

MATH 502 Notes

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References:

- Atiyah and MacDonald, *Commutative Algebra*.
- J.P. Serre, *Local Algebra*.
- Zariski and Samuel, *Commutative Algebra* Volume 1 and 2.
- Matsumura, *Commutative Algebra*.
- Bourbaki, *Commutative Algebra*.

We always assume a ring R has a multiplicative identity and is commutative.

0 NOETHERIAN, ARTINIAN, AND LOCALIZATION

Proposition 0.1. Let R be a (commutative) ring, and let M be an A -module, then the following are equivalent:

- (i) Given an infinite increasing chain of submodules of M

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq M_{n+1} \subseteq \cdots$$

then there exists some $N \in \mathbb{N}$ such that $M_N = M_{N+1} = \cdots$, i.e., for all $n \geq N$, $M_n = M_{n+1}$.

- (ii) Every non-empty family of submodules has a maximal element.

- (iii) Every submodule of M is finitely-generated.

Proof. (i) \Rightarrow (ii): This is a direct result of Zorn's lemma.

(ii) \Rightarrow (i): Obvious.

(i), (ii) \Rightarrow (iii): Take any submodule N of M and take $x_1 \in N$. If $(x_1) \neq N$, then there exists $x_2 \in N \setminus (x_1)$, so $(x_1, x_2) \subseteq N$, now we proceed inductively, but by the given property we know this stops in finite number of steps, hence we have $N = (x_1, \dots, x_n)$ for some $n \in \mathbb{N}$, thus N is finitely-generated.

(iii) \Rightarrow (i): Note that the property implies M is finitely-generated, but that means the chain of submodules must be finite. \square

Definition 0.2 (Noetherian Module). If any of the conditions in [Proposition 0.1](#) holds, then M is said to be a Noetherian module. Alternatively, we say M satisfies the ascending chain condition.

Proposition 0.3. Let R be a (commutative) ring, and let M be an A -module, then the following are equivalent:

- (i) Given an infinite decreasing chain of submodules of M

$$M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq M_{n+1} \supseteq \cdots$$

then there exists some $N \in \mathbb{N}$ such that $M_N = M_{N+1} = \cdots$, i.e., for all $n \geq N$, $M_n = M_{n+1}$.

- (ii) Every non-empty family of submodules has a minimal element.

Proof. Again, Zorn's lemma. □

Definition 0.4 (Artinian Module). If any of the conditions in [Proposition 0.3](#) holds, then M is said to be a Artinian module. Alternatively, we say M satisfies the descending chain condition.

Example 0.5. • \mathbb{Z} is Noetherian.

- \mathbb{Q}/\mathbb{Z} is not Noetherian.
- Let p be a prime. Let $\mathbb{Z}(p^\infty)$ be the union of chains (as direct limits)

$$\left\langle \frac{1}{p} \right\rangle \subseteq \left\langle \frac{1}{p^2} \right\rangle \subseteq \cdots \subseteq \left\langle \frac{1}{p^n} \right\rangle \subseteq \cdots$$

then there is an embedding $\mathbb{Z}(p^\infty) \subseteq \mathbb{Q}/\mathbb{Z}$, where \bar{a} is the image of a in \mathbb{Q}/\mathbb{Z} . With this construction, $\mathbb{Z}(p^\infty)$ is Artinian.

Exercise 0.6. Show that $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}(p^\infty)$ where p traverses through all the primes.

Proposition 0.7. Let N be a submodule of M . Suppose M satisfies ascending (respectively, descending) chain condition, then N and M/N also satisfy ascending (respectively, descending) chain condition. If, for some submodule N of M , we know N and M/N satisfy ascending (respectively, descending) chain condition, then M also satisfies ascending (respectively, descending) chain condition.

Proof. Suppose M satisfies ascending (respectively, descending) chain condition, and let N be a submodule of M . Let $\{N_i\}$ be an increasing (respectively, decreasing) sequence of submodules of N , then they can be regarded as submodules of M , therefore by the Noetherian (respectively, Artinian) condition, we know N satisfies ascending (respectively, descending) chain condition. Now let $\bar{M} = M/N$, and take $\{\bar{M}_i\}$ be an increasing (respectively, decreasing) sequence of submodules of \bar{M} . Let $\pi : M \rightarrow M/N$ be the quotient map, then the preimages give an increasing (respectively, decreasing) sequence $\{M_i\}$ of submodules of M , where $M_i = \pi^{-1}(\bar{M}_i)$, but by the Noetherian (respectively, Artinian) condition, we know the sequence stops in finite steps, therefore the original sequence stops in finite steps as well, hence \bar{M} satisfies the ascending (respectively, descending) chain condition.

Suppose a submodule N of M is such that N and M/N both satisfy ascending chain condition. Take a submodule T of M , then we have a short exact sequence

$$0 \longrightarrow T \cap N \hookrightarrow T \longrightarrow T/(T \cap N) \longrightarrow 0$$

Now $T \cap N$ is finitely-generated as N is finitely-generated, therefore we have an embedding $T/(T \cap N) \hookrightarrow M/N$, thus $T/(T \cap N)$ is finitely-generated, therefore T is also finitely-generated by a vector space argument.

Suppose we have a decreasing sequence $\{M_n\}$ of M , then we have a decreasing sequence $\{N \cap M_n\}$. Let $\bar{M} = M/N$, then $\bar{M}_n := (M_n + N)/N$ defines a decreasing sequence of submodules in \bar{M} , but N satisfies the descending chain condition, so the sequence $\{N \cap M_n\}$ stops in finite number of steps, say n_0 . Moreover, the sequence of \bar{M}_n 's also stops in finite number of steps, so by definition the sequence of $(M_n + N)/N$ stops in finite number of steps, say m_0 , but by the isomorphism theorem this shows that the sequence of $M_n/(N \cap M_n)$ stops in m_0 steps. Therefore, whenever $n \geq m_0, n_0$, then $N \cap M_n = N \cap M_{n+1}$, hence $M_n = M_{n+1} = \cdots$ for such n . □

Remark 0.8. The final argument should also work in the Noetherian case.

Definition 0.9 (Simple Module). An A -module M is simple if the submodules of M are either 0 or M .

Exercise 0.10. Let A be a commutative ring, and M is an A -module, then M is simple if and only if $M \cong A/\mathfrak{m}$ for some maximal ideal \mathfrak{m} of A .

Definition 0.11 (Jordan-Hölder Chain). Let A be a commutative ring and M be an A -module. We say M has a Jordan-Hölder chain if there exists a decreasing chain of submodules $\{M_i\}$ such that

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_{n-1} \supsetneq M_n = 0$$

such that M_i/M_{i+1} is simple. In such a situation, we know n is the length of the Jordan-Hölder chain, and such n is unique. We say M is a module of finite length, and the length is $\ell_A(M) = n$.

Exercise 0.12. Let A be a commutative ring, and let M be an A -module, then M is of finite length if and only if M is both Noetherian and Artinian.

Theorem 0.13. Let A be a commutative ring, then A is Artinian if and only if A is Noetherian and every prime ideal of A is maximal.

Proof. (\Leftarrow):

Lemma 0.14. Let A be Noetherian, then every ideal of A contains a product of prime ideals.

Subproof. Suppose, towards contradiction, that there exists some ideal I of A that does not contain a product of prime ideals. Let \mathcal{J} be the set of such ideals of A , then $\mathcal{J} \neq \emptyset$, and we can take a maximal element of \mathcal{J} , namely J .¹ By definition, J is not prime, therefore there exists $a, b \in A$ such that $a \notin J$ and $b \notin J$, but $ab \in J$. Now $J \subsetneq J + Aa$ and $J \subsetneq J + Ab$, therefore $J + Aa, J + Ab \notin \mathcal{J}$, therefore $J + Aa$ and $J + Ab$ both contain product of prime ideals. But now $(J + Aa)(J + Ab)$ should also contain products of prime ideals, but by distribution this is just $J^2 + Ja + Jb + Aab$, which is contained in J because every term is contained in J , so J contains a product of prime ideals as well, contradiction. ■

In particular, (0) contains a product of prime ideals, in particular (0) equals to this product, but every prime ideal is maximal, therefore $(0) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$ becomes the product of maximal ideals (which may not necessarily be distinct), hence we have a descending chain of ideals

$$A \supseteq \mathfrak{m}_1 \supseteq \mathfrak{m}_1 \mathfrak{m}_2 \supseteq \cdots \supseteq \mathfrak{m}_1 \cdots \mathfrak{m}_n = (0),$$

and in particular $(\mathfrak{m}_1 \cdots \mathfrak{m}_{i-1})/(\mathfrak{m}_1 \cdots \mathfrak{m}_i)$ is a finite-dimensional since A is Noetherian, and it has a natural structure as a A/\mathfrak{m}_i -vector space. From the short exact sequence

$$0 \longrightarrow \mathfrak{m}_1 \cdots \mathfrak{m}_i \longrightarrow \mathfrak{m}_1 \cdots \mathfrak{m}_{i-1} \longrightarrow (\mathfrak{m}_1 \cdots \mathfrak{m}_{i-1})/(\mathfrak{m}_1 \cdots \mathfrak{m}_i) \longrightarrow 0$$

we know the two sides of the sequence are Artinian, hence the central term is Artinian. Proceeding inductively, we know that \mathfrak{m}_1 is Artinian, and R/\mathfrak{m}_1 would also be Artinian, hence A is Artinian.

(\Rightarrow): Now suppose A is Artinian, and we want to show that every prime ideal is maximal, and (0) is a product of maximal ideals. The result then follows from the argument above.

Lemma 0.15. Every Artinian domain is a field.

Subproof. Let $0 \neq a \in A$, then consider the chain

$$(a) \supseteq (a^2) \supseteq \cdots \supseteq (a^n) \supseteq \cdots$$

and by the Artinian property, for some large enough n the descending chain stops. Hence, we have $a^n = \lambda a^{n+1}$ for some large enough n and some $\lambda \in A$. Hence, $a^n(1 - \lambda a) = 0$, by the cancellation property of a domain, since $a \neq 0$, we must have $\lambda a = 1$, therefore a is a unit, as desired. ■

Corollary 0.16. Let A be Artinian, then every prime ideal of A is maximal.

Finally, it suffices to show that $(0) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$. Let \mathfrak{J} be the set of finite products of maximal ideals, then \mathfrak{J} has a minimal element, and it suffices to show that this element is (0) . Suppose not, let $I \neq (0)$ be a minimal element of R . For any two ideals α, β of A , let $(\alpha : \beta) = \{a \in A \mid a\beta \subseteq \alpha\}$. Note that this has a natural structure as an ideal of A . Let $J = ((0) : I)$, and suppose $J = A$, then $I = 0$, contradiction, so $J \neq A$ is a proper ideal of A , now consider A/J which is Artinian, then let \mathfrak{G} be the set of all non-zero ideals of A/J , so \mathfrak{G} has a minimal element as well, call it \bar{H} . Let $H = \pi^{-1}(\bar{H})$ where $\pi : A \rightarrow A/J$, so we have $J \subsetneq H$, thus let $P = (J : H)$.

Claim 0.17. P is a prime ideal.

Subproof. Given $c, d \notin P$, we want to show that $cd \notin P$. Indeed, consider $J \subsetneq J + cH \subseteq H$, then since H is minimal, then $J + cH = H$, and similarly we have that $J + dH = H$. Therefore, we have that $J + cdH = J + c(dH + J) = J + cH = H$, hence we know $cd \notin P$, as desired. ■

¹The existence of this maximal element is the result of Zorn's lemma and ACC condition.

Now $P = (J : H)$ and $J = (0 : I)$, the by definition we have $PHI = (0)$. Since P is a prime ideal, then P is maximal, and now

$$(0 : PI) \supseteq H \supsetneq J = (0 : I)$$

Therefore $PI \subsetneq I$, where I is a minimal element, contradiction, hence (0) is a product of maximal ideals. \square

Definition 0.18 (Short Exact Sequence). Consider the sequence

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} T \longrightarrow 0$$

This is called a short exact sequence if $\ker(f) = 0$, $\text{im}(g) = T$, and $\ker(g) = \text{im}(f)$. In particular, one slot of the sequence is said to be exact if the kernel of the previous map equals to the image of the subsequent map.

Definition 0.19 (Flat Module). Let M be an A -module, then we say M is a flat A -module if for every short exact sequence

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

the tensored sequence

$$0 \longrightarrow M \otimes_A N_1 \longrightarrow M \otimes_A N_2 \longrightarrow M \otimes_A N_3 \longrightarrow 0$$

remains exact.

Remark 0.20. Recall that the properties of modules have the following implications: free \Rightarrow projective \Rightarrow flat \Rightarrow torsion-free, and in the case of finitely-generated modules, torsion-free \Rightarrow free.

Remark 0.21. We already know that the tensor functor is right exact, namely given the short exact sequence above, then

$$M \otimes_A N_1 \longrightarrow M \otimes_A N_2 \longrightarrow M \otimes_A N_3 \longrightarrow 0$$

is exact.

Exercise 0.22. Let M be an A -module, and if there exists a short exact sequence of A -modules

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

where N_1 and N_2 are finitely-generated as A -modules, and such that tensoring M preserves the short exact sequence, then M is flat.

Definition 0.23 (Multiplicatively Closed Subset). Let A be a commutative ring and M be an A -module. Let $S \subseteq A$ be a subset. We say S is a multiplicatively closed subset of A if $1 \in S$, $0 \notin S$, and whenever $s_1, s_2 \in S$, then $s_1 s_2 \in S$.

Definition 0.24 (Localization). Let $S \subseteq A$ be a multiplicatively closed subset, and let M be an A -module, then $S^{-1}M = (M \times S)/\sim$, where \sim is an equivalence relation defined by the following: $(m_1, s_1) \sim (m_2, s_2)$ if and only if there exists $t \in S$ such that $t(m_1 s_2 - m_2 s_1) = 0$. $S^{-1}M$ is said to be the localization of M at S .

Given $(m, s) \in M \times S$, we write $\overline{(m, s)}$ to be the equivalence class in $S^{-1}M$ represented by (m, s) .

Exercise 0.25. Similarly, one can define the localization $S^{-1}A$ of A at S . In fact, $S^{-1}A$ inherits a ring structure from A , namely

- $\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{a_1 s_2 + a_2 s_1}{s_1 s_2}$,
- $\frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{a_1 a_2}{s_1 s_2}$,
- $\frac{1}{s} \cdot \frac{s}{1} = \frac{1}{1} = 1$.

Remark 0.26. Note that a ring structure does not guarantee every element to have a multiplicative inverse. The localization of A at S ensures that every element of S now becomes invertible in the new ring $S^{-1}A$. In particular, this induces a ring homomorphism

$$\begin{aligned} f : A &\rightarrow S^{-1}A \\ a &\mapsto \frac{a}{1} \end{aligned}$$

This homomorphism is injective if A is a domain.

Remark 0.27. Let I be an ideal of A .

- Consider the ring homomorphism $f : A \rightarrow S^{-1}A$ above, then

$$S^{-1}I = IS^{-1}A = f(I)S^{-1}A.$$

In particular, $f^{-1}(IS^{-1}A) \supseteq I$.

- If $I \cap S \neq \emptyset$, then $IS^{-1}A = S^{-1}A$.
- If P is a prime ideal of A such that $P \cap S = \emptyset$, then $f^{-1}(PS^{-1}A) = P$.
- Let M be an A -module, then if $N \subseteq M$ is a submodule, then $S^{-1}N \subseteq S^{-1}M$. That is, given an exact sequence

$$0 \longrightarrow N \longrightarrow M$$

then we obtain an exact sequence

$$0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M$$

Indeed, given $0 \rightarrow N \xrightarrow{f} M$, say we have it sending $\frac{n}{1} \mapsto \frac{f(n)}{1} = 0$, then there exists $s \in S$ such that $sf(n) = 0$, so $f(sn) = 0$, therefore $sn = 0$ by injection, hence $\frac{n}{1} = 0$ in $S^{-1}N$ as well.

Exercise 0.28. The localization functor is exact.

Lemma 0.29. Let A be a commutative ring and S be a multiplicatively closed subset of A , then $S^{-1}A \otimes_A M \cong S^{-1}M$.

Proof. We define

$$\begin{aligned} \varphi : S^{-1}A \otimes_A M &\rightarrow S^{-1}M \\ \frac{a}{s} \otimes m &\mapsto \frac{am}{s}. \end{aligned}$$

For any $\frac{m}{s} \in S^{-1}M$, we have $\varphi\left(\frac{1}{s} \otimes m\right) = \frac{m}{s}$, so the map is onto. Now suppose $\varphi\left(\sum_{i=1}^n \frac{a_i}{s_i} \otimes m_i\right) = 0$ (since this is a finite sum), then $\varphi\left(\sum_{i=1}^n \frac{a_i}{s_i} \otimes m_i\right) = \sum_{i=1}^n \frac{a_i m_i}{s_i} = 0$. We make $s = s_1 \cdots s_n$, so

$$\frac{a_i}{s_i} \otimes m_i = \frac{a_i s_1 \cdots s_{i-1} s_{i+1} \cdots s_n}{s} \otimes m_i =: \frac{b_i}{s} \otimes m_i,$$

then $\sum_{i=1}^n \frac{a_i}{s_i} \otimes m_i = \sum_{i=1}^n \frac{b_i}{s} \otimes m_i$, therefore

$$\varphi\left(\sum_{i=1}^n \frac{a_i}{s_i} \otimes m_i\right) = \varphi\left(\sum_{i=1}^n \frac{b_i}{s} \otimes m_i\right) = \frac{\sum_{i=1}^n b_i m_i}{s} = 0,$$

so there exists $t \in S$ such that $t \sum_{i=1}^n b_i m_i = 0$, now

$$\sum_{i=1}^n \frac{a_i}{s_i} \otimes m_i = \sum_{i=1}^n \frac{b_i}{s} \otimes m_i$$

$$\begin{aligned}
 &= \sum_{i=1}^n \frac{1}{s} \otimes b_i m_i \\
 &= \frac{1}{s} \otimes \sum_{i=1}^n b_i m_i \\
 &= \frac{t}{ts} \otimes \sum_{i=1}^n b_i m_i \\
 &= \frac{1}{ts} \otimes t \sum_{i=1}^n b_i m_i \\
 &= \frac{1}{ts} \otimes 0 \\
 &= 0.
 \end{aligned}$$

□

Proposition 0.30. The map $A \rightarrow S^{-1}A$ is A -flat, i.e., $S^{-1}A$ is a flat A -module.

Proof. Consider

$$0 \longrightarrow N \longrightarrow M \longrightarrow T \longrightarrow 0$$

By [Lemma 0.29](#) (since the isomorphism is functorial), it suffices to show the exactness of

$$0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M \longrightarrow S^{-1}T \longrightarrow 0$$

and this follows from [Exercise 0.28](#). □

Definition 0.31 (Quasi-local, Local). Let A be a commutative ring. We say A is quasi-local if A has exactly one maximal ideal. In particular, if A is also Noetherian, then we say A is a local ring.

Definition 0.32 (Localization). Let A be a commutative ring and \mathfrak{p} be a prime ideal of A . Note that $S = A \setminus \mathfrak{p}$ is a multiplicatively closed subset, then we write $S^{-1}A = A_{\mathfrak{p}}$ (in general, we have $S^{-1}M = M_{\mathfrak{p}}$, where $M \otimes_A A_{\mathfrak{p}} \cong M_{\mathfrak{p}}$) to denote the localization of A away from the prime ideal \mathfrak{p} .

Exercise 0.33. $A_{\mathfrak{p}}$ is quasi-local with unique maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$.

Remark 0.34. Take $x \in M$, then the following are equivalent:

- $x = 0$;
- $\frac{x}{1} = 0$ in $M_{\mathfrak{m}}$ for any maximal ideal \mathfrak{m} of A ;
- $\frac{x}{1} = 0$ in $M_{\mathfrak{p}}$ for any prime ideal \mathfrak{p} of A .

Proof. We will prove the first two are equivalent. The (\Rightarrow) direction is obvious. Conversely, let $I = \{a \in A \mid ax = 0\}$ to be the annihilator of x in A . Suppose, towards contradiction, that $I \neq A$, then I is contained in some maximal ideal \mathfrak{m} of A , then consider $M_{\mathfrak{m}}$. Since $\frac{x}{1} = 0$ in $M_{\mathfrak{m}}$, then there exists $t \in A \setminus \mathfrak{m}$ such that $tx = 0$, but $I \subseteq \mathfrak{m}$ and $t \notin \mathfrak{m}$, then we reach a contradiction, hence $I = A$, and obviously we are done. □

Exercise 0.35. 1. Given the sequence

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} T \longrightarrow 0$$

the following are equivalent:

- the sequence is exact;
- the sequence

$$0 \longrightarrow M_{\mathfrak{m}} \xrightarrow{f_{\mathfrak{m}}} N_{\mathfrak{m}} \xrightarrow{g_{\mathfrak{m}}} T_{\mathfrak{m}} \longrightarrow 0$$

is exact for all maximal ideals \mathfrak{m} of A ;

- the sequence

$$0 \longrightarrow M_{\mathfrak{p}} \xrightarrow{f_{\mathfrak{p}}} N_{\mathfrak{p}} \xrightarrow{g_{\mathfrak{p}}} T_{\mathfrak{p}} \longrightarrow 0$$

is exact for all prime ideals \mathfrak{p} of A .

To see this, apply [Remark 0.34](#).

2. Let A be a commutative ring and M be an A -module, then the following are equivalent:

- M is A -flat;
- $M_{\mathfrak{m}}$ is $A_{\mathfrak{m}}$ -flat for all maximal ideals \mathfrak{m} of A ;
- $M_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$ -flat for all prime ideals \mathfrak{p} of A ;

Hence, exactness is a local property.

Exercise 0.36. Let A be a commutative ring, then A is Artinian if and only if A as an A -module is of finite length, i.e., $\ell_A(A) < \infty$. Indeed, note that $(0) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$, and write down the Jordan-Hölder series.

1 PRIMARY DECOMPOSITION THEOREM

Throughout [Section 1](#), the commutative ring A is always Noetherian. In [Section 1.1](#), M is a finitely-generated A -module; in [Section 1.2](#), we drop this assumption.

1.1 FINITELY-GENERATED CASE

Definition 1.1 (Coprimary). We say M is a coprimary module if for all $a \in A$, the left multiplication $m_a : M \rightarrow M$ is either injective or nilpotent (i.e., there exists $n > 0$ such that $a^n M = 0$).

Remark 1.2. (i) If M is coprimary, then N is coprimary for all $N \subseteq M$.

(ii) If M is coprimary, let $P = \{a \in A \mid a : M \rightarrow M \text{ is nilpotent}\}$, then P is a prime ideal of A .

Proof. For $a, b \notin P$, $a, b : M \rightarrow M$ are injective maps, so $ab : M \rightarrow M$ is injective, hence $ab \notin P$. □

Hence, we usually say M is P -coprimary.

(iii) Let M be P -coprimary, then there exists an injection (as M -linear map) $A/P \hookrightarrow M$.

Proof. Take any $x \neq 0$ in M , then consider

$$\begin{aligned} a_x : A &\rightarrow M \\ 1 &\mapsto x \end{aligned}$$

Let $I = \ker(a_x)$, then we have

$$\begin{aligned} A/I &\hookrightarrow M \\ \bar{1} &\mapsto x \end{aligned}$$

Now $I \subseteq P$ since I already kills x . Since A is Noetherian, P is finitely-generated, thus consider $P = (a_1, \dots, a_r)$, then $a_i^{t_i} \cdot x = 0$ for all i and some t_i 's. Let $t = t_1 + \dots + t_r$, then $P^t \cdot x = 0$ by binomial theorem, so $P^t \subseteq I \subseteq P$, hence there exists j such that $P^j \subseteq I \subsetneq P^{j-1}$. Take $y \in P^{j-1} \setminus I$, so $\bar{y} \neq 0$ in A/P , taking the injection into M , then $\text{Ann}_A(\bar{y}) = P$. We now have the composition

$$\begin{aligned} A/P &\hookrightarrow A/I \hookrightarrow M \\ \bar{1} &\mapsto \bar{y} \end{aligned}$$

to be injective. □

(iv) Suppose M is P -coprimary, and Q is a prime ideal such that $A/Q \hookrightarrow M$, then $P = Q$.

Proof. By definition of P , $Q \subseteq P$ is obvious: Q kills elements in M , therefore the mapping becomes nilpotent. The other direction is also easy. \square

Definition 1.3 (Primary). Let $N \subseteq M$ be a submodule. We say N is a primary submodule of M if M/N is coprimary. If M/N is P -coprimary, we say N is P -primary.

Remark 1.4. Let \mathfrak{p} be a prime ideal of A . We claim that \mathfrak{p}^t is P -primary. Consider

$$m_x : A/\mathfrak{p}^t \rightarrow A/\mathfrak{p}^t$$

then $x^t = 0$ on A/\mathfrak{p}^t .

Example 1.5. Let $A = k[X, Y, Z]/(Z^2 - XY)$, let $\mathfrak{p} = (x, z)$ where $x = \text{im}(X)$ and $z = \text{im}(Z)$. Now $A/\mathfrak{p} = k[Y]$. \mathfrak{p}^2 is not P -primary. Indeed, note that $A/\mathfrak{p}^2 = k[X, Y, Z]/(z^2 - xy, x^2, z^2) \cong k[X, Y, Z]/(X^2, XY, Z^2, XZ)$. Now the mapping given by multiplication by y on this map is injective, so \mathfrak{p}^2 is not P -primary.

In particular, the represented surface is not smooth, since the origin $(0, 0, 0)$ is a singularity.

Theorem 1.6 (Primary Decomposition Theorem). By assumption, A is Noetherian and M is finitely-generated. Let $N \subseteq M$ be a submodule, then there exists a decomposition

$$N = \bigcap_{i=1}^r N_i$$

where each N_i is P_i -primary, and such that

1. all P_i 's are distinct, and
2. this decomposition is irredundant, i.e., minimal. In particular, this means removing any of the N_i 's gives a different intersection, i.e., $\bigcap_{j \neq i} N_j \not\subseteq N_i$.

This is called a primary decomposition of N . Moreover, the primary decomposition is unique up to permutation of modules, that is, if there exists another primary decomposition, i.e., $N = \bigcap_{i=1}^s N'_i$ where N'_i 's are P'_i -primary, then $r = s$ and $\{N_1, \dots, N_r\} = \{N'_1, \dots, N'_s\}$.

Proof.

Definition 1.7 (Irreducible). A submodule $T \subsetneq M$ is called irreducible if $T \neq T_1 \cap T_2$, where T_1, T_2 are distinct proper submodules of M .

Claim 1.8. Every submodule T of M can be expressed by $T = T_1 \cap \dots \cap T_l$ where each T_i is irreducible.

Subproof. Suppose, towards contradiction, that there exists some T for which the claim fails, then the set of all such submodules T is a non-empty set \mathcal{T} . Since M is Noetherian, then \mathcal{T} has a maximal element W , therefore W is not irreducible. By definition, $W = W_1 \cap W_2$ where W_1, W_2 are distinct proper submodules of M , so $W_1 \notin \mathcal{T}$ and $W_2 \notin \mathcal{T}$, therefore $W_1 = T_1 \cap \dots \cap T_r$ for irreducible T_i 's, and $W_2 = T'_1 \cap \dots \cap T'_s$ where T'_i are irreducible. Therefore, W becomes an intersection of irreducible submodules, a contradiction. \blacksquare

Claim 1.9. Suppose T is irreducible in M , then T is a primary submodule of M . That is, we need to show $\bar{M} := M/T$ is coprimary.

Subproof. It suffices to show the following: for all $a \neq 0$ in A , the multiplication map $a : \bar{M} \rightarrow \bar{M}$ is either nilpotent or injective. Note that (0) in \bar{M} is irreducible. To see this, we take the ascending chain

$$\ker(a) \subseteq \ker(a^2) \subseteq \ker(a^3) \subseteq \dots$$

and since A is Noetherian we know $\ker(a^n) = \ker(a^{n+1}) = \dots$ for some large enough n , therefore for $g = a^n$ we know $\ker(g) = \ker(g^2)$.

Claim 1.10. $\ker(g) \cap \operatorname{im}(g) = (0)$ in \bar{M} .

Subproof of Subclaim. Let $x \in \ker(g) \cap \operatorname{im}(g)$, then $g(x) = 0$, and there exists $y \in \bar{M}$ such that $x = g(y)$, so $0 = g(x) = g^2(y)$, but that means $y \in \ker(g^2) = \ker(g)$, so $x = 0$. ■

Therefore, (0) is irreducible in \bar{M} , so either $\ker(g) = (0)$ or $\ker(g) = \bar{M}$. If $\ker(g) = (0)$, we have g to be injective, hence multiplication by a is injective; if $\ker(g) = \bar{M}$, we have $a^n \bar{M} = 0$, so a becomes nilpotent. ■

Claim 1.11. If N_1 and N_2 are both P -primary as submodules, then $N_1 \cap N_2$ is also P -primary.

Subproof. By definition, M/N_1 and M/N_2 are both P -coprimary, then it is easy to see that $M/N_1 \oplus M/N_2$ is also P -coprimary. We know there is an obvious inclusion

$$\begin{aligned} M/(N_1 \cap N_2) &\hookrightarrow M/N_1 \oplus M/N_2 \\ \bar{x} &\mapsto (\bar{x}, \bar{x}) \end{aligned}$$

so $M/(N_1 \cap N_2)$ is also coprimary by the inclusion, therefore $N_1 \cap N_2$ is P -primary. ■

Now by [Claim 1.8](#) we have an irreducible decomposition $N = N_1 \cap \cdots \cap N_r$ and without loss of generality let it be of the smallest length, that is, the N_i 's are irreducible modules that are irredundant. By [Claim 1.9](#), we know each of the N_i 's is primary with respect to some prime ideal. Now for any two P -primary modules N_i and N_j , we know the intersection is still P -primary according to [Claim 1.11](#), therefore we obtain an irredundant intersection $N = N'_1 \cap \cdots \cap N'_s$ where each N'_i is P_i -primary (where P_i 's are now distinct!), and this proves the existence.

For the uniqueness, suppose we have $N = N_1 \cap \cdots \cap N_r$ where N_i is P_i -primary, where P_i 's are distinct, and suppose we have $N = N'_1 \cap \cdots \cap N'_s$ where N'_i is P'_i -primary, where all P'_i are distinct as well. It is enough to show the following:

Claim 1.12. For any prime ideal p of A , $p \in \{P_1, \dots, P_r\}$ if and only if there exists an injection $A/p \hookrightarrow M/N$.

Subproof. Let $p \in \{P_1, \dots, P_r\}$, without loss of generality denote $p = P_1$, then we have an injection $A/p \hookrightarrow M/N_1$ by [Remark 1.2](#). In $\bar{M} = M/N$, we have $(0) = N_1/N \cap \cdots \cap N_r/N =: \bar{N}_1 \cap \cdots \cap \bar{N}_r$, therefore $\bar{N}_2 \cap \cdots \cap \bar{N}_r \hookrightarrow \bar{M}/\bar{N}_1 = M/N_1$. But $M/N_1 = \bar{M}/\bar{N}_1$, so this gives an injection $\bar{N}_2 \cap \cdots \cap \bar{N}_r \hookrightarrow M/N_1$, but M/N_1 is P_1 -coprimary, so $\bar{N}_2 \cap \cdots \cap \bar{N}_r$ is also P_1 -coprimary, therefore $A/P_1 \hookrightarrow \bar{N}_2 \cap \cdots \cap \bar{N}_r \hookrightarrow \bar{M} = M/N$ by [Remark 1.2](#).

Now suppose $A/p \hookrightarrow M/N$, to show $p \in \{P_1, \dots, P_r\}$, it suffices to show $A/p \hookrightarrow M/N_i$ is injective for some $1 \leq i \leq r$. We have

$$\begin{array}{ccccc} & & \varphi_i & & \\ & \nearrow & & \searrow & \\ A/p & \xhookrightarrow{\varphi} & M/N = \bar{M} & \xrightarrow{\eta_i} & \bar{M}/\bar{N}_i = M/N_i \end{array}$$

and we want to show there exists some injective φ_i . Suppose not, then $\ker(\varphi_i) \neq 0$ in A/p for all $1 \leq i \leq r$. But A/p is an integral domain, therefore $\bigcap_{i=1}^r \ker(\varphi_i) \neq 0$. Therefore, we have

$$A/p \xhookrightarrow{\varphi} M/N \xrightarrow{(\eta_1, \dots, \eta_r)} \bigoplus_{i=1}^r M/N_i$$

Thus, the defined composition above is the injection $(\varphi_1, \dots, \varphi_r)$. This implies $\bigcap_{i=1}^r \ker(\varphi_r) = \ker(\varphi_1, \dots, \varphi_r) = 0$, a contradiction. Thus, there exists some injective φ_i , and therefore $p \in \{P_1, \dots, P_r\}$. ■

□

Definition 1.13 (Zero-divisor). Let A be Noetherian and M be a finitely-generated A -module. We say $0 \neq a \in A$ is a zero-divisor on M if there exists $0 \neq x \in M$ such that $ax = 0$. Otherwise, we say a is a non-zero-divisor on M .

Definition 1.14 (Essential prime ideal, Associated prime ideal). Given a primary decomposition $N = \bigcap_{i=1}^r N_i$, the corresponding prime ideals $\{P_1, \dots, P_r\}$ are called the essential prime ideals of N . In particular, if $N = (0)$, we say these are the associated prime ideals of M , denoted by $\operatorname{Ass}_A(M) = \{P_1, \dots, P_r\}$.

Corollary 1.15. Let A be Noetherian and M be a finitely-generated A -module, and let $\text{Ass}_A(M) = \{P_1, \dots, P_r\}$, then $\bigcup_{i=1}^r P_i$ is the set of all zero-divisors on M .

Proof. If $p \in \text{Ass}_A(M)$, then there exists an injection $A/p \hookrightarrow M$ mapping $\bar{1} \mapsto x$ by [Claim 1.12](#). Therefore, $px = 0$, so elements of p are zero-divisors of M . Let a be a zero-divisor on M , i.e., let $0 \neq x \in M$ be such that $ax = 0$. Take the primary decomposition $(0) = N_1 \cap \dots \cap N_r$ in M , where N_i is P_i -primary, then there exists i such that $x \notin N_i$. Since $\bar{x} \neq 0$ in M/N_i , then $a : M/N_i \rightarrow M/N_i$ is such that $a\bar{x} = 0$, so a is nilpotent on M/N_i . Therefore, M/N_i is P_i -coprimary, and by definition $a \in P_i$. \square

Exercise 1.16. Let $\text{Ass}_A(M) = \{P_1, \dots, P_r\}$, then the set of all nilpotent elements of M is $\bigcap_{i=1}^r P_i$.

Corollary 1.17. Suppose $N \subseteq M$ is a submodule, then

$$\text{Ass}_A(N) \subseteq \text{Ass}_A(M) \subseteq \text{Ass}_A(N) \cup \text{Ass}_A(M/N).$$

Proof. The first inclusion is obvious by $A/p \hookrightarrow N \hookrightarrow M$. We now show the second inclusion. Let $p \in \text{Ass}_A(M)$, and suppose $p \notin \text{Ass}_A(N)$, and we have an inclusion $i : A/p \rightarrow M$.

Claim 1.18. $i(A/p) \cap N = (0)$.

Subproof. Suppose not, then let $0 \neq x \in i(A/p) \cap N$, then $x \in N$ and $x \in i(A/p)$, but A/p is an integral domain and is p -coprimary, so $i(A/p) \cap N$ is p -coprimary. Therefore, we have

$$A/p \hookrightarrow i(A/p) \cap N \hookrightarrow N$$

and so $p \in \text{Ass}_A(N)$, a contradiction. \blacksquare

Therefore, we have the composition $A/p \rightarrow M \rightarrow M/N$ to be injection, thus $p \in \text{Ass}_A(M/N)$. \square

Corollary 1.19. Let M be finitely-generated, and let $I = \text{Ann}_A(M)$, then the essential prime ideals of I is contained in I .

Proof. Note that the essential prime ideals of I are just $\text{Ass}_A(A/I)$, so if we write $I = I_1 \cap \dots \cap I_r$ where I_i is a P_i -primary. Therefore, we have $A/I = \bar{I}_1 \cap \dots \cap \bar{I}_r$, where $\bar{I}_i = I_i/I$, and \bar{I}_i is P_i -primary.

Now let $M = \langle \alpha_1, \dots, \alpha_n \rangle$ be given by a set of generators, so $M = \{\sum a_i \alpha_i \mid a_i \in A\}$, now we look at the map

$$\begin{aligned} \varphi : A &\rightarrow \bigoplus_{i=1}^n M \\ 1 &\mapsto (\alpha_1, \dots, \alpha_n) \end{aligned}$$

then the kernel $\ker(\varphi) = I$, so $\bar{\varphi} : A/I \hookrightarrow \bigoplus_{i=1}^n M$ is an injection. By [Corollary 1.17](#), $\text{Ass}_A(M_1 \oplus M_2) = \text{Ass}_A(M_1) \cup \text{Ass}_A(M_2)$, hence we know

$$\text{Ass}(A/I) \subseteq \bigcup_{i=1}^n \text{Ass}_A(M) = \text{Ass}_A(M).$$

\square

Definition 1.20 (Support). The support of M over A , denoted $\text{Supp}_A(M)$, is the set $\{P \mid P \subseteq \text{prime ideal such that } P \supseteq I = \text{Ann}_A(M)\}$.

Theorem 1.21 (Prime Filtration). Let M be finitely-generated, then we have a descending chain

$$M = M_0 \supseteq M_1 \supseteq \dots \supseteq M_{n-1} \supseteq M_n = (0)$$

of prime ideals such that $M_i/M_{i+1} \cong A/P_{i+1}$, $0 \leq i \leq n-1$, where P_i 's are prime ideals of A , and $\text{Ass}_A(M) \subseteq \{P_1, \dots, P_n\}$.

Proof. Note that $P \in \text{Ass}_A(M)$ if and only if $i : A/P \hookrightarrow M$, therefore $i(A/P)$ satisfies the condition stated in the theorem. Therefore, take $\mathcal{A} = \{N \subseteq M \mid N \text{ satisfies the condition of the theorem}\}$. Since A is Noetherian, we take a maximal element T of \mathcal{A} .

Claim 1.22. $T = M$.

Subproof. Suppose, towards contradiction, that $T \neq M$, then we have a short exact sequence

$$0 \longrightarrow T \longrightarrow M \longrightarrow M/T \longrightarrow 0$$

such that $M/T \neq (0)$.

Exercise 1.23. Let L be a finitely-generated A -module, then $L = 0$ if and only if $\text{Ass}_A(L) = \emptyset$.

Let $q \in \text{Ass}_A(M/T)$, then we have

$$\begin{array}{ccccccc} & & & & A/q & & \\ & & & & \downarrow j & & \\ 0 & \longrightarrow & T & \longrightarrow & M & \xrightarrow{\eta} & M/T \longrightarrow 0 \end{array}$$

and take $W = \eta^{-1}(j(A/q))$, so we have a new short exact sequence

$$0 \longrightarrow T \longrightarrow W \longrightarrow j(A/q) \cong A/q \longrightarrow 0$$

Thus, $W \supsetneq T$ satisfies the condition in the theorem. By the maximality of T , we have a contradiction. ■

□

Remark 1.24. Let A be Noetherian and $\mathfrak{m} \subseteq A$ be a maximal ideal, then for any ideal $I \subseteq A$ such that there exists n with $\mathfrak{m}^n \subseteq I \subseteq \mathfrak{m}$, then I is \mathfrak{m} -primary.

Proof. Consider the map

$$A/I \xrightarrow{\cdot x^n} A/I$$

for $x \in \mathfrak{m}$, then this is the zero map. Therefore, multiplication by x is nilpotent. Now suppose $x \notin \mathfrak{m}$, then we want to show that $A/I \xrightarrow{\cdot x} A/I$ is injective. Indeed, since $x \notin \mathfrak{m}$, then $\mathfrak{m} + Ax = A$, hence we have that $y + ax = 1$ for some $y \in \mathfrak{m}$ and $a \in A$, so $(y + ax)^n = 1$, $y^n + \mu x = 1$, but that means the map $A/I \rightarrow A/I$ is given by multiplication by μx , so $\bar{\mu}\bar{x} = \bar{1}$ since y vanishes. That is, \bar{x} is invertible over A/I , hence multiplication by x is an isomorphism. □

Exercise 1.25. Let A be a ring and S be a multiplicatively closed subset of A , and let M be an A -module, then $S^{-1}M$ is an $S^{-1}A$ -module. Let $T \subseteq S^{-1}M$ be an $S^{-1}A$ -submodule, then there exists $N \subseteq M$ such that $T = S^{-1}N$.

Remark 1.26. Localization functor is fully faithful.

Remark 1.27. Let A be Noetherian and S be a multiplicatively closed subset of A .

1. Let M be P -coprimary, then

- if $S \cap P = \emptyset$, then $S^{-1}M$ is $S^{-1}P$ -coprimary;
- if $S \cap P \neq \emptyset$, then $S^{-1}M = 0$.

Proof. Indeed, suppose $S \cap P \neq \emptyset$, let $a : M \rightarrow M$ be the multiplication map by a , so $a \in P$ gives $a^n M = 0$ for some n , and if $a \notin P$, then this is injective. Let $\frac{a}{s} : S^{-1}M \rightarrow S^{-1}M$ be the multiplication map, but $\frac{a}{s}$ is a unit, so multiplication by s or $\frac{1}{s}$ is an isomorphism, hence we can take this to be $\frac{a}{1}$ with $s = 1$. If $s \in P$, then $s^n : M \rightarrow M$ is the zero map, therefore $s^n : S^{-1}M \rightarrow S^{-1}M$ is also the zero map, so s is a unit. This only happens if $S^{-1}M = 0$. □

2. Let N be P -primary, then

- if $S \cap P = \emptyset$, then $S^{-1}N$ is $S^{-1}P$ -primary in $S^{-1}M$;
- if $S \cap P \neq \emptyset$, then $S^{-1}N = S^{-1}M$.

Remark 1.28. Consider the localization $S^{-1}M$. Take a submodule T of $S^{-1}M$, then by [Exercise 1.25](#), $T = S^{-1}N$ for some $N \subseteq M$. There is now a primary decomposition on N given by $N = N_1 \cap \cdots \cap N_t$ where N_i is P_i -primary.

Exercise 1.29. Let $W_1, W_2 \subseteq M$, then $S^{-1}(W_1 \cap W_2) = S^{-1}(W_1) \cap S^{-1}(W_2)$ in $S^{-1}M$.

Remark 1.30. This is true whenever we have a flat ring extension.

Therefore, we have

$$\begin{aligned} T &= S^{-1}N \\ &= S^{-1}N_1 \cap \cdots \cap S^{-1}N_t \\ &= S^{-1}N_{i_1} \cap \cdots \cap S^{-1}N_{i_r}, \end{aligned}$$

where $S^{-1}N_{i_j}$ is $S^{-1}P_{i_j}$ -primary, and P_{i_1}, \dots, P_{i_r} are prime ideals for which $S \cap P_j = \emptyset$, where $P_j \in \{P_1, \dots, P_t\}$.

Exercise 1.31. Let N be P -primary in M .

- if $S \cap P = \emptyset$, then $i_M : M \rightarrow S^{-1}M$ and $i_N : N \rightarrow S^{-1}N$ gives $i_M^{-1}(S^{-1}N) = N$;
- if $S \cap P \neq \emptyset$, then $i_M^{-1}(S^{-1}N) = i_M^{-1}(S^{-1}M) = M$.

Corollary 1.32. Consider a primary decomposition $N = N_1 \cap \cdots \cap N_t$ where N_i is P_i -primary. Suppose we have a different primary decomposition $N = N'_1 \cap \cdots \cap N'_t$ where N'_i is also P_i -primary. Suppose P_1 is a minimal element in $\{P_1, \dots, P_t\}$, then $N_1 = N'_1$.

Proof. Let $S = A \setminus P_1$, then $S^{-1}N = S^{-1}N_1 = S^{-1}N'_1$. Now consider $i_M : M \rightarrow S^{-1}M$, this descends to $N_1 \rightarrow S^{-1}N_1 = S^{-1}N'_1$ and $N'_1 \rightarrow S^{-1}N'_1$, so $i_M^{-1}(S^{-1}N_1 = S^{-1}N'_1) = N_1 = N'_1$. \square

Consider flat ring maps (as a ring extension) like $A \rightarrow A[x]$ and $A \rightarrow A[x_1, \dots, x_n]$ since as A -modules they are free, since we have a basis $\{x_1^{i_1}, \dots, x_n^{i_n}\}$.

Lemma 1.33. Let $A \rightarrow B$ be a flat map, and let M be an A -module. Let N_1 and N_2 be A -submodules of M , then $(N_1 \otimes_A B) \cap (N_2 \otimes_A B) = (N_1 \cap N_2) \otimes_A B$.

Proof. Consider the chain complex

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N_1 \cap N_2 & \longrightarrow & N_1 & \longrightarrow & N_1/(N_1 \cap N_2) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N_2 & \longrightarrow & M & \longrightarrow & M/N_2 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N_2/(N_1 \cap N_2) & \longrightarrow & M/N_1 & \longrightarrow & M/(N_1 + N_2) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

with exact rows and columns. We tensor this complex by $-\otimes_A B$, then since B is flat we obtain a new chain complex

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & (N_1 \cap N_2) \otimes_A B & \longrightarrow & N_1 \otimes_A B & \longrightarrow & (N/(N_1 \cap N_2)) \otimes_A B \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & N_2 \otimes_A B & \longrightarrow & M \otimes_A B & \longrightarrow & M/N_2 \otimes_A B \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & N_2/(N_1 \cap N_2) \otimes_A B & \longrightarrow & M/N_1 \otimes_A B & \longrightarrow & (M/(N_1 + N_2)) \otimes_A B \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Via diagram chasing, if $x \in (N_1 \otimes_A B) \cap (N_2 \otimes_A B)$, then $x \in (N_1 \cap N_2) \otimes_A B$. \square

Corollary 1.34. Suppose we have a primary decomposition $N = N_1 \cap \cdots \cap N_t$ in M , let $A \rightarrow A[x]$, then $N[x] = N_1[x] \cap \cdots \cap N_t[x]$ in $M[x]$ where $N_i[x] = N_i \otimes_A A[x]$.

Proof. We want to show that if N_i is P_i -primary, then $N_i[x]$ is $P_i[x]$ -primary. Take a short exact sequence

$$0 \longrightarrow P \longrightarrow A \longrightarrow A/P \longrightarrow 0$$

then we tensor it by $-\otimes_A A[x]$, then we obtain a new short exact sequence

$$0 \longrightarrow P \otimes_A A[x] \longrightarrow A[x] \longrightarrow A/P \otimes_A A[x] \longrightarrow 0$$

(Note that we are working over the commutative case, so the left tensor and the right tensor are canonically isomorphic.) We have $B \otimes_A A[x] = B[x]$, now we have $A[x] \otimes_A A/P = A[x]/PA[x] = (A/P)[x]$ which is a domain, so $PA[x]$ is a prime ideal. It now suffices to show that if M is P -coprimary, then $M[x]$ is $P[x]$ -coprimary. This simplifies to showing that:

- if $f(x) \in P[x]$, then the multiplication map $M[x] \xrightarrow{f(x)} M[x]$ is nilpotent;
- if $f(x) \notin P[x]$, $M[x] \xrightarrow{f(x)} M[x]$ is an injection.

Note that $M[x] = \sum_{i \geq 0} m_i x^i$ for some m_i 's. Since $P[x]$ is a prime ideal, then $A[x]/P[x] \cong A/P[x]$. If $f(x) \in P[x]$, we have $f(X) = p_0 + p_1 x + \cdots + p_t x^t$ for p_i 's in P . Consider the multiplication map via $[f(x)]^p : M[x] \rightarrow M[x]$, where $n = n_0 + n_1 + \cdots + n_t$ such that $p_i^{n_i} M = 0$ by the binomial theorem. Now suppose $f(x) \notin P[x]$, then let us write $f(x) = a_0 + a_1 x + \cdots + a_t x^t$, and we have two cases:

- if no a_i 's are in P , then for all i , multiplication by a_i on M is an injection. If we multiply $f(x)$ by $m_0 + m_1 x + \cdots$, then the constant term would be $a_0 m_0$, and for each term to be zero, we must have $f(x)$ equivalent to zero, hence that means multiplication by $f(x)$ on $M[x]$ would be injective as well.
- Now suppose there exists some a_i that is contained in P . We can write down $f(x) = u + v$ where u has coefficients in P and v does not have any coefficients in P . If possible, let $f(\alpha) = 0$ for $\alpha \in M[x]$, then we have $u\alpha = -v\alpha$, and so $u^2\alpha = v^2\alpha$ since $u^2\alpha = u(-v\alpha) = v(-u\alpha) = v^2\alpha$, and by induction we have $u^n\alpha = (-1)^n v^n\alpha$. Therefore, for large enough n such that $u^n\alpha = 0$, we know $v^n\alpha = 0$, and therefore we have a contradiction since v does not contain any coefficients in P . \square

Remark 1.35. Remark 1.24 would fail if P is not a maximal ideal: P^2 may not be P -primary in this case.

Let R be a Noetherian ring, we let $i_P : R \rightarrow R_P$ be the localization away from P , from R to the local ring with maximal ideal PR_P , then we have $(PR_P)^n = P^n R_P$ to be PR_P -primary. Therefore, this gives a mapping from P^n to $P^n R_P = (PR_P)^n$. We now denote $P^{(n)} := i_P^{-1}(P^n R_P)$ to be the n th symbolic power of P , then $P^{(n)}$ is P -primary. (Indeed, we note that P is disjoint from $R \setminus P$, so given $M \rightarrow S^{-1}M$ pulling $S^{-1}P$ -primary module $S^{-1}N$ back to M gives a P -primary module.) In particular, $P^{(n)} \supseteq P^n$.

Exercise 1.36. 1. • Let R be Noetherian and M be finitely-generated. Show that $\ell_R(M) < \infty$ if and only if $\text{Ass}_R(M)$ consists of maximal ideals only.

- If $\ell_A(M) < \infty$, then M is a direct sum of coprimary submodules of M .

2. Now let R be a Noetherian ring and P be a prime ideal. Prove that the following are equivalent:

- (i) P is an essential prime ideal of some submodule N of M .
- (ii) $M_P \neq 0$.
- (iii) $P \supseteq \text{Ann}_R(M)$.
- (iv) P contains some $Q \in \text{Ass}(M)$.

3. Let $R = k[x, y, z]$ for some field k , and let $P = (xz - y^2, x^3 - yz, z^2 - x^2y)$.

- Prove that P is a prime ideal of R .
- Is P^2 P -primary?

Hint: consider

$$\begin{aligned} \varphi : k[x, y, z] &\rightarrow k[t] \\ x &\mapsto t^3 \\ y &\mapsto t^4 \\ z &\mapsto t^5 \end{aligned}$$

and show that $\ker(\varphi) = P$.

1.2 INFINITELY-GENERATED CASE

Now let R be a Noetherian ring, and M is not finitely-generated.

Definition 1.37 (Coprimary). M is called coprimary if for any $a \in R$, we have multiplication map $a : M \rightarrow M$ to be either injective, or locally nilpotent, i.e., for all $x \in M$, there exists n_x such that $a^{n_x}x = 0$.

Therefore, any submodule of M is coprimary. Now we define the associated primes to be $\text{Ass}_R(M)$ to be the set of prime ideals in R such that there exists an injection $A/p \hookrightarrow M$, i.e., R/p is a cyclic submodule of M .

Theorem 1.38. Let R and M be as above. For any $P \in \text{Ass}_R(M)$, there exists a P -primary submodule $N(P)$ of M such that $(0) = \bigcap_{P \in \text{Ass}_R(M)} N(P)$, which may be infinite.

Example 1.39. Let A and B be Noetherian rings and M be a finitely-generated A -module, and we say have a ring homomorphism $\varphi : B \rightarrow A$. Via the pullback over φ , we make M into a B -module, but M may not be finitely-generated as a B -module. For instance, take $A = \mathbb{Z}$ and $B = \mathbb{Z}[x]$.

Exercise 1.40. Let $\varphi : B \rightarrow A$ be a homomorphism of Noetherian rings. If M is a finitely-generated A -module, then via the pullback of φ , M is a B -module. We write it as ${}_{\varphi}M$. Prove that $\text{Ass}_A({}_{\varphi}M) = \varphi^{-1}(\text{Ass}_A(M))$.

2 FILTERED RINGS AND MODULES, COMPLETIONS

Definition 2.1 (Topological Ring). Let R be a ring with addition φ and multiplication ψ . Suppose R has a topology such that φ and ψ are continuous, then we say R is a topological ring with respect to the given topology. That is, the topology respects the algebraic structure.

Similarly, we can define a topological group with respect to multiplication and inverse, and a topological module with respect to addition and scalar multiplication.

Remark 2.2. A topological ring R (respectively, topological group G , topological module M) is Hausdorff if and only if (0) is closed in R (respectively, (e) is closed in G , (0) is closed in M).

Let M be a topological module, consider

$$\begin{aligned}\varphi : M \times M &\rightarrow M \\ (x, y) &\mapsto x - y\end{aligned}$$

then the diagonal is given by $\varphi^{-1}(0) = \{(x, x) \mid x \in M\} = \Delta_M$. Now suppose (0) is closed, which gives Δ_M to be closed, hence M is Hausdorff.

Definition 2.3 (Pseudo-metric Space). We say (X, d) is a pseudo-metric space if we have a function $d : X \times X \rightarrow \mathbb{R}^{\geq 0}$ such that

1. $d(x, y) + d(y, z) \geq d(x, z)$,
2. $d(x, y) = d(y, x)$,
3. $d(x, x) = 0$.

This becomes a metric space if $d(x, y) = 0$ if and only if $x = y$.

Remark 2.4. A pseudo-metric space is a Hausdorff if and only if it is a metric space.

Definition 2.5 (Completion). Let (X, d) be a (pseudo-)metric space, then the completion (\hat{X}, \hat{d}) of (X, d) is a complete (all Cauchy sequences converge) metric space \hat{X} with a metric \hat{d} with a map $\varphi : X \rightarrow \hat{X}$ such that

1. φ respects both d and \hat{d} ,
2. $\varphi(X)$ is dense in \hat{X} , and
3. We have

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \hat{X} \\ & \searrow \psi & \swarrow \theta \\ & Y & \end{array}$$

that is, given any complete metric space Y and a continuous map $\psi : X \rightarrow Y$, there exists a unique map $\theta : \hat{X} \rightarrow Y$ such that the diagram commutes.

Remark 2.6. If $W \subseteq X$, then $\hat{W} \cong \overline{\varphi(W)}$.

Definition 2.7 (Directed Set). Let (I, \leq) be a poset, then I is called a directed set if for all pairs of $\alpha, \beta \in I$, there exists $\gamma \in I$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

Definition 2.8 (Inverse Limit). We say $\{X_\alpha\}_{\alpha \in I}$ is an inverse family indexed by I if for all $\alpha \leq \beta$, there exists maps $\varphi_{\alpha, \beta} : X_\beta \rightarrow X_\alpha$ such that for all $\alpha \leq \beta \leq \gamma$, we have a commutative diagram

$$\begin{array}{ccc} X_\gamma & \xrightarrow{\varphi_{\alpha\gamma}} & X_\alpha \\ & \searrow \varphi_{\beta\gamma} & \swarrow \varphi_{\alpha\beta} \\ & X_\beta & \end{array}$$

An inverse limit of $\{X_\alpha\}_{\alpha \in I}$ is an object X with maps $\varphi_\alpha : X \rightarrow X_\alpha$ for all $\alpha \in I$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi_\alpha} & X_\alpha \\ & \searrow \varphi_\beta & \nearrow \varphi_{\alpha\beta} \\ & X_\beta & \end{array}$$

commutes for all $\alpha, \beta \in I$, and for all Y such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\psi_\alpha} & X_\alpha \\ & \searrow \psi_\beta & \nearrow \varphi_{\alpha\beta} \\ & X_\beta & \end{array}$$

commutes for all $\alpha, \beta \in I$, then there exists $f : Y \rightarrow X$ such that

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ & \searrow \psi_\alpha & \nearrow \varphi_\alpha \\ & X_\beta & \end{array}$$

commutes for all α .

Remark 2.9. To construct such inverse limits, we take $\tilde{X} = \prod_{\alpha \in I} X_\alpha$, then we have an embedding $X \hookrightarrow \tilde{X}$ where

$$X = \left\{ \prod_{\alpha \in I} X_\alpha \mid \forall \alpha \leq \beta, \varphi(X_\beta) = X_\alpha \right\}.$$

3 DIMENSION THEORY

4 INTEGRAL EXTENSIONS

5 NOETHER'S NORMALIZATION LEMMA

6 HOMOLOGICAL ALGEBRA