Steenrod Operations Notes

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These notes were taken from a course on Steenrod operations taught by Dr. N. Yang in Fall 2024 at BIMSA. Any mistakes and inaccuracies would be my own. The main references are [Pus01], [Pow02][1]/[Voe03], [SV00], and [MVW06].

Contents

[1]This set of notes is the preliminary source for the construction of Voevodsky's Steenrod operations, but differs from its actual construction in [Voe03].

0 Introduction

Steenrod operations, in algebraic topology, describes natural transformations (with respect to spaces and suspension)

$$\mathrm{Sq}^i : H^*(X, \mathbb{Z}/2\mathbb{Z}) \to H^{*+i}(X, \mathbb{Z}/2\mathbb{Z})$$

with Bockstein p-homomorphisms

$$\beta : H^*(X, \mathbb{Z}/p\mathbb{Z}) \to H^{*+1}(X, \mathbb{Z}/p\mathbb{Z})$$

for prime p, and homomorphisms (whenever $p > 2$)

$$\mathcal{P}^i : H^*(X, \mathbb{Z}/p\mathbb{Z}) \to H^{n+2i(p-1)}(X, \mathbb{Z}/p\mathbb{Z})$$

for any space X. These data together form a Hopf algebra $\mathcal{A}_p$, called the Steenrod algebra.

Remark 0.1. $\mathcal{A}_p$ satisfies Adem relations and Cartan formula, therefore it has explicit additive basis.

In the context of algebraic geometry, the counterpart of singular cohomology is the motivic cohomology, namely a bigraded ring $H^{p,q}(X, R)$ for every scheme X of finite type over a field. If $p = 2q$ and $R = \mathbb{Z}/\ell\mathbb{Z}$, we have the Bloch formula

$$H^{2q,q}(X, \mathbb{Z}/\ell\mathbb{Z}) = \mathrm{CH}^q(X)/\ell,$$

where the qth Chow group $\mathrm{CH}^q(X)$ is the group freely generated by codimension-q irreducible closed subsets, quotient by the rational equivalences over $\mathrm{div}(f)$. In this case, we also have Steenrod operations that are compatible with both S^1-suspension and $\mathbb{G}_m$-suspension. For instance, the Steenrod operations are

$$\mathrm{Sq}^{2i} : H^{*,*}(X, \mathbb{Z}/2\mathbb{Z}) \to H^{*+2i,*+i}(X, \mathbb{Z}/2\mathbb{Z}),$$

Bockstein homomorphisms are

$$\beta : H^{*,*}(X, \mathbb{Z}/\ell\mathbb{Z}) \to H^{*+1,*}(X, \mathbb{Z}/\ell\mathbb{Z})$$

which determines $\mathrm{Sq}^{2i+1} = \beta\,\mathrm{Sq}^{2i}$ uniquely. In the case where $\ell > 2$, we have

$$\mathcal{P}^i : H^{*,*}(X, \mathbb{Z}/\ell\mathbb{Z}) \to H^{*+2i(\ell-1),*+i(\ell-1)}(X, \mathbb{Z}/\ell\mathbb{Z}).$$

Remark 0.2. These morphisms also satisfy Adem relations and Cartan formula, and therefore altogether they give a Hopf algebroid $\mathcal{A}_\ell$ with explicit additive basis.

Let us look at a few typical applications.

Example 0.3 (Hopf Invariant One Problem)**.** Let $f : S^{2n-1} \to S^n$ be a morphism. (In the case where $n = 1$, we require f to be a double covering.) Define $X = C(f)$ to be the mapping cone, then by the long exact sequence in cohomology

$$\cdots \longrightarrow H^i(C(f)) \longrightarrow H^i(S^n) \longrightarrow H^i(S^{2n-1}) \longrightarrow H^{i+1}(C(f)) \longrightarrow \cdots$$

one finds that

$$H^i(X, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 2n, n \\ 0, & \text{else} \end{cases}$$

Let $x \in H^n(X, \mathbb{Z})$ and $y \in H^{2n}(X, \mathbb{Z})$ be generators, then by degree argument, $x^2 = h(f)y$ for some integer $h(f)$ called the Hopf invariant. This is well-defined up to a sign change, and only depends on the homotopy class of f. One can now ask when $h(f)$ is 1. By showing that Sq^i is "indecomposable"[2] if and only if $i = 2^r$, one can show that $n = 2^r$ whenever $h(f)$ is odd.

Similarly, there are "motivic spheres" $S^{p,q} = \mathbb{G}_m^{\wedge q} \wedge S^{p-q}$ over a field k in the motivic setting, therefore consider morphisms $f : S^{2a-1,2b} \to S^{a,b}$ for $a \geqslant b \geqslant 0$[3] and $a \geqslant 1$ and mapping cone $X = C(f)$, then

$$H^{*,*}(X, \mathbb{Z}/2\mathbb{Z}) \cong H^{*,*}(k, \mathbb{Z}/2\mathbb{Z})\{x, y\}$$

given by basis $\{x, y\}$, where $x \in H^{a,b}(X, \mathbb{Z}/2\mathbb{Z})$ and $y \in H^{2a,2b}(X, \mathbb{Z}/2\mathbb{Z})$.[4] We say that f has Hopf invariant one if $x^2 = y$.

[2] That is, it cannot written as a sum of products of Sq^j's for $j < i$.
[3] In particular, if $(a, b) = (1, 0)$, then we take $f = 2$.
[4] Here the long exact sequence modifies the first parameter of the cohomology.

Example 0.4 (Adams Spectral Sequence)**.** In topology, the Adams spectral sequence is given by

$$E_2^{s,t}=\operatorname{Ext}_{\mathcal{A}_p}^{s,t}(H^*(X,\mathbb{Z}/p\mathbb{Z}),\mathbb{Z}/p\mathbb{Z})\Rightarrow\widehat{\pi_*(X)_p}$$

with differentials $d_r:E_r^{s,t}\to E_r^{s+r,t+r-1}$.

In the motivic setting, now suppose $\operatorname{char}(k)=0$ and let $\mathcal{M}_2=H^{*,*}(k,\mathbb{Z}/2\mathbb{Z})=K_*^M/2K_*^M[\tau]$ for Bott element $\tau\in H^{0,1}(k,\mathbb{Z}/2\mathbb{Z})$ that behaves like $\sqrt{-1}$, and $K_*^M/2K_*^M=\mathbb{Z}/2\mathbb{Z}\langle[x]\rangle/\langle[x][1-x]=0\rangle$ for $x\in k^\times$. This structure has its dimension bounded above by the cohomological dimension $\operatorname{cd}(2,k)$, hence it is finite in most cases we care about. For instance, the Milnor K-theory is concentrated as $\mathbb{Z}/q\mathbb{Z}$ if the field is algebraically closed, and is concentrated at K_0 and K_1 if the field is finite.

The corresponding motivic Adams spectral sequence is a trigraded spectral sequence

$$E_2^{s,t,u}=\operatorname{Ext}_{\mathcal{A}_2}^{s,t+s,u}(\mathcal{M}_2,\mathcal{M}_2)\Rightarrow\widehat{\pi_{t,u}(k)_{(2,\eta)}}$$

where $\eta:S^{1,1}\to S^{0,0}$ is the Hopf map, along with differentials $d_r:E_r^{s,t,u}\to E_r^{s+r,t-1,u}$. By applying (motivic) Adams spectral sequence, we have a full answer to the Hopf invariant one problem. That is, we can solve the Hopf invariant one problem in both the classical and the motivic case: in the classical case, there exists maps $f:S^{2n-1}\to S^n$ with Hopf invariant one if and only if $n=1,2,4,8$; in the motivic case, there exists maps $f:S^{2a-1,2b}\to S^{a,b}/\mathbb{R}$ with Hopf invariant one if and only if $(a,b)\in\{(1,0),(2,1),(4,2),(8,4),(2,0),(4,0),(8,0)\}$, c.f., [BCQ21].

Example 0.5 (Relation with Witt Group)**.** For any scheme X with underlying characteristic not 2, i.e., $2\in\mathcal{O}_X^\times$, then define $\operatorname{Sym}(X)$ to be the collection of vector bundles $V\to X$ with symmetric non-degenerate inner product. Now define the Witt group/ring to be $W(X)=\operatorname{Sym}(X)/\{\text{vector bundles } V \text{ with Lagrangian}\}$ of X.[5] When X is a regular scheme with $\dim(X)\leqslant 3$, the Witt group $W(X)$ fits into an exact sequence

$$0\longrightarrow W(X)\longrightarrow W(K(X))\longrightarrow\bigoplus_{y\in X^{(1)}}W(k(y))$$

where $X^{(1)}$ are points of codimension 1 over X and the second map is the residue map given by taking the principal divisors. Moreover, $W(X)\to W(K(X))$ is not an injection in general, e.g., Totaro constructed a counterexample in [Tot03]. Therefore, $W(X)=H^0(X,W)$ as a scheme defined locally on open subsets as the cohomology of the following Gersten complexes

$$W(K(X))\longrightarrow\bigoplus_{y\in X^{(1)}}W(k(y))\longrightarrow\bigoplus_{y\in X^{(2)}}W(k(y))\longrightarrow\cdots$$

where we start at degree 0. Now the cohomology $H^*(X,W)$ satisfies the Bockstein spectral sequence

$$E_1^{p,q}=H^{p,q}(X,\mathbb{Z}/2\mathbb{Z})\oplus H^{p+2,q}(X,\mathbb{Z}/2\mathbb{Z})\Rightarrow H^{p-q}(X,W)$$

where the differentials are

$$\begin{aligned}H^{p,q}\oplus H^{p+2,q}&\to H^{p+2,q+1}\oplus H^{p+4,q+1}\\(x,y)&\mapsto(\operatorname{Sq}^2(x)+\tau y,\operatorname{Sq}^3\operatorname{Sq}^1(x)+\operatorname{Sq}^2(y)+\rho\operatorname{Sq}^1(y))\end{aligned}$$

for $\rho=\operatorname{Sq}^1(\tau)=[-1]\in H^{1,1}(k)$ corresponding to $-1\in k^*/2k^*$. This spectral sequence is induced by the Hopf element $\eta:S^{1,1}\to S^{0,0}$ (which is 0 over $\mathbb{C}$ but -2 over $\mathbb{R}$).

Therefore, to understand the calculation to Witt groups, we need to understand Sq^2.

Example 0.6 (Sq^2 and Geometry)**.** Suppose X is a smooth projective curve over k whose étale cohomology $H^*_{\text{ét}}(X,\mathbb{Z}/2\mathbb{Z})$ has a Poincaré duality. (For instance, this works for $k=\mathbb{C},\mathbb{F}_p,\mathbb{Q}_p,\mathbb{F}_p((t))$.) In this case, we have $\operatorname{Sq}^2=0$ on X if and only if the canonical bundle $\omega_X=L\otimes L$. Any such choice of L is called a theta characteristic.

In this course, we are interested in the construction of Steenrod operations and basis properties, as well as its associated Hopf algebroid structure of $\mathcal{A}_p$.

[5] Alternatively, this can be interpreted as quotienting the metabolic vector bundles.

1 Classic Steenrod Operations

1.1 Lens Space

Definition 1.1. We define the generalized lens space $L(n,p) = S^{2n-1}/\mathbb{F}_p$ where the $\mathbb{F}_p$-action on the closed subset $S^{2n-1} \subseteq \mathbb{C}^n$ is given by multiplication by $e^{\frac{2\pi i}{p}}$. Moreover, we define $L_p^\infty = \lim\limits_{n\to\infty} L(n,1)$.

Proposition 1.2. We have $H^i(L_p^\infty, \mathbb{F}_p) = \mathbb{F}_p$ for every $i \in \mathbb{N}$. Pick a generator $\alpha \in H^1(L_p^\infty, \mathbb{F}_p)$, then $\beta := \beta(\alpha) \neq 0$ is the image of the Bockstein operation, and we have

$$H^i(L_p^\infty, \mathbb{F}_p) = \begin{cases} \beta^{\frac{i}{2}}, & i \text{ even} \\ 2\beta^{\frac{i-1}{2}}, & i \text{ odd} \end{cases}$$

and

$$\alpha^2 = \begin{cases} \beta, & p = 2 \\ 0, & p > 2 \end{cases}.$$

Proof. We have a fiber sequence

$$\mathbb{F}_p \longrightarrow S^{2n-1} \longrightarrow L(n,p)$$

and the corresponding Puppe sequence

$$\cdots \longrightarrow \pi_i(\mathbb{F}_p) \longrightarrow \pi_i(S^{2n-1}) \longrightarrow \pi_i(L(n,p)) \longrightarrow \pi_{i-1}(\mathbb{F}_p) \longrightarrow \cdots$$

This shows that $\pi_1(L(n,p)) = \pi_0(\mathbb{F}_p) = \mathbb{F}_p$, therefore $H_1(L(n,p),\mathbb{Z}) = \mathbb{F}_p$. There is another fiber sequence

$$S^1 \longrightarrow L(n,p) \longrightarrow \mathbb{C}P^{n-1}$$

with the action of $e^{2\pi i\theta}$ on $L(n,p)$. Since $\mathbb{C}P^{n-1}$ is simply connected, there is a Serre spectral sequence

$$E_2^{p,q} = H^p(\mathbb{C}P^{n-1}, H^q(S^1,\mathbb{Z})) \Rightarrow H^{p+q}(L(n,r),\mathbb{Z}).$$

The E_2-page now looks like

$$\begin{tikzcd}
H^0(\mathbb{C}P^{n-1}, H^1(S^1,\mathbb{Z})) \cong \mathbb{Z} \arrow[drr, "d_2"] & 0 & H^2(\mathbb{C}P^{n-1}, H^1(S^1,\mathbb{Z})) \arrow[drr, "d_2"] & 0 & \cdots \\
H^0(\mathbb{C}P^{n-1}, H^0(S^1,\mathbb{Z})) \cong \mathbb{Z} & 0 & H^2(\mathbb{C}P^{n-1}, H^0(S^1,\mathbb{Z})) & 0 & \cdots
\end{tikzcd}$$

and therefore the spectral sequence collapses at $E_3 = E_\infty$. By the universal coefficient theorem, we have $H^2(L(n,p),\mathbb{Z}) = \mathbb{F}_p$, but H^2 is the cokernel of $d_2 : \mathbb{Z} \to \mathbb{Z}$, therefore $d_2 = p\,\mathrm{id}_\mathbb{Z}$. In fact, we can deduce all d_2-morphisms inductively. That is, suppose $H^0(\mathbb{C}P^{n-1}, H^1(S^1,\mathbb{Z})) = \mathbb{Z}\cdot y$ and $H^2(\mathbb{C}P^{n-1}, H^0(S^1,\mathbb{Z})) = \mathbb{Z}\cdot x$, then we have $d_2(x) = 0$ and $d_2(y) = px$, therefore by Leibniz's rule,

$$d_2(x^i y) = d_2(x^i)y + x^i d_2(y) = \begin{cases} px^{i+1}, & i < n-1 \\ 0, & i = n-1 \end{cases}$$

Hence, we have shown that

$$H^i(L(n,p),\mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0, 2n-1 \\ \mathbb{F}_p, & i = 2j, 0 < j \leqslant n-1 \\ 0, & \text{otherwise} \end{cases}$$

In particular, that means $H^i(L_p^\infty, \mathbb{F}_p) = \mathbb{F}_p$ for all $i \geqslant 0$.

Now consider the Bockstein short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

which induces a long exact sequence in cohomology

$$\cdots \longrightarrow H^1(L(n,p),\mathbb{Z}) \longrightarrow H^1(L(n,p),\mathbb{F}_p) \longrightarrow H^2(L(n,p),\mathbb{Z}) \longrightarrow \cdots$$

But $H^1(L(n,p),\mathbb{Z}) = 0$, so $H^1(L(n,p),\mathbb{F}_p) \cong H^2(L(n,p),\mathbb{Z}) \cong \mathbb{F}_p$. By examining the map $H^2(L(n,p),\mathbb{Z}) \to H^2(L(n,p),\mathbb{Z})$, we know the Bockstein homomorphism $H^1(L(n,p),\mathbb{F}_p) \to H^2(L(n,p),\mathbb{F}_p)$ is an isomorphism, hence $\beta(\alpha) \neq 0$.

Let us now prove

$$H^i(L_p^\infty,\mathbb{F}_p) = \begin{cases} \beta^{\frac{i}{2}}, & i \text{ even} \\ 2\beta^{\frac{i-1}{2}}, & i \text{ odd} \end{cases}$$

by induction. If $n = 1$, then $L(n,p) = S^1$. Suppose this holds for all $m < n$, then the computation of $H_i(L(n,p),\mathbb{Z})$ which shows that $L(n,p)$ is an orientable manifold. Since $\beta \neq 0$, then there exists a non-degenerate pairing

$$H^2(L(n,p),\mathbb{F}_p) \times H^{2n-3}(L(n,p),\mathbb{F}_p) \to H^{2n-1}(L(n,p),\mathbb{F}_p),$$

hence there exists some $\gamma \in H^{2n-3}$ such that $\beta\gamma \neq 0$. By inductive hypothesis, $\gamma = \alpha\beta^{n-2}$ up to multiplication, therefore $\alpha\beta^{n-1} \neq 0 \in H^{2n-1}$. Hence, $\alpha\beta^i \neq 0$ for $i \leqslant n-1$.

If $p > 2$, the skew-commutativity tells us that $\alpha^2 = -\alpha^2$, therefore $\alpha^2 = 0$. In the case where $p = 2$, then we have $L(n,2) = \mathbb{R}P^{2n-1}$, so it suffices to show that $\alpha^2 \neq 0 \in \mathbb{R}P^2$. $\square$

For any CW-pairs (X,A) and (Y,B) of pointed spaces, there is a relative cross product

$$\times : H^*(X,A) \otimes_{\mathbb{F}_p} H^*(Y,B) \xrightarrow{\cong} H^*(X \times Y, (A \times Y) \cup (B \times X)).$$

If we use the shorthand $H^*(-) = H^*(-,\mathbb{F}_p)$, then we obtain a Künneth formula

$$\times : \tilde{H}^*(X) \otimes_{\mathbb{F}_p} \tilde{H}^*(Y) \xrightarrow{\cong} \tilde{H}^*(X \wedge Y)$$

on the reduced cohomology for suspension $X \wedge Y = (X \times Y)/(X \times *) \cup (* \times Y)$. In particular, we have $\tilde{H}^{i+1}(\Sigma X) = \tilde{H}^i(X)$ for the suspension $\Sigma X := S^1 \wedge X$.

Definition 1.3. For any topological space X, we define $\Gamma(X) = X^{\wedge p} \times S^\infty/\mathbb{F}_p$ where $S^\infty = \varinjlim_n S^n$ and $\mathbb{F}_p$ has generator τ, then the group $\mathbb{F}_p$-action is given so that

$$\tau \cdot (x_1,\ldots,x_p,z) = (x_2,\ldots,x_p,x_1,e^{\frac{2\pi i}{p}}z).$$

In particular, S^∞ is contractible with a free $\mathbb{F}_p$-action. There is an associated fiber sequence

$$\begin{aligned} X^{\wedge p} \longrightarrow \Gamma(X) &\xrightarrow{\quad\pi\quad} L_p^\infty \\ (x_1,\ldots,x_p,z) &\longmapsto [z] \end{aligned}$$

where π admits a section

$$\begin{aligned} L_p^\infty &\to \Gamma(X) \\ [z] &\mapsto (x_0,\ldots,x_0,z) \end{aligned}$$

where $x_0 \in X$ is the basepoint. Now define $\Lambda(X) = \Gamma(X)/L_p^\infty$. Note that for any $[z] \in L_p^\infty$, the composite

$$X^{\wedge p} \longrightarrow \Gamma(X) \longrightarrow \Lambda(X)$$

is an injection. Replacing S^∞ by S^n, we get $\Gamma_n(X)$ and $\Lambda_n(X)$.

Proposition 1.4. Suppose $X \subseteq Y$ be pointed spaces and Y is obtained from attaching n-cells on X where $n \geqslant n_0$, then for any map $\varphi : X \to Z$ of pointed spaces where $\pi_i(Z, z_0) = 0$ for $i \geqslant n_0 - 1$, then φ can be lifted to a map $\tilde{\varphi} : Y \to Z$.

Proof. Starting with the case where $X_0 = X$, we proceed inductively where each time we have a pushout diagram

$$\begin{tikzcd} \coprod_\alpha S^{n-1} \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ \coprod_\alpha D^n \arrow[r] & X_n \end{tikzcd}$$

where we glue (and fill in) the boundaries S^{n-1}'s on X_{n-1} in order to get X_n for $n \geqslant n_0$, then $\varinjlim X_n = Y$. We define $f_n : X_n \to Z$ inductively. For $n = n_0 - 1$, we set $X_n = X$ and $f_n = \varphi$. Suppose we defined f_{n-1} for $n \geqslant n_0$. Since $\pi_{n-1}(Z, z_0) = 0$, then the composition

$$S^{n-1} \xrightarrow{i_\alpha} \coprod_\alpha S^{n-1} \longrightarrow X_{n-1} \xrightarrow{f_{n-1}} Z$$

fills in the boundary and therefore lifts to a map $u_\alpha : D^{n-1} \to Z$ for each component α, then f_n is defined by the gluing $\left(\coprod_\alpha u_\alpha\right) \coprod_{\coprod_\alpha S^{n-1}} f_{n-1}$. Finally, set $\tilde{\varphi} = \varinjlim_n f_n$. $\square$

We now construct the Steenrod operations using the following idea. For any $\alpha \in \tilde{H}^i(X)$, we want to find a class $\lambda(\alpha) \in H^{pi}(\Lambda X)$ which pulls back to $\alpha^{\otimes p}$ on every fiber $X^{\wedge p}$. Now pullback $\lambda(\alpha)$ along the diagonal

$$\begin{aligned} \nabla : (L_p^\infty \times X)/\mathbb{F}_p &\to \Lambda X \\ (z, x) &\mapsto (x, \ldots, x, z) \end{aligned}$$

Finally, $\nabla^* \lambda(\alpha)$ decomposes as

$$\tilde{H}^{pi}(L_p^\infty \times X / L_p^\infty) = \bigoplus_{j \leqslant pi} \tilde{H}^j(X),$$

which gives a natural transformation $\tilde{H}^i(X) \to \tilde{H}^j(X)$ componentwise, and therefore defines the Steenrod operations. A lot of materials from this construction were taken from [Gul09].

Suppose we have $\alpha \in \tilde{H}^i(X) := \tilde{H}^i(X, \mathbb{F}_p)$, then we want to construct $\mathrm{Sq}^i(\alpha) \in \tilde{H}^*(\alpha)$. The given α corresponds to a map

$$X \to K_i := K(\mathbb{F}_p, i)$$

to the Eilenberg-Maclane spaces such that

$$\pi_j(K(\mathbb{F}_p, i)) = \begin{cases} \mathbb{F}_p, & j = i \\ *, & j \neq i \end{cases}$$

since $\mathrm{Hom}(X, K_i)/\sim$, quotient by homotopy relations, is isomorphic to $H^i(X, \mathbb{F}_p)$. We have

$$H_j(K_i, \mathbb{Z}) \cong \begin{cases} \mathbb{F}_p, & j = i \\ *, & 0 < j < i \end{cases}$$

by Hurewicz theorem. Therefore, $H^i(K_i) \cong \mathbb{F}_p$ by universal coefficient theorem on singular cohomology. Let us denote $0 \neq \tau_i \in \tilde{H}^i(K_i)$ to be the fundamental class of the cohomology as a generator, so

$$\tau_i^{\otimes p} : K_i^{\wedge p} \to K_{pi}$$

is non-zero (up to homotopy), and $\tau_i^{\otimes p} \circ T = \tau_i^{\otimes p}$ where $T = (1\ 2\ 3\ \cdots\ p)$. Therefore, there is a homotopy $K_i^{\wedge p} \times [0,1] \to K_{pi}$ between $\tau_i^{\otimes p}$ and $\tau_i^{\otimes p} \circ T$, which induces a map $\Gamma_1 K_i \to K_{pi}$ on quotient spaces, where $\Gamma_1 K_i = (K_i^{\wedge p} \times [0,1])/((X, 0) \sim (T(x), 1))$. This further descends to a map

$$\lambda_1 : \Lambda_1 K_i \to K_{pi}$$

where $\Lambda_1 K_i = \Gamma_1 K_i/S^1$ where S^1 acts as a section. In general, we construct $\Gamma K_i = (K_i^{\wedge p} \times S^\infty)/\mathbb{F}_p$, and that $\Lambda K_i = \Gamma K_i/L_p^\infty$. Therefore, ΛK_i can be obtained from $\Lambda_1 K_i$ by attaching n-cells where $n \geqslant pi+2$, so by Proposition 1.4, λ_1 can be lifted to $\lambda : \Lambda K_i \to K_{pi}$. Now we define $\lambda(\alpha)$ to be the composition

$$\Lambda X \xrightarrow{\Lambda\alpha} \Lambda K_i \xrightarrow{\lambda} K_{pi}$$

for $\alpha : X \to K_i$. Then $\lambda(\alpha)$ pulls back to $\alpha^{\otimes p}$ on each fiber $X^{\wedge p}$ of $\Lambda X \to L_p^\infty$, since λ is induced from $\tau_i^{\otimes p}$.

Definition 1.5. Define

$$\begin{aligned} \nabla : (L_p^\infty \times X)/L_p^\infty &\to \Lambda X := (X^{\wedge p} \times S^\infty)/\mathbb{F}_p \\ (z, x) &\mapsto (x, \ldots, x, z) \end{aligned}$$

to be the diagonal map.

Remark 1.6. By Künneth formula, we know

$$\tilde{H}^{pi}((L_p^\infty \times X)/L_p^\infty) \cong \bigoplus_{j \leqslant pi} \tilde{H}^j(X) \otimes \tilde{H}^{pi-j}(L_p^\infty),$$

therefore $\nabla^*(\lambda(\alpha)) = \sum_j \theta_j(\alpha) \otimes \omega_{(p-1)i-j}$ where $\theta_j(\alpha) \in \tilde{H}^{i+j}(X)$, and $\omega_k \in \tilde{H}^k(L_p^\infty)$ is

$$\omega_k = \begin{cases} \alpha\beta^{\frac{k}{2}}, & k \text{ odd} \\ \beta^{\frac{k}{2}}, & k \text{ even} \end{cases}$$

the generator given in Proposition 1.2.

In the case where $p = 2$, we set $\mathrm{Sq}^j(\alpha) = \theta_j(\alpha)$; for $p > 2$, a lot of terms in θ_j vanishes, and we will show that $\theta_j = 0$ unless $j = 2k(p-1)$ or $2k(p-1)+1$. We will define the Steenrod power $\mathcal{P}^k = c\theta_{2k(p-1)}$, and $\beta\mathcal{P}^k = c\theta_{2k(p-1)+1}$ for Bockstein β and some constant c that will be specified later.

1.2 Steenrod Operations for $p = 2$

Let us start by establishing the properties in the case where $p = 2$.

Theorem 1.7. The Steenrod operations satisfy the following properties in the case $p = 2$:

1. $\mathrm{Sq}^j(x) = x^2$ for $x \in \tilde{H}^j(X)$; moreover, $\mathrm{Sq}^j = 0$ if $j < 0$, and $\mathrm{Sq}^j(x) = 0$ if $x \in \tilde{H}^i(X)$ for $i < j$;
2. for any continuous morphism $f : X \to Y$, $\mathrm{Sq}^j \circ f^* = f^* \circ \mathrm{Sq}^j$;
3. they satisfy the Cartan formula, i.e., $\mathrm{Sq}^j(xy) = \sum_k \mathrm{Sq}^k(x)\,\mathrm{Sq}^{k-j}(y)$;
4. they are stable, i.e., they commute with suspension (smashing with S^1): $\mathrm{Sq}^j \circ \Sigma = \Sigma \circ \mathrm{Sq}^j$;
5. $\mathrm{Sq}^0 = \mathrm{id}$;
6. $\mathrm{Sq}^1 = \beta : H^i(X, \mathbb{Z}/2\mathbb{Z}) \to H^{i+1}(X\mathbb{Z}) \to H^{i+1}(X, \mathbb{Z}/2\mathbb{Z})$ is the Bockstein operation;
7. Sq^j is an additive group homomorphism.

Proof.

1. If $\alpha \in \tilde{H}^j(X)$, then $\mathrm{Sq}^j(\alpha)$ is obtained from the pullback $\nabla^*(\lambda(\alpha))$ along $X \to L_2^\infty \times X$ which factors through $X^{\wedge 2}$, so it is equivalent to α^2 by construction. To see $\mathrm{Sq}^j = 0$ for $j < 0$, it suffices to show this on the Eilenberg-Maclane spaces, since they are given by the pullback of these spaces in the first place. Since
$$\tilde{H}^j(K_i) \cong \begin{cases} \mathbb{F}_p, & j = i \\ 0, & j < i \end{cases}$$
then $\mathrm{Sq}^j(\tau_i) = 0$ whenever $j < 0$, as desired. The final statement follows from the definition.

2. This follows from the definition as well.

3. Note that we are taking the pullback after multiplication. Since the pullback can be calculated via the Künneth formula, so we just need to show that λ is compatible with multiplication. That is, it suffices to show that λ is multiplicative, i.e., $\lambda(\alpha\beta) = \lambda(\alpha)\lambda(\beta)$, then the statement follows easily from

$$H^*((X \times L_2^\infty)/L_2^\infty) \cong \tilde{H}^*(X)[\mathcal{O}(1)]$$

where $\mathcal{O}(1)$ is the tautological line bundle of $L_2^\infty \cong \mathbb{R}P^\infty$. Suppose $\alpha \in \tilde{H}^m(X)$ and $\beta \in \tilde{H}^n(X)$, then α and β correspond to some morphisms $X \to K_m$ and $X \to K_n$, respectively. Therefore, we have a diagram

$$\begin{tikzcd}
\Lambda X \arrow[r, "\Lambda(0)"] \arrow[dr, "\Delta"'] & \Lambda(X \wedge X) \arrow[r, "\Lambda(\alpha\wedge\beta)"] \arrow[d, "\varphi_{X,X}"] & \Lambda(K_m \wedge K_n) \arrow[r, "\Lambda(\tau_m\otimes\tau_n)"] \arrow[d, "\varphi_{K_m,K_n}"'] \arrow[dr, "\lambda(\tau_m\otimes\tau_n)"] & \Lambda K_{m+n} \arrow[d, "\lambda"] \\
 & \Lambda X \wedge \Lambda X \arrow[r, "\Lambda\alpha\wedge\Lambda\beta"'] & \Lambda K_m \wedge \Lambda K_n \arrow[r, "\lambda(\tau_m)\otimes\lambda(\tau_n)"'] & K_{2m+2n}
\end{tikzcd} \tag{1.8}$$

where τ_m, τ_n are fundamental classes, and

$$\begin{aligned} \varphi_{X,Y} : \Lambda(X \wedge Y) &\to (\Lambda X) \wedge (\Lambda Y) \\ (x, y, p) &\mapsto ((x, p), (y, p)) \end{aligned}$$

for $x \in X^{\wedge 2}, y \in Y^{\wedge 2}$, and $p \in L_2^\infty$. This diagram commutes (up to homotopy) trivially, except that we need to justify the commutativity of

$$\lambda(\tau_m \otimes \tau_n) = (\lambda(\tau_m) \otimes \lambda(\tau_n)) \circ \varphi_{K_m,K_n} \tag{1.9}$$

Remark 1.10. Note that in the case where $p > 2$, Equation (1.9) commutes up to a sign $(-1)^{\frac{p(p-1)mn}{2}}$, for a reason specified later.

Claim 1.11. The map

$$\tilde{H}^{2m+2n}(\Lambda(K_m \wedge K_n)) \to \tilde{H}^{2m+2n}((K_m \wedge K_n)^{\wedge 2})$$

is injective.

Subproof. Note that $\Lambda(K_m \wedge K_n) \cong (K_m \wedge K_n)^{\wedge 2} \times S^\infty / L_2^\infty$ is obtained from $\Lambda_1(K_m \wedge K_n)$ by attaching cells of dimension at least $2m + 2n + 2$. Therefore, we have a filtration

$$\cdots \subseteq X_i \subseteq X_{i+1} \subseteq \cdots$$

where $X_0 = \Lambda_1(K_m \wedge K_n)$ and $\bigcup_i X_i = \Lambda(K_m \wedge K_n)$, such that there is a cofiber sequence

$$X_i \longrightarrow X_{i+1} \longrightarrow \coprod_{t \geqslant 2m+2n+2} S^t \longrightarrow \Sigma X_i \cong X_i \wedge S^1$$

from each gluing given by a pushout

$$\begin{tikzcd}
X_i \arrow[r] & X_{i+1} \\
\coprod S^{t-1} \arrow[r] \arrow[u] & \coprod * \arrow[u]
\end{tikzcd}$$

By calculation the long exact sequence of cohomology and our knwoledge of cohomology of spheres, the map $\tilde{H}^{2m+2n}(X_{i+1}) \to \tilde{H}^{2m+2n}(X_i)$ is an isomorphism. Taking limits through the directed system, we get that

$$\tilde{H}^{2m+2n}(\Lambda(K_m \wedge K_n)) \to \tilde{H}^{2m+2n}(\Lambda_1(K_m \wedge K_n))$$

is an isomorphism as well. Therefore,

$$\Lambda_1(K_m \wedge K_n)/(K_m \wedge K_n)^{\wedge 2} \cong (K_m \wedge K_n)^{\wedge 2} \wedge S^1$$

as $\Gamma_1(X) = (X^{\wedge 2} \times [0,1])/((x_1, x_2, 0) \sim (x_1, x_1, 1))$, so there is an exact sequence

$$\tilde{H}^{2m+2n}((K_m \wedge K_n)^{\wedge 2} \wedge S^1) \longrightarrow \tilde{H}^{2m+2n}(\Lambda_1(K_m \wedge K_n)) \longrightarrow \tilde{H}^{2m+2n}((K_m \wedge K_n)^{\wedge 2})$$

Note that the first term is $\tilde{H}^{2m+2n-1}((K_m \wedge K_n)^{\wedge 2})$, which is zero by Künneth formula, therefore we have an injection $\tilde{H}^{2m+2n}(\Lambda_1(K_m \wedge K_n)) \to \tilde{H}^{2m+2n}((K_m \wedge K_n)^{\wedge 2})$. Now take the composition with the isomorphism we found above, we have an injection

$$\tilde{H}^{2m+2n}(\Lambda(K_m \wedge K_n)) \xrightarrow{\cong} \tilde{H}^{2m+2n}(\Lambda_1(K_m \wedge K_n)) \rightarrowtail \tilde{H}^{2m+2n}((K_m \wedge K_n)^{\wedge 2})$$

as desired. ∎

By Claim 1.11, to show the commutativity, it suffices to show that the triangle commutes when we pullback to the fiber, i.e., precompose with the map induced from $(K_m \wedge K_n)^{\wedge 2} \to \Lambda(K_m \wedge K_n)$. Therefore, we just need to show that

$$(\tau_m \otimes \tau_n)^{\otimes 2} = \tau_m^{\otimes 2} \otimes \tau_n^{\otimes 2},$$

but over $p = 2$ this is trivial.[6] The statement then follows immediately.

4. Let us consider $\mathrm{Sq}^0(\alpha)$ where α generates $H^1(S^1)$. We have a commutative diagram

$$\begin{tikzcd}
\tilde{H}^2((S^1 \times L_2^\infty)/L_2^\infty) \arrow[r, "a"] & \tilde{H}^2((S^1 \times S^1)/S^1) \\
\tilde{H}^2(\Lambda S^1) \arrow[u] \arrow[r, "b"'] & \tilde{H}^2(\Lambda_1 S^1) \arrow[u, "u^*"']
\end{tikzcd}$$

where u^* is induced from the diagonal morphism. By Künneth formula, we know a is an isomorphism; we also know that b is an isomorphism using the same argument as before. We now have two maps

$$\begin{aligned} u : S^1 \times S^1 &\to \Gamma_1 S^1 \\ (x, y) &\mapsto (x, x, y) \end{aligned}$$

where we think of $\Gamma_1 S^1 = ((S^1 \wedge S^1) \times S^1)/((x, y, z) \sim (y, x, -z))$, since the -1 acts as the square root of the identity, and a pointed map

$$\begin{aligned} v : S^1 \wedge S^1 &\to \Gamma_1 S^1 \\ (x, y) &\mapsto (x, y, x_0) \end{aligned}$$

where x_0 is the basepoint on S^1. If we think of $\Gamma_1 S^1 = (S^2 \times [0,1])/((x, y, z, 0) \sim (x, y, -z, 1))$ for $(x, y, z) \in S^1$, then this action is the same as given two nested spheres S^2's, we identify the southern hemisphere of the outer sphere with the northern hemisphere of the inner sphere. Now $\mathrm{im}(H_2(v))$ is the outer sphere and $\mathrm{im}(H_2(u))$ is the torus bounded by the two equators of the two spheres. With the identification of the semispheres, we know these two images are of the same class, i.e., $\mathrm{im}(H^2(u)) + \mathrm{im}(H^2(v)) = 0$. Therefore, $H_2(u)^* = H_2(v)^*$.

We have shown before that there is an injection

$$v^* : H^2(\Lambda_1 S^1) \to H^2(S^1 \wedge S^1) \cong \mathbb{Z}/2\mathbb{Z}$$

which is non-zero since $v^*(\lambda(\alpha)) = \alpha \otimes \alpha \neq 0$ as α is a generator. By duality, we have

$$u^* = H_2(u)^* = H_2(v)^* = v^* \neq 0,$$

hence $u^*(\lambda(\alpha))$ is the generator of $\tilde{H}^2((S^1 \times S^1)/S^1)$. In particular, $\mathrm{Sq}^0(\alpha) \neq 0$, but that forces $\mathrm{Sq}^0(\alpha) = \alpha$ from the decomposition.

[6] Again, for $p = 2$, this is true up to a sign-change term.

Now suppose $x \in H^i(X)$, then $\Sigma x = \alpha \otimes x \in H^{i+1}(\Sigma X)$, therefore

$$\begin{aligned}\operatorname{Sq}^j(\Sigma x) &= \operatorname{Sq}^j(\alpha \otimes x) \\ &= \alpha \otimes \operatorname{Sq}^j(x) \\ &= \Sigma \operatorname{Sq}^j(x)\end{aligned}$$

by definition, the property that $\operatorname{Sq}^0(\alpha) = \alpha$, and the Cartan formula.

5. Sq^0 induces the identity map on $\tilde{H}^n(S^n)$ by the previous property, therefore $\operatorname{Sq}^0(\tau_n) = \tau_n$ by the map $S^n \to K_n$ given by $\alpha^{\otimes n}$. Now $\tau_n \in K_n$ pulls back to $\alpha^{\otimes n}$ in S^n, therefore $\operatorname{Sq}^0 = \mathrm{id}$.
6. Let $\mathcal{O}(1) \in H^1(\mathbb{R}P^2)$ be the generator, then $\operatorname{Sq}^1(\mathcal{O}(1)) = \mathcal{O}(1)^2$. Recall that $\beta(\mathcal{O}(1)) = \mathcal{O}(1)^2$ in Proposition 1.2, so by the commutativity of the suspension, we know that

$$\begin{aligned}\operatorname{Sq}^1(\Sigma^n\mathcal{O}(1)) &= \Sigma^n \operatorname{Sq}^1(\mathcal{O}(1)) \\ &= \Sigma^n \beta(\mathcal{O}(1)) \\ &= \beta(\Sigma^n\mathcal{O}(1)).\end{aligned}$$

But $\Sigma^n\mathbb{R}P^2 \to K_{n+1}$ induces an isomorphism on H^{n+1}, so $\operatorname{Sq}^1(\tau_{n+1}) = \beta(\tau_{n+1})$. However, all elements on the cohomology were pulled back from τ^{n+1}, so we are done.

7. Since $\tilde{H}^*(X) = \tilde{H}^{*+1}(\Sigma X)$, it suffices to consider the classes on ΣX. Note that we may think of Sq^j as a map $K_n \to K_{n+j}$: the identity map on K_n corresponds to the fundamental class $\tau_n \in H^n(K_n)$, then we may correspond $\theta = \operatorname{Sq}^j(\tau_n) \in H^{n+j}(K_n) = [K_n, K_{n+j}]$. In particular, θ makes the diagram

$$\begin{tikzcd} K_n \arrow[r, "\theta"] & K_{n+j} \\ X \arrow[u, "\alpha"] \arrow[ur, "\operatorname{Sq}^j(\alpha)"'] & \end{tikzcd}$$

commute. From the map $\iota : S^1 \to S^1 \wedge S^1$ that twists the circle, then

$$\Sigma X \xrightarrow{\iota} (\Sigma X) \wedge (\Sigma X) \xrightarrow{\alpha \wedge \beta} K_n \xrightarrow{\theta = \operatorname{Sq}^j} K_{n+j}$$

gives

$$\begin{aligned}\operatorname{Sq}^j(\alpha+\beta) &= \theta \circ (\alpha \wedge \beta) \circ \iota \\ &= (\theta\alpha \wedge \theta\beta) \circ \iota \\ &= \theta\alpha + \theta\beta \\ &= \operatorname{Sq}^j(\alpha) + \operatorname{Sq}^j(\beta).\end{aligned}$$

□

Theorem 1.12. The Steenrod operations satisfy the Adem relations

$$\operatorname{Sq}^a \operatorname{Sq}^b = \sum_j \binom{b-j-1}{a-2j} \operatorname{Sq}^{a+b-j} \operatorname{Sq}^j$$

for $a < 2b$.

Proof. Define $\Gamma^2 X = (X^{\wedge 4} \times S^\infty \times S^\infty)/(\mathbb{F}_2 \times \mathbb{F}_2)$, where $\mathbb{F}_2$'s have generators a and b, respectively, then its action on $S^\infty \times S^\infty$ is given by $(a,b)\cdot((x_{i,j}), z_1, z_2) = ((x_{a(i),b(j)}), a\cdot z_1 = -z_1, b\cdot z_2)$ for $1 \leqslant i,j \leqslant 2.$, and $a = b = (1\,2\,\cdots\,p)$. Define $\Lambda^2 X = \Gamma^2 X/(S^\infty \times S^\infty)$, i.e., quotient by sections, then the two projections

$$\begin{tikzcd} \Gamma^2 X \arrow[r] \arrow[d] & S^\infty \times S^\infty \\ \Lambda^2 X \arrow[ur] & \end{tikzcd}$$

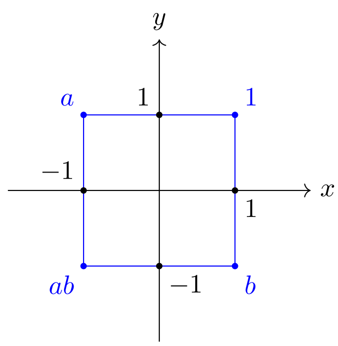
have fibers $X^{\wedge 4}$. Consider $\Lambda(\Lambda X) = (((((X^{\wedge 2} \times S^\infty)/\mathbb{F}_2) \wedge ((X^{\wedge 2} \times S^\infty)/\mathbb{F}_2))/S^\infty)/\mathbb{F}_2)/\sim$ via quotient by sections. Suppose that u and v act on the first and second term of $((X^{\wedge 2} \times S^\infty)/\mathbb{F}_2)$, respectively, and let t act on $\mathbb{F}_2$. One may easily verify that $tut = v$ as actions on $X^{\wedge 4} \times S^\infty \times S^\infty \times S^\infty$, therefore t, u, v generates a group isomorphic to D_8, where t is the reflection and tv is the rotation by $\frac{\pi}{2}$. The subgroup $\langle t, uv\rangle \subseteq D_8$ is a subgroup being isomorphic to $\mathbb{F}_2 \times \mathbb{F}_2$. Therefore, we have a map

$$\begin{aligned} \Lambda^2 X &\to \Lambda(\Lambda X) \\ ((x_{ij}), p, q) &\mapsto (((x_{1j}), q), ((x_{2j}), q), p) \end{aligned}$$

It now suffices to show the relation for every $\tau_n \in \tilde{H}^n(K_n)$. The only non-trivial map $K_n^{\wedge 4} \to K_{4n}$ is $\tau_n^{\otimes 4}$ up to homotopy, now think of the $(\mathbb{F}_2 \times \mathbb{F}_2)$-action on $K_n^{\wedge 4}$ as given by $(a, b) \cdot K_{i,j} = K_{a(i),b(j)}$ for $\mathbb{F}_2 \times \mathbb{F}_2 = \langle a\rangle \times \langle b\rangle$, then

$$\begin{aligned} \tau_n^{\otimes 4} \circ a &= \tau_n^{\otimes 4} \circ b \\ &= \tau_n^{\otimes 4} \\ &= \tau_n^{\otimes 4} \circ ab, \end{aligned}$$

and we have homotopies $\varphi_{a,1} : \tau_n^{\otimes 4} \circ a \sim \tau_n^{\otimes 4}$, $\varphi_{b,1} : K_n^{\wedge 4} \times [-1, 1] \to K_{4n}$, $\varphi_{ab,a} = (a \times \mathrm{id}) \circ \varphi_{b,1}$, and $\varphi_{ab,b} = (b \times \mathrm{id}) \circ \varphi_{a,1}$. By visualizing, we note that the four homotopies above are glued one after another in the square



and obtain a map $K_n^{\wedge 4} \times \partial[-1, 1]^2 \to K_{4n}$. By Proposition 1.4, this extends to a map $K_n^{\wedge 4} \times I^2 \to K_{4n}$, which passes down to a map $\lambda_2 : \Gamma_1^2 K_n/(\mathbb{F}_2 \times \mathbb{F}_2) \cong (K_n^{\wedge 4} \times S^1 \times S^1)/(\mathbb{F}_2 \times \mathbb{F}_2) \to K_{4n}$ by construction. By Proposition 1.4 again, we extend it to a map $\lambda_2 : \Lambda^2 K_n \cong \Gamma^2 X/(S^\infty \times S^\infty) \to K_{4n}$, which restricts to $\tau_n^{\otimes 4}$ on the fibers $X^{\wedge 4}$, where $\Gamma^2 X = (X^{\wedge 4} \times S^\infty \times S^\infty)/(\mathbb{F}_2 \times \mathbb{F}_2)$. This gives rise to a commutative diagram

$$\begin{tikzcd} L_2^\infty \times L_2^\infty \times K_n \arrow[r, "\nabla_2"] \arrow[d, "\mathrm{id} \times \nabla"'] & \Lambda^2 K_n \arrow[r, "\lambda_2"] \arrow[d] & K_{4n} \\ L_2^\infty \times \Lambda K_n \arrow[r, "\nabla"'] & \Lambda(\Lambda K_n) \arrow[ur, "\Lambda(\lambda)"'] & \end{tikzcd} \tag{1.13}$$

where

$$\begin{aligned} \nabla_2 : L_2^\infty \times L_2^\infty \times K_n &\to \Lambda^2 K_n \\ (p, q, x) &\mapsto (x, x, x, x, p, q) \end{aligned}$$

The commutativity of the square is trivial, so we need to show that the triangle also commutes up to homotopy. This is given by the commutative diagram

$$\begin{tikzcd} \tilde{H}^{4n}(K_n^{\wedge 4}) & \tilde{H}^{4n}(\Gamma^2 K_n) \arrow[l] \arrow[d] & \tilde{H}^{4n}(\Lambda^2 K_n) \arrow[l] \arrow[d, "\alpha"] & \\ & \tilde{H}^{4n}(\Gamma_1^2 K_n) \arrow[ul, "\gamma"] & \tilde{H}^{4n}(\Lambda_1^2 K_n) \arrow[l, "\beta"] & \tilde{H}^{4n-1}(S^1 \times S^1) \arrow[l] \end{tikzcd} \tag{1.14}$$

where we study the commutativity on the fibers using the fiber sequence. Since Γ_2^1 and Λ_2^1 differs by $S^1 \times S^1$, then there is an exact sequence of quotient space, which extends to the exact sequence with respect to β. In this diagram, α is an isomorphism by comparing the dimension of the cells attached. Moreover, β is an injection since $\tilde{H}^{4n-1}(S^1 \times S^1) = 0$ when $n > 0$. γ is also injective since $\Gamma_1^2 K_n/(K_n^{\wedge 4}) \cong K_n^{\wedge 4} \wedge S^1 \wedge S^1$. Therefore, the pullback $\tilde{H}^{4n}(\Lambda^2 K_n) \to \tilde{H}^{4n}(K_n^{\wedge 4})$ of the fiber is an injection as well. But $\tilde{H}^{4n}(\Lambda^2 K_n) \cong \mathbb{Z}/2\mathbb{Z}$, therefore this is an isomorphism. Moreover, there is a commutative diagram

$$\begin{tikzcd} K_n^{\wedge 4} \arrow[r] \arrow[d] & \Lambda^2 K_n \arrow[d] \\ (\Lambda K_n)^{\wedge 4} \arrow[r] & \Lambda(\Lambda K_n) \end{tikzcd}$$

where the bottom composition is an injection and hence so is the other composition. This forces an injection on the level of cohomology, which shows that $\tilde{H}^{4n}(\Lambda(\Lambda K_n)) \to \tilde{H}^{4n}(\Lambda^2 K_n) \to \tilde{H}^{4n}(K_n^{\wedge 4})$ is an isomorphism. One can observe that the map $K_n^{\wedge 4} \to K_{4n}$ gives a unique non-zero morphism induced above, then it suffices to show that when pulled back to the fiber, the triangle in Diagram 1.13 is given by a non-zero map. Indeed, $\Lambda(X)$ is non-zero by checking on its fibers $(\Lambda K_n)^{\wedge 2}$, therefore Diagram 1.13 has to be commutative.

There is now a commutative diagram

$$\begin{tikzcd} L_2^\infty \times L_2^\infty \times K_n \arrow[r, "\nabla_2"] \arrow[d, "\tau"'] & \Lambda^2 K_n \arrow[d, "\tau'"] \\ L_2^\infty \times L_2^\infty \times K_n \arrow[r, "\nabla_2"'] & \Lambda^2 K_n \end{tikzcd} \tag{1.15}$$

where

$$\begin{aligned} \tau : L_2^\infty \times L_2^\infty \times K_n &\to L_2^\infty \times L_2^\infty \times K_n \\ (p, q, x) &\mapsto (q, p, x) \end{aligned}$$

Since elements in $\Lambda^2 K_n$ is represented by a (2×2) matrix with two elements of S^∞, then

$$\begin{aligned} \tau' : \Lambda^2 K_n &\to \Lambda^2 K_n \\ ((x_{ij}), p, q) &\mapsto (x_{ji}, q, p) \end{aligned}$$

Using the commutative Diagram 1.13, pulling back via the first row should be the same as pulling back via the second row, so if we calculate the fundamental class of K_{4n}, where we suppose $\nabla_2^* \lambda_2^*(\tau_{4n}) = \sum_{r,s} \mathcal{O}(1)^r \otimes \mathcal{O}(1)^s \otimes X_{rs}$ where $\mathcal{O}(1)$ is the tautological bundle as the generator of $H^1(\mathbb{R}P^2)$, $\mathcal{O}(1)^r, \mathcal{O}(1)^s \in L_2^\infty$, and $X_{rs} \in K_n$. Using this information,

$$\begin{aligned} \sum_{r,s} \mathcal{O}(1)^s \otimes \mathcal{O}(1)^r \otimes X_{rs} &= \tau^* \nabla_2^* \lambda_2^*(\tau_{4n}) \\ &= \nabla_2^* \tau'^* \lambda_2^*(\tau_{4n}) \\ &= \nabla_2^* \lambda_2^*(\tau_{4n}) \end{aligned}$$

since τ'^* as an automorphism is the identity map on the one-dimensional space on the linear space $\tilde{H}^{4n}(\Lambda^2 K_n) \cong \mathbb{F}_2$. Therefore, $X_{rs} = X_{sr}$. Note that $\Lambda(\lambda) \circ \nabla$ gives a Steenrod operation, then then pullbacks via Diagram 1.13 gives

$$\begin{aligned} \nabla_2^* \lambda_2^*(\tau_{4n}) &= \sum_j \mathcal{O}(1)^{2n-j} \otimes \nabla^* \operatorname{Sq}^j(\lambda) \\ &= \sum_j \mathcal{O}(1)^{2n-j} \otimes \operatorname{Sq}^j(\nabla^* \lambda) \\ &= \sum_{j,k} \mathcal{O}(1)^{2n-j} \otimes \operatorname{Sq}^j(\mathcal{O}(1)^{n-k} \operatorname{Sq}^k(\tau_n)) \end{aligned} \tag{1.16}$$

where $\lambda : \Lambda K_n \to K_{2n}$. By Cartan formula,

$$\sum_j \operatorname{Sq}^j(\mathcal{O}(1)^{n-k}) t^j = (\mathcal{O}(1) + \mathcal{O}(1)^2 t)^{n-k},$$

therefore

$$\operatorname{Sq}^j(\mathcal{O}(1))^{n-k} = \binom{n-k}{j}\mathcal{O}(1)^{n-k+j}.$$

Thus, Equation (1.16) gives

$$\sum_{j,k}\mathcal{O}(1)^{2n-j}\otimes\operatorname{Sq}^j(\mathcal{O}(1)^{n-k}\operatorname{Sq}^k(\tau_n)) = \sum_{j,k,l}\mathcal{O}(1)^{2n-j}\otimes\binom{n-k}{l}\mathcal{O}(1)^{n-k+i}\otimes\operatorname{Sq}^{j-l}\operatorname{Sq}^k(\tau_n).$$

Set $n-k+l = 2n-i$, then we get

$$\sum_{j,k}\mathcal{O}(1)^{2n-j}\otimes\operatorname{Sq}^j(\mathcal{O}(1)^{n-k}\operatorname{Sq}^k(\tau_n)) = \sum_{i,j,k}\binom{n-k}{n-i+k}\mathcal{O}(1)^{2n-j}\otimes\mathcal{O}(1)^{2n-j}\otimes\operatorname{Sq}^{j+i-n-k}\operatorname{Sq}^k(\tau_n),$$

and since $X_{2n-j,2n-i} = X_{2n-i,2n-j}$, then

$$\sum_k\binom{n-k}{n-i+k}\operatorname{Sq}^{j+i-n-k}\operatorname{Sq}^k(\tau_n) = \sum_k\binom{n-k}{n-j+k}\operatorname{Sq}^{j+i-n-k}\operatorname{Sq}^k(\tau_n).$$

Lemma 1.17 (Lucas' Theorem)**.** Suppose $n,m\in\mathbb{N}$ and p is a prime. We write down binary representations $m=\sum_k a_kp^k$ and $n=\sum_k b_kp^k$ where $a_k,b_k\in\{0,\dots,p-1\}$, then

$$\binom{m}{n}\equiv\prod_k\binom{a_k}{b_k}\pmod{p}.$$

Subproof. For $1\leqslant n\leqslant p-1$, it is easy to show that $p\mid\binom{p}{n}$ since the Frobenius map $(-)^p$ on $\mathbb{F}_p$ is compatible with addition. Therefore, $(1+x)^p = 1+x^p$ in $\mathbb{F}_p[x]$. We have

$$\begin{aligned}
\sum_{n=0}^m\binom{m}{n}x^n &= (1+x)^m\\
&= \prod_{i=0}^{p-1}((1+x)^{p_i})^{a_i}\\
&= \prod_{i=0}^{p-1}(1+x^{p_i})^{a_i}\\
&= \prod_{i=0}^{p-1}\left(\sum_{n_i=0}^{a_i}\binom{a_i}{n_i}x^{n_ip_i}\right)\\
&= \prod_{i=0}^{p-1}\left(\sum_{n_i=0}^{p-1}\binom{a_i}{n_i}x^{n_ip_i}\right)\\
&= \sum_{n=0}^{m}\left(\prod_{i=0}^{p-1}\binom{a_i}{n_i}\right)x^n
\end{aligned}$$

for $n=\sum n_ip^i$ in $\mathbb{F}_p[x]$. ■

Using Lemma 1.17, $\binom{m}{n}$ is odd if and only if there is no k such that $a_k=0$ and $b_k=1$. Now suppose $n=2^r-1+s$ and $i=n+s$ to be arbitrary (but large), then

$$\binom{n-i}{n-i+k} = \binom{2^r-1-(k-s)}{k-s}.$$

Observe that this number is even unless $k-s=0$, according to Lemma 1.17. Therefore, we obtain

$$\operatorname{Sq}^j\operatorname{Sq}^s(\tau_n) = \sum_k\binom{2^r-1+s-k}{2^r-1+s-j+k}\operatorname{Sq}^{j+s-k}\operatorname{Sq}^k(\tau_n)$$

$$= \sum_k \binom{2^r - 1 + s - k}{j - 2k} \operatorname{Sq}^{j+s-k} \operatorname{Sq}^k(\tau_n).$$

Assuming $j < 2s$ and r is large, then

$$\binom{2^r - 1 + s - k}{j - 2k} \equiv \binom{s - k - 1}{j - 2k} \pmod 2.$$

To see this, note that either $j < 2k$ or $2k \leqslant j < 2s$: in the first case, both sides vanish; in the second case, since $k < s$, then $s - k - 1 > 0$, now having r large enough, adding 2^r in the binary representation adds a 1 somewhere in front of the number, but by Lemma 1.17 that does not contribute to the modulo operation, hence we have a congruence. This estaiblishes the Adems relation when $n = 2^r - 1 + s$ and when r is large. For general n, one can use the suspension to ensure $n - s = 2^r - 1$ and r is large. $\square$

This describes the Steenrod operations when $p = 2$. Now let us turn to the case where $p > 2$, and we have to establish the Steenrod powers.

1.3 Steenrod Powers for $p > 2$

Recall that given given $\alpha \in \tilde{H}^i(X)$, the pullback via ∇ satisfies a Künneth decomposition

$$\nabla^*(\lambda(\alpha)) = \sum_j \omega_{(p-1)i-j} \otimes \theta_j(\alpha)$$

where $\lambda : \Lambda(X) \to K_{pi}$ is a map pulling back to $\alpha^{\otimes p}$ on fibers $X^{\wedge p}$,

$$\begin{aligned} \nabla : (L_p^\infty \times X)/L_p^\infty &\to \Lambda(X) \\ (z, x) &\mapsto (x, \cdots, x, z) \end{aligned}$$

is the diagonal map, and $\theta_j(\alpha) \in \tilde{H}^{i+j}(X)$. We claim that θ_j is non-zero only on certain values.

Proposition 1.18. We have $\theta_j = 0$ unless $j \equiv 0 \pmod{2(p-1)}$ or $j \equiv 1 \pmod{2(p-1)}$ for $j \geqslant 0$.

Proof. Since $\mathbb{F}_p^*$ is cyclic of order $p - 1$, we assume it is generated by r. One can view $\mathbb{F}_p^*$ as an automorphism via the mapping $\mathbb{F}_p^* \to \operatorname{Aut}(\mathbb{F}_p)$, then we can define

$$\begin{aligned} \varphi : S^\infty \times X^{\wedge p} &\to S^\infty \times X^{\wedge p} \\ ((z_i), x_0, \ldots, x_{p-1}) &\mapsto \left(\left(\frac{z_i^r}{\sum (z_i)^r} \right), x_{r(0)}, \ldots, r_{r(p-1)} \right). \end{aligned}$$

Denote T to be the generator of $\mathbb{F}_p$ as an additive group, such that T acts on $X^{\wedge p} \times S^\infty$ via

$$T \cdot (x_0, \ldots, x_{p-1}, z) = (x_1, \ldots, x_{p-1}, x_0, e^{\frac{2\pi i}{p}} z).$$

Therefore

$$\begin{aligned} \varphi(T \cdot (x_0, \ldots, x_{p-1}, z)) &= \varphi(x_1, \ldots, x_{p-1}, x_0, e^{\frac{2\pi i}{p}} z) \\ &= (x_{r(1)}, \ldots, x_{r(p-1)}, x_{r(0)}, e^{\frac{2\pi i r}{p}} z^r) \\ &= T^r \cdot \varphi(x_0, \cdots, x_{p-1}, z), \end{aligned}$$

hence this induces morphisms $\varphi : \Gamma(X) \to \Gamma(X)$ and $\varphi : \Lambda(X) \to \Lambda(X)$. Now let $X = K_n$, then there is a diagram

$$\begin{tikzcd} L_p^\infty \times K_i \arrow[r, "\nabla"] \arrow[d, "(-)^r \times \mathrm{id}"'] & \Lambda(K_i) \arrow[r, "\lambda"] \arrow[d, "\varphi"] & K_{pi} \\ L_p^\infty \times K_i \arrow[r, "\nabla"'] & \Lambda(K_i) \arrow[ur, "\lambda"'] & \end{tikzcd} \tag{1.19}$$

where the triangle commutes up to a factor of $(-1)^i$. Note that the square commutes in the set-theoretical sense obviously which is then easy to verify for the general case. For the triangle, if we precompose it with $K_i^{\wedge p} \to \Lambda(K_i)$, then one can show like before that on the cohomology $\tilde{H}^{pi}$, the pullback of the morphism acts as an injection, then it suffices to show the property on the level of the fiber. That is, it suffices to prove that

$$r(\tau_i^{\otimes p}) = (-1)^i \tau_i^{\otimes p}$$

since the map $\tilde{H}^{pi}(\Lambda(K_i)) \to \tilde{H}^{pi}(K_i^{\wedge p})$ is an isomorphism, which follows from $\operatorname{sgn}(r) = -1$ since we have $r = (0)(1, r(1), \cdots, r^{p-2}(1))$ written down as a product of cycles.

- Suppose that i is even, then Diagram 1.19 comutes, therefore $\sum_j \omega_{(p-1)i-j} \otimes \theta_j(\tau_i)$ is invariant under $(-)^r \times \mathrm{id}$, so $(-)^r$ can be viewed as a L_p^∞-automorphism, i.e.,

  $$r(\omega_{(p-1)i-j}) = \omega_{(p-1)i-j}$$

  if $\theta_i(\tau_i) \neq 0$. The map $(-)^r$ induces a multiplication by r on $H_1(S^1)$, so it has the same action for $H_1(L_p^\infty)$ and $H^1(L_p^\infty)$, sending ω_1 to $r\omega_1$. Since $\omega_2 = \beta(\omega_1)$, then $r(\omega_2) = r\omega_2$, hence

  $$r(\omega_2^t) = r^t \omega_2,$$

  which is the generator of the cohomology of L_p^∞ at degree $2t$, and

  $$r(\omega_1\omega_2^t) = r^{t+1}\omega_1\omega_2^t,$$

  which is the generator of the cohomology of L_p^∞ at degree $2t+1$. Therefore, if $(p-1)i-j$ is odd and $\theta_j(\tau_i) \neq 0$, then we must have

  $$\begin{cases} (p-1)i-j & = 2t+1 \\ t+1 & = k(p-1) \end{cases}$$

  These imply that $j \equiv 1 \pmod{2(p-1)}$. Also, if $(p-1)i-j$ is even and $\theta_j(\tau_i) \neq 0$, then

  $$\begin{cases} 2t & = (p-1)i-j \\ t & = k(p-1) \end{cases}$$

  In this case, we find that $j \equiv 0 \pmod{2(p-1)}$.

- Suppose that i is odd, then Diagram 1.19 anticommutes, therefore $r(\omega_{(p-1)i-j}) = -\omega_{(p-1)i-j}$ if $\theta_j(\tau_i) \neq 0$. Therefore, if $(p-1)i-j$ is odd, then

  $$\begin{cases} 2t+1 & = (p-1)i-j \\ t+1+\frac{p-1}{2} & = k(p-1) \end{cases}$$

  where $r^{\frac{p-1}{2}} = -1$. Therefore $j \equiv 1 \pmod{2(p-1)}$. Finally, if $(p-1)i-j$ is even, then

  $$\begin{cases} 2t & = (p-1)i-j \\ t+\frac{p-1}{2} & = k(p-1) \end{cases}$$

  which implies that $j \equiv 0 \pmod{2(p-1)}$.

□

Suppose $\theta_0 : H^i \to H^i$ map τ_i to $a_i\tau_i$, then a_i determines θ_0, i.e., $\theta_0 = a_i \,\mathrm{id}$. We claim that a_i's satisfy

$$a_{i_1+i_2} = (-1)^{\frac{p(p-1)i_1 i_2}{2}} a_{i1}a_{i2} \tag{1.20}$$

$$a_i = (-1)^{\frac{p(p-1)i(i-1)}{4}} a_1^i. \tag{1.21}$$

Recall from Theorem 1.7 that
$$\lambda(\alpha\beta) = (-1)^{\frac{p(p-1)mn}{2}}\lambda(\alpha)\lambda(\beta)$$
if $\deg(\alpha) = m$ and $\deg(\beta) = n$. We may also show that $\theta_j = 0$ if $j < 0$, as in Theorem 1.7. This can be proven on Eilenberg-Maclane spaces, since the cohomology is centered at one term. Therefore, $\theta_0(\alpha\beta) = (-1)^{\frac{p(p-1)mn}{2}}\theta_0(\alpha)\theta_0(\beta)$, which proves Equation (1.20). Equation (1.21) then follows from Equation (1.20) easily by induction.

Proposition 1.22. $a_1 = \pm\left(\frac{p-1}{2}\right)!$.

Proof. It suffices to compute $\theta_0(\alpha)$ for $0 \neq \alpha \in H^1(S^1)$. Since α comes from the integral cohomology, then there are only two such choices as generators. Therefore, this determines α up to a sign. Since $H^i(S^1) = 0$ for $i > 1$, then $\theta_i(\alpha) = 0$ for $i > 0$. Therefore, we have
$$\nabla^*(\lambda(\alpha)) = \omega_{p-1} \otimes \theta_0(\alpha) = a_1\omega_{p-1} \otimes \alpha$$
in $H^p((L_p^\infty \times S^1)/L_p^\infty)$. The map
$$H^p(L_p^\infty \times S^1) \to H^p(L(p,p) \times S^1)$$
is an isomorphism, then we may substitute all L_p^∞'s by $L(p,p)$, $\Lambda(S^1)$ by $\Lambda_p(S^1) = \Gamma_p(S^1)/L(p,p)$, and $\Gamma(S^1)$ by $\Gamma_p(S^1) = ((S^1)^{\wedge p} \times S^p)/\mathbb{F}_p$, where $\Lambda_p(S^1) \to L(p,p)$ is an $(S^1)^{\wedge p} = S^p$-bundle.

By regarding S^1 as $\mathbb{R}^1 \cup \{\infty\}$, then $(S^1)^{\wedge p} = \mathbb{R}^p \cup \{\infty\}$. Its $\mathbb{F}_p$-action can therefore be written down as
$$T \cdot (x_0, \dots, x_{p-1}) = (x_1, \dots, x_{p-1}, x_0),$$
where $\mathbb{F}_p = \langle T\rangle$. Under this setting, the section $L(p,p) \to \Gamma_p(S^1)$ is the section at ∞. Now $E = \Gamma_p(S^1)\backslash L(p,p)$ is an $\mathbb{R}^p$-bundle over $L(p,p)$. Thus, $\Lambda_p(S^1) = E \cup \{\infty\}$ is the Thom space $\mathrm{Th}(E)$ of E as a compactification. Recall that a class in $H^p(\mathrm{Th}(E))$ that restricts to a generator of $H^p(\mathbb{R}^p \cup \{\infty\})$ for every fiber is called a Thom class. Therefore, $\lambda(\alpha)$ is a Thom class by construction.

We first need to construct subbundles $E_0, E_1, \dots, E_{\frac{p-1}{2}}$ of E, where $\mathrm{rank}(E_0) = 1$ and $\mathrm{rank}(E_j) = 2$ for $j > 0$. The map
$$\begin{aligned} T: \mathbb{R}^p &\to \mathbb{R}^p \\ (x_0, \dots, x_{p-1}) &\mapsto (x_1, \dots, x_{p-1}, x_0) \end{aligned}$$
has characteristic polynomial $x^p - 1$, so it decomposes to
$$(x-1)\prod_{j=1}^{\frac{p-1}{2}}\left(x^2 - 2\cos\left(\frac{2\pi j}{p}\right)x + 1\right)$$
over $\mathbb{R}$. Therefore, we have a eigensubspace decomposition $\mathbb{R}^p = V_0 \oplus V_1 \oplus \cdots \oplus V_{\frac{p-1}{2}}$ fiberwise, where $T|_{V_0} = \mathrm{id}_{V_0}$ for $\dim(V_0) = 1$, and $T|_{V_j}$ is the rotation by $\frac{2\pi j}{p}$ for $1 \leqslant j \leqslant \frac{p-1}{2}$ with $\dim(V_j) = 2$. Hence, E_j is obtained by taking V_j at each fiber of E.

In particular, E_0 is the fixed point locus of the T-action as a trivial bundle, therefore corresponding to the diagonal $(x, \dots, x) \in \mathbb{R}^p$. Therefore, the composition
$$\begin{tikzcd} E_0 \arrow[r] \arrow[d] & \Gamma_p S^1 \arrow[r] & \mathbb{R}^p \cup \{\infty\} \arrow[r, "p_1"] & \mathbb{R} \\ L(p,p) \end{tikzcd}$$
gives the isomorphism $E_0 \cong L(p,p) \times \mathbb{R}$. To calculate the Thom space, we note $\mathrm{Th}(E_0)$ is the pullback as a generator $\alpha \in H^1(S^1) \cong \mathrm{Th}(\mathbb{R})$.

Identifying E_j for $j > 0$ as $(S^p \times \mathbb{C})/\sim$ where $\sim$ is given by $(V, Z) \sim \left(e^{\frac{2\pi i}{p}}V, e^{\frac{2\pi ij}{p}}z\right)$, we get to define a $\mathbb{C}$-bundle
$$\bar{E}_j \to \mathbb{C}P^{\frac{p-1}{2}}$$

by $\bar{E}_j \cong (S^p \times \mathbb{C})/\sim$ where $(V,Z) \sim (\lambda V, \lambda^j Z)$ for $|\lambda| = 1$. We then have a commutative diagram

$$\begin{tikzcd} E_j \arrow[r, "\tilde{q}"] \arrow[d] & \bar{E}_j \arrow[r, "\tilde{f}"] \arrow[d] & \bar{E}_1 \arrow[d] \\ L(p,p) \arrow[r, "q"'] & \mathbb{C}P^{\frac{p-1}{2}} \arrow[r, "f"'] & \mathbb{C}P^{\frac{p-1}{2}} \end{tikzcd} \tag{1.23}$$

where $\tilde{q}$ is a homeomorphism on $\mathbb{C}$-fibers, and f and $\tilde{f}$ are induced by the map

$$\begin{aligned} S^p \times \mathbb{C} &\to S^p \times \mathbb{C} \\ (V_1, \cdots, V_{\frac{p-1}{2}}, Z) &\mapsto \left(\frac{V_1^j}{\sum\limits_k |V_k|^j}, \cdots, \frac{V_{p+1}^j}{\sum\limits_k |V_k|^j}, Z \right). \end{aligned}$$

The map $\tilde{f}$ is well-defined since the equivalent pairs $(V,Z) \sim (\lambda V, \lambda^j Z)$ in $\bar{E}_j$ are mapped to (V^j, Z) and $(\lambda^j V^j, \lambda^j Z)$ which are equivalent in $\bar{E}_1$. Since both $\tilde{q}$ and $\tilde{f}$ restrict to homeomorphism on the fibers, then the extension to maps of Thom spaces that pull a Thom class for $\bar{E}_1$ back to that of $\bar{E}_j$ and E_j. Therefore, to constrcut the Thom class for each bundle, it suffices to construct one for $\bar{E}_1$. Observe that $\mathrm{Th}(\bar{E}_1)$ is homeomorphic to $\mathbb{C}P^{\frac{p+1}{2}}$ where $\bar{E}_1 = \mathbb{C}P^{\frac{p+1}{2}} \backslash \{(0,\ldots,0,1)\}$ via the map

$$\begin{aligned} (S^p \times \mathbb{C})/2 &\to \mathbb{C}P^{\frac{p+1}{2}} \\ (V_1, \ldots, V_{\frac{p+1}{2}}, Z) &\mapsto (V_1 : \cdots : V_{\frac{p+1}{2}} : Z) \end{aligned}$$

Therefore, a generator of $H^2(\mathbb{C}P^{\frac{p+1}{2}})$ is $\mathrm{Th}(\bar{E}_1)$, which comes from a generator of $H^2(\mathbb{C}P^{\frac{p+1}{2}}, \mathbb{Z})$, determined up to a sign.

For each vector bundle B, we now have equations

$$\tilde{H}^*(\mathrm{Th}(E)) = H^*(\mathrm{Th}(E), \infty) = H^*(\mathrm{Th}(E), \mathrm{Th}(E)^\times) = H^*(E, E^\times)$$

by excisions, where $E^\times$ denotes E deleting the zero sections. We have projections $\pi_j : E \to E_j$. If $\tau_j \in H^*(E_j, E_j^\times)$ is the Thom class constructed above, then $\prod\limits_j \pi_j^*(\tau_j)$ is a Thom class of E. It is $\pm\lambda(\alpha)$ since both classes restrict to $\pm\lambda^{\otimes p}$ on fibers, and $\lambda(\alpha)$ is uniquely determined by its restrictions on fibers.

To finish the proof, note that the class $\nabla^*(\lambda(\alpha))$ is obtained by restricting $\lambda(\alpha) \in H^p(\mathrm{Th}(E))$ to the diagonal $\mathrm{Th}(E_0) = (L(p,p) \times S^1)/L(p,p)$, therefore we have a diagram

$$\begin{tikzcd} \prod\limits_{k=0}^{\frac{p-1}{2}} H^*(E_k, E_k^\times) \arrow[r] \arrow[dd] & H^*(E_0, E_0^\times) \times \prod\limits_{k=1}^{\frac{p-1}{2}} H^*(E_k) \arrow[d] \\ & H^*\left(E, \prod\limits_{k=0}^{\frac{p-1}{2}} E_j\right) \arrow[d] \\ H^*(E, E^\times) \arrow[r] & H^*(E_0, E_0^\times) \end{tikzcd} \tag{1.24}$$

This is given by

$$\begin{tikzcd} (\tau_0, \ldots, \tau_{\frac{p-1}{2}}) \arrow[r, mapsto] \arrow[d, mapsto] & (\tau_0, e_1, \ldots, e_{\frac{p-1}{2}}) \arrow[d, mapsto] \\ \pm\lambda(\alpha) \arrow[r, mapsto] & \pm\nabla^*(\lambda(\alpha)) \end{tikzcd} \tag{1.25}$$

To compute the E_j's, we use the diagram

$$\begin{tikzcd}
H^2(E_j, E_j^\times) & H^2(\bar{E}_j, \bar{E}_j^\times) \arrow[l, "\tilde{q}^*"'] \arrow[d] & H^2(\bar{E}_1, \bar{E}_1^\times) \arrow[l, "\tilde{f}^*"'] \arrow[d] \\
H^2(E_j) \arrow[from=u] & H^2(\bar{E}_j) \arrow[l, "\tilde{q}^*"'] & H^2(\bar{E}_1) \arrow[l, "\tilde{f}^*"'] \\
H^2(L(p,p)) \arrow[u, "\simeq"] & H^2(\mathbb{C}P^{\frac{p-1}{2}}) \arrow[l] \arrow[u, "\simeq"] & H^2(\mathbb{C}P^{\frac{p-1}{2}}) \arrow[l] \arrow[u, "\simeq"]
\end{tikzcd} \tag{1.26}$$

where we can extend to the second row, which is then homotopic to the third row. We have a mapping of elements

$$\begin{tikzcd}
\mathrm{Th}(E_j) \arrow[d, mapsto] & \mathrm{Th}(\bar{E}_j) \arrow[l, mapsto] & \mathrm{Th}(\bar{E}_1) \arrow[l, mapsto] \arrow[dd, mapsto] \\
e_j \arrow[d, mapsto] & & \\
\pm j\omega_2 & j\mathcal{O}(1) \arrow[l, mapsto] & \mathcal{O}(1) \arrow[l, mapsto]
\end{tikzcd} \tag{1.27}$$

Therefore,

$$\begin{aligned}
\nabla^*(\lambda(\alpha)) &= \tau_0 \cup e_1 \cdots \cup e_{\frac{p-1}{2}} \\
&= \pm \left(\frac{p-1}{2}\right)! \cdot \tau_0 \cup \omega_2^{\frac{p-1}{2}} \\
&= \pm \left(\frac{p-1}{2}\right)! \tau_0 \cup \omega_{p-1} \\
&= \pm \left(\frac{p-1}{2}\right)! \omega_{p-1} \otimes \alpha.
\end{aligned}$$

Therefore $a_i \neq 0$ in $\mathbb{F}_p$. $\square$

We define $\mathcal{P}^j(\alpha) = (-1)^j a_i^{-1} \theta_{2j(p-1)}(\alpha)$ for $\alpha \in H^i(X)$, therefore $\mathcal{P}_0 = \mathrm{Id}$. We know that $\theta_{2i(p-1)}(\alpha) = 2^p$ by definition, therefore we must show that $(-1)^j a_{2j}^{-1} = 1$ so that $\mathcal{P}^j(\alpha) = 2^p$ if $\deg(\alpha) = 2j$.

Lemma 1.28. We have $\left(\left(\frac{p-1}{2}\right)!\right)^2 \equiv (-1)^{\frac{p+1}{2}} \pmod p$.

Proof. The product

$$(\pm 1) \cdot (\pm 2) \cdots (\pm \frac{p-1}{2}) = \left(\left(\frac{p-1}{2}\right)!\right)^2 \cdot (-1)^{\frac{p-1}{2}}$$

is the product of all elements in $\mathbb{F}_p^\times \cong \mathbb{Z}/(p-1)\mathbb{Z}$. Since $0 + \cdots + (p-2) = \frac{(p-1)(p-2)}{2} \equiv -\frac{p-1}{2} \pmod{p-1}$, it then must be the unique element of order 2 in $\mathbb{F}_p^\times$, which is -1. Therefore, $\left(\left(\frac{p-1}{2}\right)!\right)^2 \cdot (-1)^{\frac{p+1}{2}} \equiv -1 \pmod p$. $\square$

Using the formulas Equation (1.21) and Lemma 1.28, we have

$$\begin{aligned}
a_{2i} &= (-1)^{\frac{p(p-1)2i(2i-1)}{4}} \left(\left(\frac{p-1}{2}\right)!\right)^{2i} \\
&= (-1)^{\frac{p(p-1)}{2} i(2i-1)} (-1)^{\frac{i(p+1)}{2}} \\
&= (-1)^{\frac{i(p-1)}{2}} (-1)^{\frac{i(p+1)}{2}} \\
&= (-1)^{ip} \\
&= (-1)^i.
\end{aligned}$$

Therefore, $\mathcal{P}^j(\alpha) = \alpha^p$ if $\deg(\alpha) = 2j$.

Theorem 1.29.

1. $\mathcal{P}^0 = \mathrm{Id}$;
2. $\mathcal{P}^i(\alpha) = \alpha^p$ if $\deg(\alpha) = 2i$, and $\mathcal{P}^i(\alpha) = 0$ if $\deg(\alpha) < 2i$;
3. $\mathcal{P}^i \circ f^* = f^* \circ \mathcal{P}^i$ for $f : X \to Y$;
4. $\mathcal{P}^i$ is additive as a homomorphism;
5. $\mathcal{P}^i(\alpha\beta) = \sum_j P^j(\alpha)\mathcal{P}^{i-j}(\beta)$;
6. $\mathcal{P}^i \circ \Sigma = \Sigma \circ \mathcal{P}^i$ for suspension Σ.

Proof. We have essentially shown part 1 and 2, where the last statement follows from $\omega_{(p-1)i-2j(p-1)} = 0$ whenever $i < 2j$.

3. This is obvious from construction.
4. This is the same as the case where $p = 2$.
5. For $\alpha \in H^m(X)$ and $\beta \in H^n(X)$, then $\lambda(\alpha\beta) = (-1)^{\frac{p(p-1)mn}{2}}\lambda(\alpha)\lambda(\beta)$ as remarked in Theorem 1.7. Therefore,
$$\sum_\ell \omega_{(p-1)(m+n)-\ell} \otimes \theta_\ell(\alpha\beta) = (-1)^{\frac{p(p-1)mn}{2}} \left(\sum_j \omega_{(p-1)m-j} \otimes \theta_j(\alpha)\right)\left(\sum_k \omega_{(p-1)n-k} \otimes \theta_k(\beta)\right).$$
Recall that $\omega_{2r} = \omega_1^r$ and $\omega_{2r+1} = \omega_1\omega_2^r$ with $\omega_1^2 = 0$ by Proposition 1.2. Therefore, if ℓ is even, then $\ell = (p-1)(m+n) - j - k$, $\omega_{(p-1)m-j}\omega_{(p-1)n-k} \neq 0$ implies that both j and k are even. Therefore,
$$\begin{aligned}\mathcal{P}^i(\alpha\beta) &= (-1)^i a_{m+n}^{-1}\theta_{2i}(p-1)(\alpha\beta)\\ &= (-1)^i a_{m+n}^{-1}(-1)^{\frac{p(p-1)mn}{2}}\sum_j \theta_{2(i-j)(p-1)}(\alpha)\theta_{2j(p-1)}(\beta)\\ &= \sum_j (-1)^{i-j}a_m^{-1}\theta_{2(i-j)(p-1)}(\alpha)(-1)^j a_n^{-1}\theta_{2j(p-1)}(\beta)\\ &= \sum_j \mathcal{P}^{i-j}(\alpha)\mathcal{P}^j(\beta).\end{aligned}$$
6. Suppose $t \in H^1(S^1)$ is a generator, then
$$\begin{aligned}\mathcal{P}^i(\Sigma\alpha) &= \mathcal{P}^i(t \otimes \alpha)\\ &= \mathcal{P}^0(t) \otimes \mathcal{P}^i(\alpha)\\ &= t \otimes \mathcal{P}^i(\alpha)\\ &= \Sigma\mathcal{P}^i(\alpha).\end{aligned}$$

□

Proposition 1.30. $\beta\theta_{2k} = -\theta_{2k+1}$.

Proof. Let us first reduce the problem to showing that $\beta\nabla^*(\lambda(\tau)) = 0$ for every fundamental class τ of some K_n. By the Leibniz rule of β, we have
$$\begin{aligned}\beta\nabla^*(\lambda(\tau)) &= \beta(\sum_i \omega_{(p-1)(n-i)} \otimes \theta_i(\tau))\\ &= \sum_i(\beta\omega_{(p-1)n-i} \otimes \theta_i(\tau) + (-1)^i\omega_{(p-1)n-i} \otimes \beta\theta_i(\tau)).\end{aligned}$$

Since $\beta\omega_{2j-1} = \omega_{2j}$ and $\beta\omega_{2j} = 0$ by Proposition 1.2, the terms on the right-hand side with $i = 2k$ and $i = 2k+1$ are $\sum_k \omega_{(p-1)n-2k} \otimes \beta\theta_{2k}(\tau)$ and $\sum_k (\omega_{(p-1)n-2k} \otimes \theta_{2k+1}(\tau) + (-1)^i \omega_{(p-1)n-2k-1} \otimes \beta\theta_{2k+1}(\tau))$. Thus, the coefficient of $\omega_{(p-1)n-k}$ in $\beta\nabla^*(\lambda(\tau))$ is $\beta\theta_{2k}(\tau) + \theta_{2k+1}(\tau)$. Therefore, if $\beta\nabla^*(\lambda(\tau)) = 0$, then $\beta\theta_{2k} = -\theta_{2k+1}$. This implies $\beta\theta_{2k+1} = -\beta\beta\theta_{2k} = 0$.

It now suffices to show that $\beta\nabla^*(\lambda\tau) = 0$. For this, we first compute $\beta\lambda(\tau)$. By Hurewicz's theorem, we have

$$H_i(K_n, \mathbb{Z}) = \begin{cases} \mathbb{F}_p, & i = n \\ 0, & 0 < i < n \text{ or } i = n+1 \\ \mathbb{Z} & i = 0 \end{cases}.$$

By the homology decomposition in [Hat02, Section 4.H], the K_n's are obtained by attaching cells of dimension at least $n+2$ to the suspension $\Sigma M(\mathbb{F}_p, n-1)$ of the Moore space, which is defined by a cofiber sequence

$$S^{n-1} \xrightarrow{p} S^{n-1} \longrightarrow M(\mathbb{F}_p, n-1)$$

Therefore, we may assume K_n has a single n-cell and a single $(n+1)$-cell, attached by a map of degree p. Let φ and ψ be the ceullar cochains assigning the value 1 to the n-cell and the $(n+1)$-cell, therefore the simplicial boundary $\delta(\varphi) = p\psi$. In $K_n^{\wedge p}$, we have

$$\begin{aligned} \delta(\varphi^{\otimes p}) &= \sum_i (-1)^{in} \varphi^{\otimes i} \otimes \delta(\varphi) \otimes \varphi^{\otimes(p-i-1)} \\ &= p\sum_i (-1)^{in} \varphi^{\otimes i} \otimes \psi \otimes \varphi^{\otimes(p-i-1)} \end{aligned} \tag{1.31}$$

where $\otimes$ means the cellular cross product. For instance, $\varphi^{\otimes p}$ is the cellular cochain dual to $\prod_p e_n$ of $K_n^{\wedge p}$. Recall that ΛK_n is obtained from $\Lambda_1 K_n$ by attaching cells of dimension at least $pn+2$, where $\Lambda_1 K_n = ((K_n^{\wedge p} \times [0,1])/((x, 0) \sim (T(x), 1))/(* \times [0,1]))$, where T is the permutation $(u_0, \cdots, u_{p-1}) \mapsto (u_1, \cdots, u_{p-1}, u_0)$ for $u_j \in e^n$, is obtained from $K_n^P \wedge p$ by attaching cells of dimension at least $pn+1$. There is a unique cell of dimension $pn+1$ attached, namely $(e^n)^{\times p} \times [0,1]$, by the diagram

$$\begin{tikzcd} (e^n)^{\times p} \times \{0,1\} \arrow[r, "(\mathrm{Id}, T)"] \arrow[d] & K_n^{\wedge p} \\ (e^n)^{\times p} \times [0,1] & \end{tikzcd} \tag{1.32}$$

In particular, the cell has zero cellular boundary. Hence, in ΛK_n, we have

$$\delta(\varphi^{\otimes p})(\Delta) = \begin{cases} \delta(\varphi^{\otimes p}|_{K_n^{\wedge p}})(\Delta), & \Delta \subseteq K_n^{\wedge p} \\ 0, & \text{otherwise} \end{cases}$$

for $\dim(\Delta) = np+1$. Hence, Equation (1.31) holds in ΛK_n as well. This gives rise to a commutative diagram

$$\begin{tikzcd} H^*(\Lambda K_n) \arrow[r, "\pi^*"] \arrow[d, "\nabla^*"'] & H^*((S^\infty \times K_n^{\wedge p})/S^\infty) \arrow[r, "\tau"] \arrow[d, "\nabla^*"] & H^*(\Lambda K_n) \arrow[d, "\nabla^*"] \\ H^*((L_p^\infty \times K_n)/L_p^\infty) \arrow[r, "\pi^*"'] & H^*((S^\infty \times K_n)/S^\infty) \arrow[r, "\tau"'] & H^*((L_p^\infty \times K_n)/S^\infty) \end{tikzcd} \tag{1.33}$$

where π is an $\mathbb{F}_p$-bundle. The map τ's are called the transfer homomorphisms. We recall that τ is defined as follows: if $\pi : \tilde{X} \to X$ is a p-sheeted covering space, then we define a chain map $C_*(X) \to C_*(\tilde{X})$ by sending a singular complex $\sigma : \Delta^k \to X$ to the sum of its p lifts to $\tilde{X}$, and then τ is defined as the induced map on cohomology. The key property being,

$$\tau\pi^* : H^*(X, R) \to H^*(X, R)$$

is multiplication by p, sinve the p lifts project to σ itself along π. Let us now compute the value τ on $1\otimes\psi\otimes\varphi^{\otimes(p-1)}\in C^*(S^\infty\times K_n^{\wedge p})$, where 1 being the celluar cocycle assigning the value 1 to each 0-cell of S^∞. By definition of τ, we have

$$\begin{aligned}\tau(1\otimes\psi\otimes\varphi^{\otimes(p-1)})&=\sum_i T^i(\psi\otimes\varphi^{\otimes(p-1)})\\&=\sum_i(-1)^{in}\varphi^{\otimes i}\otimes\psi\otimes\varphi^{(p-i-1)}\end{aligned}$$

which is the cycle representing $\beta\lambda(\tau)$ by Equation (1.31).

Since $\beta\lambda(\tau)\in\operatorname{im}(\tau)$, then $\nabla^*(\beta\lambda(\tau))$ lies in the image of lower τ in the diagram as well. Note that π^* is a surjection, since $\tilde{H}^*((S^\infty\times K_n)/S^\infty)=H^*(K_n)$. Therefore, $\beta\nabla^*(\lambda(\tau))=\nabla^*\beta(\lambda(\tau))\in\operatorname{im}(\tau\pi^*)=\operatorname{im}(p)=0$. □

Theorem 1.34. We have the Adem relation

$$\mathcal{P}^a\mathcal{P}^b=\sum_j(-1)^{a+j}\binom{(p-1)(b-j)-1}{a-pj}\mathcal{P}^{a+b-j}\mathcal{P}^j$$

for $a<pb$, and

$$\mathcal{P}^a\beta\mathcal{P}^b=\sum_j(-1)^{a+j}\binom{(p-1)(b-j)}{a-pj}\beta\mathcal{P}^{a+b-j}\mathcal{P}^j-\sum_j(-1)^{a+j}\binom{(p-1)(b-j)-1}{a-pj-1}\mathcal{P}^{a+b-j}\beta\mathcal{P}^j$$

for $a\leqslant pb$.

Proof. By the same constructions as in Theorem 1.12, we define $\Gamma^2X=(X^{\wedge p^2}\times S^\infty\times S^\infty)/(\mathbb{F}_p\times\mathbb{F}_p)$, with action of $(s,t)\in\mathbb{F}_p\times\mathbb{F}_p$ via

$$(s,t)\cdot(x_{ij},a,b)=(x_{s(i),t(j)},s-a,t-b)$$

given by cyclic permutations, as well as $\Lambda^2X=\Gamma^2X/(S^\infty\times S^\infty)$. Define $K_n=K(\mathbb{F}_p,n)$ where the highest cohomology degree is one-dimensional, so pick the generator $\tau_i:K_n^{\wedge p^2}\to K_{np^2}$, which is invariant under the actions of $\langle a\rangle\times\langle b\rangle=\mathbb{F}_p\times\mathbb{F}_p$, up to homotopy. Therefore, we have homotopies $\tau\sim\tau\circ a$ and $\tau\sim\tau\circ b$, which generate the homotopies

$$\tau\circ b\sim\tau\circ ab$$

and

$$\tau\circ a\sim\tau\circ ab.$$

Therefore, we obtain a map

$$K_n^{\wedge p^2}\times\partial I^2\to K_{np^2}$$

which can then be extended to a map $K_n^{\wedge p^2}\times I^2\to K_{np^2}$. This map passes down to a map

$$(K_n^{\wedge p^2}\times S^1\times S^1)/(\mathbb{F}_p\times\mathbb{F}_p)\to K_{np^2}$$

which extends to a map $\lambda_2:\Lambda^2K_n\to K_{np^2}$. By the same reasoning we have a commutative diagram

$$\begin{tikzcd} L_p^\infty\times L_p^\infty\times K_n \arrow[r, "\nabla_2"] \arrow[d, "\mathrm{id}\times\nabla"'] & \Lambda^2K_n \arrow[r, "\lambda_2"] \arrow[d] & K_{np^2} \\ L_p^\infty\times\Lambda K_n \arrow[r, "\nabla"'] & \Lambda(\Lambda K_n) \arrow[ur] & \end{tikzcd} \tag{1.35}$$

up to homotopy. The element $\nabla_2^*\lambda_2^*(\tau_{np^2})$ can be written as $\sum\limits_{r,s}\omega_r\otimes\omega_s\otimes\varphi_{rs}$ where ω_r,ω_s generates each copy of L_p^∞. This gives rise to another commutative diagram

$$\begin{tikzcd} L_p^\infty \arrow[r, "\nabla_2"] \arrow[d, "\tau"'] & \Lambda^2K_n \arrow[d, "\tau"] \\ L_p^\infty\times L_p^\infty\times K_n \arrow[r, "\nabla_2"'] & \Lambda^2K_n \end{tikzcd} \tag{1.36}$$

given by

$$\begin{tikzcd}
(p,q,x) \arrow[r, mapsto] \arrow[d, mapsto] & (x,\cdots,x,p,q)=(x_{ij},p,q) \arrow[d, mapsto] \\
(q,p,x) \arrow[r, mapsto] & (x_{ji},q,p)
\end{tikzcd}$$

On the fiber $K_n^{\wedge p^2}$ of $\Lambda^2 K_n$, τ is the map $(x_{ij})\mapsto(x_{ji})$, which is the product of $\frac{p(p-1)}{2}$ transpositions $(x_{ij}x_{ji})$ for $i<j$. The map

$$H^{np^2}(\Lambda^2 K_n)\to H^{np^2}(K_n^{\wedge p^2})$$

is an isomorphism, so

$$\begin{aligned}
\nabla_2^*\tau^*\lambda_2^* &= (-1)^{\frac{np(p-1)}{2}}\nabla_2^*\lambda_2^*(\tau_{np^2})\\
&= \tau^*\nabla_2^*\lambda_2^*(\tau_{np^2})\\
&= \sum_{r,s}(-1)^{rs}\omega_s\otimes\omega_r\otimes\varphi_{rs},
\end{aligned}$$

therefore we obtain $\varphi_{rs}=(-1)^{rs+\frac{np(p-1)}{2}}\varphi_{sr}$. From Diagram 1.35, we obtain

$$\begin{aligned}
\nabla_2^*\lambda_2^*(\tau_{np^2}) &= \sum_i\omega_{np(p-1)-i}\otimes\theta_i(\sum_j\omega_{n(p-1)-j}\otimes\theta_j(\tau_{np^2}))\\
&= \sum_{i,j}\omega_{np(p-1)-i}\otimes\theta_i(\omega_{n(p-1)-j}\otimes\theta_j(\tau_{np^2}))\\
&= \sum_{i,j}\omega_{2mpn-i}\otimes\theta_i(\omega_{2mn-j}\otimes\theta_j(\tau_{np^2})).
\end{aligned}$$

once we assume $p=2m+1$ for some m. Since the non-zero θ_i's are $\theta_{2i(p-1)}=(-1)^i a_n p^i$ and $\theta_{2i(p-1)+1}=-\beta\theta_{2i(p-1)}$, then we can group them into four terms. That is,

$$\begin{aligned}
\sum_{i,j}\omega_{2mpn-i}\otimes\theta_i(\omega_{2mn-j}\otimes\theta_j(\tau_{np^2})) &= \sum_{i,j}(-1)^{i+j}a_{2mn}a_n\omega_{2m(pn-2i)}\otimes\mathcal{P}^i(\omega_{2m(n-2j)}\otimes\mathcal{P}^j(\tau_{np^2}))\\
&\quad-\sum_{i,j}(-1)^{i+j}a_{2mn}a_n\omega_{2m(pn-2i)}\otimes\mathcal{P}^i(\omega_{2m(n-2j)-1}\otimes\beta\mathcal{P}^j(\tau_{np^2}))\\
&\quad-\sum_{i,j}(-1)^{i+j}a_{2mn}a_n\omega_{2m(pn-2i)-1}\otimes\beta\mathcal{P}^i(\omega_{2m(n-2j)}\otimes\mathcal{P}^j(\tau_{np^2}))\\
&\quad+\sum_{i,j}(-1)^{i+j}a_{2mn}a_n\omega_{2m(pn-2i)-1}\otimes\beta\mathcal{P}^i(\omega_{2m(n-2j)-1}\otimes\mathcal{P}^j(\tau_{np^2})).
\end{aligned}$$

Define the total Steenrod operation to be $\mathcal{P}(\omega_{2r})=\sum_i\mathcal{P}^i(\omega_{2r})t^i$, so by Cartan formula, we know

$$\begin{aligned}
\mathcal{P}(\omega_{2r}) &= \mathcal{P}(\omega_2^r)\\
&= \mathcal{P}(\omega_2)^r\\
&= (\omega_2+\omega_2^p t)^r\\
&= \sum_k\binom{r}{k}\omega_2^{(p-1)k+r}t^k.
\end{aligned}$$

Therefore, $\mathcal{P}^k(\omega_{2r})=\binom{r}{k}\omega_{2r+2k(p-1)}$. Similarly,

$$\begin{aligned}
\mathcal{P}(\omega_{2r+1}) &= \mathcal{P}(\omega_1)\mathcal{P}(\omega_2^r)\\
&= \mathcal{P}(\omega_1)\mathcal{P}(\omega_2)^r
\end{aligned}$$

$$\begin{aligned}&= \omega_1(\omega_2+\omega_2^p t)^r\\ &= \sum_k \binom{r}{k}\omega_1\omega_2^{(p-1)k+r}t^k.\end{aligned}$$

Therefore, $\mathcal{P}^k(\omega_{2r+1}) = \binom{r}{k}\omega_{2r+2k(p-1)+1}$. So by applying Cartan formula again, the expression becomes

$$\begin{aligned}a_{2mn}-a_n\cdot\Bigg(&\sum_{i,j,k}(-1)^{i+j}\binom{m(n-2j)}{k}\omega_{2m(np-2i)}\otimes\omega_{2m(n-2j+2k)}\otimes\mathcal{P}^{i-k}\mathcal{P}^j(\tau_{np^2})\\ &-\sum_{i,j,k}(-1)^{i+j}\binom{m(n-2j)-1}{k}\omega_{2m(np-2i)}\otimes\omega_{2m(n-2j+2k)-1}\otimes\mathcal{P}^{i-k}\beta\mathcal{P}^j(\tau_{np^2})\\ &-\sum_{i,j,k}(-1)^{i+j}\binom{m(n-2j)}{k}\omega_{2m(np-2i)-1}\otimes\omega_{2m(n-2j+2k)}\otimes\mathcal{P}^{i-k}\beta\mathcal{P}^j(\tau_{np^2})\\ &+\sum_{i,j,k}(-1)^{i+j}\binom{m(n-2j)-1}{k}\omega_{2m(np-2i)-1}\otimes\omega_{2m(n-2j+2k)}\otimes\mathcal{P}^{i-k}\beta\mathcal{P}^j(\tau_{np^2})\\ &-\sum_{i,j,k}(-1)^{i+j}\binom{m(n-2j)-1}{k}\omega_{2m(np-2i)-1}\otimes\omega_{2m(n-2j+2k)-1}\otimes\mathcal{P}^{i-k}\beta\mathcal{P}^j(\tau_{np^2})\Bigg).\end{aligned}$$

Let $\ell = mn+j-k$, then $n-2j+2k = pn-2\ell$, so

$$\begin{aligned}&\sum_{i,j,k}(-1)^{i+j}\binom{m(n-2j)}{k}\omega_{2m(np-2i)}\otimes\omega_{2m(n-2j+2k)}\otimes\mathcal{P}^{i-k}\mathcal{P}^j(\tau_{np^2})\\ &=\sum_{i,j,\ell}(-1)^{i+j}\binom{m(n-2j)}{mn+j-\ell}\omega_{2m(pn-2i)}\otimes\mathcal{P}^{i+\ell-mn-j}\mathcal{P}^j(\tau_{np^2}),\end{aligned}$$

and similarly for the other four terms. Since we have the symmetry $\varphi_{rs} = (-1)^{rs+mnp}\varphi_{sr}$, where $\nabla_2^*\lambda_2^*(\tau_{np^2}) = \sum_{r,s}\omega_r\otimes\omega_s\otimes\varphi_{rs}$, we may now assume that n is even, then by bringing the first term into the symmetry, we have

$$\sum_{i,j,\ell}(-1)^{i+j}\binom{m(n-2j)}{mn+j-\ell}\omega_{2m(pn-2i)}\otimes\mathcal{P}^{i+\ell-mn-j}\mathcal{P}^j(\tau_{np^2}) = \sum_j(-1)^{\ell+j}\binom{m(n-2j)}{mn+j-i}\mathcal{P}^{i+\ell-mn-j}\mathcal{P}^j(\tau_{np^2}) \tag{1.37}$$

since the subscripts of ω's are both even. For the terms that have ω's with subscripts both even and odd, i.e., the three terms in the middle, we can do something similar. We now have

$$\begin{aligned}\sum_j(-1)^{i+j}\binom{m(n-2j)-1}{mn+j-\ell}\mathcal{P}^{i+1-mn-j}\beta\mathcal{P}^j(\tau_{np^2}) &= \sum_j(-1)^{\ell+j}\binom{m(n-2j)}{mn+j-\ell}\beta\mathcal{P}^{i+\ell-mn-j}\mathcal{P}^j(\tau_{np^2})\\ &\quad-\sum_j(-1)^{\ell+j}\binom{m(n-2j)-1}{mn+j-i}\mathcal{P}^{i+\ell-mn-j}\mathcal{P}^j(\tau_{np^2}).\end{aligned} \tag{1.38}$$

We want to derive the first Adem relation from Equation (1.37). Choose $n = 2(1+p+\cdots+p^{r-1})+2s$ and $\ell = mn+s$, then

$$\binom{m(n-2j)}{mn+j-\ell} = \binom{p^r-1-(p-1)(j-s)}{j-s}.$$

When r is large and $j=s$, the coefficient is 1. In the case $j>s$ and $p^r-1\geqslant(p-1)(j-s)$, we have p-adic representation $j-s = \sum_{i\leqslant k}a_ip^i$ for $a_i = 0,\dots,p-1$ for $a_k>0$, then $p^r-1-(p-1)(j-s) = (p-1)(1+\cdots+p^{k-1}+\sum_{j\geqslant k}^{r-1}(1-a_j)p^j)$. Therefore, the coefficient is a multiple of $\binom{u}{a_k}$ by Lemma 1.17, where $0\leqslant u\leqslant p-1$ satisfies $u\equiv(1-a_k)(p-1)\equiv a_k-1 \pmod p$, therefore $\binom{u}{a_k} = 0$. In other cases, the coefficient vanishes by definition. We conclude that Equation (1.37) becomes

$$(-1)^{i+s}\mathcal{P}^i\mathcal{P}^s(\tau_{np^2}) = \sum_j(-1)^{\ell+j}\binom{m(n-2j)}{mn+j-i}\mathcal{P}^{i+s-j}\mathcal{P}^j(\tau_{np^2}),$$

therefore

$$\begin{aligned}
\mathcal{P}^i\mathcal{P}^s(\tau_{np^2}) &= \sum_j (-1)^{i+j}\binom{m(n-2j)}{mn+j-i}\mathcal{P}^{i+s-j}\mathcal{P}^j(\tau_{np^2})\\
&= \sum_j (-1)^{i+j}\binom{m(n-2j)}{i-pj}\mathcal{P}^{i+s-j}\mathcal{P}^j(\tau_{np^2})\\
&= \sum_j (-1)^{i+j}\binom{p^r+(p-1)(s-j)-1}{i-pj}\mathcal{P}^{i+s-j}\mathcal{P}^j(\tau_{np^2})
\end{aligned}$$

as $\ell \equiv s \pmod 2$ in our situation. If r is large and $i < ps$, then the termp^r in the coefficient can be omitted since we may assume $i \geqslant pj$, hence $j < s$, so $-1+(p-1)(s=j) \geqslant 0$, and p^r has no effect on the coefficient if r is large. This shows the first Adem relation by the stability.

The second Adem relation follows from Equation (1.38). We choose $n = 2p^r + 2s$ and $\ell = mn + s$. Reasoning as before, the left side of Equation (1.38) reduces to $(-1)^{i+s}\mathcal{P}^i\beta\mathcal{P}^s(\tau_{np^2})$, then Equation (1.38) becomes

$$\begin{aligned}
\mathcal{P}^i\beta\mathcal{P}^s(\tau_{np^2}) &= \sum_j (-1)^{i+j}\binom{(p-1)(p^r+s-j)}{i-pj}\beta\mathcal{P}^{i+s-j}\mathcal{P}^j(\tau_{np^2})\\
&\quad - \sum_j (-1)^{i+j}\binom{(p-1)(p^r+s-j)-1}{i-pj-1}\mathcal{P}^{i+s-j}\beta\mathcal{P}^j(\tau_{np^2}).
\end{aligned}$$

This time the term $\mathcal{P}^r$ can be omitted if r is large and $i \leqslant ps$. □

1.4 Steenrod Algebra and Hopf Algebra Structure

Let us now define Steenrod algebra and its dual.

Definition 1.39. We define the Steenrod algebra $\mathcal{A}_2 = \mathbb{F}_2\left\langle \mathrm{Sq}^1, \mathrm{Sq}^2, \ldots \right\rangle / \langle \text{Adem relations} \rangle$. In the case where $p > 2$, we also have $\mathcal{A}_p = \mathbb{F}_p\left\langle \beta, \mathcal{P}^1, \mathcal{P}^2, \ldots, \right\rangle / \langle \text{Adem relations} \rangle$.

Therefore, for any space X, the cohomology ring $H^*(X, \mathbb{F}_p)$ is a left $\mathcal{A}_p$-module based on the Steenrod actions.

Proposition 1.40. There is a relation

$$\mathrm{Sq}^i = \sum_{0<j<i} a_j \,\mathrm{Sq}^{i-j}\,\mathrm{Sq}^j$$

for $a_j \in \mathbb{F}_2$ and $i \neq 2^k$. Similarly, if $i \neq p^k$, we have a similar relation

$$\mathcal{P}^i = \sum_{0<j<i} a_j \mathcal{P}^{i-j}\mathcal{P}^j$$

for $a_j \in \mathbb{F}_p$ for $p > 2$.

Proof. The proof is essentially the same in these cases. Let us describe the proof for $p > 2$. The idea is to write i as the sum of $a+b$ of integers $a, b > 0$ and $a < pb$, such that in the Adem relation for $\mathcal{P}^a$ and $\mathcal{P}^b$, the coefficient on $\mathcal{P}^{a+b} = \mathcal{P}^i$ is non-zero.

Let the p-adic representation of i be $i = i_0 + i_1 p + \cdots + i_k p^k$ with $i_k > 0$. Let $b = p^k$ and $a = i - p^k$, then $a < pb$ with $a, b > 0$ if $i \neq p^k$. We claim that $\binom{(p-1)b-1}{a} \neq 0 \in \mathbb{F}_p$. The p-adic expansion of $(p-1)b - 1 = p^{k+1} - 1 - p^k$ is $(p-1) + (p-1)p + \cdots + (p-2)p^k$, and that of a is $i_0 + i_1 p + \cdots + (i_k - 1)p^k$. Hence,

$$\binom{(p-1)b-1}{a} \equiv \binom{p-1}{i_0}\cdots\binom{p-2}{i_k-1} \neq 0 \pmod p$$

by Lemma 1.17. □

Therefore, $\mathcal{A}_2$ (respectively, $\mathcal{A}_p$ for $p > 2$) is generated by Sq^{2^i} (respectively, β and $\mathcal{P}^{p^i}$) as an algebra.

There are two ways to create a basis on the Steenrod algebra.

Definition 1.41. Given a sequence of natural numbers $I = (i_1, \dots, i_k)$ of length $\ell_I := k$, we define the moment of I by $m(U) = \sum_{s=1}^{k} s i_s$. A sequence I is called admissible if $i_k \geqslant 1$ and $i_{s-1} \geqslant 2i_s$ for all $s > 1$. We then write $\mathrm{Sq}^I = \mathrm{Sq}^{i_1} \cdots \mathrm{Sq}^{i_k}$. Moreover, we say Sq^I is either $I = (0)$, i.e., $\mathrm{Sq}^I = \mathrm{Sq}^0$, or I is admissible. Finally, we define $m(\mathrm{Sq}^I) = m(I)$ and $\deg(I) = \sum_k i_k$.

We define similarly for the case where $p > 2$.

Definition 1.42. A sequence of natural numbers $I = (\varepsilon_1, i_1, \varepsilon_2, i_2, \dots, \varepsilon_k, i_k)$ with $\varepsilon_j \leqslant 1$ is called admissible if $i_k \geqslant 1$ or $\varepsilon_k = 1$, moreover $i_{s-1} \geqslant p i_s + \varepsilon_s$ for $s > 1$. We then write $\mathcal{P}^I$ as $\beta^{\varepsilon_1}\mathcal{P}^{i_1} \cdots \beta^{\varepsilon_k}\mathcal{P}^{i_k}$. We say $\mathcal{P}^I$ is admissible if either $\mathcal{P}^I = \mathcal{P}^0$ or I is admissible. Define $m(\mathcal{P}^I) = \sum_r r(i_r + \varepsilon_r)$ and $\deg(\mathcal{P}^I) = \sum_r \varepsilon_r + 2(p-1)\sum_r i_r$. The length of I is defined as

$$\ell(I) = \begin{cases} k, & i_k > 0 \\ k-1, & i_k = 0 \end{cases}.$$

Proposition 1.43. The admissible monomials form a system of generators for $\mathcal{A}_p$ as an $\mathbb{F}_p$-vector space.

Proof. In the case $p = 2$, suppose $i_r < 2i_{r+1}$ in $I = (i_1, \dots, i_k)$, then by the Adem relation, we know

$$\mathrm{Sq}^I = \cdots \mathrm{Sq}^{i_r}\,\mathrm{Sq}^{i_{r+1}} \cdots = \sum_{j=0}^{\lfloor \frac{i_r}{2} \rfloor} \lambda_j \cdots \mathrm{Sq}^{i_r + i_{r+1} - j}\,\mathrm{Sq}^j \cdots$$

for some $\lambda_j \in \mathbb{F}_2$. Therefore, the difference of the moment is

$$\begin{aligned} r(i_r + i_{r+1} - j) + (r+1)j - r i_r - (r+1) i_{r+1} &= i - i_{r+1} \\ &= \frac{i_r}{2} - i_{r+1} \\ &< 0. \end{aligned}$$

That is, if we start with a non-admissible sequence, then by Adem relations we may replace it by another sequence with smaller moment, therefore the statement follows by induction.

In the case $p > 2$, let us also start with a non-admissible sequence I. Suppose $i_r < 2i_{r+1}$ and $\varepsilon_{r+1} = 0$, then by Adem relations, we have

$$\mathcal{P}^I = \sum_{j=0}^{\lfloor \frac{i_r}{p} \rfloor} \lambda_j \cdots \mathcal{P}^{i_r + i_{r+1} - j}\mathcal{P}^j \cdots$$

for $\lambda_j \in \mathbb{F}_p$. In this situation, the difference of the moment is

$$\begin{aligned} r(i_r + i_{r+1} - j) + (r+1)j - r i_r - (r+1) i_{r+1} &= j - i_{r+1} \\ &\leqslant \frac{i_r}{p} - i_{r+1} \\ &< 0. \end{aligned}$$

Using a similar logic, we are done. In the case where $i_r \leqslant p i_{r+1}$ and $\varepsilon_{r+1} = 1$ in I, we know from Adem relation that we have

$$\mathcal{P}^I = \sum_{j=0}^{\lfloor \frac{i_r}{p} \rfloor} \lambda_j \cdots \beta\mathcal{P}^{i_r + i_{r+1} - j}\mathcal{P}^j \cdots - \sum_{j=0}^{\lfloor \frac{i_r}{p} \rfloor} \mu_j \cdots \mathcal{P}^{i_r + i_{r+1} - j}\beta\mathcal{P}^j \cdots$$

for $\lambda_j, \mu_j \in \mathbb{F}_p$. The differences of moments are

$$\begin{aligned} r(i_r + i_{r+1} - j + 1) + (r+1)j - r i_r - (r+1)(i_{r+1} + 1) &= j - i_{r+1} - 1 \\ &\leqslant \frac{i_r}{p} - i_{r+1} - 1 \end{aligned}$$

$$< 0$$

and

$$\begin{aligned} r(i_r + i_{r+1} - j) + (r+1)(j+1) - ri_r - (r+1)(i_{r+1}+1) &= j - i_{r+1} \\ &\leqslant \frac{i_r - 1}{p} - i_{r+1} \\ &\leqslant -\frac{1}{p} \\ &< 0. \end{aligned}$$

By induction on moments, we are done. □

Theorem 1.44. The admissible monomials are linearly independent in $\mathcal{A}_p$.

Proof for the case $p = 2$. Consider the element $\omega = \mathcal{O}(1)^n \in H^n((L_2^\infty)^{\times n})$ as a generator. We claim that the set $\{\mathrm{Sq}^I(\omega) : I \text{ admissible with } \deg(I) \leqslant n\}$ is linearly independent in the cohomology ring. Taking $n \to \infty$, we are done. We proceed by induction on n. If $n = 1$, the statement is clear. Suppose there is a relation

$$\sum_{\substack{\deg(I)=q\leqslant n \\ I \text{ admissible}}} a_I \,\mathrm{Sq}^I(\omega) = 0,$$

then we need to show that $a_I = 0$ for all I, which we will proceed by descending induction on the length $\ell(I)$. Suppose that $a_I = 0$ for some $\ell(I) > m > 1$, then the relation above takes the form

$$\sum_{\ell(I)=n} a_I \,\mathrm{Sq}^I(\omega) + \sum_{\ell(I)<m} a_I \,\mathrm{Sq}^I(\omega) = 0. \tag{1.45}$$

By Künneth decomposition, we have that

$$H^{q+n}((L_2^\infty)^{\times n}) = \bigoplus_s H^2(L_2^\infty) \otimes H^{q+n-s}((L_2^\infty)^{\times(n-1)}).$$

Let us denote the projection into the summand with $s = 2^m$ by g. Let $\omega' = \mathcal{O}(1)^{\times(n-1)}$, then by Cartan formula we have

$$\begin{aligned} \mathrm{Sq}^I(\omega) &= \mathrm{Sq}^I(\mathcal{O}(1) \times \omega') \\ &= \sum_{J \leqslant I} \mathrm{Sq}^J(\mathcal{O}(1)) \times \mathrm{Sq}^{I-J}(\omega') \end{aligned}$$

where $J \leqslant I$ indicates $0 \leqslant j_r \leqslant i_r$ for all r. Let J_m be the sequence $(2^{m-1}, \ldots, 2, 1)$, then we assert that

$$g(\mathrm{Sq}^I(\omega)) = \begin{cases} 0, & \ell(I) < m \\ \mathcal{O}(1)^{2^m} \times \mathrm{Sq}^{I-J_m}(\omega'), & \ell(I) = m \end{cases}. \tag{1.46}$$

Proof of Equation (1.46). Suppose $J = (j_1, \ldots, j_k)$, then if $\mathrm{Sq}^J(\mathcal{O}(1)) \neq 0$, we must have $j_{k_1} = 1$, where k_1 is the last entry of J that is positive. Inductively, if $\mathrm{Sq}^J(\mathcal{O}(1)^{2^s}) \neq 0$, we must have $j_{k_1} = 2^s$. Hence, if $\mathrm{Sq}^J(\mathcal{O}(1)) \neq 0$, we know that J is of the form $(0, \ldots, 0, 2^s, 0, \ldots, 1, 2^{s-1}, \ldots, 1, 0, \ldots, 0)$. Moreover, if $\mathrm{Sq}^J(\mathcal{O}(1)) = 2^m$, we have $s = m - 1$. If $\ell(I) < m$, then $J \leqslant I$ implies $\ell(J) < m$, thus $g(\mathrm{Sq}^I(\omega)) = 0$; if $\ell(I) = m$, then $g(\mathrm{Sq}^J(\mathcal{O}(1)) \times \mathrm{Sq}^{I-J}(\omega')) = 0$ unless $J = J_m \leqslant I$. ■

Applying g to each term of Equation (1.45), we see by Equation (1.46) that

$$\mathcal{O}(1)^{2^m} \times \sum_{\ell(I)=m} a_I \,\mathrm{Sq}^{I-J_m}(\omega') = 0.$$

If I is admissible, then so is $I - J_m$, which has a lower degree. By inductive hypothesis, $a_I = 0$ for $\ell(I) = m$. □

Proof for the case $p > 2$. Consider the element $\omega = \alpha_1 \times \cdots \times \alpha_n \times \beta_{n+1} \times \cdots \times \beta_{2n} \in H^{3n}((L_p^\infty)^{\times(2n)})$ for $\alpha_i \in H^1(L_p^\infty)$ and $\beta_i \in H^2(L_p^\infty)$. Let us write the expression as $\omega = \alpha_1 \times \beta_{n+1} \times \omega'$, then we claim that the set $\{\mathcal{P}^I(\omega) \mid I \text{ admissible of degree } \leqslant 3n\}$ is linearly independent. Once this is proven and we take $n \to \infty$, then we are done.

We proceed by induction on n. If $n = 1$, the only choices of $\mathcal{P}^I$ are $\mathcal{P}^0$ and β, so this case is trivial. Suppose

$$\sum_{\substack{\deg(I)=q\leqslant 3n\\ I \text{ admissible}}} a_I \mathcal{P}^I(\omega) = 0.$$

We prove that $a_I = 0$ for all I by descending induction on $\ell(I)$. Suppose $a_I = 0$ for $\ell(I) = m > 1$, then we have

$$\sum_{\ell(I)=m} a_I \mathcal{P}^I(\omega) + \sum_{\ell(I)<m} a_I \mathcal{P}^I(\omega) = 0. \tag{1.47}$$

Let V be the collection of elements in $H^{3n+q}((L_p^\infty)^{\times(2n)})$ of the form $\beta_1^{p^{m-1}}\beta_{n+1}^{p^m} \cdot u_m + \beta_1^{p^{m-2}}\beta_{n+1}^{p_n} \cdot u_{m-1} + \cdots + \beta_1\beta_{n+1}^{p^m}u_0 + \alpha_1\beta_{n+1}^{p^m}v + (\beta_1^{p^{m-1}}\beta_{n+1} + \beta_1\beta_{n+1}^{p^{m-1}}) \cdot z$, with projections $g : H^{3n+q}((L_p^\infty)^{\times(2n)}) \to V$. Consider the set $\{\mathcal{P}^I(\alpha, \beta_{n+1}) \neq 0\}$. We have $\mathcal{P}^0(\alpha_1) = \alpha_1$, $\beta(\alpha_1) = \beta_1$, and $\mathcal{P}^i(\alpha_1) = 0$ for $i > 1$; we also know $\mathcal{P}^0(\beta_{n+1}) = \beta_{n+1}$, $\mathcal{P}^1(\beta_{n+1}) = \beta_{n+1}^p$, and $\mathcal{P}^i(\beta_{n+1}) = 0$ for $i > 1$.

Therefore, if we have $\mathcal{P}^I(\alpha, \beta_{n+1}) \neq 0$, we can study the following cases: if there exists one β, we then return to the case $\mathcal{P}^I(\beta^{p^s}, \beta_{n+1}^{p^t})$; if there is more than one, then this is just zero; if there is no β, we have $\alpha_1\beta_{n+1}^{p^m}$.

Moreover, if we have $\mathcal{P}^{n_1}\cdots\mathcal{P}^{n_m}(\beta_1^{p^s}\beta_{n+1}^{p^t}) \neq 0$, then this is equivalent to ask for $n_i = p^{a_i} + p^{b_i}$, we assign each a_i and b_i a color 0 or 1, such that

- the color of a_i is different from the color of b_i, and that
- the elements of color 0 form a sequence $\mathcal{P}^{s+a}, \cdots \mathcal{P}^{s+1}, \mathcal{P}^3$, and similarly the elements of color 1 form a sequence $\mathcal{P}^{t+b}, \cdots, \mathcal{P}^t$.

The proof now follows a similar fashion as the case of $p = 2$. This time we should have

$$g(\mathcal{P}^I(\omega)) = \begin{cases} 0, & \ell(I) < m \\ \beta_1^{p^{m-1}}\beta_{n+1}^{p^m}\mathcal{P}^{I-J_m^{m-1}}(\omega') + \cdots + \beta_1\beta_{n+1}^{p^m}\mathcal{P}^{I-J_m^0}(\omega') & \\ +\alpha_1\beta_{n+1}^{p^m}\mathcal{P}^{I-J_m}(\omega') + (\beta_1^{p^{m-1}}\beta_{n+1} + \beta_1\beta_{n+1}^{p^m-1})\mathcal{P}^{I-J_m^{-1}}(\omega'), & \ell(I) = m \end{cases}. \tag{1.48}$$

where we have $J_m = (0, p^{m-1}, 0, p^{m-2}, \cdots, 0, 1)$, $J_m^{-1} = (0, p^{m-2}, \cdots, 0, 1, 1, 0)$, and that $J_m^t = (0, p^{m-1} + p^{m-t-2}, \cdots, 0, p^{t+1} + 1, 1, p^t, 0, p^{t-1}, \cdots, 0, 1)$ where $t = 0 \sim m-1$. $\square$

Let $A = \bigoplus_{i\in\mathbb{Z}} A_i$ be a graded vector space over a field F. The dual A' is defined by $A'_n = \mathrm{Hom}(A_m, F)$. Denote $\langle a', a\rangle = a'(a)$ for any $a' \in A'$ and $a \in A$. Similarly, we define the dual A'' of A'. Identify each $a \in A$ with the element $a'' \in A''$ which satisfies $\langle a'', a'\rangle = (-1)^{\deg(a)\deg(a')}\langle a', a\rangle$. Therefore, $A \subseteq A''$. If $\dim(A_n) < \infty$ for all n, then $A = A''$.

For every $f : A \to B$ of degree 0, we have $f' : B' \to A'$ and $f'' : A'' \to B''$. The tensor product $A \otimes B$ is defined by $(A \otimes B)_n = \bigoplus_{i+j=n}(A_i \otimes B_j)$. If A and B are of finite type and if $A_i = B_i = 0$ for $i \ll 0$ or $i \gg 0$, then $A \otimes B$ is also of finite type. In this setting, $(A \otimes B)' = A' \otimes B'$ under the rule

$$\left\langle a' \otimes b', a \otimes b\right\rangle = (-1)^{\deg(a)\deg(b')}\left\langle a', a\right\rangle\left\langle b', n\right\rangle.$$

In practice, we will use the notation A_* for a graded vector space A such that $A_i = 0$ for $i < 0$, and denote $A^* = A'$.

Definition 1.49. A pair $(A_*, \varphi_* : A_* \otimes A_* \to A_*)$ of A_* is a graded algebra if φ_* is associative and have an identity 1. Moreover, we say the algebra is connected if A_0 is spanned by 1.

Definition 1.50. A connected Hopf algebra (A_*, φ_*, ψ_*) is a connected graded algebra (A_*, φ_*) together with a map $\psi_* : A_* \to A_* \otimes A_*$ such that

1. ψ_* is a homomorphism of algebras;

2. ψ_* is coassociative and an augmentation $\varepsilon : A_* \to F$ that satisfies $(\varepsilon \otimes \mathrm{id}) \circ \psi_* = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \psi_*$, and $(\mathrm{id} \otimes \psi_*) \circ \psi_* = (\psi_* \otimes \mathrm{id}) \circ \psi_*$.

Definition 1.51. Given a connected Hopf algebra (A_*, φ_*, ψ_*), its dual Hopf algebra (A^*, ψ^*, φ^*) is a connected Hopf algebra where $\psi^* : A^* \otimes A^* \to A^*$ (respectively, $\varphi^* : A^* \to A^* \otimes A^*$) is $(\psi_*)^*$ (respectively, $(\varphi_*)^*$) and the identity $F \to A^*$ (respectively, augmentation $A^* \to F$) is the dual of the augmentation (respectively, the identity) of (A_*, φ_*, ψ_*).

We may now define the Hopf algebra structure for $\mathcal{A}_p$.

Proposition 1.52. Let X be the underlying space. For each element $\theta \in \mathcal{A}_p$, there is a unique element $\psi^*(\theta) = \sum_i \theta_i' \otimes \theta_i'' \in \mathcal{A}_p \otimes \mathcal{A}_p$ such that the equality

$$\theta(\alpha\beta) = \sum_i (-1)^{\deg(\theta_i'')\deg(\alpha)} \theta_i(\alpha)\theta_i''(\beta).$$

Furthermore, $\psi^* : \mathcal{A}_p \to \mathcal{A}_p \otimes \mathcal{A}_p$ is a ring homomorphism.

Proof. To show existence, we define $\varphi^*(\beta) = \beta \otimes 1 + 1 \otimes \beta$, and $\psi^*(\mathcal{P}^n) = \sum_{i+j=n} \mathcal{P}^i \otimes \mathcal{P}^j$, where $p > 2$. (Similarly for the case where $p = 2$.) Therefore $\beta(uv) = \beta(u)v + (-1)^{\deg(u)} u\beta(v)$ and $\mathcal{P}^n(uv) = \sum_{i+j=n} \mathcal{P}^i(u)\mathcal{P}^j(v)$ for $u, v \in H^*(X)$. Each element $\theta \in \mathcal{A}_p$ is equal to $f(\beta, \mathcal{P}^1, \mathcal{P}^2, \cdots)$, where f is a polynomial over $\mathbb{F}_p$. Suppose we have defined $\psi^*(\theta_1) = \sum a_j \otimes b_j$ and $\psi^*(\theta_2) = \sum c_j \otimes d_j$. Define $\psi^*(\theta_1\theta_2) = \psi^*(\theta_1)\psi^*(\theta_2)$, then for $u, v \in H^*(X)$, we have

$$\begin{aligned}
\psi^*(\theta_1\theta_2)(uv) &= \psi^*(\theta_1)\psi^*(\theta_2)(uv) \\
&= \sum_j \psi^*(\theta_1)(-1)^{\deg(d_j)\otimes\deg(u)} c_j(u) d_J(v) \\
&= \sum_{i,j} (-1)^{\deg(d_j)\deg(u) + \deg(b_i)\deg(c_j(u))} a_i c_j(u) b_i d_j(v) \\
&= \sum_{i,j} (-1)^{\deg(b_i)\deg(c_j)} a_i c_j \otimes b_i d_j (-1)^{\deg(b_i d_j)\deg(u)}(uv).
\end{aligned}$$

To prove unqiueness, we need the linear independence of admissible monomials. From the proof of Theorem 1.44, given an $n \in \mathbb{N}$, there is a space Y and an element $r \in H^k(Y)$ such that the map

$$\begin{aligned}
\bigoplus_{i \leqslant n} (\mathcal{A}_p)^i &\to H^*(Y) \\
\theta &\mapsto \theta(r)
\end{aligned}$$

is injective. Therefore, the map

$$\begin{aligned}
\bigoplus_{i \leqslant n} (\mathcal{A}_p \otimes \mathcal{A}_p)^i &\to H^*(Y \times Y) \\
\theta' \otimes \theta'' &\mapsto (-1)^{\deg(\theta'')\deg(r)} \theta'(r) \times \theta''(r)
\end{aligned}$$

which implies the uniqueness. □

Theorem 1.53. The homomorphisms

$$\mathcal{A}_p \xrightarrow{\psi^*} \mathcal{A}_p \otimes \mathcal{A}_p \xrightarrow{\varphi^*} \mathcal{A}_p$$

defines a Hopf algebra structure on $\mathcal{A}_p$. Furthermore, ψ^* is cocommutative.

Proof. The projection $\varepsilon : \mathcal{A}_p \to (\mathcal{A}_p)^0 = \mathbb{F}_p$ gives the augmentation, whcih is a ring homomorphism. It satisfies the counit rule for all β and $\mathcal{P}^n$, so does the counit rule for all $\mathcal{A}_p$. For the coassociativity and cocommutativity, since the maps $\psi^* : \mathcal{A}_p \to \mathcal{A}_p \otimes \mathcal{A}_p$ and

$$\begin{aligned}\tau : \mathcal{A}_p \otimes \mathcal{A}_p &\to \mathcal{A}_p \otimes \mathcal{A}_p\\ a \otimes b &\mapsto (-1)^{\deg(a)\deg(b)} b \otimes a\end{aligned}$$

are ring homomorphisms, it suffices to check for β nad $\mathcal{P}^n$. We have $\tau(\psi^*(\beta)) = \tau(\beta \otimes 1 + 1 \otimes \beta) = 1 \otimes \beta + \beta \otimes 1$, and $\tau(\psi^*(\mathcal{P}^n)) = \tau(\sum\limits_i (\mathcal{P}^i \otimes \mathcal{P}^{n-i})) = \sum\limits_i \mathcal{P}^i \otimes \mathcal{P}^{n-i}$, then

$$\begin{aligned}(\mathrm{id} \otimes \psi^*)(\psi^*(\beta)) &= (\mathrm{id} \otimes \psi^*)(\beta \otimes 1 + 1 \otimes \beta)\\ &= \beta \otimes 1 \otimes 1 + 1 \otimes \beta \otimes 1 + 1 \otimes 1 \otimes \beta\\ &= (\psi^* \otimes \mathrm{id})(\beta \otimes 1 + 1 \otimes \beta)\\ &= (\psi^* \otimes \mathrm{id})(\psi^*(\beta)),\end{aligned}$$

and

$$\begin{aligned}(\mathrm{id} \otimes \psi^*)(\psi^*(\mathcal{P}^n)) &= (\mathrm{id} \otimes \psi^*)(\sum_i (\mathcal{P}^i \otimes \mathcal{P}^{n-i}))\\ &= \sum_{i+j+k=n} \mathcal{P}^i \otimes \mathcal{P}^j \otimes \mathcal{P}^k\\ &= (\psi^* \otimes \mathrm{id})(\psi^*(\mathcal{P}^n)),\end{aligned}$$

which check the statements on the level of generators, and we are done. $\square$

Corollary 1.54. There is a dual Hopf algebra

$$\mathcal{A}_p^* \xrightarrow{\varphi_*} \mathcal{A}_p^* \otimes \mathcal{A}_p^* \xrightarrow{\psi_*} \mathcal{A}_p^*$$

which is commutative.

Let H_* and H^* denote the homology and cohomology of orientable manifolds of finite type with $\mathbb{F}_p$-coefficients. The $\mathcal{A}_p$-action on H^* gives rise to an $\mathcal{A}_p$-action on H_* defined by

$$\langle \mu \cdot \theta, \alpha \rangle = \langle \mu, \theta \cdot \alpha \rangle$$

for $\mu \in H_*, \theta \in \mathcal{A}_p, \alpha \in H^*$. This action induces a map

$$\lambda_* : H_{n+*} \otimes \mathcal{A}_p^* \to H_n$$

hence its dual

$$\lambda^* : H^n \to H^{n+*} \otimes \mathcal{A}_p^*.$$

We then have

$$\begin{aligned}\langle \mu(\theta_1\theta_2), \alpha \rangle &= \langle \mu, \theta_1\theta_2\alpha \rangle\\ &= \langle \mu\theta_1, \theta_2\alpha \rangle\\ &= \langle (\mu\theta_1)\theta_2, \alpha \rangle,\end{aligned}$$

so $\mu(\theta_1\theta_2) = (\mu\theta_1)\theta_2$, which is equivalent to a commutative diagram

$$\begin{tikzcd} H_* \otimes \mathcal{A}_p \otimes \mathcal{A}_p \arrow[r, "\mathrm{id} \otimes \varphi^*"] \arrow[d, "\lambda_* \otimes \mathrm{id}"'] & H_* \otimes \mathcal{A}_p \arrow[d, "\lambda_*"] \\ H_* \otimes \mathcal{A}_p \arrow[r, "\lambda^*"'] & H_* \end{tikzcd} \tag{1.55}$$

Taking its dual, we obtain another commutative diagram

$$\begin{tikzcd}
H^*\otimes\mathcal{A}_p^*\otimes\mathcal{A}_p^* & H^*\otimes\mathcal{A}_p^* \arrow[l, "\mathrm{id}\otimes\varphi_*"'] \\
H^*(\mathcal{A}_p^*) \arrow[u, "\lambda^*\otimes 1"] & H^* \arrow[l, "\lambda^*"] \arrow[u, "\lambda^*"']
\end{tikzcd} \tag{1.56}$$

Proposition 1.57. The map $\lambda^*: H^*\to H^*\otimes\mathcal{A}_p^*$ is a ring homomorphism.

Proof. Let K, L be spaces, $\theta\in\mathcal{A}_p$ and $\psi^*(\theta)=\sum_i\theta_i'\otimes\theta_i''$, then for any $\alpha\in H^*(K)$, $\beta\in H^*(L)$, we have

$$\theta\cdot(\alpha\times\beta)=\sum_i(-1)^{\deg(\theta_i'')\deg(\alpha)}\theta_i'(\alpha)\times\theta_i''(\beta).$$

Therefore, for any $u\in H_*(K)$ and $v\in H_*(L)$,

$$\begin{aligned}
\langle(u\times v)\cdot\theta,\alpha\times\beta\rangle &= \langle u\times v,\theta(\alpha\times\beta)\rangle\\
&=\sum_i(-1)^{\deg(\theta_i'')\deg(\alpha)}\left\langle u\times v,\theta_i'(\alpha)\times\theta_i''(\beta)\right\rangle\\
&=\sum_i(-1)^{\deg(\theta_i'')\deg(\alpha)+\deg(\theta_i')(\alpha)\deg(v)}\left\langle u,\theta_i'\alpha\right\rangle\left\langle v,\theta_i''\beta\right\rangle\\
&=\sum_i(-1)^{\deg(\theta_i')\deg(v)}\left\langle(u\cdot\theta_i')\times(v\cdot\theta_i''),\alpha\times\beta\right\rangle,
\end{aligned}$$

therefore $(u\times v)\cdot\theta=\sum_i(-1)^{\deg(\theta_i')\deg(v)}(u\theta_i')\times(v\theta_i'')$, that is, we have a commutative diagram

$$\begin{tikzcd}
H_*(K)\otimes H_*(L)\otimes\mathcal{A}_p\otimes\mathcal{A}_p \arrow[d] & H_*(K)\otimes H_*(K)\otimes\mathcal{A}_p \arrow[l, "\mathrm{id}\otimes\mathrm{id}\otimes\psi^*"'] \arrow[r, equal] & H_*(K\times L)\otimes\mathcal{A}_p \arrow[d, "\lambda_*"] \\
H_*(K)\otimes\mathcal{A}_p(X)\otimes H_*(K)\otimes\mathcal{A}_p \arrow[r, "\lambda_*\otimes\lambda_*"'] & H_*(K)\otimes H_*(L) \arrow[r, equal] & H_*(K\times L)
\end{tikzcd} \tag{1.58}$$

Taking its dual and setting $K=L$ and the diagonal map $d: K\to K\times K$, we obtain a commutative diagram

$$\begin{tikzcd}
H^*(K)\otimes H^*(K)\otimes\mathcal{A}_p^*\otimes\mathcal{A}_p^* \arrow[r] & H^*(K)\otimes H^*(K)\otimes\mathcal{A}_p^* \arrow[r, equal] & H^*(K\times K)\otimes\mathcal{A}_p^* \arrow[r, "d^*\otimes\mathrm{id}"] & H^*(K)\otimes\mathcal{A}_p^* \\
H^*(K)\otimes\mathcal{A}_p^*\otimes H^*(K)\otimes\mathcal{A}_p^* \arrow[u] & H^*(K)\otimes H^*(K) \arrow[l, "\lambda^*\otimes\lambda^*"] \arrow[r, equal] & H^*(K\times K) \arrow[u, "\lambda^*"] \arrow[r, "d^*"'] & H^*(K) \arrow[u, "\lambda^*"']
\end{tikzcd} \tag{1.59}$$

which proves the statement. ☐

Proposition 1.60. In the case $p>2$, write $\alpha\in H^1(L_p^\infty)$ and $\beta\in H^2(L_p^\infty)$ as in Proposition 1.2. We have

$$\lambda^*(\alpha)=\alpha\otimes 1+\beta\otimes\tau_0+\beta^p\otimes\tau_1+\cdots+\beta^{p^r}\otimes\tau_r$$

where $\tau_k\in\mathcal{A}_p^*$ is well-defined of degree $2p^k-1$.

In the case $p=2$, we define

$$\lambda^*(\beta)=\beta\otimes\xi_0+\beta^p\otimes\xi_1+\cdots+\beta^{p^r}\otimes\xi_r$$

where $\xi_0=1$ and $\deg(\xi_k)=2p^k-2$, then we have

$$\lambda^*(\mathcal{O}(1))=\mathcal{O}(1)\otimes\xi_0+\mathcal{O}(1)^2\otimes\xi_1+\cdots+\mathcal{O}(1)^{2^r}\otimes\xi_r$$

where $\xi_0=1$ and $\deg(\xi_i)=2^i-1$.

Proof. For any space X, suppose $H^*(X)$ has a basis given by v_i's, then we have

$$\theta \cdot v_i = \sum_j f_{ij}(\theta) v_j,$$

so

$$\lambda^*(v_i) = \sum_j v_j \otimes f_{ij}.$$

For admissible I, we have $\mathcal{P}^I(\alpha) = 0$ unless $\mathcal{P}^I = \mathcal{P}^0$ or β. Therefore, $\mathcal{P}^{p^s} \cdots \mathcal{P}^p p\beta(\alpha) = \beta^{p^{s+1}}$. ☐

Finally, we describe a ring structure on the dual Steenrod algebra $\mathcal{A}_p^*$.

Theorem 1.61. We have

$$\mathcal{A}_p^* = \begin{cases} \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes \bigwedge(\tau_0, \tau_1, \ldots), & p > 2 \\ \mathbb{F}_p[\xi_1, \xi_2, \ldots], & p = 2 \end{cases}.$$

Proof. We will prove the case where $p > 2$. The case where $p = 2$ is analogous. For any admissible $I = (\varepsilon_1, i_1, \varepsilon_2, i_2, \ldots)$, we define $r_j = i_j - pi_{j+1} - \varepsilon_{j+1}$, and $\omega(I = \tau_0^{\varepsilon_1}\xi_1^{r_1}\tau_1^{\varepsilon_2}\xi_2^{r_2}\cdots$. It suffices to show that $\omega(I)$'s describe a basis of $\mathcal{A}_p^*$. That is, we want to shwo that for admissible I and J, there is a pairing

$$\left\langle \omega(I), \mathcal{P}^J \right\rangle = \begin{cases} \pm 1, & I = J \\ 0, & I > J \end{cases}$$

Here the partial ordering on admissible sequences is given by the lexigraphical ordering from the right.

Remark 1.62. There is the issue that $\omega(I)$ and $\mathcal{P}^I$ are not dual bases of one another, which is eventually given by Milnor basis.

Therefore, the $\omega(I)$'s are linearly independent, so $\omega(I)$ has the same dimension as $\mathcal{P}^J$, but $\dim(\mathcal{A}_p^*)^i = \dim(\mathcal{A}_p)^i$, so we have a system of generators. Rercall that τ and ξ are determined by the Steenrod operations of generators of L_p^∞. By Proposition 1.60, for $0 \neq \alpha \in H^1(L_p^\infty)$ and $0 \neq \beta = \beta(\alpha)$, we have

$$\begin{cases} \mathcal{P}^J(\alpha) & = \left\langle a, \mathcal{P}^J \right\rangle \cdot \alpha + \sum\limits_i \left\langle \tau_i, \mathcal{P}^J \right\rangle \beta^{pi} \\ \mathcal{P}^J(\beta) & = \sum\limits_i \left\langle \xi_i, \mathcal{P}^J \right\rangle \beta^{p^i}. \end{cases}$$

Therefore,

$$\left\langle \tau_i, \mathcal{P}^J \right\rangle = \begin{cases} 1, & J = (0, p^{i-1}, p^{i-2}, \ldots, 0, 1, 1, 0) \\ 0, & \text{otherwise} \end{cases}$$

and

$$\left\langle \xi_i, \mathcal{P}^J \right\rangle = \begin{cases} 1, & J = (0, p^{i-1}, 0, p^{i-2}, \ldots, 0, 1) \\ 0, & \text{otherwise} \end{cases}$$

Let us $I = J$ and prove the statement by induction on the degree. Suppose $\varepsilon_m = 1$, $i_m = 0$, $\varepsilon_s = i_s = 0$ for $s > m$. In this case, $\mathcal{P}^J$ ends with a Bockstein. We then get

$$\begin{aligned} \left\langle \omega(J), \mathcal{P}^J \right\rangle &= \left\langle \omega(J - J_m^{-1})\tau_{m-1}, \mathcal{P}^J \right\rangle \\ &= \left\langle \omega(J - J_m^{-1}) \otimes \tau_{m-1}, \psi^*(\mathcal{P}^J) \right\rangle \\ &= \left\langle \omega(J - J_m^{-1}) \otimes \tau_{m-1}, \sum_{K \subseteq J} \mathcal{P}^{J-K} \otimes \mathcal{P}^K \right\rangle \\ &= \sum_K \pm \left\langle \omega(J - J_m^{-1}), \mathcal{P}^{J-K} \right\rangle \left\langle \tau_{m-1}, \mathcal{P}^K \right\rangle \end{aligned}$$

$$\begin{aligned} &= \pm\left\langle \omega(J-J_m^{-1}), \mathcal{P}^{J-J_m^{-1}} \right\rangle \\ &= \pm 1 \end{aligned}$$

by induction, where $J_m^{-1} = (0, p^{m-2}, \cdots, 0, 1, 1, 0)$. Now suppose $i_m > 0$, and $i_s = 0$ for $s > m$, then $\mathcal{P}^J$ does not end with Bockstein. Now

$$\begin{aligned} \left\langle \omega(J), \mathcal{P}^J \right\rangle &= \rangle \omega(J-J_m)\xi_m, \mathcal{P}^J \rangle \\ &= \left\langle \omega(J-J_m) \otimes \xi_m, \psi^*(\mathcal{P}^J) \right\rangle \\ &= \pm \left\langle \omega(J-J_m), \mathcal{P}^{J-J_m} \right\rangle \\ &= \pm 1, \end{aligned}$$

and the case $I > J$ follows from the same argument. □

To get a complete description of $\mathcal{A}_p^*$ as a Hopf algebra, it suffices to describe $\varphi_*(\tau_i)$ and $\varphi_*(\xi_i)$: since φ_* is a ring homomorphism, then it suffices to find the images of the generators.

Theorem 1.63. The following formula holds:

$$\varphi_*(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{pi} \otimes \xi_i$$

and

$$\varphi_*(\tau_k) = \sum_{i=0}^{k} \xi_{k-i}^{pi} \otimes \tau_i + \tau_k \otimes 1,$$

where we ignore the term if $p = 2$.

Proof. Recall that

$$\begin{cases} \lambda^*(\beta) &= \sum_j \beta^{p^j} \otimes \xi_j \\ \lambda^*(\alpha) &= \alpha \otimes 1 + \sum_j \beta^{p^j} \otimes \tau_j \end{cases}.$$

Raising to power p^i, we obtain

$$\lambda^*(\beta^{p^i}) = \sum_j \beta^{p^{i+j}} \otimes \xi_j^{p^i}$$

and

$$(\lambda^* \otimes \mathrm{id}) \circ \lambda^* = (\mathrm{id} \otimes \varphi_*) \circ \lambda^*,$$

then

$$\begin{aligned} (\lambda^* \otimes \mathrm{id})\lambda^*(\beta) &= (\lambda^* \otimes \mathrm{id})(\sum_i \beta^{p^i} \otimes \xi_i) \\ &= \sum_{i,j} \beta^{p^{i+j}} \otimes \xi_j^{p^i} \otimes \xi_i \end{aligned}$$

and

$$\begin{aligned} (\lambda^* \otimes \mathrm{id})\lambda^*(\alpha) &= (\lambda^* \otimes \mathrm{id})(\alpha \otimes 1 + \sum_i \beta^{pi} \otimes \tau_i) \\ &= \alpha \otimes 1 \otimes 1 + \beta^{pi} \otimes \tau_i \otimes 1 + \sum_{i,j} \beta^{p^{i+j}} \otimes \xi_j^{pi} \otimes \tau_i. \end{aligned}$$

We now have agreements

$$(1 \otimes \varphi_*)\lambda^*(\beta) = \sum_i \beta^{p^i} \otimes \varphi_*(\xi_i)$$

and

$$(1\otimes\varphi_*)\lambda^*(\alpha)=\alpha\otimes 1\otimes 1+\sum_i\beta^{p^i}\otimes\varphi_*(\tau_i).$$

□

2 Equidimensional Cycles

2.1 Overview and Étale Properties

For any commutative ring A, we define $\operatorname{Spec}(A)$ to be a ringed space given by the prime ideals of A. The open subsets $D(f)$ are $\{p \in \operatorname{Spec}(A) : f \notin p\}$, then the structure sheaf $\mathcal{O}_{\operatorname{Spec}(A)}(D(f)) = A_f$. Under these notion, a scheme is obtained by gluingg some $\operatorname{Spec}(A_i)$'s for commutative rings A_i via the open subsets. A morphism of schemes $X \to Y$ is localy defined by ring homomorphisms.

Definition 2.1. A scheme $X \to S$ is of finite type if there exists an open covering $\{U_i\}$ of X and an open covering $\{V_i\}$ of S such that $\mathcal{O}_{V_i} \to \mathcal{O}_{U_i}$ is a finitely-generated algebra.

A scheme is called integral if there exists an affine open covering $\{U_i\}$'s such that $\mathcal{O}_{U_i}$ is an integral domain for every i.

Remark 2.2. For schemes X and Y over S, there exists a Cartesian square

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

In the case where X, Y, and S are all afine, then $X \times_S Y$ is the spectrum of the tensor product of rings.

Definition 2.3. An open immersion is given by open subsets $U \subseteq X$ to X. A closed immersion $i : Z \to X$ gives rise to closed subsets of X, and in general there is a morphism $\mathcal{O}_X \twoheadrightarrow i_*(\mathcal{O}_Z)$ which is induced by $A \to A/I$ locally.

Let S be a scheme of finite type, then it is separated if $S \to S \times_k S$ is a closed immersion; it is reduced if there are no nilpotent elements over the field k.

Finally, we define Sch/S to be the category of separated schemes X of finite type over S; Sm/S is the category of smooth separated schemes over S.

Definition 2.4. A morphism $p : X \to S$ in Sch/k is called an equidimensional morphism of dimension r if every irreducible component of X dominates an irreducible component of S, and for every $s \in S$, $p^{-1}(s)$ is of equidimension r (or empty).

Definition 2.5. For any $X \in \mathrm{Sch}/k$, we denote $\mathrm{Cycl}(X)$ (respectively, $\mathrm{Cycl}^{\mathrm{eff}}(X)$) to be the free abelian group (respectively, the free abelian monoid) generated by points of X, i.e., the irreducible closed subsets.

For any $\sum n_i x_i \in \mathrm{Cycl}(X)$, we define its support to be $\bigcup\{\bar{x}_i : n_i \neq 0\}$.

Definition 2.6. Suppose $X \in \mathrm{Sch}/k$. For every closed subscheme $Z \subseteq X$, we define

$$\mathrm{Cycl}_X(Z) = \sum_{i=1}^{k} m_i \xi_i,$$

where the ξ_i's are generic points of Z, and $m_i = \ell_{\mathcal{O}_{Z,\xi_i}}(\mathcal{O}_{Z,\xi_i})$, since $\mathcal{O}_{Z,\xi_i}$ is a dimension-0 Artinian ring and thus has a notion of length.

Definition 2.7. Suppose $f : X \to Y$ is a map in Sch/k.

1. Suppose f is flat, i.e., locally a flat algebra. For every closed subscheme $i : Z \to Y$, we define the pullback to be $f^*(\mathrm{Cycl}_Y(Z)) = \mathrm{Cycl}(Z \times_Y X)$.

2. Suppose f is proper (and a closed mapping). For every $\sum n_i x_i \in \mathrm{Cycl}(X)$, we define the pushforward to be $f_*(\sum\limits_i n_i x_i) = \sum\limits_i n_i m_i f(x_i)$, where

$$m_i = \begin{cases} [k(x_i) : k(f(x_i))], & \dim(\bar{x}_i) = \dim(\overline{f(x_i)}) \\ 0, & \text{otherwise} \end{cases}$$

Remark 2.8. Let $T \to S$ be an arbitrary morphism, then there is a pullback $X \times_S T \to X$ for $X \to S$ a scheme. One may try to define $p^* : \mathrm{Cycl}(X) \to \mathrm{Cycl}(X \times_S T)$. This requries understanding DVRs so that we have special and generic fibers.

Definition 2.9. Suppose $S \in \mathrm{Sch}/k$. A fat point $\underline{s} \in S$ is a DVR D, a field K, and maps

$$\mathrm{Spec}(K) \xrightarrow{s_0} \mathrm{Spec}(D) \xrightarrow{s_1} S$$

such that $\mathrm{im}(s_0)$ is given by the closed point of D, and the image of s_1 at the generic point is the generic point at an irreducible compoennt of S.

Proposition 2.10 ([Har13], Exercise II.4.11)**.** For every $s \in S$, there exists a DVR D dominating $\mathcal{O}_{S,s} \subseteq D \subseteq \mathcal{O}_{S,\eta}$, i.e., $K(D) = K(\eta)$, and $\mathfrak{m}_D \cap \mathcal{O}_{S,s} = \mathfrak{m}_s$, where η is a generic point such that $s \in \bar{D}$, i.e., s is contained in some fat point.

Proposition 2.11. Let D be a DVR with $F = K(D)$. If $X \in \mathrm{Sch}/D$ and W_F is closed in the generic fiber $X \times_D \mathrm{Spec}(F)$, i.e., fiber of $\mathrm{Spec}(F)$, then there exists a unique closed subscheme W_D in X such that $W_D \times_D \mathrm{Spec}(F) = W_F$, and W_D is flat over $\mathrm{Spec}(D)$, i.e., we have a (somewhat unique) lifting of W_F.

Proof. It suffices to prove this for the case where X is affine. Suppose $W_F = \mathrm{Spec}(A \otimes_D F)/(f_1, \ldots, f_n)$ for $f_i \in A$. Define a ring $R = (A/(f_1, \ldots, f_n))/D_{\mathrm{tor}}$ by quotienting the torsions of D. Suppose $(g_1, \ldots, g_m) = (f_1, \ldots, f_m)$ as ideals in $A \otimes_D F$ for $g_j \in A$, then every g_j satisfies $ug_j \in (f_1, \ldots, f_n)$ for some $u \in D$, therefore g_j is D-torsion in $A/(f_1, \ldots, f_n)$, which is therefore zero in R. This shows that the ring is well-defined, i.e., independent of choices of f_i's. In particular, R is a flat D-module. We now define $W_D = \mathrm{Spec}(R)$. To prove unqiueness, suppose $W'_D = \mathrm{Spec}(A/(g_1, \ldots, g_m))$ satisfies the same property, then again $(f_1, \ldots, f_m) = (g_1, \ldots, g_m)$, as desired. □

Given a fat point, we want a pullback that sends cycles to the special fiber, so this requires the following definition.

Definition 2.12. Let $X \in \mathrm{Sch}/S$ and $W \subseteq X$ is a closed subscheme. For any fat point $\underline{s} \in S$ over a k-point s, we define the pullback $\underline{s}^*(w)$ as a closed subscheme in the special fiber $X \times_S \mathrm{Spec}(K)$ as the following:

- define $W_F = (W \times_S \mathrm{Spec}(D)) \times_{\mathrm{Spec}(D)} \mathrm{Spec}(F)$ for $F = K(D)$, which is the generic fiber of a closed subset of pulling X back to D;
- find $W_D \in \mathrm{Sch}/D$ such that $W_D \times_{\mathrm{Spec(D)}} \mathrm{Spec}(F) = W_F$ and W_D being flat over $\mathrm{Spec}(D)$;
- define $\underline{s}^*(w) = \mathrm{Cycl}_{X \times_S \mathrm{Spec}(K)}(W_D \times_{\mathrm{Spec}(D)} \mathrm{Spec}(K))$.

Example 2.13. For a fixed k-point $s \in S$, the choice of $\underline{s}^*(w)$ may depend on the choice of $\underline{s}$. For example, let $S = \{(x,y) \in \mathbb{A}^1 : xy = 0\}$ and $X = \mathbb{A}^1 \coprod \mathbb{A}^1 = \tilde{S}$. This is the line with double origins, and each origin $0 \in \mathbb{A}^1$ in X gives a fat point $(0,0) \in S$, with special fiber being equal to the origin itself. Therefore, the pullbacks are different.

Hence, we need to study the cases where the pullback is independent of the choice of the fat point.

Definition 2.14. Let $W = \sum_i n_i w_i \in \mathrm{Cycl(X)}$ for $X \in \mathrm{Sch}/S$. We say that W is dominant over S if each w_i is dominant over a component of S.

A relative cycle is a dominant cycle on X over S such that its pullbacks $\underline{s}^*(w)$ are independent of the choice of the fat point $\underline{s}$ for a fixed k-point s. In this case, we may abuse the notation $s = \underline{s}$ and write $s^*(w) \in \mathrm{Cycl}(X_S)$.

Definition 2.15. We define $\mathrm{Cycl}(X/S, r)$ (respectively, $\mathrm{Cycl}_{\mathrm{equi}}(X/S, r)$) to be the free abelian group of relative cycles W on X/S such that each component has dimension r over S (respectively, fiberwise).

Theorem 2.16 (Platification)**.** Let $p : S' \to S$ be a morphism of Noetherian schemes and $U \subseteq S$ is an open subset such that p is flat over U, then there exists a closed subscheme $Z \subseteq S$ such that $Z \cap U = \varnothing$ and the proper transform of S' with respect to the blow-up $\mathrm{Bl}_Z(S) \to S$ is flat over $\mathrm{Bl}_Z(S)$.

Remark 2.17. We may write $\mathrm{Bl}_Z(S)$ as a union of $S \backslash Z$ and an exceptioanl divisor $\mathbb{P}(N_{Z/S})$.

Proof. See [RG71]. □

Theorem 2.18 ([SV00], Proposition 3.1.5)**.** Suppose that S is reduced and $Z = \sum_i n_i z_i$ is a dominant cycle of relative dimension r. Every Z_j is generically flat over S, so by Theorem 2.16 there is a blow-up $S' \to S$ such that the proper transforms $\tilde{Z}_j$ are flat over S'. In this context, the following are equivalent.

1. $Z \in \mathrm{Cycl}(X/S, r)$;
2. if $x : \mathrm{Spec}(F) \to S$ is a point of S and $x' : \mathrm{Spec}(F) \to S'$ is a liftng of X, then the cycle $W \in \mathrm{Cycl}(X \times_S \mathrm{Spec}(F))$ defined by $W = \sum_j n_j \mathrm{Cycl}_{X \times_S \mathrm{Spec}(F)}(\tilde{Z}_j \times_{S'} \mathrm{Spec}(F))$, which is independent of the choice of x'.

Proof.

(1. $\Rightarrow$ 2.) If the map $X \to S$ is flat at the point x, then the map $S' \to S$ is an isomorphism around x, hence the blow-up is trivial and there is nothing to prove. Therefore, we may assume x and therefore x' is not generic. By Proposition 2.10, there exists a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(x') \arrow[r] \arrow[d] & S' \arrow[d] \\ \mathrm{Spec}(R') \arrow[r] & S \end{tikzcd} \tag{2.19}$$

where R' is a DVR with function field $K(x')$, and $\mathrm{Spec}(R') \to S$ sends $\mathfrak{m}_{R'}$ to X. Therefore, suppose we have two liftings $x_1', x_2' : \mathrm{Spec}(F) \to S'$, then we have two maps $\mathrm{Spec}(R_i') \to S'$ for $i = 1, 2$, and one can check easily that the scheme $(\mathrm{Spec}(k(R_1')) \times_S \mathrm{Spec}(k(R_2'))) \times_{S' \times_S S'} \mathrm{Spec}(F)$ is non-empty, therefore we pick a point $\mathrm{Spec}(k)$ upon it. Under this setting, we have a diagram

$$\begin{tikzcd} & \mathrm{Spec}(k(R_1')) \arrow[r] & \mathrm{Spec}(R_1) \arrow[r] & S' \arrow[dr] & \\ \mathrm{Spec}(k(L)) \arrow[ur] \arrow[rr] \arrow[dr] & & \mathrm{Spec}(F) \arrow[ur, "x_1'"] \arrow[dr, "x_2'"'] & & S \\ & \mathrm{Spec}(k(R_2')) \arrow[r] & \mathrm{Spec}(R_2') \arrow[r] & S' \arrow[ur] & \end{tikzcd}$$

which defines a point $\mathrm{Spec}(L)$ on S and two fat points $\mathrm{Spec}(L) \to \mathrm{Spec}(R_i) \to S$ for $i = 1, 2$. Now $\tilde{Z}_j$ is isomorphic to Z_j around the generic point, so on the generic fiber

$$\tilde{Z}_j \times_{S'} \mathrm{Spec}(K(R_i')) \cong Z_j \times_S \mathrm{Spec}(K(R_i'))$$

and we know $\tilde{Z}_j$ is flat over S' by definition. Therefore, $\underline{x}_i'^*(Z) = \sum_j n_j \mathrm{Cycl}((\tilde{Z}_j \times_{S'} \mathrm{Spec}(F)) \times_F L)$, which is independent of $i = 1, 2$, since the base-change maps of cycles is injective for field extensions.

(2. $\Rightarrow$ 1.) Let $x : \mathrm{Spec}(F) \to S$ be a point and $\underline{x} : \mathrm{Spec}(F) \to \mathrm{Spec}(D) \to S$ be a fat point, then it admits a lifting $\mathrm{Spec}(D) \to s'$ by the valuation criterion. Therefore,

$$\underline{x}^*(Z) = \sum_j n_j \mathrm{Cycl}(\tilde{Z}_j \times_{S'} \mathrm{Spec}(F))$$

which is independent of the choice of $\underline{x}$.

∎

Remark 2.20. If S is the spectrum of a field, we have

$$\mathrm{Cycl}(X/S, r) = \mathbb{Z}\{\text{cycles of equidimension } r/S\}.$$

Proposition 2.21 ([SV00], Corollary 3.1.8)**.** We have $\mathrm{Cycl}^{\mathrm{eff}}_{\mathrm{equi}}(X/S, r) = \mathrm{Cycl}^{\mathrm{eff}}(X/S, r)$ for any $S \in \mathrm{Sch}/k$ and $X \in \mathrm{Sch}/S$.

Proof. We want to prove that for every $\sum n_i Z_i \in \mathrm{Cycl}(X/S,r)$ such that $n_i \geqslant 0$ is an equidimensional cycle. By Chevalley Theorem, c.f., [Har13, Exercise II.3.22], every fiber of $Z_i \to S$ is either empty or of dimension at least r. Suppose that the fiber is not equidimensional, then there exists some fiber $(Z_0)_s$ has dimension higher than r for some $s \in S$. Let $\eta \in (Z_0)_s$ be the generic point such that $\dim(\overline{\eta}) > r$, then by Theorem 2.16, $\tilde{Z}_i$ is flat over the blow-up S' of S using the same notation as before, and therefore it is equidimensional of dimension r over S'. The map $\tilde{Z}_0 \to Z_0$ is proper and dominant, therefore it is surjective. Consider the diagram

$$\begin{tikzcd} \tilde{Z}_0 \arrow[r] \arrow[d] & Z_0 \arrow[d] \\ S' \arrow[r] & S \end{tikzcd} \tag{2.22}$$

for $\eta \in Z_0$, then let τ be any point of $\tilde{Z}_0$ over η. Let s_1' be the image of $\tilde{Z}_0$ in S', then we have a diagram of elementwise mappings

$$\begin{tikzcd} \tau \arrow[r, mapsto] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ s_1' \arrow[r, mapsto] & s \end{tikzcd}$$

Let s_2' be a closed point in the fiber in S' over $s \in S$, but then one can choose a field F which contains both $k(s_1')$ and $k(s_2')$, then we get two points $x_1', x_2' : \mathrm{Spec}(F) \to S'$ as images over the same point $x : \mathrm{Spec}(F) \to S$. Consider the two corresponding cycles on $X \times_S \mathrm{Spec}(F)$, namely $W_j = \sum_i n_i \,\mathrm{Cycl}_{X\times_S \mathrm{Spec}(F)}(\tilde{Z}_i \times_{s_j' \in S'} \mathrm{Spec}(F))$ for two points s_j' where $j = 1,2$. By Theorem 2.18, it suffices to show that $\pi(\mathrm{Supp}(W_1)) \neq \pi(\mathrm{Supp}(W_2))$ where $\pi : X \times_S \mathrm{Spec}(F) \to X_s$ is the natural map: once we prove this, we have a contradiction to the fact that $\sum n_i Z_i$ is a relative cycle in $\mathrm{Cycl}(X/S,r)$. By construction, $\eta \in \pi(\mathrm{Supp}(W_1)) > r$ since it contains a point of dimension higher than r when $n_i > 0$ for all i. Since the dimension $[k(s_2') : k(s)] < \infty$, then under finite extension, the dimension $\dim(\pi(\mathrm{Supp}(W_2))) \leqslant r$. ☐

2.2 Relative Cycles and Fat Point

For arbitrary $f : T \to S$ between reduced base schemes over a field k, we want to define a pullback morphism

$$f^* : \mathrm{Cycl}(X/S,r) \to \mathrm{Cycl}(X \times_S T/T, r)$$

of free abelian groups. Suppose $\tau_1, \ldots, \tau_n$ are generic points of T, and $\sigma_i = f(\tau_i)$. For any $Z \in \mathrm{Cycl}(X/S,r)$, we consider the cycle $Z_T = \sum_j \sigma_j^*(Z) \otimes_{k(\sigma_j)} k(\tau_j)$ as a candidate.

Theorem 2.23 ([SV00], Theorem 3.3.1)**.** $Z_T \in \mathrm{Cycl}(X \times_S T/T, r)$ is the unique element such that for any commutative diagram

$$\begin{tikzcd} & \mathrm{Spec}(A) \arrow[r] & T \arrow[d, "f"] \\ \mathrm{Spec}(k) \arrow[ur] \arrow[dr] & & \\ & \mathrm{Spec}(R) \arrow[r] & S \end{tikzcd}$$

of fat points $\underline{y}$ over $\mathrm{Spec}(A)$ and $\underline{x}$ over $\mathrm{Spec}(R)$, then one has $\underline{y}^*(Z_T) = \underline{x}^*(Z)$.

We will now prove Theorem 2.23. Let $Z = \sum n_i Z_i \in \mathrm{Cycl}(X/S,r)$ and let s' and $\tilde{Z}_i$ be the same as before, then by [Har13, Exercise II.4.11], there is a surjective morphism $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ for some DVR A' so that we have a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(A') \arrow[rr] \arrow[d] & & S' \arrow[d] \\ \mathrm{Spec}(K) \arrow[r] & \mathrm{Spec}(A) \arrow[r] & T \arrow[r] & S \end{tikzcd}$$

Assume τ_1 is the generic point of $\underline{y}$ and consider two elements $W = \sum_i n_i \left(\tilde{Z}_i \times_{S'} \operatorname{Spec}(A')\right)$, and

$$\begin{aligned} W_1 &= \sum_{j,\ell} n_{j,\ell} \left(\varphi(Z_{j,\ell}) \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A')\right) \\ &= \sum_{\ell} n_{1,\ell} (\varphi(Z_{1,\ell}) \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A')) \end{aligned}$$

where $Z_i = \sum_{i,\ell} n_{i,\ell} Z_{i,\ell} \in \operatorname{Cycl}(X \times_S T/T)$, and $\varphi(Z_{j,\ell})$ is the unique closed subscheme of $Z_{j,\ell} \times_T \operatorname{Spec}(A)$ being flat over $\operatorname{Spec}(A)$ in Proposition 2.11.

Definition 2.24. Define $\operatorname{Hilb}(X/S, r)$ to be the free abelian group generated by closed subschemes in X that are flat of equidimension r over S.

Proposition 2.25 ([SV00], Lemma 3.2.3)**.** Let $p : X \to S$ be a morphism of finite type of Noetherian schemes, then p is equidimensional of dimension r if and only if for any point $x \in X$, there is an open neighborhood U and a factorization of the form

$$U \xrightarrow{p_0} \mathbb{A}^r_S \cong \mathbb{A}^r \times S \longrightarrow S$$

such that p_0 is quasi-finite, i.e., every fiber has finitely many points, and that every component of U dominates a component of $\mathbb{A}^r_S$.

Proof. See [Gro66, Proposition 13.3.1(b)]. □

Proposition 2.26 (Zariski Main Theorem)**.** Let $f : X \to Y$ be a morphism between schemes of finite type over a field. Suppose f is quasi-finite, then f admits a factorization

$$X \xrightarrow{g} Z \xrightarrow{h} Y$$

where g is an open immersion and h is a finite morphism.

Proposition 2.27 ([SV00], Proposition 3.2.2)**.** Let S be a reduced scheme, and let $X \to S$ be a morphism of finite type between Noetherian schemes. Suppose S' is a reduced Noetherian scheme over S. For any $Z = \sum n_i Z_i \in \operatorname{Hilb}(X/S, r)$, if $\operatorname{Cycl}_X(Z) = 0$, then the base-change

$$\operatorname{Cycl}_{X \times_S S'}(Z \times_S S') = 0.$$

Proof. Without loss of generality, we may assume X is a union of Z_i's, then we can think of X as equidimensional of relative dimension f over S. Consider a Cartesian square

$$\begin{tikzcd} X' \arrow[r, "\mathrm{pr}_1"] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

and let $\eta' \in X' = X \times_S S'$ be any generic point, then by computing the multiplicity of η' in $\operatorname{Cycl}(Z \times_S S')$, we may replace X by any neighborhood of $\eta = \mathrm{pr}_1(\eta') \in X$. Since η is a generic point in its fiber over S, we may assume that there is a decomposition

$$X \xrightarrow{p_0} \mathbb{A}^r_S \longrightarrow S$$

where p_0 is flat and quasi-finite by Proposition 2.25, Proposition 2.26, and [GR02, Exposé IV, Corollary 5.9]. Hence, we may replace S by $\mathbb{A}^r_S$, so it suffices to treat the case where $r = 0$. Furthermore, by replacing S' with $\operatorname{Spec}(\mathcal{O}_{S',\tau'})$ and S with $\operatorname{Spec}(\mathcal{O}_{S,\tau})$, we may assume that S and S' are local as schemes, S' is Artinian, and $f : S' \to S$ is a local homomorphism.

Definition 2.28. Let A be a local ring. Suppose we take all possible extensions $A \to B$ where $(B, \mathfrak{p})$ is étale over A and $\mathfrak{p} \mapsto \mathfrak{m}_A$ on the level of prime ideals, then we define the Henselization A^h of A to be

$$A^h = \varinjlim_{\substack{(B,\mathfrak{p}) \\ k(\mathfrak{p}) = k(A)}} B$$

In particular, A^h is Henselian and $k(A^h) = k(A)$. Moreover, we define the strict Henselization A^{sh} of A to be

$$A^{sh} = \varinjlim_{(B,p)} B.$$

In particular, A^{sh} is Henselian and the residue field $k(A^{sh})$ is separably closed.

Remark 2.29. A^h and A^{sh} corresponds to the stalk of étale morphism and Nisnevich morphism, respectively. Suppose $x \in X$ is a point in the scheme. The local ring of $x \in X$ in the Nisnevich (respectively, étale) topology is the (respectively, strict) Henselization of the local ring of x in the Zariski topology.

The flat pullback shows that $\mathrm{Cycl}(Z \times_S S^{sh}) = 0$, and moreover, given a diagram

$$\begin{tikzcd} (S')^{sh} \arrow[r] \arrow[d] & S^{sh} \arrow[d] \\ S' \arrow[r] & S \end{tikzcd} \tag{2.30}$$

it suffices to check that $\mathrm{Cycl}((Z \times_S S') \times_{S'} (S')^{sh}) = \mathrm{Cycl}(Z \times_S (S')^{sh}) = 0$ since $(S')^{sh} \to S$ is an étale covering. By Diagram 2.30, it suffices to assume that S and S' are strictly Henselian.

Remark 2.31. Suppose we have a quasi-finite morphism $f : X \to S$ for Henselian ring S, then we can write $X = \coprod_{j=0}^{n} \mathrm{Spec}(R_j)$, where R_j is local for $j \geqslant 1$, and there is a finite map $\mathrm{Spec}(R_0) \to S$. Moreover, the closed point $\mathfrak{m}_s \notin f(\mathrm{Spec}(R_j))$ for $j \geqslant 1$.

In the case where k has positive characteristic, then since η lies over the closed point of S and S is Henselian, we conclude that $\mathrm{Spec}(\mathcal{O}_{X,\eta})$ is an open neighborhood of η in X, and is finite over S. For a more detailed reference of this remark, see [Mil80, Theorem 1.4.2].

Therefore, replacing X by $\mathrm{Spec}(\mathcal{O}_{X,\eta})$, we may assume that X is a local scheme that is finite over S. Under these assumptions, η is the only point over τ and the schemes Z_i are finite and flat over S, such that $\mathcal{O}_{Z_i} \cong \mathcal{O}_S^{\bigoplus n_i}$ as $\mathcal{O}_S$-modules. Here we define $\deg(Z_i/S) = n_i$.

The multiplicity of η' in $\mathrm{Cycl}_{X'}(Z \times_S S')$ should be

$$\sum_i n_i \ell(\mathcal{O}_{Z_i} \times_S S', \eta') = \sum_i n_i \frac{\deg(Z_i/S)\ell(\mathcal{O}_{S',\tau'})}{[k(\eta') : k(\tau')]}. \tag{2.32}$$

by definition.[7] To see this, note that any point in X' that lies over τ' must also lie over η in X. Therefore, $k(\eta) = k(\tau)$ as k has characteristic 0. Therefore, the fiber of η along $X' \to X$ is homeomorphic to that of τ along $S' \to S$, which is just $\{\eta'\}$. Therefore,

$$\begin{aligned} \deg(Z_i/S)\ell(\mathcal{O}_{S',\tau'}) &= \ell_{\mathcal{O}_{S'}}(\mathcal{O}_{Z_i \times_S S'}) \\ &= [k(\eta') : k(\tau')] \end{aligned}$$

by [Ful13, Lemma A.1.3]. This proves Equation (2.32). Therefore, it suffices to prove that $\sum_i n_i \deg(Z_i/S) = 0$. To do so, let τ^0 be a generic point of s and let $\eta_1^0, \cdots, \eta_k^0$ be all points of X over τ^0, then the multiplicity of η_j^0 in $\mathrm{Cycl}(Z)$ is defined by $\sum_{i: Z_i \supseteq \eta_j^0} n_i \ell(\mathcal{O}_{Z_i,\eta_j^0})$, but by our assumption we know $\mathrm{Cycl}(Z) = 0$, so $\sum_i n_i \deg(Z_i/S) = 0$. Using the same result from [Ful13], we note that

$$\begin{aligned} \sum_i n_i \deg(Z_i/S)\ell(\mathcal{O}_{S,\tau^0}) &= \sum_{\substack{i,j \\ \eta_j^0 \in Z_i}} n_i [k(\eta_j^0) : k(\tau^0)] \ell(\mathcal{O}_{Z_i,\eta_j^0}) \\ &= 0. \end{aligned}$$

We conclude that $\sum_i n_i \deg(Z_i/S) = 0$. □

[7] This recovers [SV00], Lemma 3.2.1.

We will now continue our proof of Theorem 2.23.

Lemma 2.33 ([SV00], Lemma 3.3.5)**.** Let W and W_1 be as defined above, then we have $\operatorname{Cycl}(W) = \operatorname{Cycl}(W_1)$.

Proof. Recall that we had the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}(K') \arrow[r] \arrow[d] & \operatorname{Spec}(A') \arrow[rr] \arrow[d] & & S' \arrow[d] \\
\operatorname{Spec}(K) \arrow[r] & \operatorname{Spec}(A) \arrow[r] & T \arrow[r] & S
\end{tikzcd}$$

Let F (respectively, F') be the function field of A (respectively, A'). Since the map

$$\operatorname{Hilb}((X \times_{S'} \operatorname{Spec}(A')/\operatorname{Spec}(A')), r) \to \operatorname{Hilb}((X \times_S \operatorname{Spec}(F'))/\operatorname{Spec}(F'), r)$$

is injective, then we may replace A' by F'. Furthermore,

$$\begin{aligned}
\operatorname{Cycl}(W \times_{\operatorname{Spec}(A')} \times \operatorname{Spec}(F')) &= \sum_i n_i \operatorname{Cycl}(\tilde{Z}_i \times_{S'} \operatorname{Spec}(F')) \\
&= \sigma_1^*(Z) \otimes_{k(\sigma_1)} F'
\end{aligned}$$

where σ_1 is the image of τ_1. Finally, we know that

$$\begin{aligned}
\operatorname{Cycl}(W_1 \times_{\operatorname{Spec}(A')} \operatorname{Spec}(F')) &= \sum_\ell n_{1,\ell} \operatorname{Cycl}(\varphi(Z_{1,\ell}) \times_{\operatorname{Spec}(A)} \operatorname{Spec}(A') \times_{\operatorname{Spec}(A')} \operatorname{Spec}(F')) \\
&= \sum_\ell n_{1,\ell} \operatorname{Cycl}(\varphi(Z_{1,\ell}) \times_{\operatorname{Spec}(A)} \operatorname{Spec}(F) \times_{\operatorname{Spec}(F)} \operatorname{Spec}(F')) \\
&= \sum_\ell n_{1,\ell} \operatorname{Cycl}(Z_{1,\ell} \times_T \operatorname{Spec}(F')) \\
&= \sigma_1^*(Z) \otimes_{k(\sigma_1)} F' \\
&= \operatorname{Cycl}(W) \otimes \operatorname{Spec}(F').
\end{aligned}$$

$\square$

Proof of Theorem 2.23. By Lemma 2.33, we have $\operatorname{Cycl}(W) = \operatorname{Cycl}(W_1)$. By Proposition 2.25, we have

$$\operatorname{Cycl}(W \times_{\operatorname{Spec}(A)} \operatorname{Spec}(K')) \cong \operatorname{Spec}(W_1 \times_{\operatorname{Spec}(A')} \operatorname{Spec}(K')),$$

therefore

$$\begin{aligned}
\underline{y}^*(Z_T) \otimes_K K' &= \sum_\ell n_{1\ell} \operatorname{Cycl}(\varphi(Z_{1\ell}) \times_{\operatorname{Spec}(A)} \operatorname{Spec}(K')) \\
&= \sum_i n_i \operatorname{Cycl}(\tilde{Z}_i \times_{S'} \operatorname{Spec}(K')) \\
&= \underline{x}^*(Z) \otimes_K K'.
\end{aligned}$$

This immediately implies that Z_T is a relative cycle, i.e., independent of choices. $\square$

Definition 2.34. For any $f : T \to S$ and $Z \in \operatorname{Cycl}(X/S, r)$, we define $\operatorname{Cycl}(f)(Z) = Z_T$.

Proposition 2.35. For any $X \in \operatorname{Sch}/S$, given a diagram

$$U \xrightarrow{f} T \xrightarrow{g} S$$

of reduced schemes, and $Z \in \operatorname{Cycl}(X/S, r)$, we have $\operatorname{Cycl}(g \circ f) = \operatorname{Cycl}(f) \circ \operatorname{Cycl}(g)$.

Proof. Suppose $\gamma_1, \dots, \gamma_n$ (respectively, $\tau_1, \dots, \tau_m$) are generic points on U (respectively, T), then $\mathrm{Cycl}(g \circ f)(Z) = \sum_i gf(\gamma_i)^*(Z) \otimes_{k(gf(\gamma_i))} k(\gamma_i)$, and

$$\begin{aligned}\mathrm{Cycl}(f) \circ \mathrm{Cycl}(g)(Z) &= \mathrm{Cycl}(f)\left(\sum_j g(\tau_j)^*(Z) \otimes_{k(g(\tau_j))} k(\tau_j)\right) \\ &= \sum_{i,j} f(\gamma_i)^* \left(g(\tau_j)^*(Z) \otimes_{k(g(\tau_j))} k(\tau_j)\right) \otimes k(f(\gamma_i)),\end{aligned}$$

and we have fat points $\underline{f}(\gamma_i) : \mathrm{Spec}(K) \to T$ and $\underline{\gamma}_i : \mathrm{Spec}(K) \to S$, therefore by Theorem 2.23, we have

$$\begin{aligned}\sum_j f(\gamma_i)^*(g(\tau_j)^*(Z) \otimes_{k(g(\tau_j))} k(\tau_j)) &= \underline{f}(\gamma_i)^*(Z_T) \\ &= gf(\gamma_i)^*(Z) \\ &= gf(\gamma_i)^*(Z) \otimes_{k(gf(\gamma_i))} k(gf(\gamma_i)),\end{aligned}$$

as desired. □

2.3 Equidimensional Cycles in Motivic Setting

Most of the content in this subsection can be found in my other set of notes.

Theorem 2.36. Let $S \in \mathrm{Sch}/k$ be normal and $X \in \mathrm{Sch}/S$, and suppose $Z \subseteq X$ is a closed subscheme which is equidimensional of relative dimension r over S, then $\mathrm{Cycl}_X(Z)$ is a relative cycle.

Remark 2.37. If $S \in \mathrm{Sch}/k$ is normal, then $\mathrm{Cycl}(X/S, r) = \mathbb{Z}\{\text{equidimensional cycles of } X/s \text{ of dimension } r\}$.

Definition 2.38. For any $T \in \mathrm{Sch}/k$, we define a presheaf $Z_{\text{equi}}(T, r)$ on Sm/k by

$$Z_{\text{equi}}(T, r)(S) = \mathrm{Cycl}(S \times T/S, r)$$

for $S \in \mathrm{Sm}/k$, where the functoriality with respect to S is that of the cycles.

Definition 2.39. Suppose that $X \in \mathrm{Sm}/k$ and that $U, V \subseteq X$ are irreducible closed subsets in X that intersect properly, i.e., $\mathrm{codim}(W) = \mathrm{codim}(U) + \mathrm{codim}(V)$ for any component $W \subseteq U \cap V$. We define their intersection multiplicity in $Z_i(X) = \mathbb{Z}\{$ irreducible closed subsets of dimension i in $X\}$ to be

$$m_W(U, V) = \sum_{i=0}^{\dim(X)} (-1)^i \ell_{\mathcal{O}_{X,W}} \left(\mathrm{Tor}^i(\mathcal{O}_{X,W}/P_U, \mathcal{O}_{X,W}/P_V)\right) > 0$$

which is well-defined since every term has finite length. Here P_U and P_V are prime ideals defining U and V, respectively. Moreover, if U and V of dimension a and b, respectively, intersect properly, then we define

$$U \cdot V = \sum_{W \subseteq U \cap V} m_W(U, V) \cdot W \in Z_{a+b-\dim(X)}(X)$$

is a cycle. For each proper $f : X \to Y$, we have a pushforward

$$f_* : Z_a(X) \to Z_a(Y)$$

defined as over cycles. We can also define a pullback for proper intersections. For every $f : X \to Y$ where $X, Y \in \mathrm{Sm}/k$, and that $c \in Z_a(Y)$ is such that $\dim(f^{-1}(c)) = a - \dim(Y) + \dim(X)$, then we define a graph morphism

$$\begin{aligned} j : X &\to X \times Y \\ x &\mapsto (x, f(x)) \end{aligned}$$

and the pullback as

$$f^*(c) = j_*^{-1}\left(j(X) \cdot (X \times c)\right).$$

In particular, if f is flat, then it coincides with f^* defined over cycles.

Definition 2.40. Let us define the category Cor_k of finite correspondences with objects as those of Sm/k, and $\mathrm{Cor}_k(X,Y)$ to be the free ablian group (over $\mathbb{Z}$) generated by diagrams

$$\begin{tikzcd} C \arrow[r, hook] \arrow[dr, "\text{finite surjective}"'] & X \times Y \arrow[d] \\ & X \end{tikzcd}$$

for irreducible closed subset $C \subseteq X \times Y$. For any $f : X \to Y$ and $g : X \to Z$ in Cor_k, we define $g \circ f = p_{13*}(p_{12}^*(f) \cdot p_{23}^*(Z)) \in \mathrm{Cor}_k(X,Z)$ via

$$\begin{tikzcd} & X \times Z & \\ & X \times Y \times Z \arrow[u, "p_{13}"] \arrow[dl, "p_{12}"'] \arrow[dr, "p_{23}"] & \\ X \times Y & & Y \times Z \end{tikzcd}$$

The presheaf $\mathbb{Z}(X)$ is defined by $\mathbb{Z}(X)(Y) = \mathrm{Cor}_k(Y,X)$ is an étale sheaf. For any $Y \in \mathrm{Sm}/k$, there is a natural inclusion $\mathbb{Z}(X)(Y) \subseteq Z_{\mathrm{equi}}(X,0)(Y)$. Moreover, the additive functors $\mathrm{Cor}_k^{\mathrm{op}} \to \mathrm{Ab}$ are called presheaves with transfers.

For any irreducible $c \in \mathrm{Cor}_k(Y_1,Y_2)$ and $Z \in \mathrm{equi}(X,r)(Y_2)$, we may pull back along $C \to Y_1 \times Y_2 \twoheadrightarrow Y_2$, which gives an equidimensional cycle $Z_C \in Z_{\mathrm{equi}}(X,r)(C)$ over C. Since C is a finite correspondence over Y_1, we may pushforward Z_C along the finite morphism and get a cycle $C^*(Z) \in Z_{\mathrm{equi}}(X,r)(Y_1)$. This makes $Z_{\mathrm{equi}}(X,r)$ is a presheaf with transfers.

Proposition 2.41. There is an inclusion $\mathbb{Z}(X) \subseteq Z_{\mathrm{equi}}(X,0)$ as presheaves with transfers.

Theorem 2.42 ([SV00], Theorem 3.6.1)**.** Let $f : S' \to S$ be a map of reduced schemes over k and $p : X_1 \to X_2$ in Sch/S. Set $X_i' = X_i \times_S S'$ for $i = 1,2$, and denote $p' : X_1' \to X_2'$ to be the map $p \times_S S'$. Moreover, set $Z = \sum n_i Z_i$ (respectively, $W = \sum m_j W_j$) be an element of $\mathrm{Hilb}(X_1/S,r)$ (respectively, $\mathrm{Hib}(X_2/S,r)$). Assume that Z_i^{red} are proper over p, and $p_*(\mathrm{Cycl}_{X_1}(Z)) = \mathrm{Cycl}_{X_2}(W)$, then the cycle $\mathrm{Cycl}_{X_1'}(Z \times_S S')$ is proper over p', and

$$p'_*(\mathrm{Cycl}_{X_1}(Z \times_S S')) = \mathrm{Cycl}_{X_2'}(W \times_S S').$$

Proof. Replacing X_1 by the union of Z_i's and X_2 by the union of W_j's with $p(Z_i)$'s, we may assume that p is proper, then X_1 is an equidimensional cycle of dimension r over S, and all fibers of X_2 are of dimension at most r. In fact, it can be reduced to the case when all fibers of X_2/S are of dimension r, and p is generically finite. By Proposition 2.25, we may assuem further that $X_2 \to S$ factorizes as $X_2 \to \mathbb{A}_S^r \to S$, where the first map is being of equidimensional 0 and p is finite. Therefore, the composite $X_1 \to X_2 \to \mathbb{A}_S^r$ is of equidimensional 0. By [GR02, Exposé 4, Corollary 5.9], we may assume that Z_i's and W_j's are flat over $\mathbb{A}_S^r$. We may then replace S by $\mathbb{A}_S^r$ and assume that $r = 0$. We consider

$$\begin{tikzcd} X_2' \arrow[r] \arrow[d] & X_2 \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

and fix $\eta' \in X_2'$. It has image $\eta \in X_2$ and $\tau' \in S'$ (which is a generic point), and let τ be the image of both elements in S.

$$\begin{tikzcd} \eta' \arrow[r, mapsto] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ \tau' \arrow[r, mapsto] & \tau \end{tikzcd}$$

Since this happens over S', then we may replace S' by $\mathrm{Spec}(\mathcal{O}_{S'},\tau')$. We now have a diagram

$$\begin{tikzcd} \mathrm{Spec}(\mathcal{O}_{S',\tau'}^{sh}) \arrow[r] \arrow[d] & \mathrm{Spec}(\mathcal{O}_{S,\tau}^{sh}) \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

so it suffices to prove the result after pulling back to the henselization. Since τ' is a generic point over S', then $S' = \operatorname{Spec}(K(\tau'))$, i.e., $\mathcal{O}_{S',\tau'}$ is a field, then its strict henselization $\operatorname{Spec}(\mathcal{O}^{sh}_{S',\tau'})$ is its separable closure $\operatorname{Spec}(\overline{K(\tau')})$, so we have a field extension which is an injection. Therefore, it suffices to prove the result on stalks, i.e., pulling back to $X'_2 \times_S \operatorname{Spec}(\mathcal{O}^{\mathrm{sh}}_{S',\tau'})$, we want

$$p'_*(\operatorname{Cycl}(Z \times_S \operatorname{Spec}(\mathcal{O}^{sh}_{S',\tau'}))) = \operatorname{Cycl}(W \times_S \operatorname{Spec}(\mathcal{O}^{sh}_{S',\tau'})).$$

Therefore, we may also replace S by $\operatorname{Spec}(\mathcal{O}^{sh}_{S,\tau})$, and since $r = 0$ after replacing X by $\operatorname{Spec}(\mathcal{O}_{X_2,\eta})$, then $X_2 \to S$ is a finite map between local rings. Since S is strictly henselian, then X'_2 is a singleton, namely η'. Let $\alpha_1, \ldots, \alpha'_k$ be all points of X'_2. We have

$$\begin{aligned} p'_*(\operatorname{Cycl}_{X'_1}(Z \times_S S')) &= \left(\sum_{\ell=1}^{\frac{k}{2}}[k(\alpha'_\ell) : k(\eta')]\sum_i n_i \ell(\mathcal{O}_{Z_i \times_S S', \alpha'_i})\right)\eta' \\ &= \left(\sum_i \frac{n_i}{[k(\eta') : k(\tau')]}\sum_\ell [k(\alpha'_i) : k(\tau')]\ell(\mathcal{O}_{Z_i \times_S S', \alpha'_\ell})\right)\eta' \\ &= \left(\sum_i \frac{n_i \deg(Z_i/S)}{[k(\eta') : k(\tau')]}\right)\eta' \end{aligned}$$

by Proposition 2.27, and that

$$\begin{aligned} \operatorname{Cycl}(W \times_S S') &= \left(\sum_j m_j \ell(\mathcal{O}_{W_j \times_S S', \eta'})\right)\eta' \\ &= \left(\frac{1}{[k(\eta') : k(\tau')]}\sum_j m_j \ell(\mathcal{O}_{W_j \times_S S', \eta'})[k(\eta') : k(\tau')]\right)\eta' \\ &= \left(\sum_j \frac{m_j \deg(W_j/S)}{[k(\eta') : k(\tau')]}\right)\eta' \end{aligned}$$

by Proposition 2.27 as well. Therefore, by comparing the expression termwise, we just have to show that

$$\sum_i n_i \deg(Z_i/S) = \sum_j m_j \deg(W_j/S).$$

However, this is obtained by pushing forward the conditions along the finite map $X_2 \to S$. □

Proposition 2.43 ([SV00], Proposition 3.6.2)**.** Let $p : X \to Y$ be a morphism in Sch/S where S is reduced, and $Z = \sum_i n_I Z_i \in \operatorname{Cycl}(X/S, r)$ is a relative cycle, such that it is proper over Y. Then

1. $p_*(Z) \in \operatorname{Cycl}(Y/S, r)$;
2. for any $f : S' \to S$ where S' is reduced, the cycle $\operatorname{Cycl}(f)(Z)$ is proper over $Y \times_S S'$. Moreover, denote $p' : X \times_S S' \to Y$ to be the base-change morphism, then

$$p'_*(\operatorname{Cycl}(f)(Z)) = \operatorname{Cycl}(f)(p_*(Z)).$$

Proof.

1. Define $W_i = p(Z_i)$. Replacing X by the union of Z_i's, we may assume that p is proper. Let $\underline{x} : \operatorname{Spec}(F) \to \operatorname{Spec}(R) \to S$ be a fat point over $x \in S$. Now set $Z_0 = \sum_i n_i \varphi(Z_i \times_S \operatorname{Spec}(R))$, which is flat over $\operatorname{Spec}(R)$ by definition of φ, therefore $Z_0 \in \operatorname{Hilb}(X \times_S \operatorname{Spec}(R)/\operatorname{Spec}(R), r)$; also, set $W_0 = \sum_i n_i m_i \varphi(W_i \times_S \operatorname{Spec}(R)) \in \operatorname{Hilb}(Y \times_S \operatorname{Spec}(R)/\operatorname{Spec}(R), r)$, where

$$m_i = \begin{cases} [k(Z_i) : k(p(Z_i))], & k(Z_i)/k(p(Z_i)) \text{ is a finite extension} \\ 0, & \text{otherwise} \end{cases}$$

is the coefficient we get by pushing forward. Since $\varphi(Z_i)$ (respectively, $\varphi(W_i)$) and Z_i (respectively, W_i) have the same generic fiber, then we have

$$(p \times_S \operatorname{Spec}(R))_* \operatorname{Cycl}(\varphi(Z_i)) = m_i \operatorname{Cycl}(\varphi(W_i)).$$

Hence,

$$\begin{aligned}
\underline{x}^*(p_*(Z)) &= \sum_i n_i m_i \operatorname{Cycl}(\varphi(W_i) \times_{\operatorname{Spec}(R)} \operatorname{Spec}(F)) \\
&= \sum_i n_i \operatorname{Cycl}(\varphi(Z_i) \times_{\operatorname{Spec}(R)} \operatorname{Spec}(F)) \text{ by Theorem 2.42} \\
&= (p \times_S \operatorname{Spec}(F))_* \underline{x}^*(Z),
\end{aligned}$$

but this is independent of the choice of $\underline{x}$, hence we are done.

2. Given a diagram

$$\begin{tikzcd}
 & \operatorname{Spec}(D) \arrow[r] & S' \arrow[d, "f"] \\
\operatorname{Spec}(F) \arrow[ur] \arrow[dr] & & \\
 & \operatorname{Spec}(R) \arrow[r] & S
\end{tikzcd}$$

of fat points $\underline{y}$ over $\operatorname{Spec}(D)$ and $\underline{x}$ over $\operatorname{Spec}(R)$, we have

$$\begin{aligned}
\underline{y}^* p'_*(\operatorname{Cycl}(f)(Z)) &= (p' \times_{S'} \operatorname{Spec}(F))_* \underline{y}^*(\operatorname{Cycl}((f)(Z))) \text{ by previous part} \\
&= (p \times_S \operatorname{Spec}(F))_* \underline{x}^*(Z) \text{ by Theorem 2.23} \\
&= \underline{x}^*(p_*(Z)) \text{ by previous part.}
\end{aligned}$$

The final statement then follows from Theorem 2.23.

□

Remark 2.44.

- From Proposition 2.43, we note that for any proper map $f : X \to Y$ over Sch/S, there is a map of sheaves

$$f_* : Z_{\text{equi}}(X/S, r) \to Z_{\text{equi}}(Y/S, r)$$

satisfying $(gf)_* = g_* f_*$.

- By the same method, one shows that for any $f : X \to Y$ being flat and of equidimensional n, there is a map of sheaves

$$f^* : Z_{\text{equi}}(Y/S, r) \to Z_{\text{equi}}(X/S, r).$$

In order to establish further properties of $Z_{\text{equi}}(X/S, r)$, we introduce the cdh topology.

2.4 cdh Topology

Definition 2.45. Let $X \in \operatorname{Sch}/k$ and $i : Z \to X$ be a closed immerison, then an abstract blow-up of X with center Z is a proper map $p : X' \to X$ that induces an isomorphism

$$(X' \backslash Z')^{\text{red}} = (X \backslash Z)^{\text{red}}$$

where $Z' := X' \times_X Z$.

Definition 2.46. The cdh topology on Sch/k is the minimal Grothendieck topology generated by Nisnevich coverings, i.e., an étale covering $f : X \to Y$ such that for any $y \in Y$, there exists $x \in X$ such that $f(x) = y$ and $k(x) = k(y)$, along with coverings $X' \coprod Z \to X$ corresponding to the abstract blowups.

Definition 2.47. A proper cdh covering is a proper map that is also a cdh covering.

A proper cdh covering of a reduced scheme is called a proper birational covering if it is an isomorphism over a dense open subset.

Example 2.48. Suppose $Z \subseteq X$ is a closed immersion in Sm/k, then the blow-up $\mathrm{Bl}_Z(X)$ gives rise to a covering $\mathrm{Bl}_Z(X) \coprod Z \to X$. In particular, the fiber of the blow-up given by $p : \mathrm{Bl}_Z(X) \to X$ over Z is the projective scheme of the normal bundle $\mathbb{P}(N_{Z/X}) \to Z$ over Z, which is a cdh covering since it admits sections locally. We base-change to Z and get a cdh covering, then the blow-up still gives a cdh covering, therefore p is a cdh covering. Hence, base-changing p to the blow-up gives a diagonal morphism, therefore p is a cdh covering as well.

Example 2.49. Let $X \in \mathrm{Sch}/k$, then the disjoint union of irreducible components of X also gives a cdh covering of X. Therefore, for arguments over cdh topology, we may assume without loss of generality that the scheme is integral.

In the case where X is integral and $Z \subseteq X$ is a closed subset, suppose $\tilde{X} \to X$ is a cdh covering, then by definition, we can find a closed subset $X' \subseteq \tilde{X}$ such that $X' \to X$ is a birational morphism that is an isomorphism outside of Z. Moreover, $X' \coprod (\tilde{X} \times_X Z) \to X$ is a proper birational cdh covering, and it is a refinement of $\tilde{X} \to X$.

Proposition 2.50. Consider a cdh covering of the form $T \xrightarrow{p} U \xrightarrow{q} X$ in Sch/k where p is a proper cdh cover and q is a Nisnevich cover, then such cdh covering admits a refinement of the form

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[d] & & S \arrow[d, "g"] \\ T \arrow[r, "p"'] & U \arrow[r, "q"'] & X \end{tikzcd}$$

where f is a Nisnevich covering and g is a proper cdh covering.

Proof. We may assume X is integral. Let $\{U_i\}_{i \in I}$ be the set of irreducible components of U, and $T_i = T \times_U U_i$ is the fiber of T over U_i, then we may assume the covering given by $T_i \to U_i$, after refinement, is proper and birational, since our conclusion is determined only up to refinement. By applying platification Theorem 2.16 to $T \to X$, there exists a closed subset Z of X, such that we may construct a blow-up $X' \to X$ along $Z \subseteq X$ with the property that the proper transform T_i' of T_i is flat over X'. Now we set $U_i' = U_i \times_X X'$, then there exists a commutative diagram

$$\begin{tikzcd} T_i \arrow[r] & T_i \times_X X' \arrow[r] \arrow[d] & U_i' \arrow[r, "h"] \arrow[d] & X' \arrow[d] \\ & T_i \arrow[r] & U_i \arrow[r] & X \end{tikzcd}$$

By construction, the right-hand square is Cartesian. Since $T_i \times_X X'$ is the total transform of T_i, then T_i' becomes a closed subset of it. Since $U \to X$ is étale, in particular this is unramified. then its restriction to U_i as well as its base-change $U_i' \to X'$ are both unramified. Since $U_i' \to X'$ is already flat, we conclude that $U_i' \to X'$ is étale. By Theorem 2.16 again, $T_i' \to X'$ is flat, then $g : T_i' \to U_i$ is flat as well. By the construction above, g is also proper and birational, hence g is an isomorphism. Therefore, $T_i' \to X'$ is an étale covering, and in particular it must be a Nisnevich covering, so the base-change $T \times_X X' \to X'$ has a Nisnevich covering given by refinement $\coprod T_i'$.

We proceed by induction on $\dim(X)$. In the case where $\dim(X) = 0$, the statement is trivial. For a general integral scheme X and its platification Z above, we note that $\dim(Z) < \dim(X)$. Now the cover

$$T \times_X Z \to U \times_X Z \to Z$$

admits a refinement

$$V' \to S' \to Z$$

where the first map is a Nisnevich covering, and the second map is a proper cdh covering, given by the inductive assumption above. The required refinement of $T \to X$ is then given by

$$V = V' \coprod \coprod_i T_i' \xrightarrow{f} S = S' \coprod X' \to Z \coprod X' \to X.$$

Indeed, f is Nisnevich covering since both $\coprod_i T_i' \to X'$ and $V' \to S'$ are Nisnevich coverings, and the composite of the other two morphisms is a proper cdh covering. ☐

The following proposition follows immediately.

Proposition 2.51. Every cdh cover of $X \in \mathrm{Sch}/k$ has a refinement of the form

$$U \xrightarrow{q} X' \xrightarrow{p} X,$$

where p is a proper cdh covering and q is a Nisnevich covering.

Corollary 2.52. Let F be a Nisnevich sheaf on Sch/k, then its cdh sheafification $F_{\mathrm{cdh}} = 0$ if and only if for any $a \in F(X)$, there exists a proper cdh covering $p : X' \to X$ such that $p^*(a) = 0$.

Proof. Pullback to the Nisnevich covering, then to the cdh covering. □

Theorem 2.53 (Resolution of Singularity)**.** Suppose k is a field of characteristic 0. Every integral scheme $X \in \mathrm{Sch}/k$ admits a proper birational $Y \to X$ such that $Y \in \mathrm{Sm}/k$. For any $X \in \mathrm{Sm}/k$ and an abstract blow-up $q : X' \to X$, there exists a refinement sequence of blow-ups

$$p : X_n \to X_{n-1} \to \cdots \to X_1 \to X$$

with smooth centers such that p factors through q.

Proof. Refer to [Hir64]. □

Corollary 2.54. Suppose k is a field of characteristic 0, and F is a Nisnevich sheaf on Sch/k. Then the cdh sheafification $F_{\mathrm{cdh}} = 0$ if and only if for any $a \in F(X)$ and $X \in \mathrm{Sm}/k$, there exists a refinement sequence of blow-ups

$$p : X_n \to X_{n-1} \to \cdots \to X_1 \to X$$

with smooth centers, such that $p^*(a) = 0$.

Definition 2.55. Let D^- be the derived category of sheaves with transfers. We define $\mathrm{DM}^- = D^-[(X \times \mathbb{A}^1 \to X)^{-1}]$ to be the category of effective motives.

Recall that there is an adjunction $i \dashv C_*$ given by

$$i : D^- \rightleftarrows \mathrm{DM}^- : C_*$$

between localization and Suslin complex, where for every sheaf F we may define the Suslin complex to be

$$(C_*F)_n = \underline{\mathrm{Hom}}(\Delta^n, F)$$

for the standard simplicial complex $\Delta^n = \mathrm{Spec}(k[x_0, \ldots, x_n]/(\sum_i x_i - 1)) = \mathbb{A}^n$ with differentials $\partial : (C_*F)_n \to (C_*F))_{n-1}$ by $\sum_i (-1)^i d_i$ for the ith face d_i. A map $C_1 \to C_2$ in D^- is called an $\mathbb{A}^1$-weak equivalence if its image under i is an isomorphism. We recall that this is true if and only if its image under C_* is a quasi-isomorphism.

Proposition 2.56. Let $X_r \to X_{r-1} \to \cdots \to X_1 \to X$ to be a sequence of blow-ups along smooth centers with $X \in \mathrm{Sm}/k$, and $C = \mathbb{Z}(X)/\mathbb{Z}(X_r)$ to be the cokernel of the map of sheaves. Then $C = 0$ in DM^-.

Proof. By snake lemma, it suffices to prove this for $r = 1$. Set $X_r = \mathrm{Bl}_Z(X)$. By the universal property of Cartesian product, there is an exact sequence of Nisnevich sheaves

$$\mathbb{Z}(\mathrm{Bl}_Z(X) \times_X \mathrm{Bl}_Z(X)) \xrightarrow{\pi_1 - \pi_2} \mathbb{Z}(\mathrm{Bl}_Z(X)) \xrightarrow{p} \mathbb{Z}(X)$$

by diagram chasing. Since $\mathrm{BL}_Z(X) \times_X \mathrm{BL}_Z(X) = \mathrm{Bl}_Z(X) \coprod (\mathbb{Z}(N_{Z/X}) \times_Z \mathbb{P}(N_{Z/X}))$, as the coproduct of itslef and the self-product of exceptional divisor, then the map

$$\mathbb{Z}(\mathrm{Bl}_Z(X)) \oplus \mathbb{Z}(\mathbb{P}(N_{Z/X}) \times_Z \mathbb{P}(N_{Z/X})) \to \mathbb{Z}(\mathrm{Bl}_Z(X) \times_X \mathrm{Bl}_Z(X))$$

is a surjection. We obtain a diagram of exact rows

$$\begin{tikzcd}
\mathbb{Z}(\mathbb{P}(N_{Z/X}) \times_Z \mathbb{P}(N_{Z/X})) \arrow[r] & \mathbb{Z}(\mathrm{Bl}_Z(X)) \arrow[r, "p"] & \mathbb{Z}(X) \\
\mathbb{Z}(\mathbb{P}(N_{Z/X}) \times_Z \mathbb{P}(N_{Z/X})) \arrow[r] \arrow[u, equal] & \mathbb{Z}(\mathbb{P}(N_{Z/X})) \arrow[r, "q"'] \arrow[u, hook] & \mathbb{Z}(Z) \arrow[u, hook]
\end{tikzcd}$$

with injections given by closed immersions, therefore $\ker(p) = \ker(q)$. For any Zariski covering $U_1 \cup U_2 = X$, we consider the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r, tail] & \ker(p_1) \oplus \ker(p_2) \arrow[rr] \arrow[dr] \arrow[dd, "r"'] & & \mathbb{Z}(\mathrm{Bl}_Z(X)|_{U_1}) \oplus \mathbb{Z}(\mathrm{Bl}_Z(X)|_{U_2}) \arrow[rr, "\pi_1 \oplus \pi_2"] \arrow[dd] & & \mathbb{Z}(U_1 \oplus U_2) \arrow[dd] \\
& & \mathbb{P}(N_{Z/X})|_{U_1} \oplus \mathbb{P}(N_{Z/X})|_{U_2} \arrow[ur] \arrow[rr, "q_1 \oplus q_2"] \arrow[dd] & & \mathbb{Z}(Z \cap U_1) \oplus \mathbb{Z}(Z \cap U_2) \arrow[ur] \arrow[dd] & \\
0 \arrow[r] & \ker(p) \arrow[rr] \arrow[dr] & & \mathbb{Z}(\mathrm{Bl}_Z(X)) \arrow[rr, "p"] & & \mathbb{Z}(X) \\
& & \mathbb{Z}(N_{Z/X}) \arrow[ur] \arrow[rr, "q"] & & \mathbb{Z}(Z) \arrow[ur] &
\end{tikzcd}$$

Since $\ker(p) = \ker(q)$, we may work locally on the exceptional divisor, then the map from the kernel factors over the exceptional divisor. We obtain the commutative diagram above. Since q is the structure map of a projective bundle, then it admits local sections, and so do q_1 and q_2, therefore all three of them must be surjections. By the snake lemma, r is a surjection as well. Therefore, to show that $C = 0$, the problem is really local and we get to localize. Since $C = (X/(X \backslash Z))/(\mathrm{Bl}_Z(X)/(X \backslash Z))$ is a quotient of Thom spaces, then by étale excision, we may reduce to the trivial case where $X = T \times \mathbb{A}^d$ and $Z = T \times \{0\}$ is the zero section of X. In this case, the map

$$\mathrm{Bl}_Z(X) \to \mathbb{A}^d_T \times_T \mathbb{P}^{d-1}_T \to \mathbb{P}^{d-1}_T$$

is an $\mathbb{A}^1$-bundle with a section. Consider the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \ker(p) \arrow[r] & \mathrm{Bl}_Z(X) \arrow[r] & K & 0 \\
0 \arrow[r] & \ker(q) \arrow[r] \arrow[u, equal] & T \times \mathbb{P}^{d-1} \arrow[r] \arrow[u] & T \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

On the level of DM^-, this is a morphism between two distinguished triangles. Recall from above that the blow-up is an $\mathbb{A}^1$-bundle, therefore on the level of DM^- this is an isomorphism, since it is trivial locally, i.e., of the form $X \times \mathbb{A}^1 \simeq \mathbb{A}^1$. Therefore, the middle map of the diagram above is an $\mathbb{A}^1$-weak equivalence, and so is the map $T \to K$. There is also a diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & X \arrow[r] & C \arrow[r] & 0 \\
0 \arrow[r] & T \arrow[r, equal] \arrow[u] & T \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

where $T \to X$ is a zero section of the trivial vector bundle, but that corresponds to an $\mathbb{A}^1$-weak equivalence, hence $0 \to C$ is also an $\mathbb{A}^1$-weak equivalence, thus $C = 0$ in DM^-. ☐

Theorem 2.57. Suppose that $\mathrm{char}(k) = 0$, and $F \in \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}/k)$ is a Nisnevich sheaf with transfers, and that its cdh sheafification $F_{\mathrm{cdh}} = 0$. Then $F = 0$ in DM^-.

Proof. By the hyper Ext spectral sequence, it suffices to show that $\mathrm{Ext}^n(F, H) = 0$ for all n and any homotopy-invariant sheaf H, i.e., $H(X) = H(X \times \mathbb{A}^1)$, since it converges to the hypercohomology. For a detailed explaination of this, see Proposition 5.48 and 5.49 of my other set of notes. We proceed by induction on n. Consider the surjection

$$\pi : \bigoplus_{a \in F(X)} \mathbb{Z}(X) \to F.$$

Since $F_{\text{cdh}} = 0$, by Corollary 2.54 we know that π factors thorugh $\bigoplus_\alpha C_\alpha \to F$, where each C_α is the cokernel as in Proposition 2.56. We know $C_\alpha = 0$ in DM^-, then so is its cdh sheafification. Set $K = \ker(p)$, so $K_{\text{cdh}} = 0$ as well. We conclude that $\operatorname{Ext}^n(C_\alpha, H) = 0$ by Proposition 2.56, and we are done by applying the inductive hypothesis. ☐

Theorem 2.58. Suppose that $\operatorname{char}(k) = 0$, and the complex $C \in C^-(\mathrm{Sh}_{\text{Nis}}(\mathrm{Sm}/k))$ is $\mathbb{A}^1$-local.[8] Then we have an isomorphism

$$\mathbb{H}^n_{\text{Zar}}(X, C) = \mathbb{H}^n_{\text{Nis}}(X, C) \cong \mathbb{H}^n_{\text{cdh}}(X, C_{\text{cdh}})$$

for $n \in \mathbb{Z}$ and $X \in \mathrm{Sm}/k$.

Proof. Suppose $C = F$ is a homotopy-invariant sheaf with transfers. For any cdh cover $X' \to X$, set C to be its cokernel. There is an exact sequence

$$0 \longrightarrow \operatorname{Hom}(C, F) \longrightarrow F(X) \longrightarrow F(X')$$

but $C_{\text{cdh}} = 0$, so $\operatorname{Hom}(C, F) = 0$ by Theorem 2.57. We conclude that $F(X) \to F(X')$ is an injection, so $F(X) \to F_{\text{cdh}}(X)$ is an injection as well. On the other hand, consider the exact sequence

$$0 \longrightarrow F \longrightarrow F_{\text{cdh}} \longrightarrow G \longrightarrow 0$$

By Theorem 2.57, we know $\operatorname{Ext}^1(G, F) = 0$, so the sequence splits, hence $G = 0$, so F is a sheaf of cdh topology. Let $F \to I^*$ be an injective resolution of F over cdh topology, then the restriction $I^*|_{\text{Nis}}$ over Nisnevich topology is injective as well, c.f., [Mil80, Proposition 2.12]. By the discussion above, $(F_{\text{cdh}})_{\text{Nis}} = F$, so it suffices to show that $I^*|_{\text{Nis}}$ is exact. Let $H^i(I^*|_{\text{Nis}}) = Z^i/B^i$, then we need to show that $H^i = 0$ for all $i \geqslant 0$. The case where $i = 0$ is trivial. Now suppose $H^j = 0$ for $j < i$. We know $(H^i)_{\text{cdh}} = 0$, so by Theorem 2.57, we have

$$\begin{aligned} 0 &= \operatorname{Ext}^{i+1}(H^i, F) \\ &= \operatorname{Ext}^i(H^i, B^1) \\ &= \operatorname{Ext}^i(H^i, Z^1) \\ &= \operatorname{Ext}^i(H^i, B^2) \\ &= \cdots \\ &= \operatorname{Ext}^1(H^i, B^i). \end{aligned}$$

Therefore, the sequence

$$0 \longrightarrow B^i \longrightarrow Z^i \longrightarrow H^i \longrightarrow 0$$

splits. Since Z^i is a cdh sheaf, then so is H^i, therefore $H^i = 0$. The general case follows from hypercohomological spectral sequence. ☐

Definition 2.59. The Suslin-Friedlander motivic complex is defined by

$$\mathbb{Z}^{\text{SF}}(i) = C_* Z_{\text{equi}}(\mathbb{A}^i, 0)[-2i].$$

Theorem 2.60. Let k be a field of characteristic 0, $X \in \mathrm{Sch}/k$, and $Y \subseteq X$ be a closed subscheme, and $X = U_1 \cup U_2$ is a Zariski covering. Then we have distinguished triangles

$$Z_{\text{equi}}(Y, r) \longrightarrow Z_{\text{equi}}(X, r) \longrightarrow Z_{\text{equi}}(X \backslash Y, r) \longrightarrow Z_{\text{equi}}(Y, r)[1]$$

$$Z_{\text{equi}}(X, r) \longrightarrow Z_{\text{equi}}(U_1, r) \oplus Z_{\text{equi}}(U_2, r) \longrightarrow Z_{\text{equi}}(U_1 \cap U_2, r) \longrightarrow Z_{\text{equi}}(X, r)[1]$$

in DM^-.

[8] That is, for any $\mathbb{A}^1$-weak equivalence $K \to L$, we have $\operatorname{Hom}_{D^-}(L, C) = \operatorname{Hom}_{D^-}(K, C)$, which is true if and only if $C \to C_* C$ is a quasi-isomorphism. In particular, DM^- is equivalent to the category of $\mathbb{A}^1$-local complexes.

Proof. We have exact sequences

$$0 \longrightarrow Z_{\text{equi}}(Y,r) \longrightarrow Z_{\text{equi}}(X,r) \xrightarrow{\quad i \quad} Z_{\text{equi}}(X \backslash Y, r)$$

$$0 \longrightarrow Z_{\text{equi}}(X,r) \longrightarrow Z_{\text{equi}}(U_1,r) \oplus Z_{\text{equi}}(U_2,r) \xrightarrow{j} Z_{\text{equi}}(U_1 \cap U_2, r)$$

To construct the distinguished triangle, it suffices to show that $\operatorname{coker}(i)$ and $\operatorname{coker}(j)$ are zero in DM^-. By Theorem 2.57, it suffices to show that the cokernel has zero cdh sheafification. We will prove that $\operatorname{coker}(i)_{\text{cdh}} = 0$, as the other one follows similarly. It suffices to show that for any $U \in \mathrm{Sm}/k$ and any $Z \in Z_{\text{equi}}(X \backslash Y, r)(U)$, there is a cdh covering $p : U' \to U$ where U' is smooth, such that $\mathrm{Cycl}(p)(Z)$ comes from $Z_{\text{equi}}(X,r)(U')$. One can assume that Z is an irreducible closed subset in $(X \backslash Y) \times U$. Let $\bar{Z}$ be the its closure in $X \times U$, then by platification Theorem 2.16, there is a blow-up $p : U' \to U$ such that the proper transform $\tilde{Z}$ of $\bar{Z}$ with respect to p is flat over U'. Therefore $\tilde{Z}\Big|_{(X \backslash Y) \times U} = \mathrm{Cycl}(p)(Z)$ by the disucssion in Theorem 2.18, setting $\tilde{Z} = \mathrm{Cycl}(p)(\bar{Z})$. □

Proposition 2.61. Let k be a perfect field. There is an isomorphism

$$C_* \mathbb{Z}(\mathbb{G}_m^{\wedge i})[-i] = \mathbb{Z}(i) \cong \mathbb{Z}^{\text{SF}}(i)$$

in DM^- if $\operatorname{char}(k) = 0$.

Proof. We will prove the case where k has characteristic 0. This follows from a comparison between distinguished triangles

$$\begin{tikzcd}
\mathbb{Z}(\mathbb{P}^{i-1}) \arrow[r] \arrow[d, equal] & \mathbb{Z}(\mathbb{P}^i) \arrow[r] \arrow[d] & \mathbb{Z}(i)[2i] \arrow[r] \arrow[d, equal] & \mathbb{Z}(\mathbb{P}^{i-1})[1] \arrow[d, equal] \\
Z_{\text{equi}}(\mathbb{P}^{i-1}, 0) \arrow[r] & Z_{\text{equi}}(\mathbb{P}^i, 0) \arrow[r] & \mathbb{Z}_{\text{equi}}(\mathbb{A}^i, 0) \arrow[r] & Z_{\text{equi}}(\mathbb{P}^{i-1}, 0)[1]
\end{tikzcd}$$

We conclude that the mapping cones are $\mathbb{A}^1$-weak equivalent by Theorem 2.60. □

Definition 2.62. For any $U \in \mathrm{Sm}/k$ and $X \in \mathrm{Sch}/k$, and $r \in \mathbb{N}$, we define $Z_{\text{equi}}(U, X, r)(-) = Z_{\text{equi}}(X, r)(U \times -)$.

Theorem 2.63. Suppose k is a field of characteristic 0, and $U \in \mathrm{Sm}/k$ is quasi-projective, and $X \in \mathrm{Sch}/k$, with $r \in \mathbb{N}$. Then the natural morphism

$$Z_{\text{equi}}(U, X, r) \hookrightarrow Z_{\text{equi}}(X \times U, r + \dim(U))$$

induces a quasi-isomorphism

$$C_*(Z_{\text{equi}}(U, X, r))(\mathrm{Spec}(k)) \to C_*(Z_{\text{equi}}(X \times U, r + \dim(U)))(\mathrm{Spec}(k))$$

of complexes of abelian groups.

Proof. See [FV00, Theorem 7.4]. □

By Theorem 2.58 for a field k of characteristic 0, for every $U \in \mathrm{Sm}/k$, $X \in \mathrm{Sch}/k$, $r \in \mathbb{N}$, and $i \in \mathbb{Z}$, we have

$$\begin{aligned}
\mathbb{H}^i_{\text{cdh}}(U, (C_* \mathbb{Z}_{\text{equi}}(X,r))_{\text{cdh}}) = \mathbb{H}^i_{\text{Nis}}(U, \mathbb{Z}_{\text{equi}}(X,r)) &= \mathbb{H}^i_{\text{Zar}}(U, \mathbb{Z}_{\text{equi}}(X,r)) \\
&= \mathrm{Hom}_{\mathrm{DM}^-}(U, Z_{\text{equi}}(X,r)[i]).
\end{aligned}$$

The following theorem says that the computation of hypercohomology descends to that of the cohomology on the level of sections.

Theorem 2.64 (Descent)**.** Let k be a field of characteristic 0, $U \in \mathrm{Sm}/k$ being quasi-projective, and $X \in \mathrm{Sch}/k$. We have

$$H^i(C_* Z_{\text{equi}}(X,r)(U)) = \mathbb{H}^i(U, C_* Z_{\text{equi}}(X,r)).$$

Proof. By Theorem 2.60, there is a distinguished triangle in D^-

$$C_*Z_{\mathrm{equi}}(X\times V, r+n) \longrightarrow \bigoplus_{i=1}^{2} C_*Z_{\mathrm{equi}}(X\times V_i, r+n) \xrightarrow{\pi} C_*Z_{\mathrm{equi}}(X\times V_3, r+n) \longrightarrow C_*Z_{\mathrm{equi}}(X\times V, r+n)[1]$$

for every open covering $U \supseteq V = V_1 \cup V_2$ and $V_3 = V_1 \cap V_2$. By Mayer-Vietoris property, $\operatorname{coker}(\pi)(\operatorname{Spec}(k))$ is acyclic. By Theorem 2.63 and taking sections on $\operatorname{Spec}(k)$ to the distinguished triangle above, the sequence

$$0 \longrightarrow C_*Z_{\mathrm{equi}}(X,r)(V) \longrightarrow C_*Z_{\mathrm{equi}}(X,r)(V_1) \oplus C_*Z_{\mathrm{equi}}(X,r)(V_2) \xrightarrow{\pi'} C_*Z_{\mathrm{equi}}(X,r)(V_3)$$

has the property that $\operatorname{coker}(\pi')$ is acyclic, i.e., it gives a homotopy Cartesian square

$$\begin{tikzcd} C(V) \arrow[r] \arrow[d] & C(V_1) \arrow[d] \\ C(V_2) \arrow[r] & C(V_3) \end{tikzcd}$$

where $C = C_*Z_{\mathrm{equi}}(X,r)$. By [BG06, Theorem 4], any C of this kind has the property that

$$\mathbb{H}^i(V, C) = H^i(C(V))$$

which concludes the proof. □

2.5 Motivic Cohomology over General Schemes

The following is a generalization of the definition of motivic cohomology on Sm/k.

Definition 2.65. Let $X \in \mathrm{Sch}/k$, $p \in \mathbb{Z}$, and $q \in \mathbb{N}$. We define the motivic cohomology of X to be

$$H^{p,q}(X,\mathbb{Z}) = \mathbb{H}^{p-2q}_{\mathrm{cdh}}(X, (C_*Z_{\mathrm{equi}}(\mathbb{A}^q, 0))_{\mathrm{cdh}}).$$

There is an exact sequence

$$0 \longrightarrow \mathbb{Z}(U_1 \cap U_2) \longrightarrow \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2) \longrightarrow \mathbb{Z}(X) \longrightarrow 0$$

for open covering $X = U_1 \cup U_2$, hence there is a Mayer-Vietoris sequence for $H^{p,q}(X,\mathbb{Z})$.

Theorem 2.66 (Duality)**.** Suppose k is a field of characteristic 0, $X, Y \in \mathrm{Sch}/k$, $U \in \mathrm{Sm}/k$ with $\dim(U) = n$. We have

$$\mathbb{H}^i_{\mathrm{cdh}}(Y \times U, (C_*Z_{\mathrm{equi}}(X,r))_{\mathrm{cdh}}) \cong \mathbb{H}^i_{\mathrm{cdh}}(Y, (C_*Z_{\mathrm{equi}}(X\times U, r+n))_{\mathrm{cdh}}).$$

Proof. Since U is smooth, then consider a finite open covering $\mathcal{U} = \{U_i \to V\}$ of quasi-projective schemes of U. We proceed by induction on the cardinality of the covering, and it then suffices to prove it for the case where $U = U_1 \cup U_2$ as a quasi-projective open covering. In this case, both sides of the statement admit a Mayer-Vietoris sequence, using Theorem 2.60, therefore we may assume U being quasi-projective. Set $D = C_*Z_{\mathrm{equi}}(X,r)$, then by the tensor-hom adjunction we have

$$\mathbb{H}^i_{\mathrm{cdh}}(Y \times U, D_{\mathrm{cdh}}) \cong \mathbb{H}^i_{\mathrm{cdh}}(Y, \mathrm{RHom}(U, D_{\mathrm{cdh}})).$$

We then have a natural map

$$\varphi : \underline{\mathrm{Hom}}(U, D)_{\mathrm{cdh}} \to \underline{\mathrm{Hom}}(U, D_{\mathrm{cdh}}) \to R\underline{\mathrm{Hom}}(U, D_{\mathrm{cdh}}).$$

By Theorem 2.63, the map

$$Z_{\mathrm{equi}}(X,r)(U \times -) = C_*Z_{\mathrm{equi}}(U, X, r) \to C_*Z_{\mathrm{equi}}(X \times U, r+n)$$

is an isomorphism over a field k applying sections. Applying Suslin complex, the Nisnevich sheaf with transfers becomes $\mathbb{A}^1$-local, therefore the homotopy groups are $\mathbb{A}^1$-invariant. From [MVW06, Corollary 11.2], we conclude that this is a quasi-isomorphism, since cohomology sheaves are homotopy-invariant.

Suppose Y is smooth and quasi-projective, then

$$\begin{aligned}\mathbb{H}^i_{\text{cdh}}(Y\times U,(C_*Z_{\text{equi}}(X,r))_{\text{cdh}}) &\cong \mathbb{H}^i_{\text{Zar}}(Y\times U,(C_*Z_{\text{equi}}(X,r))_{\text{cdh}})\\ &\cong H^i(C_*Z_{\text{equi}}(X,r)(Y\times U)) \text{ by Theorem 2.64 over smooth schemes}\\ &\cong H^i(C_*Z_{\text{equi}}(U,X,r)(Y))\\ &\cong \mathbb{H}^i(Y,(C_*Z_{\text{equi}}(U,X,r))_{\text{cdh}}) \text{ by Theorem 2.64 over smooth schemes}\\ &\cong \mathbb{H}^i(Y,(C_*Z_{\text{equi}}(X\times U,r+n))_{\text{cdh}}).\end{aligned}$$

Therefore $\mathbb{H}^i_{\text{cdh}}(Y,\varphi)$ is an isomorphism when Y is smooth. By resolution of singularity, it is also true for any $Y\in \mathrm{Sch}/k$, since for any complex A the presheaves $Y\mapsto \mathbb{H}^i_{\text{cdh}}(Y,A)$ and $Y\mapsto H^i(A(Y))$ have the same sheafification: this means the smooth schemes give a basis that generate all schemes over cdh topology, hence having a general scheme, the conclusion holds after taking sheafification. □

Proposition 2.67. Suppose k is a field of characteristic 0 and $X\in \mathrm{Sch}/k$. Then

1. $H^{p,q}(X,\mathbb{Z})\cong H^{p,q}(X\times\mathbb{A}^1,\mathbb{Z})$;
2. $H^{p,q}(X\times\mathbb{P}^1,\mathbb{Z})\cong H^{p,q}(X,\mathbb{Z})\oplus H^{p-2,q-1}(X,\mathbb{Z})$.

Proof.

1. We have

$$\begin{aligned}H^{p,q}(X\times\mathbb{A}^1,\mathbb{Z}) &\cong \mathbb{H}^{p-2q}(X\times\mathbb{A}^1,C_*Z_{\text{equi}}(\mathbb{A}^q,0)_{\text{cdh}})\\ &\cong \mathbb{H}^{p-2q}(X,C_*Z_{\text{equi}}(\mathbb{A}^{q+1},1)_{\text{cdh}}),\end{aligned}$$

so it suffices to prove a quasi-isomorphism

$$C_*Z_{\text{equi}}(\mathbb{A}^q,0)\simeq C_*Z_{\text{equi}}(\mathbb{A}^{q+1},1).$$

It then suffices to prove this after applying $\mathbb{H}^*(X,-)$. This is true for smooth X since they satisfy homotopy-invariance already, so then taking sheafification, the result holds for general schemes.

2. There is a Gysin triangle (with compact support)

$$Z_{\text{equi}}(\mathbb{A}^q,1)\cong Z_{\text{equi}}(\mathbb{A}^{q-1},0)\longrightarrow Z_{\text{equi}}(\mathbb{A}^q\times\mathbb{P}^1,1)\longrightarrow Z_{\text{equi}}(\mathbb{A}^{q+1},1)\longrightarrow Z_{\text{equi}}(\mathbb{A}^q,1)[1]\cong Z_{\text{equi}}(\mathbb{A}^q,0)$$

in DM^- by Theorem 2.60. After making identifications above using part 1, we note that the second morphism admits a section, therefore the Gysin triangle splits in DM^-, and we have

$$\begin{aligned}H^{p,q}(X\times\mathbb{P}^1,\mathbb{Z}) &\cong \mathbb{H}^{p-2q}_{\text{cdh}}(X\times\mathbb{P}^1,(C_*Z_{\text{equi}}(\mathbb{A}^q,0))_{\text{cdh}})\\ &\cong \mathbb{H}^{p-2q}_{\text{cdh}}(X,(C_*Z_{\text{equi}}(\mathbb{A}^q\times\mathbb{P}^1,1))_{\text{cdh}})\\ &\cong H^{p,q}(X,\mathbb{Z})\oplus H^{p-2,q-1}(X,\mathbb{Z})\end{aligned}$$

where the last isomorphism follows from the splitting above and the definition of motivic cohomology.

□

3 Motivic Steenrod Operations

From now on, we will assume k to be a field of characteristic 0. Similar constructions can be done in the case of positive characteristics, but using a different approach. We note that this is different from Voevodsky's construction in [Voe03], but the results below are comparable with references.

3.1 Equivariant Cohomology of Finite Groups

Let G be a finite group and V be a linear representation of G, with a Zariski open subset $U \subseteq V$ such that G acts freely on U.

Proposition 3.1. Suppose $X, Y \in \mathrm{Sch}/k$ are quasi-projective G-schemes, then $X \times_G U$ exists as a scheme. Suppose further that $f : X \to Y$ is G-equivariant with a property $\mathcal{P}$, which is one of the following:

i. proper;

ii. flat;

iii. smooth;

iv. regular embedding;

v. locally complete intersection, i.e., a composition of a regular embeeding followed by a smooth morphism;

vi. vector bundle.

Then $f_G : X \times_G U \to Y \times_G U$ has property $\mathcal{P}$ as well.

Proof. Refer to [EG96, Section 2, 6]. □

Proposition 3.2. Denote $Z = V \backslash U$ such that $\mathrm{codim}_V(Z) > q$, and that either

i. $X \in \mathrm{Sm}/k$ is a quasi-projective G-scheme, or

ii. $X \in \mathrm{Sch}/k$ is a quasi-projective G-scheme with trivial G-action.

Then $H^{p,q}(X \times_G U, \mathbb{Z})$ is independent of the choice of U.

Proof. Suppose $V \subseteq V$ is an open immersion in Sm/k with $U = V \backslash Z$. We have a distinguished Gysin triangle

$$C_* Z_{\mathrm{equi}}(\mathbb{A}^q \times Z, \dim(V))_{\mathrm{cdh}} \to C_* Z_{\mathrm{equi}}(\mathbb{A}^q \times V, \dim(V))_{\mathrm{cdh}} \to C_* Z_{\mathrm{equi}}(\mathbb{A}^q \times U, \dim(V))_{\mathrm{cdh}} \to C_* Z_{\mathrm{equi}}(\mathbb{A}^q \times Z, \dim(V))_{\mathrm{cdh}}[1]$$

in D^-_{cdh}, given by Theorem 2.60. Hence, for any $Y \in \mathrm{Sm}/k$, there is a long exact sequence

$$\mathbb{H}^{2p-q}_{\mathrm{cdh}}(Y, C_* Z_{\mathrm{equi}}(\mathbb{A}^q \times Z, \dim(V))_{\mathrm{cdh}}) \to H^{p,q}(Y \times V, \mathbb{Z}) \to H^{p,q}(Y \times U, \mathbb{Z}) \to \mathbb{H}^{2p-q+1}_{\mathrm{cdh}}(Y, C_* Z_{\mathrm{equi}}(\mathbb{A}^q \times Z, \dim(V))_{\mathrm{cdh}})$$

By Theorem 2.64 and Theorem 2.66, we have

$$\mathbb{H}^i_{\mathrm{cdh}}(Y, C_* Z_{\mathrm{equi}}(\mathbb{A}^q \times Z, \dim(V))_{\mathrm{cdh}}) \cong H^i(C_* Z_{\mathrm{equi}}(\mathbb{A}^q \times Z \times Y, \dim(V) + \dim(Y))(\mathrm{Spec}(k))),$$

which vanishes because all cycles vanish by definition whenever $q < \mathrm{codim}_V(Z)$. Therefore, we have

$$H^{p,q}(Y \times U, \mathbb{Z}) \cong H^{p,q}(Y \times V, \mathbb{Z})$$

if $q < \mathrm{codim}_V(Z)$ for any $Y \in \mathrm{Sch}/k$, using the same sheafification argument.

Now let (V_1, U_1) and (V_2, U_2) be two representations satisfying the given condition. Consider $W = (U_1 \times V_2) \cup (V_1 \times U_2)$, which is an open subset of $V_1 \times V_2$, and set $Z_1 = U_1 \times (V_2 \backslash U_2)$. Since $\mathrm{codim}_{V_2}(U_2) > q$, then $\mathrm{codim}_W(Z_1) > q$, so taking complement and by the discussion above, we conclude that $H^{*q}(X \times_G W, \mathbb{Z}) \cong H^{*q}(X \times_G (V_1 \times U_2), \mathbb{Z})$. By Proposition 3.1, since V_1 is an affine space, then

$$X \times_G (V_1 \times U_2) \to X \times_G U_2$$

is a vector bundle, therefore they are homotopic. By homotopy-invariance, we have

$$H^{*q}(X \times_G U_2, \mathbb{Z}) \cong H^{*q}(X \times_G (V_1 \times U_2), \mathbb{Z}) \cong H^{*q}(X \times_G W, \mathbb{Z}).$$

Substituting U_2 by U_1, one can show that

$$H^{*q}(X \times_G U_1, \mathbb{Z}) \cong H^{*q}(X \times_G W, \mathbb{Z}).$$

□

By considering V^n, the number $\operatorname{codim}_Z(V^n)$ can be arbitrarily large. The invariant $H^{p,q}(X \times_G U, \mathbb{Z})$ defined before is therefore well-defined.

Definition 3.3. We define the G-equivariant motivic cohomology $H_G^{p,q}(X, \mathbb{Z})$ to be $H^{p,q}(X \times_G U, \mathbb{Z})$. To compute this cohomology, we need a linear representation with big enough codimension. Any U such that $\operatorname{codim}_Z(V) > q$ is called an EG-model.

Now suppose that the base field k contains all roots of $x^\ell = 1$ for some prime ℓ, then the roots of unity $\mu_\ell = \mathbb{Z}/\ell\mathbb{Z}$ as ëtale sheaves. Each $\mathbb{A}^n \backslash \{0\}$ is then an $E\mu_\ell$-model for $H_{\mu_\ell}^{*,n}$, i.e., as a model for the classifying space, by diagonal multiplication by a primitive root, with

$$(\mathbb{A}^n \backslash \{0\})/\mu_\ell \cong \mathcal{O}_{\mathbb{P}^{n-1}}(\ell)^\times.$$

Quotient by the μ_ℓ-action, the model space is equivalent (as schemes) to the non-zero section of the total space considered by ℓth tensor of $\mathcal{O}(1)$ over $\mathbb{P}^{n-1}$. We then define

$$B\mu_\ell = E\mu_\ell/\mu_\ell \cong \varinjlim_n (\mathbb{A}^n \backslash \{0\})/\mu_\ell \cong \varinjlim_n \mathcal{O}_{\mathbb{P}^{n-1}}(\ell)^\times.$$

In the sense of motives, this is a limit of sheaves. There is now a natural map $B\mu_\ell \to \mathbb{P}^\infty$, since for each n in the limit, there is a map towards $\mathbb{P}^n$.

We can now compute the $\mathbb{Z}/\ell\mathbb{Z}$-motivic cohomology of $B\mu_\ell$. For a systematic treatment of this computation, see [Voe03, Section 6].

Proposition 3.4. We have

$$\mathbb{Z}/\ell\mathbb{Z}(B\mu_\ell) \cong \bigoplus_{i \geqslant 0} \mathbb{Z}/\ell\mathbb{Z}(i)[2i] \oplus \bigoplus_{i \geqslant 0} \mathbb{Z}\ell\mathbb{Z}(i+1)[2i+1]$$

in DM^-. Here $\mathbb{Z}/\ell\mathbb{Z}(i)[j]$ is defined using the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ \ell\ } \mathbb{Z} \longrightarrow \mathbb{Z}/\ell\mathbb{Z} \longrightarrow 0$$

with $\mathbb{Z}(1)[1] = \mathbb{G}_m^{\wedge 1}$, and $\mathbb{Z}/\ell\mathbb{Z}(i)[j] = \mathbb{Z}/\ell\mathbb{Z} \otimes \mathbb{Z}(i)[j]$.

In terms of cohomological classes, this is given by (v^i, uv^i), as a cohomological decomposition of lens space. Using the map $B\mu_\ell \to \mathbb{P}^\infty$, here $v \in H^{2,1}(B\mu_\ell, \mathbb{Z}/\ell\mathbb{Z})$ is the pullback of the Chern class $c_1(\mathcal{O}(1)) \in H^{2,1}(\mathbb{P}^\infty, \mathbb{Z}/\ell\mathbb{Z})$ as generator, and $u \in H^{1,1}(B\mu_\ell, \mathbb{Z}/\ell\mathbb{Z})$ is the unique element such that $\beta(u) = v$ and the restriction $u|_{\mathrm{pt}} = 0$ on rational point, where mod-ℓ Bockstein β is given by

$$H^{p,q}(X, \mathbb{Z}/\ell\mathbb{Z}) \to H^{p+1,q}(X, \mathbb{Z}/\ell^2\mathbb{Z}) \to H^{p+1,q}(X, \mathbb{Z}/\ell\mathbb{Z}).$$

Proof. This coincides with the computation of cohomology for lens space. We have a Gysin triangle of motives, i.e., an open immersion followed by a closed immersion,

$$\begin{tikzcd}
\mathbb{Z}/\ell\mathbb{Z}(B\mu_\ell) \arrow[r] & \mathbb{Z}/\ell\mathbb{Z}(\mathcal{O}_{\mathbb{P}^\infty}(\ell)) \arrow[r] \arrow[d, equal] & \mathbb{Z}/\ell\mathbb{Z}(\mathbb{P}^\infty)(1)[2] \arrow[r] & \mathbb{Z}/\ell\mathbb{Z}(B\mu_\ell)[1] \\
& \mathbb{Z}/\ell\mathbb{Z}(\mathbb{P}^\infty) \arrow[ur, "0"'] & &
\end{tikzcd}$$

Since $\mathbb{Z}/\ell\mathbb{Z}(\mathcal{O}_{\mathbb{P}^\infty}(\ell))$ is a vector bundle over $\mathbb{P}^\infty$, it is $\mathbb{A}^1$-homotopy equivalent to $\mathbb{Z}/\ell\mathbb{Z}(\mathbb{P}^\infty)$, the limit of motives of $\mathbb{P}^n$ over n. The corresponding morphism is then given by $c_1(\mathcal{O}(\ell))$, but this is just $\ell c_1(\mathcal{O}(1))$, and in characteristic ℓ this is just the zero map. Therefore, the Gysin triangle above splits. By the projective bundle theorem, we have an isomorphism

$$\bigoplus_{i=0}^{\infty} c_1(\mathcal{O}(1))^i : \mathbb{Z}/\ell\mathbb{Z}(\mathbb{P}^\infty) \cong \bigoplus_{i=0}^{\infty} \mathbb{Z}/\ell\mathbb{Z}(i)[2i]$$

given by the identification

$$\mathrm{Hom}_{\mathrm{DM}^-}(\mathbb{P}^\infty, \mathbb{Z}/\ell\mathbb{Z}(i)[2i]) \cong \mathrm{CH}^i(\mathbb{P}^\infty)/\ell \cong \mathbb{Z}/\ell\mathbb{Z} \cdot c_1(\mathcal{O}(1))^\ell.$$

Therefore

$$\begin{aligned}\mathbb{Z}/\ell\mathbb{Z}(B\mu_\ell) &\cong \mathbb{Z}/\ell\mathbb{Z}(\mathbb{P}^\infty) \oplus \mathbb{Z}/\ell\mathbb{Z}(\mathbb{P}^\infty)(1)[1] \\ &\cong \bigoplus_{i\in\mathbb{N}} \mathbb{Z}/\ell\mathbb{Z}(i)[2i] \oplus \bigoplus_{i\in\mathbb{N}} \mathbb{Z}/\ell\mathbb{Z}(i+1)[2i+1].\end{aligned}$$

We note that when $i = 0$, there is a term $\mathbb{Z}/\ell\mathbb{Z}(1)[1]$ in $\bigoplus_{i\in\mathbb{N}} \mathbb{Z}/\ell\mathbb{Z}(i+1)[2i+1]$. Here we set $u \in H^{1,1}(B\mu_\ell, \mathbb{Z}/\ell\mathbb{Z}) \cong \mathrm{Hom}(B\mu_\ell, \mathbb{Z}(1)[1])$ to be the element corresponding to $(0,1)$ in the decomposition identified above.[9] After pulling back, we see that $u|_{\mathrm{pt}} = 0$. To define v, we need to show that $\beta(u) \neq 0$. Suppose that $\beta(u) = 0$, then u comes from a class $w \in H^{1,1}(B\mu_\ell, \mathbb{Z}/\ell^2\mathbb{Z})$ from the defining sequence. We then have a diagram

$$\begin{array}{ccccc} H^{1,1}(B\mu_\ell, \mathbb{Z}/\ell^2\mathbb{Z}) & \xrightarrow{\partial} & H^{0,0}(\mathbb{P}^\infty, \mathbb{Z}/\ell^2\mathbb{Z}) \cong \mathbb{Z}/\ell^2\mathbb{Z} & \xrightarrow{c_1(\mathcal{O}(\ell))} & H^{2,1}(\mathbb{P}^\infty, \mathbb{Z}/\ell^2\mathbb{Z}) \\ \downarrow & & \downarrow & & \\ H^{1,1}(B\mu_\ell, \mathbb{Z}/\ell\mathbb{Z}) & \xrightarrow[\partial]{} & H^{0,0}(\mathbb{P}^\infty, \mathbb{Z}/\ell\mathbb{Z}) & & \end{array}$$

To build this, we take $\partial : H^{1,1}(B\mu_\ell, \mathbb{Z}/\ell^2\mathbb{Z}) \to H^{0,0}(\mathbb{P}^\infty, \mathbb{Z}/\ell^2\mathbb{Z})$ to be the boundary map of the Gysin triangle for $\mathbb{Z}/\ell^2\mathbb{Z}$, which lands in $\mathbb{P}^\infty$, which is a closed subset of $\mathcal{O}_{\mathbb{P}^\infty}(\ell)$. Multiplying by the Chern class of $\mathcal{O}(\ell)$, we then land in $H^{2,1}(\mathbb{P}^\infty, \mathbb{Z}/\ell^2\mathbb{Z})$, giving a short exact sequence. On the level of $\mathbb{Z}/\ell\mathbb{Z}$, we have a similar boundary map of the Gysin triangle that commutes using the definition of Bockstein morphism. By definition, the image of u under ∂ is 1. Set the image of w under ∂ to be $t \in H^{0,0}(\mathbb{P}^\infty, \mathbb{Z}/\ell^2\mathbb{Z})$. Since the top row is an exact sequence, we know its image in $H^{2,1}(\mathbb{P}^\infty, \mathbb{Z}/\ell^2\mathbb{Z})$ is zero; we also know its image in $H^{0,0}(\mathbb{P}^\infty, \mathbb{Z}/\ell\mathbb{Z})$ is zero since the diagram commutes. Therefore, $t \equiv 1 \pmod{\ell}$ and $\ell t \equiv 0 \pmod{\ell}^2$, which is a contradiction. Therefore, $\beta(u) \neq 0$. We conclude that $H^{2,1}(B\mu_\ell, \mathbb{Z}/\ell\mathbb{Z})$ is a one-dimensional space, so there exists some $\lambda \in \mathbb{Z}/\ell\mathbb{Z}$ such that $\beta(\lambda u) = v$, and we note that $(\lambda u)|_{\mathrm{pt}} = 0$ again. Finally, such choice of λ is unique since the space is one-dimensional. From the decomposition above given by the projective bundle theorem, we know that

$$H^{2n,n}(B\mu_\ell, \mathbb{Z}/\ell\mathbb{Z}) \cong H^{2n,n}(\mathbb{P}^\infty, \mathbb{Z}/\ell\mathbb{Z}) \cong \mathbb{Z}/\ell\mathbb{Z} \cdot v^n,$$

and applying the hom group on the decomposition gives

$$H^{2n+1,n+1}(B\mu_\ell, \mathbb{Z}/\ell\mathbb{Z}) \cong \mathbb{Z}/\ell\mathbb{Z} \oplus H^{1,1}(k, \mathbb{Z}/\ell\mathbb{Z}) \cdot H^{2n,n}(\mathbb{P}^\infty, \mathbb{Z}/\ell\mathbb{Z}).$$

We then conclude that

$$H^{2n+1,n+1}(B\mu_\ell, \mathbb{Z}/\ell\mathbb{Z}) \cong \mathbb{Z}/\ell\mathbb{Z} \cdot (uv^n) \oplus H^{1,1} \cdot v^n$$

where uv^n and $\beta(uv^n)$ are non-zero. □

We have given a direct sum decomposition for the motivic cohomology of lens space. Tensoring it by $\mathbb{Z}/\ell\mathbb{Z}(X)$, the corresponding motives are given by taking direct product with the scheme X, therefore this becomes a decomposition for that of $X \times B\mu_\ell$ instead. We thereby obtain a group structure for $H^{**}(X \times B\mu_\ell, \mathbb{Z}/\ell\mathbb{Z})$.

Similar to the computation of the lens space before, we do need to describe the ring structure. It suffices to compute u^2, since taking product with v^n only provides a free summand. If $\ell \neq 2$, then $u \cdot u = (-1)u \cdot u$, therefore $u^2 = 0$ by

[9] After applying cancellation theorem, the endomorphism of $\mathbb{Z}/\ell\mathbb{Z}(1)[1]$ is just $\mathbb{Z}/\ell\mathbb{Z}$.

graded commutativity. We can now suppose $\ell = 2$. Since k has characteristic 0, then the universal coefficient theorem shows that

$$H^{0,1}(k, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \cdot \tau,$$

where τ is called the motivic Bott element. One can show easily that

$$\beta(\tau) = \rho := -1 \in H^{1,1}(k, \mathbb{Z}/2\mathbb{Z}) \cong k^*/2.$$

Proposition 3.5. Suppose k is a field of characteristic 0, and set $\ell = 2$, then $u^2 = \tau v + \rho u$.

We note that in topology, $u^2 = v$, different from the case in algebraic geometry. In some sense, taking complex realization gives $\rho = 0$.

Proof. We have

$$u^2 \in H^{2,2}(B\mu_\ell, \mathbb{Z}/2\mathbb{Z}) \cong H^{0,1}(k, \mathbb{Z}/2\mathbb{Z}) \cdot u \oplus H^{1,1}(k, \mathbb{Z}/2\mathbb{Z}) \cdot u \oplus H^{2,2}(k, \mathbb{Z}/2\mathbb{Z})$$

where we think of the last term to be the pullback of a rational point. Since $u^2\big|_{\mathrm{pt}} = 0$, then the projection to the last factor is zero. Therefore, it suffices to consider the projection to the first two components. To compute the second factor, we use the inclusion (as closed immersion) $i : \mathcal{O}_{\mathbb{P}^1}(\ell)^\times \to \mathcal{O}_{\mathbb{P}^\infty}(\ell)^\times$, where $\mathcal{O}_{\mathbb{P}^1}(\ell)^\times \cong \mathbb{G}_m$. This is an isomorphism on the second factor, therefore it recovers the term by considering the decomposition after pulling back along i^*, i.e., computing $i^*(u)^2$. Consider $i^*(u) \in H^{1,1}(\mathbb{G}_m, \mathbb{Z}/\ell\mathbb{Z})$. We have an identification $H^{1,1}(\mathbb{G}_m, \mathbb{Z}/\ell\mathbb{Z}) \cong \mathcal{O}^*(\mathbb{G}_m)/\ell$. Since $\mathbb{G}_m$ is given by regular functions, they must be of the form $\frac{f(t)}{t^n}$, and such functions are invertible, i.e., considered in $\mathcal{O}^*(\mathbb{G}_m)$, if and only if $f(t) = \lambda \cdot t^n$ for some λ, which means elements of $\mathcal{O}^*(\mathbb{G}_m)$ are of the form $\lambda \cdot t^n$ for $n \in \mathbb{Z}$. This allows us to identify it with $k^*/\ell \oplus \mathbb{Z}/\ell\mathbb{Z}$ in terms of the constant and the degree. There is now an inclusion

$$H^{1,1}(\mathbb{G}_m, \mathbb{Z}/\ell\mathbb{Z}) \hookrightarrow H^{1,1}(\mathrm{Spec}(k[t]), \mathbb{Z}/\ell\mathbb{Z}) \cong k(t)^*/\ell.$$

Note that $\mathbb{G}_m$ has a decomposition $\mathbb{G}_m \cong \mathbb{Z}/\ell\mathbb{Z} \oplus \mathbb{Z}/\ell\mathbb{Z}(1)[1]$, where $\mathbb{Z}/\ell\mathbb{Z}$ corresponds to $k^\times$ and $\mathbb{Z}/\ell\mathbb{Z}(1)[1]$ corresponds to the degree n of t^n. Recall that u corresponds to $(0,1)$, then under this new decomposition u corresponds to t. Therefore, the image of $i^*(u)$ in $k^*/\ell \oplus \mathbb{Z}/\ell\mathbb{Z}$ is the element $(1,1)$, and therefore its image under the inclusion into $H^{1,1}(\mathrm{Spec}(k[t]), \mathbb{Z}/\ell\mathbb{Z}) \cong k(t)^*/\ell$ must be t. Let $j : \mathrm{Spec}(k(t)) \to \mathbb{G}_m$ be the generic point of $\mathbb{G}_m$. We see that $j^*i^*(u) = [t] \in K_1^M(k(t))/2$, but $[t][t] = [-1][t]$ using the identification $H^{2,2}(\mathrm{Spec}(k(t)), \mathbb{Z}/2\mathbb{Z}) \cong K_2^M(k(t))/2$, therefore $j^*i^*(u)^2 = \rho \cdot j^*i^*(u)$ for $\rho = [-1]$, and we conclude that the second factor must be ρ. Therefore, $u^2 = av + \rho u$ for some a. By applying β, we have that $0 = \beta(a)v + \rho v$, so v is a non-zero-divisor, therefore $\beta(a) = \rho$. Taking $k = \mathbb{Q}$, we see that $\rho \neq 0$, therefore $a \neq 0$, so $a = \tau$. But that means $a = \tau$ for any field k of characteristic 0, since it is invariant under base-change. □

This concludes our computation. Compare this to [Voe03, Theorem 6.10].

Corollary 3.6. For any $X \in \mathrm{Sm}/k$ where k is a field of characteristic 0, we have

$$H^{**}(X \times B\mu_\ell, \mathbb{Z}/\ell\mathbb{Z}) \cong \begin{cases} H^{**}(X, \mathbb{Z}/\ell\mathbb{Z})[u,v]/(u^2 - \tau v - \rho u), & \ell = 2 \\ H^{**}(X, \mathbb{Z}/\ell\mathbb{Z})[u,v]/u^2, & \ell \neq 2 \end{cases}.$$

We now consider the following generalized class of schemes, no longer necessarily an open subset of a linear representation of G.

Definition 3.7. For a finite group G, we define $\mathcal{E}G$ to be the category whose objects are smooth quasi-projective schemes with free G-actions, and morphisms are the maps between them that are G-equivariant.

Definition 3.8. For any $X \in \mathrm{Sch}/k$ with G-action that is described in Proposition 3.2, we define a functor

$$\begin{aligned} \alpha_{p,q}^X : \mathcal{E}G^{\mathrm{op}} &\to \mathrm{Ab} \\ E &\mapsto H^{p,q}(E \times_G X, \mathbb{Z}). \end{aligned}$$

Lemma 3.9. Let $U \subseteq V$ and $Z = V \backslash U$ as before, with $\operatorname{codim}_Z(V) \geqslant r$ and $E \in \mathcal{E}G$. Let $\pi : U \times E \to E$ be the projection, then

$$\pi^* : \alpha_{p,q}^X(E) \to \alpha_{p,q}^X(U \times E)$$

is an isomorphism whenever $r > q$.

Proof. Since V is an affine space, then the map $(V \times E \times X)/G \to E \times_G X$ is a vector bundle. Therefore we have

$$H^{p,q}(E \times_G X, \mathbb{Z}) \cong H^{p,q}((V \times E \times X)/G, \mathbb{Z}).$$

Note that the latter is isomorphic to $H^{p,q}((U \times E \times X)/G)$ whenever $r > q$. □

Proposition 3.10. There is an isomorphism

$$c : \varprojlim_{E \in \mathcal{E}G^{\mathrm{op}}} \alpha_{p,q}^X(E) \to H_G^{p,q}(X, \mathbb{Z}).$$

Proof. Take E to be an EG-model, then there is an obvious assignment c. On the other hand, for any $E \in \mathcal{E}G$ and any EG-model U, we define a map t_E to be the composite

$$t_E : H^{p,q}(U \times_G X, \mathbb{Z}) \to H^{p,q}((U \times E \times X)/G, \mathbb{Z}) \cong H^{p,q}(E \times_G X, \mathbb{Z}),$$

then this defines the inverse c^{-1}, as desired. □

The following is a twisted version of equivariant cycles.

Definition 3.11. Let $B \in \mathrm{Sch}/k$ and $V \to B$ be a vector bundle. Define $\mathrm{Sm} : B$ to be the full subcategory of B-schemes $X \to B$ in Sch/B such that $X \in \mathrm{Sm}/k$. We may define a functor

$$\begin{aligned} Z_{\mathrm{equi}}(V/B, n) : (\mathrm{Sm} : B)^{\mathrm{op}} &\to \mathrm{Ab} \\ (X \to B) &\mapsto \mathrm{Cycl}((X \times_B V)/X, n) \end{aligned}$$

which is an étale sheaf. In particular, if V is a trivial vector bundle over B of rank r, then $Z_{\mathrm{equi}}(V/B, n) = Z_{\mathrm{equi}}(\mathbb{A}^r, n)$ on $\mathrm{Sm} : B$.

Proposition 3.12. Let $V \to B$ be a vector bundle of rank r, then there is an isomorphism

$$C_* Z_{\mathrm{equi}}(V/B, n)_{\mathrm{cdh}} = C_* Z_{\mathrm{equi}}(\mathbb{A}^r, n)_{\mathrm{cdh}}$$

in $D((\mathrm{Sch}/B)_{\mathrm{cdh}})$.

Proof. Consider the pullbacks

$$\begin{tikzcd} C_* Z_{\mathrm{equi}}(\mathbb{A}^r, n)_{\mathrm{cdh}} \arrow[r, "u"] & C_* Z_{\mathrm{equi}}((\mathbb{A}^r \times V)/B, n+r)_{\mathrm{cdh}} \\ & C_* Z_{\mathrm{equi}}(V/B, n)_{\mathrm{cdh}} \arrow[u, "v"'] \end{tikzcd}$$

Then it suffices to show that both u and v are quasi-isomorphisms. We will prove this for u, as the proof for v follows in a similar fashion. It suffices to show that $H^i(u)(X)$ is a quasi-isomorphism. Again, under cdh topology, we may assume that X is a smooth scheme, as the general case follows by resolution of singularity. Moreover, we assume that X is local as a scheme, using Mayer-Vietoris sequence. Under these assumptions, we have

$$\begin{aligned} H^i(C_* Z_{\mathrm{equi}}((\mathbb{A}^r \times V)/B, n+r)(X)) &\cong H^i(C_* Z_{\mathrm{equi}}(\mathbb{A}^r \times \mathbb{A}^r, n+r)(X)) \text{ over local rings} \\ &\cong \mathbb{H}^i(X, C_* Z_{\mathrm{equi}}(\mathbb{A}^r \times \mathbb{A}^r, n+r)_{\mathrm{cdh}}) \text{ by Theorem 2.64} \\ &\cong \mathbb{H}^i(X, C_* Z_{\mathrm{equi}}(\mathbb{A}^r, n)_{\mathrm{cdh}}) \text{ by Proposition 2.67} \\ &\cong H^i(C_* Z_{\mathrm{equi}}(\mathbb{A}^r, n)(X)) \text{ by Theorem 2.64.} \end{aligned}$$

□

We can now define a twisted version of motivic cohomology.

Definition 3.13. Let $X \in \mathrm{Sch}/B$ and $V \to B$ be a vector bundle over B of rank n. We define twisted motivic cohomology

$$\begin{aligned} H^p(X, V/B, r) &= \mathbb{H}^{p-2n+2r}_{\mathrm{cdh}}(C_* Z_{\mathrm{equi}}(V/B, r)_{\mathrm{cdh}}) \\ h^p(X, V/B, r) &= H^{p-2n+2r}_{\mathrm{cdh}}(C_* Z_{\mathrm{equi}}(V/B, r)(X)) \end{aligned}$$

in terms of hypercohomology and complexes, respectively. There is now a natural morphism $h^p(X, V/B, r) \to H^p(X, V/B, r)$. Moreover, in the case where $r = 0$, we have $H^p(X, V, B) = H^p(X, V/B, 0)$.

Corollary 3.14. For any $X \in \mathrm{Sch}/B$ and $\varphi \in \mathrm{Aut}(V/B)$, the map $\varphi^* \in \mathrm{Aut}(H^*(X, V/B, *))$ is the identity.

Our next goal is to define transfer and restriction morphisms for the twisted cohomology $H^*(X, V/B, *)$.

Let $\rho : H \to G$ be a homomorphism of finite groups. For any $E \in \mathcal{E}H$, $E \times_H G$ becomes an object of $\mathcal{E}G$. If X has a K-action for some group K, then the corresponding motivic cohomology can be restricted along the K-action equivariantly. Therefore, for any G-scheme X, this induces a morphism

$$\rho^* : H^{p,q}_G(X, \mathbb{Z}) \cong \varprojlim \alpha^\times_{p,q,G} \to \varprojlim \alpha^\times_{p,q,H} \cong H^{p,q}_H(X, \mathbb{Z}).$$

where X becomes an H-scheme by pulling back along ρ. More explicitly, let U (respectively, W) be an EG-model (respectively, EH-model), then ρ^* is defined by

$$H^{p,q}(U \times_G X, \mathbb{Z}) \xrightarrow{\pi_1^*} H^{p,q}((U \times (W \times_H G)) \times_G, \mathbb{Z}) \xrightarrow{(\pi_2^*)^{-1}} H^{p,q}((W \times_H G) \times_G X, \mathbb{Z}).$$

Remark 3.15. It is easy to check that ρ^* is functorial with respect to group homomorphisms, and ρ^* is the identity map if $\rho \in \mathrm{Inn}(G)$. Moreover, if ρ is injective, then ρ^* is given by the quotient map

$$H^{p,q}(U \times_G X, \mathbb{Z}) \to H^{p,q}(U \times_H X, \mathbb{Z}).$$

Now suppose E is an EG-model, and let $V \to E$ be a G-equivariant vector bundle over E, with a free G-action on V. Then the projection $E \times_H X \to E \times_G X$ define restriction maps

$$\begin{aligned} H^p(E \times_G X, V_G/E_G, r) &\to H^p(E \times_H X, V_H/E_H, r), \\ h^p(E \times_G X, V_G/E_G, r) &\to h^p(E \times_H X, V_H/E_H, r) \end{aligned}$$

when $H \subseteq G$ is a finite subgroup. The transfer map is then defined to be the composite

$$t : h^i(E \times_H X, V_H/E_H, r) \to h^i(E \times_H X, V_H/E_G, r) \xrightarrow{\text{res}} h^i(E \times_G X, V_G/E_G, r),$$

On the level of H^p, the composite

$$H^p(E \times_G X, V_G/E_G, r) \xrightarrow{\text{res}} H^p(E \times_H X, V_H/E_H, r) \xrightarrow{t} H^p(E \times_G X, V_G/E_G, r)$$

which is just $|G| \cdot \mathrm{id}$. We will omit the proof of the following result.

Lemma 3.16. Let G be a finite group. X be a smooth quasi-projective scehme over k with trivial G-action, E be an $E\mathbb{Z}/p\mathbb{Z}$-model. Consider the trivial vector bundle $\mathbb{A}^{np} \times E$ over E, with action given by $T \cdot (x_1, \dots, x_p, s) = (x_2, \dots, x_p, x_1, s)$, where each x_i is an n-tuple, T is a generator of $\mathbb{Z}/p\mathbb{Z}$, and s is a section on E.[10] Then the composite

$$h^i(E \times X, (\mathbb{A}^{np} \times E)/E, r) \xrightarrow{t} h^i(E_{\mathbb{Z}/p\mathbb{Z}} \times X, (\mathbb{A}^{np} \times_{\mathbb{Z}/p\mathbb{Z}} E)/E_{\mathbb{Z}/p\mathbb{Z}}, r) \to H^i(E_{\mathbb{Z}/p\mathbb{Z}} \times X, (\mathbb{A}^{np} \times_{\mathbb{Z}/p\mathbb{Z}} E)/E_{\mathbb{Z}/p\mathbb{Z}}, r)$$

is zero under $\mathbb{Z}/p\mathbb{Z}$-coefficients.

[10] This is analogous to the construction of $\Gamma(X)$ for lens space, c.f., Definition 1.3.

Proof. As in the proof of Proposition 3.12, there is an isomorphism

$$H^i(E_{\mathbb{Z}/p\mathbb{Z}} \times X, (\mathbb{A}^{np} \times_{\mathbb{Z}/p\mathbb{Z}} E)/E_{\mathbb{Z}/p\mathbb{Z}}, r) \cong H^i(E_{\mathbb{Z}/p\mathbb{Z}} \times X, \mathbb{A}^{np} \times (\mathbb{A}^{np} \times_{\mathbb{Z}/p\mathbb{Z}} E)/E_{\mathbb{Z}/p\mathbb{Z}}, r+np).$$

So it suffices to show that the transfer map

$$t : H^i(E \times X, \mathbb{A}^{np} \times \mathbb{A}^{np} \times E/E, r+np) \to H^i(E_{\mathbb{Z}/p\mathbb{Z}} \times X, \mathbb{A}^{np} \times (\mathbb{A}^{np} \times_{\mathbb{Z}/p\mathbb{Z}} E)/E_{\mathbb{Z}/p\mathbb{Z}}, r+np)$$

is zero modulo p. Since $t \circ \mathrm{res} = p \cdot \mathrm{id} = 0$, it suffices to show that the restriction is surjective. But

$$\begin{aligned} H^i(E \times X, \mathbb{A}^{np} \times \mathbb{A}^{np} \times E, r+np) &\cong \mathbb{H}^{i-2np+2r}_{\mathrm{cdh}}(E \times X, c_* Z_{\mathrm{equi}}(\mathbb{A}^{2np}, r+np)_{\mathrm{cdh}}) \text{ by Proposition 3.12} \\ &\cong \mathbb{H}^{i-2np+2r}_{\mathrm{cdh}}(X, C_* Z_{\mathrm{equi}}(\mathbb{A}^{2np}, r+np)_{\mathrm{cdh}}), \end{aligned}$$

and

$$H^i(E_{\mathbb{Z}/p\mathbb{Z}} \times X, \mathbb{A}^{np} \times (\mathbb{A}^{np} \times_{\mathbb{Z}/p\mathbb{Z}} E)/E_{\mathbb{Z}/p\mathbb{Z}}, r+np) \cong \mathbb{H}^{i-2np+2r}_{\mathrm{dch}}(E_{\mathbb{Z}/p\mathbb{Z}} \times X, C_* Z_{\mathrm{equi}}(\mathbb{A}^{2np}, r+np)_{\mathrm{cdh}}).$$

Therefore the surjectivity follows by using a rational map

$$* \to E \to E_{\mathbb{Z}/p\mathbb{Z}}.$$

□

3.2 Constructing Steenrod Operations

We now move on to the construction of Steenrod operations on smooth schemes. Let $D \in \mathrm{Sm}/k$ and $X \in \mathrm{Sm} : D$. Suppose V/D is a vector bundle of rank m, and suppose E to be an ES_p-model. We want to define a map of simplicial sets

$$C_* Z_{\mathrm{equi}}(V/D, q)(X) \to C_* Z_{\mathrm{equi}}((E \times_{S_p} V^{\oplus p})/(E_{S_p} \times D), pq)(E_{S_p} \times X)$$

over $E_{S_p} \times X$, with $(C_* F)_n = F(X \times \Delta^n)$. Consider a cycle $Z \in Z_{\mathrm{equi}}(V/D, q)(\Delta^j \times X)$, we can write it down as $Z = \sum_i a_i Z_i$ where $a_i \in \mathbb{Z}/p\mathbb{Z}$ and $Z_i \in \mathrm{Cycl}(\Delta^j \times (X \times_D V)/(\Delta^j \times X), q)$. We define two diagonal morphisms

$$\Delta_X : X \to X^{\times p}, \quad \Delta_s : \Delta^j \to \Delta^{jp}.$$

Definition 3.17. We define $\tilde{Z}^p = \sum_{i_1,\dots,i_p} a_{i_1} \cdots a_{i_p} \mathrm{Cycl}(\Delta_X \times \Delta_s)(\mathrm{Cycl}(Z_{i_1} \times \cdots \times Z_{i_p})) \in Z_{\mathrm{equi}}(V^{\oplus p}/p, qi)(\Delta^j \times X)$ using external product $\times$ of cycles.

For a discussion of external products, see [SV00, Section 3.7].

Lemma 3.18.

1. Suppose $f : X \to Y$ is a morphism in $\mathrm{Sm} : E_{S_p}$ and $Z \in Z_{\mathrm{equi}}(V/D, q)(Y)$ with $\mathbb{Z}/p\mathbb{Z}$-coefficients, then
$$\mathrm{Cycl}(f)(\tilde{Z}^p) = \widetilde{\mathrm{Cycl}(f)(Z)}^p.$$
2. Note that $\tilde{Z}^p$ is a direct sum of E and $V^{\oplus p}$, then it has a natural S_p-action given by $V^{\oplus p}/p$, and in fact $\tilde{Z}^p$ is S_p-invariant.

Proposition 3.19. This follows from the compatibility of Cycl with respect to external products.

In particular, this shows that the cycle $\tilde{Z}^p \times E$ is also S_p-invariant in $Z_{\mathrm{equi}}((V^{\oplus p} \times E)/(D \times E), pq)(\Delta^j \times X \times E)$. Therefore this induces a cycle on the quotient scheme, i.e., $(\tilde{Z}^p \times E)_{S_p} \in Z_{\mathrm{equi}}((V^{\oplus p} \times_{S_p} E)/(D \times E_{S_p}), pq)(\Delta^j \times X \times E_{S_p})$. This is what we wanted, a map between simplicial sheaves

$$C_* Z_{\mathrm{equi}}(V/D, q) \to \underline{\mathrm{Hom}}(E_{S_p}, C_* Z_{\mathrm{equi}}(E \times_{S_p} V^{\oplus p}/(E_{S_p} \times D), pq)).$$

Note that this is defined as a map between simplicial sets on each section, which means it is not necessarily additive. Instead, we have to consider the homotopy sheaves. For each simplicial sheaf $F \in \mathrm{sShv}(\mathrm{Sm}/k)$, we define

$$\pi_i(X, F) = [S^i \wedge X_+, F]_{H_s(k)}$$

in the simplicial homotopy category. If F is an abelian sheaf, then this is the same as $\mathbb{H}^{-i}(X, C_* F)$, and in this case it is an abelian group.

For this map of simplicial sheaves, we apply the functor π_i, so we get

$$\pi_i(X, C_* Z_{\text{equi}}(V/D, q)) \to \pi_i(X, \underline{\mathrm{Hom}}(E_{S_p}, C_* Z_{\text{equi}}((E \times_{S_p} V^{\oplus p})/(E_{S_p} \times D), pq))).$$

When $i > 0$, this is an additive homomorphism since it has a group structure. In the case $i = 0$, π_0 does not admit a group structure. We then obtain a composition of maps of sets

$$\begin{aligned} s : H^{2m-2q-i}(X, V/D, q) &\to \mathbb{H}^{-i}(X, \underline{\mathrm{Hom}}(E_{S_p}, C_* Z_{\text{equi}}((E \times_{S_p} V^{\oplus p})/(E_{S_p} \times D)), pq)) \\ &\to \mathbb{H}^{-i}(E_{S_p} \times X, C_* Z_{\text{equi}}(E_{S_p}, C_* Z_{\text{equi}}((E \times_{S_p} V^{\oplus p})/(E_{S_p} \times D), pq))) \\ &\cong H^{2mp-2pq-i}(E_{S_p} \times X, E_{S_p}, (E \times_{S_p} V^{\oplus p})/(E_{S_p} \times D), pq) \end{aligned}$$

which is a homomorphism whenever $i > 0$. The same holds if we replace S_p by $\mathbb{Z}/p\mathbb{Z}$. The total Steenrod operation is defined using this composite s.

Definition 3.20. Let $X \in \mathrm{Sm}/k$. The natural map

$$s^{\text{cycl}} : H^{2m-2q-i}(X, V/D, q) \to H^{2mp-2pq-i}(E_{\mathbb{Z}/p\mathbb{Z}} \times X, (E \times_{\mathbb{Z}/p\mathbb{Z}} V^{\oplus p})/(E_{\mathbb{Z}/p\mathbb{Z}} \times D), pq)$$

is called the total Steenrod operation.

By Proposition 3.12, the twisted and untwisted version of cohomology agree, therefore s^{cycl} induces a map

$$H^{2(m-q)-i,2-q}(X, \mathbb{Z}/p\mathbb{Z}) \to H^{2p(m-q)-i,p(m-q)}(E_{\mathbb{Z}/p\mathbb{Z}} \times X, \mathbb{Z}/p\mathbb{Z}),$$

which is also called the total Steenrod operation.

Remark 3.21. If $i < 0$, then both sides of the total Steenrod operation vanish. Therefore, it suffices to consider $i \geqslant 0$.

Remark 3.22. The total Steenrod operation is natural under pullbacks and restrictions. In particular, $\mathrm{res} \circ s = s^{\text{cycl}}$.

Lemma 3.23. If $X \in \mathrm{Sm}/k$ is a quasi-projective smooth scheme, then s^{cycl} is a homomorphism for $i = 0$.

Proof. By Proposition 3.12, we may untwist and therefore we can think of it as the trivial bundle. That is, it suffices to prove the case where $V = \mathbb{A}^n$ and $D = *$ for the map

$$s^{\text{cycl}} : H^{2(m-q),m-q}(X, \mathbb{Z}/p\mathbb{Z}) \to H^{2p(m-q),p(m-q)}(X \times E_{\mathbb{Z}/p\mathbb{Z}}, \mathbb{Z}/p\mathbb{Z}).$$

Since X is quasi-projective, then its motivic cohomology descents to the cohomology of the cycle complex. Therefore, we consider the map

$$\begin{aligned} h^0(X, C_* Z_{\text{equi}}(\mathbb{A}^n, q)) &\to h^0(E_{\mathbb{Z}/p\mathbb{Z}} \times X, C_* Z_{\text{equi}}(E \times_{\mathbb{Z}/p\mathbb{Z}} \mathbb{A}^{np}/E_{\mathbb{Z}/p\mathbb{Z}}, pq)) \\ Z &\mapsto (E_{\mathbb{Z}/p\mathbb{Z}} \times Z^p)_{\mathbb{Z}/p\mathbb{Z}} \end{aligned}$$

We have the equation

$$\left(\sum_i (a_i + b_i) z_i\right)^{\times p} - \left(\sum_i a_i z_i\right)^{\times p} - \left(\sum_i b_i z_i\right)^{\times p} = \sum_{g \in \mathbb{Z}/p\mathbb{Z}} g \cdot w$$

where we want to find such element w. But this is just

$$\sum_{g \in \mathbb{Z}/p\mathbb{Z}} g \cdot \left(\sum_{(u_1,\ldots,u_p) \in S \setminus \{(a,\ldots,a),(b,\ldots,b)\}} (u_1)_{i_1} \cdots (u_p)_{i_p} z_{i1} \times \cdots \times z_{ip} \right),$$

where each u_j is either a_j or b_j, where the string $S = \{a, b\}^{\times p}/(\mathbb{Z}/p\mathbb{Z})$ quotient by the cyclic action. Therefore

$$\widetilde{z_1 + z_2}^p - \tilde{z}_1^p - \tilde{z}_2^p$$

is in the image of the transfer map

$$h^0(E \times X, E \times \mathbb{A}^{np}/E, pq) \to h^0(E_{\mathbb{Z}/p\mathbb{Z}} \times X, E \times_{\mathbb{Z}/p\mathbb{Z}} \mathbb{A}^{np}/E_{\mathbb{Z}/p\mathbb{Z}}, pq)$$
$$Z \mapsto \left(\sum_{g \in \mathbb{Z}/p\mathbb{Z}} g \cdot Z\right)_{\mathbb{Z}/p\mathbb{Z}}$$

But by Lemma 3.16, the image of the transfer map must be zero. □

In Proposition 3.4, we computed the generators of lens space $\mathbb{Z}/p\mathbb{Z}(B\mu_p)$ to be

$$\omega_k = \begin{cases} v^{\frac{k}{2}}, & k \text{ even} \\ uv^{\frac{k-1}{2}}, & k \text{ odd} \end{cases} \in H^{k, \lfloor \frac{k+1}{2} \rfloor}(B\mu_p, \mathbb{Z}/p\mathbb{Z}).$$

Definition 3.24. Suppose $X \in \mathrm{Sm}/k$ is quasi-projective, and let $\alpha \in H^{2n-m,n}(X, \mathbb{Z}/p\mathbb{Z})$ for $m \geqslant 0$, then there is an induced decomposition of α's total Steenrod operation via

$$s^{\mathrm{cycl}}(\alpha) \cong \sum_k \omega_k \cdot D_k(\alpha)$$

where $D_k(\alpha) \in H^{2np-m-k, np-\lfloor \frac{k+1}{2} \rfloor}(X, \mathbb{Z}/p\mathbb{Z})$.

Proposition 3.25. Suppose $\alpha \in H^{2n-m,n}(X, \mathbb{Z}/p\mathbb{Z})$ for $m \geqslant 0$. By Theorem 2.64, it can be represented by a cycle $Z \in Z_{\mathrm{equi}}(\mathbb{A}^n, 0)(\Delta^m \times X)$ such that $\delta(Z) = 0$ under simplicial differential. Then $D_0(\alpha)$ is represented by $\tilde{Z}^p \in Z_{\mathrm{equi}}(\mathbb{A}^{np}, 0)$.

Proof. By definition, $s^{\mathrm{cycl}}(\alpha) \in H^{2np-m,np}(X \times E_{\mathbb{Z}/p\mathbb{Z}}, \mathbb{Z}/p\mathbb{Z})$. Let $X \in E_{\mathbb{Z}/p\mathbb{Z}}$ be a rational point (as a non-zero section of a vector bundle), then by the decomposition of $B\mu_p$, we have $s^{\mathrm{cycl}}(\alpha)\big|_X = D_0(\alpha)$. By definition, we know $s^{\mathrm{cycl}}(\alpha)$ is represented by $\tilde{Z}^p \times_{\mathbb{Z}/p\mathbb{Z}} E \in Z_{\mathrm{equi}}((\mathbb{A}^{np} \times_{\mathbb{Z}/p\mathbb{Z}} E)/E_{\mathbb{Z}/p\mathbb{Z}}, 0)(\Delta^m \times X \times E_{\mathbb{Z}/p\mathbb{Z}})$, so pulling back E to a point in $E_{\mathbb{Z}/p\mathbb{Z}}$, we have $\tilde{Z}^p$. □

Corollary 3.26. If $\alpha \in H^{2n,n}(X, \mathbb{Z}/p\mathbb{Z}) \cong \mathrm{CH}^n(X)/p$, then $D_0(\alpha) = \alpha^p$.

Corollary 3.27. If $\alpha \in H^{m,n}(X, \mathbb{Z}/p\mathbb{Z})$ with $m < 2n$, then $D_0(\alpha) = 0$.

From this, one can show that $\mathrm{Sq}^2 = 0$ on $H^{1,1}$.

Proof. By Theorem 2.64, we represent α as a cycle $Z \in Z_{\mathrm{equi}}(\mathbb{A}^n, 0)(\Delta^{2n-m} \times X)$. The diagonal is the map

$$d : \Delta^{2n-m} \to \Delta^{(2n-m)p}$$

sending e_i for $0 \leqslant i \leqslant 2n-m$ to $((0_{(1)}, \ldots, 0_{(p)})_{(0)}, \ldots, ((e_i)_{(1)}, \ldots, (e_i)_{(p)})_{(i)}, \ldots, (0_{(1)}, \ldots, 0_{(p)})_{(2n-m)})$. We need to show that Z is a boundary as a simplicial complex. Consider $\Delta^{(2n-m)p} \supseteq \Delta^{2n-m+1} = \Delta^{2n-m} \times \mathbb{A}^1$, parametrized by $t \in \mathbb{A}^1$. Writing down the coordinates, this is given by $((e_0 + t, \ldots, e_0 + t), \ldots, (e_{2n-m} + t, \ldots, e_{2n-m+t}))$. This controls $2n - m + 2$ faces of the simplex, given by $t + e_0 = 0, \ldots, t + e_{2n-m} = 0$, as well as $t = 0$. This is the $(2n - m + 1)$-simplex we want. Now since Z has zero simplicial boundary, then we may assume Z is normalized, i.e., $\delta_i(Z) = 0$ if $i \leqslant 2n - m$, given by the Dodd-Kan correspondence. Given the inclusion, we know $\tilde{Z}^p$ can be restricted to Δ^{2n-m+1}, so we compute

$$\delta_i\left(\tilde{Z}^p\Big|_{\Delta^{2n-m+1}}\right) = \begin{cases} \tilde{Z}^p, & i = 2n - m + 1 \\ 0, & \text{else} \end{cases}.$$

Hence $\tilde{Z}^p$ is in the boundary as well. □

Proposition 3.28. We have $D_n = 0$ unless $n \equiv 0, 1 \pmod{2(p-1)}$.

This is analogous to the topological case.

Proof. Fix some $k \in \operatorname{Aut}(\mathbb{Z}/p\mathbb{Z}) = (\mathbb{Z}/p\mathbb{Z})^\times$. We define $\widetilde{\mathbb{A}^\infty\backslash\{0\}}$ to be the $\mathbb{Z}/p\mathbb{Z}$-scheme $\mathbb{A}^\infty\backslash\{0\}$ with action

$$t \cdot (\cdots x_i \cdots) = (\cdots \xi^{kt} x_i \cdots)$$

where t is the generator of $\mathbb{Z}/p\mathbb{Z}$ and ξ is a primitive pth root of unity. Therefore, there is a $\mathbb{Z}/p\mathbb{Z}$-equivariant map

$$\begin{aligned} \widetilde{\mathbb{A}^\infty\backslash\{0\}} &\to \mathbb{A}^\infty\backslash\{0\} \\ (\cdots x_i \cdots) &\mapsto (\cdots x_i^k \cdots). \end{aligned}$$

Quotient by the $\mathbb{Z}/p\mathbb{Z}$-action, this induces an equivariant automorphism

$$k^* : H^{**}(\mathcal{O}_{\mathbb{P}^\infty}(p)^\times, \mathbb{Z}/p\mathbb{Z}) \to H^{**}(\mathcal{O}_{\mathbb{P}^\infty}(p)^\times, \mathbb{Z}/p\mathbb{Z})$$

after pulling back to $\mathbb{P}^\infty$. Since $\mathcal{O}_{\mathbb{P}^\infty}(p)^\times \cong B\mu_p$, then to compute k^*, we just need to compute the images $k^*(u)$ and $k^*(v)$ of the generators. Since k^* acts by taking the kth power, then given line bundle $\mathcal{O}(1)$, the pullback would be $\mathcal{O}(k)$. It is then easy to show that $k^*(v) = kv$. Therefore $\beta(k^*(u)) = k^*\beta(u) = k^*(v) = kv$, and $k^*(u)|_{\mathrm{pt}} = 0$, so we must have $k^*(u) = ku$ by unqiueness.

Let E be an ES_p-model, then there is a commutative diagram

$$\begin{tikzcd} H^{2i-j,i}(X, \mathbb{Z}/p\mathbb{Z}) \arrow[r, "s"] \arrow[dr, "s^{\mathrm{cycl}}"'] & H^{ip-j,ip}(X \times ES_p, \mathbb{Z}/p\mathbb{Z}) \arrow[d, "\mathrm{res}"] \\ & H^{2ip-j,ip}(X \times E\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \end{tikzcd}$$

Since k^* is a $\mathbb{Z}/p\mathbb{Z}$-automorphism, then it is given by conjugation $r \in S_p$. This induces a commutative diagram

$$\begin{tikzcd} H^{2ip-j,ip}(X \times E_{S_p}, \mathbb{Z}/p\mathbb{Z}) \arrow[r, "\mathrm{Inn}(r)^*=\mathrm{id}"] \arrow[d] & H^{2ip-j,ip}(X \times E_{S_p}, \mathbb{Z}/p\mathbb{Z}) \arrow[d, "\mathrm{res}"] \\ H^{ip-j,ip}(X \times E_{\mathbb{Z}/p\mathbb{Z}}, \mathbb{Z}/p\mathbb{Z}) \arrow[r, "k^*"'] & H^{ip-j,ip}(X \times E_{\mathbb{Z}/p\mathbb{Z}}, \mathbb{Z}/p\mathbb{Z}) \end{tikzcd}$$

and therefore $\sum_j k^*(W_j)D_j(\alpha) = k^* s^{\mathrm{cycl}}(\alpha) = s^{\mathrm{cycl}}(\alpha) = \sum_j \omega_j D_j(\alpha)$ since the k^* acts as identity downstairs as well. Therefore $k^*(\omega_j) = \omega_j$. We have

$$k^*(\omega_j) = \begin{cases} k^{\frac{j}{2}}\omega_j, & j \text{ even} \\ k^{\frac{j+1}{2}}\omega_j, & j \text{ odd} \end{cases}$$

which concludes the proof. □

Proposition 3.29. Let X be a quasi-projetive smooth scheme over k, and $u \in H^{*,*}(X, \mathbb{Z}/p\mathbb{Z})$, then $\beta s^{\mathrm{cycl}}(u) = 0$.

We note that the proof is the same as in the topological case.

Proof. Suppose u is represented by a integral-coefficient cycle Z_u, then Z_u has zero simplicial boundary, i.e., $\delta(Z_u) = 0$ over $\mathbb{Z}/p\mathbb{Z}$, which means $\delta(Z_u) \equiv 0 \pmod{p}$, so there exists some well-defined element $\frac{\delta(Z_u)}{p}$. By Leibniz's rule, $\delta(\tilde{Z}_u^p) = p\left(\frac{\delta(Z_u)}{p} \times Z_u^{\times(p-1)} + Z_u \times \frac{\delta(Z_u)}{p} \times Z_u^{\times(p-2)} + \cdots + Z_u^{\times(p-1)} \times \frac{\delta(Z_u)}{p}\right)$. Note that $s^{\mathrm{cycl}}(u)$ is the mod-p reduction of $E \times_{\mathbb{Z}/p\mathbb{Z}} \tilde{Z}_u^p$, therefore $\beta s^{\mathrm{cycl}}(u) = \frac{\delta\left(E \times_{\mathbb{Z}/p\mathbb{Z}} \tilde{Z}_u^p\right)}{p}$ by snake lemma, which is the same as

$$\sum_{g \in \mathbb{Z}/p\mathbb{Z}} g \cdot \left(E \times_{\mathbb{Z}/p\mathbb{Z}} \left(\frac{\delta(Z_u)}{p} \times Z_u^{\times(p-1)}\right)\right)$$

which lies in the image of transfer, therefore it must be zero. □

Corollary 3.30. We have $\beta D_0 = 0$, and $\beta D_{2n} = -D_{2n-1}$, and $\beta D_{2n-1} = 0$.

Definition 3.31. Let $Y \to X \to S$ be maps in Sch/k, and $Z \in \mathrm{Cycl}(Y/X, r)$, and $W = \sum_i n_i w_i \in \mathrm{Cycl}(X/S, m)$. We define $\mathrm{Cor}(Z, W)$ to be

$$\sum_i n_i (\pi_Y)_* \mathrm{Cycl}(W_i \to X)(Z),$$

for $\pi_Y : W_i \times_X Y \to Y$.

This is an element in $\mathrm{Cycl}(Y/S, r+m)$, c.f., [SV00, Theorem 3.7.3]. Therefore, for any $X, Y \in \mathrm{Sch}/S$ for $S \in \mathrm{Sm}/k$, we have a pairing

$$(- \times_S Y) \times \mathrm{id} : Z_{\mathrm{equi}}(X/S, n) \times Z_{\mathrm{equi}}(Y/S, m) \to Z_{\mathrm{equi}}((X \times_S Y)/Y, n) \times Z_{\mathrm{equi}}(Y/S, m).$$

Applying the construction above, we have

$$\mathrm{Cor}(-,-) : Z_{\mathrm{equi}}((X \times_S Y)/Y, n) \times Z_{\mathrm{equi}}(Y/S, m) \to Z_{\mathrm{equi}}(X \times_S Y, n+m)$$

which gives a morphism

$$C_* Z_{\mathrm{equi}}(X/S, n) \otimes C_* Z_{\mathrm{equi}}(Y/S, m) \to C_* Z_{\mathrm{equi}}(X \times_S Y, n+m)$$

between simplicial complexes.

For the twisted case, let $B \in \mathrm{Sm}/k$, and V_1, V_2 are vectors bundles over B of rank s and t, respectively. By the same construction, we have a pairing

$$C_* Z_{\mathrm{equi}}(V_1/B, q)_{\mathrm{cdh}} \otimes C_* Z_{\mathrm{equi}}(V_2/B, t)_{\mathrm{cdh}} \to C_* Z_{\mathrm{equi}}((V_1 \otimes V_2)/B, q+r)_{\mathrm{cdh}}.$$

This induces a commutative[11] multiplication on the level of motivic cohomology: for $X, Y \in \mathrm{Sch}/B$, there is

$$H^s(X, V_1/B, q) \otimes H^t(Y, V_2/B, r) \to H^{s+t}(X \times_B Y, (V_1 \oplus V_2)/B, q+r).$$

Similar construction holds on the level of chain complexes for h^i.

Proposition 3.32. Let $X, Y \in \mathrm{Sm}/k$ be quasi-projective schemes. For $\alpha \in H^{i,s}(X, \mathbb{Z}/p\mathbb{Z})$ and $\beta \in H^{j,t}(Y, \mathbb{Z}/p\mathbb{Z})$, there is

$$\Delta^*(s^{\mathrm{cycl}}(\alpha) \times s^{\mathrm{cycl}}(\beta)) = s^{\mathrm{cycl}}(\alpha \times \beta)$$

where $\Delta : E_{\mathbb{Z}/p\mathbb{Z}} \times X \times Y \to E_{\mathbb{Z}/p\mathbb{Z}} \times X \times E_{\mathbb{Z}/p\mathbb{Z}} \times Y$ is the diagonal map.

Proof. Suppose α (respectively, β) is represented by cycle $Z_\alpha \in Z_{\mathrm{equi}}(\mathbb{A}^s, 0)(\Delta^{2s-i} \times X)$ (respectively, by cycle $Z_\beta \in Z_{\mathrm{equi}}(\mathbb{A}^t, 0)(\Delta^{2t-j} \times Y)$). By definition, $s^{\mathrm{cycl}}(\alpha)$ is defined by

$$\begin{array}{ccc} h^{2s-i}(X, \mathbb{A}^s/k, 0) & \xrightarrow{\tilde{s}^{\mathrm{cycl}}} & h^{2sp-i}(E_{\mathbb{Z}/p\mathbb{Z}} \times X, (E_{\mathbb{Z}/p\mathbb{Z}} \times \mathbb{A}^{sp})/E_{\mathbb{Z}/p\mathbb{Z}}, 0) \\ & & \downarrow \\ & & H^{2sp-i}(E_{\mathbb{Z}/p\mathbb{Z}} \times X, (E_{\mathbb{Z}/p\mathbb{Z}} \times \mathbb{A}^{sp})/E_{\mathbb{Z}/p\mathbb{Z}}, 0) \\ & & \downarrow \\ & & H^{2sp-i,sp}(X, \mathbb{Z}/p\mathbb{Z}) \end{array}$$

Then $\tilde{s}^{\mathrm{cycl}}(\alpha) = E \times_{\mathbb{Z}/p\mathbb{Z}} \tilde{Z}_\alpha^p$, and similarly $\tilde{s}^{\mathrm{cycl}}(\beta) = E \times_{\mathbb{Z}/p\mathbb{Z}} \tilde{Z}_\beta^p$. Hence

$$\begin{aligned} \tilde{s}^{\mathrm{cycl}}(\alpha \times \beta) &= E \times_{\mathbb{Z}/p\mathbb{Z}} \widetilde{Z_\alpha \times Z_\beta} \\ &= E \times_{\mathbb{Z}/p\mathbb{Z}} \left(\tilde{Z}_\alpha^p \times \tilde{Z}_\beta^p \right) \\ &= \left(E \times_{\mathbb{Z}/p\mathbb{Z}} \tilde{Z}_\alpha^p \right) \times \left(E \times_{\mathbb{Z}/p\mathbb{Z}} \tilde{Z}_\beta^p \right). \end{aligned}$$

□

[11] Usually such multiplication is only skew-commutative, but here we will record the information in the automorphism of the vector bundle, therefore upgrading it to a commutative operation.

We now move towards proving Cartan Formula.

Proposition 3.33. Consider α and β defined in Proposition 3.32.

- If $p > 2$, then $D_{2r}(\alpha \times \beta) = \sum_{t=0}^{r} D_{2t}(\alpha) \times D_{2r-2t}(\beta)$.
- If $p = 2$, then
  - ⋆ $D_{2r}(\alpha \times \beta) = \sum_{t=0}^{r} D_{2t}(\alpha) \times D_{2r-2t}(\beta) + \tau \sum_{t=0}^{r-1} D_{2t+1}(\alpha) \times D_{2r-2t-1}(\beta)$ where $\tau \in H^{0,1}(k, \mathbb{Z}/p\mathbb{Z})$ is the motivic Bott element, and
  - ⋆ $D_{2r+1}(\alpha \times \beta) = \sum_{t=0}^{2r+1} D_t(\alpha) \times D_{2r+1-t}(\beta) + \rho \sum_{t=0}^{2r+1} D_{2t+1}(\alpha) D_{2r-2r+1}(\beta)$ where $\rho = -1 \in H^{1,1}(k, \mathbb{Z}/p\mathbb{Z}) \cong k^\times/2$.

Proof. We have $s^{\text{cycl}} = \sum_m \omega_m \times D_m$ where $\omega_m \in H^{m,\lfloor \frac{m+1}{2} \rfloor}(B\mu_p, \mathbb{Z}/p\mathbb{Z})$ is a generator. Therefore $\Delta^*(s^{\text{cycl}}(\alpha) \times s^{\text{cycl}}(\beta)) = \sum_{m,n} \omega_m \omega_n D_m(\alpha) \times D_n(\beta)$. By multiplicity, it suffices to compute $\omega_m \omega_n$ and then compare the coefficients, which can be done once we see

$$\begin{cases} \omega_{2a+1}\omega_{2b+1} = 0, & p > 2 \\ \omega_{2a}\omega_{2b} = \omega_{2a+2b}, & p \text{ arbitrary} \\ \omega_{2a+1}\omega_{2b+1} = \tau\omega_{2a+2b+2} + \rho\omega_{2a+2b+1}, & p = 2 \\ \omega_{2a}\omega_{2b+1} = \omega_{2a+2b+1}, & p = 2 \end{cases}$$

using Proposition 3.5. □

Definition 3.34. For a quasi-projective smooth scheme $X \in \mathrm{Sm}/k$, suppose $x \in H^{p,q}(X, \mathbb{Z}/\ell\mathbb{Z})$ for $p \leqslant 2q$,[12] then we define the ith Steenrod power operations to be

$$\mathcal{P}^i(x) = (-1)^{q+i} D_{2(q-i)(\ell-1)}(x) \in H^{p+2i(\ell-1),i(\ell-1)}(X, \mathbb{Z}/\ell\mathbb{Z})$$

for $\ell > 2$, and the Steenrod squares to be

$$\mathrm{Sq}^{2i}(x) = D_{2(q-i)}(x) \in H^{p+2i,q+i}(X, \mathbb{Z}/\ell\mathbb{Z})$$

and

$$\mathrm{Sq}^{2i+1}(x) = \beta\,\mathrm{Sq}^{2i}(x) = D_{2(q-i)-1}(x) \in H^{p+2i+1,q+i}(X, \mathbb{Z}/\ell\mathbb{Z})$$

for $\ell = 2$.

3.3 Properties of Steenrod Operations

Proposition 3.35 (Cartan Formula, [Voe03], Proposition 9.6)**.** We have

$$\mathcal{P}^k(xy) = \sum_i \mathcal{P}^i(X)\mathcal{P}^{k-i}(y)$$
$$\beta(xy) = \beta(x)y + (-1)^{\deg(x)} x\beta(y).$$

where $\deg(x)$ is the first index of x, i.e., $\deg(p, q) = p$. Moreover,

$$\mathrm{Sq}^{2k}(xy) = \sum_i \mathrm{Sq}^{2i}(x)\,\mathrm{Sq}^{2k-2i}(y) + \tau \sum_i \mathrm{Sq}^{2i+1}(x)\,\mathrm{Sq}^{2k-2i-1}(y)$$
$$\mathrm{Sq}^{2k+1}(xy) = \sum_i \mathrm{Sq}^i(x)\,\mathrm{Sq}^{2k+1-i}(y) + \rho \sum_i \mathrm{Sq}^{2i+1}(x)\,\mathrm{Sq}^{2k-2i-1}(y).$$

[12] For $p > 2q$ this forces $x = 0$.

Remark 3.36. We see that for $\ell > 2$, the Cartan formula is the same as the topological case; for $\ell = 2$, we need to introduce elements in the motivic setting. This will always be the slogan to keep in mind.

Proof. Combine Proposition 3.28, Proposition 3.33, and Corollary 3.30. □

We now show that the Steenrod operations commute with suspension. In the motivic setting, there are a few types of suspension isomorphisms.

a. There is an isomorphism

$$\Sigma_T : H^{**}(X, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{-\times\pi} H^{*+2,*+1}((X \times \mathbb{P}^1)/(X \times \{*\}), \mathbb{Z}/p\mathbb{Z})$$

by the projective bundle theorem, where π being the generator of $H^{2,1}(\mathbb{P}^1, \mathbb{Z}/p\mathbb{Z})$.[13] This is the suspension by $\mathbb{P}^1$.

b. There is an isomorphism

$$\Sigma_t : H^{**}(X, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{-\times\lambda} H^{*+1,*+1}((X \times \mathbb{G}_m)/(X \times \{*\}), \mathbb{Z}/p\mathbb{Z})$$

where $\lambda \in H^{1,1}(\mathbb{G}_m)$ is also a generator. This is the suspension by $\mathbb{G}_m$.

c. There is an isomorphism

$$\Sigma_s : H^{**}(X, \mathbb{Z}/p\mathbb{Z}) \to H^{*+1,*}((X \times \partial\Delta^2)/X, \mathbb{Z}/p\mathbb{Z})$$

This is the simplicial suspension, and we will come back to this later.

The punchline being, the Steenrod operations commute with all of them.

For a. and b., by Cartan formula Proposition 3.35, it suffices to show that

$$\mathcal{P}^i(\pi) = \begin{cases} \pi, & i = 0 \\ 0, & i \neq 0 \end{cases}, \quad \mathcal{P}^i(\lambda) = \begin{cases} \lambda, & i = 0 \\ 0, & i \neq 0 \end{cases}$$

and similarly for Sq^i.

Proposition 3.37. We have $s^{\text{cycl}}(\pi) = -\pi\omega_{2(p-1)}$.

Proof. Let $((x : y), a)$ be a coordinate in $\mathbb{P}^1 \times \mathbb{A}^1$, then π can be represented by a flat cycle $Z = \{ax = y\} \subseteq \mathbb{P}^1 \times \mathbb{A}^1$ over $\mathbb{P}^1$ in $Z_{\text{equi}}(\mathbb{P}^1 \times \mathbb{A}^1, 1)(k)$. Note that π is an element in

$$\begin{aligned} H^{2,1}(\mathbb{P}^1, \mathbb{Z}/p\mathbb{Z}) &\cong h^0(\mathbb{P}^1, Z_{\text{equi}}(\mathbb{A}^1, 0)) \\ &\cong h^0(k, Z_{\text{equi}}(\mathbb{P}^1 \times \mathbb{A}^1, 1)) \end{aligned}$$

by duality. In the simplicial complex Z, the flat cycle is homological to to $(1:0) \times \mathbb{A}^1$ which corresponds to π. Therefore, we have written π to be a cycle of $\mathbb{P}^1 \times \mathbb{A}^1$ over k of equidimension 1. This allows us to compute the total Steenrod operation

$$\tilde{Z}^p = \{((x : y), (a_1, \ldots, a_p)) : a_1x = y, a_1 = \cdots = a_p\} \subseteq \mathbb{P}^1 \times \mathbb{A}^p.$$

Set $\Delta : E_{\mathbb{Z}/p\mathbb{Z}} \times \mathbb{A}^1 \times \mathbb{P}^1 \to (E \times_{\mathbb{Z}/p\mathbb{Z}} \mathbb{A}^p) \times \mathbb{P}^1$ be the diagonal morphism, then by direct computation we have

$$E \times_{\mathbb{Z}/p\mathbb{Z}} \tilde{Z}^p = \Delta_*(E_{\mathbb{Z}/p\mathbb{Z}} \times Z)$$

set-theoretically. Define

$$\varphi : (E_{\mathbb{Z}/p\mathbb{Z}} \times \mathbb{A}^p) \times \mathbb{P}^1 \to E_{\mathbb{Z}/p\mathbb{Z}} \times \mathbb{P}^1$$

with $\mathbb{Z}/p\mathbb{Z}$-action by permutation of coordinates, then φ is a vector bundle of rank p. Therefore, φ induces an isomorphism φ^* on the level of motivic cohomology. We have $(\varphi^*)^{-1}(\Delta_*(E_{\mathbb{Z}/p\mathbb{Z}}) \times \mathbb{Z}) \in \mathrm{CH}^p(E_{\mathbb{Z}/p\mathbb{Z}} \times \mathbb{P}^1)/p$, and this is the cycle

[13] This is essentially $\mathcal{O}(1)$ over $\mathbb{P}^1$.

corresponding to $s^{\text{cycl}}(\pi)$. Indeed, we see that $(\varphi^*)^{-1}$ untwists the existing twisting on the vector bundle, which matches the description of the total Steenrod operation on π.

Since k contains all roots of $x^p = 1$, then the vector bundle $E \times_{\mathbb{Z}/p\mathbb{Z}} \mathbb{A}^p \to E_{\mathbb{Z}/p\mathbb{Z}}$ is the direct sum $\bigoplus_{i=0}^{p-1} L^i$ where $L \in \operatorname{Pic}(B\mu_p)$ comes from $\mathcal{O}_{\mathbb{P}^\infty}(1)$, given by non-zero sections. The computation here is the same as the topological case, where we split it into a direct sum of line bundles. Denote t to be the zero section $E_{\mathbb{Z}/p\mathbb{Z}} \times \mathbb{P}^1 \to E_{\mathbb{Z}/p\mathbb{Z}} \times \mathbb{A}^1 \times \mathbb{P}^1$ of the trivial vector bundle, then we have

$$\begin{aligned}(\varphi^*)^{-1}\Delta_*(E_{\mathbb{Z}/p\mathbb{Z}} \times Z) &= t^*\Delta^*\Delta_*(E_{\mathbb{Z}/p\mathbb{Z}} \times Z) \text{ as pullback of the zero section}\\ &= t^*\Delta^*\Delta_* t_*(E_{\mathbb{Z}/p\mathbb{Z}} \times \pi) \text{ as } Z \text{ is simplicial homologous to a point over fiber}\\ &= (\pi \times E_{\mathbb{Z}/p\mathbb{Z}})t^*\Delta^*(\Delta \circ t)_*(1) \text{ by projetion formula}\\ &= \pi \cdot (p-1)! \cdot \mathcal{O}(1)^{p-1} \text{ by decomposition of Chern class}\\ &= -\pi \cdot \mathcal{O}(1)^{p-1} \text{ by Lemma 1.28}\\ &= -\pi\omega_{2(p-1)}\end{aligned}$$

on the level of motivic cohomology. □

We now get what we need for part a. by Cartan formula.

Corollary 3.38. We have

$$\mathrm{Sq}^i(\pi) = \begin{cases}\pi, & i = 0\\ 0, & i \neq 0\end{cases}, \quad \mathcal{P}^i(\pi) = \begin{cases}\pi, & i = 0\\ 0, & i \neq 0\end{cases}$$

.

To obtain the stability of λ, it suffices to consider the boundary map

$$\partial : H^{*,*}(E_{\mathbb{Z}/p\mathbb{Z}} \times \mathbb{G}_m, \mathbb{Z}/p\mathbb{Z}) \to H^{*+1,*}(E_{\mathbb{Z}/p\mathbb{Z}} \times \mathbb{P}^1, \mathbb{Z}/p\mathbb{Z})$$

induced by the Mayer-Vietoris sequence from the open cover $\mathbb{P}^1 = (\mathbb{P}^1\backslash\{0\}) \cup (\mathbb{P}^1\backslash\{\infty\})$. We can write down the boundary map exlicitly because both sides admit splittings. Under the given splitting, the boundary map is given by

$$\begin{aligned}\partial : H^{p,q}(E_{\mathbb{Z}/p\mathbb{Z}}) \oplus H^{p-1,q-1}(E_{\mathbb{Z}/p\mathbb{Z}}) \cdot \lambda &\to H^{p+1,q}(E_{\mathbb{Z}/p\mathbb{Z}}) \oplus H^{p-1,q-1}(E_{\mathbb{Z}/p\mathbb{Z}}) \cdot \pi\\ (a, b) &\mapsto (0, b).\end{aligned}$$

Suppose we want to know the total Steenrod operation, then $E_{\mathbb{Z}/p\mathbb{Z}} \times \mathbb{G}_m$ is split into two portions, with and without λ. The summand without λ can be pullback to a point, therefore the total Steenrod operation is well-understood. In the summand with λ, the boundary map above reduces the stability of λ to the stability on $\mathbb{P}^1$. It now suffices to show that λ is well-behaved with respect to the Steenrod operations.

Proposition 3.39. Suppose $\lambda \in \mathrm{Sm}/k$ is quasi-projective with open covering $\{U, V\}$. Let

$$\partial : H^{*,*}(U \cap V, \mathbb{Z}/p\mathbb{Z}) \to H^{*+1,*}(X, \mathbb{Z}/p\mathbb{Z})$$

be the boundary map of the corresponding Mayer-Vietoris sequence, then $s^{\text{cycl}} \circ \partial = \partial \circ s^{\text{cycl}}$.

Proof. Suppose $Z \in Z_{\text{equi}}(\mathbb{A}^n, 0)(\Delta^i \times X)$ is a cycle representing a class in $H^{2n-i,n}(X, \mathbb{Z}/p\mathbb{Z})$. Moreover, suppose $Z|_U = 0$ and $Z|_V = 0$, i.e., Z is a boundary on both U and V. Therefore, $Z|_U = \delta(Z'_U)$ and $Z|_V = \delta(Z'_V)$ for some Z'_U, Z'_V. Any cycle on X being zero on both U and V must be a boundary, which can be computed by snake lemma: map it to both U and V, taking the difference of restrictions on the intersection, then applying ∂. Therefore, $\partial(Z'_U|_{U\cap V} - Z'_V|_{U\cap V}) = Z$. The idea being, the restrction commutes with total Steenrod operation, so we just need to find a cycle with boundary image as Z itself on $U \cap V$, which is given by the formula above.

With the setting above, we may suppose $y \in h^*(C_* Z_{\text{equi}}(\mathbb{A}^n, 0)(U \cap V))$ can be written as $y_U|_{U\cap V} - y_V|_{U\cap V}$ where $y_U \in Z_{\text{equi}}(\mathbb{A}^n, 0)(\Delta^* \times U)$ and $y_V \in Z_{\text{equi}}(\mathbb{A}^n, 0)(\Delta^* \times V)$, then

$$\delta(y_U) = y|_U, \quad \delta(y_V) = y|_V.$$

Therefore

$$\begin{aligned}(\partial\circ s^{\mathrm{cycl}})(y_U|_{U\cap V}-y_V|_{U\cap V}) &= \partial(E\times_{\mathbb{Z}/p\mathbb{Z}}\widetilde{y_U|_{U\cap V}-y_V|_{U\cap V}}^p)\\ &= \partial(E\times_{\mathbb{Z}/p\mathbb{Z}}(\widetilde{y_U|_{U\cap V}}^p-\widetilde{y_V|_{U\cap V}}^p)) \text{ by Lemma 3.23}\\ &= E\times_{\mathbb{Z}/p\mathbb{Z}}\tilde{y}^p\\ &= (s^{\mathrm{cycl}}\circ\partial)(y_U|_{U\cap V}-y_V|_{U\cap V}).\end{aligned}$$

□

Corollary 3.40. We have

$$\mathrm{Sq}^i(\lambda)=\begin{cases}\lambda, & i=0\\ 0, & i\neq 0\end{cases},\quad \mathcal{P}^i(\lambda)=\begin{cases}\lambda, & i=0\\ 0, & i\neq 0\end{cases}$$

.

We will now introduce the simplicial suspension and demonstrate its stability. We have $\Delta^2\subseteq\mathbb{A}^3=\{(t_0,t_1,t_2)\}$ given by the subspace defined by $t_0+t_1+t_2=1$. We then define $\partial\Delta^2$ to be

$$\{t_0t_1t_2=0\}\cap\Delta^2.$$

Therefore $\partial\Delta^2=\Delta^1\cup\sigma\Delta^2$, where $\Delta^1=\{t_2=0\}$ and $\Delta^2=\{t_0t_1=0\}$, and $\Delta^1\cap\sigma\Delta^2=\partial\Delta^1=\{*\}\coprod\{*\}$. Here we have a closed cover of a scheme, and it also induced a Mayer-Vietoris sequence

$$\begin{tikzcd} \cdots \arrow[r] & H^{p,q}(X\times\partial\Delta^2,\mathbb{Z}/\ell\mathbb{Z}) \arrow[r, "\alpha"] & H^{p,q}(X\times\Delta^1,\mathbb{Z}/\ell\mathbb{Z})\oplus H^{p,q}(X\times\sigma\Delta^2,\mathbb{Z}/\ell\mathbb{Z}) \arrow[dl] \\ & H^{p,q}(X\times\partial\Delta^1,\mathbb{Z}/\ell\mathbb{Z}) \arrow[r, "\delta"] & H^{p+1,q}(X\times\partial\Delta^2,\mathbb{Z}/\ell\mathbb{Z}) \arrow[r, "\partial"] & \cdots \end{tikzcd}$$

using homology groups of simplicial schemes, where α is induced by the restriction. We see that $H^{p,q}(X\times\sigma\Delta^2,\mathbb{Z}/\ell\mathbb{Z})\cong H^{p,q}(X,\mathbb{Z}/\ell\mathbb{Z})$ because the intersection of two edges of a triangle is at a point and therefore contractible as scheme, and since Δ^1 is contractible, the second term above becomes

$$H^{p,q}(X,\mathbb{Z}/\ell\mathbb{Z})\oplus H^{p,q}(X,\mathbb{Z}/\ell\mathbb{Z}).$$

From the Mayer-Vietoris sequence above, we see that $\mathrm{im}(\alpha)=\{(y,-y):y\in H^{p,q}(X,\mathbb{Z}/\ell\mathbb{Z})\}$. Because $H^{p,q}(X\times\partial\Delta^1,\mathbb{Z}/\ell\mathbb{Z})$ is also a direct sum of two summands of motivic cohomology, then we can quotient the middle two terms of the sequence above by a summand, then we get a new exact sequence

$$0\longrightarrow H^{p,q}(X,\mathbb{Z}/\ell\mathbb{Z})\longrightarrow H^{p,q}(X\times\partial\Delta^1,\mathbb{Z}/\ell\mathbb{Z})\xrightarrow{\delta} H^{p+1,q}(X\times\partial\Delta^2,\mathbb{Z}/\ell\mathbb{Z})\xrightarrow{(0,0,0)^*} H^{p+1,q}(X,\mathbb{Z}/\ell\mathbb{Z})\longrightarrow\cdots$$

where $(0,0,0)^*$ pulls back to the origin, and since the first morphism is the pullback of the projection, then it factors through $H^{p,q}(X\times\sigma\Delta^2,\mathbb{Z}/\ell\mathbb{Z})$. The pullback $(0,0,0)^*$ must be a surjection because the existence of a rational point, therefore there is an isomorphism

$$\Sigma_s: H^{p,q}(X,\mathbb{Z}/\ell\mathbb{Z})\cong H^{p+1,q}((X\times\partial\Delta^2)/(X\times\{*\}),\mathbb{Z}/\ell\mathbb{Z}).$$

The stability follows by a similar argument as Proposition 3.39.

We now compute the Steenrod operations on $B\mu_p$.

Lemma 3.41. We have

$$\mathrm{Sq}^i(\omega_1)=\begin{cases}\omega_1, & i=0\\ \omega_2, & i=1\\ 0, & \text{else}\end{cases},\quad \mathrm{Sq}^i(\omega_2)=\begin{cases}\omega_2, & i=0\\ \omega_2^2, & i=2\\ 0, & \text{else}\end{cases},\quad \mathcal{P}^i(\omega_1)=\begin{cases}\omega_1, & i=0\\ 0, & \text{else}\end{cases},\quad \mathcal{P}^i(\omega_2)=\begin{cases}\omega_2, & i=0\\ \omega_2^p, & i=1\\ 0, & \text{else}\end{cases}$$

.

Proof. Recall that Sq^i and $\mathcal{P}^i$ were defined using D_n. By definition, we see that $\mathcal{P}^i(\omega_1), \mathcal{P}^i(\omega_2), \mathrm{Sq}^{2i}(\omega_1), \mathrm{Sq}^{2i}(\omega_2) = 0$ whenever $i \geqslant 2$. We have an embedding $j : \mathbb{G}_m \cong \mathcal{O}_{\mathbb{P}^1}(\ell)^\times \to \mathcal{O}_{\mathbb{P}^\infty}(\ell)^\times \cong B\mu_\ell$, such that $j^*(\omega_1) = \lambda$. Checking degrees in the second component, we then have $\mathcal{P}^i(\omega_1) = 0$ and $\mathcal{P}^i(\omega_2) = 0$ for $i < 0$. Moreover, we already know $\mathcal{P}^0(\lambda) = \lambda$ and $\mathrm{Sq}^0(\lambda) = \lambda$, therefore $\mathcal{P}^0(\omega_1) = \omega_1$ and $\mathrm{Sq}^0(\omega_1) = \omega_1$. We then conclude that $\mathrm{Sq}^1(\omega_1) = \omega_2$. By Corollary 3.27, we have $\mathrm{Sq}^1(\omega_1) = \mathcal{P}^1(\omega_1) = 0$. Taking the structure map $f : B\mu_p \to \mathbb{P}^\infty$, by the Gysin triangle we have f^* is injective on motivic cohomology with $\mathbb{Z}/p\mathbb{Z}$-coefficient, so to compute the image of ω_2, we just have to pullback along the embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^\infty$. We already know that $\mathcal{O}_{\mathbb{P}^\infty}(1)$ pulls back to ω_2, so to compute the Steenrod operations of ω_2, it suffices to compute that of $\mathcal{O}_{\mathbb{P}^\infty}(1)$ by naturality. By Corollary 3.26, we have $\mathcal{P}^1(\omega_2) = \omega_2^p$ and $\mathrm{Sq}^2(\omega_2) = \omega_2^2$. Since $\mathcal{O}_{\mathbb{P}^\infty}(1)$ lies in bidegree $(2,1)$, then $\mathrm{Sq}^1(\mathcal{O}_{\mathbb{P}^\infty}(1)) = 0$. Finally, the fact that $\mathrm{Sq}^0(\mathcal{O}_{\mathbb{P}^\infty}(1)) = \mathcal{O}_{\mathbb{P}^\infty}(1)$, $\mathrm{Sq}^i(\mathcal{O}_{\mathbb{P}^\infty}(1)) = 0$ and $\mathcal{P}^i(\mathcal{O}_{\mathbb{P}^\infty}(1)) = 0$ for $i < 0$ follows from Proposition 3.39. $\square$

Using Lemma 3.41 and Proposition 3.35, we get to describe Steenrod operations for all classes on $B\mu_p$.

Corollary 3.42. We have

$$
\begin{aligned}
\mathrm{Sq}^{2i}(\omega_{2t}) &= \binom{t}{i}\omega_{2t+2i}, \quad \mathrm{Sq}^{2i+1}(\omega_{2t}) = 0,\\
\mathcal{P}^i(\omega_{2t}) &= \binom{t}{i}\omega_{2t+2i(p-1)}, \quad \beta\mathcal{P}^i(\omega_{2t}) = 0,\\
\mathrm{Sq}^{2i}(\omega_{2t+1}) &= \binom{t}{i}\omega_{2t+2i+1}, \quad \mathrm{Sq}^{2i+1}(\omega_{2t+1}) = \binom{t}{i}\omega_{2t+2i+2},\\
\mathcal{P}^i(\omega_{2t+1}) &= \binom{t}{i}\omega_{2t+2i(p-1)+1}, \quad \beta\mathcal{P}^i(\omega_{2t+1}) = \binom{t}{i}\omega_{2t+2i(p-1)+2}.
\end{aligned}
$$

3.4 Adem Relations

Suppose a cycle $x \in H^{2n-i,n}(X, \mathbb{Z}/p\mathbb{Z})$, so by Theorem 2.64 it can be represented by $Z \in Z_{\text{equiu}}(\mathbb{A}^n, 0)(\Delta^i \times X)$, then $s^{\text{cycl}}s^{\text{cycl}}(x)$ is represented by

$$E \times_{\mathbb{Z}/p\mathbb{Z}} \widetilde{E \times_{\mathbb{Z}/p\mathbb{Z}} \tilde{Z}^{pp}} \in Z_{\text{equi}}((E \times_{\mathbb{Z}/p\mathbb{Z}} (E \times_{\mathbb{Z}/p\mathbb{Z}} \mathbb{A}^{np})^{\oplus p})/(E_{\mathbb{Z}/p\mathbb{Z}} \times E_{\mathbb{Z}/p\mathbb{Z}}), 0)(\Delta^i \times E_{\mathbb{Z}/p\mathbb{Z}} \times E_{\mathbb{Z}/p\mathbb{Z}} \times X).$$

Definition 3.43. We define the action of $\mathbb{Z}/p\mathbb{Z}\cdot\pi \oplus \mathbb{Z}/p\mathbb{Z}\cdot\rho \subseteq S_{p^2}$ on $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ to be generated by $\pi(i,j) = (i+1,j)$ and $\rho(i,j) = (i, j+1)$.

For any $E_{\mathbb{Z}/p\mathbb{Z}}$-model E, then $E \times E$ is an $E_{\mathbb{Z}/p\mathbb{Z}\oplus\mathbb{Z}/p\mathbb{Z}}$-model. The vector bundle $(E \times_{\mathbb{Z}/p\mathbb{Z}} (E \times_{\mathbb{Z}/p\mathbb{Z}} \mathbb{A}^{np})^{\oplus p})/(E_{\mathbb{Z}/p\mathbb{Z}} \times E_{\mathbb{Z}/p\mathbb{Z}})$ can be identified with $((E \times E) \times_{\mathbb{Z}/p\mathbb{Z}\oplus\mathbb{Z}/p\mathbb{Z}} \mathbb{A}^{np^2})/(E_{\mathbb{Z}/p\mathbb{Z}} \times E_{\mathbb{Z}/p\mathbb{Z}})$, with action given by cyclic permutation separately, and $s^{\text{cycl}}s^{\text{cycl}}(x) = E \times E \times_{\mathbb{Z}/p\mathbb{Z}\oplus\mathbb{Z}/p\mathbb{Z}} \tilde{Z}^{p^2}$, where for $Z = \sum_i a_i Z_i$, we write $\tilde{Z}^{p^2} = \sum_{i_1,\dots,i_p} a_{i_1}\cdots a_{i_p}\,\mathrm{Cycl}(\Delta_X \times \Delta_s)(Z_{i_1} \times \cdots Z_{i_{p^2}})$.

More generally, let W be an $E_{S_{p^2}}$-model, then there is a natural map of simplicial sets

$$
\begin{aligned}
C_* Z_{\text{equi}}(\mathbb{A}^n, 0)(X) &\to C_* Z_{\text{equi}}((W \times_{S_{p^2}} \mathbb{A}^{np^2})/W_{S_{p^2}}, 0)(W_{S_{p^2}} \times X)\\
Z &\mapsto W \times_{S_{p^2}} \tilde{Z}^{p^2}
\end{aligned}
$$

by S_{p^2}-invariance. Hence we obtain a diagram

$$
\begin{tikzcd}
H^{2n-i,n}(X, \mathbb{Z}/p\mathbb{Z}) \arrow[r, "s^{S_{p^2}}"] \arrow[dr, "s^{\pi\times\rho}"'] & H^{2np^2-i,np^2}(W_{S_{p^2}} \times X, \mathbb{Z}/p\mathbb{Z}) \arrow[d, "\text{res}"] \\
& H^{2np^2-i,np^2}(W_{\mathbb{Z}/p\mathbb{Z}\oplus\mathbb{Z}/p\mathbb{Z}} \times X, \mathbb{Z}/p\mathbb{Z})
\end{tikzcd}
$$

where $s^{\pi\times\rho} = s^{\text{cycl}} \circ s^{\text{cycl}}$. Similar to the topological case, we define the following.

Definition 3.44. For any $x \in H^{2n-i,n}(X, \mathbb{Z}/p\mathbb{Z})$, we define $D_{ik}(x)$ by the Künneth decomposition

$$s^{\pi\times\rho}(x) = \sum_{i,k} \omega_i \times \omega_k \cdot D_{ik}(X).$$

Lemma 3.45. We have $D_{ik} = D_{ki}$.

Proof. The swapping map

$$\begin{aligned} \sigma : \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} &\to \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \\ (a, b) &\mapsto (b, a) \end{aligned}$$

is the restriction of a conjugation action by $\tau \in S_{p^2}$. The induced pullback $\operatorname{Inn}(\tau)^*$ then must be an identity map. □

We can now prove the Adem relations for $p = 2$, c.f., [Voe03, Theorem 10.2].

Theorem 3.46 (Adem Relation for $p = 2$)**.** We have

1. $\mathrm{Sq}^{2k-1}\mathrm{Sq}^{2c-1} = \sum_i \binom{2c-2-i}{2k-1-2i} \mathrm{Sq}^{2k+2c-i-2}\mathrm{Sq}^i$ for $k < 2c$;
2. $\mathrm{Sq}^{2k}\mathrm{Sq}^{2c} = \sum_i \binom{c-i-1}{k-2i} \mathrm{Sq}^{2k+2c-2i}\mathrm{Sq}^{2i} + \tau \sum_i \binom{c-i}{k-2i+1} \mathrm{Sq}^{2k+2c-2i+1}\mathrm{Sq}^{2i-1}$ for $k < 2c$;
3. $\mathrm{Sq}^{2k}\mathrm{Sq}^{2c-1} = \sum_i \binom{2c-i-2}{2k-2i} \mathrm{Sq}^{2c+2k-i-1}\mathrm{Sq}^i + \rho \sum_i \binom{c-i-1}{k-2i+1} \mathrm{Sq}^{2c+2k-2i+1}\mathrm{Sq}^{2i-1}$ for $k < 2c - 1$;
4. $\mathrm{Sq}^{2k+1}\mathrm{Sq}^{2c} = \sum_i \binom{2c-1-i}{2k+1-2i} \mathrm{Sq}^{2k+2c-i+1}\mathrm{Sq}^i + \rho \sum_i \binom{c-i}{k-2i+1} \mathrm{Sq}^{2k+2c-2i+1}\mathrm{Sq}^{2i-1}$ for $k < 2c$.

Proof. Suppose $x \in H^{n,m}(X, \mathbb{Z}/2\mathbb{Z})$. By definition, $s^{\mathrm{cycl}}(x) = \sum_k \omega_{2m-k} \times \mathrm{Sq}^k(x)$, then

$$s^{\mathrm{cycl}} s^{\mathrm{cycl}}(x) = \sum_{j,k} \omega_{4m-j} \times \mathrm{Sq}^j(\omega_{2m-k} \times \mathrm{Sq}^k(x)).$$

By Cartan formula, we have

$$\begin{aligned}
\mathrm{Sq}^{2k}(\omega_{2m-2i} \times \mathrm{Sq}^{2i}(x)) &= \sum_j \binom{m-i}{j} \omega_{2m-2i+2j} \times \mathrm{Sq}^{2k-2j}\mathrm{Sq}^{2i}(x), \\
\mathrm{Sq}^{2k}(\omega_{2m-2i+1} \times \mathrm{Sq}^{2i-1}(x)) &= \sum_j \binom{m-i}{j} \omega_{2m-2i+2j+1} \times \mathrm{Sq}^{2k-2j}\mathrm{Sq}^{2i-1}(x) \\
&\quad + \tau \sum_j \binom{m-i}{j-1} \omega_{2m-2i+2j} \times \mathrm{Sq}^{2k-2j+1}\mathrm{Sq}^{2i-1}(x), \\
\mathrm{Sq}^{2k-1}(\omega_{2m-2i} \times \mathrm{Sq}^{2i}(x)) &= \sum_j \binom{m-i}{j} \omega_{2m-2i+2j} \times \mathrm{Sq}^{2k-2j-1}\mathrm{Sq}^{2i}(x), \\
\mathrm{Sq}^{2k-1}(\omega_{2m-2i+1} \times \mathrm{Sq}^{2i-1}(x)) &= \sum_j \binom{m-i}{j} \omega_{2m-2i+2j+1} \times \mathrm{Sq}^{2k-2j-1}\mathrm{Sq}^{2i-1}(x) \\
&\quad + \sum_j \binom{m-i}{j-1} \omega_{2m-2i+2j} \times \mathrm{Sq}^{2k-2j}\mathrm{Sq}^{2i-1}(x) \\
&\quad + \rho \sum_j \binom{m-i}{j-1} \omega_{2m-2i+2j} \times \mathrm{Sq}^{2k-2j+1}\mathrm{Sq}^{2i-1}(x).
\end{aligned}$$

Using Lemma 3.45 and the equations above, we expand and get

$$D_{2(2m-k)+1,2(2m-\ell)+1}(x) = \sum_i \binom{m-i}{m+i-\ell} \mathrm{Sq}^{2k-2(m+i-\ell)-1}\mathrm{Sq}^{2i-1}(x).$$

By Lemma 3.45, we have

$$\sum_i \binom{m-i}{m+i-\ell} \operatorname{Sq}^{2k-2(m+i-\ell)-1} \operatorname{Sq}^{2i-1}(x) = \sum_i \binom{m-i}{m+i-k} \operatorname{Sq}^{2k-2(m+i-\ell)-1} \operatorname{Sq}^{2i-1}(x).$$

Assume $m = s^2 - 1 + c$, and $\ell = m + c$, then by Lemma 1.17, we have

$$\binom{m-i}{m-i+\ell} \equiv \binom{2^s - 1 - (i-\ell)}{i-c}, \quad \binom{m-i}{m-k+i} \equiv \binom{2^s - 1 - (i-c)}{k-2i}$$

where the former is zero unless $i = c$ and s is large enough. Therefore,

$$\operatorname{Sq}^{2k-1} \operatorname{Sq}^{2c-1}(x) = \sum_i \binom{2^s - 1 - (i-\ell)}{k-2i} \operatorname{Sq}^{2k+2c-2i-1} \operatorname{Sq}^{2i-1}(x).$$

Note that we alrady assumed $k < 2c$. Also note that if $k < 2i$, then everything becomes zero, therefore we get to assume $2i \leqslant k < 2c$. In this case,

$$\binom{2^s - 1 - (i-c)}{k-2i} \equiv \binom{c-i-1}{k-2i}$$

whenever s is large enough by Lemma 1.17. Therefore

$$\begin{aligned}
\operatorname{Sq}^{2k-1} \operatorname{Sq}^{2c-1}(x) &= \sum_i \binom{c-i-1}{k-2i} \operatorname{Sq}^{2k+2c-2i-1} \operatorname{Sq}^{2i-1}(x) \\
&= \sum_i \binom{2c-2i-1}{2k-4i+1} \operatorname{Sq}^{2k+2c-2i-1} \operatorname{Sq}^{2i-1}(x) \text{ as } \binom{n}{m} \equiv \binom{2n+1}{2m+1} \pmod 2 \text{ with Lemma 1.17} \\
&= \sum_i \binom{2c-2-i}{2k-1-2i} \operatorname{Sq}^{2k+2c-2-i} \operatorname{Sq}^{i}(x) \text{ since } \binom{2n}{2m+1} \equiv 0 \pmod 2.
\end{aligned}$$

This proves part 1. From the equations above again, taking the even indices, we get

$$\begin{aligned}
D_{2(2m-k),2(2m-\ell)}(x) = &\sum_i \binom{m-i}{m+i-\ell} \operatorname{Sq}^{2k-2(m+i-\ell)} \operatorname{Sq}^{2i}(x) \\
&+ \tau \sum_i \binom{m-i}{m+i-\ell-1} \operatorname{Sq}^{2k-2(m+i-\ell)+1} \operatorname{Sq}^{2i-1}(x).
\end{aligned}$$

Interchanging k and ℓ by Lemma 3.45, this is the same as

$$\sum_i \binom{m-i}{m+i-k} \operatorname{Sq}^{2k-2(m+i-\ell)} \operatorname{Sq}^{2i}(x) + \tau \sum_i \binom{m-i}{m+i-k-1} \operatorname{Sq}^{2k-2(m+i-\ell)+1} \operatorname{Sq}^{2i-1}(x).$$

Assume $m = 2^s - 1 + c$ and $\ell = m + c$, then by Lemma 1.17 we have

$$\binom{m-i}{m+i-\ell} \equiv \begin{cases} 1, & i = \ell \\ 0, & i \neq \ell \end{cases} \text{ for large} s.$$

On the other hand,

$$\binom{m-i}{m+i-\ell-1} \equiv \begin{cases} 1, & i - c = 2^r, r = 0, \ldots, s-1 \\ 0, & \text{else} \end{cases}.$$

If $k < 2c$ and s is large enough, we have

$$\binom{m-i}{m+i-k} \equiv \binom{c-i-1}{k-2i}$$

by Lemma 1.17 as before, and

$$\binom{m-i}{m+i-k-1} \equiv \binom{m-i}{k+1-2i} \equiv \binom{2^s-1-(i-c)}{k+1-2i} \equiv \binom{c-i-1}{k+1-2i}$$

where the last step holds unless $c = i$ and $k = 2c-1$. Therefore, we have

$$\begin{aligned}\mathrm{Sq}^{2k}\mathrm{Sq}^{2c}(x) + \tau\sum_{r=1}^{s}\mathrm{Sq}^{2k-2^r+1}\mathrm{Sq}^{2c+2^r-1}(x) &= \sum_i \binom{c-i-1}{k-2i}\mathrm{Sq}^{2c+2k-2i}\mathrm{Sq}^{2i}(x)\\ &\quad + \tau\sum_i\binom{c-i-1}{k-2i+1}\mathrm{Sq}^{2k+2c-2i+1}\mathrm{Sq}^{2i-1}(x)\end{aligned}$$

unless the case $k = 2c-1$, where there is an additional term

$$\tau\,\mathrm{Sq}^{4c-1}\mathrm{Sq}^{2i-1}(x).$$

We observe that $2k-2^r+1 < 2(2c+2^r-1)$ whenever $k < 2c$, so we can apply part 1. to express

$$\mathrm{Sq}^{2k-2^r+1}\mathrm{Sq}^{2c+2^r-1}(x) = \sum_i \binom{c+2^{r-1}-i-1}{k-2^{r-1}+1-2i}\mathrm{Sq}^{2k+2c+1-2i}\mathrm{Sq}^{2i-1}(x).$$

This gives a relation

$$\begin{aligned}\mathrm{Sq}^{2k}\mathrm{Sq}^{2c}(x) &= \sum_i\binom{c-i-1}{k-2i}\mathrm{Sq}^{2c+2k-2i}\mathrm{Sq}^{2i}(x)\\ &\quad + \tau\sum_i\left(\sum_{r=0}^{s-1}\binom{c+2^r-i-1}{k-2^r+1-2i} + \binom{c-i-1}{k-2i+1}\right)\mathrm{Sq}^{2k+2c+1-2i}\mathrm{Sq}^{2i-1}(x)\end{aligned}$$

unless $k = 2c-1$, in which case there is an additional term

$$\tau\,\mathrm{Sq}^{4c-1}\mathrm{Sq}^{2c-1}.$$

We want to prove that

$$\sum_{r=0}^{s-1}\binom{c+2^r-1-i}{k-2^r+1-2i} \equiv \binom{c+2^r-1-i}{k-2^r+1-2i} \equiv \binom{c-i-1}{k-2i+1} + \binom{c-i}{k-2i+1}$$

unless $i = c$ and $k = 2c-1$. Let us denote $a = c-i-1$ and $b = k-2i+1$.

Lemma 3.47. We have

$$\sum_{r\geqslant 0}\binom{a+2^r}{b-2^r} \equiv \binom{a}{b} + \binom{a+1}{b} \pmod 2$$

whenever $a \geqslant 0$.

Subproof.

- Suppose $a = b$. We write a in the form $\sum_j(2^{i_j} + 2^{i_j+1} + \cdots + 2^{i_j+k_j})$ where $i_{j+1} > i_j + k_j + 1$, then by Lemma 1.17, $\binom{a+2^r}{a-2^r} \equiv 1 \pmod 2$ if and only if $r = i_j - 1$ or $r = i_j + k_j$ for some j. We conclude that

$$\sum_r\binom{a+2^r}{a-2^r} \equiv a \equiv \binom{a}{a} + \binom{a+1}{a}.$$

- Suppose $a > b$, then
$$\sum_{r\geqslant 0}\binom{a+2^r}{b-2^r}=\sum_{r\geqslant 0}\binom{a+2^r-1}{b-2^r}+\sum_{r\geqslant 0}\binom{a+2^r-1}{b-2^r-1}.$$
By induction on the pair $(a-b,b)$, the right-hand side is equal to
$$\binom{a-1}{b}+\binom{a}{b}+\binom{a-1}{b-1}+\binom{a}{b-1}=\binom{a}{b}+\binom{a+1}{b}.$$

- Suppose $a = b-1$.
  - ⋆ If $a\equiv 0$, then $\binom{a+2^r}{b-2^r}\equiv 0$ if $r>0$ by Lemma 1.17. In the case $r=0$, we see that $\binom{a+2^r}{b-2^r}\equiv\binom{b}{b-1}=b\equiv 1$. Therefore
$$\sum_{r\geqslant 0}\binom{a+2^r}{b-2^r}\equiv 1\equiv\binom{a}{b}+\binom{a+1}{b}.$$
  - ⋆ If $a\equiv 1$, then we write $a=\sum_j(2^{i_j}+\cdots+2^{i_j+k_j})$ as before, then $\binom{a+2^r}{b-2^r}\equiv 1$ if and only if $r=k_0+1$, where $i_0=0$.

- Suppose $a\leqslant b-2$. By Lemma 1.17, we have
$$\sum_{r\geqslant 0}\binom{a+2^r}{b-2^r}=\binom{a+1}{b-1}+\begin{cases}0, & a\equiv 0, b\equiv 1\\ \sum\limits_{r\geqslant 0}\binom{\lfloor\frac{a}{2}\rfloor+2^r}{\lfloor\frac{b}{2}\rfloor+2^r}, & \text{else}\end{cases}$$
We now proceed by induction on $b-a\geqslant 2$. If $a=b-2$, then $\lfloor\frac{a}{2}\rfloor=\lfloor\frac{b}{2}\rfloor-1$, therefore $\sum\limits_{r\geqslant 0}\binom{a+2^r}{b-2^r}\equiv 1+1\equiv 0\pmod 2$ by the third case. Now suppose $a<b-2$, then $\lfloor\frac{a}{2}\rfloor\leqslant\lfloor\frac{b}{2}\rfloor-2$, so $\sum\limits_{r\geqslant 0}\binom{a+2^r}{b-2^r}\equiv 0\pmod 2$ by induction on $b-a$.

■

We have shown that if s is large, then
$$\mathrm{Sq}^{2k}\mathrm{Sq}^{2c}(x)=\sum_i\binom{c-i-1}{k-2i}\mathrm{Sq}^{2c+2k-2i}\mathrm{Sq}^{2i}(x)+\tau\sum_i\binom{c-i}{k-2i+1}\mathrm{Sq}^{2k+2c+1-2i}\mathrm{Sq}^{2i-1}(x)$$
including the case $k=2c-1$. Now part 2. follows from stability, and part 4. follows from applying Sq^1 to part 2 via stability. It now remains to prove part 3. This uses the following relations derived from Cartan formula.
$$\begin{aligned}D_{2(2m-k),2(2m-\ell)+1}(x)&=\sum_i\binom{m-i}{m+i-\ell}\mathrm{Sq}^{2k-2(m+i-\ell)}\mathrm{Sq}^{2i-1}(x)\\ D_{2(2m-k)+1,2(2m-\ell)}(x)&=\sum_i\binom{m-i}{m+i-\ell}\mathrm{Sq}^{2k-2(m-\ell+i)-1}\mathrm{Sq}^{2i}(x)\\ &\quad+\sum_i\binom{m-i}{m+i-\ell-1}\mathrm{Sq}^{2k-2(m+i-\ell)}\mathrm{Sq}^{2i-1}(x)\\ &\quad+\rho\sum_i\binom{m-i}{m+i-\ell-1}\mathrm{Sq}^{2k-2(m+i-\ell)-1}\mathrm{Sq}^{2i-1}(x).\end{aligned}$$
Again, by Lemma 3.45, the two equations are the same. Therefore
$$\sum_i\binom{m-i}{m+i-\ell}\mathrm{Sq}^{2k-2(m+i-\ell)}\mathrm{Sq}^{2i-1}(x)=\sum_i\binom{m-i}{m+i-k}\mathrm{Sq}^{2k-2(m+i-\ell)-1}\mathrm{Sq}^{2i}(x)$$

$$+\sum_i \binom{m-i}{m+i-k-1} \operatorname{Sq}^{2k-2(m+i-\ell)} \operatorname{Sq}^{2i-1}(x)$$
$$+\rho\sum_i \binom{m-i}{m+i-k-1} \operatorname{Sq}^{2k-2(m+i-\ell)-1} \operatorname{Sq}^{2i-1}(x).$$

Assume $m = 2^s - 1 + c$ and $\ell = m + c$. By the discussion above and Lemma 1.17, we obtain when $k < 2c - 1$ and s is large that

$$\begin{aligned}
\operatorname{Sq}^{2k} \operatorname{Sq}^{2c-1}(x) &= \sum_i \binom{c-i-1}{k-2i} \operatorname{Sq}^{2k+2c-2i-1} \operatorname{Sq}^{2i}(x) \\
&+ \sum_i \binom{c-i-1}{k-2i+1} \left(\operatorname{Sq}^{2k+2c-2i} \operatorname{Sq}^{2i-1}(x) + \rho \operatorname{Sq}^{2k+2c-2i-1} \operatorname{Sq}^{2i-1}(x)\right) \\
&= \sum_i \binom{2c-i-2}{2k-2i} \operatorname{Sq}^{2k+2c-i-1} \operatorname{Sq}^{i}(x) + \rho \sum_i \binom{c-i-1}{k-2i+1} \operatorname{Sq}^{2k+2c-2i-1} \operatorname{Sq}^{2i-1}(x) \\
&\text{since } \binom{n}{m} \equiv \binom{2n+1}{2m}.
\end{aligned}$$

This proves part 3. □

For $p > 2$, the Adem relation below basically follows from the topological case using Cartan formula, c.f., [Voe03, Theorem 10.3].

Theorem 3.48 (Adem Relation for $p > 2$)**.** We have

1. $\mathcal{P}^k\mathcal{P}^c = \sum_r (-1)^{k+r}\binom{(p-1)(c-r)-1}{k-pr}\mathcal{P}^{k+c-r}\mathcal{P}^r$ for $k < pc$;
2. $\mathcal{P}^k\beta\mathcal{P}^c = \sum_r (-1)^{k+r}\binom{(p-1)(c-r)}{k-pr}\beta\mathcal{P}^{k+c-r}\mathcal{P}^r + \sum_r (-1)^{k+r+1}\binom{(p-1)(c-r)-1}{k-pr-1}\mathcal{P}^{k+c-r}\beta\mathcal{P}^r$ for $k \leqslant pc$.

Proof. This is the same as Theorem 1.34: the motivic Cartan formula is the same as the topological Cartan formula for $p > 2$. □

Proposition 3.49. We have $\mathcal{P}^i = 0$ and $\operatorname{Sq}^i = 0$ whenever $i < 0$. Also, $\mathcal{P}^0 = \mathrm{id}$ and $\operatorname{Sq}^0 = \mathrm{id}$.

Proof. This uses a comparison diagram with Voevodsky's construction. □

4 Motivic Steenrod Algebra and Its Dual

In this section, we consider a base field k of characteristic 0 and containing all pth roots of unity. Set ξ to be a primitive pth root of unity. In particular, when $p = 2$, we set $\xi = \rho = -1$.

4.1 Motivic Steenrod Algebra

Definition 4.1. Given a field k, we define its Milnor K-theory to be $K_*^M(k) = \mathbb{Z}\langle k^\times \rangle / \langle a \otimes (1-a) : a \in k^\times \backslash \{1\} \rangle$ is the tensor algebra quotient by a two-sided ideal, where each element of $k^\times$ is of degree 1 in the algebra.

Definition 4.2. The motivic Steenrod algebra $\mathcal{A}_p^{\mathrm{mot}}$ is a bigraded associative algebra defined as follows. For $p = 2$, this is

$$\mathcal{A}_2^{\mathrm{mot}} = \left(K_*^M(k)/2 \cdot \left\langle \tau, \mathrm{Sq}^1, \mathrm{Sq}^2, \ldots \right\rangle\right) / \left\langle \text{Adem relations}, \mathrm{Sq}^{2i} \tau = \tau \, \mathrm{Sq}^{2i} + \tau\rho \, \mathrm{Sq}^{2i-1}, \mathrm{Sq}^1 \tau = \rho \cdot \mathrm{id} + \tau \, \mathrm{Sq}^1 \right\rangle$$

where $\deg(K_n^M(k)/2) = (n,n)$, $\deg(\tau) = (0,1)$, $\deg(\mathrm{Sq}^1) = (1,0)$, $\deg(\mathrm{Sq}^2) = (2,1)$, and so on, c.f., Definition 3.34. For $p > 2$, this is

$$\mathcal{A}_p^{\mathrm{mot}} = \left(K_*^M(k)/p \cdot \left\langle \tau, \beta, \mathcal{P}^1, \mathcal{P}^2, \ldots \right\rangle\right) / \left\langle \text{Adem relations}, \tau \mathcal{P}^i = \mathcal{P}^i \tau, \beta\tau = \xi \cdot \mathrm{id} + \tau\beta \right\rangle.$$

We denote $h^{*,*} = H^{*,*}(k, \mathbb{Z}/p\mathbb{Z})$ to be a graded-commutative algebra.

Theorem 4.3. Suppose k is a field of characteristic not p. We have

$$h^{*,*} = K_*^M(k)/p \cdot [\tau]$$

where τ is the motivic Bott element of degree $(0,1)$, set up by $h^{n,n} = K_n^M(k)/p$.

Proof. See [Voe11, Theorem 6.17]. □

Lemma 4.4. We have $\mathrm{Sq}^i(h^{n,n}) = 0$ and $\mathcal{P}^i(h^{n,n}) = 0$ for $i \neq 0$. Moreover,

$$\mathrm{Sq}^i(\tau) = \begin{cases} \tau, & i = 0 \\ \rho = -1 \in k^\times/2, & i = 1 \\ 0, & \text{else} \end{cases}, \quad \mathcal{P}^i(\tau) = \begin{cases} \tau, & i = 0 \\ 0, & \text{else} \end{cases}, \quad \beta(\tau) = \xi$$

$$\mathrm{Sq}^i(\rho) = \begin{cases} \rho, & i = 0 \\ 0, & \text{else} \end{cases}, \quad \mathcal{P}^i(\xi) = \begin{cases} \xi, & i = 0 \\ 0, & \text{else} \end{cases}, \quad \beta(\xi) = 0.$$

Proof. All of this follows from $h^{p,q} = 0$ if $p > q$ or $q < 0$, Proposition 3.49, and Corollary 3.27. □

For simplicity, let us rewrite Theorem 3.46 in the following form, so that we can compute the motivic Steenrod algebra.

Theorem 4.5 (Theorem 3.46, Alternative Form)**.** Suppose $a < 2b$.

1. If a is even and b is odd, then

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_j \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j + \sum_{j \text{ odd}} \binom{b-1-j}{a-2j} \rho \, \mathrm{Sq}^{a+b-j-1} \mathrm{Sq}^j.$$

2. If a and b are both odd, then

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{j \text{ odd}} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j.$$

3. If a and b are both even, then

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{+} j\tau^j \pmod 2 \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j.$$

4. If a is odd and b is even, then
$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{j \text{ even}} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j + \sum_{j \text{ odd}} \binom{b-1-j}{a-2j-1} \rho \, \mathrm{Sq}^{a+b-j-1} \mathrm{Sq}^j .$$

Proposition 4.6. There is a relation $\mathrm{Sq}^i = \sum_{0<j<i} a_j \mathrm{Sq}^{i-j} \mathrm{Sq}^j$ where $a_j \in \mathbb{F}_2[\tau, \rho]$ if $i \neq 2^k$. Similarly, there is a relation $\mathcal{P}^i = \sum_{0<j<i} a_j \mathcal{P}^{i-j} \mathcal{P}^j$ where $a_j \in \mathbb{F}_p$ if $i \neq p^k$.

Proof. This is the same as the topological case, c.f., Proposition 1.40. □

We conclude that $\mathcal{A}_2^{\mathrm{mot}}$ is generated by Sq^{2^i} and τ as an $K_*^M(k)/2 \cong H_{\text{ét}}^*(k, \mathbb{Z}/2\mathbb{Z})$-algebra. Note that $\tau \in h^{*,*}$, as the cohomology of a point, is not in the center of the Steenrod algebra, which is different from the topological case. Same for $\mathcal{A}_p^{\mathrm{mot}}$ for $p > 2$.

As a linear space, we hope to construct a basis.

Definition 4.7. Suppose $p = 2$. A sequence $I = (i_1, \dots, i_k)$ is called admissible if $i_k \geqslant 1$ and $i_{s-1} \geqslant 2i_s$ for all $s > 1$ (or $I = (0)$). We write $\mathrm{Sq}^I = \mathrm{Sq}^{i_1} \cdots \mathrm{Sq}^{i_k}$ as a monomial. If I is admissible, then we say the monomial above is admissible. We define the length of such I to be $\ell(I) = k$, and the moment of I to be $m(I) = \sum_{s=1}^{k} s i_s$. We also define the degree of I to be $\deg(I) = \sum_{s=1}^{k} i_s$, as well as a technical definition $t(I) = \sum_{s=1}^{k} \lfloor \frac{i_k}{2} \rfloor$.

Suppose $p > 2$. A sequence $I = (\varepsilon_1, i_1, \varepsilon_2, i_2 \dots, \varepsilon_k, i_k)$ with $\varepsilon_j = 0, 1$ for all j is called admissible if $i_k \geqslant 1$ or $\varepsilon_k = 1$, moreover $i_{s-1} \geqslant p i_s + \varepsilon_s$ for all $s > 1$. We write $\mathcal{P}^I = \beta^{\varepsilon_1} \mathcal{P}^{i_1} \cdots \beta^{\varepsilon_k} \mathcal{P}^{i_k}$. Then we define $\mathcal{P}^0 = \mathrm{id}$ is admissible and $\mathcal{P}^I$ is admissible if I is. We define the length to be $\ell(I) = k$, moment to be $m(\mathcal{P}^I) = \sum_r r(i_r + \varepsilon_r)$, and degree to be $\deg(\mathcal{P}^I) = \sum_r \varepsilon_r + 2(p-1) \sum_r i_r$.

Theorem 4.8 ([Voe03], Lemma 11.1 and Corollary 11.5)**.** The admissible monomials form a $h^{*,*}$-basis of $\mathcal{A}_p^{\mathrm{mot}}$ (as left $h^{*,*}$-module).

Proof. By the same proof as in Proposition 1.43, the moments decreate after applying Adem relations, so they generate $\mathcal{A}_p^{\mathrm{mot}}$ as a $h^{*,*}$-module.

Now suppose $p = 2$. We have
$$H^{*,*}(B\mu_2^{\times n}, \mathbb{Z}/2\mathbb{Z}) = h^{*,*}[u_1, \dots, u_n, v_1, \dots, v_n]/(u_i^2 - \tau v_i - \rho u_i)$$
as rings with $\deg(u_i) = (1,1)$ and $\deg(v_i) = (2,1)$ by Künneth formula. Consider the element $w = u_1 \cdots u_n \in H^{n,n}(B\mu_2^{\times n}, \mathbb{Z}/2\mathbb{Z})$. We claim that the set
$$\{\mathrm{Sq}^I(w) : I \text{ admissible of degree} \leqslant n\}$$
is linearly independent, then we conclude by taking $n \to \infty$. To prove this, we proceed by induction on n. If $n = 1$, then the statement is clear. Now suppose that there is a linear relation
$$\sum_{\substack{\deg(I) = q \leqslant n \\ I \text{ admissible}}} a_I \mathrm{Sq}^I(w) = 0.$$
Set $w' = u_2 \cdots u_n$, so $w = u_1 w'$. We wish to prove that $a_I = 0$ for each I. This is proven by descending induction on $\ell(I)$. Suppose that $a_I = 0$ for $\ell(I) > m$, then the above relation takes the form
$$\sum_{\ell(I) = m} a_I \mathrm{Sq}^I(w) + \sum_{\ell(I) < m} a_I \mathrm{Sq}^I(w) = 0. \tag{4.9}$$
We observe that in order for $\mathrm{Sq}^I(u_1) \neq 0$, I must be of the form $(2^s, 0, \dots, 0, 4, \dots, 0, 2, 0, \dots, 0, 1)$ which implies $\mathrm{Sq}^I(u_1) = v_1^2$. By Cartan formula, we have
$$\mathrm{Sq}^I(w) = \mathrm{Sq}^I(u_1 \cdot w') = \sum_{I'} \rho^{\deg(I) - \deg(I')} \sum_J \tau^{t(I') - t(J) - t(I'-J)} \mathrm{Sq}^J(u_1) \mathrm{Sq}^{I'-J}(w')$$

where $I' = (i'_1, \dots, i'_m)$ satisfies $i'_s = i_s$ unless i_s is odd and $i'_s = i_s - 1$, in which case j_s must be odd. We denote g to be the projetion of $H^{*,*}(B\mu_2^{\times n}, \mathbb{Z}/2\mathbb{Z})$ onto $h^{*,*}v_1^{2^{m-1}} \times H^{*,*}(B\mu_2^{\times(n-1)}, \mathbb{Z}/2\mathbb{Z})$, then we have

$$g(\mathrm{Sq}^I(u, w')) = \begin{cases} 0, & \ell(I) < m \\ v_1^{2^{m-1}} \mathrm{Sq}^{I-J_m}(w'), & \ell(I) = m \end{cases}$$

where $J_m = (2^{m-1}, \cdots, 4, 2, 1)$. Applying g to Equation (4.9), we have

$$\sum_{\substack{\ell(I)=m \\ I \text{ admissible}}} a_I \mathrm{Sq}^{I-J_m}(w') = 0.$$

Now each $I - J_m$ is again admissible, so we have $a_I = 0$ by the inductive hypothesis.

For the case where $p > 2$, the proof is essentially the same as Theorem 1.44. □

Definition 4.10 ([Voe03], Page 46)**.** Set $A^{*,*} = (\mathcal{A}_p^{\mathrm{mot}})^{*,*}$ to be the motivic Steenrod algebra. This is a left $h^{*,*}$-module, inducing a right $h^{*,*}$-module structure as well by skew-commutativity, which we will spell out later. We have a pairing

$$A^{*,*} \otimes_{h^{*,*}} A^{*,*} \times A^{*,*} \otimes_{\mathbb{F}_p} A^{*,*} \to A^{*,*} \otimes_{h^{*,*}} A^{*,*}$$
$$(a \otimes b, c \otimes d) \mapsto ac \otimes bd$$

However, since $h^{*,*} \nsubseteq Z(\mathcal{A}_p^{\mathrm{mot}})$ is not in the center, so the $\mathbb{F}_p$-portion does not descend to $h^{*,*}$, which means $A^{*,*} \otimes_{h^{*,*}} A^{*,*}$ is not a ring: it does not have a multiplicative structure. Therefore, the motivic Steenrod algebra is not a well-defined Hopf algebra in its natural structure.

To solve this, we define the operator-like elements to be elements in $(A^{*,*} \otimes_{h^{*,*}} A^{*,*})_r$, which are the elements $x \in A^{*,*} \otimes_{h^{*,*}} A^{*,*}$ such that for any elements $y, z \in A^{*,*} \otimes_{\mathbb{F}_p} A^{*,*}$ that match over $h^{*,*}$, that is, $\pi(y) = \pi(z)$ for $\pi : A^{*,*} \otimes_{\mathbb{F}_p} A^{*,*} \to A^{*,*} \otimes_{h^{*,*}} A^{*,*}$, we have $xy = xz$. We note that this forms a ring.

The following is a combination of [Voe03, Lemmas 11.6, 11.8, 11.9].

Proposition 4.11. For any $\theta \in A^{*,*}$, there exists a unique element $\psi^*(\theta) = \sum_i \theta'_i \otimes \theta''_i \in \mathcal{A}^{*,*} \otimes h^{*,*} A^{*,*}$ such that the identity

$$\theta(\alpha\beta) = \sum_i (-1)^{\deg(\theta''_i)\deg(\alpha)} \theta'_i(\alpha)\theta''_i(\beta)$$

is satisfied for all $\alpha, \beta \in H^{*,*}(X, \mathbb{Z}/p\mathbb{Z})$ where $X \in \mathrm{Sm}/k$ is quasi-projective. Furthermore,

$$\psi^* : A^{*,*} \to (A^{*,*} \otimes_{h^{*,*}} A^{*,*})_r$$

is a ring homomorphism that is also coassociative and cocommutative.

Proof. Same as the topological case, we define the generator ψ^* by the relations

$$\psi^*(\mathrm{Sq}^{2i}) = \sum_j \tau^{j \pmod 2} \mathrm{Sq}^j \otimes \mathrm{Sq}^{2i-j}$$
$$\psi^*(\mathrm{Sq}^1) = \mathrm{Sq}^1 \otimes 1 + 1 \otimes \mathrm{Sq}^1$$
$$\psi^*(\mathcal{P}^i) = \sum_j \mathcal{P}^j \otimes \mathcal{P}^{i-j}$$
$$\psi^*(\beta) = \beta \otimes 1 + 1 \otimes \beta$$

This is well-defined by the same argument as in Proposition 1.52. We need to show its uniqueness. Similar to the linear independence argument, for any $n \in \mathbb{N}$, there is some quasi-projective $Y \in \mathrm{Sm}/k$ and cohomology class $r \in H^{*,*}(Y, \mathbb{Z}/p\mathbb{Z})$ such that the map

$$\bigoplus_{\deg = i \leqslant n} (A^{*,*})^i \to H^{*,*}(Y, \mathbb{Z}/p\mathbb{Z})$$

$$\theta \mapsto \theta(r)$$

is injective between the free $h^{*,*}$-modules by Theorem 4.8 and Proposition 3.4. Therefore, the Künneth map

$$\begin{aligned}\bigoplus_{\deg = i \leqslant n} (A^{*,*} \otimes_{h^{*,*}A^{*,*}})^i &\to H^{*,*}(Y \times Y, \mathbb{Z}/p\mathbb{Z}) \\ \theta' \otimes \theta'' &\mapsto (-1)^{\deg(\theta'')\deg(r)} \theta'(r)\theta''(r)\end{aligned}$$

is injective. To show that the defined map lands in the operator-like elements and giving a ring homomorphism, we see that for any $y, z \in A^{*,*} \otimes_{\mathbb{F}_p} A^{*,*}$ such that $y = z$ over $h^{*,*}$, with $x \in A^{*,*}$ and $w_1, w_2 \in H^{*,*}(X, \mathbb{Z}/p\mathbb{Z})$, then

$$\begin{aligned}\psi^*(x)(y(w_1 \otimes w_2)) &= x(\nabla(y(w_1 \otimes y_2))) \\ &= x(\nabla(z(w_1 \otimes w_2))) \\ &= \psi^*(x)(z(w_1 \otimes w_2)),\end{aligned}$$

where the cup product ∇ pulls back from $X \times X$, therefore $xy = xz$, so $\psi^*(x) \in (A^{*,*} \otimes_{h^{*,*}} A^{*,*})_r$. The coassociative follows from the definition directly. Finally, to prove cocommutativity, it suffices to prove it on the generators Sq^i, $\mathcal{P}^i$, and β. This then follows from the Cartan formulas. For instance, we see that

$$\begin{aligned}(\mathrm{id} \otimes \psi^*)\psi^*(\mathrm{Sq}^{2i}) &= \sum_j \tau^{j \pmod 2} \mathrm{Sq}^j \otimes \psi^*(\mathrm{Sq}^{2i-j}) \\ &= \sum_{j,k} \tau^{j \pmod 2} \mathrm{Sq}^j \otimes \mathrm{Sq}^k \otimes \mathrm{Sq}^{2i-j-k} + \sum_{j,k} \tau\rho \mathrm{Sq}^{2j-1} \otimes \mathrm{Sq}^{2k-1} \otimes \mathrm{Sq}^{2i-2k-2j+1} \\ &= (\psi^* \otimes \mathrm{id})\psi^*(\mathrm{Sq}^{2i}).\end{aligned}$$

□

4.2 Dual Structure

Definition 4.12. For every $(i, j) \in \mathbb{Z} \oplus \mathbb{Z}$, we define $A_{i,j}$ to be the family of maps

$$\varphi : A^{u,v} \to h^{u-i,v-j}$$

where $u, v \in \mathbb{Z}$, such that for any $\lambda \in h^{*,*}$ and $F \in A^{*,*}$, we have $\varphi(\lambda \cdot F) = \lambda \cdot \varphi(F)$, i.e., $h^{*,*}$-linear as a left graded module.

Remark 4.13. By definition, there is a natural pairing

$$\begin{aligned}\langle -, - \rangle : A^{p,q} \times A_{i,j} &\to h^{p-i,q-j} \\ (F, \varphi) &\mapsto \varphi(F).\end{aligned}$$

Now $A_{*,*}$ has a right $h^{*,*}$-module structure defined by

$$\langle F, \varphi \cdot u \rangle = \langle F, \varphi \rangle \cdot u$$

for $u \in h^{*,*}$.

Definition 4.14. We define a pairing

$$\begin{aligned}\langle -, - \rangle : A^{*,*} \times (A_{*,*} \otimes_{h^{*,*}} H^{*,*}(X, \mathbb{Z}/p\mathbb{Z})) &\to H^{*,*}(X, \mathbb{Z}/p\mathbb{Z}) \\ (F, \alpha \otimes X) &\mapsto \langle F, \alpha \rangle \cdot X\end{aligned}$$

into the left $A^{*,*}$-module.

Lemma 4.15. Suppose $X \in \mathrm{Sm}/k$ is quasi-projective, and $\varphi : A^{*,*} \to H^{*-i,*-j}(X, \mathbb{Z}/p\mathbb{Z})$ is a $h^{*,*}$-module morphism. There is a unique element $\alpha \in A_{*,*} \otimes H^{*,*}(X, \mathbb{Z}/p\mathbb{Z})$ such that for every $F \in A^{*,*}$, we have $\varphi(F) = \langle F, \alpha \rangle$.

More precisely, we can write explicitly that $\alpha = \sum_{I \text{ admissible}} \theta(I)^* \otimes \varphi(\mathcal{P}^I)$ as a finite sum, where $\theta(I)^* \in A_{*,*}$ is defined by $\langle \mathcal{P}^J, \theta(I)^* \rangle = \delta_{IJ}$ as Kronecker delta.

Proof. It suffices to prove that there are finitely many admissible I such that $\varphi(\mathcal{P}^I) \neq 0$. The bidegree of $\mathcal{P}^I$ is of the form $(2w+e, w)$ for $e \geqslant 0$, therefore $\varphi(\mathcal{P}^I) \in H^{2w+e-i,w-j}(X, \mathbb{Z}/p\mathbb{Z})$. If $\varphi(\mathcal{P}^I) \neq 0$, then we have $2w+e-i \leqslant w-j+\dim(X)$ by vanishing result on hypercohomological spectral sequence, therefore $w \leqslant \dim(X)+i-j$. Since w corresponds to the second entry in the bidegree, then such admissible I form a finite set. □

Remark 4.16. By Lemma 4.15, we see that $\{\theta(I)^* : I \text{ admissible}\}$ form a $h^{*,*}$-free basis of $A_{*,*}$.

Fix $x \in H^{-i,-j}(X, \mathbb{Z}/p\mathbb{Z})$. We then define a morphism φ by setting $\varphi(F) = F(x)$. This defines a corresponding α as in Lemma 4.15, and motivates the following definition.

Definition 4.17. For a scheme X of finite dimension, we fix $x \in H^{*,*}(X, \mathbb{Z}/p\mathbb{Z})$. We define $\lambda^*(x) \in A_{*,*} \otimes_{h^{*,*}} H^{*,*}(X, \mathbb{Z}/p\mathbb{Z})$ to be the element satisfying

$$\langle F, \lambda^*(x)\rangle = F(x)$$

for $F \in A^{*,*}$.

Definition 4.18. For $k \in \mathbb{N}$, we define

$$\xi_k = \begin{cases} \theta(\mathcal{P}^{p^{k-1}} \cdots \mathcal{P}^p \mathcal{P}^1)^* \in A_{2(p^k-1),p^k-1}, & p > 2 \\ \theta(\mathrm{Sq}^{2^k} \cdots \mathrm{Sq}^2), & p = 2 \end{cases}$$

and

$$\tau_k = \begin{cases} \theta(\mathcal{P}^{p^{k-1}} \cdots \mathcal{P}^p \mathcal{P}^1 \beta)^* \in A_{2(p^k-1)+1,p^k-1}, & p > 2 \\ \theta(\mathrm{Sq}^{2^k} \cdots \mathrm{Sq}^1)^*, & p = 2 \end{cases}$$

Let us compute a few examples.

Proposition 4.19. Recall that $u, v \in H^{*,*}(B\mu_p, \mathbb{Z}/p\mathbb{Z})$ are generators. We then have

$$\lambda^*(v) = \sum_{k \geqslant 0} \xi_k \otimes v^{p^k}$$
$$\lambda^*(u) = \xi_0 \otimes u + \sum_{k \geqslant 0} \tau_k \otimes v^{p^k}$$

as elements in $A_{*,*} \otimes H^{*,*}(B\mu_p, \mathbb{Z}/p\mathbb{Z})$.

Proof. The admissible sequences I such that $\mathcal{P}^I(v) \neq 0$ (respectively, $\mathcal{P}^I(u) \neq 0$) are exactly of the form $\mathcal{P}^{p^{k-1}} \cdots \mathcal{P}^p \mathcal{P}^1$ (respectively, $\mathcal{P}^{p^{k-1}} \cdots \mathcal{P}^p \mathcal{P}^1 \beta$) when $p > 2$, and similarly when $p = 2$. □

We now define another pairing. Recall that $A^{*,*}$ already has a left $h^{*,*}$-module structure, then we now define a right module structure via

$$c \cdot u \mapsto (-1)^{\deg(u)\deg(c)} u \cdot c$$

for $c \in A^{*,*}$ and $u \in h^{*,*}$. Similarly, recall that $A_{*,*}$ already has a right $h^{*,*}$-module structure, then one can also define a left module structure via skew-symmetry.

Definition 4.20. Using the bimodule structure above, we define a pairing

$$\begin{aligned} \langle -, - \rangle : A^{*,*} \otimes_{h^{*,*}} A^{*,*} \times A_{*,*} \otimes_{h^{*,*}} A_{*,*} &\to h^{*,*} \\ (C \otimes D, \alpha \otimes \beta) &\mapsto (-1)^{\deg(D)\deg(\alpha)} \langle C, \alpha\rangle \langle D, \beta\rangle. \end{aligned}$$

Definition 4.21. For any $\alpha, \beta \in A_{*,*}$, we define $\alpha\beta \in A_{*,*}$ to be the element satisfying

$$\langle F, \alpha\beta\rangle = \langle \psi^* F, \alpha \otimes \beta\rangle$$

With this, $A_{*,*}$ is now equipped with a graded commutative ring structure, where ξ_0 is the identity.

All the module structures above are induced by scalar multiplications. We now introduce a different kind of module structures, induced by composition laws.

Remark 4.22. From Definition 4.17, we have a ring homomorphism

$$\begin{aligned}\lambda^* : h^{*,*} &\to A_{*,*}\\ \lambda &\mapsto \xi_0 \cdot \lambda,\end{aligned}$$

making $A_{*,*}$ a left $h^{*,*}$-module. Note that $A^{*,*}$ also has a right $h^{*,*}$-module structure given by the composition, i.e., $F \cdot u = F \circ u$. We note that in the motivic case the product $F \times u$ is different from the composition $F \circ u$, because of the cohomological operation in $h^{*,*}$ might be non-zero.

This right $h^{*,*}$-module structure of $A^{*,*}$ defined by composition then induces another left $h^{*,*}$-module structure of $A_{*,*}$ by $\langle F, r \cdot \alpha\rangle = \langle F \circ r, \alpha\rangle$.

We claim that the two given left $h^{*,*}$-module structure of $A_{*,*}$ are the same. Suppose $F \in A^{*,*}$, $x \in h^{*,*}$, and $y \in H^{*,*}(X, \mathbb{Z}/p\mathbb{Z})$, then write $\psi^*(F) = \sum_i A_i \otimes B_i$. We then have

$$(F \circ x)(y) = F(x \cdot y) = \sum_i (-1)^{\deg(x)\deg(B_i)} A_i(x) B_i(y),$$

therefore $F \circ x = \sum_i (-1)^{\deg(x)\deg(B_i)} A_i(x) B_i$, so for any $\alpha \in A_{*,*}$, we have

$$\begin{aligned}\langle F \circ x, \alpha\rangle &= \sum_i (-1)^{deg(x)\deg(B_i)} \langle A_i(x) B_i, \alpha\rangle\\ &= \sum_i (-1)^{\deg(x)\deg(B_i)} A_i(x) \langle B_i, \alpha\rangle\\ &= \sum_i (-1)^{\deg(x)\deg(B_i)} \langle A_i, \lambda^*(x)\rangle \langle B_i, \alpha\rangle\\ &= \left\langle \sum_i A_i \otimes B_i, \lambda^*(x) \otimes \alpha \right\rangle\\ &= \langle \psi^* F, \lambda^*(x) \otimes \alpha\rangle .\end{aligned}$$

Proposition 4.23 ([Voe03], Lemma 12.2)**.** For every $X \in \mathrm{Sm}/k$ that is quasi-projetive, the map

$$\lambda^* : H^{*,*}(X, \mathbb{Z}/p\mathbb{Z}) \to A_{*,*} \otimes_{h^{*,*}} H^{*,*}(X, \mathbb{Z}/p\mathbb{Z})$$

is a ring homomorphism.

Proof. Suppose $u = \sum_i \alpha_i \otimes x_i$ and $v = \sum_j \beta_j \otimes y_j$ are elements in $A_{*,*} \otimes_{h^{*,*}} H^{*,*}(X, \mathbb{Z}/p\mathbb{Z})$, then for every $F \in A^{*,*}$, writing $\psi^*(F) = \sum_k A_k \otimes B_k$ gives

$$\begin{aligned}\langle \psi^* F, u \otimes v\rangle &= \sum_{i,j,k} (-1)^{\deg(x_i)\deg(\beta_j)} \langle A_k \otimes B_k, \alpha_i \otimes \beta_j \otimes_i y_j\rangle\\ &= \sum_{i,j,k} (-1)^{\deg(x_i)\deg(\beta_j) + \deg(\alpha_i)\deg(B_k)} \langle A_k, \alpha_i\rangle \langle B_k, \beta_j\rangle x_i y_j\\ &= \sum_{i,j} (-1)^{\deg(x_i)\deg(\beta_j)} \langle F, \alpha_i \beta_j\rangle x_i y_j\\ &= \langle F, u \cdot v\rangle .\end{aligned}$$

Therefore, for any $x_1, x_2 \in H^{*,*}(X, \mathbb{Z}/p\mathbb{Z})$, we have

$$\begin{aligned}\langle F, \lambda^*(x_1 x_2)\rangle &= F(x_1 x_2)\\ &= \sum_k (-1)^{\deg(x_i)\deg(B_k)} A_k(x_1) B_k(x_2)\end{aligned}$$

$$
\begin{aligned}
&= \sum_k \langle A_k \otimes B_k, \lambda^*(x_1) \otimes \lambda^*(x_2)\rangle \\
&= \langle \psi^* F, \lambda^*(x_1) \otimes \lambda^*(x_2)\rangle \\
&= \langle F, \lambda^*(x_1)\lambda^*(x_2)\rangle .
\end{aligned}
$$

□

Theorem 4.24 ([Voe03], Theorem 12.6)**.** $A_{*,*}$, as a graded commutative right $h^{*,*}$-algebra, is generated by ξ_k's and τ_k's over $h^{*,*}$ for $k \geqslant 0$, i.e., of the form $[\xi_0, \tau_0, \xi_1, \tau_1, \ldots]h^{*,*}$, where $\xi_0 = 1$, quotient by the relation

$$
\begin{cases} \tau_k^2 = 0, & p > 2 \\ \tau_k^2 = \xi_{k+1}\tau + \tau_0\xi_{k+1}\rho + \tau_{k+1}\rho, & p = 2 \end{cases}
$$

This will be proven in a few steps below. Let us first determine the quotient relations. If $p > 2$, this is obvious by degree reasons, where we conclude $\tau_k^2 = 0$ for $k \geqslant 0$. For $p = 2$, by Proposition 4.23 and Proposition 4.19, we see that

$$
\begin{aligned}
\sum_{k \geqslant 0} \tau_k^2 \otimes v^{2^{k+1}} + \cdots &= \lambda^*(u)\lambda^*(u) \\
&= \lambda^*(u^2) \\
&= \lambda^*(\tau v + \rho u) \\
&= (\xi_0\tau + \tau_0\rho) \sum_{k \geqslant 0} \xi_k v^{2^k} + \xi_0\rho(\xi_0 u + \sum_{k \geqslant 0} \tau_k v^{2^k}) \\
&= \sum_{k \geqslant 0} (\xi_k\tau + \tau_0\xi_k\rho + \tau_k\rho) v^{2^k} + \cdots
\end{aligned}
$$

using the structure of $B\mu_2$. By comparison of coefficients, we get

$$
\tau_k^2 = \xi_{k+1}\tau + \tau_0\xi_{k+1}\rho + \tau_{k+1}\rho.
$$

Definition 4.25. Suppose $p > 2$. For any admissible sequence $I = (\varepsilon_1, i_1, \varepsilon_2, i_2, \ldots)$,we define $r_j = i_j - p i_{j+1} - \varepsilon_{j+1}$ and $w(I) = \tau_0^{\varepsilon_1}\xi_1^{r_1}\tau_1^{\varepsilon_2}\xi_2^{r_2}\cdots$.

Suppose $p = 2$. For any admissible sequence $I = (i_1, \ldots, i_k)$, we define $r_j = \lfloor \frac{i_j}{2} \rfloor - i_{j+1}$, set $\varepsilon_j = i_j \pmod 2$, and define $w(I) = \tau_0^{\varepsilon_1}\xi_1^{r_1}\tau_1^{\varepsilon_2}\xi_2^{r_2}\cdots$.

Proposition 4.26 ([Voe03], Theorem 12.4)**.** Suppose I and J are both admissible, then we have orthogonality as in

$$
\left\langle \mathcal{P}^I, w(J) \right\rangle = \begin{cases} \pm 1, & I = J \\ 0, & I < J \end{cases}, \quad \left\langle \mathrm{Sq}^I, w(J) \right\rangle = \begin{cases} 1, & I = J \\ 0, & I < J \end{cases}
$$

where we write I to be of the form $(\varepsilon_1, r_1, \varepsilon_2, r_2, \ldots)$, and define $<$ to be the lexicographical ordering from right to left for $(\varepsilon_1, r_1, \varepsilon_2, r_2, \ldots)$.

Remark 4.27. Note that we do not assign a value for $I > J$, and it may not be zero: $w(J)$ is not $\theta(\mathcal{P}^I)^*$. Similarly, $w(I)$ may not be $(\mathrm{Sq}^I)^*$. For example, we see that $\tau_0\xi_1 = w(1,1,0,\ldots)$ corresponding to Sq^3 and $\tau_1 = w(0,0,1,0,\ldots)$ corresponding to $\mathrm{Sq}^2\mathrm{Sq}^1$. Here we have $\tau_0\xi_1 < \tau_1$ by the ordering defined, but

$$
\left\langle \mathrm{Sq}^2\mathrm{Sq}^1, \tau_0\xi_1 \right\rangle = \left\langle \mathrm{Sq}^2\mathrm{Sq}^1, \tau_1 \right\rangle = 1.
$$

Proof. If $p > 2$, then the proof is the same as in Theorem 1.61. Assume $p = 2$. Suppose $J = (\varepsilon'_1, r'_1, \ldots)$ and k is the largest number such that $r'_k \neq 0$ or $\varepsilon'_k \neq 0$. Note that we have $w(I) = \tau_0^{\varepsilon_1}\xi_1^{r_1}\cdots$ and $w(J) = \tau_0^{\varepsilon'_1}\xi_1^{r'_1}\cdots$. We will proceed by induction on $(\deg(I), \deg(J))$.

If $r'_k \neq 0$, then $\varepsilon'_j = 0 = r'_j$ whenever $j > k$. Define $J' = (\varepsilon'_1, r'_1, \ldots, \varepsilon'_k, r'_k - 1, 0, \ldots, 0)$, so $w(J') = \tau_0^{\varepsilon'_1}\xi_1^{r'_1}\cdots\tau_{k-1}^{\varepsilon'_k}\xi_k^{r'_k-1}$, therefore $w(J) = w(J')\xi_k$. For every I, we then have

$$
\left\langle \mathrm{Sq}^I, w(J) \right\rangle = \left\langle \psi^*(\mathrm{Sq}^I), w(J') \otimes \xi_k \right\rangle . \tag{4.28}
$$

Note that

$$\left\langle \mathrm{Sq}^I, \xi_k \right\rangle = \begin{cases} 1, & I = (2^k, \ldots, 2^1) =: M_k \\ 0, & \text{else} \end{cases}, \quad \left\langle \mathrm{Sq}^I, \tau_k \right\rangle = \begin{cases} 1, & I = (2^k, \ldots, 1) =: N_k \\ 0, & \text{else} \end{cases}$$

By the discussion in Theorem 4.8 and Cartan formula, we have

$$\psi^*(\mathrm{Sq}^I) = \sum_{I',K} \rho^{\deg(I)-\deg(I')} \tau^{t(I')-t(K)-t(I'-K)} \mathrm{Sq}^{I'-K} \mathrm{Sq}^K$$

where $I' = (i'_1, \ldots, i'_k)$ satisfies $i'_s = i_s$ unless i_s and k_s are odd, in which case $i'_s = i_s - 1$. Plugging this back in, we note that Equation (4.28) is the same as

$$\begin{cases} \left\langle \mathrm{Sq}^{I-M_k}, w(J') \right\rangle, & r_k > 0 \\ 0, & r_k = 0 \end{cases}.$$

Indeed, when $r_k > 0$, then $I - M_k < J'$ (respectively, $I - M_k = J'$) if $I < J$ (respectively, $I = J$). Applying the inductive hypothesis, we are done.

In the seond case, we have $\varepsilon'_k \neq 0$, $r'_j = 0$ for all $j \geqslant k$, and $\varepsilon'_j = 0$ for all $j > k$. This time we define $J' = (\varepsilon'_1, r'_1, \ldots, r'_{k-1}, 0, \ldots)$, therefore $w(J) = w(J')\tau_k$. For every I, now we have

$$\left\langle \mathrm{Sq}^I, w(J) \right\rangle = \left\langle \psi^*(\mathrm{Sq}^I), w(J') \otimes \xi_k \right\rangle = \begin{cases} \left\langle \mathrm{Sq}^{I-N_k}, w(J') \right\rangle, & \varepsilon_k = 1 \\ 0, & \varepsilon_k = 0 \end{cases}.$$

Note that $\varepsilon_k = 1$ already implies that $I - N_k < J$, so we conclude by the inductive hypothesis. □

Corollary 4.29 ([Voe03], Corollary 12.5)**.** The set

$$\{w(I) : I \text{ admissible}\}$$

is a right $h^{*,*}$-module free basis of $A_{*,*}$.

Proof. To show that the $w(I)$'s are lineraly independent, suppose $\sum_j w(J)\lambda_J = 0$, applying $\left\langle \mathcal{P}^J, - \right\rangle$ for the smallest J such that $\lambda_J \neq 0$, then we get a contradiction by Proposition 4.26. To show that they generate $A_{*,*}$, by Lemma 4.15, each $\alpha \in A_{ij}$ can be written as a sum $\sum_{I \text{ admissible}} \theta(I)^* \lambda I$. Suppose I_0 is the smallest one such that $\lambda_{I_0} \neq 0$, then $\alpha \pm w(I_0) \left\langle \alpha, \mathcal{P}^{I_0} \right\rangle$ has a larger I_0 by Proposition 4.26. □

Proof of Theorem 4.24. Let us denote the algebra defined in the statement to be $\tilde{A}_{*,*}$. Since $A_{*,*}$ already satisfies the said relations, we know there is a right $h^{*,*}$-algebra morphism $\tilde{A}_{*,*} \to A_{*,*}$. If $p > 2$, we note that $\{w(I)\}$ is also a $h^{*,*}$-free basis of $\tilde{A}_{*,*}$, so we are done. If $i = 2$, then by the relation $\tau_k^2 = \xi_{k+1}\tau + \tau_0\xi_{k+1}\rho + \tau_{k+1}\rho$, we see that the $w(I)$'s generate $\tilde{A}_{*,*}$ as a $h^{*,*}$-module, which are linearly independent themselves since their images in $A_{*,*}$ are by Corollary 4.29, and we are done. □

Let us now study some module structures given by the composition laws. For another perspective, see [Voe03, Page 52].

Definition 4.30. We define $A^{*,*} \otimes_{\rho,h^{*,*}} A^{*,*}$ to be the tensor product of $A^{*,*}$ regarded as a $h^{*,*}$-bimodule induced by the left and right compositions.

Remark 4.31. We note that the left composition is equivalent to (left) scalar multiplication, but the right composition is not equivalent to the right $h^{*,*}$-module structure defined by the left $h^{*,*}$-module structure using skew-commutativity, i.e., not by the (right) scalar multiplication.

Definition 4.32. We define $A_{*,*} \otimes_{h^{*,*},\lambda} A_{*,*}$ to be the tensor product of $A_{*,*}$ regarded as an $h^{*,*}$-bimodule with the left action induced by λ^*.

Remark 4.33. In this structure, we note that right action is given by scalar multiplication, while the left action is given by composition.

Definition 4.34. We define a pairing

$$
\begin{aligned}
[-,-]:(A^{*,*}\otimes_{\rho,h^{*,*}}A^{*,*})\times(A_{*,*}\otimes_{h^{*,*},\lambda}A_{*,*}) &\to h^{*,*}\\
(C\otimes D,\alpha\otimes\beta) &\mapsto \langle C\circ\langle D,\alpha\rangle,\beta\rangle=\langle C,\lambda^*(\langle D,\alpha\rangle)\beta\rangle\,.
\end{aligned}
$$

Lemma 4.35. Suppose $\varphi:A^{*,*}\otimes_{\rho,h^{*,*}}A^{*,*}\to h^{*,*}$ is a graded $h^{*.*}$-module map, then there exists a unique $\psi\in A_{*,*}\otimes_{h^{*,*},\lambda}A_{*,*}$ such that $\varphi=[-,\psi]$.

Proof. By the defining relations in $A^{*,*}$, we see that

$$\{\mathcal{P}^I\otimes\mathcal{P}^J: I,J \text{ admissible}\}$$

(and the argument below should work similarly for Sq's) generates $A^{*,*}\otimes_{\rho,h^{*,*}}A^{*,*}$. On the other hand,

$$[\mathcal{P}^I\otimes\mathcal{P}^J,\theta(J')^*\otimes\theta(I')^*]=\delta_{II'}\delta_{JJ'},$$

so they form a $h^{*,*}$-free basis of $A^{*,*}\otimes_{\rho,h^{*,*}}A^{*,*}$, and we see the $\theta(J)^*\otimes\theta(I)^*$'s also form a $h^{*,*}$-free basis for $A_{*,*}\otimes_{h^{*,*},\lambda}A_{*,*}$. Using this, we see that

$$\psi=\sum_{I,J}\theta(J)^*\otimes\theta(I)^*\cdot\varphi(\mathcal{P}^I\otimes\mathcal{P}^J)$$

is what we want. □

Definition 4.36. We define a comultiplication on $A_{*,*}$ as a map $\psi_*:A_{*,*}\to A_{*,*}\otimes_{h^{*,*},\lambda}A_{*,*}$ such that for every $C,D\in A^{*,*}$ and $\alpha\in A_{*,*}$, we have

$$[C\otimes D,\psi_*\alpha]=\langle CD,\alpha\rangle\,.$$

Proposition 4.37. The map

$$\psi_*:A_{*,*}\to A_{*,*}\otimes_{h^{*,*},\lambda}A_{*,*}$$

is a ring homomorphism as well as a $h^{*,*}$-bimodle homomorphism.

Proof. Suppose $C,D\in A^{*,*}$, and $\alpha,\beta\in A_{*,*}$, then set

$$
\begin{aligned}
\psi_*(\alpha)=\sum_i a_i\otimes b_i,\quad \psi_*(\beta)=\sum_j c_j\otimes d_j,\\
\psi_*(C)=\sum_k u_k\otimes v_k,\quad \psi_*(D)=\sum_\ell s_\ell\otimes d_\ell.
\end{aligned}
$$

From this, it is easy to verify that the $h^{*,*}$-bimodule strucure being compatible with ψ_*. We now show that ψ_* is a ring homomorphism. We have

$$
\begin{aligned}
[C\otimes D,\psi_*(\alpha\beta)]&=\langle CD,\alpha\beta\rangle\\
&=\langle\psi^*(CD),\alpha\otimes\beta\rangle\\
&=\left\langle\psi^*(C)\psi^(D),\alpha\otimes\beta\right\rangle\\
&=\sum_{k,\ell}(-1)^{\deg(v_k)\deg(s_\ell)}\langle u_ks_\ell\otimes v_kt_\ell,\alpha\otimes\beta\rangle\\
&=\sum_{k,\ell}(-1)^{\deg(v_k)\deg(s_\ell)}\langle u_ks_\ell,\alpha\rangle\langle v_kt_\ell,\beta\rangle\\
&=\sum_{k,\ell}(-1)^{\deg(v_k)\deg(s_\ell)}[u_k\otimes s_\ell,\psi_*\alpha][v_k\otimes t_\ell,\psi_*\beta]\\
&=\sum_{i,j,k,\ell}(-1)^{\deg(v_k)\deg(s_\ell)}[u_k\otimes s_\ell,a_i\otimes b_i][v_k\otimes t_\ell,c_j\otimes d_j]\\
&=\sum_{i,j,k,\ell}(-1)^{\deg(v_k)\deg(s_\ell)}\langle u_k\langle s_\ell,a_i\rangle,b_i\rangle\langle v_k\langle t_\ell,c_j\rangle,d_j\rangle
\end{aligned}
$$

$$\begin{aligned}
&= \sum_{i,j,k,\ell} (-1)^{\deg(v_k)\deg(s_\ell)} \langle u_k \langle s_\ell, a_i\rangle \otimes v_k \langle t_\ell, c_j\rangle, b_i \otimes d_j\rangle \\
&= \sum_{i,j} (-1)^{\deg(v_k)\deg(s_\ell)} \langle \psi^*(C)\cdot \psi^*(\langle \psi^*(D), a_i \otimes c_j\rangle), b_i \otimes d_j\rangle \\
&= \sum_{i,j} (-1)^{\deg(v_k)\deg(s_\ell)} \langle C \langle \psi^*(D), a_i \otimes c_j\rangle, b_i \otimes d_j\rangle \\
&= \sum_{i,j} (-1)^{\deg(v_k)\deg(s_\ell)} \langle C \langle D, a_i \otimes c_j\rangle, b_i \otimes d_j\rangle \\
&= [C \otimes D, \psi_*(\alpha)\psi_*(\beta)].
\end{aligned}$$

□

Proposition 4.38. The following equalities hold in $A_{*,*} \otimes_{h^{*,*},\lambda} A_{*,*}$:

$$\psi_*\xi_k = \sum_{i=0}^{k} \xi_i \otimes \xi_{k-i}^{p^i}, \quad \psi_*\tau_k = \xi_0 \otimes \tau_k + \sum_{i=0}^{k} \tau_i \otimes \xi_{k-i}^{p^i}.$$

Proof. Suppose $C, D \in A^{*,*}$. By Proposition 4.19, $\lambda^*(v) = \sum_{k\geqslant 0} \xi_k \otimes v^{p^k}$, therefore $D(v) = \sum_{i\geqslant 0} \langle D, \xi_i\rangle v^{p^i}$. We also have $\lambda^*(v^{p^i}) = \lambda^*(v)^{p^i} = \sum_{j\geqslant 0} \xi_j^{p^i} \otimes v^{p^{i+j}}$, therefore for any $F \in A^{*,*}$, we have

$$F(v^{p^i}) = \sum_{j\geqslant 0} \left\langle F, \xi_j^{p^j} \right\rangle v^{p^{i+j}}.$$

Therefore

$$\begin{aligned}
CD(v) &= \sum_{i\geqslant 0} (C\langle D, \xi_i\rangle)(v^{p^i}) \\
&= \sum_{i,j\in\mathbb{N}} \left\langle C\langle D, \xi_i\rangle, \xi_j^{p^i} \right\rangle v^{p^{i+j}} \\
&= \sum_{i,j\in\mathbb{N}} [C \otimes D, \xi_i \otimes \xi_j^{p^i}] v^{p^{i+j}}.
\end{aligned}$$

On the other hand, we have

$$CD(v) = \sum_{i\geqslant 0} \langle CD, \xi_i\rangle v^{p^i},$$

so by comparing coefficients, we have

$$\langle CD, \xi_k\rangle = \sum_{i=0}^{k} [C \otimes D, \xi_i \otimes \xi_{k-i}^{p^i}],$$

which gives the first equality. The second equality follows using the same method. □

4.3 Milnor Basis

We will now introduce the Milnor basis, given by the dual basis from $A_{*,*}$, which becomes a different basis for $A^{*,*}$.

Lemma 4.39. For every $\varphi : A_{*,*} \to h^{p-*,q-*}$, there exists a unique $F \in A^{p,q}$ such that $\varphi(\alpha) = \langle F, \alpha\rangle$.

Proof. We define $F = \sum_{I \text{ admissible}} \varphi(\theta(I)^*) \cdot \mathcal{P}^I$, which is a finite sum since $\varphi_{uv} : A_{uv} \to h^{p-u,q-v}$ is azero if $v > q$, and the set $\{I \text{ admissible} : t(I) \leqslant q\}$ is finite. □

We define $(\varepsilon_*) = (\varepsilon_0, \varepsilon_1, \ldots)$ where each $\varepsilon_i = 0, 1$, and define $(r_*) = (r_1, r_2, \ldots)$ for $r_i \in \mathbb{N}$, then we can also write $\tau_*^{\varepsilon_*} = \tau_0^{\varepsilon_0}\tau_1^{\varepsilon_1}$, and $\xi_*^{r_*} = \xi_1^{r_1}\xi_2^{r_2}\cdots$, whcih define an element $\tau_*^{\varepsilon_*}\xi_*^{r_*} \in A_{*,*}$, called the monomials.

Definition 4.40. We define $\rho(\varepsilon_*, r_*) \in A^{*,*}$ to be an element such that

$$\left\langle \rho(\varepsilon_*, r_*), \tau_*^{\varepsilon'_*}\xi_*^{r'_*} \right\rangle = \begin{cases} 1, & \varepsilon'_* = \varepsilon_*, r'_* = r_* \\ 0, & \text{else} \end{cases},$$

then such ρ's form a dual basis of $\tau_*^{\varepsilon_*}\xi_*^{r_*}$'s. Therefore, such elements form a free right $h^{*,*}$-basis of $A^{*,*}$, which we call the Milnor basis.

Definition 4.41. We define $I \subseteq A_{*,*}$ to be the free right $h^{*,*}$-module generated by $\{\tau_*^{\varepsilon_*}\xi_*^{r_*} : (r_*) \neq 0\}$, which is a bilateral ideal generated by $\{\xi_k : k \geqslant 1\}$.

Definition 4.42. We define $B^{*,*} \subseteq A^{*,*}$ to be the $h^{*,*}$-submodule fomred by elements $F \in A^{*,*}$ such that $\langle F, I\rangle = 0$.

Proposition 4.43. $B^{*,*}$ is a subalgebra fo $A^{*,*}$, with a free $h^{*,*}$-basis

$$\{Q(\varepsilon_*) := \rho(\varepsilon_*, 0) : \varepsilon_* = (\varepsilon_0, \dots, \varepsilon_k, 0, \dots, 0), \varepsilon_i = 0, 1\}.$$

Proof. The $Q(\varepsilon_*)$'s obviously form a basis of $B^{*,*}$. It now suffices to show that for every $C, D \in B^{*,*}$, $\alpha \in I$, we have $\langle CD, \alpha\rangle = [C \otimes D, \psi_*(\alpha)] = 0$. For every $\beta, \gamma \in A_{*,*}$, we have

$$[C \otimes D, \beta \otimes \gamma] = \langle C \langle D, \beta\rangle, \gamma\rangle = \langle C, \lambda^*(\langle D, \beta\rangle)\gamma\rangle,$$

which is zero if $\beta \in I$ or $\gamma \in I$. To prove $[C \otimes D, \psi_*(\alpha)] = 0$, it then suffices to show $\psi_*(\alpha) \in I \otimes A_{*,*} + A_{*,*} \otimes I$. By Proposition 4.37, it suffices to check the case $\alpha = \xi_k$ for $k \geqslant 1$, then apply Proposition 4.38. □

Motivated by Proposition 4.43, we have the following definition.

Definition 4.44. We define $\bar{\psi}_* : A_{*,*}/I \to A_{*,*}/I \otimes_{h^{*,*},\lambda} A_{*,*}/I$ to be a map induced by ψ_*. The pairing $[-,-]$ induces another pairing $[-,-] : (B^{*,*} \otimes_{\rho,h^{*,*}} B^{*,*}) \times (A_{*,*}/I \otimes_{h^{*,*},\lambda} A_{*,*}/I) \to h^{*,*}$.

Definition 4.45. For any finite subset $X \subseteq \mathbb{N}$, we define

$$\begin{aligned} \varepsilon : \mathbb{N} &\to \{0, 1\} \\ i &\mapsto \begin{cases} 0, & i \notin X, \\ 1, & i \in X \end{cases}. \end{aligned}$$

From this we define

$$\tau(X) = \tau_*^{\varepsilon_*}, \quad Q(X) = Q(\varepsilon_*).$$

We then define $Q_i = Q(\{i\}) \in A^{2(p^i-1)+1, p^i-1}$. If $p = 2$, we write $n = \sum_{i \geqslant 0} \varepsilon_i 2^i$, then note that $Q(n) = Q(\varepsilon_*)$ and $\tau(n) = \tau_*^{\varepsilon_*}$.

Proposition 4.46. For any finite subset $X \subseteq \mathbb{N}$, we have

$$\bar{\psi}_* \tau(X) = \sum_{A \coprod B = X} (-1)^{\mathrm{Inv}(A,B)} \tau(A) \otimes \tau(B),$$

where $\mathrm{Inv}(A, B) = |\{(a, b) \in A \times B : a > b\}|$.

Proof. By Proposition 4.38, $\bar{\psi}_* \tau_i = \xi_0 \otimes \tau_i + \tau_i \otimes \xi_0$. Suppose elements in X are $i_1 < \cdots < i_n$, then $\bar{\psi}_* \tau(x) = \prod_{j=1}^{n} (\xi_0 \otimes \tau_{i_j} + \tau_{i_j} \otimes \xi_0)$. Now $\mathrm{Inv}(A, B)$ is precisely the inversion number of the sequence $a_1 < \cdots < a_m, b_1 < \cdots < b_{n-m}$ where $A = \{a_1, \dots, a_m\}$ and $B = \{b_1, \dots, b_{n-m}\}$. □

Proposition 4.47. Suppose A and B are both finite subsets of $\mathbb{N}$, then

$$Q(B)Q(A) = \begin{cases} (-1)^{\mathrm{Inv}(A,B)} Q(A \coprod B), & A \cap B = \varnothing \\ 0, & A \cap B \neq \varnothing \end{cases}.$$

In particular, $Q_i^2 = 0$ for all $i \geqslant 0$.

Remark 4.48. Note that if we work with admissible monomials, then $(\mathrm{Sq}^i)^2 \neq 0$.

Proof. Since $B^{*,*}$ is a subalgebra, we write $Q(B)Q(A) = \sum_{\text{finite } X\subseteq\mathbb{N}} \lambda_X Q(X)$, where $\lambda_X = \langle Q(B)Q(A), \tau(X)\rangle$. Then

$$\begin{aligned}\lambda_X &= [Q(B)\otimes Q(A), \bar{\psi}_*\tau(X)]\\ &= \sum_{A'\coprod B'=X} (-1)^{\mathrm{Inv}(A',B')}[Q(B)\otimes Q(A), \tau(A')\otimes\tau(B')]\\ &= \begin{cases}(-1)^{\mathrm{Inv}(A,B)}, & X = A\coprod B\\ 0, & \text{else}\end{cases}.\end{aligned}$$

□

Remark 4.49. We can now write the Bockstein β in terms of the Milnor basis. For degree reasons, we must have $\beta = \langle\beta,\xi_0\rangle\,\mathrm{id} + \langle\beta,\tau_0\rangle\, Q_0 = Q_0$, since $\tau_0 = \beta^*$.

Proposition 4.50. The comultiplication $\psi^* : A^{*,*} \to A^{*,*}\otimes_{h^{*,*}} A^{*,*}$ induces a comultiplication $B^{*,*}\to B^{*,*}\otimes_{h^{*,*}} B^{*,*}$.

Proof. Since $B^{*,*}$ annihilates I, then every element in $B^{*,*}\otimes_{h^{*,*}} B^{*,*}$ annihilates the ideal $I\otimes A_{*,*} + A_{*,*}\otimes I$. If $x\in A^{*,*}\otimes_{h^{*,*}} A^{*,*}$ does so, then we write $x = \sum_{I \text{ admissible}} a_I\otimes\mathcal{P}^I$, so $\langle x, u\otimes\theta(I)^*\rangle = 0 = (-1)^{\deg(a_I)\deg(u)}\langle a_I, u\rangle$ for every $u\in I$. Hence, $a_I\in B^{*,*}$. By Proposition 4.43, we must have $x = \sum Q(\varepsilon_*)\otimes b_{\varepsilon_*}$, therefore $\left\langle x, \tau_*^{\varepsilon_*}\otimes u\right\rangle = (-1)^{\deg(b_{\varepsilon_*})\deg(u)}\left\langle b_{\varepsilon_*}, u\right\rangle = 0$ for all $u\in I$. Therefore, such x must belong to $B^{*,*}\otimes_{h^{*,*}} B^{*,*}$.

Suppose $c\in B^{*,*}$, $\beta,\gamma\in A_{*,*}$, such that one of the two of them is in I, then $\langle\psi^*(c),\beta\otimes\gamma\rangle = \langle c,\beta\gamma\rangle = 0$, therefore $\psi^*(c)\in B^{*,*}\otimes_{h^{*,*}} B^{*,*}$. This shows that the multiplication operation is closed. □

Proposition 4.51. If $p > 2$, then $\psi^*(Q_i) = Q_i\otimes 1 + 1\otimes Q_i$ for all $i\geqslant 0$. If $p = 2$, then for any $n\geqslant 0$, we have $\psi^*(Q(n)) = \sum_{n\geqslant i\geqslant 0}\rho^{\sigma(i,n-i)}Q(i)\otimes Q(n-i)$ where $\sigma(i,j)$ is the number of carries in the 2-adic addition of i and j. In particular,

$$\psi^*(Q_k) = Q_k\otimes 1 + 1\otimes Q_k + \sum_{i=1}^{2^n-1}\rho^{k-v_2(i)}Q(i)\otimes Q(2^k-i)$$

where $v_2(i) = \max\{k : 2^k\mid i\}$ is the 2-adic valuation of i.

Proof. By Proposition 4.50, for any $X\subseteq\mathbb{N}$, we can write

$$\psi^*(Q(X)) = \sum_{A,B\subseteq\mathbb{N}}\lambda_{A,B}Q(A)\otimes Q(B),$$

where A and B are both finite subsets of $\mathbb{N}$. Then

$$\lambda_{A,B} = (-1)^{|A|\cdot|B|}\langle\psi^*(Q(X)),\tau(A)\otimes\tau(B)\rangle = (-1)^{|A|\cdot|B|}\langle Q(X),\tau(A)\tau(B)\rangle.$$

If $p > 2$, then $\lambda_{A,B} = 0$ unless $A\coprod B = X$ as $\tau_i^2 = 0$ using Theorem 4.24, therefore the statement is clear by setting $X = \{i\}$.

Now suppose $p = 2$, then we have $\tau_i^2 = \tau_{i+1}\rho\in A_{*,*}/I$, so ρ keeps track of the number of carrying. From this, we see that

$$\tau(p)\tau(q) = \tau(p+q)\rho^{\sigma(p,q)}$$

in $A_{*,*}/I$, which gives the statement for $\psi^*(Q(n))$. □

Definition 4.52. Conversely, we now define $P^{(r_*)} = \rho((0,\dots,0),r_*)\in A^{*,*}$.

Proposition 4.53 ([Voe03], Proposition 13.2)**.** We have $\rho(\varepsilon_*,r_*) = Q(\varepsilon_*)P^{(r_*)}\in A^{*,*}$.

Proof. We have

$$\begin{aligned}
Q(\varepsilon_*)P^{(r_*)} &= \sum_{\varepsilon'_*, r'_*} \left\langle Q(\varepsilon_*)P^{(r_*)}, \tau_*^{\varepsilon'_*}\xi_*^{r'_*} \right\rangle \rho(\varepsilon'_*, r'_*) \\
&= \sum_{\varepsilon'_*, r'_*} [Q(\varepsilon_*) \otimes P^{(r_*)}, \psi_*(\tau_*^{\varepsilon'_*}\xi_*^{r'_*})]\rho(\varepsilon'_*, r'_*).
\end{aligned}$$

Since $Q(\varepsilon_*)$ is orthogonal (annihilating) to $I \subseteq A_{*,*}$, it suffices to compute the class of $\psi_*(\tau_*^{\varepsilon'_*}\xi_*^{r'_*})$ in $A_{*,*} \otimes_{h^{*,*},\lambda} A_{*,*}/I$ by the same proof as in Proposition 4.43. We denote this class by $\bar{\psi}_*(\tau_*^{\varepsilon'_*}\xi_*^{r'_*})$. By Proposition 4.38, we have

$$\bar{\psi}_*\xi_k = \xi_k \otimes \xi_0, \quad \bar{\psi}_*\tau_k = \xi_0 \otimes \tau_k + \tau_k \otimes \xi_0.$$

Define $I' = \{i \in \mathbb{N} : \varepsilon'_i = 1\}$, then we get

$$\bar{\psi}_*(\tau_*^{\varepsilon'_*}\xi_*^{r'_*}) = \sum_{A \coprod B = I'} (-1)^{\mathrm{Inv}(A,B)}\tau(A)\xi_*^{r'_*} \otimes \tau(B)$$

by Proposition 4.37, so

$$\begin{aligned}
[Q(\varepsilon_*) \otimes P^{(r_*)}, \psi_*(\tau_*^{\varepsilon'_*}\xi_*^{r'_*})] &= \sum_{A \coprod B = I'} (-1)^{\mathrm{Inv}(A,B)} \left\langle Q(\varepsilon_*) \left\langle P^{(r_*)}, \tau(A)\xi_*^{r'_*} \right\rangle, \tau(B) \right\rangle \\
&= \left\langle Q(\varepsilon_*) \left\langle P^{(r_*)}, \xi_*^{(r'_*)} \right\rangle, \tau(\varepsilon'_*) \right\rangle \\
&= \delta_{r_* r'_*}\delta_{\varepsilon_*, \varepsilon'_*}.
\end{aligned}$$

□

Proposition 4.54 ([Voe03], Lemma 13.5, Proposition 13.6)**.** We define $q_n = P^{(0,\dots,0,1,0,\dots,0)}$ where 1 occurs at the nth entry, to be dual to ξ_n, then

$$Q_n = [q_n, \beta] := q_n\beta - \beta q_n$$

for $n \geqslant 1$.

Proof. It suffices to write $q_n\beta$ as the sum of the Minor basis elements Q_n's and βq_n's by Proposition 4.53. As before, we have

$$q_n\beta = \sum_{\varepsilon_*, r_*} [q_n \otimes \beta, \psi_*(\tau_*^{\varepsilon_*}\xi_*^{r_*})] \cdot \rho(\varepsilon_*, r_*).$$

We denote J to be the ideal in $A_{*,*}$ generated by $\{\tau_k\}_{k \geqslant 1}$, then the pairing

$$[q_n \otimes \beta, -] : A_{*,*} \otimes_{h^{*,*},\lambda} A_{*,*} \to h^{*,*}$$

factors thorugh the quotient $A_{*,*}/(I + J) \otimes_{h^{*,*},\lambda} A_{*,*}$, since β, being dual to τ_0, annihilates I and J. We define a natural map

$$\psi_*^\vee : A_{*,*} \to A_{*,*}/(I + J) \otimes_{h^{*,*},\lambda} A_{*,*}$$

induced by ψ_*. Here we make the identification of $A_{*,*}/(I + J)$ with $\xi_0 \cdot h^{*,*} \oplus \tau_0 \cdot h^{*,*}$. Suppose $\{x_1 < \cdots < x_k\} = \{i : \varepsilon_i = \beta\}$, i.e., $\tau_*^{\varepsilon_*} = \tau_{x_1} \cdots \tau_{x_k}$ for $x_1 < \cdots < x_k$. By Proposition 4.38, we have

$$\begin{aligned}
\psi_*^\vee \xi_k &= \xi_0 \otimes \xi_k \\
\psi_*^\vee \tau_k &= \xi_0 \otimes \tau_k + \tau_0 \otimes \xi_k \\
\psi_*^\vee(\tau_*^{\varepsilon_*}\xi_*^{r_*}) &= \xi_0 \otimes \tau_0^{\varepsilon_*}\xi_*^{r_*} + \sum_{i=1}^{k} (-1)^{i-1}\tau_0 \otimes \tau_{x_1} \cdots \hat{\tau}_{x_i} \cdots \tau_{x_k}\xi_{x_i}\xi_*^{r_*}.
\end{aligned}$$

Therefore $[q_n \otimes \beta, \psi_*(\tau_*^{\varepsilon_*}\xi_*^{r_*})] = \sum_{i=1}^{k}(-1)^{i-1}\left\langle q_n, \tau_{x_1}\cdots\hat{\tau}_{x_i}\cdots\tau_{x_k}\xi_{x_i}\xi_*^{r_*}\right\rangle$, which is zero unless

$$\begin{cases} r_* = (0,\ldots,0), & \varepsilon_* = (0,\ldots,0,1,0,\ldots,0) \\ r_* = (0,\ldots,0,1,0,\ldots,0), & \varepsilon_* = (1,0,\ldots,0) \end{cases}$$

where the entry 1 in the middle only occurs at nth entry by duality. This implies that $Q_n = q_n\beta - \beta q_n$, as desired. □

Proposition 4.55 ([Voe03], Lemma 13.1)**.** For $p > 2$, we have $\mathcal{P}^n = P^{(n,0,\ldots,0)}$ for any $n \in \mathbb{N}$. (In the case where $p = 2$, the analogous statement holds if we replace $\mathcal{P}^n$ by Sq^{2n}.)

Proof. It suffices to prove that $\mathcal{P}^n$ is dual to ξ_1^n. By Proposition 4.26, we know that $\left\langle \mathcal{P}^n, \tau_*^{\varepsilon_*}\xi_*^{r_*}\right\rangle = 0$ if $(0,n,0,\ldots) < (\varepsilon_0, r_1, \varepsilon_1, r_2, \ldots)$ in lexicographical ordering from right to left. Therefore, it suffices to two other possible cases

$$(\varepsilon_0, r_1, \ldots) = \begin{cases} (0,k,0,\ldots,0), & k \leqslant n \\ (1,k,0,\ldots,0), & k < n \end{cases}.$$

Note that both of them correspond to elements in $A_{*,*}$. In the first case, the correspondence gives

$$\begin{aligned} \left\langle \mathcal{P}^n, \xi_1^k\right\rangle &= \left\langle \psi^*(\mathcal{P}^n), \xi_1^{k-1}\otimes\xi_1\right\rangle \\ &= \left\langle \mathcal{P}^{n-1}, \xi_1^{k-1}\right\rangle \\ &= \begin{cases} 1, & k = n \\ 0, & k < n \end{cases} \end{aligned}$$

by induction. In the second case, we have

$$\begin{aligned} \left\langle \mathcal{P}^n, \tau_0\xi_1^k\right\rangle &= \left\langle \psi^*(\mathcal{P}^n), \tau_0\otimes\xi_1^k\right\rangle \\ &= \begin{cases} 0, & p > 2 \\ \left\langle \tau\,\mathrm{Sq}^1\otimes\mathrm{Sq}^{2n-1}, \tau_0\otimes\xi_1^k\right\rangle, & p = 2 \end{cases} \\ &\text{In the case } p = 2, \text{ it then follows that} \\ &= \tau\left\langle \psi^*(\mathrm{Sq}^{2n-1}), \xi_1^{k-1}\otimes\xi_1\right\rangle \\ &= \tau\left\langle \mathrm{Sq}^{2n-3}, \xi - 1^{k-1}\right\rangle \\ &= 0 \end{aligned}$$

by induction. Recardless, we have $\left\langle \mathcal{P}^n, \tau_0\xi_1^k\right\rangle = 0$. □

The following is a direct result of Proposition 4.55.

Corollary 4.56. We have $q_1 = \mathcal{P}^1$.

4.4 Calculating q_n

We now discuss methods of computing $q_n \in A^{*,*}$ in general. In the topological case, this can be done using a formula involving Lie algebra, but in the algebro-geometric case this is extremely complicated. We will take [Kyl17] as a reference, and focus on the case $p = 2$. For the case $p > 2$, one can refer to [Mil58, Corollary 5], since it is the same as the topological case given by a Lie theory formula.

Proposition 4.57 ([Kyl17], Theorem 6)**.** Let us denote $t_n \in A^{*,*}$ to be the dual of $\xi_1^{2^n}\xi_n$, then we have

$$\mathrm{Sq}^{2^{n+1}} q_n = q_{n+1} + t_n + \tau\sum_{i=0}^{n-1}\rho^i Q_{n-i-1}Q_{n-i}\cdots Q_n\,\mathrm{Sq}^{2^{n-i}}$$

for every $n \geqslant 1$.

Proof. Set $b_{n,i}$ to be the sequence $(0,0,\dots,1,\dots,1,0)$ with 1's between $(n-i)$th and nth entry (inclusive), and $a_n = b_{n,0}$, then the statement is equivalent to

$$\left\langle \mathrm{Sq}^{2^{n+1}} q_n, \tau_*^{\varepsilon_*}\xi_*^{r_*}\right\rangle = \begin{cases} 1, & \varepsilon_* = (0,\dots,0), r_* = a_{n+1} \text{ or } a_n + (2^n,0,\dots,0) \\ \tau\rho^i, & \varepsilon_* = b_{n,i+1}, r_* = (2^{n-i-1},0,\dots,0), 0 \leqslant i \leqslant n-1 \\ 0, & \text{else} \end{cases}.$$

Consider the following two ideals in $A_{*,*}$:

$$\begin{aligned} I_n &:= (\xi_1^{2^n+1}, \xi_2, \xi_3, \dots, \tau_1, \tau_2, \dots) \\ J_n &:= (\xi_1, \dots, \xi_{n-1}, \xi_n^2, \xi_{n+1}, \dots, \tau_0\xi_n, \dots, \tau_{n-1}\xi_n, \tau_n, \tau_{n+1}, \dots), \end{aligned}$$

given by annihilators, then

$$\begin{aligned} [\mathrm{Sq}^{2^{n+1}} \otimes q_n, J_n \otimes A_{*,*}] &= 0 \\ [\mathrm{Sq}^{2^{n+1}} \otimes q_n, A_{*,*} \otimes I_n] &= 0. \end{aligned}$$

We now define a composite

$$\tilde{\psi}_* : A_{*,*} \xrightarrow{\psi_*} A_{*,*} \otimes_{h^{*,*,\lambda}} A_{*,*} \to A_{*,*}/J_n \otimes_{h^{*,*,\lambda}} A_{*,*}/I_n,$$

then we have for $k \geqslant 1$ that

$$\tilde{\psi}_*(\xi_k) = \begin{cases} 1 \otimes \xi_k, & k = 1, n > 1 \\ \xi_n \otimes 1, & k = n > 1 \\ \xi_n \otimes 1 + 1 \otimes \xi_n, & k = n = 1 \\ \xi_n \otimes \xi_1^{2^n}, & k = n+1 \\ 0, & k \neq 1, n, n+1 \end{cases}, \quad \tilde{\psi}_*(\tau_k) = \begin{cases} 1 \otimes \tau_0 + \tau_0 \otimes 1, & k = 0 \\ \tau_k \otimes 1 + \tau_{k-1} \otimes \xi_1^{2^{k-1}}, & k \geqslant 1 \end{cases}.$$

which proves the first case. Now

$$\left\langle \mathrm{Sq}^{2^{n+1}1} q_n, \tau_*^{\varepsilon_*}\xi_*^{r_*}\right\rangle = [\mathrm{Sq}^{2^{n+1}} q_n, \tilde{\psi}_*(\tau_*^{\varepsilon_*})\tilde{\psi}_*(\xi_*^{r_*})],$$

and we only have to consider $\tilde{\psi}_*(\tau_*^{\varepsilon_*})$, whereas the other portions being non-zero suffices. Since $\tilde{\psi}_*(\tau_k) = 0$ for any $k > n$, then we can assume $\varepsilon_* = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n, 0, \dots, 0)$. Since ψ_* is a ring homomorphism, set $n \geqslant i_1 > i_2 \cdots > i_\ell$, then $\tau_*^{\varepsilon_*} = \tau_{i_1} \cdots \tau_{i_\ell}$, where $\varepsilon_k = 1$ if and only if $k = i_j$. We can then compute

$$\tilde{\psi}_*(\tau_*^{\varepsilon_*}) = \begin{cases} \sum\limits_{A \coprod B = \{i_1,\dots,i_\ell\}} \tau(A-1)\tau(B) \otimes \xi_1^{\sum 2^{x_i - 1}}, & i_\ell \neq 0 \\ \left(\sum\limits_{A \coprod B = \{i_1,\dots,i_{\ell-1}\}} \tau(A-1)\tau(B) \otimes \xi_1^{\sum 2^{x_i - 1}} \right)(1 \otimes \tau_0 + \tau_0 \otimes 1), & i_\ell = 0 \end{cases} \tag{4.58}$$

where $A - 1 := \{x_1 - 1, \dots, x_t - 1\}$ for $A = \{x_1, \dots, x_t\}$.

Lemma 4.59. Consider $n > j_1 \geqslant j_2 \geqslant \cdots \geqslant j_\ell \geqslant 0$, where $\ell \geqslant 2$, and suppose $j_i > j_{i+2}$ for all i, then

$$\tau_{j_1} \cdots \tau_{j_\ell} \equiv \tau_{j_\ell + 1} \cdot \lambda \text{ or } \xi_n \cdot \lambda \pmod{J_n}$$

for $\lambda \in h^{*,*}$ if and only if $j_2 = j_1 - 1, \dots, j_{\ell-1} = j_1 - (\ell-2) = j_\ell$, in which case $\lambda = \rho^{\ell-1}$, or $\lambda = \tau\rho^{\ell-2}$ and $j_\ell = n-1$. In such case, we say that the sequence is decrasing. If the sequence is not decreasing, then $\tau_{j_1} \cdots \tau_{j_\ell} \equiv \rho^t \tau_{j_1'} \cdots \tau_{j_k'} \tau^\varepsilon \xi_n^\varepsilon$ where $t \geqslant 0$, $\varepsilon = 0, 1$, $k + \varepsilon \geqslant 2$, and $j_1 + 1 \geqslant j_1' \geqslant \cdots \geqslant j_k'$.

Subproof. For every $j \in \mathbb{N}$, we have

$$\tau_j^2 \equiv \begin{cases} \tau\xi_n, & j = n-1 \\ \rho\tau_{j+1}, & \text{else} \end{cases}.$$

■

We proceed by induction on ℓ. The case for $\ell = 2$ is clear, so suppose the statements are true for some $\ell - 1 \geqslant 3$.

a. If $j_1 > j_2$ and $j_2, \ldots, j_\ell$ is not a decreasing sequence, then $\tau_{j_2} \cdots \tau_{j_\ell} \equiv \rho^t \tau_{j_2'} \cdots \tau_{j_k'} \tau^\varepsilon \xi_n^\varepsilon$ for $k + \varepsilon \geqslant 3$, therefore

$$\tau_{j_1} \cdots \tau_{j_\ell} \equiv \rho^t \tau_{j_1} \tau_{j_2'} \cdots \tau_{j_k'} \tau^\varepsilon \xi_n^\varepsilon$$

since this is still not a decreasing sequence after adding j_1 to the sequence.

b. If $j_1 > j_2$ and $j_2, \ldots, j_\ell$ is a decreasing sequence, then $j_1, \ldots, j_\ell$ is decreasing if and only if $j_1 = j_2 + 1$, hence

$$\tau_{j_1} \cdots \tau_{j_\ell} \equiv \tau_{j_1} \rho^{\ell-2} \tau_{j_1} \equiv \begin{cases} \tau_{j_1} \rho^{\ell-2} \tau_{j_1} \equiv \tau_{j+1} \rho^{\ell-1}, & j_1 \neq n-1 \\ \xi_n \tau \rho^{\ell-2}, & j_1 = n-1 \end{cases}$$

if $j_1 = j_2 + 1$, otherwise $j_1 > j_2 + 1$, then $\tau_{j_1} \cdots \tau_{j_\ell} \equiv \tau_{j_1} \tau_{j_2+1} \rho^{\ell-2}$ or $\tau \tau_{j_1} \xi_n \rho^{\ell-3}$.

c. If $j_1 = j_2$, then we have $\tau_{j_1} \cdots \tau_{j_\ell} \equiv \tau_{j_1}^2 \tau_{j_3} \cdots \tau_{j_\ell} \equiv \rho \tau_{j_1+1} \tau_{j_3} \cdots \tau_{j_\ell}$ or $\tau \xi_n \tau_{j_3} \cdots \tau_{j_\ell}$. Using the fact that $j_1 > j_3$ and inductive hypothesis, we are done.

To compute the coefficient, suppose $\tilde{\psi}_*(\tau_*^{\varepsilon *}) = \sum_i a_i \otimes b_i$, then we have

$$\left\langle \operatorname{Sq}^{2^{n+1}} q_n, \tau_*^{\varepsilon *} \xi_*^{r *} \right\rangle = \sum_{i,j} \left\langle \operatorname{Sq}^{2^{n+1}} \langle q_n, \xi_n^{n_j} a_i \rangle, \xi_1^{r_j} b_i \right\rangle.$$

From Equation (4.58), b_i is of the form ξ_1^t or $\xi_1^t \cdot \tau_0$, therefore $\left\langle \operatorname{Sq}^{2^{n+1}}, \xi_1^t b_i \right\rangle = 0$ or 1, so if $\left\langle \operatorname{Sq}^{2^{n+1}} q_n, \tau_*^{\varepsilon *} \xi_*^{r *} \right\rangle \neq 0$, at least one of the a_i's must be of the form $\xi_n \cdot u$ for some $u \in h^{*,*}$.

We now have $\tau_*^{\varepsilon *} = \tau_{i_1} \cdots \tau_{i_\ell}$ where $n \geqslant i_1 > i_2 > \cdots > i_\ell \geqslant 0$.

a. Suppose $i_\ell \neq 0$, then by Lemma 4.59, $\tau(A-1)\tau(B)$ is of the form $\xi_n \cdot u$ for $u \in h^{*,*}$ if and only if the sequence $i_1 - \chi_A(1) \geqslant \cdots \geqslant i_A(\ell)$ is decreasing and $n - 1 = i_1 - \chi_A(1)$, where

$$\chi_A(x) = \begin{cases} 1, & i_x \in A \\ 0, & i_x \notin A \end{cases}$$

is a characteristic function. Therefore, for any $j < \ell - 1$, $i_j - i_{j+1} = \chi_A(j) - \chi_A(j+1) + 1$, and $i_{\ell-1} - i_\ell = \chi_A(\ell - 1) - \chi_A(\ell)$, so we have

$$\chi_A(1) \geqslant \chi_A(2) \geqslant \cdots \geqslant \chi_A(\ell-1) > \chi_A(\ell),$$

so $A = \{i_1, \ldots, i_{\ell-1}\}$ and $i_1 = n$. Assume E is such a sequence, then

$$\tau(A-1)\tau(B) \otimes \operatorname{Sq}^{\sum 2^{A-1}} = \tau \rho^{\ell-2} \xi_n \otimes \xi_1^{2^n - 2^{n-\ell+1}}$$

where we think of $\sum 2^{A-1}$ to be $\sum 2^{x_i - 1}$. The non-trivial terms must satisfy $\tilde{\psi}_*(\xi_*^{r *}) = \xi_1^{2^{n-\ell+1}}$, therefore $r_* = (2^{n-\ell+1}, 0, \ldots, 0)$.

b. If $i_{\ell-1} - 1 > i_\ell = 0$, then all terms vanish by Lemma 4.59.

c. If $i_{\ell-1} = i_\ell = 0$, then by Lemma 4.59, the only non-vanishing term occurs when $A = \{1, \ldots, n\}$, $B = \varnothing$, and $i_1 = n, \ldots, i_{\ell-1} = 1$, namely $\varepsilon_* = b_{n,n}$, and $r_* = (1, 0, \ldots, 0)$.

This proves the second and third case. □

Proposition 4.60 ([Kyl17], Lemma 9)**.** We have $q_n \operatorname{Sq}^{2^{n+1}} = t_n + \tau Q_n Q_0 \operatorname{Sq}^{2^{n+1}-2}$.

Proof. Keep I_n's and J_n's to be the same as in Proposition 4.57, now we define

$$\tilde{\psi}_* : A_{*,*} \xrightarrow{\psi_*} A_{*,*} \otimes_{h^{*,*},\lambda} A_{*,*} \to A_{*,*}/I_n \otimes_{h^{*,*},\lambda} A_{*,*}/J_n,$$

then for $k \geqslant 1$, we have

$$\tilde{\psi}_*\xi_k = \begin{cases} 0, & k > 1, k \neq n \\ 1 \otimes \xi_1 + \xi_1 \otimes 1, & k = n = 1 \\ \xi_1 \otimes 1, & k = 1, n > 1 \\ 1 \otimes \xi_n, & k = n > 1 \end{cases}, \quad \tilde{\psi}_*\tau_k = \begin{cases} 0, & k > n \\ \tau_0 \otimes \xi_n, & k = n \\ 1 \otimes \tau_k, & 0 < k < n \\ 1 \otimes \tau_0 + \tau_0 \otimes 1, & k = 0 \end{cases}.$$

Therefore, in order to have $\left\langle q_n \operatorname{Sq}^{2^{n+1}}, \tau_*^{\varepsilon_*} \xi_*^{r_*} \right\rangle \neq 0$, we must have $r_* = 0$ or $r = (1, 0, \ldots, 1, 0, \ldots, 0)$ where 1 occurs on the first and nth entry only. The statement then follows. $\square$

The following is now a combination of Proposition 4.57 and Proposition 4.60, c.f., [Kyl17, Equation 6].

Corollary 4.61. For $n \geqslant 1$, we have

$$q_{n+1} = [q_n, \operatorname{Sq}^{2^{n+1}}] + \tau \sum_{i=0}^{n-1} \rho^i Q_{n-i-1} \cdots Q_n \operatorname{Sq}^{2^{n-i}} + \tau Q_n Q_0 \operatorname{Sq}^{2^{n+1}-2}.$$

Remark 4.62. We note that in the topological case for $n \geqslant 1$, $\tau = 0$, therefore we recover

$$q_{n+1} = [q_n, \operatorname{Sq}^{2^{n+1}}].$$

4.5 Hopf Algebroid Structure

Finally, we explain the antipodal map in $A_{*,*}$ in the case $p = 2$ (again, in the case of $p > 2$, we recover the topological case, c.f., [Mil58]), so that we can define the Hopf algebroid structure on $A_{*,*}$.

Definition 4.63. Suppose $f, g : A_{*,*} \to A_{*,*}$ are linear maps (and may not be graded) such that for any $u \in h^{*,*}$ and $a, b \in A_{*,*}$, then we have

$$f(a \cdot u)g(b) = f(a)g(u \cdot b).$$

We define the convolution $f * g$ as follows: suppose $a \in A_{*,*}$ with $\psi_*(a) = \sum_i a_i' \otimes a_i''$, then we define

$$(f * g)(a) = \sum_i f(a_i')g(a_i'') \in A_{*,*},$$

which is the composite

$$A_{*,*} \xrightarrow{\psi_*} A_{*,*} \otimes_{h^{*,*},\lambda} A_{*,*} \xrightarrow{f \otimes g} A_{*,*}.$$

Definition 4.64. An antipode $c : A_{*,*} \to A_{*,*}$ is a linear map (and ring homomorphism) such that

1. for any $u \in h^{*,*}$, we have $c(u \cdot 1) = 1 \cdot u = u = \rho(u)$ via

$$\begin{aligned} \rho : h^{*,*} &\to A_{*,*} \\ u &\mapsto 1 \cdot u, \end{aligned}$$

   and $c(1 \cdot u) = u \cdot 1 = \lambda^*(u)$;

2. $c \circ c = \mathrm{id}_{A_{*,*}}$;

3. set $\varepsilon : A_{*,*} \to h^{*,*}$ to be the projection to the subspace $\xi_0 \cdot h^{*,*}$, then $c * \mathrm{id}_{A_{*,*}} = \rho \circ \varepsilon$ and $\mathrm{id}_{A_{*,*}} * c = \lambda^* \circ \varepsilon$.

Therefore, it flips the left- and right-module structure on $A_{*,*}$ (which we took to be induced by composition and scalar multiplication, respectively).

Proposition 4.65.

1. The convolution is associative.
2. The map $\lambda^* \circ \varepsilon$ (respectively, $\rho \circ \varepsilon$) is the right (respectively, left) identity of the convolution.

Proof.

1. We just have to prove the coassociativity of ψ_*, i.e., $(f * g) * h = f * (g * h)$, then the statement follows from composing the map $f \otimes g \otimes h : A_{*,*} \otimes A_{*,*} \otimes A_{*,*} \to A_{*,*}$. For this, it suffices to check on the generators ξ_k's and τ_k's because of the ring homomorphism structure. By Proposition 4.38,

$$\begin{aligned}
(\psi_* \otimes \mathrm{id})\psi_*(\xi_k) &= (\psi_* \otimes \mathrm{id}) \left(\sum_{i=0}^{k} \xi_i \otimes \xi_{k-i}^{p^i} \right) \\
&= \sum_{i=0}^{k} \sum_{j=0}^{i} \xi_j \otimes \xi_{i-j}^{p^j} \otimes \xi_{k-i}^{p^j} \\
&= \sum_{i=0}^{k} \sum_{\ell=0}^{k-i} \xi_i \otimes \xi_{\ell}^{p^i} \otimes \xi_{k-i-\ell}^{p^{\ell+i}} \\
&= (\mathrm{id} \otimes \psi_*) \left(\sum_{i=0}^{k} \xi_i \otimes \xi_{k-i}^{p^i} \right) \\
&= (\mathrm{id} \otimes \psi)\psi_*(\xi_k),
\end{aligned}$$

and the same holds for τ_k.

2. We can show that $(\mathrm{id} \otimes (\rho \circ \varepsilon)) \circ \psi_* = \mathrm{id}$ by checking on generators ξ_i and τ_i. For any $f : A_{*,*} \to A_{*,*}$ and any $x \in A_{*,*}$, set $\psi_*(x) = \sum_i a_i \otimes b_i$, then we have

$$\begin{aligned}
(f * (\lambda^* \circ \varepsilon))(x) &= \sum_i f(a_i)\lambda^*(\varepsilon(b_i)) \\
&= \sum_i f(a_i)(\varepsilon(b_i) \cdot \lambda^* \varepsilon(1)) \\
&= f\left(\sum_i a_i \cdot \varepsilon(b_i) \right) \\
&= f(x),
\end{aligned}$$

and the same holds for $\rho \circ \varepsilon$.

□

Proposition 4.66. The antipode $c : A_{*,*} \to A_{*,*}$ exists and must be unique.

Proof. We first show its uniqueness. Suppose c and c' are both antipodes, then we have

$$\begin{aligned}
c &= c * (\lambda^* \circ \varepsilon) \\
&= c * (\mathrm{id} * c') \\
&= (c * \mathrm{id}) * c' \\
&= (\rho \circ \varepsilon) * c' \\
&= c'.
\end{aligned}$$

To show its existence, we define a ring homomorphism

$$\varphi : [\xi_i, \tau_i]h^{*,*} \to A_{*,*}$$

$$\begin{aligned}
1\cdot u = u &\mapsto \lambda^*(u)\\
\xi_k &\mapsto \sum_{i=0}^{k-1}\varphi(\xi_i)\xi_{k-i}^{p^i}\\
\tau_k &\mapsto \tau_k+\sum_{i=0}^{k-1}\varphi(\tau_i)\xi_{k-i}^{p^i}
\end{aligned}$$

where $u\in h^{*,*}$, and ξ_k's for $k\geqslant 1$, and τ_k's for $k\geqslant 0$ defined inductively (with base case defined). We see this is well-defined on the polynomial ring, and since $A_{*,*}$ is a quotient of such ring, it suffices to show that it factors through $A_{*,*}$, i.e., valued as zero on the defining ideal. That is, we now prove by induction that

$$\varphi(\tau_k^2)=\varphi(\xi_{k+1}\tau+\tau_0\xi_{k+1}\rho+\tau_{k+1}\rho).$$

If $k=0$, then we have

$$\begin{aligned}
\varphi(\tau_k^2) &= \tau_0^2\\
&= \xi_1\tau+\tau_0\xi_1\rho+\tau_1\rho\\
&= \varphi(\xi_1)\varphi(\tau+\tau_0\rho)+\varphi(\tau_1)\rho\\
&= \varphi(\xi_{k+1}\tau+\tau_0\xi_{k+1}\rho+\tau_{k+1}\rho).
\end{aligned}$$

Now suppose this is true for $1,\dots,k-1$, and we show this for k. We have

$$\begin{aligned}
\varphi(\tau_k^2) &= \tau_k^2+\sum_{i=0}^{k-1}\varphi(\tau_i^2)\xi_{k-i}^{p^{i+1}}\\
&= \tau_k^2+\sum_{i=0}^{k-1}\varphi(\xi_{i+1}\tau+\tau_0\xi_{i+1}\rho+\tau_{i+1}\rho)\xi_{k-i}^{p^{i+1}}\\
&= \tau_k^2+\sum_{i=0}^{k-1}\varphi(\xi_{i+1})\xi_{k-i}^{p^{i+1}}\varphi(\tau+\tau_0\rho)+\sum_{i=0}^{k-1}\varphi(\tau_{i+1})\xi_{k-i}^{p^{i+1}}\rho\\
&= \tau_k^2+(\varphi(\xi_{k+1})+\xi_{k+1})\varphi(\tau+\tau_0\rho)+(\varphi(\tau_{k+1})+\tau_{k+1}+\varphi(\tau_0)\xi_{k+1})\rho\\
&= \varphi(\xi_{k+1}(\tau+\tau_0\rho)+\tau_{k+1}\rho)+\tau_k^2+\xi_{k+1}\tau+\tau_{k+1}\rho+\varphi(\tau_0)\xi_{k+1}\rho\\
&= \varphi(\xi_{k+1}\tau+\xi_{k+1}\tau_0\rho+\tau_{k+1}\rho).
\end{aligned}$$

Therefore, we obtained a well-defined ring map $c: A_{*,*}\to A_{*,*}$, then so is $c*\mathrm{id}$. It is now easy to verify that $c*\mathrm{id}=\rho\circ\varepsilon$ by checking on the generators and using the definition of φ. $\square$

Lemma 4.67. Suppose $\psi_*(x)=\sum_i a_i\otimes b_i$, then we have

$$\psi_*(c(x))=\sum_i c(b_i)\otimes c(a_i).$$

Proof. Again, we only have to check this on the generators. We have $\psi_*(\xi_k)=\sum_{i=0}^{k}\xi_i\otimes\xi_{k-i}^{p^i}$, then by induction on k, we see that

$$\begin{aligned}
\psi_*(c(\xi_k)) &= \psi_*\left(\sum_{i=0}^{k-1}c(\xi_i)\xi_{k-i}^{p^i}\right)\\
&= \sum_{i=0}^{k-1}\psi_*(c(\xi_i))\psi_*(\xi_{k-i}^{p^i})\\
&= \sum_{i=0}^{k-1}\sum_{j=0}^{i}\sum_{\ell=0}^{k-i}(c(\xi_{i-j}^{p^j})\otimes c(\xi_j))\cdot(\xi_\ell^{p^i}\otimes\xi_{k-i-\ell}^{p^{i+\ell}})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{k-1} \sum_{j=0}^{i} \sum_{\ell=0}^{k-i} (c(\xi_{i-j}^{p^j})\xi_\ell^{p^i}) \otimes (c(\xi_j)\xi_{k-i-\ell}^{p^{i+\ell}}) \\
&= \sum_{\ell+i=0}^{k-1} \sum_{j=0}^{k-1} \sum_{i=j}^{\ell+i} (c(\xi_{i-j})\xi_\ell^{p^{i-j}})^{p^j} \otimes c(\xi_j)\xi_{k-(\ell+i)}^{p^{\ell+i}} \\
&+ \sum_{j=0}^{k-1} \sum_{i=j}^{k-1} (c(\xi_{i-j})\xi_{k-i}^{p^{i-j}})^{p^j} \otimes c(\xi_j)
\end{aligned}
$$

now note that the first term is non-zero if and only if $\ell + i = j = i$, therefore

$$
\begin{aligned}
&= 1 \otimes c(\xi_k) + \sum_{k=0}^{k-1} c(\xi_{k-j}^{p^j}) \otimes c(\xi_j) \\
&= \sum_{j=0}^{k} c(\xi_{k-j}^{p^j}) \otimes c(\xi_j).
\end{aligned}
$$

We can do the same for τ_k. □

By the same method, one can prove that $\mathrm{id} * c = \lambda^* \circ \varepsilon$.

Lemma 4.68. We have $c \circ c = \mathrm{id}$.

Proof. Suppose $x \in A_{*,*}$, and write $\psi_*(x) = \sum_i a_i \otimes b_I$, then we have

$$
\begin{aligned}
((c \circ c) * c)(x) &= \sum_i (c \circ c)(a_i) c(b_i) \\
&= (\mathrm{id} * c)(c(x)) \text{ by Lemma 4.67} \\
&= (\lambda^* \circ \varepsilon)(c(x)) \\
&= (\lambda^* \circ \varepsilon)(x),
\end{aligned}
$$

therefore $(c \circ c) * c = \lambda^* \circ \varepsilon$, then

$$
\begin{aligned}
c \circ c &= (c \circ c) * (\lambda^* \circ \varepsilon) \\
&= (c \circ c) * (\mathrm{id} * c) \\
&= (c \circ c) * \mathrm{id} * c \\
&= (c \circ c) * ((\rho \circ \varepsilon) * \mathrm{id}) * c \\
&= (c \circ c) * (c * \mathrm{id}) * \mathrm{id} * c \\
&= ((c \circ c) * c) * \mathrm{id} * \mathrm{id} * c \\
&= (\lambda^* \circ \varepsilon) * \mathrm{id} * (\lambda^* \circ \varepsilon) \\
&= (\lambda^* \circ \varepsilon) * \mathrm{id} \\
&= \mathrm{id}.
\end{aligned}
$$

□

We have shown the existence of the antipode map, and by the discussion above, identified all the required structure for the Hopf algebroid.

Theorem 4.69. $A_{*,*}$ is a Hopf algebroid with left unit $\lambda^* \circ \varepsilon$, right unit $\rho \circ \varepsilon$, coproduct ψ_*, counit ε, and antipode c.

Remark 4.70. $A^{*,*}$ is not a Hopf algebra.

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