MATH 518 Notes

Jiantong Liu

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1 Aug 21, 2023

Definition 1.1. Let M be a topological space. An *atlas* on M is a collection $\{\varphi_\alpha : U_\alpha \to W_\alpha\}_{\alpha \in A}$ of homeomorphisms called *coordinate charts*, so that

1. $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M,
2. for all $\alpha \in A$, W_α is an open subset of some $\mathbb{R}^{n_\alpha}$,
3. for all $\alpha, \beta \in A$, the induced map $\varphi_\beta \circ \varphi_\alpha^{-1}|_{U_\alpha \cap U_\beta}$ is C^∞, i.e., smooth.

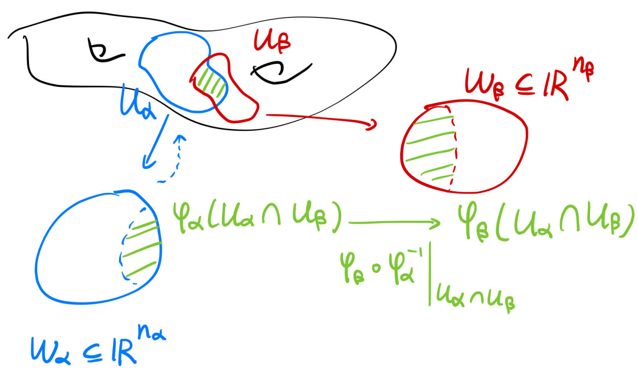


Figure 1: Atlas and Coordinate Chart

Example 1.2. Let $M = \mathbb{R}^n$ be equipped with standard topology, and let $A = \{*\}$, so $U_* = \mathbb{R}^n$ is the open cover of itself. Now the identity map

$$\begin{aligned} \varphi_* : U_* &\to \mathbb{R}^n \\ u &\mapsto u \end{aligned}$$

is an atlas on $\mathbb{R}^n$.

Example 1.3. Let $M = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be equipped with subspace topology. Let $U_\alpha = S^1 \backslash \{(1, 0)\}$ and $U_\beta = S^1 \backslash \{(-1, 0)\}$, and let $A = \{\alpha, \beta\}$. Let $W_\alpha = (0, 2\pi)$ and $W_\beta = (-\pi, \pi)$. We define $\varphi_\alpha^{-1}(\theta) = (\cos(\theta), \sin(\theta))$ and $\varphi_\beta^{-1}(\theta) = (\cos(\theta), \sin(\theta))$, then

$$(\varphi_\beta \circ \varphi_\alpha^{-1})(\theta) = \begin{cases} \theta, 0 < \theta < \pi \\ \theta - 2\pi, \pi < \theta < 2\pi \end{cases}$$

is smooth.

Example 1.4. Let X be a topological space with discrete topology, and let $A = X$, then $\{\varphi_x : \{x\} \to \mathbb{R}^0\}_{x \in X}$ gives an atlas.

Example 1.5. Let V be a finite-dimensional real vector space of dimension n. Pick a basis $\{v_1, \ldots, v_n\}$ of V, then there is a linear bijection φ with inverse

$$\begin{aligned} \varphi^{-1} : \mathbb{R}^n &\to V \\ (x_1, \ldots, x_n) &\mapsto \sum_{i=1}^n x_i v_i. \end{aligned}$$

The topology on V needs to make φ^{-1} a homeomorphism, and the obvious choice is just the collection of preimages, namely

$$\mathcal{T} = \{\varphi^{-1}(W) \mid W \subseteq \mathbb{R}^n \text{ open}\},$$

then $\varphi : V \to \mathbb{R}^n$ becomes an atlas.

Definition 1.6. Two atlases $\{\varphi_\alpha : U_\alpha \to W_\alpha\}_{\alpha \in A}$ and $\{\psi_\beta : V_\beta \to O_\beta\}_{\beta \in B}$ on a topological space M are *equivalent* if for all $\alpha \in A$ and $\beta \in B$,

$$\psi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap V_\beta) \subseteq \mathbb{R}^{n_\alpha} \to \psi_\beta(U_\alpha \cap V_\beta) \subseteq \mathbb{R}^{n_\beta}$$

is always C^∞, with C^∞-inverses. Such continuous maps are called *diffeomorphisms*. Alternatively, the two atlases are equivalent if their union $\{\varphi_\alpha\}_{\alpha \in A} \cup \{\psi_\beta\}_{\beta \in B}$ is always an atlas.

Exercise 1.7. Equivalence of atlases is an equivalence condition.

Definition 1.8. A (smooth) *manifold* is a topological space together with an equivalence class of atlases.

Convention. All manifolds are assumed to be smooth of C^∞, but not necessarily *Haudorff* and/or *second countable*.

Example 1.9. Continuing from Example 1.5, now suppose $\{w_1, \ldots, w_n\}$ gives another basis of V, with

$$\begin{aligned} \psi^{-1} : \mathbb{R}^n &\to V \\ (y_1, \ldots, y_n) &\mapsto \sum_{i=1}^n y_i w_i. \end{aligned}$$

This gives a change-of-basis matrix, so it is automatically C^∞ as a multiplication of invertible matrices. Therefore, the topology here does not depend on the chosen basis.

Recall. A topological space X is *Hausdorff* if for all distinct points $x, y \in X$, there exists open neighborhoods $U \ni x$ and $V \ni y$ such that $U \cap V = \varnothing$.

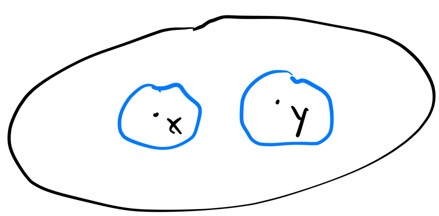


Figure 2: Hausdorff Condition

Convention. Via our definition (Definition 1.8), not all manifolds are Hausdorff.

Example 1.10. Let $Y = \mathbb{R} \times \{0, 1\}$, i.e., a space with two parallel lines, with a fixed topology. Define $\sim$ to be the smallest equivalence relation on Y such that $(x, 0) \sim (x, 1)$ for $x \neq 0$, and define $X = Y/\sim$. X is called the *line with two origins*, and it is second countable but not Hausdorff.

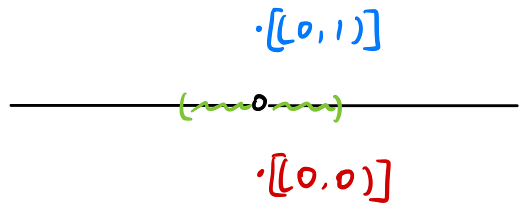


Figure 3: Line with Two Origins

Example 1.11. Take charts

$$\begin{aligned}\{\varphi : M = \mathbb{R} &\to \mathbb{R}\} \\ x &\mapsto x\end{aligned}$$

and

$$\begin{aligned}\{\psi : M = \mathbb{R} &\to \mathbb{R}\} \\ x &\mapsto x^3\end{aligned}$$

on $M = \mathbb{R}$, then

$$\begin{aligned}\varphi \circ \psi^{-1} : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^{\frac{1}{3}}\end{aligned}$$

is not C^∞, so φ and ψ are two different charts, hence give two different manifolds.

Definition 1.12. A map $F : M \to N$ between two manifolds is *smooth* if

1. F is continuous, and
2. for all charts $\varphi : U \to \mathbb{R}^m$ on M and charts $\psi : V \to \mathbb{R}^n$ on N, $\psi^{-1} \circ F \circ \varphi|_{\varphi(U \cap F^{-1}(V))}$ is C^∞.

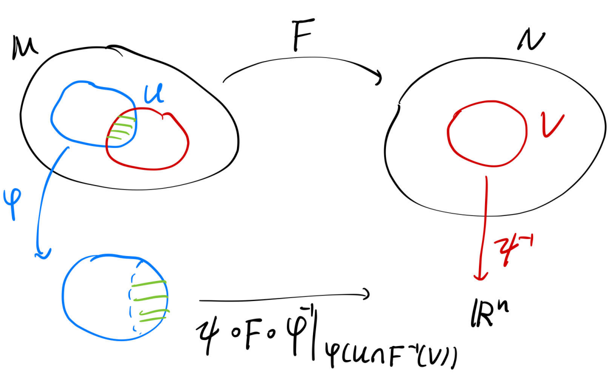


Figure 4: Smooth Map between Manifolds

2 Aug 23, 2023

Exercise 2.1.
1. $\mathrm{id} : M \to M$ is smooth.
2. If $f : M \to N$ and $g : N \to Q$ are smooth maps between manifolds, then so is $gf : M \to Q$.

Punchline. The manifolds and the smooth maps between manifolds form a category.

Recall. A smooth map $f : M \to N$ is called a *diffeomorphism*, as seen in Definition 1.6, if it has a smooth inverse. This is the notion of an isomorphism in the category of manifolds.

Warning.

1. Following Example 1.11,
$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^3 \end{aligned}$$
has an inverse
$$\begin{aligned} f^{-1} : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^{\frac{1}{3}}, \end{aligned}$$
but f^{-1} is not differentiable at $x = 0$. Hence, f is not a diffeomorphism.
2. Take $\mathbb{R}$ with discrete topology, then all singletons are open sets, then the map
$$\begin{aligned} f : \mathbb{R}_{\text{dis}} &\to \mathbb{R}_{\text{std}} \\ x &\mapsto x \end{aligned}$$
is a smooth bijection, but f^{-1} is not continuous.

Example 2.2. Consider $M = (\mathbb{R}, \{\psi = \text{id} : \mathbb{R} \to \mathbb{R}\})$ and $N = (\mathbb{R}, \{\psi : \mathbb{R} \to \mathbb{R}, x \mapsto x^3\})$ as two manifolds on $\mathbb{R}$ with standard topology. To see that they are equivalent, consider the homeomorphism
$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^{\frac{1}{3}}, \end{aligned}$$
then $(\psi \circ f \circ \varphi^{-1})(x) = \psi(f(x)) = (x^{\frac{1}{3}})^3 = x$, so f is smooth, and $(\psi \circ f \circ \varphi^{-1})^{-1} = \varphi \circ f^{-1} \circ \psi^{-1} = \text{id}$, therefore f^{-1} is also smooth. Hence, f is a diffeomorphism.

We will now consider the real projective space $\mathbb{R}P^{n-1}$ and the quotient map $\pi : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}P^{n-1}$.

Definition 2.3. Define a binary relation on $\mathbb{R}^n \backslash \{0\}$ by $v_1 \sim v_2$ if and only if there exists $\lambda \neq 0$ such that $v_1 = \lambda v_2$. This is an equivalence relation, and we identify the equivalence class $[v]$ of $v \in \mathbb{R}^n \backslash \{0\}$ as a line $\mathbb{R}v = \text{span}_{\mathbb{R}}\{v\}$ through v. Then we define the *real projective space* $\mathbb{R}P^{n-1} = (\mathbb{R}^n \backslash \{0\})/\sim$.

The natural topology on $\mathbb{R}P^{n-1}$ is the quotient topology, where $\pi : \mathbb{R}^n \backslash \{0\} \twoheadrightarrow \mathbb{R}P^{n-1}$ is surjective and continuous, so we define $U \subseteq \mathbb{R}P^{n-1}$ to be open if and only if $\pi^{-1}(U)$ is open in $\mathbb{R}^n \backslash \{0\}$.

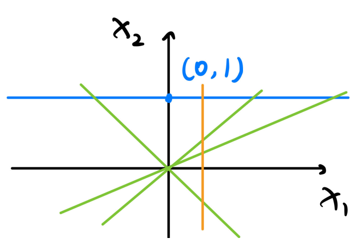


Figure 5: Stereographical Projection

Claim 2.4. $\mathbb{R}P^{n-1}$ is a manifold.

Proof. Define
$$\begin{aligned} \varphi_i : U_i &\to \mathbb{R}^{n-1} \\ [v_1, \dots, v_n] &\mapsto \left(\frac{v_1}{v_i}, \dots, \frac{v_{i-1}}{v_i}, \frac{v_{i+1}}{v_i}, \dots, \frac{v_n}{v_i} \right), \end{aligned}$$

then

$$\begin{aligned}\varphi_i^{-1}:\mathbb{R}^{n-1}&\mapsto U_i\\(x_1,\dots,x_{n-1})&\mapsto[(x_1,\dots,x_{i-1},1,x_i,\dots,x_{n-1})],\end{aligned}$$

therefore

$$\begin{aligned}\varphi_j\circ\varphi_i^{-1}:\varphi_i(U_i\cap U_j)&\to\varphi_j(U_i\cap U_j)\\(x_1,\dots,x_{n-1})&\mapsto\varphi_j([(x_1,\dots,x_{i-1},1,x_i,\dots,x_{n-1})])\\&=\begin{cases}\left(\frac{x_1}{x_j},\dots,\frac{x_{j-1}}{x_j},\frac{x_{j+1}}{x_j},\dots,\frac{x_{n-1}}{x_j}\right), & j<i\\(x_1,\dots,x_{n-1}), & j=i\\\left(\frac{x_1}{x_{j-1}},\dots,\frac{1}{x_{j-1}},\dots,\frac{x_{j-2}}{x_{j-1}},\frac{x_j}{x_{j-1}},\dots,\frac{x_{n-1}}{x_{j-1}}\right), & j>i\end{cases}\end{aligned}$$

Therefore, this is C^∞ as a rational map on $\varphi_i(U_i\cap U_j)$, and so this gives an atlas, hence $\mathbb{R}P^{n-1}$ is a manifold. ☐

Claim 2.5. $\pi:\mathbb{R}^n\backslash\{0\}\to\mathbb{R}P^{n-1}$ is smooth.

Proof. Note that

$$\begin{aligned}\psi:\mathbb{R}^n\backslash\{0\}&\hookrightarrow\mathbb{R}^n\\x&\mapsto x\end{aligned}$$

is an atlas on $\mathbb{R}^n\backslash\{0\}$, and

$$\begin{aligned}\varphi_i\circ\pi\circ\varphi_i^{-1}:\mathbb{R}^n\backslash\{0\}&\to\mathbb{R}^{n-1}\\(v_1,\dots,v_n)&\mapsto\varphi_i([(v_1,\dots,v_n)])\\&=\left(\frac{v_1}{v_i},\dots,\frac{v_{i-1}}{v_i},\frac{v_{i+1}}{v_i},\dots,\frac{v_n}{v_i}\right).\end{aligned}$$

This is C^∞ on $\pi^{-1}(U_i)=\{(v_1,\dots,v_n)\mid v_i\neq 0\}$, so π is smooth. ☐

Definition 2.6. A *smooth function* on a manifold M is a function $f:M\to\mathbb{R}$ so that for any coordinate chart $\varphi:U\to\varphi(U)$ open in $\mathbb{R}^m$, the function $f\circ\varphi^{-1}:\varphi(U)\to\mathbb{R}$ is smooth.

Remark 2.7. $f:M\to\mathbb{R}$ is smooth if and only if $f:M\to(\mathbb{R},\{\mathrm{id}:\mathbb{R}\to\mathbb{R}\})$, usually called the *standard manifold structure on* $\mathbb{R}$, is smooth.

Notation. We denote $C^\infty(M)$ to be the set of all smooth functions $f:M\to\mathbb{R}$.

Remark 2.8. $C^\infty(M)$ is a smooth $\mathbb{R}$-vector space, that is, for all $\lambda,\mu\in\mathbb{R}$ and $f,g\in C^\infty(M)$,

- $(\lambda f+\mu g)(x)=\lambda f(x)+\mu g(x)$ for all $x\in M$,
- $(f\cdot g)(x)=f(x)g(x)$ for all $x\in M$.

Therefore, $C^\infty(M)$ becomes a (commutative, associative) $\mathbb{R}$-algebra.

Fact. Connecting manifolds have the notion of dimension. That is, the dimensions of open subsets induced by coordinate charts are the same.

3 Aug 25, 2023

Definition 3.1. Let M be a manifold, then for every point $q\in M$, there exists a well-defined non-negative integer $\dim_M(q)$, so that for any coordinate chart $\varphi:U\to\mathbb{R}^m$ for $U\ni q$, we have $\dim_M(q)=m$ for some non-negative integer m that only depend on M. Consequently, $\dim_M:M\to\mathbb{Z}^{\geqslant 0}$ is a locally constant function. This integer m is called the *dimension* of M.

Proof. Indeed, say $\psi : V \to \mathbb{R}^n$ is another chart with $U \cap V \ni q$, then $\psi \circ \varphi^{-1}|_{\varphi(U \cap V)} : \varphi(U \cap V) \subseteq \mathbb{R}^m \to \psi(U \cap V) \subseteq \mathbb{R}^n$ is a diffeomorphism, therefore the Jacobian $D(\psi \circ \varphi^{-1})(\varphi(a)) : \mathbb{R}^m \to \mathbb{R}^n$ is a linear isomorphism, thus $m = n$. □

Definition 3.2. Suppose $(M, \{\varphi_\alpha : U_\alpha \to \mathbb{R}^m\}_{\alpha \in A})$ and $(N, \{\psi_\alpha : V_\beta \to \mathbb{R}^n\}_{\beta \in B})$ are two manifolds. One can give a manifold structure to the product set $M \times N$, called the *product manifold*, as follows:

- give $M \times N$ the product topology,
- let $\{\varphi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \to \mathbb{R}^m \times \mathbb{R}^n\}_{(\alpha,\beta) \in A \times B}$ to be the atlas on $M \times N$. This is well-defined since the transition maps of $\alpha, \alpha' \in A$ and $\beta, \beta' \in B$ are over $(U_\alpha \times V_\beta) \cap U_{\alpha'} \times V_{\beta'} = (U_\alpha \cap U_{\alpha'}) \times (V_\beta \cap V_{\beta'})$ with $(\varphi_{\alpha'} \times \psi_{\beta'}) \circ (\varphi_\alpha \times \psi_\beta)^{-1} = (\varphi_{\alpha'} \circ \varphi_\alpha^{-1}, \psi_{\beta'} \circ \psi_\beta^{-1})$. This is smooth since products of smooth maps are smooth.

Punchline. The product construction of manifolds gives the categorical product in the category of manifolds.

Property.

1. The projection maps
$$\begin{aligned} p_M : M \times N &\to M \\ (m, n) &\mapsto m \end{aligned}$$
and
$$\begin{aligned} p_N : M \times N &\to N \\ (m, n) &\mapsto n \end{aligned}$$
are C^∞.
2. *Universal Property of Product*: for any manifold Q and smooth maps $f_M : Q \to M$ and $f_N : Q \to N$, there exists a unique map
$$\begin{aligned} g : Q &\to M \times N \\ q &\mapsto (f(q), g(q)) \end{aligned}$$
such that $p_M \circ g = f_M$, and $p_N \circ g = f_N$.

$$\begin{tikzcd}
& Q \arrow[ddl, bend right, "f_M"'] \arrow[d, dashed, "g"', "\exists!"] \arrow[ddr, bend left, "f_N"] & \\
& M \times N \arrow[dl, "p_M"] \arrow[dr, "p_N"'] & \\
M & & N
\end{tikzcd}$$

Figure 6: Universal Property of Product

Recall.

- A topological space X is *second countable* if the topology has a countable basis: there exists a collection $\mathcal{B} = \{B_i\}_{i \in \mathbb{N}}$ of open sets so that any open set of X is a union of some B_i's.
- A cover $\{U_\alpha\}_{\alpha \in A}$ of a topological space is *locally finite* if for all $x \in X$, there exists a neighborhood N of X such that $N \cap U_\alpha = \varnothing$ for all but finitely many α's.

Example 3.3. Let $X = \mathbb{R}$, then

- $\{U_n = (-n, n)\}_{n \geqslant 0}$ is an open cover, but is not locally finite,
- $\{U_n = (n, n+2)\}_{n \in \mathbb{Z}}$ is a locally finite open cover of $\mathbb{R}$,
- $\{U_n = (n, n+2]\}_{n \in \mathbb{Z}}$ is a locally finite cover of $\mathbb{R}$, but is not an open cover.

Recall. An (open) cover $\{V_\beta\}_{\beta\in B}$ is a *refinement* of a cover $\{U_\alpha\}_{\alpha\in A}$ if for all β, there exists $\alpha=\alpha(\beta)$ such that $V_\beta\subseteq U_{\alpha(\beta)}$.

Definition 3.4. A Hausdorff topological space is *paracompact* if every open cover has a locally finite open refinement.

Fact. A connected Hausdorff manifold is paracompact if and only if it is second countable.

Corollary 3.5. A Haudorff manifold is paracompact if and only if its connected components are second countable.

Example 3.6. $\mathbb{R}$ with discrete topology is paracompact but not second countable.

Convention. Usually, we assume manifolds are paracompact, except when we need a non-Haudorff manifold. This condition is required for the existence of *partition of unity* (i.e., constant function id).

Recall. If X is a space, and $Y\subseteq X$ is a subset, then the *closure* $\bar{Y}$ of Y is the smallest closed set containing Y.

Definition 3.7. Given a topological space X and a function $f:X\to\mathbb{R}$, the *support* of f over X is

$$\operatorname{supp}(f)=\overline{\{x\in X\mid f(x)\neq 0\}}.$$

Example 3.8. The function

$$f(x)=\begin{cases} e^{-\frac{1}{x}}, & x>0\\ 0, & x\leqslant 0\end{cases}$$

is C^∞, with support $\overline{(0,\infty)}=[0,\infty)$.

Definition 3.9. Let M be a topological space and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover. A *partition of unity* subordinate to the cover is a collection of continuous functions $\{\psi_\alpha:M\to[0,1]\}_{\alpha\in A}$ such that

1. $\operatorname{supp}(\psi_\alpha)\subseteq U_\alpha$ for all $\alpha\in A$,
2. $\{\operatorname{supp}(\psi_\alpha)\}_{\alpha\in A}$ is a locally finite closed cover of M,
3. $\sum\limits_{\alpha\in A}\psi_\alpha(x)=1$ for all $x\in M$.

Remark 3.10. For all $x\in M$, there exists $\alpha_1,\dots,\alpha_n$ such that $x\in\operatorname{supp}(\psi_{\alpha_i})$. Hence, for $\alpha\neq\alpha_1,\dots,\alpha_n$, $\psi_\alpha(x)=0$. Therefore, the summation in Definition 3.9 is finite.

Theorem 3.11. Let M be a paracompact manifold with open cover $\{U_\alpha\}_{\alpha\in A}$, then there exists a partition of unity $\{\psi_\alpha: U_\alpha\to[0,1]\}_{\alpha\in A}\subseteq C^\infty(M)$ subordinate to the cover.

Example 3.12. Let $M=\mathbb{R}$ and consider for $n>0$ the open sets $\{U_n=(-n,n)\}_{n\in\mathbb{N}}$. This is not locally finite at one point.

Example 3.13. Let $M=\mathbb{R}^n$, then for all $x\in\mathbb{R}^n$ and for $r>0$, we have $B_r(x)=\{x'\in\mathbb{R}^n\mid \lvert\lvert x-x'\rvert\rvert<r\}$ and so $\{B_r(x)\}_{r>0,x\in\mathbb{R}^n}$ is an open cover, but this is not locally finite everywhere.

4 Aug 28, 2023

We will start to talk about tangent vectors.

Recall. For any point $q\in\mathbb{R}^n$ and any vector $v\in\mathbb{R}^n$, and any $f\in C^\infty(\mathbb{R}^n)$, the *directional derivative* of q in direction v with respect to f is

$$D_vf(q)=\frac{d}{dt}|_0 f(q+tv).$$

This gives a map $D_v(-)(q):C^\infty(\mathbb{R}^n)\to\mathbb{R}$ which is

- linear, and

- Leibniz rule holds, i.e.,
$$D_v(fg)(q) = D_v(f)(q) \cdot g(q) + f(q)D_v(g)(q).$$

In other words, $D_v(-)(q) : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation.

Definition 4.1. Let q be a point of a manifold M. A *tangent vector* to M at q is an $\mathbb{R}$-linear map $v : C^\infty(M) \to \mathbb{R}$ such that for all $f, g \in C^\infty(M)$,
$$v(fg) = v(f)g(q) + f(q)v(g).$$

Remark 4.2. v gives smooth vector fields over M an $C^\infty(M)$-module structure via evaluation.

Lemma 4.3. The set T_qM of all tangent vectors to M at q is an $\mathbb{R}$-vector space.

Lemma 4.4. Suppose $c \in C^\infty(M)$ is a constant function, then for all q and all $v \in T_qM$, $v(c) = 0$.

Proof. We have $v(1) = v(1 \cdot 1) = 1(q)v(1) + v(1)1(q) = 2v(1)$, so $v(1) = 0$. For a constant function c, we have
$$v(c) = v(c \cdot 1) = cv(1) = c(0) = 0.$$
□

Lemma 4.5 (Hadamard)**.** For any $f \in C^\infty(\mathbb{R}^n)$, there exists $g_1, \dots, g_n \in C^\infty(\mathbb{R}^n)$ such that

- $f(x) = f(0) + \sum\limits_{i=1}^n x_i g_i(x)$, and
- $g_i(0) = \left(\frac{\partial}{\partial x_i} f\right)(0)$.

Proof. We have
$$\begin{aligned} f(x) - f(0) &= \int_0^1 \frac{d}{dt}(f(tx))dt \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(tx) \cdot x_i dt \\ &= \sum_{i=1}^n x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx)dt \\ &= \sum_{i=1}^n x_i g_i(x). \end{aligned}$$

Therefore, $g_i(0) = \int_0^1 \frac{\partial f}{\partial x_i}(t \cdot 0)dt = \frac{\partial f}{\partial x_i}(0)$. □

Remark 4.6. For $1 \leqslant i \leqslant n$, we have canonical tangent vectors to $\mathbb{R}^n$ at 0 given by
$$\begin{aligned} \frac{\partial}{\partial x_i}|_0 : C^\infty(\mathbb{R}^n) &\to \mathbb{R} \\ f &\mapsto \frac{\partial f}{\partial x_i}(0). \end{aligned}$$

Lemma 4.7. $\left\{\frac{\partial}{\partial x_1}|_0, \dots, \frac{\partial}{\partial x_n}|_0\right\}$ is a basis of $T_0\mathbb{R}^n$.

Proof. Suppose $\sum c_i \frac{\partial}{\partial x_i}|_0 = 0$, then
$$0 = \left(\sum_i c_i \frac{\partial}{\partial x_i}|_0\right)(x_j) = \sum_i c_i \delta_{ij} = c_j.$$

Therefore, $c_j = 0$ for all j, thus we have linear independence. For all $v \in T_0\mathbb{R}^n$, i.e., $v : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is a derivation, then $v = \sum\limits_i v(x_i)\frac{\partial}{\partial x_i}|_0$. Let $f \in C^\infty(\mathbb{R}^n)$, then $f(X) = f(0) + \sum x_i g_i(x)$, thus

$$\begin{aligned}
v(f) &= v(f(0)) + \sum_{i=1}^n v(x_i g_i(x)) \\
&= \sum_{i=1}^n v(x_i g_i(x)) \\
&= \sum_{i=1}^n (v(x_i)g_i(0) + x_i(0)v(g_i)) \\
&= \sum_{i=1}^n v(x_i)g_i(0) \\
&= \sum_{i=1}^n v(x_i)\frac{\partial f}{\partial x_i}(0).
\end{aligned}$$

□

Remark 4.8. This shows $\dim(T_0\mathbb{R}^n) = n$ with the basis above.

Now let V be a finite-dimensional vector space with a basis $e_1, \dots, e_n$, then

$$\begin{aligned}
\varphi : \mathbb{R}^n &\to V \\
(t_1, \dots, t_n) &\mapsto \sum_{i=1}^n t_i e_i
\end{aligned}$$

is a linear bijection, with linear inverse

$$\begin{aligned}
\psi : V &\to \mathbb{R}^n \\
v &\mapsto (\psi_1(v), \dots, \psi_n(v))
\end{aligned}$$

where $\psi_i(v)$'s are linear maps. To describe this with a basis, we have $\psi(\sum\limits_i a_i e_i) = (a_1, \dots, a_n)$, i.e., $\psi_i(e_j) = \delta_{ij}$.

Claim 4.9. $\{\psi_1, \dots, \psi_n\}$ is a basis of $V^* = \mathrm{Hom}(V, \mathbb{R})$, called the *dual basis* of $\{e_1, \dots, e_n\}$, denoted $e_j^* = \psi_j$.

Proof. Linear independence follows from $e_j^*(e_i) = \delta_{ij}$. Given $\ell : V \to \mathbb{R}$ to be a linear map, then $\ell = \sum \ell(e_i)e_i^*$ since $\left(\sum\limits_i \ell(e_i)e_i^*\right)(e_j) = \ell(e_j)$. Given $v \in T_0\mathbb{R}^n$, $v(f) = \sum a_i \left(\frac{\partial}{\partial x_i}|_0 f\right)$ for all $f \in C^\infty(\mathbb{R}^n)$. Note that $\frac{\partial}{\partial x_i}|_0(x_j) = \delta_{ij}$, so $v(x_j) = \sum a_i \frac{\partial}{\partial x_i}|_0(x_j) = \sum\limits_i a_i\delta_{ij} = a_j$. Therefore, we have $a_i = v(x_i)$ for all i, thus $v(f) = \sum v(x_i)\left(\frac{\partial}{\partial x_i}|_0 f\right)$. Thus, the dual basis to $\frac{\partial}{\partial x_1}|_0, \dots, \frac{\partial}{\partial x_n}|_0$ is $\{d(x_i)_0\}_{i=1}^n$ where $(dx_i)_0(v) = v(x_i)$ for all i. Hence, we have $v = \sum (dx_i)_0(v)\frac{\partial}{\partial x_i}|_0$. □

Remark 4.10. Via a change of basis, this works at every point q on the local chart, so we can describe the tangent space on any point on a local chart.

5 Aug 30, 2023

Let M be a manifold and $x \in M$. Recall that a tangent vector $v : C^\infty(M) \to \mathbb{R}$ is a derivation, i.e., linear map, and the set of tangent vectors at q gives the tangent space.

Example 5.1. Let $M = \mathbb{R}^n$, and $q = 0$, then $\left\{\frac{\partial}{\partial x_1}|_0, \dots, \frac{\partial}{\partial x_n}|_0\right\}$ is a basis of $T_0\mathbb{R}^n$. Moreover, for all $v \in T_0\mathbb{R}^n$, $v = \sum v(x_i)\frac{\partial}{\partial x_i}|_0$, thus $\{v \mapsto v(x_i)\}_{i=1}^n$ is the dual basis, with $v(x_i) = (dx_i)_0(v)$ for all $1 \leqslant i \leqslant n$.

Remark 5.2. The proof used Hadamard's lemma (Lemma 4.5) and the fact that for all $x \in \mathbb{R}^n$ and all $t \in [0,1]$, $f(tx)$ is defined. Thus, the same argument should work for a version of Hadamard's lemma for star-shaped open subsets $U \subseteq \mathbb{R}^n$.

Definition 5.3. We say an open subset $U \subseteq \mathbb{R}^n$ is a *star-shaped domain* if for all $t \in [0,1]$ and all $x \in U$, $tx \in U$.

Definition 5.4. Let $F : M \to N$ be a smooth map between two manifolds, and $q \in M$ is a point, then

$$\begin{aligned} T_qF : T_qM &\to T_qN \\ v(f) &\mapsto v(f \circ F) \end{aligned}$$

via the pullback.

Exercise 5.5. Check that the definition makes sense, in particular:

(i) $(T_qF)(v)$ is a tangent vector to N of $F(q)$, and

(ii) T_qF is a derivation.

Remark 5.6. (a) It is easy to deduce the *chain rule*. That is, given $M \xrightarrow{F} N \xrightarrow{G} Q$ with $q \in M$, then $T_q(G \circ F) = T_{F(q)}G \circ T_qF$ because for all $f \in C^\infty(Q)$ and all $v \in T_qM$, we have

$$(T_q(G \circ F)(v))(f) = v(f \circ (G \circ F))$$

and

$$(T_{F(q)}G(T_qF(v))) = (T_qF)(v)(f \circ G) = v((f \circ G) \circ F).$$

(b) $T_q(\mathrm{id}_M) = \mathrm{id}_{T_qM}$.

As a result, we know T is a functor from the category of pointed manifolds to the category of $\mathbb{R}$-vector spaces.

Corollary 5.7. If $F : M \to N$ is a diffeomorphism, then for all $q \in M$, $T_qF : T_qM \to T_{F(q)}N$ is an isomorphism.

Proof. Since F is a diffeomorphism, then it has a smooth inverse $G : N \to M$, so

$$\mathrm{id}_{T_qM} = T_q(\mathrm{id}_M) = T_q(G \circ F) = T_{F(q)}G \circ T_qF$$

and

$$\mathrm{id}_{T_{F(q)}N} = T_{F(q)}(\mathrm{id}_N) = T_{F(q)}(F \circ G) = T_{F(q)}F \circ T_{F(q)}G.$$

$\square$

We also need to show that $\dim(T_qM) = \dim_q(M)$, which is a result of Lemma 5.8, whose proof will be postponed till next time.

Lemma 5.8. Let M be a manifold and $q \in M$, and let U be an open neighborhood of q in M, and let $i : U \hookrightarrow M$ be an inclusion, then

$$\begin{aligned} I = T_qi : T_qU &\to T_qM \\ v(f) &\mapsto v(f|_U) \end{aligned}$$

is an isomorphism for all $v \in T_qM$ and all $U \subseteq M$.

Notation. We denote $r_1, \ldots, r_n : \mathbb{R}^m \to \mathbb{R}$ to be the standard coordinates on $\mathbb{R}^m$.

Let M be a manifold, $q_0 \in M$, and $\varphi : U \to \mathbb{R}^m$ is a coordinate chart with $q_0 \in U$. Now let $x_i = r_i \circ \varphi$, then $\varphi(q) = (x_1(q), \ldots, x_m(q))$.

We may now assume that

- $\varphi(q_0) = 0$, otherwise, we replace $\varphi(q)$ by $\varphi(q) := \varphi(q) - \varphi(q_0)$, and
- $\varphi(U)$ is an open ball $B_R(0) = \{r \in \mathbb{R}^m \mid ||r|| < R\}$ because there exists $R > 0$ such that $B_R(0) \subseteq \varphi(U)$, and we can then replace U with $\varphi^{-1}(B_R(0))$ and restrict the charts φ to $\varphi|_{\varphi^{-1}(B_R(0))}$.

We now define

$$\frac{\partial}{\partial x_j}|_{q_0} : C^\infty(U) \to \mathbb{R}$$
$$f \mapsto \frac{\partial}{\partial r_j}|_0(f \circ \varphi^{-1})$$

Claim 5.9. $\left\{\frac{\partial}{\partial x_j}|_{q_0}\right\}_{j=1}^m$ is a basis of T_qM and for all $v \in T_{q_0}M$, $v = \sum v(x_j)\frac{\partial}{\partial x_j}|_{q_0}$.

Proof. By Hadamard's lemma Lemma 4.5 on $B_R(0)$, for all $f \in C^\infty(U)$, we have $f \circ \varphi^{-1} \in C^\infty(B_R(0))$, so there exists $g_1, \dots, g_m \in C^\infty(B_R(0))$ such that $(f \circ \varphi^{-1})(r) = f(\varphi^{-1}(0)) + \sum r_i g_i(r)$. Therefore, $f(q) = f(q_0) + \sum (r_i \circ \varphi)(q)(g_i \circ \varphi)(q)$, hence $f = f(q_0) + \sum x_i(g_i \circ \varphi)$, and $(g_i \circ \varphi)(q_0) = g_i(0) = \frac{\partial}{\partial r_i}|_0(f \circ \varphi^{-1}) = \frac{\partial}{\partial x_i}|_0(f)$.

Hence, for all $v \in T_{q_0}(U)$, we know

$$\begin{aligned} v(f) &= v(f(q_0)) + v\left(\sum x_i \cdot (g_i \circ \varphi)\right) \\ &= \sum_i v(x_i)(g_i \circ \varphi)(q_0) \\ &= \sum v(x_i)\frac{\partial}{\partial x_i}|_{q_0}(f). \end{aligned}$$

□

Remark 5.10.

1. The linear functionals
$$(dx_i)_{q_0} : T_{q_0}U \to \mathbb{R}$$
$$v \mapsto v(x_i)$$
is the basis of $(T_{q_0}U)^*$ dual to $\left\{\frac{\partial}{\partial x_i}|_{q_0}\right\}$.

2. $(T_0\varphi^{-1})\left(\frac{\partial}{\partial r_i}|_0\right) = \frac{\partial}{\partial x_i}|_{q_0}$ by definition. Since $\left\{\frac{\partial}{\partial x_i}|_0\right\}_{i=1}^n$ is a basis of $T_0(B_R(0))$, then $\left\{\frac{\partial}{\partial x_i}|_{q_0}\right\}$ has to be a basis.

Lemma 5.11. Let M be a manifold and $q \in M$ a point. Let $U \ni q$ be anopen neighborhood, and $f \in C^\infty(M)$ such that $f|_U = 0$, then for all $v \in T_qM$, we have $v(f) = 0$.

Proof. We have shown the existence of a bump function $\rho \in C^\infty(M)$ in homework 1, that is, $0 \leqslant \rho(x) \leqslant 1$, $\operatorname{supp}(\rho) \subseteq U$ and $\rho \equiv 1$ near q.

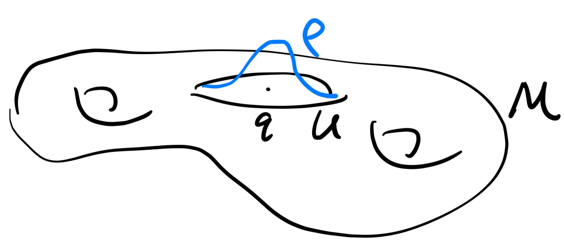


Figure 7: Bump Function

Therefore, $\rho f \equiv 0$, so $v(f) = v(\rho)f(q) + \rho(q)v(f) = v(\rho f) = 0$. □

6 Sept 1, 2023

Recall. Given a coordinate chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$, and $q \in U$ with $f(q) = 0$, we defined $\left\{ \frac{\partial}{\partial x_i}|_q \right\}_{i=1}^m \subseteq T_qU$ by $\frac{\partial}{\partial x_i}|_q f = \frac{\partial}{\partial r_i}(f \circ \varphi^{-1})|_{\varphi(q)}$ where $\frac{\partial}{\partial r_i}$'s are the standard partials on $C^\infty(\mathbb{R}^m)$.

We know this is a basis with dual basis

$$\begin{aligned}(dx_i)_q : T_qM &\to \mathbb{R}\\ v &\mapsto v(x_i)\end{aligned}$$

therefore $v = \sum v(x_i) \frac{\partial}{\partial x_i}|_q$ for all v. Note that

$$\begin{aligned}C^\infty(M) &\to C^\infty(U)\\ f &\mapsto f|_U\end{aligned}$$

is not surjective.

Also, we know $v \in T_qM$ is local, if $f, g \in C^\infty(M)$ agree on a neighborhood of q, then $v(f) = v(g)$.

Finally, given $F : M \to N$, this induces

$$\begin{aligned}T_qF : T_qM &\to T_{F(q)}N\\ v &\mapsto v(f \circ F).\end{aligned}$$

Lemma 6.1. Given a manifold M and $q \in M$, open neighborhood $q \in U \subseteq M$ and $i : U \hookrightarrow M$ inclusion, then

$$I \equiv T_q i : T_qU \to T_qM$$

is an isomorphism with $(I(v))(f) = v(f|_U)$ for all $f \in C^\infty(M)$.

Proof. Suppose $v \in \ker(I)$, then $v(f|_U) = 0$ for all $f \in C^\infty(M)$. We want $v(h) = 0$ for all $h \in C^\infty(U)$. We first choose bump function $\rho : M \to [0, 1]$ that is C^∞, and $\rho \equiv 1$ near q, and suppose $\operatorname{supp}(\rho) \subseteq U$, hence $\rho|_{M \setminus U} \equiv 0$. Then define $\rho h \in C^\infty(M)$ via

$$\rho h(x) = \begin{cases} \rho(x)h(x), & x \in U \\ 0, & x \notin U \end{cases}$$

Now $\rho h|_U \equiv h$ near q, i.e., identically 1. Therefore, $v(h) = v(\rho h|_U) = 0$, so $v \equiv 0$.

It remains to show that for all $w \in T_qM$, there exists $v \in T_qU$ such that $I(v) = w$, i.e., for all $f \in C^\infty(M)$, $w(f) = v(f|_U)$. Take the same $\rho \in C^\infty(M, [0.1])$ as above, define $v(h) = w(\rho h)$ for all $h \in C^\infty(M)$, and we can check that

- $v \in T_qM$, and
- for all $f \in C^\infty(M)$, $v(f|_U) = w(f)$.

Note that v is $\mathbb{R}$-linear, and for all $f, g \in C^\infty(W)$ we have $v(fg) = w(\rho fg) = w(\rho^2 fg)$ since $\rho fg = \rho^2 fg$ near q, then we have

$$\begin{aligned}v(fg) &= w(\rho^2 fg)\\ &= w((\rho f)(\rho g))\\ &= v(\rho f) \cdot (\rho g)(g) + \rho(f)(q) \cdot v(\rho g)\\ &= v(f)g(q) + f(q)v(g).\end{aligned}$$

Finally, for all $f \in C^\infty(M)$, we have $v(f|_U) = w(\rho f) = w(f)$ since $\rho f = f$ near q. □

Notation. We now suppress the isomorphisms $I : T_qU \to T_qM$. In particular, given a chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$, we view $\left\{ \frac{\partial}{\partial x_i}|_q \right\}_{i=1}^m$ as a basis of T_qM.

Lemma 6.2. Let V be a finite-dimensional vector space with $q \in V$, then

$$\begin{aligned}\varphi : V &\to T_q V\\ v(f) &\mapsto \frac{d}{dt}|_0 f(q+tv)\end{aligned}$$

for all $f \in C^\infty(V)$, is an isomorphism.

Proof. One can see this is linear, so it suffices to show injectivity. We have

$$\ker(\varphi) = \{v \in V \mid \frac{d}{dt}|_0(q+tv) = 0\ \forall f \in C^\infty(V)\}.$$

If $0 \neq v \in \ker(\varphi)$, then there exists $\ell : V \to \mathbb{R}$ such that $\ell(V) \neq 0$, so

$$0 \neq \frac{d}{dt}|_0(\ell(q+tv)) = \frac{d}{dt}|_0(\ell(q)+t\ell(v)) = \ell(v).$$

□

Definition 6.3. A *curve* through a point $q \in M$ on a manifold M is a C^∞-map $\gamma : (a,b) \to M$ with $0 \in (a,b)$ such that $\gamma(0) = q$.

Definition 6.4. Given $\gamma : (a,b) \to M$ with $\gamma(0) = q$, we define $\dot\gamma(0) \in T_q M$ by $\dot\gamma(0)f = \frac{d}{dt}|_0 f(\gamma(t)) = \frac{d}{dt}|_0(f \circ \gamma)$ for all $f \in C^\infty(M)$.

Remark 6.5.

$$\begin{aligned}t : (a,b) &\to \mathbb{R}\\ x &\mapsto x\end{aligned}$$

is a coordinate chart on (a,b), where $\frac{d}{dt}|_0 \in T_0(a,b)$ is a basis vector. Since γ is C^∞,

$$\begin{aligned}T_0\gamma : T_0(a,b) &\to T_{\gamma(0)}M \equiv T_q M\\ ((T_0\gamma)(\frac{d}{dt}|_0))f &= \frac{d}{dt}|_0(f\circ\gamma) = \dot\gamma(0),\end{aligned}$$

so $\dot\gamma(0) = (T_0\gamma)\left(\frac{d}{dt}|_0\right)$.

Let $\mathscr{C} = \{\gamma : I \to M \mid \gamma(0) = q, I \text{ interval depending on } \gamma\}$, then we have a map

$$\begin{aligned}\Phi : \mathscr{C} &\to T_q M\\ \gamma &\mapsto \dot\gamma(0)\end{aligned}$$

Note that Φ is not injective. However, there is an equivalence relation $\sim$ on $\mathscr{C}$ defined by $\gamma \sim \sigma$ if and only if $\Phi(\gamma) = \Phi(\sigma)$, so this gives an injection

$$\begin{aligned}\tilde\Phi : \mathscr{C}/\sim &\to T_q M\\ [\gamma] &\mapsto \dot\gamma(0).\end{aligned}$$

Claim 6.6. $\tilde\Phi$ is onto.

Proof. Choose coordinates $\varphi = (x_1,\dots,x_m) : U \to \mathbb{R}^m$ near q such that $(x_1,\dots,x_m)(q) = 0$. Now, for all $v \in T_q M$, we have $v = \sum v(x_i)\frac{\partial}{\partial x_i}|_q$. Consider $\gamma(t) = \varphi^{-1}(tv(x_1),\dots,tv(x_m))$, then $\gamma(0) = \varphi^{-1}(0) = q$ and for any $f \in C^\infty(M)$, we have

$$\begin{aligned}\dot\gamma(0)f &= \frac{d}{dt}|_0(f\circ\varphi^{-1})(tv(x_1),\dots,tv(x_m))\\ &= \sum \frac{\partial}{\partial r_i}(f\circ\varphi^{-1})|_0 \cdot v(x_i)\\ &= \sum v(x_i)\frac{\partial}{\partial x_i}|_q f\\ &= v(f).\end{aligned}$$

□

Lemma 6.7. For any smooth map $F : M \to N$ between manifolds, for all $q \in M$, we have

$$T_qF(\dot{\gamma}(0)) = (F \circ \gamma)^{\cdot}(0).$$

Proof.

$$\begin{aligned} T_qF(\dot{\gamma}(0)) &= T_qF(T_0\gamma\left(\frac{d}{dt}|_0\right)) \\ &= T_0(F \circ \gamma)\left(\frac{d}{dt}|_0\right) \\ &= (F \circ \gamma)^{\cdot}(0). \end{aligned}$$

□

Example 6.8. Let $M = N = \mathbb{C}$ and $F(z) = e^z$. We claim that $(T_zF)(v) = e^z v$, which uses $\mathbb{C} \cong T_w\mathbb{C}$ for all $w \in \mathbb{C}$. Indeed, since $\frac{d}{dt}|_0 e^{tv} = v$, then

$$\begin{aligned} (T_zF)(v) &= \frac{d}{dt}|_0 F(z + tv) \\ &= \frac{d}{dt}|_0 e^{z+tv} \\ &= \frac{d}{dt}|_0 (e^z e^{tv}) \\ &= e^z v. \end{aligned}$$

Note that T_zF is an isomorphism for all z, given by

$$\begin{tikzcd} T_z\mathbb{C} \arrow[r, "T_zF"] \arrow[d, "\cong"] & T_{F(z)}\mathbb{C} \arrow[d, "\cong"] \\ \mathbb{C} \arrow[r, "e^z\cdot -"'] & \mathbb{C} \end{tikzcd}$$

Also note that this is not a diffeomorphism, since the inverse is the complex logarithm function, which is only well-behaved on the principal branches.

7 Sept 6, 2023

Definition 7.1. Given a manifold M, $q \in M$, and $f \in C^\infty(M)$, we define the *exact differential* to be a linear map

$$\begin{aligned} df_q : T_qM &\to \mathbb{R} \\ v &\mapsto v(f) \end{aligned}$$

in $\mathrm{Hom}(T_qM, \mathbb{R}) =: T_q^*M$, the cotangent space.

Exercise 7.2.

- df_q is linear,
- $f \equiv g$ near q, then $df_q = dg_q$.

We have seen differentials before: given a coordinate chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$ is a coordinate chart, then $\{(dx_i)_q\}_{i=1}^m$ is a basis of T_q^*M dual to $\{\frac{\partial}{\partial x_i}|_q\}_{i=1}^m$. Note that for all $\eta \in T_q^*M \equiv (T_qM)^*$, then $\eta = \sum \eta\left(\frac{\partial}{\partial x_i}|_q\right)(dx_i)_q$.

Lemma 7.3. Let M be a manifold, $q \in M$, and $f \in C^\infty(M)$, then the derivative

$$(T_qf)(v) = df_q(v)\frac{d}{dt}|_{f(q)}.$$

Proof. Note that $\{dt_{f(q)}\}$ is a basis of $T^*_{f(q)}\mathbb{R}$, then

$$dt_{f(q)}(T_q f(v)) = (T_q f(v))t = v(t \circ f) = v(f) = df_q(v),$$

so $(T_q f)(v) = df_q(v)\frac{d}{dt}|_{f(q)}$. □

Recall. Let $T : V \to W$ be a linear map, and let $\{e_1, \dots, e_n\}$ be a basis of V, and let $\{f_1, \dots, f_n\}$ be a basis of W, with dual basis $\{f_1^*, \dots, f_n^*\}$ in W^*. Then let $t_{ij} = f_i^*(Te_j)$, then

$$T(e_j) = \sum_i f_i^*(Te_j) f_i = \sum_i t_{ij} f_i.$$

For all $F : \mathbb{R}^m \to \mathbb{R}^n$, consider the coordinates $(x_1, \dots, x_m) : \mathbb{R}^m \to \mathbb{R}$ and $(y_1, \dots, y_n) : \mathbb{R}^n \to \mathbb{R}$, which gives coordinates $\{(\frac{\partial}{\partial x_i}|_q)\}$ and $\{(\frac{\partial}{\partial y_i}|_{F(q)})\}$, respectively. With $T = T_q F$, we have

$$t_{ij} = (dy_i)_{F(q)}(T_q F(\frac{\partial}{\partial x_j}|_q)) = (T_q F(\frac{\partial}{\partial x_j}|_q))y_i = \frac{\partial}{\partial x_j}|_q(y_i \circ F).$$

If we denote $F = (F_1, \dots, F_n)$ where $F_i = y_i \circ F$ then this is just $\frac{\partial F_i}{\partial x_j}(q)$, so $\left(\frac{\partial F_i}{\partial x_j}(q)\right)$ is the matrix of $T_q F$.

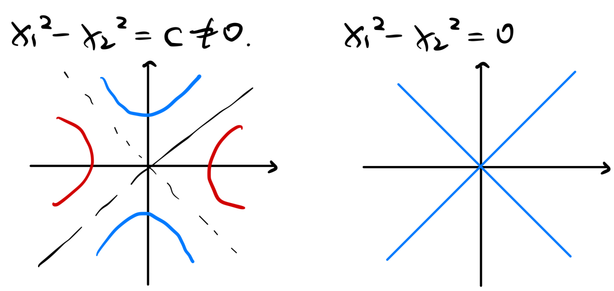
Definition 7.4. Let $F : M \to N$ be a smooth map, we say $c \in N$ is a *regular value* of F if either $F^{-1}(c) = \varnothing$, or for all $q \in F^{-1}(c)$, $T_q F : T_q M \to T_{F(q)} N = T_c N$ is onto.

We say $c \in N$ is a *singular value* if it is not a regular value.

Example 7.5. Consider

$$F : \mathbb{R}^2 \to \mathbb{R}$$
$$(x_1, x_2) \mapsto x_1 - x_2^2$$

for all $q = (x_1, x_2) \in \mathbb{R}^2$, then $T_q F$ is the matrix $\left(\frac{\partial F}{\partial x_1}(q), \frac{\partial F}{\partial x_1}(q)\right) = (2x_1, 2x_2)$. Hence, $c \neq 0$ is a regular value, and $c = 0$ is a singular value.



Definition 7.6. An *embedded submanifold* (of dimension k) of a manifold M is a subspace $Z \subseteq M$ such that for all $q \in Z$ there exists a coordinate chart $\varphi = (x_1, \dots, x_k, x_{k+1}, \dots, x_m) : U \to \mathbb{R}^m$ with $\varphi(U \cap Z) = \{(r_1, \dots, r_m) \in \varphi(U) \mid r_k = \dots = r_m = 0\}$.

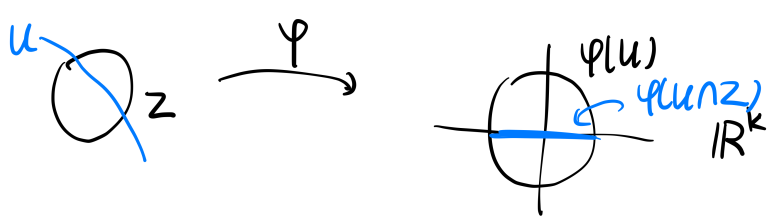


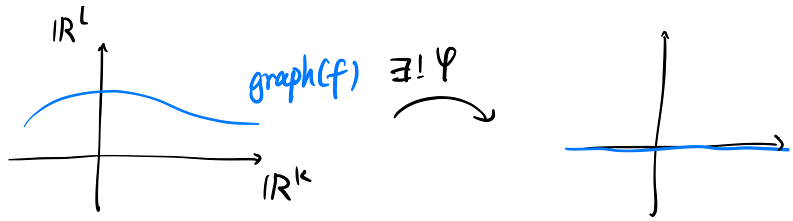
Figure 8: Embedded Submanifold

Remark 7.7.
- Any open subset $U \subseteq M$ is an embedded submanifold.
- Any singleton in M is an embedded submanifold.

Example 7.8. Let $f : \mathbb{R}^k \to \mathbb{R}^l$ be C^∞, then the graph of f is

$$\operatorname{graph}(f) = \{(x, f(x) \in \mathbb{R}^k \times \mathbb{R}^l \mid x \in \mathbb{R}^k\}$$

is an embedded submanifold of $\mathbb{R}^k \times \mathbb{R}^l$.



Here $\varphi(x, y) = (x, y - f(x))$ is a coordinate chart of $\mathbb{R}^k \times \mathbb{R}^l$ with inverse $\varphi^{-1}(x, y') = (x, y' + f(x))$.

Theorem 7.9 (Regular Value Theorem)**.** Let $c \in N$ be a regular value of smooth function $F : M \to N$. If $F^{-1}(c) = \varnothing$, then for all $q \in F^{-1}(c)$, $T_qF : T_qM \to T_qN$ is onto, so $F^{-1}(c)$ is an embedded submanifold of M. Moreover, $T_qF^{-1}(c) = \ker(T_qF)$ and $\dim(F^{-1}(c)) = \dim(M) - \dim(N)$.

Example 7.10. Consider

$$\begin{aligned} F : \mathbb{R}^m &\to \mathbb{R} \\ x &\mapsto \sum x_i^2 = ||x||^2 \end{aligned}$$

Now T_qF gives a local chart with $(2x_1, \dots, 2x_m)$. Any $c \neq 0$ is a regular value. We have $F^{-1}(c) = \{x \mid ||x||^2 = c\}$ is the sphere of radius $\sqrt{c}$ for $c > 0$. Moreover, $F^{-1}(0) = \{0\}$, an embedded submanifold, but $\dim(\{0\}) \neq \dim(\mathbb{R}^m) - \dim(\mathbb{R})$.

8 Sept 8, 2023

Recall. A subset Z of a manifold M is an embedded submanifold (of dimension k and codimension $m - k$ for $m = \dim(M)$) if for all $z \in Z$, there exists a coordinate chart $\varphi : U \to \mathbb{R}^m$ and $z \in U$ which is adapted to Z, i.e., $\varphi(U \cap Z) = \varphi(U) \cap (\mathbb{R}^k \times \{0\})$.

Remark 8.1.
- Submanifolds of codimension 0 are open subsets.
- Submanifolds of codimension $m = \dim(M)$ are discrete sets of points.

We will proceed to prove Theorem 7.9.

Remark 8.2. Once we proved $F^{-1}(c)$ is embedded and $\dim(F^{-1}(c)) = \dim(M) - \dim(N)$, then the last statement follows. Indeed, given $v \in T_q(F^{-1}(c))$, there exists $\gamma : (a, b) \to F^{-1}(c)$ such that $\gamma(0) = q$, $\gamma'(0) = v$, and $F(\gamma(t)) = c$ for all t. Therefore,

$$0 = \frac{d}{dt}|_0 F(\gamma(t)) = T_qF(\gamma'(0)) = T_qFv,$$

so $v \in \ker(T_qF)$, and so $T_qF^{-1}(c) \subseteq \ker(T_qF)$. By dimension argument, we have equality.

We will introduce inverse function theorem and implicit function theorem.

Theorem 8.3 (Inverse Function Theorem)**.** Let $U \subseteq \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^n$ be C^∞ with $q \in U$ such that $T_qf = Df(q) : T_qU = \mathbb{R}^n \to \mathbb{R}^n = T_{F(q)}\mathbb{R}^n$ is an isomorphism. Then there exists an open neighborhood $q \in V \subseteq U$ and $f(q) \in W$ such that $f : V \to W$ is a diffeomorphism.

Notation. Given $F : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^m$ for $(a,b) \in \mathbb{R}^k \times \mathbb{R}^l$, then we denote

- $\frac{\partial F}{\partial x}(a,b) = T_{(a,b)}F|_{\mathbb{R}^k \times \{0\}} = DF(a,b)|_{\mathbb{R}^k \times \{0\}}$,
- $\frac{\partial F}{\partial y}(a,b) = T_{(a,b)}F|_{\{0\} \times \mathbb{R}^l} = DF(a,b)|_{\{0\} \times \mathbb{R}^l}$.

Theorem 8.4 (Implicit Function Theorem)**.** Let $F : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ be C^∞, let $(a,b) \in \mathbb{R}^k \times \mathbb{R}^l$. Suppose $\frac{\partial F}{\partial y}(a,b) : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, then there exists a neighborhood $W \ni (a,b)$ and $U \ni a$ in $\mathbb{R}^k$, as well as C^∞-map $g : U \to \mathbb{R}^n$ such that $F^{-1}(c) \cap W = \mathrm{graph}(g) \cap W$.

Remark 8.5. inverse function theorem and implicit function theorem are equivalent.

Proof. Consider

$$\begin{aligned} H : \mathbb{R}^k \times \mathbb{R}^n &\to \mathbb{R}^k \times \mathbb{R}^n \\ (x,y) &\mapsto (x, F(x,y)) \end{aligned}$$

then $H(a,b) = (a, F(a,b)) = (a,c)$. The partials give

$$DH(a,b) = \begin{pmatrix} I & 0 \\ \frac{\partial F}{\partial x}(a,b) & \frac{\partial F}{\partial y}(a,b) \end{pmatrix}$$

As $\frac{\partial F}{\partial y}(a,b)$ is invertible, so is $DH(a,b)$, so there exists neighborhoods $(a,b) \in W \subseteq \mathbb{R}^n \times \mathbb{R}^k$ and $a \in U \subseteq \mathbb{R}^k$, $c \in V \subseteq \mathbb{R}^n$, such that $H : W \to U \times V$ is a diffeomorphism. Consider

$$\begin{aligned} G = H^{-1} : U \times V &\to W \subseteq \mathbb{R}^n \times \mathbb{R}^l \\ (u,v) &\mapsto (G_1(u,v), G_2(u,v)) \end{aligned}$$

therefore

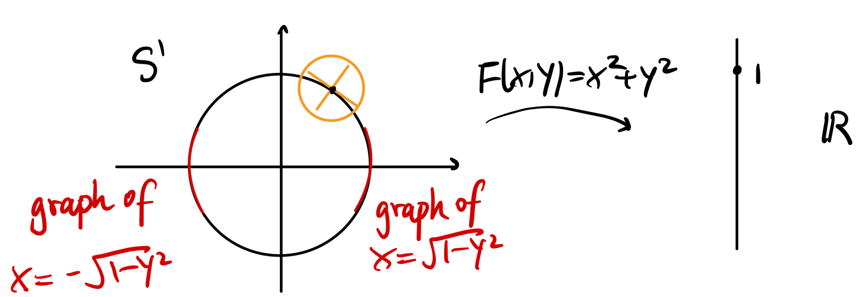
$$(u,v) = H(H^{-1}(u,v)) = H(G_1(u,v), G_2(u,v)) = (G_1(u,v), F(G_1(u,v), G_2(u,v))$$

so $G_1(u,v) = u$, and $v = F(u, G_2(u,v))$ for all u, v, hence $c = F(u, G_2(u,c))$ for all u. Now let $g(u) = G_2(u,c)$, then $F(u, g(u)) = c$ for all u. Hence, $\mathrm{graph}(g) \subseteq F^{-1}(c)$. □

Proof of Regular Value Theorem. Let $F : M \to N$, $c \in N$, $F^{-1}(c) \neq \varnothing$. Now for all $q \in F^{-1}(c)$, then $T_qF : T_qM \to T_qN$ is onto. Given $q \in F^{-1}(c)$, we want a chart T from a neighborhood of q to $\mathbb{R}^m$, adapted to $F^{-1}(c)$. Let $\varphi : U \to \mathbb{R}^m$ and $\psi : V \to \mathbb{R}^m$ be charts such that $q \in U$, $c \in V$, then

$$\tilde{F} = \psi \circ F \circ \varphi^{-1}|_{\varphi(F^{-1}(V) \cap U} : \varphi(F^{-1}(V) \cap U) \subseteq \mathbb{R}^m \to \mathbb{R}^n$$

is C^∞. Now $\psi(c)$ is a regular value in $\tilde{F}$, Let $r = \varphi(q)$, then we have $D\tilde{F}(r) : \mathbb{R}^m \twoheadrightarrow \mathbb{R}^n$. Let $X = \ker(D\tilde{F}(r))$ and Y be a complement in $\mathbb{R}^m$. So $\mathbb{R}^m = X \otimes Y$ and $D\tilde{F}(r)|_Y : Y \to \mathbb{R}^n$ is an isomorphism. Apply inverse function theorem to $\tilde{F}$ from the intersection of $X \times Y$ and the open subset to $\mathbb{R}^n$.



□

Example 8.6. Let $\mathrm{Sym}^2(\mathbb{R}^n)$ be the $n \times n$ symmetric real matrices, also known as $\mathbb{R}^{\frac{n^2-n}{2}+n}$. There is

$$\begin{aligned} F : \mathrm{GL}_(n, \mathbb{R}) &\to \mathrm{Sym}^2(\mathbb{R}^n) \\ A &\mapsto A^T A \end{aligned}$$

$$F^{-1}I = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T A = I\} \leftarrow\!\shortmid I$$

Remark 8.7. We have $F = F \circ L_A$ for all $A \in O(U)$, then for all A, we have $T_A F$ onto.

Claim 8.8. 1 is a regular value of F, so $O(n)$ is an embedded submanifold of $\mathrm{GL}(n, \mathbb{R})$.

Proof.

$$\begin{aligned} (T_I F)(v) &= \frac{d}{dt}|_0 (I + tv)^T (I + tv) \\ &= \frac{d}{dt}|_0 (I^2 + tv^T + tv + t^2 v^T v) \\ &= v^T + v \end{aligned}$$

and this is surjective since for all $Y \in \mathrm{Sym}^2(\mathbb{R})$, we have $Y = \frac{1}{2}(Y^T + Y)$, so $Y = (T_I F)(\frac{1}{2}Y)$. □

9 Sept 11, 2023

Recall. Let $F : M \to N$ be C^∞, let $c \in N$ be a regular value such that $F^{-1}(c) \neq \varnothing$. (For all $q \in F^{-1}(c)$, $T_q F : T_q M \to T_q N$ is onto.) Then:

i $F^{-1}(c)$ is an embedded submanifold of M.

ii $\dim(M) = \dim(F^{-1}(c)) = \dim(N)$.

iii for all $q \in F^{-1}(c)$, $T_q F^{-1}(c) = \ker(T_q F)$.

The proof uses inverse function theorem and/or implicit function theorem, and the key is to note that locally $f^{-1}(c)$ is a graph.

Also, $O(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T A = I\}$ is an embedded submanifold.

Definition 9.1. A *Lie group* G is a group and a manifold so that

i the multiplication map

$$\begin{aligned} m : G \times G &\to G \\ (a, b) &\mapsto (a, b) \end{aligned}$$

is C^∞.

ii the inverse map

$$\begin{aligned} \mathrm{inv} : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

is C^∞.

Notation. $e_G = 1_G$ is the identity element.

Example 9.2. $G = \mathbb{R}^n$ with $m(v, w) = v + w$, and $\mathrm{inv}(v) = -v$ gives a Lie group.

Example 9.3. Let $G = \mathrm{GL}(n, \mathbb{R})$ be with $e_G = \mathrm{diag}(1, \dots, 1) = I$, with maps $m(A, B) = AB$ and $\mathrm{inv}(A) = A^{-1}$.

Remark 9.4. One can think of a Lie group G as four pieces of data:

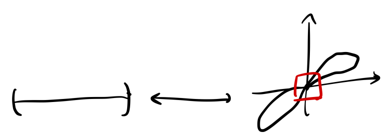
- manifold G,
- map $m : G \times G \to G$,
- map $\text{inv} : G \to G$,
- $e_G \in G$.

Note that a subgroup H of a Lie group G is not necessarily a Lie group. The sufficient condition would be H is an embedded submanifold of G, i.e.,

- $m|_{H\times H} : H \times H \to H$ are C^∞,
- $\text{inv}|_H : H \to H$

are C^∞. Note $m|_{H\times H} : H \times H \to G$ is C^∞ since $i : H \hookrightarrow G$ is C^∞ and $m|_{H\times H} = m(i \times i)$.

Example 9.5. For example, think of the embedding



but at the origin the preimage is split into three pieces, because the inverse is not continuous, which does not embed into a submanifold.

Lemma 9.6. If $i : Q \hookrightarrow M$ is an embedded submanifold, and $f : N \to M$ is a smooth map such that $f(N) \subseteq Q$, then $g : N \to Q$ with $g(n) = f(n)$ is C^∞.

$$\begin{tikzcd} N \arrow[r, "f"] \arrow[dr, dashed, "g"'] & M \\ & Q \arrow[u, hook, "i"'] \end{tikzcd}$$

Proof. Since $Q \hookrightarrow M$ is embedded, for all $q \in Q$, there exists an adapted chart $\varphi = (x_1, \dots, x_n, x_{k+1}, \dots, x_m) : U \to \mathbb{R}^m$ such that $Q \cap U = \{x_k = \cdots = x_n = 0\}$. Consider $\varphi \circ f|_{f^{-1}(U)} : f^{-1}(U) \to \mathbb{R}^m$, then $f(f^{-1}(U)) \subseteq Q \cap U$.

Then $\varphi \circ f|_{f^{-1}(U)} = \varphi(U \cap Q) = \{(r_1, \dots, r_k, r_{k+1}, \dots, r_m) \mid r_{k+1} = \cdots = r_n = 0\}$, so $\varphi \circ f = (h_1, \dots, h_k, 0, \dots, 0)$ where $h_1, \dots, h_k \in C^\infty(f^{-1}(U))$. Therefore, $\varphi|_{U\cap Q}\, g|_{f^{-1}(U)} = (h_1, \dots, h_k)$. $\square$

Example 9.7. $O(n) \subseteq \text{GL}(n, \mathbb{R})$ is embedded, thus a Lie group.

Example 9.8. $\text{SL}(n, \mathbb{R}) = \{A \in \text{GL}(n, \mathbb{R}) \mid \det(A) = 1\}$ is also a Lie group.

Claim 9.9. $1 \in \mathbb{R}$ is a regular value of $\det : \text{GL}(n, \mathbb{R}) \to \mathbb{R}$.

Proof. The key fact is that $T_I(\det) : \mathbb{R}^{n^2} \to \mathbb{R}$ is an $(n \times n)$-matrix given by $A \mapsto \text{tr}(A)$. Indeed, note that the trace is the differential of the determinant. $\square$

Definition 9.10. A (real) *Lie algebra* is a (real) vector space $\mathfrak{g}$ with an $\mathbb{R}$-bilinear map

$$\begin{aligned}[\cdot,\cdot]: \mathfrak{g}\times\mathfrak{g} &\to \mathfrak{g}\\ (X,Y) &\mapsto [X,Y]\end{aligned}$$

such that for all $X, Y, Z \in \mathfrak{g}$,

- $[Y,X] = -[X,Y]$,
- $[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$.

Example 9.11. Let $\mathfrak{g} = M_n(\mathbb{R})$, $[X,Y] = XY - YX$ is the anti-commutator.

Example 9.12. Let M be a manifold, $\mathfrak{g} = \mathrm{Der}(C^\infty(M)) = \{X : C^\infty(M) \to C^\infty(M) \mid X(fg) = X(f)\cdot g + f\cdot X(g)\}$. Therefore, $\mathfrak{g}$ is a Lie algebra with the bracket $[X,Y](f) = X(Y(f)) - Y(X(f))$ for all $f \in C^\infty(M)$. This is the Lie algebra of vector fields on M.

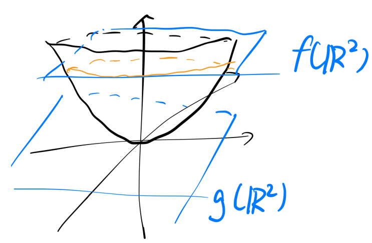
Example 9.13. Let $\mathfrak{g} = \mathbb{R}^3$, then $[v,w] := v \times w$ is a Lie algebra with cross product.

We will see that for all Lie group G, $\mathfrak{g} = \mathrm{Lie}(G) = T_eG$ is naturally a Lie algebra.

Definition 9.14. Let $F : M \to N$ be a C^∞-map, $Z \subseteq N$ be an embedded submanifold. We say F is *transverse* to Z, denoted $F \pitchfork Z$, if for all $x \in F^{-1}(Z)$, $T_xF(T_xM) + T_{F(x)}Z = T_{F(x)}N$.

Example 9.15. If $Z = \{c\}$, then $F \pitchfork c$ if and only if for all $q \in F^{-1}(c)$, $(T_xF)(T_xN) + T_cc = T_cN$, if and only if for all $q \in F^{-1}(c)$, $(T_xF)(T_xN) = T_cN$, if and only if c is a regular value of F.

Example 9.16. Let $M = \mathbb{R}^2$, $N = \mathbb{R}^3$, $Z = \{(x,y,z) \mid z = x^2 + y^2\}$, with $f(x,y) = (x,y,1)$ and $g(x,y) = (x,y,0)$, then $f \pitchfork Z$ but $g \not\pitchfork Z$.



10 Sept 13, 2023

Theorem 10.1. Suppose $f : M \to N$ is transverse to an embedded submanifold $Z \subseteq N$, then

(i) $f^{-1}(z)$ is an embedded submanifold of M.

(ii) If $f^{-1}(z) \neq \varnothing$, then $\dim(M) - \dim(f^{-1}(z)) = \dim(N) - \dim(Z)$, i.e., $\mathrm{codim}(f^{-1}(Z)) = \mathrm{codim}(Z)$.

Proof. Fix $z_0 \in Z$ with $f^{-1}(z_0) \neq \varnothing$, let $\psi : V \to \mathbb{R}^n$ be a coordinate chart on N, adapted to Z such that $\psi(V \cap Z) = \psi(V) \cap (\mathbb{R}^k \backslash \{0\})$. Let $\pi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ be the canonical projection, then

$$(\pi\circ\psi)^{-1}(0) = \psi^{-1}(\pi^{-1}(0)) = \psi^{-1}(\psi(V) \cap (\mathbb{R}^k \times \{0\})) = Z \cap V,$$

therefore

$$(\pi\circ\psi\circ f)^{-1}(0) = f^{-1}(Z\cap V) = f^{-1}(Z) \cap f^{-1}(V).$$

Claim 10.2. 0 is a regular value of $\pi\circ\psi\circ f|_{f^{-1}}(V)$.

Subproof. Take arbitrary $x \in (\pi \circ \psi \circ f)^{-1}(0) = f^{-1}(V) \cap f^{-1}(Z)$, then $T_x f(T_x M) + T_{f(x)} Z = T_{f(x)} N$. Note that $T_x M = T_x(f^{-1}(V))$. Therefore,

$$\mathbb{R}^k \times \mathbb{R}^{n-k} = T_{f(x)}\psi(T_{f(x)}N) = T_{f(x)}\psi(T_x f(T_x f^{-1}(V))) + T_{f(x)}\psi(T_{f(x)}Z)$$

by applying $T_{f(x)}\psi$ on both sides. Now apply $T_{\psi(f(x))}\psi$ on both sides, then $T_{f(x)}\psi(T_{f(x)}Z)$ vanishes, so we get

$$\begin{aligned} \mathbb{R}^{n-k} &= T_{\psi(f(x))}\pi(T_{f(x)}\psi(T_x f(T_x f^{-1}(V)))) \\ &= T_x(\pi \circ \psi \circ f)(T_x f^{-1}(V)). \end{aligned}$$

■

□

Definition 10.3. A C^∞-map $f : Q \to M$ is an *embedding* if

(i) $f(Q) \subseteq M$ is an embedded submanifold, and

(ii) $f : Q \to f(Q)$ is a diffeomorphism.

Remark 10.4. We know $f : Q \to f(Q)$ is C^∞ since $f(Q) \subseteq M$ is embedded and $f : Q \to M$ is given by the composition of $i : f(Q) \hookrightarrow M$ and $f : Q \to f(Q)$.

Remark 10.5.
1. Since $f : Q \to f(Q)$ is a diffeomorphism, then it is a homeomorphism. Thus $f : Q \to M$ is a topological embedding.
2. For all $q \in Q$, then $T_q f : T_q Q \to T_{f(q)} M$ is injective, i.e., $T_q f(T_q Q) = T_{f(q)} f(Q)$.

Example 10.6 (Non-example)**.** Let $Q = \mathbb{R}$ with discrete topology, then Q is a paracompact but not second countable as a 0-dimensional manifold. Consider

$$\begin{aligned} f : Q &\to \mathbb{R}^2 \\ x &\mapsto (x, 0) \end{aligned}$$

be a C^∞-map, then this is not an embedding.

Example 10.7. Let M be a manifold with $f \in C^\infty(M)$, then

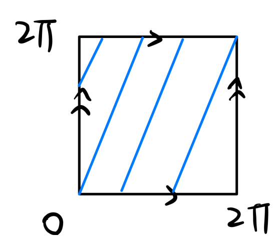
$$\begin{aligned} g : M &\to M \times \mathbb{R} \\ q &\mapsto (q, f(q)) \end{aligned}$$

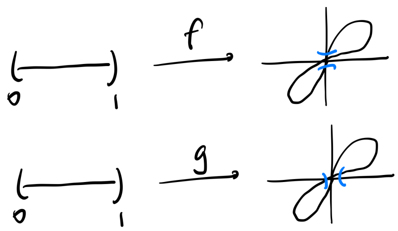
gives an embedding of M into $R \times \mathbb{R}$, as the graph of f.

Definition 10.8. A C^∞-map $f : Q \to M$ is an *immersion* if for all $q \in Q$, $T_q f : T_q Q \to T_{f(q)} M$ is injective.

Example 10.9. Consider

$$\begin{aligned} f : \mathbb{R} &\to S^1 \times S^1 \\ \theta &\mapsto (e^{i\theta}, e^{i\sqrt{2}\theta}) \end{aligned}$$





Example 10.10. Now $g \circ f^{-1} : (0,1) \to (0,1)$ is not an embedding, as it is not continuous.

Definition 10.11. The *rank* of a C^∞-map $f : M \to N$ at a point $q \in M$ is the rank of the linear map $T_q f : T_q M \to T_{f(q)} N$, i.e., $\operatorname{rank}_q(f) = \dim(T_q f(T_q M))$.

Example 10.12. If $f : M \to N$ is an immersion, then $\operatorname{rank}_q(f) = \dim_q(M)$.

Remark 10.13. Immersions are embeddings.

Theorem 10.14 (Rank Theorem)**.** Let $F : M \to N$ be a C^∞-map of constant rank k. Then for all $q \in M$, there exists coordinates $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ on M with $q \in U$, and $\psi = (y_1, \ldots, y_n) : V \to \mathbb{R}^n$ with $F(q) \in V$ such that $(\psi \circ F \circ \varphi^{-1})(r_1, \ldots, r_m) = (r_1, \ldots, r_k, 0, \ldots, 0)$ for all $r = (r_1, \ldots, r_m) \in \varphi(F^{-1}(V) \cap U)$.

Notation. Given a collection of sets $\{S_\alpha\}_{\alpha \in A}$, $\coprod\limits_{\alpha \in A} S_\alpha$ is the disjoint union of the collection.

We will give the following construction of a tangent bundle.

Remark 10.15. Given a manifold M, we form a set $TM = \coprod\limits_{q \in M} T_q M$. Given a chart $\varphi = (x_1, \ldots, x_n) : U \to \mathbb{R}^m$ on M, the corresponding candidate chart is $\tilde{\varphi} : TU = \coprod\limits_{q \in U} T_q M \to \varphi(U) \times \mathbb{R}^m$. One can check that if $\varphi : U \to \mathbb{R}^m$ and $\psi : V \to \mathbb{R}^m$ are charts on M with $U \cap V \neq \varnothing$, then $\tilde{\psi}\tilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^m \to \psi(U \cap V) \times \mathbb{R}^m$ is C^∞. Now we give TM the topology making $\tilde{\varphi}$'s homeomorphic onto their images, then $\{\tilde{\varphi} : TU \to \varphi(U) \times \mathbb{R}^m\}$ will be an atlas on TM.

11 Sept 15, 2023

Definition 11.1. A map $f : M \to N$ is a *submersion* if for all $p \in M$, the differential $T_q f : T_q M \to T_{f(q)} N$ is onto.

Remark 11.2. Every value over a submersion is regular.

Recall. For a manifold M, we defined the set $TM = \coprod\limits_{q \in M} T_q M = \bigcup(\{q\} \times T_q M)$, which is a called a tangent bundle, with additional structures. We will show that TM is a manifold, and

$$\begin{aligned} \pi : TM &\to M \\ (q, v) &\mapsto q \end{aligned}$$

is C^∞ and a submersion.

Proof. Let $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ be a coordinate chart on M. For any $q \in U$, let $\left\{ \frac{\partial}{\partial x_1}\Big|_q, \ldots, \frac{\partial}{\partial x_m}\Big|_q \right\}$ be a basis of $T_q M$. The dual basis is $\{(dx_1)_q, \ldots, (dx_m)_q\}$. For any $v \in T_q M$, we have $v = \sum v(x_i) \left.\frac{\partial}{\partial x_i}\right|_q := \sum (dx_i)_q(v) \left.\frac{\partial}{\partial x_i}\right|_q$, and

$$\begin{aligned} T_q M &\to \mathbb{R} \\ v &\mapsto ((dx_1)_q(v), \ldots, (dx_m)_q(v)) \end{aligned}$$

is a linear isomorphism. Define

$$\begin{aligned}\tilde{\varphi}: TU = \coprod_{q\in M} T_qM &\to \mathbb{R}^m \times \mathbb{R}^m\\ (q,v) &\mapsto (x_1(q),\dots,x_m(q),(dx_1)_q(v),\dots,(dx_m)_q(v)).\end{aligned}$$

Suppose $\psi=(y_1,\dots,y_m):V\to\mathbb{R}^m$ is another chart, we then have

$$\begin{aligned}\tilde{\psi}: TV &\to \mathbb{R}^m\times\mathbb{R}^m\\ (q,v) &\mapsto (y_1(q),\dots,y_m(q),(dy_1)_q(v),\dots,(dy_m)_q(v)).\end{aligned}$$

Claim 11.3. For any $(r,w)\in\varphi(U\cap V)\times\mathbb{R}^m$, we have

$$\begin{aligned}(\tilde{\psi}\circ\tilde{\varphi}^{-1})(r,w) &= ((\psi\circ\varphi^{-1})(r),\sum_j\frac{\partial y_1}{\partial x_j}(\varphi^{-1}(r))w_i,\dots,\sum_j\frac{\partial y_m}{\partial x_j}(\varphi^{-1}(r))w_i)\\ &= \left((\psi\circ\varphi^{-1})(r),\left(\frac{\partial y_i}{\partial x_j}(\varphi^{-1}(r))\right)\begin{pmatrix}w_1\\ \vdots\\ w_m\end{pmatrix}\right)\end{aligned}$$

Subproof.

Recall. If $T:A\to B$ is a linear map, with $\{e_1,\dots,e_n\}$ basis of A, $\{f_1,\dots,f_n\}$ is a basis of B, with dual basis $\{f_1^*,\dots,f_n^*\}$, then we set $t_{ij}=f_u^*(Te_j)$, i.e.,

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "(t_{ij})"] \arrow[d, "(v_1,\dots,v_n)\mapsto\sum v_ie_i"'] & \mathbb{R}^n \arrow[d] \\ A \arrow[r, "T"'] & B \end{tikzcd}$$

In our case, we have $A=B=T_qM$ with $T=\mathrm{id}$, with basis $\left\{\frac{\partial}{\partial x_i}\Big|_q\right\}$ of A, $\{f_1,\dots,f_n\}=\left\{\frac{\partial}{\partial y_1}\Big|_q,\dots,\frac{\partial}{\partial y_m}\Big|_q\right\}$ and dual basis $\{f_1^*,\dots,f_m^*\}=\{(dy_1)_q,\dots,(dy_m)_q\}$, then

$$\begin{aligned}t_{ij} &= (dy_i)_q\left(\frac{\partial}{\partial x_j}\Big|_q\right)\\ &= \frac{\partial}{\partial x_j}(y_i)(q)\\ &= \frac{\partial y_i}{\partial x_j}(\varphi^{-1}(\gamma)).\end{aligned}$$

■

We define the topology on TM to be the topology generated by the sets of form $\tilde{\varphi}^{-1}(W)$ where $\varphi:U\to\mathbb{R}^m$ is a coordinate chart with open subset $W\subseteq\mathbb{R}^m\times\mathbb{R}^m$. Given an atlas $\{\varphi_\alpha:U_\alpha\to\mathbb{R}^m\}$ on M, we get an induced atlas $\{\tilde{\varphi}_\alpha:TU_\alpha\to\mathbb{R}^m\times\mathbb{R}^m\}$ on TM. One can check that the choice of an atlas on M does not matter. □

Exercise 11.4.
- If M is Hausdorff, then so is TM.
- If M is second countable, then so is TM.

Lemma 11.5. The canonical projection $\pi:TM\to M$ is C^∞ and is a submersion.

Proof. Let $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$ be a coordinate chart, $\tilde{\varphi} : TU \to \mathbb{R}^m \times \mathbb{R}^m$ be the induced chart on TM, then

$$\begin{aligned}(\varphi \circ \pi \circ \tilde{\varphi}^{-1})(r, w) &= \varphi \circ \pi \left(\varphi^{-1}(r), \sum_i w_i \left. \frac{\partial}{\partial x_i} \right|_q \right) \\ &= \varphi(\varphi^{-1}(r)) \\ &= r.\end{aligned}$$

Moreover,

$$\left(T_{(r,w)}(\varphi \circ \pi \circ \tilde{\varphi}^{-1})\right)(v, w') = v$$

where $(v, w') \in T_{(r,w)}(\varphi(U) \times \mathbb{R}^m) \cong \mathbb{R}^n \times \mathbb{R}^m$. Therefore, $T_{(q,v)}\pi : T_{(q,v)}TM \to T_qM$ is onto, hence a submersion. $\square$

Definition 11.6. A *(algebraic) vector field* on a manifold M is a derivation $v : C^\infty(M) \to C^\infty(M)$, i.e., v is $\mathbb{R}$-linear and $v(fg) = v(f)g + fv(g)$ for all $f, g \in C^\infty(M)$.

Definition 11.7. A *(geometric) vector field* on a manifold M is a section of the tangent bundle TM of M, i.e., $X : M \to TM$ is C^∞ with $\pi \circ X = \mathrm{id}_M$. Geometrically, this depicts tangent vectors over a point with directions in $X(q)$.

Notation.
- $\mathrm{Der}(C^\infty(M))$ is the set of all derivations of $C^\infty(M)$.
- $\mathfrak{X}(M) = \Gamma(TM)$ is the set of sections of $\pi : TM \to M$.

Proposition 11.8. Given a section $v : M \to TM$ in $\mathfrak{X}(M)$, we can try and define

$$\begin{aligned}D_v : C^\infty(M) &\to C^\infty(M) \\ (D_v(f))(q) &\mapsto v(q)f\end{aligned}$$

and this assignment $v \mapsto D_v$ is a linear isomorphism.

12 Sept 18, 2023

Recall. $TM = \coprod_{q \in M} T_qM$ is a manifold. To show this, given chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$ on M, we set

$$\begin{aligned}\tilde{\varphi} = (x_1, \dots, x_m, dx_1, \dots, dx_m) : TU \equiv \coprod_{q \in U} T_qM &\to \mathbb{R}^m \to \mathbb{R}^m \\ (q, v) &\mapsto (\varphi(q), (dx_1)_q(v), \dots, (dx_m)_q(v))\end{aligned}$$

with inverse

$$\tilde{\varphi}^{-1}(r, u) = (\varphi^{-1}(r), \sum w_i \left. \frac{\partial}{\partial q_i} \right|_{\varphi(r)}.$$

Also,

$$\begin{aligned}\pi : TM &\to M \\ (q, v) &\mapsto q\end{aligned}$$

is a C^∞-submersion.

We defined vector fields in two ways,

- as sections of tangent bundle $\pi : TM \to M$, i.e., as C^∞-maps $X : M \to TM$ such that $\pi X = \mathrm{id}$, i.e., $X(q) \in T_qM$, and
- as derivations $c : C^\infty(M) \to C^\infty(M)$, i.e., as $\mathbb{R}$-linear maps such that $v(fg) = fv(g) + v(f)g$ for all $f, g \in C^\infty(M)$.

Remark 12.1. Both $\Gamma(TM)$ and $\mathfrak{X}(M)$ are $\mathbb{R}$-vector spaces, and $C^\infty(M)$-modules.

We now prove Proposition 11.8.

Proof. Given $v \in \Gamma(TM)$ and $f \in C^\infty(M)$, consider a function

$$\begin{aligned} D_v f &: M \to \mathbb{R} \\ (D_v(f))(q) &= v(q)f \end{aligned}$$

To go back, given $X \in \mathrm{Der}(C^\infty(M))$, for any $q \in M$, we have $\mathrm{ev}_q : C^\infty(M) \to \mathbb{R}$, and then $\mathrm{ev}_q \circ X : C^\infty(M) \to \mathbb{R}$ is a tangent vector. Define $v_X(q) = \mathrm{ev} \circ X$, and we can check other requirements like C^∞ and so on.

Claim 12.2. $D_v f$ is C^∞.

Subproof. Given a chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$, we have

$$\begin{aligned} \tilde{\varphi} &: TU \to \mathbb{R}^m \times \mathbb{R}^m \\ (q, v) &\mapsto (\varphi(q), dx_1(v), \dots, dx_m(v)) \end{aligned}$$

Since v is C^∞, the map $\tilde{\varphi} \circ v|_U : U \to \mathbb{R}^m \times \mathbb{R}^m$, defined by $(\tilde{\varphi} \circ v)(q) = (\varphi(q), (dx_1)_q(v(q)), \dots, (dx_m)_q(v(q)))$, is C^∞. Therefore, the assignment $q \mapsto (dx_i)_q(v(q))$ are C^∞ on U. Hence, $v = \sum v_i \frac{\partial}{\partial x_i}$ where $v_i(q) = (dx_i)_q(v(q))$ for all i. So $(D_v f)|_U = \left(\sum v_i \frac{\partial}{\partial x_i}\right) f = \sum v_i \frac{\partial f}{\partial x_i}$. This concludes the proof. ■

Also, for all $f, g \in C^\infty(M)$ and all q, we have

$$\begin{aligned} (D_v(fg))(q) &= v(q)(fg) \\ &= (v(q)f)g(q) + f(q)(v(q)g) \\ &= ((D_v f)g + f(D_v g))(q). \end{aligned}$$

Recall that derivations are local, i.e., for $X \in \mathrm{Der}(C^\infty(M))$ and $f \in C^\infty(M)$ and $f|_U \equiv 0$, then $Xf|_U \equiv 0$. As a consequence, for $U \subseteq M$ open, define $X|_U : C^\infty(U) \to C^\infty(U)$ such that $(X|_U)(f|_U) = (Xf)|_U$ for all $f \in C^\infty(M)$. Now given a chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$, we know x_i's are in $C^\infty(U)$, then $(X|_U)(x_i)$ is a smooth function on U. Therefore,

$$\begin{aligned} v_X|_U &= \sum (dx_i)(v_X) \frac{\partial}{\partial x_i} \\ &= \sum v_X X(x_i) \frac{\partial}{\partial x_i} \\ &= \sum X|_U (x_i) \frac{\partial}{\partial x_i}, \end{aligned}$$

and thus $v_X|_U : U \to TU$ is C^∞, and since U is arbitrary, then $v_X \in \Gamma(TM)$. □

Recall. For any $X, Y \in \mathrm{Der}(C^\infty(M))$, $[X, Y] \in \mathrm{Der}(C^\infty(M))$. Therefore, $\mathrm{Der}(C^\infty(M))$ is a real Lie algebra with bracket $(X, Y) \mapsto [X, Y]$. Note that $\mathrm{Der}(C^\infty(M)) \subseteq \mathrm{Hom}_{\mathbb{R}}(C^\infty(M), C^\infty(M))$.

Recall. If $(A, \circ)$ is a real associative algebra, then $[a, b] := a \circ b - b \circ a$ gives A the structure of a Lie algebra, and $\mathrm{Der}(C^\infty(M)) \subseteq \mathrm{Hom}_{\mathbb{R}}(C^\infty(M), C^\infty(M))$.

Now given a C^∞-map $f : M \to N$ of manifolds, we get a map

$$\begin{aligned} Tf &: TM \to TN \\ (q, v) &\mapsto (f(q), T_q f v) \end{aligned}$$

Exercise 12.3. Tf is C^∞.

Remark 12.4. Given $f : M \to N$ and $v \in \Gamma(TM)$, we may not have a commutative diagram:

$$\begin{tikzcd} TM \arrow[r, "Tf"] & TN \\ M \arrow[u, "v"] \arrow[r, "f"'] & N \arrow[u, dashed, "?"'] \end{tikzcd}$$

Definition 12.5. Let $f : M \to N$ be a smooth map on manifolds, then $v \in \Gamma(TM)$ and $w \in \Gamma(TN)$ are *f-related* if we have a commutative diagram

$$\begin{tikzcd} TM \arrow[r, "Tf"] & TN \\ M \arrow[u, "v"] \arrow[r, "f"'] & N \arrow[u, "w"'] \end{tikzcd}$$

That is, for any $q \in M$, $w(f(q)) = (f(q), T_q f(v(q)))$.

Equivalently, for $f : M \to N$, we say $X \in \mathrm{Der}(C^\infty(M))$ is *f-related* to $Y \in \mathrm{Der}(C^\infty(N))$ if for all $h \in C^\infty(N)$, we have $Y(h) \circ f = X(h \circ f)$ in $C^\infty(M)$.

13 Sept 20, 2023

Recall. Let M be a manifold, we have a bijection

$$\begin{aligned} \Gamma(TM) &\to \mathrm{Der}(C^\infty(M)) \\ v &\mapsto D_v : (Dvf)(q) = v_q(f)\ \forall f, q \end{aligned}$$

with inverse by assignment $X \mapsto v_X$ where $v_X(q)f = (Xf)(q)$.

Lemma 13.1. Let $f : M \to N$, then $v \in \Gamma(TM)$ is f-related to $w \in \Gamma(TN)$ if and only if $D_v \in \mathrm{Der}(C^\infty(M))$ is f-related to $D_w \in \mathrm{Der}(C^\infty(N))$.

Proof. v is f-related to w if and only if $(T_q f)(v(q)) = w(f(q))$ for all q, if and only if $((T_q f)(v(q)))h = (w(f(q)))h$ for all q and all h, if and only if $(D_v(h \circ f))(q) = (D_w h)(f(q))$, if and only if $D_v(h \circ f) = D_w(h \circ f)$. $\square$

Lemma 13.2. Suppose $f : M \to N$, let $X_1, X_2 \in \mathrm{Der}(C^\infty(M))$, and $Y_1, Y_2 \in \mathrm{Der}(C^\infty(N))$ such that X_i is f-related to Y_i for $i = 1, 2$, then $[X_1, X_2]$ is f-related to $[Y_1, Y_2]$.

Proof. For any $h \in C^\infty(N)$, $X_i(h \circ f) = Y_i(h) \circ f$ for $i = 1, 2$. Therefore,

$$\begin{aligned} ([X_1, X_2])(h \circ f) &= X_1(X_2(h \circ f)) - X_2(X_1(h \circ f)) \\ &= X_1(Y_2(h) \circ f) - X_2(Y_1(h) \circ f) \\ &= Y_1(Y_2(h)) \circ f - Y_2(Y_1(h)) \circ f \\ &= ([Y_1, Y_2](h)) \circ f. \end{aligned}$$

$\square$

Definition 13.3. Let $Q \subseteq M$ be an embedded submanifold. A vector field $Y \in \Gamma(TM)$ is *tangent* to Q if for all $q \in Q$, $Y(q) \in T_q Q$.

Example 13.4. If $M = \mathbb{R}^2$, let $Q = \mathbb{R} \times \{0\}$, then $Y(x_1, x_2) = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$, so $Y(x, 0) = x_1 \frac{\partial}{\partial x_1} + 0 \in T_{(x,0)} Q$. Equivalently, we have $i : Q \hookrightarrow M$ to be an inclusion, so $Ti : TQ \hookrightarrow TM$ is an embedding since i is, as $Y(q) \in T_q Q$ for all $q \in Q$ indicates $(Y \circ i)(Q) \subseteq TQ$:

$$\begin{tikzcd} Q \arrow[r, "i"] \arrow[d, dashed, "Y \circ i"'] & M \arrow[d, "Y"] \\ TQ \arrow[r, hook, "Ti"'] & TM \end{tikzcd}$$

Hence, $Y \circ i : Q \to TQ$ is a vector field on Q, and $Y \circ i$ is i-related to Y.

Lemma 13.5. Let $Q \subseteq M$ be an embedded submanifold, let $Y_1, Y_2 \in \Gamma(TM)$ which are tangent to Q, then $[Y_1, Y_2]$ is tangent to Q.

Proof. Since $Y_i|_Q$ is i-related to Y_i, then $[Y_1, Y_2]|_Q$ is i-related to $[Y_1, Y_2]$. $\square$

Definition 13.6. Let G be a Lie group, then we give T_eG the structure of a Lie algebra. A vector field $X : G \to TG$ is *left-invariant* if for all $a \in G$, $TL_a(X(g)) = X(L_a g)$ for all $g \in G$ and all $a \in G$, that is, X is L_a-related to X where $L_a(g) = ag$ is the left translation.

Recall.
- $(La)^{-1} = L_{a^{-1}}$.
- By Lemma 13.2, if X and Y are left-invariant, then so is $[X, Y]$.

Notation. We denote $\mathfrak{g} = \mathrm{Lie}(G)$ to be the Lie algebra of the left-invariant vector fields.

Lemma 13.7. Let G be a Lie group, let $\mathfrak{g}$ be the space of left-invariant vector fields, then the evaluation map

$$\begin{aligned} \mathrm{ev}_e : \mathfrak{g} &\to T_eG \\ X &\mapsto X(e) \end{aligned}$$

is an $\mathbb{R}$-linear bijection. In particular, they have the same dimension.

Proof. Obviously ev_e is linear. If $X(e) = 0$, then for all $a \in G$, $X(a) = X(L_a e) = (TL_a)_e(X(e)) = 0$, so ev_e is injective. Conversely, given $v \in T_eG$, define

$$\begin{aligned} \tilde{v} : G &\to TG \\ a &\mapsto (TL_a)_e v \end{aligned}$$

then $\tilde{v}$ is left-invariant. We know

$$\begin{aligned} m : G \times G &\to G \\ (a, b) &\mapsto ab \end{aligned}$$

is C^∞, so $T_m : TG \times TG \to TG$ is C^∞. Consider

$$\begin{aligned} f : G &\to TG \times TG \\ a &\mapsto ((a, 0), (e, v)). \end{aligned}$$

Claim 13.8. $(T_m \circ f)(a) = (T_e L_a)(v)$.

Subproof. Pick $\gamma : I \to G$ such that $\gamma(0) = e$ and $\dot{\gamma}(0) = v$, then

$$\begin{aligned} \sigma : I &\to G \times G \\ t &\mapsto (a, \gamma(t)) \end{aligned}$$

is C^∞ where $\sigma(0) = (a, e)$, and $\frac{d}{dt}\big|_0 (a, \gamma(t)) = (0, v) \in T_{(a,e)}(G \times G)$. Now

$$\begin{aligned} T_m(f(a)) &= (T_m)_{(a,e)}(0, v) \\ &= \frac{d}{dt}\bigg|_0 m(\sigma(t)) \\ &= \frac{d}{dt}\bigg|_0 a\gamma(t) \\ &= \frac{d}{dt}\bigg|_0 L_a(\gamma(t)) \\ &= (T_e L_a)(\dot{\gamma}(0)) \\ &= (T_e L_a)(v) \\ &= \tilde{v}(a). \end{aligned}$$

■

□

Therefore, the left-invariant vector field $\mathrm{Lie}(G)$ is isomorphic to T_eG as $\mathbb{R}$-vector spaces.

Definition 13.9. Let $X : M \to TM$ be a vector field. An *integral curve* $\gamma : I \to M$ of X passing through q at $t = 0$ is a C^∞-map $\gamma : I \to M$ such that $\gamma(0) = q$ and $\dot{\gamma}(t) = X(\gamma(t))$ for all $t \in I$. Here $\dot{\gamma}(t) = (T_t\gamma)\left(\frac{d}{dt}\big|_t\right) \in T_{\gamma(t)}M$. Equivalently, $\dot{\gamma}(t)f = X(\gamma(t))f = \frac{d}{dt}\big|_t (f \circ \gamma)$ for all $f \in C^\infty(M)$.

14 Sept 22, 2023

Remark 14.1. if $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ is a coordinate chart and v is a vector field on U, so $v = \sum v_i \frac{\partial}{\partial x_i}$ for $v_1, \ldots, v_m$ in $C^\infty(U)$. This is a section $q \mapsto \sum v_i(q) \left.\frac{\partial}{\partial x_i}\right|_q \in \Gamma(TU)$ and for all $f \in C^\infty(U)$, $f \mapsto \sum v_i \frac{\partial f}{\partial x_i} \in C^\infty(U)$ which is a derivation.

Recall. An integral curve of $X \in \Gamma(TM)$ is a curve $\gamma : I \to M$ with $\gamma(0) = q$ such that $\left.\frac{d\gamma}{dt}\right|_t = X(\gamma(t))$.

Example 14.2. Let $M = U$ be open in $\mathbb{R}^m$, and $X = \sum x_i \frac{\partial}{\partial r_i}$. Let $\gamma(t) = (\gamma_1(t), \ldots, \gamma_m(t))$ for $\gamma_i \in C^\infty(I)$, then $\left.\frac{\partial \gamma}{\partial t}\right|_t = \sum \gamma_i'(t) \frac{\partial}{\partial \gamma_i}$. Therefore, $\frac{\partial \gamma}{\partial t} = X(\gamma(t))$ amounts to $\sum \gamma_i'(t) \frac{\partial}{\partial \gamma_i} = \sum x_i(\gamma(t)) \frac{\partial}{\partial \gamma_i}$. Therefore, $\gamma_i'(t) = x_i(\gamma_1(t), \ldots, \gamma_m(t))$.

Hence, γ is an integral curve of X if and only if γ solves such a system of equations with initial condition $\gamma(0) = q$.

Theorem 14.3. Let $U \subseteq \mathbb{R}^m$ be open, $X = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ be C^∞, then for all $q_0 \in U$, there exists an open neighborhood V of q_0 in U and $\varepsilon > 0$, and a C^∞-map $\Phi : V \times (-\varepsilon, \varepsilon) \to U$ such that for all $q \in V$, $\gamma_q(t) := \Phi(q, t)$ solves $\gamma_i'(t) = x_i(\gamma_1(t), \ldots, \gamma_m(t))$ with initial condition $\gamma_q(0) = q$. Moreover, such mapping Φ is unique.

Proof. Apply contraction mapping principle. ☐

Example 14.4. Say $U = (-1, 1)$, let

$$\begin{aligned} X : (-1,1) &\to \mathbb{R} \\ x &\mapsto \frac{d}{dx} \end{aligned}$$

with $X(q) = 1$ be the ODE, i.e., $\frac{dX}{dt} = 1$ with $X(0) = q$, then $\Phi(q, t) = q + t$. The domain of definition of Φ is $W = \{(q, t) \mid q \in (-1, 1), q + t \in (-1, 1)\}$.

Remark 14.5. We need to keep track of the initial conditions. Say $\gamma : (a, b) \to M$ is an integral curve of vector field X on M with $\gamma(0) = q$, then for all $t_0 \in (a, b)$, we know

$$\begin{aligned} \sigma : (a - t_0, b - t_0) &\to M \\ s &\mapsto \gamma(s + t_0) \end{aligned}$$

is also an integral curve. Therefore, γ and σ has the same image.

Proof.

$$\begin{aligned} \left.\frac{d}{dt}\right|_t \sigma &= \left.\frac{d}{ds}\right|_t \gamma(s + t_0) \\ &= \left.\frac{d}{du}\right|_{u=t+t_0} \gamma(u) \\ &= X(\gamma(t + t_0)) \\ &= X(\sigma(t)). \end{aligned}$$

☐

Lemma 14.6. Let $X : M \to TM$ be a vector field, $\varphi = (x_1, \ldots, x_m) : U \to \mathbb{R}^m$ be a coordinate chart and $X = \sum x_i \frac{\partial}{\partial x_i}$ where $x_i \in C^\infty(U)$, then $\gamma : I \to U$ with $\gamma(0) = q$ is an integral curve of X if and only if $(x_1 \circ \gamma, \ldots, x_m \circ \gamma) : I \to \mathbb{R}^m$ solves $y_i' = Y_i(Y_1, \ldots, y_m)$ with $y_i(0) = x_i(\gamma(0))$. Here $Y_i = X_i \circ \varphi^{-1} \in C^\infty(\varphi^{-1}(U))$.

Proof. We have $\dot\gamma(t) = \sum dx_i(\dot\gamma(t)) \frac{\partial}{\partial x_i} = \sum (x_i \circ \gamma)'(t) \frac{\partial}{\partial x_i}$. Therefore, $\dot\gamma(t) = X(\gamma(t))$ if and only if $(X_i \circ \gamma)' = X_i(\gamma(t)) = (X_i \circ \varphi^{-1})(\varphi(\gamma(t))) = Y_i(X_1 \circ \gamma(t), \ldots, X_m \circ \gamma(t))$ for all i. ☐

Corollary 14.7. Let $X : M \to TM$ be a vector field, then for all $q \in M$, there exists an integral curve $\gamma : I \to M$ of X such that $\gamma(0) = q$. Moreover, γ depends smoothly on q, and is locally unique: for all integral curve $\sigma : J \to M$ of X mapping $0 \mapsto q$, there exists $\delta > 0$ such that $(-\delta, \delta) \in I \cap J$ and $\gamma|_{(-\delta,\delta)} = \sigma|_{(-\delta,\delta)}$.

Remark 14.8. It may not be the case that $\gamma|_{I\cap J} = \sigma|_{I\cap J}$. This is true if M is Hausdorff.

Example 14.9. Consider line with two origins in Example 1.10, with translations that agree before the origins.

Lemma 14.10. Suppose $\gamma : I \to M$ and $\sigma : J \to M$ are continuous curves, and M is Hausdorff, then the set $Z = \{t \in I \cap J \mid \gamma(t) = \sigma(t)\}$ is closed in $I \cap J$.

Proof. Note that

$$\begin{aligned}(\gamma, \sigma) : I \cap J &\to M \times M \\ t &\mapsto (\gamma(t), \sigma(t))\end{aligned}$$

is continuous, and $Z = (\gamma, \sigma)^{-1}(\Delta_M)$. □

Lemma 14.11. Let $\gamma : I \to M$ and $\sigma : J \to M$ be two integral curves of a vector field X on M with $\sigma(0) = \gamma(0)$, then $W = \{t \in I \cap J \mid \gamma(t) = \sigma(t)\}$ is open in $I \cap J$.

Proof. Given $t_0 \in W$, then $t_0 \in I \cap J$ and $\sigma(t_0) = \gamma(t_0)$, and we consider $\tilde{\sigma}(t) := \sigma(t + t_0)$ and $\tilde{\gamma}(t) = \gamma(t + t_0)$, then $\tilde{\sigma}(0) = \sigma(t_0) = \gamma(t_0) = \tilde{\gamma}(0)$. Both $\tilde{\gamma}$ and $\tilde{\sigma}$ are integral curves of X with $\tilde{\sigma}(0) = \tilde{\gamma}(0)$, therefore by Corollary 14.7, there exists $\delta > 0$ such that $\tilde{\sigma}|_{(-\delta,\delta)} = \tilde{\gamma}|_{(-\delta,\delta)}$, then $t_0 + (-\delta, \delta) = (t_0 - \delta, t_0 + \delta) \subseteq W$. □

Lemma 14.12. Let M be a Hausdorff manifold, $X \in \Gamma(TM)$, $\gamma : I \to M$ and $\sigma : J \to M$ be two integral curves with $\gamma(0) = \sigma(0)$, then $\gamma|_{I\cap J} = \sigma|_{I\cap J}$.

Proof. Since I and J are intervals, then $I \cap J$ is connected. By Lemma 14.11 and Lemma 14.10, $W = \{t \in I \cap J \mid \gamma(t) = \sigma(t)\}$ is clopen, thus $W = I \cap J$. □

15 Sept 25, 2022

Recall. We introduced integral curves of vector fields, and in particular we introduced Lemma 14.12.

Corollary 15.1. For any vector field $X \in \Gamma(TM)$ and any $q \in M$, there exists a unique maximal integral curve $\gamma_q : I_q \to M$ of X with $\gamma_q(0) = q$. Here *maximal* means that if $\sigma : J \to M$ is another integral curve of X with $\sigma(0) = q$, then $J \subseteq I_q$ and $\sigma = \gamma_q|_J$.

Proof. Consider the subset $\Gamma \subseteq \mathbb{R} \times M$ defined as follows: let Y be the set of all integral curves γ of X with $\gamma(0) = q$, then define $\Gamma = \bigcup\limits_{\gamma\in Y} \operatorname{graph}(\gamma)$. By Lemma 14.12, Γ is a graph of a smooth curve, which is the desired maximal integral curve γ_q of X with $\gamma_q(0) = q$. □

Lemma 15.2. Let $f : M \to N$ be a map of manifolds, with $X \in \Gamma(TM)$ and $Y \in \Gamma(TY)$, and $Tf \circ X = Y \circ f$, i.e., X and Y are f-related, then for any integral curve γ of X, $f \circ \gamma$ is an integral curve of Y.

Proof. We have

$$\begin{aligned}\frac{d}{dt} (f \circ \gamma)|_t &= T_t(f \circ \gamma) \left(\frac{d}{dt}\right) \\ &= T_{\gamma(t)} f \left(T_t\gamma \left(\frac{d}{dt}\right)\right) \\ &= T_{\gamma(t)} f(X(\gamma(t))) \\ &= Y(f(\gamma(t))) \\ &= Y((f \circ \gamma)(t)).\end{aligned}$$

□

Example 15.3. Let $M = (-1,1)$, $N = \mathbb{R}$, $f : (-1,1) \hookrightarrow \mathbb{R}$ be the inclusion. Let $X = \frac{d}{dt}$ and $Y = \frac{d}{dt}$, then

$$\begin{aligned} \gamma : (-1,1) &\to M \\ t &\mapsto t \end{aligned}$$

is a maximal integral curve of X with $\gamma(0) = 0$. Note that it is not a maximal integral curve of Y because $f \circ \gamma$ is not an integral curve of Y that is not maximal.

Example 15.4. Let $M = \mathbb{R}^2$ and $N = \mathbb{R}$, then consider $f(x,y) = x$ with $X = \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$, with $Y(x) = \frac{d}{dx}$, then $\gamma_x(t) = x + t$ is the integral curve of Y with $\gamma_x(0) = x$. it is defined for all $t \in \mathbb{R}$.

To compute integral curves of X, we solve

$$\begin{cases} \dot{x} = 1, x(0) = x_0 \\ \dot{y} = y^2, y(0) = y_0, \end{cases}$$

then $x(t) = x_0 + t$ and $\frac{1}{y}^2 \frac{dy}{dt} = 1$, therefore

$$\int_0^t \frac{1}{y^2} \frac{dy}{dt} dt = \int_0^t dt$$

and so $t = -\left.\frac{1}{y}\right|_0^t = \frac{1}{y_0} - \frac{1}{y(t)}$, hence $y(t) = \frac{y_0}{1 - y_0 t}$. Thus, $t \in (-\infty, \frac{1}{y_0})$. That is, the curve runs off to ∞ in finite time.

Definition 15.5. Let X be a vector field on a (Hausdorff) manifold M, and let $\gamma_q : I_q \to M$ be the unique maximal integral curve with $\gamma_q(0) = q$. Let $W = \bigcup\limits_{q \in M} \{q\} \times I_q \subseteq M \times \mathbb{R}$, then the *(local) flow* of X is the map

$$\begin{aligned} \Phi : W &\to M \\ (q,t) &\mapsto \gamma_q(t) \end{aligned}$$

We say Φ is a *global flow* if $W = M \times \mathbb{R}$, and in this case we say X is *complete*.

Theorem 15.6. Let $\Phi : M \to M$ be a flow of a vector field, then

1. $M \times \{0\} \subseteq W$,
2. W is open, and
3. Φ is C^∞.

Proof. See Lee. ☐

Example 15.7. Let $X = y^2 \frac{d}{dy} \in \Gamma(\mathbb{R})$, then $W = \{(y,t) \in \mathbb{R} \times \mathbb{R} \mid t < \frac{1}{y} \text{ when } y > 0, t \text{ arbitrary when } y = 0, t > \frac{1}{y} \text{ if } y < 0\}$. The flow is $\Phi(y,t) = \frac{y}{1-yt}$.

Lemma 15.8. Let $\Phi : W \to M$ be a local flow of a vector field X, then $\Phi(q, s+t) = \Phi(\Phi(q,s),t)$ whenever both sides are defined.

Remark 15.9. Note that if $s = -t$, then the left-hand side is defined, but the right-hand side is not.

Proof. Fix q and fix s such that $(q,s) \in W$. Consider $\sigma(t) = \Phi(q, s+t) = \gamma_q(s+t)$, and $\tau(t) = \Phi(\Phi(q,s),t) = \gamma_{\Phi(q,s)}(t)$, then $\tau(0) = \Phi(q,s) = \gamma_q(s) = \sigma(0)$. Both $\sigma(t)$ and $\tau(t)$ are integral curves, and that they agree at $t = 0$, then $\sigma(t) = \tau(t)$ for all t in the intersection of their domains of definition. Therefore, the two equations agree whenever both sides are defined. ☐

Definition 15.10. An (left) *action* of a Lie group G on a manifold M is a C^∞-map

$$\begin{aligned} G \times M &\to M \\ (g,q) &\mapsto g \cdot q \end{aligned}$$

such that

1. $e \cdot q = q$ for all q, and
2. $g_1 \cdot (g_2 \cdot q) = (g_1 g_2) \cdot q$.

Claim 15.11. If X is complete, then its flow is an action of the Lie group $(\mathbb{R}, +, \cdot)$.

Proof. Define $t \cdot q = \Phi(q, t)$, then

$$\begin{aligned} t \cdot (s \cdot q) &= \Phi(\Phi(q, s), t) \\ &= \Phi(q, s+t) \\ &= (t+s) \cdot q \end{aligned}$$

and $0 \cdot q = \Phi(q, 0) = q$. □

Remark 15.12. If we have a group action, we determine the groupoid structure, and therefore we recover the groupoid version of the lemma.

Remark 15.13. For a Lie group G, the multiplication $m : G \times G \to G$ is a left action of G on G, with $e \cdot g = g$ and $a \cdot (b \cdot g) = (a \cdot b) \cdot g$.

Remark 15.14. For any manifold, there exists a group $\mathrm{Diff}(M) = \{f : M \to M \mid f \text{ is a diffeomorphism}\}$, where the operation is function composition, and the identity is the identity map.

Exercise 15.15. An (left) action $G \times M \to M$ of a Lie group G on a manifold M gives rise to a homomorphism

$$\begin{aligned} \rho : G &\to \mathrm{Diff}(M) \\ (\rho(g))(q) &\mapsto g \cdot q \end{aligned}$$

In particular, the multiplication $m : G \times G \to G$ gives rise to

$$\begin{aligned} L : G &\to \mathrm{Diff}(G) \\ a &\mapsto L_a \end{aligned}$$

Definition 15.16. An *abstract local flow* on a manifold M is a C^∞-map $\psi : W \to M$, where W is an open neighborhood of $M \times \{0\}$ in $M \times \mathbb{R}$, so that $\psi(q, 0) = q$ for all $q \in M$ and $\psi(q, s+t) = \psi(\psi(q, s), t)$ whenever both sides are defined.

We will show that any abstract local flow is part of a flow on a vector field.

16 Sept 27, 2023

Recall. Given a vector field X on a manifold M, we define the flow to be $\Phi : W \to \mathbb{R}$ for some open neighborhood of $M \times \{0\}$ in $M \times \mathbb{R}$. The defining property of Φ would be that for every $q \in M$, $W \cap (\{q\} \times \mathbb{R}) = \{q\} \times I_q$ and $I_q \ni t \mapsto \Phi(q, t)$ is the maximal integral curve of X. We also proved that $\Phi(q, t+s) = \Phi(\Phi(q, t), s)$ for all q, t, s such that both sides are defined.

We say the flow is a global flow if $W = M \times \mathbb{R}$, that is, for all $q \in M$, the maximal integral curve $\gamma_q \in I_q \to M$ of X with $\gamma_q(0) = q$ is defined for all $t \in \mathbb{R}$, i.e., $I_q = \mathbb{R}$.

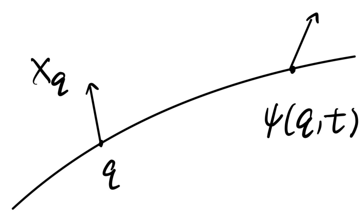
Lemma 16.1. Let M be a manifold, $U \subseteq M \times \mathbb{R}$ be an open neighborhood of $M \times \{0\}$ with $U \cap (\{q\} \times \mathbb{R})$ connected for all $q \in M$, and $\psi : U \to M$ a smooth map such that

1. $\psi(q, 0) = q$ for all q, and
2. $\psi(q, s+t) = \psi(\psi(q, s), t)$ whenever both sides are defined,

then there exists a vector field X on M such that for all $q \in M$, the assignment $t \mapsto \psi(q, t)$ is an integral (but not necessarily maximal) curve of X with $\psi(q, 0) = q$.

Proof. For all $q \in M$, we define $X(q) = \frac{d}{dt}\big|_0 \psi(q,t)$, then

$$\begin{aligned}\frac{d}{dt}\bigg|_t \psi(q,t) &= \frac{d}{dt}\bigg|_0 \psi(q,t+s)\\ &= \frac{d}{ds}\bigg|_0 \psi(\psi(q,t),s)\\ &= X(\psi(q,t)).\end{aligned}$$



□

Lemma 16.2. Let $\Phi : W \to M$ be a flow of a vector field X on a manifold M. Suppose there exists $\varepsilon > 0$ such that $M \times [-\varepsilon,\varepsilon] \subseteq W$, then $W = M \times \mathbb{R}$, i.e., the vector field X is complete.

Proof. We want to show that for all $q \in M$, $I_q := \{t \in \mathbb{R} \mid (q,t) \mid W\}$ is $\mathbb{R}$. Since I_q is connected, then it suffices to show that I_q is unbounded. By assumption, $\varphi_\varepsilon(q) := \varphi(q,\varepsilon)$ and $\varphi_{-\varepsilon}(q) := \varphi(q,-\varepsilon)$ are defined for all $q \in M$, since $q = \varphi(q,0) = \varphi(\varphi(q,\varepsilon),-\varepsilon) = \varphi(\varphi(q,-\varepsilon),\varepsilon)$, therefore $(\varphi_\varepsilon)^{-1}$ exists and is just $\varphi_{-\varepsilon}$.

Given $q \in M$, we consider $\mu(t) = \varphi(q,t+\varepsilon) = \gamma_q(\varepsilon+t)$, and it is easy to check that $\mu'(t) = X(\mu(t))$, therefore μ is an integral curve of X with $\mu(0) = \gamma_\varepsilon(\varepsilon)$. Since γ_q is defined on I_q, then μ is defined for all t such that $t+\varepsilon \in I_q$, that is, $t \in I_q - \varepsilon$. Since $\gamma_{\varphi_\varepsilon(q)} : I_{\varphi_\varepsilon(q)} \to M$ is a maximal integral curve of X such that $\gamma_{\varphi_\varepsilon(q)}(0) = \Phi_\varepsilon(q) = \gamma_q(\varepsilon)$, so $I_q - \varepsilon \subseteq I_{\varphi_\varepsilon(q)}$, and similarly $I_q + \varepsilon \subseteq I_{\varphi_{-\varepsilon}(q)}$, therefore $I_{\varphi_\varepsilon(q)} + \varepsilon \subseteq I_{\varphi_{-\varepsilon}}(\varphi_\varepsilon(q)) = I_q$. Therefore, $I_q - \varepsilon = I_{\varphi_\varepsilon(q)}$. By induction, we conclude that for all $n > 0$, $I_q - n\varepsilon = I_{(\varphi_\varepsilon)^n(q)}$. Since $0 \in I_{q'}$ for all q', and $0 \in I_q - n\varepsilon$, so $n\varepsilon \in I_q$ for all $n \in \mathbb{N}$. Similar argument shows that $-n\varepsilon \in I_q$ for all $n \in \mathbb{N}$. That is, I_q is neither bounded above nor bounded below. □

Definition 16.3. The *support* of a vector field $X \in \Gamma(TM)$ is $\operatorname{supp}(X) = \overline{\{q \in M \mid X(q) \neq 0\}}$.

Corollary 16.4. Suppose $X \in \Gamma(TM)$ has compact support, then X is complete: its flow exists for all time.

Proof. Note that $X \equiv 0$ on $M \setminus \operatorname{supp}(X)$, so for all $q \in M \setminus \operatorname{supp}(X)$. Note that $\gamma_q(t) = q$ is the maximal integral curve of X, which exists for all t, so $(M \setminus \operatorname{supp}(X)) \times \mathbb{R} \subseteq W$, which is the domain of the flow φ. Since $\operatorname{supp}(X)$ is compact, then $(\operatorname{supp}(X) \times \{0\}) \subseteq W$ is compact. Since W is open, then by tube lemma, there exists $\varepsilon > 0$ such that $\operatorname{supp}(X) \times (-2\varepsilon, 2\varepsilon) \subseteq W$, hence $\operatorname{supp}(X) \times [-\varepsilon,\varepsilon] \subseteq W$. Therefore,

$$(M \setminus \operatorname{supp}(X)) \times [-\varepsilon,\varepsilon] \subseteq (M \setminus \operatorname{supp}(X)) \times \mathbb{R} \subseteq W,$$

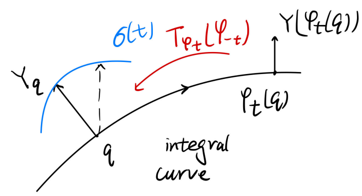
so $M \times [-\varepsilon,\varepsilon] \subseteq W$. Now apply Lemma 16.2. □

We will start talking about Lie derivatives. Let $X, Y \in \Gamma(TM)$ be two vector fields. For simplicity we assume X and Y have global flow $\varphi(q,t) = \varphi_t(q)$, and $\psi(q,t) = \psi_t(q)$, respectively. (It suffices to have the flow maintained for small neighborhood of time.) Fix $q \in M$. Consider

$$\begin{aligned}\sigma : \mathbb{R} &\to T_qM\\ t &\mapsto (T_{\varphi_t(q)}\varphi_{-t})(Y(\varphi_t(q)))\end{aligned}$$

Remark 16.5. For any curve $\gamma : \mathbb{R}$, $\dot\gamma(t) \in T_{\gamma(t)}(T_qM) = T_qM$ since γ_qM is a vector space. In particular,

$$\frac{d\sigma}{dt}\bigg|_0 = \frac{d}{dt}\bigg|_0 (T_{\varphi_t(q)}\Phi_{-t}(Y(\varphi_t(q)))) \in T_qM.$$



Definition 16.6. The *Lie derivative* L_XY of Y with respect to X is defined by

$$(L_XY)(q) = \frac{d}{dt}\bigg|_0 T_{\varphi_t(q)}\varphi_{-t}(Y(\varphi_t(q))) = \lim_{t\to 0}\frac{1}{t}\left(T_{\varphi_t(q)}\varphi_{-t}(Y(\varphi_t(q))) - Y_q\right).$$

Theorem 16.7. For any two vector fields $X, Y \in \Gamma(TM)$, $L_XY = [X, Y]$.

To prove this, we will prove the following.

Lemma 16.8. Let M be a manifold and $\gamma : I \to T_qM$ be a curve. Let $f \in C^\infty(M)$, then

$$\frac{d}{dt}\bigg|_0 (\gamma(t)f) = \left(\frac{d\gamma}{dt}\bigg|_0\right) f.$$

Proof. Choose a chart $(x_1, \ldots, x_n) : U \to \mathbb{R}^m$ with $q \in U$, then $\gamma(t) = \sum \gamma_i(t)\,\frac{\partial}{\partial x_i}\Big|_q$, where each $\gamma_i : I \to \mathbb{R}$ is C^∞. Now $\frac{d\gamma}{dt}\Big|_0 = \sum \gamma_i'(0)\,\frac{\partial f}{\partial x_i}\Big|_q$. We also know that $\gamma(t)f = \sum \gamma_i(t)\,\frac{\partial f}{\partial x_i}\Big|_q$, therefore $\frac{d}{dt}\big|_0 \gamma(t) = \frac{d}{dt}\big|_0\left(\sum \gamma_i(t)\,\frac{\partial f}{\partial x_i}\Big|_q\right) = \sum \gamma_i'(0)\,\frac{\partial f}{\partial x_i}\Big|_q$ as well. □

Lemma 16.9. Let X and Y be two vector fields with flows $\{\varphi_t\}$ and $\{\psi_t\}$, viewed as family of diffeomorphisms with $\mathbb{R}$-actions. For any $f \in C^\infty(M)$,

$$(L_XY)(q)f = \frac{\partial^2}{\partial s\partial t}\bigg|_{(0,0)} (f \circ \varphi_{-t} \circ \psi_s \circ \varphi_t)(q).$$

Proof. We have

$$\begin{aligned}
(L_XY)(q)f &= \left(\frac{d}{dt}\bigg|_0 T\varphi_{-t}(Y(\varphi_t(q)))\right) f \\
&= \frac{d}{dt}\bigg|_0 (T\varphi_{-t}(Y(\varphi_t(q))f)) \\
&= \frac{d}{dt}\bigg|_0 Y(\varphi_t(q))(f \circ \varphi_{-t}) \\
&= \frac{d}{dt}\bigg|_0 \frac{\partial}{\partial s}\bigg|_0 (f \circ \varphi_{-t})(\psi_s(\varphi_t(q))) \\
&= \frac{\partial^2}{\partial t\partial s}\bigg|_{(0,0)} (f \circ \varphi_{-t} \circ \psi_s \circ \varphi_t)(q) \\
&= \frac{\partial^2}{\partial s\partial t}\bigg|_{(0,0)} (f \circ \varphi_{-t} \circ \psi_s \circ \varphi_t)(q).
\end{aligned}$$

□

17 Sept 29, 2023

Recall. Let $X, Y \in \Gamma(TM)$ be two vector fields, and we assume for simplicity that X, Y have global flows $\{\varphi_t\}_{t\in\mathbb{R}}$ and $\{\psi_s\}_{s\in\mathbb{R}}$. We define the Lie derivative $L_X Y$ of Y with respect to X by

$$(L_X Y)(q) = (L_X Y)(q) = \left.\frac{d}{dt}\right|_0 T_{\varphi_t(q)}\varphi_{-t}(Y(\varphi_t(q))).$$

Theorem 17.1. $L_X Y = [X, Y]$.

Proof. It suffices to show that for all $f \in C^\infty(M)$ and all $q \in M$,

$$((L_X Y)(q))f = ([X, Y](q))f = ([X, Y]f)(q).$$

Consider

$$\begin{aligned} H : \mathbb{R}^3 &\to \mathbb{R} \\ (x, y, z) &\mapsto (f \circ \Phi_x \circ \psi_y \circ \Phi_z)(q), \end{aligned}$$

then by Lemma 16.8,

$$((L_X Y)(q))f = \left.\frac{\partial^2}{\partial t \partial s}\right|_{(0,0)} (H(-t, s, t)) = \left.\frac{d}{ds}\right|_{s=0} \left(\left.\frac{\partial}{\partial t}\right|_{(0,s)} H(-t, s, t) \right),$$

and by the chain rule,

$$\left.\frac{\partial}{\partial t}\right|_{(0,s)} H(-t, s, t) = -\frac{\partial H}{\partial x}(0, s, 0) + \frac{\partial H}{\partial z}(0, s, 0).$$

Hence,

$$\begin{aligned} ((L_X Y)(q))f &= \left.\frac{d}{ds}\right|_0 \left(-\left.\frac{\partial}{\partial x}\right|_{(0,s)} (f \circ \varphi_x \circ \psi_s \circ \varphi_0)(q) + \left.\frac{\partial}{\partial z}\right|_{(0,s)} (f \circ \psi_s \circ \varphi_z)(q) \right) \\ &= \left.\frac{d}{ds}\right|_0 \left(-\left.\frac{\partial}{\partial x}\right|_{(0,s)} (f \circ \varphi_x \circ \psi_s \circ \varphi_0)(q) + \left.\frac{\partial}{\partial z}\right|_{(0,s)} (f \circ \psi_s \circ \varphi_z)(q) \right) \\ &= \left.\frac{d}{ds}\right|_0 (-(Xf)(\psi_s(q)) + \left.\frac{d}{dz}\right|_0 (Yf)(\varphi_z(q)) \\ &= (-Y(Xf))(q) + (X(Yf))(q) \\ &= ((XY - YX)f)(q) \\ &= ([X, Y](q))f. \end{aligned}$$

□

Corollary 17.2. Let $X, Y \in \Gamma(TM)$ be two complete vector fields with flows $\{\varphi_t\}_{t\in\mathbb{R}}$, $\{\psi_s\}_{s\in\mathbb{R}}$, then $[X, Y] = 0$ if and only if $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$ for all s and t.

Proof. ($\Leftarrow$): Suppose $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$ for all t, s, then for all $f \in C^\infty(M)$, we have

$$\begin{aligned} ([X, Y]f)(q) = (L_X Y)(q)f \\ &= \left.\frac{\partial^2}{\partial t \partial s}\right|_{(0,0)} (f \circ \varphi_{-t} \circ \psi_s \circ \varphi_t)(q) \\ &= \left.\frac{\partial^2}{\partial s \partial t}\right|_{(0,0)} (f \circ \psi_s \circ \varphi_{-t} \circ \varphi_t)(q) \\ &= \left.\frac{\partial^2}{\partial s \partial t}\right|_{(0,0)} (f \circ \psi_s)(q) \\ &= 0. \end{aligned}$$

($\Rightarrow$): Suppose $0 = [X, Y] = L_X Y$, consider $\sigma(t) = (T\varphi_{-t})_{\varphi_t(q)}(Y(\varphi_t(q)))$, then we have $\sigma(0) = (T\varphi_0)(Y(q)) = Y(q)$, therefore

$$\begin{aligned}
\sigma'(t) &= \left.\frac{d}{ds}\right|_{s=0} \sigma(t+s) \\
&= \left.\frac{d}{ds}\right|_{0} (T\varphi_{-t-s})(Y(\varphi_s(q))) \\
&= \left.\frac{d}{ds}\right|_{s=0} (T\varphi_{-t})(T\varphi_{-s})(Y(\varphi_s(\varphi_t(q)))) \\
&= (T\varphi_{-t})\left(\left.\frac{d}{ds}\right|_{0} (T\varphi_{-s})_{\varphi_t(q)}(Y(\varphi_s(\varphi_t(q))))\right) \\
&= (T\varphi_{-t})\left(\left.\frac{d}{ds}\right|_{0} (T\varphi_{-s})_{q'}(Y(\varphi_s(q')))\right)
\end{aligned}$$

where $(T\varphi_{-s})(Y(\varphi_s(\varphi_t(q))))$ is a path in $T_{\varphi_t(q)}(M)$. Therefore, the expression is just applying a linear map onto $(L_X Y)(q')$, but this term is now just zero.

Therefore, for all t, we know that

$$Y(q) = \sigma(0) = \sigma(t) = (T\varphi_{-t})_{\varphi_t(q)}(Y(\varphi_t(q)),$$

so $(T\varphi_t)_q(Y(q)) = Y(\varphi_t(q))$, therefore $T\varphi_t \circ Y = Y \circ \varphi_t$, therefore this means Y is φ_t-related to Y, that means for all q, we know $\varphi_t(\psi_s(q)) = \psi_s(\varphi_t(q))$ for all s, t. □

We will now talk about linear algebra a bit. The blanket assumption is that all vector spaces are real and has finite dimensions.

Recall. Given vector spaces $V_1, \ldots, V_n$ and U, we say $f : V_1 \times \cdots \times V_n \to U$ is multi-linear if it is linear in each slot, that is , for all i, the assignment $v \mapsto f(v_1, \ldots, v_{i-1}, v, \ldots, v_n)$ is a linear map.

Example 17.3.

$$\begin{aligned}
\det : (\mathbb{R}^n)^n &\to \mathbb{R} \\
(v_1, \ldots, v_n) &\mapsto \det(v_1, \ldots, v_n)
\end{aligned}$$

is n-linear.

Example 17.4. For any inner product g on a vector space V, the map

$$\begin{aligned}
g : V &\to V \times \mathbb{R} \\
(v_1, v_2) &\mapsto g(v_1, v_2)
\end{aligned}$$

is bilinear.

Example 17.5. If $\mathfrak{g}$ is a Lie algebra, then the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is bilinear.

Notation. We say $\mathrm{Mult}(V_1, \ldots, V_n; U)$ is the set of n-linear maps $f : V_1 \times \cdots \times V_n \to U$.

Fact. $\mathrm{Mult}(V_1, \ldots, V_n; U)$ is an $\mathbb{R}$-vector space.

Lemma 17.6. Let V, W, U be three vector spaces with bases $\{v_i\}$, $\{w_j\}$, and $\{u_k\}$, respectively, and let $\{v_i^*\}$, $\{w_j^*\}$, and $\{u_k^*\}$ be their duals, respectively. We now define

$$\begin{aligned}
\varphi_{ij}^k : V \times W &\to U \\
(v, w) &\mapsto v_i^*(v) \cdot w_j^*(w) \cdot u_k \\
(-, \cdot) &\mapsto v_i^*(-) \cdot w_j^*(\cdot) u_k,
\end{aligned}$$

then $\{\varphi_{ij}^k\}$ is a basis of $\mathrm{Mult}(V, W; U)$.

Proof. Given a bilinear map $b : V \times W \to U$ with $(x, y) \in V \times W$, then

$$\begin{aligned} b(x,y) &= b(\sum v_i^*(x)y_j, \sum w_j^*(y)w_j) \\ &= \sum_{i,j} v_i^*(x)w_j^*(y)b(v_i,w_j) \\ &= \sum_{i,j,k} v_j^*(x)w_j^*(y)u_k^*(b(v_i,w_j))u_k \\ &= \sum_{i,j,k} u_k^*(b(v_i,w_j))\varphi_{ij}^k(x,y), \end{aligned}$$

therefore $\{\varphi_{ij}^k\}$ spans $\mathrm{Mult}(V,W;U)$.

Suppose $\sum\limits_{i,j,k} c_k^{ij}\varphi_{ij}^k = 0$, then for all r, l, we know $\varphi_{ij}^k(v_r,w_l) = v_i^*(v_r)w_j^*(w_l)u_k = \delta_{ir}\delta_{jl}u_k$, so

$$0 = \sum c_k^{ij}\varphi_{ij}^k(v_r,w_l) = \sum_{i,j,k} c_k^{ij}\delta_{ir}\delta_{il}u_k = \sum_k c_k^{rl}u_k.$$

□

18 Oct 2, 2023

Definition 18.1. Let V and W be two (finite-dimensional) vector spaces over $\mathbb{R}$. The tensor product $V \otimes W$ of V and W is a vector space together with a unique bilinear map

$$\begin{aligned} \otimes : V \times W &\to V \otimes W \\ (v,w) &\mapsto v \otimes w \end{aligned}$$

with the following universal property: for any bilinear map $b : V \times W \to U$, there exists a unique linear map $\bar{b} : V \otimes W \to U$ so that the diagram

$$\begin{tikzcd} V \otimes W \arrow[r, "\bar{b}"] & U \\ V \times W \arrow[u, "\otimes"] \arrow[ur, "b"'] & \end{tikzcd}$$

commutes, i.e., $b(v, w) = \bar{b}(v \otimes w)$ for all $(v, w) \in V \times W$.

Lemma 18.2. For any two vector spaces V and W, the tensor product $V \otimes W$ with respect to $\otimes : V \times W \to V \otimes W$ exists and is unique up to unique isomorphism.

Corollary 18.3. For any three vector spaces U, V, and W, the map

$$\begin{aligned} \varphi : \mathrm{Hom}(V \otimes W, U) &\to \mathrm{Mul}(V, W; U) \\ A &\mapsto \varphi(A) = A \circ \otimes \end{aligned}$$

is an isomorphism of vector spaces.

Proof. The uniqueness follows from the universal property. To prove existence, recall that for any set X, there is a construction of free vector space which has a copy of X as a basis. Define the tensor product to be the categorical product quotiented out by the obvious equivalence relations, given by additions and scalar multiplications, then this gives a tensor product construction over the free vector space. To prove the universal property, write down the canonical mapping, then the bilinear map $b : V \times W \to U$ induces $\bar{b} : F(V \times W) \to U$, then it satisfies the universal property and we are done. □

Lemma 18.4. For any two finite-dimensional vector spaces V and W, then $V \otimes W$ is a finite-dimensional vector space and $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$.

Proof. We know $\operatorname{Hom}(V \otimes W, \mathbb{R}) = \operatorname{Mult}(V, W; \mathbb{R})$, and we know that $\dim(\operatorname{Mult}(V, W; \mathbb{R})) = \dim(V) \cdot \dim(W) \cdot \dim(\mathbb{R})$, therefore $\dim(\operatorname{Hom}(V \otimes W, \mathbb{R})) < \infty$, so $\dim(V \otimes W) < \infty$, and then $\dim(V \otimes W) = \dim(\operatorname{Hom}(V \otimes W, \mathbb{R})) = \dim(V) \cdot \dim(W)$. ☐

Corollary 18.5. If $\{v_i\}_{i=1}^n$ is a basis of V and $\{w_j\}_{j=1}^m$ a basis of W, then $\{v_i \otimes w_j\}$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$ is a basis of $V \otimes W$.

Proof. By construction of the tensor product, we know this set spans $V \otimes W$ already. For any element $x \otimes y \in V \otimes W$, then write down each element with respect to the basis, reorder them, then we get a sum with respect to the given basis $\{v_i \otimes w_j\}$, and we know this spans indeed. Moreover, the dimension matches and we are done. ☐

Lemma 18.6. There exists a unique linear map

$$\begin{aligned} T : V \otimes W &\to W \otimes V \\ v \otimes w &\mapsto w \otimes v \end{aligned}$$

for all $v \in V$ and $w \in W$.

Proof. The uniqueness is easy: this is given by the assignment. To show the existence, consider

$$\begin{aligned} b : V \times W &\to W \otimes V \\ (v, w) &\mapsto w \otimes v \end{aligned}$$

which is a bilinear map and then take the universal property and we are done. ☐

Remark 18.7. T is an isomorphism, and the tensor product $\otimes$ gives rise to a symmetric monoidal category structure on the category of vector spaces.

Lemma 18.8. For any two finite-dimensional vector space V and W, there exists a unique linear map

$$\begin{aligned} \varphi : V^* \otimes W &\to \operatorname{Hom}(V, W) \\ l \otimes w &\mapsto l(-)w. \end{aligned}$$

Proof. Consider the bilinear map

$$\begin{aligned} b : V^* \times W &\to \operatorname{Hom}(V, W) \\ (l, w) &\mapsto l(-)w \end{aligned}$$

then by the universal property φ is the unique linear map as specified above. This is an isomorphism if we check the basis. ☐

19 Oct 4, 2023

Remark 19.1. The universal property of $\otimes$ can be explained by 1) the universal property over bilinear maps; 2) the universal property over categorical product; 3) the natural bijection between bilinear maps to U and homomorphisms to U.

Remark 19.2. If V and W are finite-dimensional, then there exists a natural transformation

$$\begin{aligned} V^* \otimes W^* &\xrightarrow{\sim} \operatorname{Mult}(V, W; \mathbb{R}) \\ l \otimes \eta &\mapsto l(-)\eta(-) \end{aligned}$$

Remark 19.3. Since $\operatorname{Mult}(V, W; \mathbb{R}) \cong \operatorname{Hom}(V \otimes W, \mathbb{R}) = (V \otimes W)^*$, so $(V \otimes W)^* \cong V^* \otimes W^*$.

Recall. An $\mathbb{R}$-algebra is a vector space A with a bilinear map $\circ : A \times A \to A$. An algebra A is associative if $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in A$.

Definition 19.4. An $(\mathbb{Z}_{\geqslant 0})$-*graded vector space* A is a sequence of vector spaces $\{V_i\}_{i \geqslant 0}$. Equivalently, a graded vector space V is a direct sum $V = \bigoplus_{i=0}^{\infty} V_i$.

Recall.

$$\bigoplus_{i=0}^{\infty} V_i = \{\{v_i\}_{i=0}^{\infty} \mid v_i \in V_i, v_i = 0 \text{ for all but finitely many } i\}.$$

Definition 19.5. A $(\mathbb{Z}_{\geqslant 0})$-*graded algebra* is a graded vector space $A = \bigoplus\limits_{i \geqslant 0} A_i$ together with a bilinear map $\circ : A \times A \to A$ such that for all i, j, $a_i \in A_i$ and $a_j \in A_j$, $a_i \circ a_j \in A_{i+j}$.

We are mostly interested in two types of graded associative algebras:

- the tensor algebra of a vector space V, given by $\mathcal{T}(V) = \bigoplus\limits_{k=0}^{\infty} V^{\otimes k}$, and
- the Grassmannian/exterior algebra $\bigwedge^*(V) = \bigoplus\limits_{i=0}^{\infty} \bigwedge^k V$.

Definition 19.6. We define the *exterior algebra* as follows: $V^{\otimes 0} = \mathbb{R}$, $V^{\otimes 1} = V$, and $V^{\otimes 2} = V \otimes V$. For $k > 2$, there exists a unique (up to isomorphism) vector space $V^{\otimes k}$ together with a k-linear map

$$\begin{aligned} \otimes^k : V^k &\to V^{\otimes k} \\ (v_1, \ldots, v_k) &\mapsto v_1 \otimes v_2 \otimes \cdots \otimes v_k, \end{aligned}$$

so that it satisfies the following universal property, that is, for any vector space U, we have $\mathrm{Hom}(V^{\otimes k}, U) = \mathrm{Mult}(V^k = (V, \ldots, V); U)$. To define each of them, we can

- either define it inductively, using the fact that tensor products are associative up to unique isomorphism, or
- we construct it using the free vector space, that is, $V^{\otimes k} = F(V^k)/S$ where S is an appropriate subspace, imitating the construction of the tensor product. Therefore, we want $\bigotimes^k(v_1, \ldots, v_k) = \delta_{(v_1,\ldots,v_k)} + S \cdots$

Remark 19.7. Consider the tensor product $\mathbb{R}^2 \otimes \mathbb{R}^2$. We have

$$\begin{aligned} (1,1) \otimes (1,-1) &= ((1,0) + (0,1)) \otimes ((1,0) + (0,-1)) \\ &= (1,0) \otimes (1,0) - (0,1) \otimes (0,1) - (1,0) \otimes (0,1) + (0,1) \otimes (1,0) \\ &= \cdots \end{aligned}$$

Definition 19.8. To make $\mathcal{T}(V) = \bigoplus V^{\otimes k}$ into an (associative) algebra, we need bilinear maps $\circ_{k,l} : V^{\otimes k} \times V^{\otimes l} \to V^{\otimes (k+l)}$. We would want

$$(v_1 \otimes \cdots \otimes v_k) \circ_{k,l} (v_{k+1} \otimes \cdots \otimes v_{k+l}) = v_1 \otimes \cdots \otimes v_k \otimes v_{k+1} \otimes \cdots \otimes v_{k+l}.$$

To start with, we take $k, l \geqslant 1$,

$$\begin{aligned} \varphi : V^k \times V^l &\to V^{\otimes (k+l)} \\ ((v_1, \ldots, v_k), (v_{k+1}, \ldots, v_{k+l})) &\mapsto v_1 \otimes \cdots \otimes v_k \otimes \cdots \otimes v_{k+l}, \end{aligned}$$

then this is a $(k+l)$-linear map. We now fix $(v_{k+1}, \ldots, v_{k+l}) \in V^l$, then

$$\begin{aligned} \varphi_{(v_{k+1},\ldots,v_{k+l})} : V^k &\to V^{\otimes (k+l)} \\ (v_1, \ldots, v_k) &\mapsto v_1 \otimes \cdots \otimes v_k \otimes v_{k+1} \otimes \cdots \otimes v_{k+l} \end{aligned}$$

which is k-linear, then by universality there exists a unique map $\bar{\varphi}_{(v_{k+1},\ldots,v_{k+l})} : V^{\otimes k} \to V^{\otimes (k+l)}$, then for any each fixed t in $V^{\otimes k}$, we get a map

$$\begin{aligned} V^l &\to V^{\otimes (k+l)} \\ (v_{k+1}, \ldots, v_{k+l}) &\mapsto \bar{\varphi}_{(v_{k+1},\ldots,v_{k+l})}(t) \end{aligned}$$

and therefore we get a bilinear map

$$\circ_{k,l} : V^{\otimes k} \times V^{\otimes l} \to V^{\otimes(k+l)}$$

with $(v_1 \otimes \cdots v_k) \circ_{k,l} (v_{k+1}, \dots, v_{k+l}) = v_1 \otimes \cdots \otimes v_{k+l}$. It now remains to check that for all k, l, m, we have

$$\begin{tikzcd} & V^{\otimes k} \times V^{\otimes l} \times V^{\otimes m} \arrow[dl, "\circ_{k,l} \times \mathrm{id}"'] \arrow[dr, "\mathrm{id} \times \circ_{l,m}"] & \\ V^{\otimes(k+l)} \times V^{\otimes m} \arrow[dr, "\circ_{k+l,m}"'] & & V^{\otimes k} \times V^{\otimes(l+m)} \arrow[dl, "\circ_{k,l+m}"] \\ & V^{\otimes(k+l+m)} & \end{tikzcd}$$

To show this, we just have to check on the generators, since all maps are already well-defined. It is enough to check on generators, given by

$$\begin{tikzcd} & (v_1 \otimes \cdots \otimes v_k, v_{k+1} \otimes \cdots \otimes v_{k+l}, v_{k+l+1} \otimes \cdots \otimes v_{k+l+m}) \arrow[dl] \arrow[dr] & \\ (v_1 \otimes \cdots \otimes v_{k+l}, v_{k+l+1} \otimes \cdots \otimes v_{k+l+m}) \arrow[dr] & & (v_1 \otimes \cdots \otimes v_k, v_{k+1} \otimes \cdots \otimes v_{k+l+m}) \arrow[dl] \\ & v_1 \otimes \cdots \otimes v_{k+l+m} & \end{tikzcd}$$

Therefore, this proves associativity.

Remark 19.9. We can think of TV as an associative algebra freely generated by elements in degree 1, which is just V.

Definition 19.10. The *Grassmannian/exterior algebra* on a vector space V is a graded-commutative associative algebra $\bigwedge^* V = \bigoplus_{k=0}^{\infty} \bigwedge^k V$ with an injective linear map $i : V \hookrightarrow \bigwedge^* V$ so that $\bigwedge^0 V = \mathbb{R}$, $i(V) = \bigwedge^1 V$, that has the following universal property: for any associative algebra A, for all linear map $j : V \to A$ such that $j(v) \cdot j(v) = 0$ for all $v \in V$, there exists a unique map of algebras (i.e., linear map that preserves multiplications) $\bar{j} : \bigwedge^* V \to A$ such that

$$\begin{tikzcd} \bigwedge^* V \arrow[r, dashed, "\exists ! \bar{j}"] & A \\ V \arrow[u, hook, "i"] \arrow[ur, "j"'] & \end{tikzcd}$$

Remark 19.11. The pair $(\bigwedge^* V, i : V \hookrightarrow \bigwedge^* V)$ is unique up to a unique isomorphism.

20 Oct 6, 2023

Definition 20.1. A graded associative algebra $A = \bigoplus_{k \geqslant 0} A_k$ is *graded-commutative* if for all k, l, $a \in A_k$, $b \in A_l$, then $ab = (-1)^{kl} ba$.

Definition 20.2. Let V be a finite-dimensional vector space, the *Grassmannian/exterior algebra* $\bigwedge^* V = \bigoplus_{k \geqslant 0} \bigwedge^k V$ of V is a graded-commutative algebra freely generated by $\bigwedge^1 V = V$. The term "freely generated" has the following universal property: for any unital associative algebra A and any linear map $j : V \to A$ such that $(j(v))^2 = 0$ for all $v \in V$, then there exists a unique map of algebras $\bar{j} : \bigwedge^* V \to A$ such that the restriction $\bar{j}|_{\bigwedge^1 V = V} = j$. That is, we have a commutative diagram

$$\begin{tikzcd} \bigwedge^* V \arrow[r, dashed, "\exists ! \bar{j}"] & A \\ V \arrow[u, hook] \arrow[ur, "j"'] & \end{tikzcd}$$

Remark 20.3. Analogously, the tensor algebra $\mathcal{T}(V)$ is the associative algebra freely generated by elements in $V^{\otimes 1} = V$.

Remark 20.4.

- Being unital means there exists $1_A \in A$ such that $1_A a = a 1_A = a$ for all $a \in A$.

- $(j(v))^2 = 0$ for all v implies that $j(v_1)j(v_2) = -j(v_2)j(v_1)$ for all $v_1, v_2 \in V$. Indeed, we have

$$\begin{aligned}0 &= j(v_1+v_2)j(v_1+v_2)\\ &= (j(v_1)+j(v_2))(j(v_1)+j(v_2))\\ &= (j(v_1))^2 + j(v_2)j(v_1) + j(v_1)j(v_2) + (j(v_2))^2\\ &= j(v_2)j(v_1) + j(v_1)j(v_2).\end{aligned}$$

Remark 20.5 (Existence of $\bigwedge^* V$)**.** Consider the two-sided ideal I in $\mathcal{T}(V)$ generated by $\{v \otimes v \mid v \in V\}$. Therefore, I is the $\mathbb{R}$-span of elements of the form $a \otimes v \otimes v \otimes b$ where $v \in V$, $a, b \in \mathcal{T}(V)$. Since I is generated by elements of degree 2, then $I = \bigoplus_{k \geqslant 0} I_k$ where $I_k = I \cap V^{\otimes k}$ is a graded ideal of degree k. Note $I_0 = I \cap V^{\otimes 0} = 0$; $I_1 = I \cap V = 0$. We construct $\bigwedge^* V = \mathcal{T}(V)/I$ to be an associative algebra. Denote the multiplication of $\bigwedge^* V$ by $\wedge$ where $(a+I)\wedge(b+I) = a \otimes b + I$ for all $a, b \in V$. In particular, $\bigwedge^k V = V^{\otimes k}/I_k$, and so $\bigwedge^* V = \bigoplus_{k \geqslant 0} \bigwedge^k V$.

Notation. We denote $v_1 \wedge \cdots \wedge v_k := v_1 \otimes \cdots \otimes v_k + I$ for all $v_1, \dots, v_k \in V$. This identifies $v \mapsto v + I$. With this abuse of notation, $v \wedge v + I = 0 + I = 0$. Therefore, $v \wedge w = -w \wedge v$ for all $v, w \in V$, which satisfies graded-commutativity.

Remark 20.6 (Uniqueness of $\bigwedge^* V$)**.** Suppose A is a unital associative algebra, and $j : V \to A$ is a linear map with $(j(v))^2 = 0$ for all $v \in V$. Consider

$$\begin{aligned}V^n &\to A\\ (v_1, \dots, v_n) &\mapsto j(v_1)\cdots j(v_n).\end{aligned}$$

This is n-linear, hence gives rise to a unique linear map $\tilde{j}^n : V^{\otimes n} \to A$ with $\tilde{j}^n(v_1 \otimes \cdots \otimes v_n) = j(v_1)\cdots j(v_n)$, hence we get a morphism $\tilde{j} : \bigoplus_{n \geqslant 0} V^{\otimes n} \to A$ of algebras. For all $v \in V$, $\tilde{j}(v \otimes v) = j(v)j(v) = 0$, so there exists a unique $\bar{j} : \bigoplus_{n \geqslant 0} V^{\otimes n}/I \to A$ such that $\bar{j}(v_1 \wedge \cdots \wedge v_n) = j(v_1)\cdots j(v_n)$, by the first isomorphism theorem.

Remark 20.7. Recall in $\bigwedge^* V$ we have $v \wedge w = (-1) w \wedge v$ for $v, w \in V$ since they have degree 1. In general, we have

$$\begin{aligned}(v_1 \wedge \cdots \wedge v_k) \wedge (v_{k+1} \wedge \cdots \wedge v_{k+l}) &= v_1 \wedge \cdots \wedge v_k \wedge v_{k+1} \wedge \cdots \wedge v_{k+l}\\ &= (-1)^k v_{k+1} \wedge v_1 \wedge \cdots \wedge v_k \wedge v_{k+2} \wedge \cdots \wedge v_{k+l}\\ &= (-1)^{kl}(v_{k+1} \wedge \cdots \wedge v_{k+l}) \wedge (v_1 \wedge \cdots \wedge v_k)\end{aligned}$$

and therefore $\bigwedge^* V$ is graded-commutative.

Recall. The permutation group S_n is generated by transpositions $(i\ j)$ for $1 \leqslant i < j \leqslant n$. In fact, it is generated by $(1\,2), (2\,3), \cdots, (n-1\ n)$.

Lemma 20.8. Let V be a finite-dimensional vector space and let $n \geqslant 2$, then take $v_1, \dots, v_n \in V$. For any permutation $\sigma \in S_n$, we have $v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)} = (\operatorname{sgn}(\sigma)) v_1 \wedge \cdots \wedge v_n$.

Proof. It suffices to check when $\sigma = (i\ i+1)$, which is obvious. □

Corollary 20.9. Let $v_1, \dots, v_n$ be a basis of a finite-dimensional vector space V, then

1. $\bigwedge^k V = 0$ for $k > n$,
2. elements of kth exterior power $\{v_{i_1} \wedge \cdots \wedge v_{i_k} \mid i_1 < i_2 < \cdots < i_k\}$ spans $\bigwedge^k V$.

Proof. We know $\{v_i \otimes v_j \mid 1 \leqslant i, j \leqslant n\}$ is a basis of $V \otimes V = V^{\otimes 2}$. Proceeding by induction on k, we know $\{v_{i_1} \otimes \cdots \otimes v_{i_k} \mid 1 \leqslant i_1, \dots, i_k \leqslant n\}$ is a basis of $V^{\otimes k}$, therefore $\{v_{i_1} \wedge \cdots \wedge v_{i_k} \mid 1 \leqslant i_1, \dots, i_k \leqslant n\}$ spans $\bigwedge^k V = V^{\otimes n}/I_k$.

If $k > n$, we must have repeated indices in $v_{i_1} \wedge \cdots \wedge v_{i_k}$, therefore this is zero: if we permute the indices, we can ask the two repeated indices stand next to each other, and in particular their wedge is zero, therefore the entire term would be zero. We will prove that $\{v_{i_1} \wedge \cdots \wedge v_{i_k} \mid 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n\}$ is a basis of $\bigwedge^k V$. The key is $v_1 \wedge \cdots \wedge v_n \neq 0$. □

21 Oct 9, 2023

First, we will show that $v_1 \wedge \cdots \wedge v_n \neq 0$.

Definition 21.1. Let V, U be two vector spaces. A k-linear map $f : V^k \to U$ is said to be *alternating* if for all $\sigma \in S_k$, $f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma) f(v_1, \dots, v_k)$.

Example 21.2. For all $l_1, l_2 : V \to \mathbb{R}$, the map

$$\begin{aligned} f : V \times V &\to \mathbb{R} \\ (v_1, v_2) &\mapsto l_1(v_1)l_2(v_2) - l_1(v_2)l_2(v_1) = \det \begin{pmatrix} l_1(v_1) & l_1(v_2) \\ l_2(v_1) & l_2(v_2) \end{pmatrix} \end{aligned}$$

Notation. We denote $\operatorname{Alt}^n(V; U)$ to be the set of maps $f : V^n \to U$ where f is alternating.

Proposition 21.3. For any $n \geqslant 2$, for all $f \in \operatorname{Alt}^n(V; U)$, there exists a unique linear map $\bar{f} : \bigwedge^n V \to U$ such that $\bar{f}(v_1 \wedge \cdots \wedge v_n) = f(v_1, \dots, v_n)$.

Proof. Since f is n-linear, there exists a unique linear map $\tilde{f} : V^{\otimes n} \to U$ such that $\tilde{f}(v_1 \otimes \cdots \otimes v_n) = f(v_1, \dots, v_n)$. Recall that $\bigwedge^n V = V^{\otimes n}/I_n$ where I_n is the intersection of $V^{\otimes n}$ and the ideal generated by $\{v \otimes v \mid v \in V\}$. Since f is alternating, then $\tilde{f}\Big|_{I_n} = 0$, so there exists a linear map $\bar{f} : \bigwedge^n V \to U$ such that $\bar{f}(v_1 \wedge \cdots \wedge v_n) = f(v_1, \dots, v_n)$ for all $v_1, \dots, v_n \in U$. Since $\{v_1 \wedge \cdots \wedge v_n \mid v_i \in V\}$ generates $\bigwedge^n V$, then $\bar{f}$ is unique. □

Lemma 21.4. Suppose $\{v_1, \dots, v_n\}$ is a basis of V, then $v_1 \wedge \cdots \wedge v_n \neq 0$, hence $\dim(\bigwedge^n V) = 1$ and $\bigwedge^n V \cong \mathbb{R}$.

Proof. Take the dual basis $\{v_1^*, \dots, v_n^*\}$, and consider

$$\begin{aligned} f : V^n &\to \mathbb{R} \\ (x_1, \dots, x_n) &\mapsto \det \begin{pmatrix} v_1^*(x_1) & \cdots & v_1^*(x_n) \\ \vdots & & \vdots \\ v_n^*(x_1) & \cdots & v_n^*(x_n) \end{pmatrix} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n v_i^*(x_{\sigma(i)}). \end{aligned}$$

Therefore, f is alternating. Hence, there exists a unique

$$\begin{aligned} \bar{f} : \bigwedge^n V &\to \mathbb{R} \\ x_1 \wedge \cdots \wedge x_n &\mapsto \det(v_i^*(x_j)) \end{aligned}$$

and such that $\bar{f}(v_1 \wedge \cdots \wedge v_n) = \det(\operatorname{diag}(1, \dots, 1)) = 1$, hence $v_1 \wedge \cdots \wedge v_n \neq 0$. □

Corollary 21.5. Let $\{v_1, \dots, v_n\}$ be a basis of V, then for any $1 \leqslant k \leqslant n$, the generating set $\mathcal{B} = \{v_{i_1} \wedge \cdots \wedge v_{i_k} \mid 1 \leqslant i_1 < \cdots < i_k \leqslant n\}$ is a basis of $\bigwedge^k V$.

Proof. We know $\mathcal{B}$ spans $\bigwedge^k V$. Suppose $\sum_{i_1 < \cdots < i_k} a_{i_1, \dots, i_k} v_{i_1} \wedge \cdots \wedge v_{i_k} = 0$. Fix $1 \leqslant i_1^\circ < \cdots < i_k^\circ \leqslant n$. Let $1 \leqslant j_{k+1} < \cdots < j_n \leqslant n$ denote the complementary set of the indices, i.e., $\{i_1^\circ, \dots, i_k^\circ\} \cap \{j_{k+1}, \dots, j_n\} = \varnothing$, then for all $1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant n$, we have

$$v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge v_{j_{k+1}} \wedge \cdots \wedge v_{j_n} = \begin{cases} 0, & (i_1, \dots, i_k) \neq (i_1^\circ, \dots, i_k^\circ) \\ \pm v_1 \wedge \cdots \wedge v^n, & (i_1, \dots, i_k) = (i_1^\circ, \dots, i_k^\circ) \end{cases}$$

Therefore, $\left(\sum_{i_1 < \cdots < i_k} a_{i_1, \dots, i_k} v_{i_1} \wedge \cdots \wedge v_{i_k} \right) \wedge (v_{j_{k+1}} \wedge \cdots \wedge v_{j_n}) = \pm a_{i_1^\circ, \dots, i_k^\circ} v_1 \wedge \cdots \wedge v_k$, but $v_1 \wedge \cdots \wedge v_k \neq 0$, so $a_{i_1^\circ, \dots, i_k^\circ} = 0$ since $\{v_1 \wedge \cdots \wedge v_n\}$ is a basis. □

Corollary 21.6. Suppose $\dim(V) = n$, then for all $1 \leqslant k \leqslant n$, $\dim(\bigwedge^k V) = \binom{n}{k}$. Consequently, $\dim(\bigwedge^* V) = 2^n$.

Lemma 21.7. Let $f : V \to W$ be a linear map, then there exists a unique $\bigwedge^*(f) : \bigwedge^* V \to \bigwedge^* W$ of graded commutative algebras so that for all k and for all $v_1, \ldots, v_k \in V$,

$$(\bigwedge^* f)(v_1 \wedge \cdots \wedge v_k) = f(v_1) \wedge \cdots \wedge f(v_k).$$

In particular, note that $\bigwedge^*(f)(\bigwedge^k V) \subseteq \bigwedge^k W$.

Proof. Note that $V \xrightarrow{f} W = \bigwedge^1 W \subseteq \bigwedge^* W$, and for any $v \in V$, $f(v) \wedge f(v) = 0$. Therefore, there exists a unique map $\bigwedge^* f : \bigwedge^* V \to \bigwedge^* W$ such that $\bigwedge^* f\big|_{\bigwedge^1 V} = f$. Moreover, for any k, and any $v_1, \ldots, v_k \in V$, we know

$$\begin{aligned}\bigwedge^* f(v_1 \wedge \cdots \wedge v_k) &= \bigwedge^* f(v_1) \wedge \cdots \wedge \bigwedge^* f(v_k) \\ &= f(v_1) \wedge \cdots \wedge f(v_K).\end{aligned}$$

□

Remark 21.8. Uniqueness of $\bigwedge^* f$ implies that if we have two linear maps

$$V \xrightarrow{f} W \xrightarrow{g} U$$

and then $\bigwedge^*(g \circ f) = \bigwedge^*(g) \circ \bigwedge^*(f)$. Moreover, $\bigwedge^*(\mathrm{id}_V) = \mathrm{id}_{\bigwedge^* V}$. In other words, there is a functor

$$\bigwedge^*(-) : \mathbf{Vect} \to \mathbf{CGA},$$

from the category of finite-dimensional real vector spaces with linear maps as morphisms, to the category of graded commutative algebras over $\mathbb{R}$.

Remark 21.9. The map $V \to \mathcal{T}(V)$ also extends to a functor

$$\mathcal{T}(-) : \mathbf{Vect} \to \mathbf{GAA}$$

from the category of finite-dimensional real vector spaces to the category of graded associative algebras. In particular, it sends $f : V \to W$ to $\mathcal{T} : \mathcal{T}(V) \to \mathcal{T}(W)$ that maps $v_1 \otimes \cdots \otimes v_n$ to $f(v_1) \otimes \cdots \otimes f(v_k)$ for all k and for all $v_1, \ldots, v_K \in V$.

Remark 21.10. For each $k \geqslant 0$, we also have functors

$$\bigwedge^k(-) : \mathbf{Vect} \to \mathbf{Vect}$$

that takes a linear map $f : V \to W$ and sends it to $\bigwedge^k f : \bigwedge^k V \to \bigwedge^k W$, as well as

$$(-)^{\otimes k} : \mathbf{Vect} \to \mathbf{Vect}$$

that sends $f : V \to W$ to $f^{\otimes k} : V^{\otimes k} \to W^{\otimes k}$.

Lemma 21.11. For any two finite-dimensional vector spaces V and U, for all k, we have an isomorphism

$$\begin{gathered}\mathrm{Hom}(\bigwedge^k V, U) \to \mathrm{Alt}^k(V; U) \\ (\varphi : \bigwedge^k V \to U) \mapsto (\varphi \circ i^{(k)} : V^k \to U)\end{gathered}$$

where

$$\begin{gathered}i^{(k)} : V^k \to \bigwedge^k V \\ (v_1, \ldots, v_k) \mapsto v_1 \wedge \cdots \wedge v_k.\end{gathered}$$

Proof. Same as Proposition 21.3. □

22 Oct 13, 2023

Recall. If $\{d_1, \dots, d_n\}$ is a basis of V, then for all $1 \leqslant k \leqslant n$, $\{\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k} \mid 1 \leqslant i_1 < \cdots < i_k \leqslant n$ is a basis$\}$ of $\bigwedge^k V$.

For vector spaces V and U, we have

$$\begin{aligned} i^{(k)} : V^k &\to \bigwedge^k V \\ (v_1, \dots, v_k) &\mapsto v_1 \wedge \cdots \wedge v_k, \end{aligned}$$

then

$$\begin{aligned} \mathrm{Hom}(\bigwedge^k V, U) &\to \mathrm{Alt}^k(V, U) \\ \varphi &\mapsto \varphi \circ i^{(k)} \end{aligned}$$

is an isomorphism. The inverse if $f \mapsto \bar{f}$ where $\bar{f}(v_1 \wedge \cdots \wedge v_k) = f(v_1, \dots, v_k)$ for all v_i's.

Remark 22.1. Lemma 21.11 says that $(\bigwedge^k V)^* = \mathrm{Hom}(\bigwedge^k V, \mathbb{R}) \cong \mathrm{Alt}^k(V, \mathbb{R})$.

Lemma 22.2. let V be a finite-dimensional vector space, then for all $1 \leqslant k \leqslant n$, we have

$$\bigwedge^k V^* \cong (\bigwedge^k V)^* \cong \mathrm{Alt}^k(V; \mathbb{R}).$$

Proof. Consider $\mathrm{Map}(V^k, \mathbb{R})$ be the set of all maps from V^k to $\mathbb{R}$. Note that the multilinear maps $\mathrm{Alt}^k(V, \mathbb{R}) \subseteq \mathrm{Mult}^k(V, \mathbb{R}) \subseteq \mathrm{Map}(V^k, \mathbb{R})$. Consider

$$\begin{aligned} \varphi : (V^*)^k &\to \mathrm{Map}(V^k, \mathbb{R}) \\ (\varphi(l_1, \dots, l_k))(v_1, \dots, v_k) &= l_1(v_1) \cdots l_k(v_k) =: \det(l_i(v_j)). \end{aligned}$$

for all $v_1, \dots, v_k \in V$ and $l_1, \dots, l_k \in V^*$. For fixed $l_1, \dots, l_k$, $\varphi(l_1, \dots, l_k)$ is k-linear and alternating, so $\varphi(l_1, \dots, l_k) \in \mathrm{Alt}^k(V, \mathbb{R})$. Thus we have

$$\begin{aligned} \varphi : (V^*)^k &\to \mathrm{Alt}^k(V; \mathbb{R}) \\ (l_1, \dots, l_k) &\mapsto ((v_1, \dots, v_k) \mapsto \det(l_i(v_j))) \end{aligned}$$

Since φ is k-linear in $l_1, \dots, l_k$, therefore we have another map

$$\begin{aligned} \tilde{\varphi} : (V^*)^{\otimes k} &\to \mathrm{Alt}^k(V, \mathbb{R}) \\ (\tilde{\varphi}(l_1 \otimes \cdots \otimes l_k))(v_1, \dots, v_k) &= \det(l_i(v_j)). \end{aligned}$$

Note that $\tilde{\varphi}$ vanishes if any two l_i's are repeated, so there is a unique map

$$\begin{aligned} \bar{\varphi} : \bigwedge^k V^k &\to \mathrm{Alt}^k(V, \mathbb{R}) \\ \bar{\varphi}(l_1 \wedge \cdots \wedge l_k)(v_1, \dots, v_k) &= \det(l_i(v_j)). \end{aligned}$$

Composing with the isomorphism $\mathrm{Alt}^k(V, \mathbb{R}) \to (\bigwedge^k V)^*$, we get

$$\begin{aligned} \psi : \bigwedge^k (V^k) &\to (\bigwedge^k V)^* \\ (\psi(l_1 \wedge \cdots \wedge l_k))(v_1 \wedge \cdots \wedge v_k) &= \det(l_i(v_j)). \end{aligned}$$

It remains to show that ψ is an isomorphism. Pick a basis $\{\alpha_1, \ldots, \alpha_n\}$ of V, with dual basis $\{\alpha_1^*, \ldots, \alpha_n^*\}$ of V^*. Let $\mathcal{A} = \{\alpha_{j_1}^* \wedge \cdots \wedge \alpha_{j_k}^* \mid 1 \leqslant j_1 < \cdots < j_k \leqslant n\}$ and let $\mathcal{B} = \{\alpha_{i_1} \wedge \cdots \alpha_{i_k} \mid 1 \leqslant i_1 < \cdots < i_k \leqslant n\}$ of $\bigwedge^k(V)$. We have

$$\begin{aligned}(\psi(\alpha_{j_1}^* \wedge \cdots \wedge \alpha_{j_k}^*))(\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k}) &= \det(\alpha_{j_r}^*(\alpha_{i_s}))_{s,r} \\ &= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \prod_{r=1}^{k} \alpha_{j_r}^*(\alpha_{i_{\sigma(r)}}) \\ &= \begin{cases} 1, & (j_1, \ldots, j_k) = (l_1, \ldots, l_k) \\ 0, & \text{otherwise} \end{cases}\end{aligned}$$

Hence, ψ is an isomorphism. □

Remark 22.3. For $\alpha \in \operatorname{Alt}^k(V, \mathbb{R})$ and $\beta \in \operatorname{Alt}^l(V, \mathbb{R})$, then we have

$$\begin{aligned}\alpha\beta : V^k \times V^l &\to \mathbb{R} \\ v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+l} &\mapsto \alpha(v_1, \ldots, v_k)\beta(v_{k+1}, \ldots, v_{k+l})\end{aligned}$$

which is $k + l$-linear but not alternating.

Example 22.4. For $k = l = 1$, $\operatorname{Alt}^1(V, \mathbb{R}) = V^*$, so $(\alpha \cdot \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) \neq -\alpha(v_2)\beta(v_1)$. On the other hand, $\operatorname{Alt}^k(V, \mathbb{R}) \cong \bigwedge^k(V^*)$, so $\bigwedge_{k \geqslant 0} \operatorname{Alt}^k(V, \mathbb{R}) \cong \bigoplus_{k=0}^{\infty} \bigwedge^k(V^*) = \bigwedge^*(V^*)$ which is a graded commutative algebra. (We set $\operatorname{Alt}^0(V, \mathbb{R}) = \mathbb{R}$.) Therefore, there is a graded commutative algebra structure on the direct sum of alternating maps.

Remark 22.5. For each $n \geqslant 1$, there exists a projection

$$\begin{aligned}\pi : \operatorname{Mult}^n(V, \mathbb{R}) &\to \operatorname{Alt}^n(V, \mathbb{R}) \\ (\pi(\gamma))(v_1, \ldots, v_n) &= \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\gamma(v_{\gamma(1)}, \ldots, v_{\sigma(n)}).\end{aligned}$$

Therefore, we could have defined a multiplication

$$\begin{aligned}\wedge : \operatorname{Alt}^k(V, \mathbb{R}) \times \operatorname{Alt}^l(V, \mathbb{R}) &\to \operatorname{Alt}^k(V, \mathbb{R}) \\ \alpha \wedge \beta &= \pi(\alpha\beta).\end{aligned}$$

The issue is, we do not have associativity: $\pi(\pi(\alpha\beta)\gamma) \neq \pi(\alpha\pi(\beta\gamma))$.

Note that for any k,

$$\begin{aligned}\bigwedge^k(V^*) &\to \operatorname{Alt}^k(V, \mathbb{R}) \\ l_1 \wedge \cdots \wedge l_k &\mapsto k!\pi(k_1(-)l_2(-) \cdots l_k(-))\end{aligned}$$

for all $l_1, \ldots, l_k \in V^*$. One should be cautious because

$$\begin{aligned}\bigwedge^k(V^*) &\to \operatorname{Alt}^k(V, \mathbb{R}) \\ l_1 \wedge \cdots \wedge l_k &\mapsto \pi(k_1(-)l_2(-) \cdots l_k(-))\end{aligned}$$

is also used in literature.

We now want to define the cotangent bundle, but first we need to redefine the charts.

Recall. Recall the construction of charts on TM is as follows:

- Given a chart $\varphi = (x_1, \dots, x_m) : U \to \mathbb{R}^m$ on M, we define
$$\begin{aligned}\tilde{\varphi} : TU = TM &\to \mathbb{R}^m \times \mathbb{R}^m \\ \tilde{\varphi}(q, v) &= (x_1(q), \dots, x_n(q), (dx_1)_q(v), \dots, (dx_n)_q(v))\end{aligned}$$
Given another chart $\psi = (y_1, \dots, y_m) \to \mathbb{R}^n$, we have
$$\begin{aligned}F : \tilde{\psi} \circ (\tilde{\varphi}|_{\varphi(U \cap V) \times \mathbb{R}^m})^{-1} : \varphi(U \cap V) \times \mathbb{R}^m &\to \psi(U \cap V) \times \mathbb{R}^m \\ (a_1, \dots, a_m, w_1, \dots, w_m) &\mapsto (\psi(\varphi^{-1}(a_1, \dots, a_m)), D(\psi \circ \varphi^{-1})(a) \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix})\end{aligned}$$

- From a better point of view, let $\varphi = (x_1, \dots, x_m)$, then the map $\tilde{\varphi} : TU \to \varphi(U) \times \mathbb{R}^m$ "is" $T\varphi : TU \to T(\varphi(U))$. To see this, for all $f \in C^\infty(U)$, we have $\frac{\partial}{\partial x_i}\Big|_q f = \frac{\partial}{\partial r_i}\Big|_{\varphi(q)} (f \circ \varphi^{-1}) = ((T_{\varphi(q)}\varphi^{-1}) \left(\frac{\partial}{\partial r_i}\Big|_{\varphi(q)}\right))f$, that is, $T\varphi^{-1}\left(\frac{\partial}{\partial r_i}\right) = \frac{\partial}{\partial x_i}$, dropping the basepoint. Therefore, $T\varphi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial r_i}$. Hence, $(T_q\varphi)(\sum v_i \frac{\partial}{\partial x_i}\Big|_q) = \sum v_i \frac{\partial}{\partial r_i}\Big|_{\varphi(q)}$, From this point of view, $T\varphi : TU \to T(\varphi(U))$ is $(q, \sum v_i \frac{\partial}{\partial x_i}\Big|_q) \mapsto (\varphi(q), \sum v_i \frac{\partial}{\partial r_i}\Big|_{\varphi(q)})$. Now identify
$$\begin{aligned}T\varphi(U) &\cong \varphi(U) \times \mathbb{R}^m \\ (r_1, \dots, r_m, \sum v_i \frac{\partial}{\partial r_i}) &\mapsto (r_1, \dots, r_m, v_1, \dots, v_m),\end{aligned}$$
and given this we can write $\tilde{\psi}\tilde{\varphi}^{-1} : T\psi \circ (T\varphi)^{-1} = T\psi \circ T\varphi^{-1} = T(\psi \circ \varphi^{-1})$ by the functoriality.

Definition 22.6. The *cotangent bundle* is defined by $T^*M = \coprod_{q \in M} (T_qM)^*$.