MATH 214B Notes

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1 Lecture 1

Globally, every divisor (codimension-1 subvariety) of $\mathbb{P}^n$ is defined by a single polynomial hom in $n+1$ variables.

Definition 1.1 (Complete Intersection)**.** A *complete intersection* in $\mathbb{P}_k^n$ is a subvariety of codimension-m that is defined by m equations.

A subscheme in $\mathbb{P}_k^n$ of codimension m is said to be set-theoretic a complete intersection if there is an ideal $I \subseteq K[x_0, \dots, x_n]$ generated by m elements whose vanishing locus is exactly the subvariety.

Remark 1.2 (Hartshorne's Conjecture)**.** Every closed curve in $\mathbb{P}^3$ is a set-theoretic complete intersection.

Example 1.3 (Normal Rational Curve of Degree n)**.**

$$\begin{aligned} \varphi_n : \mathbb{P}_k^1 &\to \mathbb{P}_k^n \\ [s : t] &\mapsto [s^n : s^{n-1}t : \cdots : t^n] \end{aligned}$$

This is an embedding with $C_n := \varphi_n(\mathbb{P}_k^1)$.

Theorem 1.4 (Perron, 1941)**.** If $\frac{2^{n-1}}{n} \in \mathbb{Z}$, then C_n is a set-theoretic complete intersection of $n-1$ quadrics.

Theorem 1.5 (Gallarati-Rollero, 1988)**.** C_n is the set-theoretic complete intersection of $s-1$ quadrics and $n-s$ forms of degree $s+1, \dots, n$ where $s = \max\{k \in \mathbb{N} \mid 2^k \leq n\}$.

Therefore, although the case of codimension-1 is easy, it becomes increasingly difficult for higher codimensions.

Locally, every codimension-1 subvariety of a smooth variety is locally cut out by a single equation.

Example 1.6. The cone over $C_3 \hookrightarrow \mathbb{P}^3$ is given by

$$\mathrm{Cone}(C_3) \hookrightarrow \mathbb{A}^4,$$

where $\mathrm{Cone}(C_3)$ is a surface.

Remark 1.7. There is a correspondence (equivalence of categories) between projective varieties and germs of a singularity: given projective variety $X \hookrightarrow \mathbb{P}^n$, we obtain the germ $\bar{X} \hookrightarrow \mathbb{A}^{n+1}$ by looking at the cones; given a germ $\bar{X} \hookrightarrow \mathbb{A}^{n+1}$, we recover the projective variety by considering the collection of tangent directions.

The automorphisms (bijectives) of projective n-space (i.e., $\mathrm{Aut}(\mathbb{P}_k^n)$) is well-studied, and is known to be $\mathrm{PGL}(n+1)$. However, it is hard to study that for affine $\mathbb{A}_k^n$ spaces.

Theorem 1.8 (Lefschetz Principle, 1960s)**.** A first order logic proposition is true on $\mathbb{C}$ if it is true for $\mathbb{F}_p$ for infinitely many p's.

Theorem 1.9 (Ax-Grothendieck)**.** If $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ is a injective morphism, then it is surjective.

To do projective geometry, we have the following strategy: given a projective variety X, understand all its codimension-1 subvarieties.

Definition 1.10 (Irreducible Divisor)**.** Let X be an integral Noetherian scheme. An *irreducible divisor* on X is a closed irreducible subvariety of codimension 1.

Remark 1.11. If η_Y is the generic point of an irreducible subvariety $Y \hookrightarrow X$, then $\dim(\mathcal{O}_{X,Y}) = 1$ is a DVR with $v_Y : k(X)^* \to \mathbb{Z}$ such that $v_Y(f)$ is the order of vanishing of f at Y.

Example 1.12. $f = \frac{x}{y} \in k(\mathbb{A}_k^2)$, then the order of f at $x = 0$ is 1, and the order of f at $y = 0$ is -1.

Lemma 1.13. If X is an integral regular Northerian scheme with $f \in k(X)^*$, then $\mathrm{ord}_Y(f) = 0$ for all but finitely $Y \subseteq X$.

The *group of divisors* of X is the collection of finite formal sums $\sum_{i \in I} \alpha_i D_i$ for $\alpha_i \in \mathbb{Z}$ and D_i are irreducible divisors. For $f \in k(X)^*$, $\mathrm{div}(f) = (f) = \sum_{Y \subseteq X} \mathrm{ord}_Y(f) Y$. The group of divisors is denoted by $\mathrm{Div}(X)$ or Weil divisors.

Definition 1.14 (Principal Divisor)**.** A *principal divisor* is a divisor of the form $\operatorname{div}(f)$ for $f \in k(X)^*$. This forms a subgroup $\operatorname{Prin}(X)$ in the group of divisors, since $\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$.

Proof of Lemma. For $f \in k(X)^*$, there exists open subset $U \subseteq X$ such that $f\,|_U$ is regular. If $Y \cap U \neq \varnothing$, then $\operatorname{ord}_Y(f) \geq 0$, but $X \setminus U$ contains finitely many divisors. For the other part, consider f^{-1}. $\square$

Remark 1.15. The order is well-defined from the condition.

Definition 1.16 (Divisor Class Group)**.** The *divisor class group* of X is defined to be $\operatorname{Div}(X)/\operatorname{Prin}(X)$.

Definition 1.17 (Linearly Equivalent)**.** Two divisors D_1 and D_2 on X are said to be *linearly equivalent* (denoted by $D_1 \sim D_2$) if $D_1 - D_2 = \operatorname{div}(f)$ for some rational function f.

We denote the class group of X by $\operatorname{Cl}(X)$.

Example 1.18. $\operatorname{Cl}(\mathbb{P}_k^n) = \mathbb{Z}$. Let $H_d \subseteq \mathbb{P}^n$ be the vanishing locus of f_d polynomial hom of degree d. At x_0 (the first coordinate), the function $\frac{f_d}{x_0^d}$ is rational, and $\operatorname{div}\left(\frac{f_d}{x_0^d}\right) = H_d - dH_0$.

2 Lecture 2

Proposition 2.1. Let A be a normal Noetherian domain. Let $U = \operatorname{Spec}(A)$. Then $\operatorname{Cl}(U) = 0$ if and only if A is a UFD.

Proof. First suppose A is a UFD. Let $Y \subseteq U$ be a codimension-1 subvariety. Y corresponds to a prime ideal of $\mathfrak{p} \subseteq A$ with height 1, hence $\mathfrak{p} = (f)$ for some $f \in A$. Thus, $Y = (f)$ in $\operatorname{Div}(U)$. $\square$

Exercise 2.2. Describe the class group of $U_n = \operatorname{Spec}(k[x, y, z]/\langle xy - z^n\rangle)$. Compute $\operatorname{Cl}(U_n)$. How does it depend on n?

In fact, $\operatorname{Cl}(U_n) = \mathbb{Z}_n$.

Example 2.3. For any field k and $n \geq 1$, $\operatorname{Cl}(\mathbb{A}_k^n) = 0$.

Definition 2.4 (Factorial Variety)**.** A variety is called *factorial* if every divisor D is locally cut out by a single equation. In other words, for $p \in X$, there exists a neighborhood $U \ni p$ for which $[D|_U] = 0 \in \operatorname{Cl}(U)$.

A variety is called $\mathbb{Q}$-*factorial* if for every divisor D and $p \in X$, there exists a neighborhood U of p in X and an integer $m \neq 0$, for which $[mD|_U] = 0$ in $\operatorname{Cl}(U)$.

Remark 2.5. If $\mathrm{Cl}(X)$ is finitely-generated, then $\mathrm{Cl}(X) \simeq \mathbb{Z}^k \oplus T$ where T is finite, abelian, and torsion.

To construct a pullback for a map $f : X \to Y$ with respect to $D = \{g = 0\}$,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left] & Y \arrow[r, "g"] & \mathbb{P}^1 \end{tikzcd}$$

we define it by defining it locally with respect to the definition of D.

Remark 2.6. Suppose Y is $\mathbb{Q}$-factorial and D is on Y, and mD is *Cartier* (locally cut out by a single equation), then

$$f * D = f^*(mD)/m.$$

Theorem 2.7 (Algebraic Hartog Lemma)**.** Let R be a normal Noetherian domain, then

$$R = \bigcap \{R_p \mid p \subseteq R \text{ prime ideal of codimension-1}\} \subseteq \mathrm{Frac}(R).$$

Corollary 2.8. Let X be an affine normal integral Noetherian scheme, then a rational function f on X is regular if and only if $\mathrm{ord}_Y(f) \geq 0$ for each $Y \subseteq X$ is an irreducible divisor.

Corollary 2.9. Let X be a normal scheme and $Y \subseteq X$ be a closed subset of codimension ≥ 2. Then the restriction $\mathcal{O}(X) \to \mathcal{O}(X \backslash Y)$ is an isomorphism. In particular, $\mathcal{O}(\mathbb{A}^n_k \backslash \{0\}) \cong \mathcal{O}(\mathbb{A}^n_k) = k[x_1, \ldots, x_n]$.

Remark 2.10. Normal singularities are the "worst" class of singularities for which Hartog's lemma holds.

Example 2.11.

$$\{x = y = 0\} \cup \{z = w = 0\} \subseteq \mathbb{A}^4_k$$

is not normal.

Theorem 2.12. Let k be a field and $n \geq 1$, then $\mathrm{Cl}(\mathbb{P}^n_k) \cong \mathbb{Z}$ and is generated by the class of a hyperplane.

Proof. Consider

$$\psi : \mathbb{Z} \to \mathrm{Cl}(\mathbb{P}^n_k)$$
$$1 \mapsto [H]$$

We first show surjectivity. Let $Y \subset \mathbb{P}^n_k$ be a codimension-1 closed subvariety that corresponds to a homogeneous prime ideal $p \subseteq k[x_0, \ldots, x_n]$, then $p = (f)$ because we are in a

UFD, so $\deg(f) = d$ and $Y = \{f = 0\} \subseteq \mathbb{P}^n_k$. Let $H = \{x_0 = 0\} \subseteq \mathbb{P}^n_j$, then $\frac{f}{x_0^d}$ is rational on $\mathbb{P}^n_k$ and $\left(\frac{f}{x_0^d}\right) = Y \setminus dH$, thus $Y = dH$.

We now show injectivity. Assume $dH \sim 0$ and $d > 0$, then $dH = (f)$. Since all the orders of vanishing of the rational function f are ≥ 0, by corollary of the algebraic Hartog Lemma, f must be regular. Therefore, f must be a non-zero constant, contradiction. □

Theorem 2.13 (Auslander–Buchsbaum, 1950s)**.** A regular local ring is a UFD.

Definition 2.14 (Sheaf associated to a Divisor)**.** Let D be a divisor on a normal Noetherian scheme X. Consider open set $U \subseteq X$, then

$$\mathcal{O}_X(D)(U) = \{f \in k(X) \mid (\operatorname{div}(f) + D)|_U \geq 0\}$$

gives a coherent sheaf of $\mathcal{O}_X$-modules. Furthermore, $\mathcal{O}_X(0) = \mathcal{O}_X$. The *sheaf associated to* D is denoted $\mathcal{O}_X(D)$.

Proposition 2.15. If divisors $D_1 \sim D_2$, then $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$ as $\mathcal{O}_X$-modules.

Proof. $D_1 \setminus D_2 = (\varphi)$ for $\varphi \in k(X)^*$, then there is an isomorphism

$$\mathcal{O}_X(D_1) \xrightarrow{\cong} \mathcal{O}_X(D_2)$$
$$f \mapsto fg$$

. □

Proposition 2.16. If X is a regular scheme, then $\mathcal{O}_X(D)$ is a line bundle for every D.

Proof. Fix $p \in X$, and write $D = \sum a_i D_i$, so $D_i = (f_i)$ near p. Then $D \sim 0$ near p, hence $\mathcal{O}_X(D) \cong \mathcal{O}_X$ near p, so $D = \operatorname{div}(f_i^{a_i})$. □

Theorem 2.17. Let X be a normal Noetherian integral scheme, then there is an injective homomorphism (first Chern class)

$$c_1 : \operatorname{Pic}(X) \to \operatorname{Cl}(X).$$

Moreover, if X is regular, then c_1 is an isomorphism.

Remark 2.18. To construct c_1, consider $\mathcal{L} \to X$ where $\mathcal{L}$ is a line bundle and let s be a rational section of $\mathcal{L}$. We think of s as an element $\mathcal{L}_{\eta_X}$ defined by $(s) = \sum_{Y \subseteq X} \operatorname{ord}_Y(s) Y \in \operatorname{Div}(X)$. Note that $\mathcal{L}_{\eta_Y}$ is free of rank-1 over $\mathcal{O}_{X,\eta_Y} = \mathcal{P}_{X,Y}$, then $\operatorname{ord}_Y(s) = -\min\{r \in \mathbb{Z} \mid t^r s \in \mathcal{L}_Y\}$, then t is a local uniformizer of $\mathcal{O}_{X,Y}$. Now $(fs) = (s) + (f)$ for any $f \in k(X)^*$, so this is a homomorphism.

3 Lecture 3

Proof. Let X be normal and $\mathcal{L}$ be a line bundle over X, then $c_1(\mathcal{L}) = 0$. Let s be a rational section of $\mathcal{L}$, then $(s) = (f)$ for some $f \in k(X)$. Define $t = \frac{s}{f}$ of $\mathcal{L}$, so $(t) = 0$. Since $(t) \geq 0$, we have a morphism

$$\begin{tikzcd} \mathcal{L}(\mathcal{O}_X) \arrow[rr, "t"] \arrow[dr] & & \mathcal{L} \arrow[dl] \\ & X & \end{tikzcd}$$

Since $(t) \leq 0$, we have a morphism

$$\begin{tikzcd} \mathcal{L} \arrow[rr, "t^{-1}"] \arrow[dr] & & \mathcal{L}(\mathcal{O}_X) \arrow[dl] \\ & X & \end{tikzcd}$$

This proves injectivity. Suppose X is regular furthermore, then for every divisor $Y \subseteq X$, $\mathcal{O}(Y)$ is a line bundle, then $c_1(\mathcal{O}(Y)) = Y$. □

Proposition 3.1. Let X be a smooth proper curve over k. Let $f : X \to Y$ be a morphism over k. Then, either

1. $f(X) = p$, or
2. $f : X \to Y$ is a surjective finite morphism, $k(X) \supseteq k(Y)$ is a finite field extension, and Y is proper over k.

Lemma 3.2. Let $f : X \to Y$ be a dominant rational map of varieties of the same dimension over k, then $k(X) \supseteq k(Y)$ is finite.

Lemma 3.3 (Hartshorne, 4.4)**.** If $f : X \to Y$ is a morphism over k, where X and Y are over k. If X is proper over k and Y is separated over k, then $f(X)$ is proper over k.

Proof of Proposition. Observe that $f(X)$ is closed and irreducible. Suppose $f : X \to Y$ is surjective, and $V = \mathrm{Spec}(B)$ is open in Y. Here, $B \subseteq K(Y)$ and in fact $\mathrm{Frac}(B) = k(Y)$. Let A be an integral closure of B in $k(X)$. Since $k(X) \supseteq k(Y)$ is finite, we conclude that A is a finitely-generated k-algebra and a domain. Denote $U = \mathrm{Spec}(A)$. We obtain a finite morphism $f : U \to V$ as open sets, where U is a smooth curve over k and X is its unique proper model. If $f^{-1}(V) = U$, then we are done. If this was true, then for any point $x \in X$, there is a neighborhood on which f is finite, which is a local property. Let U_0 be adding a point onto U (i.e., supposing there is some point not defined on U that cannot attain image in V). The valuative criteria for properness implies that $U_0 \dashrightarrow U$ (inclusion map) extends to a rational morphism $U_0 \to U$, which implies $f^{-1}(V) = U$. □

Proposition 3.4 (Algebraic Version of Liouville's Theorem)**.** Let X be a projective variety over k, and $f \in k(X)^*$ for which $(f) \geq 0$, then f is constant.

Proof. Consider $X \hookrightarrow \mathbb{P}_k^N$. There is a map $f : X \dashrightarrow \mathbb{P}_k^1$ that is defined at codimension-1 points. Let Z be the loci where f f is not defined and $n = \dim(X)$. Let $H_1, \ldots, H_{n-1}$ be general hyperplanes on $\mathbb{P}_k^N$ with $\operatorname{codim}(Z, X) \geq 2$. Then $C := (H_1 \cap \cdots \cap H_{n-1} \cap X) \cap Z = \varnothing$. We obtain a restricted map $f|_C : C \to \mathbb{P}_k^1$, but since $(f) \geq 0$ so $(f|_C) \geq 0$. Therefore, f is constant. $\square$

Definition 3.5 (Degree)**.** Let $f : X \dashrightarrow Y$ be a dominant rational map of varieties of the same dimension, then $\deg(f) = [k(X) : k(Y)] < \infty$. If X is a curve and $p \in X$ is a point, then the *degree* of the point is $[k(p) : k] < \infty$. If k is algebraically closed and $p \in X$ is closed, then $\deg(p) = 1$.

Suppose $D = \sum_{i \in I} \alpha_i p_i$, where $\alpha_i \in \mathbb{Z}$ and $p_i \in C$be closed points in C over k. Then the *degree* of the divisor is defined by $\deg(D) = \sum_{i \in I} \alpha_i \deg(p_i)$.

Definition 3.6 (Pullback)**.** Let $f : X \to Y$ be a finite morphism of smooth curves over k, then there is a map $f^* : \operatorname{Div}(Y) \to \operatorname{Div}(X)$ defined by $f^*(p) = \sum_{q \in f^{-1}(p)} \alpha_q \cdot q$. For $t \in \mathcal{O}_{X,p}$ a uniformizer (i.e., $\operatorname{ord}_p(f) = 1$), then $f^*(t) = v^{\alpha_q}$ in $\mathcal{O}_{Y,q}$ where v is a uniformizer of $\mathcal{O}_{Y,q}$.

Example 3.7. Consider $f : \mathbb{P}_k^1 \to \mathbb{P}_k^1$ defined by $[s : t] \mapsto [s^2 : t^2]$. The pullback satisfies $f^*[1 : 0] = 2[1 : 0]$ and $f^*[1 : 1] = [1 : -1] + [1 : 1]$.

Theorem 3.8 (Degree of Pullback)**.** Let $f : X \to Y$ be a finite morphism between smooth curves over k. Let $D \in \operatorname{Div}(Y)$, then $\deg(f^*D) = \deg(f) \cdot \deg(D)$.

Proof. First suppose $D = p$ is closed in Y. (Note that the linearity of both sides takes care of the general case.) Suppose $V = \operatorname{Spec}(B)$ and $f^{-1}(V) = \operatorname{Spec}(A) = U$, and A is finite over B. Consider $A \otimes_B \mathcal{O}_{Y,p}$ be a module over $\mathcal{O}_{Y,p}$ is a finitely-generated torsion-free $\mathcal{O}_{Y,p}$-module, free of rank d. So $A \otimes_B k(Y)$ is free of rank d over $k(Y)$ and $d = \deg(f)$. Thus, $A \otimes_B k(p)$ is free of rank d over $k(p)$. This implies that $f^*(p) = (\deg(f))(\deg(p))$. $\square$

Corollary 3.9. Let X be a smooth proper curve over k and $f \in k(X)^*$, then $\deg((f)) = 0$.

Proof. Consider $f : X \dashrightarrow \mathbb{P}_k^1$. Since X is smooth and $\mathbb{P}_k^1$ is proper, then f extends to a morphism $f : X \to \mathbb{P}_k^1$. If f is constant, then $(f) = 0$. If f is not constant, it must be a finite surjective morphism of degree d by Proposition 3.1. Moreover, by Theorem 3.8, $\deg(f^*(0)) = \deg(f) = \deg(f^*(\infty))$, and $(f) = f^*(0) - f^*(\infty)$, so $\deg((f)) = 0$. $\square$

Remark 3.10. This defines a degree map $\deg : \operatorname{Cl}(X) \cong \operatorname{Pic}(X) \to \mathbb{Z}$ for any smooth curve X over k, which agrees with the concept in complex geometry.

4 Lecture 4

Consider the projective morphisms, usually given by

$$X \to \mathbb{P}^n_A = \mathrm{Proj}(A[x_0, \ldots, x_n])$$

which is basically equivalent to $\mathcal{L}$ on X and global sections $s_1, \ldots, s_n \in \Gamma(X, \mathcal{L})$.

Let A be a ring, consider $\mathbb{P}^n_A$ projective, and a scheme X over A. There is a map $\varphi : X \to \mathbb{P}^n_A$ as sending line bundles $\varphi^*(\mathcal{O}(1))$ (such that $\varphi^*(x_i) = s_i \in \Gamma(X, \mathcal{L})$ as described) to line bundles $\mathcal{O}(1)$ generated by global sections $x_0, \ldots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$.

Theorem 4.1. Let A be a ring and X be a scheme over A. If $\mathcal{L}$ describes line bundles on X and $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ generating $\mathcal{L}$. Then there exists a unique $\varphi : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong \varphi^*(\mathcal{O}(1))$ and $s_i = \varphi^*(x_i)$.

Proof. Consider $X_i = \{P \in X \mid (s_i)_p \notin m_p\mathcal{L}_p\}$ which are open in X. Since s_i's generate $\mathcal{L}$, then $X = \bigcup X_i$. Define $U_i = \{x_i \neq 0\}$, then $U = \mathrm{Spec}(A[y_0, \ldots, \tilde{y}_i, \ldots, y_n] = \mathrm{Spec}\left(A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right]\right)$. Therefore, a map $X_i \to U_i$ is equivalent to a morphism $A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right] \to \Gamma(X_i, \mathcal{O}_i)$ global sections, where the map is defined by $\frac{x_j}{x_i} \mapsto \frac{s_j}{s_i}$ and thus gives a morphism ψ in the global section. In particular, these maps glue together. To see uniqueness, the morphism has to be exactly ψ locally on X_i. Therefore, φ is unique. $\square$

Remark 4.2 (Automorphisms of $\mathbb{P}^n_k$)**.** Let k be a field, consider the $n \times n$-matrices in $\mathrm{GL}_k(n+1)$. This gives $\mathrm{Aut}(k[x_0, \ldots, x_n])$ and therefore gives $\mathrm{Aut}(\mathbb{P}(k[x_0, \ldots, x_n])$. Note that $M \sim N$ gives the same automorphism if and only if $M = \lambda N$ for $\lambda \in K \setminus \{0\}$. Therefore, if $M = \lambda N$ then they have the same automorphism group of rings; if $M \neq \lambda N$, then use the coordinates and look at points $(1 : 0 : \cdots : 0), (0 : 1 : 0 : \cdots : 0), \ldots, (0 : \cdots : 0 : 1)$. Hence, $\mathrm{PGL}_k(n+1) = \mathrm{GL}_k(n+1)/\lambda < \mathrm{Aut}(\mathbb{P}^n_k)$. On the other hand, let $\varphi : \mathbb{P}^n_k \to \mathbb{P}^n_k$ be an automorphism of projective space, then note that $\mathrm{Pic}(\mathbb{P}^n_k) \cong \mathbb{Z}$ with generators $\mathcal{O}(1)$. Thus, φ induces $\varphi^* : \mathrm{Pic}(\mathbb{P}^n_k) \to \mathrm{Pic}(\mathbb{P}^n_k)$ defined by $\mathcal{O}(1) \mapsto \varphi^*(\mathcal{O}(1))$. Now $\varphi^*(\mathcal{O}(1))$ is either $\mathcal{O}(1)$ or $\mathcal{O}(-1)$, i.e., has no global sections. Therefore, $\varphi^*(\mathcal{O}(1)) = \mathcal{O}(1)$. Therefore, the global sections of the sheaf $\Gamma(\mathbb{P}^n_k, \mathcal{O}(1))$ is a k-vector space with basis $x_0, \ldots, x_n$. The morphisms to $\mathbb{P}^n_k$ corresponds to the global sections s_i's. Each $\varphi^*(\lambda_0) = s_i = \sum a_i x_i$. Hence, the only choice that corresponds to picking the morphism is this matrix, and corresponds to the original construction.

Proposition 4.3. Let A be a ring and let $\varphi : X \to \mathbb{P}^n_A$ corresponds to $\mathcal{L}$, with $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$. Then φ is a closed immersion if

1. $X_i = \{p \in X \mid (s_i)_p \notin m_p\mathcal{L}_p\}$ is affine, and

2. $A[y_0, \ldots, y_n] \to \Gamma(x_i, \mathcal{O}_i)$ are surjective maps.

Proof. If φ is a closed immersion, with $X \subseteq \mathbb{P}^N_A$. Let $X_i = X \cap U_i$ be closed subschemes of U_i, an affine scheme. In particular, the closed subschemes of $\mathrm{Spec}(B)$, i.e., $\mathrm{Spec}(B/b) \hookrightarrow \mathrm{Spec}(B)$, should correspond to $B \twoheadrightarrow B/b$, hence 1) and 2). Also, given 1) and 2), we know that $X_i = \mathrm{Spec}(C)$, then $\Gamma(X_i, \mathcal{O}_i) = C$, therefore corresponds to a map $A[x_0, \ldots, x_n] \twoheadrightarrow C = A[x_0, \ldots, x_n]/b$, therefore corresponds to a closed subscheme of $U_i \subseteq \mathbb{P}^N_A$, hence gives a closed subscheme $X \subseteq \mathbb{P}^N_A$, as the inclusion glues to one inclusion. $\square$

Proposition 4.4. Let $k = \bar{k}$ and X be a projective scheme over k. Define $\varphi : X \to \mathbb{P}^n_k$, and consider $\mathcal{L}$ with $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$. Let $V \subseteq \Gamma(X, \mathcal{L})$ be generated by $s_0, \ldots, s_n$. Now φ is a closed immersion if

1. V separates points, i.e., for P, Q closed, there exists $s \in V$ such that $s \in m_P \mathcal{L}_P$ but $s \notin m_Q \mathcal{L}_Q$ (and vice versa), and

2. V separates tangent vectors, i.e., for P closed, $\{s \in V \mid s_p \in m_p \mathcal{L}_p\}$ spans $m_p \mathcal{L}_p / m_p^2 \mathcal{L}_p$.

Remark 4.5. Geometrically, this comes from hyperplanes in $\mathbb{P}^n$ separating points on tangent directions. On the other hand, φ as a closed map gives a surjection $\mathcal{O}_{\mathbb{P}^n} \twoheadrightarrow \varphi_* \mathcal{O}_X$, which comes from an algebraic lemma, and the closedness comes from properness since X is projective.

5 Lecture 5

Proof. ($\Rightarrow$): Separate by hyperplanes on $\mathbb{P}^n_k$.

($\Leftarrow$): Since X is projective over k, then it is proper over k, so $\varphi(X) \subseteq \mathbb{P}^n$ is closed and so φ is proper. Therefore, φ is a closed map. It now suffices to show that $\mathcal{O}_{\mathbb{P}^n} \twoheadrightarrow \varphi_* \mathcal{O}_X$, which can be shown on stalks, by using the following lemma on the immersion case. $\square$

Lemma 5.1. If $f : A \to B$ is a local homomorphism of locally Noetherian rings. If

1. $A/m_A \to B/m_B$ is an isomorphism,

2. $m_A \to m_B/m_B^2$ is a surjection, and

3. B is a finitely-generated A-module,

then $f : A \twoheadrightarrow B$ is a surjection.

Definition 5.2. Let $\mathcal{L}$ be a line bundle on X. We say $\mathcal{L}$ is *very ample* with respect to Y if there exists an immersion $i : X \to \mathbb{P}^n_Y$ such that $\mathcal{L} \cong i^*(\mathcal{O}(1))$.

Remark 5.3. We say examples of very ample line bundles and criterions over $\mathrm{Spec}(k)$.

Proposition 5.4. Let X be a projective space over A, and $\mathcal{L}$ a very ample line bundle over $\mathrm{Spec}(A)$. For all coherent sheaves $\mathcal{F}$ on X, there exists $n_0 > 0$ such that $n \geq 0$ and that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections.

Definition 5.5. Let $\mathcal{L}$ be a line bundle on X. We say $\mathcal{L}$ is *ample* if for all coherent sheaves $\mathcal{F}$, there exists $n_0 > 0$ such that $n \geq n_0$ and $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections.

Proposition 5.6. Let $\mathcal{L}$ be a line bundle on X. The following are equivalent:

1. $\mathcal{L}$ is ample,
2. $\mathcal{L}^m$ is ample for any $m > 0$,
3. $\mathcal{L}^m$ is ample for some $m > 0$.

Proof. (2) $\Rightarrow$ (3): pick $m = 1$.

(1) $\Rightarrow$ (2): from definition.

(3) $\Rightarrow$ (1): Suppose $\mathcal{L}^m$ is ample for some coherent sheaf $\mathcal{F}$. Then there exists n_0 such that whenever $n > n_0$, $\mathcal{F} \otimes (\mathcal{L}^m)^n = \mathcal{F} \otimes \mathcal{L}^{mn}$ is generated by global sections. But $\mathcal{F} \otimes \mathcal{L}$ is coherent since $\mathcal{L}$ is a line bundle, so there exists n_1 such that for any $n > n_1$, $(\mathcal{F} \otimes \mathcal{L}^n) \otimes \mathcal{L}^{mn}$ is generated by global sections. Continuing for all $0 \leq i < n$, we have $\mathcal{F} \otimes \mathcal{L}^i$ and can pick some n_i correspondingly, such that $\mathcal{F} \otimes \mathcal{L}^{mn+i}$ is generated by global sections. Therefore, for $\mathcal{F}$ and $\mathcal{L}$, pick $(\max_i n_i) \cdot m =: N$, so $n \geq N$, therefore $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections, by some case in the argument above. $\square$

Theorem 5.7. Let X be a scheme of finite type over a Noetherian ring A, and $\mathcal{L}$ is a line bundle on X. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^m$ is very ample over $\mathrm{Spec}(A)$ for some $m > 0$.

Proof. Suppose $\mathcal{L}^m$ is very ample, then it is ample, and so $\mathcal{L}$ is ample.

Suppose $\mathcal{L}$ is ample, then for all $P \in X$, let there is an open affine neighborhood $P \in U \subseteq X$ such that $\mathcal{L}|_U$ is free. Let $\bar{Y} := X \backslash U$, and let $\mathcal{I}_Y$ be a sheaf of ideals of Y. Since $\mathcal{L}$ is ample, then for some n we know $\mathcal{I}_Y \otimes \mathcal{L}^n$ is generated by global sections. Therefore, there exists $s \in \Gamma(X, \mathcal{I}_Y \otimes \mathcal{L}^n)$ such that $s_p \notin m_p(\mathcal{I} \times \mathcal{L}^n)_p$. We can think of $s \in \Gamma(X, \mathcal{L}^n)$ since the global section $\Gamma(X, \mathcal{I}_Y \otimes \mathcal{L}^n)$ is a subsheaf of $\mathcal{O}_X$. Define $X_s = \{Q \in X \mid s_Q \notin m_Q(\mathcal{I}_Y \otimes \mathcal{L}^n)_Q\}$, then $P \in X_s \subseteq U$ such that $\mathcal{L}_U$ is trivial, and s induces a section $f \in \Gamma(U, \mathcal{O}_U)$. So we have $X_s = U_{\mathcal{F}}$ is also affine. Therefore, for all $P \in X$, there exists $n > 0$ and $s \in \Gamma(X, \mathcal{L}^n)$ such that $P \in X_s$ affine is of finite type over Noetherian ring, so there exists $N > 0$ such that for all $P \in X$ and affine $P \in X_s$ with $s \in \Gamma(X, \mathcal{L}^N = \mathcal{L}^1)$, and $X_1, \ldots, X_k$ over X_s.

Let $B_i = \Gamma(X_i, \mathcal{O}_i)$, all B_i are finitely-generated A-algebras with generators $\{b_{ij}\}$ over B_i. By definition of X_i, for all $b_{ij} \in \Gamma(X_i, \mathcal{O}_i)$, there exists some m such that $s_i^m \cdot b_{ij} \in \Gamma(X, \mathcal{L}_1^m)$. Pick M such that for all i, j, $s_i^M b_{ij} \in \Gamma(X, \mathcal{L}_1^M)$, so $\mathcal{L}_1^M$ are line bundles and sections $\big\{\{s_1^M, \ldots, s_n^M\}, \{s_i^M b_{ij}\}\big\}$, which defines a morphism to $\mathbb{P}_A$ as an immersion. $\square$

6 Lecture 6

Let X be projective and non-singular, and k is algebraically closed. The line bundles $\mathcal{L}$ are in one-to-one correspondence with the classes of of divisors.

Let $s \in \Gamma(X, \mathcal{L})$. Then $D = (s)_0$ is the divisor of zeros of s. For all $U \subseteq X$, where $\mathcal{L}$ is trivial, there is a map $\varphi : \mathcal{L}|_U \cong \mathcal{O}(U)$ where $\varphi(s) \in \Gamma(U, \mathcal{O}_U)$. We call $\{U, \varphi(s)\}$ an effective Cartier divisor.

Proposition 6.1. Let D_0 be a divisor and $\mathcal{L} = \mathcal{L}(D_0)$.

(a) For all $s \in \Gamma(X, \mathcal{L})$, $(s)_0 \sim D_0$.

(b) For all D effective such that $D \sim D_0$, there exists $s \in \Gamma(X, \mathcal{L})$ such that $(s)_0 = D$.

(c) For any $s, s' \in \Gamma(X, \mathcal{L})$, $(s)_0 = (s')_0$ if and only if there exists $\lambda \in k^*$ such that $s' = \lambda s$.

Definition 6.2 (Complete Linear System)**.** For D_0, all the effective divisors $D \geq 0$ such that $D \sim D_0$ form a collection called *complete linear system* $|D_0|$. There is a one-to-one correspondence between $|D_0|$ and $(\Gamma(X, \mathcal{L}) \setminus \{0\})/k^*$.

Linear systems are k-linear subsets of complete linear systems.

Example 6.3. For a line bundle $\mathcal{L}$, if we take some $s_1, \ldots, s_n$, they form a linear system.

The conics of $\mathbb{P}^2$ form a complete linear system.

Definition 6.4 (Basepoint)**.** Let $|D_0|$ be a complete linear system and ∂ be a linear system in it. The *basepoint* ∂_0 is a linear system of $P \in X$ such that $P \in \mathrm{supp}(D)$ for all $D \in \partial$. We say a linear system is *basepoint-free* if it has no basepoints.

We say $\mathcal{L}$ is *basepoint-free* if the complete linear system given by $s_1, \ldots, s_n$ is basepoint free. The line bundle being basepoint-free implies that there is a morphism to the projective space.

Definition 6.5. We say $\mathcal{L}$ is *semi-ample* if $\mathcal{L}^{\otimes n}$ is generated by global sections for some $n \geq 0$.

$\mathcal{L}$ is *nef* if for all curves $C \subseteq X$, $\deg(\mathcal{L}|_C) \geq 0$.

Remark 6.6. Very ample implies ample and globally generated. Ample implies semi-ample. Globally generated implies semiample and nef.

Definition 6.7 (Proj Bundle)**.** Let $\mathscr{J}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Let λ be the sheaf, then $\mathscr{J}$ is the sheaf of graded $\mathcal{O}_X$-algebras, so $\mathscr{J} \equiv \bigoplus\limits_{d\geq 0} \mathscr{J}_d$. Therefore, $\mathscr{J}_0 = \mathcal{O}_X$, $\mathscr{J}_1$ is coherent, and $\mathscr{J}$ is locally generated by $(\mathscr{J}_1)$ as $\mathcal{O}_X$-algebra. For all $U = \operatorname{Spec}(A) \subseteq X$, we define $\mathscr{J}(U) = \Gamma(U, \mathscr{J}|_U)$, which defines a proj of graded ring and morphism $\operatorname{Proj}(\mathscr{J}(U)) \to U$. By gluing the morphisms, we obtain $\pi : \operatorname{Proj}(\varphi) \to X$. For all $\operatorname{Proj}(\varphi(U))$ we have $\mathcal{O}(1)$ and they glue to $\mathcal{O}(1)$ on $\operatorname{Proj}(\varphi)$.

Proposition 6.8. Let X be a space and $\mathscr{J}$ be a sheaf of graded $\mathcal{O}_X$-algebras.

(a) π is proper,

(b) if there exists $\mathcal{L}$ ample line bundle on X and π is projective, then $\mathcal{O}_{\operatorname{Proj}(\varphi)} \otimes \pi^*\mathcal{L}^n$ is very ample for some $n > 0$.

Definition 6.9 (Symmetric Algebra)**.** Let X be a Noetherian scheme and $\mathcal{E}$ be a locally free coherent sheaf. Then $\varphi = S(\mathcal{E})$ is a *symmetric algebra* of $\mathcal{E}$ and $\mathbb{P}(\mathcal{E}) = \operatorname{Proj}(\varphi)$.

Definition 6.10 (Blow Up)**.** Let X be a space and $\mathscr{J}$ a coherent sheaf of ideals. Denote $\varphi = \bigoplus\limits_{d\geq 0} \mathscr{J}^d$ with $\mathscr{J}^0 = \mathcal{O}_X$. Then $\tilde{X} = \operatorname{Proj}(\varphi)$ is the *blow up* with respect to $\mathscr{J}$.

Remark 6.11. Classically blow up is defined at a point or at a closed subvariety $Y \subseteq X$, denoted $\operatorname{Bl}_Y(X)$. This corresponds to $\tilde{X} = \operatorname{Proj}(\varphi)$ when $\mathscr{J}$ corresponds to Y.

7 Lecture 7

Definition 7.1. Let X be a topological space and $\mathcal{A}$ is an abelian group. We denote by $\mathcal{A}_x$ the constant sheaf on X with values on $\mathcal{A}$, defining $\mathcal{A}_X(U)$ to be the locally constant functions $f : U \to \mathcal{A}$. We just write $H^i_{\text{Sheaf}}(X, \mathcal{A})$ instead of $H^i_{\text{Sheaf}}(X, \mathcal{A}_X)$ for the cohomology groups.

Remark 7.2. Whenever X is a topological manifold or it has the structure of a CW complex, we have that

$$H^i_{\text{Sheaf}}(X, \mathcal{A}) \cong H^i_{\text{Sing}}(X, \mathcal{A}).$$

Note that $H^0_{\text{Sheaf}}(X, \mathcal{A})$ is the set of locally constant functions $f : X \to \mathcal{A}$, and $H^0_{\text{Sing}}(X, \mathcal{A})$ is the set of functions from X to $\mathcal{A}$.

Example 7.3. Consider the cantor set $\mathcal{A} = \{0, 1\}^{\mathbb{N}}$, so

$$H^0_{\text{Sing}}(\mathcal{C}, \mathbb{Z})$$

is the set of functions $\mathcal{C}$ to $\mathbb{Z}$, whereas

$$H^0_{\text{Sing}}(\mathcal{C}, \mathbb{Z})$$

is the set of locally constant functions $\mathcal{C} \to \mathbb{Z}$. Therefore, the first set has cardinality $2^{2^{\mathbb{N}}}$, and the second set has cardinality $2^{\mathbb{N}}$.

Definition 7.4. The *sheaf cohomology* is the right derived functor of the functor from the sheaves of Abelian groups on X to the abelian groups, given by $\mathcal{E} \mapsto \mathcal{E}(X)$.

Definition 7.5. An *abelian category* $\mathcal{Q}$ is a category such that for any objects $A, B \in \mathbb{Q}$, $\mathbf{Hom}(A, B)$ is given an abliean group structure with bilinear map $(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$.

Remark 7.6. Finite direct sums, coproducts, kernels, and cokernels all exist in abelian categories.

Every monomorphsim is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel.

Any morphism can be factored as an epimorphism and then a monomorphism.

Example 7.7.

- R-modules over a left Noetherian ring R.
- $\text{Shv}(X)$, the sheaves of abelian groups on a topologicla space X.
- $\text{Shv}(\mathcal{O}_X)$, the $\mathcal{O}_X$-modules over ringed spaces.
- $\text{Qcho}(X)$, the quasi-coherent sheaves on schemes.
- $\text{Coh}(X)$, the coherent sheaves on Noetherian schemes.

Definition 7.8. Let $\mathcal{Q}$ be an abelian category. A *cochain complex* on $\mathcal{Q}$ is a sequence of maps

$$A := \cdots \to A^i \xrightarrow{d} A^{i+1} \xrightarrow{d} A^{i+2} \xrightarrow{d} \cdots$$

where $A^i \in \mathbb{Q}$ and the d's are morphisms in the category such that $d^2 = 0$.

To the cochain complex A, we can associate cohomological objects as

$$H^i(A) = \ker(d : A^i \to A^{i+1}) / \operatorname{im}(d : A^{i-1} \to A^i).$$

We say that a cochain complex in $\mathcal{Q}$ is exact if $H^*(A) = 0$.

A *chain map* of maps of complexes is a commutative diagram is a series of commutative squares between two chain complexes.

If $f : A \to B$ is a chain map, we get an induced homomorphism of cohomology groups $f_* : H^i(A) \to H^i(B)$ for each $i \in \mathbb{Z}$. A homotopy between maps $f_*, g_* : H^i(A) \to H^i(B)$ is a sequence of morphisms F making the square commutes:

$$\begin{tikzcd}
A^{i-1} \arrow[r, "d"] \arrow[d] & A^i \arrow[r, "d"] \arrow[d] \arrow[dl] & A^{i+1} \arrow[d] \arrow[dl] \\
B^{i-1} \arrow[r, "d"] & B^i \arrow[r, "d"] & B^{i+1}
\end{tikzcd}$$

such that $dF + Fd = f - g$.

We say that f is homotopic to g, written as $f \sim g$, if there is a homotopy from f to g.

Remark 7.9. If $f \sim g$, then $f_* = g_* : H^i(A) \to H^i(B)$ for every $i \in \mathbb{Z}$.

Exercise 7.10. Find two maps f, g not homotopic with $f_* = g_*$ for every $i \in \mathbb{Z}$.

Lemma 7.11. A short exact sequence of cochain complexes induce a long exact sequence of cohomology groups: given $0 \to A \to B \to C \to 0$ a short exact sequence, there is a long exact sequence

$$\cdots \to H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \cdots$$

Lemma 7.12. Let X be an integral Noetherian scheme and let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of cohomology sheaves on X. Then we have a long exact sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \to H^1(X, \mathcal{F}) \to \cdots$$

Definition 7.13. Let $X \subseteq \mathbb{P}^n$ be a smooth projective variety, and let $\omega_X = \mathcal{O}_X(K_X) = \bigwedge^n \Omega_x$ where $\Omega_X = T_X^v$. The *canonical ring* is denoted

$$\bigoplus_{m \in \mathbb{Z}} H^0(X, \omega_X^{\otimes m}).$$

Lemma 7.14. Let S be a smooth divisor on X and $\omega_S = \omega_X(S)$. Then there is a short sequence

$$0 \to \omega_X \to \omega_X \otimes S \to \omega_X \otimes S|_S \to 0$$

where $\omega_X \otimes S|_S \cong \omega_S$.

8 Lecture 8

Throughout this lecture we consider abelian categories.

Definition 8.1 (Exact Functor)**.** A functor $F : \mathcal{A} \to \mathcal{B}$ is said to be *additive* if the induced morphism $F : \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{B}}(FX, FY)$ is a homomorphism of abelian groups for all X, Y.

An additive functor is *left exact* if $0 \to FA \to FB \to FC$ is exact whenever $0 \to A \to B \to C \to 0$ is exact. Likewise, we say the functor is right exact if $FA \to FB \to FC \to 0$ is exact whenever $0 \to A \to B \to C \to 0$ is exact.

An additive functor is exact if it is both left and right exact.

Example 8.2. For abelian category $\mathcal{A}$ and $X \in \mathcal{A}$, then

$$\begin{aligned} \mathcal{A} &\to \mathbf{Ab} \\ Y &\mapsto \mathbf{Hom}(X, Y) \end{aligned}$$

is left exact, and

$$\begin{aligned} \mathcal{A}^{\mathrm{op}} &\to \mathbf{Ab} \\ Y &\mapsto \mathbf{Hom}(Y, X) \end{aligned}$$

is left exact.

Example 8.3. For $\mathcal{A} = \mathbf{Shv}(X)$, the category of abelian sheaves on a topological space X, then $\mathcal{E} \to \mathcal{E}(X)$ is a left exact functor and $\mathcal{E}(X) = \mathbf{Hom}_{\mathbf{Shv}(X)}(\mathbb{Z}_X, \mathcal{E})$.

Definition 8.4 (Injective Object)**.** An object I in an abelian category $\mathcal{A}$ is *injective* if $\mathbf{Hom}(-, I)$ is exact. Equivalently, for any monomorphism $A \hookrightarrow B$ in $\mathcal{A}$, every $A \to I$ extends to a map $B \to I$.

Analogously, P is projective if $\mathbf{Hom}(P, -)$ is exact. Equivalently, $B \twoheadrightarrow C$ gives a surjection $\mathbf{Hom}(P, B) \to \mathbf{Hom}(P, C)$.

Definition 8.5 (Resolution)**.** An injective *resolution* of an object $A \in \mathcal{A}$ is a complex

$$0 \to I^0 \to I^1 \to \cdots$$

of injective objects with a map $A \to I^0$ such that the induced complex

$$0 \to A \to I^0 \to I^1 \to \cdots$$

is exact. That is,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & A \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots
\end{tikzcd}$$

is a quasi-isomorphism, i.e., induces an isomorphism of cohomology groups.

Definition 8.6. An abelian category $\mathcal{A}$ has *enough injectives* if for all $A \in \mathcal{A}$ there is a monomorphism $A \to I$ where I is injective.

Lemma 8.7. If $\mathcal{A}$ has enough injectives, then every element admits an injective resolution.

Proof. $0 \to A \to I^0$ by definition of enough injectives, $I^0/A \hookrightarrow I^1$ by definition of enough injectives, where I^1 is injective, so this induces a map $I^0 \to I^1$, and we continue the resolution inductively. $\square$

Definition 8.8. Two complexes A^* and B^* in $\mathcal{A}$ are *homotopic* or homotopy equivalent if there are $f : A^* \to B^*$ and $g : B^* \to A^*$ such that $fg \sim \mathrm{id}_{B^*}$ and $gf \sim \mathrm{id}_{A^*}$

Remark 8.9. This implies $f_* : H^i(A^*) \to H^i(B^*)$ are isomorphisms for all $i \in \mathbb{Z}$.

Lemma 8.10. Two injective resolutions of the same element $A \in \mathcal{A}$ are homotopic.

Proof. Consider I^* and J^* to be

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "1_A"] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \\
0 \arrow[r] & A \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & \cdots
\end{tikzcd}$$

Since I^0 is injective, then there exists $f_0 : J^0 \to I^0$ via A, so analogously I^0/A induces a map $f_1 : I^1 \to J^1$, and analogously we have all the maps we need. Similarly, there are maps g_i's from J^i to I^i. Now $fg - 1$ is a map $I^0/A \to I^0$, since I^0 is injective, then this induces $F : I^1 \to I^0$ that satisfies $Fd = fg - 1$ as maps between $I^0 \to I^0$. We proceed inductively to get all such F's.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "d"] \arrow[d, "0"] & I^0 \arrow[r, "d"] \arrow[d, "fg-1"] & I^1 \arrow[r, "d"] \arrow[d, "fg-1"] \arrow[dl, "F"] & \cdots \\
0 \arrow[r] & A \arrow[r, "d"] & I^0 \arrow[r, "d"] & I^1 \arrow[r, "d"] & \cdots
\end{tikzcd}$$

$\square$

Corollary 8.11. I^* and J^* are injective resolutions of A, then $H^k(I^*) \cong H^k(J^*)$ for every $k \in \mathbb{Z}$.

Definition 8.12. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Suppose $\mathcal{A}$ has enough injective, then the *right derived functor* of F are functors $R^iF : \mathcal{A} \to \mathcal{B}$ defined as follows: given injective resolution I^* as $0 \to A \to I^0 \to I^1 \to \cdots$, we apply F, and obtain a resolution FI^* as $0 \to FA \to FI^0 \to FI^1 \to \cdots$. Let $(R^iF)(A) = H^i(FI^*)$ in $\mathcal{B}$. In particular, $(R^iF)(A) = 0$ for all $i < 0$.

Remark 8.13. $(R^0F)(A) \cong F(A)$.

Theorem 8.14. If $0 \to A \to B \to C \to 0$ is exact in $\mathcal{A}$ and $F : \mathcal{A} \to \mathcal{B}$ is left exact, then we have a long exact sequence

$$0 \to FA \to FB \to FC \to (R^1F)(A) \to (R^1F)(B) \to (R^1F)(C) \to (R^2F)(A) \to \cdots$$

Proof. Look at their injective resolutions I^*, J^*, K^* and produce a complex with vertical maps. We want the vertical sequences between the resolutions to be split. Choose $J^0 \supseteq (I^0 \oplus B)/A$, then this gives a vertical diagram after applying the functor F. Again, we want this to be exact on vertical sequences, so we apply the theorem from last time. $\square$

Definition 8.15. Given $\mathbf{Shv}(X) \to \mathbf{Ab}$ given by $\mathcal{E} \to \mathcal{E}(X)$, then the right derived functors are called *sheaf cohomologies*, denoted by $H^i(X, \mathcal{E})$.

Theorem 8.16. If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of sheaves on X, then there is a long exact sequence

$$0 \to H^0(\mathcal{F}) \to H^0(\mathcal{G}) \to H^0(\mathcal{H}) \to H^1(\mathcal{F}) \to \cdots$$

9 Lecture 9

Let $\mathbb{P}^1_k$ be a field, $S = \{0, \infty\}$ in $\mathbb{P}^1_k$. Then there is

$$0 \to I_{S/\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^1} \to i_*(\mathcal{O}_S) \to 0.$$

We then get a long exact sequence

$$\cdots \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \to H^0(\mathbb{P}^1, i_*(\mathcal{O}_S) \to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2) \to \cdots$$

where $I_{S/\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ and $H^0(\mathbb{P}^1, i_*(\mathcal{O}_S) \cong H^0(S, \mathcal{O}_S) \cong H^0(p, \mathcal{O}_p) \oplus H^0(q, \mathcal{O}_q)$. The diagonal map $k \to k \otimes k$ given by $1 \mapsto (1, 1)$ induces the fact that $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \neq 0$.

Now there is a complex

$$\begin{tikzcd}
0 \arrow[r] & \omega^v_{\mathbb{P}^1} \otimes I_{S/\mathfrak{P}^1} \arrow[r] \arrow[d, "\cong"] & \omega^v_{\mathbb{P}^1} \arrow[r] \arrow[d, "\cong"] & \omega^v_{\mathbb{P}^1} \otimes i_*(\mathcal{O}_S) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \omega^v_{\mathbb{P}^1}(-2) \arrow[r] \arrow[d, "\cong"] & \omega^v_{\mathbb{P}^1} \arrow[r] & i_*(\mathcal{O}_S) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1} \arrow[r] & \omega^v_{\mathbb{P}^1} \arrow[r] & i_*(\mathcal{O}_S) \arrow[r] & 0
\end{tikzcd}$$

In particular, we have

$$H^0(\mathbb{P}^1, \omega^v_{\mathbb{P}}) \to k \otimes k \to H^1(\mathcal{O}_{\mathbb{P}^1}) \cong H^0(\omega_{\mathbb{P}^1})$$

where $H^1(\mathcal{O}_{\mathbb{P}^1}) = 0$, and the final isomorphism is given by Serre duality.

Remark 9.1. For every two values $\alpha_0, \alpha_\infty \in k$, there is a section $\Gamma \in H^0(\mathbb{P}^1, \omega^v_{\mathbb{P}})$ taking value α_0 at $\{0\}$ and taking value α_∞ and $\{\infty\}$.

The question is, does $\mathbf{Shv}(X)$ have enough injectives? Let R be a ring, the abelian groups $\mathbf{Ext}^i_R(M, N)$ for R-modules M and N and $i \in \mathbb{Z}$ are the right derived functors of the left exact functor $N \mapsto \mathbf{Hom}_R(M, N)$. There is an isomorphism $H^i(X, \mathcal{E}) \cong \mathbf{Ext}^i_{\mathbf{Shv}(X)}(\mathbb{Z}_X, \mathcal{E})$. In particualr, $H^0(X, \mathcal{E}) \cong \mathbf{Hom}_{\mathbf{Shv}(X)}(\mathbb{Z}_X, \mathcal{E})$.

Lemma 9.2. A $\mathbb{Z}$-module is injective if and only if M is divisible.

Proof. $(\Rightarrow)$: Let M be an injective $\mathbb{Z}$-module, pick $m \in M$, $n \in \mathbb{Z} > 0$. Then

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\times n"] \arrow[d, "1 \mapsto M"] & \mathbb{Z} \arrow[dl, dashed] \\
M &
\end{tikzcd}$$

$(\Leftarrow)$: Let M be an abelian group, and let $A \subseteq B$ be an inclusion of abelian groups with a homomorphism $f : A \to M$. Consider the poset of abelian groups $A \subseteq S \subseteq B$ and homomorphisms $g_S : S \to M$ making

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "f"] & S \arrow[r, hook] \arrow[dl, "g_S"] & B \\
M & &
\end{tikzcd}$$

commutative.

Now $\{(S_\alpha, g_\alpha)\}$ is a totally ordered subset of this poset, then $\{(U_\alpha S_\alpha, U_\alpha g_\alpha)\}$ is an element of this poset, meaning that every totally ordered subset has an upper bound $S_\alpha \leq U_\alpha S_\alpha$. Zorn's Lemma implies that this poset has a maximal element

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "f"] & H \arrow[r, hook] \arrow[dl, "h"] & B \\ M & & \end{tikzcd}$$

Claim 9.3. $H = B$.

Subproof. Pick $b \in B$ not in H, $\bar{H} = \langle H, b\rangle$. Consider $\varphi : H \otimes \mathbb{Z} \twoheadrightarrow \bar{H}$ by $(\alpha, 1) \mapsto \alpha + b$. Let $K = \ker(\varphi)$, then K injects into $\mathbb{Z}$ by projection. If $K = 0$, then we can extend $h : H \to M$ to $\bar{h} : \bar{H} \to M$, contradiction. ■

Therefore, $K \cong \langle n\rangle$ for some n, so $nb = s$ in the group H. Since M is divisible, then $h(b)$ is divisible in M, so there exists $nm = h(b)$ in M. Hence, we can extend h to $\bar{H}$ by mapping b to m, contradiction. □

Remark 9.4. Any product of an injective is an injective. Moreover,

$$\mathbf{Hom}_{\mathcal{A}}(A, \prod_{\alpha \in S} B_\alpha) \cong \prod_{\alpha \in S} \mathbf{Hom}_{\mathcal{A}}(A, B_\alpha).$$

Example 9.5. $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective.

Lemma 9.6. The category of $\mathbb{Z}$-modules has enough injective.

Proof. Let M be an abelian group. Pick $m \in M$, if m is torsion, set $I_m = \mathbb{Q}/\mathbb{Z}$, otherwise set $I_m = \mathbb{Q}$. Consider

$$\begin{tikzcd} \langle m\rangle \arrow[r, hook] \arrow[d, hook] & I_m \\ M \arrow[ur, dashed] & \end{tikzcd}$$

so we have $M \to I_m$ an abelian group homomorphism. Then we have a homomorphism $M \to \prod_{m \in M} I_m$ where the group in the right is injective. This is a group monomorphism. Indeed, pick $m_0 \in M$, then $M \to \prod_{m \in M} I_m \to I_{m_0}$, the image of m_0 in the composition is non-zero as soon as $m_0 \neq 0$. Therefore, $M \to \prod_{m \in M} I_m$ is injective. □

Exercise 9.7. A $\mathbb{Z}$-module is projective if and only if it is free. What is the analogous of Remark 9.4? Can we prove this without axiom of choice?

Lemma 9.8. If $\mathbb{T}$ is an injective $\mathbb{Z}$-module, then $\mathbf{Hom}_{\mathbb{Z}}(R, T)$ is an injective R-module.

Proof. Let $X_{\scriptscriptstyle<} \to X_2$ be an injection of R-modules. We want to prove that

$$\mathbf{Hom}_R(X_2, \mathbf{Hom}_{\mathbb{Z}}(R, T)) \to \mathbf{Hom}_R(X_1, \mathbf{Hom}_{\mathbb{Z}}(R, T))$$

is surjective.

Exercise 9.9. There is an isomorphism $\mathbf{Hom}_R(X_2, \mathbf{Hom}_{\mathbb{Z}}(R, T)) \cong \mathbf{Hom}_{\mathbb{Z}}(X_2, T)$.

Therefore, we have a commutative diagram

$$\begin{tikzcd}
\mathbf{Hom}_R(X_2, \mathbf{Hom}_{\mathbb{Z}}(R, T)) \arrow[r] \arrow[d, "\cong"] & \mathbf{Hom}_R(X_1, \mathbf{Hom}_{\mathbb{Z}}(R, T)) \arrow[d, "\cong"] \\
\mathbf{Hom}_{\mathbb{Z}}(X_2, T) \arrow[r] & \mathbf{Hom}_{\mathbb{Z}}(X < T)
\end{tikzcd}$$

$\square$

Theorem 9.10. Let R be a ring, then the category of R-modules has enough injectives.

Proof. Let M be an R-module. Let $M \hookrightarrow T$ be an injection into an injective $\mathbb{Z}$-module. By the lemma, $\mathbf{Hom}_{\mathbb{Z}}(R, T)$ is an injective R-module. Define $m \mapsto f_m$, with $f_m(a) = f(am) \in T$ for all $a \in R$. This gives an injection of R-modules $M \hookrightarrow \mathbf{Hom}_{\mathbb{Z}}(T, R)$. $\square$

Theorem 9.11. Let X be a locally ringed space, then $\mathbf{Shv}(\mathcal{O}_X)$ has enough injectives.

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Pick $x \in X$, then $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module. There is now an injection $\mathcal{F}_x \hookrightarrow \mathcal{J}_x$ where $\mathcal{J}_x$ is an injective $\mathcal{O}_{X,x}$-module. Define $j_x : \{x\} \hookrightarrow X$, then we define the $\mathcal{O}_X$-sheaf. Therefore, $\mathcal{J} = \prod_{x \in X} (j_x)_* J_x$. We have a map of sheaves $\mathcal{F} \to \mathcal{J}$ that on open subsets $U \subseteq X$, there is

$$\mathcal{F}(U) \hookrightarrow \prod_{x \in X} \mathcal{F}_x \to \prod_{x \in U} \mathcal{J}_x = \mathcal{J}(U).$$

Note that for an $\mathcal{O}_X$-sheaf $\mathcal{G}$, we have $\mathbf{Hom}_{\mathcal{O}_X}(\mathcal{G}, (j_x)_* \mathcal{J}_X) \cong \mathbf{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, \mathcal{J}_x)$. Let $\mathcal{G}_1 \subseteq \mathcal{G}_2$ be an injection of $\mathcal{O}_X$-modules, then we have a diagram

$$\begin{tikzcd}
\mathbf{Hom}_{\mathbf{Shv}(\mathcal{O}_X)}(\mathcal{G}_2, (j_x)_* \mathcal{J}_x) \arrow[r] \arrow[d, "\cong"] & \mathbf{Hom}_{\mathbf{Shv}(\mathcal{O}_X)}(\mathcal{G}_1, (j_x)_* \mathcal{J}_X) \arrow[d, "\cong"] \\
\mathbf{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_{2,x}, \mathcal{J}_x) \arrow[r] & \mathbf{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_{1,x}, \mathcal{J}_x)
\end{tikzcd}$$

which is an surjection of $\mathcal{O}_{X,x}$-modules, so $\mathbf{Hom}_{\mathbf{Shv}(\mathcal{O}_X)}(\mathcal{G}_2, \mathcal{J}) \twoheadrightarrow \mathbf{Hom}_{\mathbf{Shv}(\mathcal{O}_X)}(\mathcal{G}_1, \mathcal{J})$. $\square$

10 Lecture 10

Lemma 10.1. Let X be a topological space. Assume X contains a point $x \in X$ such that for every open neighborhood U of $x \in X$ there exists a proper connected open neighborhood $x \in V \subseteq U \subseteq X$, then $\mathrm{Shv}(X)$ does not have enough projectives.

Proof. Let $i : \{x\} \hookrightarrow X$ be the inclusion, consider $i_*\mathbb{Z}_x$. We will prove that $i_*\mathbb{Z}_x$ is not the image of a projective sheaf, which is evaluated as $\mathbb{Z}$ on U whenever $x \in U$, and is 0 on U whenever $x \notin U$.

For $j : V \hookrightarrow X$ and a sheaf $\mathcal{E}$ on V, the extension by zeros, denoted by $j!\mathcal{E}$, is the sheafification of the presheaf defined by $W \mapsto \mathcal{E}(W)$ if $W \subseteq X$, and is 0 otherwise. The stalks of $j!\mathcal{E}$ are $\mathcal{E}x$ if $x \in V$ and 0 otherwise. Hence, for any open $W \subseteq X$, $(j!\mathcal{E})(W)$ is the set $\{s \in \mathcal{E}(V \cap W) \mid s = 0 \in V \cap N, N \text{ is a neighborhood of } W \setminus V\}$. Note that

$$\mathbf{Hom}_X(j!\mathcal{E}, \mathcal{F}) = \mathbf{Hom}_V(\mathcal{E}, j^*\mathcal{F}).$$

Let U be a connected neighborhood of $x \in X$. Let $V \subsetneq U$ be a connected neighborhood of $x \in X$. Write $\mathbb{Z}_V$ for the extension by zero of the sheaf $\mathbb{Z}_V$ on V ($j : V \hookrightarrow X$, $\mathbb{Z}_V := j!\mathbb{Z}_V$), so $\mathbb{Z}_V(W)$ is the set of constant functions $W \to \mathbb{Z}$ that are zero on an open neighborhood of $W \setminus V$. We have that $\mathbb{Z}_V \to i_*\mathbb{Z}_x \leftarrow P$, then by the projective assumption there is a map $P \to \mathbb{Z}_V$, then applying this to U we get

$$\begin{tikzcd} \mathbb{Z}_V(U) \arrow[r] & i_*\mathbb{Z}_x(U) \\ & P(U) \arrow[ul] \arrow[u] \end{tikzcd}$$

but $\mathbb{Z}_V(U) = 0$ by definition, therefore this lift is the zero map for every U connected neighborhood of $x \in X$. From that, any neighborhood of $x \in X$ contains a connected neighborhood of $x \in X$, we conclude that this is true for every neighborhood of $U \ni x \in X$, then taking the inverse limit, we conclude that $P_x \to (i_*\mathbb{Z}_x)_x \cong \mathbb{Z}$ is zero. Therefore, $P \to i_*\mathbb{Z}_x$ is not surjective, otherwise it is surjective on stalks. $\square$

Definition 10.2. A sheaf $\mathcal{E}$ on X is *flasque* (or flabby) if for every open $U \subseteq X$, $\mathcal{E}(X) \to \mathcal{E}(U)$ is onto.

Lemma 10.3. Let $(X, \mathcal{O}_X)$ be a locally ringed space, then every injective $\mathcal{O}_X$-module is flasque.

Proof. Let $V \subseteq X$ be open, $\mathcal{I}$ is an injective $\mathcal{O}_X$-modules. For $j : V \hookrightarrow X$ open, write $\mathcal{O}_V$ for $j!\mathcal{O}_V$ on X. Observe that for any open $V \subseteq X$, we have an injection of sheaves $0 \to$

$\mathcal{O}_V \to \mathcal{O}_X$. Since $\mathcal{I}$ is injective, we have $\mathbf{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{I}) \to \mathbf{Hom}_{\mathcal{O}_X}(\mathcal{O}_V, I)$ a surjection. Note $\mathbf{Hom}_{\mathcal{O}_X}(j!\mathcal{O}_V, \mathcal{I}) \cong \mathbf{Hom}_{\mathcal{O}_V}(\mathcal{O}_V, j^*\mathcal{I}) \cong \mathcal{I}(V)$, so we have a commutative diagram

$$\begin{tikzcd} \mathbf{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{I}) \arrow[r] \arrow[d, "\sim"] & \mathbf{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{I}) \arrow[d, "\sim"] \\ \mathcal{I}(X) \arrow[r] & \mathcal{I}(V) \end{tikzcd}$$

□

Example 10.4. Let $X = \mathbb{R}^1$ and $x = \{0\}$, with $i : X \hookrightarrow \mathbb{R}$ and A an abelian group. Then i_*A is a flasque sheaf. Not injective unless A is an injective $\mathbb{Z}$-module.

Definition 10.5. A sheaf $\mathcal{F}$ is called *acyclic* if $H^i(X, \mathcal{F}) = 0$ for $i > 0$.

Example 10.6. Consider $X \hookrightarrow \mathbb{P}^n$ projective variety and $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^N}(H)|_X$ and $\mathcal{O}_X(m) = \mathcal{O}_X(1)^{\otimes m}$, then $\mathcal{O}_X(m)$ is acyclic for $m \gg 0$.

Proposition 10.7. Let $\mathcal{F}$ be a flasque sheaf on a space X, then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$ (therefore $H^i(U, \mathcal{F}) = 0$ for all $U \subseteq X$ open and $i > 0$, which implies $H^i(U, \mathcal{F}|_U = 0$ is flasque).

Proof. $\mathcal{F} \hookrightarrow \mathcal{I}$ where $\mathcal{I}$ is injective hence flasque, and $0 \to \mathcal{F} \to \mathcal{I} \to \mathbb{Q} \to 0$ is exact.

Claim 10.8. $\mathbb{Q}$ is also flasque.

Subproof. $U \subseteq X$ and $s \in \mathbb{Q}(U)$, we first lift s to a section of $\mathcal{I}(U)$. Consider the poset (V, t) with $V \subseteq U$, t a section of $\mathbb{Q}(V)$ lifting to $\mathcal{I}(V)$. Every totally ordered subset has an upper bound (gluing axiom on sheaves). By Zorn's lemma, there exists a maximal with respect to W and $W \subseteq U$. Assume by contradiction, $W \neq U$, since $I \to \mathbb{Q}$ is surjective, there is an open cover U_α of U such that $I(U_\alpha) \twoheadrightarrow \mathbb{Q}(U_\alpha)$ for all α. We can choose $t_\alpha \in I(U_\alpha)$ mapping to $s_\alpha = s|_{U_\alpha}$ for every α. Since $W \subsetneq U$, then there exists $U_\alpha \not\subseteq W$. Note $t_W - t_\alpha \in \mathcal{F}(W \cap U_\alpha)$. Since $\mathcal{F}$ is flasque, $t_W - t_\alpha$ extends to $b \in \mathcal{F}(U_\alpha)$. Replace t_W with $t_\alpha + b$ so now t_W and t_α agree on the intersection $W \cap U_\alpha$. This gives a section on $I(W \cup U_\alpha)$ that maps to $w|_{W|cupU_\alpha} \in \mathbb{Q}(W \cup U_\alpha)$, contradiction, so $W = U$. ■

We note that there is now a commutative diagram

$$\begin{tikzcd} I(X) \arrow[r] \arrow[d, two heads] & \mathbb{Q}(X) \arrow[d] \\ I(U) \arrow[r, two heads] & \mathbb{Q}(U) \end{tikzcd}$$

then in the claim above we proved $0 \to H^0(X, \mathcal{F}) \to H^0(X, I) \to H^0(X, \mathbb{Q}) \to 0$. Hence $H^i(X, \mathcal{F}) = 0$. We proved that the first cohomology of a flasque is zero. Therefore, looking at the long exact sequence, all cohomology of the flasque is zero. $\square$

Remark 10.9. Injective implies flasque implies acyclic.

Proposition 10.10. Consider a long exact sequence $0 \to \mathcal{E} \to \mathcal{A}_0 \to \mathcal{A}_1 \to \cdots$ where $\mathcal{A}_i$'s are acyclic, then $H^*(X, \mathcal{E}) \cong H^*(0 \to H^0(X, \mathcal{A}_0) \to H^0(X, \mathcal{A}^1) \to \cdots)$.

11 Lecture 11

Let R be a commutative ring and M be an R-module, then we have an associated $\mathcal{O}_X$-module $\tilde{M}$ on $X = \operatorname{Spec}(R)$. There is a natural isomorphism $H^0(X, \tilde{M}) \cong M$ as $\mathcal{O}_X$-modules, and as R-modules.

For every $f \in R$, $V_f = \{f \neq 0\} \subseteq X$ is affine and $H^0(V_f, M) \cong M[\frac{1}{f}]$. So $\tilde{M}_p = M_p$ for every $p \in \operatorname{Spec}(R)$.

The question is, in the setting with $\mathcal{E}$ quasi-compact and X affine, what can we say about $H^i(X, \mathcal{E})$?

Theorem 11.1. For an affine scheme X, there is an equivalence of categories between $\mathcal{O}(X)$-modules and quasi-coherent sheaves on X, given by $M \mapsto \tilde{M}$ and $H^0(X, \mathcal{E}) \leftarrow\!\shortmid \mathcal{E}$.

Corollary 11.2. Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be a short exact sequence of quasi-coherent sheaves on affine scheme X. Then the sequence $0 \to H^0(X, \mathcal{A}) \to H^0(X, \mathcal{B}) \to H^0(X, \mathcal{C}) \to 0$ is exact.

Theorem 11.3 (Serre, 1955)**.** For a quasi-coherent sheaf $\mathcal{E}$ on X an affine scheme, we have that $H^i(X, \mathcal{E}) = 0$ for $i > 0$.

Remark 11.4. There are three major approaches:

- Hartshorne (in 1990s) shows that

  Theorem 11.5. For X a Noetherian scheme, the following are equivalent:

  1. X is affine,
  2. $H^i(X, \mathcal{F}) = 0$ for $i > 0$ and $\mathcal{F}$ quasi-coherent,
  3. $H^1(X, \mathcal{I}) = 0$ for $\mathcal{I}$ coherent.

  Exercise 11.6. Write down the proof and point out where Noetherian is used.

- In EGA (in 1960s) and the Stack Project, one shows that: Čech cohomology and sheaf cohomology ar e the same thing and use it to prove Serre vanishing.
- We will look at the proof of Kempf in *Algebraic Varieties* in 1970s.

Proof. Let X be a topological space, $\mathcal{E}$ be a sheaf on X, and $j : U \hookrightarrow X$ be an open embedding. Define $u\mathcal{E} := j_* j^* \mathcal{E}$, that is for any $V \subseteq X$ we have $u\mathcal{E}(V) = \mathcal{E}(U \cap V)$. There is a natural map of sheaves $\mathcal{E} \to u\mathcal{E}$ that gives a homology

$$H^i(X, \mathcal{E}) \to H^i(X, u\mathcal{E}) \to H^i(U, j^*\mathcal{E}) := H^i(U, \mathcal{E}).$$

Proposition 11.7. Let $\mathcal{U}$ be a basis of opens of X a topological space. Assume $\mathcal{U}$ is closed under finite intersections. Let $i \in \mathbb{Z}_{>0}$. Let $\mathcal{F}$ be a sheaf of abelian groups on X. Suppose that $H^j(U, \mathcal{F}) = 0$ for all $0 < j < i$ and $U \in \mathcal{U}$. Then for any element $\alpha \in H^i(X, \mathcal{F})$, there is an open covering $X = U_\sigma W_\sigma$ with $W_\sigma \in \mathcal{U}$ such that $\alpha \mapsto 0 \in H^I(X, W_\sigma \mathcal{F})$ induced by the sequence above.

Subproof. We proceed by induciton on i. First, we prove the case $i = 1$. We embed $\mathcal{F} \hookrightarrow \mathcal{L}$ flasque that vies a short exact sequence, then for any open $W \subseteq X$ we have

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{L} \arrow[r] \arrow[d] & \mathcal{L}/\mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W\mathcal{F} \arrow[r] & W\mathcal{I} \arrow[r] & W\mathcal{I}/W\mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

and this gives $H^0(\mathcal{I}/\mathcal{F}) \to H^1(\mathcal{F}) \to H^1(\mathcal{I}) = 0$. Let $\alpha \in H^1(X, \mathcal{F})$, so $\alpha = d\beta$ for some $\beta \in H^0(X, \mathcal{I}/\mathcal{F})$. We take an open cover $\{W_\sigma\}$ where $W_\sigma \in \mathcal{U}$, then β lifts to $\mathcal{L}(W_\sigma)$ for each σ and gives $\mathcal{L} \twoheadrightarrow \mathcal{L}/\mathcal{F}$. The image of β in $H^0(X, W_\sigma \mathcal{I}/W_\sigma \mathcal{F})$ lifts to $H^0(X, W_\sigma \mathcal{I})$ for each σ. Therefore, $d\beta = \alpha$ is zero in $H^1(X, W_\sigma \mathcal{F})$ for every σ. This proves the case for $i = 1$.

Claim 11.8.

- There is a short exact sequence of sheaves

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{L} \arrow[r] \arrow[d] & \mathcal{L}/\mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W\mathcal{F} \arrow[r] & W\mathcal{L} \arrow[r] & W(\mathcal{L}/\mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

- The sheaf $\mathcal{L}/\mathcal{F}$ satisfies all the assumptions of the proposition for $i = 1$. More precisely, $H^j(U, \mathcal{F}/\mathcal{L}) = 0$ for all $0 < j < i - 1$ and $U \in \mathcal{U}$.

Proof of Claim. If $V \in \mathcal{U}$ is an open different from W, using the corollary and the assumption on $\mathcal{U}$, we get that

$$0 \to H^0(W \cap V, \mathcal{F}) \to H^0(W \cap V, \mathcal{I}) \to H^0(W \cap V, \mathcal{L}/\mathcal{F}) \to 0$$

which is exact. Note that the sequence above is just equivalent to

$$0 \to H^0(V, W\mathcal{F}) \to H^0(V, W\mathcal{I}) \to H^0(V, W(\mathcal{L}/\mathcal{F})) \to 0$$

This proves the first part. Observe that $H^j(V, \mathcal{L}/\mathcal{F}) \cong H^{j+1}(V, \mathcal{F})$ for all $j \geq 1$, by restricting from

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{I}/\mathcal{F} \to 0$$

and the fact that $\mathcal{F}$ is flasque (restricting it to V). ■

Now we prove that the claim implies the proposition. Consider the complex

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{L} \arrow[r] \arrow[d] & \mathcal{L}/\mathcal{F} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & W\mathcal{F} \arrow[r] & W\mathcal{I} \arrow[r] & W(\mathcal{I}/\mathcal{F}) \arrow[r] & 0 \end{tikzcd}$$

with $W\mathcal{I}$ flasque, so by the first part of the claim, we have a commutative diagram

$$\begin{tikzcd} H^{i-1}(X, \mathcal{I}/\mathcal{F}) \arrow[r, "\cong"] \arrow[d] & H^i(X, \mathcal{F}) \arrow[d] \\ H^{i-1}(X, W(\mathcal{I}/\mathcal{F})) \arrow[r, "\cong"] & H^i(X, W\mathcal{F}) \end{tikzcd}$$

Note that the right map gives $\alpha \mapsto 0$, then we lift it back to β on the top-left corner, therefore this forces the map on the left to be $\beta \mapsto 0$, which proves the proposition. ■

We will now prove the main theorem. Consider $V_f = \{f \neq 0\}$ for $f \in \mathcal{O}(X)$ and $W_\sigma = \{f_\sigma \neq 0\}$ is a finite basis of X quasi-compact. We may assume $\alpha \in H^i(X, \mathcal{F})$ goes to zero in $H^i(X, W_\sigma \mathcal{F})$ by the proposition. Observe $W_\sigma \mathcal{F}$ is quasi-coherent on X as $W_\sigma \mathcal{F} = (\widetilde{M[\frac{1}{f_\sigma}]})$. Consider the sequence

$$0 \to \mathcal{F} \to \bigoplus_\sigma W_\sigma \mathcal{F} \to \mathfrak{g} \to 0$$

where $\mathfrak{g}$ is quasi-coherent. Take $\alpha \in H^i(X, \mathcal{F})$ and take the long exact sequence in homology, we obtain

$$H^{i-1}(X, \mathfrak{g}) \to^i (X, \mathcal{F}) \to H^i(X, \bigoplus_{W_\sigma} \mathcal{F}) \to H^i(X, \mathfrak{g}) \to \cdots$$

Take $\alpha \in H^i(X, \mathcal{F})$ which maps to 0, then there is a lifting to $\beta \mapsto \alpha$ in the homology. Since $H^{i-1}(X, \mathcal{G}) = 0$ by induction on i, we get $\alpha = 0$ so $H^i(X, \mathcal{F}) = 0$. The induction works provided $H^1 = 0$. To show this, we have

$$\begin{tikzcd} 0 \arrow[r] & H^0(X, \mathcal{F}) \arrow[r] & H^0(X, \bigoplus_\alpha W_\alpha \mathcal{F}) \arrow[r] & H^0(X, \mathcal{G}) \arrow[r] \arrow[d] & 0 \\ & & & H^1(X, \mathcal{F}) & \end{tikzcd}$$

Take $\beta \in H^0(X, \mathcal{G})$ mapping to $\alpha = H^1(X, \mathcal{F})$, but the mapping to β is surjective, so as $\beta = d\gamma$, this forces $\alpha = d^2\gamma = 0$, therefore the homology there is zero as desired. □

12 Lecture 12

Recall the following results:

Theorem. Category of quasi-coherent sheaves admits enough injectives and cannot be used to define sheaf cohomology $H^i(X, \mathcal{F})$.

Theorem. Sheaf $\mathcal{F}$ being injective implies being flabby implies being acyclic.

Theorem (Serre's Vanishing)**.** If $\mathcal{F}$ is quasi-coherent on an affine scheme X, then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.

The question is, what about cohomology of projective schemes?

Theorem 12.1 (Mayer-Vietoris)**.** Let $\mathcal{E}$ be a sheaf of abelian groups on X, let $X = U \cup V$ be an open cover, then there is a long exact sequence

$$\cdots \to H^i(X, \mathcal{E}) \to H^i(U, \mathcal{E}) \oplus H^i(V, \mathcal{E}) \to H^i(U \cap V, \mathcal{E}) \to H^{i+1}(X, \mathcal{E})$$

Proof. Consider an (flabby) injective resolution

$$0 \to \mathcal{E} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \cdots$$

and restricts to a flabby resolution

$$0 \to \mathcal{E}|_V \to \mathcal{I}_0|_V \to \mathcal{I}_1|_V \to \cdots$$

Observe that restricting on open sets and intersection of open sets preserves the flabby property for all I_i. Therefore, we have a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{I}_0(X) \arrow[r] \arrow[d] & \mathcal{I}_1(X) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{I}_0(U) \oplus \mathcal{I}_0(V) \arrow[r] \arrow[d] & \mathcal{I}_1(U) \oplus \mathcal{I}_1(V) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{I}_0(U \cap V) \arrow[r] & \mathcal{I}_1(U \cap V) \arrow[r] & \cdots
\end{tikzcd}$$

The only non-trivial part of the diagram is $\mathcal{I}_j(U) \oplus \mathcal{I}_j(V) \to \mathcal{I}_j(U \cap V)$ being surjective. This follows from the flasque condition, so we take LES in cohomology associated to the short exact sequence of complexes. □

Example 12.2. Compute $H^*(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$, $U = \mathbb{P}^1 \setminus [0:1] \simeq \mathbb{A}^1$ and $V = \mathbb{P}^1 \setminus [1:0] \simeq \mathbb{A}^1$, then $U \cap V \simeq \mathbb{G}_m := \mathbb{A}^1 \setminus \{0\}$. From Theorem 12.1, we have a long exact sequence

$$0 \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1} \to H^0(U, \mathcal{O}_{\mathbb{P}^1}|_U) \oplus H^0(V, \mathcal{O}_V) \to H^0(U \cap V, \mathcal{O}_{U\cap V}) \to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \to 0$$

Indeed, we have

- $H^i(U, \mathcal{O}_U) = 0$ for all $i \geq 1$,
- $H^i(V, \mathcal{O}_V) = 0$ for all $i \geq 1$, and
- $H^i(U \cap V, \mathcal{O}_{U\cap V}) = 0$ for all $i \geq 1$.

Therefore, $H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$ for $i \geq 2$.

Moreover, we have a short exact sequence

$$0 \to k \to k[x^{-1}] \oplus k[x] \to k[x^{\pm 1}] \to 0$$

defined by $1 \mapsto (1,1)$ and $(a,b) \mapsto a+b$ and therefore $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$.

Theorem 12.3 (Grothendieck, 1975)**.** If X is a noetherian topological space of finite Krull dimension n, then $H^i(X, \mathcal{E}) = 0$ if $i > n$ and $\mathcal{E}$ is any sheaf of abelian groups.

Theorem 12.4. If X is a noetherian projective scheme and $\mathcal{E}$ is a quasi-coherent sheaf on X, then $H^i(X, \mathcal{E}) = 0$ for $i > \dim(X)$.

Remark 12.5 (Čech Cohomology)**.** Let X be a topological space and let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open covering of X. Fix an order in I. For any sequence $i_0, \ldots, i_p \in I$, write $U_{i_0,\ldots,i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$. Let $\mathcal{F}$ be a sheaf of abelian groups on X, define the Čech complex to be $\mathcal{C}^*(X, \mathcal{F})$: for $p \geq 0$, let $\mathcal{C}^p(\mathcal{U}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0,\ldots,i_p})$. This induces a long sequence

$$0 \to \mathcal{C}^0 \to \mathcal{C}^1 \to \mathcal{C}^2 \to \cdots$$

also known as

$$\mathcal{F}(X) \to \bigoplus_{i \in I} \mathcal{F}(U_i) \to \bigoplus_{i<j} \mathcal{F}(U_{i,j}) \to \cdots$$

The differentials are $d : \mathcal{C}^p \to \mathcal{C}^{p+1}$ given by

$$(d\alpha)_{i_0,\ldots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\ldots,\tilde{i}_k,\ldots,i_{p+1}}|_{U_{i_0,\ldots,i_{p+1}}}.$$

For $X = U_0 \cup U_1$ and $\mathcal{U} = \{U_0, U_1\}$, we have a sequence

$$0 \to \mathcal{F}(U_0) \oplus \mathcal{F}(U_1) \to \mathcal{F}(U_0 \cap U_1) \to \cdots$$

defined by $(s_0, s_1) \mapsto (s_0 - s_1)|_{U_0 \cap U_1}$.

Definition 12.6 (Čech Cohomology)**.** The *Čech cohomology* of $\mathcal{F}$ in X with respect to the cover $\mathcal{U} = \{U_i \mid i \in I\}$ is $H^*(\mathcal{U}, \mathcal{F}) = H^*(\mathcal{C}(X, \mathcal{F}))$.

Remark 12.7.

1. There is a natural homomorphism $H^i(X, \mathcal{F}) \to H^i(\mathcal{U}, \mathcal{F})$, from sheaf cohomology to Čech cohomology.

2. If $\mathcal{F}$ is acyclic on $U_{i_0,\ldots,i_p}$ for all $i_0, \ldots, i_p$, then the homomorphism is an isomorphism.

Theorem 12.8. Let X be a noetherian separated scheme. Let $\mathcal{U}$ be an affine open cover of X. Let $\mathcal{F}$ be a quasi-coherent sheaf on X, then $H^i(X, \mathcal{F}) \simeq H^i(\mathcal{C}(X, \mathcal{F}))$ with respect to $\mathcal{U}$.

Proof. If suffices to show that $H^i(U_{i_0,\ldots,i_p}, \mathcal{F}) = 0$ for all $i_0, \ldots, i_p$ (by Remark 12.7). This would be true if finite intersections of affine is affine, which uses Lemma 12.9. $\square$

Lemma 12.9. Let X be a separated scheme and let U and V be affine on X. Then $U \cap V$ is affine.

Proof. The diagonal functor $\Delta : X \to X \times X$ gives a closed embedding and a commutative diagram

$$\begin{tikzcd}
U \cap V \arrow[r] \arrow[d] & U \times V \arrow[d] & \\
X \arrow[r, "\Delta"] & X \times X \arrow[r] & X
\end{tikzcd}$$

Since the bottom map is a closed immersion, then so is the top one. Thus, since $U \times V$ is affine, we conclude that the image in the first $\mathrm{Proj}(X)$ is affine. $\square$

Note that intersections of affines in general is not affine.

Example 12.10.

- Consider $X = \mathbb{A}^2_k \cup_{\mathbb{A}^2 \setminus \{0\}} \mathbb{A}^2_k$ with $U = \mathbb{A}^2_k$ and $V = \mathbb{A}^2_k$, then $U \cap V = \mathbb{A}^2_k \setminus \{0\}$ is not affine.

- $\mathbb{Z} = \bigcap_{\lambda \in \mathbb{C} \setminus \mathbb{Z}} V(x - \lambda) \subseteq \mathbb{A}^1_{\mathbb{C}}$ is not affine. Every countable affine variety over $\mathbb{C}$ is finite.

Lemma 12.11. Let X be a projective scheme with $X \hookrightarrow \mathbb{P}^n$. Let H be a hyperplane such that $X \not\subseteq H$, then $X \setminus H$ (set-theoretically) is an affine variety.

Proof. $X \setminus H \hookrightarrow \mathbb{P}^n \setminus H = \mathbb{A}^n$. Assume $H = \{x_0 = 0\}$, let $I(X)$ be the hom ideal defining X, and let the dehomogenization of $I(X)$ with respect to x_0 be the ideal $k[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}]$ defining $X \setminus H$. $\square$

Theorem 12.12. Let X be a projective scheme (separated) of dimension n. Let $\mathcal{F}$ be a quasi-coherent sheaf on X, then $H^i(X, \mathcal{F}) = 0$ for $i \geq n$.

Proof. $X \hookrightarrow \mathbb{P}^n$, and consider $H_1, \dots, H_{n+1}$ be general hyperplanes, so $H_1 \cap \cdots \cap H_{n+1} \cap X = \varnothing$. This gives an affine open cover $U_1, \dots, U_{n+1}$ of X by Lemma 12.11. Set $\mathcal{U} = \{U_1, \dots, U_{n+1}\}$, then since X is separated, we have $H^i(X, \mathcal{F}) \simeq H^i(\mathcal{C}(X, \mathcal{F}))$ with respect to $\mathcal{U}$, then the Čech complex stabilize at $n+1$, so $H^i(\mathcal{C}(X, \mathcal{F})) = 0$ for $i \geq n+1$. □

Proposition 12.13. Let C be a smooth projective curve, $c \in C$ a closed point, then $C \setminus \{c\}$ is affine.

Remark 12.14. Suppose C has genus at least 1, then there is an embedding $C \hookrightarrow \mathbb{P}^2$. Therefore, let d be such that $g = \frac{d(d+1)}{2}$, there are d intersections on $\mathbb{P}^2$. However, by Proposition 12.13, if we embed P as $C \hookrightarrow \mathbb{P}^N$ for some large enough N, then there exists a hyperplane H of $\mathbb{P}^N$ such that $H \cap C = p$ intersects at a point.

Example 12.15. Let $L \subseteq \mathbb{P}^2$ and $P \in E$ for E elliptic, $3P \sim 0$, then $E \setminus \{p\}$ is affine.

Exercise 12.16. Let E be an elliptic curve and $p \in E$, prove that $E \setminus \{p\}$ is affine.

13 Lecture 13

Theorem 13.1. Let k be a field and n be a positive integer. Then for $r \in \mathbb{Z}$,

$$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = \begin{cases} \bigoplus_k x_0^{a_0} \cdots x_n^{a_n}, a_k \geq 0, \sum a_i = r, & \text{if } i = 0 \\ 0 & \text{if } i \neq \{0, n\} \\ \bigoplus_k x_0^{a_0} \cdots x_n^{a_n}, a_k < 0, \sum a_i = r, & \text{if } i = n \end{cases}$$

Remark 13.2. We will see that for any projective scheme X over k and any coherent sheaf $\mathcal{E}$ on X, the cohomology group $H^i(X, \mathcal{E})$ is a finite-dimensional vector space.

If X over k is proper and $\mathcal{E}$ is coherent, we write $h^i(X, \mathcal{E}) = \dim_k(H^i(X, \mathcal{E}))$.

Corollary 13.3. We have

$$h^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = \begin{cases} \binom{n+r}{n}, & \text{if } i = 0 \text{ and } r \geq 0 \\ 0, & \text{otherwise} \\ \binom{-r-1}{n}, & \text{if } i = n, r < 0 \end{cases}$$

Remark 13.4. If $-n \leq r \leq -1$, then $\mathcal{O}(r)$ has no cohomologies in any degree in $\mathbb{P}^n$.

Remark 13.5. Recall $\omega_{\mathbb{P}^n}$. Hence, for every line bundle $\mathcal{L}$ on $\mathbb{P}^n$, we have an equality of dimensions $h^i(\mathbb{P}^n, \mathcal{L}) = h^{n-i}(\mathbb{P}^n, \omega_{\mathbb{P}^n} \otimes \mathcal{L}^\vee)$.

Example 13.6. On $\mathbb{P}^n$, we have the following table:

	$\mathcal{O}(-3)$	$\mathcal{O}(-2)$	$\mathcal{O}(-1)$	$\mathcal{O}(0)$	$\mathcal{O}(1)$	$\mathcal{O}(2)$	$\mathcal{O}(3)$
h^0	0	0	0	1	2	3	4
h^1	2	1	0	0	0	0	0

Definition 13.7 (Tensor Product)**.** Let A and B be two complex of abelian groups, then the *tensor product* of A and B is defined by $(A \otimes B)^j = \bigoplus_{i \in \mathbb{Z}} A^i \otimes B^{j-i}$, such that $d(a \otimes b) = (da) \otimes b + (-1)^{|a|} a \otimes (db)$.

Exercise 13.8. Show that this is indeed a complex.

Exercise 13.9. Prove the Kunneth formula, that $H^i(A \otimes B) = H^i(A) \otimes_k H^{j-i}(B)$.

Proof of Theorem. We use Čech cohomology for affine open cover of $\mathbb{P}^n$ given by $U_i = \{x_i \neq 0\}$ for $0 \leq i \leq n$. Since $\mathbb{P}^n$ is separated, we have

$$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = \check{H}^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r))$$

So we have a complex

$$\cdots \to \bigoplus_{i=1}^{n} \mathcal{O}(r)(U_i) \to \bigoplus_{0 \leq i_0 < i_1 \leq n} \mathcal{O}(r)(U_{i_0, i_1}) \to \cdots$$

Now observe that

$$\bigoplus_{i \in \mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) \cong H^i(\mathbb{P}^n, \bigoplus_{r \in \mathbb{Z}} \mathcal{O}(r)).$$

Set $\mathcal{F} = \bigoplus_{r \in \mathbb{Z}} \mathcal{O}(r)$, and set $U = U_{i_0, \ldots, i_p}$. Consider the map

$$\begin{aligned} \pi : \mathbb{A}^{n+1} \setminus \{0\} &\to \mathbb{P}^n \\ (x_0, \ldots, x_n) &\mapsto [x_0 : \ldots : x_n] \end{aligned}$$

and denote $\mathcal{F}(U) = \mathcal{F}(\pi^{-1}(U)) = k[x_{i_0}, \ldots, x_{i_n}, x_{i_0}^{-1}, \ldots, x_{i_p}^{-1}]$. This is a k-vector space with a basis $k\{x_0^{a_0} \cdots x_n^{a_n} \mid a_k \in \mathbb{Z}, a_k \geq 0, \text{ if } k \notin \{i_0, \ldots, i_p\}\}$. Therefore,

$$H^*(\mathbb{P}^n, \mathcal{F}) = H^*(0 \to \bigoplus_{i=0}^{n} k[x_0, \ldots, x_n, x_i^{-1}] \to \bigoplus_{0 \leq i < j \leq n} k[x_0, \ldots, x_n, x_i^{-1}, x_j^{-1}] \to \cdots)$$

This complex looks similar to the following complex T:

$$(0 \to k[x_0] \to k[x_0, x_0^{-1}] \to 0) \otimes_k (0 \to k[x_0] \to k[x_1, x_1^{-1}] \to 0) \otimes_k \cdots \otimes_k (0 \to k[x_0] \to k[x_n, x_n^{-1}] \to 0)$$

The difference is that we have to remove the group in degree) and shift by 1. By Kunneth formula,

$$H^i(T) = \begin{cases} k[x_0, x_0^{-1}]/k[x_0] \otimes_k \cdots \otimes_k k[x_n, x_n^{-1}]/k[x_n], & \text{if } j = n-1 \\ 0 & \text{if } j \neq n-1 \end{cases}$$

Therefore,

$$H^j(\mathbb{P}^n, \mathcal{F}) = \begin{cases} k[x_0, \ldots, x_n] & \text{if } j = 0 \\ k[x_0, x_0^{-1}]/k[x_0] \otimes_k \cdots \otimes_k k[x_n, x_n^{-1}]/k[x_n] & \text{if } j = n \\ 0 & \text{if } j \neq \{0, n\} \end{cases}$$

□

Remark 13.10. Most Riemann surfaces do not embed into $\mathbb{P}^2_{\mathbb{C}}$, not even analytically.

Remark 13.11. Every smooth projective curve C admits an embedding into $\mathbb{P}^3_{\mathbb{C}}$.

Lemma 13.12. Let $i : S \hookrightarrow X$ be a closed set of a topological space. Then

1. the functor $i_* : \mathrm{Shv}(S) \to \mathrm{Shv}(X)$, where $i_*(\mathcal{E}(U)) = \mathcal{E}(U \cap S)$, is exact;

2. for a sheaf $\mathcal{E}$ of abelian groups on S, we have

$$H^j(S, \mathcal{E}) \cong H^j(X, i_*(\mathcal{E})).$$

Proof. 1. Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be exact on S. It suffices to show $0 \to (i_*\mathcal{A})_x \to (i_*\mathcal{B})_x \to (i_*\mathcal{C})_x \to 0$ is exact on X for every $x \in X$. If $x \in S$, then $(i_*\mathcal{A})_x \cong \mathcal{A}_x$. If $x \notin S$, then since S is closed, we get $(i_*\mathcal{A})_x = (i_*\mathcal{B})_x = (i_*\mathcal{C})_x = 0$.

2. Consider an injective resolution $0 \to \mathcal{E} \to I^0 \to I^1 \to \cdots$ where each I^j is flabby. Now

$$H^*(S, \mathcal{E}) = H^*(0 \to I^0(S) \to I^1(S) \to \cdots)$$

then the sheaves i_*I^j are flabby, now

$$H^*(X, i_*\mathcal{E}) = H^*(0 \to i_*I^0(X) \to i_*I^1(X) \to \cdots)$$

and the two cohomologies agree degreewise.

□

Definition 13.13 (Genus)**.** Let C be a smooth projective curve over k, then the *genus* of C is $g(C) = \dim_k(H^1(C, \mathcal{O}_C))$.

Proposition 13.14. Let $C_j \subseteq \mathbb{P}^2$ be a smooth projective curve of degree d, then $g(C_d) = \frac{(d-1)(d-2)}{2}$.

Proof. We have
$$0 \to I_{\mathcal{C}/\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2} \to i_*\mathcal{O}_C \to 0$$
where $I_{\mathcal{C}/\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-d)$. Taking the first cohomology, we have
$$\cdots \to H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \to H^1(C, \mathcal{O}_C) \to H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d)) \to H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2} \to \cdots$$
where $H^1(C, \mathcal{O}_C) \cong H^1(\mathbb{P}^2, i_*\mathcal{O}_C)$, and by the previous result, the first and last term are zero, so the middle two terms agree, therefore $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d))$ has dimension $\binom{n-1}{2}$. $\square$

14 Lecture 14

Theorem 14.1. Consider $\mathbb{P}^n_k$ for some $n \geq 0$ and k a field. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_k$, then $\mathcal{F}(m) = \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n_k}(m)$ is globally generated for some $m > 0$. That is, there is a surjection $\mathcal{O}^{\oplus r} \twoheadrightarrow \mathcal{F}(m)$ over $\mathbb{P}^n$.

Remark 14.2. For simplicity , we have $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^n_k}(H)$ for H hyperplane and $\mathcal{O}(m) = \mathcal{O}(1)^{\oplus m}$. Then $\mathcal{F}(k)$ is globally generated for $k \geq m$.

Proof. Observe that only coherent sheaf on affine scheme is globally generated. For $\mathcal{E}$ coherent sheaf on X affine, there is an equivalence of categories between quasi-coherent sheaves on X and $\mathcal{O}(X)$-modules. Every finitely-generated $\mathcal{O}(X)$-module admits a surjection $\mathcal{O}(X)^{\oplus r} \twoheadrightarrow H^0(X, \mathcal{E})$. Sheafifying this surjection of modules gives $\mathcal{O}^{\oplus r} \cong \tilde{\mathcal{O}(X)}^{\oplus r} \twoheadrightarrow H^0(\tilde{X}, \mathcal{E}) \cong \mathcal{E}$. In conclusion, $\mathcal{F}|_{U_i}$ spanned by sections $s_1, \ldots, s_j$ in $H^0(U_i, \mathcal{F}|_{U_i})$ where $U_i = \{x_i \neq 0\}$. Note that for $\mathcal{E}$ quasi-coherent on X affine and $f \in \mathcal{O}(X)$, we have $H^0(\{f \neq 0\}, \mathcal{E}) \cong H^0(X, \mathcal{E}) \left[\frac{1}{f}\right]$. Then $(\frac{x_1}{x_j})^m s_1 \cdots (\frac{x_r}{x_j})^m s_r$ extends to sections of $\mathcal{F}$ in U_j. Equivalently, $x_i^m s_1 \cdots x_i^m s_r$ extends to sections of $\mathcal{F}(m)$ on $U_i \cup U_j$. To extend to $\mathbb{P}^n_k$, we need to agree on overlaps $U_i \cap U_j$. If $s \in H^0(X, \mathcal{E})[\frac{1}{f}] \cong H^0(\{f \neq 0\}, \mathcal{E})$, then there exists $m \geq 0$ such that $f^m \in H^0(X, \mathcal{E})$. Furthermore, if $s \in \frac{a}{f^m} = \frac{b}{f^m} \in M[\frac{1}{f}]$ where $M = H^0(X, \mathcal{E})$, there exists $j \geq 0$ for which $f^j a = f^j b$. Hence, if m is large enough, the sections $x_i^m s_r \cdots x_i^m s_r$ extend to sections of $\mathcal{F}(m)$ on $\mathbb{P}^n$. $\square$

Theorem 14.3 (Serre)**.** Let $i : X \hookrightarrow \mathbb{P}^n_k$ a projective variety and let $\mathcal{F}$ be a coherent sheaf on X. Then

1. $H^i(X, \mathcal{F})$ is a finite-dimensional k-vector space.

2. there exists $m_0 = m(\mathcal{F})$ such that $H^0(X, \mathcal{F}(m)) = 0$ for $m \geq m_0$.

Proof. We know $H^i(X, \mathcal{F}) \cong H^i(\mathbb{P}^n, i_*\mathcal{F})$ and $i_*\mathcal{F} \otimes \mathcal{O}(m) \cong i_*(\mathcal{F} \otimes i^*\mathcal{O}(m)) = i_*(\mathcal{F}(m))$, therefore $X = \mathbb{P}^n_k$ and $\mathcal{F}$ is coherent in $\mathbb{P}^n_k$.

Also, since $\mathcal{F}(m)$ is globally generated for $m \gg 0$, then there is a surjection $\mathcal{O}^{\oplus r} \twoheadrightarrow \mathcal{F}(m)$, and we have a short exact sequence

$$0 \to K \to \mathcal{O}^{\oplus r} \to \mathcal{F}(m) \to 0$$

and by Serre twisting we have

$$0 \to K(-m) \to \mathcal{O}(-m)^{\oplus r} \to \mathcal{F} \to 0$$

This gives a long exact sequence in cohomology:

$$\cdots \to H^j(\mathbb{P}^n, \mathcal{O}(-m)^{\oplus r} \to H^j(\mathbb{P}^n, \mathcal{F}) \to H^{j+1}(\mathbb{P}^n, K(-m)) \to \cdots$$

We know $H^i(\mathbb{P}^n, \mathcal{E}) = 0$ for $j > n$ and $\mathcal{E}$ coherent. We conclude that $H^j(\mathbb{P}^n, \mathcal{F})$ is a finite-dimensional k-vector space by descending induction. This proves the first part.

Fix s and let $m \gg s$. We know $\mathcal{F}(m)$ is globally generated for $s \gg 0$, then we have $\mathcal{O}^{\oplus r} \twoheadrightarrow \mathcal{F}(s)$ and therefore $\mathcal{O}(-s)^{\oplus r} \twoheadrightarrow \mathcal{F}$. Similar as above by Serre twisting we have

$$0 \to K(m) \to \mathcal{O}(m-s)^{\oplus r} \to \mathcal{F}(m) \to 0$$

We get a long exact sequence of finite-dimensional k-vector spaces, such that for $m \gg 0$,

$$0 = H^j(\mathbb{P}^n, \mathcal{O}(m-s)^{\oplus r}) \to H^i(\mathbb{P}^n, \mathcal{F}(m)) \to H^{j+1}(\mathbb{P}^n, K(m)) \to \cdots$$

We know $H^{n+1}(\mathbb{P}^n, K(m)) = 0$ and so $H^n(\mathbb{P}^n, \mathcal{F}(m)) = 0$. By descending induction, we are done. $\square$

Corollary 14.4 (Serre)**.** Let $\mathcal{L}$ be ample on X, then for every coherent sheaf on X we have a constant $m(\mathcal{F}) = m_0$ for which

1. $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ is globally generated and

2. $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$ for $i > 0$.

Proof. Because $\mathcal{O}^{\otimes a}$ is very ample, then $i : X \hookrightarrow \mathbb{P}^N$ such that $i^*(\mathcal{O}(1)) \cong \mathcal{L}^{\otimes a}$. Think of $\mathcal{O}(1) = \mathcal{L}^{\otimes a}$, now apply the previous theorem to this $\mathcal{O}(1)$ for $\mathcal{F}, \mathcal{F} \otimes \mathcal{L}, \ldots, \mathcal{F} \otimes \mathcal{L}^{\otimes(a-1)}$. Note that $\mathcal{F} \otimes \mathcal{O}(m)$ is globally generated for $m \gg 0$, so $\mathcal{F} \otimes \mathcal{L}^{\otimes m_0}$ is globally generated. Therefore, there exists m_0 such that for all $0 \leq b \leq a-1$, $(\mathcal{F} \otimes \mathcal{L}^b) \otimes \mathcal{L}^{\otimes m_0}$ satisfies the two properties. $\square$

Theorem 14.5 (Cartan-Serre-Grothendieck)**.** Let $\mathcal{L}$ be a line bundle on a complete scheme X. The following are equivalent:

1. $\mathcal{L}$ is ample,
2. $\mathcal{L}^{\otimes m}$ is very ample for some $m \geq 0$,
3. for every coherent sheaf $\mathcal{F}$, there is $m_0 = m_0(\mathcal{F})$ such that $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ is globally generated for $m \geq m_0$,
4. for every coherent sheaf $\mathcal{F}$, there is $m_1 = m_1(\mathcal{F})$ such that $H^i(\mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$ for $i > 0$ and $m \geq m_1$.

Corollary 14.6. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are ample, then $\mathcal{L}_1^{\otimes a} \otimes \mathcal{L}_2^{\otimes b}$ is ample for $a > 0$ and $b > 0$.

Proof. Since $\mathcal{L}_1$ is ample, then $\mathcal{L}_1^{\otimes am}$ is globally generated for $m \gg 0$. Similarly, $\mathcal{L}_2^{\otimes bm} \otimes \mathcal{F}$ is globally generated for $m \gg 0$. Therefore, $\mathcal{F} \otimes (\mathcal{L}_1^{\otimes a} \otimes \mathcal{L}_2^{\otimes b})^{\otimes m}$ is globally generated for $m \gg 0$. ☐

Remark 14.7 (Where is the cone?)**.** Let X be a smooth projective scheme, then we have

$$0 \to 2\pi i\mathbb{Z}|to\mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

then in cohomology we have

$$\cdots \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}^{)}_X \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to \cdots$$

where

$$\begin{tikzcd} H^1(X, \mathcal{O}^{)}_X \arrow[r] \arrow[d, "\cong"] & H^2(X, \mathbb{Z}) \\ \mathrm{Pic}(X) \arrow[r, "\mathrm{ch}_1"] & \mathbb{Z}^\rho \arrow[u, hook] \end{tikzcd}$$

and extends to a short exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \cong H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathbb{Z}^\rho \cong \mathrm{NS}(X) \longrightarrow 0$$

where $\mathrm{Pic}^0(X)$ is the complex tori.

Theorem 14.8. $\mathcal{L} \in \mathrm{Pic}^0(X)$ if and only if $\mathcal{L}$ deforms to $\mathcal{O}_X$ if and only if $\mathcal{L}.C = 0$ for all curves $C \subseteq X$.

Remark 14.9. The space of divisors $N_1(X) = \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Theorem 14.10. Ampleness is a numerical condition. That is, the ample property forms a cone on the plane, and further inside the cone there is no higher cohomology.

15 Lecture 15

Proposition 15.1. Let X be a separated quasi-compact scheme over a field k, and let $\mathcal{E}$ be a quasi-coherent sheaf on X. Let F/k be a field extension, and write $X_F = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(F)$. Therefore we have

$$\begin{tikzcd} X_F \arrow[r] \arrow[d, "\pi"] & \mathrm{Spec}(F) \arrow[d] \\ X \arrow[r] & \mathrm{Spec}(k) \end{tikzcd}$$

Then $H^i(X_F, \pi^*\mathcal{E}) \cong H^i(X, \mathcal{E}) \otimes_k F$.

Proof. $\pi^*(\mathcal{E}) = (\pi^{-1}(\mathcal{E})) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_F}$ sheafified. If X is affine, then $H^0(X_F, \mathcal{E}) \cong H^0(X, \mathcal{E}) \otimes_k F$. Since X is quasi-compact, we write $X = \bigcup_{i=1}^{n} U_i$ such that U_i is an affine open variety of X for all i. Now $H^*(X, \mathcal{E}) = H^*(0 \to \prod_i H^0(U_i, \mathcal{E}) \to \prod_{i<j} H^0(U_{ij}, \mathcal{E}) \to \cdots)$ by Cech homology. Here, we are using $\mathcal{E}$ quasi-compact and X separable. We get $H^*(X_F, \mathcal{E} := \pi^*(\mathcal{E})) = H^*(0 \to \prod_i [H^0(U_i, \mathcal{E}) \otimes_k F] \to \cdots)$. It now suffices to tensor with F over k, which is an exact functor on k-vector spaces and commutes with finite products. □

Remark 15.2. If we drop the quasi-compact condition, this would not work. Say $X = \coprod_{n \geq 0} \mathrm{Spec}(k)$ then $\mathcal{O}(X) = \prod_{n \geq 0} k$ and $\mathcal{O}(X_F) = \prod_{n \geq 0} F$, then $(\prod k) \otimes_k F \ncong \prod F$ if the product is not finite.

Remark 15.3. A property of schemes that is preserved under taking field extensions is called "geometric". For instance, $\dim_k(H^1(X, \mathcal{O}_X))$ is geometric for quasi-coherent sheaves.

Remark 15.4. Suppose X is a projective scheme defined over an algebraically closed field k, then $\mathcal{O}(X) \cong k$.

Proposition 15.5. Let X be a projective variety defined over a field k, then $\mathcal{O}(X)$ is a finite field extension of k.

Proof. It suffices to check that the field extension is finite: $\mathcal{O}(X)$ is a domain and we can invert elements. Note that $\mathcal{O}(X) = H^0(X, \mathcal{O}_X)$ is finite-dimensional over k. □

Definition 15.6 (Geometrically Integral)**.** Let X be a scheme defined over k. X is *geometrically integral* if $X_{\bar{k}}$ is integral where $\bar{k}$ is the algebraic closure of k.

Remark 15.7. Given a property P that is not geometric, we say that that X is "geometrically P" if $X_{\bar{k}}$ satisfies P.

Proposition 15.8. Let X be a smooth projective variety over k, then the following are equivalent:

1. X is geometrically integral,
2. X is geometrically connected,
3. The map $k \to \mathcal{O}(X)$ is an isomorphism.

Proof. If X is smooth over k, then $X_{\bar{k}}$ is smooth over $|bark$. Therefore, $X_{\bar{k}}$ is connected with local rings that are domains, which means $X_{\bar{k}}$ is a domain. This proves (2) $\Rightarrow$ (1). (1) $\Rightarrow$ (2) is trivial.

We now show (1) $\Rightarrow$ (3). The extension $\bar{k} \to \mathcal{O}(X_{\bar{k}})$ is an isomorphism. Observe that $\bar{k} = k \otimes_k \bar{k}$ and $\mathcal{O}(X_{\bar{k}}) = \mathcal{O}(X) \otimes_k \bar{k}$, therefore $k \to \mathcal{O}(X)$ is an isomorphism. To see (3) $\Rightarrow$ (2), if $k \to \mathcal{O}(X)$ is an isomorphism, then $\bar{k} \to \mathcal{O}(X_{\bar{k}})$ is an isomorphism, then X is geometrically connected. $\square$

Proposition 15.9. If X is a smooth projective variety over k and $X(k) \neq \varnothing$, then X is geometrically integral and $k \to \mathcal{O}(X)$ is an isomorphism.

Proof. Consider $X \to \operatorname{Spec}(k)$ and a section $s : \operatorname{Spec}(k) \to X$, this corresponds to a ring homomorphism $\mathcal{O}(X) \to k$ that is injective, so $k \cong \mathcal{O}(X)$. Hence, X is geometrically integral by the previous proposition. $\square$

Let X be a smooth projective complex variety over $\mathbb{C}$, we will view it as a scheme over $\mathbb{R}$. We have $\mathcal{O}(X) \cong \mathbb{C}$, then X is not geometrically integral as a scheme over $\mathbb{R}$. To calculate $X_{\mathbb{C}}$, we have $\operatorname{Spec}(\mathbb{C}) \otimes_{\operatorname{Spec}(\mathbb{R})} \operatorname{Spec}(\mathbb{C}) = \operatorname{Spec}(\mathbb{C} \otimes \mathbb{R}\mathbb{C}) \cong \operatorname{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[x]/(x^2+1)) \cong \operatorname{Spec}(\mathbb{C}[x]/(x^2+1)) \cong \operatorname{Spec}(\mathbb{C}) \coprod \operatorname{Spec}(\mathbb{C})$. Therefore, $X_{\mathbb{C}} = X \coprod X^{\text{conj}}$, where X^{conj} is defined by conjugating equations of X.

Definition 15.10 (Genus)**.** Let X be a smooth projective geometrically integral curve, then the genus of X is $\dim_k(H^1(X, \mathcal{O}_X))$.

Theorem 15.11. Let X be a smooth projective curve over a field k, assume $g(C) = 0$, then X is isomorphic to a conic in $\mathbb{P}^2_k$.

Theorem 15.12 (Diophantus, 200 AD)**.** Let X be a smooth projective curve over a field k. Assume $g(C) = 0$, then $X \cong \mathbb{P}^1_k$ if and only if $X(k) \neq \varnothing$.

Example 15.13. $\{x^2 + y^2 + z^2 = 0\} \subseteq \mathbb{P}^2_{\mathbb{R}}$ as an $\mathbb{R}$-scheme.

Remark 15.14. If $k = \bar{k}$, then every genus-0 smooth projective curve is isomorphic to $\mathbb{P}^1_k$.

Proof. Let $p \in X$ be a k-point, then we have a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}(p) \to i_* i^* \mathcal{O}(p) \to 0$$

where $i : \{p\} \hookrightarrow X$. We have an induced long exact sequence of homology

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}(p)) \to H^0(X, i_* i^* \mathcal{O}(p)) \to H^1(X, \mathcal{O}_X) \to \cdots$$

where $H^0(X, i_* i^* \mathcal{O}(p)) \cong H^0(p, i^* \mathcal{O}(p)) \cong k$, $H^0(X, \mathcal{O}_X) \cong k$, and $H^1(X, \mathcal{O}_X) \cong 0$. We conclude that $H^0(X, \mathcal{O}(p))$ has a non-trivial element $f \in H^0(X, \mathcal{O}(p))$ has a simple pole at p. We consider $f : X \to \mathbb{P}^1$ to be $x \mapsto [g_1(x), g_2(x)]$ where $f = \frac{g_1}{g_2}$. But $\mathbb{P}^1$ is projective, and X is smooth, so X is a morphism. But the pullback $f^*([\infty]) = [p]$ as divisors, so $\deg(f) = 1$, and $\deg[k(X); k(\mathbb{P}^1)] = 1$, so $k(\mathbb{P}^1) \cong k(X)$, therefore f is birational, and so f is an isomorphism since X and $\mathbb{P}^1$ are smooth. □

16 Lecture 16

Theorem 16.1. Let C be a smooth projective curve of genus $g \geq 1$ over a field k. If $x \neq y$ are two k-points, then $x \not\sim y$.

Proof. Assume $x \sim y$, then $x - y = \operatorname{div}(f)$. We can think of f as a morphism $f : C \to \mathbb{P}^1$. By assumption, $f^*([\infty]) = [y]$, therefore $\deg(f) = 1$ and $\operatorname{trdeg}(k(C), k(\mathbb{P}^1)) = 1$, hence f is an isomorphism, contradiction. □

Proposition 16.2. Let C be a smooth projective curve of genus $g \geq 1$ over a field k. Then we have an injection $\varphi : C(k) \hookrightarrow \operatorname{Pic}^\circ(C)$, to the set of line bundles of degree 0.

Proof. Recall $\operatorname{Pic}^\circ(X) = \ker(\deg(\operatorname{Pic}(C) \to \mathbb{Z}))$, so it suffices to show that there is no k-point. Suppose there exists $p_0 \in C(k)$, now consider $C(k) \to \operatorname{Pic}^\circ(C)$ by $x \mapsto [x - p_0]$. Assume $\varphi(x) = \varphi(y)$, then $[x - p_0] = [y - p_0]$ in $\operatorname{Pic}(C)$, which means $x - p_0 \sim y - p_0$, so $x - y \sim 0$. But we have genus at least 1, then $x = y$. □

Proposition 16.3. Let C be a smooth projective curve of genus $g \geq 1$ over an algebraically closed field k. Then the cardinality of $\operatorname{Pic}^\circ(X)$ equals to the cardinality of k.

Proof. The $\geq$ directions uses injectivity, and the $\leq$ direction uses the fact that a line bundle $\mathcal{L}$ on C is isomorphic to $\mathcal{O}_C(\sum\limits_i \alpha_i p_i)$. □

Corollary 16.4. Let C be a smooth curve over $\mathbb{C}$. If $g(C) \geq 1$, then $\mathcal{O}(C)$ is not a UFD.

Proof. Let $C \hookrightarrow \bar{C}$ where $\bar{C}$ is a smooth projective curve with $g(\bar{C}) \geq 1$. Now $\mathrm{Pic}(C) = \mathrm{Pic}(\bar{C})/\sum_{i=1}^{r} \mathbb{Z}p_i$ where $\bar{C} \setminus C = \{p_1, \ldots, p_r\}$. $\square$

Proposition 16.5. Let $p \in C$ be a point in a smooth curve C with $g(C) \geq 1$. Assume $\bar{C} \setminus C = \{p_0\}$, then $\mathcal{L}_{p/C}$ is non-trivial.

Proof. If $\mathcal{O}_C(-p)$ was trivial on C, then we would have $p \sim ap_0$ in $\bar{C}$. Taking degree we have $a = 1$, contradiction. $\square$

Let $\mathcal{E}$ be a coherent sheaf on projective scheme X over a field k. Define $\chi(X, \mathcal{E}) = \sum_{i=1}^{\infty} (-1)^i h^i(X, \mathcal{E})$ where $h^i(X, \mathcal{E}) = \dim_k(H^i(X, \mathcal{E}))$.

Lemma 16.6. If $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is a short exact sequence of coherent sheaves on X, then $\chi(X, \mathcal{B}) = \chi(X, \mathcal{A}) + \chi(X, \mathcal{C})$.

Proof. Take the corresponding long exact sequences

$$0 \to H^0(X, \mathcal{A}) \to H^0(X, \mathcal{B}) \to \cdots \to H^n(X, \mathcal{E}) \to 0$$

and if we label the terms as $V_1, \cdots V_{3n}$, then $\sum_{i=1}^{3} n(-1)^i \dim(V_i) = 0$, this concludes the proof. $\square$

Theorem 16.7 (Riemann-Roch)**.** Let $\mathcal{L}$ be a line bundle over a smooth projective curve C, where C is defined over a field k. Then $\chi(C, \mathcal{L}) = \deg(\mathcal{L}) - g + 1$. Moreover, there is $h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = d - g + 1$.

Corollary 16.8. $h^0(X, \mathcal{L}) \geq \deg 9\mathcal{L}) - g + 1$.

Example 16.9. $X = \mathbb{P}^1_k$, we know that every line bundle $\mathcal{L}$ is isomorphic to $\mathcal{O}_X(j)$ for some $j \in \mathbb{Z}$.

	$\mathcal{O}(-3)$	$\mathcal{O}(-2)$	$\mathcal{O}(-1)$	$\mathcal{O}(0)$	$\mathcal{O}(1)$	$\mathcal{O}(2)$	$\mathcal{O}(3)$	$\mathcal{O}(4)$
$h^0(X, \mathcal{O}(j))$	0	0	0	1	2	3	4	5
$h^1(X, \mathcal{O}(j))$	2	1	0	0	0	0	0	0

so $h^0 - h^1 = d + 1$.

Proof. Assume k is algebraically closed, then $\mathcal{L} \cong \mathcal{O}(D)$ where $D = \sum_{i=1}^{r} \alpha_i p_i$ where $p_i \in C$ and $\alpha_i \in \mathbb{Z}$. Observe the theorem holds for $D = 0$. Indeed, $h^0(C, \mathcal{O}_C) - h^1(C, \mathcal{O}_C) = 1 - g = 0 - g + 1 = \deg(\mathcal{O}) - g + 1$.

Claim 16.10. Riemann-Roch holds for $\mathcal{L}$ if and only if Riemann-Roch holds for $\mathcal{L}(-p)$ for any $p \in C$.

Subproof. We have a short exact sequence $0 \to \mathcal{O}_C(-p) \to \mathcal{O}_C \to \mathcal{O}_C|_p \to 0$. By tensoring with $\mathcal{L}$, we have

$$0 \to \mathcal{L}(-p) \to \mathcal{L} \to \mathcal{L}|_p \to 0$$

Taking the long exact sequence in homology, we get $0 \to H^0(C, \mathcal{L}(-p)) \to H^0(C, \mathcal{L}) \to H^0(C, \mathcal{L}|_p) \cong H^0(p, \mathcal{O}_p) \to H^1(C, \mathcal{L}(-p)) \to H^1(C, \mathcal{L}) \to H^1(p, \mathcal{O}_p) \to 0$. Therefore, $\chi(C, \mathcal{L}) = \chi(C, \mathcal{L}(-p)) + \chi(p, \mathcal{O}_p) = \chi(C, \mathcal{L}(-p) + 1$. From this equality, we get both implications. ■

□

Definition 16.11 (Elliptic Curve)**.** An *elliptic curve* is a smooth projective curve C of genus $g = 1$ over a field k together with a k-point $p_0 \in C(k)$.

Theorem 16.12. Let C be an elliptic curve. There is a bijection $C(k) \to \mathrm{Pic}^0(C)$ that sends p_0 to 0. In particular, $C(k)$ is an abelian group.

Proof. Note that

$$\begin{aligned} \varphi : C(k) &\to \mathrm{Pic}^0(C) \\ x &\mapsto [x - p_0] \end{aligned}$$

is an injection. We need to prove that every line bundle $\mathcal{L}$ in C of degree 0 in $\mathrm{Pic}^0(X)$ is isomorphic to $\mathcal{O}_C(X - p_0)$ for some $x \in C$. This is the same as proving that every $\mathcal{L}$ with $\deg(\mathcal{L}) = 1$ is isomorphic to $\mathcal{O}_C(x)$ for some $x \in C$. Let us take a line bundle $\mathcal{L}$ of degree 1 on C. Now $h^0(C, \mathcal{L}) \geq \deg(\mathcal{L}) - g + 1 = 1$. Let $s \in H^0(X, \mathcal{L})$ and $\mathcal{L} \cong \mathcal{O}_C(D)$ with D a divisor on C. By definition, $\mathrm{div}(s) + D = H \geq 0$. By assumption, $H \geq 0$ and $\deg(H) = 1$, so $H = x$. Now $H = \sum_{i=1}^{r} \alpha_i x_i$, where $\alpha_i \in \mathbb{Z}_{>0}$, so $\deg(H) = \sum \alpha_i = 1$. We now know that $\mathcal{L} \cong \mathcal{O}_C(D) \cong \mathcal{O}_C(x)$. □

If G is an algebraic group, then we have a short exact sequence $1 \to A \to G \to L \to 1$ where A is an abelian variety and L is a linear algebraic group. No algebraic group G can act on curve C with $g(C) \geq 2$.

17 Lecture 17

Remark 17.1. In any dimension, we can understand $\chi(\mathcal{L}^{\otimes m})$ as a polynomial and compute the leading term and the second term.

For curves, $\chi(\mathcal{L}^{\otimes m}) = m\deg(\mathcal{L}) - g(C) = 1$. In dimension-2, we have a precise formula

$$\chi(X,\mathcal{L}) = \frac{\mathcal{L}^2 + \mathcal{L}(-K_X)}{2} + \chi(X,\mathcal{O}_X)$$

and

$$\chi(X,\mathcal{L}^{\otimes m}) = m^2\cdot\frac{\mathcal{L}^2}{2} + m\cdot\frac{\mathcal{L}\cdot(-K_X)}{2} + \chi(X,\mathcal{O}_X)$$

We know $\Omega_x = T_x^*$ is the cotangent bundle and $\omega_x = \bigwedge^n\Omega_x$. Then ω_x is called canonical line bundle, and $\mathcal{O}_X(K_X)\cong\omega_X$ where K_X is called the canonical divisor.

In dimension-3, we still have a formula; in dimension-4, things break down.

We also have Riemann-Roch for singular varieties (which gives extra terms). This works for curves, normal surfaces and 3-dimensional manifolds. This works for curves, normal surfaces, and 3-dimensional orbfolds.

Theorem 17.2 (Serre Duality)**.** Let X be a smooth projective variety of dimension n over a field k. Then there is a natural trace map

$$\mathrm{tr}: H^n(X,K_X)\to K$$

that is an isomorphism if X is geometrically connected. For every vector bundle $\mathcal{E}$ on X, the product

$$H^i(X,\mathcal{E})\times H^{p-i}(X,K_X\otimes\mathcal{E}^*)\to H^n(X,K_X)\xrightarrow{\mathrm{tr}} K$$

is a dual pairing.

Corollary 17.3. $h^0(C,\mathcal{L}) - h^0(C,K_C\otimes\mathcal{L}^*) = \deg(\mathcal{L}) - g(C) + 1$.

Corollary 17.4. If $\deg(\mathcal{L}) > \deg(K_C)$, then $h^0(X,\mathcal{L}) = \deg(\mathcal{L}) - g(C) + 1$.

Definition 17.5 (Cup Product)**.** For sheaves $\mathcal{E}$ and $\mathcal{F}$ of $\mathcal{O}_X$-modules on a ringed space X, there is a product $H^i(X,\mathcal{E})\times H^j(X,\mathcal{F})\to H^{i+j}(X,\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{F})$ that is $\mathcal{O}_X$-bilinear. If X is separated and if $\mathcal{E}$ and $\mathcal{F}$ are quasi-coherent, then we can use affine covers to define this product.

Let $\{U_i\}$ be an affine cover of X. Let $U_{m_0,\dots,m_i} = U_{m_0}\cap\cdots\cap U_{m_i}$. There is $\mathcal{C}^i(X,\mathcal{E})\times\mathcal{C}^i(X,\mathcal{F})\to\mathcal{C}^{i+j}(X,\mathcal{E}\otimes\mathcal{F}$, with $(\alpha\times\beta)_{m_0,\dots,m_{i+j}} = \alpha_{m_0,\dots,m_i}|_{U_{m_0,\dots,m_{i+j}}}\cdot\beta_{m_{i+1},\dots,m_{i+j}}|_{U_{m_0,\dots,m_{i+j}}}$ induces a term in $\mathcal{E}\otimes\mathcal{F}(U_{m_0,\dots,m_{i+j}})$.

Exercise 17.6. Check that the differential map in this case is $d(\alpha\beta) = d\alpha\cdot\beta + (-1)^{|\alpha|}\alpha d\beta$.

The previous equality implies that the product descends to cohomology.

Exercise 17.7. $K_{\mathbb{P}^n} = -(n+1)H$ and $\omega_{\mathbb{P}^n} \cong \mathcal{O}(-n-1)$. So $H^n(\mathbb{P}^n, K_{\mathbb{P}^n}) \cong H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1)) \cong K$.

Definition 17.8. Let $f : X \to Y$ be a continuous map of topological spaces Then $R^i f_* L\mathbf{Shv}(X) \to \mathbf{Shv}(Y)$ are the right derived functors of $f_* : \mathbf{Shv}(X) \to \mathbf{Shv}(Y)$. Here $f_*\mathcal{E}(U) = \mathcal{E}(f^{-1}(U))$ where U is open in Y so that the preimage is open in X. Each sheaf $\mathcal{E}$ on X gives sheaves $R^i f_* \mathcal{E}$ on Y.

We would hope for $R^i f_*(\mathcal{E}(U)) = H^i(f^{-1}(U), \mathcal{E})$.

Proposition 17.9. For $i \in \mathbb{Z}$, $f : X \to Y$ continuous, $\mathcal{E}$ a sheaf of abelian groups in X, then $R^i f_*(\mathcal{E})$ is the sheafification of the presheaf $U \mapsto H^i(f^{-1}(U), \mathcal{E})$.

The previous equality implies that the product descends to cohomology. In particular,

$$H^i(X, \mathcal{E}) \otimes H^{n-i}(X, K_X \otimes \mathcal{E}^*) \to H^n(X, \mathcal{E} \otimes K_X \otimes \mathcal{E}^*) \xrightarrow{s} H^n(X, K_X) \xrightarrow{\mathrm{tr}} K$$

Lemma 17.10. Let M be a line bundle on C with $H^0(C, K_C \otimes M^*) = 0$, then $H^1(X, M) = 0$.

Proof. Let $D \geq 0$ be an effective divisor. Define $\mathcal{F} = M(D) = M \otimes \mathcal{O}_C(D)$. $H^0(C, K_C \otimes \mathcal{F}^*) = H^0(C, K_C \otimes M^* \otimes \mathcal{O}_C(-D)) \subseteq H^0(C, K_C \otimes M^*) = 0$. Then $\mathcal{F}$ satisfies the same assumption as M. For $c \in C(k)$, and consider $0 \to \mathcal{F} \to \mathcal{F}(c) \to \mathcal{F}(c)|_c \to 0$, then

$$K \cong H^0(c, \mathcal{F}(c)|_c) \xrightarrow{\delta_c} H^1(C, \mathcal{F}) \to H^1(C, \mathcal{F}(c)) \to 0$$

We claim that δ_c is the zero map. Indeed, let Δ be a square of sides c and a diagonal of Δ, then there are maps π_1 and π_2 from this structure to C. We have

$$0 \to \pi_1^* \mathcal{F} \to \pi_1^* \mathcal{F}(\Delta) \to \pi_1^* \mathcal{F}(\Delta)|_\Delta \to 0$$

We consider the long exact sequence of $R^j(\pi_2)_*$. This gives

$$\begin{tikzcd}
\pi_{2_*}\pi_1^*\mathcal{F}(\Delta)|_\Delta \arrow[r, "\delta"] \arrow[d] & R^1(\pi_{2_*}\pi_1^*(\mathcal{F})) \arrow[r] & \cdots \\
k \cong H^0(C, \mathcal{F}(c)|_c) \arrow[r, "\delta_c"'] & H^1(C, \mathcal{F}) \otimes_k \mathcal{O}_C &
\end{tikzcd}$$

Now Δ is a divisor in $C \times C$, so $\mathcal{O}(\Delta)|_\Delta \cong \mathcal{N}_\Delta|_{C \times C}$. We have a short exact sequence of tangents and normals:

$$0 \to T_\Delta \cong T_C \to T_{C\times C}|_\Delta \cong N_C \times T_C \to \mathcal{N}_\Delta|_{C\times C} \to 0$$

This forces $\mathcal{N}_\Delta|_{C\times C} \cong T_C \cong K_C^*$. Hence, $(\pi_1^*\mathcal{F}(\Delta))|_\Delta \cong \mathcal{F}\otimes K_C^*$. Then δ is a map of vector bundles over C and its dual map goes from $H^1(C,\mathcal{F})^*\otimes C$ to $K_C\otimes\mathcal{F}^*$. Since $H^0(C, K_C\otimes\mathcal{F}^*) = 0$, then δ must be zero. Therefore, $\delta_c = 0$ for all c by the commutativity of the diagram. We then conclude that $H^1(C,\mathcal{F}) \cong H^1(C,\mathcal{F}(c))$ for every point $c\in C(k)$.

We know $H^1(C,\mathcal{F})\cong H^1(C,\mathcal{F}(c))$ for any point $c\in C(k)$, then $H^1(C,\mathcal{F})\cong H^1(C,\mathcal{F}(D))$ for every $D\geq 0$. Therefore, $H^1(C,\mathcal{F}) = \varprojlim_{D\geq 0}(C,\mathcal{F}(D)) \cong H^1(C,\varprojlim_{D\geq 0}\mathcal{F}(D))\cong H^1(C,K(c))$ where K is the function field and we pass in the limit because the Noetherian property. Therefore, this is flasque, so this is just 0. $\square$

18 Lecture 18

Recall:

Theorem (Riemann-Roch)**.** Let C be a smooth projective curve over an algebraically closed field k. Let $\mathcal{L}$ be a line bundle on C, then $h^0(\mathcal{L}) - h^1(\mathcal{L}) = \deg(\mathcal{L}) - g(C) + 1$.

Theorem (Serre Duality)**.** Let C be a smooth projective curve over an algebraically closed field k. There is an isomorphism $\mathrm{tr} : H^1(C, K_C)\to K$. For any line bundle $\mathcal{L}$, there is an induced isomorphism $H^1(C,\mathcal{L})\to h^0(C, K_C\otimes\mathcal{L}^*)^*$ by using the trace map.

Theorem. Any smooth projective curve admits a finite map to $\mathbb{P}^1$.

Theorem (Riemann-Hurwitz)**.** Let $f : C_1\to C_2$ be a finite separable map between smooth projective curves, then $2g(C_1)-2 = \deg(f)(2g(C_2)-2)+\deg(R_f)$ where $R_f = \sum_{x\in C_1}(e_x-1)x$ is the ramification.

Proof of Serre Duality. Let $\mathcal{F}$ be a line bundle on C. We compared $H^1(C,\mathcal{F})$ and $H^1(C,\mathcal{F}(C))$. We have a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}(C) \longrightarrow \mathcal{F}(C)|_C \to 0.$$

Therefore,

$$K \cong H^0(C,\mathcal{F}(C)|_C) \longrightarrow H^1(C,\mathcal{F}) \longrightarrow H^1(C,\mathcal{F}(C)) \longrightarrow 0$$

If $\mathbf{Hom}_C(\mathcal{F}, K_C) = 0$, then $H^1(C,\mathcal{F}) = 0$. Therefore, taking π_1*, we have

$$0 \longrightarrow \pi_1^*\mathcal{F} \longrightarrow \pi_1^*\mathcal{F}(\Delta) \longrightarrow \pi_1^*\mathcal{F}(\Delta)|_\Delta \longrightarrow 0$$

Apply $R^i\pi_{2_*}$. Therefore we have

$$\begin{tikzcd} (\pi_2)_*[(\pi_1^*\mathcal{F}(\Delta))|_\Delta] \arrow[r] \arrow[d, "\cong"] & R^1(\pi_2)_*(\pi_1^*\mathcal{F}) \arrow[d, "\cong"] \\ \mathcal{F} \otimes K_C^* \arrow[r, "\Delta"] & H^1(C,\mathcal{F}) \otimes \mathcal{O}_C \end{tikzcd}$$

Now δ is given by elements of $H^0(C, K_C \otimes \mathcal{F}^*) = 0$, so $\delta = 0$, then $\delta_c = 0$ for all $c \in C(k)$. Therefore, $H^1(C,\mathcal{F}) = 0$, whenever $H^0(C, K_C \otimes \mathcal{F}^*) = 0$.

Lemma 18.1. $H^1(C, K_C) \cong K$.

Subproof. If $\deg(M) < \deg(K_C)$, then $H^0(C, K_C \otimes M^*) = 0$ since $\deg(K_C \otimes M^*) = \deg(K_C) - \deg(M) < 0$, therefore $H^1(C, M) = 0$. If $\deg(M) < g-1$, then $H^1(C,M) \neq 0$. Now $h^0(M) - h^1(M) = \deg(M) - g + 1 < 0$. Now let M be a line bundle with maximal degree for which $H^1(C,M) \neq 0$, so $\deg(M) \geq g$, then $H^1(C, M(c)) = 0$ for any $c \in C(k)$. Therefore, we have

$$0 \longrightarrow M \longrightarrow M(c) \longrightarrow M(c)|_c \longrightarrow 0$$

and so

$$H^0(c, M(c)|_c) \cong k \xrightarrow{\delta_c} H^1(C, M) \longrightarrow H^1(C, M(c)) = 0 \longrightarrow 0$$

Therefore, we have $H^1(C, M) \cong k$. Moreover,

$$\delta : M \otimes K_c^* \to H^1(C,M) \otimes \mathcal{O}_C \cong k \otimes \mathcal{O}_C \cong \mathcal{O}_C$$

and therefore $M \cong K_C$. ■

Now let C be smooth projective curve over an algebraically closed field k. We have the following information:

Theorem. If $H^0(C, K_C \otimes \mathcal{F}^*) = 0$, then $H^1(C, \mathcal{F}) = 0$.

Theorem. If M is of maximal degree with $H^1(C,M) \neq 0$, then $M \cong K_C$ and $H^1(C, M) \cong k$.

We have

$$H^0(C, K_C \otimes \mathcal{F}^*) \times H^1 9C, \mathcal{F}) \longrightarrow H^1(C, K_C) \xrightarrow[\cong]{tr} k$$

Then we have $\varphi : \mathbf{Hom}_C(\mathcal{F}, K_C) \to H^1(C, \mathcal{F})^*$. We want to show that φ is an isomorphism.

Pick $\alpha \in \mathbf{Hom}_C(\mathcal{F}, K_C)$, then we have a diagram

$$\begin{tikzcd} \mathcal{F} \otimes K_C^* \arrow[r, "\delta"] \arrow[d, "\alpha"] & H^1(C,\mathcal{F}) \otimes \mathcal{O}_C \arrow[d, "\varphi(\alpha)"] \\ K_C \otimes K_C^* \arrow[r, "\cong", "\delta"'] & H^1(C, K_C) \otimes \mathcal{O}_C \cong \mathcal{O}_C \end{tikzcd}$$

So we have $\varphi : \mathbf{Hom}_C(\mathcal{F}, K_C) \to H^1(C, \mathcal{F})^*$. Assume $\varphi(\alpha) = 0$ from the diagram and δ being an isomorphism, it follows that $\alpha = 0$.

For every $\beta \in H^1(C, \mathcal{F})^*$, we can consider the following diagram

$$\begin{tikzcd} \mathcal{F} \otimes K_C^* \arrow[r] \arrow[d, dashed, "\alpha"] & H^1(C, \mathcal{F}) \otimes \mathcal{O}_C \arrow[d, "\beta"] \\ K_C \otimes K_C^* \arrow[r, "\delta", "\cong"'] & H^1 * C, K_C) \otimes \mathcal{O}_C \cong \mathcal{O}_C \end{tikzcd}$$

Then there is a unique $\alpha : \mathcal{F} \otimes K_C^* \to \mathcal{O}_C$ making the diagram commutative. Note that if $\deg(\mathcal{F}) > \deg(K_C)$, then φ is an isomorphism.

We proceed by descending induction on $\deg(\mathcal{F})$. There is $\mathcal{F} \hookrightarrow \mathcal{F}(c)$ for $c \in C(k)$, then the corresponding φ of $\mathcal{F}(c)$ is an isomorphism $H^1(C, \mathcal{F}) \twoheadrightarrow H^1(C, \mathcal{F}(c)))$. If we dualize $H^1(C, \mathcal{F}(c))^* \hookrightarrow H^1(C, \mathcal{F})^*$, every element in $H^1(C, \mathcal{F}(c))^*$ comes from $\mathcal{F}(c) \to K_c$ over C, hence it comes from $\mathcal{F} \to \mathcal{F}(c) \to K_c$ over C. Given $\beta \in H^1(C, \mathcal{F})^*$, we found α so that both β and $\varphi(\alpha)$ fits in the commutative diagram for β. Therefore, $\varphi(\alpha)$ and β in $H^1(C, \mathcal{F})^*$ are equal in the image of $\delta_c : K \to H^1(C, \mathcal{F})$. Hence $\varphi(\alpha) - \beta$ lies in the image of φ. Therefore, β is in the image of φ. $\square$

As corollaries, we have

Theorem. $h^0(\mathcal{L}) - h^0(K_c \otimes \mathcal{L}^*) = \deg(\mathcal{L}) - g(C) + 1$.

Theorem. $h^0(K_c) = g$ and $\deg(K_c) = 2g - 2$.

Proof. Set $\mathcal{L} = K_c$, $h^0(K_c) - h^0(\mathcal{O}_c) = \deg(K_c) - g + 1$, and $h^0(K_c) - 1 = \deg(K_c) - g + 1$. Let $\mathcal{L} = \mathcal{O}_C$, we also know that $h^0(\mathcal{O}_C) - h^0(K_C) = 0 - g + 1$, therefore $h^0(K_c) = g$, so $\deg(K_c) = 2g - 2$. $\square$

Curves	g	$\deg(K_C)$	π_1	Aut	$C(\mathbb{Q})$	Dynamics
$\mathbb{P}^1$	0	-2	$\{1\}$	PGL(2)	dense/empty	complicated
Elliptic	1	0	$\mathbb{Z}^2$	Elliptics normal in G	Mordell-Weil	complicated
Canonically Polarized	≥ 2	≥ 2	$\langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$	finite	finite	finite

Proposition 18.2. If $\mathcal{L}$ has degree 0 and $H^0(C, \mathcal{L}) \neq 0$, then $\mathcal{L} \cong \mathcal{O}_C$. Otherwise if $H^0(C, \mathcal{L}) = 0$, then $\mathcal{L} \ncong \mathcal{O}_C$.

Proof. Assume $H^0(C, \mathcal{L}) \ncong 0$, so there exists $s \in K(C)$, for which $\mathcal{L} + \operatorname{div}(s) = H \geq 0$, and since $\deg(\mathcal{L}) + \deg(\operatorname{div}(s)) = \deg(H) = 0$, so $\mathcal{L} = \operatorname{div}(s^{-1})$. $\square$

Proposition 18.3. Let C be elliptic, then $K_C \cong \mathcal{O}_C$.

Proof. We know $\deg(K_C) = 0$, then $h^0(K_C) - h^0(\mathcal{O}_C) = \deg(K_C) - g + 1$, so $h^0(K_C) = 1$. $\square$

Proposition 18.4. Let $\mathcal{L}$ be a line bundle over C elliptic, then

$\deg(\mathcal{L})$	-3	-2	-1	0	1	2	3
$h^0(\mathcal{L})$	0	0	0	a	1	2	3
$h^1(\mathcal{L})$	-3	-2	-1	b	0	0	0

where $a = 1 - b = \begin{cases} 1, & \text{if } \mathcal{L} \cong \mathcal{O}_C \\ 0, & \text{if } \mathcal{L} \ncong \mathcal{O}_C \end{cases}$

19 Lecture 19

Definition 19.1. Let $f : X \to Y$ be a finite morphism of smooth curves over an algebraically closed field. Then we say that f is separable if either

1. $df : T_x X \to T_{f(x)} Y$ is not identically zero, or
2. the field extension $K(X) \supseteq K(Y)$ is separable.

Recall: let E/F a field extension generated by a single element t, then $E \cong F[t]/(g(t))$ for $g(t) \in F(t)$ irreducible, i.e., $g(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$ for $a_i \in F$, then E is separable if and only if $\gcd(g, g') = 1$. In characteristic 0, every finite extension is separable.

We will now show that the two statements are equivalent.

Proof. Let $a_0, \ldots, a_{n-1} \in \mathcal{O}(Y)$ be regular functions with $X = \{(y,t) \mid t^n + a_{n-1}(y)t^{n-1} + \cdots + a_0(y) = 0\} \subseteq Y \times \mathbb{A}^1$, and let $X \to Y$ be a projection defined as $(y,t) \mapsto y$. The previous description holds locally around a point $y \in Y(k)$ (as well as both statements). We have $\frac{\partial}{\partial t}(t^n + \cdots + a_0(y)) = 0$ if and only if $\frac{\partial g}{\partial t} = 0$ anywhere $g = 0$, so (1) holds if and only if $g'(t) = 0$ anywhere $g = 0$ if and only if (2) holds. $\square$

Remark 19.2. Let $f : X \to Y$ be a finite separable morphism, $df : f^* K_X \to K_Y$ is not identically zero, so we get an exact sequence

$$0 \longrightarrow f^* K_Y \longrightarrow K_X \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

Here $\Omega^1_{X/Y}$ is isomorphic to $\mathcal{O}_{R_f}$ for some effective divisor R_f in X. Then R_f is called the ramification divisor of f.

The local completion gives $\hat{\mathcal{O}_{X,x}} \cong k[[t]]$ where t is local coordinate with $\operatorname{ord}_x(t) = 1$. Let $y = f(x)$, then $\hat{\mathcal{O}_{Y,y}} \cong K[[u]]$ where u is a local parameter around $y = f(x)$. The map $f : X \to Y$ induces $f^* : K[[u]] \to K[[x]]$ defined by $f^*(u) = a_e t^e + a_{e+1} t^{e+1} + \cdots$ for

$a_e \neq 0$. Hence, $f^*[y] = \sum_{x \in f^{-1}(y)} e_x[x] \in \mathrm{Div}(X)$. So $du = (ea_e t^{e-1} + (e+1)a_{e+1}t^e + \cdots)dt$. If $\mathrm{char}(k) = 0$, then the coefficient of R_f is $e_x - 1$; if $\mathrm{char}(k) = p$ and $p \nmid e_x$, then the coefficient of R_f is $e_x - 1$; if $p \mid e_x$, then f is wildly ramified at x: the coefficient of R_f is finite.

Theorem 19.3 (Riemann-Hurwitz)**.** Let $f : X \to Y$ be a finite separable morphism between smooth projective curves over an algebraically closed field k. Then

$$2(g(X) - 1) = 2(g(Y) - 1) \cdot \deg(f) + \deg(R_f)$$

Proof. We know that $2(g(X) - 1) = \deg(K_X)$ and $2(g(Y) - 1) = \deg(K_Y)$. Now $f^*K_Y = K_X(-R_f)$, so

$$\deg(f)\deg(K_X) = \deg(f^*K_X) = \deg(K_X) - \deg(R_f).$$

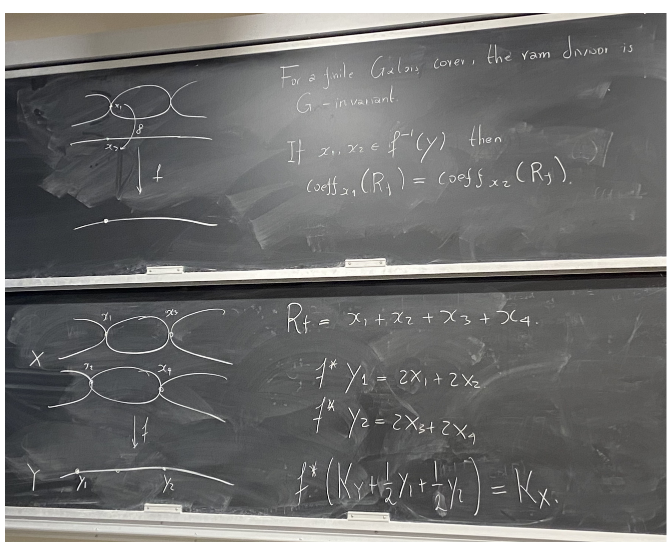
$\square$

Remark 19.4. Any finite group G acts on some smooth projective curve C. Indeed, we have $G \hookrightarrow \mathbb{P}GL_n(\mathbb{C})$, then G acts on $\mathbb{P}^n_k$. We choose a general G-invariant hypersurfaces $H_1, \ldots, H_{n-1}$.

Theorem 19.5 (Nagata)**.** Let R be a normal ring and G be a finite group acting on R, then R^G is normal. Thus, let $X = \mathrm{Spec}(R)$, then $X/G = \mathrm{Spec}(R^G)$.

Corollary 19.6. If X is a smooth projective curve and G is a finite group acting on X, then $Y = X/G$ is smooth. In this case, we say Y is Galois.

Remark 19.7. For a finite Galois cover, the ramification divisor is G-invariant. If $x_1, x_2 \in f^{-1}(y)$, then the coefficient of R_1 with respect to x_1 is equal to the coefficient of R_2 with respect to x_2.



Theorem 19.8 (Riemann-Hurwitz)**.** Let $X \to Y = X/G$ be a finite Galois quotient of smooth projective curves over an algebraically closed field of characteristic 0. Then we can write $f^*(K_Y.\Delta_Y) = K_X$ where $\Delta_Y = \sum\limits_{p \in Y}(1 - \frac{1}{n_p})p$ for a finite set of positive integers n_p.

Theorem 19.9. Let $f : X \to Y$ be a finite separable morphism between smooth projective curves, there exists a finite separable morphism $g : Z \to Y$ such that

$$\begin{tikzcd} X \arrow[d, "f"'] & Z \arrow[l, "f"'] \arrow[dl] \\ Y & \end{tikzcd}$$

where g and fg are Galois.

Definition 19.10. A hyperelliptic curve is a curve with $g \geq 2$ that admits a finite morphism to $\mathbb{P}_1$ of degree 2.

To describe the double covers of $\mathbb{P}^1_X$, we have $K(\mathbb{P}^1) \cong k(x)$ with a morphism $Y \to \mathbb{P}^1$. Now $f^*[p] = x_1 + x_2$ with $x_1 \neq x_2$ and is $2x$ if $x = x_1 = x_2$. Then $K(Y) \cong k(X)[\sqrt{f}]$ where $f = c(x - \alpha_1) \cdots (x - \alpha_s)$ and so $K(Y) \cong k(X)[t]/(t^2 = (x - \alpha_1) \cdots (x - \alpha_s))$. Let $U = \{(x, t) \in \mathbb{A}^2_k \mid t^2 = (x - \alpha_1) \cdots (x - \alpha_s)\} \to \mathbb{A}^1_k$ with variable x. Now U is smooth and Y is the unique smooth projective curve containing U: $U \subseteq \mathbb{A}^1 \times \mathbb{A}^1$, $Y = \bar{U} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$. This gives a 2-to-1 map $\pi_1 : Y \to \mathbb{P}^1_x$, which ramifies at ∞ if and only if s is odd. Therefore, $s = 2g + 2$ or $s = 2g + 1$.

Theorem 19.11. Let C be a hyperelliptic curve of genus g, then there is an embedding $C \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(2, g+1)$. Furthermore, there is an embedding $C \hookrightarrow \mathbb{P}^3$.

Proof. For the "furthermore" part, just use the Segre embedding

$$\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$$
$$([x : y], [z : w]) \mapsto ([[xz : xw : yz : yw]])$$

where the image if a quadric defined by $ad - bc = 0$. □

20 Lecture 20

Theorem 20.1. Let C be a smooth projective over k, with characteristic 0 and algebraically closed. Then there exists a finite separable morphism $f : C \to \mathbb{P}^1$.

Theorem 20.2 (Alexander, 1920)**.** Let M be a n-dimensional smooth manifold. There exists a finite map $M \to S^n$ ramified along a subset of $\mathrm{colim}_{\mathbb{R}} \geq 2$.

Example 20.3. Let S be a Riemann surface, then $f : S \to S^2$; let M be a smooth 3-manifold, then $f : M \to S^3$. We can always choose such f so it ramifies along Borromean circles. (2015)

Over what do you need to ramify in dimension 4?

Theorem 20.4 (Noether, 1950)**.** Let X be a smooth projective variety of dimension n over $\mathbb{C}$. There exists a finite surjective morphism $f : X \to \mathbb{P}^n_{\mathbb{C}}$ only ramified along divisors.

Definition 20.5. Let X be a normal variety and $\mathcal{L}$ be a line bundle on X. We say that $\mathcal{L}$ is basepoint-free if for every $x \in X$, there is $s \in H^0(X, \mathcal{L})$ for which $s(x) \neq 0$.

Remark 20.6. If $\mathcal{L}$ is basepoint-free, and $s_0, \ldots, s_N \in H^0(X, \mathcal{L})$, then

$$\begin{aligned} f : X &\to \mathbb{P}^N \\ x &\mapsto [s_0(x), \ldots, s_N(x)] \end{aligned}$$

is a morphism, then $\mathcal{L} \cong f^*\mathcal{O}(1)$.

Let $\mathcal{L}$ be a line bundle over a smooth projective curve C over a field k.

Lemma 20.7. $\mathcal{L}$ is basepoint free if and only if for all $p \in C(k)$, $h^0(\mathcal{L}) - 1 = h^0(\mathcal{L}(-p))$.

Proof. We have a short exact sequence

$$0 \longrightarrow \mathcal{L}(-p) \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}|_p \longrightarrow 0$$

and therefore gives

$$H^0(\mathcal{L}(-p)) \longrightarrow H^0(\mathcal{L}) \longrightarrow H^0(\mathcal{L}|_p)$$

note that $H^0(\mathcal{L}|_p) \cong k$, with the second map given by $s \mapsto s(p)$. We know $h^0(\mathcal{L}) - 1 = h^0(\mathcal{L}(-p))$ if and only if there is a section s of $H^0(X, \mathcal{L})$ not vanishing at p. $\square$

Corollary 20.8. If $\deg(\mathcal{L}) \geq 2g(C)$, then $\mathcal{L}$ is basepoint-free.

Proof. Observe that if $\deg(\mathcal{F}) \geq 2g-1$, then $h^0(\omega_C(\mathcal{F}^\vee)) = 0$. Therefore, by Riemann-Roch, $h^0(\mathcal{F}) = \deg(\mathcal{F}) - g + 1$. Apply this to $\mathcal{F} = \mathcal{L}(-p)$ and $\mathcal{F} = \mathcal{L}$. $\square$

Corollary 20.9. Let k be an algebraically closed field and let C be a smooth curve of genus 1 over k. Then C is a double cover of $\mathbb{P}^1$. Furthermore, if $\mathrm{char}(k) \neq 2$, then this double cover ramifies along 4 points.

Proof. By Corollary 20.8, $2p_0$ is basepoint-free, and $\mathcal{O}_C(2p_0)$ is basepoint-free. Therefore, it defines a morphism $C \to \mathbb{P}^N$, where $N = h^0(C, \mathcal{O}_C(2p_0))$. Now $h^0(2p_0) - h^1(2p_0) = 2-1+1 = 2$ by Riemann-Roch, and $h^1(2p_0) = h^0(\omega_C(-2p_0)) = h^0(-2p_0) = 0$ by Serre duality. Then $f : C \to \mathbb{P}^1$ gives $\mathcal{O}_C(2p_0) \cong f^*\mathcal{O}(1)$. Then $2 = \deg(f^*\mathcal{O}(1)) = \deg(f) \cdot \deg(\mathcal{O}(1)) = \deg(f)$. By Riemann-Hurwitz, $2(g(C) - 1) = \deg(f)(2g(\mathbb{P}^1) - 1) + \deg(R_f)$ where $R_f = \sum_{i=1}^{s} x_i$. Now $0 = 2 \cdot (-2) + \deg(R_f)$, so $s = 4$. □

Exercise 20.10. Let $x, y, z \in \mathbb{P}^1$ be three different points, there exists $g \in \mathrm{Aut}(\mathbb{P}^1)$ such that $g(x) = [1 : 0]$, $g(y) = [1 : 1]$, and $g(z) = [0 : 1]$.

Theorem 20.11. The moduli space paarametrizing elliptic curves is 1-dimensional.

Proposition 20.12. Let X be a proper scheme over an algebraically closed field k. Let $f : X \to \mathbb{P}^n_k$ be a morphism over k. Let $\mathcal{L} := f^*\mathcal{O}(1)$ be basepoint-free. Let $V = k\{s_0, \ldots, s_N\} = H^0(X, \mathcal{L})$, then f is an embedding if:

1. V separates points: for every $p \neq q$, there is $s \in V$ with $s(p) = 0$ and $s(q) \neq 0$.
2. V separates tangents: for every $x \in X(k)$, the set $\{s \in V \mid s_p \in \mathfrak{m}_p\mathcal{L}_p\}$ spans the vector space $\mathfrak{m}_p\mathcal{L}_p/\mathfrak{m}_p^2\mathcal{L}_p$.

Lemma 20.13. Let $\mathcal{L}$ on a smooth projective curve C over k and k is algebraically closed. Then $\mathcal{L}$ is very ample if and only if for all $p, q \in C(k)$, $h^0(\mathcal{L}(-p-q)) = h^0(\mathcal{L}) - 2$ (*).

Proof. ($\Rightarrow$): Left as extra homework exercise.

($\Leftarrow$): (*) implies $\mathcal{L}$ is basepoint-free, so there is an induced morphism $f : C \to \mathbb{P}^N$. Let $p \in C(k)$, and there exists $s \in \mathfrak{m}_p\mathcal{L}_p \setminus \mathfrak{m}_p^2\mathcal{L}_p$. Take the exact sequence

$$0 \longrightarrow \mathcal{L}(-2p) \longrightarrow \mathcal{L}(-p) \longrightarrow \mathcal{L}(-p)|_p \longrightarrow 0$$

and this gives

$$H^0(\mathcal{L}(-2p)) \longrightarrow H^0(\mathcal{L}(-p)) \longrightarrow k$$

Note that $\mathfrak{m}_p\mathcal{L}_p/\mathfrak{m}_p^2\mathcal{L}_p$ as tangent only has dimension 1, so $\mathcal{L}$ separates tangents. □

Corollary 20.14. $\deg(\mathcal{L}) \geq 2g + 1$, then $\mathcal{L}$ is very ample.

Proof. $\deg(\mathcal{F}) \geq 2g - 1$, then $h^0(\mathcal{F}) = \deg(\mathcal{F}) - g + 1$. Apply this statement to $\mathcal{L}$, $\mathcal{L}(-p)$, and $\mathcal{L}(-p-q)$. □

Proposition 20.15. An elliptic curve can be embedded in $\mathbb{P}^2$ as a cubic.

Proof. Let $p_0 \in C(k)$ and $\mathcal{L} = \mathcal{O}_C(3p_0)$. Then $\mathcal{L}$ is very ample. By Riemann-Roch, $h^0(3p_0) - h^1(3p_0) = 3 - 1 + 1 = 3$, and $h^1(3p_0) = h^0(\omega_C(-3p_0)) = 0$ by Serre duality. Let $f : C \hookrightarrow \mathbb{P}^2$ and $\mathcal{L} = f^*\mathcal{O}(1)|_C$, then $\deg(f)\deg(\mathcal{O}(1)|_C) = \deg(f^*\mathcal{O}(1))|_C = \deg(\mathcal{L}) = 3$, then since $\deg(f) = 1$, so $\deg(\mathcal{O}(1)|_C) = 3$. $\square$

Theorem 20.16. Let C be a smooth projective curve of genus $g \geq 2$ over k algebraically closed. Then one of the following happens:

1. C is hyperelliptic and it admits an embedding in $\mathbb{P}^3$ as a curve of degree $2g + 2$, or

2. ω_C is very ample, and $C \hookrightarrow \mathbb{P}^{g-1}$ as a curve of degree $2g - 2$.

Proof. Let $p, q \in C(k)$, now $h^0(p + q) - h^0(\omega_C(-p - q)) = 3 - g$, and $h^0(\omega_C) = g - 1$, so either $h^0(\omega_C(-p - q)) = g - 3$ for all p, q (ω_C is very ample), or $h^0(p + q) > 0$ for some p, q, which means $f : C \to \mathbb{P}^1$ is of degree 2. $\square$

Theorem 20.17. Every curve C of genus $g \geq 2$ satisfies that $\omega_C^{\otimes 3}$ is very ample, and $C \hookrightarrow \mathbb{P}^{5g-5}$ of degree $6g - 6$.

21 Lecture 21

Let C be an elliptic curve with $C \hookrightarrow \mathbb{P}^2_k$ embedding as a plane cubic.

Theorem 21.1. For $x, y, z \in C(k)$, we have $x + y + z = 0$ in $C(k)$ if and only if there is a line $L \subseteq \mathbb{P}^2_k$ for which $L \cap C = \{x\} + \{y\} + \{z\}$ in $\mathrm{Div}(C)$.

Proof. Recall the group structure of $C(k)$ is given by

$$\begin{aligned} C(k) &\to \mathrm{Pic}^0(C) \\ x &\mapsto [x - p_0] \end{aligned}$$

with $C \hookrightarrow \mathbb{P}^2_k$ defined by $\mathcal{O}_C(3p_0)$. Then $x + y + z = 0$ in $C(k)$ if and only if $(x - p_0) + (y - p_0) + (z - p_0) \sim 0$ if and only if $x + y + z \sim 3p_0$. There exists $s \in H^0(C, \mathcal{O}_C(3p_0))$ that only vanishes on x, y, z. Indeed, the zero locus of s is $[x] + [y] + [z]$. Note that $H^0(C, \mathcal{O}_C(3p_0)) \cong H^0(\mathbb{P}^2, \mathcal{O}(1))$. Let $s_{\mathbb{P}^2}$ be the corresponding element of s, then $s_{\mathbb{P}^2}|_C = s$ and the zero set of $s_{\mathbb{P}^2}$ is a line $L \subseteq \mathbb{P}^2$. We conclude that $L \cap C = [x] + [y] + [z]$. $\square$

Corollary 21.2. The image of p_0 in $\mathbb{P}^2$ is on inflection point of C.

Remark 21.3. For $k = \mathbb{C}$, $C(\mathbb{C})[3] = (\mathbb{Z}_3)^2$, so $C \hookrightarrow \mathbb{P}^2$ has 9 inflection points.

Lemma 21.4. Let C be a smooth projective curve of genus $g \geq 1$, then ω_C is basepoint-free.

Proof. $h^0(C, \omega_C) = g$. So $h^0(C, \omega_C(-p)) = g-1$ or g. If it is $g-1$ for all p, then ω_C is basepoint-free. Assume $h^0(C, \omega_C(-p)) = g$ for some k-point o. Hence, by Riemann-Roch, $h^0(p) \geq 3$. Now $f \in H^0(C, \mathcal{O}_C(p))$ of $C \dashrightarrow \mathbb{P}^1$ gives $C \to \mathbb{P}^1$ of degree 1, so $C \cong \mathbb{P}^1$. □

Remark 21.5. So far, for smooth projective curves over algebraically closed fields, if $g = 0$, then $C \cong \mathbb{P}^1_k$; if $g = 1$, then C is elliptic. $C \to \mathbb{P}^1$ is finite ramified along 4 points of index 2. We have $C \hookrightarrow \mathbb{P}^0$ as a cube, and M_1 is one-dimensional.

Theorem 21.6. A curve of genus 2 is hyperelliptic.

Proof. We define 2 morphisms $C \to \mathbb{P}^{N-1}$ where $N = h^0(C, \omega_C) = 2$ and $f : C \to \mathbb{P}^1$ where $\omega_C \cong f^*(\mathcal{O}(1))$, and $2 = \deg(\omega_C) = \deg(f)\deg(\mathcal{O}(1)) = \deg(f)$, so $\deg(f) = 2$. By Riemann-Hurwitz, $2g_C - 2 = 2(2g(\mathbb{P}^1) - 2) + \deg(R_f)$, so $\deg(R_p) = 6$. □

Remark 21.7. When $g = 2$, C is hyperelliptic and $C \to \mathbb{P}^1$ ramifies along 6 points of ramification index 2, so M_2 is 3-dimensional.

Definition 21.8. Let $g \geq 2$, the subspace of M_g parametrizing hyperelliptic curves is the hyperelliptic loci and denoted by $\mathcal{H}_g$.

Theorem 21.9. $\dim(M_g) = 3g - 3$, $\dim(H_g) = 2g - 1$, $\operatorname{codim}_{M_g}(\mathcal{H}_g) = g - 2$.

Proof Sketch. Deformations of an object X are understood by looking at T_X.

Remark 21.10. Fact: infinitesimal deformations of smooth affine varieties are trivial. ($T_X$coherent, $h^i(T_X) = 0$ for $i > 0$.)

Remark 21.11. Fact: Infinitesimal deformations of smooth projective X are parametrized by $H^1(X, T_X)$. (There is some obstructions in $H^2(X, T_X)$.)

Now we have $\dim_C(M_g)$ as the dimension of deformations of C, which is just $\dim(H^1(C, T_C))$, where $T_C = \omega_C^\vee$. By Riemann-Roch, $h^0(\omega_C^\vee) - h^1(\omega_C^\vee) = \deg(\omega_C^\vee) - g + 1$. By Serre duality, $h^1(\omega_C^\vee) = h^0(\omega_C \otimes (\omega_C^\vee)^\vee) = h^0(\omega_C^{\otimes 2})$. We note $h^0(\omega_C^\vee) = 0$, so $h^1(\omega_C^\vee) = h^0(\omega_C^{\otimes 2}) = g - 1 - \deg(\omega_C^\vee) = g - 1 + 2g - 2 = 3g - 3$. □

Remark 21.12. We have the following table:

g	0	1	2	3
$\dim(\mathcal{H}_g)$	-	-	3	5
$\dim(M_g)$	0	1	3	6

Exercise 21.13. A curve of genus 3 is either hyperelliptic or a smooth plane quartic.

Definition 21.14. A variety X over a field k is said to be unirational if it admits a dominant rational map $\mathbb{P}^n \dashrightarrow X$ for some n. This implies $n = \dim(X)$.

Theorem 21.15. Let $f : X \dashrightarrow Y$ be a domainant rational map between smooth projective curves, then $g_X \geq g_Y$.

Proof. Since X is smooth and Y is separable, then φ can be extended to a finite morphism. By Riemann-Hurwitz, $2(g_X - 1) = \deg(\varphi)(2g_Y - 1) + \deg(R_f) \geq \deg(\varphi)(2g_Y - 2) \geq 2g_Y - 2$, so $g_X \geq g_Y$. □

22 Lecture 22

We start the study of surfaces by looking at the divisors, and in particular the intersection of divisors. Here we try to mimic the case for curves on $\mathbb{P}^2$. Given a surface S and divisors C and D, the intersection $C.D$ should satisfy:

- for transversed intersections, $(C.D)$ is the number of intersection points.
- symmetry.
- additive.
- invariant under linear equivalences.

Lemma 22.1. Let C be an irreducible smooth curve on a surface S, and let D be a curve intersecting C transversally. Then the number of intersections of C and D is $\deg(\mathcal{O}_S(D) \otimes \mathcal{O}_C)$.

Proof. Take

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0$$

and tensor by $\mathcal{O}_C$, we obtain

$$0 \to \mathcal{O}_S(-D) \otimes \mathcal{O}_C \to \mathcal{O}_C \to \mathcal{O}_{C \cap D} \to 0$$

Therefore $\mathcal{O}_S(-d) \otimes \mathcal{O}_C$ corresponds to the points $C \cap D$ in the reduced case. □

Lemma 22.2. Let $C_1, \ldots, C_r$ be irreducible curves and let D be a very ample divisor. Then almost all $D' \in |D|$ are irreducible smooth and transversal to C_i.

Proof. Consider $S \hookrightarrow \mathbb{P}^N$ which corresponds to $|D|$ and H. By Bertini on $S, C_1, \ldots, C_r$ on each step, we still have a dense open of $|H|$, and this restricts to S. □

Proposition 22.3. $C.D$ is uniquely defined and well-defined.

Proof. Uniqueness: we show that any divisor can be written as a combination of ample divisors. Let $C.D$ be divisors and fix ample H. By definition, $C + nH$ and $D + nH$ is globally generated for large enough n. Also, nH is very ample. Then $C + 2nH$, $D + 2nH$, $2nH$ are all very ample. Take non-singular "mutually" transversal sections $C^1 \in |C + 2nH|$, $D^1 \in |D + 2nH|$, $E^1 \in |2nH|$ and $F^1 \in |2nH|$, then $C \sim C^1 - E^1$, $D \sim D^1 - F^1$, so $C.D$ is just $\#(C^1 \cap D^1) - \#(C^1 \cap F^1) - \#(E^1 \cap D^1) + \#(E^1 \cap F^1)$, which shows uniqueness.

Existence: Let C and D be very ample, then choose $C^1 \in |C|$ and $D^1 \in |D|$, then $C.D. = \#(D^1 \cap C^1) = \deg(\mathcal{O}(D) \otimes \mathcal{O}_C) = \deg(\mathcal{O}(C) \otimes \mathcal{O}_D)$. By linear equivalence, we can change C' by C'' and D' by D'', so this is well-defined. To show existence, if we have $C' \sim C''$ and $E' \sim E''$, and $C'' - E'' \sim C$, then $C'' + E' \sim C' + E''$ is very ample, therefore $C''.D' + E''.D' = C'.D' + E''.D'$ is very ample and so is $C''.D' - D''/.D' = C'.D' - E'.D'$. Therefore, this holds according to the very ample case. □

Remark 22.4 (Positivity)**.** $\deg(D) > 0$ if and only if D is very ample in curves. In surfaces, D is ample if $D.C > 0$ for C effective and $D^2 > 0$. If D is nef, then $D.C > 0$.

Definition 22.5. $D \equiv C$ is numerically equivalent if $D.E. = C.E.$ for any divisor E.

Proposition 22.6 (Adjunction)**.** Let C be a smooth curve of genus g on S, then $2g - 2 = C.(C + K_S)$.

Proof. By adjunction, we have $\omega_C \cong \omega_S \otimes \mathcal{O}(C) \otimes \mathcal{O}_C = \mathcal{O}(C + K_S)$, then $\deg(\omega_C) = 2g - 2 = [C.(C + K_S)]$. □

By the degree-genus formula, $K_{\mathbb{P}^2} = -3L$, so $g = \frac{(d-1)(d-2}{2}$.

23 Lecture 23

Theorem 23.1 (Riemann-Roch)**.** Let D be a divisor on surface S, then $\chi(\mathcal{O}(D)) = \frac{1}{2}(D - K_S).D + \chi(\mathcal{O}_S)$.

Proof. Note that $D \sim C \setminus E$ which is non-singular as a transverse intersection. We have short exact sequences $0 \to \mathcal{O}(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$ and $0 \to \mathcal{O}(-E) \to \mathcal{O}_S \to \mathcal{O}_E \to 0$, by tensoring $\mathcal{O}(C)$, we have $0 \to \mathcal{O}_S \to \mathcal{O}(C) \to \mathcal{O}(C) \otimes \mathcal{O}_C \to 0$, and $0 \to \mathcal{O}(C \setminus E) \to \mathcal{O}(C) \to \mathcal{O}(C) \otimes \mathcal{O}_E \to 0$. Therefore,

$$\chi(\mathcal{O}(C \setminus E)) + \chi(\mathcal{O}(C) \otimes \mathcal{O}_E) = \chi(\mathcal{O}(C)) = \chi(\mathcal{O}_S) + \chi(\mathcal{O}(C) \otimes \mathcal{O}_C).$$

Now

$$\begin{aligned}\chi(\mathcal{O}(C)\otimes\mathcal{O}_E &= \deg(\mathcal{O}(C)\otimes\mathcal{O}_E)+\chi(\mathcal{O}_E)\\ &= C.E+1-g_E\\ &= C.E+\frac{1}{2}(E^2+E.K_S)\end{aligned}$$

and similarly $\chi(\mathcal{O}(C)\otimes\mathcal{O}_C)=C^2+\frac{1}{2}(C^2\setminus C.K_S)$. Now we are done because $\chi=h^0-h^1+h^2=h^0-h^1+h^0(K\setminus D)$. In curves we have $h^0(K\setminus nD)$ for $n\gg 0$ then this is 0 because D has positive degree. In surfaces we have $D.H>0$ being very ample, so for $n\gg 0$ $K-nD.H<0$ is not effective. $\square$

Lemma 23.2. Let H be ample, if $D.H>K_S.H$, then $H^2(\mathcal{O}(D))=0$.

Proof. If $H^2(\mathcal{O}(D))=H^0(\mathcal{O}(K\setminus\ D))\geq 1$, then $(K\setminus D)$ is some effective divisor, so $(K\setminus D).H>0$, therefore $K.H>D.H$, contradiction. $\square$

Corollary 23.3. Let H be ample and $D.H>0$ with $D^2>0$. Then for $n\gg 0$, nD is an effective divisor and $h^0(nD)\geq 1$.

Proof. For $n\gg 0$, $nD.H>K_s.H$, so $h^0(nD)\geq\chi(\mathcal{O}(nD))=\frac{1}{2}n^2D^2-\frac{1}{2}nD.K+\chi(\mathcal{O}_S)$, ans this goes up to ∞ as n goes to ∞. $\square$

Theorem 23.4 (Hodge Index Theorem)**.** Let H be ample and $D\not\equiv 0$, such that $D.H=0$, then $D^2<0$.

Proof. Suppose $D^2>0$, and let $H'=D+nH$ for large enough $n\gg 0$, so that H is ample. Then $D.H'=D.(D+nH)=D^2>0$, so some mD is "effective", i.e., $mD.H>0$, contradiction. If $D^2=0$, then there exists E such that $D.E\neq 0$. Therefore $E\not\sim\alpha H$ is not equivalent for all α. Now $\alpha E+\beta H\not\sim 0$, therefore $E':=H^2E\setminus(E.H)H$, then we have $E'.H=0$. Now $D'=nD+E'$, so $D'.H=0$, and so $D'^2=2nD.E'+E'^2>0$ for some $n\in\mathbb{Z}$. Do the case for $D^2>0$ again and we are done. $\square$

Theorem 23.5 (Nakai-Moishezon criterion)**.** D is ample if and only if $D^2>0$ and $D.C>0$ for all irreducible curves C.

Definition 23.6. We say S is a geometrically ruled surface if $\pi:S\to C$ for some smooth C such that fibers are $\mathbb{P}^2$ (with a section) (C_0).

Proposition 23.7. Let $\pi:S^1\to C$, then there exists ε vector bundle of rank 2 on C such that $S\cong\mathbb{P}(\mathcal{E})$. Therefore, $\mathbb{P}(\mathcal{E})\cong\mathbb{P}(\mathcal{E}')$ if and only if $\mathcal{E}'\cong\mathcal{E}\otimes\mathcal{L}$ for some line bundle $\mathcal{L}$.

Definition 23.8. Birationally ruled surfaces S are surfaces birational to $C \times \mathbb{P}^1$ for some curve C.

Remark 23.9. Geomtetrically ruled implies birationally ruled surfaces.

Proposition 23.10. Let $\pi : S \to C$ with section C_0 and let F be a fiber. Then $\operatorname{Pic}(S) \cong \mathbb{Z}C_0 \oplus \pi^* \operatorname{Pic}(C)$, where $\pi^* \operatorname{Pic}(C)$ is the fibers over the points of C. Now $S \cong \mathbb{Z} \oplus \mathbb{Z}$.

Proof. Let D be a divisor in S, let $F^2 = 0$ with $C_0.F = 1$. Denote $m := D.F$. We claim that $D' := D - mC_0$, then $D' = \pi^*G$ for divisor G on C. Now $F.K_S = -2$, so $D' + nF$ implies $(D_n)^2 = D'^2$ for intersection D_n. By Riemann-Roch, we have an effective divisor. □

24 Lecture 24

Proposition 24.1. Let $\pi : S \to C$ be such that $\operatorname{Pic}(S) = \operatorname{Pic}(C) \oplus \mathbb{Z}[C_0]$ where C_0 is the section, then $\operatorname{Num}(S) = \mathbb{Z}[F] \oplus \mathbb{Z}[C_0]$.

Remark 24.2. $C \times \mathbb{P}^1$ is geometrically ruled, but $C \times C$ does not decompose nicely in general.

Proof. Take $D \in \operatorname{Div}(S)$, let $m = D.F$.

Claim 24.3. Let $D' = D - mC_0$, then $D' \sim \pi^*(G)$ for $G \in \operatorname{Div}(C)$.

$D_n = D' + nF$, notice $D'.F = 0$, then $D_n^2 = D'^2$. The fibers on $\mathbb{P}^1$ gives $2g-2 = F^2 + F.K_S$, so $F.K_S = -2$. Now $D_n.K_S = D'.K_S - 2n$ and by Riemann-Roch, $\chi(\mathcal{O}(D_n)) = \chi(\mathcal{O}_S) + \frac{1}{2}(D_n^2 - D_n.K_S)$. Now $h^0(\mathcal{O}(D_n)) - h^1(\mathcal{O}(D_n)) + h^0(K \setminus D_n) = \frac{1}{2}(D'^2 - D'.K_S + 2n) > 0$ for $n \gg 0$, since for $n \gg 0$ $h^0(K \setminus D_n)$ intersects negatively an ample, so this is not effective (numerical equation), hence $D_n \sim E$ effective. Now $E.F = 0$, then $\pi(E) \subsetneq C$, so $E = \pi^*(G')$ is a divisor, and therefore $D \sim D_n - nF \sim \pi^*(G' + \operatorname{Div}_C)$. □

Proposition 24.4. Let C be of genus ≥ 1, then $\operatorname{Num}(C \times C) > \mathbb{Z}[C_1] \oplus \mathbb{Z}[C_2]$.

Proof. Consider the diagonal curve, then $g(\Delta) = g$, so $\Delta.K_{C\times C} + \Delta^2 = 2g - 2$ where $\Delta.K_{C\times C} = (2g-2)(C_1 + C_2)$, then $\Delta.C_1 = 1 = \Delta.C_2$. Assume $\Delta = \alpha C_1 + \beta C_2$, then for $\Delta = C_1 + C_2$ we have $g = 0$ by the adjunction formula. □

We now try to blow up points in surfaces. Let $\tilde{X} = \operatorname{Bl}_p(X)$.

Remark 24.5. Let $\pi : \tilde{X} \to X$, then $\pi^{-1}(p) = E$ is exceptional divisor. Let $\pi : \tilde{X} \setminus E \to X \setminus \{p\}$ be an isomorphism with $E \cong \mathbb{P}^1$ with $E^2 = -1$.

Proposition 24.6. Let $\pi : \tilde{X} \to X$, then $\text{Pic}(\tilde{X}) \cong \text{Pic}(X) \oplus \mathbb{Z}[E]$. The intersection numbers satisfies:

(a) for $C, D \in \text{Pic}(X)$, then $\pi^*(C).\pi^*(D) = C.D.$,

(b) for $C \in \text{Pic}(X)$, then $\pi^*(C).E = 0$,

(c) $E^2 = -1$,

(d) $\pi^* C.D = C.\pi_* D$ for $C \in \text{Pic}(X)$ and $D \in \text{Pic}(\tilde{X})$.

Proof. (a), (b), and (d) follows from taking $C = A_1 - A_2$ as moving them away from p.

Note $\text{Pic}(X) \cong \text{Pic}(X \setminus \{p\})$ then $X \setminus \{p\} \cong \tilde{X} \setminus E$. We have a short exact sequence $0 \to \mathbb{Z} \to \text{Pic}(\tilde{X}) \to \text{Pic}(\tilde{X} \setminus E) \to 0$ with a section π^*. If $nE \sim 0$, then $(nE)^2 = 0$, but $(nE)^2 = -n^2 \neq 0$, contradiction. $\square$

Lemma 24.7. For projective surface, D and $-D$ cannot be both effective.

Proof. There exists A ample and $A.D > 0$ and $A.(-D) > 0$. $\square$

Proposition 24.8. $K_{\tilde{X}} = \pi^* K_X + E$.

Proof. $K_{\tilde{X}} = \pi^* K_X + nE$ and the adjunction formula gives $n = 1$. $\square$

Suppose C is effective on X and $\tilde{C} = \pi^{-1}(\tilde{C} \setminus \{p\})$ is the strict transformation.

Proposition 24.9. $\pi^*(C) = \tilde{C} + rE$ where r is the multiplicity $M_p(C)$.

Proof. $\tilde{C} = \pi^*(C) - rE = d\tilde{L} - rE = d\tilde{L} - M_p(C) \leq d\tilde{L} - rE$ with $r \leq d$. Now $\pi^*(L) = \tilde{C}_1 + E$, so $\tilde{C}_1 = \pi^* L - E = L - E$. $\square$

25 Lecture 25

Theorem 25.1. Let $f : X \dashrightarrow Y$ be a rational map of k-varieties with X smooth and Y proper, then f is a morphism of codimension 1. That is, there is a closed subset $S \subsetneq X$ of codimension ≥ 2 such that $X \setminus S \to Y$ is a morphism.

Example 25.2. Consider $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ defined by $[x_0 : x_1 : x_2] \mapsto [x_0 : x_1]$. Outside $p = [0 : 0 : 1]$, $\pi : \mathbb{P}^2 \setminus \{p\} \to \mathbb{P}^1$ is a morphism. Then $\pi^{-1}([1 : 1]) = [1 : 1 : t] = [\frac{1}{t} : \frac{1}{t} : 1]$, a line through p. This transforms to $\pi' : \text{bl}_p(\mathbb{P}^2) \to \mathbb{P}^1$ with $E \cong \mathbb{P}^1$. Therefore,

$$\begin{tikzcd} \text{Bl}_p(\mathbb{P}^2) \arrow[d] \arrow[dr, "\pi'"] & \\ \mathbb{P}^2 \arrow[r, dashed, "\pi"'] & \mathbb{P}^1 \end{tikzcd}$$

with $\text{Ex}(\pi) = p$.

Theorem 25.3. Let $X \dashrightarrow Y$ be a rational map, then there exists a projective birational morphism $\varphi : X' \to X$ and a morphism $\pi' : X' \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X' \arrow[d, "\varphi"'] \arrow[dr, "\pi'"] & \\ X \arrow[r, dashed, "\pi"'] & Y \end{tikzcd}$$

Furthermore, the projective birational morphism $X' \to X$ is obtained by a sequence of blow-ups of $\mathrm{Ex}(\pi)$.

Corollary 25.4. If $f : X \dashrightarrow Y$ is a rational map of k-varieties, X is smooth over k and Y is smooth and projective over k, then there is a homomorphism $f^* : H^0(Y, \Omega_Y^j) \to H^0(X, \Omega_X^j)$ for $j \geq 0$. Furthermore, if f is birational, then f^* is an isomorphism.

Proof. By Theorem, there exists $S \subseteq X$ of codimension at least 2, such that $f : X \setminus S \to Y$ is a morphism so we have a homomorphism $f^* : H^0(Y, \Omega_Y^j) \to H^0(X \setminus S, \Omega^j_{X\setminus S})$ for $j \geq 0$. But the restriction $H^0(X, \Omega_X^j) \to H^0(X \setminus S, \Omega^j_{X\setminus S})$ is an isomorphism. To see this is an isomorphism, see the following lemma. $\square$

Lemma 25.5. If $\mathcal{E}$ is a vector bundle on X normal and $S \subseteq X$ has codimension at least 2, then $H^0(X, \mathcal{E}) \to H^0(X \setminus S, \mathcal{E}|_{X\setminus S}$ is an isomorphism.

Proof. Pick $s \in S$, around $s \in X$ we know that $\mathcal{E} \cong \mathcal{O}_X^{\oplus r}$. $\square$

Example 25.6. For $X \subseteq \mathbb{P}^{n+1}$ a smooth hypersurface of degree $\geq n+2$ over a field k, then X is not rational. Indeed, $K_{\mathbb{P}^n} \cong \mathcal{O}(-n-1)$, and by the adjunction formula $K_X \cong (K_{\mathbb{P}^{n+1}} + X)|_X = (-(n+2)H + dH)|_X = ((d-n-2)H)|_X$. Then $\omega_{\mathbb{P}^n} \cong \mathcal{O}(-n-1)$ and $\omega_X \cong \mathcal{O}(d-n-2)$ with $d-n-2 \geq 0$. Assume $f : \mathbb{P}^n \dashrightarrow X$ is a birational map, then $f^* : H^0(\mathbb{P}^n, K_{\mathbb{P}^n}) \to H^0(X, K_X)$ but this is a map from a zero set to a non-zero set, contradiction. If $d-n-2 = 0$, then $\omega_X \cong \mathcal{O}_X$, so it has constant sections. If $d > n+2$, then take $H_1 + \cdots + H_{d-n-2}$, so $H_1 + \cdots + H_{d-n-2}|_X \sim K_X$.

Remark 25.7. $H^0(\Omega_X), H^0(\Omega_X^2), \ldots, H^0(\Omega_X^n)$ are birational invariants for smooth projective varieties. The question is, for which i, j are the groups $H^i(\Omega_X^j)$ birational invariants among smooth projective varieties?

Theorem 25.8 (Hodge)**.** For X a smooth projective variety over $\mathbb{C}$,

$$H^j_{\mathrm{Sing}}(X, \mathbb{C}) \cong \bigoplus_{i=0}^{j} H^i(X, \Omega_X^j)$$

where the right side is sheaf cohomology, computed on Zariski or classical topology. We define $h^{i,j} = \dim_{\mathbb{C}}(H^i(X, \Omega_X^j))$ to be the Hodge number.

Remark 25.9 (Hodge Number of Curves)**.** Let C be a smooth projective curve of genus g, then $H^0(C, \mathcal{O}_C) = 1$, $H^1(C, \omega_C) = 1$, $H^0(C, \omega_C) = g$, and $H^1(C, \mathcal{O}_C) = g$. Therefore, $h^{1,1} = H^2$, $h^{1,0}, h^{0,1} = H^1$, and $h^{0,0} = H^0$:

$$\begin{array}{ccccc} & h^{1,1} = 1 & & & = H^2 \\ h^{1,0} & & h^{0,1} & & = H^1 \\ & h^{0,0} & & & = H^0 \end{array}$$

For smooth projective surfaces, we have

$$\begin{array}{cccccc} & & h^{2,2} = 1 & & & = H^4 \\ & h^{1,2} & & h^{2,1} & & = H^3 \\ h^{0,2} & & h^{1,1} & & h^{2,0} & = H^2 \\ & h^{0,1} & & h^{2,1} & & = H^1 \\ & & h^{0,0} & & & = H^0 \end{array}$$

Theorem 25.10 (Serre)**.** $H^i(X, \Omega^j) \cong H^{2-i}(X, \Omega^{2-j})^*$.

Theorem 25.11. $\dim_{\mathbb{C}}(H^p(X, \Omega_X^q)) = \dim_{\mathbb{C}} H^q(X, \Omega_X^p)$.

Therefore, the top left row and the bottom right row are birational invariant. $h^{2,1}$ and $h^{0,1}$ are also birational invariant, respectively, but $h^{1,1}$ is not birational invariant.

Proposition 25.12. $\mathrm{Bl}_x(X) \to X$ where X is a smooth projective surface, then $H^2_{\mathrm{Sing}}(X, \mathbb{C})$ goes up in the dimension exactly by 1.

Remark 25.13. For $\mathbb{P}^2$, we have

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

For $E \times F$, we have

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 2 & & 4 & & 2 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

For $\mathrm{Bl}_p(\mathbb{P}^2)$, we have

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

For $\mathrm{Bl}_{p_1,\ldots,p_k}(\mathbb{P}^2)$, we have

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & k+1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Remark 25.14. If $\mathcal{E}$ is a covariant functor from vector space to vector space, then $H^0(X, \mathcal{E}(\Omega_X^1))$ is a birational invariant for smooth projective varieties.

Example 25.15. $H^0(X, \bigwedge^j(\Omega_X^1))$, $H^0(X, S^j(\Omega_X^1))$, and $H^0(X, K_X^{\otimes a})$. For instance, for $f \in H^0(X, K_X^{\otimes a})$ and $g \in H^0(X, K_X^{\otimes b})$, then $fg \in H^0(X, K_X^{\otimes(a+b)})$.

Definition 25.16. The canonical ring of a smooth projective variety X is

$$R(X) = R(X, K_X) = \bigoplus_{a \geq 0} H^0(X, K_X^{\otimes a}).$$

Theorem 25.17. The canonical ring is a birational invariant for smooth projective varieties.

Theorem 25.18. Let X and Y be smooth projective varieties, assume X is birationally equivalent to Y and both K_X and K_Y are ample, then $X \cong Y$.

Proof. $X \cong \mathrm{Proj}(R(X, K_X))$ from the fact that K_X is ample, then since X and Y are birational, then this is isomorphic to $\mathrm{Proj}(R(Y, K_Y))$, and since K_Y is ample, this is isomorphic to Y. $\square$

26 Lecture 26

Recall:

Proposition 26.1. The canonical ring is a birational invariant for smooth projective varieties. For $a \geq 0$, $f : X \dashrightarrow Y$ birational map of smooth projective k-varieties, then $f^* : H^0(Y, aK_Y) \to H^0(X, aK_X)$ is an isomorphism of k-vector spaces, and $f^* : \bigoplus_{a \geq 0} H^0(Y, aK_Y) \to \bigoplus_{a \geq 0} H^0(X, aK_X)$ is an isomorphism of $\mathbb{Z}_{\geq 0}$-graded rings.

We defined multiplication in $R(X)$ inside $K(X) \cong K(Y)$.

Proposition 26.2. Let X and Y be two smooth projective varieties. If K_X and K_Y are ample and $X \cong Y$ as birational, then $X \cong Y$ as abstract k-varieties.

Example 26.3. Let $H_d \subseteq \mathbb{P}^n$ be a smooth projective hypersurface, if $d < n+1$, then $K_{H_d}^\vee$ is ample; if $d = n+1$, then $K_{H_d} \sim 0$; if $d > n+1$, then K_{H_d} is ample.

Definition 26.4. Let X be a complex manifold. A Hermitian metric on X is a Riemannian metric (X is viewed as a real manifold) such that $i : T_pX \to T_pX$ is a isometry with respect to this metric. The metric is Kahler if parallel translations associated to paths $C : [0,1] \to X$ gives isomorphism $T_{C(0)}X \cong T_{C(1)}X$ that are $\mathbb{C}$-linear.

Proposition 26.5. Submanifolds of Kahler manifolds are Kahler.

Theorem 26.6. $\mathbb{P}^n_{\mathbb{C}}$ with the FS metric is Kahler.

Corollary 26.7. Any smooth projective variety is Kahler.

Theorem 26.8. A smooth complex manifold that is a deformation of a smooth projective variety is Kahler.

Remark 26.9. Conjecture: Any complex Kahler manifold is a deformation of a smooth projective variety. False in dimension 4 due to Voisin.

Definition 26.10. The Rucci curvature of a Kahler metric is a real closed 2-form on X that represents $c_1(T_X) = -c_1(K_X) \in H^2(X, \mathbb{R})$.

Theorem 26.11 (Yau)**.** Let X be a smooth projective variety.

- X admits a Kahler metric with Ricci curvature > 0 if and only if $K_X^\vee$ is ample (Fano variety)

- X admits a Kahler metric with Ricci curvature $= 0$ if and only if $K_X \equiv 0$ (Calabi-Yau)

- X admits a Kahler metric with Ricci curvature < 0 if and only if K_X is ample (canonically polarized)

For Fano varieties in $\mathbb{P}^n$, we have $K_{\mathbb{P}^n} \cong \mathcal{O}(-n-1)$ anti-ample; $\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_k}$ would be Fano. Over dimension 1, C is Fano if and only if $C \cong \mathbb{P}^1_k$ over algebraically closed field k.

$H_d \subseteq \mathbb{P}^3$ if $d \leq 3$ then H_d is Fano (supposing H_d is smooth). For $d = 1$, then $H_d \cong \mathbb{P}^2$; if $d = 2$, then $H_d \cong \mathbb{P}^1 \times \mathbb{P}^1$; if $d = 3$, then X is rational, indeed, $X \cong \mathrm{Bl}_{p_1,\ldots,p_c}(\mathbb{P}^2)$ and X contains exactly 27 lines.

What can we say about the Picard group of a surface? $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$.

Theorem 26.12 (Gorthendieck-Lefschetz Hyperplane Theorem)**.** Let Y be a smooth projective variety over a field, and $X \subseteq Y$ a smooth ample hypersurface with $\dim(X) \geq 3$. Then the homomorphism $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is an isomorphism.

Corollary 26.13. If $H_d \subseteq \mathbb{P}^n$ with $n \geq 4$ is a smooth hypersurface, then $\mathrm{Pic}(H_d) \cong \mathbb{Z}$. Furthermore, it is generated by $\mathcal{O}(1)|_{H_d}$.

Corollary 26.14. Let X and Y be two smooth hypersurfaces in $\mathbb{P}_k^{n+1}$ and $\deg(X) \geq n+3$, $\deg(Y) \geq n+3$. If $X \sim Y$ in birational, then there is $L \in \mathrm{PGL}(n+2)$ with $L(X) = Y$. In particular, X and Y have the same degree.

Proof. Observe K_X and K_Y are ample. If $X \sim Y$ in birational sense, then $X \cong Y$ as k-varieties. Now let $f : X \to Y$ be such an isomorphism, then $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot \mathcal{O}(1)|_X$ and $\mathrm{Pic}(Y) \cong \mathbb{Z} \cdot \mathcal{O}(1)|_Y$, so $f^*\mathcal{O}(1)|_Y = \mathcal{O}(\pm 1)|_X$ by positivity we get $f^*\mathcal{O}(1)|_X \mathcal{O}(1)|_Y$. Therefore, the isomorphism is defined by linear polynomials. □

Theorem 26.15. Let $X \subseteq \mathbb{P}^3$ be a smooth hypersurface. Then $\mathrm{Pic}(X)$ is torsion-free. Let $X, Y \subseteq \mathbb{P}^3$ be hypersurfaces of degree $d_X \geq 5$ and $d_Y \geq 5$, then if $X \sim Y$ in birational sense, then $X \cong Y$. Let $f : X \to Y$ be such an isomorphism, then $f^*(K_X) = K_Y$, and $f^*(\mathcal{O}(d_Y - 4)) = \mathcal{O}(d_X - 4)$. If $d_X = d_Y$, then $f^*\mathcal{O}(1)|_Y = \mathcal{O}(1)|_X$, so X and Y differ by a linear automorphism of $\mathbb{P}^3$.

Theorem 26.16. Let X be a smooth projective surface, let $Y \to X$ be a blow up at a point $x \in X$, then K_Y is not ample.

Proof. $K_Y \cong f^*K_X \otimes \mathcal{O}_Y(\alpha E)$, for $df : f^*K_X \to K_Y$, the $f^*(dx \vee dy) = y dx \vee dy \in K_Y$, so $\alpha = 1$, now $K_Y E = f^*(K_X)E + E^2 = 0 + (-1)$. □

27 Lecture 27

Proposition 27.1. Let Y be a smooth projective surface. Let $y \in Y$ be a point and $\pi : X \to Y$ be the blow-up of $y \in Y$. Then the following holds:

1. π is an isomorphism on $Y \setminus y$,

2. $\pi^{-1}(y) \cong E$ is a smooth rational curve,

3. $K_X E = E^2 = -1$.

Proof. The first two follow from the definition of blow-up. For (3), take two smooth curves C_1 and C_2 with different tangent directions through y. Assume $C_1 \sim C_2$ up to shrinking around $y \in Y$. Since Y is smooth at y, then $C_1 \sim 0$ on a neighborhood of $y \in Y$. Further, assume $y = C_1 \cap C_2$. We now have $C_1 \sim C_2$, $\pi^* C_1 = C_1' + E$, $\pi^* C_2 = C_2' + E$, and $C_1 \cdot C_2 = 1$. Now $\pi^*(C_1) \sim \pi_1^*(C_2)$, then $\pi^*(C_1) \cdot \pi^*(C_2) = C_1 C_2 = 1$, and $(C_1' + E)(C_2' + E) = 1$, therefore $E^2 = -1$ by rearranging terms.

We now have $(K_X + E).E = \deg_E((K_X + E)|_E) = \deg(K_E) = -2$ by adjunction formula since E is smooth rational, then $K_X.E + E^2 = -2$, so $K_X.E = -1$. □

Definition 27.2. Let X be a smooth projective surface. A (-1)-curve on X is a smooth rational curve $E \subseteq X$ with $E^2 = -1$.

Definition 27.3. A smooth projective surface with no (-1)-curves is said to be minimal.

Theorem 27.4 (Castelnuvo, 1920)**.** Let X be a smooth projective surface with a (-1)-curve E, then there exists a projective birational morphism $\pi : X \to Y$ satisfying the following:

1. Y is a smooth projective surface,

2. π is an isomorphism on $X \setminus E$,

3. $\pi(E) = y \in Y$,

4. using the isomorphism

$$\begin{tikzcd} \mathbb{P}^1 \subseteq \mathrm{Bl}_y(Y) \arrow[rr, "\sim"] \arrow[dr] & & X \supseteq E \arrow[dl] \\ & Y & \end{tikzcd}$$

   X is isomorphic to $\mathrm{Bl}_y(Y)$,

5. $E^2 = -1$ and $K_X \cdot E = -1$,

6. $h^{1,1}(Y) = h^{1,1}(X) - 1$.

Remark 27.5. Any (-1)-curve can be blown-down.

Proof. We want a basepoint-free line bundle $\mathcal{L}$ for which $\mathcal{L}.E = 0$. X admits a very ample line bundle H, then $H.E = k > 0$, and define $\mathcal{L} = \mathcal{O}_X(kE + H)$. For $s_0, \ldots, s_N \in H^0(X, \mathcal{O}_X(H))$, and a parameter t of E, we have

$$0 \longrightarrow \mathcal{O}_X(-E) \xrightarrow{\cdot t} \mathcal{O}_X 7 \mathcal{O}_E \longrightarrow 0$$

taking $\mathcal{O}_X(H+iE)$, we obtain

$$0 \longrightarrow \mathcal{O}_X(H+(i-1)E) \xrightarrow{\cdot t} \mathcal{O}_X(H+iE) \longrightarrow \mathcal{O}_E(k-i) \longrightarrow 0$$

and note that $\mathcal{O}_X(H)|_E \cong \mathcal{O}_k$ and $\mathcal{O}(E)|_E \cong \mathcal{O}(-1)$. Therefore,

$$0 \rightarrow H^0(X, H+(i-1)E) \rightarrow H^0(X, H+iE) \rightarrow H^0(E, k-i) \rightarrow H^1(X, H+(i-1)E) \rightarrow H^1(X, H+iE) \rightarrow 0$$

The sections of $H^0(H+E)$ are $t_{s_0}, \dots, t_{s_N}$ and $a_{1,0}, \dots, a_{k-1,0}$ which comes from $H^0(E, k-1)$ from the exact sequence for $i = 1$. Moreover, the sections of $H^0(H+2E)$ can be written as $t^2 s_0, \dots, t^2 s_N, ta_{1,0}, \dots, ta_{k-1,0}, a_{2,0}, \dots, a_{2,k-2}$. Let $\mathcal{L} = \mathcal{O}_X(H+kE)$. Therefore, the sections of $H^0(H+kE)$ are $t^k s_0, \dots, t^k s_N, t^{k-1} a_{1,0}, \dots, t^{k-1} a_{k-1,0}, \dots, a_{k,0}$. By definition, given $s_0, \dots, s_N$, $\mathcal{L}$ is basepoint-free on $X \setminus E$ and it also separates tangent directions on $X \setminus E$. $a_{k,0}$ does not vanish on E, so $\mathcal{L}$ is basepoint-free and it separates tangents on $X \setminus E$. This induces

$$\begin{aligned} \varphi_{\mathcal{L}} : X &\to \mathbb{P}^M \\ E &\mapsto [0 : \cdots : 0 : 1] \end{aligned}$$

which is an isomorphism onto its image on $X \setminus E$. The point $[0 : \cdots : 0 : 1]$ is smooth in $\varphi_{\mathcal{L}}(X)$. Let U be an open neighborhood of E in X defined by $a_{k,0} \neq 0$. Define $x, y \in \mathcal{O}_X(-E)$ by $x = \frac{a_{k-1,0}}{a_{k,0}}$ and $y = \frac{a_{k-1,1}}{a_{k,0}}$, and we know $a_{k-1,0}$ and $a_{k-1,1}$ form a basis of $H^0(E, \mathcal{O}_E(1))$. We may assume that x and y do not vanish simultaneously on U by shrinking again. Define

$$\begin{aligned} h_1 : U &\to \mathbb{A}^2 \\ u &\mapsto (tx(u), ty(u)) \end{aligned}$$

and

$$\begin{aligned} h_2 : U &\to \mathbb{P}^1 \\ u &\mapsto [x(u) : y(u)] \end{aligned}$$

Therefore we have

$$\begin{aligned} (h_1, h_2) : U &\to \mathbb{A}^2_{a,b} \times \mathbb{P}^1_{u,v} \\ u &\mapsto ((tx(u), ty(u)), [x(u) : y(u)]) \end{aligned}$$

where $av - bw = 0$ is the equation defining $\mathrm{Bl}_0(\mathbb{A}^2) \subseteq \mathbb{A}^2 \times \mathbb{P}^1$. This is a map $h : U \to \mathrm{Bl}_0(\mathbb{A}^2)$. Therefore, h induces an isomorphism from E to $\mathbb{P}^1$, and h is etale at any point of U, i.e., locally analytic isomorphism. Take $q \in E$, and assume $h(q) = ((0,0), [0,1]) =: ((a,b), [w : v])$. Now $h(q)$ has local coordinates b and $\frac{w}{v}$. Now $h^*(b) = ty$ and $h^*(\frac{w}{v}) = \frac{x}{y}$. Therefore the local coordinates at $h(p)$ pullback via h^* to loca lcoordinates at q, therefore h is etale at q.

Lemma 27.6. Let $f : X \to Y$ be a continuous map between Hausdorff spaces $K \subseteq X$ compact. Assume

1. $f|_K$ is a homeomorphism, and

2. for every $x \in K$, af is a local homeomorphism around x.

Then there is an open $U \supseteq K$ such that $f|_U$ is a homeomorphism into its image.

We now have a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\hat{h}"] \arrow[d, "\varphi_{\mathcal{L}}"'] & \mathrm{Bl}_0(\mathbb{A}^3) \arrow[d, "\pi"] \\ \varphi_{\mathcal{L}}(U) \arrow[r, "\bar{h}"'] & \mathbb{A}^2 \end{tikzcd}$$

sending x, y to $x_{\mathcal{L}}, y_{\mathcal{L}}$. Now $\varphi_{\mathcal{L}}^*(x_{\mathcal{L}}) = tx$ and $\varphi_{\mathcal{L}}^*(y_{\mathcal{L}}) = ty$. Now $\hat{h}$ is an isomorphism on E and a local isomorphism at each point of E, then with the lemma above, $\hat{h}$ is an isomorphism, so $\bar{h}$ is an isomorphism.

Lemma 27.7 (Univesal Property of Blow-up)**.** If $f : X \to S$ is a birational map between surfaces, and f^{-1} is undefined at $p \in S$, then we have

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "f"'] & \mathrm{Bl}_p(S) \arrow[r, "\varepsilon"] & S \end{tikzcd}$$

and

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "\varphi_{\mathcal{L}}"'] & & \mathrm{Bl}_y(Y) \arrow[dl] \\ & Y \ni y & \end{tikzcd}$$

We know (6) is true by $h^{1,1}(X) = h^{1,1}(Y) + 1$. Now we have $h^{1,1}(X) = h^{1,1}(\mathrm{Bl}_y(Y)) = h^{1,1}(Y) + 1$. If g is not an isomorphism, then by lemma again, we blow up at y' on Y, then we have a diamond diagram with $X \to \mathrm{Bl}_{Y'}(\mathrm{Bl}_y(Y))$.

Exercise 27.8. If $X \to Y$ is a projective birational morphism of smooth projective surfaces, then $h^{1,1}(X) \geq h^{1,1}(Y)$.

This gives a contradiction and so g is an isomorphism. $\square$

28 Lecture 28

Definition 28.1. A minimal surface is a smooth projective surface with no (-1)-curves. Therefore, X is minimal if and only if there is no blow-down $X \to Y$ with Y smooth.

Example 28.2. $k(\mathbb{P}^n) = -\infty$, $k(E) = 0$, $k(E) \sim 0$, $h^0(mK_E) = h^0(\mathcal{O}_E) = 1$, $k(C_g) = 1$ if $g \geq 2$.

Definition 28.3. Let X be a normal projective variety. The Kodaira dimension of X, denoted by $k(X)$, is $\max\{k \in \mathbb{N} \lim\limits_{m\to\infty} h^0(X, mK_x)/m^k > 0\}$. If $h^0(X, mK_X) = 0$ for $m \geq 0$, we set $k(X) = -\infty$.

Remark 28.4. $k(X) \in \{-\infty, 0, 1, \ldots, \dim(X)\}$, if $k(X) = \dim(X)$, then we say that X is of general type.

Example 28.5. $k(\mathbb{P}^1 \times C_g) = -\infty$, $k(\mathbb{P}^1 \times X) = -\infty$, $k(C_g \times E) = 1$ for $g \geq 2$, and $K_{\mathbb{P}^1 \times C_g \sim F} \geq 0$.

Definition 28.6. A surface X is said to be rational if it admits a birational map $\mathbb{P}^2 \dashrightarrow X$. A surface X is said to be ruled if it admits a birational map $\mathbb{P}^1 \times C \dashrightarrow X$ for any curve C.

Remark 28.7. There is a birational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ with restriction on to $\varphi : \mathbb{C}^2_m \to \mathbb{C}^2_m$, so rational implies ruled.

Definition 28.8. The ith Betti number of X is $b_i = \dim_{\mathbb{C}} H^i(X, \mathbb{C}) = \sum\limits_{i+k=j} h^{i,k}$.

The Euler characteristic of X is $e(X) = b_0 - b_1 + b_2 - b_3 + b_4 - \cdots$

The second Chern number of X is $c_2(X) = e(X)$.

Irregularity of X is $q(X) = h^{1,0}(X)$.

$p_g(X) = h^{(0,2)}(X)$ and $p_u(X) = p_g(X) - q(X)$.

$c_1(X)^2 = K_X^2$.

Suppose we have a minimal surface with $k(X) = -\infty$, then it is either

- rational, then

  - $\mathbb{P}^2$. Therefore, the Hodge number would be

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

  - $\Sigma_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(n))$ with $n \geq 0$ or $n \geq 2$. Therefore, the Hodge number would be

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

- or ruled, then

  - $\varphi : X \to C$ as a smooth morphism be a curve all whose fibers are isomorphic to $\mathbb{P}^1$. Therefore, the Hodge number would be

$$\begin{array}{ccccc} & & 1 & & \\ & g & & g & \\ 0 & & 2 & & 0 \\ & g & & g & \\ & & 1 & & \end{array}$$

For $\Sigma_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}_n)$, where $\mathcal{E}_n = \mathcal{O} \oplus \mathcal{O}(n)$, and $s_0^2 = -n$ and $s_\infty^2 = n$, and $s_0 \cong s_\infty = \mathbb{P}^1$.

Suppose we have a minimal surface with $k(X) = 0$.

Definition 28.9. A $K3$ surface is a smooth projective surfaces with $K_X \sim 0$ and $\pi_1(X) = \{1\}$.

Remark 28.10. All $K3$ surfaces are diffeomorphic to each other, but not necessarily isomorphic.

A $K3$ surface is a spin simply connected 4-manifold.

$H^2(X, \mathbb{Z}) = II_{3,10}$ is the unique even unimodular lattice with dim $= 22$ and sign $= -16$.

$K3$ surfaces (compact Kahler) form a 20-dimensional modular. The algebraic $K3$ surface in compact $K3$ is a divisor with countably many components.

The Hodge diamond of $K3$ surface would be

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 20 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Example 28.11. Any smooth surface in $\mathbb{P}^3$ of degree 4.

Abelian surfaces would be $\mathbb{C}^2/\Lambda$ that is algebraic. All of them are diffeomorphic to $(S^1)^4$. For instance, take $E \times F$ where both E and F are elliptic.

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 2 & & \end{array}$$

Enriques surfaces are ones with $q(X) = 0$, $K_X \not\sim 0$, and $2K_X \sim 0$. These are quotients of $K3$'s by involutions.

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 10 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

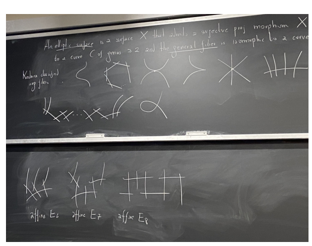
The bi-elliptic surfaces are quotients of Abelian surfaces by abelian groups of order at most 8.

This gives a complete classification of minimal surfaces with codimension 0.

The minimal surfaces are not unique. If we consider $\mathbb{P}^1 \times \mathbb{P}^1$ as blow-up of two copies of $\mathbb{P}^1$ with fiber F_1 and G_1, respectively, with $F_1^2 = G_1^2 = 0$. Therefore, taking the blow-up again, we have F_1', E, G_1' with E intersecting the other two curves. Therefore $\pi^*(F_1) = F_1' + E$ and $0 = F_1^2 = (\pi_1^*(F_1))^2 = (F_1')^2 + 2F_1' \cdot E + E^2$, but $2F_1' \cdot E = 2$ and $E^2 = -1$, so $(F_1')^2 = -1$. By contraction on both sides of E, this is just equivalent to E, isomorphic to $\mathbb{P}^2$ by the classification above.

Definition 28.12. An elliptic surface is a surface X that admits a surjective projective morphism $\varphi : X \to C$ to a curve C of genus ≥ 2 and the general fiber is isomorphic to a curve of $g = 1$.

Kodarra classifies singular fibers to be the following ones:



Let X be a smooth minimal surface of general type, then

- $c_1^2, c_2 > 0$,

- $c_1^3 \leq 3c_2$ (Bogomolov-Miyaoka-Yau inequality). The equality here holds if and only if $X = B/\Gamma \subseteq \mathbb{C}^2$,
- $5c_1^2 - c_2 + 36 \geq 0$,
- $c_1^2 + c - 2 \equiv 0 \pmod{12}$.

Theorem 28.13 (Gieseker)**.** Let $c_1^{`}c_2 > 0$, then there is a coarse (parametrizes all objects up to isomorphism) moduli space $M_{c_1^2,c_2}$ that parametrizes minimal surfaces X of general type with $c_1(X)^2 = c_1^2$, and $c_2(X) = c_2$.

Remark 28.14. Remaining unsolved question: for what c_1^2, c_2 is $M_{c_1^2,c_2} \neq \varnothing$?

Theorem 28.15 (Rolleaux-Urgua)**.** Let $r \in [1, 3]$, there is a sequence X_i of minimal smooth surfaces of general type with

$$\lim_{i\to\infty} \frac{c_1(X_i)^2}{c_2(X_i)} = r$$

As a conclusion, to classify minimal surfaces as a whole,

- if $k(X) = -\infty$, then X is rational or ruled, so it becomes $\mathbb{P}^2$ or Σ_n for $n = 0$ or $n \geq 2$, or $\varphi : X \to C$ with fibers $\mathbb{P}^1$.
- if $k(X) = 0$, then we know $K3$-surfaces map to Enriques, and Abelian surfaces map to Bi-elliptics...
- if $k(X) = 1$, then these are minimal elliptic surfaces $\varphi : X \to C$ and singular fibers are classified by Kodaira.
- if $k(X) = 2$, then

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