## MATH 595 (Group Cohomology) Notes

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## 1 Aug 21, 2023: Introduction

Group cohomology works over different settings of groups, like finite groups, profinite groups, and topological groups. The course will develop towards

- duality in  $H^*(G, -)$ , and
- focus on computations, e.g., spectral sequences.

We first establish some notations.

- Let G be a group. If G has a topology, that would also be part of the information of G.
- A (left) G-module is an abelian group M with an action map

$$G \times M \to M$$
  
 $(g, m) \mapsto g \cdot m = gm$ 

satisfying

- $-1 \cdot m = m$
- $-(gh) \cdot m = g \cdot (hm),$
- q(m+m') = qm + qm'.

Remark 1.1. If G is a finite group, then the associated (non-commutative) group ring  $\mathbb{Z}[G] := \bigoplus_{g \in G} \mathbb{Z}e_g$ , where the multiplication is determined by  $e_g e_h = e_{gh}$ . Therefore, a G-module is just a  $\mathbb{Z}[G]$ -module.

**Example 1.2.** • Trivial module  $\mathbb{Z}$ , or any abelian group with the trivial action  $g \cdot a = a$ .

- $C_2$ , or any group with  $f: G \to C_2$ , then G with  $C_2$  as a quotient gives the sign representation  $\mathbb{Z}_{sgn}$ , with  $g \cdot (a) = (-1)^{\rho(g)}a$ .
- $\mathbb{Z}[G]$  is a G-module via the left multiplication action, and/or the conjugation action.

**Definition 1.3** (Fixed points/Invariants). The set of fixed points of M over G is  $M^G = \{m \in M \mid gm = m \ \forall g \in G\}$ .

**Definition 1.4** (Orbits/Coinvariants). The set of orbits of M over G is  $M_G = M/(gm-m)$ .

**Example 1.5.** If  $M = \mathbb{Z}_{sgn}$ , then everything gets multiplied by -1, so there are no fixed points. The orbits of M over G would be  $\mathbb{Z}_{sgn}/(-2) \cong \mathbb{Z}/2\mathbb{Z}$ .

**Example 1.6.** If 
$$M=\mathbb{Z}[G]$$
, then the fixed points are  $\mathbb{Z}\left\{\sum_{g\in G}e_g\right\}$ .

Thinking in a categorical setting, we have a trivial action function  $\mathbb{Z}\text{-Mod} \to G\text{-Mod}$ , sending  $ga \mapsto a$  for all  $g \in G$  and  $a \in A$ . This gives an exact functor from Ab to G-Mod. Then this functor has a right adjoint () $^G: G\text{-Mod} \to Ab$ , and a left adjoint () $_G: Ab \to G\text{-Mod}$ . More specifically,  $M^G$  becomes the maximal trivial action submodule of M, namely  $Hom_G(\mathbb{Z}, M)$ ;  $M_G$  becomes the largest quotient of M with trivial action, namely  $\mathbb{Z} \otimes_{\mathbb{Z}[G]} M$ . This simplifies to the tensor-hom adjunction in some sense. For a more detailed derivation of this, see Chapter 6.1 of Weibel.

**Remark 1.7.** In general, as in the category of G-sets, we have the orbit functor  $X \mapsto X/G$  and the fixed point functor  $X \mapsto X^G$ . The orbit functor is left adjoint to the free G-set functor, and the fixed point functor is the right adjoint of the trivial G-set functor.

Remark 1.8. Read more about the setting in profinite groups with their topologies in Neukirch-Schmidt-Wingberg.

**Definition 1.9** (Profinite Group). A profinite group of a collection of groups is  $G = \varprojlim_i G_i$  as an inverse limit, where each  $G_i$  is a finite group of the form  $G/U_i$  for some open  $U_i$ . This gives a topology to the profinite group.

Remark 1.10. The groups rings  $\mathbb{Z}[[G]] = \varprojlim_i \mathbb{Z}[G_i]$ . For instance, let  $G = \mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n \mathbb{Z}$ , then  $\mathbb{Z}_p[[G]] = \varprojlim_n \mathbb{Z}_p[\mathbb{Z}/p^n \mathbb{Z}]$ , where each  $\mathbb{Z}[\mathbb{Z}/p^n \mathbb{Z}] \cong \bigoplus_{i=0}^{n-1} \mathbb{Z}\{e_i\}$  where  $e_i \cdot e_j = e_{ij}$ . Therefore,  $\mathbb{Z}_p[[G]]$  is now equivalent to  $\varprojlim_n \mathbb{Z}_p[t]/(t^{p^n} - 1_e)$ , and hence becomes a power series.

**Remark 1.11.** By a change of variables, this becomes  $\varprojlim_n \mathbb{Z}_p[x]/(x^{p^n})$ , but this only works in the finite group  $\mathbb{Z}_p$  case, and not in general for  $\mathbb{Z}$ .

Example 1.12.  $\mathbb{Z}[C_n] \cong \mathbb{Z}\{e\} \oplus \mathbb{Z}\{g\} \oplus \mathbb{Z}\{g^2\} \oplus \cdots \oplus \mathbb{Z}\{g^{n-1}\} \cong \mathbb{Z}[g]/(g^n - 1_e)$ .

2 Aug 23, 2021: Cohomology of groups

3 Aug 25, 2021: Cohomology of groups, continued