \frac{\partial}{\partial x_i}$ for $v_1, \ldots, v_m$ in $C^\infty(U)$. This is a section $q \mapsto \sum v_i(q) \left.\frac{\partial}{\partial x_i}\right|_q \in \Gamma(TU)$ and for all $f \in C^\infty(U)$, $f \mapsto \sum v_i \frac{\partial f}{\partial x_i} \in C^\infty(U)$ which is a derivation.

Recall. An integral curve of $X \in \Gamma(TM)$ is a curve $\gamma : I \to M$ with $\gamma(0) = q$ such that $\left.\frac{d\gamma}{dt}\right|_t = X(\gamma(t))$.

Example 14.2. Let $M = U$ be open in $\mathbb{R}^m$, and $X = \sum x_i \frac{\partial}{\partial r_i}$. Let $\gamma(t) = (\gamma_1(t), \ldots, \gamma_m(t))$ for $\gamma_i \in C^\infty(I)$, then $\left.\frac{\partial \gamma}{\partial t}\right|_t = \sum \gamma_i'(t) \frac{\partial}{\partial \gamma_i}$. Therefore, $\frac{\partial \gamma}{\partial t} = X(\gamma(t))$ amounts to $\sum \gamma_i'(t) \frac{\partial}{\partial \gamma_i} = \sum x_i(\gamma(t)) \frac{\partial}{\partial \gamma_i}$. Therefore, $\gamma_i'(t) = x_i(\gamma_1(t), \ldots, \gamma_m(t))$.

Hence, γ is an integral curve of X if and only if γ solves such a system of equations with initial condition $\gamma(0) = q$.

Theorem 14.3. Let $U \subseteq \mathbb{R}^m$ be open, $X = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ be C^∞, then for all $q_0 \in U$, there exists an open neighborhood V of q_0 in U and $\varepsilon > 0$, and a C^∞-map $\Phi : V \times (-\varepsilon, \varepsilon) \to U$ such that for all $q \in V$, $\gamma_q(t) := \Phi(q, t)$ solves $\gamma_i'(t) = x_i(\gamma_1(t), \ldots, \gamma_m(t))$ with initial condition $\gamma_q(0) = q$. Moreover, such mapping Φ is unique.

Proof. Apply contraction mapping principle. □

Example 14.4. Say $U = (-1, 1)$, let

$$\begin{aligned} X : (-1,1) &\to \mathbb{R} \\ x &\mapsto \frac{d}{dx} \end{aligned}$$

with $X(q) = 1$ be the ODE, i.e., $\frac{dX}{dt} = 1$ with $X(0) = q$, then $\Phi(q, t) = q + t$. The domain of definition of Φ is $W = \{(q, t) \mid q \in (-1, 1), q + t \in (-1, 1)\}$.

Remark 14.5. We need to keep track of the initial conditions. Say $\gamma : (a, b) \to M$ is an integral curve of vector field X on M with $\gamma(0) = q$, then for all $t_0 \in (a, b)$, we know

$$\begin{aligned} \sigma : (a - t_0, b - t_0) &\to M \\ s &\mapsto \gamma(s + t_0) \end{aligned}$$

is also an integral curve. Therefore, γ and σ has the same image.

Proof.

$$\begin{aligned} \left.\frac{d}{dt}\right|_t \sigma &= \left.\frac{d}{ds}\right|_t \gamma(s + t_0) \\ &= \left.\frac{d}{du}\right|_{u=t+t_0} \gamma(u) \\ &= X(\gamma(t + t_0)) \\ &= X(\sigma(t)). \end{aligned}$$

□

Lemma 14.6. Let $X : M \to TM$ be a vector field, $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ be a coordinate chart and $X = \sum x_i \frac{\partial}{\partial x_i}$ where $x_i \in C^\infty(U)$, then $\gamma : I \to U$ with $\gamma(0) = q$ is an integral curve of X if and only if $(x_1 \circ \gamma, \ldots, x_m \circ \gamma) : I \to \mathbb{R}^m$ solves $y_i' = Y_i(Y_1, \ldots, y_m)$ with $y_i(0) = x_i(\gamma(0))$. Here $Y_i = X_i \circ \varphi^{-1} \in C^\infty(\varphi^{-1}(U))$.

Proof. We have $\dot\gamma(t) = \sum dx_i(\dot\gamma(t)) \frac{\partial}{\partial x_i} = \sum (x_i \circ \gamma)'(t) \frac{\partial}{\partial x_i}$. Therefore, $\dot\gamma(t) = X(\gamma(t))$ if and only if $(X_i \circ \gamma)' = X_i(\gamma(t)) = (X_i \circ \varphi^{-1})(\varphi(\gamma(t))) = Y_i(X_1 \circ \gamma(t), \ldots, X_m \circ \gamma(t))$ for all i. □

Corollary 14.7. Let $X : M \to TM$ be a vector field, then for all $q \in M$, there exists an integral curve $\gamma : I \to M$ of X such that $\gamma(0) = q$. Moreover, γ depends smoothly on q, and is locally unique: for all integral curve $\sigma : J \to M$ of X mapping $0 \mapsto q$, there exists $\delta > 0$ such that $(-\delta, \delta) \in I \cap J$ and $\gamma|_{(-\delta,\delta)} = \sigma|_{(-\delta,\delta)}$.

Remark 14.8. It may not be the case that $\gamma|_{I\cap J} = \sigma|_{I\cap J}$. This is true if M is Hausdorff.

Example 14.9. Consider line with two origins in Example 1.10, with translations that agree before the origins.

Lemma 14.10. Suppose $\gamma : I \to M$ and $\sigma : J \to M$ are continuous curves, and M is Hausdorff, then the set $Z = \{t \in I \cap J \mid \gamma(t) = \sigma(t)\}$ is closed in $I \cap J$.

Proof. Note that

$$\begin{aligned}(\gamma, \sigma) : I \cap J &\to M \times M \\ t &\mapsto (\gamma(t), \sigma(t))\end{aligned}$$

is continuous, and $Z = (\gamma, \sigma)^{-1}(\Delta_M)$. □

Lemma 14.11. Let $\gamma : I \to M$ and $\sigma : J \to M$ be two integral curves of a vector field X on M with $\sigma(0) = \gamma(0)$, then $W = \{t \in I \cap J \mid \gamma(t) = \sigma(t)\}$ is open in $I \cap J$.

Proof. Given $t_0 \in W$, then $t_0 \in I \cap J$ and $\sigma(t_0) = \gamma(t_0)$, and we consider $\tilde{\sigma}(t) := \sigma(t + t_0)$ and $\tilde{\gamma}(t) = \gamma(t + t_0)$, then $\tilde{\sigma}(0) = \sigma(t_0) = \gamma(t_0) = \tilde{\gamma}(0)$. Both $\tilde{\gamma}$ and $\tilde{\sigma}$ are integral curves of X with $\tilde{\sigma}(0) = \tilde{\gamma}(0)$, therefore by Corollary 14.7, there exists $\delta > 0$ such that $\tilde{\sigma}|_{(-\delta,\delta)} = \tilde{\gamma}|_{(-\delta,\delta)}$, then $t_0 + (-\delta, \delta) = (t_0 - \delta, t_0 + \delta) \subseteq W$. □

Lemma 14.12. Let M be a Hausdorff manifold, $X \in \Gamma(TM)$, $\gamma : I \to M$ and $\sigma : J \to M$ be two integral curves with $\gamma(0) = \sigma(0)$, then $\gamma|_{I\cap J} = \sigma|_{I\cap J}$.

Proof. Since I and J are intervals, then $I \cap J$ is connected. By Lemma 14.11 and Lemma 14.10, $W = \{t \in I \cap J \mid \gamma(t) = \sigma(t)\}$ is clopen, thus $W = I \cap J$. □

15 Sept 25, 2022

Recall. We introduced integral curves of vector fields, and in particular we introduced Lemma 14.12.

Corollary 15.1. For any vector field $X \in \Gamma(TM)$ and any $q \in M$, there exists a unique maximal integral curve $\gamma_q : I_q \to M$ of X with $\gamma_q(0) = q$. Here *maximal* means that if $\sigma : J \to M$ is another integral curve of X with $\sigma(0) = q$, then $J \subseteq I_q$ and $\sigma = \gamma_q|_J$.

Proof. Consider the subset $\Gamma \subseteq \mathbb{R} \times M$ defined as follows: let Y be the set of all integral curves γ of X with $\gamma(0) = q$, then define $\Gamma = \bigcup\limits_{\gamma \in Y} \operatorname{graph}(\gamma)$. By Lemma 14.12, Γ is a graph of a smooth curve, which is the desired maximal integral curve γ_q of X with $\gamma_q(0) = q$. □

Lemma 15.2. Let $f : M \to N$ be a map of manifolds, with $X \in \Gamma(TM)$ and $Y \in \Gamma(TY)$, and $Tf \circ X = Y \circ f$, i.e., X and Y are f-related, then for any integral curve γ of X, $f \circ \gamma$ is an integral curve of Y.

Proof. We have

$$\begin{aligned}\frac{d}{dt}\left.(f \circ \gamma)\right|_t &= T_t(f \circ \gamma)\left(\frac{d}{dt}\right) \\ &= T_{\gamma(t)} f\left(T_t\gamma\left(\frac{d}{dt}\right)\right) \\ &= T_{\gamma(t)} f(X(\gamma(t))) \\ &= Y(f(\gamma(t))) \\ &= Y((f \circ \gamma)(t)).\end{aligned}$$

□

Example 15.3. Let $M = (-1, 1)$, $N = \mathbb{R}$, $f : (-1, 1) \hookrightarrow \mathbb{R}$ be the inclusion. Let $X = \frac{d}{dt}$ and $Y = \frac{d}{dt}$, then

$$\begin{aligned} \gamma : (-1,1) &\to M \\ t &\mapsto t \end{aligned}$$

is a maximal integral curve of X with $\gamma(0) = 0$. Note that it is not a maximal integral curve of Y because $f \circ \gamma$ is not an integral curve of Y that is not maximal.

Example 15.4. Let $M = \mathbb{R}^2$ and $N = \mathbb{R}$, then consider $f(x, y) = x$ with $X = \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$, with $Y(x) = \frac{d}{dx}$, then $\gamma_x(t) = x + t$ is the integral curve of Y with $\gamma_x(0) = x$. it is defined for all $t \in \mathbb{R}$.

To compute integral curves of X, we solve

$$\begin{cases} \dot{x} = 1, x(0) = x_0 \\ \dot{y} = y^2, y(0) = y_0, \end{cases}$$

then $x(t) = x_0 + t$ and $\frac{1}{y}^2 \frac{dy}{dt} = 1$, therefore

$$\int_0^t \frac{1}{y^2} \frac{dy}{dt} dt = \int_0^t dt$$

and so $t = -\left.\frac{1}{y}\right|_0^t = \frac{1}{y_0} - \frac{1}{y(t)}$, hence $y(t) = \frac{y_0}{1-y_0 t}$. Thus, $t \in (-\infty, \frac{1}{y_0})$. That is, the curve runs off to ∞ in finite time.

Definition 15.5. Let X be a vector field on a (Hausdorff) manifold M, and let $\gamma_q : I_q \to M$ be the unique maximal integral curve with $\gamma_q(0) = q$. Let $W = \bigcup_{q \in M} \{q\} \times I_q \subseteq M \times \mathbb{R}$, then the *(local) flow* of X is the map

$$\begin{aligned} \Phi : W &\to M \\ (q, t) &\mapsto \gamma_q(t) \end{aligned}$$

We say Φ is a *global flow* if $W = M \times \mathbb{R}$, and in this case we say X is *complete*.

Theorem 15.6. Let $\Phi : M \to M$ be a flow of a vector field, then

1. $M \times \{0\} \subseteq W$,
2. W is open, and
3. Φ is C^∞.

Proof. See Lee. □

Example 15.7. Let $X = y^2 \frac{d}{dy} \in \Gamma(\mathbb{R})$, then $W = \{(y, t) \in \mathbb{R} \times \mathbb{R} \mid t < \frac{1}{y}$ when $y > 0, t$ arbitrary when $y = 0, t > \frac{1}{y}$ if $y < 0\}$. The flow is $\Phi(y, t) = \frac{y}{1-yt}$.

Lemma 15.8. Let $\Phi : W \to M$ be a local flow of a vector field X, then $\Phi(q, s + t) = \Phi(\Phi(q, s), t)$ whenever both sides are defined.

Remark 15.9. Note that if $s = -t$, then the left-hand side is defined, but the right-hand side is not.

Proof. Fix q and fix s such that $(q, s) \in W$. Consider $\sigma(t) = \Phi(q, s + t) = \gamma_q(s + t)$, and $\tau(t) = \Phi(\Phi(q, s), t) = \gamma_{\Phi(q,s)}(t)$, then $\tau(0) = \Phi(q, s) = \gamma_q(s) = \sigma(0)$. Both $\sigma(t)$ and $\tau(t)$ are integral curves, and that they agree at $t = 0$, then $\sigma(t) = \tau(t)$ for all t in the intersection of their domains of definition. Therefore, the two equations agree whenever both sides are defined. □

Definition 15.10. An (left) *action* of a Lie group G on a manifold M is a C^∞-map

$$\begin{aligned} G \times M &\to M \\ (g, q) &\mapsto g \cdot q \end{aligned}$$

such that

1. $e \cdot q = q$ for all q, and
2. $g_1 \cdot (g_2 \cdot q) = (g_1 g_2) \cdot q$.

Claim 15.11. If X is complete, then its flow is an action of the Lie group $(\mathbb{R}, +, \cdot)$.

Proof. Define $t \cdot q = \Phi(q, t)$, then

$$\begin{aligned} t \cdot (s \cdot q) &= \Phi(\Phi(q, s), t) \\ &= \Phi(q, s + t) \\ &= (t + s) \cdot q \end{aligned}$$

and $0 \cdot q = \Phi(q, 0) = q$. $\square$

Remark 15.12. If we have a group action, we determine the groupoid structure, and therefore we recover the groupoid version of the lemma.

Remark 15.13. For a Lie group G, the multiplication $m : G \times G \to G$ is a left action of G on G, with $e \cdot g = g$ and $a \cdot (b \cdot g) = (a \cdot b) \cdot g$.

Remark 15.14. For any manifold, there exists a group $\mathrm{Diff}(M) = \{f : M \to M \mid f \text{ is a diffeomorphism}\}$, where the operation is function composition, and the identity is the identity map.

Exercise 15.15. An (left) action $G \times M \to M$ of a Lie group G on a manifold M gives rise to a homomorphism

$$\begin{aligned} \rho : G &\to \mathrm{Diff}(M) \\ (\rho(g))(q) &\mapsto g \cdot q \end{aligned}$$

In particular, the multiplication $m : G \times G \to G$ gives rise to

$$\begin{aligned} L : G &\to \mathrm{Diff}(G) \\ a &\mapsto L_a \end{aligned}$$

Definition 15.16. An *abstract local flow* on a manifold M is a C^∞-map $\psi : W \to M$, where W is an open neighborhood of $M \times \{0\}$ in $M \times \mathbb{R}$, so that $\psi(q, 0) = q$ for all $q \in M$ and $\psi(q, s + t) = \psi(\psi(q, s), t)$ whenever both sides are defined.

We will show that any abstract local flow is part of a flow on a vector field.

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Recall. Given a vector field X on a manifold M, we define the flow to be $\Phi : W \to \mathbb{R}$ for some open neighborhood of $M \times \{0\}$ in $M \times \mathbb{R}$. The defining property of Φ would be that for every $q \in M$, $W \cap (\{q\} \times \mathbb{R}) = \{q\} \times I_q$ and $I_q \ni t \mapsto \Phi(q, t)$ is the maximal integral curve of X. We also proved that $\Phi(q, t + s) = \Phi(\Phi(q, t), s)$ for all q, t, s such that both sides are defined.

We say the flow is a global flow if $W = M \times \mathbb{R}$, that is, for all $q \in M$, the maximal integral curve $\gamma_q \in I_q \to M$ of X with $\gamma_q(0) = q$ is defined for all $t \in \mathbb{R}$, i.e., $I_q = \mathbb{R}$.

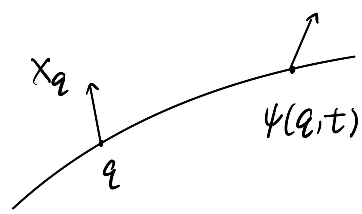
Lemma 16.1. Let M be a manifold, $U \subseteq M \times \mathbb{R}$ be an open neighborhood of $M \times \{0\}$ with $U \cap (\{q\} \times \mathbb{R})$ connected for all $q \in M$, and $\psi : U \to M$ a smooth map such that

1. $\psi(q, 0) = q$ for all q, and
2. $\psi(q, s + t) = \psi(\psi(q, s), t)$ whenever both sides are defined,

then there exists a vector field X on M such that for all $q \in M$, the assignment $t \mapsto \psi(q, t)$ is an integral (but not necessarily maximal) curve of X with $\psi(q, 0) = q$.

Proof. For all $q \in M$, we define $X(q) = \frac{d}{dt}\big|_0 \psi(q,t)$, then

$$\begin{aligned}\frac{d}{dt}\bigg|_t \psi(q,t) &= \frac{d}{dt}\bigg|_0 \psi(q,t+s)\\ &= \frac{d}{ds}\bigg|_0 \psi(\psi(q,t),s)\\ &= X(\psi(q,t)).\end{aligned}$$



□

Lemma 16.2. Let $\Phi : W \to M$ be a flow of a vector field X on a manifold M. Suppose there exists $\varepsilon > 0$ such that $M \times [-\varepsilon, \varepsilon] \subseteq W$, then $W = M \times \mathbb{R}$, i.e., the vector field X is complete.

Proof. We want to show that for all $q \in M$, $I_q := \{t \in \mathbb{R} \mid (q,t) \mid W\}$ is $\mathbb{R}$. Since I_q is connected, then it suffices to show that I_q is unbounded. By assumption, $\varphi_\varepsilon(q) := \varphi(q,\varepsilon)$ and $\varphi_{-\varepsilon}(q) := \varphi(q,-\varepsilon)$ are defined for all $q \in M$, since $q = \varphi(q,0) = \varphi(\varphi(q,\varepsilon),-\varepsilon) = \varphi(\varphi(q,-\varepsilon),\varepsilon)$, therefore $(\varphi_\varepsilon)^{-1}$ exists and is just $\varphi_{-\varepsilon}$.

Given $q \in M$, we consider $\mu(t) = \varphi(q, t+\varepsilon) = \gamma_q(\varepsilon + t)$, and it is easy to check that $\mu'(t) = X(\mu(t))$, therefore μ is an integral curve of X with $\mu(0) = \gamma_\varepsilon(\varepsilon)$. Since γ_q is defined on I_q, then μ is defined for all t such that $t + \varepsilon \in I_q$, that is, $t \in I_q - \varepsilon$. Since $\gamma_{\varphi_\varepsilon(q)} : I_{\varphi_\varepsilon(q)} \to M$ is a maximal integral curve of X such that $\gamma_{\varphi_\varepsilon(q)}(0) = \Phi_\varepsilon(q) = \gamma_q(\varepsilon)$, so $I_q - \varepsilon \subseteq I_{\varphi_\varepsilon(q)}$, and similarly $I_q + \varepsilon \subseteq I_{\varphi_{-\varepsilon}(q)}$, therefore $I_{\varphi_\varepsilon(q)} + \varepsilon \subseteq I_{\varphi_{-\varepsilon}}(\varphi_\varepsilon(q)) = I_q$. Therefore, $I_q - \varepsilon = I_{\varphi_\varepsilon(q)}$. By induction, we conclude that for all $n > 0$, $I_q - n\varepsilon = I_{(\varphi_\varepsilon)^n(q)}$. Since $0 \in I_{q'}$ for all q', and $0 \in I_q - n\varepsilon$, so $n\varepsilon \in I_q$ for all $n \in \mathbb{N}$. Similar argument shows that $-n\varepsilon \in I_q$ for all $n \in \mathbb{N}$. That is, I_q is neither bounded above nor bounded below. □

Definition 16.3. The *support* of a vector field $X \in \Gamma(TM)$ is $\operatorname{supp}(X) = \overline{\{q \in M \mid X(q) \neq 0\}}$.

Corollary 16.4. Suppose $X \in \Gamma(TM)$ has compact support, then X is complete: its flow exists for all time.

Proof. Note that $X \equiv 0$ on $M \backslash \operatorname{supp}(X)$, so for all $q \in M \backslash \operatorname{supp}(X)$. Note that $\gamma_q(t) = q$ is the maximal integral curve of X, which exists for all t, so $(M \backslash \operatorname{supp}(X)) \times \mathbb{R} \subseteq W$, which is the domain of the flow φ. Since $\operatorname{supp}(X)$ is compact, then $(\operatorname{supp}(X) \times \{0\}) \subseteq W$ is compact. Since W is open, then by tube lemma, there exists $\varepsilon > 0$ such that $\operatorname{supp}(X) \times (-2\varepsilon, 2\varepsilon) \subseteq W$, hence $\operatorname{supp}(X) \times [-\varepsilon, \varepsilon] \subseteq W$. Therefore,

$$(M \backslash \operatorname{supp}(X)) \times [-\varepsilon, \varepsilon] \subseteq (M \backslash \operatorname{supp}(X)) \times \mathbb{R} \subseteq W,$$

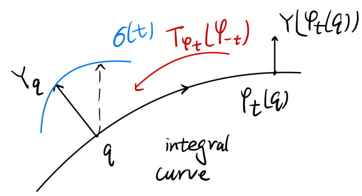
so $M \times [-\varepsilon, \varepsilon] \subseteq W$. Now apply Lemma 16.2. □

We will start talking about Lie derivatives. Let $X, Y \in \Gamma(TM)$ be two vector fields. For simplicity we assume X and Y have global flow $\varphi(q,t) = \varphi_t(q)$, and $\psi(q,t) = \psi_t(q)$, respectively. (It suffices to have the flow maintained for small neighborhood of time.) Fix $q \in M$. Consider

$$\begin{aligned}\sigma : \mathbb{R} &\to T_qM\\ t &\mapsto (T_{\varphi_t(q)}\varphi_{-t})(Y(\varphi_t(q)))\end{aligned}$$

Remark 16.5. For any curve $\gamma : \mathbb{R}$, $\dot{\gamma}(t) \in T_{\gamma(t)}(T_qM) = T_qM$ since γ_qM is a vector space. In particular,

$$\frac{d\sigma}{dt}\bigg|_0 = \frac{d}{dt}\bigg|_0 (T_{\varphi_t(q)}\Phi_{-t}(Y(\varphi_t(q)))) \in T_qM.$$



Definition 16.6. The *Lie derivative* $L_X Y$ of Y with respect to X is defined by

$$(L_X Y)(q) = \left.\frac{d}{dt}\right|_0 T_{\varphi_t(q)}\varphi_{-t}(Y(\varphi_t(q))) = \lim_{t\to 0}\frac{1}{t}\left(T_{\varphi_t(q)}\varphi_{-t}(Y(\varphi_t(q))) - Y_q\right).$$

Theorem 16.7. For any two vector fields $X, Y \in \Gamma(TM)$, $L_X Y = [X, Y]$.

To prove this, we will prove the following.

Lemma 16.8. Let M be a manifold and $\gamma : I \to T_q M$ be a curve. Let $f \in C^\infty(M)$, then

$$\left.\frac{d}{dt}\right|_0 (\gamma(t)f) = \left(\left.\frac{d\gamma}{dt}\right|_0\right) f.$$

Proof. Choose a chart $(x_1, \dots, x_n) : U \to \mathbb{R}^m$ with $q \in U$, then $\gamma(t) = \sum \gamma_i(t) \left.\frac{\partial}{\partial x_i}\right|_q$, where each $\gamma_i : I \to \mathbb{R}$ is C^∞. Now $\left.\frac{d\gamma}{dt}\right|_0 = \sum \gamma_i'(0) \left.\frac{\partial f}{\partial x_i}\right|_q$. We also know that $\gamma(t)f = \sum \gamma_i(t) \left.\frac{\partial f}{\partial x_i}\right|_q$, therefore $\left.\frac{d}{dt}\right|_0 \gamma(t) = \left.\frac{d}{dt}\right|_0 \left(\sum \gamma_i(t) \left.\frac{\partial f}{\partial x_i}\right|_q\right) = \sum \gamma_i'(0) \left.\frac{\partial f}{\partial x_i}\right|_q$ as well. $\square$

Lemma 16.9. Let X and Y be two vector fields with flows $\{\varphi_t\}$ and $\{\psi_t\}$, viewed as family of diffeomorphisms with $\mathbb{R}$-actions. For any $f \in C^\infty(M)$,

$$(L_X Y)(q)f = \left.\frac{\partial^2}{\partial s \partial t}\right|_{(0,0)} (f \circ \varphi_{-t} \circ \psi_s \circ \varphi_t)(q).$$

Proof. We have

$$\begin{aligned}
(L_X Y)(q)f &= \left(\left.\frac{d}{dt}\right|_0 T\varphi_{-t}(Y(\varphi_t(q)))\right) f \\
&= \left.\frac{d}{dt}\right|_0 (T\varphi_{-t}(Y(\varphi_t(q))f)) \\
&= \left.\frac{d}{dt}\right|_0 Y(\varphi_t(q))(f \circ \varphi_{-t}) \\
&= \left.\frac{d}{dt}\right|_0 \left.\frac{\partial}{\partial s}\right|_0 (f \circ \varphi_{-t})(\psi_s(\varphi_t(q))) \\
&= \left.\frac{\partial^2}{\partial t \partial s}\right|_{(0,0)} (f \circ \varphi_{-t} \circ \psi_s \circ \varphi_t)(q) \\
&= \left.\frac{\partial^2}{\partial s \partial t}\right|_{(0,0)} (f \circ \varphi_{-t} \circ \psi_s \circ \varphi_t)(q).
\end{aligned}$$

$\square$

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Recall. Let $X, Y \in \Gamma(TM)$ be two vector fields, and we assume for simplicity that X, Y have global flows $\{\varphi_t\}_{t\in\mathbb{R}}$ and $\{\psi_s\}_{s\in\mathbb{R}}$. We define the Lie derivative $L_X Y$ of Y with respect to X by

$$(L_X Y)(q) = (L_X Y)(q) = \left.\frac{d}{dt}\right|_0 T_{\varphi_t(q)}\varphi_{-t}(Y(\varphi_t(q))).$$

Theorem 17.1. $L_X Y = [X, Y]$.

Proof. It suffices to show that for all $f \in C^\infty(M)$ and all $q \in M$,

$$((L_X Y)(q))f = ([X, Y](q))f = ([X, Y]f)(q).$$

Consider

$$\begin{aligned} H : \mathbb{R}^3 &\to \mathbb{R} \\ (x, y, z) &\mapsto (f \circ \Phi_x \circ \psi_y \circ \Phi_z)(q), \end{aligned}$$

then by Lemma 16.8,

$$((L_X Y)(q))f = \left.\frac{\partial^2}{\partial t\partial s}\right|_{(0,0)} (H(-t, s, t)) = \left.\frac{d}{ds}\right|_{s=0} \left(\left.\frac{\partial}{\partial t}\right|_{(0,s)} H(-t, s, t)\right),$$

and by the chain rule,

$$\left.\frac{\partial}{\partial t}\right|_{(0,s)} H(-t, s, t) = -\frac{\partial H}{\partial x}(0, s, 0) + \frac{\partial H}{\partial z}(0, s, 0).$$

Hence,

$$\begin{aligned} ((L_X Y)(q))f &= \left.\frac{d}{ds}\right|_0 \left(-\left.\frac{\partial}{\partial x}\right|_{(0,s)} (f \circ \varphi_x \circ \psi_s \circ \varphi_0)(q) + \left.\frac{\partial}{\partial z}\right|_{(0,s)} (f \circ \psi_s \circ \varphi_z)(q)\right) \\ &= \left.\frac{d}{ds}\right|_0 \left(-\left.\frac{\partial}{\partial x}\right|_{(0,s)} (f \circ \varphi_x \circ \psi_s \circ \varphi_0)(q) + \left.\frac{\partial}{\partial z}\right|_{(0,s)} (f \circ \psi_s \circ \varphi_z)(q)\right) \\ &= \left.\frac{d}{ds}\right|_0 (-(Xf)(\psi_s(q)) + \left.\frac{d}{dz}\right|_0 (Yf)(\varphi_z(q)) \\ &= (-Y(Xf))(q) + (X(Yf))(q) \\ &= ((XY - YX)f)(q) \\ &= ([X, Y](q))f. \end{aligned}$$

□

Corollary 17.2. Let $X, Y \in \Gamma(TM)$ be two complete vector fields with flows $\{\varphi_t\}_{t\in\mathbb{R}}$, $\{\psi_s\}_{s\in\mathbb{R}}$, then $[X, Y] = 0$ if and only if $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$ for all s and t.

Proof. ($\Leftarrow$): Suppose $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$ for all t, s, then for all $f \in C^\infty(M)$, we have

$$\begin{aligned} ([X, Y]f)(q) &= (L_X Y)(q)f \\ &= \left.\frac{\partial^2}{\partial t\partial s}\right|_{(0,0)} (f \circ \varphi_{-t} \circ \psi_s \circ \varphi_t)(q) \\ &= \left.\frac{\partial^2}{\partial s\partial t}\right|_{(0,0)} (f \circ \psi_s \circ \varphi_{-t} \circ \varphi_t)(q) \\ &= \left.\frac{\partial^2}{\partial s\partial t}\right|_{(0,0)} (f \circ \psi_s)(q) \\ &= 0. \end{aligned}$$

($\Rightarrow$): Suppose $0 = [X, Y] = L_X Y$, consider $\sigma(t) = (T\varphi_{-t})_{\varphi_t(q)}(Y(\varphi_t(q)))$, then we have $\sigma(0) = (T\varphi_0)(Y(q)) = Y(q)$, therefore

$$\begin{aligned}
\sigma'(t) &= \left.\frac{d}{ds}\right|_{s=0} \sigma(t+s) \\
&= \left.\frac{d}{ds}\right|_0 (T\varphi_{-t-s})(Y(\varphi_s(q))) \\
&= \left.\frac{d}{ds}\right|_{s=0} (T\varphi_{-t})(T\varphi_{-s})(Y(\varphi_s(\varphi_t(q)))) \\
&= (T\varphi_{-t})\left(\left.\frac{d}{ds}\right|_0 (T\varphi_{-s})_{\varphi_t(q)}(Y(\varphi_s(\varphi_t(q))))\right) \\
&= (T\varphi_{-t})\left(\left.\frac{d}{ds}\right|_0 (T\varphi_{-s})_{q'}(Y(\varphi_s(q')))\right)
\end{aligned}$$

where $(T\varphi_{-s})(Y(\varphi_s(\varphi_t(q))))$ is a path in $T_{\varphi_t(q)}(M)$. Therefore, the expression is just applying a linear map onto $(L_X Y)(q')$, but this term is now just zero.

Therefore, for all t, we know that

$$Y(q) = \sigma(0) = \sigma(t) = (T\varphi_{-t})_{\varphi_t(q)}(Y(\varphi_t(q)),$$

so $(T\varphi_t)_q(Y(q)) = Y(\varphi_t(q))$, therefore $T\varphi_t \circ Y = Y \circ \varphi_t$, therefore this means Y is φ_t-related to Y, that means for all q, we know $\varphi_t(\psi_s(q)) = \psi_s(\varphi_t(q))$ for all s, t. $\square$

We will now talk about linear algebra a bit. The blanket assumption is that all vector spaces are real and has finite dimensions.

Recall. Given vector spaces $V_1, \ldots, V_n$ and U, we say $f : V_1 \times \cdots \times V_n \to U$ is multi-linear if it is linear in each slot, that is , for all i, the assignment $v \mapsto f(v_1, \ldots, v_{i-1}, v, \ldots, v_n)$ is a linear map.

Example 17.3.

$$\begin{aligned}
\det : (\mathbb{R}^n)^n &\to \mathbb{R} \\
(v_1, \ldots, v_n) &\mapsto \det(v_1, \ldots, v_n)
\end{aligned}$$

is n-linear.

Example 17.4. For any inner product g on a vector space V, the map

$$\begin{aligned}
g : V &\to V \times \mathbb{R} \\
(v_1, v_2) &\mapsto g(v_1, v_2)
\end{aligned}$$

is bilinear.

Example 17.5. If $\mathfrak{g}$ is a Lie algebra, then the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is bilinear.

Notation. We say $\mathrm{Mult}(V_1, \ldots, V_n; U)$ is the set of n-linear maps $f : V_1 \times \cdots \times V_n \to U$.

Fact. $\mathrm{Mult}(V_1, \ldots, V_n; U)$ is an $\mathbb{R}$-vector space.

Lemma 17.6. Let V, W, U be three vector spaces with bases $\{v_i\}$, $\{w_j\}$, and $\{u_k\}$, respectively, and let $\{v_i^*\}$, $\{w_j^*\}$, and $\{u_k^*\}$ be their duals, respectively. We now define

$$\begin{aligned}
\varphi_{ij}^k : V \times W &\to U \\
(v, w) &\mapsto v_i^*(v) \cdot w_j^*(w) \cdot u_k \\
(-, \cdot) &\mapsto v_i^*(-) \cdot w_j^*(\cdot) u_k,
\end{aligned}$$

then $\{\varphi_{ij}^k\}$ is a basis of $\mathrm{Mult}(V, W; U)$.

Proof. Given a bilinear map $b : V \times W \to U$ with $(x, y) \in V \times W$, then

$$\begin{aligned} b(x,y) &= b(\sum v_i^*(x)y_j, \sum w_j^*(y)w_j) \\ &= \sum_{i,j} v_i^*(x)w_j^*(y)b(v_i, w_j) \\ &= \sum_{i,j,k} v_j^*(x)w_j^*(y)u_k^*(b(v_i, w_j))u_k \\ &= \sum_{i,j,k} u_k^*(b(v_i, w_j))\varphi_{ij}^k(x,y), \end{aligned}$$

therefore $\{\varphi_{ij}^k\}$ spans $\mathrm{Mult}(V, W; U)$.

Suppose $\sum\limits_{i,j,k} c_k^{ij}\varphi_{ij}^k = 0$, then for all r, l, we know $\varphi_{ij}^k(v_r, w_l) = v_i^*(v_r)w_j^*(w_l)u_k = \delta_{ir}\delta_{jl}u_k$, so

$$0 = \sum c_k^{ij}\varphi_{ij}^k(v_r, w_l) = \sum_{i,j,k} c_k^{ij}\delta_{ir}\delta_{il}u_k = \sum_k c_k^{rl}u_k.$$

□

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Definition 18.1. Let V and W be two (finite-dimensional) vector spaces over $\mathbb{R}$. The tensor product $V \otimes W$ of V and W is a vector space together with a unique bilinear map

$$\begin{aligned} \otimes : V \times W &\to V \otimes W \\ (v, w) &\mapsto v \otimes w \end{aligned}$$

with the following universal property: for any bilinear map $b : V \times W \to U$, there exists a unique linear map $\bar{b} : V \otimes W \to U$ so that the diagram

$$\begin{tikzcd} V \otimes W \arrow[r, "\bar{b}"] & U \\ V \times W \arrow[u, "\otimes"] \arrow[ur, "b"'] & \end{tikzcd}$$

commutes, i.e., $b(v, w) = \bar{b}(v \otimes w)$ for all $(v, w) \in V \times W$.

Lemma 18.2. For any two vector spaces V and W, the tensor product $V \otimes W$ with respect to $\otimes : V \times W \to V \otimes W$ exists and is unique up to unique isomorphism.

Corollary 18.3. For any three vector spaces U, V, and W, the map

$$\begin{aligned} \varphi : \mathrm{Hom}(V \otimes W, U) &\to \mathrm{Mul}(V, W; U) \\ A &\mapsto \varphi(A) = A \circ \otimes \end{aligned}$$

is an isomorphism of vector spaces.

Proof. The uniqueness follows from the universal property. To prove existence, recall that for any set X, there is a construction of free vector space which has a copy of X as a basis. Define the tensor product to be the categorical product quotiented out by the obvious equivalence relations, given by additions and scalar multiplications, then this gives a tensor product construction over the free vector space. To prove the universal property, write down the canonical mapping, then the bilinear map $b : V \times W \to U$ induces $\bar{b} : F(V \times W) \to U$, then it satisfies the universal property and we are done. □

Lemma 18.4. For any two finite-dimensional vector spaces V and W, then $V \otimes W$ is a finite-dimensional vector space and $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$.

Proof. We know $\mathrm{Hom}(V \otimes W, \mathbb{R}) = \mathrm{Mult}(V, W; \mathbb{R})$, and we know that $\dim(\mathrm{Mult}(V, W; \mathbb{R})) = \dim(V) \cdot \dim(W) \cdot \dim(\mathbb{R})$, therefore $\dim(\mathrm{Hom}(V \otimes W, \mathbb{R})) < \infty$, so $\dim(V \otimes W) < \infty$, and then $\dim(V \otimes W) = \dim(\mathrm{Hom}(V \otimes W, \mathbb{R})) = \dim(V) \cdot \dim(W)$. ☐

Corollary 18.5. If $\{v_i\}_{i=1}^n$ is a basis of V and $\{w_j\}_{j=1}^m$ a basis of W, then $\{v_i \otimes w_j\}$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$ is a basis of $V \otimes W$.

Proof. By construction of the tensor product, we know this set spans $V \otimes W$ already. For any element $x \otimes y \in V \otimes W$, then write down each element with respect to the basis, reorder them, then we get a sum with respect to the given basis $\{v_i \otimes w_j\}$, and we know this spans indeed. Moreover, the dimension matches and we are done. ☐

Lemma 18.6. There exists a unique linear map

$$\begin{aligned} T : V \otimes W &\to W \otimes V \\ v \otimes w &\mapsto w \otimes v \end{aligned}$$

for all $v \in V$ and $w \in W$.

Proof. The uniqueness is easy: this is given by the assignment. To show the existence, consider

$$\begin{aligned} b : V \times W &\to W \otimes V \\ (v, w) &\mapsto w \otimes v \end{aligned}$$

which is a bilinear map and then take the universal property and we are done. ☐

Remark 18.7. T is an isomorphism, and the tensor product $\otimes$ gives rise to a symmetric monoidal category structure on the category of vector spaces.

Lemma 18.8. For any two finite-dimensional vector space V and W, there exists a unique linear map

$$\begin{aligned} \varphi : V^* \otimes W &\to \mathrm{Hom}(V, W) \\ l \otimes w &\mapsto l(-)w. \end{aligned}$$

Proof. Consider the bilinear map

$$\begin{aligned} b : V^* \times W &\to \mathrm{Hom}(V, W) \\ (l, w) &\mapsto l(-)w \end{aligned}$$

then by the universal property φ is the unique linear map as specified above. This is an isomorphism if we check the basis. ☐

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Remark 19.1. The universal property of $\otimes$ can be explained by 1) the universal property over bilinear maps; 2) the universal property over categorical product; 3) the natural bijection between bilinear maps to U and homomorphisms to U.

Remark 19.2. If V and W are finite-dimensional, then there exists a natural transformation

$$\begin{aligned} V^* \otimes W^* &\xrightarrow{\sim} \mathrm{Mult}(V, W; \mathbb{R}) \\ l \otimes \eta &\mapsto l(-)\eta(-) \end{aligned}$$

Remark 19.3. Since $\mathrm{Mult}(V, W; \mathbb{R}) \cong \mathrm{Hom}(V \otimes W, \mathbb{R}) = (V \otimes W)^*$, so $(V \otimes W)^* \cong V^* \otimes W^*$.

Recall. An $\mathbb{R}$-algebra is a vector space A with a bilinear map $\circ : A \times A \to A$. An algebra A is associative if $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in A$.

Definition 19.4. An $(\mathbb{Z}_{\geqslant 0})$-*graded vector space* A is a sequence of vector spaces $\{V_i\}_{i \geqslant 0}$. Equivalently, a graded vector space V is a direct sum $V = \bigoplus_{i=0}^{\infty} V_i$.

Recall.

$$\bigoplus_{i=0}^{\infty} V_i = \{\{v_i\}_{i=0}^{\infty} \mid v_i \in V_i, v_i = 0 \text{ for all but finitely many } i\}.$$

Definition 19.5. A $(\mathbb{Z}_{\geqslant 0})$-*graded algebra* is a graded vector space $A = \bigoplus_{i \geqslant 0} A_i$ together with a bilinear map $\circ : A \times A \to A$ such that for all i, j, $a_i \in A_i$ and $a_j \in A_j$, $a_i \circ a_j \in A_{i+j}$.

We are mostly interested in two types of graded associative algebras:

- the tensor algebra of a vector space V, given by $\mathcal{T}(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$, and
- the Grassmannian/exterior algebra $\bigwedge^*(V) = \bigoplus_{i=0}^{\infty} \bigwedge^k V$.

Definition 19.6. We define the *exterior algebra* as follows: $V^{\otimes 0} = \mathbb{R}$, $V^{\otimes 1} = V$, and $V^{\otimes 2} = V \otimes V$. For $k > 2$, there exists a unique (up to isomorphism) vector space $V^{\otimes k}$ together with a k-linear map

$$\begin{aligned} \otimes^k : V^k &\to V^{\otimes k} \\ (v_1, \dots, v_k) &\mapsto v_1 \otimes v_2 \otimes \cdots \otimes v_k, \end{aligned}$$

so that it satisfies the following universal property, that is, for any vector space U, we have $\operatorname{Hom}(V^{\otimes k}, U) = \operatorname{Mult}(V^k = (V, \dots, V); U)$. To define each of them, we can

- either define it inductively, using the fact that tensor products are associative up to unique isomorphism, or
- we construct it using the free vector space, that is, $V^{\otimes k} = F(V^k)/S$ where S is an appropriate subspace, imitating the construction of the tensor product. Therefore, we want $\bigotimes^k(v_1, \dots, v_k) = \delta_{(v_1,\dots,v_k)} + S \cdots$

Remark 19.7. Consider the tensor product $\mathbb{R}^2 \otimes \mathbb{R}^2$. We have

$$\begin{aligned} (1,1) \otimes (1,-1) &= ((1,0) + (0,1)) \otimes ((1,0) + (0,-1)) \\ &= (1,0) \otimes (1,0) - (0,1) \otimes (0,1) - (1,0) \otimes (0,1) + (0,1) \otimes (1,0) \\ &= \cdots \end{aligned}$$

Definition 19.8. To make $\mathcal{T}(V) = \bigoplus V^{\otimes k}$ into an (associative) algebra, we need bilinear maps $\circ_{k,l} : V^{\otimes k} \times V^{\otimes l} \to V^{\otimes (k+l)}$. We would want

$$(v_1 \otimes \cdots \otimes v_k) \circ_{k,l} (v_{k+1} \otimes \cdots \otimes v_{k+l}) = v_1 \otimes \cdots \otimes v_k \otimes v_{k+1} \otimes \cdots \otimes v_{k+l}.$$

To start with, we take $k, l \geqslant 1$,

$$\begin{aligned} \varphi : V^k \times V^l &\to V^{\otimes (k+l)} \\ ((v_1, \dots, v_k), (v_{k+1}, \dots, v_{k+l})) &\mapsto v_1 \otimes \cdots \otimes v_k \otimes \cdots \otimes v_{k+l}, \end{aligned}$$

then this is a $(k+l)$-linear map. We now fix $(v_{k+1}, \dots, v_{k+l}) \in V^l$, then

$$\begin{aligned} \varphi_{(v_{k+1},\dots,v_{k+l})} : V^k &\to V^{\otimes (k+l)} \\ (v_1, \dots, v_k) &\mapsto v_1 \otimes \cdots \otimes v_k \otimes v_{k+1} \otimes \cdots \otimes v_{k+l} \end{aligned}$$

which is k-linear, then by universality there exists a unique map $\bar{\varphi}_{(v_{k+1},\dots,v_{k+l})} : V^{\otimes k} \to V^{\otimes (k+l)}$, then for any each fixed t in $V^{\otimes k}$, we get a map

$$\begin{aligned} V^l &\to V^{\otimes (k+l)} \\ (v_{k+1}, \dots, v_{k+l}) &\mapsto \bar{\varphi}_{(v_{k+1},\dots,v_{k+l})}(t) \end{aligned}$$

and therefore we get a bilinear map

$$\circ_{k,l} : V^{\otimes k} \times V^{\otimes l} \to V^{\otimes (k+l)}$$

with $(v_1 \otimes \cdots v_k) \circ_{k,l} (v_{k+1}, \ldots, v_{k+l}) = v_1 \otimes \cdots \otimes v_{k+l}$. It now remains to check that for all k, l, m, we have

$$\begin{tikzcd} & V^{\otimes k} \times V^{\otimes l} \times V^{\otimes m} \arrow[dl, "\circ_{k,l} \times \mathrm{id}"'] \arrow[dr, "\mathrm{id} \times \circ_{l,m}"] & \\ V^{\otimes (k+l)} \times V^{\otimes m} \arrow[dr, "\circ_{k+l,m}"'] & & V^{\otimes k} \times V^{\otimes (l+m)} \arrow[dl, "\circ_{k,l+m}"] \\ & V^{\otimes (k+l+m)} & \end{tikzcd}$$

To show this, we just have to check on the generators, since all maps are already well-defined. It is enough to check on generators, given by

$$\begin{tikzcd} & (v_1 \otimes \cdots \otimes v_k, v_{k+1} \otimes \cdots \otimes v_{k+l}, v_{k+l+1} \otimes \cdots \otimes v_{k+l+m}) \arrow[dl] \arrow[dr] & \\ (v_1 \otimes \cdots \otimes v_{k+l}, v_{k+l+1} \otimes \cdots \otimes v_{k+l+m}) \arrow[dr] & & (v_1 \otimes \cdots \otimes v_k, v_{k+1} \otimes \cdots \otimes v_{k+l+m}) \arrow[dl] \\ & v_1 \otimes \cdots \otimes v_{k+l+m} & \end{tikzcd}$$

Therefore, this proves associativity.

Remark 19.9. We can think of TV as an associative algebra freely generated by elements in degree 1, which is just V.

Definition 19.10. The *Grassmannian/exterior algebra* on a vector space V is a graded-commutative associative algebra $\bigwedge^* V = \bigoplus_{k=0}^{\infty} \bigwedge^k V$ with an injective linear map $i : V \hookrightarrow \bigwedge^* V$ so that $\bigwedge^0 V = \mathbb{R}$, $i(V) = \bigwedge^1 V$, that has the following universal property: for any associative algebra A, for all linear map $j : V \to A$ such that $j(v) \cdot j(v) = 0$ for all $v \in V$, there exists a unique map of algebras (i.e., linear map that preserves multiplications) $\bar{j} : \bigwedge^* V \to A$ such that

$$\begin{tikzcd} \bigwedge^* V \arrow[r, dashed, "\exists ! \bar{j}"] & A \\ V \arrow[u, "i"] \arrow[ur, "j"'] & \end{tikzcd}$$

Remark 19.11. The pair $(\bigwedge^* V, i : V \hookrightarrow \bigwedge^* V)$ is unique up to a unique isomorphism.

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Definition 20.1. A graded associative algebra $A = \bigoplus_{k \geqslant 0} A_k$ is *graded-commutative* if for all k, l, $a \in A_k$, $b \in A_l$, then $ab = (-1)^{kl} ba$.

Definition 20.2. Let V be a finite-dimensional vector space, the *Grassmannian/exterior algebra* $\bigwedge^* V = \bigoplus_{k \geqslant 0} \bigwedge^k V$ of V is a graded-commutative algebra freely generated by $\bigwedge^1 V = V$. The term "freely generated" has the following universal property: for any unital associative algebra A and any linear map $j : V \to A$ such that $(j(v))^2 = 0$ for all $v \in V$, then there exists a unique map of algebras $\bar{j} : \bigwedge^* V \to A$ such that the restriction $\bar{j}|_{\bigwedge^1 V = V} = j$. That is, we have a commutative diagram

$$\begin{tikzcd} \bigwedge^* V \arrow[r, dashed, "\exists ! \bar{j}"] & A \\ V \arrow[u, hook] \arrow[ur, "j"'] & \end{tikzcd}$$

Remark 20.3. Analogously, the tensor algebra $\mathcal{T}(V)$ is the associative algebra freely generated by elements in $V^{\otimes 1} = V$.

Remark 20.4.

- Being unital means there exists $1_A \in A$ such that $1_A a = a 1_A = a$ for all $a \in A$.

- $(j(v))^2 = 0$ for all v implies that $j(v_1)j(v_2) = -j(v_2)j(v_1)$ for all $v_1, v_2 \in V$. Indeed, we have
$$\begin{aligned} 0 &= j(v_1+v_2)j(v_1+v_2) \\ &= (j(v_1)+j(v_2))(j(v_1)+j(v_2)) \\ &= (j(v_1))^2 + j(v_2)j(v_1) + j(v_1)j(v_2) + (j(v_2))^2 \\ &= j(v_2)j(v_1) + j(v_1)j(v_2). \end{aligned}$$

Remark 20.5 (Existence of $\bigwedge^* V$)**.** Consider the two-sided ideal I in $\mathcal{T}(V)$ generated by $\{v \otimes v \mid v \in V\}$. Therefore, I is the $\mathbb{R}$-span of elements of the form $a \otimes v \otimes v \otimes b$ where $v \in V$, $a, b \in \mathcal{T}(V)$. Since I is generated by elements of degree 2, then $I = \bigoplus_{k \geqslant 0} I_k$ where $I_k = I \cap V^{\otimes k}$ is a graded ideal of degree k. Note $I_0 = I \cap V^{\otimes 0} = 0$; $I_1 = I \cap V = 0$. We construct $\bigwedge^* V = \mathcal{T}(V)/I$ to be an associative algebra. Denote the multiplication of $\bigwedge^* V$ by $\wedge$ where $(a+I)\wedge(b+I) = a \otimes b + I$ for all $a, b \in V$. In particular, $\bigwedge^k V = V^{\otimes k}/I_k$, and so $\bigwedge^* V = \bigoplus_{k \geqslant 0} \bigwedge^k V$.

Notation. We denote $v_1 \wedge \cdots \wedge v_k := v_1 \otimes \cdots \otimes v_k + I$ for all $v_1, \dots, v_k \in V$. This identifies $v \mapsto v + I$. With this abuse of notation, $v \wedge v + I = 0 + I = 0$. Therefore, $v \wedge w = -w \wedge v$ for all $v, w \in V$, which satisfies graded-commutativity.

Remark 20.6 (Uniqueness of $\bigwedge^* V$)**.** Suppose A is a unital associative algebra, and $j : V \to A$ is a linear map with $(j(v))^2 = 0$ for all $v \in V$. Consider
$$\begin{aligned} V^n &\to A \\ (v_1, \dots, v_n) &\mapsto j(v_1) \cdots j(v_n). \end{aligned}$$

This is n-linear, hence gives rise to a unique linear map $\tilde{j}^n : V^{\otimes n} \to A$ with $\tilde{j}^n(v_1 \otimes \cdots \otimes v_n) = j(v_1) \cdots j(v_n)$, hence we get a morphism $\tilde{j} : \bigoplus_{n \geqslant 0} V^{\otimes n} \to A$ of algebras. For all $v \in V$, $\tilde{j}(v \otimes v) = j(v)j(v) = 0$, so there exists a unique $\bar{j} : \bigoplus_{n \geqslant 0} V^{\otimes n}/I \to A$ such that $\bar{j}(v_1 \wedge \cdots \wedge v_n) = j(v_1) \cdots j(v_n)$, by the first isomorphism theorem.

Remark 20.7. Recall in $\bigwedge^* V$ we have $v \wedge w = (-1) w \wedge v$ for $v, w \in V$ since they have degree 1. In general, we have
$$\begin{aligned} (v_1 \wedge \cdots \wedge v_k) \wedge (v_{k+1} \wedge \cdots \wedge v_{k+l}) &= v_1 \wedge \cdots \wedge v_k \wedge v_{k+1} \wedge \cdots \wedge v_{k+l} \\ &= (-1)^k v_{k+1} \wedge v_1 \wedge \cdots \wedge v_k \wedge v_{k+2} \wedge \cdots \wedge v_{k+l} \\ &= (-1)^{kl} (v_{k+1} \wedge \cdots \wedge v_{k+l}) \wedge (v_1 \wedge \cdots \wedge v_k) \end{aligned}$$

and therefore $\bigwedge^* V$ is graded-commutative.

Recall. The permutation group S_n is generated by transpositions $(i\ j)$ for $1 \leqslant i < j \leqslant n$. In fact, it is generated by $(1\ 2), (2\ 3), \cdots, (n-1\ n)$.

Lemma 20.8. Let V be a finite-dimensional vector space and let $n \geqslant 2$, then take $v_1, \dots, v_n \in V$. For any permutation $\sigma \in S_n$, we have $v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)} = (\operatorname{sgn}(\sigma)) v_1 \wedge \cdots \wedge v_n$.

Proof. It suffices to check when $\sigma = (i\ i+1)$, which is obvious. □

Corollary 20.9. Let $v_1, \dots, v_n$ be a basis of a finite-dimensional vector space V, then

1. $\bigwedge^k V = 0$ for $k > n$,
2. elements of kth exterior power $\{v_{i_1} \wedge \cdots \wedge v_{i_k} \mid i_1 < i_2 < \cdots < i_k\}$ spans $\bigwedge^k V$.

Proof. We know $\{v_i \otimes v_j \mid 1 \leqslant i, j \leqslant n\}$ is a basis of $V \otimes V = V^{\otimes 2}$. Proceeding by induction on k, we know $\{v_{i_1} \otimes \cdots \otimes v_{i_k} \mid 1 \leqslant i_1, \dots, i_k \leqslant n\}$ is a basis of $V^{\otimes k}$, therefore $\{v_{i_1} \wedge \cdots \wedge v_{i_k} \mid 1 \leqslant i_1, \dots, i_k \leqslant n\}$ spans $\bigwedge^k V = V^{\otimes n}/I_k$.

If $k > n$, we must have repeated indices in $v_{i_1} \wedge \cdots \wedge v_{i_k}$, therefore this is zero: if we permute the indices, we can ask the two repeated indices stand next to each other, and in particular their wedge is zero, therefore the entire term would be zero. □

We will prove that $\{v_{i_1} \wedge \cdots \wedge v_{i_k} \mid 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n\}$ is a basis of $\bigwedge^k V$. The key is $v_1 \wedge \cdots \wedge v_n \neq 0$.