

# MATH 540 Notes

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## 1 ABSTRACT MEASURE THEORY

### 1.1 INTRODUCTION

**Definition 1.1.** Let  $X$  be an (non-empty) underlying space we are working over. We denote  $\mathcal{P}(X)$  to be the power set of  $X$ , i.e., the set of all subsets of  $X$ .

**Example 1.2.** Let  $X = \{1, 2\}$ , then  $\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ .

**Remark 1.3.** If  $X$  is a finite set of size  $n$ , then  $\mathcal{P}(X)$  is a finite set of size  $2^n$ .

We will consider a subcollection  $\mathcal{A}$  of subsets of  $X$ , i.e., a subset of the power set. We will try to define this as an algebra. Note that an algebra is just a ring with a module structure with respect to some other ring.

**Definition 1.4.**  $\mathcal{A} \subseteq \mathcal{P}(X)$  is an algebra on  $X$  if it is

- a. closed under finite union, i.e., given  $E_1, E_2 \in \mathcal{A}$ , then  $E_1 \cup E_2 \in \mathcal{A}$ , and
- b. closed under complements, i.e., if  $E \in \mathcal{A}$ , then the complement  $E^c \in \mathcal{A}$  as well.

**Remark 1.5.** An algebra  $\mathcal{A}$  would be closed under finite intersection. Indeed, for any  $E_1, E_2 \in \mathcal{A}$ , we have  $E_1 \cap E_2 \in \mathcal{A}$  if and only if  $(E_1 \cap E_2)^c \in \mathcal{A}$ , if and only if  $E_1^c \cup E_2^c \in \mathcal{A}$ , which is true by definition.

**Lemma 1.6.** If  $\mathcal{A}$  is a non-empty algebra on  $X$ , then  $\emptyset \in \mathcal{A}$  and  $X \in \mathcal{A}$ .

*Proof.* Since  $\mathcal{A}$  is non-empty, take  $E \in \mathcal{A}$ , then  $\emptyset = E \cap E^c \in \mathcal{A}$  as well. Also,  $X = E \cup E^c \in \mathcal{A}$ . □

**Example 1.7.** Let  $X$  be a set, and let  $\mathcal{A} = \{\emptyset, X\} \subseteq \mathcal{P}(X)$ . It is easy to verify that  $\mathcal{A}$  is an algebra.

**Definition 1.8.** Let  $\emptyset \neq \mathcal{A} \subseteq \mathcal{P}(X)$  be an algebra, then we say  $\mathcal{A}$  is a  $\sigma$ -algebra on  $X$  if

- a. closed under countable union, i.e., if  $E_j \in \mathcal{A}$  for all  $j \in \mathbb{N}$ , then  $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$ ;
- b. if  $E \in \mathcal{A}$ , then  $E^c \in \mathcal{A}$ .

**Lemma 1.9.** If  $\mathcal{A} \neq \emptyset$  is a  $\sigma$ -algebra on  $X$ , then  $\{\emptyset, X\} \subseteq \mathcal{A}$  is a  $\sigma$ -algebra.

**Example 1.10.** Let  $X$  be an uncountable set, let  $\mathcal{A} = \{E \subseteq X : E \text{ is countable or } E^c \text{ is countable}\}$ , then  $\mathcal{A}$  is a  $\sigma$ -algebra on  $X$ .

**Theorem 1.11.** Suppose a non-empty algebra  $\mathcal{A} \subseteq \mathcal{P}(X)$  such that,

- if  $E_j \in \mathcal{A}$  for all  $j \in \mathbb{N}$ , and  $E_j$ 's are pairwise disjoint, then  $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$ ,

then  $\mathcal{A}$  is a  $\sigma$ -algebra on  $X$ .

*Proof.* Take  $E_j \in \mathcal{A}$  for all  $j \in \mathbb{N}$ , we will show that  $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$ . To do this, we will rearrange the sets. Let  $F_1 = E_1$ , let  $F_2 = E_2 \setminus E_1$ , let  $F_3 = E_3 \setminus (E_1 \cup E_2)$ , and so on, such that let  $F_k = E_k \setminus \bigcup_{i=1}^{k-1} E_i$ . We note

$$\begin{aligned} F_k &= E_k \cap \left( \bigcup_{j=1}^{k-1} E_j \right)^c \\ &= E_k \cap \left( \bigcap_{j=1}^{k-1} E_j^c \right) \in \mathcal{A}. \end{aligned}$$

One can also verify that  $\bigcup_{j=1}^{\infty} E_j = \bigcup_{k=1}^{\infty} F_k$ , and that  $F_k$ 's are disjoint from the definition.  $\square$

**Definition 1.12.** Let  $X$  be a non-empty space. A topology on  $X$  is a family  $\mathcal{F}$  of subsets of  $X$  satisfying the following conditions:

- i.  $\emptyset, X \in \mathcal{F}$ ;
- ii.  $\mathcal{F}$  is closed under arbitrary union;
- iii.  $\mathcal{F}$  is closed under finite intersection.

Every member of  $\mathcal{F}$  is now called an open subset of  $X$ . A complement of an open subset of  $X$  is called a closed subset.

**Definition 1.13.** Let  $\mathcal{A}_1, \mathcal{A}_2$  be  $\sigma$ -algebras. We say  $\mathcal{A}_1$  is smaller than  $\mathcal{A}_2$  if  $\mathcal{A}_1 \subseteq \mathcal{A}_2$ , and equivalently  $\mathcal{A}_2$  is larger than  $\mathcal{A}_1$ .

**Definition 1.14.** Let  $\mathcal{F}$  be a family of subsets of  $X$ , the smallest  $\sigma$ -algebra containing  $\mathcal{F}$  is called the  $\sigma$ -algebra generated by  $\mathcal{F}$ . This is denoted by  $\mathcal{M}(\mathcal{F})$ .

**Lemma 1.15.** Let  $\mathcal{F}$  be a family of subsets of  $X$ . Suppose  $\mathcal{F} \subseteq \mathcal{A}$  where  $\mathcal{A}$  is a  $\sigma$ -algebra, then  $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{A}$ .

*Proof.* Obvious.  $\square$

**Definition 1.16.** Let  $\mathcal{F}$  be a topology on  $X$ , then we say  $(X, \mathcal{F})$  is a topological space. We say  $\mathcal{M}(\mathcal{F})$  is the Borel  $\sigma$ -algebra on  $X$ , denoted by  $\mathcal{B}_X = \mathcal{B}_{X, \mathcal{F}}$ . Any member of  $\mathcal{B}_X$  is called a Borel set.

**Example 1.17.** Let  $X = \mathbb{R}$ , we denote the corresponding Borel  $\sigma$ -algebra to be  $\mathcal{B}_{\mathbb{R}}$ .

**Definition 1.18.** A  $G_{\delta}$ -set is a countable intersection of open subsets of  $X$ . A  $F_{\sigma}$ -set is a countable union of closed subsets of  $X$ .

**Theorem 1.19.** Both  $G_{\delta}$ -sets and  $F_{\sigma}$ -sets are Borel sets, that is,  $G_{\delta}, F_{\sigma} \subseteq \mathcal{B}_X$ .

*Proof.* We will prove that any  $G_{\delta}$ -set  $E$  is a Borel set, and similarly any  $F_{\sigma}$ -set is a Borel set. By definition  $E = \bigcap_{j=1}^{\infty} O_j$ , where each  $O_j$  is an open subset. To show  $E \in \mathcal{B}_X$ , we show that  $E^c \in \mathcal{B}_X$ . Note that  $E^c = \left( \bigcap_{j=1}^{\infty} O_j \right)^c = \bigcup_{j=1}^{\infty} O_j^c$ . Since  $O_j \in \mathcal{B}_X$  for all  $j$ , then  $O_j^c \in \mathcal{B}_X$  as well. Therefore,  $E^c \in \mathcal{B}_X$  since a  $\sigma$ -algebra  $\mathcal{B}_X$  is closed under countable unions.  $\square$

**Definition 1.20.** Let  $X_1, \dots, X_n$  be non-empty spaces. The product space is  $\prod_{j=1}^n X_j$ . Define  $\pi_j : \prod_{i=1}^n X_i \rightarrow X_j$  by  $\pi_j(x_1, \dots, x_n) = x_j$ . Let  $\mathcal{A}_j$  be a  $\sigma$ -algebra on  $X_j$ , the product  $\sigma$ -algebra on  $\prod_{i=1}^n X_i$  is the  $\sigma$ -algebra generated by  $\{\pi_j^{-1}(E_j) : E_j \in \mathcal{A}_j \forall j \in \{1, \dots, n\}\}$ . The product  $\sigma$ -algebra is denoted by  $\bigotimes_{j=1}^n \mathcal{A}_j = \prod_{j=1}^n \mathcal{A}_j$ .

**Example 1.21.**  $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_{j=1}^n \mathcal{B}_{\mathbb{R}}$ .

## 1.2 MEASURES

**Definition 1.22.** Let  $\mathcal{A}$  be a  $\sigma$ -algebra on  $X$ . A measure  $\mu$  on  $X$  and  $\mathcal{A}$  is a function  $\mu : \mathcal{A} \rightarrow [0, \infty]$  such that

- a.  $\mu(\emptyset) = 0$ ;
- b. if  $E_j \in \mathcal{A}$  for all  $j \in \mathbb{N}$  and  $E_j$ 's are disjoint, then  $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$ .

We then say  $(X, \mathcal{A})$  is a measureable space. A measureable space is a triple  $(X, \mathcal{A}, \mu)$  with measure  $\mu$  specified.

**Definition 1.23.** Let  $\mu$  be a measure on  $(X, \mathcal{A})$ .

1. If  $\mu(X) < \infty$ , then we say  $\mu$  is a finite measure. In particular, if  $\mu(X) = 1$ , this is a probability measure.
2. If  $X = \bigcup_{j=1}^{\infty} E_j$  such that  $\mu(E_j) < \infty$  for all  $j \in \mathbb{N}$ , then we say  $\mu$  is  $\sigma$ -finite.
3. If for all  $E \in \mathcal{A}$  with  $\mu(E) = \infty$ , there is  $F \in \mathcal{A}$  such that  $F \subseteq E$  and  $0 < \mu(F) < \infty$ , then we say  $\mu$  is semi-finite.

**Remark 1.24.** A  $\sigma$ -finite measure is semi-finite. However, the converse is not true.

**Example 1.25.** Let  $f : X \rightarrow [0, \infty]$  be a function. For any  $E \subseteq \mathcal{P}(E)$ , we can define a measure  $\mu(E) = \sum_{x \in E} f(x)$ . Note that the summation makes sense only when  $E$  is finite. In case  $E$  is infinite, we should define  $\sum_{x \in E} f(x) = \sup \left\{ \sum_{x \in F} f(x) : F \subseteq E \text{ for finite } F \right\}$ . Let  $\mu$  be a measure on  $\mathcal{P}(X)$ .

- If  $f(x) \equiv 1$  for all  $x \in X$ , then  $\mu(E) = \sum_{x \in E} 1 = \text{Card}(E)$ . In this case,  $\mu$  is called a counting measure.
- Suppose  $x_0 \in X$  is fixed. Define

$$f(x) = \begin{cases} 1, & \text{if } x = x_0 \\ 0, & \text{if } x \neq x_0 \end{cases}$$

then for any  $E \in \mathcal{P}(X)$ ,

$$\mu(E) = \begin{cases} 1, & \text{if } x_0 \in E \\ 0, & \text{if } x_0 \notin E \end{cases}$$

This is called the Dirac measure of  $x_0$ .

**Definition 1.26.** Let  $(X, \mathcal{A}, \mu)$  be a measure space. A set  $E \subseteq \mathcal{A}$  is called a null set if  $\mu(E) = 0$ .

If a statement about points  $x \in X$  is true except for null sets, then we say the statement is true almost everywhere.

**Example 1.27.** Suppose  $f(x) \leq 1$  for all  $x \in X$ , then we say  $f$  is bounded above by 1 everywhere. If we want to weaken this statement, we can say  $f(x) \leq 1$  almost everywhere  $x \in X$ , which is true if and only if  $\mu(\{x \in X : f(x) > 1\}) = 0$ .

**Theorem 1.28.** Let  $E, F \in \mathcal{A}$  be such that  $E \subseteq F$ , then  $\mu(E) \leq \mu(F)$ .

*Proof.* We can write  $F = E \cup (E \setminus F)$ , then

$$\begin{aligned} \mu(F) &= \mu(E) + \mu(F \setminus E) \\ &\geq \mu(E) \end{aligned}$$

since  $\mu(F \setminus E) \geq 0$ . □

**Theorem 1.29** (Sub-additivity). Let  $E_j \in \mathcal{A}$  for all  $j \in \mathbb{N}$ , then  $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu(E_j)$ .

*Proof.* Set  $F_1 = E_1$  and let  $F_k = E_k \setminus \left( \bigcup_{j=1}^{k-1} E_j \right)$  be defined inductively, then  $\bigcup_{k \in \mathbb{N}} F_k = \bigcup_{j \in \mathbb{N}} E_j$ . Since  $F_k$ 's are disjoint, we have

$$\begin{aligned} \mu \left( \bigcup_{j \in \mathbb{N}} E_j \right) &= \mu \left( \bigcup_{k \in \mathbb{N}} F_k \right) \\ &= \sum_{k=1}^{\infty} \mu(F_k) \\ &= \sum_{k=1}^{\infty} \mu(E_k) \\ &= \sum_{j=1}^{\infty} \mu(E_j) \end{aligned}$$

by [Theorem 1.28](#). □

**Theorem 1.30.** Let  $E_j \in \mathcal{A}$  for all  $j \in \mathbb{N}$ .

a. (Continuity from below): If  $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_j \subseteq \cdots$  for all  $j$ , then  $\mu \left( \bigcup_{j=1}^{\infty} E_j \right) = \lim_{j \rightarrow \infty} \mu(E_j)$ .

b. (Continuity from above): If  $E_1 \supseteq E_2 \supseteq \cdots \supseteq E_j \supseteq \cdots$  for all  $j \in \mathbb{N}$ , then  $\mu \left( \bigcap_{j=1}^{\infty} E_j \right) = \lim_{j \rightarrow \infty} \mu(E_j)$  if  $\mu(E_1) < \infty$ .

In particular, the limits on the right exist on  $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ .

**Example 1.31.** Let  $\mu$  be the counting measure on  $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ . For each  $j \in \mathbb{N}$ , we define  $E_j = \{n \in \mathbb{N} : n > j\}$ . Therefore  $E_1 \supseteq E_2 \supseteq \cdots$  is a decreasing sequence of sets. Note that  $\mu(E_1) = \mu(\{n \in \mathbb{N}\}) = \mathbb{N} = \infty$ , and  $\lim_{j \rightarrow \infty} \mu(E_j) =$

$\lim_{j \rightarrow \infty} \infty = \infty$ , but  $\mu \left( \bigcap_{j=1}^{\infty} E_j \right) = \mu(\emptyset) = 0$ .

*Proof.* a. Set  $E_0 = \emptyset$ . Now

$$\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} (E_j \setminus E_{j-1})$$

and therefore

$$\begin{aligned} \mu \left( \bigcup_{j=1}^{\infty} E_j \right) &= \mu \left( \bigcup_{j=1}^{\infty} (E_j \setminus E_{j-1}) \right) \\ &= \sum_{j=1}^{\infty} \mu(E_j \setminus E_{j-1}) \\ &= \lim_{k \rightarrow \infty} \sum_{j=1}^k \mu(E_j \setminus E_{j-1}) \\ &= \lim_{k \rightarrow \infty} \mu \left( \bigcup_{j=1}^k E_j \setminus E_{j-1} \right) \\ &= \lim_{k \rightarrow \infty} \mu(E_k) \\ &= \lim_{j \rightarrow \infty} \mu(E_j). \end{aligned}$$

- b. For any  $j \in \mathbb{N}$ , set  $F_j = E_1 \setminus E_j$ . Note that  $F_j \subseteq F_{j+1}$  since  $E_j \supseteq E_{j-1}$ . This is now an increasing sequence as in part a. By part a., we know  $\mu\left(\bigcup_{j=1}^{\infty} F_j\right) = \lim_{j \rightarrow \infty} \mu(F_j)$ . Now note that

$$\begin{aligned}
 \bigcup_{j=1}^{\infty} F_j &= \bigcup_{j=1}^{\infty} (E_1 \setminus E_j) \\
 &= \bigcup_{j=1}^{\infty} (E_1 \cap E_j^c) \\
 &= E_1 \cap \bigcup_{j=1}^{\infty} E_j^c \\
 &= E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j\right)^c \\
 &= \left(\left(E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j\right)^c\right) \cup \left(\bigcap_{j=1}^{\infty} E_j\right)\right) \cap \left(\bigcap_{j=1}^{\infty} E_j\right)^c \\
 &= \left(E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j\right)^c\right) \cup \left(\bigcap_{j=1}^{\infty} E_j\right).
 \end{aligned}$$

Note that  $E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j\right)^c$  and  $\bigcap_{j=1}^{\infty} E_j$  are disjoint, therefore by property of measure we have

$$\begin{aligned}
 \mu(E_1) &= \mu\left(E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j\right)^c\right) + \mu\left(\bigcap_{j=1}^{\infty} E_j\right) \\
 &= \mu\left(\bigcup_{j=1}^{\infty} F_j\right) + \mu\left(\bigcap_{j=1}^{\infty} E_j\right) \\
 &= \lim_{j \rightarrow \infty} \mu(F_j) + \mu\left(\bigcap_{j=1}^{\infty} E_j\right).
 \end{aligned}$$

Recall that  $F_j = E_1 \setminus E_j$  for all  $j$ , therefore  $E_1 = F_j \cup F_j^c = F_j \cup E_j$ , where  $F_j$  and  $E_j$  are disjoint, therefore  $\mu(E_1) = \mu(F_j) + \mu(E_j)$ . Since  $\mu(E_1) < \infty$ , and  $F_j$  is a subset of  $E_1$  and hence also a real number, then  $\mu(E_1)$  is a sum of two real numbers. Therefore, we have  $\mu(E_1) - \mu(E_j) = \mu(F_j)$ . With this, we have

$$\begin{aligned}
 \mu(E_1) &= \lim_{j \rightarrow \infty} (\mu(E_1) - \mu(E_j)) + \mu\left(\bigcap_{j=1}^{\infty} E_j\right) \\
 &= \mu(E_1) - \lim_{j \rightarrow \infty} (\mu(E_j)) + \mu\left(\bigcap_{j=1}^{\infty} E_j\right).
 \end{aligned}$$

In particular, we get

$$\lim_{j \rightarrow \infty} (\mu(E_j)) = \mu\left(\bigcap_{j=1}^{\infty} E_j\right).$$

□

### 1.3 OUTER MEASURE

**Definition 1.32.** An outer measure  $\mu^*$  on  $X$  (or  $\mathcal{P}(X)$ ) is a function  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$  such that

- i.  $\mu^*(\emptyset) = 0$ ,
- ii.  $\mu^*(A) \leq \mu^*(B)$  for all  $A \subseteq B \subseteq X$ ,
- iii.  $\sigma$ -subadditivity:  $\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$ .

**Example 1.33.** Let  $\rho : \mathcal{A} \rightarrow [0, \infty]$  be such that  $\rho(\emptyset) = 0$ , where  $\mathcal{A} \subseteq \mathcal{P}(X)$  is a subcollection (but not necessarily an algebra) such that  $\emptyset, X \in \mathcal{A}$ .

For all  $A \in \mathcal{P}(X)$ , i.e.,  $A \subseteq X$ , we define

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \rho(E_j) : E_j \in \mathcal{A} \text{ and } A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}.$$

**Theorem 1.34.**  $\mu^*$  defined in Example 1.33 is an outer measure.

*Proof.* i. Let  $E_j = \emptyset$  for all  $j \in \mathbb{N}$ , then  $\emptyset \subseteq \bigcup_{j=1}^{\infty} E_j$ , and so

$$\sum_{j=1}^{\infty} \rho(E_j) = \sum_{j=1}^{\infty} \rho(\emptyset) = \sum_{j=1}^{\infty} 0 = 0$$

and therefore  $\mu^*(\emptyset) = 0$ .

ii. Let  $A \subseteq B \subseteq X$ . If  $B \subseteq \bigcup_{j=1}^{\infty} E_j$ , we have  $A \subseteq \bigcup_{j=1}^{\infty} E_j$ , then

$$\left\{ \sum_{j=1}^{\infty} \rho(E_j) : E_j \in \mathcal{A}, B \subseteq \bigcup_{j=1}^{\infty} E_j \right\} \subseteq \left\{ \sum_{j=1}^{\infty} \rho(E_j) : E_j \in \mathcal{A}, A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}.$$

In particular, given subsets  $S_1 \subseteq S_2$ , then  $\inf S_2 \leq \inf S_1$  and  $\sup S_1 \leq \sup S_2$ . This implies  $\mu^*(A) \leq \mu^*(B)$ .

iii. We want to show  $\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$ . Now for any  $j \in \mathbb{N}$ , we have

$$\mu^*(A_j) = \inf \left\{ \sum_{k=1}^{\infty} \rho(E_k) : E_k \in \mathcal{A} \text{ and } A_j \subseteq \bigcup_{k=1}^{\infty} E_k \right\}.$$

For any  $\varepsilon > 0$ , we note that  $\mu^*(A_j) + \varepsilon \cdot 2^{-j}$  is not a lower bound of  $\left\{ \sum_{k=1}^{\infty} \rho(E_k) : E_k \in \mathcal{A} \text{ and } A_j \subseteq \bigcup_{k=1}^{\infty} E_k \right\}$ .

Then there exists  $E_k^{(j)} \in \mathcal{A}$  for  $k \in \mathbb{N}$  such that  $A_j \subseteq \bigcup_{k=1}^{\infty} E_k^{(j)}$  and  $\sum_{k=1}^{\infty} \rho(E_k^{(j)}) \leq \mu^*(A_j) + \varepsilon \cdot 2^{-j}$ . Summing with respect to  $j$ , we get

$$\begin{aligned} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \rho(E_k^{(j)}) &\leq \sum_{j=1}^{\infty} \mu^*(A_j) + \sum_{j=1}^{\infty} \varepsilon \cdot 2^{-j} \\ &= \sum_{j=1}^{\infty} \mu^*(A_j) + \varepsilon. \end{aligned}$$

Note that

$$\bigcup_{j=1}^{\infty} A_j \subseteq \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} E_k^{(j)}$$

is a countable union of subsets of  $\mathcal{A}$ . We will calculate the value over  $\mu^*$ . By definition of  $\mu^*$ , we have

$$\begin{aligned}\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) &\leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \rho(E_k^{(j)}) \\ &\leq \sum_{j=1}^{\infty} \mu^*(A_j) + \varepsilon.\end{aligned}$$

Since this is true for all  $\varepsilon > 0$ , then take  $\varepsilon \rightarrow 0$ , we are done.  $\square$

**Definition 1.35.** Let  $\mu^*$  be an outer measure on  $(X, \mathcal{P}(X))$ . A set  $A \subseteq X$  is called  $\mu^*$ -measurable if  $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$  for all  $E \subseteq X$ .

**Remark 1.36.** First note that  $\mu^*(E) = \mu^*((E \cap A) \cup (E \cap A^c))$ , therefore  $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$  for all  $E \subseteq X$ .

**Theorem 1.37** (Fundamental Theorem of Measure Theory). Let  $\mu^*$  be an outer measure on  $X$ . Let  $\mathcal{A}$  be the collection of all  $\mu^*$ -measurable set, then  $\mathcal{A}$  is a  $\sigma$ -algebra, and  $\mu^*|_{\mathcal{A}}$  is a measure on  $\mathcal{A}$ , i.e.,  $(X, \mathcal{A}, \mu^*)$  is a measure space.

*Proof.* We first prove that  $\mathcal{A}$  is an algebra. To see  $\mathcal{A}$  is closed under complement, we have  $A \in \mathcal{A}$  if and only if  $A^c \in \mathcal{A}$  by the definition of measurable set. To show  $\mathcal{A}$  is closed under finite union, suppose  $A, B \in \mathcal{A}$ , and we want to show  $A \cup B \in \mathcal{A}$ , which is true if and only if  $\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$  for all  $E \subseteq X$ , hence it suffices to show that  $\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$ . We have

$$\begin{aligned}\mu^*(E \cap (A \cup B)) &= \mu^*(E \cap (A \cup B) \cap A) + \mu^*(E \cap (A \cup B) \cap A^c) \\ &= \mu^*(E \cap A) + \mu^*(E \cap B \cap A^c)\end{aligned}$$

and

$$\begin{aligned}\mu^*(E \cap (A \cup B)^c) &= \mu^*(E \cap (A \cup B)^c \cap A) + \mu^*(E \cap (A \cup B)^c \cap A^c) \\ &= \mu^*(\emptyset) + \mu^*(E \cap A^c \cap B^c) \\ &= \mu^*(E \cap A^c \cap B^c).\end{aligned}$$

Therefore

$$\begin{aligned}\mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c) &= \mu^*(E \cap A) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\ &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E)\end{aligned}$$

where the last two steps follow from the fact that  $A, B \in \mathcal{A}$  are  $\mu^*$ -measurable. Therefore,  $\mathcal{A}$  is an algebra. We now want to show that it is a  $\sigma$ -algebra. It suffices to prove that  $\mathcal{A}$  is closed under disjoint  $\sigma$ -unions. Let  $A_j \in \mathcal{A}$  for all  $j \in \mathbb{N}$  where they are pairwise disjoint, and we want to show that  $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$ . That is,

$$\mu^*(E) = \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)\right) + \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)^c\right)$$

for all  $E \subseteq X$ .

**Lemma 1.38.** For a pairwise disjoint family  $A_1, \dots, A_n \in \mathcal{A}$ ,

$$\mu^*\left(E \cap \bigcup_{j=1}^n A_j\right) = \sum_{j=1}^n \mu^*(E \cap A_j).$$

*Subproof.* We proceed by induction. For  $n = 1$ , this is obviously true. Now suppose  $n > 1$ . To simplify the notation, let  $B_n = \bigcup_{j=1}^n A_j$ , and use the convention that  $B_0 = \emptyset$ . Now

$$\begin{aligned}\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) \\ &= \sum_{i=1}^n (E \cap A_i) + \mu^*(E \cap B_0) \\ &= \sum_{i=1}^n (E \cap A_i) \\ &= \sum_{i=1}^n (E \cap A_i)\end{aligned}$$

for all  $n \in \mathbb{N}$ . This finishes the proof. ■

Now for any  $E \subseteq X$ , we have

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \\ &= \sum_{j=1}^n \mu^*(E \cap A_j) + \mu^*(E \cap B_n^c) \\ &\geq \sum_{j=1}^n \mu^*(E \cap A_j) + \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)^c\right)\end{aligned}$$

since  $B_n = \bigcup_{j=1}^n A_j \subseteq \bigcup_{j=1}^{\infty} A_j$ . Now take  $n \rightarrow \infty$ , we get

$$\begin{aligned}\mu^*(E) &\geq \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*\left(\left(\bigcup_{j=1}^{\infty} A_j\right)^c\right) \\ &\geq \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)\right) + \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)^c\right) \\ &\geq \mu^*(E).\end{aligned}$$

This forces all inequalities here to be equality, therefore

$$\mu^*(E) = \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)\right) + \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)^c\right)$$

as desired. Finally, we need to show that the restriction still gives a measure. We already know

$$\mu^*(E) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*\left(\left(\bigcup_{j=1}^{\infty} A_j\right)^c\right)$$

for any  $E \subseteq X$ , then in particular take  $E = \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$  to be the disjoint union, then this forces

$$\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*(\emptyset) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j).$$

Therefore  $\mu^*|_{\mathcal{A}}$  is a measure. □



**Definition 1.39.** A measure  $\mu$  is said to be complete if its domain contains all subsets of null sets.

**Example 1.40.** Let  $X = \{a, b\}$ ,  $\mathcal{A} = \{\emptyset, \{a, b\}\}$ . Define  $\mu : \mathcal{A} \rightarrow [0, \infty]$  by setting  $\mu^*(X) = 0$ ,  $\mu^*(\emptyset) = 0$ . This is not a complete measure because  $\{a\} \notin \mathcal{A}$ .

**Theorem 1.41.** Let  $\mathcal{A}$  be the collection of all  $\mu^*$ -measurable sets, then the measure  $\mu^*|_{\mathcal{A}}$  is complete.

*Proof.* Let  $N$  be any null set in  $\mathcal{A}$ , i.e.,  $\mu^*(N) = 0$ . Take an arbitrary subset  $A \subseteq N$ , we need to show  $A \in \mathcal{A}$ . Since  $\mu^*(N) = 0$ , then  $\mu^*(A) = 0$  as well. For any  $E \subseteq X$ , we prove  $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ . It is clear that

$$\begin{aligned} \mu^*(E) &\leq \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &\leq \mu^*(A) + \mu^*(E \cap A^c) \\ &\leq \mu^*(N) + \mu^*(E \cap A^c) \\ &= \mu^*(E \cap A^c) \\ &= \mu^*(E). \end{aligned}$$

by the subadditivity of  $\mu^*$ . □

**Definition 1.42.** Let  $\mathcal{A} \subseteq \mathcal{P}(X)$  be an algebra. A function  $\mu_0 : \mathcal{A} \rightarrow [0, \infty]$  is a pre-measure if

i.  $\mu_0(\emptyset) = 0$ ,

ii. if  $A_j \in \mathcal{A}$  for all  $j \in \mathbb{N}$  with  $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$ , and they are pairwise disjoint, then  $\mu_0\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu_0(A_j)$ .

Therefore, the difference of a pre-measure from a measure is that a pre-measure is not defined on a  $\sigma$ -algebra.

**Theorem 1.43.** Let  $\mu_0$  be a pre-measure, then  $\mu_0(A) \leq \mu_0(B)$  if  $A, B \in \mathcal{A}$  are such that  $A \subseteq B$ .

*Proof.* We write  $B = (B \setminus A) \cup A$ , where  $B \setminus A = B \cap A^c \in \mathcal{A}$ , therefore

$$\begin{aligned} \mu_0(B) &= \mu_0(B \setminus A) + \mu_0(A) \\ &\geq \mu_0(A). \end{aligned}$$

□

**Definition 1.44.** Given a pre-measure  $\mu_0$ , we extend it to an outer measure as follows: for any  $E \subseteq X$ , define  $\mu^*(E) = \inf\left\{\sum_{j=1}^{\infty} \mu_0(A_j) : E \subseteq \bigcup_{j=1}^{\infty} A_j, A_j \in \mathcal{A}\right\}$ .

**Theorem 1.45** (Carathéodory's Extension Theorem). Let  $\mu^*$  be the outer measure induced by  $\mu_0$  specified in Definition 1.44, then

- i.  $\mu^*|_{\mathcal{A}} = \mu_0$ , or equivalently, for any  $A \in \mathcal{A}$ , we have  $\mu^*(A) = \mu_0(A)$ ;
- ii. if  $A \in \mathcal{A}$ , then  $A$  is  $\mu^*$ -measurable.

*Proof.* i. We want to show that for any  $E \in \mathcal{A}$ ,  $\mu^*(E) = \mu_0(E)$ . To show  $\mu^*(E) \leq \mu_0(E)$ , we choose  $A_1 = E \in \mathcal{A}$ , and  $A_j = \emptyset$  for all  $j \geq 2$ , then  $E \subseteq \bigcup_{j=1}^{\infty} A_j$ , therefore

$$\begin{aligned} \mu^*(E) &\leq \sum_{j=1}^{\infty} \mu_0(A_j) \\ &= \mu_0(E). \end{aligned}$$

It now suffices to show that  $\mu_0(E)$  is a lower bound of  $\{\sum_{j=1}^{\infty} \mu_0(A_j) : E \subseteq \bigcup_{j=1}^{\infty} A_j, A_j \in \mathcal{A}\}$ . Let  $A_j \in \mathcal{A}$  and

$\bigcup_{j=1}^{\infty} A_j \supseteq E$ . We prove that  $\mu_0(E) \leq \sum_{j=1}^{\infty} \mu_0(A_j)$ . For any  $n \in \mathbb{N}$ , define  $B_n = E \cap \left(A_n \setminus \bigcup_{j=1}^{n-1} A_j\right)$ , therefore

$\bigcup_{n=1}^{\infty} B_n = E \cap \left(\bigcup_{j=1}^{\infty} A_j\right) = E$  where  $B_n$ 's are disjoint. We have

$$\begin{aligned} \mu_0(E) &= \mu_0\left(\bigcup_{n=1}^{\infty} B_n\right) \\ &= \sum_{n=1}^{\infty} \mu_0(B_n) \\ &\leq \sum_{n=1}^{\infty} \mu_0(A_n) \\ &= \sum_{j=1}^{\infty} \mu_0(A_j). \end{aligned}$$

- ii. For any  $A \in \mathcal{A}$ , we want to prove that  $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$  for all  $E \subseteq X$ . It suffices to show that for any  $E \subseteq X$ , we have  $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ .

Pick arbitrary  $\varepsilon > 0$ , then  $\mu^*(E) + \varepsilon$  is not a lower bound of  $\{\sum_{j=1}^{\infty} \mu_0(A_j) : E \subseteq \bigcup_{j=1}^{\infty} A_j, A_j \in \mathcal{A}\}$ . Therefore, there

exists some  $A_j \in \mathcal{A}$  such that  $E \subseteq \bigcup_{j=1}^{\infty} A_j$  and  $\sum_{j=1}^{\infty} \mu_0(A_j) \leq \mu^*(E) + \varepsilon$ . Since  $\mu_0(A_j) = \mu_0(A_j \cap A) + \mu_0(A_j \cap A^c)$ , then

$$\begin{aligned} \sum_{j=1}^{\infty} \mu_0(A_j) &= \sum_{j=1}^{\infty} \mu_0(A_j \cap A) + \sum_{j=1}^{\infty} \mu_0(A_j \cap A^c) \\ &= \sum_{j=1}^{\infty} \mu^*(A_j \cap A) + \sum_{j=1}^{\infty} \mu^*(A_j \cap A^c) \\ &\geq \mu^*\left(\left(\bigcup_{j=1}^{\infty} A_j\right) \cap A\right) + \mu^*\left(\left(\bigcup_{j=1}^{\infty} A_j\right) \cap A^c\right) \\ &\geq \mu^*(E \cap A) + \mu^*(E \cap A^c). \end{aligned}$$

Let  $\varepsilon \rightarrow 0$ , then  $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ , as desired. □

**Theorem 1.46.** Let  $\mathcal{A} \subseteq \mathcal{P}(X)$  be an algebra, and let  $\mu_0$  be a pre-measure on  $\mathcal{A}$ . Define  $\mathcal{M}(\mathcal{A})$  to be the  $\sigma$ -algebra generated by  $\mathcal{A}$ .

- a. The outer measure  $\mu^*$  induced by  $\mu_0$  defines a measure function on  $\mathcal{M}(\mathcal{A})$ , and  $\mu^*|_{\mathcal{A}} = \mu_0$ .
- b. If  $\tilde{\mu}$  is another measure on  $\mathcal{M}(\mathcal{A})$  that extends  $\mu_0$ , then  $\tilde{\mu}(E) \leq \mu^*(E)$  for all  $E \subseteq \mathcal{M}(\mathcal{A})$ , with equality if and only if  $\mu^*(E) < \infty$ .
- c. If  $\mu_0$  is  $\sigma$ -finite, i.e.,  $X = \bigcup_{j=1}^{\infty} A_j$  with  $A_j \in \mathcal{A}$  and  $\mu_0(A_j) < \infty$  for all  $j$ , then  $\mu^*|_{\mathcal{M}(\mathcal{A})}$  is the unique extension of  $\mu_0$  to a measure on  $\mathcal{M}(\mathcal{A})$ .

*Proof.* a. Let  $\mathcal{B}$  be the set of all  $\mu^*$ -measurable sets, then  $\mu^*|_{\mathcal{B}}$  is a measure on  $\mathcal{B}$  that extends  $\mu_0$ . By the fundamental theorem of measure theory, we know  $\mathcal{B}$  is a  $\sigma$ -algebra. In particular,  $\mathcal{B} \supseteq \mathcal{A}$ , therefore  $\mathcal{B} \supseteq \mathcal{M}(\mathcal{A})$ . That means  $\mu^*|_{\mathcal{M}(\mathcal{A})}$  is a measure as well.

b. Let  $\tilde{\mu}$  be any measure on  $\mathcal{M}(\mathcal{A})$  that extends  $\mu_0$ . We first show that for all  $E \in \mathcal{M}(\mathcal{A})$ , then  $\tilde{\mu}(E) \leq \mu^*(E)$ .

Recall that  $\mu^*(E) = \inf\{\sum_{j=1}^{\infty} \mu_0(A_j) : E \subseteq \bigcup_{j=1}^{\infty} A_j, A_j \in \mathcal{A}\}$ . Given a cover  $E \subseteq \bigcup_{j=1}^{\infty} A_j$  and fix  $A_j \in \mathcal{A}$ .

Therefore,

$$\begin{aligned} \tilde{\mu}(E) &\leq \tilde{\mu}\left(\bigcup_{j=1}^{\infty} A_j\right) \\ &\leq \sum_{j=1}^{\infty} \tilde{\mu}(A_j) \\ &= \sum_{j=1}^{\infty} \mu_0(A_j), \end{aligned}$$

therefore  $\tilde{\mu}(E) \leq \mu^*(E)$ . Assume we have  $\mu^*(E) < \infty$ , and we want to show that  $\tilde{\mu}(E) = \mu^*(E)$ . It suffices to show  $\mu^*(E) \leq \tilde{\mu}(E)$ .

**Claim 1.47.** Let  $A_j \in \mathcal{A}$  for all  $j \in \mathbb{N}$ , then  $\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) = \tilde{\mu}\left(\bigcup_{j=1}^{\infty} A_j\right)$ .

*Subproof.* Note that  $\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}(\mathcal{A})$ , then we can just work on  $\mathcal{M}(\mathcal{A})$ . Consider  $\mu^*|_{\mathcal{M}(\mathcal{A})}$  and  $\tilde{\mu}$  are measures on

$\mathcal{M}(\mathcal{A})$ . Let  $E_n = \bigcup_{j=1}^n A_j$  for all  $n \in \mathbb{N}$ , then we have a nested increasing sequence of  $E_n$ 's. In particular, we know

$\bigcup_{n=1}^{\infty} E_n = \bigcup_{j=1}^{\infty} A_j$ . Therefore

$$\begin{aligned} \mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) &= \mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \\ &= \lim_{n \rightarrow \infty} \mu^*(E_n) \\ &= \lim_{n \rightarrow \infty} \mu^*\left(\bigcup_{j=1}^n A_j\right) \\ &= \lim_{n \rightarrow \infty} \mu_0\left(\bigcup_{j=1}^n A_j\right) \\ &= \lim_{n \rightarrow \infty} \tilde{\mu}\left(\bigcup_{j=1}^n A_j\right) \\ &= \tilde{\mu}\left(\bigcup_{j=1}^{\infty} A_j\right) \end{aligned}$$

by continuity from below and closure of finite union. ■

We know from the claim that

$$\begin{aligned} \mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) &= \lim_{n \rightarrow \infty} \mu_0\left(\bigcup_{j=1}^n A_j\right) \\ &\leq \lim_{n \rightarrow \infty} \sum_{j=1}^n \mu_0(A_j) \end{aligned}$$

$$= \sum_{j=1}^{\infty} \mu_0(A_j).$$

Take arbitrary  $\varepsilon > 0$ , then consider  $\mu^*(E) + \varepsilon$ , which is not a lower bound of the set anymore. Therefore, there exists  $A_j \in \mathcal{A}$  for each  $j \in \mathbb{N}$  such that  $E \subseteq \bigcup_{j=1}^{\infty} A_j$  and that  $\sum_{j=1}^{\infty} \mu_0(A_j) \leq \mu^*(E) + \varepsilon$ . In particular, this means

$$\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \mu^*(E) + \varepsilon. \text{ Since } \mu^*(E) < \infty, \text{ then}$$

$$\begin{aligned} \mu^*\left(\bigcup_{j=1}^{\infty} A_j \setminus E\right) &= \mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) - \mu^*(E) \\ &< \varepsilon. \end{aligned}$$

Now that

$$\begin{aligned} \mu^*(E) &\leq \mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \\ &= \tilde{\mu}\left(\bigcup_{j=1}^{\infty} A_j\right) \\ &= \tilde{\mu}(E) + \tilde{\mu}\left(\bigcup_{j=1}^{\infty} A_j \setminus E\right) \\ &< \tilde{\mu}(E) + \varepsilon \end{aligned}$$

by the claim. Therefore, for any  $\varepsilon > 0$ , we have  $\mu^*(E) \leq \tilde{\mu}(E) + \varepsilon$  whenever  $\mu^*(E) < \infty$ . Take  $\varepsilon \rightarrow 0$ , we get  $\mu^*(E) \leq \tilde{\mu}(E)$ .

- c. Since  $\mu_0$  is  $\sigma$ -finite, then there exists a decomposition  $X = \bigcup_{j=1}^{\infty} A_j$  for  $A_j \in \mathcal{A}$  and that  $\mu_0(A_j) < \infty$ . For any  $E \in \mathcal{M}(\mathcal{A})$ , then

$$\begin{aligned} E &= E \cap X \\ &= E \cap \left(\bigcup_{j=1}^{\infty} A_j\right) \\ &= \bigcup_{j=1}^{\infty} (E \cap A_j) \end{aligned}$$

and

$$\begin{aligned} \mu^*(E) &= \mu^*\left(\bigcup_{j=1}^{\infty} (E \cap A_j)\right) \\ &= \sum_{j=1}^{\infty} \mu^*(E \cap A_j) \\ &= \sum_{j=1}^{\infty} \tilde{\mu}(E \cap A_j) \\ &= \tilde{\mu}\left(\bigcup_{j=1}^{\infty} (E \cap A_j)\right) \\ &= \tilde{\mu}(E) \end{aligned}$$

since  $\mu^*(E \cap A_j) \leq \mu^*(A_j) = \mu_0(A_j) < \infty$ .

□

## 1.4 BOREL MEASURE

Recall that the Borel  $\sigma$ -algebra  $\mathcal{B}_{\mathbb{R}}$  is the smallest  $\sigma$ -algebra containing all open sets. Let  $\mathcal{G}$  be the set of all open sets in  $\mathbb{R}$  with respect to the standard topology. Therefore  $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{G})$ . We can in fact use something smaller than  $\mathcal{G}$ .

**Theorem 1.48.**  $\mathcal{B}_{\mathbb{R}}$  is a  $\sigma$ -algebra generated by

- a.  $\mathcal{A}_0 = \{(a, b) : a, b \in \mathbb{R}, a < b\}$ , or by
- b.  $\mathcal{A}_1 = \{(a, b] : a, b \in \mathbb{R}, -\infty \leq a < b < \infty\} \cup \{(a, \infty) : -\infty \leq a < \infty\} \cup \{\emptyset\}$ .

Any member in  $\mathcal{A}_1$  is called an  $h$ -interval.

*Proof.* a. We want to show that  $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{A})$ . Obviously  $\mathcal{A}_0 \subseteq \mathcal{G}$ , then  $\mathcal{M}(\mathcal{G})$  is a  $\sigma$ -algebra containing  $\mathcal{A}_0$ , then  $\mathcal{M}(\mathcal{A}_0) \subseteq \mathcal{M}(\mathcal{G}) = \mathcal{B}_{\mathbb{R}}$ . Conversely, recall that any open subset in  $\mathbb{R}$  is a  $\sigma$ -union of open intervals, therefore  $\mathcal{G} \subseteq \mathcal{M}(\mathcal{A})$ , so  $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{G}) \subseteq \mathcal{M}(\mathcal{A}_0)$ , therefore  $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{A}_0)$ .

b. We first show that  $\mathcal{M}(\mathcal{A}_1) \subseteq \mathcal{B}_{\mathbb{R}}$ . Since  $\mathcal{M}(\mathcal{A}_1)$  is the smallest  $\sigma$ -algebra containing  $\mathcal{A}_1$ , then it suffices to show that  $\mathcal{A}_1 \subseteq \mathcal{B}_{\mathbb{R}}$ . It is easy to see that  $(a, b] = \bigcap_{n=1}^{\infty} (a, b + \frac{1}{n}) \in \mathcal{B}_{\mathbb{R}}$ , and  $(a, \infty) = \bigcup_{n=1}^{\infty} (a, n) \in \mathcal{B}_{\mathbb{R}}$ .

We now verify that  $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{A}_1)$ . By a. we know  $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{A}_0)$ , so it suffices to show that  $\mathcal{A}_0 \subseteq \mathcal{M}(\mathcal{A}_1)$ . For  $a < b$ , we have  $(a, b) = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}]$ , therefore the right-hand side is a  $\sigma$ -union of intervals, hence belongs to  $\mathcal{M}(\mathcal{A}_1)$ , and we are done.  $\square$

**Definition 1.49.** We define  $\mathcal{A}_2$  to be the collection of finite disjoint unions of  $h$ -intervals, e.g.,  $\bigcup_{j=1}^n (a_j, b_j]$ , then  $\mathcal{A}_2$  is an algebra.

**Definition 1.50.** A function on  $\mathbb{R}$  is said to be right continuous if  $\lim_{x \rightarrow x_0^+} F(x) = F(x_0)$ .

**Theorem 1.51.** Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be increasing and right continuous. Let  $I_j = (a_j, b_j]$  for  $j = 1, \dots, n$  be disjoint  $h$ -intervals. We define the pre-measure  $\mu_0$  on  $\mathcal{A}_2$  by  $\mu_0(\emptyset) = 0$  and  $\mu_0\left(\bigcup_{j=1}^n (a_j, b_j]\right) = \sum_{j=1}^n [F(b_j) - F(a_j)]$ .

*Proof.* First one can check that  $\mu_0$  is well-defined, that is, given any partition of  $h$ -interval, the  $\mu_0$ -measurements on the interval are the same.

Second, we need to show that  $\mu_0$  satisfies  $\sigma$ -additivity, that is, if  $\bigcup_{j=1}^{\infty} I_j \in \mathcal{A}_2$  such that  $I_j$ 's are disjoint, then

$$\mu_0\left(\bigcup_{j=1}^{\infty} I_j\right) = \sum_{j=1}^{\infty} \mu_0(I_j). \text{ It is easy to verify finite additivity, so we now assume}$$

$$\bigcup_{j=1}^{\infty} I_j = I = (a, b] \in \mathcal{A}_2$$

for  $-\infty \leq a < b < \infty$ , then we will show that

$$F(b) - F(a) = \mu_0(I) = \sum_{j=1}^{\infty} \mu_0(I_j)$$

for  $I_j = (a_j, b_j]$ .

To show  $\mu_0(I) \geq \sum_{j=1}^{\infty} \mu_0(I_j)$ , we know  $F(b) - F(a) \geq \sum_{j=1}^n [F(b_j) - F(a_j)]$ , therefore taking the limit of  $n \rightarrow \infty$  gives  $F(b) - F(a) \geq \sum_{j=1}^{\infty} \mu_0(I_j)$ .

To show  $\mu_0(I) \leq \sum_{j=1}^{\infty} \mu(I_j)$ , since  $F$  is right continuous, then for all  $\varepsilon > 0$ , there exist  $\delta > 0$  such that  $F(a + \delta) - F(a) < \varepsilon$ . Therefore, for every  $j > 0$ , there exists  $\delta_j > 0$  such that  $F(b_j + \delta_j) - F(b_j) < 2^{-j}\varepsilon$ , then

$$\begin{aligned} [a + \delta, b] &\subseteq (a, b] \\ &= \bigcup_{j=1}^{\infty} (a_j, b_j] \\ &= \bigcup_{j=1}^{\infty} (a_j, b_j + \delta_j). \end{aligned}$$

By compactness, there exists some  $N \in \mathbb{N}$  such that  $[a + \delta, b] \subseteq \bigcup_{j=1}^N (a_j, b_j + \delta_j)$ . Assume  $b_j + \delta_j \in (a_{j+1}, b_{j+1}]$ , then

$$\begin{aligned} \mu_0(I) &= \mu_0((a, b]) \\ &= F(b) - F(a) \\ &\leq F(b) - F(a + \delta) + \varepsilon \\ &\leq F(b_N + \delta_N) - F(a + \delta) + \varepsilon \\ &= F(b_N + \delta_N) - F(a_N) + F(a_N) - F(a + \delta) + \varepsilon \\ &= F(b_N + \delta_N) - F(a_N) + \sum_{j=1}^{N-1} [F(a_{j-1}) - F(a_j)] + \varepsilon \\ &\leq F(b_N + \delta_N) - F(a_N) + \sum_{j=1}^{N-1} [F(b_j + \delta_j) - F(a_j)] + \varepsilon \\ &= \sum_{j=1}^N [F(b_j + \delta_j) - F(a_j)] + \varepsilon \\ &= \sum_{j=1}^N [F(b_j + \delta_j) - F(b_j) + F(b_j) - F(a_j)] + \varepsilon \\ &= \sum_{j=1}^N [F(b_j + \delta_j) - F(b_j)] + \sum_{j=1}^N [F(b_j) - F(a_j)] + \varepsilon \\ &\leq \sum_{j=1}^N 2^{-j}\varepsilon + \sum_{j=1}^N \mu_0(I_j) + \varepsilon \\ &\leq 2\varepsilon + \sum_{j=1}^{\infty} \mu_0(I_j) \end{aligned}$$

since  $F$  is increasing. Let  $\varepsilon \rightarrow 0$  and we are done.  $\square$

**Theorem 1.52.** Let  $F$  be increasing and right-continuous, then

- there is a unique measure  $\mu_F$  on  $\mathbb{R}$  such that  $\mu_F((a, b]) = F(b) - F(a)$  for all  $a, b \in \mathbb{R}$ ;
- if  $G$  is another increasing and right-continuous function, then  $\mu_F = \mu_G$  if and only if  $F - G$  is a constant function;
- if  $\mu$  is a Borel measure on  $\mathbb{R}$  that is finite on all bounded Borel sets, i.e., a set  $S \subseteq \mathbb{R}$  contained in  $[-M, M]$  for some  $M \in \mathbb{R}$ , then

$$F(x) = \begin{cases} \mu((0, x]), & x > 0 \\ 0, & x = 0 \\ -\mu((x, 0]), & x < 0 \end{cases}$$

is an increasing function and right-continuous, and  $\mu_F = \mu$ .

*Proof.* a. Consider  $\mathbb{R} = \bigcup_{j=-\infty}^{\infty} (j, j+1]$ , then the pre-measure  $\mu_0((j, j+1]) = F(j+1) - F(j) < \infty$  defined on  $h$ -intervals is  $\sigma$ -finite. Therefore there exists a unique extension of measure  $\mu$  of  $\mu_0$  on  $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{A}_2)$  such that  $\mu|_{\mathcal{A}_2} = \mu_0$ .

b. We have  $\mu_F((a, b]) = F(b) - F(a)$  and  $\mu_G((a, b]) = G(b) - G(a)$ , then

$$\begin{aligned}\mu_F((a, b]) = \mu_G((a, b]) &\iff F(b) - F(a) = G(b) - G(a) \\ &\iff F(b) - G(b) = G(a) - F(a) \\ &\iff F - G \text{ is constant.}\end{aligned}$$

c. First note that  $F$  is an increasing function since the measure function is increasing. Take any  $x_0 \in \mathbb{R}$ , we want to show that  $\lim_{x \rightarrow x_0^+} F(x) = F(x_0)$ . We prove this by cases, either  $x_0 = 0$ ,  $x_0 > 0$ , or  $x_0 < 0$ . We will only prove the first case, but the two other cases are analogous. Suppose  $x_0 = 0$ , take a nested sequence of intervals  $E_n = (0, \frac{1}{n}]$ , with  $E_n \supseteq E_{n+1}$  for all  $n \in \mathbb{N}$ , then

$$\begin{aligned}\lim_{x \rightarrow 0^+} F(x) &= \lim_{x \rightarrow 0^+} \mu((0, x]) \\ &= \lim_{n \rightarrow \infty} \mu((0, \frac{1}{n}]) \\ &= \lim_{n \rightarrow \infty} \mu(E_n) \\ &= \mu\left(\bigcap_{n=1}^{\infty} E_n\right) \\ &= \mu(\emptyset) \\ &= 0 \\ &= F(0)\end{aligned}$$

since  $\mu(E_1) < \infty$ .

□

**Definition 1.53.** Suppose  $F$  is increasing and right-continuous, then we can use  $F$  to create  $\mu_0$  on  $\mathcal{A}_2$ , and get an outer measure  $\mu^*$  induced by  $\mu_0$ . Let  $\mathcal{A}$  be the collection of all  $\mu^*$ -measurable sets, then  $\mu^*|_{\mathcal{A}}$  is a measure. Note that  $\mathcal{A} \supseteq \mathcal{B}_{\mathbb{R}}$ : since  $\mu_F$  is only defined on  $\mathcal{B}_{\mathbb{R}}$ , then  $\mu^*|_{\mathcal{A}}$  becomes the extension of  $\mu_F$  on  $\mathcal{A}$ . We denote this measure to be  $\bar{\mu}_F$ , as the extension of  $\mu_F$ , called the Lebesgue-Stieltjes measure.

**Remark 1.54.** In particular, if  $F(x) = x$  for all  $x \in \mathbb{R}$ , then  $\bar{\mu}_F$  is called a Lebesgue measure, denoted by  $\mathbf{m}$ , with  $\mathbf{m}((a, b]) = F(b) - F(a) = b - a$ .

**Definition 1.55.** Let  $\mu$  be a Lebesgue-Stieltjes measure associated to an increasing and right-continuous function  $F$ . Let  $\mathcal{M}_{\mu}$  be the domain of the measure  $\mu$ , which gives the collection of measurable sets. For any measurable set  $E \in \mathcal{M}_{\mu}$ , we have

$$\begin{aligned}\mu(E) &= \inf \left\{ \sum_{j=1}^{\infty} [F(b_j) - F(a_j)] : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j] \right\} \\ &= \inf \left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j]) : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j] \right\}.\end{aligned}$$

**Theorem 1.56.** For all  $E \in \mathcal{M}_{\mu}$ , we have

$$\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j]) : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j] \right\}.$$

*Proof.* Let  $\tilde{\mu}(E)$  be the right-hand side of this equation, so we will show that  $\mu(E) = \tilde{\mu}(E)$ . Note that we have a partition

$$(a_j, b_j) = \bigcup_{k=1}^{\infty} I_k^{(j)},$$

where  $I_k^{(j)} = (b_j - \frac{1}{2^k}(b_j - a_j), b_j - \frac{1}{2^{k+1}}(b_j - a_j)]$ . Now  $E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j)$ , so  $E \subseteq \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} I_k^{(j)}$ , and thus

$$\begin{aligned} \sum_{j=1}^{\infty} \mu((a_j, b_j)) &= \sum_{j=1}^{\infty} \mu\left(\bigcup_{k=1}^{\infty} I_k^{(j)}\right) \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mu(I_k^{(j)}). \end{aligned}$$

Because  $\left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j)) : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j) \right\} \subseteq \left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j]) : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j] \right\}$ , then  $\tilde{\mu}(E) \geq \mu(E)$ .

We now show that  $\mu(E) \geq \tilde{\mu}(E)$ . Pick arbitrary  $\varepsilon > 0$ , then we know  $\mu(E) + \varepsilon$  is not a lower bound of the set  $\left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j]) : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j] \right\}$ , hence there exists  $(a_j, b_j]$  for  $j \geq 1$  such that  $E \subseteq \bigcup_{j \geq 1} (a_j, b_j]$ . Therefore  $\sum_{j=1}^{\infty} \mu((a_j, b_j]) \leq \mu(E) + \varepsilon$ . By the right continuity of  $F$ , for  $\varepsilon \cdot 2^{-j} > 0$ , there exists  $\delta_j > 0$  such that  $F(b_j + \delta_j) - F(b_j) < \varepsilon \cdot 2^{-j}$ , then  $E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j + \delta_j)$ . We know

$$\begin{aligned} \tilde{\mu}(E) &\leq \sum_{j=1}^{\infty} \mu((a_j, b_j + \delta_j)) \\ &= \sum_{j=1}^{\infty} [F(b_j + \delta_j) - F(a_j)] \\ &= \sum_{j=1}^{\infty} [F(b_j + \delta_j) - F(b_j) + F(b_j) - F(a_j)] \\ &\leq \sum_{j=1}^{\infty} [F(b_j + \delta_j) - F(b_j)] + \sum_{j=1}^{\infty} [F(b_j) - F(a_j)] \\ &< \sum_{j=1}^{\infty} \varepsilon \cdot 2^{-j} + \sum_{j=1}^{\infty} \mu((a_j, b_j]) \\ &< \varepsilon + \mu(E) + \varepsilon \\ &= \mu(E) + 2\varepsilon. \end{aligned}$$

Taking small enough  $\varepsilon$  finishes the proof.  $\square$

**Remark 1.57.** The union of  $h$ -intervals may not be open, so often times we use the characterization in [Theorem 1.56](#) instead.

**Theorem 1.58.** For any  $E \subseteq \mathcal{M}_{\mu}$ , we have

$$\mu(E) = \inf\{\mu(U) : \text{open } U \supseteq E\} = \sup\{\mu(K) : \text{compact } K \subseteq E\}.$$

*Proof.* Let  $\tilde{\mu}(E) = \inf\{\mu(U) : \text{open } U \supseteq E\}$ . First,  $\mu(E) \leq \tilde{\mu}(E)$ : since  $E \subseteq U$ , then  $\mu(E) \leq \mu(U)$ , therefore  $\mu(E) \leq \tilde{\mu}(E)$ . To see  $\tilde{\mu}(E) \leq \mu(E)$ , we have  $\mu(E) + \varepsilon$  is not a lower bound of  $\left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j]) : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j] \right\}$ ,



then there exists  $(a_j, b_j)$  for each  $j \in \mathbb{N}$  such that  $E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j)$ , and that  $\sum_{j=1}^{\infty} \mu((a_j, b_j)) \leq \mu(E) + \varepsilon$ . Therefore, take  $U$  to be the open set  $\bigcup_{j=1}^{\infty} (a_j, b_j)$ , then

$$\tilde{\mu}(E) \leq \mu(U) \leq \sum_{j=1}^{\infty} \mu((a_j, b_j)) \leq \mu(E) + \varepsilon$$

as desired.

Now let  $\nu(E) = \sup\{\mu(K) : \text{compact } K \subseteq E\}$ . We note that if  $K \subseteq E$ , then  $\mu(K) \leq \mu(E)$ , therefore  $\nu(E) \leq \mu(E)$ . To prove the reverse inequality, we consider the following cases:

- $E$  is bounded.
  - $E$  is closed. Since  $E$  is bounded and closed, it is compact over  $\mathbb{R}$ , thus  $\mu(E) \leq \nu(E)$ .
  - $E$  is bounded but not closed. We have  $\mu(\bar{E} \setminus E) = \inf\{\mu(U) : \text{open } U \supseteq \bar{E} \setminus E\}$ . For any  $\varepsilon > 0$ , there exists an open set  $U$  such that  $U \supseteq \bar{E} \setminus E$  and  $\mu(U) \leq \mu(\bar{E} \setminus E) + \varepsilon$ . Set  $K = \bar{E} \setminus U$ , then  $K$  is compact. Since all measures here are finite, we have

$$\begin{aligned} \mu(K) &= \mu(E) - \mu(E \cap U) \\ &= \mu(E) - [\mu(U) - \mu(U \setminus E)] \\ &\geq \mu(E) - \mu(U) + \mu(\bar{E} \setminus E) \\ &\geq \mu(E) - \varepsilon. \end{aligned}$$

Therefore  $\nu(E) \geq \mu(E) - \varepsilon$ , and we are done by taking  $\varepsilon \rightarrow 0$ .

- $E$  is not bounded. Suppose  $E = \bigcup_{j=-\infty}^{\infty} ((j, j+1] \cap E)$ , then denote  $E_j = E \cap (j, j+1]$ , which is bounded. Therefore, we know the statement is true for each  $E_j$  for  $j \geq 1$ , thus  $\mu(E_j) = \sup\{\mu(K) : \text{compact } K \subseteq E_j\}$ . Take arbitrary  $\varepsilon > 0$ , then  $\mu(E_j) - \frac{1}{3}\varepsilon \cdot 2^{-|j|}$  is not the upper bound of  $\{\mu(K) : \text{compact } K \subseteq E_j\}$ , then there exists a compact set  $K_j \subseteq E_j$  such that  $\mu(K_j) \geq \mu(E_j) - \frac{1}{3}\varepsilon \cdot 2^{-|j|}$ . Since  $K_j \subseteq E_j$  and  $E_j$ 's are disjoint, then  $K_j$ 's are disjoint. Therefore, for  $n \in \mathbb{N}$ , set  $H_n = \bigcup_{j=-n}^n K_j$ , which is a finite disjoint union of compact sets, so this is a compact set. But  $H_n \subseteq E$ , then

$$\begin{aligned} \mu(H_n) &= \mu\left(\bigcup_{j=-n}^n K_j\right) \\ &= \sum_{j=-n}^n \mu(K_j) \\ &\geq \sum_{j=-n}^n \mu(E_j) - \frac{\varepsilon}{3} \sum_{j=-n}^n 2^{-|j|} \\ &\geq \sum_{j=-n}^n \mu(E_j) - \frac{\varepsilon}{3} \sum_{j=-\infty}^{\infty} 2^{-|j|} \\ &\geq \sum_{j=-n}^n \mu(E_j) - \varepsilon. \end{aligned}$$

Note that  $H_n$  still depends on  $n$ , so we should not take  $n \rightarrow \infty$  here. Since  $\nu(E)$  is the upper bound of  $\mu(K)$ 's for compact  $K \subseteq E$ , then  $\nu(E) \geq \mu(H_n)$ , therefore

$$\nu(E) \geq \sum_{j=-n}^n \mu(E_j) - \varepsilon$$

$$= \mu \left( \bigcup_{j=-n}^n E_j \right) - \varepsilon.$$

Take  $n \rightarrow \infty$ , then

$$\begin{aligned} \nu(E) &\geq \lim_{n \rightarrow \infty} \mu \left( \bigcup_{j=-n}^n E_j \right) - \varepsilon \\ &= \mu \left( \bigcup_{j=-\infty}^{\infty} E_j \right) - \varepsilon \\ &= \mu(E) - \varepsilon. \end{aligned}$$

Let  $\varepsilon \rightarrow 0$ , we are done. □

**Theorem 1.59.** Let  $E \subseteq \mathbb{R}$ , then the following are equivalent:

- a.  $E \in \mathcal{M}_\mu$ ;
- b.  $E = V \setminus N_1$ , where  $V$  is a  $G_\delta$ -set and  $\mu(N_1) = 0$ ;
- c.  $E = H \cup N_2$ , where  $H$  is a  $F_\sigma$ -set and  $\mu(N_2) = 0$ .

*Proof.* •  $b. \Rightarrow a.$ : note that  $\mathcal{M}_\mu \supseteq \mathcal{B}_\mathbb{R}$ , then both  $V$  and  $N_1$  are measurable, therefore  $E$  is measurable, i.e.,  $E \in \mathcal{M}_\mu$ .

•  $c. \Rightarrow a.$ : similar to the case above.

•  $a. \Rightarrow b.$ :

- If  $\mu(E) < \infty$ , recall  $\mu(E) = \inf\{\mu(U) : \text{open } U \supseteq E\}$ . For any  $k \in \mathbb{N}$ , consider  $2^{-k} > 0$ , then there exists open subset  $U_k \supseteq E$  such that  $\mu(U_k) \leq \mu(E) + 2^{-k}$ . Let  $V = \bigcap_{k=1}^{\infty} U_k$  be a  $G_\delta$ -set, then  $V \supseteq E$  as well. It suffices to show that  $V \setminus E$  is a null set. We know

$$\begin{aligned} \mu(V) &= \mu \left( \bigcap_{k=1}^{\infty} U_k \right) \\ &\leq \mu(U_k) \\ &\leq \mu(E) + 2^{-k} \end{aligned}$$

for all  $k \in \mathbb{N}$ . Since  $\mu(V)$  and  $\mu(E)$  are independent of  $k$ , then take  $k \rightarrow \infty$ , therefore  $\mu(V) \leq \mu(E)$ . But since  $E \subseteq V$ , then  $\mu(E) \leq \mu(V)$ , therefore this gives equality. Since  $\mu(E) < \infty$ , then  $\mu(V) - \mu(E) = 0$ , then  $\mu(V \setminus E) = 0$  by additivity.

- If  $\mu(E) = \infty$ , then the proof can be done using the previous case.

•  $a. \Rightarrow c.$ : the proof is similar to the case above. □

**Theorem 1.60.** Let  $E \in \mathcal{M}_\mu$ , and suppose  $\mu(E) < \infty$ . For any  $\varepsilon > 0$ , there exists some set  $A$  that is a finite union of open intervals such that  $\mu(E \Delta A) = \mu((E \setminus A) \cup (A \setminus E)) < \varepsilon$ .

*Proof.* Note that  $\mu(E) = \sup\{\mu(K) : \text{compact } K \subseteq E\}$ . For any  $\varepsilon > 0$ , there exists compact  $K \subseteq E$  such that  $\mu(E) - \frac{\varepsilon}{2} < \mu(K)$ , which is equivalent to having  $\mu(E \setminus K) < \frac{\varepsilon}{2}$ . Similarly, recall that  $\mu(E) = \inf\{\mu(U) : \text{open } U \supseteq E\}$ , but open set  $U$  on  $\mathbb{R}$  is characterized as a union of open intervals, therefore this is just  $\mu(E) = \inf\left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j)) : \right.$

$\bigcup_{j=1}^{\infty} (a_j, b_j) \supseteq E$ . Therefore, there exists  $\bigcup_{j=1}^{\infty} I_j \supseteq E$ , where  $I_j$  is open interval for each  $j$ , such that  $\mu\left(\bigcup_{j=1}^{\infty} I_j\right) < \mu(E) + \frac{\varepsilon}{2}$ . Since  $\mu(E)$  is finite, then  $\mu\left(\bigcup_{j=1}^{\infty} I_j \setminus E\right) < \frac{\varepsilon}{2}$ . Now  $K \subseteq E \subseteq \bigcup_{j=1}^{\infty} I_j$ , but  $K$  is compact, so there exists  $I_1, \dots, I_n$  such that their union cover  $K$ . Set  $A = \bigcup_{j=1}^m I_j$ , and we are done.  $\square$

**Definition 1.61.** Let  $F(x) = x$  be a function for all  $x \in \mathbb{R}$ , then  $\mu_F$  is called the Lebesgue measure defined by  $m((a, b]) = b - a$ . The domain of  $m$  is  $\mathcal{L}$ .

For  $E \subseteq \mathbb{R}$  and  $s, r \in \mathbb{R}$ , we denote  $E + s = \{x + s : x \in E\}$  and  $rE = \{rx : x \in E\}$ .

**Theorem 1.62.** If  $E \in \mathcal{L}$ , then  $m(E + s) = m(E)$  and  $m(rE) = |r|m(E)$ .

*Proof.* We prove the first claim. For any  $E \in \mathcal{L}$  and  $s \in \mathbb{R}$ , define  $m_s = m(E + s)$ , then this is a measure.

**Claim 1.63.** For any  $E \in \mathcal{L}$ ,  $m_s(E) = m(E)$ .

*Subproof.* First note that this is true if  $E$  is a finite (disjoint) union of  $h$ -intervals of  $m_s$ , as  $m$  extends the pre-measure  $\mu_0$ . On  $\mathcal{B}_{\mathbb{R}}$ , the extension is unique, so  $m_s(E) = m(E)$  if  $E \in \mathcal{B}_{\mathbb{R}}$ . Moreover, recall  $E \in \mathcal{L}$  if and only if  $E = V \setminus N_1$  for  $V \in \mathcal{B}_{\mathbb{R}}$ . Therefore this is true for all  $E \in \mathcal{L}$ .  $\blacksquare$

$\square$

**Definition 1.64.** The Cantor set  $\mathcal{C}$  is constructed iteratively from the interval  $[0, 1]$ , that for any remaining connected interval  $[m, n]$ , we delete the subinterval  $(m + \frac{1}{3}(n - m), m + \frac{2}{3}(n - m))$  from  $[m, n]$ .

**Remark 1.65.** Note that

$$\begin{aligned} m(\mathcal{C}) &= m([0, 1]) - \frac{1}{3} - \frac{2}{3^2} - \frac{2^2}{3^3} - \dots \\ &= 1 - \sum_{j=0}^{\infty} \frac{2^j}{3^{j+1}} \\ &= 1 - 1 \\ &= 0. \end{aligned}$$

**Remark 1.66.** If  $E$  is countable, then

$$\begin{aligned} m(E) &= \sum_{j=1}^{\infty} m(\{a_j\}) \\ &= 0. \end{aligned}$$

**Theorem 1.67.** The Cantor set  $\mathcal{C}$  is uncountable.

*Proof.* Alternatively, the Cantor set  $\mathcal{C}$  can be represented as

$$\mathcal{C} = \{x \in [0, 1] : x = \sum_{j=1}^{\infty} a_j 3^{-j}, a_j \in \{0, 2\}\}.$$

To prove that  $\mathcal{C}$  is uncountable, it suffices to build a surjection  $f : \mathcal{C} \rightarrow [0, 1]$ . For  $x \in \mathcal{C}$ , we have  $x = \sum_{j=1}^{\infty} a_j 3^{-j}$ ,  $a_j \in \{0, 2\}$ . Set  $f(x) = \sum_{j=1}^{\infty} \frac{a_j}{2} 2^{-j}$  for  $\frac{a_j}{2} \in \{0, 1\}$ , therefore this gives a decimal representation with base 2, so any real number in  $[0, 1]$  can be represented in this form, therefore we have a surjection.  $\square$

**Theorem 1.68.** Let  $F \subseteq \mathbb{R}$  be such that every subset of  $F$  is Lebesgue measurable, then  $m(F) = 0$ .

**Corollary 1.69.** If  $m(F) > 0$ , then there exists a subset  $S$  of  $F$  such that  $S \notin \mathcal{L}$ .

**Remark 1.70** (Banach-Tarski Paradox). Given a ball  $B = S^2$ , then there exists some  $m \in \mathbb{N}$  such that  $B = V_1 \cup \cdots \cup V_m$  is a union of subsets  $V_i$  that are not Lebesgue measurable and  $m(B) \neq m(V_1 \cup \cdots \cup V_m)$ .

**Definition 1.71.** For any  $x \in \mathbb{R}$ , we defined the cosets over  $\mathbb{Q}$  to be  $\mathbb{Q} + x = \{r + x : r \in \mathbb{Q}\}$  for any  $x$ . This is called the coset of an additive group  $\mathbb{R}$ .

Let  $E$  be the set that contains exactly one point from each coset of  $\mathbb{Q}$  as representations, which requires the axiom of choice.

Now  $E$  allows us make a partition on  $\mathbb{R}$ .

**Lemma 1.72.** 1.  $(E + r_1) \cap (E + r_2) = \emptyset$  if  $r_1 \neq r_2$  and  $r_1, r_2 \in \mathbb{Q}$ .

$$2. \mathbb{R} = \bigcup_{r \in \mathbb{Q}} (E + r).$$

*Proof.* 1. Suppose  $x \in (E + r_1) \cap (E + r_2)$ , then  $x = e_1 + r_1 = e_2 + r_2$  for some  $e_1, e_2 \in E$ . Therefore  $e_1 - e_2 = r_2 - r_1$ , which is a non-zero rational number, therefore  $0 \neq e_1 - e_2 \in \mathbb{Q}$ . Therefore  $e_1$  and  $e_2$  are in the same coset, so  $e_1 = e_2$ , contradiction.

2. Obviously  $\mathbb{R} \supseteq \bigcup_{r \in \mathbb{Q}} (E + r)$ . Take any  $x \in \mathbb{R}$ , then  $E$  contains a point  $y$  from the coset  $\mathbb{Q} + x$ , therefore  $y - x \in \mathbb{Q}$ , so take  $r = y - x$ , then  $x \in E + r$ .

□

*Proof of Theorem 1.68.*

□