132H Final Revision

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1 Chapter 1: Preliminary Knowledge

Definition . We say a function is **holomorphic** at z_0 if the limit at that point $f'(z_0) := \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists.

From directional derivative, we conclude that

$$f' = \frac{\partial f}{\partial x} = \frac{1}{i} \cdot \frac{\partial f}{\partial y}.$$

We say a function f satisfies Cauchy-Riemann condition if the following holds: by writing the function $f(z)$ as $f(x,y) = u(x,y) + iv(x,y)$ where u, v are real-valued functions, we have:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

We can then conclude that 1) a function satisfies Cauchy-Riemann condition if it is holomorphic, and 2) a function is holomorphic if the function satisfies Cauchy-Riemann condition and all partial derivatives exists and are continuous.

Proof. Suppose we know the function is holomorphic, then we have $\frac{\partial f}{\partial x} = \frac{1}{i} \cdot \frac{\partial f}{\partial y}$, so when writing $f = u + iv$, we have

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} \\ &= \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \end{aligned}$$

Then since u, v are real-valued functions, the partial derivatives are real-valued, then we conclude with Cauchy-Riemann equations.

We now show the converse. We can write

$$u(x+h_1,y+h_2)-u(x,y)=\frac{\partial u}{\partial x}h_1+\frac{\partial u}{\partial y}h_2+|h|\psi_1(h)$$
$$v(x+h_1,y+h_2)-v(x,y)=\frac{\partial v}{\partial x}h_1+\frac{\partial v}{\partial y}h_2+|h|\psi_2(h)$$

where $\psi_j(h)\to 0$ whenever $|h|\to 0$, where $h=h_1+ih_2$. By Cauchy-Riemann equations, we have

$$f(z+h)-f(z)=(\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y})(h_1+ih_2)+|h|\psi(h)$$

where $\psi(h)=\psi_1(h)+\psi_2(h)$, so it converges to 0 whenever $h\to 0$ as well. Therefore, f is holomorphic by definition. □

We can then define two "partial derivatives": although they have the form of partial derivatives, they do not make sense as partial derivatives:

$$\frac{\partial}{\partial z}=\frac{1}{2}(\frac{\partial}{\partial x}+\frac{1}{i}\frac{\partial}{\partial y})$$
$$\frac{\partial}{\partial \bar{z}}=\frac{1}{2}(\frac{\partial}{\partial x}-\frac{1}{i}\frac{\partial}{\partial y})$$

We then have

Proposition 2.3. If f is holomorphic at z_0, then $\frac{\partial f}{\partial \bar{z}}(z_0)=0$, and $f'(z_0)=\frac{\partial f}{\partial z}(z_0)=2\frac{\partial u}{\partial z}(z_0)$.

Proof. Note that

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}}&=\frac{1}{2}(\frac{\partial f}{\partial x}-\frac{1}{i}\frac{\partial f}{\partial y})\\&=0\end{aligned}$$

because f is holomorphic and so $f'=\frac{\partial f}{\partial x}=\frac{1}{i}\cdot\frac{\partial f}{\partial y}$.

Now by previous observation we clearly have $f'(z_0)=\frac{\partial f}{\partial z}(z_0)$. Moreover, note that

$$\begin{aligned}2\frac{\partial u}{\partial z}&=\frac{\partial u}{\partial x}+\frac{1}{i}\frac{\partial u}{\partial y}\\&=\frac{\partial u}{\partial x}-\frac{1}{i}\frac{\partial v}{\partial x}\\&=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\\&=\frac{\partial f}{\partial x}.\end{aligned}$$

Then this concludes the proof. □

Although we haven't proven this, a function is smooth (C^1) if and only if it is holomorphic, if and only if it is infinitely-differentiable (C^∞).

For power series $\sum_{n=0}^{\infty} a_n z^n$, there exists some $R \geq 0$ as its radius of convergence (**Theorem** 2.5): for any $|z| < R$, the series converges absolutely; for any $|z| > R$, the series diverges. Moreover, if we are less worried of 0 and ∞, we can obtain $\frac{1}{R} = \limsup_{n\geq 0} |a_n|^{\frac{1}{n}}$. Furthermore, power series is infinitely differentiable in the radius of convergence, with term-wise differentiation. (**Theorem** 2.6)

Proof. We only prove the case for $|z| < R$.

Denote $L = \frac{1}{R}$, and let us assume $L \neq 0, \infty$. Now whenever $|z| < R$, there exists some $\varepsilon > 0$ such that

$$(L+\varepsilon)|z| < 1.$$

Because $\frac{1}{R} = \limsup_{n\geq 0} |a_n|^{\frac{1}{n}}$, then $|a_n|^{\frac{1}{n}} \leq L + \varepsilon$ for large n by definition. Therefore, $|a_n| < (L+\varepsilon)^n$. Now

$$\begin{aligned}
\sum_{n=0}^{\infty} |a_n z^n| &= \sum_{n=0}^{\infty} |a_n||z^n| \\
&\leq \sum_{n=0}^{\infty} (L+\varepsilon)^n |z^n| \\
&= \sum_{n=0}^{\infty} ((L+\varepsilon)|z|)^n
\end{aligned}$$

Note that $(L+\varepsilon)|z| < 1$, so by comparison test the series $\sum_{n=0}^{\infty} |a_n z^n|$ converges. □

Proof. Denote $g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$. We show that this is the derivative of f. We write $f = S_N(z) + E_N(z)$, where $S_N(z) = \sum_{n=0}^{N} a_n z^n$ and $E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n$. Whenever we pick h such that $|z_0 + h| < r < R$, we have

$$\begin{aligned}
\frac{f(z_0+h) - f(z_0)}{h} - g(z_0) &= \left(\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0)\right) \\
&\quad + \left(S_N'(z_0) - g(z_0)\right) \\
&\quad + \left(\frac{E_N(z_0+h) - E_N(z_0)}{h}\right).
\end{aligned}$$

We claim that all three parts converge to 0 for small h. Note that we can bound

$$\begin{aligned}\left(\frac{E_N(z_0+h)-E_N(z_0)}{h}\right) &\leq \sum_{n=N+1}^{\infty}|a_n|\left|\frac{(z_0+h)^n-z_0^n}{h}\right| \\ &= \sum_{n=N+1}^{\infty}|a_n|\left|\frac{h((z_0+h)^{n-1}+(z_0+h)^{n-2}z_0+\cdots+z_0^{n-1})}{h}\right| \\ &= \sum_{n=N+1}^{\infty}|a_n|\left|(z_0+h)^{n-1}+(z_0+h)^{n-2}z_0+\cdots+z_0^{n-1}\right| \\ &\leq \sum_{n=N+1}^{\infty}|a_n|nr^{n-1}\end{aligned}$$

Since $r<R$, then the series on the last line converges absolutely, so for any $\varepsilon>0$ there exists $N_1\in\mathbb{N}$ such that whenever $N\geq N_1$, we have

$$\left|\frac{E_N(z_0+h)-E_N(z_0)}{h}\right|<\varepsilon.$$

Moreover, the term $S_N'(z_0)-g(z_0)$ converges to 0 as $N\to\infty$, so for any $\varepsilon>0$ there exists some $N_2\in\mathbb{N}$ such that whenever $N\geq N_2$, we have

$$\left|S_N'(z_0)-g(z_0)\right|<\varepsilon.$$

Finally, fix $N_3=\max(N_1,N_2)$, then for any $\varepsilon>0$, there exists some $\delta>0$ such that whenever $|h|<\delta$, we have

$$\left|\frac{S_N(z_0+h)-S_N(z_0)}{h}-S_N'(z_0)\right|<\varepsilon$$

because the derivative of a polynomial is obtained by term-wise differentiation. For $N\geq N_3$, we have

$$\left|\frac{f(z_0+h)-f(z_0)}{h}-g(z_0)\right|<3\varepsilon.$$

Therefore, we have convergence. $\square$

We say a function is **analytic** at z_0 if there exists a power series expansion centered at z_0.

A parametrized curve is a function $z(t)$ which maps a closed interval $[a,b]\subseteq\mathbb{R}$ to $\mathbb{C}$. The parametrized curve is smooth if $z'(t)$ exists and is continuous on $[a,b]$, and $z'(t)\neq 0$ for $t\in[a,b]$. Two parametrizations $z:[a,b]\to\mathbb{C}$ and $\tilde{z}:[c,d]\to\mathbb{C}$ are equivalent if there exists a continuously differentiable bijection $s\mapsto t(s)$ from $[a,b]$ to $[c,d]$ so that $t'(s)>0$ and $\tilde{z}(s)=z(t(s))$. A parametrization corresponds to a curve γ with direction.

If we have a curve γ with parametrization $z:[a,b]\to\mathbb{C}$, there is a parametrization in the opposite direction, namely γ^- with $z^-:[a,b]\to\mathbb{C}$ such that $z^-(t)=z(a+b-t)$.

In particular, for a circle, we say the curve has positive orientation if it is counterclockwise, with $z(t) = z_0 + e^{it}$ where $t \in [0, 2\pi]$, and has negative orientation if it is clockwise, with $z(t) = z_0 + e^{-it}$ where $t \in [0, 2\pi]$.

The integral of a function f along the curve γ with parametrization $z : [a, b] \to \mathbb{C}$ is defined by

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dz.$$

The length of a (smooth) curve γ is length$(\gamma) = \int_a^b |z'(t)|dt$. (**Theorem** 3.1) Note that such integration satisfies:

- for $a, b \in \mathbb{C}$, $\int_\gamma (af(z) + bg(z))dz = a \int_\gamma f(z)dz + b \int_\gamma f(z)dz$.
- $\int_{\gamma^-} f(z)dz = -\int_\gamma f(z)dz$.
- $|\int_\gamma f(z)dz| \leq \sup\limits_{z\in\gamma} |f(z)| \cdot \text{length}(\gamma)$.

Proof.
- Find a parametrization and uses the linearity of the integral.
- We write $\int_{\gamma^-} f(z)dz = \int_a^b f(z(a+b-t))z'(a+b-t)dt = -\int_b^a f(z(w))(-z'(w))dw = \int_a^b f(z(w))(-z'(w))dw$.
- We have $|\int_\gamma f(z)dz| \leq \sup\limits_{t\in[a,b]} |f(z(t))| \cdot \int_a^b |z'(t)|dt = \sup\limits_{z\in\gamma} |f(z)| \cdot \text{length}(\gamma)$.

□

A **primitive** of a function f on Ω is a holomorphic function F on Ω such that $F'(z) = f(z)$ for all $z \in \Omega$.

Theorem 3.2. If a continuous function f has a primitive F on Ω, and γ is a curve in Ω that begins at ω_1 and ends at ω_2, then $\int_\gamma f(z)dz = F(\omega_2) - F(\omega_1)$.

Proof. By parametrization of γ such that $z(a) = \omega_1$ and $z(b) = \omega_2$, we have

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_a^b f(z(t))z'(t)dt \\
&= \int_a^b F'(z(t))z'(t)dt \\
&= \int_a^b \frac{d}{dt}F(z(t))dt \\
&= F(z(b)) - F(z(a)) \\
&= F(\omega_2) - F(\omega_1).
\end{aligned}$$

□

Corollary 3.3. If γ is a closed curve on an open set Ω, and f is a continuous function with a primitive in Ω, then

$$\int_\gamma f(z)dz = 0.$$

Proof. This is immediate by taking the same endpoints. □

Corollary 3.4. If f is a holomorphic function on a region Ω with $f'(z) \equiv 0$, then f is a constant function.

Proof. Consider an arbitrary point ω. We claim that for any $z \in \Omega$, we have $f(z) = f(\omega)$. Indeed, the region Ω is connected and so there exists a curve γ from ω to z. Observe that f is holomorphic, so f is a primitive of f', and so we have

$$\int_\gamma f'(z)dz = f(z) - f(\omega).$$

However, $f'(z) \equiv 0$, so the integral on the left is 0, and so $f(z) = f(\omega)$, which is true for all $z \in \Omega$. □

2 Chapter 2: Cauchy's Theorem and Its Applications

Theorem 1.1 (Goursat). Suppose f is a holomorphic function on an open set Ω, and $T \subseteq \Omega$ is a triangle with its interior contained in Ω as well. Then

$$\int_T f(z)dz = 0.$$

Proof. By geometric trick, we divide the triangle $T = T^{(0)}$ into four triangles of the same shape by connecting the midpoints of three sides. Therefore, the integral over the original triangle is the sum of integrals of the four triangles. In particular, one of the four triangles, namely $T^{(1)}$, must satisfy

$$\left|\int_{T^{(0)}} f(z)dz\right| \leq 4\left|\int_{T^{(1)}} f(z)dz\right|,$$

We repeat the process and obtain a sequence of triangles $T^{(0)}, T^{(1)}, \cdots, T^{(n)}, \cdots$. By induction, we have

$$\left|\int_{T^{(0)}} f(z)dz\right| \leq 4^n\left|\int_{T^{(n)}} f(z)dz\right|,$$

for any $n \in \mathbb{N}$. In particular, since the triangles are nested, then there exists a point z_0 that is contained in all the triangles. Therefore, f is holomorphic at z_0, so we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$$

where $\psi(z) \to 0$ as $z \to z_0$. By corollary, since the first two terms have primitives, the integration over them is 0. Hence, we have

$$\int_{T^{(n)}} f(z)dz = \int_{T^{(n)}} \psi(z)(z - z_0)dz.$$

Therefore, we can bound the integral by using the diameter $d^{(n)}$ and the perimeter $p^{(n)}$ of the triangle $T^{(n)}$:

$$\left|\int_{T^{(n)}} f(z)dz\right| \leq \sup_{z \in T^{(n)}} |\psi(z)| d^{(n)} p^{(n)}$$

because $|z - z_0| \leq d^{(n)}$. Therefore,

$$\left|\int_{T^{(n)}} f(z)dz\right| \leq \sup_{z \in T^{(n)}} |\psi(z)| 4^{-n} d^{(0)} p^{(0)}$$

Therefore,

$$\left|\int_{T^{(0)}} f(z)dz\right| \leq 4^n\left|\int_{T^{(n)}} f(z)dz\right| \leq \sup_{z \in T^{(n)}} |\psi(z)| d^{(0)} p^{(0)}$$

and note that the term $\sup_{z \in T^{(n)}} |\psi(z)| \to 0$ whenever $n \to \infty$, then taking $n \to \infty$ now yields

$$\left|\int_{T^{(0)}} f(z)dz\right| = 0.$$

$\square$

Corollary 1.2. Suppose f is a holomorphic function on an open set Ω, and $R \subseteq \Omega$ is a rectangle with its interior contained in Ω as well. Then

$$\int_R f(z)dz = 0.$$

Proof. This is immediate because we have

$$\int_R f(z)dz = \int_{T_1} f(z)dz + \int_{T_2} f(z)dz.$$

□

Theorem 2.1. Suppose f is holomorphic on an open disk, then f has a primitive on the disk.

Proof. By performing multiple cancellations, we have

$$F(z+h) - F(z) = \int_\eta f(w)dw$$

where η is the straight line from z to $z+h$. We can then write $f(z+h) = f(z) + \psi(w)$ for some $\psi(w) \to 0$ as $w \to z$. Then

$$F(z+h) - F(z) = \int_\eta f(z)dz + \int_\eta \psi(w)dw = f(z)\int_\eta 1dw + \int_\eta \psi(w)dw.$$

Then we have an estimation

$$\left| \int_\eta \psi(w)dw \right| \leq \sup_{w \in \eta} |\psi(w)||h|.$$

Since the supremum above goes to 0 as h tends to 0, we conclude that

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z),$$

thereby proving that F is a primitive for f on the disc. □

Theorem 2.2 (Cauchy). If f is holomorphic on a disk, then for any closed curve γ on the disk we have

$$\int_\gamma f(z)dz = 0.$$

Proof. Since f has a primitive, then by corollary 3.3, we conclude the proof. □

Corollary 2.3. Suppose f is a holomorphic function on an open set and contains a circle C and its interior, then

$$\int_C f(z)dz = 0.$$

Proof. Let D be the disk with boundary C, then there exists a slightly larger circle D' containing D such that f is holomorphic on D. We conclude by applying Cauchy's theorem on D'. $\square$

Theorem 4.1 (Cauchy). Suppose f is a holomorphic function on an open set that contains the closure of a closed disk D. Let γ is be the boundary circle of the disk (i.e. ∂D) with positive orientation, then for any point $z \in D$ we have

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} d\xi.$$

Proof. We consider the keyhole structure with corridor width δ and circle radius r, with the function $F(\xi) = \frac{f(\xi)}{\xi - z}$, which is holomorphic away from the point z. By Cauchy's theorem, this structure has integral 0. We now take $\delta \to 0$, then the two sides of the corridor cancels out, so it suffices to check the large circle with positive orientation and the small circle with negative orientation together, which have a sum of 0. Note that for

$$F(\xi) = \frac{f(\xi) - f(z)}{\xi - z} + \frac{f(z)}{\xi - z}$$

the first term is bounded because f is holomorphic, so the integral over the small circle converges to 0 as $\varepsilon \to 0$. The second term, when taking the integral over the small circle, is $-f(z)2\pi i$ by parametrization on circle with negative orientation. Therefore,

$$0 = \int_C \frac{f(\xi)}{\xi - z} d\xi - 2\pi i f(z),$$

where C is the large circle. $\square$

Corollary 4.2. Suppose f is a holomorphic function on an open set Ω, then f is infinitely-differentiable on Ω. Moreover, if $C \subseteq \Omega$ is a circle whose interior is also contained in Ω, then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^{n-1}} d\xi.$$

Proof. The proof is by induction on n, the case $n = 0$ being simply the Cauchy integral formula. It suffices to show the inductive case. Suppose that f has up to $n - 1$ complex derivatives and that

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^n} d\zeta.$$

Now for h small, the difference quotient for $f^{(n-1)}$ takes the form

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} = \frac{(n-1)!}{2\pi i} \int_C f(\zeta) \frac{1}{h} \left[\frac{1}{(\zeta - z - h)^n} - \frac{1}{(\zeta - z)^n} \right] d\zeta.$$

We now recall that

$$A^n - B^n = (A - B)[A^{n-1} + A^{n-2}B + \cdots + AB^{n-2} + B^{n-1}].$$

With $A = \frac{1}{\zeta - z - h}$ and $B = \frac{1}{\zeta - z}$, then the term above in the bracket is just

$$\frac{h}{(\zeta - z - h)(\zeta - z)}[A^{n-1} + A^{n-2}B + \cdots + AB^{n-2} + B^{n-1}].$$

But observe that if h is small, then $z + h$ and z stay at a finite distance from the boundary circle C, so in the limit as h tends to 0, we find that the quotient converges to

$$\frac{(n-1)!}{2\pi i}\int_C f(\zeta)\left[\frac{1}{(\zeta - z)^2}\right]\left[\frac{n}{(\zeta - n)^{n-1}}\right]d\zeta = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n-1}}d\zeta.$$

□

Corollary 4.3 (Cauchy Inequality). If f is a holomorphic function in an open set that contains the closure of a disk D centered at z_0 and of radius R, then

$$|f^{(n)}(z_0)| \leq \frac{n! \sup_{z\in\partial D} |f(z)|}{R^n}.$$

Proof. Apply Cauchy integral formula on $f^{(n)}(z_0)$. We have a bound

$$\begin{aligned}
|f^{(n)}(z_0)| &= \left|\frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta\right| \\
&= \frac{n!}{2\pi}\left|\int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})}{(Re^{i\theta})^{n+1}}Rie^{i\theta}\right| \\
&\leq \frac{n!}{2\pi}\frac{||f||_C}{R^n}2\pi.
\end{aligned}$$

□

Theorem 4.4. For any holomorphic function f on open set Ω, if D is a disk that is centered at z_0 and whose closure is contained in Ω, then f has a power series expansion centered at z_0:

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

for all $z \in D$, and the coefficients are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

for all $n \geq 0$.

Proof. Fix $z \in D$. By the Cauchy integral formula, we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta,$$

where C denotes the boundary of the disc and $z \in D$. The idea is to write

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}},$$

and use the geometric series expansion. Since $\zeta \in C$ and $z \in D$ is fixed, there exists $0 < r < 1$ such that

$$\left| \frac{z - z_0}{\zeta - z_0} \right| < r,$$

therefore

$$\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n,$$

where the series converges uniformly for $\zeta \in C$. Then combining the results above, we have

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right) \cdot (z - z_0)^n.$$

This proves the power series expansion; further the use of the Cauchy integral formulas for the derivatives (or simply differentiation of the series) proves the formula for a_n. □

Corollary 4.5 (Liouville). Suppose f is an entire function (i.e. holomorphic on $\mathbb{C}$) and is bounded, then f is constant.

Proof. It suffices to show that $f' \equiv 0$, then by corollary 3.4, since $\mathbb{C}$ is connected, we know f is constant.

For arbitrary $z_0 \in \mathbb{C}$ and all $R > 0$, the Cauchy inequality yields

$$|f'(z_0)| \leq \frac{B}{R}$$

where B is a bound for f. By taking $R \to \infty$, we conclude the proof. □

Corollary 4.6. Every non-constant polynomial $P(z) = a_n z^n + \cdots + a_0$ with complex coefficients has a root in $\mathbb{C}$.

Proof. If P has no roots, then $\frac{1}{P(z)}$ is a bounded holomorphic function. To see this, we can of course assume that $a_n \neq 0$, and write

$$\frac{P(z)}{z^n} = a_n + \left(\frac{a_{n-1}}{z} + \cdots \frac{a_0}{z^n} \right)$$

whenever $z \neq 0$. Since each term in the parentheses goes to 0 as $|z| \to \infty$ we conclude that there exists $R > 0$ so that if $c = \frac{|a_n|}{2}$, then $|P(z)| \geq c|z|^n$ whenever $|z| > R$.

In particular, P is bounded from below when $|z| > R$. Since P is continuous and has no roots in the disc $|z| \leq R$, it is bounded from below in that disc as well, thereby proving our claim.

By Liouville's theorem we then conclude that $\frac{1}{P}$ is constant. This contradicts our assumption that P is non-constant and proves the corollary. □

Theorem 4.8. Suppose f is a holomorphic function in a region Ω that vanishes on a sequence of distinct points with a limit point in Ω. Then f is identically 0.

Corollary (Analytic Continuation). Suppose f and g are holomorphic in a region Ω and $f(z) = g(z)$ for all z in some non-empty open subset of Ω (or more generally for z in some sequence of distinct points with limit point in Ω). Then $f(z) = g(z)$ throughout Ω.

Suppose we are given a pair of functions f and F analytic in regions Ω and Ω', respectively, with $\Omega \subseteq \Omega'$. If the two functions agree on the smaller set Ω, we say that F is an analytic continuation of f into the region Ω'. The corollary then guarantees that there can be only one such analytic continuation, since F is uniquely determined by f.

Theorem 5.1 (Morera). Suppose f is a continuous function on an open disk D such that for any triangle T contained D, we have

$$\int_T f(z)dz = 0,$$

then f is holomorphic.

Proof. We can construct a primitive F of f over the open disk D. (This is similar to that of Theorem 2.1) Then F is holomorphic, so F is infinitely-differentiable, and so f is holomorphic. □

Theorem 5.2. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω, then f is holomorphic in Ω.

Proof. Let D be any disk whose closure is contained in Ω and T is any triangle in the disk. Therefore, since each f_n is holomorphic, by Goursat's theorem,

$$\int_T f_n(z)dz = 0$$

for all n. By assumption, $f_n \to f$ uniformly in the closure of D, so f is continuous and

$$\int_T f_n(z)dz \to \int_T f(z)dz.$$

As a result, we have $\int_T f(z)dz = 0$, and by Morera's theorem, weconclude that f is holomorphic in D. Since this is true for all D disk whose closure is contained in Ω, then this is true on Ω. □

Theorem 5.3. Under the hypotheses of the previous theorem, the sequence of derivatives $\{f_n'\}_{n=1}^{\infty}$ converges uniformly to f' on every compact subset of Ω.

Proof. This proof is from Homework 4, Problem 7.

We prove the following special case:

Claim . Suppose f_n and f are holomorphic functions on a region Ω, such that f_n converges to f uniformly on all of Ω. Then f_n' converges to f', uniformly on compact subsets of Ω.

Suppose that we proved this statement. In particular, for every compact subset $K \subseteq \Omega$, there exists some $U \supseteq K$ such that $\bar{U} \subseteq \Omega$ is compact. But since $\bar{U}$ is compact, then $\{f_n'\}_{n\geq 1}^{\infty} \xrightarrow[n\to\infty]{u} f'$ holds on $\bar{U}$ as well, and so $\{f_n'\}_{n\geq 1}^{\infty} \xrightarrow[n\to\infty]{u} f'$ holds on K. Therefore, the original statement of the problem is proven.

Subproof. Take arbitrary $\delta > 0$. Now define

$$\Omega_\delta = \{z \in \Omega : \overline{D_\delta}(z) \subseteq \Omega\}.$$

It now suffices to show that $\{f_n'\}_{n\geq 1}^{\infty} \xrightarrow[n\to\infty]{u} f'$ on Ω_δ. Indeed, if this is the case, every compact set can be covered in one of these sets for some $\delta > 0$, and we are done.

Claim . If F is holomorphic on Ω, we have $\sup\limits_{z\in\Omega_\delta} |F'(z)| \leq \frac{1}{\delta} \sup\limits_{\zeta\in\Omega} |F(\zeta)|$.

Subproof. For every $z \in \Omega_\delta$, we have $\overline{D_\delta(z)} \subseteq \Omega$ and

$$F'(z) = \frac{1}{2\pi i} \int_{\partial D_\delta(z)} \frac{F(\zeta)}{(\zeta - z)^2} d\zeta.$$

We then conclude that

$$\begin{aligned} |F'(z)| &\leq \frac{1}{2\pi} \int_{\partial D_\delta(z)} \frac{|F(\zeta)|}{|\zeta - z|^2} |d\zeta| \\ &\leq \frac{1}{2\pi} \sup_{\zeta\in\Omega} \frac{|F(\zeta)|}{|\zeta - z|^2} |d\zeta| \\ &= \frac{1}{\delta} \sup_{\zeta\in\Omega} |F(\zeta)|. \end{aligned}$$

■

Apply $F = f_n - f$, and notice that since $f_n \xrightarrow[n\to\infty]{u} f$ and f is holomorphic on Ω, then $F(z)$ is holomorphic on Ω. In particular, we have $F'(z) = f_n'(z) - f'(z)$. By **claim 2**, we have $\sup\limits_{z\in\Omega_\delta} |f_n'(z) - f'(z)| \leq \frac{1}{\delta} \sup\limits_{\zeta\in\Omega} |f_n(\zeta) - f(\zeta)|$. Furthermore, note that $\frac{1}{\delta} \sup\limits_{\zeta\in\Omega} |f_n(\zeta) - f(\zeta)| \to 0$ whenever $n \to \infty$, so we have $f_n'(z) \xrightarrow[n\to\infty]{u} f'(z)$ on Ω_δ. ■

Following by the discussion at the beginning, we conclude the proof. □

3 Chapter 3: Meromorphic Functions and Complex Logarithm

Theorem 1.1. Suppose that f is holomorphic in a connected open set Ω, has a zero at a point $z_0 \in \Omega$, and does not vanish identically in Ω. Then there exists a neighborhood $U \subseteq \Omega$ of z_0, a non-vanishing holomorphic function g on U, and a unique positive integer n such that $f(z) = (z - z_0)^n g(z)$ for all $z \in U$.

Proof. Since Ω is connected and f is not identically zero, we conclude that f is not identically zero in a neighborhood of z_0. In a small disc centered at z0 the function f has a power series expansion

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

Since f is not identically zero near z_0, there exists a smallest integer n such that $a_n \neq 0$. Then, we can write

$$f(z) = (z - z_0)^n [a_n + a_{n+1}(z - z_0) + \cdots] = (z - z_0)^n g(z),$$

where g is defined by the series in brackets, and hence is holomorphic, and is nowhere vanishing for all z close to z_0 (since $a_n \neq 0$). To prove the uniqueness of the integer n, suppose that we can also write

$$f(z) = (z - z_0)^n g(z) = (z - z_0)^m h(z)$$

where $h(z_0) \neq 0$. If $m > n$, then we may divide by $(z - z_0)^n$ to see that $g(z) = (z - z_0)^{m-n} h(z)$, and letting $z \to z_0$ yields $g(z_0) = 0$, a contradiction. If $m < n$ a similar argument gives $h(z_0) = 0$, which is also a contradiction. We conclude that $m = n$, thus $h = g$, and the theorem is proved. □

In the case of the above theorem, we say that f has a **zero of order** n (or multiplicity n) at z_0. If a zero is of order 1, we say that it is simple. We observe that, quantitatively, the order describes the rate at which the function vanishes.

The importance of the previous theorem comes from the fact that we can now describe precisely the type of singularity possessed by the function $\frac{1}{f}$ at z_0.

For this purpose, it is now convenient to define a deleted neighborhood of z_0 to be an open disc centered at z_0, minus the point z_0, that is, the set

$$\{z : 0 < |z - z_0| < r\}$$

for some $r > 0$. Then, we say that a function f defined in a deleted neighborhood of z_0 has a **pole** at z_0, if the function $\frac{1}{f}$, defined to be zero at z_0, is holomorphic in a full neighborhood of z_0.

Theorem 1.2. If f has a pole at $z_0 \in \Omega$, then in a neighborhood of that point there exist a non-vanishing holomorphic function h and a unique positive integer n such that $f(z) = (z - z_0)^{-n} h(z)$.

Proof. By the previous theorem we have $\frac{1}{f(z)} = (z-z_0)^n g(z)$, where g is holomorphic and non-vanishing in a neighborhood of z_0, so the result follows with $h(z) = \frac{1}{g(z)}$. □

The integer n is called the **order** (or **multiplicity**) of the pole, and describes the rate at which the function grows near z_0. If the pole is of order 1, we say that it is **simple**.

The next theorem should be reminiscent of power series expansion, except that now we allow terms of negative order, to account for the presence of a pole.

Theorem 1.3. If f has a pole of order n at z_0, then

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0} + G(z),$$

where G is a holomorphic function in a neighborhood of z_0.

Proof. The proof follows from the multiplicative statement in the previous theorem. Indeed, the function h has a power series expansion

$$h(z) = A_0 + A_1(z-z_0) + \cdots$$

so that

$$\begin{aligned} f(z) &= (z-z0)^{-n}(A_0 + A_1(z-z_0) + \cdots) \\ &= \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0} + G(z). \end{aligned}$$

□

The sum

$$\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0}$$

is called **the principal part** of f at the pole z_0, and the coefficient a_{-1} is **the residue** of f at that pole. We write res $z_0(f) = a_{-1}$. The importance of the residue comes from the fact that all the other terms in the principal part, that is, those of order strictly greater than 1, have primitives in a deleted neighborhood of z_0. Therefore, if $P(z)$ denotes the principal part above and C is any circle centered at z_0, we get

$$\frac{1}{2\pi i}\int_C P(z)dz = a_{-1}.$$

Theorem 1.4. If f has a pole of order n at z_0, then

$$\text{res}_{z_0}(f) = \lim_{z\to z_0} \frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}(z-z_0)^n f(z).$$

Proof. This is an immediate consequence of the previous theorem. Note that we can do it inductively starting from $\text{res}_{z_0}(f) = \lim_{z\to z_0}(z-z_0)f(z)$. □

Theorem 2.1. Suppose that f is holomorphic in an open set containing a circle C and its interior, except for a pole at z_0 inside C. Then

$$\int_C f(z)dz = 2\pi i \operatorname{res}_{z_0}(f).$$

Proof. Once again, we may choose a keyhole contour that avoids the pole, and let the width of the corridor go to zero to see that

$$\int_C f(z)dz = \int_{C_\varepsilon} f(z)dz$$

where C_ε is the small circle centered at the pole z_0 and of radius ε. Now we observe that

$$\frac{1}{2\pi i}\int_{C_\varepsilon} \frac{a_{-1}}{z-z_0}dz = a_{-1}$$

is an immediate consequence of the Cauchy integral formula (Theorem 4.1 of the previous chapter), applied to the constant function $f = a_{-1}$. Similarly,

$$\frac{1}{2\pi i}\int_{C_\varepsilon} \frac{a_{-k}}{(z-z_0)^k}dz = 0$$

when $k > 1$, by using the corresponding formulae for the derivatives (Corollary 4.2 also in the previous chapter). But we know that in a neighborhood of z_0 we can write

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0} + G(z),$$

where G is holomorphic. By Cauchy's theorem, we also know that $\int_{C_\varepsilon} G(z)dz = 0$, hence $\int_{C_\varepsilon} f(z)dz = a_{-1}$. This implies the desired result. □

Corollary 2.2. Suppose that f is holomorphic in an open set containing a circle C and its interior, except for poles at the points $z_1, \cdots, z_N$ inside C. Then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^N \operatorname{res}_{z_k}(f).$$

Proof. For the proof, consider a multiple keyhole which has a loop avoiding each one of the poles. Let the width of the corridors go to zero. In the limit, the integral over the large circle equals a sum of integrals over small circles to which Theorem 2.1 applies. □

Corollary 2.3 (Residue Formula). Suppose that f is holomorphic in an open set containing a toy contour γ and its interior, except for poles at the points $z_1, \cdots, z_N$ inside γ. Then

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^N \operatorname{res}_{z_0}(f).$$

Proof. In the above, we take γ to have positive orientation. The proof consists of choosing a keyhole appropriate for the given toy contour, so that, as we have seen previously, we can reduce the situation to integrating over small circles around the poles where **Theorem 2.1** applies. $\square$

Definition . Let f be a function holomorphic in an open set Ω except possibly at one point z_0 in Ω. If we can define f at z_0 in such a way that f becomes holomorphic in all of Ω, we say that z_0 is a **removable singularity** for f.

Theorem 3.1 (Riemann's theorem on removable singularities). Suppose that f is holomorphic in an open set Ω except possibly at a point z_0 in Ω. If f is bounded on $\Omega\backslash\{z_0\}$, then z_0 is a removable singularity.

Proof. $\square$

Corollary 3.2. Suppose that f has an isolated singularity at the point z_0. Then z_0 is a pole of f if and only if $|f(z)| \to \infty$ as $z \to z_0$.

Proof. If z_0 is a pole, then we know that $\frac{1}{f}$ has a zero at z_0, and therefore $|f(z)| \to \infty$ as $z \to z_0$. Conversely, suppose that this condition holds. Then $\frac{1}{f}$ is bounded near z_0, and in fact $\frac{1}{|f(z)|} \to 0$ as $z \to z_0$. Therefore, $\frac{1}{f}$ has a removable singularity at z_0 and must vanish there. This proves the converse, namely that z_0 is a pole. $\square$

Isolated singularities belong to one of three categories:

- Removable singularities (f bounded near z_0)
- Pole singularities ($|f(z)| \to \infty$ as $z \to z_0$)
- Essential singularities.

By default, any singularity that is not removable or a pole is defined to be an essential singularity.

Theorem 3.3 (Casorati-Weierstrass). Suppose f is holomorphic in the punctured disc $D_r(z_0)\backslash\{z_0\}$ and has an essential singularity at z_0. Then, the image of $D_r(z_0)\backslash\{z_0\}$ under f is dense in the complex plane.

Proof. We argue by contradiction. Assume that the range of f is not dense, so that there exists $w \in \mathbb{C}$ and $\delta > 0$ such that

$$|f(z) - w| > \delta \text{ for all } z \in D_r(z_0)\backslash\{z_0\}.$$

We may therefore define a new function on $D_r(z_0)\backslash\{z_0\}$ by $g(z) = \frac{1}{f(z)-w}$, which is holomorphic on the punctured disc and bounded by $\frac{1}{\delta}$. Hence g has a removable singularity at z_0 by Theorem 3.1. If $g(z_0) \neq 0$, then $f(z) - w$ is holomorphic at z_0, which contradicts the assumption that z_0 is an essential singularity. In the case that $g(z_0) = 0$, then $f(z) - w$ has a pole at z_0 also contradicting the nature of the singularity at z_0. The proof is complete. $\square$

Definition . A function f on an open set Ω is meromorphic if there exists a sequence of points $\{z_0, z_1, z_2, \cdots\}$ that has no limit points in Ω, and such that

- the function f is holomorphic in $\Omega \backslash \{z_0, z_1, z_2, \cdots\}$, and
- f has poles at the points $\{z_0, z_1, z_2, \cdots\}$.

If a function is holomorphic for all large values of z, we can describe its behavior at infinity using the tripartite distinction we have used to classify singularities at finite values of z. Thus, if f is holomorphic for all large values of z, we consider $F(z) = f(\frac{1}{z})$, which is now holomorphic in a deleted neighborhood of the origin. We say that f has a **pole at infinity** if F has a pole at the origin. Similarly, we can speak of f having **an essential singularity at infinity**, or a **removable singularity** (hence holomorphic) **at infinity** in terms of the corresponding behavior of F at 0. A meromorphic function in the complex plane that is either holomorphic at infinity or has a pole at infinity is said to be **meromorphic in the extended complex plane**.

Theorem 3.4. The meromorphic functions in the extended complex plane are the rational functions.

Proof. Suppose that f is meromorphic in the extended plane. Then $f(\frac{1}{z})$ has either a pole or a removable singularity at 0, and in either case it must be holomorphic in a deleted neighborhood of the origin, so that the function f can have only finitely many poles in the plane, say at $z_1, \cdots, z_n$. The idea is to subtract from f its principal parts at all its poles including the one at infinity. Near each pole $z_k \in \mathbb{C}$ we can write

$$f(z) = f_k(z) + g_k(z),$$

where $f_k(z)$ is the principal part of f at z_k and g_k is holomorphic in a (full) neighborhood of z_k. In particular, f_k is a polynomial in $\frac{1}{z-z_k}$. Similarly, we can write

$$f(\frac{1}{z}) = \tilde{f}_\infty(z) + \tilde{g}_\infty(z),$$

where $\tilde{g}_\infty(z)$ is holomorphic in a neighborhood of the origin and $\tilde{f}_\infty(z)$ is the principal part of $f(\frac{1}{z})$ at 0, that is, a polynomial in $\frac{1}{z}$. Finally, let $f_\infty(z) = \tilde{f}_\infty(\frac{1}{z})$.

We contend that the function $H = f - f_\infty - \sum\limits_{k=1}^{n} f_k$ is entire and bounded. Indeed, near the pole z_k we subtracted the principal part of f so that the function H has a removable singularity there. Also, $H(\frac{1}{z})$ is bounded for z near 0 since we subtracted the principal part of the pole at ∞. This proves our contention, and by Liouville's theorem we conclude that H is constant. From the definition of H, we find that f is a rational function, as was to be shown. $\square$

Theorem 4.1 (Argument Principle). Suppose f is meromorphic in an open set containing a circle C and its interior. If f has no poles and never vanishes on C, then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = (\text{number of zeros of } f \text{ inside } C) - (\text{number of poles of } f \text{ inside } C),$$

where the zeros and poles are counted with their multiplicities.

Corollary 4.2. The above theorem holds for toy contours.

Theorem 4.3 (Rouché's Theorem). Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If $|f(z)| > |g(z)|$ for all $z \in C$, then f and $f+g$ have the same number of zeros inside the circle C.

Proof. For $t \in [0,1]$ define $f_t(z) = f(z) + tg(z)$ so that $f_0 = f$ and $f_1 = f+g$. Let n_t denote the number of zeros of f_t inside the circle counted with multiplicities, so that in particular, n_t is an integer. The condition $|f(z)| > |g(z)|$ for $z \in \mathbb{C}$ clearly implies that f_t has no zeros on the circle, and the argument principle implies

$$n_t = \frac{1}{2\pi i}\int_C \frac{f_t'(z)}{f_t(z)}dz.$$

To prove that n_t is constant, it suffices to show that it is a continuous function of t. Then we could argue that if n_t were not constant, the intermediate value theorem would guarantee the existence of some $t_0 \in [0,1]$ with n_{t_0} not integral, contradicting the fact that $n_t \in \mathbb{Z}$ for all t.

To prove the continuity of n_t, we observe that $\frac{f_t'(z)}{f_t(z)}$ is jointly continuous for $t \in [0,1]$ and $z \in \mathbb{C}$. This joint continuity follows from the fact that it holds for both the numerator and denominator, and our assumptions guarantee that $f_t(z)$ does not vanish on $\mathbb{C}$. Hence n_t is integer-valued and continuous, and it must be constant. We conclude that $n_0 = n_1$, which is Rouché's theorem. $\square$

Theorem 4.4 (Open Mapping Theorem). If f is holomorphic and non-constant in a region Ω, then f is open.

Proof. Let w_0 belong to the image of f, say $w_0 = f(z_0)$. We must prove that all points w near w0 also belong to the image of f.

Define $g(z) = f(z) - w$ and write

$$\begin{aligned} g(z) &= (f(z) - w_0) + (w0 - w) \\ &= F(z) + G(z). \end{aligned}$$

Now choose $\delta > 0$ such that the disc $|z - z0| \leq \delta$ is contained in Ω and $f(z) \neq w_0$ on the circle $|z - z_0| = \delta$. We then select $\varepsilon > 0$ so that we have $|f(z) - w_0| \geq \varepsilon$ on the circle $|z - z_0| = \delta$. Now if $|w - w_0| < \varepsilon$ we have $|F(z)| > |G(z)|$ on the circle $|z - z_0| = \delta$, and by Rouché's theorem we conclude that $g = F + G$ has a zero inside the circle since F has one. $\square$

Theorem 4.5 (Maximum Modulus Principle). If f is a non-constant holomorphic function in a region Ω, then f cannot attain a maximum in Ω.

Proof. Suppose that f did attain a maximum at z_0. Since f is holomorphic it is an open mapping, and therefore, if $D \subseteq \Omega$ is a small disc centered at z_0, its image $f(D)$ is open and contains $f(z_0)$. This proves that there are points in $z \in D$ such that $|f(z)| > |f(z_0)|$, a contradiction. $\square$

Definition . Let γ_0 and γ_1 be two curves in an open set Ω with common end-points. So if $\gamma_0(t)$ and $\gamma_1(t)$ are two parametrizations defined on $[a, b]$, we have $\gamma_0(a) = \gamma_1(a) = \alpha$ and $\gamma_0(b) = \gamma_1(b) = \beta$. These two curves are said to be **homotopic** in Ω if for each $0 \leq s \leq 1$ there exists a curve $\gamma_s \subseteq \Omega$, parametrized by $\gamma_s(t)$ defined on $[a, b]$, such that for every s, $\gamma_s(a) = \alpha$ and $\gamma_s(b) = \beta$, and for all $t \in [a, b]$, $\gamma_s(t)|_{s=0} = \gamma_0(t)$ and $\gamma_s(t)|_{s=1} = \gamma_1(t)$. Moreover, $\gamma_s(t)$ should be jointly continuous in $s \in [0, 1]$ and $t \in [a, b]$.

Theorem 5.1. If f is holomorphic in Ω, then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

whenever the two curves γ_0 and γ_1 are homotopic in Ω.

Definition . A region Ω in the complex plane is simply connected if any two pair of curves in Ω with the same end-points are homotopic.

Theorem 5.2. Any holomorphic function in a simply connected domain has a primitive.

Proof. Fix a point z_0 in Ω and define $F(z) = \int_\gamma f(w)dw$ where the integral is taken over any curve in Ω joining z_0 to z. This definition is independent of the curve chosen, since Ω is simply connected, and if $\tilde{\gamma}$ is another curve in Ω joining z_0 and z, we would have

$$\int_\gamma f(w)dw = \int_{\tilde{\gamma}} f(w)dw$$

by Theorem 5.1. Now we can write $F(z + h) - F(z) = \int_\eta f(w)dw$ where η is the line segment joining z and $z + h$. Arguing as in the proof of Theorem 2.1 in Chapter 2, we find that

$$\lim_{h \to 0} \frac{F(z + h) - F(z)}{h} = f(z).$$

□

Corollary 5.3. If f is holomorphic in the simply connected region Ω, then

$$\int_\gamma f(z)dz = 0$$

for any closed curve γ in Ω.

Proof. This is immediate from the existence of a primitive. □

Suppose we wish to define the logarithm of a non-zero complex number. If $z = re^{i\theta}$, and we want the logarithm to be the inverse to the exponential, then it is natural to set $\log z = \log r + i\theta$. Here and below, we use the convention that $\log r$ denotes the standard logarithm of the positive number r. The trouble with the above definition is that θ is unique only up to an integer multiple of 2π. However, for given z we can fix a choice of θ, and if z varies only a little, this determines the corresponding choice of θ uniquely (assuming we require that θ varies continuously with z). Thus "locally" we can give an unambiguous definition of the logarithm, but this will not work "globally." For example, if z starts at 1, and then winds around the origin and returns to 1, the logarithm does not return to its original value, but rather differs by an integer multiple of $2\pi i$, and therefore is not "single-valued." To make sense of the logarithm as a single-valued function, we must restrict the set on which we define it. This is the so-called choice of a **branch** or sheet of the logarithm.

For example, in the slit plane $\Omega = \mathbb{C}\backslash\{(-\infty, 0]\}$ we have the principal branch of the logarithm $\log z = \log r + i\theta$ where $z = re^{i\theta}$ with $|\theta| < \pi$.

Theorem 6.2. If f is a nowhere vanishing holomorphic function in a simply connected region Ω, then there exists a holomorphic function g on Ω such that $f(z) = e^{g(z)}$.

The function $g(z)$ in the theorem can be denoted by $\log f(z)$, and determines a "branch" of that logarithm.

Proof. Fix a point z_0 in Ω, and define a function

$$g(z) = \int_\gamma \frac{f'(w)}{f(w)}dw + c_0,$$

where γ is any path in Ω connecting z_0 to z, and c_0 is a complex number so that $e^{c_0} = f(z_0)$. This definition is independent of the path γ since Ω is simply connected. Arguing as in the proof of Theorem 2.1, Chapter 2, we find that g is holomorphic with $g'(z) = \frac{f'(z)}{f(z)}$, and a simple calculation gives $\frac{d}{dz}(f(z)e^{-g(z)}) = 0$, so that $f(z)e^{-g(z)}$ is constant. Evaluating this expression at z_0 we find $f(z_0)e^{-c_0} = 1$, so that $f(z) = e^{g(z)}$ for all $z \in \Omega$, and the proof is complete. $\square$

4 Chapter 6 & 7: Zeta Function and the Prime Number Theorem

Definition . The Riemann zeta function is initially defined for real $s > 1$ by the convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Proposition 2.1. The series defining $\zeta(s)$ converges for $\mathrm{Re}(s) > 1$, and the function ζ is holomorphic in this half-plane.

Proof. $\square$

Theorem 2.4. The zeta function has a meromorphic continuation into the entire complex plane, whose only singularity is a simple pole at $s = 1$.

Proof. $\square$

Proposition 2.7. Suppose $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$. Then for each σ_0, $0 \leq \sigma_0 \leq 1$, and every $\varepsilon > 0$, there exists a constant c_ε so that

1. $|\zeta(s)| \leq c_\varepsilon |t|^{1-\sigma_0+\varepsilon}$, if $\sigma_0 \leq \sigma$ and $|t| \geq 1$.
2. $|\zeta'(s)| \leq c_\varepsilon |t|^{\varepsilon}$, if $1 \leq \sigma$ and $|t| \geq 1$.

Proof. $\square$

5 Chapter 8: Conformal Mappings

Definition . Suppose $U, V \subseteq \mathbb{C}$ open, $f : U \to V$ holomorphic. We say f is biholomorphic (conformal) if

1. f is bijective.
2. $f^{-1} : V \to U$ is also holomorphic.

Remark . Note that 1) implies 2).

Definition . Mapping of the form $z \mapsto \frac{az+b}{cz+d}$ are called fractional linear transformation. These are exactly all the automorphisms on $\mathbb{CP}^1$.

Theorem 3.1 (Riemann Mapping Theorem). Suppose $U \subsetneq \mathbb{C}$ is a simply connected region and is neither $\varnothing$ nor $\mathbb{C}$. Then U is biholomorphic to the unit disk.

Proof. Consider all injective functions $f : U \to \mathbb{D}$. Apply some sort of compactness to the set of all such functions. □

Lemma 2.1 (Schwarz Lemma). Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then

1. $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.
2. If for some $z_0 \neq 0$ we have $|f(z_0)| = |z_0|$, then f is a rotation, i.e. $f(z) = cz$ for some c with $|c| = 1$.
3. $|f'(0)| \leq 1$, and if equality holds, then f is a rotation.

Proof. □

Claim . $f(z) = \frac{z-a}{\bar{a}z-1}$ for $|a| < 1$ gives a biholomorphism $f : \mathbb{D} \to \mathbb{D}$ with $f(a) = 0$.

Theorem . Every biholomorphic map on $\mathbb{D}$ is given by $f(z) = c(\frac{z-a}{\bar{a}z-1})$ for some $|c| = 1$ and $|a| < 1$.

6 Chapter 9: Elliptic Functions