# MATH 540 Notes

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## 1 Abstract Measure Theory

### 1.1 Introduction

**Definition 1.1.** Let $X$ be an (non-empty) underlying space we are working over. We denote $\mathcal{P}(X)$ to be the power set of $X$, i.e., the set of all subsets of $X$.

**Example 1.2.** Let $X = \{1, 2\}$, then $\mathcal{P}(X) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$.

**Remark 1.3.** If $X$ is a finite set of size $n$, then $\mathcal{P}(X)$ is a finite set of size $2^n$.

We will consider a subcollection $\mathcal{A}$ of subsets of $X$, i.e., a subset of the power set. We will try to define this as an algebra. Note that an algebra is just a ring with a module structure with respect to some other ring.

**Definition 1.4.** $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra on $X$ if it is

a. closed under finite union, i.e., given $E_1, E_2 \in \mathcal{A}$, then $E_1 \cup E_2 \in \mathcal{A}$, and

b. closed under complements, i.e., if $E \in \mathcal{A}$, then the complement $E^c \in \mathcal{A}$ as well.

**Remark 1.5.** An algebra $\mathcal{A}$ would be closed under finite intersection. Indeed, for any $E_1, E_2 \in \mathcal{A}$, we have $E_1 \cap E_2 \in \mathcal{A}$ if and only if $(E_1 \cap E_2)^c \in \mathcal{A}$, if and only if $E_1^c \cup E_2^c \in \mathcal{A}$, which is true by definition.

**Lemma 1.6.** If $\mathcal{A}$ is an non-empty algebra on $X$, then $\varnothing \in \mathcal{A}$ and $X \in \mathcal{A}$.

*Proof.* Since $\mathcal{A}$ is non-empty, take $E \in \mathcal{A}$, then $\varnothing = E \cap E^c \in \mathcal{A}$ as well. Also, $X = E \cup E^c \in \mathcal{A}$. $\square$

**Example 1.7.** Let $X$ be a set, and let $\mathcal{A} = \{\varnothing, X\} \subseteq \mathcal{P}(X)$. It is easy to verify that $\mathcal{A}$ is an algebra.

**Definition 1.8.** Let $\varnothing \neq \mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra, then we say $\mathcal{A}$ is a $\sigma$-algebra on $X$ if

a. closed under countable union, i.e., if $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, then $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$;

b. if $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$.

**Lemma 1.9.** If $\mathcal{A} \neq \varnothing$ is a $\sigma$-algebra on $X$, then $\{\varnothing, X\} \subseteq \mathcal{A}$ is a $\sigma$-algebra.

**Example 1.10.** Let $X$ be an uncountable set, let $\mathcal{A} = \{E \subseteq X : E \text{ is countable or } E^c \text{ is countable}\}$, then $\mathcal{A}$ is a $\sigma$-algebra on $X$.

**Theorem 1.11.** Suppose a non-empty algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ such that,

- if $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, and $E_j$'s are pairwise disjoint, then $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$,

then $\mathcal{A}$ is a $\sigma$-algebra on $X$.

*Proof.* Take $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, we will show that $\bigcup\limits_{j=1}^{\infty} E_j \in \mathcal{A}$. To do this, we will rearrange the sets. Let $F_1 = E_1$, let $F_2 = E_2 \backslash E_1$, let $F_3 = E_3 \backslash (E_1 \cup E_2)$, and so on, such that let $F_k = E_k \backslash \bigcup\limits_{i=1}^{k-1} E_i$. We note

$$\begin{aligned} F_k &= E_k \cap \left( \bigcup_{j=1}^{k-1} E_j \right)^c \\ &= E_k \cap \left( \bigcap_{j=1}^{k-1} E_j^c \right) \in \mathcal{A}. \end{aligned}$$

One can also verify that $\bigcup\limits_{j=1}^{\infty} E_j = \bigcup\limits_{k=1}^{\infty} F_k$, and that $F_k$'s are disjoint from the definition. □

**Definition 1.12.** Let $X$ be a non-empty space. A topology on $X$ is a family $\mathcal{F}$ of subsets of $X$ satisfying the following conditions:

i. $\varnothing, X \in \mathcal{F}$;

ii. $\mathcal{F}$ is closed under arbitrary union;

iii. $\mathcal{F}$ is closed under finite intersection.

Every member of $\mathcal{F}$ is now called an open subset of $X$. A complement of an open subset of $X$ is called a closed subset.

**Definition 1.13.** Let $\mathcal{A}_1, \mathcal{A}_2$ be $\sigma$-algebras. We say $\mathcal{A}_1$ is smaller than $\mathcal{A}_2$ if $\mathcal{A}_1 \subseteq \mathcal{A}_2$, and equivalently $\mathcal{A}_2$ is larger than $\mathcal{A}_1$.

**Definition 1.14.** Let $\mathcal{F}$ be a family of subsets of $X$, the smallest $\sigma$-algebra containing $\mathcal{F}$ is called the $\sigma$-algebra generated by $\mathcal{F}$. This is denoted by $\mathcal{M}(\mathcal{F})$.

**Lemma 1.15.** Let $\mathcal{F}$ be a family of subsets of $X$. Suppose $\mathcal{F} \subseteq \mathcal{A}$ where $\mathcal{A}$ is a $\sigma$-algebra, then $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{A}$.

*Proof.* Obvious. □

**Definition 1.16.** Let $\mathcal{F}$ be a topology on $X$, then we say $(X, \mathcal{F})$ is a topological space. We say $\mathcal{M}(\mathcal{F})$ is the Borel $\sigma$-algebra on $X$, denoted by $\mathcal{B}_X = \mathcal{B}_{X,\mathcal{F}}$. Any member of $\mathcal{B}_X$ is called a Borel set.

**Example 1.17.** Let $X = \mathbb{R}$, we denote the corresponding Borel $\sigma$-algebra to be $\mathcal{B}_{\mathbb{R}}$.

**Definition 1.18.** A $G_\delta$-set is a countable intersection of open subsets of $X$. A $F_\sigma$-set is a countable union of closed subsets of $X$.

**Theorem 1.19.** Both $G_\delta$-sets and $F_\sigma$-sets are Borel sets, that is, $G_\delta, F_\sigma \subseteq \mathcal{B}_X$.

*Proof.* We will prove that any $G_\delta$-set $E$ is a Borel set, and similarly any $F_\sigma$-set is a Borel set. By definition $E = \bigcap\limits_{j=1}^{\infty} O_j$, where each $O_j$ is an open subset. To show $E \in \mathcal{B}_X$, we show that $E^c \in \mathcal{B}_X$. Note that $E^c = \left( \bigcap\limits_{j=1}^{\infty} O_j \right)^c = \bigcup\limits_{j=1}^{\infty} O_j^c$. Since $O_j \in \mathcal{B}_X$ for all $j$, then $O_j^c \in \mathcal{B}_X$ as well. Therefore, $E^c \in \mathcal{B}_X$ since a $\sigma$-algebra $\mathcal{B}_X$ is closed under countable unions. □

**Definition 1.20.** Let $X_1, \ldots, X_n$ be non-empty spaces. The product space is $\prod\limits_{j=1}^{n} X_j$. Define $\pi_j : \prod\limits_{i=1}^{n} X_i \to X_j$ by $\pi_j(x_1, \ldots, x_n) = x_j$. Let $\mathcal{A}_j$ be a $\sigma$-algebra on $X_j$, the product $\sigma$-algebra on $\prod\limits_{i=1}^{n} X_j$ is the $\sigma$-algebra generated by $\{\pi_j^{-1}(E_j) : E_j \in \mathcal{A}_j \; \forall j \in \{1, \ldots, n\}\}$. The product $\sigma$-algebra is denoted by $\bigotimes\limits_{j=1}^{n} \mathcal{A}_j = \prod\limits_{j=1}^{n} \mathcal{A}_j$.

**Example 1.21.** $\mathcal{B}_{\mathbb{R}^n} = \bigotimes\limits_{j=1}^{n} \mathcal{B}_{\mathbb{R}}$.

## 1.2 Measures

**Definition 1.22.** Let $\mathcal{A}$ be a $\sigma$-algebra on $X$. A measure $\mu$ on $X$ and $\mathcal{A}$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that

a. $\mu(\varnothing) = 0$;

b. if $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$ and $E_j$'s are disjoint, then $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$.

We then say $(X, \mathcal{A})$ is a measureable space. A measureable space is a triple $(X, \mathcal{A}, \mu)$ with measure $\mu$ specified.

**Definition 1.23.** Let $\mu$ be a measure on $(X, \mathcal{A})$.

1. If $\mu(X) < \infty$, then we say $\mu$ is a finite measure. In particular, if $\mu(X) = 1$, this is a probability measure.
2. If $X = \bigcup_{j=1}^{\infty} E_j$ such that $\mu(E_j) < \infty$ for all $j \in \mathbb{N}$, then we say $\mu$ is $\sigma$-finite.
3. If for all $E \in \mathcal{A}$ with $\mu(E) = \infty$, there is $F \in \mathcal{A}$ such that $F \subseteq E$ and $0 < \mu(F) < \infty$, then we say $\mu$ is semi-finite.

**Remark 1.24.** A $\sigma$-finite measure is semi-finite. However, the converse is not true.

**Example 1.25.** Let $f : X \to [0, \infty]$ be a function. For any $E \subseteq \mathcal{P}(E)$, we can define a measure $\mu(E) = \sum_{x \in E} f(x)$. Note that the summation makes sense only when $E$ is finite. In case $E$ is infinite, we should define $\sum_{x \in E} f(x) = \sup\{\sum_{x \in F} f(x) : F \subseteq E \text{ for finite } F\}$. Let $\mu$ be a measure on $\mathcal{P}(X)$.

- If $f(x) \equiv 1$ for all $x \in X$, then $\mu(E) = \sum_{x \in E} 1 = \mathrm{Card}(E)$. In this case, $\mu$ is called a counting measure.
- Suppose $x_0 \in X$ is fixed. Define

  $$f(x) = \begin{cases} 1, & \text{if } x = x_0 \\ 0, & \text{if } x \neq x_0 \end{cases}$$

  then for any $E \in \mathcal{P}(X)$,

  $$\mu(E) = \begin{cases} 1, & \text{if } x_0 \in E \\ 0, & \text{if } x_0 \notin E \end{cases}$$

  This is called the Dirac measure of $x_0$.

**Definition 1.26.** Let $(X, \mathcal{A}, \mu)$ be a measure space. A set $E \subseteq \mathcal{A}$ is called a null set if $\mu(E) = 0$.

If a statement about points $x \in X$ is true except for null sets, then we say the statement is true almost everywhere.

**Example 1.27.** Suppose $f(x) \leqslant 1$ for all $x \in X$, then we say $f$ is bounded above by 1 everywhere. If we want to weaken this statement, we can say $f(x) \leqslant 1$ almost everywhere $x \in X$, which is true if and only if $\mu(\{x \in X : f(x) > 1\} = 0$.

**Theorem 1.28.** Let $E, F \in \mathcal{A}$ be such that $E \subseteq F$, then $\mu(E) \leqslant \mu(F)$.

*Proof.* We can write $F = E \cup (E \backslash F)$, then

$$\begin{aligned} \mu(F) &= \mu(E) + \mu(F \backslash E) \\ &\geqslant \mu(E) \end{aligned}$$

since $\mu(F \backslash E) \geqslant 0$. □

**Theorem 1.29** (Sub-additivity)**.** Let $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, then $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leqslant \sum_{j=1}^{\infty} \mu(E_j)$.

*Proof.* Set $F_1 = E_1$ and let $F_k = E_k \backslash \left( \bigcup_{j=1}^{k-1} E_j \right)$ be defined inductively, then $\bigcup_{k \in \mathbb{N}} F_k = \bigcup_{j \in \mathbb{N}} E_j$. Since $F_k$'s are disjoint, we have

$$\begin{aligned} \mu\left( \bigcup_{j \in \mathbb{N}} E_j \right) &= \mu\left( \bigcup_{k \in \mathbb{N}} F_k \right) \\ &= \sum_{k=1}^{\infty} \mu(F_k) \\ &= \sum_{k=1}^{\infty} \mu(E_k) \\ &= \sum_{j=1}^{\infty} \mu(E_j) \end{aligned}$$

by Theorem 1.28. □

**Theorem 1.30.** Let $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$.

a. (Continuity from below): If $E_1 \subseteq E_2 \subseteq \cdots E_j \subseteq \cdots$ for all $j$, then $\mu\left( \bigcup_{j=1}^{\infty} E_j \right) = \lim_{j \to \infty} \mu(E_j)$.

b. (Continuity from above): If $E_1 \supseteq E_2 \supseteq \cdots E_j \supseteq \cdots$ for all $j \in \mathbb{N}$, then $\mu\left( \bigcap_{j=1}^{\infty} E_j \right) = \lim_{j \to \infty} \mu(E_j)$ if $\mu(E_1) < \infty$.

In particular, the limits on the right exist on $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.

**Example 1.31.** Let $\mu$ be the counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. For each $j \in \mathbb{N}$, we define $E_j = \{n \in \mathbb{N} : n > j\}$. Therefore $E_1 \supseteq E_2 \supseteq \cdots$ is a decreasing sequence of sets. Note that $\mu(E_1) = \mu(\{n \in \mathbb{N}\}) = \mathbb{N} = \infty$, and $\lim_{j \to \infty} \mu(E_j) = \lim_{j \to \infty} \infty = \infty$, but $\mu\left( \bigcap_{j=1}^{\infty} E_j \right) = \mu(\varnothing) = 0$.

*Proof.* a. Set $E_0 = \varnothing$. Now

$$\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} (E_j \backslash E_{j-1})$$

and therefore

$$\begin{aligned} \mu\left( \bigcup_{j=1}^{\infty} E_j \right) &= \mu\left( \bigcup_{j=1}^{\infty} (E_j \backslash E_{j-1}) \right) \\ &= \sum_{j=1}^{\infty} \mu(E_j \backslash E_{j-1}) \\ &= \lim_{k \to \infty} \sum_{j=1}^{k} \mu(E_j \backslash E_{j-1}) \\ &= \lim_{k \to \infty} \mu\left( \bigcup_{j=1}^{k} E_j \backslash E_{j-1} \right) \\ &= \lim_{k \to \infty} \mu(E_k) \\ &= \lim_{j \to \infty} \mu(E_j). \end{aligned}$$

b. For any $j \in \mathbb{N}$, set $F_j = E_1 \backslash E_j$. Note that $F_j \subseteq F_{j+1}$ since $E_j \supseteq E_{j-1}$. This is now an increasing sequence as in part a. By part a., we know $\mu\left(\bigcup_{j=1}^{\infty} F_j\right) = \lim_{j\to\infty} \mu(F_j)$. Now note that

$$\begin{aligned}
\bigcup_{j=1}^{\infty} F_j &= \bigcup_{j=1}^{\infty}(E_1 \backslash E_j) \\
&= \bigcup_{j=1}^{\infty}(E_1 \cap E_j^c) \\
&= E_1 \cap \bigcup_{j=1}^{\infty} E_j^c \\
&= E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j\right)^c \\
&= \left(\left(E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j\right)^c\right) \cup \left(\bigcap_{j=1}^{\infty} E_j\right)\right) \cap \left(\bigcap_{j=1}^{\infty} E_j\right)^c \\
&= \left(E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j\right)^c\right) \cup \left(\bigcap_{j=1}^{\infty} E_j\right).
\end{aligned}$$

Note that $E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j\right)^c$ and $\bigcap_{j=1}^{\infty} E_j$ are disjoint, therefore by property of measure we have

$$\begin{aligned}
\mu(E_1) &= \mu\left(E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j\right)^c\right) + \mu\left(\bigcap_{j=1}^{\infty} E_j\right) \\
&= \mu\left(\bigcup_{j=1}^{\infty} F_j\right) + \mu\left(\bigcap_{j=1}^{\infty} E_j\right) \\
&= \lim_{j\to\infty} \mu(F_j) + \mu\left(\bigcap_{j=1}^{\infty} E_j\right).
\end{aligned}$$

Recall that $F_j = E_1 \backslash E_j$ for all $j$, therefore $E_1 = F_j \cup F_j^c = F_j \cup E_j$, where $F_j$ and $E_j$ are disjoint, therefore $\mu(E_1) = \mu(F_j) + \mu(E_j)$. Since $\mu(E_1) < \infty$, and $F_j$ is a subset of $E_1$ and hence also a real number, then $\mu(E_1)$ is a sum of two real numbers. Therefore, we have $\mu(E_1) - \mu(E_j) = \mu(F_j)$. With this, we have

$$\begin{aligned}
\mu(E_1) &= \lim_{j\to\infty} (\mu(E_1) - \mu(E_j)) + \mu\left(\bigcap_{j=1}^{\infty} E_j\right) \\
&= \mu(E_1) - \lim_{j\to\infty} (\mu(E_j)) + \mu\left(\bigcap_{j=1}^{\infty} E_j\right).
\end{aligned}$$

In particular, we get

$$\lim_{j\to\infty} (\mu(E_j)) = \mu\left(\bigcap_{j=1}^{\infty} E_j\right).$$

□

## 1.3 Outer Measure

**Definition 1.32.** An outer measure $\mu^*$ on $X$ (or $\mathcal{P}(X)$) is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that

i. $\mu^*(\varnothing) = 0$,

ii. $\mu^*(A) \leqslant \mu^*(B)$ for all $A \subseteq B \subseteq X$,

iii. $\sigma$-subaddivity: $\mu^*\left(\bigcup\limits_{j=1}^{\infty} A_j\right) \leqslant \sum\limits_{j=1}^{\infty} \mu^*(A_j)$.

**Example 1.33.** Let $\rho : \mathcal{A} \to [0, \infty]$ be such that $\rho(\varnothing) = 0$, where $\mathcal{A} \subseteq \mathcal{P}(X)$ is a subcollection (but not necessarily an algebra) such that $\varnothing, X \in \mathcal{A}$.

For all $A \in \mathcal{P}(X)$, i.e., $A \subseteq X$, we define

$$\mu^*(A) = \inf\left\{\sum_{j=1}^{\infty} \rho(E_j) : E_j \in \mathcal{A} \text{ and } A \subseteq \bigcup_{j=1}^{\infty} E_j\right\}.$$

**Theorem 1.34.** $\mu^*$ defined in Example 1.33 is an outer measure.

*Proof.* i. Let $E_j = \varnothing$ for all $j \in \mathbb{N}$, then $\varnothing \subseteq \bigcup\limits_{j=1}^{\infty} E_j$, and so

$$\sum_{j=1}^{\infty} \rho(E_j) = \sum_{j=1}^{\infty} \rho(\varnothing) = \sum_{j=1}^{\infty} 0 = 0$$

and therefore $\mu^*(\varnothing) = 0$.

ii. Let $A \subseteq B \subseteq X$. If $B \subseteq \bigcup\limits_{j=1}^{\infty} E_j$, we have $A \subseteq \bigcup\limits_{j=1}^{\infty} E_j$, then

$$\left\{\sum_{j=1}^{\infty} \rho(E_j) : E_j \in \mathcal{A}, B \subseteq \bigcup_{j=1}^{\infty} E_j\right\} \subseteq \left\{\sum_{j=1}^{\infty} \rho(E_j) : E_j \in \mathcal{A}, A \subseteq \bigcup_{j=1}^{\infty} E_j\right\}.$$

In particular, given subsets $S_1 \subseteq S_2$, then $\inf S_2 \leqslant \inf S_1$ and $\sup S_1 \leqslant \sup S_2$. This implies $\mu^*(A) \leqslant \mu^*(B)$.

iii. We want to show $\mu^*\left(\bigcup\limits_{j=1}^{\infty} A_j\right) \leqslant \sum\limits_{j=1}^{\infty} \mu^*(A_j)$. Now for any $j \in \mathbb{N}$, we have

$$\mu^*(A_j) = \inf\left\{\sum_{k=1}^{\infty} \rho(E_k) : E_k \in \mathcal{A} \text{ and } A_j \subseteq \bigcup_{k=1}^{\infty} E_k\right\}.$$

For any $\varepsilon > 0$, we note that $\mu^*(A_j) + \varepsilon \cdot 2^{-j}$ is not a lower bound of $\left\{\sum\limits_{k=1}^{\infty} \rho(E_k) : E_k \in \mathcal{A} \text{ and } A_j \subseteq \bigcup\limits_{k=1}^{\infty} E_k\right\}$. Then there exists $E_k^{(j)} \in \mathcal{A}$ for $k \in \mathbb{N}$ such that $A_j \subseteq \bigcup\limits_{k=1}^{\infty} E_k^{(j)}$ and $\sum\limits_{k=1}^{\infty} \rho(E_k^{(j)}) \leqslant \mu^*(A_j) + \varepsilon \cdot 2^{-j}$. Summing with respec to $j$, we get

$$\begin{aligned}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \rho(E_k^{(j)}) &\leqslant \sum_{j=1}^{\infty} \mu^*(A_j) + \sum_{j=1}^{\infty} \varepsilon \cdot 2^{-j} \\ &= \sum_{j=1}^{\infty} \mu^*(A_j) + \varepsilon.\end{aligned}$$

Note that

$$\bigcup_{j=1}^{\infty} A_j \subseteq \bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty} E_k^{(j)}$$

is a countable union of subsets of $\mathcal{A}$. We will calculate the value over $\mu^*$. By definition of $\mu^*$, we have

$$\begin{aligned}\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) &\leqslant \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\rho(E_k^{(j)})\\ &\leqslant \sum_{j=1}^{\infty}\mu^*(A_j)+\varepsilon.\end{aligned}$$

Since this is true for all $\varepsilon > 0$, then take $\varepsilon \to 0$, we are done.

□

**Definition 1.35.** Let $\mu^*$ be an outer measure on $(X, \mathcal{P}(X))$. A set $A \subseteq X$ is called $\mu^*$-measurable if $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for all $E \subseteq X$.

**Remark 1.36.** First note that $\mu^*(E) = \mu^*((E \cap A) \cup (E \cap A^c))$, therefore $\mu^*(E) \leqslant \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for all $E \subseteq X$.

**Theorem 1.37** (Fundamental Theorem of Measure Theory)**.** Let $\mu^*$ be an outer measure on $X$. Let $\mathcal{A}$ be the collection of all $\mu^*$-measurable set, then $\mathcal{A}$ is a $\sigma$-algebra, and $\mu^*|_{\mathcal{A}}$ is a measure on $\mathcal{A}$, i.e., $(X, \mathcal{A}, \mu^*)$ is a measure space.

*Proof.* We first prove that $\mathcal{A}$ is an algebra. To see $\mathcal{A}$ is closed under complement, we have $A \in \mathcal{A}$ if and only if $A^c \in \mathcal{A}$. by the definition of measurable set. To show $\mathcal{A}$ is closed under finite union, suppose $A, B \in \mathcal{A}$, and we want to show $A \cup B \in \mathcal{A}$, which is true if and only if $\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$ for all $E \subseteq X$, hence it suffices to show that $\mu^*(E) \geqslant \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$. We have

$$\begin{aligned}\mu^*(E \cap (A \cup B)) &= \mu^*(E \cap (A \cup B) \cap A) + \mu^*(E \cap (A \cup B) \cap A^c)\\ &= \mu^*(E \cap A) + \mu^*(E \cap B \cap A^c)\end{aligned}$$

and

$$\begin{aligned}\mu^*(E \cap (A \cup B)^c) &= \mu^*(E \cap (A \cup B)^c \cap A) + \mu^*(E \cap (A \cup B)^c \cap A^c)\\ &= \mu^*(\varnothing) + \mu^*(E \cap A^c \cap B^c)\\ &= \mu^*(E \cap A^c \cap B^c).\end{aligned}$$

Therefore

$$\begin{aligned}\mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c) &= \mu^*(E \cap A) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c)\\ &= \mu^*(E \cap A) + \mu^*(E \cap A^c)\\ &= \mu^*(E)\end{aligned}$$

where the last two steps follow from the fact that $A, B \in \mathcal{A}$ are $\mu^*$-measurable. Therefore, $\mathcal{A}$ is an algebra. We now want to show that it is a $\sigma$-algebra. It suffices to prove that $\mathcal{A}$ is closed under disjoint $\sigma$-unions. Let $A_j \in \mathcal{A}$ for all $j \in \mathbb{N}$ where they are pairwise disjoint, and we want to show that $\bigcup\limits_{j=1}^{\infty} A_j \in \mathcal{A}$. That is,

$$\mu^*(E) = \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)\right) + \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)^c\right)$$

for all $E \subseteq X$.

**Lemma 1.38.** For a pairwise disjoint family $A_1, \ldots, A_n \in \mathcal{A}$,

$$\mu^*\left(E \cap \bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n}\mu^*(E \cap A_j).$$

*Subproof.* We proceed by induction. For $n = 1$, this is obviously true. Now suppose $n > 1$. To simplify the notation, let $B_n = \bigcup_{j=1}^{n} A_j$, and use the convention that $B_0 = \varnothing$. Now

$$\begin{aligned}
\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\
&= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) \\
&= \sum_{i=1}^{n} (E \cap A_i) + \mu^*(E \cap B_0) \\
&= \sum_{i=1}^{n} (E \cap A_i) \\
&= \sum_{i=1}^{n} (E \cap A_i)
\end{aligned}$$

for all $n \in \mathbb{N}$. This finishes the proof. ■

Now for any $E \subseteq X$, we have

$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \\
&= \sum_{j=1}^{n} \mu^*(E \cap A_j) + \mu^*(E \cap B_n^c) \\
&\geqslant \sum_{j=1}^{n} \mu^*(E \cap A_j) + \mu^* \left( E \cap \left( \bigcup_{j=1}^{\infty} A_j \right)^c \right)
\end{aligned}$$

since $B_n = \bigcup_{j=1}^{n} A_j \subseteq \bigcup_{j=1}^{\infty} A_j$. Now take $n \to \infty$, we get

$$\begin{aligned}
\mu^*(E) &\geqslant \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^* \left( \left( \bigcup_{j=1}^{\infty} A_j \right)^c \right) \\
&\geqslant \mu^* \left( E \cap \left( \bigcup_{j=1}^{\infty} A_j \right) \right) + \mu^* \left( E \cap \left( \bigcup_{j=1}^{\infty} A_j \right)^c \right) \\
&\geqslant \mu^*(E).
\end{aligned}$$

This forces all inequalities here to be equality, therefore

$$\mu^*(E) = \mu^* \left( E \cap \left( \bigcup_{j=1}^{\infty} A_j \right) \right) + \mu^* \left( E \cap \left( \bigcup_{j=1}^{\infty} A_j \right)^c \right)$$

as desired. Finally, we need to show that the restriction still gives a measure. We already know

$$\mu^*(E) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^* \left( \left( \bigcup_{j=1}^{\infty} A_j \right)^c \right)$$

for any $E \subseteq X$, then in particular take $E = \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$ to be the disjoint union, then this forces

$$\mu^* \left( \bigcup_{j=1}^{\infty} A_j \right) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*(\varnothing) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j).$$

Therefore $\mu^*|_{\mathcal{A}}$ is a measure. □

**Definition 1.39.** A measure $\mu$ is said to be complete if its domain contains all subsets of null sets.

**Example 1.40.** Let $X = \{a, b\}$, $\mathcal{A} = \{\varnothing, \{a, b\}\}$. Define $\mu : \mathcal{A} \to [0, \infty]$ by setting $\mu^*(X) = 0$, $\mu^*(\varnothing) = 0$. This is not a complete measure because $\{a\} \notin \mathcal{A}$.

**Theorem 1.41.** Let $\mathcal{A}$ be the collection of all $\mu^*$-measurable sets, then the measure $\mu^*|_{\mathcal{A}}$ is complete.

*Proof.* Let $N$ be any null set in $\mathcal{A}$, i.e., $\mu^*(N) = 0$. Take an arbitrary subset $A \subseteq N$, we need to show $A \in \mathcal{A}$. Since $\mu^*(N) = 0$, then $\mu^*(A) = 0$ as well. For any $E \subseteq X$, we prove $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$. It is clear that

$$\begin{aligned}\mu^*(E) &\leqslant \mu^*(E \cap A) + \mu^*(E \cap A^c)\\ &\leqslant \mu^*(A) + \mu^*(E \cap A^c)\\ &\leqslant \mu^*(N) + \mu^*(E \cap A^c)\\ &= \mu^*(E \cap A^c)\\ &= \mu^*(E).\end{aligned}$$

by the subadditivity of $\mu^*$. □

**Definition 1.42.** Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra. A function $\mu_0 : \mathcal{A} \to [0, \infty]$ is a pre-measure if

i. $\mu_0(\varnothing) = 0$,

ii. if $A_j \in \mathcal{A}$ for all $j \in \mathbb{N}$ with $\bigcup\limits_{j=1}^{\infty} A_j \in \mathcal{A}$, and they are pairwise disjoint, then $\mu_0\left(\bigcup\limits_{j=1}^{\infty} A_j\right) = \sum\limits_{j=1}^{\infty} \mu_0(A_j)$.

**Theorem 1.43.** Let $\mu_0$ be a pre-measure, then $\mu_0(A) \leqslant \mu_0(B)$ if $A, B \in \mathcal{A}$ are such that $A \subseteq B$.

*Proof.* We write $B = (B\backslash A) \cup A$, where $B\backslash A = B \cap A^c \in \mathcal{A}$, therefore

$$\begin{aligned}\mu_0(B) &= \mu_0(B\backslash A) + \mu_0(A)\\ &\geqslant \mu_0(A).\end{aligned}$$

□

**Definition 1.44.** Given a pre-measure $\mu_0$, we extend it to an outer measure as follows: for any $E \subseteq X$, define $\mu^*(E) = \inf\{\sum\limits_{j=1}^{\infty} \mu_0(A_j) : E \subseteq \bigcup\limits_{j=1}^{\infty}, A_j \in \mathcal{A}\}$.

**Theorem 1.45.** Let $\mu^*$ be the outer measure induced by $\mu_0$ specified in Definition 1.44, then

i. $\mu^*|_{\mathcal{A}} = \mu_0$, or equivalently, for any $A \in \mathcal{A}$, we have $\mu^*(A) = \mu_0(A)$;

ii. if $A \in \mathcal{A}$, then $A$ is $\mu^*$-measurable.

*Proof.* i. We want to show that for any $E \in \mathcal{A}$, $\mu^*(E) = \mu_0(E)$. To show $\mu^*(E) \leqslant \mu_0(E)$, we choose $A_1 = E \in \mathcal{A}$, and $A_j = \varnothing$ for all $j \geqslant 2$, then $E \subseteq \bigcup\limits_{j=1}^{\infty} A_j$, therefore

$$\begin{aligned}\mu^*(E) &\leqslant \sum_{j=1}^{\infty} \mu_0(A_j)\\ &= \mu_0(E).\end{aligned}$$

It now suffices to show that $\mu_0(E)$ is a lower bound of $\{\sum\limits_{j=1}^{\infty} \mu_0(A_j) : E \subseteq \bigcup\limits_{j=1}^{\infty}, A_j \in \mathcal{A}\}$. Let $A_j \in \mathcal{A}$ and $\bigcup\limits_{j=1}^{\infty} A_j \supseteq E$. We prove that $\mu_0(E) \leqslant \sum\limits_{j=1}^{\infty} \mu_0(A_j)$. For any $n \in \mathbb{N}$, define $B_n = E \cap \left(A_n \backslash \bigcup\limits_{j=1}^{n-1} A_j\right)$, therefore

$\bigcup\limits_{n=1}^{\infty} B_n = E \cap \left(\bigcup\limits_{j=1}^{\infty} A_j\right) = E$ where $B_n$'s are disjoint. We have

$$\begin{aligned}\mu_0(E) &= \mu_0\left(\bigcup_{n=1}^{\infty} B_n\right)\\ &= \sum_{n=1}^{\infty} \mu_0(B_n)\\ &\leqslant \sum_{n=1}^{\infty} \mu_0(A_n)\\ &= \sum_{j=1}^{\infty} \mu_0(A_j).\end{aligned}$$

ii. For any $A \in \mathcal{A}$, we want to prove that $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for all $E \subseteq X$. It suffices to show that for any $E \subseteq X$, we have $\mu^*(E) \geqslant \mu^*(E \cap A) + \mu^*(E \cap A^c)$.

Pick arbitrary $\varepsilon > 0$, then $\mu^*(E) + \varepsilon$ is not a lower bound of $\{\sum\limits_{j=1}^{\infty} \mu_0(A_j) : E \subseteq \bigcup\limits_{j=1}^{\infty}, A_j \in \mathcal{A}\}$. Therefore, there exists some $A_j \in \mathcal{A}$ such that $E \subseteq \bigcup\limits_{j=1}^{\infty} A_j$ and $\sum\limits_{j=1}^{\infty} \mu_0(A_j) \leqslant \mu^*(E) + \varepsilon$. Since $\mu_0(A_j) = \mu_0(A_j \cap A) + \mu_0(A_j \cap A^c)$, then

$$\begin{aligned}\sum_{j=1}^{\infty} \mu_0(A_j) &= \sum_{j=1}^{\infty} \mu_0(A_j \cap A) + \sum_{j=1}^{\infty} \mu_0(A_j \cap A^c)\\ &= \sum_{j=1}^{\infty} \mu^*(A_j \cap A) + \sum_{j=1}^{\infty} \mu^*(A_j \cap A^c)\\ &\geqslant \mu^*\left(\left(\bigcup_{j=1}^{\infty} A_j\right) \cap A\right) + \mu^*\left(\left(\bigcup_{j=1}^{\infty} A_j\right) \cap A^c\right)\\ &\geqslant \mu^*(E \cap A) + \mu^*(E \cap A^c).\end{aligned}$$

Let $\varepsilon \to 0$, then $\mu^*(E) \geqslant \mu^*(E \cap A) + \mu^*(E \cap A^c)$, as desired.

□

**Theorem 1.46.** Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra, and let $\mu_0$ be a pre-measure on $\mathcal{A}$. Define $\mathcal{M}(\mathcal{A})$ to be the $\sigma$-algebra generated by $\mathcal{A}$.

a. The outer measure $\mu^*$ induced by $\mu_0$ defines a measure function on $\mathcal{M}(\mathcal{A})$, and $\mu^*|_{\mathcal{A}} = \mu_0$.

b. If $\tilde{\mu}$ is another measure on $\mathcal{M}(\mathcal{A})$ that extends $\mu_0$, then $\tilde{\mu}(E) \leqslant \mu^*(E)$ for all $E \subseteq \mathcal{M}(\mathcal{A})$, with equality if and only if $\mu^*(E) < \infty$.

c. If $\mu_0$ is $\sigma$-finite, i.e., $X = \bigcup\limits_{j=1}^{\infty} A_j$ with $A_j \in \mathcal{A}$ and $\mu_0(A_j) < \infty$ for all $j$, then $\mu^*|_{\mathcal{M}(\mathcal{A})}$ is the unique extension of $\mu_0$ to a measure on $\mathcal{M}(\mathcal{A})$.

*Proof.* a. Let $\mathcal{B}$ be the set of all $\mu^*$-measurable sets, then $\mu^*|_{\mathcal{B}}$ is a measure on $\mathcal{B}$ that extends $\mu_0$. By the fundamental theorem of measure theory, we know $\mathcal{B}$ is a $\sigma$-algebra. In particular, $\mathcal{B} \supseteq \mathcal{A}$, therefore $\mathcal{B} \supseteq \mathcal{M}(\mathcal{A})$. That means $\mu^*|_{\mathcal{M}(\mathcal{A})}$ is a measure as well.

b. Let $\tilde{\mu}$ be any measure on $\mathcal{M}(\mathcal{A})$ that extends $\mu_0$. We first show that for all $E \in \mathcal{M}(\mathcal{A})$, then $\tilde{\mu}(E) \leqslant \mu^*(E)$. Recall that $\mu^*(E) = \inf\{\sum\limits_{j=1}^{\infty} \mu_0(A_j) : E \subseteq \bigcup\limits_{j=1}^{\infty} A_j, A_j \in \mathcal{A}\}$. Given a cover $E \subseteq \bigcup\limits_{j=1}^{\infty} A_j$ and fix $A_j \in \mathcal{A}$.

Therefore,

$$\begin{aligned}\tilde{\mu}(E) &\leqslant \tilde{\mu}\left(\bigcup_{j=1}^{\infty} A_j\right)\\ &\leqslant \sum_{j=1}^{\infty} \tilde{\mu}(A_j)\\ &= \sum_{j=1}^{\infty} \mu_0(A_j),\end{aligned}$$

therefore $\tilde{\mu}(E) \leqslant \mu^*(E)$. Assume we have $\mu^*(E) < \infty$, and we want to show that $\tilde{\mu}(E) = \mu^*(E)$. It suffices to show $\mu^*(E) \leqslant \tilde{\mu}(E)$.

**Claim 1.47.** Let $A_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, then $\mu^*\left(\bigcup\limits_{j=1}^{\infty} A_j\right) = \tilde{\mu}\left(\bigcup\limits_{j=1}^{\infty} A_j\right)$.

*Subproof.* Note that $\bigcup\limits_{j=1}^{\infty} A_j \in \mathcal{M}(\mathcal{A})$, then we can just work on $\mathcal{M}(\mathcal{A})$. Consider $\mu^*|_{\mathcal{M}(\mathcal{A})}$ and $\tilde{\mu}$ are measures on $\mathcal{M}(\mathcal{A})$. Let $E_n = \bigcup\limits_{j=1}^{n} A_j$ for all $n \in \mathbb{N}$, then we have a nested increasing sequence of $E_n$'s. In particular, we know $\bigcup\limits_{n=1}^{\infty} E_n = \bigcup\limits_{j=1}^{\infty} A_j$. Therefore

$$\begin{aligned}\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) &= \mu^*\left(\bigcup_{j=1}^{\infty} A_j\right)\\ &= \lim_{n\to\infty} \mu^*(E_n)\\ &= \lim_{n\to\infty} \mu^*\left(\bigcup_{j=1}^{n} A_j\right)\\ &= \lim_{n\to\infty} \mu_0\left(\bigcup_{j=1}^{n} A_j\right)\\ &= \lim_{n\to\infty} \tilde{\mu}\left(\bigcup_{j=1}^{n} A_j\right)\\ &= \tilde{\mu}\left(\bigcup_{j=1}^{\infty} A_j\right)\end{aligned}$$

by continuity from below and closure of finite union. ■

We know from the claim that

$$\begin{aligned}\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) &= \lim_{n\to\infty} \mu_0\left(\bigcup_{j=1}^{n} A_j\right)\\ &\leqslant \lim_{n\to\infty} \sum_{j=1}^{n} \mu_0(A_j)\\ &= \sum_{j=1}^{\infty} \mu_0(A_j).\end{aligned}$$

Take arbitrary $\varepsilon > 0$, then consider $\mu^*(E) + \varepsilon$, which is not a lower bound of the set anymore. Therefore, there exists $A_j \in \mathcal{A}$ for each $j \in \mathbb{N}$ such that $E \subseteq \bigcup_{j=1}^{\infty} A_j$ and that $\sum_{j=1}^{\infty} \mu_0(A_j) \leqslant \mu^*(E) + \varepsilon$. In particular, this means $\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leqslant \mu^*(E) + \varepsilon$. Since $\mu^*(E) < \infty$, then

$$\begin{aligned}\mu^*\left(\bigcup_{j=1}^{\infty} A_j \backslash E\right) &= \mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) - \mu^*(E)\\ &< \varepsilon.\end{aligned}$$

Now that

$$\begin{aligned}\mu^*(E) &\leqslant \mu^*\left(\bigcup_{j=1}^{\infty} A_j\right)\\ &= \tilde{\mu}\left(\bigcup_{j=1}^{\infty} A_j\right)\\ &= \tilde{\mu}(E) + \tilde{\mu}\left(\bigcup_{j=1}^{\infty} A_j \backslash E\right)\\ &< \tilde{\mu}(E) + \varepsilon\end{aligned}$$

by the claim. Therefore, for any $\varepsilon > 0$, we have $\mu^*(E) \leqslant \tilde{\mu}(E) + \varepsilon$ whenever $\mu^*(E) < \infty$. Take $\varepsilon \to 0$, we get $\mu^*(E) \leqslant \tilde{\mu}(E)$.

c. Since $\mu_0$ is $\sigma$-finite, then there exists a decomposition $X = \bigcup_{j=1}^{\infty} A_j$ for $A_j \in \mathcal{A}$ and that $\mu_0(A_j) < \infty$. For any $E \in \mathcal{M}(\mathcal{A})$, then

$$\begin{aligned}E &= E \cap X\\ &= E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)\\ &= \bigcup_{j=1}^{\infty} (E \cap A_j)\end{aligned}$$

and

$$\begin{aligned}\mu^*(E) &= \mu^*\left(\bigcup_{j=1}^{\infty} (E \cap A_j)\right)\\ &= \sum_{j=1}^{\infty} \mu^*(E \cap A_j)\\ &= \sum_{j=1}^{\infty} \tilde{\mu}(E \cap A_j)\\ &= \tilde{\mu}\left(\bigcup_{j=1}^{\infty} (E \cap A_j)\right)\\ &= \tilde{\mu}(E)\end{aligned}$$

since $\mu^*(E \cap A_j) \leqslant \mu^*(A_j) = \mu_0(A_j) < \infty$.

$\square$

## 1.4 Borel Measure

Recall that the Borel $\sigma$-algebra $\mathcal{B}_{\mathbb{R}}$ is the smallest $\sigma$-algebra containing all open sets. Let $\mathcal{G}$ be the set of all open sets in $\mathbb{R}$ with respect to the standard topology. Therefore $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{G})$. We can in fact use something smaller than $\mathcal{G}$.

**Theorem 1.48.** $\mathcal{B}_{\mathbb{R}}$ is a $\sigma$-algebra generated by

a. $\mathcal{A}_0 = \{(a,b) : a, b \in \mathbb{R}, a < b\}$;

b. $\mathcal{A}_1 = \{(a,b] : a, b \in \mathbb{R}, -\infty \leqslant a < b < \infty\} \cup \{(a,\infty) : -\infty \leqslant a < \infty\} \cup \{\varnothing\}$.

Any member in $\mathcal{A}_1$ is called an $h$-interval.

*Proof.* a. We want to show that $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{A})$. Obviously $\mathcal{A}_0 \subseteq \mathcal{G}$, then $\mathcal{M}(\mathcal{G})$ is a $\sigma$-algebra containing $\mathcal{A}_0$, then $\mathcal{M}(\mathcal{A}_0) \subseteq \mathcal{M}(\mathcal{G}) = \mathcal{B}_{\mathbb{R}}$. Conversely, recall that any open subset in $\mathbb{R}$ is a $\sigma$-union of open intervals, therefore $\mathcal{G} \subseteq \mathcal{M}(\mathcal{A})$, so $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{G}) \subseteq \mathcal{M}(\mathcal{A}_0)$, therefore $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{A}_0)$.

b. We first show that $\mathcal{M}(\mathcal{A}_1) \subseteq \mathcal{B}_{\mathbb{R}}$. Since $\mathcal{M}(\mathcal{A}_1)$ is the smallest $\sigma$-algebra containing $\mathcal{A}_1$, then it suffices to show that $\mathcal{A}_1 \subseteq \mathcal{B}_{\mathbb{R}}$. It is easy to see that $(a,b] = \bigcap\limits_{n=1}^{\infty}(a, b + \frac{1}{n} \in \mathcal{B}_{\mathbb{R}})$, and $(a,\infty) = \bigcup\limits_{n=1}^{\infty}(a,n) \in \mathcal{B}_{\mathbb{R}}$.

We now verify that $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{A}_1)$. By a. we know $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{A}_0)$, so it suffices to show that $\mathcal{A}_0 \subseteq \mathcal{M}(\mathcal{A}_1)$. For $a < b$, we have $(a,b) = \bigcup\limits_{n=1}^{\infty}(a, b - \frac{1}{n}]$, therefore the right-hand side is a $\sigma$-union of intervals, hence belongs to $\mathcal{M}(\mathcal{A}_1)$, and we are done.

□

**Definition 1.49.** We define $\mathcal{A}_2$ to be the collection of finite disjoint unions of $h$-intervals, e.g., $\bigcup\limits_{j=1}^{n}(a_j, b_j]$, then $\mathcal{A}_2$ is an algebra.

**Definition 1.50.** A function on $\mathbb{R}$ is said to be right continuous if $\lim\limits_{x \to x_0^+} F(x) = F(x_0)$.

**Theorem 1.51.** Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous. Let $I_j = (a_j, b_j]$ for $j = 1, \ldots, n$ be disjoint $h$-intervals. We define the premeasure $\mu_0$ on $\mathcal{A}_2$ by $\mu_0(\varnothing) = 0$ and $\mu_0\left(\bigcup\limits_{j=1}^{n}(a_j, b_j]\right) = \sum\limits_{j=1}^{n}[F(b_j) - F(a_j)]$.

*Proof.* First one cancheck that $\mu_0$ is well-defined, that is, given any partition of $h$-interval, the $\mu_0$-measurements on the interval are the same.

Second, we need to show that $\mu_0$ satisfies $\sigma$-additivity, that is, if $\bigcup\limits_{j=1}^{\infty} I_j \in \mathcal{A}_2$ such that $I_j$'s are disjoint, then $\mu_0\left(\bigcup\limits_{j=1}^{\infty} I_j\right) = \sum\limits_{j=1}^{\infty}\mu_0(I_j)$. It is easy to verify finite additivity, so we now assume

$$\bigcup_{j=1}^{\infty} I_j = I = (a,b] \in \mathcal{A}_2$$

for $-\infty \leqslant a < b < \infty$, then we will show that

$$F(b) - F(a) = \mu_0(I) = \sum_{j=1}^{\infty}\mu_0(I_j)$$

for $I_j = (a_j, b_j]$.

To show $\mu_0(I) \geqslant \sum\limits_{j=1}^{\infty}\mu(I_j)$, we know $F(b) - F(a) \geqslant \sum\limits_{j=1}^{n}[F(b_j) - F(a_j)]$, therefore taking the limit of $n \to \infty$ gives $F(b) - F(a) \geqslant \sum\limits_{j=1}^{\infty}\mu_0(I_j)$.

To show $\mu_0(I) \leqslant \sum_{j=1}^{\infty} \mu(I_j)$, since $F$ is right continuous, then for all $\varepsilon > 0$, there exist $\delta > 0$ such that $F(a+\delta) - F(a) < \varepsilon$. Therefore, for every $j > 0$, there exists $\delta_j > 0$ such that $F(b_j + \delta_j) - F(b_j) < 2^{-j}\varepsilon$, then

$$\begin{aligned}
[a+\delta, b] &\subseteq (a, b] \\
&= \bigcup_{j=1}^{\infty} (a_j, b_j] \\
&= \bigcup_{j=1}^{\infty} (a_j, b_j + \delta_j).
\end{aligned}$$

By compactness, there exists some $N \in \mathbb{N}$ such that $[a+\delta, b] \subseteq \bigcup_{j=1}^{N} (a_j, b_j + \delta_j)$. Assume $b_j + \delta_j \in (a_{j+1}, b_{j+1}]$, then

$$\begin{aligned}
\mu_0(I) &= \mu_0((a, b])) \\
&= F(b) - F(a) \\
&\leqslant F(b) - F(a+\delta) + \varepsilon \\
&\leqslant F(b_N + \delta_N) - F(a+\delta) + \varepsilon \\
&= F(b_N + \delta_N) - F(a_N) + F(a_N) - F(a+\delta) + \varepsilon \\
&= F(b_N + \delta_N) - F(a_N) + \sum_{j=1}^{N-1} [F(a_{j-1}) - F(a_j)] + \varepsilon \\
&\leqslant F(b_N + \delta_N) - F(a_N) + \sum_{j=1}^{N-1} [F(b_j + \delta_j) - F(a_j)] + \varepsilon \\
&= \sum_{j=1}^{N} [F(b_j + \delta_j) - F(a_j)] + \varepsilon \\
&= \sum_{j=1}^{N} [F(b_j + \delta_j) - F(b_j) + F(b_j) - F(a_j)] + \varepsilon \\
&= \sum_{j=1}^{N} [F(b_j + \delta_j) - F(b_j)] + \sum_{j=1}^{N} [F(b_j) - F(a_j)] + \varepsilon \\
&\leqslant \sum_{j=1}^{N} 2^{-j}\varepsilon + \sum_{j=1}^{N} \mu_0(I_j) + \varepsilon \\
&\leqslant 2\varepsilon + \sum_{j=1}^{\infty} \mu_0(I_j)
\end{aligned}$$

since $F$ is increasing. Let $\varepsilon \to 0$ and we are done. □

**Theorem 1.52.** Let $F$ be increasing and right-continuous, then

a. there is a unique measure $\mu_F$ on $\mathbb{R}$ such that $\mu_F((a, b]) = F(b) - F(a)$ for all $a, b \in \mathbb{R}$;

b. if $G$ is another increasing and right-continuous function, then $\mu_F = \mu_F$ if and only if $F - G$ is a constant function;

c. if $\mu$ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets, i.e., a set $S \subseteq \mathbb{R}$ contained in $[-M, M]$ for some $M \in \mathbb{R}$, then

$$F(x) = \begin{cases} \mu((0, x]), & x > 0 \\ 0, & x = 0 \\ -\mu(x, 0]), & x < 0 \end{cases}$$

is an increasing function and right-continuous, and $\mu_F = \mu$.

*Proof.* a. Consider $\mathbb{R} = \bigcup_{j=-\infty}^{\infty}(j, j+1]$, then the pre-measure $\mu_0((j, j+1]) = F(j+1) - F(j) < \infty$ defined on $h$-intervals is $\sigma$-finite. Therefore there exists a unique extension of measure $\mu$ of $\mu_0$ on $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{A}_2)$ such that $\mu|_{\mathcal{A}_2} = \mu_0$.

b. We have $\mu_F((a, b]) = F(b) - F(a)$ and $\mu_G((a, b]) = G(b) - G(a)$, then

$$\begin{aligned}\mu_F((a,b]) = \mu_G((a,b]) &\iff F(b) - F(a) = G(b) - G(a)\\ &\iff F(b) - G(b) = G(a) - F(a)\\ &\iff F - G \text{ is constant.}\end{aligned}$$

c. First note that $F$ is an increasing function since the measure function is increasing. Take any $x_0 \in \mathbb{R}$, we want to show that $\lim_{x\to x_0^+} F(x) = F(x_0)$. We prove this by cases, either $x_0 = 0$, $x_0 > 0$, or $x_0 < 0$. We will only prove the first case, but the two other cases are analogous. Suppose $x_0 = 0$, take a nested sequence of intervals $E_n = (0, \frac{1}{n}]$, with $E_n \supseteq E_{n+1}$ for all $n \in \mathbb{N}$, then

$$\begin{aligned}\lim_{x\to 0^+} F(x) &= \lim_{x\to 0^+} \mu((0, x])\\ &= \lim_{n\to 0} \mu((0, \frac{1}{n}])\\ &= \lim_{n\to\infty} \mu(E_n)\\ &= \mu\left(\bigcap_{n=1}^{\infty} E_n\right)\\ &= \mu(\varnothing)\\ &= 0\\ &= F(0)\end{aligned}$$

since $\mu(E_1) < \infty$.

$\square$

**Definition 1.53.** Suppose $F$ is increasing and right-continuous, then we can use $F$ to create $\mu_0$ on $\mathcal{A}_2$, and get an outer measure $\mu^*$ induced by $\mu_0$. Let $\mathcal{A}$ be the collection of all $\mu^*$-measurable sets, then $\mu^*|_{\mathcal{A}}$ is a measure. Note that $\mathcal{A} \supseteq \mathcal{B}_{\mathbb{R}}$: since $\mu_F$ is only defined on $\mathcal{B}_{\mathbb{R}}$, then $\mu^*|_{\mathcal{A}}$ becomes the extension of $\mu_F$ on $\mathcal{A}$. We denote this measure to be $\bar{\mu}_F$, as the extension of $\mu_F$, called the Lebesgue-Stieltjes measure.

**Remark 1.54.** In particular, if $F(x) = x$ for all $x \in \mathbb{R}$, then $\bar{\mu}_F$ is called a Lebesgue measure, denoted by $\mathfrak{m}$, with $\mathfrak{m}((a, b]) = F(b) - F(a) = b - a$.

**Definition 1.55.** Let $\mu$ be a Lebesgue-Stieltjes measure associated to an increasing and right-continuous function $F$. Let $\mathcal{M}_\mu$ be the domain of the measure $\mu$, which gives the collection of measurable sets. For any measurable set $E \in \mathcal{M}_\mu$, we have

$$\begin{aligned}\mu(E) &= \inf\left\{\sum_{j=1}^{\infty}[F(b_j) - F(a_j)] : E \subseteq \bigcup_{j=1}^{\infty}(a_j, b_j]\right\}\\ &= \inf\left\{\sum_{j=1}^{\infty}\mu((a_j, b_j]) : E \subseteq \bigcup_{j=1}^{\infty}(a_j, b_j]\right\}.\end{aligned}$$

**Theorem 1.56.** For all $E \in \mathcal{M}_\mu$, we have

$$\mu^*(E) = \inf\left\{\sum_{j=1}^{\infty}\mu((a_j, b_j)) : E \subseteq \bigcup_{j=1}^{\infty}(a_j, b_j)\right\}.$$

*Proof.* Let $\tilde{\mu}(E)$ be the right-hand side of this equation, so we will show that $\mu(E) = \tilde{\mu}(E)$. Note that we have a partition

$$(a_j, b_j) = \bigcup_{k=1}^{\infty} I_k^{(j)},$$

where $I_k^{(j)} = (b_j - \frac{1}{2^k}(b_j - a_j), b_j - \frac{1}{2^{k+1}}(b_j - a_j)]$. Now $E \subseteq \bigcup\limits_{j=1}^{\infty}(a_j, b_j)$, so $E \subseteq \bigcup\limits_{j=1}^{\infty}\bigcup\limits_{k=1}^{\infty} I_k^{(j)}$, and thus

$$\begin{aligned}\sum_{j=1}^{\infty}\mu((a_j, b_j)) &= \sum_{j=1}^{\infty}\mu\left(\bigcup_{k=1}^{\infty} I_k^{(j)}\right)\\ &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\mu(I_k^{(j)}).\end{aligned}$$

Because $\left\{\sum\limits_{j=1}^{\infty}\mu((a_j, b_j)) : E \subseteq \bigcup\limits_{j=1}^{\infty}(a_j, b_j)\right\} \subseteq \left\{\sum\limits_{j=1}^{\infty}\mu((a_j, b_j]) : E \subseteq \bigcup\limits_{j=1}^{\infty}(a_j, b_j]\right\}$, then $\tilde{\mu}(E) \geqslant \mu(E)$.

We now show that $\mu(E) \geqslant \tilde{\mu}(E)$. Pick arbitrary $\varepsilon > 0$, then we know $\mu(E) + \varepsilon$ is not a lower bound of the set $\left\{\sum\limits_{j=1}^{\infty}\mu((a_j, b_j]) : E \subseteq \bigcup\limits_{j=1}^{\infty}(a_j, b_j]\right\}$, hence there exists $(a_j, b_j]$ for $j \geqslant 1$ such that $E \subseteq \bigcup\limits_{j\geqslant 1}(a_j, b_j]$. Therefore $\sum\limits_{j=1}^{\infty}\mu((a_j, b_j]) \leqslant \mu(E) + \varepsilon$. By the right continuity of $F$, for $\varepsilon \cdot 2^{-j} > 0$, there exists $\delta_j > 0$ such that $F(b_j + \delta_j) - F(b_j) < \varepsilon \cdot 2^{-j}$, then $E \subseteq \bigcup\limits_{j=1}^{\infty}(a_j, b_j + \delta_j)$. We know

$$\begin{aligned}\tilde{\mu}(E) &\leqslant \sum_{j=1}^{\infty}\mu((a_j, b_j + \delta_j))\\ &= \sum_{j=1}^{\infty}[F(b_j + \delta_j) - F(a_j)]\\ &= \sum_{j=1}^{\infty}[F(b_j + \delta_j) - F(b_j) + F(b_j) - F(a_j)]\\ &\leqslant \sum_{j=1}^{\infty}[F(b_j + \delta_j) - F(b_j)] + \sum_{j=1}^{\infty}[F(b_j) - F(a_j)]\\ &< \sum_{j=1}^{\infty}\varepsilon \cdots 2^{-j} + \sum_{j=1}^{\infty}\mu((a_j, b_j])\\ &< \varepsilon + \mu(E) + \varepsilon\\ &= \mu(E) + 2\varepsilon.\end{aligned}$$

Taking small enough $\varepsilon$ finishes the proof. □

**Remark 1.57.** The union of $h$-intervals may not be open, so often times we use the characterization in Theorem 1.56 instead.

**Theorem 1.58.** For any $E \subseteq \mathcal{M}_\mu$, we have

$$\mu(E) = \inf\{\mu(U) : \text{ open } U \supseteq E\} = \sup\{\mu(K) : \text{ compact } K \subseteq E\}.$$

*Proof.* Let $\tilde{\mu}(E) = \inf\{\mu(U) : \text{ open } U \supseteq E\}$. First, $\mu(E) \leqslant \tilde{\mu}(E)$: since $E \subseteq U$, then $\mu(E) \leqslant \mu(U)$, therefore $\mu(E) \leqslant \tilde{\mu}(E)$. To see $\tilde{\mu}(E) \leqslant \mu(E)$, we have $\mu(E) + \varepsilon$ is not a lower bound of $\left\{\sum\limits_{j=1}^{\infty}\mu((a_j, b_j]) : E \subseteq \bigcup\limits_{j=1}^{\infty}(a_j, b_j)\right\}$, then there exists $(a_j, b_j)$ for each $j \in \mathbb{N}$ such that $E \subseteq \bigcup_{j=1}^{\infty}(a_j, b_j)$, and that $\sum_{j=1}^{\infty}\mu((a_j, b_j)) \leqslant \mu(E)+\varepsilon$. Therefore, take $U$ to be the open set $\bigcup_{j=1}^{\infty}(a_j, b_j)$, then

$$\tilde{\mu}(E) \leqslant \mu(U) \leqslant \sum_{j=1}^{\infty}\mu((a_j, b_j)) \leqslant \mu(E)+\varepsilon$$

as desired.

Now let $\nu(E) = \sup\{\mu(K) : \text{ compact } K \subseteq E\}$. We note that if $K \subseteq E$, then $\mu(K) \leqslant \mu(E)$, therefore $\nu(E) \leqslant \mu(E)$. To prove the reverse inequality, we consider the following cases:

- $E$ is bounded.
  - $E$ is closed. Since $E$ is bounded and closed, it is compact over $\mathbb{R}$, thus $\mu(E) \leqslant \nu(E)$.
  - $E$ is bounded but not closed. We have $\mu(\bar{E}\backslash E) = \inf\{\mu(U) : \text{ open } U \supseteq \bar{E}\backslash E\}$. For any $\varepsilon > 0$, there exists an open set $U$ such that $U \supseteq \bar{E}\backslash E$ and $\mu(U) \leqslant \mu(\bar{E}\backslash E)+\varepsilon$. Set $K = \bar{E}\backslash U$, then $K$ is compact. Since all measures here are finite, we have

$$\begin{aligned}\mu(K) &= \mu(E) - \mu(E \cap U) \\ &= \mu(E) - [\mu(U) - \mu(U\backslash E)] \\ &\geqslant \mu(E) - \mu(U) + \mu(\bar{E}\backslash E) \\ &\geqslant \mu(E) - \varepsilon.\end{aligned}$$

    Therefore $\nu(E) \geqslant \mu(E) - \varepsilon$, and we are done by taking $\varepsilon \to 0$.

- $E$ is not bounded. Suppose $E = \bigcup_{j=-\infty}^{\infty}((j, j+1] \cap E)$, then denote $E_j = E \cap (j, j+1]$, which is bounded. Therefore, we know the statement is true for each $E_j$ for $j \geqslant 1$, thus $\mu(E_j) = \sup\{\mu(K) : \text{ compact } K \subseteq E_j\}$. Take arbitrary $\varepsilon > 0$, then $\mu(E_j) - \frac{1}{3}\varepsilon \cdot 2^{-|j|}$ is not the upper bound of $\{\mu(K) : \text{ compact } K \subseteq E_j\}$, then there exists a compact set $K_j \subseteq E_j$ such that $\mu(K_j) \geqslant \mu(E_j) - \frac{1}{3}\varepsilon \cdot 2^{-|j|}$. Since $K_j \subseteq E_j$ and $E_j$'s are disjoint, then $K_j$'s are disjoint. Therefore, for $n \in \mathbb{N}$, set $H_n = \bigcup_{j=-n}^{n} K_j$, which is a finite disjoint union of compact sets, so this is a compact set. But $H_n \subseteq E$, then

$$\begin{aligned}\mu(H_n) &= \mu\left(\bigcup_{j=-n}^{n} K_j\right) \\ &= \sum_{j=-n}^{n}\mu(K_j) \\ &\geqslant \sum_{j=-n}^{n}\mu(E_j) - \frac{\varepsilon}{3}\sum_{j=-n}^{n} 2^{-|j|} \\ &\geqslant \sum_{j=-n}^{n}\mu(E_j) - \frac{\varepsilon}{3}\sum_{j=-\infty}^{\infty} 2^{-|j|} \\ &\geqslant \sum_{j=-n}^{n}\mu(E_j) - \varepsilon.\end{aligned}$$

  Note that $H_n$ still depends on $n$, so we should not take $n \to \infty$ here. Since $\nu(E)$ is the upper bound of $\mu(K)$'s for compact $K \subseteq E$, then $\nu(E) \geqslant \mu(H_n)$, therefore

$$\nu(E) \geqslant \sum_{j=-n}^{n}\mu(E_j) - \varepsilon$$

$$= \mu\left(\bigcup_{j=-n}^{n} E_j\right) - \varepsilon.$$

Take $n \to \infty$, then

$$\begin{aligned}\nu(E) &\geqslant \lim_{n\to\infty} \mu\left(\bigcup_{j=-n}^{n} E_j\right) - \varepsilon \\ &= \mu\left(\bigcup_{j=-\infty}^{\infty} E_j\right) - \varepsilon \\ &= \mu(E) - \varepsilon.\end{aligned}$$

Let $\varepsilon \to 0$, we are done.

□

**Theorem 1.59.** Let $E \subseteq \mathbb{R}$, then the following are equivalent:

a. $E \in \mathcal{M}_\mu$;

b. $E = V\backslash N_1$, where $V$ is a $G_\delta$-set and $\mu(N_1) = 0$;

c. $E = H \cup N_2$, where $H$ is a $F_\sigma$-set and $\mu(N_2) = 0$.

*Proof.*
- $b. \Rightarrow a.$: note that $\mathcal{M}_\mu \supseteq \mathcal{B}_\mathbb{R}$, then both $V$ and $N_1$ are measurable, therefore $E$ is measurable, i.e., $E \in \mathcal{M}_\mu$.
- $c. \Rightarrow a.$: similar to the case above.
- $a. \Rightarrow b.$:
  - If $\mu(E) < \infty$, recall $\mu(E) = \inf\{\mu(U) : \text{ open } U \supseteq E\}$. For any $k \in \mathbb{N}$, consider $2^{-k} > 0$, then there exists open subset $U_k \supseteq E$ such that $\mu(U_k) \leqslant \mu(E) + 2^{-k}$. Let $V = \bigcap_{k=1}^{\infty} U_k$ be a $G_\delta$-set, then $V \supseteq E$ as well. It suffices to show that $V\backslash E$ is a null set. We know

    $$\begin{aligned}\mu(V) &= \mu\left(\bigcap_{k=1}^{\infty} U_k\right) \\ &\leqslant \mu(U_k) \\ &\leqslant \mu(E) + 2^{-k}\end{aligned}$$

    for all $k \in \mathbb{N}$. Since $\mu(V)$ and $\mu(E)$ are independent of $k$, then take $k \to \infty$, therefore $\mu(V) \leqslant \mu(E)$. But since $E \subseteq V$, then $\mu(E) \leqslant \mu(V)$, therefore this gives equality. Since $\mu(E) < \infty$, then $\mu(V) - \mu(E) = 0$, then $\mu(V\backslash E) = 0$ by additivity.
  - If $\mu(E) = \infty$, then the proof can be done using the previous case.
- $a. \Rightarrow c.$: the proof is similar to the case above.

□

**Theorem 1.60.** Let $E \in \mathcal{M}_\mu$, and suppose $\mu(E) < \infty$. For any $\varepsilon > 0$, there exists some set $A$ that is a finite union of open intervals such that $\mu(E\Delta A) = \mu((E\backslash A) \cup (A\backslash E)) < \varepsilon$.

*Proof.* Note that $\mu(E) = \sup\{\mu(K) : \text{ compact } K \subseteq E\}$. For any $\varepsilon > 0$, there exists compact $K \subseteq E$ such that $\mu(E) - \frac{\varepsilon}{2} < \mu(K)$, which is equivalent to having $\mu(E\backslash K) < \frac{\varepsilon}{2}$. Similarly, recall that $\mu(E) = \inf\{\mu(U) : \text{ open } U \supseteq E\}$, but open set $U$ on $\mathbb{R}$ is characterized as a union of open intervals, therefore this is just $\mu(E) = \inf\{\sum_{j=1}^{\infty} \mu((a_j, b_j)) :$

$\bigcup_{j=1}^{\infty}(a_j, b_j) \supseteq E\}$. Therefore, there exists $\bigcup_{j=1}^{\infty} I_j \supseteq E$, where $I_j$ is open interval for each $j$, such that $\mu\left(\bigcup_{j=1}^{\infty} I_j\right) < \mu(E) + \frac{\varepsilon}{2}$. Since $\mu(E)$ is finite, then $\mu\left(\bigcup_{j=1}^{\infty} I_j \backslash E\right) < \frac{\varepsilon}{2}$. Now $K \subseteq E \subseteq \bigcup_{j=1}^{\infty} I_j$, but $K$ is compact, so there exists $I_1, \ldots, I_n$ such that their union cover $K$. Set $A = \bigcup_{j=1}^{m} I_j$, and we are done. □