Advanced Functions Notes

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Reference: Chapter 1.1, 1.2 and part of 1.3.

Definition 1.1 (Function)**.** A *function* is a relation/rules between some input and some output in which there is a unique (one and only one) output for each input.

Definition 1.2 (Interval)**.** An *interval* is a connected portion of the real number line. It usually takes one of the following four forms, with corresponding notations:

- $x \in [a, b]$, i.e., $a \leq x \leq b$;
- $x \in (a, b)$, i.e., $a < x < b$;
- $x \in (a, b]$, i.e., $a < x \leq b$;
- $x \in [a, b)$, i.e., $a \leq x < b$.

Remark 1.3.

- A function is sometimes written as $f(x)$, as in a function relation f with respect to x.
- The expression $y = f(x)$ says given the input value x, the output value with respect to f is y.
- The domain and the range of the function are also crucial information that one should not omit.

  Example 1.4. For example, look at $f(x) = \sin(\pi x)$ when the domain is $\mathbb{Z}$ and when the domain is $\mathbb{R}$. If the domain is $\mathbb{R}$, then the range is $[-1, 1]$, i.e., $y = f(x)$ satisfies $-1 \leq y \leq 1$. Moreover, the function is oscillating with period 2. If the domain is $\mathbb{Z}$, then the range is $\{0\}$, so the function $f(x)$ is equivalent to the zero function!

- For now, if the domain and range are not specified, we assume the domain to be wherever it makes sense on $\mathbb{R}$. One may need to restrict the domain and range in actual problems for it to make sense. For example, the number of objects, the number of years.

Remark 1.5 (Representation of a function)**.**

- Algebraic: the algebraic expression like $f(x) = 2\sin(3x) + 4$. This is what you would see most of the time. This representation is concise, but it can be hard to find its properties.
- Graphic: draw the graph of the function in the Cartesian plane, which tells you the behavior of the function. This gives the most information, but if the function is too complicated we may not have this in hand.
- Numerical: use i) table of values, ii) mapping diagram, etc, to find corresponding y-values of particular x-values. This is easy to obtain, but note that this only provides a handful of them, therefore this does not give a general representation of the function. This is useful if you cannot get the graph, because you get the idea of the general behavior.

The algebraic model is the most useful and most accurate. If you know the value of one variable, you can substitute this value into the function to create an equation, which can then be solved using an appropriate strategy. This leads to an accurate answer. Both numerical and graphical models are limited in their use because they represent the function for only small intervals of the domain and range. When using a graphical model, it may be necessary to interpolate or extrapolate. This can lead to approximate answers.

Remark 1.6 (Vertical line test)**.** How to see if a relation is a function in graph? Any vertical line drawn on the graph of a function passes through, at most, a single point.

Exercise 1.7. Is $y = \frac{1}{x^2}$ a function? If so, what is the domain and range?

Proof. This is a function with domain $\{x \in \mathbb{R} : x \neq 0\}$ and range $\{y \in \mathbb{R} : y > 0\}$. Drawing the graph, we see a *vertical asymptote* at $x = 0$ and a *horizontal asymptote* at $y = 0$. □

Exercise 1.8. Is $x^2 + y^2 = 1$ a function? If so, what is the domain and range?

Proof. This is the expression of a circle, so no, it is not. □

1.1 Properties of graphs of functions

How do we describe the behavior of a function's graph?

- Domain and range.
- x-intercepts (i.e., zeros/roots) and y-intercepts. If we are thinking of a polynomial, then this tells you the roots and the constant coefficient.
- Intervals of increase/decrease.
  - An *interval of increase* of $f(x)$ is an interval $[a, b]$ of the domain where $f(x)$ (strictly) increases as x increases;

  - An *interval of decrease* of $f(x)$ is an interval $[a, b]$ of the domain where $f(x)$ (strictly) decreases as x increases;
  - A *turning point* is a point (x, y) on the function $f(x)$ where the function changes from increasing to decreasing, or vice versa.

- Symmetries: odd/even functions.
  - An *odd function* has the graph symmetric via the origin $(0, 0)$. Algebraically, this says that $f(-x) = -f(x)$.
  - An *even function* has the graph symmetric via the y-axis $x = 0$. Algebraically, this says that $f(-x) = f(x)$.

  Remark 1.9. Suppose f is an odd function with 0 contained in the domain, then $f(0) = 0$.

  Exercise 1.10. When is a function both odd and even?

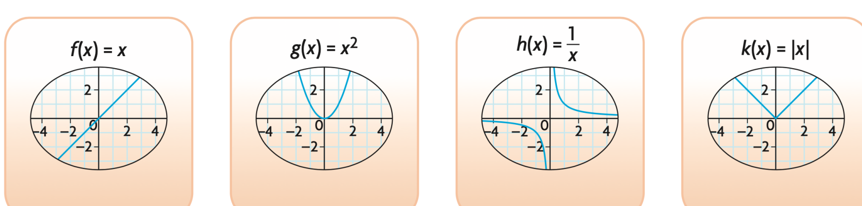
  Proof. This requires $-f(x) = f(-x) = f(x)$ for all x, so $f(x) = 0$ for all x. □

- Horizontal/Vertical Asymptotes. An asymptote can be local!
- Continuity.
  - A *continuous function* is a function that does not contain any holes/breaks over its entire domain.
  - A break in the graph is called a point of discontinuity.

  Remark 1.11. The current definition says that continuity is when you can draw the graph of the function with one stroke, which is different from the definition in calculus.

- Behavior at infinity ($\pm\infty$). Does it converge to 0? Does it goes to infinity? Does it oscillate? What happens near the breaks?

1.2 Example: Polynomial Functions, Reciprocal Functions, and Absolute Value Functions



Function	x^{2n+1}	x^{2n}	$\frac{1}{x^{2n+1}}$	$\frac{1}{x^{2n}}$	$\|x-a\|$
Restriction	$n \geq 0$	$n \geq 1$	$n \geq 0$	$n \geq 1$	$a \in \mathbb{R}$
Prototype	x	x^2	$\frac{1}{x}$	$\frac{1}{x^2}$	$\|x\|$
Domain	$\mathbb{R}$	$\mathbb{R}$	$\{x \in \mathbb{R} : x \neq 0\}$	$\{x \in \mathbb{R} : x \neq 0\}$	$\mathbb{R}$
Range	$\mathbb{R}$	$\{y \in \mathbb{R} : y \geq 0\}$	$\{y \in \mathbb{R} : y \neq 0\}$	$\{y \in \mathbb{R} : y > 0\}$	$\{y \in \mathbb{R} : y \geq 0\}$
x-intercept	$x = 0$	$x = 0$	N/A	N/A	$x = 0$
y-intercept	$y = 0$	$y = 0$	N/A	N/A	$y = 0$
$\nearrow$	$\mathbb{R}$	$[0, \infty)$	N/A	$(-\infty, 0)$	$[0, \infty)$
$\searrow$	N/A	$(-\infty, 0]$	$(-\infty, 0)$, $(0, \infty)$	$(0, \infty)$	$(-\infty, 0]$
Symmetry	Odd	Even	Odd	Even	Even
Asymptote	N/A	N/A	$x = 0$ and $y = 0$	$x = 0$ and $y = 0$	N/A
Continuity	Yes	Yes	Not at $x = 0$	Not at $x = 0$	Yes
$x \to \infty$	$y \to \infty$	$y \to \infty$	$y \searrow 0$	$y \searrow 0$	$y \to \infty$
$x \to -\infty$	$y \to -\infty$	$y \to \infty$	$y \nearrow 0$	$y \searrow 0$	$y \to \infty$

Remark 1.12. A function $f(x) = x^n$ is odd if and only if n is odd, and is even if and only if n is even.

2 July 21, 2023

Reference: Chapter 1.3-1.7, and Chapter 9.1-9.4.

2.1 Revision

Exercise 2.1. Suppose $f(x) = |x - a|$ is an absolute value function such that

- it increases on the interval $(2, \infty)$;
- it decreases on the interval $(-\infty, -2)$;
- it has a y-intercept at $(0, 4)$.

Does such function exist? Is it unique? What if we suppose $f(x)$ to be a quadratic function of the form $f(x) = (x - a)^2$?

Remark 2.2. Note that both functions should be symmetric with respect to the vertical line $x = a$, and the vertex of both functions is the point $(a, 0)$.

Solution. First suppose $f(x) = |x-a|$. Since the y-intercept is at $(0, 4)$, then $f(0) = 4$, so $|-a| = 4$, thus $a = \pm 4$. We now just have to look at the functions $|x - 4|$ and $|x + 4|$. By the characteristic we gave, we know the function $f(x) = |x - a|$ increases when $x \geq a$ and decreases when $x \leq a$, therefore both are not suitable.

Now suppose $f(x) = (x - a)^2$. Again, we have $f(0) = 4$, so $a^2 = 4$, thus $a = \pm 2$. We now just have to look at the functions $(x - 2)^2$ and $(x + 2)^2$. Both turn out to work. $\square$

2.2 Addendum: Continuity of Piecewise Function

We will continue to focus on the "piecewise" function $f(x) = |x - a|$ for $a \in \mathbb{R}$. This function tells you the distance to a. (That also means the function is symmetric with respect to the vertical line $x = a$.) To make sense of what is happening, the function is depicted by

$$f(x) = \begin{cases} x - a, & \text{if } x \geq a, \\ -(x - a), & \text{if } x \leq a. \end{cases}$$

This is called a *piecewise function*, i.e., a function with two or more pieces, where each piece is defined for a specific interval in the domain.

Exercise 2.3. Suppose we have a piecewise function

$$f(x) = \begin{cases} g(x), & \text{if } x \leq a, \\ h(x), & \text{if } a \leq x \leq b, \\ k(x), & \text{if } x \geq b. \end{cases}$$

When is $f(x)$ continuous?

Proof. $f(x)$ is continuous if $g(x)$, $h(x)$, and $k(x)$ are all continuous, and $g(a) = h(a)$, $h(b) = k(b)$, i.e., all the branches connect. □

Remark 2.4. This works in general if we have multiple branches on the domain. Draw the graph by drawing each piece on the given interval, and check continuity piecewise.

Exercise 2.5. What is a value of $a \in \mathbb{R}$ such that the function

$$f(x) = \begin{cases} 5x, & \text{if } x < -1, \\ x + a, & \text{if } -1 \leq x \leq 3, \\ 2x^2, & \text{if } x > 3 \end{cases}$$

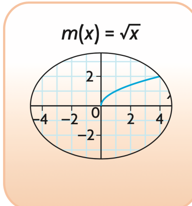
is continuous?

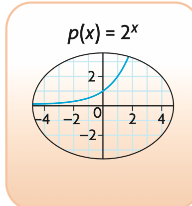
Proof. We want $-1 + a = 5 \times (-1)$ and $3 + a = 2 \times 3^2 = 18$, and such a does not exist. □

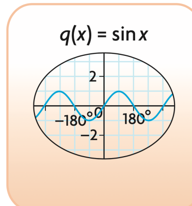
Problem 2.6. What does the greatest integer function $f(x) = [x]$ look like? This is the function that takes a real number x as input, and then outputs the largest integer n such that $n \leq x$.

Remark 2.7. Such functions are often called "step functions".

2.3 Example: Exponential Functions, Root Functions, Trigonometric Functions



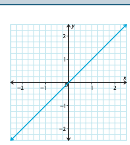
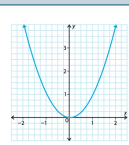
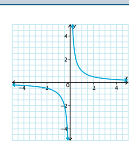
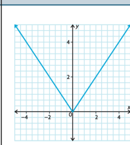
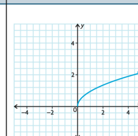
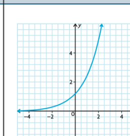
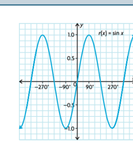




Function	a^x	$\sqrt[n]{x}$	Trigonometric
Restriction	$a > 1$	$n \geq 2$	N/A

Prototype	2^x	$\sqrt{x}$	$\sin(x)$
Domain	$\mathbb{R}$	$x \geq 0$	$\mathbb{R}$
Range	$\{y \in \mathbb{R} : y > 0\}$	$\{y \in \mathbb{R} : y \geq 0\}$	$\{y \in \mathbb{R} : -1 \leq y \leq 1\}$
x-intercept	N/A	$x = 0$	$x = n\pi,\ n \in \mathbb{Z}$
y-intercept	$y = 1$	$y = 0$	$y = 0$
$\nearrow$	$\mathbb{R}$	$\{x \in \mathbb{R} : x \geq 0\}$	$x \in [(2n - \frac{1}{2})\pi, (2n + \frac{1}{2})\pi]$
$\searrow$	N/A	N/A	$x \in [(2n + \frac{1}{2})\pi, (2n + \frac{3}{2})\pi]$
Symmetry	N/A	N/A	Odd
Asymptote	$y = 0$	N/A	N/A
Continuity	Yes	Yes	Yes
$x \to \infty$	$y \to \infty$	$y \to \infty$	Oscillation
$x \to -\infty$	$y \searrow 0$	N/A	Oscillation

Here is a table of the characteristics from the textbook pg. 615:

Parent Function	$f(x) = x$	$g(x) = x^2$	$h(x) = \frac{1}{x}$	$k(x) = \lvert x \rvert$	$m(x) = \sqrt{x}$	$p(x) = 2^x$	$r(x) = \sin x$
Sketch							
Domain	$\{x \in \mathbf{R}\}$	$\{x \in \mathbf{R}\}$	$\{x \in \mathbf{R} \mid x \neq 0\}$	$\{x \in \mathbf{R}\}$	$\{x \in \mathbf{R} \mid x \geq 0\}$	$\{x \in \mathbf{R}\}$	$\{x \in \mathbf{R}\}$
Range	$\{f(x) \in \mathbf{R}\}$	$\{f(x) \in \mathbf{R} \mid f(x) \geq 0\}$	$\{f(x) \in \mathbf{R} \mid f(x) \neq 0\}$	$\{f(x) \in \mathbf{R} \mid f(x) \geq 0\}$	$\{f(x) \in \mathbf{R} \mid f(x) \geq 0\}$	$\{f(x) \in \mathbf{R} \mid f(x) > 0\}$	$\{f(x) \in \mathbf{R} \mid -1 \leq f(x) \leq 1\}$
Intervals of Increase	$(-\infty, \infty)$	$(0, \infty)$	None	$(0, \infty)$	$(0, \infty)$	$(-\infty, \infty)$	$[90(4k + 1), 90(4k + 3)]$ $K \in \mathbf{Z}$
Intervals of Decrease	None	$(-\infty, 0)$	$(-\infty, 0)$ $(0, \infty)$	$(-\infty, 0)$	None	None	$[90(4k + 3), 90(4k + 1)]$ $K \in \mathbf{Z}$
Location of Discontinuities and Asymptotes	None	None	$y = 0$ $x = 0$	None	None	$y = 0$	None
Zeros	(0, 0)	(0, 0)	None	(0, 0)	(0, 0)	None	$180k$ $K \in \mathbf{Z}$
y-Intercepts	(0, 0)	(0, 0)	None	(0, 0)	(0, 0)	(0, 1)	(0, 0)
Symmetry	Odd	Even	Odd	Even	Neither	Neither	Odd
End Behaviours	$x \to \infty, y \to \infty$ $x \to -\infty, y \to -\infty$	$x \to \infty, y \to \infty$ $x \to -\infty, y \to \infty$	$x \to \infty, y \to 0$ $x \to -\infty, y \to 0$	$x \to \infty, y \to \infty$ $x \to -\infty, y \to \infty$	$x \to \infty, y \to \infty$	$x \to \infty, y \to \infty$ $x \to -\infty, y \to 0$	Oscillating

2.4 Transformation of Graphs of Functions

Given a function $f(x)$, one obvious way of creating a new function is to adjust its coefficients. For instance, what does the function $g(x) = af(b(x + c)) + d$ looks like for $a, b \neq 0$? Intuitively, this should look similar to the given function. We will play around with the function $f(x) = x^2$, and see what would happen.

- $a \neq 0$ gives a vertical stretch/compression by a factor of $|a|$ (depending on $|a| > 1$ or $|a| < 1$), where the sign of a determines having a reflection via the x-axis or not.
- $b \neq 0$ gives a horizontal stretch/compression by a factor of $\frac{1}{|b|}$ (depending on $|b| < 1$ or $|b| > 1$), where the sign of b determines having a reflection via the y-axis or not.
- c gives a vertical translation to the left (if $c > 0$) or to the right (if $c < 0$).
- d gives a horizontal translation upwards (if $d > 0$) or downwards (if $d < 0$).

In particular, a point (x, y) on the function, is now transformed to $(\frac{x}{b} - c, ay + d)$.

To determine the order of transformations, note that the stretches/compressions/reflections are with respect to the axes, which is harder to control compared to the translations. Therefore, we first work on those, and then the translations.

In Summary

Key Ideas

- Transformations on a function $y = af(k(x - d)) + c$ must be performed in a particular order: horizontal and vertical stretches/compressions (including any reflections) must be performed before translations. All points on the graph of the parent function $y = f(x)$ are changed as follows: $(x, y) \rightarrow \left(\frac{x}{k} + d, ay + c\right)$
- When using transformations to graph, you can apply a and k together, and then c and d together, to get the desired graph in the fewest number of steps.

Need to Know

- The value of a determines whether there is a vertical stretch or compression, or a reflection in the x-axis:
  - When $|a| > 1$, the graph of $y = f(x)$ is stretched vertically by the factor $|a|$.
  - When $0 < |a| < 1$, the graph is compressed vertically by the factor $|a|$.
  - When $a < 0$, the graph is also reflected in the x-axis.
- The value of k determines whether there is a horizontal stretch or compression, or a reflection in the y-axis:
  - When $|k| > 1$, the graph is compressed horizontally by the factor $\frac{1}{|k|}$.
  - When $0 < |k| < 1$, the graph is stretched horizontally by the factor $\frac{1}{|k|}$.
  - When $k < 0$, the graph is also reflected in the y-axis.
- The value of d determines whether there is a horizontal translation:
  - For $d > 0$, the graph is translated to the right.
  - For $d < 0$, the graph is translated to the left.
- The value of c determines whether there is a vertical translation:
  - For $c > 0$, the graph is translated up.
  - For $c < 0$, the graph is translated down.

Exercise 2.8. What does the function $g(x) = \sqrt{2(x - 6)}$ look like?

Proof. To relate this to a known function, we let $f(x) = \sqrt{x}$. Therefore, $g(x) = f(2(x-6))$. This gives a horizontal compression by a factor of $\frac{1}{2}$, and then moving to the right by 6 units. □

Exercise 2.9. Given $f(x) = 2^x$, what is the function $g(x)$ where we stretch $f(x)$ vertically by a factor of 3, reflected in the x-axis, and shifted 5 units to the right?

Proof. This is $g(x) = -3 \times 2^{(x-5)}$. □

Exercise 2.10. Suppose we know the point $(3, 6)$ is on the graph of $y = 2f(x+1) - 4$. What is the corresponding point on $y = f(x)$?

Proof. Note that the new graph we have in hand is obtained from $f(x)$ by a horizontal stretch of a factor of 2, moving to the left by 1 unit, and then moving downwards by 4 units. Therefore, reversing these operations, we have $(3, 6) \to (3, 10) \to (4, 10) \to (4, 5)$. □

Exercise 2.11. Given $f(x) = \sin(x)$, how do we obtain $g(x) = \cos(x)$?

Proof. Recall that $\sin(x + \frac{\pi}{2}) = \cos(x)$, so this is just moving the graph of sine function leftwards by $\frac{\pi}{2}$. □

2.5 Inverse relations

Suppose we have a function $f(x)$, then this tells us how some variable $y = f(x)$ behaves with respect to x. What if we want to know how x behaves with respect to y? We would want to write down a relation of x with respect to y.

Example 2.12. Suppose we have a function $y = x^2$, and we want to know what the inverse relation is, i.e., what is a relation of x with respect to y. We look at $x^2 = y$, then we need to get rid of the power of 2, so we take the square root on both sides. We obtain $x = \pm\sqrt{y}$, which tells you the relation of x with respect to y.

Remark 2.13. The inverse relation of a function may not be a function.

Exercise 2.14. What is the inverse of the formula $V(r) = \frac{4}{3}\pi r^3$?

Proof. We want to write r with respect to V, so we need to clear the right-hand side a little bit. We have $\frac{3}{4\pi}V = r^3$, so $r(V) = \sqrt[3]{\frac{3V}{4\pi}}$. □

Exercise 2.15. Suppose we have a function $y = f(x)$ and we know its graph, then what does the inverse relation of x with respect to y look like graphically?

Proof. The two graphs are symmetric with respect to $y = x$! For $y = f(x)$, we know given an input x, we receives an output value y. To obtain the inverse relation, we would expect that given an input y, the output should be x! Therefore, given any point (x, y) on the graph of $f(x)$, the point (y, x) is on the graph of the inverse relation. □

Definition 2.16 (Inverse Function)**.** If the inverse relation of $f(x)$ is a function as well, then we say the relation is the *inverse function* of $f(x)$, denoted $f^{-1}(x)$.

Exercise 2.17. Suppose the domain and range of a function $f(x)$ are A and B, respectively, and suppose $f(x)$ has an inverse function. What are the domain and range of the inverse function?

Proof. B and A, respectively. $\square$

Exercise 2.18. We know an inverse relation of a function is not necessarily a function, but when would it be a function?

Proof. We need the inverse relation of $f(x)$ to satisfy the vertical line test, i.e., given any vertical line, it passes through at most one point on the graph of the function. Mirroring it through $y = x$, we know given any horizontal line, it passes through at most one point on the graph of $f(x)$. Therefore, $f(x)$ is a function such that given any point y in the range, there exists exactly one point x in the domain that is mapped to y. Such functions are said to be *injective*. Thus, for the inverse of $f(x)$ to be a function, we need $f(x)$ to be injective. $\square$

2.6 Pointwise Operations on Function Structures

Given a function $f(x)$, we already saw how moving its graph around give rise to new functions $g(x)$'s based on $f(x)$ alone. We will now look at a more algebraic perspective of this, namely creating new function structures based on algebraic operations.

Given multiple functions with the same domain, we define the sum/difference/product/quotient of functions on the intersection as the pointwise sum/difference/product/quotient. For instance, let $f(x)$ and $g(x)$ be functions with domains and ranges as $\mathbb{R}$, then the pointwise-defined sum "$h = f + g$" is a function defined by $h(x) = f(x) + g(x)$ for any $x \in \mathbb{R}$.

In general, we can define such functions *at most* on the intersection of two domains. The function is only defined on x-values where both functions are defined!

Example 2.19. The function $f(x) = \frac{1}{x^2}$ is the product of $g(x) = h(x) = \frac{1}{x}$. Note that $g(x) = h(x)$ is defined on $\{x \in \mathbb{R} : x \neq 0\}$, so the product $f(x)$ is defined on the intersection, which is also $\{x \in \mathbb{R} : x \neq 0\}$.

Exercise 2.20. Given $f(x) = \sqrt{x - 10}$ and $g(x) = \sin(3x)$, what is the domain of $(f \times g)(x)$?

Proof. This is the intersection of the domains, so $x \geq 10$. $\square$

Exercise 2.21. Suppose $f(x)$ and $g(x)$ are functions with domains A and B, respectively. What is the domain of the quotient "$\frac{f}{g}(x)$"?

Proof. The quotient is defined pointwise by $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$, and this is defined wherever $g(x) \neq 0$. Therefore, the domain of this function is just $\{x \in A \cap B : g(x) \neq 0\}$. $\square$

Exercise 2.22. Given $f(x) = 10x^2$ and $g(x) = x^3 - 3x$, what is the domain of $\frac{f}{g}(x)$?

Proof. This is $\frac{10x^2}{x^3-3x}$, where $g(x) = 0$ implies $x = \sqrt{3}$ or $x = 0$. Therefore, this is defined on $x \in \mathbb{R}$ for $x \neq \sqrt{3}$, $-\sqrt{3}$, or 0. $\square$

Remark 2.23. This is a function different from $\frac{10x}{x^2-3}$!

The range of two such functions is usually defined to be the set of points expressed as the sum/difference/product/quotient of two functions, wherever they both are defined. What else can we say about the sum/difference/product/quotient of two functions, with respect to the characteristics we developed?

Example 2.24. The sum/difference/product of continuous functions is continuous. The quotient $\frac{f}{g}(x)$ of continuous functions f and g has a jump whenever $x \in \mathbb{R}$ is a point such that $g(x) = 0$.

3 August 31, 2023

Reference: Chapter 9.1-9.5.

3.1 Revision

First, we should talk about notations. We have introduced the notion of intervals in Definition 1.2 (click for review). Now, we denote a function f as $f : A \to B$ if A is the domain of f, and B contains the range of f (the range may not be B). For instance,

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^2 \end{aligned}$$

is a quadratic function with domain $\mathbb{R}$ and range $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$, so the range is contained in $\mathbb{R}$.

Now, recall we defined pointwise operations on functions as follows:

- the sum of two functions f and g is given by $(f + g)(x) = f(x) + g(x)$;
- the difference of two functions f and g is given by $(f - g)(x) = f(x) - g(x)$
- the product of two functions f and g is given by $(f \times g)(x) = f(x) \times g(x)$
- the quotient of two functions f and g is given by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$.

This is called *pointwise* because we define it using the two functions in a parallel sense. We should also define the domain of such functions. Suppose $f : A \to B$ and $g : C \to D$, then for the value $(f \star g)(x)$ to make sense (here $\star$ is any of the $+, -, \times, \div$ operations), we need both $f(x)$ and $g(x)$ to make sense. Therefore, the domain of $f \star g$ is the intersection of A and C, denoted $A \cap C$, which denotes the points contained in both A and C.

Moreover, for division to make sense, we need the result $\frac{f(x)}{g(x)}$ to make sense for all x in the domain. To do so, we additionally require x in $A \cap C$ to satisfy $g(x) \neq 0$.

Problem 3.1.
- Let $f : [0,1] \to \mathbb{R}$ and $g : [-1,2] \to \mathbb{R}$ be functions. Suppose $f(x)$ has an x-intercept at $x = 0$ and $g(x)$ has an x-intercept at $x = -1$. What can you say about the x-intercepts of the product function $(f \times g)(x)$?
- Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are functions with y-intercept at $y = 1$ and $y = 2$, respectively. What is the y-intercept of the sum/difference/product/quotient of two functions?
- Is the sum/difference of two even functions even? What about odd functions? What about those of an even and an odd function?
- Is the product/quotient of two even functions even? What about odd functions? What about those of an even and an odd function?

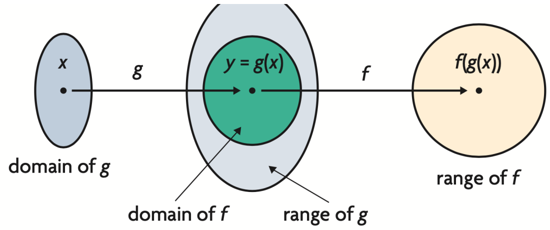
Proof.
- We know the x-intercept of $f \times g$ is just the points $(x, 0)$ such that $(f \times g)(x) = 0$. By definition, we know $f(x) \times g(x) = (f \times g)(x) = 0$, so we need either $f(x) = 0$ or $g(x) = 0$. We know $f(x) = 0$ when $x = 0$, and $g(x) = 0$ when $x = -1$, so we know two values of x such that $(f \times g)(x) = 0$, namely $x = -1$ and $x = 0$. However, not both points are defined, i.e., in the domain of function $f \times g$. Indeed, note that $x = 0$ is defined in the domain of f and domain of g, so it is defined on the domain of $f \times g$; but $x = -1$ is defined in the domain of g but not in the domain of f, so it is not in the domain of $f \times g$. Therefore, all we can say is that $(0, 0)$ is an x-intercept of $(f \times g)(x)$. We can not say $(-1, 0)$ is an x-intercept.
- This is just the pointwise sum/difference/product/quotient of two functions, i.e., we have y-intercepts at 3, -1, 2, $\frac{1}{2}$, respectively.
- Even; odd; inconclusive. The easiest way to see this is to check the algebraic definition of even/odd function.
- Even; even; odd. The easiest way to see this is to check the algebraic definition of even/odd function.

□

3.2 Function Composition

Suppose f is a function with domain A and range B, and g is a function with domain C and range D, **such that D contains A**, then the composition of f and g is defined by

$$f \circ g : C \to B$$
$$(f \circ g)(x) = f(g(x))$$



In this case, the domain of $f \circ g$ is C, the domain of g; the range of $f \circ g$ is B, the range of f. Recall that we define pointwise operations in a "parallel" sense. Opposite to that, we define composition operation in a "serial" sense.

This definition may not be beneficial due to the restriction of function domains and ranges. We will try to remove this condition, and define composition functions whenever D and A intersects (overlaps), i.e., $A \cap D \neq \varnothing$. In general, the domain of $f \circ g$ is a subset of the domain of g, such that $g(x)$ lands in the domain of f; the image of $f \circ g$ is a subset of the image of f. However, we need to restrict the domain of g so that the function makes sense, that means we need to get rid of **points in C that are mapped to points in D that is not in A**. Hence, domain of g is often larger than domain of $f \circ g$.

Example 3.2.
- $k(x) = x^4$ is the composition of $f(x) = x^2$ with itself: $(x^2)^2 = x^4$.
- $k(x) = |x-a|$ is the composition of $f(x) = |x|$ and $g(x) = x-a$. In particular, the composition operation of functions is not commutative.

Exercise 3.3. Suppose a function $f(x) : A \to B$ has an inverse function $f^{-1}(x) : B \to A$. What is $f \circ f^{-1}$? What is $f^{-1} \circ f$?

Proof. Suppose f sends $x \mapsto y$, then f^{-1} sends $y \mapsto x$, therefore $f \circ f^{-1}(y) = f(x) = y$, and $f^{-1} \circ f(x) = f^{-1}(y) = x$, therefore they are both identity functions id that sends $a \mapsto a$. □

Remark 3.4. The takeaway is that although function composition is usually not commutative, it holds for inverse functions.

Exercise 3.5. Suppose $y = 2x + 5$, $x = \sqrt{3t-1}$, and $t = 3k - 5$. Find the expression of function $y(k)$.

Proof. We have $y = 2x + 5 = 2\sqrt{3t-1} + 5 = 2\sqrt{3(3k-5)-1} + 5 = 2\sqrt{9k-16} + 5$. So the expression is $y(k) = 2\sqrt{9k-16} + 5$. □

Problem 3.6.
- Let $f(x) = x - 3$, then what is the function g given by f composing itself n times?
- Let $f(x) = \frac{1}{\sqrt{x+1}}$ and $g(x) = x^2 + 3$. What are the domains and ranges of $f \circ g$ and $g \circ f$? Also, find $(f \circ g)(0)$ and $(g \circ f)(0)$.

- Let $f(x) = x^2 - 2$, then what is the function $f\left(\frac{1}{x}\right)$? *Hint*: think about $f\left(\frac{1}{x}\right)$ as applying f on a (somewhat implicit) function defined by $x \mapsto \frac{1}{x}$, so you can regard it as a composition of functions.

3.3 Summary of Chapter 1 and 9

- Most of the functions are combinations of more basic functions.
- We studied seven basic functions in Chapter 1 and the ways of characterizing them using graphic and algebraic properties.
- To obtain new functions, there are several approaches:
  - transformation of graphs, in a graphical sense;
  - taking the inverse function, when applicable;
  - consider pointwise operations on functions with restricted domains, in an algebraic and parallel sense;
  - consider composition operation on functions with restricted domain, in an algebraic and serial sense.

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