

HONORS ANALYSIS NOTES

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PRELIMINARIES

This document is the notes based on Professor Monica Visan's teaching at UCLA's 131AH and 131BH in winter and spring 2021. The corresponding textbook is Baby Rudin.

1 LECTURE 1: STATEMENTS

In Rubin's notation, natural numbers start with 1, i.e. $\mathbb{N} = \{1, 2, \dots\}$.

Let A and B be two statements. We use the following notations:

- We write " A " if A is true.
- We write "not A " if A is false.
- We write " A and B " if both A and B are true.
- We write " A or¹ B " if A is true or B is true or both A and B are true.
- We write " $A \Rightarrow B$ " if " A and B " or "not A ". We read this as " A implies B " or "if A then B ". In this case, B is at least as true as A . In particular, A , a false statement A can imply anything.

We usually write shorthand notation " T " and " F " to represent "true" and "false".

Example 1.1. Consider the following statement:

If x is a natural number, i.e. $x \in \mathbb{N} = \{1, 2, 3, \dots\}$, then $x \geq 1$.

In this case, A is the statement " x is a natural number" and B is the statement " $x \geq 1$ ".

¹The notation "or" in mathematics is inclusive. We distinguish it from the exclusive or, usually called "xor", which means "either A or B "

- Taking $x = 3$, we get $T \Rightarrow T$.
- Taking $x = \pi$, we get $F \Rightarrow T$.
- Taking $x = 0$, we get $F \Rightarrow F$.

Example 1.2. Consider the statement:

If a number is less than 10, then it is less than 20.

The statement is of the form “if... then...”, where A is the statement “a number is less than 10”, and B is the statement “it is less than 20”.

- Taking a number 5, we get $T \Rightarrow T$.
- Taking a number 15, we get $F \Rightarrow T$.
- Taking a number 25, we get $F \Rightarrow F$.

We also write “ $A \iff B$ ” if A and B are true together or false together. We read this as “ A is equivalent to B ” or “ A if and only if B ”.

We can now compare these notions in logic to similar ones from set theory. Let X be an ambient space. Let A and B be subsets of X . Then

- $^cA = \{x \in X : x \notin A\}$.
- $A \cap B = \{x \in X : x \in A \text{ and } x \in B\}$.
- $A \cup B = \{x \in X : x \in A \text{ or } x \in B \text{ or } x \in A \cap B\}$.
- $A \subseteq B$ corresponds to $A \Rightarrow B$.
- $A = B$ corresponds to $A \iff B$.

We now can use truth tables to check the statements.

A	B	not A	A and B	A or B	$A \Rightarrow B$	$A \iff B$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Example 1.3. We can use the truth table to show that $A \Rightarrow B$ is logically equivalent to (not A) or B . Indeed, by considering the following truth table,

A	B	$A \Rightarrow B$	not A	(not A) or B
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

we realize that the column of $A \Rightarrow B$ and (not A) or B are the same.

Exercise 1.4. Use the truth table to prove De Morgan's laws:

$$\text{not } (A \text{ and } B) = (\text{not } A) \text{ or } (\text{not } B)$$

$$\text{not } (A \text{ or } B) = (\text{not } A) \text{ and } (\text{not } B)$$

One can compare these statements to

$$^c(A \cap B) = ^cA \cup ^cB$$

$$^c(A \cup B) = ^cA \cap ^cB$$

Example 1.5. Negate the following statement:

If A then B .

Note that the negation is “not $(A \Rightarrow B)$ ”, then it is equivalent to not $((\text{not } A) \text{ or } B)$, which is equivalent to $[\text{not}(\text{not } A)]$ and $(\text{not } B)$, and that is just A and $(\text{not } B)$.

Therefore, the negation is “ A is true and B is false”.

Example 1.6. Negate the following statement:

If I speak in front of the class, I am nervous.

That would be I speak in front of the class and I am not nervous.

We now introduce quantifiers.

- \forall reads “for all ” or “for any”.
- \exists reads “there is” or “there exists”.
- The negation of “ $\forall A, B$ is true” is “ $\exists A$ such that B is false”.
- The negation of “ $\exists A$ such that B is true” is “ $\forall A, B$ is false”.

Example 1.7. Negate the following:

Every student had coffee or is late for class.

This statement is represented as

$$\forall \text{ student } (\text{had coffee}) \text{ or } (\text{is late for this})$$

and so the negation would be

$$\exists \text{ student such that not (had coffee) and not (is late for class)}$$

Writing this out, we get “there is a student that did not have coffee and is not late for class”.

2 LECTURE 2: PEANO AXIOM AND MATHEMATICAL INDUCTION

Definition 2.1 (Peano Axiom). The natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ satisfy the Peano axioms:

1. $1 \in \mathbb{N}$.
2. If a number $n \in \mathbb{N}$, then its successor $n + 1 \in \mathbb{N}$.
3. 1 is not the successor of any natural number.
4. If two numbers $n, m \in \mathbb{N}$ are such that they have the same successor, i.e. $n + 1 = m + 1$, then they are the same, i.e. $n = m$.
5. Let $S \subseteq \mathbb{N}$. Assume that S satisfies the following two conditions:
 - (i) $1 \in S$,
 - (ii) and if $n \in S$ then $n + 1 \in S$,
 then $S = \mathbb{N}$.

Axiom number 5 forms the basis for mathematical induction.

Definition 2.2 (Mathematical Induction). Assume we want to prove that a property $P(n)$ holds for all $n \in \mathbb{N}$. Then it suffices to verify two steps:

- Step 1 (Base Step): $P(1)$ holds.
- Step 2 (Inductive Step): If $P(n)$ is true for some $n \geq 1$, then $P(n + 1)$ is true, i.e. $P(n) \Rightarrow P(n + 1) \forall n \geq 1$.

Indeed, if we let

$$S = \{n \in \mathbb{N} : P(n) \text{ holds}\},$$

then Step 1 implies $1 \in S$ and Step 2 implies if $n \in S$ then $n + 1 \in S$. By axiom 5, we deduce that $S = \mathbb{N}$.

Example 2.3. Prove that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad \forall n \in \mathbb{N}.$$

We argue that mathematical induction. For $n \in \mathbb{N}$, let $P(n)$ denote the statement

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Step 1 (Base Step): $P(1)$ is the statement $1^2 = \frac{1 \cdot 2 \cdot 3}{6}$, which is true, so $P(1)$ holds.

Step 2 (Inductive Step): Assume that $P(n)$ holds for some $n \in \mathbb{N}$, we want to show that $P(n+1)$ holds. We know

$$1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

then we have

$$\begin{aligned} 1^2 + \cdots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= (n+1) \left[\frac{n(2n+1)}{6} + n+1 \right] \\ &= (n+1) \cdot \frac{2n^2 + 7n + 6}{6} \\ &= \frac{(n+1) \cdot [2n(n+2) + 3n + 6]}{6} \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \end{aligned}$$

So $P(n+1)$ holds.

Collecting the two steps, we conclude $P(n)$ holds $\forall n \in \mathbb{N}$.

Example 2.4. Prove that $2^n > n^2$ for all $n \geq 5$.

We argue by mathematical induction. For $n \geq 5$, let $P(n)$ denote the statement $2^n > n^2$.

Step 1 (Base Step): $P(5)$ is the statement

$$32 = 2^5 > 5^2 = 25$$

which is true. So $P(5)$ holds.

Step 2 (Inductive Step): Assume $P(n)$ is true for some $n \geq 5$ and we want to prove $P(n+1)$. We know $2^n > n^2$, then

$$\begin{aligned} 2^{n+1} &> 2n^2 \\ &= (n+1)^2 + n^2 - 2n - 1 \\ &= (n+1)^2 + (n-2)^2 - 2 \end{aligned}$$

For $n \geq 5$, we have $(n-1)^2 - 2 \geq 4^2 - 2 = 14 \geq 0$, so we know $2^{n+1} > (n+1)^2$. Therefore, $P(n+1)$ holds.

Collecting the two steps, we conclude $P(n)$ holds $\forall n \geq 5$.

Remark 2.5. Each of the two steps are essential when arguing by induction. Note that $P(1)$ is true. However, our proof of the second step fails if $n = 1$: $(1-1)^2 - 2 = -2 < 0$. Also note that our proof of the second step is valid as soon as

$$(n-1)^2 - 2 \geq 0 \iff (n-1)^2 \geq 2 \iff n-1 \geq 2 \iff n \geq 3.$$

However, $P(3)$ fails.

Example 2.6. Prove by mathematical induction that the number $4^n + 15n - 1$ is divisible by 9 for all $n \geq 1$.

We will argue by induction. For $n \geq 1$, let $P(n)$ denote the statement that “ $4^n + 15n - 1$ is divisible by 9”. We write this as $9 \mid (4^n + 15n - 1)$.

Step 1: $4^1 + 15 \cdot 1 - 1 = 18 = 9 \cdot 2$. This is divisible by 9, so $P(1)$ holds.

Step 2: Assume $P(n)$ is true for some $n \geq 1$, we want to show $P(n+1)$ holds.

$$\begin{aligned} 4^{n+1} + 15(n+1) - 1 &= 4 \cdot (4^n + 15n - 1) - 60n + 4 + 15n + 14 \\ &= 4 \cdot (4^n + 15n - 1) - 45n + 18 \\ &= 4 \cdot (4^n + 15n - 1) - 9 \cdot (5n - 2). \end{aligned}$$

By the inductive hypothesis, $9 \mid (4^n + 15n - 1)$ implies $9 \mid 4 \cdot (4^n + 15n - 1)$. Also we know $9 \mid 9 \cdot (5n - 2)$ since $5n - 2 \in \mathbb{N}$. Therefore, we know $9 \mid [4 \cdot (4^n + 15n - 1) - 9 \cdot (5n - 2)]$. Hence, $9 \mid [4 \cdot (4^n + 15n - 1) - 9 \cdot (5n - 2)]$, so $P(n+1)$ holds.

Collecting the two steps, we conclude $P(n)$ holds $\forall n \in \mathbb{N}$.

Example 2.7. Compute the following sum and then use mathematical induction to prove your answer: for $n \geq 1$,

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)}.$$

Note that $\frac{1}{(2n-1)(2n+1)} = \frac{1}{2} \left[\frac{1}{2n-1} - \frac{1}{2n+1} \right]$ for all $n \geq 1$. So

$$\begin{aligned} \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} &= \frac{1}{2} \left(\frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{2n-1} - \frac{1}{2n+1} \right) \\ &= \frac{1}{2} \cdot \frac{2n}{2n+1} \\ &= \frac{n}{2n+1}. \end{aligned}$$

For $n \geq 1$, let $P(n)$ denote the statement

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.$$

Step 1: $P(1)$ becomes $\frac{1}{1 \cdot 3} = \frac{1}{3}$, which is true. So $P(1)$ holds.

Step 2: Assume $P(n)$ holds for some $n \geq 1$. We want to show $P(n+1)$. We know

$$\frac{1}{1 \cdot 3} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1},$$

and so

$$\begin{aligned} \frac{1}{1 \cdot 3} + \cdots + \frac{1}{(2n+1)(2n+3)} &= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} \\ &= \frac{2n^2 + 3n + 1}{(2n+1)(2n+3)} \\ &= \frac{(n+1)(2n+1)}{(2n+1)(2n+3)} \\ &= \frac{n+1}{2n+3}. \end{aligned}$$

So $P(n+1)$ holds.

Collecting the two steps, we conclude $P(n)$ holds $\forall n \geq 1$.

3 LECTURE 3: EQUIVALENCE RELATION

We now extend \mathbb{N} and construct the set of integers $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n : n \in \mathbb{N}\}$.

Definition 3.1 (Equivalence Relation). An equivalence relation \sim on a non-empty set A satisfies the following three properties:

1. Reflexivity: $a \sim a \ \forall a \in A$.
2. Symmetry: If $a, b \in A$ are such that $a \sim b$, then $b \sim a$.
3. Transitivity: If $a, b, c \in A$ are such that $a \sim b$ and $b \sim c$, then $a \sim c$.

Example 3.2. The equal relation $=$ is an equivalence relation on \mathbb{Z} .

Example 3.3. Let $q \in \mathbb{N}$ and $q > 1$. For $a, b \in \mathbb{Z}$ we write $a \sim b$ if $q \mid (a - b)$. This is an equivalence relation on \mathbb{Z} . Indeed, it suffices to check the three properties:

- Reflexivity: If $a \in \mathbb{Z}$, then $a - a = 0$, which is divisible by q . So $q \mid (a - a)$, by definition we know $a \sim a$.

- Symmetry: Let $a, b \in \mathbb{Z}$ such that $a \sim b$, then by definition we know $q \mid (a - b)$. Therefore, there exists some $k \in \mathbb{Z}$ such that $a - b = kq$, so $b - a = (-k) \cdot q$. Note that $-k \in \mathbb{Z}$, so $q \mid (b - a)$, and by definition we know $b \sim a$.
- Transitivity: Let $a, b, c \in \mathbb{Z}$ such that $a \sim b$ and $b \sim c$. Now $a \sim b$ indicates $q \mid (a - b)$, so there exists $n \in \mathbb{Z}$ such that $a - b = qn$. Similarly there exists $m \in \mathbb{Z}$ such that $b - c = qm$. Therefore, $a - c = q(n + m)$, where $n + m \in \mathbb{Z}$. Therefore, $q \mid (a - c)$, so by definition $a \sim c$.

Definition 3.4 (Equivalence Class). Let \sim denote an equivalence relation on a non-empty set A . The equivalence class of an element $a \in A$ is given by

$$C(a) = \{b \in A : a \sim b\}.$$

Proposition 3.5 (Properties of Equivalence Classes). Let \sim denote an equivalence relation on a non-empty set A . Then

1. $a \in C(a)$ for all $a \in A$.
2. If $a, b \in A$ are such that $a \sim b$, then $C(a) = C(b)$.
3. If $a, b \in A$ are such that $a \not\sim b$, then $C(a) \cap C(b) = \emptyset$.
4. $A = \bigcup_{a \in A} C(a)$.

Proof. 1. By reflexivity, $a \sim a$ for all $a \in A$, then $a \in C(a)$ for all $a \in A$.

2. Assume $a, b \in A$ with $a \sim b$. Let us show $C(a) \subseteq C(b)$. Let $c \in C(a)$ be arbitrary, then $a \sim c$. Because $a \sim b$, by symmetry we have $b \sim a$, then by transitivity we know $b \sim c$, and so $c \in C(b)$. This proves that $C(a) \subseteq C(b)$. A similar argument shows that $C(b) \subseteq C(a)$, and so $C(a) = C(b)$.
3. We argue by contradiction. Assume that $a, b \in A$ are such that $a \not\sim b$, but $C(a) \cap C(b) \neq \emptyset$. Let $c \in C(a) \cap C(b)$, then $c \in C(a)$ and $c \in C(b)$. The first property implies $a \sim c$, and the second property implies $b \sim c$, so $c \sim b$, and therefore by transitivity we have $a \sim b$. This contradicts the hypothesis $a \not\sim b$. Therefore, if $a \not\sim b$, then $C(a) \cap C(b) = \emptyset$.
4. Clearly, as $C(a) \subseteq A$ for all $a \in A$, we get $\bigcup_{a \in A} C(a) \subseteq A$. Then conversely, $A = \bigcup_{a \in A} \{a\} \subseteq \bigcup_{a \in A} C(a)$, and therefore $A = \bigcup_{a \in A} C(a)$.

□

Example 3.6. Take $q = 2$ in our previous example: for $a, b \in \mathbb{Z}$, we write $a \sim b$ if $2 \mid (a - b)$. The equivalence classes are

$$\begin{aligned} C(0) &= \{a \in \mathbb{Z} : 2 \mid (a - 0)\} = \{2n : n \in \mathbb{Z}\} \\ C(1) &= \{a \in \mathbb{Z} : 2 \mid (a - 1)\} = \{2n + 1 : n \in \mathbb{Z}\} \end{aligned}$$

and $\mathbb{Z} = C(0) \cup C(1)$.

Example 3.7. Let $F = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : b \neq 0\}$. If $(a, b), (c, d) \in F$ we write $(a, b) \sim (c, d)$ if $ad = bc$. Then for example, we have $(1, 2) \sim (2, 4) \sim (3, 6) \sim (-4, -8)$.

Lemma 3.8. \sim is an equivalence relation on F .

Proof. We have to check the three properties.

Reflexivity: Fix $(a, b) \in F$. As $ab = ba$, we have $(a, b) \sim (b, a)$.

Symmetry: Let $(a, b), (c, d) \in F$ such that $(a, b) \sim (c, d)$, then by definition we know $ad = bc$, and so $cb = da$, and by definition $(c, d) \sim (a, b)$.

Transitivity: Let $(a, b), (c, d), (e, f) \in F$ such that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Now $(a, b) \sim (c, d)$ implies $ad = bc$, then $adf = bcf$. Similarly, $cfb = deb$. Therefore, $adf = deb$. Now $d(af - be) = 0$, and because $d \neq 0$ by definition, we know $af = be$, and by definition we have $(a, b) \sim (e, f)$ as desired. \square

For $(a, b) \in F$, we denote its equivalence class by $\frac{a}{b}$. We define addition and multiplication of equivalence classes as follows:

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} &= \frac{ad + bc}{bd} \\ \frac{a}{b} \cdot \frac{c}{d} &= \frac{ac}{bd} \end{aligned}$$

We have to check that these operations are well-defined. Specifically, if $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, then we should have

$$\begin{cases} (ad + bc, bd) \sim (a'd' + b'c', b'd') \\ (ac, bd) \sim (a'c', b'd') \end{cases}$$

We now check the first property and left the second property as an exercise to the readers. We want to show $(ad + bc)b'd' = bd(a'd' + b'c')$. We know that $(a, b) \sim (a', b')$, so $ab' = ba'$, and therefore $ab'dd' = badd'$. Similarly we know $(c, d) \sim (c', d')$, so $cd' = dc'$, and therefore $cd'bb' = dc'bb'$. Now we get

$$ab'dd' + cd'bb' = ba'dd' + dc'bb',$$

and so

$$(ad + bc)b'd' = bd(a'd' + b'c').$$

This proves addition is well-defined.

Now the set of rational numbers is exactly the set of equivalence classes on F , i.e.

$$\mathbb{Q} = \left\{ \frac{a}{b} : (a, b) \in F \right\}.$$

4 LECTURE 4: FIELD

Definition 4.1 (Field). A field is a set F with at least two elements equipped with two operations: addition (denoted $+$) and multiplication (denoted \cdot) that satisfies the following:

1. (A1) Closure: if $a, b \in F$, then $a + b \in F$.
2. (A2) Commutativity: if $a, b \in F$, then $a + b = b + a$.
3. (A3) Associativity: if $a, b, c \in F$, then $(a + b) + c = a + (b + c)$.
4. (A4) Identity: $\exists 0 \in F$ such that $a + 0 = 0 + a = a \forall a \in F$.
5. (A5) Inverse: $\forall a \in F, \exists (-a) \in F$ such that $a + (-a) = -a + a = 0$.
6. (M1) Closure: if $a, b \in F$, then $a \cdot b \in F$.
7. (M2) Commutativity: if $a, b \in F$, then $a \cdot b = b \cdot a$.
8. (M3) Associativity: if $a, b, c \in F$, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
9. (M4) Identity: $\exists 1 \in F$ such that $a \cdot 1 = 1 \cdot a = a \forall a \in F$.
10. (M5) Inverse: $\forall a \in F \setminus \{0\}, \exists a^{-1} \in F$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.
11. (D) Distributivity: if $a, b, c \in F$, then $(a + b) \cdot c = a \cdot c + b \cdot c$.

Example 4.2. $(\mathbb{N}, +, \cdot)$ is not a field because (A_4) fails.

Example 4.3. $(\mathbb{Z}, +, \cdot)$ is not a field because (M_5) fails.

Example 4.4. $(\mathbb{Q}, +, \cdot)$ is a field.

Recall $\mathbb{Q} = \{\frac{a}{b} : (a, b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})\}$ where $\frac{a}{b}$ denotes the equivalence class of $(a, b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ with respect to the equivalence relation \sim , where $(a, b) \sim (c, d)$ if and only if $a \cdot d = b \cdot c$. We defined two operations

$$\begin{aligned}\frac{a}{b} + \frac{c}{d} &= \frac{ad + bc}{bd} \\ \frac{a}{b} \cdot \frac{c}{d} &= \frac{ac}{bd}\end{aligned}$$

Then the additive identity $\frac{0}{1}$ is the equivalence class of $(0, 1)$, and the multiplicative identity $\frac{1}{1}$ is the equivalence class of $(1, 1)$.

The additive inverse of $\frac{a}{b} \in \mathbb{Q}$ is given by $\frac{-a}{b}$, and for $\frac{a}{b} \in \mathbb{Q} \setminus \{\frac{0}{1}\}$, the multiplicative inverse is given by $\frac{b}{a}$.

Proposition 4.5. Let $(F, +, \cdot)$ be a field. Then

1. The additive and multiplicative identities are unique.
2. The additive and multiplicative inverses are unique.
3. If $a, b, c \in F$ such that $a + b = a + c$, then $b = c$. In particular, if $a + b = a$, then $b = 0$.
4. If $a, b, c \in F$ such that $a \neq 0$ and $a \cdot b = a \cdot c$, then $b = c$. In particular, if $a \neq 0$ and $a \cdot b = a$, then $b = 1$.
5. $a \cdot 0 = 0 \cdot a = 0 \forall a \in F$.
6. If $a, b \in F$, then $(-a) \cdot b = a \cdot (-b) = -(a \cdot b)$.
7. If $a, b \in F$, then $(-a) \cdot (-b) = a \cdot b$.
8. If $a \cdot b = 0$, then $a = 0$ or $b = 0$.

Proof. 1. We will show the additive identity is unique. Assume $\exists 0, 0' \in F$ such that $a + 0 = 0 + a = a$ and $a + 0' = 0' + a = a$ for all $a \in F$. Take $a = 0'$ in the first equation and $a = 0$ in the second equation yields $0' + 0 = 0'$ and $0' + 0 = 0$, so $0 = 0'$.

2. We will show that the additive inverse is unique. Let $a \in F$. Assume there exists $-a, a' \in F$ such that $-a + a = a + (-a) = 0$ and $a' + a = a + a' = 0$. Because $a' + a = 0$, then $(a' + a) + (-a) = 0 + (-a)$, so $a' + (a + (-a)) = -a$, but that means $a' + 0 = -a$, so $a' = -a$.

3. Assume $a + b = a + c$. Then $-a + (a + b) = -a + (a + c)$. Therefore, $(-a + a) + b = (-a + a) + c$, so $0 + b = 0 + c$, which means $b = c$. So if $a + b = a = a + 0$, then $b = 0$.

4. We have a proof similar as above.
5. $a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$, so $a \cdot 0 = 0$. Similarly, $0 \cdot a = (0 + 0) \cdot a = 0 \cdot a + 0 \cdot a$, we have $0 \cdot a = 0$.
6. $(-a) \cdot b + a \cdot b = (-a + a) \cdot b = 0 \cdot b = 0$, and so $(-a) \cdot b = -(a \cdot b)$. Similarly, we have $a \cdot (-b) = -(a \cdot b)$.
7. $(-a) \cdot (-b) + [-(a \cdot b)] = (-a) \cdot (-b) + (-a) \cdot b = (-a)(-b + b) = (-a) \cdot 0 = 0$. Therefore, $(-a) \cdot (-b) = a \cdot b$.
8. Assume $a \cdot b = 0$. Assume $a \neq 0$, then $\exists a^{-1} \in F$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$. Now because $a \cdot b = 0$, then $a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0$, and so $(a^{-1} \cdot a) \cdot b = 0$, then $1 \cdot b = 0$, so $b = 0$.

□

Definition 4.6 (Order Relation). An order relation $<$ on a non-empty set A satisfies the following properties:

- Trichotomy: If $a, b \in A$, then one and only one of the following statements holds: $a < b$, or $a = b$, or $b < a$.
- Transitivity: If $a, b, c \in A$ such that $a < b$ and $b < c$, then $a < c$.

Example 4.7. For $a, b \in \mathbb{Z}$, we write $a < b$ if $b - a \in \mathbb{N}$. This is an order relation.

We write $a > b$ if $b < a$, we write $a \leq b$ if $[a < b \text{ or } a = b]$, and we write $a \geq b$ if $b \leq a$.

Definition 4.8 (Ordered Field). Let $(F, +, \cdot)$ be a field. We say $(F, +, \cdot)$ is an ordered field if it is equipped with an order relation $<$ that satisfies the following:

- (O1): If $a, b, c \in F$ such that $a < b$, then $a + c < b + c$.
- (O2): If $a, b, c \in F$ such that $a < b$ and $0 < c$, then $a \cdot c < b \cdot c$.

5 LECTURE 5: ORDERED FIELD

Proposition 5.1. Let $(F, +, \cdot, <)$ be an ordered field. Then,

1. $a > 0 \iff -a < 0$.
2. if $a, b, c \in F$ are such that $a < b$ and $c < 0$, then $a \cdot c > b \cdot c$.
3. if $a \in F \setminus \{0\}$, then $a^2 = a \cdot a > 0$. In particular, $1 > 0$.

4. if $a, b \in F$ are such that $0 < a < b$, then $0 < b^{-1} < a^{-1}$.

Proof. 1. (\Rightarrow): assume $a > 0$, then $a + (-a) > 0 + (-a)$, so $0 > -a$.

(\Leftarrow): assume $-a < 0$, then $-a + a < 0 + a$, then $0 < a$.

2. Assume $a < b$ and $c < 0$, then $-c > 0$, so $a \cdot (-c) < b \cdot (-c)$, which means $-a \cdot c < -b \cdot c$. Therefore, $-ac + (ac + bc) < -bc + (ac + bc)$. We then see $(-ac + ac) + bc < -bc + (bc + ac)$, so $0 + bc < (-bc + bc) + ac$, and so $bc < 0 + ac$, which means $bc < ac$.

3. By trichotomy, exactly one of the following holds:

- if $a > 0$, then $a \cdot a > 0 \cdot a$, so $a^2 > 0$.
- if $a < 0$, then $a \cdot a > 0 \cdot a$, so $a^2 > 0$.

4. First we show that if $a > 0$ then $a^{-1} > 0$. Let us argue by contradiction. Assume $\exists a \in F$ such that $a > 0$ but $a^{-1} \leq 0$. Note $a^{-1} \neq 0$ since a^{-1} has a multiplicative inverse a . Since $a > 0$ and $a^{-1} < 0$, then $a \cdot a^{-1} < 0$, so $1 < 0$. This contradicts the previous part. So if $a > 0$, then $a^{-1} > 0$. Because $0 < a < b$, then $0 \cdot (a^{-1} \cdot b^{-1}) < a \cdot (a^{-1} \cdot b^{-1}) < b \cdot (a^{-1} \cdot b^{-1})$, and so $0 < (a \cdot a^{-1}) \cdot b^{-1} < b \cdot (b^{-1} \cdot a^{-1})$, therefore $0 < 1 \cdot b^{-1} < (b \cdot b^{-1}) \cdot a^{-1}$. Then we have $0 < b^{-1} < 1 \cdot a^{-1}$, therefore $0 < b^{-1} < a^{-1}$. \square

Theorem 5.2. Let $(F, +, \cdot)$ be a field. The following are equivalent:

1. F is an ordered field.
2. There exists $P \subseteq F$ that satisfies the following properties:
 - ($O1'$): For every $a \in F$, one and only one of the following statements holds: $a \in P$, or $a = 0$, or $-a \in P$.
 - ($O2'$): If $a, b \in P$, then $a + b \in P$, and $a \cdot b \in P$.

Proof. Let us show that (1) \Rightarrow (2). Define $P = \{a \in F : a > 0\}$. Let us check ($O1'$). Fix $a \in F$. By trichotomy for the order relation on F , we get that exactly one of the following statements is true: $a > 0$, which implies $a \in P$, or $a = 0$, or $a < 0$, which implies $-a > 0$, so $-a \in P$. We can now check ($O2'$). Fix $a, b \in P$. Because $a \in P$, then $a > 0$, and similarly $b > 0$. Therefore, $a + b > 0 + b = b > 0$, so $a + b \in P$. Also, we know $a \cdot b > 0 \cdot b = 0$, so $a \cdot b \in P$.

We now show that (2) \Rightarrow (1). For $a, b \in F$, we write $a < b$ if $b - a \in P$. Let us check that this is an order relation.

Trichotomy: fix $a, b \in F$. By $(O1')$, exactly one of the following hold: $b - a \in P$, which means $a < b$, or $b - a = 0$, which means $a = b$, or $-(b - a) \in P$, which means $a - b \in P$ and so $b < a$.

Transitivity: assume $a, b, c \in F$ such that $a < b$ and $b < c$. Therefore, $b - a \in P$ and $c - b \in P$, so $(b - a) + (c - b) = c - a \in P$, and so $a < c$.

We now check that with this order relation, F is an ordered field. We have to check $(O1)$ and $(O2)$.

$(O1)$: fix $a, b, c \in F$ such that $a < b$, then $b - a \in P$, so $(b + c) - (a + c) \in P$, which means $a + c < b + c$.

$(O2)$: fix $a, b, c \in F$ such that $a < b$ and $0 < c$. Because $a < b$, then $b - a \in P$, and because $0 < c$, then $c - 0 = c \in P$. Therefore, $(b - a) \cdot c \in P$, and so $b \cdot c - a \cdot c \in P$, therefore $a \cdot c < b \cdot c$. \square

We extend the order relation $<$ from \mathbb{Z} to the field $(\mathbb{Q}, +, \cdot)$ by writing $\frac{a}{b} > 0$ if $a \cdot b > 0$.

Let us show that this is well-defined. Specifically, we need to show that if $\frac{a}{b} = \frac{c}{d}$, i.e. $(a, b) \sim (c, d)$, and $a \cdot b > 0$, then $c \cdot d > 0$. Now if $(a, b) \sim (c, d)$, then $a \cdot d = b \cdot c$, so $0 < (ad)^2 = (a \cdot b) \cdot (c \cdot d)$.² Therefore, $0 < (ab) \cdot (cd)$ and because $0 < ab$, so $cd > 0$, and therefore $\frac{c}{d} > 0$.

Let $P = \{\frac{a}{b} \in \mathbb{Q} : \frac{a}{b} > 0\}$. By the theorem, to prove that \mathbb{Q} is an ordered field, it suffices to show that P satisfies $(O1')$ and $(O2')$, which is left as an exercise to the readers.

6 LECTURE 6: BOUNDS

Definition 6.1. Let $(F, +, \cdot, <)$ be an ordered field. Let $\emptyset \neq A \subseteq F$.

- We say that A is bounded above if $\exists M \in F$ such that $a \leq M \forall a \in A$. Then M is called an upper bound for A . If moreover, $M \in A$, then we say that M is the maximum of A .
- We say that A is bounded below if $\exists m \in F$ such that $m \leq a \forall a \in A$. Then m is called a lower bound for A . If moreover, $m \in A$, then we say that m is the minimum of A .
- We say that A is bounded if A is bounded both above and below.

Example 6.2. • $A = \{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\}$ is a bounded set. 3 is an upper bound for A , $\frac{3}{2}$ is the maximum of A , 0 is a lower bound for A , and 0 is the minimum of A .

²Note that $a \cdot d \neq 0$ since $d \neq 0$ and $a \cdot b > 0$, and so $a \neq 0$.

- $A = \{x \in \mathbb{Q} : 0 < x^4 \leq 16\}$ is a bounded set. 2 is the maximum of A , and -2 is the minimum of A .
- $A = \{x \in \mathbb{Q} : x^2 < 2\}$ is a bounded set. 2 is an upper bound for A , and -2 is a lower bound for A . But A does not have a maximum. Indeed, let $x \in A$. We will construct $y \in A$ such that $y > x$.

Define $y = x + \frac{2-x^2}{2+x}$. Because $x \in A$, then $x \in \mathbb{Q}$, so $2 - x^2, 2 + x \in \mathbb{Q}$. Moreover, because $x \in A$, then $2 + x > 0$, and so $\frac{1}{2+x} \in \mathbb{Q}$. Therefore, $\frac{2-x^2}{2+x} \in \mathbb{Q}$. Hence, we know $y \in \mathbb{Q}$.

Also note that $2 - x^2 > 0$ since $x \in A$, and $2 + x > 0$ indicates $\frac{1}{2+x} > 0$, so $\frac{2-x^2}{2+x} > 0$. Therefore, $y = x + \frac{2-x^2}{2+x} > x$.

Let us compute y^2 . Note that

$$\begin{aligned}
 y^2 &= \frac{2x + x^2 + 2 - x^2}{2 + x} \\
 &= \frac{4(x+1)^2}{(2+x)^2} \\
 &= \frac{4x^2 + 8x + 4}{x^2 + 4x + 4} \\
 &= \frac{2(x^2 + 4x + 4) + 2x^2 - 4}{x^2 + 4x + 4} \\
 &= 2 + \frac{2 \cdot (x-2)}{(x+2)^2} \\
 &< 2.
 \end{aligned}$$

Collecting the properties above, we constructed $y \in A$ and $y > x$ as desired.

Exercise 6.3. Show that the maximum and minimum of a set are unique, if they exist.

Definition 6.4. Let $(F, +, \cdot, <)$ be an ordered field. Let $\emptyset \neq A \subseteq F$ and assume A is bounded above. We say that L is the least upper bound of A if it satisfies:

1. L is an upper bound of A .
2. If M is an upper bound of A , then $L \leq M$.

We write $L = \sup(A)$ and we say L is the supremum of A .

Lemma 6.5. The least upper bound of a set is unique, if it exists.

Proof. Say that a set A , satisfies $\emptyset \neq A \subseteq F$ and is bounded above, admits two least upper bounds L and M . Because L is a least upper bound, then L is an upper bound for A . But because M is a least upper bound for A , we have $M \leq L$. Similarly we conclude that $L \leq M$, and so $L = M$. \square

Definition 6.6. Let $(F, +, \cdot, <)$ be an ordered field. Let $\emptyset \neq A \subseteq F$ and assume A is bounded below. We say that l is the greatest lower bound of A if it satisfies:

1. l is a lower bound of A .
2. If m is a lower bound of A then $m \leq l$.

We write $l = \inf(A)$ and we say l is the infimum of A .

Exercise 6.7. Show that the greatest lower bound of a set is unique, if it exists.

Definition 6.8. Let $(F, +, \cdot, <)$ be an ordered field. Let $\emptyset \neq S \subseteq F$.

We say that S has the least upper bound property if it satisfies the following: for any non-empty subset A of S that is bounded above, there exists a least upper bound of A and $\sup(A) \in S$.

We say that S has the greatest lower bound property if it satisfies the following: $\forall \emptyset \neq A \subseteq S$ with A bounded below, $\exists \inf(A) \in S$.

Example 6.9. $(\mathbb{Q}, +, \cdot, <)$ is an ordered field. Note that

1. Consider $\emptyset \neq A \subseteq \mathbb{Q}$, \mathbb{N} has the least upper bound property. Indeed, if $\emptyset \neq A \subseteq \mathbb{N}$, A bounded above, then the largest element in A is the least upper bound of A and $\sup(A) \in \mathbb{N}$. \mathbb{N} also has the greatest lower bound property.
2. Consider $\emptyset \neq A \subseteq \mathbb{Q}$, but \mathbb{Q} does not have the least upper bound property. Indeed, $\emptyset \neq A = \{x \in \mathbb{Q} : x \geq 0, x^2 < 2\} \subseteq \mathbb{Q}$. Note that A is bounded above by 2. However, $\sup(A) = \sqrt{2} \notin \mathbb{Q}$.

Proposition 6.10. Let $(F, +, \cdot, <)$ be an ordered field. Then F has the least upper bound property if and only if it has the greatest lower bound property.

Proof. We will only prove the (\Rightarrow) direction: the opposite direction has a similar proof.

Assume F has the least upper bound property. Let $\emptyset \neq A \subseteq F$ bounded below. We want to show that $\exists \inf(A) \in F$. Because A is bounded below, then $\exists m \in F$ such that $m \leq a \forall a \in A$. Let $B = \{b \in F : b \text{ is a lower bound for } A\}$. Note $B \neq \emptyset$ because $m \in B$, and we know $B \subseteq F$, and B is bounded above (in fact, every element in A is an upper bound for B), and F has the least upper bound property. Therefore, $\exists \sup(B) \in F$.

Claim 6.11. $\sup(B)$ is a lower bound for A .

Subproof. Indeed, let $a \in A$. We know $a \geq b \forall b \in B$, and $\sup(B)$ is the least upper bound for B , so $a \geq \sup(B)$. As $a \in A$ was arbitrary, we conclude that $\sup(B) \leq a \forall a \in A$, and so $\sup(B)$ is a lower bound for A . ■

Claim 6.12. If l is a lower bound for A , then $l \leq \sup(B)$.

Subproof. Because l is a lower bound for A , then $l \in B$. Also, because $\sup(B)$ is an upper bound for B , we know $l \leq \sup(B)$. ■

Using the two claims above, we find that $\inf(A) = \sup(B)$. □

7 LECTURE 7: ARCHIMEDEAN PROPERTY

We present an alternative proof of [Proposition 6.10](#).

Remark 7.1 (Alternative Proof). Let $\emptyset \neq A \subseteq F$ be such that A is bounded below. Let $B = \{-a : a \in A\}$. Note $B \subseteq F$ by (A5), and $B \neq \emptyset$ because $A \neq \emptyset$, and B is bounded above: indeed, if m is a lower bound for A , then $-m$ is an upper bound for B .³ Also note that F has the least upper bound property. Collecting these properties above, we know $\exists \sup(B) \in F$. The reader can easily show that $-\sup(B) = \inf(A) \in F$.

Theorem 7.2. There exists an ordered field with the least upper bound property. We denote it \mathbb{R} and we call it the set of real numbers. \mathbb{R} contains \mathbb{Q} as a subfield. (We will prove this statement in [Theorem 8.4](#).) Moreover, we have the following uniqueness property: if $(F, +, \cdot, <)$ is an ordered field with the least upper bound property, then F is order isomorphic with \mathbb{R} , that is, there exist a bijection $\varphi : \mathbb{R} \rightarrow F$ such that

- (i) $\varphi(x + y) = \varphi(x) + \varphi(y)$.
- (ii) $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$.
- (iii) if $x < y$, then $\varphi(x) < \varphi(y)$.

Theorem 7.3. \mathbb{R} has the Archimedean property, that is, $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $x < n$.

Proof. We argue by contradiction. Assume $\exists x_0 \in \mathbb{R}$ such that $x_0 \geq n \forall n \in \mathbb{N}$. Then we know $\emptyset \neq \mathbb{N} \subseteq \mathbb{R}$, \mathbb{N} is bounded above by x_0 , and \mathbb{R} has the least upper bound property. Therefore, $\exists L = \sup(\mathbb{N}) \in \mathbb{R}$.

³Note that $m \leq a \forall a \in A$ implies $-m \geq -a \forall a \in A$.

Now we know $L = \sup(\mathbb{N})$ and $L - 1 < L$, so $L - 1$ is not an upper bound for \mathbb{N} . That means $\exists n_0 \in \mathbb{N}$ such that $n_0 > L - 1$, so $\sup(\mathbb{N}) = L < n_0 + 1 \in \mathbb{N}$. We therefore have a contradiction. \square

Remark 7.4. \mathbb{Q} has the Archimedean property. If $r \in \mathbb{Q}$ is such that $r \leq 0$, then choose $n = 1$. If $r \in \mathbb{Q}$ is such that $r > 0$, then write $r = \frac{p}{q}$ for $p, q \in \mathbb{N}$, and we can choose $n = p + 1$ since $\frac{p}{q} < p + 1$.

Corollary 7.5. If $a, b \in \mathbb{R}$ are such that $a > 0, b > 0$, then there exists $n \in \mathbb{N}$ such that $n \cdot a > b$.

Proof. Apply the Archimedean property to $x = \frac{b}{a}$. \square

Corollary 7.6. If $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.

Proof. Apply the Archimedean property to $x = \frac{1}{\varepsilon}$. \square

Lemma 7.7. For any $a \in \mathbb{R}$ there exists $N \in \mathbb{Z}$ such that $N \leq a < N + 1$.

Proof. If $a = 0$, then we can just take $N = 0$.

If $a > 0$. Consider $A = \{n \in \mathbb{Z} : n \leq a\} \subseteq \mathbb{R}$. Obviously $A \neq \emptyset$, as $0 \in A$. We also know A is bounded above by a , and \mathbb{R} has the least upper bound property. Therefore, there exists $L = \sup(A) \in \mathbb{R}$. Now consider $L - 1 < L = \sup(A)$, then $L - 1$ is not an upper bound for A , so there exists $N \in A$ such that $L - 1 < N$, and so $L < N + 1$. But $L = \sup(A)$, so $N + 1 \notin A$. Therefore, $N \in A$, so $N \leq a$, and as $N + 1 \notin A$, then $N + 1 > a$. Therefore, $N \leq a < N + 1$.

If $a < 0$, then $-a > 0$. Then by the case $a > 0$, $\exists n \in \mathbb{Z}$ such that $n \leq -a < n + 1$, so $-n - 1 < a \leq -n$. If $a = -n$, let $N = -n$ and so $N \leq a < N + 1$. If $a < -n$, let $N = -n - 1$, and so $N \leq a < N + 1$. Either way, we conclude the proof. \square

Definition 7.8 (Dense). We say that a subset A of \mathbb{R} is dense in \mathbb{R} if for every $x, y \in \mathbb{R}$ such that $x < y$, there exists $a \in A$ such that $x < a < y$.

Lemma 7.9. \mathbb{Q} is dense in \mathbb{R} .

Proof. Let $x, y \in \mathbb{R}$ such that $x < y$. Since $y - x > 0$, by [Corollary 7.6](#), $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < y - x$, so $\frac{1}{n} + x < y$.

Consider $nx \in \mathbb{R}$. By [Lemma 7.7](#), $\exists m \in \mathbb{Z}$ such that $m \leq nx < m + 1$, so $\frac{m}{n} \leq x < \frac{m+1}{n}$. Therefore,

$$x < \frac{m+1}{n} = \frac{m}{n} + \frac{1}{n} \leq x + \frac{1}{n} < y.$$

\square

8 LECTURE 8: CONSTRUCTION OF REAL NUMBERS

Remark 8.1. For any two rational numbers $r_1, r_2 \in \mathbb{Q}$ such that $r_1 < r_2$, there exists $s \in \mathbb{Q}$ such that $r_1 < s < r_2$. Indeed, if $r_1 < 0 < r_2$, then we may take $s = 0 \in \mathbb{Q}$. Assume $0 < r_1 < r_2$, write $r_1 = \frac{a}{b}$ and $r_2 = \frac{c}{d}$ with $a, b, c, d \in \mathbb{N}$. Take $s = \frac{ad+bc}{2bd} \in \mathbb{Q}$. Note $r_1 < s < r_2$:

$$r_1 < s \iff \frac{a}{b} < \frac{ad+bc}{2bd} \iff 2ad < ad+bc \iff ad < bc \iff \frac{a}{b} < \frac{c}{d} \iff r_1 < r_2.$$

We leave the construction of s in the remaining cases as an exercise to the readers.

Lemma 8.2. $\mathbb{R} \setminus \mathbb{Q}$ is dense in \mathbb{R} .

Proof. Let $x, y \in \mathbb{R}$ such that $x < y$, then $x + \sqrt{2} < y + \sqrt{2}$. Because we know \mathbb{Q} is dense in \mathbb{R} , we know $\exists q \in \mathbb{Q}$ such that $x + \sqrt{2} < q < y + \sqrt{2}$, so $x < q - \sqrt{2} < y$. It now suffices to prove the following claim.

Claim 8.3. $q - \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.

Subproof. Otherwise, $\exists r \in \mathbb{Q}$ such that $q - \sqrt{2} = r$, so $\sqrt{2} = q - r \in \mathbb{Q}$, contradiction. ■

□

Theorem 8.4. There exists an ordered field with the least upper bound property. We denote it \mathbb{R} and call it the set of real numbers. \mathbb{R} contains \mathbb{Q} as a subfield.

Remark 8.5. The rest of the statement in [Theorem 7.2](#) is left as an exercise for the readers.

Proof. We will construct an ordered field with the least upper bound property using Dedekind cuts.

The element of the field are certain subsets of \mathbb{Q} called cuts.

Definition 8.6 (Cut). A cut is a set $\alpha \subseteq \mathbb{Q}$ that satisfies

- (i) $\emptyset \neq \alpha \neq \mathbb{Q}$,
- (ii) if $q \in \alpha$ and $p \in \mathbb{Q}$ such that $p < q$, then $p \in \alpha$.
- (iii) for every $q \in \alpha$, there exists $r \in \alpha$ such that $r > q$, i.e. α has no maximum.

Intuitively, we think of a cut as $\mathbb{Q} \cap (-\infty, a)$.⁴ Note that if $\mathbb{Q} \ni q \notin \alpha$, then $q > p$ for all $p \in \alpha$. Indeed, otherwise, if $\exists p_0 \in \alpha$ such that $q \leq p_0$, then by (ii) we would have $q \in \alpha$, contradiction.

⁴Of course, at this point we have not yet constructed \mathbb{R} .

We define

$$F = \{\alpha : \alpha \text{ is a cut}\}$$

and we will show that F is an ordered field with the least upper bound property.

Subproof on Order. We first show that there is an order relation on F . For $\alpha, \beta \in F$, we write $\alpha < \beta$ if α is a proper subset of β , i.e. $\alpha \subsetneq \beta$.

- Transitivity: if $\alpha, \beta, \gamma \in F$ are such that $\alpha < \beta$ and $\beta < \gamma$, then $\alpha \subsetneq \beta \subsetneq \gamma$, and so $\alpha \subsetneq \gamma$, so $\alpha < \gamma$.
- Trichotomy: first note that at most one of the following holds: $\alpha < \beta$, or $\alpha = \beta$, or $\beta < \alpha$.

To prove trichotomy, it thus suffices to show that at least one of the following holds: $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$. We show this by contradiction. Assume that $\alpha < \beta$, $\alpha = \beta$, $\beta < \alpha$ all fail. Then we know that α is not a proper subset of β , $\alpha \neq \beta$, and β is not a proper subset of α , which means $\exists p \in \alpha \setminus \beta$ and $\exists q \in \beta \setminus \alpha$. Therefore, $p > r$ for all $r \in \beta$ and $q > s$ for all $s \in \alpha$. Taking $r = q$ and $s = p$, we get $p > q > p$, which is a contradiction.

Therefore, $<$ defines an order relation on F . ■

We now show that F has the least upper bound property. Let $\emptyset \neq A \subseteq F$ be bounded above by $\beta \in F$. Define $\gamma = \bigcup_{\alpha \in A} \alpha$.

Claim 8.7. $\gamma \in F$.

Subproof of Claim. • $\gamma \neq \emptyset$ because $A \neq \emptyset$ and $\emptyset \neq \alpha \in A$.

- β being an upper bound for A indicates $\beta \geq \alpha$ for all $\alpha \in A$, and so $\beta \supseteq \alpha$ for all $\alpha \in A$, and therefore $\beta \supseteq \bigcup_{\alpha \in A} \alpha = \gamma$, but since $\beta \neq \mathbb{Q}$, we know that $\gamma \neq \mathbb{Q}$.
- Let $q \in \gamma$ and let $p \in \mathbb{Q}$ such that $p < q$. As $q \in \gamma$, we know $\exists \alpha \in A$ such that $q \in \alpha$. We also know that $\mathbb{Q} \ni p < q$, so $p \in \alpha$ and therefore $p \in \gamma$.
- Consider $q \in \gamma$, then there exists $\alpha \in A$ such that $q \in \alpha$, which means that there exists $r \in \alpha$ such that $q < r$, so $r \in \gamma$ and $q < r$.

Collecting the properties above, we deduce $\gamma \in F$. ■

Claim 8.8. $\gamma = \sup(A)$.

Subproof of Claim. Note $\alpha \subseteq \gamma$ for all $\alpha \in A$, so $\alpha \leq \gamma$ for all $\alpha \in A$. Therefore, γ is an upper bound for A . Moreover, let δ be an upper bound for A , so $\delta \geq \alpha$ for all $\alpha \in A$, but that means $\delta \supseteq \alpha$ for all $\alpha \in A$, and we can deduce that $\delta \supseteq \bigcup_{\alpha \in A} \alpha = \gamma$. Therefore, $\delta \geq \gamma$. ■

We will continue the proof next time. ■

9 LECTURE 9: CONSTRUCTION OF REAL NUMBERS, CONTINUED

Proof, Continued. We now define addition on the structure F to be

$$\alpha + \beta = \{p + q : p \in \alpha, q \in \beta\}.$$

We now check the axioms and start by (A1), namely, $\alpha + \beta \in F$.

- Note that $\alpha + \beta \neq \emptyset$ because $\alpha \neq \emptyset$ which means $\exists p \in \alpha$, and $\beta \neq \emptyset$, which means $\exists q \in \beta$, and so there exists $p + q \in \alpha + \beta$.
- Note that $\alpha + \beta \neq \mathbb{Q}$. Indeed, $\alpha \neq \mathbb{Q}$, so $\exists r \in \mathbb{Q} \setminus \alpha$, so $r > p$ for all $p \in \alpha$; similarly, because $\beta \neq \mathbb{Q}$, so $\exists s \in \mathbb{Q} \setminus \beta$, so $s > q$ for all $q \in \beta$. Therefore, $r + s > p + q$ for all $p \in \alpha$ and $q \in \beta$, and so $r + s \notin \alpha + \beta$.
- Let $r \in \alpha + \beta$ and $s \in \mathbb{Q}$ such that $s < r$. Because $r \in \alpha + \beta$, we know $r = p + q$ for some $p \in \alpha$ and $q \in \beta$. Because $s < r$, then $s < p + q$, and so $\mathbb{Q} \ni s - p < q \in \beta$, therefore $s - p \in \beta$, which means $s = p + (s - p) \in \alpha + \beta$.
- Let $r \in \alpha + \beta$, and so $r = p + q$ for some $p \in \alpha$ and some $q \in \beta$. Because $\alpha \in F$, so $\exists p' \in \alpha$ such that $p' > p$. Similarly, because $\beta \in F$, so $\exists q' \in \beta$ such that $q' > q$. Therefore, $\alpha + \beta \ni p' + q' > p + q = r$. Therefore, $p' + q' \in \alpha + \beta$ is such that $p' + q' > r$.

Collecting all these properties above, we see that $\alpha + \beta \in F$.

We now check (A2): for $\alpha, \beta \in F$, we have

$$\begin{aligned} \alpha + \beta &= \{p + q : p \in \alpha, q \in \beta\} \\ &= \{q + p : q \in \beta, p \in \alpha\} \\ &= \beta + \alpha. \end{aligned}$$

We now check (A3): for $\alpha, \beta, \gamma \in F$, we have

$$\begin{aligned}
 (\alpha + \beta) + \gamma &= \{s + r : s \in \alpha + \beta, r \in \gamma\} \\
 &= \{(p + q) + r : p \in \alpha, q \in \beta, r \in \gamma\} \\
 &= \{p + (q + r) : p \in \alpha, q \in \beta, r \in \gamma\} \\
 &= \{p + t : p \in \alpha, t \in \beta + \gamma\} \\
 &= \alpha + (\beta + \gamma).
 \end{aligned}$$

We now check (A4): let $0^* = \{q \in \mathbb{Q} : q < 0\}$.

Claim 9.1. $0^* \in F$.

Subproof. • Note $p^* \neq \emptyset$ because $-1 \in 0^*$.

- Note that $0^* \neq \mathbb{Q}$ because $2 \notin 0^*$.
 - Let $q \in 0^*$ and let $p \in \mathbb{Q}$ and $p < q$. Then $q \in 0^*$ implies that $q < 0$, and because $p < q$, then $p < 0$, so $p \in 0^*$.
 - Let $q \in 0^*$, then $q < 0$, so $\exists r \in \mathbb{Q}$ such that $q < r < 0$. Therefore, $r \in 0^*$ and $r > q$.
- Collecting all these properties, we get $0^* \in F$. ■

Claim 9.2. $\alpha + 0^* = \alpha \quad \forall \alpha \in F$.

Proof. • We check $\alpha + 0^* \subseteq \alpha$. Let $r \in \alpha + 0^*$, so $r = p + q$ for some $p \in \alpha$ and some $q \in 0^*$. Therefore, $q < 0$. So we know $\mathbb{Q} \ni r = p + q < p$, and because $p \in \alpha \in F$, so $r \in \alpha$. As r was arbitrary in $\alpha + 0^*$, we find $\alpha + 0^* \subseteq \alpha$.

- We now check $\alpha \subseteq \alpha + 0^*$. Let $p \in \alpha$, so there exists $r \in \alpha$ such that $r > p$. We now write $p = r + (p - r) \in \alpha + 0^*$. As $p \in \alpha$ was arbitrary, this shows that $\alpha \subseteq \alpha + 0^*$.

Collecting the properties above, we get $\alpha + 0^* = \alpha$. ■

We now check (A5): fix $\alpha \in F$. We now define

$$\beta = \{q \in \mathbb{Q} : \exists r \in \mathbb{Q} \text{ with } r > 0 \text{ such that } -q - r \notin \alpha\}.$$

Claim 9.3. $\beta \in F$.

Subproof. • Note that $\beta \neq \emptyset$. As $\alpha \neq \emptyset$, there exists $p \in \mathbb{Q} \setminus \alpha$, then $(-\beta + 1) \in \beta$ because $-[-(p + 1)] - 1 = (p + 1) - 1 = p \notin \alpha$.

- Note that $\beta \neq \emptyset$. As $\alpha \neq \emptyset$, there exists $p \in \alpha$. Then $-p \notin \beta$ because $\forall r \in \mathbb{Q}, r > 0$, we have $-(-p) - r = p - r < p$, and because $p \in \alpha \in F$. Therefore, $p - r \in \alpha$, and so $-p \notin \beta$.
- Let $q \in \beta$ and let $p \in \mathbb{Q}$ such that $p < q$. Because $q \in \beta$, there exists $r \in \mathbb{Q}$ such that $r > 0$ and $-q - r \notin \alpha$, therefore $-q - r > s$ for all $s \in \alpha$. Hence, $-p - r > -q - r > s$ for all $s \in \alpha$, and so $-p - r \notin \alpha$, which means $p \in \beta$.
- Let $q \in \beta$. We want to find $s \in \beta$ such that $s > q$. Because $q \in \beta$, so there exists $r \in \mathbb{Q}$ such that $r > 0$ and $-q - r \notin \alpha$. Therefore, $-(q + \frac{r}{2}) - \frac{r}{2} = -q - r \notin \alpha$, and so $q + \frac{r}{2} \in \beta$. We then define $s = q + \frac{r}{2}$.

Collecting all the properties, we get $\beta \in F$. ■

Claim 9.4. $\alpha + \beta = 0^*$.

Subproof. • We first check $\alpha + \beta \subseteq 0^*$. Let $s \in \alpha + \beta$, then $s = p + q$ with $p \in \alpha$ and $q \in \beta$. Because $q \in \beta$, so there exists $r \in \mathbb{Q}$ with $r > 0$ such that $-q - r \notin \alpha$, so $-q - r > p$, which means $\mathbb{Q} \ni p + q < -r < 0$. Therefore, $s = p + q \in 0^*$, and so $\alpha + \beta \subseteq 0^*$.

- We now check $0^* \subseteq \alpha_\beta$. Let $r \in 0^*$, then $r \in \mathbb{Q}$ and $r < 0$.

Claim 9.5. $\exists N \in \mathbb{N}$ such that $N \cdot (-\frac{r}{2}) \in \alpha$, but $(N + 1)(-\frac{r}{2}) \notin \alpha$.

Subproof. We prove this by contradiction. Assume

$$\{n \cdot (-\frac{r}{2}) : n \in \mathbb{N}\} \subseteq \alpha.$$

We will show that in this case $\mathbb{Q} \subseteq \alpha$ and thus reach a contradiction.

Fix $q \in \mathbb{Q}$. By the Archimedean property for \mathbb{Q} , $\exists n \in \mathbb{N}$ such that $n > q \cdot (-\frac{2}{r}) \in \mathbb{Q}$. Therefore, $n \cdot (-\frac{r}{2}) > q$, and because $n \cdot (-\frac{r}{2}) \in \alpha \in F$, and so $q \in \alpha$. As $q \in \mathbb{Q}$ was arbitrary, this shows $\mathbb{Q} \subseteq \alpha$, contradiction. ■

We now write $r = N(-\frac{r}{2}) + (N + 2) \cdot \frac{r}{2}$, and note that $(N + 2)\frac{r}{2} \in \beta$ since

$$-(N + 2) \cdot \frac{r}{2} - \frac{r}{2} = (N + 1) \cdot (-\frac{r}{2}) \notin \alpha.$$

As $r \in 0^*$ was arbitrary, this shows $0^* \subseteq \alpha_\beta$.

Therefore, $\alpha + \beta = 0^*$. ■

We now check (O1). If $\alpha, \beta, \gamma \in F$ such that $\alpha < \beta$, so $\alpha \subsetneq \beta$, then $\alpha + \gamma \subsetneq \beta + \gamma$, and so $\alpha + \gamma < \beta + \gamma$.

We define multiplication on F as follows: for $\alpha, \beta \in F$ with $\alpha > 0$ and $\beta > 0$, we define

$$\alpha \cdot \beta = \{q \in \mathbb{Q} : q < r \cdot s \text{ for some } 0 < r \in \alpha \text{ and some } 0 < s \in \beta\}.$$

For $\alpha \in F$, we define $\alpha \cdot 0^* = 0^*$. We define

$$\alpha \cdot \beta = \begin{cases} (-\alpha) \cdot (-\beta), & \text{if } \alpha < 0, \beta < 0 \\ -[(-\alpha) \cdot \beta], & \text{if } \alpha < 0, \beta > 0 \\ -[\alpha \cdot (-\beta)], & \text{if } \alpha > 0, \beta < 0 \end{cases}$$

We leave the proof of properties (M1) through (M5), as well as (D) and (O2) as an exercise for the readers. \square

We identify a rational number $r \in \mathbb{Q}$ with the Dedekind cut

$$r^* = \{q \in \mathbb{Q} : q < r\}.$$

One can check that

$$\begin{aligned} r^* + s^* &= (r + s)^* \\ r^* \cdot s^* &= (r \cdot s)^* \\ r < s &\iff r^* < s^* \end{aligned}$$

10 LECTURE 10: SEQUENCES

Definition 10.1 (Sequence). A sequence of real number is a function $f : \{n \in \mathbb{Z} : n \geq m\} \rightarrow \mathbb{R}$ where m is a fixed integer⁵. We write the sequence as $f(m), f(m+1), f(m+2), \dots$ or as $\{f(n)\}_{n \geq m}$ or as $\{f_n\}_{n \geq m}$.

Definition 10.2 (Bounded Sequence). We say that a sequence $\{a_n\}_{n \geq 1}$ of real numbers is bounded below (respectively, bounded above, bounded) if the set $\{a_n : n \geq 1\}$ is bounded below (respectively, bounded above, bounded).

We say that the sequence $\{a_n\}_{n \geq 1}$ is

- (monotonically) increasing if $a_n \leq a_{n+1} \quad \forall n \geq 1$.
- strictly increasing if $a_n < a_{n+1} \quad \forall n \geq 1$.

⁵ m is usually 1 or 0.

- (monotonically) decreasing if $a_n \geq a_{n+1} \quad \forall n \geq 1$.
- strictly decreasing if $a_n > a_{n+1} \quad \forall n \geq 1$.
- monotone if it is either increasing or decreasing.

Example 10.3. 1. $\{a_n\}_{n \geq 1}$ with $a_n = 3 - \frac{1}{n}$ is bounded and strictly increasing.

2. $\{a_n\}_{n \geq 1}$ with $a_n = (-1)^n$ is bounded but not monotone.

3. $\{a_n\}_{n \geq 0}$ with $a_n = n^2$ is bounded below and strictly increasing.

4. $\{a_n\}_{n \geq 0}$ with $a_n = \cos(\frac{n\pi}{3})$ is bounded but not monotone.

To define the notion of convergence of a sequence, we need a notion of distance between two real numbers.

Definition 10.4 (Absolute Value). For $x \in \mathbb{R}$, the absolute value of x is

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

This function satisfies the following:

1. $|x| \geq 0$ for all $x \in \mathbb{R}$.
2. $|x| = 0 \iff x = 0$.
3. $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.⁶
4. $|x \cdot y| = |x| \cdot |y|$ for all $x, y \in \mathbb{R}$.
5. $||x| - |y|| \leq |x - y|$ for all $x, y \in \mathbb{R}$.⁷

We think of $|x - y|$ as the distance between $x, y \in \mathbb{R}$.

Definition 10.5 (Converge, Limit, Diverge). We say that a sequence $\{a_n\}_{n \geq 1}$ of real numbers converges if $\exists a \in \mathbb{R}$ such that $\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}$ such that $|a_n - a| < \varepsilon \quad \forall n \geq n_\varepsilon$.

If this is the case, we say that a is the limit of $\{a_n\}_{n \geq 1}$ and we write $a = \lim_{n \rightarrow \infty} a_n$ or $a_n \xrightarrow[n \rightarrow \infty]{} a$.

If the sequence does not converge, we say it diverges.

Lemma 10.6. The limit of a convergent sequence is unique.

Proof. We argue by contradiction. Assume that $\{a_n\}_{n \geq 1}$ is a convergent sequence and assume that there exists $a, b \in \mathbb{R}$ such that $a \neq b$ and $a = \lim_{n \rightarrow \infty} a_n$ and $b = \lim_{n \rightarrow \infty} a_n$. Let $0 < \varepsilon < \frac{|b-a|}{2}$.⁸

⁶This is known as the triangle inequality.

⁷This is known as the inverse triangle inequality.

⁸We can choose such an ε because \mathbb{Q} is dense in \mathbb{R} .

Because $a = \lim_{n \rightarrow \infty} a_n$, then there exists $n_1(\varepsilon) \in \mathbb{N}$ such that $|a_n - a| < \varepsilon \forall n \geq n_1(\varepsilon)$. Similarly, because $b = \lim_{n \rightarrow \infty} a_n$, then there exists $n_2(\varepsilon) \in \mathbb{N}$ such that $|a_n - b| < \varepsilon \forall n \geq n_2(\varepsilon)$. Now set $n_\varepsilon = \max\{n_1(\varepsilon), n_2(\varepsilon)\}$. Then for $n \geq n_\varepsilon$, we have

$$|b - a| = |b - a_n + a_n - a| \leq |b - a_n| + |a_n - a| < 2\varepsilon < |b - a|.$$

This is a contradiction. □

Example 10.7. We can show that the sequence given by $a_n = \frac{1}{n}$ for all $n \geq 1$ converges to 0.

Let $\varepsilon > 0$. By the Archimedean property, there exists $n_\varepsilon \in \mathbb{N}$ such that $n_\varepsilon > \frac{1}{\varepsilon}$. Then for $n \geq n_\varepsilon$, we have

$$|0 - \frac{1}{n}| = \frac{1}{n} \leq \frac{1}{n_\varepsilon} < \varepsilon.$$

By definition, $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

Example 10.8. We can show that the sequence given by $a_n = (-1)^n$ for all $n \geq 1$ does not converge.

We argue by contradiction. Assume $\exists a \in \mathbb{R}$ such that $a = \lim_{n \rightarrow \infty} (-1)^n$. Let $0 < \varepsilon < 1$. Then $\exists n_\varepsilon \in \mathbb{N}$ such that $|a - (-1)^n| < \varepsilon$ for all $N \geq n_\varepsilon$. By taking $n = 2n_\varepsilon$, we get $|a - 1| < \varepsilon$, and by taking $n = 2n_\varepsilon + 1$, we get $|a + 1| < \varepsilon$. By the triangle inequality,

$$2 = |1 + 1| = |1 - a + a + 1| \leq |1 - a| + |a - 1| < 2\varepsilon < 2.$$

This is a contradiction.

Lemma 10.9. A convergent sequence is bounded.

Proof. Let $\{a_n\}_{n \geq 1}$ be a convergent sequence and let $a = \lim_{n \rightarrow \infty} a_n$. There exists $n_1 \in \mathbb{N}$ such that $|a - a_n| < 1$ for all $n \geq n_1$. So $|a_n| \leq |a_n - a| + |a| < 1 + |a|$ for all $n \geq n_1$. Let $M = \max\{1 + |a|, |a_1|, |a_2|, \dots, |a_{n_1-1}|\}$. Clearly, $|a_n| \leq M$ for all $n \geq 1$, so $\{a_n\}_{n \geq 1}$ is bounded. □

Theorem 10.10. Let $\{a_n\}_{n \geq 1}$ be a convergent sequence and let $a = \lim_{n \rightarrow \infty} a_n$. Then for any $k \in \mathbb{R}$, the sequence $\{ka_n\}_{n \geq 1}$ converges and $\lim_{n \rightarrow \infty} ka_n = ka$.

Proof. If $k = 0$, then $ka_n = 0$ for all $n \geq 1$, and so $\lim_{n \rightarrow \infty} ka_n = 0 = ka$.

If $k \neq 0$, let $\varepsilon > 0$. As $a = \lim_{n \rightarrow \infty} a_n$, there exists $n_{\varepsilon, k} \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{|k|}$ for all $n \geq n_{\varepsilon, k}$. Therefore, $|ka_n - ka| = |k| \cdot |a_n - a| < |k| \cdot \frac{\varepsilon}{|k|} = \varepsilon$ for all $n \geq n_{\varepsilon, k}$. By definition, $\lim_{n \rightarrow \infty} ka_n = ka$. □

Remark 10.11. The idea is that we want to find $n_\varepsilon \in \mathbb{N}$ such that $\forall n \geq n_\varepsilon, |ka_n - ka| < \varepsilon$. But that is equivalent to having $|a_n - a| < \frac{\varepsilon}{|k|}$.

11 LECTURE 11: SEQUENCES, CONTINUED

Theorem 11.1. Let $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$ be two convergent sequences of real numbers and let $a = \lim_{n \rightarrow \infty} a_n$ and $b = \lim_{n \rightarrow \infty} b_n$. Then

1. the sequence $\{a_n + b_n\}_{n \geq 1}$ converges and $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$.
2. the sequence $\{a_n \cdot b_n\}_{n \geq 1}$ converges and $\lim_{n \rightarrow \infty} (a_n b_n) = a \cdot b$.
3. if $a \neq 0$ and $a_n \neq 0$ for all $n \geq 1$, then $\{\frac{1}{a_n}\}_{n \geq 1}$ converges and $\lim_{n \rightarrow \infty} \frac{1}{a_n} = \frac{1}{a}$.
4. if $a \neq 0$ and $a_n \neq 0$ for all $n \geq 1$, then $\{\frac{b_n}{a_n}\}_{n \geq 1}$ converges and $\lim_{n \rightarrow \infty} \frac{b_n}{a_n} = \frac{b}{a}$.

Proof. 1. Let $\varepsilon > 0$. We want to find $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$,

$$|(a + b) - (a_n + b_n)| < \varepsilon.$$

Then it suffices to find large enough n such that $|a - a_n| < \frac{\varepsilon}{2}$ and $|b - b_n| < \frac{\varepsilon}{2}$, which means

$$|(a + b) - (a_n + b_n)| < |a - a_n| + |b - b_n| < \varepsilon.$$

As $\lim_{n \rightarrow \infty} a_n = a$, then there exists $n_1(\varepsilon) \in \mathbb{N}$ such that $|a - a_n| < \frac{\varepsilon}{2}$ for all $n \geq n_1(\varepsilon)$. Similarly, as $\lim_{n \rightarrow \infty} b_n = b$, then there exists $n_2(\varepsilon) \in \mathbb{N}$ such that $|b - b_n| < \frac{\varepsilon}{2}$ for all $n \geq n_2(\varepsilon)$.

Now let $n_\varepsilon = \max\{n_1(\varepsilon), n_2(\varepsilon)\}$. Then for $n \geq n_\varepsilon$, we have

$$|(a + b) - (a_n + b_n)| \leq |a - a_n| + |b - b_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

By definition, $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$.

2. Let $\varepsilon > 0$. As $\{a_n\}_{n \geq 1}$ converges, it is bounded. Let $M > 0$ be such that $|a_n| \leq M$ for all $n \geq 1$.

We want to find $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$, $|ab - a_n b_n| < \varepsilon$. To find such n_ε , it suffices to make it large enough so that $|a - a_n| \cdot |b| < \frac{\varepsilon}{2}$ and $|a_n| \cdot |b - b_n| < \frac{\varepsilon}{2}$, then we know that

$$|ab - a_n b_n| = |(a - a_n) \cdot b + a_n(b - b_n)| \leq |a - a_n| \cdot |b| + |a_n| \cdot |b - b_n| < \varepsilon.$$

To do so, it suffices to take $|a - a_n| < \frac{\varepsilon}{2(|b|+1)}$ and $|b - b_n| < \frac{\varepsilon}{2M}$, where $M > 0$ is such that $|a_n| \leq M$ for all $n \geq 1$.⁹

⁹While the obvious choice for $|b - b_n|$ is to bound it by $\frac{\varepsilon}{|a_n|}$, note that this does not guarantee us to shrink to less than $\frac{\varepsilon}{2}$.

As $\lim_{n \rightarrow \infty} a_n = a$, there exists $n_1(\varepsilon) \in \mathbb{N}$ such that $|a - a_n| < \frac{\varepsilon}{2(|b|+1)}$ for all $n \geq n_1(\varepsilon)$. Similarly, as $\lim_{n \rightarrow \infty} b_n = b$, there exists $n_2(\varepsilon) \in \mathbb{N}$ such that $|b - b_n| < \frac{\varepsilon}{2M}$ for all $n \geq n_2(\varepsilon)$.

Set $n_\varepsilon = \max\{n_1(\varepsilon), n_2(\varepsilon)\}$. For $n \geq n_\varepsilon$, we have

$$\begin{aligned} |ab - a_nb_n| &= |(a - a_n)b + a_n(b - b_n)| \\ &\leq |a - a_n| \cdot |b| + |a_n| \cdot |b - b_n| \\ &< \frac{\varepsilon}{2(|b|+1)} \cdot |b| + M \cdot \frac{\varepsilon}{2M} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

By definition, $\lim_{n \rightarrow \infty} (a_nb_n) = ab$.

3. Let $\varepsilon > 0$. We want to find $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$, $|\frac{1}{a} - \frac{1}{a_n}| < \varepsilon$. Note that

$$\left| \frac{1}{a} - \frac{1}{a_n} \right| = \frac{|a_n - a|}{|a| \cdot |a_n|} < \varepsilon$$

and so we want $|a_n - a| < \varepsilon|a| \cdot |a_n|$.

As $a = \lim_{n \rightarrow \infty} a_n$, there exists $n_1(a) \in \mathbb{N}$ such that $|a - a_n| < \frac{|a|}{2}$ for all $n \geq n_1(a)$. Then, for all $n \geq n_1$, we have

$$|a_n| \geq |a| - |a - a_n| > |a| - \frac{|a|}{2} = \frac{|a|}{2}.$$

Moreover, there exists $n_2(\varepsilon, a) \in \mathbb{N}$ such that $|a - a_n| < \frac{\varepsilon|a|^2}{2}$ for all $n \geq n_2(\varepsilon, a)$.

Now let $n_\varepsilon = \max\{n_1(a), n_2(\varepsilon, a)\}$. For $n \geq n_\varepsilon$, we have

$$\left| \frac{1}{a} - \frac{1}{a_n} \right| = \frac{|a - a_n|}{|a| \cdot |a_n|} < \frac{\varepsilon|a|^2}{2|a|} \cdot \frac{2}{|a|} = \varepsilon.$$

By definition, $\lim_{n \rightarrow \infty} \frac{1}{a_n} = \frac{1}{a}$.

4. We leave this as an exercise.

□

Example 11.2.

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{n^3 + 5n + 8}{3n^3 + 2n^2 + 7} &= \lim_{n \rightarrow \infty} \frac{1 + \frac{5}{n^2} + \frac{8}{n^3}}{3 + \frac{2}{n} + \frac{7}{n^3}} \\
&= \frac{1 + 5 \cdot \lim_{n \rightarrow \infty} \frac{1}{n^2} + 8 \cdot \lim_{n \rightarrow \infty} \frac{1}{n^3}}{3 + 2 \cdot \lim_{n \rightarrow \infty} \frac{1}{n} + 7 \cdot \lim_{n \rightarrow \infty} \frac{1}{n^3}} \\
&= \frac{1 + 5 \cdot 0 + 8 \cdot 0}{3 + 2 \cdot 0 + 7 \cdot 0} \\
&= \frac{1}{3}.
\end{aligned}$$

Theorem 11.3. Every bounded monotone sequence converges.

Proof. We will show that an increasing sequence bounded above converges. A similar argument can be used to show that a decreasing sequence bounded below converges.

Let $\{a_n\}_{n \geq 1}$ be a sequence of real numbers that is bounded above and $a_{n+1} \geq a_n$ for all $n \geq 1$. As $\emptyset \neq \{a_n : n \geq 1\} \subseteq \mathbb{R}$ is bounded above and \mathbb{R} has the least upper bound property, there exists $a \in \mathbb{R}$ such that $a = \sup\{a_n : n \geq 1\}$. It now suffices to prove that this number is the point of convergence we want.

Claim 11.4. $a = \lim_{n \rightarrow \infty} a_n$.

Subproof. Let $\varepsilon > 0$. Then $a - \varepsilon$ is not an upper bound for $\{a_n : n \geq 1\}$. Therefore, there exists $n_\varepsilon \in \mathbb{N}$ such that $a - \varepsilon < a_{n_\varepsilon}$. Therefore, for $n \geq n_\varepsilon$, we have

$$a - \varepsilon < a_{n_\varepsilon} \leq a_n \leq a < a + \varepsilon,$$

which means $|a_n - a| < \varepsilon$. This proves the claim. ■

□

Definition 11.5. Let $\{a_n\}_{n \geq 1}$ be a sequence of real numbers.

We write $\lim_{n \rightarrow \infty} a_n = \infty$ and say that $\{a_n\}_{n \geq 1}$ diverges to $+\infty$ if $\forall M > 0, \exists n_M \in \mathbb{N}$ such that $a_n > M$ for all $n \geq n_M$.

We write $\lim_{n \rightarrow \infty} a_n = -\infty$ and say that $\{a_n\}_{n \geq 1}$ diverges to $-\infty$ if $\forall M < 0, \exists n_M \in \mathbb{N}$ such that $a_n < M$ for all $n \geq n_M$.

Exercise 11.6. 1. Show that $\lim_{n \rightarrow \infty} (\sqrt[3]{n} + 1) = \infty$.

2. Show that the sequence given by $a_n = (-1)^n n$ for all $n \geq 1$ does not diverge to ∞ or to $-\infty$.

3. Let $\{a_n\}_{n \geq 1}$ be a sequence of positive real numbers. Show that

$$\lim_{n \rightarrow \infty} a_n = \infty \iff \lim_{n \rightarrow \infty} \frac{1}{a_n} = 0.$$

12 LECTURE 12: CAUCHY SEQUENCE

Example 12.1. We can show that $\lim_{n \rightarrow \infty} \frac{n^2+1}{n+3} = \infty$.

Let $M > 0$. We want to find $n_M \in \mathbb{N}$ such that for all $n \geq n_M$ we have $\frac{n^2+1}{n+3} > M$. Note that it suffices to ask $\frac{n}{4} > M$, and then

$$\frac{n^2+1}{n+3} > \frac{n^2}{n+3} > \frac{n^2}{4n} = \frac{n}{4} > M.$$

By the Archimedean property, there exists $n_M \in \mathbb{N}$ such that $n_M > 4M$, then for $n \geq n_M$, we have the desired equation above. By the definition, $\lim_{n \rightarrow \infty} \frac{n^2+1}{n+3} = \infty$.

Definition 12.2. We say that a sequence of real numbers $\{a_n\}_{n \geq 1}$ is a Cauchy sequence if

$$\forall \varepsilon > 0 \exists n_\varepsilon \in \mathbb{N} \text{ such that } |a_n - a_m| < \varepsilon \quad \forall n, m \geq n_\varepsilon.$$

Theorem 12.3 (Cauchy Criterion). A sequence of real numbers is Cauchy if and only if it converges.

We will split the proof of this theorem into various lemmas and properties.

Proposition 12.4. Any convergent sequence is a Cauchy sequence.

Proof. Let $\{a_n\}_{n \geq 1}$ be a convergent sequence and let $a = \lim_{n \rightarrow \infty} a_n$. Let $\varepsilon > 0$. As $a_n \xrightarrow[n \rightarrow \infty]{a}$, there exists $n_\varepsilon \in \mathbb{N}$ such that $|a - a_n| < \frac{\varepsilon}{2}$ for all $n \geq n_\varepsilon$. Then for $n, m \geq n_\varepsilon$, we have

$$|a_n - a_m| \leq |a_n - a| + |a - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Lemma 12.5. A Cauchy sequence is bounded.

Proof. Let $\{a_n\}_{n \geq 1}$ be a Cauchy sequence. Then there exists $n_1 \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $n, m \geq n_1$. So taking $m = n_1$, we get

$$|a_n| \leq |a_{n_1}| + |a_n - a_{n_1}| < |a_{n_1}| + 1$$

for all $n \geq n_1$. Now let $M = \max\{|a_1|, |a_2|, \dots, |a_{n_1-1}|, |a_{n_1}| + 1\}$. Clearly, $|a_n| \leq M$ for all $n \geq 1$. □

Definition 12.6 (Subsequence). Let $\{k_n\}_{n \geq 1}$ be a sequence of natural numbers such that $k_1 \geq 1$ and $k_{n+1} > k_n$ for all $n \geq 1$. Using induction, it is easy to see that $k_n \geq n$ for all $n \geq 1$. If $\{a_n\}_{n \geq 1}$ is a sequence, we say that $\{a_{k_n}\}_{n \geq 1}$ is a subsequence of $\{a_n\}_{n \geq 1}$.

Example 12.7. The following are subsequences of $\{a_n\}_{n \geq 1}$:

- $\{a_{2n}\}_{n \geq 1}$.
- $\{a_{2n-1}\}_{n \geq 1}$.
- $\{a_{n^2}\}_{n \geq 1}$.
- $\{a_{p_n}\}_{n \geq 1}$ where p_n denotes the n th prime.

Theorem 12.8. Let $\{a_n\}_{n \geq 1}$ be a sequence of real numbers. Then $\lim_{n \rightarrow \infty} a_n = a \in \mathbb{R} \cup \{\pm\infty\}$ if and only if every subsequence $\{a_{k_n}\}_{n \geq 1}$ of $\{a_n\}_{n \geq 1}$ satisfies $\lim_{n \rightarrow \infty} a_{k_n} = a$.

Proof. We will consider $a \in \mathbb{R}$. The cases $a \in \{\pm\infty\}$ can be handled by an analogous argument.

(\Leftarrow): Take $k_n = n$ for all $n \geq 1$.

(\Rightarrow): Assume $\lim_{n \rightarrow \infty} a_n = a$ and let $\{a_{k_n}\}_{n \geq 1}$ be a subsequence of $\{a_n\}_{n \geq 1}$. Let $\varepsilon > 0$. As $a_n \xrightarrow[n \rightarrow \infty]{} a$, $\exists n_\varepsilon \in \mathbb{N}$ such that $|a - a_n| < \varepsilon$ for all $n \geq n_\varepsilon$. Recall that $k_n \geq n$ for all $n \geq 1$. So for $n \geq n_\varepsilon$ we have $k_n \geq n \geq n_\varepsilon$ and so $|a - a_{k_n}| < \varepsilon$ for all $n \geq n_\varepsilon$. By definition, $\lim_{n \rightarrow \infty} a_{k_n} = a$. \square

Proposition 12.9. Every sequence of real numbers has a monotone subsequence.

Proof. Let $\{a_n\}_{n \geq 1}$ be a sequence of real numbers. We say that the n th term is dominant if $a_n > a_m$ for all $m > n$. We distinguish two cases:

Case 1: There are infinitely many dominant terms. Then a subsequence formed by these dominant terms is strictly decreasing.

Case 2: There are none of finitely many dominant terms. Let N be larger than the largest index of the dominant terms. So for all $n \geq N$, a_n is not dominant. Set $k_1 = N$, $a_{k_1} = a_N$. Because a_{k_1} is not dominant, there exists $k_2 > k_1$ such that $a_{k_2} \geq a_{k_1}$. Now $k_2 > k_1 = N$, then a_{k_2} is not dominant, so there exists $k_3 > k_2$ such that $a_{k_3} \geq a_{k_2}$. Proceeding inductively, we construct a subsequence $\{a_{k_n}\}_{n \geq 1}$ such that $a_{k_{n+1}} \geq a_{k_n}$ for all $n \geq 1$. \square

Theorem 12.10 (Bolzano-Weierstrass). Any bounded sequence has a convergent subsequence.

Proof. Let $\{a_n\}_{n \geq 1}$ be a bounded sequence. By the previous proposition, there exists $\{a_{k_n}\}_{n \geq 1}$ monotone subsequence of $\{a_n\}_{n \geq 1}$. As $\{a_n\}_{n \geq 1}$ is bounded, so is $\{a_{k_n}\}_{n \geq 1}$. As bounded monotone sequences converge, $\{a_{k_n}\}_{n \geq 1}$ converges. \square

Corollary 12.11. Every Cauchy sequence has a convergent subsequence.

Lemma 12.12. A Cauchy sequence with a convergent subsequence converges.

Proof. Let $\{a_n\}_{n \geq 1}$ be a Cauchy sequence such that $\{a_{k_n}\}_{n \geq 1}$ is a convergent subsequence. Let $a = \lim_{n \rightarrow \infty} a_{k_n}$. Let $\varepsilon > 0$. As $a_{k_n} \xrightarrow{n \rightarrow \infty} a$, there exists $n_1(\varepsilon)$ such that $|a - a_{k_n}| < \frac{\varepsilon}{2}$ for all $n \geq n_1(\varepsilon)$. As $\{a_n\}_{n \geq 1}$ is Cauchy, there exists $n_2(\varepsilon)$ such that $|a_n - a_m| < \frac{\varepsilon}{2}$ for all $n, m \geq n_2(\varepsilon)$. Let $n_\varepsilon = \max\{n_1(\varepsilon), n_2(\varepsilon)\}$. Then for $n \geq n_\varepsilon$, we have

$$|a - a_n| \leq |a - a_{k_n}| + |a_{k_n} - a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

because $k_n \geq n \geq n_\varepsilon$. By definition, $\lim_{n \rightarrow \infty} a_n = a$. \square

Combining the last two results, we see that a Cauchy sequence of real numbers converges.

13 LECTURE 13: LIMIT SUPERIOR AND LIMIT INFERIOR

Let $\{a_n\}_{n \geq 1}$ be a bounded sequence of real numbers (convergent or not). The asymptotic behavior of $\{a_n\}_{n \geq 1}$ depends on sets of the form $\{a_n : n \geq N\}$ for $N \in \mathbb{N}$.

As $\{a_n\}_{n \geq 1}$ bounded, the set $\{a_n : n \geq N\}$ (where $N \in \mathbb{N}$ is fixed) is a non-empty bounded subset of \mathbb{R} .

As \mathbb{R} has the least upper bound property (and so also the greatest lower bound property), the set $\{a_n : n \geq N\}$ has an infimum and a supremum in \mathbb{R} .

For $N \geq 1$, let $u_N = \inf\{a_n : n \geq N\}$ and $v_N = \sup\{a_n : n \geq N\}$. Clearly, $u_N \leq v_N$ for all $N \geq 1$.

Notice that for $N \geq 1$, we have $\{a_n : n \geq N\} \supseteq \{a_n : n \geq N+1\}$, therefore

$$\begin{cases} \inf\{a_n : n \geq N\} \leq \inf\{a_n : n \geq N+1\} \\ \sup\{a_n : n \geq N\} \geq \sup\{a_n : n \geq N+1\} \end{cases}$$

So $u_N \leq u_{N+1}$ and $v_{N+1} \leq v_N$ for all $N \geq 1$. Thus, $\{u_N\}_{N \geq 1}$ is increasing and $\{v_N\}_{N \geq 1}$ is decreasing. Moreover, for all $N \geq 1$, we have

$$u_1 \leq u_2 \leq \cdots \leq u_N \leq v_N \leq \cdots \leq v_2 \leq v_1$$

So the two sequences are bounded. As monotone bounded sequences converge, we know the two sequences must converge.

Let

$$u = \lim_{N \rightarrow \infty} u_N = \sup\{u_N : N \geq 1\} =: \sup_N u_N$$

and

$$v = \lim_{N \rightarrow \infty} v_N = \sup\{v_N : N \geq 1\} =: \inf_N v_N$$

Because of the boundedness, we see that $u_M \leq v_N$ for all $M, N \geq 1$, and so $\lim_{M \rightarrow \infty} u_M \leq v_N$ for all $N \geq 1$. Therefore, $u \leq v_N$ for all $N \geq 1$, and therefore $u \leq \lim_{N \rightarrow \infty} v_N$, which means $u \leq v$.

Moreover, if $\lim_{n \rightarrow \infty} a_n$ exists, then for all $N \geq 1$, we have

$$u_N = \inf\{a_n : n \geq N\} \leq a_n \leq \sup\{a_n : n \geq N\} = v_N$$

for all $n \geq N$. Therefore, $u_N \leq \lim_{n \rightarrow \infty} a_n \leq v_N$, and so

$$u = \lim_{N \rightarrow \infty} u_N \leq \lim_{n \rightarrow \infty} a_n \leq \lim_{N \rightarrow \infty} v_N = v.$$

Definition 13.1. Let $\{a_n\}_{n \geq 1}$ be a sequence of real numbers. We define

$$\limsup_{n \rightarrow \infty} a_n = \lim_{N \rightarrow \infty} \sup\{a_n : n \geq N\} = \lim_{N \rightarrow \infty} v_N = \inf_N v_N = \inf_N \sup_{n \geq N} a_n$$

and

$$\liminf_{n \rightarrow \infty} a_n = \lim_{N \rightarrow \infty} \inf\{a_n : n \geq N\} = \lim_{N \rightarrow \infty} u_N = \sup_N u_N = \sup_N \inf_{n \geq N} a_n$$

with the convention that if $\{a_n\}_{n \geq 1}$ is unbounded above, then $\limsup_{n \rightarrow \infty} a_n = \infty$ and if $\{a_n\}_{n \geq 1}$ is unbounded below then $\liminf_{n \rightarrow \infty} a_n = -\infty$.

Remark 13.2. We have

$$\inf\{a_n : n \geq 1\} \leq \liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n \leq \sup\{a_n : n \geq 1\}.$$

Note that $\liminf_{n \rightarrow \infty} a_n$ is the smallest value that infinitely many a_n get close to, and $\limsup_{n \rightarrow \infty} a_n$ is the largest value that infinitely many a_n get close to.

Example 13.3. Consider $a_n = 3 + \frac{(-1)^n}{n}$, then $\lim_{n \rightarrow \infty} a_n = 3$, and therefore $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = 3$. Observe that $\inf\{a_n : n \geq 1\} = 2 \neq 3$ and $\sup\{a_n : n \geq 1\} = \frac{7}{2} \neq 3$.

Theorem 13.4. Let $\{a_n\}_{n \geq 1}$ be a sequence of real numbers.

1. If $\lim_{n \rightarrow \infty} a_n$ exists in $\mathbb{R} \cup \{\pm\infty\}$, then $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} a_n$.
2. If $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n \in \mathbb{R} \cup \{\pm\infty\}$, then $\lim_{n \rightarrow \infty} a_n$ exists and

$$\lim_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n.$$

Proof. 1. We distinguish three cases.

- Case 1: $\lim_{n \rightarrow \infty} a_n = -\infty$. It is enough to show $\limsup_{n \rightarrow \infty} a_n = -\infty$ since $\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$.
Fix $M < 0$. As $\lim_{n \rightarrow \infty} a_n = -\infty$, there exists $n_M \in \mathbb{N}$ such that $a_n < M$ for all $n \geq n_M$, then for $N \geq n_M$, we have $v_N = \sup\{a_n : n \geq N\} \leq M$.¹⁰ Now by definition, $\limsup_{n \rightarrow \infty} a_n = \lim_{N \rightarrow \infty} v_N = -\infty$.
- Case 2: $\lim_{n \rightarrow \infty} a_n = \infty$. The proof is essentially the same as above, and we leave this as an exercise.
- Case 3: $\lim_{n \rightarrow \infty} a_n = a \in \mathbb{R}$. Fix $\varepsilon > 0$. Then $\exists n_\varepsilon \in \mathbb{N}$ such that $|a - a_n| < \frac{\varepsilon}{2}$ for all $n \geq n_\varepsilon$. So we know

$$a - \frac{\varepsilon}{2} < a_n < a + \frac{\varepsilon}{2}$$

for all $n \geq n_\varepsilon$. Thus, for $N \geq n_\varepsilon$, we have

$$a - \frac{\varepsilon}{2} \leq \inf\{a_n : n \geq N\} \leq \sup\{a_n : n \geq N\} \leq a + \frac{\varepsilon}{2}$$

which means $a - \frac{\varepsilon}{2} \leq u_N \leq v_N \leq a + \frac{\varepsilon}{2}$.

Therefore, for all $N \geq n_\varepsilon$, we have $|u_N - a| \leq \frac{\varepsilon}{2} < \varepsilon$ and $|v_N - a| \leq \frac{\varepsilon}{2} < \varepsilon$ for all $N \geq n_\varepsilon$. By definition, that means $\liminf_{n \rightarrow \infty} a_n = \lim_{N \rightarrow \infty} u_N = a$ and $\limsup_{n \rightarrow \infty} a_n = \lim_{N \rightarrow \infty} v_N = a$.

2. Again, we distinguish three cases.

- Case 1: $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = -\infty$. We will use $\limsup_{n \rightarrow \infty} a_n = -\infty$. Fix $M < 0$. Then since $\limsup_{n \rightarrow \infty} a_n = \lim_{N \rightarrow \infty} v_N = -\infty$, then there exists $N_M \in \mathbb{N}$ such that $v_N < M$ for all $N \geq N_M$. In particular, $v_{N_M} = \sup\{a_n : n \geq N_M\} < M$, which means $a_n < M$ for all $n \geq N_M$. By definition, that means $\lim_{n \rightarrow \infty} a_n = -\infty$.
- Case 2: $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = \infty$. The proof is essentially the same as above, and we leave this as an exercise.
- Case 3: $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = a \in \mathbb{R}$. Fix $\varepsilon > 0$. Because $a = \liminf_{n \rightarrow \infty} a_n = \lim_{N \rightarrow \infty} u_N$, then there exists $N_1(\varepsilon) \in \mathbb{N}$ such that $|u_N - a| < \varepsilon$ for all $N \geq N_1$. Therefore, $a - \varepsilon < u_{N_1} = \inf\{a_n : n \geq N_1\} < a + \varepsilon$, and we have $a - \varepsilon < a_n$ for all $n \geq N_1$.

¹⁰Note that when taking supremum, the $<$ sign can be changed to \leq . For example, $a_n = 3 - \frac{1}{n}$ has the property of $a_n < 3$ for all $n \geq 1$, but $\sup_{n \geq 1} a_n = 3$.

Similarly, considering the limit supremum, there exists $N_2(\varepsilon) \in \mathbb{N}$ such that $|v_N - a| < \varepsilon$ for all $N \geq N_2$, and so $a - \varepsilon < v_{N_2} = \inf\{a_n : n \geq N_2\} < a + \varepsilon$, which means $a_n < a + \varepsilon$ for all $n \geq N_2$.

Thus, for $n \geq \max\{N_1, N_2\}$, we have $a - \varepsilon < a_n < a + \varepsilon$, which means $|a_n - a| < \varepsilon$.

By definition, $\lim_{n \rightarrow \infty} a_n = a$.

□

14 LECTURE 14: LIMIT SUPERIOR AND LIMIT INFERIOR, CONTINUED

Theorem 14.1. Let $\{a_n\}_{n \geq 1}$ be a sequence of real numbers. Then there exists a monotonic subsequence of $\{a_n\}_{n \geq 1}$ whose limit is $\limsup_{n \rightarrow \infty} a_n$. Also, there exists a monotonic subsequence of $\{a_n\}_{n \geq 1}$ whose limit is $\liminf_{n \rightarrow \infty} a_n$.

Proof. We will prove the statement about $\limsup_{n \rightarrow \infty} a_n$. One can use a similar argument to show the statement about $\liminf_{n \rightarrow \infty} a_n$.

Note that it suffices to find a subsequence $\{a_{k_n}\}_{n \geq 1}$ of $\{a_n\}_{n \geq 1}$ such that $\lim_{n \rightarrow \infty} a_{k_n} = \limsup_{n \rightarrow \infty} a_n$. As every sequence has a monotone subsequence, $\{a_{k_n}\}_{n \geq 1}$ has a monotone subsequence $\{a_{p_{k_n}}\}_{n \geq 1}$. Then as $\lim_{n \rightarrow \infty} a_{k_n}$ exists, $\lim_{n \rightarrow \infty} a_{p_{k_n}}$ exists and

$$\lim_{n \rightarrow \infty} a_{p_{k_n}} = \lim_{n \rightarrow \infty} a_{k_n} = \limsup_{n \rightarrow \infty} a_n.$$

Finally, note that $\{a_{p_{k_n}}\}_{n \geq 1}$ is a subsequence of $\{a_n\}_{n \geq 1}$.

Let us find a subsequence of $\{a_n\}_{n \geq 1}$ whose limit is $\limsup_{n \rightarrow \infty} a_n$.

Case 1: $\limsup_{n \rightarrow \infty} a_n = -\infty$. We showed that in this case, $\lim_{n \rightarrow \infty} a_n = -\infty$. Choose $\{a_{k_n}\}_{n \geq 1}$ to be $\{a_n\}_{n \geq 1}$.

Case 2: $\limsup_{n \rightarrow \infty} a_n = a \in \mathbb{R}$. By definition, $a = \limsup_{n \rightarrow \infty} a_n = \lim_{N \rightarrow \infty} v_N$, then $\exists N_1 \in \mathbb{N}$ such that $|a - v_N| < 1$ for all $N \geq N_1$. In particular, $a - 1 < v_{N_1} < a + 1$, and note that $a - 1 < \sup\{a_n : n \geq N_1\}$ and there exists $k_1 \geq N_1$ such that $a - 1 < a_{k_1}$. Therefore, $a - 1 < a_{k_1} \leq v_{N_1} < a + 1$. Hence, $|a - a_{k_1}| < 1$.

Similarly, as $a = \lim_{N \rightarrow \infty} v_N$, there exists $N_2 \in \mathbb{N}$ such that $|a - v_N| < \frac{1}{2}$ for all $N \geq N_2$. Let $\tilde{N}_2 = \max\{N_2, k_1 + 1\}$, then in particular, $a - \frac{1}{2} < v_{\tilde{N}_2} < a + \frac{1}{2}$. Then we know $a - \frac{1}{2} < \sup\{a_n : n \geq \tilde{N}_2\}$, and because there exists $k_2 \geq \tilde{N}_2 > k_1$ such that $a - \frac{1}{2} < a_{k_2}$, we conclude that $a - \frac{1}{2} < a_{k_2} \leq v_{\tilde{N}_2} < a + \frac{1}{2}$. Hence, $|a - a_{k_2}| < \frac{1}{2}$.

To construct our subsequence, we proceed inductively. Assume we have found $k_1 < k_2 < \dots < k_n$ and a_{k_1}, \dots, a_{k_n} such that $|a - a_{k_j}| < \frac{1}{j}$ for all $1 \leq j \leq n$. As $a = \lim_{N \rightarrow \infty} v_N$, there exists $N_{n+1} \in \mathbb{N}$ such that $|a - v_N| < \frac{1}{n+1}$ for all $N \geq N_{n+1}$. Now we can let

$N_{n+1} = \max\{N_{n+1}, k_n + 1\}$. Then $a - \frac{1}{n+1} < v_{N_{n+1}} < a + \frac{1}{n+1}$. Therefore, we have $a - \frac{1}{n+1} < \sup\{a_n : n \geq N_{n+1}\}$, and there exists $k_{n+1} \geq N_{n+1} > k_n$ such that $a - \frac{1}{n+1} < a_{k_{n+1}}$. Therefore, $a - \frac{1}{n+1} < a_{k_{n+1}} \leq v_{N_{n+1}} < a + \frac{1}{n+1}$, and so $|a_{k_{n+1}} - a| < \frac{1}{n+1}$.

Case 3: $\limsup n \rightarrow \infty a_n = \infty$. We leave this as an exercise. \square

Definition 14.2 (Subsequential Limit). Let $\{a_n\}_{n \geq 1}$ be a sequence of real numbers. A subsequential limit of $\{a_n\}_{n \geq 1}$ is any $a \in \mathbb{R} \cup \{\pm\infty\}$ that is the limit of a subsequence of $\{a_n\}_{n \geq 1}$.

Example 14.3. 1. For $a_n = n(1 + (-1)^n)$, the subsequential limits are $0 = \lim_{n \rightarrow \infty} a_{2n+1}$ and $\infty = \lim_{n \rightarrow \infty} a_{2n}$.

2. For $a_n = \cos(\frac{n\pi}{3})$. The subsequential limits are $1 = \lim_{n \rightarrow \infty} a_{6n}$, $\frac{1}{2} = \lim_{n \rightarrow \infty} a_{6n+1} = \lim_{n \rightarrow \infty} a_{6n+5}$, $-\frac{1}{2} = \lim_{n \rightarrow \infty} a_{6n+2} = \lim_{n \rightarrow \infty} a_{6n+4}$, and $-1 = \lim_{n \rightarrow \infty} a_{6n+3}$.

Theorem 14.4. Let $\{a_n\}_{n \geq 1}$ be a sequence of real numbers and let A denote its set of subsequential limits:

$$A = \{a \in \mathbb{R} \cup \{\pm\infty\} : \exists \{a_{k_n}\}_{n \geq 1} \text{ subsequence of } \{a_n\}_{n \geq 1} \text{ such that } \lim_{n \rightarrow \infty} a_{k_n} = a\}.$$

Then

1. $A \neq \emptyset$.
2. $\lim_{n \rightarrow \infty} a_n$ exists in $\mathbb{R} \cup \{\pm\infty\}$ if and only if A has exactly one element.
3. $\inf(A) = \liminf_{n \rightarrow \infty} a_n$ and $\sup(A) = \limsup_{n \rightarrow \infty} a_n$.

Proof. 1. By [Theorem 14.1](#), $\liminf_{n \rightarrow \infty} a_n, \limsup_{n \rightarrow \infty} a_n \in A$. Therefore, $A \neq \emptyset$.

2. (\Rightarrow): Assume $\lim_{n \rightarrow \infty} a_n$ exists. Then if $\{a_{k_n}\}_{n \geq 1}$ is a subsequence of $\{a_n\}_{n \geq 1}$, we have $\lim_{n \rightarrow \infty} a_{k_n} = \lim_{n \rightarrow \infty} a_n$. So $A = \{\lim_{n \rightarrow \infty} a_n\}$.

(\Leftarrow): If A has a single element, then $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n$ and so $\lim_{n \rightarrow \infty} a_n$ exists.

3. It suffices to prove the following claim.

Claim 14.5. $\liminf_{n \rightarrow \infty} a_n \leq a \leq \limsup_{n \rightarrow \infty} a_n \quad \forall a \in A$.

Assuming the claim, we can first see how to finish the proof. The claim implies

- Because $\liminf_{n \rightarrow \infty} a_n$ is a lower bound for A , so $\liminf_{n \rightarrow \infty} a_n \geq \inf(A)$. On the other hand, $\liminf_{n \rightarrow \infty} a_n \in A$, and so $\liminf_{n \rightarrow \infty} a_n \geq \inf(A)$. Therefore, $\liminf_{n \rightarrow \infty} a_n = \inf(A)$.

- Similarly, we can show that $\limsup_{n \rightarrow \infty} a_n = \sup(A)$.

We now prove the claim.

Subproof. Fix $a \in A$, then there exists a subsequence $\{a_{k_n}\}_{n \geq 1}$ of $\{a_n\}_{n \geq 1}$ such that $\lim_{n \rightarrow \infty} a_{k_n} = a$. Because of the nature of the subsequence, we know there is

$$\inf\{a_n : n \geq N\} \leq \inf\{a_{k_n} : n \geq N\} \leq \sup\{a_{k_n} : n \geq N\} \leq \sup\{a_n : n \geq N\}$$

where the first two sequences are increasing and the last two sequences are decreasing. By taking the limit, we know

$$\begin{aligned} \lim_{N \rightarrow \infty} \inf\{a_n : n \geq N\} &\leq \lim_{N \rightarrow \infty} \inf\{a_{k_n} : n \geq N\} \\ &\leq \lim_{N \rightarrow \infty} \sup\{a_{k_n} : n \geq N\} \\ &\leq \lim_{N \rightarrow \infty} \sup\{a_n : n \geq N\}, \end{aligned}$$

which means

$$\liminf_{n \rightarrow \infty} a_n \leq \liminf_{n \rightarrow \infty} a_{k_n} \leq \limsup_{n \rightarrow \infty} a_{k_n} \leq \limsup_{n \rightarrow \infty} a_n.$$

Because the subsequence converges, we have $a = \lim_{a_{k_n}} = \liminf_{n \rightarrow \infty} a_{k_n} = \limsup_{n \rightarrow \infty} a_{k_n}$.

Therefore,

$$\liminf_{n \rightarrow \infty} a_n \leq a \leq \limsup_{n \rightarrow \infty} a_n.$$

■

□

15 LECTURE 15: CESARO-STOLZ THEOREM, SERIES AND CONVERGENCE TESTS

Theorem 15.1 (Cesaro-Stolz). Let $\{a_n\}_{n \geq 1}$ be a sequence of non-zero real numbers. Then

$$\liminf_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \leq \liminf_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

In particular, if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists, then $\lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$ exists and

$$\lim_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Example 15.2. We can apply this theorem to find $\lim_{n \rightarrow \infty} \sqrt[n]{n} = \lim_{n \rightarrow \infty} n^{\frac{1}{n}}$.

If we let $a_n = n$, then $\left| \frac{a_{n+1}}{a_n} \right| = \frac{n+1}{n} \xrightarrow{n \rightarrow \infty} 1$. By Cesaro-Stolz, we get $\{\sqrt[n]{n}\}_{n \geq 1}$ converges and

$$\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1.$$

Proof. It suffices to prove the last inequality, i.e.

$$\limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

One can prove the first inequality with a similar proof.

Let $l = \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} \geq 0$ and $L = \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \geq 0$. We want to show $l \leq L$. If $L = \infty$, then it is clear. Henceforth, we assume $L \in \mathbb{R}$. We will prove the following claim.

Claim 15.3. l is the lower bound for the set

$$(L, \infty) = \{M \in \mathbb{R} : M > L\}.$$

Assuming the claim for now, we can see how to finish the proof. Note (L, ∞) is a non-empty subset of \mathbb{R} which is bounded below by L . As \mathbb{R} has the least upper bound property, $\inf(L, \infty)$ exists in \mathbb{R} . In fact, $\inf(L, \infty) = L$. As l is a lower bound for (L, ∞) , we must have $l \leq L$. We now prove the claim.

Subproof. Fix $M \in (L, \infty)$. We will show $l \leq M$. We have $M > L = \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \inf_N \sup_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right|$. Therefore, there exists $N_0 \in \mathbb{N}$ such that $\sup_{n \geq N_0} \left| \frac{a_{n+1}}{a_n} \right| < M$, and so $\left| \frac{a_{n+1}}{a_n} \right| < M$ for all $n \geq N_0$. Therefore, $|a_{n+1}| < M \cdot |a_n|$ for all $n \geq N_0$.

A simple inductive argument then yields

$$|a_n| < M^{n-N_0} |a_{N_0}| \quad \forall n \geq N_0,$$

so $|a_n|^{\frac{1}{n}} < M \left(\frac{|a_{N_0}|}{M^{N_0}} \right)^{\frac{1}{n}}$ for all $n > N_0$. We can conclude that

$$l = \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n \rightarrow \infty} M \cdot \left(\frac{|a_{N_0}|}{M^{N_0}} \right)^{\frac{1}{n}} = M \cdot \limsup_{n \rightarrow \infty} \left(\frac{|a_{N_0}|}{M^{N_0}} \right)^{\frac{1}{n}}.$$

We need to apply the following claim to the inequality above.

Claim 15.4. For $r > 0$, we have $\lim_{n \rightarrow \infty} r^{\frac{1}{n}} = 1$.

Subproof. Indeed, if $r \geq 1$, we have

$$0 \leq r^{\frac{1}{n}} - 1 = \frac{r - 1}{r^{n-1} + r^{n-2} + \dots + 1} \leq \frac{r - 1}{n} \xrightarrow{n \rightarrow \infty} 0.$$

If $r < 1$, then $r^{\frac{1}{n}} = \frac{1}{(\frac{1}{r})^{\frac{1}{n}}} \xrightarrow{n \rightarrow \infty} \frac{1}{1} = 1$. ■

We now take $r = \frac{|a_{N_0}|}{M^{N_0}}$ in the inequality, then $l \leq M$. ■

□

Definition 15.5. Let $\{a_n\}_{n \geq 1}$ be a sequence of real numbers. For $n \geq 1$, we define the partial sum $s_n = a_1 + \dots + a_n$.

The series $\sum_{n=1}^{\infty} a_n$, sometimes denoted $\sum_{n \geq 1} a_n$, is said to converge if $\{s_n\}_{n \geq 1}$ converges.

We say that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely if the series $\sum_{n=1}^{\infty} |a_n|$ converges.¹¹

Theorem 15.6 (Cauchy Criterion). A series $\sum_{n \geq 1} a_n$ converges if and only if

$$\forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \text{ such that } \left| \sum_{k=n+1}^{n+p} a_k \right| < \varepsilon \quad \forall n \geq n_\varepsilon \quad \forall p \in \mathbb{N}.$$

Proof. Note that

$$\text{the series } \sum_{n \geq 1} a_n \text{ converges} \iff \text{the sequence } \{s_n\}_{n \geq 1} \text{ converges}$$

$$\iff \{s_n\}_{n \geq 1} \text{ is Cauchy}$$

$$\iff \forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \text{ such that } |s_m - s_n| < \varepsilon \quad \forall m, n \geq n_\varepsilon.$$

Without loss of generality, we may assume $m > n$ and write $m = n + p$ for $p \in \mathbb{N}$. Note

$$|s_m - s_n| = \left| \sum_{k=1}^{n+p} a_k - \sum_{k=1}^n a_k \right| = \left| \sum_{k=n+1}^{n+p} a_k \right|,$$

so $\sum_{n \geq 1} a_n$ converges if and only if

$$\forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \text{ such that } \left| \sum_{k=n+1}^{n+p} a_k \right| < \varepsilon \quad \forall n \geq n_\varepsilon \quad \forall p \in \mathbb{N}.$$

□

¹¹Note that $\sum_{n=1}^{\infty} |a_n|$ either converges or it diverges to ∞ .

Corollary 15.7. If $\sum_{n \geq 1} a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.

Proof. Taking $p = 1$, we find $\sum_{n \geq 1} a_n$ converges implies

$$\forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \text{ such that } |a_{n+1}| < \varepsilon \quad \forall n \geq n_\varepsilon.$$

□

Corollary 15.8. If $\sum_{n \geq 1} a_n$ converges absolutely, then it converges.

Proof. If $\sum_{n \geq 1} a_n$ converges absolutely, $\sum_{n \geq 1} |a_n|$ converges. By definition,

$$\forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \text{ such that } \sum_{k=n+1}^{n+p} |a_k| < \varepsilon \quad \forall n \geq n_\varepsilon \quad \forall p \in \mathbb{N}.$$

Note that

$$\left| \sum_{k=n+1}^{n+p} a_k \right| \leq \sum_{k=n+1}^{n+p} |a_k| < \varepsilon \quad \forall n \geq n_\varepsilon \quad \forall p \in \mathbb{N}.$$

Therefore, $\sum_{n \geq 1} a_n$ converges by the Cauchy criterion. □

Theorem 15.9 (Comparison Test). Let $\sum_{n \geq 1} a_n$ be a series of real numbers with $a_n \geq 0 \quad \forall n \geq 1$.

1. If $\sum_{n \geq 1} a_n$ converges and $|b_n| \leq a_n \quad \forall n \geq 1$, then $\sum_{n \geq 1} b_n$ converges.
2. If $\sum_{n \geq 1} a_n$ diverges and $b_n \geq a_n \quad \forall n \geq 1$, then $\sum_{n \geq 1} b_n$ diverges.

Proof. 1. Because $\sum_{n \geq 1} a_n$ converges, then

$$\forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \text{ such that } \left| \sum_{k=n+1}^{n+p} a_k \right| < \varepsilon \quad \forall n \geq n_\varepsilon \quad \forall p \in \mathbb{N}.$$

Then

$$\left| \sum_{k=n+1}^{n+p} b_k \right| \leq \sum_{k=n+1}^{n+p} |b_k| \leq \sum_{k=n+1}^{n+p} a_k < \varepsilon \quad \forall n \geq n_\varepsilon \quad \forall p \in \mathbb{N}.$$

Therefore, by the Cauchy criterion, $\sum_{n \geq 1} b_n$ converges.

2. Note that $b_1 + \cdots + b_n \geq a_1 + \cdots + a_n \xrightarrow{n \rightarrow \infty} \infty$, and so $\sum_{n \geq 1} b_n$ diverges.

□

Lemma 15.10. Let $r \in \mathbb{R}$. The series $\sum_{n \geq 0} r^n$ converges if and only if $|r| < 1$. If $|r| < 1$, then

$$\sum_{n \geq 0} r^n = \frac{1}{1-r}.$$

Proof. First note that if $|r| \geq 1$, then $|r^n| = |r|^n \geq 1$, therefore $r^n \not\rightarrow 0$ as $n \rightarrow \infty$. By [Corollary 15.7](#), $\sum_{n \geq 0} r^n$ does not converge. Assume now that $|r| < 1$, then $|r^n| = |r|^n \xrightarrow{n \rightarrow \infty} 0$.

Also note that $\sum_{k=0}^n r^k = \frac{1-r^{n+1}}{1-r} \xrightarrow{n \rightarrow \infty} \frac{1}{1-r}$. □

16 LECTURE 16: CONVERGENCE TESTS, CONTINUED

Proposition 16.1 (The Dyadic Criterion). Let $\{a_n\}_{n \geq 1}$ be a decreasing sequence of real numbers with $a_n \geq 0$ for all $n \geq 1$. Then the series $\sum_{n \geq 1} a_n$ converges if and only if the series $\sum_{n \geq 0} 2^n a_{2^n}$ converges.

Proof. For $n \geq 1$, let $s_n = \sum_{k=1}^n a_k = a_1 + \cdots + a_n$, and let $t_n = \sum_{k=0}^n 2^k a_{2^k} = a_1 + 2a_2 + \cdots + 2^n a_{2^n}$.

Note that both sequences are increasing, thus $\sum_{n \geq 1} a_n$ converges if and only if $\{s_n\}_{n \geq 1}$ is bounded, and $\sum_{n \geq 0} 2^n a_{2^n}$ converges if and only if $\{t_n\}_{n \geq 0}$ is bounded. It now suffices to prove that $\{s_n\}_{n \geq 1}$ is bounded if and only if $\{t_n\}_{n \geq 1}$ is bounded.

Consider the summation $\sum_{l=2^k+1}^{2^{k+1}} a_l$. Because $\{a_n\}_{n \geq 1}$ is decreasing, we know that

$$\frac{1}{2}(2^{k+1} a_{2^{k+1}}) = 2^k a_{2^{k+1}} \leq \sum_{l=2^k+1}^{2^{k+1}} a_l \leq 2^k a_{2^k+1} \leq 2^k a_{2^k}$$

and therefore

$$\frac{1}{2} \sum_{k=0}^n 2^{k+1} a_{2^{k+1}} \leq \sum_{k=0}^n \sum_{l=2^k+1}^{2^{k+1}} a_l \leq \sum_{k=0}^n 2^k a_{2^k},$$

and so $\frac{1}{2} \sum_{l=1}^{n+1} 2^l a_{2^l} \leq \sum_{l=2}^{2^{n+1}} a_l \leq t_n$. That is to say, $\frac{1}{2}(t_{n+1} - a_1) \leq s_{2^{n+1}} - a_1 \leq t_n$. We conclude that $t_{n+1} \leq 2s_{2^{n+1}} - a_1$ and $s_n \leq s_{2^{n+1}} \leq t_n + a_1$ since $n \leq 2^{n+1}$ for all $n \geq 1$.

In particular, if $\{s_n\}_{n \geq 1}$ is bounded, then there exists $M > 0$ such that $|s_n| \leq M$ for all $n \geq 1$, and so $t_{n+1} \leq 2M + a_1$ for all $n \geq 1$. Similarly, if $\{t_n\}$ is bounded, then there exists $L > 0$ such that $|t_n| \leq L$ for all $n \geq 0$, which is to say $s_n \leq L + a_1$ for all $n \geq 1$. □

Corollary 16.2. The series $\sum_{n \geq 1} \frac{1}{n^\alpha}$ converges if and only if $\alpha > 1$.

Proof. If $\alpha \leq 0$, then $\frac{1}{n^\alpha} = n^{-\alpha} \geq 1$ for all $n \geq 1$. In particular, $\frac{1}{n^\alpha} \not\rightarrow_{n \rightarrow \infty} 0$ so $\sum_{n \geq 1} \frac{1}{n^\alpha}$ cannot converge. Assume $\alpha > 0$, then $\{\frac{1}{n^\alpha}\}_{n \geq 1}$ is a decreasing sequence of positive real numbers. By the dyadic criterion, $\sum_{n \geq 1} \frac{1}{n^\alpha}$ converges if and only if $\sum_{n \geq 0} 2^n \frac{1}{(2^n)^\alpha}$ converges. Note that $\sum_{n \geq 0} \frac{2^n}{(2^n)^\alpha} = \sum_{n \geq 0} (2^{1-\alpha})^n = \sum_{b \geq 0} r^n$ where $r = 2^{1-\alpha}$, and this term converges if and only if $r < 1$ if and only if $2^{1-\alpha} < 1$ if and only if $1 - \alpha < 0$ if and only if $\alpha > 1$. \square

Theorem 16.3 (The Root Test). Let $\sum_{n \geq 1}$ be a series of real numbers.

1. If $\limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} < 1$, then $\sum_{n \geq 1} a_n$ converges absolutely.
2. If $\liminf_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} < 1$, then $\sum_{n \geq 1} a_n$ diverges.
3. The test is inconclusive if $\liminf_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} \leq 1 \leq \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$.

Proof. 1. Let $L = \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$. Since $L < 1$, then $1 - L < 0$, and because \mathbb{Q} is dense in \mathbb{R} , there exists $\varepsilon \in \mathbb{R}$ such that $0 < \varepsilon < 1 - L$, and so $L < L + \varepsilon < 1$. Therefore, $L + \varepsilon > L = \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = \inf_{N \rightarrow \infty} \sup_{n \geq N} |a_n|^{\frac{1}{n}}$. In particular, there exists $N_0 \in \mathbb{N}$ such that $\sup_{n \geq N_0} |a_n|^{\frac{1}{n}} < L + \varepsilon$. Therefore, $|a_n|^{\frac{1}{n}} < L + \varepsilon$ for all $n \geq N_0$, and so $|a_n| < (L + \varepsilon)^n$ for all $n \geq N_0$.

As $L + \varepsilon < 1$, when denote $n = N_0 + k$, we have the series $\sum_{n \geq N_0} (L + \varepsilon)^n = \sum_{k \geq 0} (L + \varepsilon)^{N_0+k} = (L + \varepsilon)^{N_0} \sum_{k \geq 0} (L + \varepsilon)^k = (L + \varepsilon)^{N_0} \cdot \frac{1}{1-(L+\varepsilon)}$. By the comparison test, $\sum_{n \geq N_0} a_n$ converges absolutely. Note that $|a_1| + \cdots + |a_{N_0-1}| \in \mathbb{R}$. Therefore, $\sum_{n \geq 1} a_n$ converges absolutely.

2. Let $\{a_{k_n}\}_{n \geq 1}$ be a subsequence of $\{a_n\}_{n \geq 1}$ such that $\lim_{n \rightarrow \infty} |a_{k_n}|^{\frac{1}{k_n}} = \liminf_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} > 1$. Therefore, there exists $n_0 \in \mathbb{N}$ such that $|a_{k_n}|^{\frac{1}{k_n}} > 1$ for all $n \geq n_0$. Therefore, $|a_{k_n}| > 1$ for all $n \geq n_0$. That is to say, $|a_{k_n}| > 1$ for all $n \geq n_0$. In particular, $a_{k_n} \not\rightarrow_{n \rightarrow \infty} 0$, that is to say $a_n \not\rightarrow_{n \rightarrow \infty} 0$, and so $\sum_{n \geq 1} a_n$ diverges.

3. Consider $a_n = \frac{1}{n}$ for all $n \geq 1$. The series $\sum_{n \geq 1} a_n = \sum_{n \geq 1} \frac{1}{n}$ diverges. However, by Cesaro-Stolz theorem,

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{n}} = \frac{1}{\lim_{n \rightarrow \infty} \frac{n+1}{n}} = 1.$$

Therefore, $\liminf_{n \rightarrow \infty} \sqrt[n]{a_n} = \limsup_{n \rightarrow \infty} \sqrt[n]{a_n} = 1$.

Now consider $a_n = \frac{1}{n^2}$ for all $n \geq 1$. The series $\sum_{n \geq 1} a_n = \sum_{n \geq 1} \frac{1}{n^2}$ converges. However, by Cesaro-Stolz theorem,

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{n^2}} = \frac{1}{\lim_{n \rightarrow \infty} \frac{(n+1)^2}{n^2}} = 1.$$

Therefore, $\liminf_{n \rightarrow \infty} \sqrt[n]{a_n} = \limsup_{n \rightarrow \infty} \sqrt[n]{a_n} = 1$.

□

Theorem 16.4 (The Ratio Test). Let $\sum_{n \geq 1} a_n$ be a series of non-zero real numbers.

1. If $\limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, the series $\sum_{n \geq 1} a_n$ converges absolutely.
2. If $\liminf_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, the series $\sum_{n \geq 1} a_n$ diverges.
3. The test is inconclusive if $\liminf_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \leq 1 \leq \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$.

Proof. The first two conclusions follow from the root test and the Cesaro-Stolz theorem:

$$\liminf_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \leq \liminf_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} \leq \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

The last conclusion is true by applying the same examples as in the theorem above. □

Theorem 16.5 (The Abel Criterion). Let $\{a_n\}_{n \geq 1}$ be a decreasing sequence with $\lim_{n \rightarrow \infty} a_n = 0$.

0. Let $\{b_n\}_{n \geq 1}$ be a sequence so that $\{\sum_{k=1}^n b_k\}_{n \geq 1}$ is bounded. Then $\sum_{n \geq 1} a_n b_n$ converges.

Corollary 16.6 (The Leibniz Criterion). Let $\{a_n\}_{n \geq 1}$ be a decreasing sequence with $\lim_{n \rightarrow \infty} a_n = 0$.

0. Then $\sum_{n \geq 1} (-1)^n a_n$ converges.

Proof of the Abel Criterion. Let $t_n = \sum_{k=1}^n b_k$ for $n \geq 1$. As $\{t_n\}_{n \geq 1}$ is bounded, there exists $M > 0$ such that $|t_n| \leq M$ for all $n \geq 1$. We will use the Cauchy criterion to prove convergence of $\sum_{n \geq 1} a_n b_n$. Let $\varepsilon > 0$. As $\lim_{n \rightarrow \infty} a_n = 0$, then there exists $n_\varepsilon \in \mathbb{N}$ such that

$|a_n| < \frac{\varepsilon}{2M}$ for all $n \geq n_\varepsilon$. For $n \geq n_\varepsilon$ and $p \in \mathbb{N}$, we have

$$\begin{aligned}
\left| \sum_{k=n+1}^{n+p} a_k b_k \right| &= \left| \sum_{k=n+1}^{n+p} a_k (t_k - t_{k-1}) \right| \\
&= \left| \sum_{k=n+1}^{n+p} a_k t_k - \sum_{k=n+1}^{n+p} a_k t_{k-1} \right| \\
&= \left| \sum_{k=n+1}^{n+p} a_k t_k - \sum_{k=n}^{n+p-1} a_{k+1} t_k \right| \\
&= \left| \sum_{k=n}^{n+p} t_k (a_k - a_{k+1}) - a_n t_n + a_{n+p+1} t_{n+p} \right| \\
&\leq \sum_{k=n}^{n+p} |t_k| |a_k - a_{k+1}| + |a_n| \cdot |t_n| + |a_{n+p+1}| \cdot |t_{n+p}| \\
&\leq \sum_{k=n}^{n+p} M(a_k - a_{k+1}) + a_n M + a_{n+p+1} M \\
&= M(a_n - a_{n+p+1}) + a_n M + a_{n+p+1} M \\
&= 2M a_n \\
&< \varepsilon.
\end{aligned}$$

□

17 LECTURE 17: REARRANGEMENT

Definition 17.1 (Rearrangement). Let $k : \mathbb{N} \rightarrow \mathbb{N}$ be a bijective function. For a sequence $\{a_n\}_{n \geq 1}$ of real numbers, we denote $\tilde{a}_n = a_{k(n)} = a_{k_n}$. Then $\sum_{n \geq 1} \tilde{a}_n$ is called a rearrangement of $\sum_{n \geq 1} a_n$.

Example 17.2. Consider $a_n = \frac{(-1)^{n-1}}{n}$ for all $n \geq 1$. The series looks like

$$\sum_{n \geq 1} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Note that the sequence $\{\frac{1}{n}\}_{n \geq 1}$ is decreasing and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$. Thus, by the Leibniz criterion, $\sum_{n \geq 1} a_n$ converges.

However, we can also write the series as follows:

$$\sum_{n \geq 1} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \sum_{k \geq 2} \left(\frac{1}{2k} - \frac{1}{2k+1} \right)$$

Note that for $k \geq 2$, we have $0 < \frac{1}{2k} - \frac{1}{2k+1} = \frac{1}{2k(2k+1)} < \frac{1}{4k^2}$.

Recall that the series $\sum_{k \geq 2} \frac{1}{4k^2}$ converges by the dyadic criterion. By the comparison test, the series $0 < \sum_{k \geq 2} \left(\frac{1}{2k} - \frac{1}{2k+1} \right)$ converges. So $\sum_{n \geq 1} a_n < 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. Consider next the following rearrangement:

$$\frac{1}{1} + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots = \sum_{k \geq 1} \left(\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} \right).$$

Now we have

$$\begin{aligned} 0 &< \frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} \\ &= \frac{8k^2 - 2k + 8k^2 - 6k - (16k^2 - 16k + 3)}{(4k-3)(4k-1) \cdot 2k} \\ &= \frac{8k-3}{(4k-3)(4k-1) \cdot 2k} \\ &< \frac{8k}{k \cdot 3k \cdot 2k} \\ &= \frac{4}{3k^2}. \end{aligned}$$

As the series $\sum_{k \geq 1} \frac{4}{3k^2}$ converges, we deduce that the series $\sum_{k \geq 1} \left(\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} \right)$ converges.

Moreover,

$$\sum_{k \geq 1} \left(\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} \right) = 1 + \frac{1}{3} - \frac{1}{2} + \sum_{k \geq 2} \left(\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} \right) > 1 + \frac{1}{3} - \frac{1}{2} = \frac{5}{6}.$$

Therefore, we see that the two series converge to two different numbers.

Theorem 17.3 (Riemann). Let $\sum_{n \geq 1} a_n$ be a series that converges, but it does not converge absolutely. Let $-\infty \leq \alpha \leq \beta \leq \infty$. Then there exists a rearrangement $\sum_{n \geq 1} \tilde{a}_n$ with partial sums $\tilde{s}_n = \sum_{k=1}^n \tilde{a}_k$ such that $\liminf_{n \rightarrow \infty} \tilde{s}_n = \alpha$ and $\limsup_{n \rightarrow \infty} \tilde{s}_n = \beta$.

Proof. For $n \geq 1$, let $b_n = \frac{|a_n|+a_n}{2}$ and $c_n = \frac{|a_n|-a_n}{2}$. This means that $b_n = \begin{cases} a_n, & a_n \geq 0 \\ 0, & a_n < 0 \end{cases}$

and $c_n = \begin{cases} 0, & a_n \geq 0 \\ -a_n, & a_n < 0 \end{cases}$. Therefore, $b_n \geq 0$ for all n , and $c_n \geq 0$ for all n .

Claim 17.4. The series $\sum_{n \geq 1} b_n$ and \sum_{c_n} both diverge.

Subproof. Note $\sum_{k=1}^n b_k - \sum_{k=1}^n c_k = \sum_{k=1}^n (b_k - c_k) = \sum_{k=1}^n a_k$. Therefore, $\sum_{k=1}^n b_k = \sum_{k=1}^n c_k + \sum_{k=1}^n a_k$. Because $\sum_{k=1}^n a_k$ converges as $n \rightarrow \infty$, then we know $\{\sum_{k=1}^n b_k\}_{n \geq 1}$ converges if and only if $\{\sum_{k=1}^n c_k\}_{n \geq 1}$ converges.

On the other hand, if $\sum_{n \geq 1} b_n$ and $\sum_{n \geq 1} c_n$ both converged, then

$$\sum_{k=1}^n b_k + \sum_{k=1}^n c_k = \sum_{k=1}^n (b_k + c_k) = \sum_{k=1}^n |a_k|$$

which diverges as $n \rightarrow \infty$. However, we know the sum of these two series would converge as $n \rightarrow \infty$, contradiction. Therefore, both $\sum_{n \geq 1} b_n$ and $\sum_{n \geq 1} c_n$ diverge to ∞ . \blacksquare

Note that $\sum_{n \geq 1} a_n$ converges, to $\lim_{n \rightarrow \infty} a_n = 0$ and so $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} c_n = 0$.

Let B_1, B_2, B_3, \dots denote the non-negative terms in $\{a_n\}_{n \geq 1}$ in the order in which they appear. Let C_1, C_2, C_3, \dots denote the absolute values of the negative terms in $\{a_n\}_{n \geq 1}$ in the order in which they appear.

Note $\sum_{n \geq 1} B_n$ differs $\sum_{n \geq 1} b_n$ only by the terms that are zero. Therefore, $\sum_{n \geq 1} B_n = \infty$. Similarly, $\sum_{n \geq 1} C_n$ differs $\sum_{n \geq 1} c_n$ only by the terms that are zero. Therefore, $\sum_{n \geq 1} C_n = \infty$.

We now choose sequences $\{\alpha_n\}_{n \geq 1}$ and $\{\beta_n\}_{n \geq 1}$ so that $\alpha_n \xrightarrow{n \rightarrow \infty} \alpha$, $\beta_n \xrightarrow{n \rightarrow \infty} \beta$, $\alpha_n < \beta_n$ for all $n \geq 1$, and $\beta_1 > 0$.¹² Next, we construct increasing sequences $\{k_n\}_{n \geq 1}$ and $\{j_n\}_{n \geq 1}$ as follows:

1. Choose k_1 and j_1 to be the smallest natural numbers so that $x_1 = B_1 + \dots + B_{k_1} > \beta_1$ and $y_1 = B_1 + \dots + B_{k_1} - C_1 - C_2 - \dots - C_{j_1} < \alpha_1$. Note that both choices are possible because the series summation goes to infinity.
2. Choose k_2 and j_2 to be the smallest natural numbers so that $x_2 = B_1 + \dots + B_{k_1} - C_1 - C_2 - \dots - C_{j_1} + B_{k_1+1} + \dots + B_{k_2} > \beta_2$, and $y_2 = B_1 + \dots + B_{k_1} - C_1 - \dots - C_{j_1} + B_{k_1+1} + \dots + B_{k_2} - C_{j_1+1} - \dots - C_{j_2} < \alpha_2$.
3. Proceed inductively.

Note that by definition, $x_n - B_{k_n} \leq \beta_n$, and so $\beta_n - B_{k_n} < \beta_m < x_n \leq \beta_n + B_{k_n}$. In particular, $|x_n - \beta_n| \leq B_{k_n} \xrightarrow{n \rightarrow \infty} 0$ as $\beta_n \xrightarrow{n \rightarrow \infty} \beta$. Therefore, $\lim_{n \rightarrow \infty} x_n = \beta$. Similarly, we have

¹²This requirement says that α_i 's approach α from the left, β_i 's approach β from the right, and β_i 's start from somewhere large enough.

$y_n + C_{j_n} \geq \alpha_n$, and so $\alpha_n - C_{j_n} \leq y_n < \alpha_n < \alpha_n + C_{j_n}$, therefore $|y_n - \alpha_n| \leq C[j_n] \xrightarrow{n \rightarrow \infty} 0$ as $\alpha_n \xrightarrow{n \rightarrow \infty} \alpha$. We conclude that $\lim_{n \rightarrow \infty} y_n = \alpha$.

Finally, note that x_n and y_n are partial sums in the rearrangement

$$B_1 + B_2 + \cdots + B_{k+1} - C_1 - \cdots - C_{j_1} + B_{k_1+1} + \cdots + B_{k_2} - C_{j_1+1} - \cdots - C_{j_2} + \cdots$$

By construction, no number less than α or larger than β can occur as a subsequential limit of the partial sums. \square

Theorem 17.5. If a series $\sum_{n \geq 1} a_n$ converges absolutely, then any rearrangement $\sum_{n \geq 1} \tilde{a}_n$ converges to $\sum_{n \geq 1} a_n$.

Proof. For $n \geq 1$ let $s_n = \sum_{k=1}^n a_k$ and $\tilde{s}_n = \sum_{k=1}^n \tilde{a}_k$. As $\sum_{n \geq 1} a_n$ converges absolutely, $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ such that

$$\sum_{k=n+1}^{n+p} |a_k| < \varepsilon \quad \forall n \geq n_\varepsilon, \forall p \in \mathbb{N}.$$

Choose N_ε sufficiently large so that $a_1, \dots, a_{n_\varepsilon}$ belong to the set $\{\tilde{a}_1, \dots, \tilde{a}_N\}$. Then for $n > N_\varepsilon$, the terms $a_1, \dots, a_{n_\varepsilon}$ cancel in $s_n - \tilde{s}_n$, then

$$|s_n - \tilde{s}_n| \leq \sum_{k=n_\varepsilon+1}^n |a_k| + \sum_{\substack{1 \leq k \leq n \\ \tilde{a}_k \notin \{a_1, \dots, a_{n_\varepsilon}\}}} |\tilde{a}_k| < \varepsilon$$

because the sum holds finitely many terms and all indices are greater than n_ε . Therefore, as $\lim_{n \rightarrow \infty} s_n = s \in \mathbb{R}$, we deduce that $\lim_{n \rightarrow \infty} \tilde{s}_n = s$. \square

18 LECTURE 18: FUNCTIONS, CARDINALITY

Definition 18.1 (Function). Let A, B be two non-empty sets. A function $f : A \rightarrow B$ is a way of associating to each element $a \in A$ exactly one element in B denoted $f(a)$.

We say A is the domain of f and B is the range (alternatively, codomain) of f . The set $f(A) = \{f(a) : a \in A\}$ is the image of A under f .

If $A' \subseteq A$, then $f(A') = \{f(a) : a \in A'\}$ is called the image of A' under f . If $f(A) = B$, then we say that f is surjective, or onto. In this case, $\forall b \in B, \exists a \in A$ such that $f(a) = b$. We say that f is injective if it satisfies: if $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$, then $a_1 = a_2$. We say that f is bijective if f is injective and surjective.

Remark 18.2. Injectivity and surjectivity of a function depend not only on the law f , but also on the domain and the codomain.

Example 18.3. Consider $f : \mathbb{Z} \rightarrow \mathbb{Z}$ such that $f(n) = 2n$. This function is injective but not surjective.

Consider $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $g(x) = 2x$, then g is a bijection.

Example 18.4. Consider $f : [0, \infty) \rightarrow [0, \infty)$ such that $f(x) = x^2$. This function is a bijection.

Consider $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $g(x) = x^2$. This function is surjective but not injective.

Definition 18.5 (Composition). Let A, B, C be non-empty sets and $f : A \rightarrow B$ and $g : B \rightarrow C$ be two functions. The composition of g with f is a function $g \circ f : A \rightarrow C$ defined by $(g \circ f)(a) = g(f(a))$.

Remark 18.6. The composition of two functions need not be commutative. For example, consider $f, g : \mathbb{Z} \rightarrow \mathbb{Z}$ where $f(n) = 2n$ and $g(n) = n + 1$. Then $g \circ f : \mathbb{Z} \rightarrow \mathbb{Z}$ is defined by $(g \circ f)(n) = g(f(n)) = g(2n) = 2n + 1$, and $f \circ g : \mathbb{Z} \rightarrow \mathbb{Z}$ is defined by $(f \circ g)(n) = f(g(n)) = f(n + 1) = 2n + 2$.

Exercise 18.7. The composition of functions is associative: if $f : A \rightarrow B$, $g : B \rightarrow C$ and $h : C \rightarrow D$ are three functions, then $(h \circ g) \circ f = h \circ (g \circ f)$.

Definition 18.8 (Inverse). Let $f : A \rightarrow B$ be a bijective function. The inverse of f is a function $f^{-1} : B \rightarrow A$ defined as follows: if $b \in B$ then $f^{-1}(b) = a$ where a is the unique element in A such that $f(a) = b$. The existence of a is generated by surjectivity and the uniqueness by injectivity. Therefore, $f \circ f^{-1} : B \rightarrow B$ is defined by $(f \circ f^{-1})(b) = b$, and $f^{-1} \circ f : A \rightarrow A$ is defined by $(f^{-1} \circ f)(a) = a$.

Exercise 18.9. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two bijective functions. Then $g \circ f : A \rightarrow C$ is a bijection and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Definition 18.10 (Preimage). Let $f : A \rightarrow B$ be a function. If $B' \subseteq B$ then the preimage of B' is $f^{-1}(B') = \{a \in A : f(a) \in B'\}$. The preimage of a set is well-defined whether or not f is bijective. In fact, if $B' \subseteq B$ such that $B' \cap f(A) = \emptyset$, then $f^{-1}(B') = \emptyset$.

Exercise 18.11. Let $f : A \rightarrow B$ be a function and let $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$. Then

1. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
2. $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.
3. $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.
4. $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

5. The following are equivalent:

- f is injective.
- $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ for all subsets $A_1, A_2 \subseteq A$.

Definition 18.12 (Cardinality). We say that two sets A and B have the same cardinality (or the same cardinal number) if there exists a bijection $f : A \rightarrow B$. In this case, we write $A \sim B$.

Exercise 18.13. Show that \sim is an equivalence relation on sets.

Definition 18.14 (Finite, Countable, At Most Countable). We say that a set A is finite if $A = \emptyset$ (in which case we say that it has cardinality 0) or $A \sim \{1, \dots, n\}$ for some $n \in \mathbb{N}$ (in which case we say that A has cardinality n).

We say that A is countable if $A \sim \mathbb{N}$. In this case, we say that A has cardinality \aleph_0 .

We say that A is at most countable if A is finite or countable. If A is not at most countable we say that A is uncountable.

Lemma 18.15. Let A be a finite set and let $B \subseteq A$. Then B is finite.

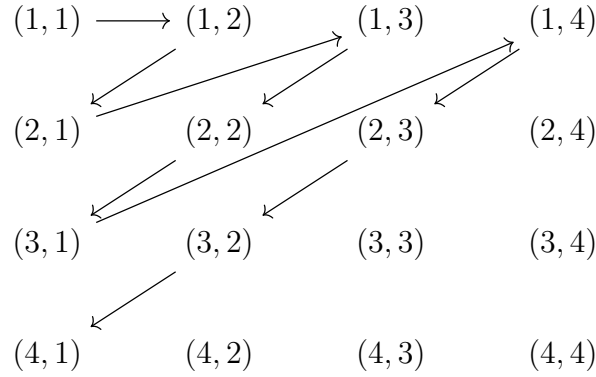
Proof. If $B = \emptyset$, then B is finite. Assume now that $B \neq \emptyset$, then $A \neq \emptyset$. As A is finite, $\exists n \in \mathbb{N}$, and $\exists f : A \rightarrow \{1, \dots, n\}$ a bijection, then $f|_B : B \rightarrow f(B)$ is a bijection. We merely have to relabel the elements in $f(B)$. Let $b_1 \in B$ be such that $f(b_1) = \min(f(B))$. Define $g(b_1) = 1$. If $B \setminus \{b_1\} \neq \emptyset$, let $b_2 \in B$ be such that $f(b_2) = \min(f(B \setminus \{b_1\}))$. Define $g(b_2) = 2$, and proceed by induction. The process terminates in at most n steps. \square

Example 18.16. Consider $f : \mathbb{N} \cup \{0, \dots, -k\} \rightarrow \mathbb{N}$ where $k \in \mathbb{N}$ and $f(n) = n + l + 1$ is bijective. So the cardinality of $\mathbb{N} \cup \{0, -1, \dots, -k\}$ is \aleph_0 .

Example 18.17. Consider $f : \mathbb{Z} \rightarrow \mathbb{N}$ where $f(n) = \begin{cases} 2n + 2, & n \geq 0 \\ -2n - 1, & n < 0 \end{cases}$ is bijective. So the cardinality of \mathbb{Z} is \aleph_0 .

Example 18.18. Consider $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that $f(n, m) = \frac{(n+m-1)(n+m-2)}{2} + n$ is bijective, so the cardinality of \mathbb{N} is \aleph_0 . This can be done by labeling the following n/m value

table with a diagonal argument.



19 LECTURE 19: CARDINALITY, CONTINUED

Example 19.1 (Continued). We prove that f is surjective by induction. For $k \in \mathbb{N}$, let $P(k)$ denote the statement

$$\exists(n, m) \in \mathbb{N} \times \mathbb{N} \text{ such that } f(n, m) = k.$$

Base Case: Note that $f(1, 1) = \frac{1 \cdot 0}{2} + 1 = 1$, so $P(1)$ holds.

Inductive Step: Fix $k \geq 1$ and assume that $P(k)$ holds. Then $\exists(n, m) \in \mathbb{N} \times \mathbb{N}$ such that $f(n, m) = k$. Therefore,

$$\frac{(n + m - 1)(n + m - 2)}{2} + n + 1 = k + 1.$$

Therefore, $\frac{[(n+1)(m+1)-1][(n+1)+(m-1)-2]}{2} + n + 1 = k + 1$, so $f(n + 1, m - 1) = k + 1$. This works if $(n + 1, m - 1) \in \mathbb{N} \times \mathbb{N}$ if and only if $m - 1 \in \mathbb{N}$ and if and only if $m \geq 2$.

Hence, if $m \geq 2$, we found $(n + 1, m - 1) \in \mathbb{N} \times \mathbb{N}$ such that $f(n + 1, m - 1) = k + 1$. Assume now $m = 1$. Then $f(n, 1) = k$ if and only if $\frac{n(n-1)}{2} + n = k$ if and only if $\frac{(n+1)n}{2} = k$. We now know $\frac{(n+1)n}{2} + 1 = k + 1$. We now have

$$\frac{[1 + (n + 1) - 1][1 + (n + 1) - 2]}{2} + 1 = k + 1,$$

so $f(1, n + 1) = k + 1$. Therefore, if $m = 1$, we found $(1, n + 1) \in \mathbb{N} \times \mathbb{N}$ such that $f(1, n + 1) = k + 1$. This proves $P(k + 1)$ holds. By induction, $\forall k \in \mathbb{N}$, $\exists(n, m) \in \mathbb{N} \times \mathbb{N}$ such that $f(n, m) = k$, i.e. f is surjective.

Let $(n, m), (a, n) \in \mathbb{N} \times \mathbb{N}$ such that $f(n, m) = f(a, n)$. We want to show that $(n, m) = (a, n)$, thus proving that f is injective.

Case 1: $\frac{(n+m-1)(n+m-2)}{2} = \frac{(a+b-1)(a+b-2)}{2}$, then because $f(n, m) = f(a, b)$, so $n = a$. Then $(n+m-1)(n+m-2) = (n+b-1)(n+b-2)$. By simplification, we have $(m-b)(2n+m+b-3) = 0$, but note that $2n+m+b-3 \geq 2+1+1-3 \geq 1$, then $m = b$.

Case 2: $\frac{(n+m-1)(n+m-2)}{2} = \frac{(a+b-1)(a+b-2)}{2} + r$ for some $r \in \mathbb{N}$.

Exercise 19.2. Show that this cannot occur.

Lemma 19.3. Let A be a countable set. Let B be an infinite subset of A . Then B is countable.

Proof. Since A is countable, then $\exists f : \mathbb{N} \rightarrow A$ is a bijection. This means we may enumerate the elements of A , i.e. denote $A = \{a_1, a_2, a_3, \dots\}$ where $f(n) = a_n$. Let $k_1 = \min\{n : a_n \in B\}$. Define $g(1) = a_{k_1}$. Then $B \setminus \{a_{k_1}\} \neq \emptyset$. Let $k_2 = \min\{n : a_n \in B \setminus \{a_{k_1}\}\}$. Define $g(2) = a_{k_2}$.

We proceed inductively. Assume we found $k_1 < \dots < k_j$ such that $a_{k_1}, \dots, a_{k_j} \in B$, and $g(1) = a_{k_1}, \dots, g(j) = a_{k_j}$. Then $B \setminus \{a_{k_1}, \dots, a_{k_j}\} \neq \emptyset$. Let $k_{j+1} = \min\{n : a_n \in B \setminus \{a_{k_1}, \dots, a_{k_j}\}\}$. Define $g(j+1) = a_{k_{j+1}}$. By construction, $g : \mathbb{N} \rightarrow B$ is bijective. \square

Lemma 19.4. Let A be a finite set and let B be a proper subset of A . Then A and B are not equipotent, that is, there is no bijective function $f : A \rightarrow B$.

Proof. Suppose $B = \emptyset$, then $A \neq \emptyset$. There is no function $f : A \rightarrow B$. Assume $B \neq \emptyset$. Assume towards a contradiction that there exists a bijection $f : A \rightarrow B$. As $B \subsetneq A$, $\exists a_0 \in A \setminus B$. For $n \geq 1$, let $a_n = f^n(a_0) = (f \circ f \circ \dots \circ f)(a_0)$. Note $a_{n+1} = f(a_n)$ for all $n \geq 0$, and that $a_n \in B$ for all $n \geq 1$. We will show the following claim.

Claim 19.5. $a_n \neq a_m$ for $n \neq m$.

If the claim holds, then B (and so A) would contain countably many elements. This is a contradiction since A is finite.

Subproof. Assume that there exists $n, k \in \mathbb{N}$ such that $a_{n+k} = a_n$. We write $a_{n+k} = f^n(a_k)$ and $a_n = f^n(a_0)$, then since f is injective, so f^n is injective, then $a_k = a_0$. However, $a_k \in B$ and $a_0 \in A \setminus B$, contradiction. \blacksquare

This proves the claim and completes the proof of the lemma. \square

Lemma 19.6. Every infinite set has a countable subset.

Proof. Let A be an infinite set, then $A \neq \emptyset$, so $\exists a_1 \in A$. Then $A \setminus \{a_1\} \neq \emptyset$, so $\exists a_2 \in A \setminus \{a_1\}$. We proceed inductively. Having found $a_1, \dots, a_n \in A$ distinct elements, then $A \setminus \{a_1, \dots, a_n\} \neq \emptyset$, so there exists $a_{n+1} \in A \setminus \{a_1, \dots, a_n\}$. This gives a sequence $\{a_n\}_{n \geq 1}$ of distinct elements in A . \square

Theorem 19.7. A set A is infinite if and only if there is a bijection between A and a proper subset of A .

Proof. (\Leftarrow): Assume that there is a bijection $f : A \rightarrow B$ where $B \subsetneq A$. By [Lemma 19.4](#), A must be infinite.

(\Rightarrow): Assume that A is infinite, then by [Lemma 19.6](#), there exists a countable subset B of A . Write $B = \{a_1, a_2, \dots\}$ with $a_n \neq a_m$ if $n \neq m$. Then $A \setminus \{a_1\}$ is a proper subset of A .

Define $f : A \rightarrow A \setminus \{a_1\}$ via

$$f(a) = \begin{cases} a, & \text{if } a \in A \setminus B \\ a_{j+1}, & \text{if } a = a_j \text{ for some } j \geq 1 \end{cases}$$

This is a bijective function. Assume $f(a) = f(b)$.

Case 1: $a, b \in A \setminus B$. Then $f(a) = a$ and $f(b) = b$, and so $f(a) = f(b)$ implies $a = b$.

Case 2: $a, b \in B$, then $\exists i, j \in \mathbb{N}$ such that $a = a_i$ and $b = a_j$. Then $f(a) = f(b)$ implies $a_{i+1} = a_{j+1}$, so $i + 1 = j + 1$, and therefore $i = j$ and so $a = b$.

Case 3: $a \in A \setminus B$ and $b \in B$. Then $f(a) \in A \setminus B$, and $f(b) \in B$, which cannot occur.

Case 4: $a \in B$ and $b \in A \setminus B$. Argue as for case 3.

Exercise 19.8. f is surjective.

□

Theorem 19.9 (Schroder-Bernstein). Assume that A and B are two sets such that there exists two injective functions $f : A \rightarrow B$ and $g : B \rightarrow A$. Then A and B are equipotent.

Example 19.10. Suppose $f : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ with $f(n) = (1, n)$, then f is injective. Suppose $g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ with $g(n, m) = 2^n \cdot 3^m$, then g is injective. Therefore, by Schroder-Bernstein Theorem, $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$.

20 LECTURE 20: CARDINALITY, CONTINUED

We first prove Schroder-Bernstein Theorem.

Proof. We will decompose each of the sets A and B into disjoint subsets:

$$A = A_1 \cup A_2 \cup A_3$$

with $A_i \cap A_j = \emptyset$ if $i \neq j$, and

$$B = B_1 \cup B_2 \cup B_3$$

with $B_i \cap B_j = \emptyset$ if $i \neq j$, and we will show that $f : A_1 \rightarrow B_1$, $f : A_2 \rightarrow B_2$ and $g : B_3 \rightarrow A_3$ are bijections.

If this is the case, then $h : A \rightarrow B$ given by

$$h(a) = \begin{cases} f(a), & \text{if } a \in A_1 \cup A_2 \\ (g|_{B_3})^{-1}(a), & \text{if } a \in A_3 \end{cases}$$

is a bijection. We leave this as an exercise. For $a \in A$, we consider the set

$$S_a = \{a, g^{-1}(a), f^{-1} \circ g^{-1}(a), g^{-1} \circ f^{-1} \circ g^{-1}(a), \dots\}$$

where elements are in A and in B , alternatively. Note that the preimage under f or g is either \emptyset or it contains exactly one point (since f and g are injective). There are three possibilities.

1. The process defining S_a does not terminate. We can always find a preimage.
2. The process defining S_a terminates in A , that is, the last element $x \in S_a$ is $x = a$ or $x = f^{-1} \circ g^{-1} \circ \dots \circ g^{-1}(a)$ and $g^{-1}(x) = \emptyset$.
3. The process defining S_a terminates in B , that is, the last element $x \in S_a$ is $x = g^{-1}(a)$ or $x = g^{-1} \circ f^{-1} \circ g^{-1} \circ \dots \circ g^{-1}(a)$ and $f^{-1}(x) = \emptyset$.

We define the sets A_1, A_2, A_3 as the set of elements $a \in A$ where the process defining S_a does not terminate, where the process defining S_a terminates in A , and where the process defining S_a terminates in B , respectively.

Similarly, for $b \in B$, we define the set

$$T_b = \{b, f^{-1}(b), g^{-1} \circ f^{-1}(b), f^{-1} \circ g^{-1} \circ f^{-1}(b), \dots\}$$

where elements are in B and in A , alternatively. As before, we define the sets B_1, B_2, B_3 as the set of elements $b \in B$ where the process defining T_b does not terminate, where the process defining T_b terminates in A , and where the process defining T_b terminates in B , respectively.

We now show that $f : A_1 \rightarrow B_1$ is a bijection. Injectivity is inherited from $f : A \rightarrow B$ is injective. Let $b \in B_1$. Then the process defining $T_b = \{b, f^{-1}(b), g^{-1} \circ f^{-1}(b), f^{-1} \circ g^{-1} \circ f^{-1}(b), \dots\}$ does not terminate. In particular, $\exists a \in A$ such that $f^{-1}(b) = a$. Note that $S_a = \{a, g^{-1}(a), f^{-1} \circ g^{-1}(a), g^{-1} \circ f^{-1} \circ g^{-1}(a), \dots\} = \{f^{-1}(b), g^{-1} \circ f^{-1}(b), f^{-1} \circ g^{-1} \circ f^{-1}(b), \dots\}$ does not terminate, so $a \in A_1$. This proves $f : A_1 \rightarrow B_1$ is surjective.

We now show that $f : A_2 \rightarrow B_2$ is a bijection. Again, injectivity is inherited from $f : A \rightarrow B$ is injective. Let $b \in B_2$. Then the process defining $T_b = \{b, f^{-1}(b), g^{-1} \circ$

$f^{-1}(b), f^{-1} \circ g^{-1} \circ f^{-1}(b), \dots$ terminates in A . In particular, $\exists a \in A$ such that $f^{-1}(b) = a$. Note that $S_a = \{a, g^{-1}(a), f^{-1} \circ g^{-1}(a), g^{-1} \circ f^{-1} \circ g^{-1}(a), \dots\} = \{f^{-1}(b), g^{-1} \circ f^{-1}(b), f^{-1} \circ g^{-1} \circ f^{-1}(b), \dots\}$ terminates in A , so $a \in A_2$. Therefore, $f : A_2 \rightarrow B_2$ is surjective.

Exercise 20.1. $g : B_3 \rightarrow A_3$ is bijective.

□

Theorem 20.2. Let $\{A_n\}_{n \geq 1}$ be a sequence of countable sets. Then $\bigcup_{n \geq 1} A_n = \{a : a \in A_n \text{ for some } n \geq 1\}$ is countable.

Proof. We define $B_1 = A_1$ and $B_{n+1} = A_{n+1} \setminus \bigcup_{k=1}^n A_k$ for all $n \geq 1$. By construction, we have $B_n \cap B_m = \emptyset$ for all $n \neq m$, and $\bigcup_{n \geq 1} B_n = \bigcup_{n \geq 1} A_n$. Note that each B_n is at most countable. Let $I = \{n \in \mathbb{N} : B_n \neq \emptyset\}$. Then $\bigcup_{n \geq 1} B_n = \bigcup_{n \in I} B_n$. For $n \in I$, let $f_n : B_n \rightarrow J_n$ bijection where J_n is an at most countable subset of \mathbb{N} . In particular, $f_1 : B_1 \rightarrow \mathbb{N}$ bijective and therefore $f_1^{-1} : \mathbb{N} \rightarrow B_1$ is bijective.

To show $\bigcup_{n \in I} B_n$ is countable, we will use the Schroder-Bernstein Theorem. Let $g : \mathbb{N} \rightarrow \bigcup_{n \in I} B_n$ be defined by $g(n) = f_1^{-1}(n) \in B_1 \subseteq \bigcup_{n \in I} B_n$, then it is injective. Let $h : \bigcup_{n \in I} B_n \rightarrow \mathbb{N} \times \mathbb{N}$ be defined as follows: if $b \in \bigcup_{n \in I} B_n$, then there exists a unique $n \in I$ such that $b \in B_n$. Define $h(b) = (n, f_n(b))$. Note that h is injective. Indeed, if $h(b_1) = h(b_2)$, then $(n_1, f_{n_1}(b_1)) = (n_2, f_{n_2}(b_2))$, then $n_1 = n_2$, and $f_{n_1}(b_1) = f_{n_2}(b_2)$. Because f_{n_1} is injective, then $b_1 = b_2$. Recall that there exists a bijection $\varphi : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, so $\varphi \circ h : \bigcup_{n \in I} B_n \rightarrow \mathbb{N}$ is injective. By Schroder-Bernstein theorem, $\bigcup_{n \in I} B_n = \bigcup_{n \geq 1} A_n \sim \mathbb{N}$. □

Proposition 20.3. Let $\{A_n\}_{n \geq 1}$ be a sequence of sets such that for each $n \geq 1$, A_n has at least two elements. Then $\prod_{n \geq 1} A_n = \{\{a_n\}_{n \geq 1} : a_n \in A_n \ \forall n \geq 1\}$ is uncountable.

Proof. We argue by contradiction. Assume that $\prod_{n \geq 1} A_n$ is countable. Thus, we may enumerate the elements of $\prod_{n \geq 1} A_n$:

$$\begin{aligned} a_1 &= (a_{11}, a_{12}, a_{13}, \dots) \\ a_2 &= (a_{21}, a_{22}, a_{23}, \dots) \\ &\vdots \\ a_n &= (a_{n1}, a_{n2}, a_{n3}, \dots) \\ &\vdots \end{aligned}$$

Let $x = \{x_n\}_{n \geq 1} \in \prod_{n \geq 1} A_n$ such that $x_n \in A_n \setminus \{a_{nn}\}$, as we look at the diagonal elements. Then $x \neq a_n$ for all $n \geq 1$ since $x_n \neq a_{nn}$. This gives a contradiction. \square

Remark 20.4. The same argument using binary expansion shows that the set $(0, 1)$ is uncountable.

21 LECTURE 21: CARDINALITY, CONTINUED, METRIC SPACES

Proposition 21.1. Let $\{A_n\}_{n \geq 1}$ be a sequence of sets such that $\forall n \geq 1$, the set A_n has at least two elements. Then $\prod_{n \geq 1} A_n$ is uncountable.

Remark 21.2. 1. The Cantor diagonal argument can be used to show that the set $(0, 1)$ is uncountable. This can be proved by binary expansion.

2. We can identify

$$\begin{aligned} & \left\{ \{a_n\}_{n \geq 1} : a_n \in \{0, 1\}, \forall n \geq 1 \right\} \\ &= \{f : \mathbb{N} \rightarrow \{0, 1\} : f \text{ function}\} \\ &= \{0, 1\} \times \{0, 1\} \times \cdots \\ &= \{0, 1\}^{\mathbb{N}}. \end{aligned}$$

By [Proposition 21.1](#), this set is uncountable. We say it has cardinality 2^{\aleph_0} .

Theorem 21.3. Let A be any set. Then there exists no bijection between A and the power set of A , $\mathcal{P}(A) = \{B : B \subseteq A\}$.

Proof. If $A = \emptyset$, then $\mathcal{P}(A) = \{\emptyset\}$. So the cardinality of A is 0, but the cardinality of $\mathcal{P}(A)$ is 1. Thus, A is not equipotent with $\mathcal{P}(A)$.

Assume $A \neq \emptyset$. We argue by contradiction. Assume that there exists $f : A \rightarrow \mathcal{P}(A)$ a bijection. Let $B = \{a \in A : a \notin f(a)\} \subseteq A$. Because f is a surjection, then there exists $b \in A$ such that $f(b) = B$. We now distinguish two cases:

- Case 1: $b \in B = f(b)$, then $b \notin B$, we have a contradiction.
- Case 2: $b \notin B = f(b)$, then $b \in B$, we have a contradiction.

Therefore, A is not equipotent to $\mathcal{P}(A)$. \square

Theorem 21.4. The set $[0, 1)$ has cardinality 2^{\aleph_0} .

Proof. We write $x \in [0, 1)$ using the binary expansion $x = 0.x_1x_2x_3\cdots$ with $x_n \in \{0, 1\}$ for all $n \geq 1$. Therefore, we can write $x = \frac{x_1}{2} + \frac{x_2}{2^2} + \frac{x_3}{2^3} + \cdots = \sum_{n \geq 1} \frac{x_n}{2^n}$, with the convention that no expansion ends in all ones. For example, for $x = 0.x_1x_2x_3\cdots x_n0111\cdots = \frac{x_1}{2} + \cdots + \frac{x_n}{2^n} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+3}} + \cdots = \frac{x_1}{2} + \cdots + \frac{x_n}{2^n} + \frac{1}{2^{n+2}} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{x_1}{2} + \cdots + \frac{x_n}{2^n} + \frac{1}{2^{n+1}} = 0.x_1x_2\cdots x_n1000\cdots$. Note that we can identify $[0, 1)$ with $\mathcal{F} = \{f : \mathbb{N} \rightarrow \{0, 1\} : \forall n \in \mathbb{N}, \exists m > n \text{ such that } f(m) = 0\} \subseteq \{f : \mathbb{N} \rightarrow \{0, 1\} : f \text{ function}\}$.

In particular, we have an injection $\varphi : [0, 1) \rightarrow \{f : \mathbb{N} \rightarrow \{0, 1\}\}$. To prove the theorem, by Schroder-Bernstein theorem, it suffices to construct an injective function $\psi : \{f : \mathbb{N} \rightarrow \{0, 1\}\} \rightarrow [0, 1)$. For $f : \mathbb{N} \rightarrow \{0, 1\}$, we define $\psi(f) = 0.0f(1)0f(2)0f(3)\cdots = \frac{f(1)}{2^2} + \frac{f(2)}{2^4} + \frac{f(3)}{2^6} + \cdots = \sum_{n \geq 1} \frac{f(n)}{2^{2n}}$.

We now show that ψ is injective. Let $f_1, f_2 : \mathbb{N} \rightarrow \{0, 1\}$ such that $f_1 \neq f_2$. Let $n_0 = \min\{n : f_1(n) \neq f_2(n)\}$. Say, $f_1(n_0) = 1$ and $f_2(n_0) = 0$. Now

$$\begin{aligned}
& \psi(f_1) - \psi(f_2) \\
&= \sum_{n \geq 1} \frac{f_1(n)}{2^{2n}} - \sum_{n \geq 1} \frac{f_2(n)}{2^{2n}} \\
&= \frac{f_1(n_0) - f_2(n_0)}{2^{2n_0}} + \sum_{n \geq n_0+1} \frac{f_1(n) - f_2(n)}{2^{2n}} \\
&\geq \frac{1}{2^{2n_0}} - \sum_{n \geq n_0+1} \frac{1}{2^{2n}} \\
&= \frac{1}{2^{2n_0}} - \frac{1}{2^{2(n_0+1)}} \cdot \frac{1}{1 - \frac{1}{2}} \\
&= \frac{1}{2^{2n_0}} - \frac{1}{2^{2n_0+1}} \\
&= \frac{1}{2^{2n_0+1}} \\
&> 0,
\end{aligned}$$

therefore $\psi(f_1) > \psi(f_2)$. Hence, ψ is an injective function. By Schroder-Bernstein, $[0, 1) \sim \{f : \mathbb{N} \rightarrow \{0, 1\}\}$ and so it has cardinality 2^{\aleph_0} . \square

Definition 21.5 (Metric Space). Let X be a non-empty set. A metric on X is a map $d : X \times X \rightarrow \mathbb{R}$ such that

1. $d(x, y) \geq 0 \quad \forall x, y \in X$.
2. $d(x, y) = 0$ if and only if $x = y$.
3. $d(x, y) = d(y, x) \quad \forall x, y \in X$.

$$4. d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in X.$$

Then we say (X, d) is a metric space.

Example 21.6. 1. $X = \mathbb{R}$, $d(x, y) = |x - y|$.

$$2. X = \mathbb{R}^n, d_2(x, y) = \sqrt{\sum_{k=1}^n |x_k - y_k|^2} \text{ is a metric, called the } l^2 \text{ norm.}$$

$$3. \text{ Let } X \text{ be any non-empty set, the discrete metric is defined by } d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}.$$

4. Let (x, d) be a metric space. Then $\tilde{d} : X \times X \rightarrow \mathbb{R}$ defined by $\tilde{d}(x, y) = \frac{d(x, y)}{1+d(x, y)}$ is a metric.

We now show that this metric satisfies the fourth property. Since d is a metric, fix $x, y, z \in X$, then $d(x, y) \leq d(x, z) + d(z, y)$. Note $a \mapsto \frac{a}{1+a} = 1 - \frac{1}{1+a}$ is increasing on $[0, \infty)$. Then $\tilde{d}(x, y) = \frac{d(x, y)}{1+d(x, y)} \leq \frac{d(x, z) + d(z, y)}{1+d(x, z) + d(z, y)} \leq \frac{d(x, z)}{1+d(x, z)} + \frac{d(z, y)}{1+d(z, y)} = \tilde{d}(x, z) + \tilde{d}(z, y)$.

Definition 21.7 (Bounded). We say that a metric space (X, d) is bounded if $\exists M > 0$ such that $d(x, y) \leq M \quad \forall x, y \in X$.

If (X, d) is not bounded, we say that it is unbounded.

Remark 21.8. If (X, d) is an unbounded metric space, then (X, \tilde{d}) is a bounded metric space where $\tilde{d}(x, y) = \frac{d(x, y)}{1+d(x, y)}$.

Definition 21.9 (Distance). Let (X, d) be a metric space and let $A, B \subseteq X$. The distance between A and B is $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.

Remark 21.10. This does not define a metric on subsets of X . In fact, $d(A, B) = 0$ does not even imply $A \cap B \neq \emptyset$.

For example, $(X, d) = (\mathbb{R}, |\cdot|)$, let $A = (0, 1)$, $B = (-1, 0)$, now $d(A, B) = 0$ but $A \cap B = \emptyset$.

Definition 21.11 (Distance). Let (X, d) be a metric space, $A \subseteq X$, $x \in X$. The distance from x to A is $d(x, A) = \inf\{d(x, a) : a \in A\}$. Again, $d(x, A) = 0$ does not imply $x \in A$.

22 LECTURE 22: HÖLDER'S INEQUALITY, BASIC TOPOLOGY

Proposition 22.1 (Hölder's Inequality). Fix $1 \leq p \leq \infty$ and let q denote the dual of p , that is, $\frac{1}{p} + \frac{1}{q} = 1$. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and let $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. Then

$$\sum_{k=1}^n |x_k y_k| \leq \left(\sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^n |y_k|^q \right)^{\frac{1}{q}}$$

with the convention that if $p = \infty$, then $(\sum_{k=1}^n |x_k|^p)^{\frac{1}{p}} = \sup_{1 \leq k \leq n} |x_k|$.

Remark 22.2. If $p = 2$ (and so $q = 2$), we call this the Cauchy-Schwarz inequality.

Proof. If $p = 1$, then $q = \infty$. Now we have

$$\sum_{k=1}^n |x_k y_k| \leq \sum_{k=1}^n |x_k| \cdot \sup_{1 \leq l \leq n} |y_l| \leq \left(\sum_{k=1}^n |x_k| \right) \cdot \sup_{1 \leq l \leq n} |y_l|.$$

If $p = \infty$, then $q = 1$, a similar argument yields the claim.

We now assume $1 < p < \infty$. We will use the fact that $f(0, \infty) \rightarrow \mathbb{R}$, $f(x) = \log(x)$ is a concave function. This tells us that for any $a > b$, at the point $b < b + t(a - b) = ta(1 - t)b < a$, we have $f(b + t(a - b)) > f(b) + t(f(a) - f(b))$. Therefore, for any $(a, b) \in (0, \infty)$ and $t \in (0, 1)$, we have $tf(a) + (1 - t)f(b) \leq f(ta + (1 - t)b)$. Therefore, we have $t \log(a) + (1 - t) \log(b) \leq \log(ta + (1 - t)b)$, and so $\log(a^t b^{1-t}) \leq \log(ta + (1 - t)b)$, and so $a^t b^{1-t} \leq ta + (1 - t)b$.

We now apply this inequality with $a = \frac{|x_k|^p}{\sum_{l=1}^n |x_l|^p}$ and $b = \frac{|y_k|^q}{\sum_{l=1}^n |y_l|^q}$, now $t = \frac{1}{p}$, so $1 - t = 1 - \frac{1}{p} = \frac{1}{q}$. We get

$$\frac{|x_k|}{\left(\sum_{l=1}^n |x_l|^p \right)^{\frac{1}{p}}} \cdot \frac{|y_k|}{\left(\sum_{l=1}^n |y_l|^q \right)^{\frac{1}{q}}} \leq \frac{1}{p} \frac{|x_k|^p}{\sum_{l=1}^n |x_l|^p} + \frac{1}{q} \frac{|y_k|^q}{\sum_{l=1}^n |y_l|^q}$$

and by summing over $1 \leq k \leq n$,

$$\sum_{k=1}^n \frac{|x_k|}{\left(\sum_{l=1}^n |x_l|^p \right)^{\frac{1}{p}}} \cdot \frac{|y_k|}{\left(\sum_{l=1}^n |y_l|^q \right)^{\frac{1}{q}}} \leq \frac{1}{p} \sum_{k=1}^n \frac{1}{p} \frac{|x_k|^p}{\sum_{l=1}^n |x_l|^p} + \frac{1}{q} \sum_{k=1}^n \frac{1}{q} \frac{|y_k|^q}{\sum_{l=1}^n |y_l|^q} = \frac{1}{p} + \frac{1}{q} = 1.$$

Therefore,

$$\sum_{k=1}^n |x_k y_k| \leq \left(\sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^n |y_k|^q \right)^{\frac{1}{q}}.$$

□

Corollary 22.3 (Minkowski's Inequality). Fix $1 \leq p \leq \infty$ and let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. Then

$$\left(\sum_{k=1}^n |x_k + y_k|^p \right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^n |y_k|^p \right)^{\frac{1}{p}}$$

Proof. If $p = 1$, this follows from the triangle inequality: indeed, the left-hand side is just $\sum_{k=1}^n |x_k + y_k|$, and that is bounded above by $\sum_{k=1}^n |x_k| + |y_k|$, which is the right-hand side.

If $p = \infty$, then the left-hand side is just $\sup_{1 \leq k \leq n} |x_k + y_k|$, which is bounded above by $\sup_{1 \leq k \leq n} |x_k| + \sup_{1 \leq k \leq n} |y_k|$, which is the right-hand side.

From now on, we can just assume $1 < p < \infty$. We observe that

$$\begin{aligned}
\sum_{k=1}^n |x_k + y_k|^p &= \sum_{k=1}^n |x_k + y_k| |x_k + y_k|^{p-1} \\
&\leq \sum_{k=1}^n (|x_k| + |y_k|) |x_k + y_k|^{p-1} \\
&= \sum_{k=1}^n |x_k| \cdot |x_k + y_k|^{p-1} + \sum_{k=1}^n |y_k| |x_k + y_k|^{p-1} \\
&\leq \left(\sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{k=1}^n |x_k + y_k|^{(p-1) \cdot q} \right)^{\frac{1}{q}} \\
&\quad + \left(\sum_{k=1}^n |y_k|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{k=1}^n |x_k + y_k|^{(p-1) \cdot q} \right)^{\frac{1}{q}} \\
&= \left[\left(\sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^n |y_k|^p \right)^{\frac{1}{p}} \right] \cdot \left(\sum_{k=1}^n |x_k + y_k|^{(p-1) \cdot q} \right)^{\frac{1}{q}}
\end{aligned}$$

where the last inequality follows from Hölder's inequality. Because $\frac{1}{p} + \frac{1}{q} = 1$, then $\frac{1}{q} = \frac{p-1}{p}$, so $q = \frac{p}{p-1}$. We then get that

$$\sum_{k=1}^n |x_k + y_k|^p \leq \left[\left(\sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^n |y_k|^p \right)^{\frac{1}{p}} \right] \cdot \left(\sum_{k=1}^n |x_k + y_k|^p \right)^{1 - \frac{1}{p}}$$

and therefore

$$\left(\sum_{k=1}^n |x_k + y_k|^p \right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^n |y_k|^p \right)^{\frac{1}{p}}.$$

□

Corollary 22.4. For $1 \leq p < \infty$, let $d_p : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ be such that $d_p(x, y) = \left(\sum_{k=1}^n |x_k - y_k|^p \right)^{\frac{1}{p}}$. For $p = \infty$, let $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ be such that $d_\infty(x, y) = \sup_{1 \leq k \leq n} |x_k - y_k|$. Then d_p is a metric on \mathbb{R}^n for all $1 \leq p \leq \infty$.

Proof. The triangle inequality follows from Minkowski's inequality. □

Remark 22.5. Hölder's and Minkowski's inequalities generalize to sequences. For example, say $\{x_n\}_{n \geq 1}$ and $\{y_n\}_{n \geq 1}$ are sequences of real numbers such that $\left(\sum_{n \geq 1} |x_n|^p \right)^{\frac{1}{p}} < \infty$ and

$\left(\sum_{n \geq 1} |y_n|^q\right)^{\frac{1}{q}} < \infty$. Then for each fixed $N \geq 1$, we have

$$\sum_{n=1}^N |x_k y_k| \leq \left(\sum_{n=1}^N |x_n|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{n=1}^N |y_n|^q\right)^{\frac{1}{q}} \leq \left(\sum_{n \geq 1} |x_n|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{n \geq 1} |y_n|^q\right)^{\frac{1}{q}} < \infty.$$

Note that this is an increasing sequence indexed by N . So

$$\sum_{n \geq 1} |x_k y_k| \leq \left(\sum_{n \geq 1} |x_n|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{n \geq 1} |y_n|^q\right)^{\frac{1}{q}}.$$

A similar argument gives Minkowski for sequences.

Definition 22.6 (Neighborhood). Let (X, d) be a metric space. A neighborhood of a point $a \in X$ is $B_r(a) = \{x \in X : d(a, x) < r\}$ for some $r > 0$.

Example 22.7. 1. (\mathbb{R}^2, d_2) is a metric space. The ball of radius 1 is represented by $B_1(0) = \{(x, y) \in \mathbb{R}^2 : d_2((x, y), (0, 0))\} < 1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, which is just the unit disc.

2. (\mathbb{R}^2, d_1) is a metric space. The ball of radius 1 is represented by $B_1(0) = \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\}$, which is a square of side length $\sqrt{2}$.

3. (\mathbb{R}^2, d_∞) is a metric space. The ball of radius 1 is represented by $B_1(0) = \{(x, y) \in \mathbb{R}^2 : \max\{|x|, |y|\} < 1\}$, which is a square of side length 2.

Definition 22.8 (Interior, Open). Let (X, d) be a metric space and let $\emptyset \neq A \subseteq X$. We say that a point $a \in X$ is an interior point of A if $\exists r > 0$ such that $B_r(a) \subseteq A$. The set of all interior points of A is denoted by \mathring{A} and is called the interior of A . We say that A is open if $A = \mathring{A}$.

Example 22.9. 1. \emptyset, X are open sets.

2. $B_r(a)$ is an open set for all $a \in X$, for all $r > 0$. Indeed, let $x \in B_r(a)$, then $d(x, a) < r$ by definition. Hence, $l = r - d(x, a) > 0$.

Claim 22.10. $B_l(x) \subseteq B_r(a)$ and so $x \in B_r(a)$.

Subproof. Let $y \in B_l(x)$, then $d(x, y) < l$. Now $d(y, a) \leq d(y, x) + d(x, a) < l + d(x, a) = r$, then $y \in B_r(a)$ by definition. \square

Remark 22.11. $\mathring{A} \subseteq A$. To prove A is open, it suffices to show $A \subseteq \mathring{A}$.

23 LECTURE 23: BASIC TOPOLOGY, CONTINUED

Proposition 23.1. Let (X, d) be a metric space and let $A, B \subseteq X$. Then

1. If $A \subseteq B$ then $\overset{\circ}{A} \subseteq \overset{\circ}{B}$.
2. $\overset{\circ}{A} \cup \overset{\circ}{B} \subseteq (A \cup B)^\circ$.
3. $\overset{\circ}{A} \cap \overset{\circ}{B} \subseteq (A \cap B)^\circ$.
4. $\overset{\circ}{\overset{\circ}{A}} = \overset{\circ}{A}$. In particular, $\overset{\circ}{A}$ is an open set.
5. $\overset{\circ}{A}$ is the largest open set contained in A .
6. A finite intersection of open sets is an open set.
7. An arbitrary union of open sets is an open set.

Remark 23.2. An arbitrary intersection of open sets need not be open. For example, consider $\bigcap_{n \geq 1} (-\frac{1}{n}, \frac{1}{n}) = \{0\} = B_{\frac{1}{n}}(0)$ in $(\mathbb{R}, |\cdot|)$. Note that $\{0\}$ is not an open set because it does not contain any neighborhood of 0.

Proof. 1. If $\overset{\circ}{A} = \emptyset$, then this is clear. Assume $\overset{\circ}{A} \neq \emptyset$. Let $a \in \overset{\circ}{A}$, then there exists $r > 0$ such that $B_r(a) \subseteq A$, and because $A \subseteq B$, then $B_r(a) \subseteq B$. Hence, $a \in \overset{\circ}{B}$.

2. Because $A \subseteq A \cup B$, then by (1), $\overset{\circ}{A} \subseteq (A \cup B)^\circ$. Similarly, $\overset{\circ}{B} \subseteq (A \cup B)^\circ$. Therefore, $\overset{\circ}{A} \cup \overset{\circ}{B} \subseteq (A \cup B)^\circ$.

3. Because $A \cap B \subseteq A$, then $(A \cap B)^\circ \subseteq \overset{\circ}{A}$. Similarly, $(A \cap B)^\circ \subseteq \overset{\circ}{B}$, and so $(A \cap B)^\circ \subseteq \overset{\circ}{A} \cap \overset{\circ}{B}$. Now let $x \in \overset{\circ}{A} \cap \overset{\circ}{B}$, then there exists $r_1, r_2 > 0$ such that $B_{r_1}(x) \subseteq A$ and $B_{r_2}(x) \subseteq B$. Let $r = \min\{r_1, r_2\} > 0$, then $B_r(x) \subseteq B_{r_1}(x) \cap B_{r_2}(x) \subseteq A \cap B$, therefore $x \in (A \cap B)^\circ$. Hence, $\overset{\circ}{A} \cap \overset{\circ}{B} \subseteq (A \cap B)^\circ$.

4. Because $\overset{\circ}{A} \subseteq A$, then $\overset{\circ}{\overset{\circ}{A}} \subseteq \overset{\circ}{A}$. Let $x \in \overset{\circ}{\overset{\circ}{A}}$, then there exists $r > 0$ such that $B_r(x) \subseteq \overset{\circ}{A}$. By (1), $B_r(x) = B_r^\circ(x) \subseteq \overset{\circ}{A}$, so $x \in \overset{\circ}{A}$. Therefore, $\overset{\circ}{A} \subseteq \overset{\circ}{\overset{\circ}{A}}$.

5. By (4), $\overset{\circ}{A}$ is an open set contained in A . Let $B \subseteq A$ be an open set, then by (1), $B = \overset{\circ}{B} \subseteq \overset{\circ}{A}$.

6. Using (3) and (4), we see that if $A = \overset{\circ}{A}$ and $B = \overset{\circ}{B}$, then $A \cap B = (A \cap B)^\circ$ is an open set. A simple inductive argument then yields the claim.

7. Let $\{A_i\}_{i \in I}$ be a family of open sets. Let us show $\left(\bigcup_{i \in I} A_i\right)^\circ = \bigcup_{i \in I} A_i^\circ$. Let $x \in \bigcup_{i \in I} A_i$, then there exists $i_0 \in I$ such that $x \in A_{i_0}$. But $A_{i_0} = A_{i_0}^\circ$, then there exists $r > 0$ such that $B_r(x) \subseteq A_{i_0}$. Therefore, $B_r(x) \subseteq \bigcup_{i \in I} A_i$, so $x \in \left(\bigcup_{i \in I} A_i\right)^\circ$. Thus, $\left(\bigcup_{i \in I} A_i\right)^\circ \supseteq \bigcup_{i \in I} A_i^\circ$, which concludes the proof. \square

Definition 23.3 (Closed). Let (X, d) be a metric space. A set $A \subseteq X$ is closed if ${}^c A$ is open.

Example 23.4. 1. \emptyset, X are closed.

2. If $a \in X$, $r < 0$, then ${}^c B_r(a) = \{x \in X : d(a, x) \geq r\}$ is a closed set.
3. If $a \in X$, $r < 0$, then $K_r(a) = \{x \in X : d(a, x) \leq r\}$ is a closed set. Indeed, let us show that ${}^c K_r(a) = \{x \in X : d(a, x) > r\}$ is open. Let $x \in {}^c K_r(a)$, then $d(a, x) > r$. Now let $l = d(a, x) - r > 0$.

Claim 23.5. $B_l(x) \subseteq {}^c K_r(a)$.

Subproof. Let $y \in B_l(x)$, then $d(x, y) < l$. By triangle inequality, $d(a, y) \geq d(a, x) - d(x, y) > d(a, x) - l = r$. Therefore, $y \in {}^c K_r(a)$. This concludes the proof. \square

By definition, $x \in ({}^c K_r(a))^\circ$. Thus, ${}^c K_r(a)$ is an open set.

4. There are sets that are neither open nor closed. For example, $(0, 1]$.

Definition 23.6 (Adherent Point, Closure, Isolation, Accumulation Point). Let (X, d) be a metric space and let $A \subseteq X$. A point $a \in X$ is an adherent point for A if for all $r > 0$, we have $B_r(a) \cap A \neq \emptyset$. The set of all adherent points of A is denoted \bar{A} and is called the closure of A .

An adherent point is called isolated if $\exists r > 0$ such that $B_r(a) \cap A = \{a\}$. In particular, $a \in A$. If every point in A is an isolated point of A , then A is called an isolated set.

An adherent point a of A that is not isolated is called an accumulation point for A . The set of accumulation points of A is denoted A' .

Note that $a \in A'$ if and only if $\forall r > 0$, $B_r(a) \cap A \setminus \{a\} \neq \emptyset$.

Example 23.7. Consider $(\mathbb{R}, |\cdot|)$ with $A = \{\frac{1}{n} : n \geq 1\}$. Now A is isolated. Indeed, $B_{\frac{1}{n(n+1)}}(\frac{1}{n}) \cap A = \{\frac{1}{n}\}$, and $A' = \{0\}$ since $\forall r > 0$, $B_r(0) = (-r, r)$ intersects $A \setminus \{0\} = A$.

Remark 23.8. 1. $A \subseteq \bar{A}$.

2. $\bar{A} = A' \cup A$.

24 LECTURE 24: BASIC TOPOLOGY, CONTINUED; COMPLETE METRIC SPACE

Proposition 24.1. Let (X, d) be a metric space and let $A, B \subseteq X$. Then

1. ${}^c(\bar{A}) = ({}^cA)^\circ$.
2. ${}^c({}^\circ A) = \overline{{}^cA}$.
3. A is a closed set if and only if $A = \bar{A}$.
4. If $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$.
5. $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$.
6. $\bar{A} \cup \bar{B} = \overline{A \cup B}$.
7. $\bar{\bar{A}} = \bar{A}$. In particular, \bar{A} is a closed set.
8. \bar{A} is the smallest closed set containing A .
9. A finite union of closed sets is a closed set.
10. An arbitrary intersection of closed sets is a closed set.

Remark 24.2. An arbitrary union of closed sets need not be a closed set. For example,

$$\bigcup_{n \geq 1} [\frac{1}{n}, 1] = (0, 1].$$

Proof. 1. $x \in {}^c(\bar{A})$ if and only if $x \notin \bar{A}$, if and only if $\exists r > 0$ such that $B_r(x) \cap A = \emptyset$, if and only if $\exists r > 0$ such that $B_r(x) \subseteq {}^cA$, if and only if $x \in {}^cA$.

2. Apply part (1) to cA .

3. A is closed if and only if cA is open if and only if ${}^cA = ({}^cA)^\circ$ if and only if (by part 1) that ${}^cA = {}^c(\bar{A})$, if and only if $A = \bar{A}$.

4. If $\bar{A} = \emptyset$, then clearly $\bar{A} \subseteq \bar{B}$. Assume $\bar{A} \neq \emptyset$. Let $a \in \bar{A}$, then for all $r > 0$ we have $B_r(a) \cap A \neq \emptyset$, but since $A \subseteq B$, then $B_r(a) \cap B \neq \emptyset$ for all $r > 0$ as well. hence, $a \in \bar{B}$. Therefore, $\bar{A} \subseteq \bar{B}$.

5. Since $A \cap B \subseteq A$, by part (4) we know $\overline{A \cap B} \subseteq \bar{A}$, and similarly we know $\overline{A \cap B} \subseteq \bar{B}$. Therefore, $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$.

6. Because of part (1), ${}^c(\overline{A \cup B}) = {}^c(A \dot{\cup} B) = {}^cA \dot{\cap} {}^cB = {}^cA \cap {}^cB$, then by part (1) we know that ${}^c(\bar{A}) \cap {}^c(\bar{B})$, which is just ${}^c(\bar{A} \cup \bar{B})$. Therefore, $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
7. Clearly $A \subseteq \bar{A}$, then by part (4), we have $\bar{A} \subseteq \bar{\bar{A}}$. We want to show the other inclusion $\bar{\bar{A}} \subseteq \bar{A}$. Let $a \in \bar{\bar{A}}$. We show that for all $r > 0$, $B_r(a) \cap A \neq \emptyset$.
Fix $r > 0$. Since $a \in \bar{\bar{A}}$, then $B_r(a) \cap \bar{A} \neq \emptyset$. Let $x \in B_r(a) \cap \bar{A}$, then $x \in \bar{A}$, so for any $l > 0$, $B_l(x) \cap A \neq \emptyset$. We now choose $l = r - d(a, x) > 0$, then $B_l(x) \subseteq B_r(a)$. But since $B_l(x) \cap A \neq \emptyset$, then $B_r(a) \cap A \neq \emptyset$. Therefore, $a \in \bar{A}$.
8. Note that \bar{A} is a closed set containing A . Let B be a closed set containing A , then if $A \subseteq B$, then $\bar{A} \subseteq \bar{B} = B$, according to part (3).
9. Let $\{A_n\}_{n=1}^N$ be closed sets. Then cA_n is an open set for all $1 \leq n \leq N$. Therefore, $\bigcap_{n=1}^N {}^cA_n$ is an open set. Now $\bigcup_{n=1}^N {}^cA_n = {}^c(\bigcup_{n=1}^N A_n)$ is open, therefore $\bigcup_{n=1}^N A_n$ is closed.
10. Let $\{A_i\}_{i \in I}$ be a family of closed sets. Then cA_i is open for all $i \in I$. Therefore, $\bigcup_{i \in I} {}^cA_i = {}^c(\bigcap_{i \in I} A_i)$ is open, and so $\bigcap_{i \in I} A_i$ is closed.

□

Definition 24.3 (Induced Metric, Subspace). Let (X, d) be a metric space and let $\emptyset \neq Y \subseteq X$. Then $d_1 : Y \times Y \rightarrow \mathbb{R}$ defined by $d_1(x, y) = d(x, y)$ for all $x, y \in Y$ is a metric on Y , and is called the induced metric on Y . Now (Y, d_1) is called a subspace of (X, d) .

Proposition 24.4. Let (X, d) be a metric space and let $\emptyset \neq Y \subseteq X$ equipped with the induced metric d_1 .

1. A set $D \subseteq Y$ is open in (Y, d_1) if and only if there exists $O \subseteq X$ open in (X, d) such that $D = O \cap Y$.
2. A set $F \subseteq Y$ is closed in (Y, d_1) if and only if there exists $C \subseteq X$ closed in (X, d) such that $F = C \cap Y$.

Proof. 1. (\Rightarrow): Let $D \subseteq Y$ be open in (Y, d_1) . Then for all $a \in D$, there exists $r_a > 0$ such that $B_{r_a}^Y(a) = \{y \in Y : d(a, y) < r_a\} \subseteq D$. Note that $B_{r_a}^Y(a) = B_{r_a}^X(a) \cap Y$. Therefore, $D = \bigcup_{a \in D} B_{r_a}^Y(a) = \bigcup_{a \in D} [B_{r_a}^X(a) \cap Y] = \left(\bigcup_{a \in D} B_{r_a}^X(a) \right) \cap Y$, with $\bigcup_{a \in D} B_{r_a}^X(a)$ open in (X, d) .

(\Leftarrow): Assume that $D = O \cap Y$ for O open in (X, d) . Let $a \in D \subseteq O$, then there exists $r > 0$ such that $B_r^X(a) \subseteq O$. Therefore, $B_r^Y(a) = B_r^X(a) \cap Y \subseteq O \cap Y = D$. Therefore, a is an interior point of D in (Y, d_1) . Therefore, D is open in (Y, d_1) .

2. Note that $F \subseteq Y$ is closed in (Y, d_1) if and only if $Y \setminus F$ is open in (Y, d_1) , if and only if (by part (1)) there exists an open set O in (X, d) such that $Y \setminus F = O \cap Y$. But $F = Y \setminus (Y \setminus F) = Y \setminus (O \cap Y) = Y \cap {}^c(O \cap Y) = Y \cap ({}^cO \cup {}^cY) = (Y \cap {}^cO) \cup (Y \cap {}^cY) = Y \cap {}^cO$, where cO is closed in (X, d) .

□

Example 24.5. 1. $[0, 1)$ is not an open set in $(\mathbb{R}, |\cdot|)$, but it is open in $([0, 2], |\cdot|)$. Say $[0, 1) = (-1, 1) \cap [0, 2)$.

2. $(0, 1]$ is not a closed set in $(\mathbb{R}, |\cdot|)$, but it is closed in $([0, 2], |\cdot|)$. Say $(0, 1] = [-1, 1] \cap (0, 2)$.

Proposition 24.6. Let (X, d) be a metric space and let $\emptyset \neq Y \subseteq X$ equipped with the induced metric. The following are equivalent:

1. Any $A \subseteq Y$ that is open (respectively, closed) in Y is also open (respectively, closed) in X .
2. Y is open (respectively, closed) in X .

Proof. (1) \Rightarrow (2): Take $A = Y$.

(2) \Rightarrow (1): Assume Y is closed in X . Let $A \subseteq Y$ be open in Y , then there exists an open set O in X such that $A = O \cap Y$, but both sets are open in X , then A is also open in X . □

Proposition 24.7. Let (X, d) be a metric space and let $\emptyset \neq Y \subseteq X$ equipped with the induced metric. For a set $A \subseteq Y$, $\bar{A}^Y = \bar{A}^X \cap Y$.

Proof. $a \in \bar{A}^Y$ if and only if for all $r > 0$, $B_r(a) \cap A \neq \emptyset$, if and only if for all $r > 0$, $B_r^X(a) \cap Y \cap A \neq \emptyset$, if and only if $a \in \bar{A}^X \cap Y$. □

Definition 24.8 (Limit). Let (X, d) be a metric space and let $\{x_n\}_{n \geq 1} \subseteq X$. We say $\{x_n\}_{n \geq 1}$ converges to a point $x \in X$ if for all $\varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq n_\varepsilon$. Then x is called the limit of $\{x_n\}_{n \geq 1}$ and we write $x = \lim_{n \rightarrow \infty} x_n$ or $x_n \xrightarrow[n \rightarrow \infty]{d} x$.

Exercise 24.9. The limit of a convergent sequence is unique.

Exercise 24.10. A sequence $\{x_n\}_{n \geq 1}$ converges to $x \in X$ if and only if every subsequence of $\{x_n\}_{n \geq 1}$ converges to x .

Remark 24.11. If $x_n \xrightarrow[n \rightarrow \infty]{d} x$ and $y_n \xrightarrow[n \rightarrow \infty]{d} y$, then $d(x_n, y_n) \xrightarrow[n \rightarrow \infty]{} d(x, y)$. Indeed, $|d(x_n, y_n) - d(x, y)| \leq |d(x_n, y_n) - d(x_n, y)| + |d(x_n, y) - d(x, y)| \leq d(y_n, y) + d(x_n, x) \xrightarrow[n \rightarrow \infty]{} 0$.

Definition 24.12 (Cauchy Sequence). Let (X, d) be a metric space. We say that $\{x_n\}_{n \geq 1} \subseteq X$ is Cauchy if for all $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq n_\varepsilon$.

Exercise 24.13. Every convergent sequence is Cauchy.

Remark 24.14. Not every Cauchy sequence is convergent in an arbitrary metric space.

Example 24.15. $(X, d) = ((0, 1), |\cdot|)$, $x_n = \frac{1}{n}$ for all $n \geq 2$ is Cauchy but does not converge in X .

Example 24.16. $(X, d) = (\mathbb{Q}, |\cdot|)$, $x_1 = 3$, $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ for all $n \geq 1$. Then $\{x_n\}_{n \geq 1}$ is Cauchy, but does not converge in X .

Definition 24.17 (Complete Metric Space). A metric space (X, d) is complete if every Cauchy sequence in X converges in X .

Example 24.18. $(\mathbb{R}, |\cdot|)$ is a complete metric space.

Exercise 24.19. Show that a Cauchy sequence with a convergent subsequence converges.

25 LECTURE 25: COMPLETE METRIC SPACE, CONTINUED

Lemma 25.1. Let (X, d) be a metric space and let $\emptyset \neq F \subseteq X$. The following are equivalent:

1. $a \in \bar{F}$.
2. There exists $\{a_n\}_{n \geq 1} \subseteq F$ such that $a_n \xrightarrow[n \rightarrow \infty]{d} a$.

Proof. (1) \Rightarrow (2): Assume $a \in \bar{F}$. Then for all $r > 0$, $B_r(a) \cap F \neq \emptyset$. For $n \geq 1$, take $r = \frac{1}{n}$. Then $B_{\frac{1}{n}}(a) \cap F \neq \emptyset$. Let $a_n \in B_{\frac{1}{n}}(a) \cap F$. Consider $\{a_n\}_{n \geq 1} \subseteq F$. We have for all $n \geq 1$ that $d(a_n, a) < \frac{1}{n} \xrightarrow[n \rightarrow \infty]{} 0$, and so $a_n \xrightarrow[n \rightarrow \infty]{d} a$.

(2) \Rightarrow (1): Assume there exists a sequence $\{a_n\}_{n \geq 1}$ of F such that $a_n \xrightarrow[n \rightarrow \infty]{d} a$. Fix $r > 0$. Then there exists $n_r \in \mathbb{N}$ such that $d(a_n, a) < r$ for all $n \geq n_r$. In particular, for all $n \geq n_r$, $a_n \in B_r(a) \cap F$, so $B_r(a) \cap F \neq \emptyset$. Since r is arbitrary, we get $a \in \bar{F}$. \square

the following definition is from the homework

Definition 25.2 (Diameter). Let (X, d) be a metric space. The diameter of a set $\emptyset \neq A \subseteq X$ is given by $\delta(A) = \sup\{d(x, y) : x, y \in A\}$ with the convention that $\delta(A) = \infty$ if the set $\{d(x, y) : x, y \in A\}$ is unbounded.

Theorem 25.3. Let (X, d) be a metric space. The following are equivalent:

1. (X, d) is a complete metric space.
2. For every sequence $\{F_n\}_{n \geq 1}$ of non-empty closed subsets of X that is nested, i.e., $F_{n+1} \subseteq F_n \forall n \geq 1$, and satisfies $\delta(F_n) \xrightarrow{n \rightarrow \infty} 0$, we have $\bigcap_{n \geq 1} F_n = \{a\}$ for some $a \in X$.

Proof. (1) \Rightarrow (2): Assume (X, d) is complete. As $F_n \neq \emptyset$ for all $n \geq 1$, there exists $a_n \in F_n$.

Claim 25.4. $\{a_n\}_{n \geq 1}$ is Cauchy.

Subproof. Let $\varepsilon > 0$. As $\delta(F_n) \xrightarrow{n \rightarrow \infty} 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\delta(F_n) < \varepsilon$ for all $n \geq n_\varepsilon$. Let $m, n \geq n_\varepsilon$, then since $\{F_n\}_{n \geq 1}$ is nested, $F_n \subseteq F_{n_\varepsilon}, F_m \subseteq F_{n_\varepsilon}$, so $d(a_n, a_m) \leq \delta(F_{n_\varepsilon}) < \varepsilon$. ■

As (X, d) is complete, there exists $a \in X$ such that $a_n \xrightarrow{n \rightarrow \infty} a$. But because for all $n \geq 1$ we have $\{a_m\}_{m \geq n} \subseteq F_n$, then $a \in \bar{F}_n = F_n$. Therefore, $a \in \bigcap_{n \geq 1} F_n$.

It remains to show that a is the only point in $\bigcap_{n \geq 1} F_n$. Assume, towards contradiction, that $\exists y \neq a$ such that $y \in \bigcap_{n \geq 1} F_n$. Then $y \in F_n$ for all $n \geq 1$, so $d(y, a) \leq \delta(F_n) \xrightarrow{n \rightarrow \infty} 0$, so $y = a$, contradiction.

(2) \Rightarrow (1): We want to show that (X, d) is complete. Let $\{x_n\}_{n \geq 1} \subseteq X$ be a Cauchy sequence. To prove that $\{x_n\}_{n \geq 1}$ converges in X , it suffices to show that $\{x_n\}_{n \geq 1}$ admits a subsequence that converges in X . Because the sequence is Cauchy, then there exists $n_1 \in \mathbb{N}$ such that $d(x_n, x_m) < \frac{1}{2^2}$ for all $n, m \geq n_1$. Now let $k_1 = n_1$ and select x_{k_1} . For similar reasons, we can find $n_2 \in \mathbb{N}$ such that $d(x_n, x_m) < \frac{1}{2^3}$ for all $n, m \geq n_2$. Now let $k_2 = \max\{n_2, k_1 + 1\}$ and we select x_{k_2} .

Proceeding inductively, we find a strictly increasing sequence $\{k_n\}_{n \geq 1} \subseteq \mathbb{N}$ such that $d(x_l, x_m) < \frac{1}{2^{n+1}}$ for all $l, m \geq k_n$. For $n \geq 1$, let $F_n = K_{\frac{1}{2^n}} = \{x \in X : d(x, x_{k_n}) \leq \frac{1}{2^n}\}$. Note $\emptyset \neq F_n = \bar{F}_n$, and $\delta(F_n) \leq 2 \cdot \frac{1}{2^n} \xrightarrow{n \rightarrow \infty} 0$.

Claim 25.5. $F_{n+1} \subseteq F_n$ for all $n \geq 1$.

Subproof. Let $y \in F_{n+1}$, then $d(y, x_{k_{n+1}}) \leq \frac{1}{2^{n+1}}$. By the triangle inequality, $d(y, x_{k_n}) \leq d(y, x_{k_{n+1}}) + d(x_{k_{n+1}}, x_{k_n}) \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n}$. Therefore, $y \in F_n$. As $y \in F_{n+1}$ was arbitrary, we get $F_{n+1} \subseteq F_n$. ■

By hypothesis, $\bigcap_{n \geq 1} F_n = \{a\}$ for some $a \in X$.

As for all $n \geq 1$, $a \in F_n$, we have $d(a, x_{k_n}) \leq \frac{1}{2^n} \xrightarrow{n \rightarrow \infty} 0$, then $x_{k_n} \xrightarrow{n \rightarrow \infty} a$, and because $\{x_n\}_{n \geq 1}$ is Cauchy, then $x_n \xrightarrow{n \rightarrow \infty} a$. □

We now look at some examples of complete metric spaces. For instance, we know $(\mathbb{R}, |\cdot|)$ is a complete metric space.

Lemma 25.6. Assume (A, d_1) and (B, d_2) are complete metric spaces. We define $d : (A \times B) \times (A \times B) \rightarrow \mathbb{R}$ via $d((a_1, b_1), (a_2, b_2)) = \sqrt{d_1^2(a_1, a_2) + d_2^2(b_1, b_2)}$. Then $(A \times B, d)$ is a complete metric space.

Proof.

Exercise 25.7. Show that d is a metric on $A \times B$.

Let us show that $A \times B$ is complete. Let $\{(a_n, b_n)\}_{n \geq 1} \subseteq A \times B$ be a Cauchy sequence. Fix $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $d((a_n, b_n), (a_m, b_m)) < \varepsilon$ for all $n, m \geq n_\varepsilon$. Therefore, for all $n, m \geq n_\varepsilon$, we have $\sqrt{d_1^2(a_n, a_m) + d_2^2(b_n, b_m)} < \varepsilon$. Therefore, $d_1(a_n, a_m), d_2(b_n, b_m) < \varepsilon$ for all $n, m \geq n_\varepsilon$. Therefore, $\{a_n\}_{n \geq 1}$ is a Cauchy sequence in A , and $\{b_n\}_{n \geq 1}$ is a Cauchy sequence in B . As A and B are complete metric spaces, there exists $a \in A, b \in B$ such that $a_n \xrightarrow[n \rightarrow \infty]{d_1} a$ and $b_n \xrightarrow[n \rightarrow \infty]{d_2} b$.

Claim 25.8. $(a_n, b_n) \xrightarrow[n \rightarrow \infty]{d} (a, b)$.

Subproof. Indeed, $d((a_n, b_n), (a, b)) = \sqrt{d_1^2(a_n, a) + d_2^2(b_n, b)} \leq d_1(a_n, a) + d_2(b_n, b) \xrightarrow[n \rightarrow \infty]{} 0$.

Therefore, $(a_n, b_n) \xrightarrow[n \rightarrow \infty]{d} (a, b)$. ■

□

Corollary 25.9. For $n \geq 2$, (\mathbb{R}^n, d_2) is a complete metric space.

Proof. Make use of induction. Left as an exercise. □

Exercise 25.10. Show that for all $n \geq 2$, (\mathbb{R}^n, d_p) is a complete metric space for all $1 \leq p \leq \infty$.

Example 25.11. We define

$$l^2 = \{\{x_n\}_{n \geq 1} \subseteq \mathbb{R} : \sum_{n \geq 1} |x_n|^2 < \infty\}.$$

We define a metric on l^2 as follows: for $x = \{x_n\}_{n \geq 1} \in l^2$ and $y = \{y_n\}_{n \geq 1} \in l^2$,

$$d_2(x, y) = \sqrt{\sum_{n \geq 1} |x_n - y_n|^2} < \varepsilon.$$

The fact that this is a metric follows from Minkowski's inequality.

26 LECTURE 26: COMPLETE METRIC SPACE, CONTINUED; SEPARATION

Claim 26.1. (l^2, d_2) is a complete metric space.

Proof. Let $\{x^{(k)}\}_{k \geq 1}$ be a Cauchy sequence in l^2 . Then we denote $x^{(n)} = \{x_1^{(n)}, x_2^{(n)}, \dots\}$.

By definition, for all $\varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}$ such that $d_2(x^{(k)}, x^{(l)}) < \varepsilon$ for all $k, l \geq k_\varepsilon$. Therefore,

$$d_2(x^{(k)}, x^{(l)}) = \sqrt{\sum_{n \geq 1} |x_n^{(k)} - x_n^{(l)}|^2} < \varepsilon$$

for all $k, l \geq k_\varepsilon$. Hence, $\sum_{n \geq 1} |x_n^{(k)} - x_n^{(l)}|^2 < \varepsilon^2$ for all $k, l \geq k_\varepsilon$. That is, for all $n \geq 1$, we have $|x_n^{(k)} - x_n^{(l)}| < \varepsilon$ for all $k, l \geq k_\varepsilon$.

So whenever $n \geq 1$, the sequence $\{x_n^{(k)}\}_{k \geq 1}$ is Cauchy in $(\mathbb{R}, |\cdot|)$. Since $(\mathbb{R}, |\cdot|)$ is complete, then there exists $x_n \in \mathbb{R}$ such that $x_n^{(k)} \xrightarrow[k \rightarrow \infty]{\mathbb{R}} x_n$. Let $x = \{x_n\}_{n \geq 1}$.

Claim 26.2. $x \in l^2$ and $x^{(k)} \xrightarrow[k \rightarrow \infty]{l^2} x$.

Subproof. Note $d_2(x^{(k)}, x) = \sqrt{\sum_{n \geq 1} |x_n^{(k)} - x_n|^2}$. While $|x_n^{(k)} - x_n| \xrightarrow[k \rightarrow \infty]{0}$ for all $n \geq 1$, the limit theorems do not apply to yield $\sum_{n \geq 1} |x_n^{(k)} - x_n|^2 \xrightarrow[k \rightarrow \infty]{} 0$. Instead, we argue as follows.

Fix $\varepsilon > 0$. As $\{x^{(k)}\}_{k \geq 1}$ is Cauchy in l^2 , there exists $k_\varepsilon \in \mathbb{N}$ such that $d_2(x^{(k)}, x^{(l)}) < \varepsilon$ for all $k, l \geq k_\varepsilon$. In particular, $\sum_{n \geq 1} |x_n^{(k)} - x_n^{(l)}|^2 < \varepsilon^2$ for all $k, l \geq k_\varepsilon$. So for each fixed $N \in \mathbb{N}$ we have

$$\sum_{n=1}^N |x_n^{(k)} - x_n^{(l)}|^2 < \varepsilon^2$$

for all $k, l \geq k_\varepsilon$. Note $\lim_{l \rightarrow \infty} |x_n^{(k)} - x_n^{(l)}| = |x_n^{(k)} - x_n|$ for all $n \geq 1$ and all $k \geq k_\varepsilon$. By the limit theorems, $\lim_{l \rightarrow \infty} \sum_{n=1}^N |x_n^{(k)} - x_n^{(l)}|^2 \leq \varepsilon^2$ for all $k \geq k_\varepsilon$. Therefore, $\sum_{n=1}^N |x_n^{(k)} - x_n|^2 \leq \varepsilon^2$ for all $k \geq k_\varepsilon$.

Note that $\{\sum_{n=1}^N |x_n^{(k)} - x_n|^2\}_{N \geq 1}$ is an increasing sequence bounded above by ε^2 . So $\sum_{n \geq 1} \sum_{n=1}^N |x_n^{(k)} - x_n|^2 \leq \varepsilon^2$ for all $k \geq k_\varepsilon$. That is to say, $d_2(x^{(k)}, x) \leq \varepsilon$ for all $k \geq k_\varepsilon$.

Finally, $x \in l^2$ if and only if $d_2(x, 0) < \infty$. But $d_2(x, 0) \leq d_2(x, x^{(k)}) + d_2(x^{(k)}, 0) < \infty$, where the first term is bounded above by ε for all $k \geq k_\varepsilon$, and the second term is finite since $x^{(k)} \in l^2$. ■

□

Exercise 26.3. 1. Fix $1 \leq p < \infty$ and let

$$l^p = \{ \{x_n\}_{n \geq 1} \subseteq \mathbb{R} : \sum_{n \geq 1} |x_n|^p < \infty \}.$$

We define $d_p : l^p \times l^p \rightarrow \mathbb{R}$ via

$$d_p(\{x_n\}_{n \geq 1}, \{y_n\}_{n \geq 1}) = \left(\sum_{n \geq 1} |x_n - y_n|^p \right)^{\frac{1}{p}}.$$

Then (l^p, d_p) is a complete metric space.

2. Define

$$l^\infty = \{ \{x_n\}_{n \geq 1} \subseteq \mathbb{R} : \sup_{n \geq 1} |x_n| < \infty \}.$$

We define $d_\infty : l^\infty \times l^\infty \rightarrow \mathbb{R}$ via

$$d_\infty(\{x_n\}_{n \geq 1}, \{y_n\}_{n \geq 1}) = \sup_{n \geq 1} |x_n - y_n|.$$

Then (l^∞, d_∞) is a complete metric space.

Definition 26.4 (Separated). Let (X, d) be a metric space and let $A, B \subseteq X$. We say that A and B are separated if $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$.

Remark 26.5. Separated sets are disjoint: $A \cap B \subseteq \bar{A} \cap B = \emptyset$. But disjoint sets need not be separated: consider $(X, d) = (\mathbb{R}, |\cdot|)$ where $A = (-1, 0)$ and $B = [0, 1)$. Then $A \cap B = \emptyset$, but $\bar{A} \cap B = \{0\} \neq \emptyset$, so A and B are not separated.

Remark 26.6. If A and B are separated and $A_1 \subseteq A$ and $B_1 \subseteq B$, then A_1 and B_1 are separated.

Lemma 26.7. Let (X, d) be a metric space and let $A, B \subseteq X$. If $d(A, B) > 0$, then A and B are separated.

Proof. Assume, towards contradiction, that A and B are not separated. Then $\bar{A} \cap B \neq \emptyset$ or $A \cap \bar{B} \neq \emptyset$. Say $\bar{A} \cap B \neq \emptyset$, then let $a \in \bar{A} \cap B$. Therefore, $a \in B$ and $a \in \bar{A}$, so $d(a, A) = 0$, and so $d(A, B) = 0$, contradiction. \square

Remark 26.8. Two sets A and B can be separated even if $d(A, B) = 0$. For example, let $A = (0, 1)$ and $B = (1, 2)$ in the usual metric space \mathbb{R} . The two sets are separated, but $d(A, B) = 0$.

Proposition 26.9. 1. Two closed sets A and B are separated if and only if $A \cap B = \emptyset$.

2. Two open sets A and B are separated if and only if $A \cap B = \emptyset$.

Proof. Two separated sets are disjoint. Therefore, we only have to prove the (\Leftarrow) direction in both statements.

1. Assume $A \cap B = \emptyset$, then since A is closed, $\bar{A} = A$, so $\bar{A} \cap B = A \cap B = \emptyset$. Similarly, since B is closed, then $\bar{B} = B$ and so $\bar{B} \cap A = B \cap A = \emptyset$. Therefore, A and B are separated.
2. Assume $A \cap B = \emptyset$, then $A \subseteq {}^c B$, where ${}^c B$ is closed. Therefore, $\bar{A} \subseteq \overline{{}^c B} = {}^c B$, so $\bar{A} \cap B = \emptyset$. A similar argument shows that $\bar{B} \cap A = \emptyset$, and so A and B are separated.

□

Proposition 26.10. 1. If an open set D is the union of two separated sets A and B , then A and B are both open.

2. If a closed set F is the union of two separated sets A and B , then A and B are both closed.

Proof. 1. If $A = \emptyset$, then since $D = A \cup B$, we have $B = D$ and so both A and B are open. Assume $A \neq \emptyset$. We want to show that A is open, which is equivalent to the statement $A = \mathring{A}$.

Let $a \in A \subseteq D$, since D is open, then there exists $r_1 > 0$ such that $B_{r_1}(a) \subseteq D$. As A and B are separated, then $A \cap \bar{B} = \emptyset$, so $a \in A \subseteq {}^c(\bar{B}) = {}^c\bar{B}$, so there exists $r_2 > 0$ such that $B_{r_2}(a) \subseteq {}^c B$. Let $r = \min\{r_1, r_2\}$, then $B_r(a) \subseteq D \cap {}^c B = (A \cup B) \cap {}^c B = A$, and so $a \in \mathring{A}$. This shows that A is open. A similar argument shows that B is open.

2. Let us show that A is closed, which is to show $\bar{A} = A$. We have $A \subseteq F$, and since F is closed, i.e., $F = \bar{F}$, then $\bar{A} \subseteq \bar{F} = F$, so $\bar{A} = \bar{A} \cap F = \bar{A} \cap (A \cup B) = (\bar{A} \cap A) \cup (\bar{A} \cap B) = A$ since A and B are separated. Similarly, one shows that $\bar{B} = B$, and so B is closed.

□

27 LECTURE 27: CONNECTEDNESS