

# MATH 595 (Group Cohomology) Notes

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## 1 AUG 21, 2023: INTRODUCTION

Group cohomology works over different settings of groups, like finite groups, profinite groups, and topological groups. The course will develop towards

- duality in  $H^*(G, -)$ , and
- focus on computations, e.g., spectral sequences.

We first establish some notations.

- Let  $G$  be a group. If  $G$  has a topology, that would also be part of the information of  $G$ .
- A (left)  $G$ -module is an abelian group  $M$  with an action map

$$\begin{aligned} G \times M &\rightarrow M \\ (g, m) &\mapsto g \cdot m = gm \end{aligned}$$

satisfying

- $1 \cdot m = m$ ,
- $(gh) \cdot m = g \cdot (hm)$ ,
- $g(m + m') = gm + gm'$ .

**Remark 1.1.** If  $G$  is a finite group, then the associated (non-commutative) group ring  $\mathbb{Z}[G] := \bigoplus_{g \in G} \mathbb{Z}e_g$ , where the multiplication is determined by  $e_g e_h = e_{gh}$ . Therefore, a  $G$ -module is just a  $\mathbb{Z}[G]$ -module.

**Example 1.2.** • Trivial module  $\mathbb{Z}$ , or any abelian group with the trivial action  $g \cdot a = a$ .

- $C_2$ , or any group with  $f : G \twoheadrightarrow C_2$ , then  $G$  with  $C_2$  as a quotient gives the sign representation  $\mathbb{Z}_{\text{sgn}}$ , with  $g \cdot (a) = (-1)^{\rho(g)} a$ .
- $\mathbb{Z}[G]$  is a  $G$ -module via the left multiplication action, and/or the conjugation action.

**Definition 1.3** (Fixed points/Invariants). The set of fixed points of  $M$  over  $G$  is  $M^G = \{m \in M \mid gm = m \ \forall g \in G\}$ .

**Definition 1.4** (Orbits/Coinvariants). The set of orbits of  $M$  over  $G$  is  $M_G = M/(gm - m)$ .

**Example 1.5.** If  $M = \mathbb{Z}_{\text{sgn}}$ , then everything gets multiplied by  $-1$ , so there are no fixed points. The orbits of  $M$  over  $G$  would be  $\mathbb{Z}_{\text{sgn}}/(-2) \cong \mathbb{Z}/2\mathbb{Z}$ .

**Example 1.6.** If  $M = \mathbb{Z}[G]$ , then the fixed points are  $\mathbb{Z} \left\{ \sum_{g \in G} e_g \right\}$ .

Thinking in a categorical setting, we have a trivial action function  $\mathbb{Z}\text{-Mod} \rightarrow G\text{-Mod}$ , sending  $ga \mapsto a$  for all  $g \in G$  and  $a \in A$ . This gives an exact functor from  $\mathbf{Ab}$  to  $G\text{-Mod}$ . Then this functor has a right adjoint  $( )^G : G\text{-Mod} \rightarrow \mathbf{Ab}$ , and a left adjoint  $( )_G : \mathbf{Ab} \rightarrow G\text{-Mod}$ . More specifically,  $M^G$  becomes the maximal trivial action submodule of  $M$ , namely  $\text{Hom}_G(\mathbb{Z}, M)$ ;  $M_G$  becomes the largest quotient of  $M$  with trivial action, namely  $\mathbb{Z} \otimes_{\mathbb{Z}[G]} M$ . This simplifies to the tensor-hom adjunction in some sense. For a more detailed derivation of this, see Chapter 6.1 of Weibel.

**Remark 1.7.** In general, as in the category of  $G$ -sets, we have the orbit functor  $X \mapsto X/G$  and the fixed point functor  $X \mapsto X^G$ . The orbit functor is left adjoint to the free  $G$ -set functor, and the fixed point functor is the right adjoint of the trivial  $G$ -set functor.

**Remark 1.8.** Read more about the setting in profinite groups with their topologies in Neukirch-Schmidt-Wingberg.

**Definition 1.9** (Profinite Group). A profinite group of a collection of groups is  $G = \varprojlim_i G_i$  as an inverse limit, where each  $G_i$  is a finite group of the form  $G/U_i$  for some open  $U_i$ . This gives a topology to the profinite group.

**Remark 1.10.** The groups rings  $\mathbb{Z}[[G]] = \varprojlim_i \mathbb{Z}[G_i]$ . For instance, let  $G = \hat{\mathbb{Z}}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ , then  $\mathbb{Z}_p[[G]] = \varprojlim_n \mathbb{Z}_p[\mathbb{Z}/p^n\mathbb{Z}]$ , where each  $\mathbb{Z}[\mathbb{Z}/p^n\mathbb{Z}] \cong \bigoplus_{i=0}^{n-1} \mathbb{Z}\{e_i\}$  where  $e_i \cdot e_j = e_{ij}$ . Therefore,  $\mathbb{Z}_p[[G]]$  is now equivalent to  $\varprojlim_n \mathbb{Z}_p[t]/(t^{p^n} - 1_e)$ , and hence becomes a power series.

**Remark 1.11.** By a change of variables, this becomes  $\varprojlim_n \mathbb{Z}_p[x]/(x^{p^n})$ , but this only works in the finite group  $\mathbb{Z}_p$  case, and not in general for  $\mathbb{Z}$ .

**Example 1.12.**  $\mathbb{Z}[C_n] \cong \mathbb{Z}\{e\} \oplus \mathbb{Z}\{g\} \oplus \mathbb{Z}\{g^2\} \oplus \cdots \oplus \mathbb{Z}\{g^{n-1}\} \cong \mathbb{Z}[g]/(g^n - 1_e)$ .

## 2 AUG 23, 2023: COHOMOLOGY OF GROUPS

**Definition 2.1.** Let  $G$  be a group, then we have a diagram

$$EG : \cdots \rightrightarrows G \times G \rightrightarrows G$$

where the arrows are given by

$$EG^n = G^{n+1} \xrightarrow{d_i} G^n$$

for all  $0 \leq i \leq n$ . In the sense of simplicial sets, we have  $d_i(g_0, \dots, g_n) = (g_0, \dots, \hat{g}_i, \dots, g_n)$ .

Now let  $M$  be a  $G$ -module, then we define  $X^n = X^n(G, M) = \text{Map}_{\text{Set}}(G^{n+1}, M)$ .  $G$  now has an action on this set, given by

$$(g \circ f)(g_0, \dots, g_n) = gf(g^{-1}g_0, \dots, g^{-1}g_n).$$

The action on  $d^i$ 's are contravariant, namely we obtain  $d_i^* : X_n \rightarrow X^{n+1}$  with an inherited structure. Note that  $M$  sits inside  $X^0$ , therefore we have a complex  $(*)$ :

$$0 \longrightarrow M \xleftarrow{\partial_0} X^0 \xrightarrow{\partial_1} X^1 \xrightarrow{\partial_2} X^2 \xrightarrow{\partial_3} \cdots$$

Here  $\partial_0$  includes  $M$  as the constant functions into  $X$ , namely  $\partial_0(m) = f$  for  $f(g) = m$ , and so on. In general, for  $n > 0$ , we have

$$\partial_n = \sum_{i=0}^n (-1)^i d_i^*.$$

**Lemma 2.2.** The complex  $(*) : M \rightarrow X^\cdot$  is an exact complex of  $G$ -modules, i.e.,  $\partial^2 = 0$  and  $\ker(\partial_{n+1}) = \text{im}(\partial_n)$ , and the  $\partial_i$ 's preserves the  $G$ -action. This is called the standard resolution of  $M$  as a  $G$ -module.

*Proof.* Exercise. □

**Definition 2.3.** The  $G$ -fixed points of the  $X^n$ 's are defined by  $C^n(G, M) = (X^n(G, M))^G$ , called the homogeneous  $n$ -cochains of  $G$  with coefficients in  $M$ . Because the complex preserves  $G$ -actions, then we obtain a complex of  $C^n(G, M)$ 's, given by

$$0 \longrightarrow C^0(G, M) \xrightarrow{\partial_0} C^1(G, M) \xrightarrow{\partial_1} \dots$$

**Remark 2.4.** To see what the induced mapping is, suppose  $A \rightarrow B$  is a  $G$ -module map, then there is an induced map of fixed points  $A^G \rightarrow B^G$  by the restriction. In particular, let  $a \in A$  be fixed with  $ga = a$  for all  $g \in G$ , then  $f(a) = f(ga) = gf(a)$ .

**Remark 2.5.** In the complex of Definition 2.3,  $\partial^2 = 0$  as well, but in general this is not an exact sequence.

**Definition 2.6** (Group Cohomology). The group cohomology of  $G$  with coefficients in  $M$  is the collection

$$\{H^n(G, M)\}_{n \geq 0},$$

where  $H^n(G, M) := H^n(C^\cdot(G, M)) = \ker(\partial : C^n \rightarrow C^{n+1}) / \text{im}(\partial : C^{n-1} \rightarrow C^n)$ . We usually use the notion of cocycles  $Z^n(G, M) = \ker(\partial : C^n \rightarrow C^{n+1})$  and coboundaries  $B^n(G, M) = \text{im}(\partial : C^{n-1} \rightarrow C^n)$ .

**Exercise 2.7.** Show that  $H^0(G, M)$  is isomorphic to  $M^G$ .

**Definition 2.8.** The inhomogeneous cochains  $C_i^\cdot(G, M)$  are given by

- $C_i^0 = M$ , and
- for  $n > 0$ ,  $C_i^n = \text{Map}(G^n, M)$ ,

with coboundary maps  $\partial^{n+1} : C_i^n \rightarrow C_i^{n+1}$ , given by

- $\partial^1(m)(g) = gm - m$ ,
- $\partial^2(f)(g_1, g_2) = g_1 f(g_2) - f(g_1 g_2) + f(g_1)$ , and so on, with
- $\partial^{n+1}(f)(g_1, \dots, g_{n+1}) = g_1 f(g_2, \dots, g_{n+1}) + \sum_{i=1}^n (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) + (-1)^{n+1} f(g_1, \dots, g_n)$ .

This gives the inhomogeneous setting of this cochain.

**Lemma 2.9.** The maps

$$\begin{aligned} C^n(G, M) &\rightarrow C_i^n(G, M) \\ (\varphi : G^{n+1} \rightarrow M) &\mapsto (f : G^n \rightarrow M) \\ f(g_1, \dots, g_n) &:= \varphi(1, g_1, g_1 g_2, \dots, g_1 g_2 \cdots g_n) \end{aligned}$$

give a cochain homotopy equivalence  $C^\cdot(G, M) \xrightarrow{\sim} C_i^\cdot(G, M)$ , and hence this is a quasi-isomorphism.

**Corollary 2.10.** The cohomology  $H^*(C_i^\cdot(G, M)) \cong H^*(G, M)$ .

**Remark 2.11.** Any cohomology class can be represented by a normalized inhomogeneous cocycle  $f : G^n \rightarrow M$ , i.e.,  $f(g_1, \dots, g_n) = 0$  where  $g_i = 1$  for some  $i$ .

**Remark 2.12.** Even for  $G = C_2$ ,  $C_i^n$  or  $C^n$  get large as  $n$  grows.

**Remark 2.13.** • Using homological algebra, we can find other cochain complexes which computes group cohomology  $H^*(G, M)$ .

- We would also understand  $H^*(G, M)$  as the failure of exactness of  $(\ )^G : G\text{-}\mathbf{Mod} \rightarrow \mathbf{Ab}$ . Therefore, when taking the fixed points, the exact sequence may not be mapped to another exact sequence. In particular, if we take an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of  $G$ -modules, the induced sequence

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G$$

do not give a surjection at  $B^G \rightarrow C^G$ . One needs to take higher cohomology to obtain a long exact sequence. Hence,  $(\ )^G : G\text{-}\mathbf{Mod} \rightarrow \mathbf{Ab}$  is a left exact functor, but not necessarily right exact.

## 3 AUG 25, 2023: COHOMOLOGY OF GROUPS, CONTINUED

**Example 3.1.** Let  $G$  be  $C_2$ , or any group with a surjection  $p$  onto  $C_2$ , then it has an action on  $\mathbb{Z}_{\text{sgn}}$  given by  $g \cdot a = (-1)^{p(g)}a$ , therefore we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}_{\text{sgn}} \xrightarrow{\times 2} \mathbb{Z}_{\text{sgn}} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

and taking the fixed point functor we have

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

**Remark 3.2.** Higher homologies measure the failure of exactness.

**Remark 3.3.** The collection  $\{H^n(G, -)\}_{n \in \mathbb{Z}}$  satisfies

- $H^n(G, -) = 0$  for  $n < 0$ ;
- for short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  in  $G\text{-Mod}$ , we have a long exact sequence

$$0 \longrightarrow H^0(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, C) \xrightarrow{\delta} H^1(G, A) \longrightarrow \cdots$$

where  $\delta$  is the connecting homomorphism.

- the connecting homomorphisms  $\delta$  are natural, i.e., given a commuting diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \longrightarrow 0 \end{array}$$

the induced diagram

$$\begin{array}{ccc} H^n(G, C) & \xrightarrow{\delta} & H^{n+1}(G, A) \\ \downarrow & & \downarrow \\ H^n(G, C') & \xrightarrow{\delta} & H^{n+1}(G, A') \end{array}$$

also commutes, and  $\{H^n(G, -)\}_{n \in \mathbb{Z}}$  is a cohomological  $\delta$ -functor. Note that a  $\delta$ -functor is additive, and usually occurs in abelian categories.

**Definition 3.4** ( $\delta$ -functor). A map of  $\delta$ -functors  $T^* \rightarrow F^*$  is a collection of natural transformations  $T^n \rightarrow F^n$ , commuting with the  $\delta$ 's, i.e.,

$$\begin{array}{ccc} T^n & \longrightarrow & F^n \\ \delta_T \downarrow & & \downarrow \delta_F \\ T^{n+1} & \longrightarrow & F^{n+1} \end{array}$$

A  $\delta$ -functor  $T^*$  is universal if, given any other  $\delta$ -functor  $F^*$ , a map  $T^* \rightarrow F^*$  is uniquely determined by  $T^0 \rightarrow F^0$ .

**Proposition 3.5.**  $H^*(G, -) : G\text{-Mod} \rightarrow \mathbf{Ab}$  is a  $\delta$ -functor.

*Proof.* We need to show:

- each  $H^n(G, -)$  is a well-defined functor,
- the connecting homomorphisms  $\delta$ 's gives a long exact sequence,
- the naturality of  $\delta$ .

First, let  $f : A \rightarrow B$  be in  $G\text{-Mod}$ , then  $C^*(G, A) \rightarrow C^*(G, B)$  is equivalent to  $\text{Map}(G^{*+1}, A)^G \rightarrow \text{Map}(G^{*+1}, B)^G$  by composition with  $f$ . One can show that this is equivariant, i.e., respects the  $G$ -action, so it is well-defined to take the fixed points, and thus commutes with  $\partial$ 's.

Second, we need to apply the snake lemma. Given a short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ , we claim:

**Claim 3.6.**  $0 \longrightarrow C^*(G, A) \longrightarrow C^*(G, B) \longrightarrow C^*(G, C) \longrightarrow 0$  is a short exact sequence of cochain complexes, i.e.,  $C^*(G, -) : G\text{-Mod} \rightarrow \mathbf{coCh}$  is an exact functor.

*Subproof.* Exercise. ■

Now take the complex

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^n(G, A) & \longrightarrow & C^n(G, B) & \longrightarrow & C^n(G, C) \longrightarrow 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\ 0 & \longrightarrow & C^{n+1}(G, A) & \longrightarrow & C^{n+1}(G, B) & \longrightarrow & C^{n+1}(G, C) \longrightarrow 0 \end{array}$$

and quotient the boundaries everywhere (and thus lose the injectivity/surjectivity when applicable)

$$\begin{array}{ccccccc} C^n(G, A)/B^n(G, A) & \longrightarrow & C^n(G, B)/B^n(G, B) & \longrightarrow & C^n(G, C)/B^n(G, C) & \longrightarrow & 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\ 0 & \longrightarrow & Z^{n+1}(G, A) & \longrightarrow & Z^{n+1}(G, B) & \longrightarrow & Z^{n+1}(G, C) \end{array}$$

Taking the kernels and cokernels on  $\partial$ 's, we obtain a complex

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & H^n(G, A) & \longrightarrow & H^n(G, B) & \longrightarrow & H^n(G, C) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & C^n(G, A)/B^n(G, A) & \longrightarrow & C^n(G, B)/B^n(G, B) & \longrightarrow & C^n(G, C)/B^n(G, C) \longrightarrow 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\ 0 & \longrightarrow & Z^{n+1}(G, A) & \longrightarrow & Z^{n+1}(G, B) & \longrightarrow & Z^{n+1}(G, C) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & H^{n+1}(G, A) & \longrightarrow & H^{n+1}(G, B) & \longrightarrow & H^{n+1}(G, C) \end{array}$$

By the snake lemma, we obtain the long exact sequence. □

**Proposition 3.7.** If  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is a short exact sequence such that  $H^*(G, B) = 0$  for  $* > 0$  (or at least  $H^n(G, B) = 0 = H^{n+1}(G, B)$ ), then  $\delta : H^n(G, C) \rightarrow H^{n+1}(G, A)$  is an isomorphism.

**Definition 3.8** (Acyclic, Cohomologically Trivial). A  $G$ -module  $M$  is

- acyclic if  $H^*(G, M) = 0$  for  $* > 0$ ,
- cohomologically trivial if  $H^*(H, M) = 0$  for  $* > 0$  and any (closed) subgroup  $H \subseteq G$ .

**Definition 3.9** (Induced Module). Given any  $G$ -module  $M$ , the induced module  $\text{ind}_G(M) = \text{Map}(G, M) = X^0(G, M)$ .

**Example 3.10.**  $M$  could have the trivial action.

**Exercise 3.11.** For any  $M$ , the induced module of  $M$  over  $G$  is isomorphic (under the  $G$ -action) to the induced module of module given by forgetful action over  $G$ .

**Remark 3.12.** •  $\text{Ind}_G(-) : G\text{-Mod} \rightarrow G\text{-Mod}$  is exact.

- We say  $A$  is an induced module if  $A \cong \text{Ind}_G(M)$  for some module  $M$ . If  $A$  is an induced  $G$ -module, then  $A$  is induced as an  $H$ -module for any subgroup  $H \subseteq G$ .

**Lemma 3.13.** Induced modules are cohomologically trivial.

*Proof.* There is an isomorphism

$$C^*(G, \text{Ind}_G(M)) \cong X^*(G, M).$$

□

**Remark 3.14.** We have an equivariant inclusion of fixed points

$$M \hookrightarrow \text{Ind}_G(M)$$

which is an embedding, and we take  $Q \cong \text{Ind}_G(M)/M$ , then this extends to a short exact sequence

$$0 \longrightarrow M \hookrightarrow \text{Ind}_G(M) \longrightarrow Q \longrightarrow 0$$

then  $H^{n+1}(G, M) \cong H^n(G, Q)$ . One say that  $H^*(G, -)$  is effaceable. By Tohoku, an effaceable is universal.

#### 4 AUG 28, 2023: FIRST COHOMOLOGY OF GROUPS

There are three ways to think about  $H^1(G, M)$ .

##### 4.1 CROSSED HOMOMORPHISMS

Recall that  $H^1(G, M) = Z_i^1(G, M)/B_i^1(G, M)$  as inhomogeneous cochains, where

- $Z_i^1(G, M) = \ker(\text{Map}(G, M) \rightarrow \text{Map}(G \times G, M))$  where the map sends  $f \mapsto (g, h) \mapsto gf(h) - f(gh) + f(g)$ . The kernel of this is exactly the maps  $f$  such that  $f(gh) = gf(h) + f(g)$ , and note that this is not a group homomorphism.
- $B_i^1(G, M) = \text{im}(M \rightarrow \text{Map}(G, M))$  given by  $m \mapsto (g \mapsto gm - m)$ , where the image is called a principal crossed homomorphism.

**Exercise 4.1.**  $B_i^1(G, M) \cong M/M^G$  as an isomorphism of  $\mathbb{Z}[G]$ -modules.

**Remark 4.2.** If the  $G$ -action is trivial, then  $H^1(G, M) = \text{Hom}_{\text{Grp}}(G, M)$ .

**Corollary 4.3.** If  $G$  is a finite group with trivial action, then  $H^1(G, \mathbb{Z}) = 0$ .

**Theorem 4.4** (Hilbert's Theorem 90). Let  $L/K$  be a Galois extension with (finite or profinite) Galois group  $G$ , then  $H^1(G, L^\times) = 0$ .

*Proof.* Let  $f : G \rightarrow L^\times$  be a crossed homomorphism. We know the addition is given by  $f(gh) = gf(h) + f(g)$ , and the multiplication is given by  $f(gh) = (g \cdot f(h))f(g)$ , where  $\cdot$  represents the group action. Now for any  $l \in L^\times$ , the multiplication with respect to  $l$  is given by  $m_l = \sum_{h \in G} f(h)(h \cdot l)$ . We can first choose  $l$  so that  $m_l \neq 0$ , since the Galois conjugates  $h \cdot l$  over  $l \in L$  are linearly independent. For  $g \in G$ , we have

$$\begin{aligned} g \cdot m_l &= \sum_{h \in G} (g \cdot f(h))(gh \cdot l) \\ &= \sum_{h \in G} \frac{f(gh)}{f(g)} (gh \cdot l) \\ &= \frac{1}{f(g)} \sum_{h \in G} f(gh)(gh \cdot l) \\ &= \frac{1}{f(g)} m_l. \end{aligned}$$

Therefore,  $f(g) = \frac{m_l}{g \cdot m_l}$ . For any crossed homomorphism, there exists  $m \in L^\times$  such that  $f(g) = \frac{gm}{m}$ , so every crossed homomorphism is principal. □

**Exercise 4.5.** Let  $G$  acts over a commutative ring  $R$ , then  $H^1(G, R^\times)$  classifies invariant  $R$ -modules with a compatible  $G$ -action.

4.2 NON-ABELIAN  $H^1$  AND TORSORS

Let  $A$  be a group with  $G$ -action, so let the action  $g \cdot a = {}^g a$ . Hence,  $g \cdot (ab) = {}^g a {}^g b$ . Define the  $G$ -cocycles to be  $f : G \rightarrow A$  such that  $f(gh) = f(g) {}^g f(h)$ . Two cocycles  $f$  and  $f'$  are said to be cohomologous as  $f \sim f'$  if there exists  $a \in A$  such that for all  $g \in G$ ,  $f'(g) = a^{-1} f(g) {}^g a$ . This becomes an equivalence relation on the set of  $G$ -cocycles with coefficients in  $A$ , then  $H^1(G, A)$  is the set of equivalence classes of  $G$ -cocycles. Now the first cohomology  $H^1(G, A)$  has only a pointed set structure with distinguished point  $f \equiv 1$ , the constant function at 1.

**Exercise 4.6.** This definition is equivalent to the inhomogeneous cochain definition in the abelian case.

**Definition 4.7.** An  $A$ -torsor is a  $G$ -set  $X$  with action

$$\begin{aligned} X \times A &\rightarrow A \\ (x, a) &\mapsto xa \end{aligned}$$

that is free and transitive, i.e., for any  $x, y \in X$ , there exists a unique  $a \in A$  such that  $y = xa$ . Moreover, the action  $X \times A \rightarrow X$  respects the  $G$ -action, i.e.,  ${}^g(xa) = {}^g x {}^g a$ .

**Remark 4.8.** •  $A$  is an  $A$ -torsor.

- An isomorphism of  $A$ -torsors is a bijection that respects the  $G$ - and  $A$ - action.
- If  $A \subseteq B$  is a sub- $G$ -group, then  $bA$  is an  $A$ -torsor.
- An  $A$ -torsor is a principal  $A$ -bundle on the classifying space  $BG$ .

**Theorem 4.9.** There is a canonical bijection of pointed sets

$$H^1(G, A) \cong \text{Torsor}(G, A)$$

*Proof.* • The backwards map  $\lambda : \text{Torsor}(G, A) \rightarrow H^1(G, A)$  is defined as follows: for  $x \in \text{Torsor}(G, A)$ , we want to define a cocycle  $f(X) : G \rightarrow A$ . For arbitrary  $x \in X$ , note that for any  $g \in G$ , there exists a unique  $f_x(g) \in A$  such that  ${}^g x = x f_x(g)$  by the simple transitivity of the  $A$ -action on  $X$ . To see this is well-defined, if we have another  $y \in X$ , then  $y = xb$  for some  $b \in A$ , then  $f_y(g) = b^{-1} f_x(g) {}^g b$ , so  $f_x$  and  $f_y$  are cohomologous and define the same class in  $H^1(G, A)$ , which is defined to be the image  $\lambda(X)$ .

- To define  $\mu : H^1(G, A) \rightarrow \text{Torsor}(G, A)$ , given a cocycle  $f : G \rightarrow A$ , let  $X_f$  be the group  $A$ , then the action of  $A$  on  $X_f$  is by multiplication on the right, and one can twist the  $G$ -action on it using cocycle  $f : G \rightarrow A$  with  ${}^g x = f(g)gx$ , which defines an  $A$ -torsor. This is well-defined.

□

**Remark 4.10.** Suppose

$$1 \longrightarrow A \longrightarrow B \xrightarrow{p} C \longrightarrow 1$$

is a short exact sequence of  $G$ -groups, i.e.,  $A$  is a sub- $G$ -group and  $C \cong B/A$ , then there is a long exact sequence

$$1 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \xrightarrow{\delta} H^1(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, C)$$

where  $\delta$  is given by  $\delta(c) = p^{-1}(c)$ . For the exactness in the sense of pointed sets to work, the kernel is the subset mapping to the distinguished element.

## 4.3 EXTENSION SPLITTING

Consider the a split extension

$$1 \longrightarrow A \longrightarrow E \xrightarrow{p} G \longrightarrow 1$$

That is,  $E$  is the direct product  $A \times G$  with group action  $(a, g)(a', g') = (a {}^g a', gg')$ , and by definition  $E$  is the semidirect product  $A \rtimes G$ . Equivalently, there exists a section (as group homomorphism)  $s : G \rightarrow E$ .

There is an equivalence relation on the set of sections to the projection  $p : E \rightarrow G$ , where the sections  $s, s' : G \rightarrow E$  are conjugates if there exists  $a \in A$  such that  $s'(g) = a^{-1} s(g) a$ . We denote  $\text{sec}(E \rightarrow G)$  to be the conjugacy class of sections of  $p$ . Note that the class of trivial section  $s : g \mapsto (1, g) \in E$  is the distinguished element.

**Proposition 4.11.** The pointed set  $H^1(G, A)$  is isomorphic to  $\text{sec}(E \rightarrow G)$ .

*Proof.* Take  $\varphi \in \text{sec}(E \rightarrow G)$ , then the composition  $G \xrightarrow{\varphi} E \xrightarrow{\pi_1} A$ , where  $\pi_1$  is the set-theoretic projection to the first component, defines a cocycle  $G \rightarrow A$ . Conversely, given a cocycle  $f : G \rightarrow A$ , the section is given by  $g \mapsto (f(g), g)$ .  $\square$

**Exercise 4.12.** Expand the proof above.

**Exercise 4.13.** Describe  $\mathbb{Z} \rtimes C_2$  where  $C_2$  acts on  $\mathbb{Z}$  by inversion. How many sections are there of  $\mathbb{Z} \rtimes C_2 \rightarrow C_2$ ?

**Exercise 4.14.** How many sections are there to the projection  $D_{2n} \rightarrow C_2$ ?

## 5 AUG 30, 2023: $H^2$ , ABELIAN EXTENSIONS, AND BRAUER GROUP

Suppose we have an abelian extension, that is, let  $A$  be abelian, the short exact sequence of group extensions

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1$$

is such that  $E/i(A) \cong G$ . Note that  $A$  can be regarded as a normal subgroup in  $E$  given this notation.

Note that two extensions are equivalent if there exists a group isomorphism  $\varphi : E \rightarrow E'$  such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G \longrightarrow 1 \\ & & \parallel & & \downarrow \varphi & & \parallel \\ 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G \longrightarrow 1 \end{array}$$

commutes.

Consider the continuous functions

$$\varphi : G \times G \rightarrow A$$

such that  $\varphi(g_1g_2, g_3) + \varphi(g_1, g_2) = \varphi(g_1, g_2g_3) + g_1\varphi(g_2, g_3)$ . We know  $H^2(G, M)$  is the quotient of all such functions over the coboundaries, i.e., the functions  $\varphi$  such that  $\varphi(g_1, g_2) = f(g_1) - f(g_1g_2) + g_1f(g_2)$ .

Now  $E \cong A \times G$  can be considered as a bijection, so we pick a set-theoretic section  $s : G \rightarrow E$  with  $s(1) = 1$ , and now every element in  $E$  is written as  $as(g)$  uniquely for some  $a \in A$  and  $g \in G$ , we have

$$s(g)a = s(g)as(g)^{-1}s(g) = {}^g as(g).$$

Note that  $s$  may not be a homomorphism, but we have  $s(g)s(h) = f(g, h)s(gh)$  since  $s(g)s(h)$  and  $s(gh)$  are both lifts of  $gh$ .

As a consequence, we have

$$(s(g_1)s(g_2))s(g_3) = f(g_1, g_2)s(g_1g_2)s(g_3) = f(g_1, g_2)f(g_1g_2, g_3)s(g_1g_2g_3)$$

and

$$s(g_1)(s(g_2)s(g_3)) = s(g_1)f(g_2, g_3)s(g_2, g_3) = {}^{g_1}f(g_2, g_3)s(g_1)s(g_2g_3) = {}^{g_1}f(g_2, g_3)f(g_1, g_2g_3)s(g_1g_2g_3).$$

In additive notation, we have

$$f(g_1, g_2) + f(g_1g_2, g_3) = g_1f(g_2, g_3) + f(g_1, g_2g_3).$$

Therefore,  $f$  becomes an inhomogeneous 2-cocycle.

**Proposition 5.1.** The induced map  $\lambda : \text{ext}(G, A) \rightarrow H^2(G, A)$  is a well-defined bijection between the set of equivalence classes of extensions and  $H^2(G, A)$ .



**Example 5.2.** The two elements in  $H^2(C_2, \mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$  are given by non-split extension of  $Q_8$

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow Q_8 \longrightarrow C_2 \longrightarrow 1$$

and the identity element given by  $D_8 \cong \mathbb{Z}/4\mathbb{Z} \rtimes C_2$

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow D_8 \longrightarrow C_2 \longrightarrow 1$$

where  $D_8$  has the action of  $C_2$  over  $\mathbb{Z}/4\mathbb{Z}$ .

**Proposition 5.3.** An associative finite-dimensional  $K$ -algebra  $A$  is a CSA if and only if one of the following equivalent conditions hold:

1. Base-changed to the separable closure  $\bar{K}$  of  $K$  via  $\bar{K} \otimes_K A$ ,  $A \cong M_n(\bar{K})$  for some integer  $n \geq 1$ .
2. there exists a finite Galois extension  $L/K$  such that base-changed to  $L$  via  $L \otimes_K A$ ,  $A$  becomes isomorphic to a matrix algebra  $M_n(L)$  for some integer  $n \geq 1$ .
3.  $A \cong M_n(D)$  matrix algebra for some  $m \geq 1$  and some finite division algebra  $D$  over  $K$ .

A CSA  $A$  over  $K$  is said to be split over  $L$  if the above holds, i.e.,  $A \otimes_K L \cong M_n(L)$ . One can define an equivalence class on CSAs, such that  $A \sim B$  if and only if  $A \otimes_K M_n(K) \cong B \otimes_K M_m(K)$ . Now the Brauer group of  $K$  is the abelian group of equivalence classes of CSAs over  $K$  equipped with tensor product.

Suppose  $L/K$  is an extension, then there exists a homomorphism of base-change of algebras  $\text{Br}(K) \rightarrow \text{Br}(L)$ . We say the kernel  $\text{Br}(L | K)$  is the relative Brauer group of  $K$ -CSAs that split over  $L$ . The absolute Brauer group is  $\text{Br}(\bar{K} | K) = \text{Br}(K)$ , then

$$\text{Br}(K) = \bigcup_{L/K \text{ finite}} \text{Br}(L | K).$$

Now let  $L/K$  be a finite Galois extension with Galois group  $G$ , and we pick a normalized inhomogeneous 2-cycle  $\varphi : G \times G \rightarrow L^\times$  as the representative of its class, and we can construct  $A_\varphi$  as a  $K$ -CSA, then  $A_\varphi = \bigoplus_{g \in G} L e_g$  has dimension  $|G|^2$ , where  $e_g$ 's are the generators, with a multiplication operation  $(l e_g)(m e_h) = l(g \cdot m) \varphi(g, h) e_{gh}$  which can be extended via distribution.  $A_\varphi$  is said to be the crossed product of  $L$  and  $G$  via  $\varphi$ .

**Theorem 5.4.** 1.  $A_\varphi$  is a split algebra over  $L$ .

2. If  $\varphi, \varphi'$  are two normalized inhomogeneous 2-cocycles, then  $A_\varphi \sim A_{\varphi'}$  if and only if  $\varphi \sim \varphi'$ .
3.  $A_{\varphi\varphi'} \sim A_\varphi \otimes_K A_{\varphi'}$ .
4. Any  $K$ -CSA which is split over  $L$  is similar to a crossed product  $A_\varphi$  for some  $\varphi : G \times G \rightarrow L^\times$ .

**Corollary 5.5.**  $H^2(G, L^\times)$  is isomorphic to  $\text{Br}(L | K)$ , and  $H^2(\text{Gal}(\bar{K}/K), \bar{K}^\times)$  is isomorphic to  $\text{Br}(K)$ .

## 6 SEPT 1, 2023: COHOMOLOGY OF CYCLIC AND FREE GROUPS

Recall that we can compute  $H^*(G, M)$  using any acyclic resolution of  $M$ . We want to describe  $H^*(G, M)$  for specific  $G$  using nice resolutions.

We have

$$\dots \rightarrow G^3 \xrightarrow{\delta} G^2 \xrightarrow{\delta} G$$

and to obtain  $H^*(G, M)$  we map out of the resolution and into  $M$ , so  $\text{Map}(G, M) \cong \text{Hom}(\mathbb{Z}[G], M)$  as  $G$ -modules, and in general we obtain

$$\text{Map}(G^k, M) \cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]^{\otimes k}, M)$$

as  $\mathbb{Z}$ -modules.

We denote  $F^{\text{st}}$  to be the standard free resolution given by

$$\mathbb{Z}[G]^{\otimes k} \xrightarrow{d} \mathbb{Z}[G]^{\otimes(k-1)} \rightarrow \dots \rightarrow \mathbb{Z}[G]^{\otimes 2} \xrightarrow{d_1-d_0} \mathbb{Z}[G]$$

To obtain  $X^*(G, M)$ , we can map this into  $M$ . Now the standard resolution becomes an augmentation of  $\mathbb{Z}$  that makes  $X^*(G, M)$  exact, free, and acyclic. The kernel of  $\mathbb{Z}[G] \rightarrow \mathbb{Z}$  is the augmentation ideal of  $G$  as of  $\mathbb{Z}[G]$ . Since this is a  $G$ -equivariant map, then the augmentation ideal is a  $G$ -submodule of  $\mathbb{Z}[G]$ , as a free abelian group generated by the set  $\{(g-1) \mid 1 \neq g \in G\}$ .

**Lemma 6.1.** If  $P_* \rightarrow \mathbb{Z}$  is any free resolution of  $\mathbb{Z}$  as a  $G$ -module, then for a  $G$ -module  $M$ , we have  $H^*(G, M) \cong H^*(\text{Hom}(P_*, M))^G$ .

*Proof.* Since each  $P_i$  is free, then  $\text{Hom}(P_i, M)$  is an acyclic module, so  $M \rightarrow \text{Hom}(P_*, M)$  is an acyclic resolution of  $M$ . Now apply Proposition 2.28 in the notes.  $\square$

**Remark 6.2.**  $H^*(G, M) \cong \text{Ext}_{\mathbb{Z}[G]}^*(\mathbb{Z}, M)$  as universal  $\delta$ -functors.

Now let  $C_n$  be the cyclic group of order  $n$ , generated by element  $g$ , then  $\mathbb{Z}[C_n] \cong \mathbb{Z}[g]/(g^n - 1)$ , so we have  $0 = g^n - 1 = (g-1)N_g$  in  $\mathbb{Z}[C_n]$  where  $N_g$  is the norm element  $N_g = 1 + g + \cdots + g^{n-1}$ , so we have a free resolution of  $\mathbb{Z}$ :

$$\cdots \longrightarrow \mathbb{Z}[C_n] \xrightarrow{1-g} \mathbb{Z}[C_n] \xrightarrow{N_g} \mathbb{Z}[C_n] \xrightarrow{1-g} \mathbb{Z}[C_n] \xrightarrow{\varepsilon} \mathbb{Z}$$

where augmentation  $\varepsilon$  sends  $g$  to 1. This allows us to compute the cohomology of any  $C_n$ -modules.

**Proposition 6.3.** Let  $M$  be an  $C_n$ -module, then

$$H^i(G, M) = \begin{cases} M^G, & i = 0 \\ \{m \in M \mid N_g m = 0\}/(1-g)M, & i > 0 \text{ odd} \\ M^G/N_g M, & i > 0 \text{ even} \end{cases}$$

*Proof.* Taking  $\text{Hom}(P_*, M)^G$  gives

$$\cdots \longleftarrow M \xleftarrow{1-g} M \xleftarrow{N_g} M \xleftarrow{1-g} M \longleftarrow \cdots$$

$\square$

**Remark 6.4.** If  $M$  has trivial action, then

$$H^i(G, M) = \begin{cases} M, & i = 0 \\ M[n], & i > 0 \text{ odd} \\ M/n, & i > 0 \text{ even} \end{cases}$$

where  $M[n]$  is the  $n$ -torsion in  $M$ .

Now if  $T = \mathbb{Z}$  be with generator  $t$ , then  $\mathbb{Z}[T]$  is isomorphic to the Laurent polynomials, so we have a resolution

$$0 \longrightarrow \mathbb{Z}[T] \xrightarrow{1-t} \mathbb{Z}[T] \longrightarrow \mathbb{Z}$$

since  $(1-t)$  is not a zero-divisor of  $\mathbb{Z}[T]$ . Therefore, taking  $\text{Hom}(P_*, M)^T$  gives

$$0 \longleftarrow M \xleftarrow{1-t} M$$

$$H^i(T, M) = \begin{cases} M^T, & i = 0 \\ M_T, & i = 1 \\ 0, & \text{otherwise} \end{cases}$$

Now let  $X$  be a set, and let  $G_X$  be the free group on  $X$ .

**Proposition 6.5.** The augmentation ideal  $I_X$  is a free  $\mathbb{Z}[G_X]$ -module, generated by the set  $\{(x-1) \mid x \in X\}$ , and so the exact sequence

$$0 \longrightarrow I_X \longrightarrow \mathbb{Z}[G_X] \longrightarrow \mathbb{Z} \longrightarrow 0$$

is a free resolution of  $\mathbb{Z}$  as a  $G_X$ -module.

*Proof.* As  $\mathbb{Z}$ -bases of  $I_X$ , we have  $\{(g-1) \mid g \in G_X\}$ , but  $\{h(x-1) \mid h \in G, x \in X\}$  is also a  $\mathbb{Z}$ -linear basis for  $I_X$ .  $\square$

**Remark 6.6.** Groups are free if and only if they have cohomological dimension 1.

## 7 SEPT 6, 2023: CUP PRODUCT

**Remark 7.1.** 1. A crossed homomorphism would be a group homomorphism when  $G$  has trivial action on  $M$ .

2. If  $X$  is an  $A$ -torsor, then there is a given  $G$ -action and a right  $A$ -action so that  $X \times A \rightarrow X$  is given by a diagonal action compatible to the  $G$ -action. Therefore,  ${}^g(x \cdot a) = {}^gx \cdot {}^ga$ .

**Definition 7.2.** Let  $A$  and  $B$  be  $G$ -modules, then there is a notion of tensor product  $A \otimes_G B$  as a  $G$ -module via the diagonal action  $g(a \otimes b) = ga \otimes gb$ . On the level of cochain, we have a cup product

$$\begin{aligned} C^p(G, A) \otimes C^q(G, B) &\xrightarrow{\sim} C^{p+q}(G, A \otimes B) \\ (\alpha : G^{p+1} \rightarrow A) \otimes (\beta : G^{q+1} \rightarrow B) &\mapsto (\alpha \smile \beta) \\ (g_0, \dots, g_{p+q}) &\mapsto \alpha(g_0, \dots, g_p) \otimes \beta(g_p, \dots, g_{p+q}) \end{aligned}$$

**Proposition 7.3.**  $\partial(\alpha \smile \beta) = (\partial\alpha) \cup \beta + (-1)^{|\alpha|} \alpha \smile \partial\beta$ .

**Corollary 7.4.** • If  $\alpha$  and  $\beta$  are cocycles, then  $\alpha \smile \beta$  is also a cocycle.

• If  $\alpha$  is a cocycle  $\beta$  is a coboundary, or vice versa, then  $\alpha \smile \beta$  is a coboundary. Indeed, if  $\beta = \partial\gamma$ , then  $\partial(\alpha \smile \gamma) = (-1)^{|\alpha|} \alpha \smile \beta$ .

Therefore, on the level of cohomology, we have a (bilinear) cup product as well:

$$H^p(G, A) \otimes H^q(G, B) \rightarrow H^{p+q}(G, A \otimes B)$$

**Example 7.5.** • If  $p = q = 0$ , then

$$\begin{aligned} H^0(G, A) \otimes H^0(G, B) &\cong A^G \otimes B^G \rightarrow H^0(G, A \otimes B) \cong (A \otimes B)^G \\ a \otimes b &\mapsto a \otimes b \end{aligned}$$

• By extending this property, we get a  $G$ -equivariant pairing  $A \otimes B \rightarrow C$  and therefore

$$H^p(G, A) \otimes H^q(G, B) \xrightarrow{\sim} H^{p+q}(G, C).$$

**Example 7.6.** Let  $R$  be a commutative ring, and if there is a  $G$ -action on  $R$ , then the multiplication  $m : R \otimes R \rightarrow R$  is  $G$ -equivariant, so we have a cup product

$$\smile : H^p(G, R) \otimes H^q(G, R) \rightarrow H^{p+q}(R)$$

This has the following properties:

1. This is natural in  $A, B$ , and  $C$ .
2. This is compatible with connecting homomorphism and exact sequences, that is,
  - Given short exact sequences

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

and

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

and pairing  $A \otimes B \rightarrow C$ , then this induces  $A \otimes B \rightarrow C'$  and in the quotients we have  $A'' \otimes B \rightarrow C''$ , so  $\delta(\alpha \smile \beta) = \delta\alpha \smile \beta$ , so we have a commutative diagram<sup>1</sup>

$$\begin{array}{ccccccc} A' \otimes B & \longrightarrow & A \otimes B & \longrightarrow & A'' \otimes B & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' \longrightarrow 0 \end{array}$$

<sup>1</sup>This may require the assumption that the modules are flat.

and thus

$$\begin{array}{ccc} H^o(G, A'') \otimes H^q(G, B) & \longrightarrow & H^{p+q}(G, A'' \otimes B) \\ \downarrow \delta \otimes 1 & & \downarrow \delta \\ H^{p+1}(G, A') \otimes H^q(G, B) & \longrightarrow & H^{p+q+1}(G, A' \otimes B) \end{array}$$

• Given

$$0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$$

and

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

and pairings

$$\begin{array}{ccccccc} A \otimes B' & \longrightarrow & A \otimes B & \longrightarrow & A \otimes B'' & & \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' \longrightarrow 0 \end{array}$$

$$\text{so } \delta(\alpha \smile \beta) = (-1)^{|\alpha|} \alpha \smile \delta\beta.$$

*Proof.* Let  $\alpha = [a]$  for  $a : G^{p+1} \rightarrow A$  and  $\beta = [b]$  for  $b : G^{q+1} \rightarrow B''$ , then there is a lift  $\tilde{b} : G^{q+1} \rightarrow B \rightarrow B''$ . Then we have

$$\begin{array}{ccccccc} C^q/B^q(B') & \longrightarrow & C^q/B^q(B) & \longrightarrow & C^q/B^q(B'') & \longrightarrow & 0 \\ \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\ 0 & \longrightarrow & Z^q(B') & \longrightarrow & Z^{q+1}(B) & \longrightarrow & Z^{q+1}(B'') \end{array}$$

and by the snake lemma we have a connecting homomorphism over group cohomologies.  $\square$

## 8 SEPT 8, 2023: RESTRICTION AND TRANSFER

Recall that we have a chain-level cup product, and we extend it to the level of cohomology. The cup product has the following properties:

1. If  $p = q = 0$ , then the cup product is the natural composition

$$A^G \otimes B^G \rightarrow (A \otimes B)^G \rightarrow C^G$$

2. Functoriality.

3. We have  $\delta(\alpha \smile \beta) = \delta(\alpha) \smile \beta$ , and incorporating this with the exact sequence, we have  $\delta(\alpha \smile \beta) = (-1)^{|\alpha|} \alpha \smile \delta(\beta)$ .

By the universal property of the tensor product, there exists a unique bilinear pairing that also satisfies these properties. To prove this, we use dimension-shifting.

**Remark 8.1.** Let  $M$  be a module, and map it into the induced module with an extended short exact sequence

$$0 \longrightarrow M \longrightarrow \text{Ind}^G(M) = \text{Map}(G, M) \cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], M) \longrightarrow M_1 \longrightarrow 0$$

Taking the fixed points, we have

$$0 \longrightarrow M^G \longrightarrow (\text{Ind}^G(M))^G \longrightarrow (M_1)^G \longrightarrow H^1(G, M) \longrightarrow 0 \longrightarrow \dots$$

$$\dots \longrightarrow 0 \longrightarrow H^k(G, M_1) \xrightarrow{\cong} H^{k+1}(G, M)$$

Here  $(M_1)^G \rightarrow H^1(G, M)$  is a surjection. Now we know  $\delta : H^i(G, M_1) \rightarrow H^{i+1}(G, M)$  is a surjection for  $i = 0$ , and is an isomorphism for  $i > 0$ .

Proceeding inductively, we define

$$0 \longrightarrow M_i \longrightarrow \text{Ind}^G(M) \longrightarrow M_{i+1} \longrightarrow 0$$

If we start with  $A \otimes B \rightarrow C$ , then use property (3) repeatedly to the short exact sequence above, we get the uniqueness.

**Example 8.2.** Consider  $G = C_2$ , and consider the cohomology ring  $H^*(C_2, \mathbb{F}_2)$ . The action is obviously trivial. This induced the sequence with augmentation

$$0 \longrightarrow \mathbb{F}_2 \longrightarrow \mathbb{F}_2[C_2] \longrightarrow \mathbb{F}_2 \longrightarrow 0$$

The boundary map is  $\delta : H^i(C_2, \mathbb{F}_2) \rightarrow H^{i+1}(C_2, \mathbb{F}_2)$  is an isomorphism for all  $i$ .

We know  $H^i(C_2, \mathbb{F}_2) = \mathbb{F}_2\{x_i\}$ , so we can write  $x_{i+1} = \delta x_i$ . The product  $x_i \smile x_j = \delta^i x_0 \smile \delta^j x_0 = \delta^{i+j} x_0 \smile x_0 = \delta^{i+j} x_0 = x_{i+j}$ . Hence,  $H^*(C_2, \mathbb{F}_2) \cong \mathbb{F}_2[x]$  where  $x = |x_1|$ .

Note that

$$H^i(C_2, M) = \begin{cases} M^{C_2}, & i = 0 \\ \ker(N)/(\sim), & i \text{ odd} \\ M^{C_2}/N, & i > 0 \text{ even} \end{cases}$$

**Remark 8.3.** For odd prime  $p$ , we want to use the same method to calculate  $H^i(C_p, \mathbb{F}_p)$  with trivial action, then this is  $\{\mathbb{F}_p, i \geq 0\}$ . For instance, if we look at  $x_1 \smile x_1$ , then this is  $(-1)^{|x_1|} x_1 \smile x_1$ , so this gives  $2x_1 \smile x_1 = 0 \in H^2 = \mathbb{F}_p$ , so this gives  $x_1 \smile x_1 = 0$ . Note that  $H^*(C_p, \mathbb{F}_p) \cong \bigwedge(x_1) \otimes \mathbb{F}_p[y]$ .

We now talk about the functoriality in  $G$ . Given  $G_1$  acting on  $M_1$  and  $G_2$  acting on  $M_2$ , and say  $\varphi : G_1 \rightarrow G_2$  is a group homomorphism, and a map of modules  $f : M_2 \rightarrow M_1$ , then we say  $\varphi$  and  $f$  is a compatible pair of morphisms if for any  $g \in G_1$ , the diagram

$$\begin{array}{ccc} M_2 & \xrightarrow{f} & M_1 \\ \varphi(g) \downarrow & & \downarrow g \\ M_2 & \xrightarrow{f} & M_1 \end{array}$$

This gives a map  $C^*(G_2, M_2) \rightarrow C^*(G_1, M_1)$ , and hence a map on cohomology  $H^*(G_2, M_2) \rightarrow H^*(G_1, M_1)$ . For instance, if  $\varphi = \text{id}$ , we obtain the functoriality in  $M$ , as we previously saw. Similarly, if  $f = \text{id}$ , and  $M = M_2$  is a  $G_2$ -module, on which  $g_1 \cdot m = \varphi(g_1) \cdot m$ .

There are some special situations from the relations above.

1. Conjugation: let  $H \subseteq G$  be a subgroup, and we consider  $A$  to be a  $G$ -module, then there is restriction of  $G$ -action on  $A$  to  $H$ , so  $A$  becomes a  $H$ -module. Let  $B \subseteq A$  be a  $H$ -submodule in this sense. This is preserved by the action of  $A$ , but not necessarily by the action of  $G$ . For any  $g \in G$ , let the right conjugation be  $h^g = g^{-1}hg$  on  $h$ , and let  ${}^gH = gHg^{-1}$  on subgroup  $H$ . The compatible morphisms are now

$$\begin{aligned} {}^gH &\rightarrow H \\ h &\mapsto h^g \end{aligned}$$

and

$$\begin{aligned} B &\rightarrow gB \\ b &\mapsto gb \end{aligned}$$

Therefore, the induced maps on conjugation is given by  $(g)_* = H^*(H, B) \rightarrow H^*({}^gH, gB)$ . Therefore,  $(g_1g_2)_* = (g_1)_*(g_2)_*$ .

2. Inflation: suppose  $H \triangleleft G$  is a normal subgroup. We have the canonical map  $G \rightarrow G/H$ . Let  $A$  be a  $G$ -module, then  $G/H$  acts on  $A^H$ , and we look at the inclusion  $A^H \hookrightarrow A$ . Now  $\varphi : G \rightarrow G/H$  and  $f : A^H \hookrightarrow A$  are compatible, so on the level of cohomology, we get an inflation map

$$\inf_G^{G/H} : H^*(G/H, A^H) \rightarrow H^*(G, A).$$

If we look at  $H_1 \subseteq H_2 \triangleleft G$  where  $H_i \triangleleft G$ , we have  $G \rightarrow G/H_1 \rightarrow G/H_2 \cong (G/H_1)/(H_2/H_1)$ , then the inflation is

$$\inf_G^{G/H_1} \circ \inf_{G/H_1}^{G/H_2} = \inf_G^{G/H_2}.$$

3. Restriction: Let  $\varphi : H \hookrightarrow G$  and consider  $A$  as  $G$ -module and  $H$ -module respectively. There is now a restriction map

$$\text{res}_H^G : H^*(G, A) \rightarrow H^*(H, A)$$

Now if  $H_1 \subseteq H_2 \subseteq G$ , then

$$\text{res}_{H_1}^G = \text{res}_{H_1}^{H_2} \circ \text{res}_{H_2}^G$$

Inflation and restriction fit in a long exact sequence.

Finally, we discuss corestriction/transfer/norm. Let  $G$  be a finite group and let  $M$  be a  $G$ -module, then we have  $M^G \hookrightarrow M$  as inclusion. On the other way around, we have

$$\begin{aligned} \text{tr}/N : M &\rightarrow M^G \\ m &\mapsto \sum_{g \in G} gm. \end{aligned}$$

9 SEPT 11, 2023:

Let  $\varphi : G_1 \rightarrow G_2$  and  $f : M_2 \rightarrow M_1$  be compatible, then we denote  $(\varphi, f)^* = H^*(G_2, M_2) \rightarrow H^*(G_1, M_1)$ , with

$$G_1^{\times(*+1)} \longrightarrow G_2^{\times(*+1)} \longrightarrow M_2 \xrightarrow{f} M_1$$

such that it follows composition, and  $(\varphi, f)^*$  commutes with  $\delta$ , i.e.,

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_2' & \longrightarrow & M_2 & \longrightarrow & M_2'' \longrightarrow 0 \\ & & \downarrow f & & \downarrow f & & \downarrow f \\ 0 & \longrightarrow & M_1' & \longrightarrow & M_1 & \longrightarrow & M_1'' \longrightarrow 0 \end{array}$$

and therefore we have a commutative square

$$\begin{array}{ccc} H^k(G, M_2'') & \xrightarrow{\delta} & H^{k+1}(G_2, M_2') \\ (\varphi, f)^* \downarrow & & \downarrow (\varphi, f)^* \\ H^k(G_1, M_1'') & \xrightarrow{\delta} & H^{k+1}(G, M_1') \end{array}$$

For  $\alpha \in C^k(M_2'')/B^k$ , we trace it back to  $\tilde{\alpha} \in C^k(M_2)/B_k$ , and  $\alpha$  is sent to  $Z^{k+1}(M_2'')$ , but now that means  $\tilde{\alpha}$  lands in the kernel of  $Z^{k+1}(M_2) \rightarrow Z^{k+1}(M_2')$ , so this is in  $Z^{k+1}(M_2')$ .

$$\begin{array}{ccccccc} C^k(M_2)/B_k & \longrightarrow & C^k(M_2'')/B_k & \longrightarrow & 0 \\ \partial \downarrow & & \downarrow \partial & & \\ 0 & \longrightarrow & Z^{k+1}(M_2') & \longrightarrow & Z^{k+1}(M_2) & \longrightarrow & Z^{k+1}(M_2'') \end{array}$$

Moreover, we have  $(\varphi, f)^*(\alpha \smile \beta) = (\varphi, f)^*\alpha \smile (\varphi, f)^*\beta$ , whenever the modules are compatible.

For transfer/corestriction, if  $H \subseteq G$  is a subgroup with finite index, and  $M$  is a  $G$ -module, then we have

$$\begin{aligned} \mathrm{tr}_G^H : M^H &\rightarrow M^G \\ m &\mapsto \sum_{g \in G/H} gm \end{aligned}$$

For instance, we have  $\mathrm{tr} : \mathbb{Z}^H = \mathbb{Z} \rightarrow \mathbb{Z}^G = \mathbb{Z}$  is multiplication by  $[G : H]$ . Note that  $H^*(X^*(G, M)^G) = H^*(G, M)$ , but  $H^*(X^*(G, M)^H) = H^*(H, M)$ , and the latter maps to the former cohomology structure via the transfer mapping. Hence, we have  $\mathrm{tr}_G^H : X^*(G, M)^H \rightarrow X^*(G, M)^G$  giving  $\mathrm{tr}_G^H \equiv \mathrm{cores}_G^H : H^*(H, M) \rightarrow H^*(G, M)$ . This is not a ring homomorphism.

**Remark 9.1** (Properties). 1.  $\mathrm{tr}$  commutes with  $\delta$ , that is, for a short exact sequence of  $G$ -modules (hence a short exact sequence of  $H$ -modules),

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

then we have

$$\begin{array}{ccc} H^k(H, C) & \xrightarrow{\delta} & H^{k+1}(H, A) \\ \mathrm{tr} \downarrow & & \downarrow \mathrm{tr} \\ H^k(G, C) & \xrightarrow{\delta} & H^{k+1}(G, A) \end{array}$$

2. If  $H_1 \subseteq H_2 \subseteq G$  are subgroups with finite indices, then  $\mathrm{tr}_G^{H_1} = \mathrm{tr}_G^{H_2} \mathrm{tr}_{H_2}^{H_1}$ .

3.  $\mathrm{tr}(\mathrm{res}(\alpha) \smile \beta) = \alpha \smile \mathrm{tr}(\beta)$ . Now given a pairing  $A \otimes B \rightarrow C$  of  $G$ -modules, with  $H \subseteq G$ , then

$$\begin{array}{ccccc} H^i(H, A) & \otimes & H^j(H, B) & \xrightarrow{\smile} & H^{i+j}(H, C) \\ \mathrm{res} \uparrow & & \downarrow \mathrm{tr} & & \downarrow \mathrm{tr} \\ H^i(G, A) & \otimes & H^j(G, B) & \xrightarrow{\smile} & H^{i+j}(G, C) \end{array}$$

*Proof Idea.* By dimension shifting, we reduce the case  $H^0$ , in which we have an explicit description. We have  $A^H \otimes B^H \rightarrow C^H$ , so for  $\alpha \in A^G$  and  $\beta \in B^H$ , we have  $\mathrm{tr}(\alpha \otimes \beta) = \sum_{g \in G/H} g(\alpha \otimes \beta) = \sum g\alpha \otimes g\beta = \alpha \otimes \sum_{g \in G/H} g\beta$ .  $\square$

**Example 9.2.** Let  $R$  be a commutative ring with a  $G$ -action, then the restriction  $\mathrm{res} : H^*(G, R) \rightarrow H^*(H, R)$  is a ring homomorphism, so  $H^*(H, R)$  is a  $H^*(G, R)$ -algebra. The opposite side has  $\mathrm{tr}$  is a map of  $H^*(G, R)$ -modules where the cohomology of  $H$  is given the module structure from the restriction. This induces the Frobenius reciprocity.

**Remark 9.3** (Other compatibilities). Let  $K \subseteq H \subseteq G$  be (normal) subgroups, then  $G \rightarrow G/K \rightarrow G/H$  are quotient maps. The restrictions of inclusions correspond to inflations of surjections: if  $K \triangleleft G$ , then  $G \rightarrow G/K$  and  $H \rightarrow H/K$ , so  $\mathrm{inf}_H^{H/K} \circ \mathrm{res}_{H/K}^{G/K} = \mathrm{res}_H^G \circ \mathrm{inf}_G^{G/K}$ . Note that the maps are contravariants. Moreover, we have  $\mathrm{inf}_G^{G/K} \circ \mathrm{cores}_{G/K}^{H/K} = \mathrm{cores}_G^H \circ \mathrm{inf}_H^{H/K}$ .

If  $H \triangleleft G$ , then  $\mathrm{res}_H^G \circ \mathrm{cor}_G^H = N_{G/H}$ ; also,  $\mathrm{cor}_G^H \circ \mathrm{res}_H^G = [G : H]$ .

## 10 SEPT 13, 2023: SPECTRAL SEQUENCE

Whenever  $G$  is not cyclic or  $Q_8$ , the group cohomology  $H^*(G, M)$  would not have a small resolution. We know there is a pullback diagram

$$\begin{array}{ccc} M & \longrightarrow & \prod_p M_p^n \\ \downarrow & & \downarrow \\ M_{\mathbb{Q}} & \longrightarrow & \prod_p (M_p^n)_{\mathbb{Q}} \end{array}$$

Here  $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$  is the base-change, and  $M_p^n = \varprojlim_i M/p^i$  is the completion. For finite group  $G$ , we have  $H^*(G, M_{\mathbb{Q}}) = M_{\mathbb{Q}}^G$  if  $*$  = 0 and is trivial otherwise. Now we have the diagram

$$\begin{array}{ccc} H^*(G, M) & \xrightarrow{\text{res}} & H^*(\{e\}, M) \\ & \searrow |G| & \downarrow \text{tr} \\ & & H^*(G, M) \end{array}$$

where  $H^*(\{e\}, M)$  is  $M$  if  $*$  = 0 and is otherwise trivial. Note that if  $*$  > 0, then  $H^*(G, M)$  is annihilated by  $|G|$ . Let  $P \subseteq G$  be a Sylow  $p$ -subgroup, then if  $P$  is normal, then  $H^*(G, M_p^n) \cong H^*(P, M_p^n)^{G/P}$ . Therefore we have a normal series  $\cdots \triangleleft P_2 \triangleleft P_1 \triangleleft P$  with simple enough quotients, e.g., as abelian series. Therefore, we need ways to reassemble the cohomology.

For  $H \triangleleft G$  we know there is a  $G/H$ -action on  $H^*(H, M)$  via conjugation, so we can calculate  $H^*(G/H, H^*(H, M))$ , hence calculate  $H^*(G, M)$  using Lyndon-Hochschild-Serre spectral sequences.

We will first look at Bockstein spectral sequences. We start by looking at the sequence

$$\cdots \subseteq p^2\mathbb{Z} \subseteq p\mathbb{Z} \subseteq \mathbb{Z}$$

and factors each inclusion  $p^k\mathbb{Z} \subseteq p^{k-1}\mathbb{Z}$  via  $p^k(\mathbb{Z}/p\mathbb{Z})$ , then we have cohomology  $H^*(G, M/p)[p]$ , thus calculate  $H^*(G, M)$ . (Here the attachment by  $p$  is given by tensoring  $\mathbb{Z}[v_0]$  with grading  $p$ .) In general, we construct the abstract version as filtered cochain complex, with

$$\cdots \subseteq F^{p+1}C^* \subseteq F^pC^* \subseteq \cdots \subseteq C^*$$

so we can map each term to the graded version  $\text{gr}^p C^*$ . We denote the inclusions by  $i$  and the projections to the graded versions by  $\pi$ . The goal is to understand  $H^*(C^*)$  from the building blocks  $H^*(\text{gr}^* C^*)$ . There exists the factoring

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^q(F^{p+2}) & \xrightarrow{i} & H^q(F^{p+1}) & \xrightarrow{i} & H^q(F^p) \longrightarrow \cdots \\ & & \delta \uparrow & \swarrow \pi & \delta \uparrow & \swarrow \pi & \\ & & H^q(\text{gr}^{p+1}) & & H^q(\text{gr}^p) & & \end{array}$$

This is the  $E_1$ -page of the spectral sequence, given by  $E_1^{p,q} = H^q(\text{gr}^p)$ . We denote  $d_1 : H^q(\text{gr}^p) \rightarrow H^{q+1}(\text{gr}^{p+1})$  as the composition. Obviously  $d_1^2 = 0$ .

Now the  $E_2$ -page is given by  $H^*(E_1, d_1)$ . For  $a \in \ker(d_1)$ , the map  $i$  induces  $\tilde{\delta} \mapsto \delta a$  by lifting, so  $\pi(\tilde{\delta}a) \in H^{q+1}(\text{gr}^{p+2}) = E_1^{p+2, q+1}$ , with  $d_1(\pi(\tilde{\delta}a)) = \pi\delta\pi(\tilde{\delta}a) = 0$ . We then define  $d_2([a]) = [\pi(\tilde{\delta}a)] \in E_2$ . We then proceed inductively and find higher pages. This is usually done by calculating derived pages.