Motivic Homotopy Theory Notes

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May 10, 2024

These notes were taken from a course on Motivic Homotopy Theory taught by Dr. P. Du in Spring 2024 at BIMSA. Any mistakes and inaccuracies would be my own. References for this course include [BH21], [EH23], [Lur18], [Lur09], and others mentioned in the references.

Contents

1 Commutative Monoids and Commutative Semirings as Functors

The materials from this section can be found in [EH23], Chapter 1.1-1.2.

1.1 Spans and Monoids

Definition 1.1. A commutative monoid $(M, \times, 1)$ has a multiplication operation

$$\begin{aligned} \times : M \times M &\to M \\ (a, b) &\mapsto a \times b =: ab \end{aligned}$$

that satisfies $ab = ba$, as well as the associativity by the pentagon axiom

$$\begin{tikzcd} & & (ab)(cd) \arrow[drr, dash] & & \\ ((ab)c)d \arrow[urr, dash] \arrow[dr, dash] & & & & a(b(cd)) \\ & (a(bc))d \arrow[rr, dash] & & a((bc)d) \arrow[ur, dash] & \end{tikzcd}$$

Definition 1.2. Denote $\mathbb{F} = \mathbf{FinSet}$ to be the finite category of finite sets, then a commutative monoid M induces a contravariant functor

$$\begin{aligned} \bar{M} : \mathbb{F}^{\mathrm{op}} &\to \mathbf{Set} \\ I &\mapsto M^I \\ (I \xleftarrow{f} S) &\mapsto (M^I \xrightarrow{f^*} M^S) \\ (a_i)_{i \in I} &\mapsto (a_{f(s)})_{s \in S} \end{aligned}$$

and similarly a covariant functor

$$\begin{aligned} \bar{M}' : \mathbb{F} &\to \mathbf{Set} \\ I &\mapsto M^I \\ (s \xrightarrow{g} I) &\mapsto (M^S \xrightarrow{g_\otimes} M^I) \\ (b_s)_{s \in S} &\mapsto \left(\prod_{s \in g^{-1}(j)} b_s \right)_{j \in J} \end{aligned}$$

Now given the construction in Definition 1.2 above, suppose we have a zigzag

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ I & & J \end{tikzcd} \tag{1.3}$$

we can use $\bar{M}$ and $\bar{M}'$ and obtain f^* and $g_\otimes$. One can map Diagram 1.3 to a morphism $g_\otimes f^* : M^I \to M^J$.

Remark 1.4. To define a functor precisely, we need to specify what category Diagram 1.3 lies in. As we will see later, we want a category with the same objects as $\mathbb{F}$, and morphisms are the zigzags of the form Diagram 1.3, which are called spans (or correspondences).

To define the composition of spans as morphisms, we should think of a diagram

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & & T \arrow[dl, "u"'] \arrow[dr, "v"] & \\ I & & J & & K \end{tikzcd} \tag{1.5}$$

The two zigzags give rise to $g_{\otimes}f^*$ and $v_{\otimes}u^*$. For compositions to be well-defined, we should map this diagram to $v_{\otimes}u^*g_{\otimes}f^*$. In order to obtain functoriality, we would hope

$$v_{\otimes}u^*g_{\otimes}f^* = v_{\otimes}g_{\otimes}u^*f^* = (vg)_{\otimes}(fu)^*.$$

This is certainly not true. As a remedy, we complete Diagram 1.5 to

$$\begin{tikzcd} & & A \arrow[dl, "u'"'] \arrow[dr, "g'"] & & \\ & S \arrow[dl, "f"'] \arrow[dr, "g"] & & T \arrow[dl, "u"'] \arrow[dr, "v"] & \\ I & & J & & K \end{tikzcd} \tag{1.6}$$

as we obtain $u^*g_{\otimes} : M^S \to M^T$ defined by the composition

$$(b_s)_{s\in S} \mapsto \left(\prod_{s\in g^{-1}(j)} b_s\right)_{j\in J} \mapsto \left(\prod_{s\in g^{-1}(u(t))} b_s\right)_{t\in T}.$$

Remark 1.7. If Diagram 1.6 is a commutative diagram, then there is a restriction of u' given by $u' : g'^{-1}(t) \to g^{-1}(u(t))$. In particular, if Diagram 1.6 is a pullback diagram, then this restriction map is a bijection. In this setting, the map $u^*g_{\otimes}$ sends $(b_s)_{s\in S}$ to

$$\left(\prod_{s\in g^{-1}(u(t))} b_s\right)_{t\in T} = \left(\prod_{a\in g'^{-1}(t)} b_{u'(a)}\right)_{t\in T} = g'_{\otimes}u'^*(b_s)_{s\in S}.$$

Therefore,

$$v_{\otimes}u^*g_{\otimes}f^* = v_{\otimes}g'_{\otimes}u'^*f^* = (vg')_{\otimes}(fu')^*.$$

Definition 1.8. We define $\mathbf{Span}(\mathbb{F})$ to be the category of span of $\mathbb{F}$, where objects are finite sets as in $\mathbb{F}$, and morphisms of the form $I \to J$ are the zigzag of the form $I \leftarrow S \to J$. The composition of morphisms $I \to J \to K$ on the zigzag is now defined by $I \leftarrow A \to K$ using the diagram

$$\begin{tikzcd} & & A \arrow[dl] \arrow[dr] \arrow[ddll, dashed, bend right] \arrow[ddrr, dashed, bend left] & & \\ & S \arrow[dl] \arrow[dr] & & T \arrow[dl] \arrow[dr] & \\ I \arrow[rr, dashed] & & J \arrow[rr, dashed] & & K \end{tikzcd}$$

whenever A is constructed as the pullback, otherwise known as the outer span $S \times_K T$.

Remark 1.9. One issue that persists from this construction is the fact that the pullback A is not unique. (This may be unique up to unique isomorphism.) With this in mind, $\mathbf{Span}(\mathbb{F})$ admits a $(2,1)$-category structure instead of an ordinary category.

The 2-morphisms of $\mathbf{Span}(\mathbb{F})$ are defined by $S \to S'$ via

$$\begin{tikzcd}
& S \arrow[dl] \arrow[dr] \arrow[dd] & \\
I \arrow[rr, dashed] & & J \\
& S' \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Moreover, these 2-morphisms are isomorphisms (of spans) and hence invertible, therefore admitting the $(2,1)$-category structure.

Remark 1.10. The functors we defined in Definition 1.2 can be extended to a functor

$$\tilde{M} : \mathbf{Span}(\mathbb{F}) \to \mathbf{Set}$$

such that $\tilde{M}\Big|_{\mathbb{F}^{\mathrm{op}}} \in \mathbf{Fun}^{\times}(\mathbb{F}^{\mathrm{op}}, \mathbf{Set})$. To see this, recall that there is a natural inclusion

$$\begin{aligned}
\mathbb{F}^{\mathrm{op}} &\hookrightarrow \mathbf{Span}(\mathbb{F}) \\
A &\mapsto A \\
(I \leftarrow S) &\mapsto (I \leftarrow S \xrightarrow{=} S)
\end{aligned}$$

then the extension $\tilde{M}$ is the functor we want, as the product and coproduct of the 2-category $\mathbf{Span}(\mathbb{F})$ are both the coproduct on $\mathbf{FinSet}$, i.e., the disjoint union.

Remark 1.11. In fact, given any category $\mathscr{C}$ with finite products, then there is an identification of commutative monoids on $\mathscr{C}$ with product-preserving functors $\mathbf{Span}(\mathbb{F}) \to \mathscr{C}$. Moreover, this is true homotopically, c.f., [Cra09] and [Cra11].

This is the story of how we induce functors from commutative monoids, and we will see below that there is a similar one for commutative semirings.

1.2 Bispans and Semirings

Definition 1.12. A commutative semiring $(R, +, \times, 0, 1)$ is a set R equipped with operations $+$ and $\times$ as well as additive identity 0 and multiplicative identity 1. However, we do not assume the existence of additive inverse and/or multiplicative inverse. Therefore, R is both an additive monoid and a multiplicative monoid.

Using the same construction in Definition 1.2, we have a functor

$$\begin{aligned}
\mathbb{F} &\to \mathbf{Set} \\
I &\mapsto R^I
\end{aligned}$$

which induces a functor

$$\begin{aligned}
\tilde{R}_{\times} : \text{“}\mathbf{Span}(\mathbb{F})\text{”} &\to \mathbf{Set} \\
(I \xleftarrow{f} S \xrightarrow{g} J) &\mapsto g_{\otimes} f^*
\end{aligned}$$

Now note that we still have an additive monoidal structure on R, so we would hope to define a functor of the form

$$\begin{aligned}
\tilde{R}_{+} : \text{“}\mathbf{Span}(\mathbb{F})\text{”} &\to \mathbf{Set} \\
? &\mapsto g_{\oplus} f^*
\end{aligned}$$

for some unknown category "$\mathbf{Span}(\mathbb{F})$". These two functors altogether shall define a desired functor $\tilde{R}$: "$\mathbf{Span}(\mathbb{F})$" $\to$ $\mathbf{Set}$. In particular, admitting two different structures here already tells us that the spans are no longer suitable, and a natural adaptation would be bispans.

Definition 1.13. A bispan (or a polynomial diagram) from I to J is given by a diagram

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[dl, "p"'] & Y \arrow[dr, "q"] & \\ I & & & J \end{tikzcd}$$

The category of bispans, denoted $\mathbf{Bispan}(\mathbb{F})$, has objects (again) the same with objects of $\mathbb{F}$, and morphisms are bispans.

Given a semiring R, we would want to construct a functor

$$\begin{aligned} \mathbf{Bispan}(R) &\to \mathbf{Set} \\ I &\mapsto R^I \\ (I \xleftarrow{p} X \xrightarrow{f} Y \xrightarrow{q} J) &\mapsto q_\oplus f_\otimes p^* \end{aligned}$$

where

$$\begin{aligned} p^* : R^I &\to R^X \\ p^*(\varphi)(x) &= \varphi(px), \end{aligned}$$

$$\begin{aligned} f_\otimes : R^X &\to R^Y \\ f_\otimes(\varphi)(y) &= \prod_{x \in f^{-1}(y)} \varphi(x), \end{aligned}$$

and

$$\begin{aligned} q_\oplus : R^Y &\to R^J \\ q_\oplus(\varphi)(j) &= \sum_{y \in q^{-1}(j)} \varphi(y), \end{aligned}$$

which represent composition (as pullback), fiberwise multiplication (as pushforward), and fiberwise addition (as pushforward), respectively. Altogether, this gives

$$\begin{aligned} q_\oplus f_\otimes p^* : M^I &\to M^J \\ (a_i)_{i \in I} &\mapsto \left(\sum_{y \in q^{-1}(j)} \prod_{x \in f^{-1}(y)} a_{p(x)} \right)_{j \in J}. \end{aligned}$$

Again, to construct such a functor, we need to consider the composition of bispans:

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[dl, "p"'] & Y \arrow[dr, "q"] & & X' \arrow[r, "g"] \arrow[dl, "u"'] & Y' \arrow[dr, "v"] & \\ I & & & J & & & K \end{tikzcd}$$

As we have seen previously, we need to study the pullback structure so that we can resolve $v_\oplus g_\otimes u^* q_\oplus f_\otimes p^*$. Using similar construction, we have

$$v_\oplus g_\otimes u^* q_\oplus f_\otimes p^* = v_\oplus g_\otimes q'_\oplus u'^* f_\otimes p^*$$

$$
\begin{aligned}
&= v_{\oplus}q''_{\oplus}g'_{\otimes}u'^{*}f_{\otimes}p^{*}\\
&= v_{\oplus}q''_{\oplus}g'_{\otimes}f'_{\otimes}u''^{*}p^{*}\\
&= (vq'')_{\oplus}(g'f')_{\otimes}(pu'')^{*}
\end{aligned}
$$

assuming we can construct $g'_{\otimes}$ and $q''_{\oplus}$ such that $g_{\otimes}q'_{\oplus} = q''_{\oplus}g'_{\otimes}$. That is, we have constructed two pullback squares

$$
\begin{tikzcd}
 & & A \arrow[dl, "u''"'] \arrow[r, "f'"] & Y \times_K X' \arrow[dl, "u'"'] \arrow[dr, "q'"] & & & \\
 & X \arrow[dl, "p"'] \arrow[r, "f"] & Y \arrow[dr, "q"] & & X' \arrow[dl, "u"'] \arrow[r, "g"] & Y' \arrow[dr, "v"] & \\
I & & & J & & & K
\end{tikzcd}
\tag{1.14}
$$

To deal with this, recall that addition distributes over multiplication, therefore given any

$$I \xrightarrow{u} J \xrightarrow{v} K$$

we know $v_{\otimes}u_{\oplus} : R^I \to R^K$ is the mapping defined by

$$
(a_i)_{i\in I} \mapsto \left(\prod_{j\in v^{-1}(k)} \sum_{i\in u^{-1}(j)} a_j\right)_{k\in K} = \left(\sum_{(i_j)\in \prod\limits_{j\in v^{-1}(k)} u^{-1}(j)} \prod_{t\in v^{-1}(k)} a_{i_t}\right)_{k\in K}. \tag{1.15}
$$

The goal is to identify the said image from Equation (1.15). Recall that the slice categories $\mathbf{FinSet}/\mathbf{K}$ and $\mathbf{FinSet}/\mathbf{J}$ are involved in a pullback/pushforward adjunction

$$
\begin{tikzcd}
\mathbf{FinSet}/\mathbf{K} \arrow[r, "\cong"] \arrow[d, shift right, "v^*"'] & \mathbf{Fun}(\mathbf{K}, \mathbf{Set}) \\
\mathbf{FinSet}/\mathbf{J} \arrow[r, "\cong"] \arrow[u, shift right, "v_*"'] & \mathbf{Fun}(\mathbf{J}, \mathbf{Set})
\end{tikzcd}
\tag{1.16}
$$

where

- $\mathbf{FinSet}/\mathbf{J} \cong \mathbf{Fun}(\mathbf{J}, \mathbf{Set})$ is a Grothendieck correspondence, where given $u : I \to J$, we obtain a functor

$$
\begin{aligned}
J &\to \mathbf{FinSet}\\
j &\mapsto u^{-1}(j)
\end{aligned}
$$

- $\mathbf{FinSet}/\mathbf{K} \cong \mathbf{Fun}(\mathbf{K}, \mathbf{Set})$ is a Grothendieck correspondence, where given $v : J \to K$, we obtain a functor

$$
\begin{aligned}
K &\to \mathbf{FinSet}\\
k &\mapsto v^{-1}(k)
\end{aligned}
$$

- $h = v_* u \in \mathbf{Set}/\mathbf{K}$ is a functor, and by the correspondence we obtain a functor

$$
\begin{aligned}
h' : K &\to \mathbf{Set}\\
k &\mapsto \prod_{j\in v^{-1}(k)} u^{-1}(j) = \prod_{j\in h^{-1}(k)} u^{-1}(j)
\end{aligned}
$$

- v^* is the pullback along $v : J \to K$. In particular, consider the counit $\varepsilon : v^*v_*I \to I$ of the adjunction, then for $X = v_*I$, the pullback $v^*X = v^*v_*I$ gives a counit ε in the diagram

$$\begin{tikzcd}
& v^*v_*I \arrow[r, "\tilde{v}"] \arrow[dl, "\varepsilon"'] \arrow[dd] & v_*I \arrow[dd, "h"] \\
I \arrow[dr, "u"'] & & \\
& J \arrow[r, "v"'] & K
\end{tikzcd} \tag{1.17}$$

We now make an effort to show that Diagram 1.17 actually commutes.

For any $k \in K$, we pullback $\alpha \in X$ such that $h(\alpha) = k$, but by the correspondence we know α is in the image of k along h'. Similarly, for $k \in K$, we pullback $j \in J$, and using the same argument we would then conclude that the pullback element in v^*X is just a pair (α, j).

Now let $i = \varepsilon(\alpha, j)$, but one can identify i to be the image of α under the projection $h^{-1}(k) \to \prod_{j' \in v^{-1}(k)} u^{-1}(j') \to u^{-1}(j)$. Therefore, $\alpha = (i_j)_{j \in v^{-1}(k)}$. For any fixed α, one can then identify

$$\prod_{t \in v^{-1}(k)} a_{it} = \prod_{j \in v^{-1}(k)} a_{\varepsilon(\alpha,j)}.$$

Therefore, the image of Equation (1.15) is

$$\left(\sum_{\alpha \in h^{-1}(k)} \prod_{j \in v^{-1}(k)} a_{\varepsilon(v,j)} \right)_{k \in K} = h_\oplus \tilde{v}_\otimes \varepsilon^* (a_i)_{i \in I}.$$

In particular, we obtain

$$v_\otimes u_\oplus = h_\oplus \tilde{v}_\otimes \varepsilon^*,$$

i.e., Diagram 1.17 commutes, which describes the distributivity.

Let us go back to Diagram 1.14. Using Diagram 1.17, we extend the diagram to

$$\begin{tikzcd}
& & A \arrow[r, "f'"] \arrow[dl, "u''"'] & Y \times_K X' \arrow[dl, "u'"'] \arrow[dr, "q'"] & B \arrow[l, "\varepsilon"'] \arrow[r, "\tilde{g}"] \arrow[d, "g^*g_*"] & Z \arrow[d, "h=g_*q'"] & \\
& X \arrow[r, "f"] \arrow[dl, "p"'] & Y \arrow[dr, "q"] & & X' \arrow[r, "g"] \arrow[dl, "u"'] & Y' \arrow[dr, "v"] & \\
I & & & J & & & K
\end{tikzcd}$$

which can be extended by taking one last pullback

$$\begin{tikzcd}
& & & G \arrow[dl, "\varepsilon'"'] \arrow[dr, "f''"] \arrow[dlll, dashed, bend right] \arrow[rr, dashed, bend left, "\otimes"] & & & \\
& & A \arrow[r, "f'"] \arrow[dl, "u''"'] & Y \times_K X' \arrow[dl, "u'"'] \arrow[dr, "q'"] & B \arrow[l, "\varepsilon"'] \arrow[r, "\tilde{g}"] \arrow[d, "g^*g_*"] & Z \arrow[d, "g_*q'"] \arrow[ddr, dashed, bend left, "\oplus"] & \\
& X \arrow[r, "f"] \arrow[dl, "p"'] & Y \arrow[dr, "q"] & & X' \arrow[r, "g"] \arrow[dl, "u"'] & Y' \arrow[dr, "v"] & \\
I \arrow[uuurrr, dashed, bend left, "(-)^*"] & & & J & & & K
\end{tikzcd}$$

and we define the composition to be the outer bispan in this diagram.

Remark 1.18. An explicit construction of this $(2,1)$-category $\mathbf{Bispan}(\mathbb{F})$ can be found in [Cra09], where it is proven that the category has a product structure given by coproducts of $\mathbf{FinSet}$. In this sense, commutative semirings in a category $\mathscr{S}$ correspond to functors $\mathbf{Bispan}(\mathbb{F}) \to \mathscr{S}$ that preserve finite products.

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