MATH 595 (Group Cohomology) Notes

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1 Aug 21, 2023: Introduction

Group cohomology works over different settings of groups, like finite groups, profinite groups, and topological groups. The course will develop towards

- duality in $H^*(G, -)$, and
- focus on computations, e.g., spectral sequences.

We first establish some notations.

- Let G be a group. If G has a topology, that would also be part of the information of G.
- A (left) G-module is an abelian group M with an action map

$$\begin{aligned} G \times M &\to M \\ (g, m) &\mapsto g \cdot m = gm \end{aligned}$$

  satisfying
  - $1 \cdot m = m$,
  - $(gh) \cdot m = g \cdot (hm)$,
  - $g(m + m') = gm + gm'$.

Remark 1.1. If G is a finite group, then the associated (non-commutative) group ring $\mathbb{Z}[G] := \bigoplus_{g \in G} \mathbb{Z}e_g$, where the multiplication is determined by $e_g e_h = e_{gh}$. Therefore, a G-module is just a $\mathbb{Z}[G]$-module.

Example 1.2.
- Trivial module $\mathbb{Z}$, or any abelian group with the trivial action $g \cdot a = a$.
- C_2, or any group with $f : G \twoheadrightarrow C_2$, then G with C_2 as a quotient gives the sign representation $\mathbb{Z}_{\text{sgn}}$, with $g \cdot (a) = (-1)^{\rho(g)} a$.
- $\mathbb{Z}[G]$ is a G-module via the left multiplication action, and/or the conjugation action.

Definition 1.3 (Fixed points/Invariants)**.** The set of fixed points of M over G is $M^G = \{m \in M \mid gm = m \ \forall g \in G\}$.

Definition 1.4 (Orbits/Coinvariants)**.** The set of orbits of M over G is $M_G = M/(gm - m)$.

Example 1.5. If $M = \mathbb{Z}_{\text{sgn}}$, then everything gets multiplied by -1, so there are no fixed points. The orbits of M over G would be $\mathbb{Z}_{\text{sgn}}/(-2) \cong \mathbb{Z}/2\mathbb{Z}$.

Example 1.6. If $M = \mathbb{Z}[G]$, then the fixed points are $\mathbb{Z}\left\{\sum_{g \in G} e_g\right\}$.

Thinking in a categorical setting, we have a trivial action function $\mathbb{Z}$-**Mod** $\to$ G-**Mod**, sending $ga \mapsto a$ for all $g \in G$ and $a \in A$. This gives an exact functor from **Ab** to G-**Mod**. Then this functor has a right adjoint $(\ )^G : G$-**Mod** $\to$ **Ab**, and a left adjoint $(\ )_G$: **Ab** $\to$ G-**Mod**. More specifically, M^G becomes the maximal trivial action submodule of M, namely $\mathrm{Hom}_G(\mathbb{Z}, M)$; M_G becomes the largest quotient of M with trivial action, namely $\mathbb{Z} \otimes_{\mathbb{Z}[G]} M$. This simplifies to the tensor-hom adjunction in some sense. For a more detailed derivation of this, see Chapter 6.1 of Weibel.

Remark 1.7. In general, as in the category of G-sets, we have the orbit functor $X \mapsto X/G$ and the fixed point functor $X \mapsto X^G$. The orbit functor is left adjoint to the free G-set functor, and the fixed point functor is the right adjoint of the trivial G-set functor.

Remark 1.8. Read more about the setting in profinite groups with their topologies in Neukirch-Schmidt-Wingberg.

Definition 1.9 (Profinite Group)**.** A profinite group of a collection of groups is $G = \varprojlim_i G_i$ as an inverse limit, where each G_i is a finite group of the form G/U_i for some open U_i. This gives a topology to the profinite group.

Remark 1.10. The groups rings $\mathbb{Z}[[G]] = \varprojlim_i \mathbb{Z}[G_i]$. For instance, let $G = \hat{\mathbb{Z}}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$, then $\mathbb{Z}_p[[G]] = \varprojlim_n \mathbb{Z}_p[\mathbb{Z}/p^n\mathbb{Z}]$, where each $\mathbb{Z}[\mathbb{Z}/p^n\mathbb{Z}] \cong \bigoplus_{i=0}^{n-1} \mathbb{Z}\{e_i\}$ where $e_i \cdot e_j = e_{ij}$. Therefore, $\mathbb{Z}_p[[G]]$ is now equivalent to $\varprojlim_n \mathbb{Z}_p[t]/(t^{p^n} - 1_e)$, and hence becomes a power series.

Remark 1.11. By a change of variables, this becomes $\varprojlim_n \mathbb{Z}_p[x]/(x^{p^n})$, but this only works in the finite group $\mathbb{Z}_p$ case, and not in general for $\mathbb{Z}$.

Example 1.12. $\mathbb{Z}[C_n] \cong \mathbb{Z}\{e\} \oplus \mathbb{Z}\{g\} \oplus \mathbb{Z}\{g^2\} \oplus \cdots \oplus \mathbb{Z}\{g^{n-1}\} \cong \mathbb{Z}[g]/(g^n - 1_e)$.

2 Aug 23, 2021: Cohomology of groups

3 Aug 25, 2021: Cohomology of groups, continued