Motivic Cohomology Notes

Jiantong Liu

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These notes were taken from a course on Motivic Cohomology taught by Dr. N. Yang in Spring 2024 at BIMSA. Any mistakes and inaccuracies would be my own. References for this course include [MVW06] and [Ros96], and others mentioned in the references.

Contents

0 Introduction

Let $X \in \mathrm{Sm}/k$ be a smooth separated scheme over a field k. The study of motivic cohomology started with the hope that

Conjecture 0.1 (Beilinson and Lichtenbaum, 1982-1987)**.** There exists some complexes $\mathbb{Z}(n)$ for $n \in \mathbb{N}$ of sheaves in Zariski topology on Sm/k such that

1. $\mathbb{Z}(0)$ is (quasi-isomorphic to) the constant sheaf $\mathbb{Z}$, i.e., the complex
$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$
concentrated at degree 0;
2. $\mathbb{Z}(1)$ is the complex $\mathcal{O}^*[-1]$, i.e., the complex
$$\cdots \longrightarrow 0 \longrightarrow \mathcal{O}^* \longrightarrow 0 \longrightarrow \cdots$$
concentrated at degree 1;
3. for every field F/k, the hypercohomology over Zariski topology satisfies[1]
$$\mathbb{H}^n_{\mathrm{Zar}}(F, \mathbb{Z}(n)) = H^n(\mathbb{Z}(n)(\mathrm{Spec}(F))) = K_n^M(F),$$
where $K_n^M(F)$ is the nth Milnor K-theory of a field F, given by the quotient of the tensor algebra $T(F^*)/\{x \otimes (1-x) : x \in F^*\}$ over $\mathbb{Z}$, c.f., [MVW06], Theorem 5.1;

   Example 0.2.

   a. $K_0^M(F) = K_0(F) = \mathbb{Z}$;
   b. $K_1^M(F) = K_1(F) = F^\times$;
   c. $K_2^M(F) = K_2(F)$.
4. $\mathbb{H}^{2n}_{\mathrm{Zar}}(X, \mathbb{Z}(n)) = \mathrm{CH}^n(X)$, c.f., [MVW06], Corollary 19.2, where the nth classical Chow group $\mathrm{CH}^n(X)$ is the free group given by
$$\mathrm{CH}^n(X) = \mathbb{Z}\{\text{cycles of codimension } n\}/\text{rational equivalences};$$
5. there is a natural Atiyah–Hirzebruch spectral sequence
$$E_2^{p,q} = \mathbb{H}^p_{\mathrm{Zar}}(X, \mathbb{Z}(q)) \Rightarrow K_{2q-p}(X).$$
Moreover, tensoring with $\mathbb{Q}$, the spectral sequence degenerates and one has
$$\mathbb{H}^i_{\mathrm{Zar}}(X, \mathbb{Z}(n))_{\mathbb{Q}} = \mathrm{gr}^n_\gamma(K_{2n-i}(X)_{\mathbb{Q}})$$
where gr^n_γ's are the quotients (graded pieces) of γ-filtration. ([Lev94]; [Lev99], Theorem 11.7)

Remark 0.3. Such choice of complexes $\mathbb{Z}(q)$ exists, and is called the motivic complex. For a clear definition of these complexes, see Definition 3.1 of [MVW06]. Moreover, by convention $\mathbb{Z}(q) = 0$ for $q < 0$.

Definition 0.4. The motivic cohomology of X is defined by $H^{p,q}(X, \mathbb{Z}) = \mathbb{H}^p_{\mathrm{Zar}}(X, \mathbb{Z}(q))$, the hypercohomology of the motivic complexes with respect to the Zariski topology.[2]

[1] Here we use the convention that the (hyper)cohomology of F should be interpreted as of $\mathrm{Spec}(F)$, the corresponding space.

[2] This is not exactly correct as illustrated in the notes. The original definition of hypercohomology is with respect to Nisnevich topology, c.f., Definition 2.44, but one can show that it is the same as taking Zariski topology, c.f., Corollary 7.13.

Remark 0.5. In general, a motivic cohomology with coefficient in an abelian group A is a family of contravariant functors $H^{p,q}(-,A) : \mathrm{Sm}/k \to \mathrm{Ab}$.

Remark 0.6. The motivic cohomology of X satisfies the cancellation property: set $\mathbb{G}_m = \mathbb{A}^1 \backslash \{0\}$, then

$$H^{p,q}(X \times \mathbb{G}_m, \mathbb{Z}) = H^{p,q}(X, \mathbb{Z}) \oplus H^{p-1,q-1}(X, \mathbb{Z}).$$

Remark 0.7. It turns out that the group remains unchanged if we replace the Zariski topology by Nisnevich topology.[3] If one uses étale topology instead, we retrieve Lichtenbaum motivic cohomology $H_L^{p,q}(X,\mathbb{Z})$. If $\mathrm{char}(k) \nmid n$, it admits the comparison

$$H_L^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) = H_{\text{étale}}(X, \mathbb{Z}/n\mathbb{Z}(q)),$$

where $\mathbb{Z}/n\mathbb{Z}(q)$ is the q-twist $\mu_n^{\otimes q}$.

We may compare Lichtenbaum motivic cohomology with motivic cohomology by the following theorem, formerly known as Beilinson-Lichtenbaum Conjecture[4]:

Theorem 0.8 ([Voe11])**.** The natural map

$$H^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) \to H_L^{p,q}(X, \mathbb{Z}/n\mathbb{Z})$$

is an isomorphism if $p \leqslant q$, is a monomorphism if $p = q+1$, and gives a spectral sequence for any pair of p, q.

Corollary 0.9. For $p \leqslant q$, we have

$$H^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) = H^p_{\text{étale}}(X, \mathbb{Z}/n\mathbb{Z}(q)).$$

In particular, for $X = \mathrm{Spec}(k)$ as a point, this is the theorem formerly known as Milnor conjecture:

Corollary 0.10 ([Voe97], [Voe03a], [Voe03b])**.**

- $H^{p,p}(k, \mathbb{Z}/n\mathbb{Z}) = K_p^M(k)/n = H^p_{\text{étale}}(X, \mathbb{Z}/n\mathbb{Z}(p))$ as the Galois cohomology;
- in general,

$$H^{p,q}(k, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} 0, & p > q \\ H^{p,p}(k, \mathbb{Z}/n\mathbb{Z}) \cdot \tau^{q-p}, & p \leqslant q \end{cases}$$

  where $\tau \in \mu_n(k) = H^{0,1}(k, \mathbb{Z})$ is a primitive nth root of unity.

Remark 0.11. Unlike the case with finite coefficients, $H^{p,q}(k,\mathbb{Z})$ is quite hard to compute for small $p < q$; for $p \geqslant q$, this is 0.

A current long-standing conjecture is

Conjecture 0.12 (Beilinson-Soulé Vanishing Conjecture, [Lev93])**.** $H^{p,q}(k, \mathbb{Z}) = 0$ if $p < 0$.

Remark 0.13. Here are a few known cases:

- for $\mathrm{char}(k) = 0$, this is known for number fields ([Bor74]), function fields of genus 0 ([Dég08]), curves over number fields, and their inductive limits; ([DG05])

[3] Recall that the Nisnevich topology is a Grothendieck topology on the category of schemes that is finer than the Zariski topology but coarser than the étale topology.

[4] This is also known as the norm residue isomorphism theorem, or (formerly) Bloch-Kato conjecture.

- for $\mathrm{char}(k) > 0$, this is known for finite fields ([Qui72]) and global fields ([Har77]).

Remark 0.14. The motivic cohomology could be realized in a tensor triangulated category, namely the (triangulated, derived) category of effective motives $\mathrm{DM}(k)$. For any pair of p, q, we can find an Eilenberg-Maclane space and a corresponding representable functor so that

$$H^{p,q}(X, \mathbb{Z}) = \mathrm{Hom}_{DM}(\mathbb{Z}(X), \mathbb{Z}(q)[p])$$

where $\mathbb{Z}(X)$ is the motive of X and $\mathbb{Z}(q)[p] = \mathbb{G}_m^{\wedge q}[p-q]$.[5]

Remark 0.15. Dually, we can define the motivic homology by

$$H_{p,q}(X, \mathbb{Z}) = \mathrm{Hom}_{DM}(\mathbb{Z}(q)[p], \mathbb{Z}(X)).$$

Remark 0.16 ([MVW06] Properties 14.5, page 110)**.** By taking the hom functor from the aspect of motives, we can derive theorems for all (co)homologies which can be represented in DM. The main derives are the following:

1. If $E \to X$ is an $\mathbb{A}^n$-bundle, then motives $\mathbb{Z}(E) = \mathbb{Z}(X)$ in DM.
2. If $\{U, V\}$ is a Zariski open covering of X, we have a Mayer-Vietoris sequence

$$\mathbb{Z}(U \cap V) \longrightarrow \mathbb{Z}(U) \oplus \mathbb{Z}(V) \longrightarrow \mathbb{Z}(X) \longrightarrow \mathbb{Z}(U \cap V)[1]$$

   in the form of a distinguished triangle in DM.
3. If $Y \subseteq X$ is a closed embedding of codimension c in Sm/k, then we have a Gysin triangle

$$\mathbb{Z}(X \backslash Y) \longrightarrow \mathbb{Z}(X) \longrightarrow \mathbb{Z}(Y)(c)[2c] \longrightarrow \mathbb{Z}(X \backslash Y)[1]$$

   which is a distinguished triangle where $\mathbb{Z}(Y)(c)[2c] := \mathbb{Z}(Y) \otimes \mathbb{Z}(c)[2c]$.
4. For any vector bundle of rank n on X, we have the projective bundle formula

$$\mathbb{Z}(\mathbb{P}(E)) = \bigoplus_{i=0}^{n} \mathbb{Z}(X)(i)[2i]$$

   which defines the Chern class of E.
5. Let X be a proper smooth scheme and let d_X be its dimension, then $\mathbb{Z}(X)$ has a strong dual $\mathbb{Z}(X)(-d_X)[-2d_X]$ in DM by stabilization. This gives a Poincaré duality[6]

$$H^{p,q}(X, \mathbb{Z}) \cong H_{2d_X - p, d_X - q}(X, \mathbb{Z}).$$

[5] Again, this notation goes back to the concise definition of the motivic complexes: see Lecture 3 from [MVW06] as well as the concept of presheaves with transfers.

[6] We can use cohomology with compact support for this.

1 Intersection Theory

1.1 Cycles of Scheme

Definition 1.1. Let X be a scheme of finite type over k. We define the ith cycle on the scheme X to be a free abelian group

$$Z_i(X) = \bigoplus_{\substack{\text{irreducible closed } c \subseteq X \\ \text{with } \dim(c)=i}} \mathbb{Z} \cdot c$$

and set $Z(X) = \bigoplus_i Z_i(X)$. Define a set $K_i(X)$ to be the set of coherent sheaves $\mathcal{F}$ on X with $\dim(\operatorname{supp}(F)) \leqslant i$.[7]

Remark 1.2. Let $(A, \mathfrak{m})$ be a Noetherian local ring and M be an A-module, then by the dimension theorem, we know $\dim(M) = d(M) = \dim(\operatorname{supp}(M))$, where $d(M)$ is the degree of the Hilbert-Samuel polynomial $P_{\mathfrak{m}}(M, n)$.

Definition 1.3. Let $X \in \mathrm{Sm}/k$ and let $U, V \subseteq X$ be irreducible and closed. Suppose $W \subseteq U \cap V$ is a irreducible and closed component. If $\dim(W) = \dim(U) + \dim(V) - \dim(X)$, i.e., $\operatorname{codim}(W) = \operatorname{codim}(U) + \operatorname{codim}(V)$, we say that U and V intersect properly at W.

Remark 1.4. This condition is weaker than saying they intersect transversely: we do not require information about tangent spaces.

Theorem 1.5. Let $A \supseteq k$ be a Noetherian regular ring, M, N be finitely-generated A-modules, and suppose $\ell(M \otimes_A N) < \infty$, then

1. $\ell(\operatorname{Tor}_i^A(M, N)) < \infty$ for all $i \geqslant 0$;
2. the Euler-Poincaré characteristic $\chi(M, N) := \sum_{i=0}^{\dim(A)} (-1)^i \ell(\operatorname{Tor}_i^A(M, N)) \geqslant 0$;
3. by Remark 1.2, we have $\dim(M) + \dim(N) \leqslant \dim(A)$;
4. in particular, we have $\dim(M) + \dim(N) < \dim(A)$ if and only if $\chi(M, N) = 0$.

Proof. See [Ser12], page 106. □

Remark 1.6. Part 3. from Theorem 1.5 implies that $\dim(W) \geqslant \dim(U) + \dim(V) - \dim(X)$, i.e., $\operatorname{codim}(W) \leqslant \operatorname{codim}(U) + \operatorname{codim}(V)$ in the notation of Definition 1.3.

Definition 1.7. Let X, U, V, W be as in Definition 1.3, then we define the intersection multiplicity $m_W(U, V)$ of U and V at W by

$$m_W(U, V) = \chi^{\mathcal{O}_{X,W}}(\mathcal{O}_{X,W}/P_U, \mathcal{O}_{X,W}/P_V)$$

where P_U and P_V are prime ideals defining U and V, respectively.

Remark 1.8. By Theorem 1.5, we know $m_W(U, V) \geqslant 0$, and $m_W(U, V) = 0$ if and only if U and V do not intersect properly at W.

[7] Despite the notation, this has nothing to do with a K-theory.

1.2 Intersection Product and Cross Product

Definition 1.9. Let $X \in \mathrm{Sm}/k$, and let $U \in Z_a(X)$ and $V \in Z_b(X)$. If U and V intersect properly at every component, then we define the intersection product to be the cycle

$$U \cdot V = \sum_{\substack{W \subseteq U \cap V \\ \dim(W) = a+b-d_X}} m_W(U,V) \cdot W \in Z_{a+b-d_X}(X).$$

Example 1.10. Let X be a smooth projective surface, and let C and D be divisors on X. For any point $x \in C \cap D$, locally we think of $C = \{f = 0\}$ and $D = \{g = 0\}$ around x, then $m_x(C,D) = \ell_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(f,g))$.

Definition 1.11. Suppose X is a scheme of finite type over k, and $\mathcal{F} \in K_n(X)$ is a coherent sheaf, then we define $Z_a(\mathcal{F}) = \sum_{\dim(\bar{\eta})=a} (\mathcal{O}_{X,\eta}(\mathcal{F}_\eta) \cdot \bar{\eta}) \in Z_a(X)$.

Therefore, we define the cycle of F as an element of the cycle of X.

Definition 1.12 ([Har13], Exercise III.6.9)**.** Every coherent sheaf $\mathcal{F}$ on $X \in \mathrm{Sm}/k$ has a resolution

$$0 \longrightarrow E_k \longrightarrow E_{k-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

where E_i's are locally free of finite rank. Therefore, for any coherent sheaf $\mathcal{G}$, we can define the Tor functor[8] of coherent sheaves by

$$\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) = H_i(E_* \otimes_{\mathcal{O}_X} \mathcal{G}).$$

Proposition 1.13. Let $X \in \mathrm{Sm}/k$. Suppose $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(X)$ intersect properly, then

$$Z_a(\mathcal{F}) \cdot Z_b(\mathcal{G}) = \sum_{i=0}^{d_X} (-1)^i \cdot Z_{a+b-d_X}(\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F},\mathcal{G})).$$

Proof. We only have to do it locally, so we can assume X to be affine, and count the coefficients of $\bar{\xi}$ where $\dim(\xi) = a+b-d_X$. It suffices to show that the stalks at ξ satisfies

$$\chi(F_\xi, G_\xi) = \sum_{\substack{\dim(\bar{\lambda})=a \\ \dim(\beta\eta)=b \\ \xi \in \bar{\lambda} \cap \bar{\eta}}} \ell(\mathcal{F}_\lambda) \cdot \ell(G_\eta) \cdot m_{\bar{\xi}}(\bar{\lambda},\bar{\eta}).$$

Because our ring is Noetherian, then $\mathcal{F}$ admits a filtration

$$0 = M_0 \subseteq \cdots \subseteq M_d = \mathcal{F}$$

such that $M_i/M_{i-1} \cong \mathcal{O}_X/\mathcal{I}$ is coherent for prime ideal $\mathcal{I}$. By the additivity of both sides of the isomorphism, we may assume $\mathcal{F} = \mathcal{O}_X/\mathfrak{p}$ with dimension at most a, where $\mathfrak{p} \sim \lambda \in X$. Similarly, we may assume $\mathcal{G} = \mathcal{O}_X/\mathfrak{q}$ with dimension at most b, where $\mathfrak{q} \sim \eta \in X$. Moreover, set $\xi \in \bar{\lambda} \cap \bar{\eta}$. By definition, we now have $\chi(\mathcal{F}_\xi, \mathcal{G}_\xi) = m_{\bar{\xi}}(\bar{\lambda},\bar{\eta})$.

- If $\dim(\bar{\lambda}) = a$ and $\dim(\bar{\eta}) = b$, then the equality follows from the fact that $\ell(\mathcal{F}_\lambda) = \ell(\mathcal{G}_\eta) = 1$.
- If not, then either $\dim(\bar{\lambda}) < a$ or $\dim(\bar{\eta}) < b$, then $\bar{\lambda}$ and $\bar{\eta}$ do not intersect properly at $\bar{\xi}$, so both the left-hand side and the right-hand side become 0.

□

[8] Since we are working over sheaves of $\mathcal{O}_X$-modules, using the same argument on the level of modules shows that the Tor functor is independent from the choice of resolution.

Proposition 1.14. The intersection product is commutative.

Proof. This is obvious since the Tor functor is commutative. □

Proposition 1.15. The intersection product is associative.

Proof. Suppose we pick $\mathcal{F} \in K_a(X)$, $\mathcal{G} \in K_b(X)$, and $\mathcal{H} \in K_c(X)$ with support dimension at most a, b, c, respectively, and they intersect properly. Let L_* and M_* be free resolutions of $\mathcal{F}$ and $\mathcal{H}$, respectively. Define a double complex $N_{ij} = L_i \otimes \mathcal{G} \otimes M_j$, then the associativity of tensor product allows us to calculate triple Tor

$$H_i(L_i \otimes H_j(\mathcal{G}) \otimes M_j)) \cong \mathrm{Tor}_i(\mathcal{F}, \mathcal{G}, \mathcal{H}) \cong H_i(H_j(L_i \otimes \mathcal{G}) \otimes M_j)$$

as the homology of two (tensor) double complexes. We obtain two spectral sequences

$$\begin{aligned} {}^I E^2_{p,q} &= \mathrm{Tor}_p(\mathcal{F}, \mathrm{Tor}_q(\mathcal{G}, \mathcal{H})) \Rightarrow \mathrm{Tor}_{p+q}(\mathcal{F}, \mathcal{G}, \mathcal{H}) \\ {}^{II} E^2_{p,q} &= \mathrm{Tor}_p(\mathrm{Tor}_q(\mathcal{F}, \mathcal{G}), \mathcal{H}) \Rightarrow \mathrm{Tor}_{p+q}(\mathcal{F}, \mathcal{G}, \mathcal{H}). \end{aligned}$$

Recall Euler-Poincaré characteristic is invariant with respect to taking spectral sequence $(*)$, then

$$\begin{aligned} Z_a(\mathcal{F}) \cdot ((Z_b\mathcal{G}) \cdot Z_c(\mathcal{H})) &= Z_a(\mathcal{F}) \cdot \sum_q (-1)^q Z_{b+c-d_X}(\mathrm{Tor}_q(\mathcal{G}, \mathcal{H})) \text{ by Proposition 1.13} \\ &= \sum_{p,q} (-1)^{p+q} Z_{a+b+c-2d_X}({}^I E^2_{p,q}) \text{ by Proposition 1.13} \\ &= \sum_i (-1)^i Z_{a+b+c-2d_X}(\mathrm{Tor}_i(\mathcal{F}, \mathcal{G}, \mathcal{H})) \text{ by } (*) \\ &= \sum_{p,q} (-1)^{p+q} Z_{a+b+c-2d_X}({}^{II} E^2_{p,q}) \text{ by } (*) \\ &= \sum_p Z_{a+b-d_X}(\mathrm{Tor}_p(\mathcal{F}, \mathcal{G})) \cdot Z_c(\mathcal{H}) \text{ by Proposition 1.13} \\ &= (Z_a(\mathcal{F}) \cdot Z_b(\mathcal{G})) \cdot Z_c(\mathcal{H}) \text{ by Proposition 1.13.} \end{aligned}$$

□

Definition 1.16. Suppose $X_1, X_2 \in \mathrm{Sm}/k$, with $\mathcal{F}_1 \in K_{a_1}(X_1)$ and $\mathcal{F}_2 \in K_{a_2}(X_2)$. We define the cross product of cycles to be

$$Z_{a_1}(\mathcal{F}_1) \times Z_{a_2}(\mathcal{F}_2) = Z_{a_1+d_{X_2}}(p_1^*\mathcal{F}_1) \cdot Z_{a_2+d_{X_1}}(p_2^*\mathcal{F}_2),$$

where $p_i : X_1 \times X_2 \to X_i$ is the projection for $i = 1, 2$.

Exercise 1.17. One should check that this is well-defined.

Remark 1.18. Suppose $X_1, X_2 \in \mathrm{Sm}/k$, with $\mathcal{F}_1 \in K_{a_1}(X_1)$, $\mathcal{F}_2 \in K_{b_1}(X_1)$, $\mathcal{G}_1 \in K_{a_2}(X_2)$ and $\mathcal{G}_2 \in K_{a_2}(X_2)$. Suppose $Z_{a_1}(\mathcal{F}_1) \cdot Z_{a_2}(\mathcal{G}_1)$ and $Z_{b_1}(\mathcal{F}_2) \cdot Z_{b_2}(\mathcal{G}_2)$ are defined, then

- $Z_{a_1}(\mathcal{F}_1) \times Z_{a_2}(\mathcal{G}_1)$ and $Z_{b_1}(\mathcal{F}_2) \times Z_{b_2}(\mathcal{G}_2)$ intersect properly on $X_1 \times X_2$, and
- $(Z_{a_1}(\mathcal{F}_1) \times Z_{a_2}(\mathcal{G}_1)) \cdot (Z_{b_1}(\mathcal{F}_2) \times Z_{b_2}(\mathcal{G}_2)) = (Z_{a_1}(\mathcal{F}_1) \cdot Z_{b_1}(\mathcal{F}_2)) \times (Z_{a_2}(\mathcal{G}_1) \cdot Z_{b_2}(\mathcal{G}_2))$.

1.3 Pushout and Pullback

Definition 1.19. Suppose X, Y are schemes of finite type over k, and let $f : X \to Y$ be a proper map. For every irreducible closed subset $c \subseteq X$ of dimension a, we define the direct image to be

$$f_*c = \begin{cases} [k(c) : k(f(c))] \cdot f(c) \in Z_a(Y), & \dim(f(c)) = a \\ 0, & \dim(f(c)) < a \end{cases}$$

to be the direct image of c under f.

Lemma 1.20. Suppose X and Y are schemes of finite type over k of the same dimension n, and that $f : X \to Y$ is proper, then there exists an open subset $U \subseteq Y$ such that $\dim(Y \backslash U) < n$ and $f : f^{-1}(U) \to U$ is a finite morphism.

Proof. Suppose $\xi \in Y$ has $\dim(\bar{\xi}) = n$. We can find $U \ni \xi$ such that $f|_U$ has finite fibers by Exercise II.3.7 from [Har13]. By Exercise III.11.2 in [Har13], such f is finite. □

Proposition 1.21. Let $f : X \to Y$ be a proper morphism between schemes over k of finite type, and let $\mathcal{F} \in K_a(X)$, then

1. $f_*\mathcal{F} \in K_a(Y)$ and the right derived $R^i f_*\mathcal{F} \in K_{a-1}(Y)$ for $i > 0$.
2. $f_*Z_a(\mathcal{F}) = Z_a(f_*\mathcal{F})$.

Proof.

1. By Theorem III.8.8 from [Har13], $R^i f_*\mathcal{F}$ is coherent for all $i \geqslant 0$. We have $\operatorname{supp}(R_i f_*\mathcal{F}) \subseteq \operatorname{supp}(\mathcal{F})$. If f is finite, then f_* is exact, so $R^i f_*\mathcal{F} = 0$ for $i > 0$. For general cases, we may assume $\dim(f(\operatorname{supp}(\mathcal{F}))) = a$ and set $W = \operatorname{supp}(\mathcal{F})$. We have a commutative diagram

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, "i"'] & f(W) \arrow[d, "j"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where h is also proper. By Lemma 1.20, there exists $V \subseteq f(W)$ such that $\dim(f(W) \backslash V) < a$ and $h|_V$ is finite. Let $\mathcal{J}$ be the ideal sheaf of W, then $\mathcal{J}^s\mathcal{F}/\mathcal{J}^{s+1}\mathcal{F} = i_*i^*\mathcal{J}^s\mathcal{F}/\mathcal{J}^{s+1}\mathcal{F}$. By the long exact sequence, it suffices to prove the case for $\mathcal{F} = i_*\mathcal{G}$. Then

$$(R^k f_*)i_*\mathcal{G} = R^k(fi)_*\mathcal{G} = j_*R^k h_*\mathcal{G}.$$

It suffices to consider h, but

$$(R^k h_*\mathcal{G})V = R^k h(\mathcal{G}|_{f^{-1}(V)}) = 0$$

for $k > 0$, so $\operatorname{supp}(R^k h_*\mathcal{G}) \subseteq f(W) \backslash V$ if $k > 0$.

2. If f is finite, let us write down the coefficients of ξ of dimension a on both sides, namely

$$\ell((f_*\mathcal{F})_\xi) = \sum_{\substack{\eta \in f^{-1}(\xi) \\ \dim(\bar{\eta}) = a}} \ell(F_\eta) \cdot [k(\bar{\eta}) : k(\overline{f(\eta)})].$$

By additivity, one reduces to the case when X is affine and $F = \mathcal{O}_X/\mathfrak{p}$. For the general case, use Lemma 1.20, and the case where f is finite.

□

Definition 1.22. Suppose $f : X \to Y$ where $Y \in \mathrm{Sm}/k$ and X is closed in $Z \in \mathrm{Sm}/k$. Define $j : X \to Z \times Y$ to be the graph map. For any $C \in Z_a(X)$ and $D \in Z_b(Y)$ such that C and $f^{-1}(D)$ intersect properly, define the intersection cycle to be

$$C \cdot_f D = j_*^{-1}(j(C) \cdot (Z \times D)) \in Z_{a+b-d_Y}(X)$$

In particular, $f^*(D) = X \cdot_f D$ for $C = X$.

Proposition 1.23. Using the notation above, for $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(Y)$, if $\mathcal{F}$ and $f^*\mathcal{G}$ intersect properly, we have

$$Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G}) = \sum_{i=0}^{d_Y} (-1)^i Z_{a+b-d_Y}(L_i(\mathcal{F} \otimes f^*)\mathcal{G})$$

Proof. Denote $p_2 : Z \times Y \to Y$ to be the projection onto the second coordinate. By linearity, $Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G}) = j_*^{-1}(Z_a(j_*\mathcal{F}) \cdot Z_{b+d_Z}(p_2^*\mathcal{G})$ for $j : X \to Z \times Y$. Suppose $L_* \to \mathcal{G}$ is the locally free resolution of $\mathcal{G}$. Note that for all $i \geqslant 0$, we have

$$j^*(j_*\mathcal{F} \otimes p_2^* L_i) + F \otimes f^* L_i,$$

which induces an isomorphism

$$j_*\mathcal{F} \otimes p_2^* L_i = j_*(\mathcal{F} \otimes f^* L_i).$$

Hence $\mathrm{Tor}_i^{\mathcal{O}_{Z \times Y}}(j_*\mathcal{F}, p_2^*\mathcal{G}) = j_* L_i(F \otimes f^*)\mathcal{G}$. So

$$j_*^{-1} Z_{a+b-d_Y}(\mathrm{Tor}_i^{\mathcal{O}_{Z \times Y}}(j_*\mathcal{F}, p_2^*\mathcal{G})) = Z_{a+b-\dim(Y)}(L_i(F \otimes f^*)\mathcal{G}).$$

Therefore the statement follows. $\square$

Proposition 1.24. Let $X \in \mathrm{Sm}/k$, $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(X)$ such that $\mathcal{F}$ and $\mathcal{G}$ intersect properly. Let $\Delta : X \to X \times X$ be the diagonal map, then

$$\Delta^*(Z_a(\mathcal{F}) \times Z_b(\mathcal{G})) = Z_a(\mathcal{F}) \cdot Z_b(\mathcal{G}).$$

Proof. See page 115 of [Ser12]. $\square$

Proposition 1.25. f^* is compatible with intersection product, and $f^*g^* = (gf)^*$.

Proof. See page 119 of [Ser12]. $\square$

Lemma 1.26. Let $\mathcal{A}$ be an abelian category with enough projectives (respectively, injectives) and F be a right (respectively, left) exact functor from $\mathcal{A}$. Suppose C is chain complex in $\mathcal{A}$, then there exists a double complex $M_{*,*}$ in $\mathcal{A}$ such that

$${}^I E^2_{p,q} = L_p F H_q(C) \quad \text{(respectively, } R^{-p} F(H_q(C))\text{)}.$$

Proof. To do this when F is right exact, use the Cartan-Eilenberg resolution[9] $C_* \to C$ and consider the double complex FC_*. $\square$

Proposition 1.27. Suppose $f : X \to Y$ is in Sm/k, suppose $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(Y)$, then

$$Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G}) = Z_a(\mathcal{F}) \cdot f^* Z_b(\mathcal{G})$$

if both sides are defined.

[9] See Proposition 11 on page 210 of [GM13].

Proof. We may assume X is affine. Let $L_* \to \mathcal{G}$ be a free resolution and apply Lemma 1.26 to f^*L_* and $F \otimes -$, then we find a double complex such that

$$\begin{aligned} {}^I E^2_{p,q} &= \operatorname{Tor}_p(\mathcal{F}, L_q f^*\mathcal{G}) \\ {}^{II} E^2_{p,q} &= L_p(F \otimes f^*)\mathcal{G}. \end{aligned}$$

□

Proposition 1.28. Let $X \subseteq Z$ and $Y, Z \in \mathrm{Sm}/k$ and $f : X \to Y$ be proper. Suppose $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(Y)$, and suppose $\mathcal{F}$ and $f^*\mathcal{G}$ intersect properly, then

$$f_*(Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G})) = (f_* Z_a(\mathcal{F})) \cdot Z_b(\mathcal{G}).$$

Proof. Pick $L_* \to \mathcal{G}$ to be a resolution and apply Lemma 1.26 to $F \otimes f^*L_*$ and f_*, then we have a double complex $M_{*,*}$ such that

$${}^I E^2_{p,q} = R^{-p} f_* L_q(F \otimes f^*)\mathcal{G}).$$

On the other hand, $H_q(M_{*,n}) = R^{-q} f_*(F \otimes f^* L_n) = (R^{-q} f_* \mathcal{F}) \otimes L_n.$, therefore

$${}^{II} E^2_{p,q} = \operatorname{Tor}_p(R^{-q} f_* \mathcal{F}, \mathcal{G}).$$

□

Corollary 1.29. Under the same hypothesis as Proposition 1.28, we have

$$f_*(Z_a(\mathcal{F}) \cdot f^*(Z_b(\mathcal{G}))) = f_*(Z_a(\mathcal{F})) \cdot Z_b(\mathcal{G}).$$

2 Sheaves with Transfers

We fix a base scheme $S \in \mathrm{Sm}/k$.

2.1 Algebra of Correspondences

Definition 2.1. Let $X, Y \in \mathrm{Sm}/S$, then we define the group of finite correspondences

$$\mathrm{Cor}_S(X, Y) = \mathbb{Z}\{\text{irreducible closed } C \subseteq X \times_S Y \mid C \to X \text{ finite}, \dim(C) = \dim(X)\}$$

to be the free abelian group generated by elementary correspondences from X to Y.

Example 2.2. For any $f : X \to Y$, the graph $\Gamma_f = (x, f(x)) \subseteq X \times_S Y$ is a finite correspondence from $X \to Y$.

Example 2.3. If $f : X \to Y$ is finite and $\dim(X) = \dim(Y)$, then the graph Γ_f is also a finite correspondence from $Y \to X$.

Definition 2.4. Define an additive category Cor_S whose objects are the same as Sm/S, and the hom sets defined as $\mathrm{Hom}_{\mathrm{Sm}/S}(X, Y) = \mathrm{Cor}_S(X, Y)$ as in Definition 2.1. The contravariant additive functors

$$F : \mathrm{Cor}_S^{\mathrm{op}} \to \mathrm{Ab}$$

are called presheaves with transfers. The corresponding category is denoted by $\mathrm{PSh}(S) = \mathrm{PSh}(\mathrm{Cor}_S)$, which is abelian with enough injectives and projectives. We have a functor $\gamma : \mathrm{Sm}/S \to \mathrm{Cor}_S$ by Example 2.3.

Remark 2.5. For any additive F and $X, Y \in \mathrm{Sm}/S$, there is a pairing

$$\mathrm{Cor}_S(X, Y) \otimes F(Y) \to F(X).$$

Restricting to Sm/S over Cor_S, we note that F is a presheaf of abelian groups over Sm/S with transfer map $F(Y) \to F(X)$ indexed by finite correspondences from X to Y.

Example 2.6. Every $X \in \mathrm{Sm}/S$ gives an element $\mathbb{Z}(X) \in \mathrm{PSh}(S)$ defined by $\mathbb{Z}(X)(Y) = \mathrm{Cor}_S(Y, X)$. Therefore, we say $\mathbb{Z}(X)$ is the presheaf with transfers represented by X. By Yoneda Lemma we know there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{PST}}(\mathbb{Z}(X), F) \cong F(X).$$

Moreover, representable functors give embeddings of Sm/S and Cor_S into $\mathrm{PSh}(S)$ via

$$\begin{array}{ccccc} \mathrm{Sm}/S & \xrightarrow{\gamma} & \mathrm{Cor}_S & \longrightarrow & \mathrm{PSh}(S) \\ X & \longmapsto & X & \longmapsto & \mathbb{Z}(X) \end{array}$$

In particular, $\mathbb{Z}(S) = \mathbb{Z}$.

Example 2.7. The presheaves $\mathcal{O}$ and $\mathcal{O}^*$ are in $\mathrm{PSh}(S)$. For any $C \in \mathrm{Cor}_S(X, Y)$ and $f \in \mathcal{O}(Y)$ (respectively, $\mathcal{O}^*(Y)$), we have a diagram

$$\begin{tikzcd} C \arrow[r, "i"] & X \times_S Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \\ & X & \end{tikzcd}$$

and can define $\mathcal{O}(C)(f) = \mathrm{Tr}_{C/X}((p_2 \circ i)^*(f))$ (respectively, $\mathcal{O}^*(C)(f) = \mathrm{N}_{C/X}((p_2 \circ i)^*(f))$).

We study the properties of finite correspondence through Chapter 16.1 in [Ful13].

Definition 2.8. Let us describe the composition in Cor_S. Suppose $f \in \mathrm{Cor}_S(X,Y)$ and $g \in \mathrm{Cor}_S(Y,Z)$, then from the diagram

$$\begin{tikzcd} X \times_S Z & \\ X \times_S Y \times_S Z \arrow[u, "p_{13}"] \arrow[r, "p_{23}"] \arrow[d, "p_{12}"] & Y \times_S Z \\ X \times_S Y & \end{tikzcd}$$

we define the composition $g \circ f = p_{13*}(p_{23}^*(g)p_{12}^*(f))$.

Exercise 2.9. One should check that all intersections are proper.

Remark 2.10. Using this language, given a correspondence $\alpha \in \mathrm{Cor}_S(X,Y)$, we can define pullbacks and pushouts on the cycles as homomorphisms

$$\begin{aligned} \alpha_* : Z(X) &\to Z(Y) \\ x &\mapsto p_{Y*}^{XY}(\alpha \cdot p_X^{XY*}(x)) \end{aligned}$$

and

$$\begin{aligned} \alpha^* : Z(Y) &\to Z(X) \\ y &\mapsto p_{X*}^{XY}(\alpha \cdot p_Y^{XY*}(y)) \end{aligned}$$

Remark 2.11 ([Ful13], Proposition 1.7, Base-change Formula)**.** Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

be a fiber square where f is proper and g is flat, then f' is proper and g' is flat, and that $f'_*g'^* = g^*f_*$ over Y'.

Proposition 2.12 ([Ful13], Proposition 16.1.1)**.** The composition law is associative.

Proof. Suppose

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

are morphisms in Cor_S, then we have two Cartesian squares

$$\begin{tikzcd} X \times_S Y \times_S Z \times_S W \arrow[r] \arrow[d] & X \times_S Z \times_S W \arrow[d] \\ X \times_S Y \times_S Z \arrow[r] & X \times_S Z \end{tikzcd}$$

and

$$\begin{tikzcd} X \times_S Y \times_S Z \times_S W \arrow[r] \arrow[d] & X \times_S Y \times_S W \arrow[d] \\ Y \times_S Z \times_S W \arrow[r] & Y \times_S W \end{tikzcd}$$

Now using the base-change formula, we know

$$h \circ (g \circ f) = p_{XW*}^{XZW}(p_{ZW}^{XZW*}(h)p_{XZ}^{XZW*}p_{XZ*}^{XYZ}(p_{YZ}^{XYZ*}(g)p_{XY}^{XYZ*}(f)))$$

$$
\begin{aligned}
&= p^{XZW}_{XW*}(p^{XZW*}_{ZW}(h)p^{XYZW}_{XZW*}p^{XYZW*}_{XYZ}(p^{XYZ*}_{YZ}(g)p^{XYZ*}_{XY}(f)))\\
&= p^{XZW}_{XW*}(p^{XZW*}_{ZW}(h)p^{XYZW}_{XZW*}(p^{XYZW*}_{YZ}(g)p^{XYZW*}_{XY}(f)))\\
&= p^{XZY}_{XW*}p^{XYZW}_{XZW*}(p^{XYZW*}_{ZW}(h)p^{XYZW*}_{YZ}(g)p^{XYZW}_{XY}(f))\\
&= p^{XYW}_{XW*}p^{XYZW}_{XYW*}(p^{XYZW*}_{ZW}(h)p^{XYZW*}_{YZ}(g)p^{XYZW}_{XY}(f))\\
&= p^{XYW}_{XW*}(p^{XYZW}_{XYW*}(p^{XYZW*}_{ZW}(h)p^{XYZW*}_{YZ}(g))p^{XYW*}_{XY}(f))\\
&= p^{XYW}_{XW*}(p^{XYZW}_{XYW*}p^{XYZW*}_{YZW}(p^{YZW*}_{ZW}(h)p^{YZW}_{YZ}(g))p^{XYW*}_{XY}(f))\\
&= p^{XYW}_{XW*}(p^{XYW*}_{YW}p^{YZW}_{YW*}(p^{YZW*}_{ZW}(h)p^{YZW}_{YZ}(g))p^{XYW*}_{XY}(f))\\
&= p^{XYW}_{XW*}(p^{XYW*}_{YW}p^{YZW}_{YW*}(p^{YZW*}_{ZW}(h)p^{YZW}_{YZ}(g))p^{XYW*}_{XY}(f))\\
&= h \circ g \circ f.
\end{aligned}
$$

□

Theorem 2.13. We have $\mathcal{O}(g \circ f) = \mathcal{O}(f) \circ \mathcal{O}(g)$ and $\mathcal{O}^*(g \circ f) = \mathcal{O}^*(f) \circ \mathcal{O}^*(g)$.

Proof. We sketch the proof for $\mathcal{O}$. Pick $X \in \mathrm{Sm}/k$. For every $a \in \mathbb{N}$, define $\mu_a(x) = \bigoplus_{\dim(\bar{V})=a} K(V)$. Therefore, we have a pairing

$$
\begin{aligned}
\mathcal{O}(X) \times Z_a(X) &\to \mu_a(X)\\
(s, V) &\mapsto s|_V
\end{aligned}
$$

by restricting the regular function on the closed subset. For any map $f : X \to Y$ where X contains irreducible and closed C, suppose C is finite over Y and $s \in K(C)$, then we define $f_*(s) = \mathrm{Tr}_{K(C)/K(f(C))}(s)$.[10] Therefore, for any finite correspondence $C \in \mathrm{Cor}(Y, X)$ and $s \in \mathcal{O}(X)$, we have

$$
\begin{tikzcd}
C \arrow[r, hook] & X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y\\
& X &
\end{tikzcd}
$$

and thus $\mathcal{O}(C)(s) = p_{2*}(p_1^*(s)|_C)$.

Now suppose we have closed subsets $C \subseteq X$ and $D \subseteq Y$, with

$$
\begin{tikzcd}
X \arrow[r, "f"] & Y\\
C \arrow[u] \arrow[ur, "\text{finite}"'] &
\end{tikzcd}
$$

and that C and $f^{-1}(D)$ intersect properly, then one can show that

$$f_*(s|_C)|_D = f_*(s|_{C \cdot_f D})$$

by Tor formula. Moreover, for diagrams like

$$
\begin{tikzcd}
X \times_S Y \times_S Z \arrow[r, "p_{23}"] \arrow[d, "p_{12}"'] & YZ \arrow[d, "p_1"]\\
X \times_S Y \arrow[r, "p_2"'] & Y
\end{tikzcd}
$$

where $C \subseteq YZ$ and C is finite over Y, then one can show that for all $s \in \mathcal{O}(Y \times_S Z)$ and C finite over Y, we have

$$p_2^* p_{1*}(s|_C) = p_{12*}(p_{23}^*(s)|_{p_{23}^*(C)}).$$

We finish the proof by working with formal calculation. □

[10] Here $f(C)$ is closed since f is finite.

Remark 2.14 ([Ful13], Proposition 16.1.2)**.** For $\alpha \in \mathrm{Cor}_S(X,Y)$ and $\beta \in \mathrm{Cor}_S(Y,Z)$, we have

$$(\beta \circ \alpha)_* = \beta_* \circ \alpha_*$$

and

$$(\beta \circ \alpha)^* = \alpha^* \circ \beta^*.$$

2.2 Operations on Presheaves with Transfers

Definition 2.15. Suppose $\mathcal{F}_1, \mathcal{F}_2, \mathcal{G} \in \mathrm{PSh}(S)$ be presheaves with transfers. A bilinear function $\varphi : \mathcal{F}_1 \times \mathcal{F}_2 \to \mathcal{G}$ is a collection of bilinear maps

$$\varphi_{x_1,x_2} : \mathcal{F}_1(x_1) \times \mathcal{F}_2(x_2) \to \mathcal{G}(x_1 \times_S x_2)$$

for every $x_1, x_2 \in \mathrm{Sm}/S$ any any morphisms $f_i \in \mathrm{Cor}_S(x_i, x_i')$ for $i = 1, 2$, such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{F}_1(x_1') \times \mathcal{F}_2(x_2) \arrow[r, "\varphi_{x_1',x_2}"] \arrow[d, "\mathcal{F}_1(f_1)\times \mathrm{id}"'] & \mathcal{G}(x_1' \times_S x_2) \arrow[d, "(f_1 \times \mathrm{id})"] \\
\mathcal{F}_1(x_1) \times \mathcal{F}_2(x_2) \arrow[r, "\varphi_{x_1,x_2}"'] & \mathcal{G}(x_1 \times_S x_2)
\end{tikzcd}$$

for f_1 and similarly there is a diagram that commutes for f_2.

Definition 2.16. Define the tensor product $\mathcal{F}_1 \otimes \mathcal{F}_2$ to be the presheaf such that for every $\mathcal{G}$, the hom set $\mathrm{Hom}(\mathcal{F}_1 \otimes \mathcal{F}_2, G)$ is the same as the collection of bilinear functions $\mathcal{F}_1 \times \mathcal{F}_2 \to \mathcal{G}$.

Proposition 2.17. The tensor product $\mathcal{F}_1 \otimes \mathcal{F}_2$ exists.

Proof. For every $Z \in \mathrm{Sm}/S$, define

$$(\mathcal{F}_1 \otimes \mathcal{F}_2)(Z) = \bigoplus_{X,Y \in \mathrm{Sm}/S} \mathcal{F}_1(X) \otimes_{\mathbb{Z}} \mathcal{F}_2(Y) \otimes_{\mathbb{Z}} \mathrm{Cor}_S(Z, X \times_S Y)/\sim$$

where $\sim$ is the subgroup generated by the relations $\varphi \otimes \psi(f \times \mathrm{id}_Y) \circ h = f^*(\varphi) \otimes \psi \otimes h$ where $f \in \mathrm{Cor}_S(X', X)$, $\varphi \in \mathcal{F}_1(X)$, $\psi \in \mathcal{F}_2(Y)$, $h \in \mathrm{Cor}_S(Z, X' \times Y)$, and the relations $\varphi \otimes \psi \otimes (\mathrm{id}_X \times g) \circ h = \varphi \otimes g^*(\psi) \otimes h$ where $g \in \mathrm{Cor}_S(Y', Y)$, $\varphi \in \mathcal{F}_1(X)$, $\psi \in \mathcal{F}_2(Y)$, $h \in \mathrm{Cor}_S(Z, X \times Y')$. ☐

Definition 2.18. A pointed presheaf $(\mathcal{F}, x)$ is a split injective map given by the constant presheaf $x : \mathbb{Z} \to \mathcal{F}$ for some $\mathcal{F} \in \mathrm{PSh}(S)$. We set $\mathcal{F}^{\wedge 1} = \mathcal{F}/x$. For any two pointed presheaves $(\mathcal{F}_1, x_1)$ and $(\mathcal{F}_2, x_2)$, define $\mathcal{F}_1 \wedge \mathcal{F}_2 = (\mathcal{F}_1 \otimes \mathcal{F}_2)/((\mathcal{F}_1 \otimes x_2) \oplus (x_1 \otimes \mathcal{F}_2))$. This allows us to define $\mathcal{F}^{\wedge n}$ inductively as a cokernel, c.f., Definition 2.12 from [MVW06].

Proposition 2.19.

- $\mathbb{Z}(X) \otimes \mathbb{Z}(Y) = \mathbb{Z}(X \times Y)$;
- $\mathcal{F}^{\wedge 1} \otimes \mathcal{G}^{\wedge 1} = \mathcal{F} \wedge \mathcal{G}$.

Definition 2.20. For any $\mathcal{F} \in \mathrm{PSh}(S)$ and $X \in \mathrm{Sm}/S$, define $\mathcal{F}^X \in \mathrm{PSh}(S)$ by $\mathcal{F}^X(Y) = \mathcal{F}(X \times_S Y)$. For any $\mathcal{F}, \mathcal{G} \in \mathrm{PSh}(S)$, define the internal hom $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}) \in \mathrm{PSh}(S)$ by $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})(X) = \mathrm{Hom}(\mathcal{F}, \mathcal{G}^X)$.

Proposition 2.21. We have a tensor-hom adjunction

$$\mathrm{Hom}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \cong \mathrm{Hom}(\mathcal{F}, \underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{H})).$$

2.3 Nisnevich Topology

Let us give a brief introduction to Nisnevich topology, c.f., section 3 and 4 from Chapter I of [Mil80].

Definition 2.22. Suppose $f : Y \to X$ is a morphism between schemes that are locally of finite type.

1. It is called unramified if for all $y \in Y$, the maximal ideals satisfy $\mathfrak{m}_{f(y)}\mathcal{O}_{Y,y} = \mathfrak{m}_y$, and $k(y)/k(f(y))$ is a finite separable field extension of function fields.

2. It is called étale if it is both flat and unramified.

3. It is called Nisnevich if for all $x \in X$, there is some $y \in Y$ such that $f(y) = x$, $k(y) = k(x)$, and f is étale.

Definition 2.23. A morphism $f : Y \to X$ is called a Nisnevich covering if f is Nisnevich and surjective.

Definition 2.24. Suppose $\mathcal{F} \in \mathrm{PSh}(S)$. We say that it is a Nisnevich sheaf with transfers if for any $X \in \mathrm{Sm}/S$ and Nisnevich covering $\pi : Y \to X$, the sequences

$$0 \longrightarrow \mathcal{F}(X) \xrightarrow{\pi^*} \mathcal{F}(Y) \xrightarrow{p_1^* - p_2^*} \mathcal{F}(Y \times_X Y) \xrightarrow{p_1} Y$$

$$0 \longrightarrow \mathcal{F}(X) \xrightarrow{\pi^*} \mathcal{F}(Y) \xrightarrow{p_1^* - p_2^*} \mathcal{F}(Y \times_X Y) \xrightarrow{p_2} Y$$

are exact. The category of Nisnevich sheaves with transfers is denoted by $\mathrm{Sh}(S)$.

Definition 2.25. A local ring is called Henselian if for any monic polynomial $f \in A[t]$ such that its image $\bar{f}$ in the residue field satisfies $\bar{f} = g_0 h_0$ in $k(A)[T]$ where g_0, h_0 are monic and relatively prime, there are monic $g, h \in A[T]$ such that $\bar{g} = g_0$, $\bar{h} = h_0$ in the residue fields, and $f = gh$.

Example 2.26. Complete local rings are Henselian.

Theorem 2.27 ([Mil80], Theorem I.4.2)**.** Let A be a local ring, $X = \mathrm{Spec}(A)$, and $x \in X$ be the closed point, then the following are equivalent:

1. A is Henselian;

2. any finite A-algebra B is a direct product of local rings $B \cong \prod_{i \in I} B_i$, where each B_i is of the form $B_{\mathfrak{m}_i}$ for some maximal ideal $\mathfrak{m}_i$ of B;

3. if $f : Y \to X$ has finite fibers and is separated, then $Y = \coprod_{i=0}^{n} Y_i$ where $X \notin f(Y_0)$, and for $i \geqslant 1$, Y_i is finite over X and is the spectrum of a local ring;

4. if $f : Y \to X$ is étale and there exists $y \in Y$ such that $f(y) = x$ and $k(y) = k(x)$, then f has a section $s : X \to Y$ such that $f \circ s = \mathrm{id}_X$.

Now let A be a Noetherian ring and $\mathfrak{p} \in \mathrm{Spec}(A)$. Consider the set I whose elements are pairs $(B, \mathfrak{q})$, where B is a connected étale A-algebra, $\mathfrak{q} \in \mathrm{Spec}(B)$, $\mathfrak{q} \cap A = \mathfrak{p}$, i.e., $\mathfrak{q}$ lies over $\mathfrak{p}$, and $k(\mathfrak{p}) = k(\mathfrak{q})$. We say that $(B_1, \mathfrak{q}_1) \preceq (B_2, \mathfrak{q}_2)$ if there is an A-morphism $f : B_1 \to B_2$ such that $f^{-1}(\mathfrak{q}_2) = \mathfrak{q}_1$. This gives a poset structure.

Proposition 2.28. The set I is a directed set and the ring $\varinjlim_{(B,\mathfrak{q})} B = A_{\mathfrak{p}}^h$, i.e., the Henselization of $A_{\mathfrak{p}}$, is Henselian and admits the following universal property: for any Henselian A-algebra C such that $\mathfrak{m}_C \cap A = \mathfrak{p}$, there is a unique morphism $\varphi : A_{\mathfrak{p}}^h \to C$ (as a local homomorphism) such that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \\ A_{\mathfrak{p}}^h \arrow[ur, dashed, "\exists!\varphi"'] & \end{tikzcd}$$

Proof. This makes use of Lemma I.4.8 from [Mil80]. □

Let φ_X be the smallest Nisnevich site on X. Suppose X is Noetherian, pick $x \in X$, and $\mathcal{F} \in \mathrm{PSh}(\varphi_X)$. We write $\mathcal{F}_x = \mathcal{F}(\mathcal{O}^h_{X,x}) = \varinjlim_{(V,u)} \mathcal{F}(V)$ as the stalk of F at x, taking all the pairs (V, u) with étale morphism

$$\begin{aligned} V &\to X \\ u &\mapsto x \end{aligned}$$

with $k(u) = k(x)$.

Proposition 2.29. Let

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

be a complex in $\mathrm{Sh}(\varphi_X)$. The following are equivalent:

1. the complex is exact;
2. for every $x \in X$, the complex

$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{H}_x \longrightarrow 0$$

is exact.

Proof. This mimics the idea in the usual sheaf theory with Zariski topology. To do so, we need to construct a sheafification in the sense of Nisnevich, explained as follows: suppose $\mathcal{F} \in \mathrm{PSh}(\varphi_X)$, define $\mathcal{F}^+$ as the following: for every Nisnevich covering $\{V_i\}$ of U, define

$$\mathcal{F}(U) = \{(s_i) \in \prod_i \mathcal{F}(V_i) : s_i|_{V_i \times_X V_j} = s_j|_{V_i \times_S V_j}\}.$$

Now let $\mathcal{F}^+(U) = \varinjlim_{V \supseteq U} \mathcal{F}(V)$, then $\mathcal{F}^{++}$ is a Nisnevich sheaf with the same stalks as $\mathcal{F}$, with a map $\mathcal{F} \to \mathcal{F}^{++}$.

If the complex is exact, then the sequence of stalks is also exact because the direct limit functor is exact. Conversely, if we have an exact sequence of stalks, then we prove that the given sequence is exact using the usual proof in the Zariski case. □

For any Noetherian scheme X with $\dim(X) < \infty$, we define the cochain to be

$$C^p(X) = \{Y \subseteq X \mid \mathrm{codim}(Y) \geqslant p\} = \bigoplus_{\substack{y \in X \\ \mathrm{codim}(\bar{y}) \geqslant p}} \mathbb{Z} \cdot \bar{y}.$$

Fir $\mathcal{F} \in \mathrm{Sh}(\varphi_X)$. For closed subschemes $Z \subseteq W$ of X where $Z \in C^{p+1}(X)$ and $W \in C^p(X)$, we have a long exact sequence

$$\cdots \longrightarrow H^i_Z(X, \mathcal{F}) \longrightarrow H^i_W(X, \mathcal{F}) \longrightarrow H^i_{W \backslash Z}(X \backslash Z, \mathcal{F}) \longrightarrow H^{i+1}_Z(X, \mathcal{F}) \longrightarrow \cdots$$

with supports specified as subscripts, using the exactness of

$$0 \longrightarrow \mathcal{F}_Z(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}(X \backslash Z)$$

and defining $H^i_Z = R^i\Gamma_Z(X, -) : D(X_{\text{étale}}) \to D(\mathrm{Ab})$ as the right exact functor, where

$$\Gamma_Z(X, \mathcal{F}) = \{s \in \mathcal{F}(X) \mid \mathrm{supp}(s) \subseteq Z\}$$

for closed subscheme $Z \subseteq X$. Now define $H^i(C^p(X), \mathcal{F}) = \varinjlim_{Z \in C^p(X)} H^i_Z(X, \mathcal{F})$, then

$$H^i(C^p(X)/C^{p+1}(X), \mathcal{F}) = \varinjlim_{\substack{Z \subseteq W \\ W \in C^p, Z \in C^{p+1}}} H^i_{W \backslash Z}(X \backslash Z, \mathcal{F}).$$

Taking limit with respect to pairs $Z \subseteq W$ where $W \in C^p(X)$ and $Z \in C^{p+1}(X)$, we get a long exact sequence

$$\cdots \longrightarrow H^i(C^{p+1}(X), \mathcal{F}) \longrightarrow H^i(C^p(X), \mathcal{F}) \longrightarrow H^i(C^p(X)/C^{p+1}(X), \mathcal{F}) \longrightarrow H^{i+1}(C^{p+1}(X), \mathcal{F}) \longrightarrow \cdots$$

Set the pth filtration to be $F^p H^i(X, \mathcal{F}) = \operatorname{im}(H^i(C^p(X), \mathcal{F}) \to H^i(X, \mathcal{F}))$, then we obtain the Coniveau spectral sequence

$$E_1^{p,q} = H^{p+q}(C^p(X)/C^{p+1}(X), \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

Remark 2.30. $E_1^{p,q} = 0$ if $p > \dim(X)$ and $q > 0$.

Definition 2.31. Suppose $x \in X$. Define the local cohomology

$$H^i_x(X, \mathcal{F}) = \varinjlim_{\text{open } x \in V \subseteq X} H^i_{\bar{x} \cap V}(V, \mathcal{F}).$$

This allows us to calculate $E_1^{p,q}$ as

$$E_1^{p,q} = \bigoplus_{\operatorname{codim}(\bar{x}) = p} H^{p+q}_x(X, \mathcal{F}).$$

Proposition 2.32 (Étale Excision)**.** Suppose $\varphi : Y \to X$ is a étale morphism of sheaves, and suppose $Z \subseteq X$ is a closed subset such that $\varphi^{-1}(Z) = Z$. For any $\mathcal{F} \in \operatorname{Sh}(\varphi_X)$, we have

$$H^i_Z(Y, \varphi^* \mathcal{F}) = H^i_Z(X, \mathcal{F}).$$

Proof. The morphism

$$Y \coprod (X \backslash Z) \to X$$

is a Nisnevich covering, but by the (Nisnevich) sheaf condition on $\mathcal{F}$, we have a Cartesian square

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(Y) \\ \downarrow & & \downarrow \\ \mathcal{F}(X \backslash Z) & \longrightarrow & \mathcal{F}(Y \backslash Z) \end{array}$$

which shows the result for $i = 0$. The map φ^* is exact and has a left adjoint $\varphi_!$, namely the extension by zero, which is the sheafification of the presheaf defined by

$$(\varphi_! \mathcal{F})(U) = \begin{cases} \mathcal{F}(U), & U \subseteq Y \\ 0, & U \nsubseteq Y \end{cases}$$

In particular, φ^* preserves injective objects. Using the case where $i = 0$ and the δ-functor, we prove that the case for $i > 0$ follows. □

Corollary 2.33. The local cohomology (think of $x \in X$ as a point) agrees with the supported cohomology (think of $x \in X$ as a maximal ideal in $\operatorname{Spec}(\mathcal{O}^h_{X,x})$), i.e.,

$$H^i_x(X, \mathcal{F}) \cong H^i_x(\operatorname{Spec}(\mathcal{O}^h_{X,x}), \mathcal{F}_x).$$

Theorem 2.34. For all $n > \dim(X)$, $H^n(X, \mathcal{F}) = 0$.

Proof. We proceed by induction on $\dim(X)$. If $\dim(X) = 0$, then X is a disjoint union of spectra of Henselian rings[11], but over each Henselian, the higher cohomology vanishes since henselization is an exact functor. so the statement holds. Now suppose the statement is true for any scheme Y such that $\dim(Y) < \dim(X)$, then we have a long exact sequence

$$\cdots \to H^i(\operatorname{Spec}(\mathcal{O}^h_{X,x}), \mathcal{F}_x) \to H^i(\operatorname{Spec}(\mathcal{O}^h_{X,x}\backslash\{x\}), \mathcal{F}_x) \to H^{i+1}_x(\operatorname{Spec}(\mathcal{O}^h_{X,x}), \mathcal{F}_x) \to H^{i+1}(\operatorname{Spec}(\mathcal{O}^h_{X,x}), \mathcal{F}_x) \to \cdots$$

For $i > 0$, we know that $H^i(\operatorname{Spec}(\mathcal{O}^h_{X,x}), \mathcal{F}_x) = H^{i+1}(\operatorname{Spec}(\mathcal{O}^h_{X,x}), \mathcal{F}_x) = 0$, therefore

$$H^i(\operatorname{Spec}(\mathcal{O}^h_{X,x}\backslash\{x\}), \mathcal{F}_x) \cong H^{i+1}_x(\operatorname{Spec}(\mathcal{O}^h_{X,x}), \mathcal{F}_x)$$

for $i > 0$. By induction, $H^{n-1}(\operatorname{Spec}(\mathcal{O}^h_{X,x}\backslash\{x\}), \mathcal{F}_x) = 0$ if $n > \dim(\bar{x})$,[12] therefore $H^n_x(\operatorname{Spec}(\mathcal{O}^h_{X,x}, \mathcal{F}_x) = 0$ if $n > \dim(\bar{x})$. This tells us that the Coniveau spectral sequence satisfies

$$E_1^{p,q} \cong \bigoplus_{\operatorname{codim}(\bar{x})=p} H^{p+q}_x(\operatorname{Spec}(\mathcal{O}^h_{X,x}), \mathcal{F}_x) = 0$$

when $p + q > \dim(X)$ (since $n > \dim(\bar{x})$). Therefore the spectral sequence collapses, i.e., $H^n(X, \mathcal{F}) = 0$ for $n > \dim(X)$. □

Theorem 2.35. Let $X, U \in \operatorname{Sm}/S$ and $p : U \to X$ be a Nisnevich covering. Denote the n-fold product $A \times_B A \times_B \cdots \times_B A$ by A^n_B, then the Čech complex of sheaves (associated to the complex over Sm/S)

$$\check{C}(U/X) = (\cdots \longrightarrow \mathbb{Z}(U^n_X) \xrightarrow{d_n} \cdots \longrightarrow \mathbb{Z}(U \times_X U) \xrightarrow{d_2} \mathbb{Z}(U) \longrightarrow \mathbb{Z}(X) \longrightarrow 0)$$

is exact[13], where $d_n = \sum_i (-1)^{i-1}\mathbb{Z}(p_i)$ for ith omission map $p_i : U^n_X \to U^{n-1}_X$.

Proof. It suffices to show exactness stalkwise, so to do things locally, we suppose $Y = \operatorname{Spec}(A)$ where A is Henselian, regular and local, and $a \in \operatorname{Cor}_S(Y, U^n_X) = \mathbb{Z}(U^n_X)(Y)$ such that $d_n(a) = 0$. Define $T = \operatorname{supp}(a)$ and $R = T \times_{X\times Y} (U \times Y)$. Since U is Nisnevich over X, then R is Nisnevich over T. Since a is a finite correspondence, and $T \subseteq Y \times_S U^n_X$ is a closed subset, then T is finite over Y. But Y is Henselian, then T is the spectrum of a disjoint union of Henselian rings by Theorem 2.27. Since R is a Nisnevich covering of T, so the map $R \to T$ admits a section $s : T \to R$,[14] where s is both an open immersion and a closed immersion, i.e., T is clopen in R. This gives a diagram of Cartesian squares

$$\begin{tikzcd}
R^n_T \arrow[r] \arrow[d, "\mathrm{id}^n \times s"'] \arrow[dd, bend right=60, equal] & (U \times Y)^n_{X\times Y} \times_{X\times Y} ((U \times Y)\backslash(R\backslash T)) \arrow[d, "j_{n+1}"] \\
R^{n+1}_T \arrow[r] \arrow[d, "p_{n+1}"'] & (U \times Y)^{n+1}_{X\times Y} \arrow[d, "p_{n+1}"] \\
R^n_T \arrow[r] & (U \times Y)^n_{X\times Y}
\end{tikzcd}$$

where j_{n+1} is a closed immersion. But note that the composition of the left column is just identity, so we define

$$b = (j_{n+1*}(p_{n+1} \circ j_{n+1})^*)(a) \in \operatorname{Cor}_S(Y, U^{n+1}_X).$$

By intersection theory, one can check that $d_{n+1}(b) = a$. □

[11] Being Artinian local rings, they should be complete and therefore Henselian.

[12] Note that removing the closure of the point (as a maximal ideal) reduces the length by 1, therefore drops the dimension by 1, so the inductive hypothesis still works.

[13] To be precise, we consider this sequence to be the sheafification of Nisnevich presheaves restricted on Nisnevich sites.

[14] We have an étale morphism $R \to T$ that is Nisnevich at the maximal ideal of T, so we admit a section by Theorem 2.27.

Theorem 2.36. There is a unique sheafification function $a : \mathrm{PSh}(S) \to \mathrm{Sh}(S)$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{PSh}(S) \arrow[r, "a"] \arrow[d] & \mathrm{Sh}(S) \arrow[d] \\ \mathrm{PSh}(\mathrm{Sm}/S) \arrow[r, "+"'] & \mathrm{Sh}(\mathrm{Sm}(S)) \end{tikzcd}$$

Proof. Take $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Sh}(S)$. We first prove uniqueness. Suppose $\mathcal{F}_1|_{\mathrm{Sm}/S} = \mathcal{F}_2|_{\mathrm{Sm}/S} = (\mathcal{F}|_{\mathrm{Sm}/S})^+$, set $s \in \mathcal{F}_1(Y) = \mathcal{F}_2(Y)$ and $T \in \mathrm{Cor}_S(X, Y)$ where X is Henselian, then there is a Nisnevich covering $p : U \to Y$ such that $s|_U = t^+$ where $t \in \mathcal{F}(U)$. Consider the Cartesian square

$$\begin{tikzcd} T_U \arrow[r] \arrow[d] & X \times U \arrow[d] \\ T \arrow[r] & X \times Y \end{tikzcd}$$

then since T is irreducible so T is the spectrum of some Henselian ring, which gives a section s of the map $T_U \to T$. Denote $D = \mathrm{im}(s)$, then $D \in \mathrm{Cor}_S(X, U)$. Therefore $p \circ D = T$, so we have a commutative diagram

$$\begin{tikzcd} \mathcal{F}_1(X) \arrow[r, equal] & \mathcal{F}_2(X) \\ \mathcal{F}_1(U) \arrow[r, equal] \arrow[u, "\mathcal{F}_1(D)"] & \mathcal{F}_2(U) \arrow[u, "\mathcal{F}_2(D)"'] \\ \mathcal{F}_1(Y) \arrow[r, equal] \arrow[u, "\mathcal{F}_1(p)"] \arrow[uu, bend left=60, "\mathcal{F}_1(T)"] & \mathcal{F}_2(Y) \arrow[u, "\mathcal{F}_2(p)"'] \arrow[uu, bend right=60, "\mathcal{F}_2(T)"'] \end{tikzcd}$$

In particular, $\mathcal{F}_1 = \mathcal{F}_2$, so we have uniqueness. To prove existence, we make $(\mathcal{F}|_{\mathrm{Sm}/S})^+$ a sheaf with transfers. Suppose $y \in (\mathcal{F}|_{\mathrm{Sm}/S})^+(Y)$, and $y|_U = Z^+$, where $p : U \to Y$ is a Nisnevich covering and $Z \in \mathcal{F}(U)$ (and so Z^+ is the image of Z over sheafification). By shrinking U, we allow Z to agree on the intersection, i.e., we may assume that Z is mapped to 0 in $\mathcal{F}(U \times_Y U)$. This gives a sequence

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}(Y), (\mathcal{F}|_{\mathrm{Sm}/S})^+ \longrightarrow \mathrm{Hom}(\mathbb{Z}(0), (\mathcal{F}|_{\mathrm{Sm}/S})^+ \longrightarrow \mathrm{Hom}(\mathbb{Z}(U \times_X U), (\mathcal{F}|_{\mathrm{Sm}/S})^+)$$

which is exact by Theorem 2.35. We know that $p^*(Z) = 0$, so there exists $[y] : \mathbb{Z}(Y) \to (\mathcal{F}|_{\mathrm{Sm}/S})^+$ such that $[y]|_U = y|_U$. Take $f \in \mathrm{Cor}_S(X, Y)$, then by Yoneda lemma we know the composition

$$\mathbb{Z}(X) \xrightarrow{f} \mathbb{Z}(Y) \xrightarrow{[y]} (\mathcal{F}|_{\mathrm{Sm}/S})^+$$

of Nisnevich sheaves produces the transfer of y with respect to f. $\square$

Remark 2.37. The category $\mathrm{Sh}(S)$ is an abelian category, then the statement in Proposition 2.29 holds for $\mathrm{Sh}(S)$.

Proposition 2.38. Suppose $X \in \mathrm{Sm}/S$ and $\{U_1, U_2\}$ is a Zariski covering of X, then we have an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}(U_1 \cap U_2) & \longrightarrow & \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2) & \longrightarrow & \mathbb{Z}(X) & \longrightarrow & 0 \\ & & s & \longmapsto & (s|_{U_1}, -s|_{U_2}) & & & & \\ & & & & (s_1, s_2) & \longmapsto & s_1 + s_2 & & \end{array}$$

Proof. Note that $U_1 \coprod U_2$ is a Nisnevich covering of X. Applying the Čech complex of X in Theorem 2.35, we obtain an exact sequence

$$\mathbb{Z}(U_1) \oplus \mathbb{Z}(U_1 \cap U_2)^{\oplus 2} \oplus \mathbb{Z}(U_2) \xrightarrow{d} \mathbb{Z}(U_1) \oplus \mathbb{Z}(U_2) \xrightarrow{+} \mathbb{Z}(X) \longrightarrow 0$$

where $d(x, y, a, b) = (a - y, y - a)$. $\square$

Definition 2.39. Define Sim to be the category of simplicial sets $[n] = \{0, \cdots, n\}$ for $n \in \mathbb{N}$, where $\mathrm{Hom}_{\mathrm{Sim}}([n], [m])$ is the set of non-decreasing simplicial maps $[n] \to [m]$.

For any category $\mathscr{C}$, we define a simplicial (respectively, cosimplicial) object in $\mathscr{C}$ to be a functor $\mathrm{Sim}^{\mathrm{op}} \to \mathscr{C}$ (respectively, $\mathrm{Sim} \to \mathscr{C}$).

For any $n \in \mathbb{N}$, we define a scheme $\Delta^n = \mathrm{Spec}(k[x_0, \ldots, x_n])/\{(x_0, \ldots, x_n) : \sum_{i=0}^{n} x_i = 1\}$ that is isomorphic to $\mathbb{A}^n$. This is a cosimplicial object in Sm/k. For any $f : [n] \to [m]$, we have

$$\Delta(f)(x_i) = (y_j)$$

where $y_j = \sum_{i \in f^{-1}(j)} x_i$.

Definition 2.40. For any $F \in \mathrm{PSh}(S)$, we define a simplicial object

$$(C_* F)_n = F^{\Delta^n}$$

which associates to the Suslin complex of F

$$C_* F : \cdots \longrightarrow F^{\Delta^n} \xrightarrow{d_n} F^{\Delta^{n-1}} \longrightarrow \cdots \longrightarrow F^{\Delta^1} \xrightarrow{d_1} F \longrightarrow 0$$

with $d_n = \sum_i (-1)^{i-1} \partial_i$, where $\partial_i : \Delta^{n-1} \to \Delta^n$ is the ith face map.

Remark 2.41. In Theorem 2.34, we showed that the cohomological dimension of Nisnevich topology on X is just $\dim(X)$, so for every bounded-above (cochain) complex $C \in C^-(\mathrm{Sh}(S))$, we could find a quasi-isomorphism $i : C \to I^*$ where $H^n(X, I^m) = 0$ for any m and any $n > 0$. Therefore, we can define the nth hypercohomology of C with respect to X as

$$\mathbb{H}^n(X, C) = H^n(I^*(X)).$$

It is a standard argument to show that $\mathbb{H}^n(X, C)$ is independent of I^*.

Definition 2.42. For every $q \in \mathbb{N}$, we define the motivic complex to be

$$\mathbb{Z}(q) = C_*(\mathbb{Z}(\mathbb{G}_m^{\wedge q})[-q]),$$

given by the augmentation of the smashing with a shifting by $-q$, where $\mathbb{Z}(\mathbb{G}_m^{\wedge q}) = (\mathbb{Z}(\mathbb{G}_m), 1)^{\wedge q}$, and $\mathbb{Z}(q)^i = C_{q-i}\mathbb{Z}(\mathbb{G}_m^{\wedge q})$ in the Suslin complex for $q \geqslant 0$. For $q < 0$, we define $\mathbb{Z}(q) = 0$.

For any group A, we write $\mathbb{Z}(q) \otimes_{\mathbb{Z}}^{L} A$ as $A(q)$.

Example 2.43. $\mathbb{Z}(0)$ is the constant sheaf $\mathbb{Z}$.

Definition 2.44. For every $X \in \mathrm{Sm}/k$, we define the motivic cohomology to be the hypercohomology with respect to Nisnevich topology

$$H^{p,q}(X, A) = \mathbb{H}^p_{\mathrm{Nis}}(X, A(q))$$

with coefficients in A.

Remark 2.45. It turns out that this is equivalent to giving the hypercohomology the Zariski topology instead.

Proposition 2.46. For any $X \in \mathrm{Sm}/k$, we have

$$H^{p,q}(X, A) = 0$$

if $p > \dim(X) + q$. In particular, if A is a field, then $H^{p,q}(X, A) = 0$ if $p > q$.

Proof. Using Lemma 1.26, we obtain a spectral sequence

$$H^s(X, H^t(A(q))) \Rightarrow \mathbb{H}^{s+t}(X, A(q)) = H^{s+t,q}(X, A)$$

Let $p = s + t$. If $p > \dim(X) + q$, then either $t > q$ or $s > \dim(X)$. This gives $H^s(X, H^t(A(q))) = 0$. $\square$

3 Milnor K-theory

3.1 K-theory of Residue Field

Definition 3.1. For any field F, define the Milnor K-theory to be the graded algebra

$$K_*^M(F) = T(F^\times)/\{x \otimes (1-x) : x \in F\backslash\{0,1\}\},$$

defined as the tensor algebra of $F^\times$ quotient by the Steinberg relation.

Example 3.2.

- $K_0^M(F) = \mathbb{Z}$;
- $K_1^M(F) = F^\times$.

Proposition 3.3. For any $x \in F^\times$, let $[x] \in K_1^M(F)$ be its representative, then obviously $[xy] = [x] + [y]$. Moreover,

1. $[x][y] + [y][x] = 0$ for all $x, y \in F^\times$;
2. $[x][x] = [x][-1]$ for all $x \in F^\times$.

Proof.

1. We have

$$\begin{aligned}[x][-x] &= [x]\left[\frac{1-x}{1-x^{-1}}\right] \\ &= [x][1-x] + [x^{-1}][1-x^{-1}] \\ &= 0 + 0 \\ &= 0,\end{aligned}$$

therefore

$$\begin{aligned}[x][y] + [y][x] &= [x][-x] + [x][y] + [y][x] + [y][-y] \\ &= [x][-xy] + [y][-xy] \\ &= [xy][-xy] \\ &= 0.\end{aligned}$$

2. Using the previous part, we know

$$\begin{aligned}[x][x] &= [x][-1] + [x][-x] \\ &= [x][-1].\end{aligned}$$

□

Proposition 3.4 ([Hes05], Proposition 1)**.** Let k be a field and ν be a normalized discrete valuation on k. We define the residue field of k with respect to ν as $k(\nu) = \mathcal{O}_\nu/\mathfrak{m}_\nu$, then there exists a unique homomorphism (known as the Milnor residue map)

$$\partial_\nu : K_n^M(k) \to K_{n-1}^M(k(\nu))$$

such that for all $u_1, \dots, u_{n-1} \in \mathcal{O}_\nu^\times$ and $x \in k^\times$,

$$\partial_\nu([x][u_1]\cdots[u_{n-1}]) = \nu(x)\cdot[\bar{u}_1]\cdots[\bar{u}_{n-1}]$$

where $\bar{u}_i \in k(\nu)^\times$ is the image of u_i in the residue field.

Proof. The uniqueness is obvious by the universal property, so we shall prove existence. We choose a uniformizer π, and define a graded ring morphism

$$\begin{aligned}\theta_\pi : K_*^M(k) &\to K_*^M(k(\nu))[\varepsilon]/(\varepsilon^2 - \varepsilon[-1])\\ [\pi^i u] &\mapsto [\bar{u}] + i\varepsilon\end{aligned}$$

for $u \in \mathcal{O}_\nu^\times$ and some variable ε of degree 1, then this morphism satisfies the Steinberg relation. Now if we decompose it into

$$\theta_\pi(z) = s_\pi(z) + \partial_\nu(z)\varepsilon,$$

then

$$\begin{aligned}\theta_\pi([\pi^i u][u_1]\cdots[u_{n-1}]) &= ([u] + i\varepsilon)[\bar{u}_1]\cdots[\bar{u}_{n-1}]\\ &= [\bar{u}][\bar{u}_1]\cdots[\bar{u}_{n-1}] + i[\bar{u}_1]\cdots[\bar{u}_{n-1}]\varepsilon.\end{aligned}$$

In particular, the ∂_ν map does what we want. □

Theorem 3.5 ([Hes05], Theorem 5)**.** There is a split exact sequence

$$0 \longrightarrow K_*^M(k) \xrightarrow{i} K_*^M(k(T)) \xrightarrow{(\partial_p)} \bigoplus_{\text{irreducible monic } p} K_{*-1}^M(k[T]/p) \longrightarrow 0$$

where each ∂_p is given by evaluation of p using the partial map defined in Proposition 3.4.

Proof. It is easy to see that this is an exact sequence, and that we have $s_\pi \circ i = \mathrm{id}$. Now we want to construct an isomorphism

$$\tau_{n,p} : \bigoplus_{\text{irreducible monic } p} K_n^M(k[T]/p) \to K_{n+1}^M(k(T))/K_{n+1}^M(k)$$

with inverse $(\partial_p)_p$. We define $\tau_{n,p}$ inductively on $\deg(p)$. Suppose $p = T - \lambda$, then we define $\tau_{n,p}$ as the composite

$$K_n^M(k[T]/p) \xrightarrow{\text{ev}} K_n^M(k) \xrightarrow{[p]} K_{n+1}^M(k(T))/K_{n+1}^M(k).$$

Let $f_i \in k[T]$ for each i, then this composite maps $[\bar{f}_1]\cdots[\bar{f}_n]$ to $[p][f_1(\lambda)]\cdots[f_n(\lambda)]$. Moreover,

$$\partial_q \circ \tau_{n,p} = \begin{cases} \mathrm{id}, & q = p \\ 0, & q \neq p. \end{cases}$$

For general polynomial p and general $f_1, \ldots, f_n \in k[T]$ such that $\deg(f_i) < \deg(p)$ for all i[15], then we define

$$\tau_{n,p}([\bar{f}_1]\cdots[\bar{f}_n]) = [\bar{p}] \cdot [f_1]\cdots[f_n] - \sum_{\substack{\text{irreducible monic } q\\ \text{such that } \deg(q)<\deg(p)}} \tau_{n,q}(\partial_q([p][f_1]\cdots[f_n])),$$

then by inductive hypothesis we are done. It remains to check that this is well-defined, and that

- $$\partial_q \circ \tau_{n,p} = \begin{cases} \mathrm{id}, & q = p \\ 0, & q \neq p. \end{cases}$$

[15] This makes sense since we can pick it in the residue field.

- $\sum_{\deg(q)<\deg(p)} \tau_{n,q}(\partial_q(x)) = x$ if $x = \sum_{\substack{i \\ \deg(f_{ij})<\deg(p)}} [f_{i1}]\cdots[f_{in}]$.

$\square$

Remark 3.6. The map $-d_\infty$ from

$$\begin{tikzcd} K_*^M(k(T)) \arrow[r, "-\partial_\infty"] & K_{*-1}^M(k) \\ K_*^M(k) \arrow[u, "i"] \arrow[ur, "0"'] & \end{tikzcd}$$

and the exact sequence from Theorem 3.5 together induce a norm map

$$(N_p) : \bigoplus_p K_*^M(k[T]/p) \to K_*^M(k)$$

with $N_\infty = \mathrm{id}$.

Definition 3.7. Suppose $k(a)/k$ is a finite simple extension and the minimal polynomial of a is p. Define the norm

$$N_{a/k} : K_*^*M(k(a)) \to K_*^M(k)$$

to be N_p. In general, suppose K/k is a finite extension where $K = k(a_1, \dots, a_n)$, then define the norm map to be

$$N_{a_1,\dots,a_r/k} = N_{a_1/k} \circ N_{a_1/k(a_1)} \circ \cdots \circ N_{a_r/k(a_1,\dots,a_{r-1})}.$$

Theorem 3.8 ([Hes05], Theorem 3)**.** The norm map $N_{a_1,\dots,a_r/k}$ is independent from the choices of $a_1, \dots, a_r$. In particular, this gives rise a well-defined norm map

$$N_{K/k} : K_*^M(K) \to K_*^M(k)$$

on all finite extensions K/k.

3.2 Proof of Theorem 3.8

Proposition 3.9 ([Hes05], Lemma 10)**.** Let k be a field and p be a prime, then there exists an algebraic extension L/k such that every finite extension of L has order a power of p, and localization at p gives a map

$$K_*^M(k)_{(p)} \to K_*^M(L)_{(p)}$$

is injective.

Proof. Recall an ordinal W is a limit ordinal if and only if $W = \bigcup_{\alpha<W} \alpha$. Define a poset

$$S = \{(\alpha, \{L_\beta \mid \beta \leqslant \alpha\}) : \alpha \text{ ordinal number}, p \nmid [L_\beta : k] < \infty, [L_{\beta+1} : L_\beta] > 1, L_W = \bigcup_{\alpha<W} L_\alpha \text{ for limit ordinal } W\}$$

for some field extensions $k \subseteq L_\beta \subseteq \bar{k}$ in the algebraic closure $\bar{k}$. The partial order on S is given by $(\alpha, \{L_\beta \mid \beta \leqslant \alpha\}) \leqslant (\alpha', \{L_{\beta'} \mid \beta' \leqslant \alpha'\}$ if and only if $\alpha \leqslant \alpha'$, $L_\beta = L_{\beta'}$, $\beta \leqslant \alpha$. We note that $\mathrm{card}(\alpha) \leqslant \mathrm{card}(\bar{k})$, so S must be a set. Every totally ordered subset of S has a maximal element by taking the union, therefore there is a maximal element $(\alpha, \{L_\beta \mid \beta \leqslant \alpha\})$ in S. Now $L = L_\alpha$ does not have an extension with order prime to p, hence every finite extension of L has order a power of p. For any simple extension $k(a)/k$, the composite

$$K_*^M(k) \longrightarrow K_*^M(k(a)) \xrightarrow{N_{a/k}} K_*^M(k)$$

is the multiplication by $[k(a) : k]$ by direct computation. Therefore, for any $\beta \leqslant \alpha$, the composite

$$K_*^M(L_\beta)_{(p)} \longrightarrow K_*^M(L_{\beta+1})_{(p)} \xrightarrow{N} K_*^M(L_\beta)_{(p)}$$

is an injection, hence $K_*^M(k)_{(p)} \to K_*^M(L)_{(p)}$ is also injective by transfinite induction. $\square$

Proposition 3.10 ([Hes05], Lemma 2)**.** Suppose k'/k is a field extension and ν (respectively, ν') is a discrete valuation on k (respectively, k') such that $\nu'|_k = \nu$. Then there is a commutative diagram

$$\begin{tikzcd} K_*^M(k) \arrow[r, "\partial_\nu"] \arrow[d] & K_{*-1}^M(k(\nu)) \arrow[d] \\ K_*^M(k') \arrow[r, "e\partial_{\nu'}"'] & K_{*-1}^M(k(\nu')) \end{tikzcd}$$

where e is the ramification index, i.e., $\pi_\nu = u \cdot \pi_{\nu'}^e$ for some uniformizer $u \in \mathcal{O}_{\nu'}^*$.

Proposition 3.11 ([Hes05], Lemma 11)**.** Let $k' = k(a)$ be a finite extension of k, and let p be the minimal polynomial of a over k. Let L/k be a field extension and suppose $p = \prod_i p_i^{e_i}$ is the prime decomposition for some polynomials p_i in L, then for each i we define $L_i' \supseteq k'$ to be $L[t]/p_i$, and set $a_i = \bar{t} \in L_i'$, then we have a commutative diagram

$$\begin{tikzcd} K_*^M(k') \arrow[r, "(e_i)"] \arrow[d, "N_{a/k}"'] & \bigoplus_i K_*^M(L_i') \arrow[d, "\sum_i N_{a_i/L}"] \\ K_*^M(k) \arrow[r, "\text{base-change}"'] & K_*^M(L) \end{tikzcd}$$

where e_i is the (multiplication of) ramification index of L_i' over k'.

Proof. Let $f_1, \ldots, f_n \in k[i]$ be prime to p, then $\partial_{p_i}([p][f_1]\cdots[f_n]) = e_i[\bar{f}_1]\cdots[\bar{f}_n]$. Therefore, there is a commutative diagram

$$\begin{tikzcd} K_*^M(k(t)) \arrow[r] \arrow[d] & K_*^M(L(t)) \arrow[d] \\ \bigoplus_R K_{*-1}^M(k[T]/R) \arrow[r, "\varphi_{R,Q}"'] & \bigoplus_Q K_{*-1}^M(L[T]/Q) \end{tikzcd}$$

where

$$\varphi_{R,Q} = \begin{cases} \mathrm{ord}_Q(p), & R = p \\ 0, & R \neq p \end{cases}$$

The statement follows from the definition of the map $(N_p) : \bigoplus_p K_*^M(k[T]/p) \to K_*^M(k)$. $\square$

Corollary 3.12 ([Hes05], Corollary 12)**.** Let $k \subseteq k' \subseteq K$ be extensions of fields, then

1. for any $x \in K_*^M(k')$ and $y \in K_*^M(k)$, we have a projection formula
$$N_{k'/k}(x \cdot y) = N_{k'/k}(x) \cdot y;$$
2. if k'/k is normal and $x \in K_*^M(k')$, then the base-change of norm over K is $N_{k'/k}(x)_K = [k' : k]_{\text{insep}} \sum_{j:k' \to K} j_*(x)$, where $[k' : k]_{\text{insep}}$ is the inseparable degree of k'/k;
3. $N_{k'/k} \circ N_{K/k'} = N_{K/k}$.

Proof.

1. It suffices to assume $k' = k(a)$ by choosing generators of k'/k and Theorem 3.8, then the statement follows from the construction in Theorem 3.5.

2. If k'/k is separable where $k' = k(a)$, then Proposition 3.11 gives a diagram

$$\begin{tikzcd} K_*^M(k') \arrow[r] \arrow[d, "N_{k'/k}"'] & \bigoplus_{j:k'\to k} K_*^M(k') \arrow[d, "(j_*)"] \\ K_*^M(k) \arrow[r] & K_*^M(K) \end{tikzcd}$$

which gives the statement.

If k'/k is purely inseparable, it suffices to assume $k' = k(a)$ and proceed inductively. We have $k(a) \otimes_k K = k[t]/(t - a^{\frac{1}{d}})^d$ where $d = [k(a) : k]$. The statement now follows from Proposition 3.11.

For general k'/k, denote by k^s the separable closure of k'/k. The map $\mathrm{Hom}(k', k) \to \mathrm{Hom}(k^s, k)$ is an isomorphism. Therefore, the base-change over K gives

$$\begin{aligned} N_{k'/k}(x)_K &= N_{k^s/k}(N_{k'/k^s}(x))_K \\ &= \sum_{j:k^s\to K} j_* N_{k'/k^s}(x) \\ &= \sum_{j:k'\to K} j_*(N_{k'/k^s}(x)_{k'}) \\ &= [k' : k]_{\mathrm{insep}} \sum_{k'\to K} j_*(x). \end{aligned}$$

3. This will be obvious once we prove Theorem 3.8.

□

Proposition 3.13 ([Hes05], Proposition 13)**.** Let k be a field and set $k' = k(a)$ to be such that the extension $k(a)/k$ has prime degree, then the map

$$N_{a/k} : K_n^M(k') \to K_n^M(k)$$

is independent of the choice of the generator a.

Proof. Suppose all finite extensions of k have order a power of k, then we write $k' = k[T]/p$ where the image $\bar{T} = a$ and $\deg(p) = p$. For any monic $f, g \in k[T]$ of the same degree, we get to write $f = g + h$ for $\deg(h) < \deg(f)$. If $h = 0$, then $[f][g] = [f][-1]$, otherwise we have $([h] - [f])([g] - [f]) = \left[\frac{h}{f}\right]\left[\frac{g}{f}\right] = \left[\frac{h}{f}\right]\left[1 - \frac{h}{f}\right] = 0$ by the Steinberg relation. Therefore, $[f][g] = [h][g] - [h][f] + [f][-1]$, hence every element in $K_n^M(k')$ is a sum of the form $[f_1]\cdots[f_n]$ where f_i's are irreducible or constant and that $p > \deg(f_1) > \cdots > \deg(f_n)$. But we know $f_2, \ldots, f_n$ are constant by the condition on k, so for any choice of a, we must have

$$N_{a/k}([f_1]\cdots[f_n]) = N_{k'/k}(f_1)[\bar{f}_2]\cdots[\bar{f}_n]$$

according to the projection formula and the Weil reciprocity formula, therefore it is independent of a.

For a general field k, it suffices to show that

$$N_{a/k} : K_*^M(k')_{(l)} \to K_*^M(k)_{(l)}$$

does not depend on a for every prime l. By Proposition 3.9, there exists some extension L/k such that every finite extension of L has degree a power of l and $K_*^M(k)_{(l)} \to K_*^M(L)_{(l)}$ is injective. Since $[k' : k]$ is prime, then the extension k'/k is either separable or purely inseparable.

- Suppose k'/k is separable, then $L' = L \otimes_k k'$ is étale over L by base-change, therefore it is a reduced Artinian ring, hence it is a field of p products of L.

  - If L' is a field, $[k':k]=p$, otherwise L'/L would be a finite extension of degree prime to p. In particular $l=p$, so by Proposition 3.11 we know there is a commutative diagram

$$\begin{tikzcd} K_*^M(k') \arrow[r, "\partial_\nu"] \arrow[d, "N_{a/k}"'] & K_*^M(L') \arrow[d, "N_{L'/L}"] \\ K_*^M(k) \arrow[r, "e\partial_{\nu'}"'] & K_*^M(L) \end{tikzcd}$$

  - If L' is a product of p fields, then by Proposition 3.11 we know there is a commutative diagram

$$\begin{tikzcd} K_*^M(k') \arrow[r, "(\partial_\nu)"] \arrow[d, "N_{a/k}"'] & \bigoplus K_*^M(L) \arrow[d, "\sum \mathrm{id}=\sum N_{L/L}"] \\ K_*^M(k) \arrow[r, "e\partial_{\nu'}"'] & K_*^M(L) \end{tikzcd}$$

    over all possible embeddings of k' in L.

  Regardless, $N_{a/k}$ is independent of a.

- Suppose k'/k is purely separable, so we can write $k'=k[t]/(t^p-a)$. If $a^{\frac{1}{p}}\notin L$, then L' is a field; it not, then $L'=L^{\times p}$. By applying Proposition 3.11 to both cases, we are done.

□

Definition 3.14. Suppose K is a field with discrete valuation ν. Fix $a\in(0,1)$, then we can define an absolute value $||x||=a^{\nu(x)}$ for every $x\in K$. Taking the completion $\hat{K}$ of K, we obtain a metric space, at the same time getting a field with discrete valuation. In particular, if $\hat{K}=K$, then we say the valuation is complete.

Remark 3.15. Recall from section II.2 of [Ser13] that if K is a complete discrete valuation field and if L/K is a finite extension, then the discrete valuation on K extends uniquely to a discrete valuation on L, and L is complete with respect to the valuation. Moreover, we have $[L:K]=e_{L/K}\cdot[k(\mathcal{O}_L):k(\mathcal{O}_K)]$ where $e_{L/K}$ is the ramification index.

Proposition 3.16 ([Hes05], Lemma 14)**.** Let K be a complete discrete valuation field, and let K'/K be a normal extension of prime degree p. Let k and k' be the residue field of K and K', respectively. Since the extension has prime degree, then the norm $N_{K'/K}$ is well-defined, and the following diagram commutes.

$$\begin{tikzcd} K_n^M(K') \arrow[r, "\partial K'"] \arrow[d, "N_{K'/K}"'] & K_{n-1}^M(k') \arrow[d, "N_{k'/k}"] \\ K_n^M(K) \arrow[r, "\partial_K"'] & K_{n-1}^M(k) \end{tikzcd}$$

Proof. We show that $\delta_{K'/K}:=\partial_K\circ N_{K'/K}-N_{k'/k}\circ\partial_{K'}$ is 0. We first show that $p\delta_{K'/K}=0$.

- Suppose that K'/K is unramified, i.e., $e_{K'/K}=1$.
  - If K'/K is separable, then k'/k is normal by Proposition 20 in section 1.7 of [Ser13].
    * If, in addition, that k'/k is separable, then $\mathrm{Gal}(K'/K)\cong\mathrm{Gal}(k'/k)$. By the fact that $e_{K'/K}=1$ and by Corollary 3.12, we know

$$\begin{aligned} K_*^M(k'/k)\circ\delta_{K'/K} &= K_*^M(k'/k)\circ(\partial_K\circ N_{K'/K}-N_{K'/K}\circ\partial_{K'}) \\ &= \partial_{K'}\circ K_*^M(K'/K)\circ N_{K'/K}-K_*^M(k'/k)\circ N_{k'/k}\circ\partial_{K'} \\ &= \sum_{\sigma\in\mathrm{Gal}(K'/K)}\partial_{K'}\circ\sigma-\sum_{\bar{\sigma}\in\mathrm{Gal}(k'/k)}\bar{\sigma}\circ\partial_{K'} \\ &= 0. \end{aligned}$$

  * If k'/k is purely inseparable instead, then by Corollary 3.12 we know that

$$K_*^M(k'/k) \circ \delta_{K'/K} = \sum_{\sigma \in \mathrm{Gal}(K'/K)} \partial_{K'} \circ \sigma - p\partial_{K'}. \tag{3.17}$$

    However, since k'/k is purely inseparable, then $\sigma \in \mathrm{Gal}(K'/K)$ induces identity map on k', hence Equation (3.17) must be zero.

- If K'/K is purely inseparable instead, then k'/k is also purely inseparable by Proposition 16 in section 1.6 of [Ser13], therefore the same argument shows that

$$K_*^M(k'/k) \circ \delta_{K'/K} = p\partial_{K'} - p\partial_{K'} = 0$$

  since K'/K is unramified. Finally, by Corollary 3.12 we know $N_{k'/k} \circ K_*^M(k'/k) = p$, so we have proven the claim for the case where K'/K is unramified.

- Now suppose K'/K is totally ramified, i.e., $e_{K'/K} = p$.
  - If K'/K is Galois, then

$$\begin{aligned} p\delta_{K'/K} &= p\partial_K \circ N_{K'/K} - p\partial_{K'} \\ &= \partial_{K'} \circ K_*^M(k'/k) \circ N_{K'/K} - p\partial_{K'} \\ &= \sum_{\sigma \in \mathrm{Gal}(K'/K)} \partial_{K'} \circ \sigma - p\partial_{K'} \\ &= 0. \end{aligned}$$

  - If K'/K is purely inseparable, then by Corollary 3.12 we have

$$\begin{aligned} p\delta_{K'/K} &= \partial_{K'} \circ K_*^M(K'/K) \circ N_{K'/K} - p\partial_{K'} \\ &= p\partial_{K'} - p\partial_{K'} \\ &= 0. \end{aligned}$$

This shows that $p\delta_{K'/K} = 0$. It now suffices to show that, for every $Z \in K_n^M(K')$, there exists some integer m coprime to p such that $m\delta_{K'/K}(Z) = 0$.

Suppose that L is an extension of K of degree prime to p, and let $L' = [L, K'] = L \otimes_K K'$ (since they are linearly disjoint) be a field generated by L and K' in $\bar{K}$, therefore $[L' : L] = p$. By Proposition 3.11, the diagram

$$\begin{tikzcd} K_n^M(K') \arrow[r] \arrow[d, "N_{K'/K}"'] & K_n^M(L') \arrow[d, "N_{L'/L}"] \\ K_n^M(K) \arrow[r] & K_n^M(L) \end{tikzcd}$$

commutes. Here we have $e_{L'/L}[k(\mathcal{O}_{L'}) : k(\mathcal{O}_L)] = p$ and $e_{K'/K}[k(\mathcal{O}_{K'}) : k(\mathcal{O}_K)] = p$, therefore $e_{L'/L}e_{L/K} = e_{L'/K} = e_{K'/K}e_{L'/K'}$. Therefore, $e_{L'/L} = e_{K'/K}$ and $k(\mathcal{O}_L) \otimes_k k' = k(\mathcal{O}_{L'})$ since $[L : K]$ and $[K' : K]$ are coprime. Therefore, we have a commutative diagram

$$\begin{tikzcd} K_n^M(k') \arrow[r] \arrow[d, "N_{k'/k}"'] & K_n^M(k(\mathcal{O}_{L'})) \arrow[d, "N_{k(\mathcal{O}_{L'})/k(\mathcal{O}_L)}"] \\ K_n^M(k) \arrow[r] & K_n^M(k(\mathcal{O}_L)) \end{tikzcd}$$

by Proposition 3.11. Now fix $Z \in K_n^M(K')$, then

$$\begin{aligned} e_{L/K} \cdot K_*^M(k(\mathcal{O}_L)/k) \circ \delta_{K'/K} &= e_{L/K} \cdot K_*^M(k(\mathcal{O}_L)/k) \circ (\partial_K - N_{K'/K} - N_{k'/k} \circ \partial_{K'}) \\ &= \partial_L \circ K_*^M(L/K) \circ N_{K'/K} - e_{L/K} \circ K_*^M(k(\mathcal{O}_L)/k) \circ N_{k'/k} \circ \partial_{K'} \\ &= \partial_L \circ K_*^M(L/K) \circ N_{K'/K} - e_{L/K} \cdot N_{k(\mathcal{O}_{L'})/k(\mathcal{O}_L)} \circ K_*^M(k(\mathcal{O}_{L'})/k') \circ \partial_{K'} \\ &= \partial_L \circ K_*^M(L/K) \circ N_{K'/K} - N_{k(\mathcal{O}_{L'})/k(\mathcal{O}_L)} \circ \partial_{L'} \circ K_*^M(L'/K') \\ &= \partial_L \circ N_{L'/L} \circ K_*^M(L'/K') - N_{k(\mathcal{O}_{L'})-k(\mathcal{O}_L)} \circ \partial_{L'} \circ K_*^M(L'/K') \\ &= \delta_{L'/L} \circ K_*^M(L'/K'). \end{aligned} \tag{3.18}$$

We claim that for our fixed Z, there exists some extension L/K such that Equation (3.18) is 0. If this is true, then by applying $N_{k(\mathcal{O}_L)/k}$, we obtain that $[L:K]\delta_{K'/K}(Z) = 0$ since $[L:K]$ is coprime to p, and we are done.

To find such extension, suppose $\bar{L}$ is the algebraic extension of K obtained in Proposition 3.9 with respect to p, then since $[K':K] = p$, hence we know $K' \otimes_K \bar{L}$ is also a field. Now $K_*^M(K' \otimes_K \bar{L})(Z)$ can be written in the form of $\sum [x][y_2] \cdots [y_n]$ where $x \in K' \otimes_K \bar{L}$ and $y_i \in \bar{L}$, using statements similar to Proposition 3.13.

Therefore, there exists some subextension $K \subseteq L \subseteq \bar{L}$ where $p \nmid [L:K]$ such that $K_*^M(L'/K')(Z)$ has similar properties where $L' = K' \otimes_K L$. Therefore we may assume that we are working over K' already, so $Z = \sum [x][y_2] \cdots [y_n]$ where $x \in K'$ and $y_i \in K$ for all i. By considering the cases where K'/K is either totally ramified or unramified, the projective formula gives $\delta_{K'/K}(Z) = 0$. $\square$

Proposition 3.19 ([Hes05], Proposition 15)**.** Let k be a field and let k' be a finite normal extension of k of prime degree p. Let $F = k(a)$ be a finite extension, and suppose that $F = k'(a)$ is a field, then the following diagram commutes.

$$\begin{tikzcd} K_n^M(F') \arrow[r, "N_{a/k'}"] \arrow[d, "N_{F'/F}"'] & K_n^M(k') \arrow[d, "N_{k'/k}"] \\ K_n^M(F) \arrow[r, "N_{a/k}"'] & K_n^M(k) \end{tikzcd}$$

Proof. We first talk about homotopy invariance. Let ν be a discrete valuation on $k(t)/k$, and let $k(t)_\nu$ be the completion of $k(t)$ at ν. Since $k(t)_\nu/k(t)$ (respectively, $k'(t)_\nu/k'(t)$) is separable, then the minimal polynomial $\pi \in k(t)[x]$ where $k' = k(\alpha)$ with respect to a generator α of $k'(t)/k(t)$ (which gives a correspondence w/ν). Since $k(t)_\nu/k(t)$ is separable, then we have a decomposition of α as a product

$$\pi = \prod_{w/\nu} \pi_{w/\nu}$$

where $\pi_{w/v} \in k(t)_\nu[x]$ are distinct monic irreducible polynomials, and where w ranges over the possible extensions of ν to a discrete valuation on $k'(t)/k'$. We then consider the following diagram:

$$\begin{tikzcd} K_{n+1}^M(k'(t)) \arrow[r, "(j_{k'(t)/k'(t)*})"] \arrow[d, "N_{k'(t)/k(t)}"'] & \bigoplus_{w/\nu} K_{n+1}^M(k'(t)_w) \arrow[r, "\bigoplus \partial_w"] \arrow[d, "\sum N_{k'(t)_w/k(t)_\nu}"] & \bigoplus_{w/v} K_n^M(k(\mathcal{O}_w)) \arrow[d, "\sum N_{k(\mathcal{O}_w)/k(\mathcal{O}_\nu)}"] \\ K_{n+1}^M(k(t)) \arrow[r, "j_{k(t)_\nu/k(t)*}"'] & K_{n+1}^M(k(t)_\nu) \arrow[r, "\partial_\nu"'] & K_n^M(k(\mathcal{O}_\nu)) \end{tikzcd} \tag{3.20}$$

The commutativity of the left-hand square follows from Proposition 3.10, and the commutativity of the right-hand square follows from Proposition 3.16. Let $\theta \in k[t]$ and $\theta' \in k'[t]$ be the minimal polynomial of a over k and k', respectively. Given $x' \in K_n^M(F')$, Theorem 3.5 shows that there exists $y' \in K_{n+1}^M(k'(t))$ such that $\partial_{w_{\theta'}}(y') = x'$ and $\partial_w(y') = 0$ if $w \neq w_{\theta'}, w_\infty$, then by definition we know

$$N_{a/k}(x') = -\partial_\infty(y').$$

We now define $x = N_{F'/F}(x')$ and $y = N_{k'(t)/k(t)}(y')$. Therefore, Equation (3.20) shows that $\partial_{\nu_\theta}(y) = x$ and $\partial_\nu(y) = 0$ if $\nu \neq \nu_\theta, \nu_\infty$, and this gives

$$N_{a/k'}(x) = -\partial_{\nu_\infty}(y).$$

Again, applying Equation (3.20) to $\nu = \nu_\infty$ shows that

$$\begin{aligned} N_{a/k}(N_{F'/F}(x')) &= N_{a/k}(x) \\ &= -\partial_{\nu_\infty}(y) \\ &= -\partial_{\nu_\infty}(N_{k'(t)/k(t)}(y')) \\ &= -N_{k'/k}(\partial_{w_\infty}(y')) \\ &= N_{k'/k}(N_{a/k'}(x')) \end{aligned}$$

as desired. □

Proof of Theorem 3.8. Let $K = k(a_1, \ldots, a_r)$, then we claim that $N_{K/k}$ is independent of $a_1, \ldots, a_r$. We proceed by induction on $[K : k]$ and prove the statement after localizing at a prime p. Choose L/k as in Proposition 3.9, and define $L' = L \otimes_k K$ which is finite over L and therefore Artinian. Therefore, L' has finitely many prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$. Suppose $e_i = \ell_{L'_{\mathfrak{p}_i}}(L'_{\mathfrak{p}_i})$.

1. By Proposition 3.11, we can show that the diagram

$$\begin{tikzcd} K_*^M(K)_{(p)} \arrow[r, "(e_i)"] \arrow[d, "N_{a_1,\ldots,a_r/k}"'] & \bigoplus_{i=1}^m K_*^M(L'_i)_{(p)} \arrow[d, "\sum N_{L'_i/L}"] \\ K_*^M(k)_{(p)} \arrow[r] & K_*^M(L)_{(p)} \end{tikzcd}$$

   commutes. Therefore, if $m > 1$, we conclude by induction that the composition does not depend on the choice of elements. Taking the localization gives what we want.

2. Hence we suppose L' is a field, and choose M/L' to be a Galois extension, now $\mathrm{Gal}(M/L)$ is a p-group, hence we have a composition series

$$\mathrm{Gal}(M/L) = G_1 \supseteq \cdots \supseteq G_n = \mathrm{Gal}(M/L'),$$

   i.e., $G_{i+1} \lhd G_i$ and $G_i/G_{i+1} \cong \mathbb{Z}/p\mathbb{Z}$ for all i. Therefore, set $E_i = M^{G_{n-i+1}}$, so we obtain a sequence

$$L' = E_1 \supseteq \cdots \supseteq E_n = L$$

   such that $[E_i : E_{i+1}] = p$ and E_i/E_{i+1} is normal.

3. We need to show that the norm over L'/L satisfies $N_{a_1,\ldots,a_r/k} = N_{E_1/E_0} \circ \cdots \circ N_{E_n/E_{n-1}}$, where the right-hand side is independent of the choice of $a_1, \ldots, a_r \in k'$. Therefore we are done by Proposition 3.13.

□

Example 3.21. Suppose $r = n = 2$, and take

$$\begin{tikzcd} & E_2 = L' \arrow[dl, dash] \arrow[dr, dash] & \\ E_1 \arrow[dr, dash] & & L(a_1) \arrow[dl, dash] \\ & L & \end{tikzcd}$$

If $a_1 \in E_1$, then $N_{a_2/L(a_1)} = N_{E_2/E_1}$ and $N_{a_1/L} = N_{E_1/L}$; if $a_1 \notin E_1$, then $N_{a_2/L(a_1)} = N_{E_2/L(a_1)}$ and $N_{a_1/L} \circ N_{E_2/L(a_1)} = N_{E_1/L} \circ N_{a_1/E_1} = N_{E_1/L} \circ N_{E_2/E_1}$ by Proposition 3.19.

Remark 3.22 ([Hes05], Remark 4)**.** By homotopy invariance, Theorem 3.5 says that $\sum\limits_{\nu \in \mathbb{P}^1_k} N_{k(\nu)/k} \circ \partial_\nu = 0$ on $K^M_*(k(t))$, c.f., residue theorem. Moreover, this holds for any algebraic function field L/k.

Theorem 3.23 ([BT06], Theorem 5.6; [MVW06], Theorem 5.4; Weil Reciprocity)**.** For any algebraic function field L/k, we have $\sum\limits_{\nu \in \mathrm{DV}(L/k)} N_{k(\nu)/k} \circ \partial_\nu = 0$ on $K^M_*(L)$.

Proof. The key idea is that there is a finite map from every curve to $\mathbb{P}^1_k$, c.f., [Sus83], where we want to show the statement on the fibers. That is, we want to show that for every finite extension E/F between algebraic function fields and $w \in \mathrm{DV}(F/k)$ as a discrete valuation, then we have

$$\sum_{\substack{\nu \in \mathrm{DV}(E/k) \\ \nu \text{ lying over } w}} N_{k(\nu)/k(w)} \circ \partial_\nu = \partial_w \circ N_{E/F} \tag{3.24}$$

on $K^M_*(E)$ using Proposition 3.16. Since L is a finite extension of $k(t)$, we have

$$\begin{aligned} \sum_{\nu \in \mathrm{DV}(L/k)} N_{k(\nu)/k} \circ \partial_\nu &= \sum_{w \in \mathbb{P}^1_k} \sum_{\nu / w} N_{k(\nu)/k(w)} \circ \partial_\nu \\ &= \left(\sum_{w \in \mathbb{P}^1_k} N_{k(w)/k} \circ \partial_w \right) \circ N_{L/k(t)} \\ &= 0 \end{aligned}$$

by homotopy invariance, c.f., Remark 3.22. □

3.3 Rost Complex

Definition 3.25. Suppose X is an integral scheme. We define the Milnor K-theory $K^M_*(X)$ of a scheme to be the kernel of ∂^x_y[16] on the exact sequence

$$0 \longrightarrow K^M_*(X) \longrightarrow K^M_*(K(X)) \xrightarrow{\partial^{\xi_x}_y} \bigoplus_{y \in X^{(1)}} K^M_{*-1}(k(y))$$

Here for any point $x \in X^{(n)}$ with codimension n, we have a divisor $y \in X^{(n+1)} \cap \bar{X}$, we define $\partial^x_y : K^M_*(k(x)) \to K^M_*(k(y))$ as the following: let $Z = \overline{\{x\}}$ and $p : \tilde{Z} \to Z \ni y$ be the normalization, then define

$$\partial^x_y = \sum_{\substack{u \in \tilde{Z} \\ p(u) = y}} N_{k(u)/k(\nu)} \circ \partial_u$$

with u running through finitely many points of $\tilde{Z}$ lying over y.[17]

Definition 3.26. For any scheme X, define the Rost complex by

$$C^p(X, K^M_n) = \bigoplus_{x \in X^{(p)}} K^M_{n-p}(k(x)).$$

[16]Here we denote $\partial^x_y = \partial^{\xi_x}_y$ to be the map from the K-theory of the function field $K(X) = \mathcal{O}_{X,\xi_x}$ where ξ_x corresponds to the generic point, therefore this map is the corresponding residue homomorphism.

[17]Without making precise formalization, the K-groups above are just cycle premodules over X, c.f., [Ros96], page 337. Moreover, the whole "$(-)^x_y$" notation means a specific component for $x \in X$ and $y \in Y$ of the $(-)$-map about direct sums.

Define

$$\begin{aligned} d_X : C^p(X, K_n^M) &\to C^{p+1}(X, K_n^M) \\ x \in X^{(p)} &\mapsto y \in X^{(p+1)} \\ d_X &= \begin{cases} \partial_y^x, & y \in \bar{x} \\ 0, & y \notin \bar{x} \end{cases} \end{aligned}$$

Remark 3.27. Note that the last two terms of the complex is given by the principal divisor map

$$\operatorname{div} : C^{n-1}(X, K_n^M) = \bigoplus_{x \in X^{(n-1)}} k(x)^\times \to C^n(X, K_n^m) = \bigoplus_{x \in X^{(n)}} \mathbb{Z},$$

and its cokernel is the classical Chow groups $\mathrm{CH}^n(X)$ of n-dimensional cocycles on X. In particular, the nth cohomology of the complex agrees with the nth Chow group, i.e.,

$$H^n(C^*(X, K_n^M)) = \mathrm{CH}^n(X).$$

Therefore, the Rost complex gives rise to a notion of higher Chow group, which is a bit different from the usual notion.

Our main goal is to show that $C^*(X; K_n^M)$ is indeed a complex, which shows how Rost complex connects Milnor K-theory with the Chow group, c.f., [Ros96].

Definition 3.28. Suppose $f : X \to Y$ is a proper morphism between schemes of finite type over a field k. We define the pushforward of the Rost complex to be

$$f_* : C^p(X, K_n^M) = \bigoplus_{x \in X^{(p)}} K_{n-p}^M(k(x)) \to C^{p+\dim(Y)-\dim(X)}(Y, K^M_{n+\dim(Y)-\dim(X)})$$

such that for $x \in X$ and $y \in Y$, if $y = f(x)$ and $[k(x) : k(y)] < \infty$, then $(f_*)_y^x = N_{k(x)/k(y)}$, otherwise $(f_*)_y^x = 0$.

Definition 3.29. Suppose $f : Y \to X$ is a flat morphism between schemes of finite type over k, and define $Y_x = Y \times_X \mathrm{Spec}(k(x)) = f^{-1}(x)$ to be the fiber of x. We define the pullback of the Rost complex to be

$$f^* : C^p(X, K_n^M) \to C^p(Y, K_n^M)$$

using the following procedure. Suppose we have $x \in X$ and a generic point $y \in Y_x^{(0)}$ on the fiber, then the localization $Y_{x,(y)}$ of Y_x in y is the spectrum of an Artinian ring $R = \mathcal{O}_{Y_x, y}$, the stalk of the local ring $\mathcal{O}_{Y_x}$ at $y \in Y$, with unique residue class field $k(y)$. As a module over itself, we obtain a notion of length $\ell_{\mathcal{O}_{Y_x,y}}(\mathcal{O}_{Y_x,y})$. With the embedding $K_*^M(k(x))$ into $K_*^M(k(y))$, we define $(f^*)_y^x = \ell_{\mathcal{O}_{Y_x,y}}(\mathcal{O}_{Y_x,y}) \cdot (K_*^M(k(x)) \to K_*^M(k(y)))$. For other choices of $x \in X$ and $y \in Y$, we have $(f^*)_y^x = 0$.

Proposition 3.30 ([Ros96], Proposition 4.4 & 4.6)**.** Let d_X, d_Y be the differential of the Rost complex of X and of Y respectively.

1. If $f : X \to Y$ is proper, then $d_Y \circ f_* = f_* \circ d_X$.
2. If $f : Y \to X$ is flat, then $d_Y \circ f^* = f^* \circ d_X$.

Proof.

1. We show that $\delta(f_*) := d_Y \circ f_* - f_* \circ d_X = 0$. Let $x \in X^{(p)}$ and $y \in Y^{(p+\dim(Y)-\dim(X))}$. We should discuss the possible relations between $f(x)$ and y.

- if $y \notin \overline{f(x)}$, since the differential maps a closed subset to some subset of the closed subset, we know the image $\delta(f_*)^x_y = 0$ by definition;
- if $f(x) = y$, we can perform base-change, given by the pullback square

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(k(y)) \arrow[r] & Y \end{tikzcd}$$

so that we get a proper curve $\bar{x} \subseteq X_y \to \mathrm{Spec}(X_y)$ according to the dimension condition, now apply Theorem 3.23;
- if $y \in \overline{f(x)}^{(1)}$, then $[k(x) : k(f(x))] < \infty$, so we can use the compatibility between ∂'s and norms from Equation (3.24).

2. Define $\delta(f^*) := d_Y \circ f^* - f^* \circ d_X$, and suppose $y \in Y^{(p+1)}$ and $x \in X^{(p)}$, The only non-trivial case is when $f(y) \in \bar{X}^{(1)}$. By normalization and localization at y and $f(y)$, we reduce to the case when $X = \mathrm{Spec}(R)$ for some discrete valuation ring R and $Y = \mathrm{Spec}(S)$ for some local ring S of dimension at most 1. By definition,

$$\delta(f^*)^x_y = \sum_{u \in Y^{(0)}} \ell_{\mathcal{O}_{Y,u}}(\mathcal{O}_{Y,u}) \cdot \partial^u_y \circ K^M_*(k(u)/k(x)) - \ell_{\mathcal{O}_Y}(\mathcal{O}_Y/\mathfrak{m}_x\mathcal{O}_Y) K^M_*(k(u)/k(f(y))) \circ \partial^x_{f(y)}$$

Suppose $\hat{S}$ is the normalization of S. We have

$$\begin{aligned} \partial^u_y \circ K^M_*(k(u)/k(x)) &= \sum_{w \in \hat{S}^{(1)}} N_{k(w)/k(u)} \circ \partial^u_w \circ K^M_*(k(u)/k(x)) \\ &= \sum_{w \in \hat{S}^{(1)}} \ell_{\hat{S}_{(u)}}(\hat{S}_{(u)}/\mathfrak{m}_x \hat{S}_{(u)}) N_{k(w)/k(y)} \circ K^M_*(k(w)/k(f(y))) \circ \partial^x_{f(y)} \\ &= \sum_{w \in \hat{S}^{(1)}} \ell_{\hat{S}_{(u)}}(\hat{S}_{(u)}/\mathfrak{m}_x \hat{S}_{(u)})[k(u) : k(y)] K^M_*(k(y)/k(f(y))) \circ \partial^x_{f(y)} \end{aligned}$$

by Proposition 3.11. Then the question is reduced to computations of lengths.

□

Theorem 3.31 ([Ros96], Lemma 3.3 & page 355[18]). For any scheme X of finite type over k, we have $d_X \circ d_X = 0$.

Proof. For any choice of $x \in X^{(p)}$ and $y \in X^{(p+1)}$, we know it suffices to prove the statement for any integral affine scheme $X = \mathrm{Spec}(R)$ over k, where R is a local ring of dimension 2. Let x_0 be the[19] closed point on X, i.e., maximal ideal $\mathfrak{m}_x$, and ξ_x be the generic point of X, then on the divisor,

$$\sum_{x \in X^{(1)}} \partial^x_{x_0} \circ \partial^{\xi_x}_x = 0.$$

We choose a lift of a transcendental basis of $k(x_0)/k$ to R using the extension $k(x_0)/R/k$, so we can find a field $k \subseteq K \subseteq R$ such that $k(x_0)/K$ is finite, by defining K to be the extension on k attaching the said transcendental basis. Choose $u \in X \times_K k(x_0) =: X'$ which lies over x_0 with respect to the projection $p : X' \to X$, then $k(u) = k(x_0)$. Moreover, we know that $d_X \circ d_X = 0$ if and only if at u, we have $(p^* \circ d_X \circ d_X)_u = 0$, if and only if at u, $(d_{X'} \circ d_{X'} \circ p^*)_u = 0$, so it suffices to prove the case when $k(x_0) = k$.

[18] The proof has been reorganized: also see page 342-343 of [Ros96].

[19] There is a unique point of codimension 2.

Suppose X is a localization of $Y \subseteq \mathbb{P}_k^n$ at a rational point y, where Y is closed and $\dim(Y) = 2$. We define the flag variety $H = \mathrm{Gr}(1, n-2, n+1) \subseteq \mathbb{P}^n \times \mathrm{Gr}(n-2, n+1)$ as a Grassmannian in V of dimension $n+1$ by the set $\{(x, V) \mid X \subseteq V\}$, i.e., x is a line and $V \in \mathrm{Gr}(n-2, n+1)$, then we have a map

$$\begin{aligned} \mathbb{P}^n \times \mathrm{Gr}(n-2, n+1) \backslash H &\to \mathbb{P}^2 \times \mathrm{Gr}(n-2, n+1) \\ (x, V) &\mapsto (P_V(x), V) \end{aligned}$$

where $x \nsubseteq V$ and $P_V(x)$ is the projection of x centered at V. Let F be the function field $K(\mathrm{Gr}(n-2, n+1))$, then a dimension argument shows that $H_F \cap Y_F = \varnothing$.[20] To see this, note that $\dim(Y) = 2$, and note that for any $v \in \mathrm{Gr}(n-2, n+1)$, $\mathrm{codim}(v) = 3$, and we want to find $V \in \mathrm{Gr}(n-2, n+1)$ such that $Y \cap V = \varnothing$. We now have a projection $\varphi : H = \mathrm{Gr}(1, n-2, n+1) \to \mathbb{P}^n$ by forgetting the line. We know the flag variety H has dimension $4n-9$, moreover, it not only has a projection φ but also a projection to $\mathrm{Gr}(n-2, n+1)$ by definition.

$$\begin{tikzcd} H = \mathrm{Gr}(1, n-2, n+1) \arrow[r, two heads, "\psi"] \arrow[d, "\varphi"'] & \mathrm{Gr}(n-2, n+1) \\ \mathbb{P}^n \end{tikzcd}$$

We know φ has relative dimension $3n-9$ as a flat morphism, therefore the subspace $Y \subseteq \mathbb{P}^n$ satisfies

$$\dim(\varphi^{-1}(Y)) = 2 + 3n - 9 = 3n - 7 < 3n - 6 = (n-2) \cdot 3 = \dim(\mathrm{Gr}(n-2, n+1)).$$

Therefore, the composite

$$Y \xrightarrow{\varphi^{-1}} \varphi^{-1}(Y) \xrightarrow{\psi} \mathrm{Gr}(n-2, n+1)$$

is a proper subset of $\mathrm{Gr}(n-2, n+1)$, i.e., the composition

$$\varphi^{-1}(Y) \xhookrightarrow{\subseteq} H \longrightarrow \mathrm{Gr}(n-2, n+1)$$

is not surjective. Pick any $V \notin \mathrm{im}(\varphi^{-1}(Y))$, then it satisfies $V \cap Y = \varnothing$. Therefore, we have maps $\pi : \mathbb{P}_F^2 \backslash H_F \to \mathbb{P}_F^2$ and a diagram

$$\begin{tikzcd} Y_F \arrow[r, hook, "\subseteq"] \arrow[rr, bend left, "p"] \arrow[d, "q"'] & \mathbb{P}_F^2 \backslash H_F \arrow[r, "\pi"] & \mathbb{P}_F^2 \\ Y \end{tikzcd}$$

where p is proper. Since $y \in Y$ is a rational point, then it has a unique fiber, therefore we identify $q(y) = y \in Y$. Hence, $p^{-1}(p(y)) \cap Y_{\{0\}} = \{y\}$, so for any $s \in K_*^M(K(y))$, we have

$$\begin{aligned} \sum_{\substack{p \in (\mathbb{P}_F^2)^{(1)} \\ p(y) \in \bar{u}}} \partial_{p(y)}^u \partial_u^{\xi_{\mathbb{P}_F^2}} (p_* q^*(s)) &= (d_{\mathbb{P}_F^2} \circ d_{\mathbb{P}_F^2}(p_* q^*(s)))_{p(y)} \\ &= (p_* q^*(d_Y \circ d_Y(s)))_{p(y)} \\ &= K_*^M(F/k) \circ (d_Y \circ d_Y(s))_y. \end{aligned}$$

However, because the Grassmannian is a rational variety, then F/k is purely transcendental, therefore $K_*^M(F/k)$ is injective, hence the embedding $K_*^M(k) \hookrightarrow K_*^M(k(t))$ is an injection by homotopy invariance. Therefore, we may assume

[20] Here $H_F = H \times_{\mathrm{Gr}(n-2,n+1)} F$ is the base-change.

$X = \operatorname{Spec}(\mathcal{O}_{\mathbb{A}^2,(0,0)}) = \mathbb{A}^2_{(0,0)}$. Suppose $\mathbb{A}^2$ has coordinates of the form (s,t). By Theorem 3.5, we have a split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & K_*^M(k(s)) \arrow[r, "i"] & K_*^M(k(s,t)) \arrow[r] & \bigoplus_{x\in\mathbb{A}^1_{k(s)}} K^M_{*-1?}(k(x)) \arrow[l, bend right, "(\tau_x)"'] \arrow[r] & 0 \end{tikzcd}$$

Since the transfer is defined, we have a explicit description of the splitting $(\tau_x)_{x\in X}$, which is defined componentwise to be

$$\tau_x(a) = N_{k(x)(t)/k(s,t)}([t-t(x)]K_*^M(k(x)(t)/k(x))(a))$$

where $t(x)$ is the canonical generator of $k(x)/k(s)$. Fix $y \in \mathbb{A}^1_{k(s)}$, then we can check that

$$\partial_y \circ \tau_x = \begin{cases} \mathrm{id}, & y = x \\ 0, & y \neq x \end{cases}$$

since $[t-t(x)]K_*^M(k(x)(t)/k(x))(a)$ has non-zero valuation only at $t-t(x)$, which lies over the valuation at $x \in k(s)$.

Suppose $b \in K_*^M(k(s,t))$, $y = \{s=0\} \in X = \mathbb{A}^2$, ν runs through valuations $k(x)(t)/k(t)$ and $\bar{\nu} = \nu/k(x)$. If $b \in \operatorname{im}(i)$, then $d_X \circ d_X(b) = 0$ by naturality of the pullback along

$$\begin{aligned} p: \mathbb{A}^2 &\to \mathbb{A}^2 \\ (s,t) &\mapsto s \end{aligned}$$

which is given by

$$p^*: k(s) \to k(s,t),$$

and now $i = K_*^M(p^*)$. Therefore, it suffices to prove the statement $d_X \circ d_X = 0$ for $X = \mathbb{P}^1$. Since $\dim(\mathbb{A}^1) = 1$, then assume $b = \tau_x(a)$ for $x \in (\mathbb{A}^2)^{(1)}$, i.e., $x \in \mathbb{A}^1_{k(s)}$ corresponds to a divisor $\bar{x}$ in $\mathbb{A}^2$ different from $\{s=0\}$, then it suffices to prove that

$$\partial_0^y \circ \partial_y^{\xi_{\mathbb{A}^2}} \circ \tau_x = -\partial^x_{(0,0)} \tag{3.32}$$

where $p(x) = \xi_{\mathbb{A}^1}$ and $\bar{x} \ni (0,0)$.

We know that

$$\sum_{0\in y} \partial_0^y \circ \partial_y^{\xi_{\mathbb{A}^2}} = 0. \tag{3.33}$$

However, one can show that Equation (3.32) and Equation (3.33) are equivalent: a divisor in $\mathbb{A}^2$ is either $\{s=0\}$ or a point $x \in \mathbb{A}^1_{k(s)}$ corresponding to an irreducible polynomial, then note that the first case corresponds to $-\partial^x_{(0,0)}$, and the second case corresponds to $\partial_0^y \circ \partial_y^{\xi_{\mathbb{A}^2}} \circ \tau_x$. □

4 Comparison Theorem of Milnor K-theory and Motivic Cohomology

In this section, we will compute $H^{n,n}(\mathrm{Spec}(F),\mathbb{Z})$, the nth motivic cohomology of F, where F/k is a field. This requires studying the connections between motivic cohomology and Milnor K-theory, c.f., lecture 5 of [MVW06].

Proposition 4.1 ([MVW06], Lemma 5.2)**.**

$$H^{p,q}(\mathrm{Spec}(F),\mathbb{Z}) = H_{q-p}(C_*\mathbb{Z}(\mathbb{G}_m^{\wedge q})(\mathrm{Spec}(F)))$$

for all p,q, where C_* is the Suslin complex, c.f., Definition 2.40. In particular, if $p=q=n$, then

$$\begin{aligned} H^{n,n}(\mathrm{Spec}(F),\mathbb{Z}) &= \mathrm{coker}(\mathbb{Z}(\mathbb{G}_m^{\wedge n})(\mathbb{A}^1) \xrightarrow{\partial_0-\partial_1} \mathbb{Z}(\mathbb{G}_m^{\wedge n})(\mathrm{Spec}(F))) \\ &= \mathrm{coker}(\mathrm{Cor}_k(\mathbb{A}^1,\mathbb{G}_m^{\wedge n}) \xrightarrow{\partial_0-\partial_1} \mathrm{Cor}_k(\mathrm{Spec}(F),\mathbb{G}_m^{\wedge n}))\end{aligned}$$

Proof. By definition, we have $H^{p,q}(\mathrm{Spec}(F),\mathbb{Z}) = \mathbb{H}^p(\mathrm{Spec}(F), C_*(\mathbb{Z}(\mathbb{G}_m^{\wedge q})[-q]))$. Since the functor defined by $\mathcal{G}\mapsto\mathcal{G}(\mathrm{Spec}(F))$ is exact, then we retrieve

$$\mathbb{H}^p(\mathrm{Spec}(F), C_*\mathbb{Z}(\mathbb{G}_m^{\wedge q})[-q]) = H^p(C_*\mathbb{Z}[\mathbb{G}_m^{\wedge q}][-q](\mathrm{Spec}(F))) = H_{q-p}(C_*\mathbb{Z}(\mathbb{G}_m^{\wedge q})(\mathrm{Spec}(F)))$$

using the duality $H^n = H_{-n}$. □

Now suppose E/F is a finite field extension over k, then by Proposition 4.1, the pushforward of cycles gives a map

$$N_{E/F}: H^{n,n}(\mathrm{Spec}(E),\mathbb{Z}) \to H^{n,n}(\mathrm{Spec}(F),\mathbb{Z}).$$

Proposition 4.2 ([MVW06], Lemma 5.3)**.** Suppose $x\in H^{n,n}(\mathrm{Spec}(E),\mathbb{Z})$ and $y\in H^{m,m}(\mathrm{Spec}(F),\mathbb{Z})$, then

1. $N_{E/F}(y_E\cdot x) = y\cdot N_{E/F}(x)$ and $N_{E/F}(x\cdot y_E) = N_{E/F}(x)\cdot y$;
2. suppose $F\subseteq E\subseteq K$ are finite extensions, where K/F is normal, then similar to Corollary 3.12, we have
$$N_{E/F}(x)_K = [E:F]_{\mathrm{insep}} \sum_{j\in\mathrm{Hom}(E,K)} j(x);$$
3. if $F\subseteq E'\subseteq E$, then $N_{E/F} = N_{E'/F}\circ N_{E/E'}$.

Proof. The finite correspondence $\mathbb{Z}(\mathbb{G}_m^{\wedge n})(\mathrm{Spec}(F))$ is the quotient of free abelian group given by closed points of $\mathbb{G}_m^{\times n}$ over those of the form $(x_1,\dots,1,\dots,x_n)$. The exterior product

$$\mathrm{Cor}(\mathrm{Spec}(F),\mathbb{G}_m^{\times n}) \times \mathrm{Cor}(\mathrm{Spec}(F),\mathbb{G}_m^{\times m}) \to \mathrm{Cor}(\mathrm{Spec}(F),\mathbb{G}_m^{\times(n+m)})$$

gives a ring structure on $\bigoplus_n H^{n,n}(\mathrm{Spec}(F),\mathbb{Z})$.

1. This comes from the projection formula of cycles, c.f., Proposition 1.28.
2. We have a Cartesian square
$$\begin{tikzcd} (\mathbb{G}_m^{\times n})_E \arrow[d] & (\mathbb{G}_m^{\times n})\otimes_F(E\otimes_F K) \arrow[l] \arrow[d] \\ (\mathbb{G}_m^{\times n})_F & (\mathbb{G}_m^{\times n})_K \arrow[l] \end{tikzcd}$$
We have a similar property as Proposition 3.11, as in
$$\begin{tikzcd} K_*^M(E) \arrow[r] \arrow[d, "N"'] & \bigoplus_{p\in\mathrm{Spec}(E)\otimes_F K} K_*^M(k(p)) \arrow[d, "\sum N_{k(p)/K}"] \\ K_*^M(F) \arrow[r] & K_*^M(K). \end{tikzcd}$$
then the proof is the same as Corollary 3.12.

3. This follows from the same transitivity statement for the pushforward of cycles.

□

Proposition 4.3 ([MVW06], Corollary 5.5)**.** Let $p : Z \to \mathbb{A}^1_F$ be a finite surjective morphism of schemes and suppose that Z is integral. Let $f_1, \dots, f_n \in \mathcal{O}^*(Z)$ are invertible functions on Z, and define the pullbacks $p^*(0) = \sum_i n_i^0 z_i^0$ and $p^*(1) = \sum_i n_i^1 z_i^1$ in $Z_0(Z)$, i.e., $n_i^t \in \mathbb{N}$ is the multiplicity at z_i^t and $z_i^t \in Z$ for $t = 0, 1$. For $t = 0, 1$, we define

$$\varphi_t = \sum_i n_i^t N_{k(z_i^t)/F}([f_1] \cdots [f_n]|_{z_i^t}),$$

then $\varphi_0 = \varphi_1 \in K_n^M(F)$.

Proof. The extension $k(Z)/F$ is an algebraic function field. Denote $\mathbb{A}^1_F = \mathrm{Spec}(F(t))$, then $x = \left[\frac{t}{t-1}\right][f_1] \cdots [f_n]$, i.e., t is a parameter in $\mathbb{A}^1_F$, and $\nu \in \mathrm{DV}(k(Z)/F)$. We consider the valuation of ν at $K(\mathbb{A}^1_F)$.

- If $\nu|_{k(\mathbb{A}^1_F)} = \nu_\infty$ is the valuation at ∞, then $\left[\frac{t}{t-1}\right] = 1$ at $t = \infty$ since other functions $[f_i]$ has no evaluation as invertible functions, so $\partial_\nu(x) = 0$.

- If $\nu|_{k(\mathbb{A}^1_F)} \neq \nu_0, \nu_1, \nu_\infty$, then $\frac{t}{t-1}, f_1, \dots, f_n \in \mathcal{O}_\nu^\times$, i.e., valuated to be 0 via ν. Therefore, $\partial_\nu(x) = 0$ at all finite places except those at 0 or 1.

- For $t = 0, 1$, if $\nu|_{k(\mathbb{A}^1_F)} = \nu_t$, then ν centers at some fibers z_i^t of t. Let $p_Z : \tilde{Z} \to Z$ be the normalization, then for any i, we have
$$\sum_{p_Z(\nu) = z_i^t} N_{k(\nu)/k(z_i^t)}(\partial_\nu(x)) = (-1)^t\ n_i^t[f_1] \cdots [f_n]|_{z_i^t}$$
for $t = 0, 1$. For the case when $t = 0$, we have $\left[\frac{t}{t-1}\right] = -1$, then the valuation here is 1 since t has a zero of order 1. Taking $\partial_\nu(x)$ gives $[f_1] \cdots [f_n]$, and by definition the degree of extension $k(\nu)/k(z_i^0)$ is just n_i^0, and now the formula follows from the projection formula. When $t = 1$, the valuation of $\left[\frac{t}{t-1}\right]$ at $t = 1$ is -1, which contributes to the difference.

We find that $\sum_{p_Z(\nu)} N_{k(\nu)/F}(\partial_\nu(x)) = \varphi_0 - \varphi_1$ since the only non-zero valuations are at 0 and at 1, then by Weil reciprocity Theorem 3.23 we know the difference is 0, as desired. □

Now we define a map

$$\theta : H^{n,n}(\mathrm{Spec}(F), \mathbb{Z}) \to K_n^M(F)$$

as follows: every closed point x of $(\mathbb{G}_m^{\times n})_F$ corresponds to an n-tuple $(x_1, \dots, x_n)$ where each $x_i \in k(x)^\times$, then define

$$\begin{aligned} f : \mathbb{Z}(\mathbb{G}_m^{\wedge n})(\mathrm{Spec}(F)) &\to K_n^M(F) \\ x &\mapsto N_{k(x)/F}([x_1] \cdots [x_n]) \end{aligned}$$

If one of x_i's is 1, then $f(x) = 0$,[21] so it is well-defined.

Recall that $H^{n,n}(\mathrm{Spec}(F), \mathbb{Z}) = \mathrm{coker}(\mathbb{Z}(\mathbb{G}_m^{\wedge n})(\mathbb{A}^1) \xrightarrow{\partial_1 - \partial_0} \mathbb{Z}(\mathbb{G}_m^{\wedge n})(\mathrm{Spec}(F)))$, so to construct θ, we need to show that f vanishes on $\mathrm{im}(\partial_1 - \partial_0)$. In particular, this induces a unique θ via

$$\begin{tikzcd} \mathbb{Z}(\mathbb{G}_m^{\wedge n})(\mathbb{A}^1) \arrow[r, "\partial_1 - \partial_0"] & \mathbb{Z}(\mathbb{G}_m^{\wedge n})(\mathrm{Spec}(F)) \arrow[r, two heads] \arrow[dr, "f"'] & H^{n,n}(\mathrm{Spec}(F), \mathbb{Z}) \arrow[d, dashed, "\theta"] \\ & & K_n^M(F) \end{tikzcd}$$

[21] To see this, again recall that $\mathbb{Z}(\mathbb{G}_m^{\wedge n})$ is the free abelian group generated by adjoining $\mathbb{Z}$ with the closed points in $\mathbb{G}_m^{\times n}$, and then quotienting the equivalence relation on the n-tuple $(x_1, \dots, x_n)$ where $x_i = 1$ for some i.

Now $\mathbb{Z}(\mathbb{G}_m^{\wedge n})(\mathbb{A}^1)$ is generated by irreducible subset $C \subseteq \mathbb{A}^1 \times \mathbb{G}_m^{\times n}$, such that C is finite and surjective over $\mathbb{A}^1$:

$$\begin{tikzcd} C \arrow[r, hook, "\subseteq"] \arrow[dr, "\text{finite surjective}"'] & \mathbb{A}^1 \times \mathbb{G}_m^{\times n} \arrow[d] \\ & \mathbb{A}^1 \end{tikzcd}$$

Since C has a projection to $\mathbb{G}_m^{\wedge n}$, then it gives invertible functions $f_1, \dots, f_n \in \mathcal{O}^*(C)$ via pullback of the parameter of each copy of $\mathbb{G}_m$. Since C is surjective and finite over $\mathbb{A}^1$, then by Proposition 4.3, we have $f \circ (\partial_0 - \partial_1) = 0$, hence θ is defined.

Conversely, we define a map $\lambda_F : K_n^M(F) \to H^{n,n}(\operatorname{Spec}(F), \mathbb{Z})$ as follows: every $x \in F^\times$ corresponds to a map $x : \operatorname{Spec}(F) \to \mathbb{G}_m$, which gives $\lambda_F([x]) \in H^{1,1}(\operatorname{Spec}(F), \mathbb{Z})$, since ∂ is an isomorphism on $H^{1,1}(\operatorname{Spec}(F), \mathbb{Z})$ already. Recall that the Milnor K-group is given by a tensor algebra, therefore we need to show that λ_F is well-defined. To show this, we define $[x : 1-x] = \lambda_F([x]) \cdot \lambda_F([1-x]) \in H^{2,2}(\operatorname{Spec}(F), \mathbb{Z})$, then we need to show that $[x : 1-x] = 0$ if $x \neq 0, 1$.

Remark 4.4.

- Note that $[ab : c] = [a : c] + [b : c]$, which follows from the linearity of λ_F. Similar fact holds on the second coordinate as well.
- In particular, $[1 : x] = 0$. Indeed, we know $[1 : x] + [1 : x] = [1 : x]$.

Proposition 4.5 ([MVW06], Lemma 5.8)**.** Suppose there exists $n > 0$, such that $n[x : 1-x] = 0$ for all finite extensions of F and all $x \neq 0, 1$, then $[x : 1-x] = 0 \in H^{2,2}(\operatorname{Spec}(F), \mathbb{Z})$ for all $x \neq 0, 1$.

Proof. We proceed by induction on the number of factors of n. Essentially, we just need to suppose $n = mp$ where p is prime, and then show that $m[x : 1-x] = 0$. In the cases below, let $y = \sqrt[p]{x}$, i.e., $y^p = x$.

- First suppose $y \notin F$, then we define $E = F(y)$, then $0 = mp[y : 1-y] = m[x : 1-y]$, and $1-x = N_{E/F}(1-y)$. By the projection formula, we have

$$0 = N_{E/F}(m[x : 1-y]) = m[x : N_{E/F}(1-y)] = m[x : 1-x].$$

  In the following cases, suppose $y \in F$.

- Suppose F is a splitting field of the polynomial $T^p - x \in F[T]$, i.e., there is a primitive pth root of unity ω in F, so we know all the pth root of unity of x, given by the collection $\{y\omega^i\}$, is contained in F. By linearity and distributivity we know

$$\begin{aligned} m[x : 1-x] &= \sum_i m[x : 1 - y\omega^i] \\ &= \sum_i mp[y\omega^i : 1 - y\omega^i] \\ &= \sum_i n[y\omega^i : 1 - y\omega^i] \\ &= 0. \end{aligned}$$

  Alternatively, we have

$$m[x : 1-x] = mp[y : 1-y^p]$$

$$
\begin{aligned}
&= \sum_i mp[y : \omega^i - y] \\
&= \sum_i mp \left([y : \omega^i] + \left[\frac{y}{\omega^i} : 1 - \frac{y}{\omega^i} \right] + \left[\omega^i : 1 - \frac{y}{\omega^i} \right] \right) \\
&= \sum_i mp \left([y : \omega^i] + \left[\omega^i : 1 - \frac{y}{\omega^i} \right] \right) \\
&= \sum_i m \left([y : \omega^{ip}] + \left[\omega^{ip} : 1 - \frac{y}{\omega^i} \right] \right) \\
&= \sum_i m \left([y : 1] + \left[1 : 1 - \frac{y}{\omega^i} \right] \right) \\
&= 0
\end{aligned}
$$

by Remark 4.4.

- Now suppose F is not a splitting field of $T^p - x \in F[T]$, then the primitive pth root of unity $\omega \notin F$.[22] In this case, let $E = F(\omega)$. It is easy to show that $N_{E/F}(\omega - y) = (-1)^p(y^p - 1)$, which is just $1 - x$ since $p \neq 2$, then by projection formula

$$
\begin{aligned}
m[x : 1 - x] &= m[x : N_{E/F}(\omega - y)] \\
&= N_{E/F}(m[x : \omega - y]) \\
&= N_{E/F}(n[y : \omega - y]) \\
&= 0.
\end{aligned}
$$

□

Proposition 4.6 ([MVW06], Proposition 5.9)**.** For $x \neq 0, 1$ in F, $[x : 1 - x] = 0$.

Proof. Recall that we can represent a finite correspondence $Z \in \mathrm{Cor}(\mathbb{A}^1, \mathbb{G}_m)$ as a cycle in $\mathbb{A}^1 \times \mathbb{G}_m$. Using the parametrization $(t, a) \in \mathbb{A}^1 \times \mathbb{G}_m$, we define Z to be

$$a^3 - t(x^3 + 1)a^2 + t(x^3 + 1)a - x^3 = 0.$$

Let ω be a root of $a^2 + a + 1$, so $\omega^3 = 1$. Suppose $E = F(\omega)$, then the fibers over $t = 0$ are $\{a = x\}$, $\{a = \omega x\}$, and $\{a = \omega^2 x\}$, and the fibers over $t = 1$ are $\{a = x^3\}$, $\{a = -\omega\}$, and $\{a = -\omega^2\}$. Since $x \neq 0$, then $x^3 \neq 0$, therefore $a \neq 0$.

Suppose $x^3 \neq 1$, we know $a \neq 1$, then Z is a pushforward coming from $\mathrm{Cor}(\mathbb{A}^1, \mathbb{G}_m \backslash \{1\})$. Solving t with respect to a, we know $Z \cong \mathbb{G}_m \backslash \{1\}$. In particular, Z is integral. In the category of finite correspondence, Z is just a morphism $\mathbb{A}^1 \to \mathbb{G}_m \backslash \{1\}$, so composing Z with the map

$$
\begin{aligned}
\mathbb{G}_m \backslash \{1\} &\to \mathbb{G}_m^{\times 2} \\
a &\mapsto (a, 1 - a),
\end{aligned}
$$

we obtain $Z' \in \mathrm{Cor}(\mathbb{A}^1, \mathbb{G}_m^{\times 2})$. Recall that the motivic cohomology in $H^{2,2}(\mathrm{Spec}(E), \mathbb{Z})$ is defined as the cokernel of the map $\partial_1 - \partial_0$, so $(\partial_1 - \partial_0)(Z') = 0$ in the cohomology. Therefore, $\partial_0(Z')$ and $\partial_1(Z')$ should give the same motivic cohomology class in $H^{2,2}(\mathrm{Spec}(E), \mathbb{Z})$, namely

$$\partial_0(Z') = [x : 1 - x] + [\omega x : 1 - \omega x] + [\omega^2 x : 1 - \omega^2 x]$$

[22] In particular, this implies $p \neq 2$. If $p = 2$, then we have $y^2 = (-y)^2 = x$, therefore y and $-y$ are the only roots of unity of x. Since $y \in F$, then $-y \in F$, contradiction.

$$
\begin{aligned}
&= [x : 1-x] + ([x : 1-\omega x] + [\omega : 1-\omega x]) + ([\omega^2 : 1-\omega^2 x] + [x : 1-\omega^2 x]) \\
&= [x : 1-x] + ([x : 1-\omega x] + [\omega : 1-\omega x]) + ([\omega : (1-\omega^2 x)^2] + [x : 1-\omega^2 x]) \\
&= [x : 1-x^3] + [\omega : (1-\omega x)(1-\omega^2 x)^2]
\end{aligned}
$$

is the same as

$$\partial_1(Z') = [x^3 : 1-x^3] + [-\omega : 1+\omega] + [-\omega^2 : 1+\omega^2]$$

over $F(\omega)$. By Remark 4.4, we have

$$
\begin{aligned}
[x^3 : 1-x^3] &= 3\partial_0(Z') \\
&= 3\partial_1(Z') \\
&= 3[x^3 : 1-x^3],
\end{aligned}
$$

thus $2[x^3 : 1-x^3] = 0$ over E, so

$$
\begin{aligned}
0 &= N_{E/F}(2[x^3 : 1-x^3]) \\
&= 2[N_{E/F}(x^3) : 1-x^3] \\
&= 2[x^6 : 1-x^3] \\
&= 4[x^3 : 1-x^3] \qquad (4.7)
\end{aligned}
$$

over F.

Claim 4.8. For arbitrary element $x \neq 0, 1$ in F such that $x^3 \neq 1$, we have $12[x : 1-x] = 0$ over F.

Subproof.

- If $x = y^3$ for some $y \in F$, then Equation (4.7) shows that $4[x : 1-x] = 0$ over F.
- If all $y \in \bar{F}$ such that $y^3 = x$ are not in F, then we set $K = F(y)$, so $N_{K/F}(1-y) = 1-x$, thus $4[x : 1-x] = 0$ over K by Equation (4.7), hence $0 = N_{K/F}(4[x : 1-x]) = 4[N_{K/F}(x) : 1-x] = 12[x : 1-x]$ over F.

■

By Claim 4.8 and Proposition 4.5 we know $[x : 1-x] = 0$ for arbitrary element $x \neq 0, 1$ in F such that $x^3 \neq 1$.

Finally, suppose $x^3 = 1$. By Remark 4.4, $3[x : 1-x] = [x^3 : 1-x] = [1 : 1-x] = 0$ in F, and again $[x : 1-x] = 0$ by Proposition 4.5. □

To show that λ_F is an isomorphism, since $\theta \circ \lambda_F = \mathrm{id}$, it suffices to show that λ_F is surjective.

Proposition 4.9 ([MVW06], Lemma 5.11)**.** For any finite extension E/F, the diagram

$$
\begin{tikzcd}
K_n^M(E) \arrow[r, "\lambda_E"] \arrow[d, "N_{E/F}"'] & H^{n,n}(\mathrm{Spec}(E), \mathbb{Z}) \arrow[d, "N_{E/F}"] \\
K_n^M(F) \arrow[r, "\lambda_F"'] & H^{n,n}(\mathrm{Spec}(F), \mathbb{Z})
\end{tikzcd}
$$

commutes.

Proof. Assume that all finite extensions of F has order l^n for l prime. Suppose we have an extension $[E : F] = l$. By the statement in Proposition 3.13, we know that $K_n^M(E)$ is generated by $[f_1] \cdots [f_n]$ where $f_1 \in E$, $f_2, \ldots, f_n \in F$, then the statement follows from projection formulas in Corollary 3.12 and Proposition 4.2.

If $[E:F]=l^m$, then we take an extension M/E such that M/F is Galois, then $\mathrm{Gal}(M/F)$ is an l-group. Using the decomposition series

$$E=E_n\supseteq\cdots\supseteq E_0=F$$

where $E_{i-1}\lhd E_i$ and $[E_i:E_{i-1}]=l$, we know the transitivity of norms reduces the question to the former case.

For general fields F, using Proposition 3.9, we know the maps $K_n^M(F)_{(l)}\to K_n^M(L)_{(l)}$ and $H^{n,n}(\mathrm{Spec}(F),\mathbb{Z})_{(l)}\to H^{n,n}(\mathrm{Spec}(L),\mathbb{Z})_{(l)}$ are injective for some algebraic extension L/F such that every finite extension of L has degree a power of l. Moreover, we may assume E/F is a simple extension which is either separable or purely inseparable. In both cases, we could apply Proposition 3.11, which reduces the proof to the previous case. □

Theorem 4.10 ([MVW06], Lemma 5.10)**.** The map $\lambda_F:K_n^M(F)\to H^{n,n}(\mathrm{Spec}(F),\mathbb{Z})$ is an isomorphism of rings.

Proof. If $x\in(\mathbb{G}_m^{\times n})_F$ is a rational point, it is in $\mathrm{im}(\lambda_F)$ by construction. In general, a closed point $x\in(\mathbb{G}_m^{\times n})^{(1)}$ is the pushforward of a rational point of $(\mathbb{G}_m^{\times n})_{k(x)}$, so the statement follows from Proposition 4.9. □

5 Effective Motivic Categories over Smooth Bases

We discuss a few categorical results formulated via Grothendieck's six operations on the level of sheaves, as discussed in [CD19], section A.5.

5.1 Grothendieck's Six-functor Formalism

Lemma 5.1. Let $\varphi : \mathscr{C} \to \mathscr{D}$ be a functor of small categories and $\mathcal{M}$ be a category with arbitrary colimits, then the functor

$$\begin{aligned} \varphi_* : \mathrm{PSh}(\mathscr{D}, \mathcal{M}) &\to \mathrm{PSh}(\mathscr{C}, \mathcal{M}) \\ \mathcal{F} &\mapsto \mathcal{F} \circ \varphi \end{aligned}$$

has a left adjoint φ^*.

Proof. Suppose that $\mathcal{G} \in \mathrm{PSh}(\mathscr{C}, \mathcal{M})$. For every $Y \in \mathscr{D}$, define C_Y to be the category whose objects are $\{Y \to \varphi(X) \mid X \in \mathscr{C}\}$ and morphisms from $a_1 : Y \to \varphi(X_1)$ to $a_2 : Y \to \varphi(X_2)$ are those $b : X_1 \to X_2$ such that $a_2 = \varphi(b) \circ a_1$. We have a contravariant functor

$$\begin{aligned} \theta_Y : C_Y &\to \mathcal{M} \\ (Y \mapsto \varphi(X)) &\mapsto \mathcal{G}X \end{aligned}$$

then define $(\varphi^*\mathcal{G})(Y) = \varinjlim \theta_Y$. For any morphism $c : Y_1 \to Y_2$ in $\mathscr{D}$, we define $(\varphi^*\mathcal{G})(c)$ by the commutative diagram

$$\begin{tikzcd} \theta_{Y_2}(Y_2 \to \varphi(X)) \arrow[r, "c^*"] \arrow[d] & \theta_{Y_2}(Y_1 \to \varphi(X)) \arrow[d] \\ \varinjlim \theta_{Y_2} \arrow[r, "(\varphi^*\mathcal{G})(c)"'] & \varinjlim \theta_{Y_1} \end{tikzcd}$$

$\square$

Definition 5.2. Suppose that $f : S \to T$ is a morphism in Sm/k. We have a functor

$$\begin{aligned} \varphi^f : \mathrm{Cor}_T &\to \mathrm{Cor}_S \\ X &\mapsto X \times_T S \\ f &\mapsto f \times_T S \end{aligned}$$

then Lemma 5.1 provides adjunction pairs

$$\begin{tikzcd} \mathrm{PSh}(T) \arrow[d, shift right, "f^*"'] \\ \mathrm{PSh}(S) \arrow[u, shift right, "f_*"'] \end{tikzcd}$$

where $f_* = (\varphi^f)_*$, and

$$\begin{tikzcd} \mathrm{Sh}(T) \arrow[d, shift right, "f^*"'] \\ \mathrm{Sh}(S) \arrow[u, shift right, "f_*"'] \end{tikzcd}$$

Proposition 5.3. Suppose that $f : S \to T$ is a morphism in Sm/k.

1. $f^*\mathbb{Z}_T(Y) = \mathbb{Z}_S(Y \times_T S)$ for any $Y \in \mathrm{Sm}/T$, where $\mathbb{Z}_T(Y)(X) = \mathrm{Cor}_T(X, Y)$.

2. $(f_*\mathcal{F})^Y = f_*(\mathcal{F}^{Y\times_T S})$ for any $\mathcal{F} \in \mathrm{Sm}(S)$ and $Y \in \mathrm{Sm}/T$, where $\mathcal{F}^X(Y) = \mathcal{F}(X \times_S Y)$ for $X, Y \in \mathrm{Sm}/S$ is defined in Definition 2.20.

3. $\underline{\mathrm{Hom}}_T(\mathcal{F}, f_*\mathcal{G}) = f_*\underline{\mathrm{Hom}}_S(f^*\mathcal{F}, \mathcal{G})$ for any $\mathcal{F} \in \mathrm{Sh}(T)$ and $\mathcal{G} \in \mathrm{Sh}(S)$, and for internal hom $\underline{\mathrm{Hom}}$ defined as in Definition 2.20.

4. $f^*\mathcal{F} \otimes_S f^*\mathcal{G} = f^*(\mathcal{F} \otimes_S \mathcal{G})$ for $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(T)$.

Proof.

1. We have

$$\begin{aligned}\mathrm{Hom}_S(f^*\mathbb{Z}_T(Y), -) &= \mathrm{Hom}_T(\mathbb{Z}_T(Y), f_*(-)) \\ &= \mathrm{Hom}_S(\mathbb{Z}_S(Y \times_T S), -).\end{aligned}$$

2. Note that

$$\begin{aligned}(f_*\mathcal{F})^Y(Z) &= \mathcal{F}((Y \times_T Z) \times_T S) \\ &= \mathcal{F}((Z \times_T S) \times_S (Y \times_T S)) \\ &= (f_*(\mathcal{F}^{Y\times_T S})) \\ &= (f_*(\mathcal{F}^{Y\times_T S})(Z))\end{aligned}$$

for $Z \in \mathrm{Sm}/T$.

3. For any $Y \in \mathrm{Sm}/T$, we have

$$\begin{aligned}\underline{\mathrm{Hom}}_T(\mathcal{F}, f_*\mathcal{G})(Y) &= \mathrm{Hom}_T(\mathcal{F}, (f_*\mathcal{G})^Y) \\ &= \mathrm{Hom}_T(\mathcal{F}, f_*(\mathcal{G}^{Y\times_T S})) \\ &= \mathrm{Hom}_S(f^*\mathcal{F}, \mathcal{G}^{Y\times_T S}) \\ &= (f_*\underline{\mathrm{Hom}}_S(f^*\mathcal{F}, \mathcal{G}^{Y\times_T S})) \\ &= (f_*\underline{\mathrm{Hom}}_S(f^*\mathcal{F}, \mathcal{G}))(Y).\end{aligned}$$

4. For any $\mathcal{H} \in \mathrm{Sh}(S)$, we have

$$\begin{aligned}\mathrm{Hom}_S(f^*\mathcal{F} \otimes_S f^*\mathcal{G}, \mathcal{H}) &= \underline{\mathrm{Hom}}_S(f^*\mathcal{G}, \underline{\mathrm{Hom}}_S(f^*\mathcal{F}, \mathcal{H})) \\ &= \mathrm{Hom}_T(\mathcal{G}, f_*\underline{\mathrm{Hom}}_S(f^*\mathcal{F}, \mathcal{H})) \\ &= \mathrm{Hom}_T(\mathcal{G}, \underline{\mathrm{Hom}}_T(\mathcal{F}, f_*\mathcal{H})) \\ &= \mathrm{Hom}_T(\mathcal{F} \otimes_T \mathcal{G}, f_*\mathcal{H}) \\ &= \mathrm{Hom}_S(f^*(\mathcal{F} \otimes_T \mathcal{G}), \mathcal{H}).\end{aligned}$$

□

Definition 5.4. Suppose that $f : S \to T$ is a smooth morphism in Sm/k, then every $X \in \mathrm{Sm}/S$ is naturally an object in Sm/T. Moreover, for $X_1, X_2 \in \mathrm{Sm}/S$, then the Cartesian diagram

$$\begin{tikzcd} X_1 \times_S X_2 \arrow[r, "q_f"] \arrow[d] & X_1 \times_T X_2 \arrow[d] \\ S \arrow[r, "\Delta"'] & S \times_T S \end{tikzcd}$$

commutes, so q_f is a closed immersion. Thus we define

$$\begin{aligned}\varphi_f : \mathrm{Cor}_S &\to \mathrm{Cor}_T\\ X &\mapsto X\\ g &\mapsto (q_f)_*(g).\end{aligned}$$

So by Lemma 5.1, we obtain adjunction pairs

$$\begin{tikzcd} \mathrm{PSh}(T) \arrow[d, shift right, "(q_f)_*"'] \\ \mathrm{PSh}(S) \arrow[u, shift right, "f_\#"'] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{Sh}(T) \arrow[d, shift right, "(q_f)_*"'] \\ \mathrm{Sh}(S) \arrow[u, shift right, "f_\#"'] \end{tikzcd}$$

Proposition 5.5. We have $(q_f)_* = f^*$ for smooth $f : S \to T$.

Proof. For any $Y \in \mathrm{Sm}/S$, $\mathrm{id}_Y \in \mathrm{Cor}_T(Y, Y) = \mathrm{Cor}_S(Y, Y \times_T S)$ is the initial element of C_Y in Lemma 5.1. Because $f^*(Y) = \varinjlim C_Y$, then the direct limit is itself. Therefore, for any $\mathcal{F} \in \mathrm{PSh}(T)$, we have $(f^*\mathcal{F})(Y) = \mathcal{F}Y = (q_{f*}\mathcal{F})(Y)$. □

Proposition 5.6. Let $f : S \to T$ be smooth.

1. $f_\#\mathbb{Z}_S(X) = \mathbb{Z}_T(X)$ for any $X \in \mathrm{Sm}/S$.
2. $f^*(\mathcal{F}^Y) = (f^*\mathcal{F})^{Y\times_T S}$ for any $\mathcal{F} \in \mathrm{Sm}(T)$ and $Y \in \mathrm{Sm}/T$.
3. $\underline{\mathrm{Hom}}_T(f_\#\mathcal{F}, \mathcal{G}) = f_*\underline{\mathrm{Hom}}_S(\mathcal{F}, f^*\mathcal{G})$ for any $\mathcal{F} \in \mathrm{Sm}(S)$ and $\mathcal{G} \in \mathrm{Sm}(T)$.
4. $f_\#(\mathcal{F} \otimes_S f^*\mathcal{G}) = (f_\#\mathcal{F}) \otimes_T \mathcal{G}$, where $\mathcal{F} \in \mathrm{Sm}(S)$ and $\mathcal{G} \in \mathrm{Sm}(T)$.

Proof.

1. For any $\mathcal{F} \in \mathrm{Sh}(T)$, we have

$$\begin{aligned}\mathrm{Hom}_T(f_\#\mathbb{Z}_S(X), \mathcal{F}) &= \mathrm{Hom}_S(\mathbb{Z}_S(X), f^*\mathcal{F})\\ &= (f^*\mathcal{F})(X)\\ &= \mathcal{F}X.\end{aligned}$$

2. For any $X \in \mathrm{Sm}/S$, we have

$$\begin{aligned}(f^*(\mathcal{F}^Y))(X) &= \mathcal{F}(Y \times_T X)\\ &= \mathcal{F}((Y \times_T S) \times_S X)\\ &= (f^*\mathcal{F})^{Y\times_T S}(X).\end{aligned}$$

3. For $Y \in \mathrm{Sm}/T$, we have

$$\begin{aligned}\underline{\mathrm{Hom}}_T(f_\#\mathcal{F}, \mathcal{G})(Y) &= \mathrm{Hom}_T(f_\#\mathcal{F}, \mathcal{G}^Y)\\ &= \mathrm{Hom}_S(\mathcal{F}, f^*(\mathcal{G}^Y))\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Hom}_S(\mathcal{F}, (f^*\mathcal{G})^{Y\times_T S}) \\
&= \underline{\mathrm{Hom}}_S(\mathcal{F}, f^*\mathcal{G})(Y\times_T S) \\
&= (f_*\underline{\mathrm{Hom}}_S(\mathcal{F}, f^*\mathcal{G}))(Y).
\end{aligned}
$$

4. For $\mathcal{H} \in \mathrm{Sh}(T)$, we know

$$
\begin{aligned}
\mathrm{Hom}_T(f_\#(\mathcal{F}\otimes_S f^*\mathcal{G}), \mathcal{H}) &= \mathrm{Hom}_S(\mathcal{F}\otimes_S f^*\mathcal{G}, f^*\mathcal{H}) \\
&= \mathrm{Hom}_S(f^*\mathcal{G}, \underline{\mathrm{Hom}}_S(\mathcal{F}, f^*\mathcal{H})) \\
&= \mathrm{Hom}_S(\mathcal{G}, f_*\underline{\mathrm{Hom}}_S(\mathcal{F}, f^*\mathcal{H})) \\
&= \mathrm{Hom}_T(\mathcal{G}, \underline{\mathrm{Hom}}_T(f_\#\mathcal{F}, \mathcal{H})) \\
&= \mathrm{Hom}_T((f_\#\mathcal{F})\otimes_T \mathcal{G}, \mathcal{H})
\end{aligned}
$$

□

Given these operations on the level of sheaves, namely $\otimes_S$, $f_\#$, and f^*, we want to define them again on the level of derived categories.

Definition 5.7. Let $C^-(S)$ be the category of bounded-above complexes, then we define $K^-(S) = C^-(S)/\sim$, quotient by the chain homotopy equivalences, to be the homotopy category of bounded-above complexes of $\mathrm{Sh}(S)$.

We also define $D^-(S) = K^-(S)[\text{invertion of quasi-isomorphisms}]$ to be the derived category of bounded-above complexes of $\mathrm{Sh}(S)$.

Definition 5.8. A presheaf $F \in \mathrm{PSh}(S)$ is free if it is a direct sum of Yoneda presheaves $\mathbb{Z}_S(X)$, and is projective if it is a direct summand of a free presheaf.

A sheaf $F \in \mathrm{Sh}(S)$ is free (respectively, projective) if it is a sheafification of a free (respectively, projective) presheaf.

A bounded-above complex of $\mathrm{Sh}(S)$ is free (respectively, projective) if all its terms are free (respectively, projective).

Definition 5.9. A projective resolution of $K \in C^-(S)$ is a quasi-isomorphism $P \to K$ with P being projective.

Now suppose $S, T \in \mathrm{Sm}/k$, and let Y be a scheme with morphisms $S \xleftarrow{f} Y \xrightarrow{g} T$ where g is smooth (but f may not be smooth in general). We consider functors

$$
\begin{tikzcd}
\mathrm{Cor}_S \arrow[r, "\varphi^f"] \arrow[rr, bend left, "\varphi"] & \mathrm{Cor}_Y \arrow[r, "\varphi_g"] & \mathrm{Cor}_T \\
X \arrow[rr, mapsto] & & X\times_S Y
\end{tikzcd}
$$

and

$$
\begin{aligned}
\mathrm{Sm}/S &\xrightarrow{\psi} \mathrm{Sm}/T \\
X &\longmapsto X\times_S Y
\end{aligned}
$$

determined by the triple (Y, S, T).

Recall that $\mathrm{PSh}(S)$ has enough projectives, then it is possible to derive any left exact functor, e.g., to Ab. Moreover we obtain φ_* defined by the composition, and its left adjoint φ^* by Lemma 5.1.

Proposition 5.10. For any $\mathcal{F} \in \mathrm{PSh}(S)$, the sheafification $(L_i\varphi^*(\mathcal{F}^+))^+ = (L_i\varphi^*(\mathcal{F}))^+$ for $i \geqslant 0$, where $L_i\varphi^*$ is the ith left derived functor of φ^*.

Proof. It suffices to show that for any $\mathcal{F} \in \mathrm{PSh}(S)$ with $\mathcal{F}^+ = 0$, we have $((L_i\varphi^*)\mathcal{F})^+ = 0$ for all $i \geqslant 0$. Suppose this is true, then for any presheaf $\mathcal{F}$, we consider the natural morphism $\theta : \mathcal{F} \to \mathcal{F}^+$ satisfying $(\mathrm{coker}(\theta))^+ = (\ker(\theta))^+ = 0$ by the properties of sheafification. Hence for all $i \geqslant 0$, we know by the long exact sequence that

$$((L_i\varphi^*)\mathcal{F}^+)^+ = ((L_i\varphi^*)\,\mathrm{im}(\theta))^+ = ((L_i\varphi^*)\mathcal{F})^+,$$

so the proposition follows.

To prove the statement, we proceed by induction on i. The case when $i = 0$ is trivial, as φ^* commutes with the sheafification functor. Now we may suppose that it is true for $i < n$, and we want to show it for $i = n$. For any $\mathcal{F} \in \mathrm{PSh}(S)$, we cover it by presheaves

$$(i_\alpha) : \bigoplus_{\alpha \in \mathcal{F}(X)} \mathbb{Z}_S(X) \to \mathcal{F}$$

for sections $\alpha \in \mathcal{F}(X)$ over X. By the Yoneda lemma, this construction is a surjection. Since $\mathcal{F}^+ = 0$, then for any $\alpha \in \mathcal{F}(X)$ where $X \in \mathrm{Sm}/S$, there is a Nisnevich covering $U_\alpha \to X$ of X such that $\alpha|_{U_\alpha} = 0$, therefore the composite

$$\bigoplus_{\alpha \in \mathcal{F}(X)} \mathbb{Z}_S(U_\alpha) \longrightarrow \bigoplus_{\alpha \in \mathcal{F}(X)} \mathbb{Z}_S(X) \xrightarrow{(i_\alpha)} \mathcal{F}$$

is zero. Since the composite is zero, then each i_α factors through the cokernel of $\bigoplus_{\alpha \in \mathcal{F}(X)} \mathbb{Z}_S(U_\alpha) \to \bigoplus_{\alpha \in \mathcal{F}(X)} \mathbb{Z}_S(X)$, which is the direct sum of Čech complexes of form

$$\check{C}(U_\alpha/X) : (\cdots \longrightarrow \mathbb{Z}_S(U_\alpha \times_X U_\alpha) \longrightarrow \mathbb{Z}_S(U_\alpha) \longrightarrow \mathbb{Z}(X) \longrightarrow \cdots)$$

Taking the cokernel gives the cokernel of the Cech complex, and using the fact that the sheafification of this Čech complex gives an exact sequence, we first obtain a surjective map $\bigoplus_{\alpha \in \mathcal{F}(X)} H_0(\check{C}(U_\alpha/X)) \to \mathcal{F}$ since (i_α) factors through it, then taking the kernel K gives an exact sequence of presheaves

$$0 \longrightarrow K \longrightarrow \bigoplus_{\alpha \in \mathcal{F}(X)} H_0(\check{C}(U_\alpha/X)) \longrightarrow \mathcal{F} \longrightarrow 0$$

By Theorem 2.35, we know that $H_p(\check{C}(U_\alpha/X))^+ = 0$ for every $\alpha \in \mathcal{F}(X)$ and $p \in \mathbb{Z}$. Since the two other terms in the short exact sequence are zero after sheafification, then $K^+ = 0$ as well. By Lemma 1.26, we have a hypercohomology spectral sequence

$$(L_p\varphi^*)H_q(\check{C}(U_\alpha/X)) \Rightarrow (L_{p+q}\varphi^*)\check{C}(U_\alpha/X).$$

Since $H_q(\check{C}(U_\alpha/X))^+ = 0$, then by the inductive hypothesis, if $q < n$, then $(L_p\varphi^* H_q(\check{C}(U_\alpha/X)))^+ = 0$. Therefore, taking sheafification on both sides gives $(L_n\varphi^* H_0(\check{C}(U_\alpha/X)))^+ = (L_n\varphi^*\check{C}(U_\alpha/X))^+$. Since $\check{C}(U_\alpha/X)$ is a projective complex, we also have

$$\begin{aligned}(L_n\varphi^*\check{C}(U_\alpha/X))^+ &= H_n(\varphi^+\check{C}(U_\alpha/X))^+\\ &= H_n(\check{C}(\psi U_\alpha/\psi X))^+\\ &= 0.\end{aligned}$$

Therefore, we have $(L_n\varphi^*)H_0(\check{C}(U_\alpha/X)) = 0$, so $(L_n\varphi^*\mathcal{F})^+ = (L_{n-1}\varphi^* K)^+ = 0$ by the long exact sequence and the inductive hypothesis with $K^+ = 0$. $\square$

Proposition 5.11. The functor φ^* takes acyclic projective complexes of sheaves to acyclic projective complexes of sheaves.

Proof. For any projective sheaf $\mathcal{F} \in \operatorname{Sh}(S)$, we know $\mathcal{F} = \mathcal{G}^+$ for some projective $\mathcal{G} \in \operatorname{PSh}(S)$, therefore

$$(L_i\varphi^*\mathcal{F})^+ = (L_i\varphi^*\mathcal{G})^+ = 0$$

for any $i > 0$ by Proposition 5.10. Given a short exact sequence

$$0 \longrightarrow K \longrightarrow \mathcal{F} \longrightarrow P \longrightarrow 0$$

be a short exact sequence in $\operatorname{Sh}(S)$ with $(L_i\varphi^*P)^+ = 0$ for $i > 0$, then the sequence is still exact after applying φ^*. Moreover, if $\mathcal{F}$ is projective, we have $(L_i\varphi^*K)^+ = 0$ for $i > 0$. This concludes the proof. $\square$

Proposition 5.12. We have an exact functor

$$\begin{aligned} L\varphi^* : D^-(S) &\to D^-(T) \\ K &\mapsto \varphi^*P \end{aligned}$$

where $P \to K$ is a projective resolution.

Proof. By Proposition 5.11, the class of projective complexes is adapted, c.f., III.6.3 of [GM13], to the functor φ^*. We conclude the proof by using applying III.6.6 from [GM13]. $\square$

In the following, we write φ^* in place of $L\varphi^*$ for convenience.

Proposition 5.13.

1. The category $D^-(S)$ is endowed with a tensor product defined by

$$\begin{aligned} \otimes_S : D^-(S) \times D^-(S) &\to D^-(S) \\ (K, L) &\mapsto P \otimes_S Q \end{aligned}$$

where P, Q are projective resolutions of K and L, respectively, and $P \otimes_S Q = \operatorname{Tot}(\{P_i \otimes_S Q_j\})$. Moreover, for any $K \in D^-(S)$, the functor $K \otimes_S -$ is exact.

2. Suppose that $f : S \to T$ is smooth, then there is an exact functor

$$\begin{aligned} f_\# : D^-(S) &\to D^-(T) \\ K &\mapsto f_\# P \end{aligned}$$

where $P \to K$ is a projective resolution.

3. Suppose that $f : S \to T$ is in Sm/k, there is an exact functor

$$\begin{aligned} f^* : D^-(S) &\to D^-(T) \\ K &\mapsto f^* P \end{aligned}$$

where $P \to K$ is a projective resolution.

Proof.

1. Let $Y \in \operatorname{Sm}/S$. From the definition of φ, we take $(Y, S, T) = (Y, S, S)$, then $\varphi^*\mathcal{F} = \mathcal{F} \otimes_S \mathbb{Z}_S(Y)$ for any $\mathcal{F} \in \operatorname{Sh}(S)$ by the projection formula in Proposition 5.6. Given an acyclic projective complex P and a projective sheaf $\mathcal{F}$, then the complex $\mathcal{F} \otimes_S P$ is also acyclic by Proposition 5.11. Therefore, for any projective complex K, the complex $P \otimes_S K$ is also acyclic by the spectral sequence of the double complex $\{P_i \otimes K_j\}$. Then for any projective complexes P, Q, R with a quasi-isomorphism $a : P \to Q$, we have a projective complex $C(a \otimes_S R)$ where C is a mapping cone given by $C(a)^i = P^{i+1} \oplus Q^i$, then $C(a \otimes_S R) = C(a) \otimes_S R$. Since a is a quasi-isomorphism, then $C(a)$ is acyclic, so $C(a \otimes_S R) = C(a) \otimes_S R$ is acyclic as well. Therefore, $a \otimes_S R$ is also a quasi-isomorphism.

2. Take $(Y, S, T) = (S, S, T)$ and apply Proposition 5.12.
3. Take $(Y, S, T) = (T, S, T)$ and apply Proposition 5.12.

$\square$

Remark 5.14. There is no f_* in the current system of bounded above complexes. However, we can consider the category of unbounded complexes instead and construct it using model categories instead, c.f., [CD09].

Definition 5.15 ([GM13], III.2.9)**.** Let $\mathcal{A}$ be a category and $\mathcal{S}$ be a localizing class of morphisms in $\mathcal{A}$.

- A left roof between M and N is a diagram

$$\begin{tikzcd} & L \arrow[dl, "s"'] \arrow[dr, "f"] & \\ M & & N \end{tikzcd}$$

where $s \in \mathcal{S}$.

- A right roof between M and N is a diagram

$$\begin{tikzcd} & L & \\ M \arrow[ur, "g"] & & N \arrow[ul, "t"'] \end{tikzcd}$$

where $t \in \mathcal{S}$.

Proposition 5.16. Let $f : S \to T$ be a smooth morphism in Sm/k, then we have an adjunction

$$\begin{tikzcd} D^-(T) \arrow[d, shift left, "f^*"] \\ D^-(S) \arrow[u, shift left, "f_\#"] \end{tikzcd}$$

Proof. We have an adjunction

$$\begin{tikzcd} K^-(T) \arrow[d, shift left, "f^*"] \\ K^-(S) \arrow[u, shift left, "f_\#"] \end{tikzcd}$$

by the adjunction between sheaves. Since f^* has both left and right adjoints, it is exact, hence $Lf^* = f^*$. Suppose that $K \in D^-(S)$ and $L \in D^-(T)$, and that $p : P \to K$ is a projective resolution. We construct a morphism

$$\alpha : \mathrm{Hom}_{D^-(S)}(f_\# K, L) \to \mathrm{Hom}_{D^-(T)}(K, f^* L)$$

as follows: suppose that $s \in \mathrm{Hom}_{D^-(S)}(f_\# K, L)$ is written as a right roof

$$\begin{tikzcd} & R & \\ f_\# P \arrow[ur, "a"] \arrow[rr, dashed, "s"'] & & L \arrow[ul, Rightarrow, "b"'] \end{tikzcd}$$

where b is a quasi-ismorphism, denoted by $\Rightarrow$. We now define $\theta(s)$ to be the map

$$\begin{tikzcd} & & f^* R & \\ K & P \arrow[l, Leftarrow] \arrow[ur, "a'"] & & f^* L \arrow[ul, Rightarrow, "f^* b"'] \end{tikzcd}$$

where f^*b is also a quasi-isomorphism since f^* is exact.

Next, we construct a morphism

$$\xi : \mathrm{Hom}_{D^-(T)}(K, f^*L) \to \mathrm{Hom}_{D^-(S)}(f_\# K, L)$$

Suppose that $t \in \mathrm{Hom}_{D^-(T)}(K, f^*L)$ and $t \circ p$ is written as a left roof

$$\begin{tikzcd} & R \arrow[dl, Rightarrow, "a"'] \arrow[dr, "b"] & \\ P & & f^*L \end{tikzcd}$$

Without loss of generality, here we can take R to be projective, or to take a projective resolution. Define $\xi(t)$ as

$$\begin{tikzcd} & f_\# R \arrow[dl, Rightarrow, "f_\# a"'] \arrow[dr, "b'"] & \\ f_\# P & & L \end{tikzcd}$$

with $f_\# a$ being a quasi-isomorphism by Proposition 5.13. One checks that θ and ξ are inverses to each other. □

5.2 Homotopy Invariant Presheaves

Now let us get the homotopy relation $X \times \mathbb{A}^1 \sim X$ involved. Being a derived category is not enough for motives, as we also have to fully invert those relations.

Definition 5.17. An $\mathcal{F} \in \mathrm{PSh}(S)$ is called homotopy invariant if for every $X \in \mathrm{Sm}/S$, the map $p^* : \mathcal{F}(X) \to \mathcal{F}(X \times \mathbb{A}^1)$, induced from the projection $p : X \times \mathbb{A}^1 \to X$, is an isomorphism.

Remark 5.18. Since $p : X \times \mathbb{A}^1 \to X$ has a section, then p^* above is split injective. Hence, $\mathcal{F}$ is homotopy invariant if and only if p^* is surjective.

Remark 5.19. The homotopy invariant presheaves of abelian groups form a Serre subcategory of presheaves. In particular, if $\mathcal{F}$ and $\mathcal{G}$ are homotopy invariant presheaves with transfers, then the kernel and cokernel of every map $f : \mathcal{F} \to \mathcal{G}$ are homotopy invariant presheaves with transfers.

Lemma 5.20 ([MVW06], Lemma 2.16)**.** For any $\mathcal{F} \in \mathrm{PSh}(S)$, we have

$$\begin{aligned} i_\alpha : X &\hookrightarrow X \times \mathbb{A}^1 \\ x &\mapsto (x, \alpha) \end{aligned}$$

for $\alpha = 0, 1$, then the maps

$$i_0^*, i_1^* : (C_*(\mathcal{F}))(X \times \mathbb{A}^1) \to (C_*\mathcal{F})(X),$$

are defined as $\mathcal{F}(i_0)$ and $\mathcal{F}(i_1)$. $\mathcal{F}$ is homotopy invariant if and only if $i_0^* = i_1^* : \mathcal{F}(X \times \mathbb{A}^1) \to \mathcal{F}(X)$ for all X.

Proof. One direction is obvious. Now suppose $i_0^* = i_1^*$, we want to show that $\mathcal{F}$ is homotopy invariant. Denote by

$$\begin{aligned} m : \mathbb{A}^1 \times \mathbb{A}^1 &\to \mathbb{A}^1 \\ (x, y) &\mapsto Ky \end{aligned}$$

the multiplication map, we have a commutative diagram

$$\begin{tikzcd} & \mathcal{F}(X \times \mathbb{A}^1) \arrow[r, "i_0^*"] \arrow[dl, "\mathrm{id}_{X \times \mathbb{A}^1}"', "\cong"] \arrow[d, "(\mathrm{id}_X \times m)^*"] & \mathcal{F}(X) \arrow[d, "p^*"] \\ \mathcal{F}(X \times \mathbb{A}^1) & \mathcal{F}(X \times \mathbb{A}^1 \times \mathbb{A}^1) \arrow[l, "(i_1 \times \mathrm{id}_{\mathbb{A}^1})^*"] \arrow[r, "(i_0 \times \mathrm{id}_{\mathbb{A}^1})^*"'] & \mathcal{F}(X \times \mathbb{A}^1) \end{tikzcd}$$

By the condition, we have $p^* i_0^* = (\mathrm{id}_{\mathbb{A}^1} \times i_0)^* m^* = (\mathrm{id}_{\mathbb{A}^1} \times i_1)^* m^* = \mathrm{id}_{X \times \mathbb{A}^1}$. Since $i_0^* p^* = \mathrm{id}_X$, then p^* is an isomorphism. □

Lemma 5.21 ([MVW06], Lemma 2.18)**.** For any $\mathcal{F} \in \mathrm{PSh}(S)$, the maps

$$i_0^*, i_1^* : (C_*\mathcal{F})(X \times \mathbb{A}^1) \to (C_*\mathcal{F})(X)$$

are chain homotopic, where $(C_*\mathcal{F})_n = \mathcal{F}^{\Delta^n}$.

Proof. For any $i = 0, \dots, n$, we define

$$\begin{aligned} \theta_i : \Delta^{n+1} &\to \Delta^n \times \mathbb{A}^1 \\ v_j &\mapsto \begin{cases} (v_i, 0), & 0 \leqslant j \leqslant i \\ (v_{j-1}, 1), & i < j \leqslant n+1 \end{cases} \end{aligned}$$

where v_j is $(0, \dots, 0, 1, 0, \dots, 0)$ at the jth coordinate.[23] Each θ_i induces a map

$$h_i = \mathcal{F}(\mathrm{id}_X \times \theta_i) : \mathcal{F}^{\Delta^n}(X \times \mathbb{A}^1) = \mathcal{F}(X \times \mathbb{A}^1 \times \Delta^n) \to F^{\Delta^{n+1}}(X) = \mathcal{F}(X \times \Delta^{n+1})$$

Then by using a technique similar to the proof of prism decomposition in topology, we can show that $s_n = \sum_i (-1)^i h_i$ is a chain homotopy from i_1^* to i_0^*, c.f., [Wei94], Lemma 8.3.13. □

Proposition 5.22 ([MVW06], Corollary 2.19)**.** For any $\mathcal{F} \in \mathrm{PSh}(S)$, the homology presheaves $H_n(C_*\mathcal{F})$ defined by $X \mapsto H_n(C_*\mathcal{F}(X))$ are homotopy invariant.

Proof. By Lemma 5.21, we know i_0^* and i_1^* of $H_n(C_*\mathcal{F})(X)$ are equal, so we conclude by Lemma 5.20. □

Example 5.23. $\mathcal{O}^*$ is also a homotopy invariant presheaf.

Definition 5.24. An additive full subcategory $\mathscr{D}$ of a triangulated category $\mathscr{C}$ is thick if

1. it satisfies two-out-of-three, i.e., for any distinguished triangle $A \to B \to C \to A[1]$, any two of A, B, and C are in $\mathscr{D}$, then so is the third;
2. if $A \oplus B \in \mathscr{D}$, then A and B are in $\mathscr{D}$.

Definition 5.25. Let $\mathscr{C}$ be a category and let $S \subseteq \mathscr{C}$ be a class of maps. We say S is a (left) localizing system if

1. given any $x \in \mathscr{C}$, we have $\mathrm{id}_X \in S$; given any $f, g \in S$, then $g \circ f \in S$;
2. any diagram
$$\begin{tikzcd} X \arrow[r] \arrow[d, Rightarrow] & Y \\ X' & \end{tikzcd}$$
where $X \Rightarrow X'$ is in S can be completed to
$$\begin{tikzcd} X \arrow[r] \arrow[d, Rightarrow] & Y \arrow[d, Rightarrow] \\ X' \arrow[r] & Y' \end{tikzcd}$$
where the two quasi-isomorphisms are in S;

[23] These are the algebraic analogues of the top-dimensional simplicies in the standard simplicial decomposition of the polyhedron $\Delta^n \times \Delta^1$, c.f., [MVW06], Definition 2.17.

3. given
$$X' \xrightarrow{\sigma} X \underset{\beta}{\overset{\alpha}{\rightrightarrows}} Y$$
where $\sigma \in S$ such that $\alpha\sigma = \beta\sigma$, then there exists $\gamma : Y \Rightarrow Y'$ such that $\gamma\alpha = \gamma\beta$.

We say S is a (right) localizing system if it is (left) localizing in $\mathscr{C}^\circ$.

Proposition 5.26 ([Nee01], Theorem 2.1.8; [Wei94], Proposition 10.4.1)**.** Let $\mathscr{D}$ be a thick subcategory of a triangulated category $\mathscr{C}$, and define $W_{\mathscr{D}}$ to be those maps whose cones lie in $\mathscr{D}$, then $W_{\mathscr{D}}$ is a left and right localizing system.

Consider the category $\mathscr{C}[W_{\mathscr{D}}^{-1}]$ with objects being those of $\mathscr{C}$, and morphisms being left or right roofs, then $\mathscr{C}[W_{\mathscr{D}}^{-1}]$ is a triangulated category, with distinguished triangles given by those isomorphic to distinguished triangles coming from $\mathscr{C}$.

Moreover, if $\mathscr{E}$ is another triangulated category with functor $F : \mathscr{C} \to \mathscr{E}$ is an exact functor such that $F(\alpha) = 0$ for all $\alpha \in \mathscr{D}$, then there exists a unique exact functor $\mathscr{C}[W_{\mathscr{D}}^{-1}] \to \mathscr{E}$ such that the diagram

$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[d, "F"'] & \mathscr{C}[W_{\mathscr{D}}^{-1}] \arrow[dl, dashed, "\exists!"] \\ \mathscr{E} & \end{tikzcd}$$

commutes.

Proof. See [GM13], Exercise IV.2.4. ◻

Definition 5.27. Define $\mathscr{E}_{\mathbb{A}}$ to be the smallest[24] thick subcategory of $D^-(S)$ such that

1. the cone of $\mathbb{Z}(X \times \mathbb{A}^1) \to \mathbb{Z}(X)$ is in $\mathscr{E}_{\mathbb{A}}$ for every $X \in \mathrm{Sm}/S$;
2. $\mathscr{E}_{\mathbb{A}}$ is closed under any direct sum that exists in $D^-(S)$.

We say that $f \in D^-(S)$ is an $\mathbb{A}^1$-weak equivalence if $f \in W_{\mathscr{E}_{\mathbb{A}}}$. We define $\mathrm{DM}^{\mathrm{eff},-}(S) = D^-(S)[W_{\mathscr{E}_{\mathbb{A}}}^{-1}]$ to be the (triangulated, derived) category of effective motives over S.[25]

Remark 5.28. Therefore, the category of effective motives is given by localizing the homotopy relation $X \times \mathbb{A}^1 \sim X$ over the derived category.

We should now try to define the six functors on the category of effective motives.

Lemma 5.29 ([MVW06], Lemma 9.4)**.** The smallest class in $D^-(S)$ that contains all $\mathbb{Z}(X)$ and closed under quasi-isomorphisms, direct sums, shifts, and cones is all of $D^-(S)$.

Proof. First we show that for any complex D_*, if all D_n are in the class, so is D_*. Let $\beta_n D$ be the truncation $0 \to D_n \to D_{n-1} \to \cdots$ of D_*, then $D_* = \varinjlim_n \beta_n D$. We have a distinguished triangle

$$D_n[-1] \longrightarrow \beta_{n-1}D_* \longrightarrow \beta_n D_* \longrightarrow D_n$$

so each $\beta_n D_*$ belongs to the class. Since there is an exact sequence

$$0 \longrightarrow \bigoplus \beta_n D_* \longrightarrow \bigoplus \beta_n D_* \longrightarrow D_* \longrightarrow 0$$

[24] Note that if $\mathscr{D}$ is a full (triangulated) subcategory of triangulated category $\mathscr{C}$, then the intersection of all thick subcategories of $\mathscr{C}$ containing $\mathscr{D}$ is also a thick subcategory.

[25] According to notations in [MVW06], this is equipped with étale topology.

it follows that D_* is in the class. Finally, for each sheaf $\mathcal{F}$, there is a free resolution $L_* \to \mathcal{F}$: by Yoneda lemma, we know there is a surjection $(i_\alpha) : \bigoplus_{\alpha \in \mathcal{F}(X)} \mathbb{Z}(X) \to \mathcal{F}$, then taking the kernel K gives the free resolution

$$\cdots \longrightarrow \bigoplus_{\alpha \in \mathcal{F}(X)} \mathbb{Z}(X) \longrightarrow K \longrightarrow \bigoplus_{\alpha \in \mathcal{F}(X)} \mathbb{Z}(X) \xrightarrow{(i_\alpha)} \mathcal{F} \longrightarrow 0$$

$\square$

Proposition 5.30. The functor $\varphi = g_\# \circ f^*$, induced by $S \xleftarrow{f} Y \xrightarrow{g} T$ where g is smooth, induces an exact functor $\varphi^* : \mathrm{DM}^{\text{eff},-}(S) \to \mathrm{DM}^{\text{eff},-}(T)$ which is determined by the diagram

$$\begin{tikzcd} D^-(S) \arrow[r, "\varphi^*"] \arrow[d] & D^-(T) \arrow[d] \\ \mathrm{DM}^{\text{eff},-}(S) \arrow[r, "\varphi^*"'] & \mathrm{DM}^{\text{eff},-}(T) \end{tikzcd}$$

Proof. Let $\mathscr{C}$ be the full subcategory of $D^-(S)$, with objects consisting of those complexes $K \in D^-(S)$ satisfying $\varphi^* K \in \mathscr{E}_{\mathbb{A}}$. It is a thick subcategory of $D^-(S)$. For any $X \in \mathrm{Sm}/S$, we have

$$\varphi^*(\mathbb{Z}_S(X \times \mathbb{A}^1) \to \mathbb{Z}_S(X)) = (\mathbb{Z}_T(\psi(X) \times \mathbb{A}^1) \to \mathbb{Z}_T(\psi(X)))$$

where

$$\begin{aligned} \psi : \mathrm{Sm}/S &\to \mathrm{Sm}/T \\ X &\mapsto X \times_S Y. \end{aligned}$$

Note that the cone of both homotopy relations are $\mathbb{A}^1$-equivalent to 0, and $\mathscr{E}_{\mathbb{A}}$ is generated by the cone of homotopy relation, so $\mathscr{E}_{\mathbb{A}} \subseteq \mathscr{C}$ by definition of $\mathscr{E}_{\mathbb{A}}$ and exactness of φ^*. We conclude by Proposition 5.26. $\square$

Proposition 5.31.

1. There is a tensor product

   $$\otimes_S : \mathrm{DM}^{\text{eff},-}(S) \times \mathrm{DM}^{\text{eff},-}(S) \to \mathrm{DM}^{\text{eff},-}(S)$$

   which is determined by the following diagram

   $$\begin{tikzcd} D^-(S) \times D^-(S) \arrow[r] \arrow[d] & D^-(S) \arrow[d] \\ \mathrm{DM}^{\text{eff},-}(S) \times \mathrm{DM}^{\text{eff},-}(S) \arrow[r] & \mathrm{DM}^{\text{eff},-}(S) \end{tikzcd}$$

   of descents. Furthermore, for any $K \in \mathrm{DM}^{\text{eff},-}(S)$, the functor $K \otimes_S -$ is exact.

2. Suppose that $f : S \to T$ is a smooth morphism in Sm/k, there is an exact functor

   $$f_\# : \mathrm{DM}^{\text{eff},-}(S) \to \mathrm{DM}^{\text{eff},-}(T)$$

   which is determined by the following diagram

   $$\begin{tikzcd} D^-(S) \times D^-(S) \arrow[r, "f_\#"] \arrow[d] & D^-(T) \arrow[d] \\ \mathrm{DM}^{\text{eff},-}(S) \arrow[r, "f_\#"'] & \mathrm{DM}^{\text{eff},-}(T) \end{tikzcd}$$

   of descents.

3. Suppose that $f : S \to T$ is a map in Sm/k, there is an exact functor
$$f^* : \mathrm{DM}^{\mathrm{eff},-}(T) \to \mathrm{DM}^{\mathrm{eff},-}(S)$$
which is determined by the following diagram
$$\begin{tikzcd} D^-(T) \times D^-(S) \arrow[r, "f^*"] \arrow[d] & D^-(S) \arrow[d] \\ \mathrm{DM}^{\mathrm{eff},-}(T) \arrow[r, "f^*"'] & \mathrm{DM}^{\mathrm{eff},-}(S) \end{tikzcd}$$
of descents.

Proof. We will prove the first part, as the second and third are obvious by applying the same technique. Suppose $Y \in \mathrm{Sm}/S$, then in the definition of φ, we take $(Y, S, T) := (Y, S, S)$, i.e., we have φ to be the diagram $S \leftarrow Y \to S$, then $\varphi^*\mathcal{F} = \mathcal{F} \otimes_S \mathbb{Z}_S(Y)$ as in Proposition 5.13. Now, given an $\mathbb{A}^1$-weak equivalence a, then $\mathbb{Z}_S(Y) \otimes a$ is also an $\mathbb{A}^1$-weak equivalence by applying Proposition 5.30 to φ. Moreover, for any $C \in \mathscr{E}_{\mathbb{A}}$, the full subcategory of all $D \in D^-(S)$ such that $C \otimes_S D \in \mathscr{E}_{\mathbb{A}}$ constitutes a thick subcategory of $D^-(S)$ containing $\mathbb{Z}_S(Y)$ for all $Y \in \mathrm{Sm}/S$. So this category is just $D^-(S)$ by Lemma 5.29, then we conclude the proof after applying Proposition 5.26. □

Proposition 5.32. Let $f : S \to T$ be a morphism in Sm/k.

1. For any $K, L \in \mathrm{DM}^{\mathrm{eff},-}(T)$, we have $f^*(K \otimes_T L) = (f^*K) \otimes_S (f^*L)$.
2. If f is smooth, then for any $K \in \mathrm{DM}^{\mathrm{eff},-}(S)$ and $L \in \mathrm{DM}^{\mathrm{eff},-}(T)$, we have $f_\#(K \otimes_S f^*L) = (f_\#K) \otimes_S L$.
3. If f is smooth, there is an adjunction
$$\begin{tikzcd} \mathrm{DM}^{\mathrm{eff},-}(S) \arrow[d, shift left, "f^*"] \\ \mathrm{DM}^{\mathrm{eff},-}(S) \arrow[u, shift left, "f^\#"] \end{tikzcd}$$

Proof.

1. This follows from Proposition 5.3.
2. This follows from Proposition 5.6.
3. This follows from the same technique as in Proposition 5.16.

□

Definition 5.33. Two morphisms $f, g : \mathcal{F} \to \mathcal{G}$ in $\mathrm{Sh}(S)$ are called $\mathbb{A}^1$-homotopic if there is a map $h : \mathcal{F} \otimes_S \mathbb{Z}_S(\mathbb{A}^1) \to \mathcal{G}$ of sheaves so that $hi_0 = f$ and $hi_1 = g$.

Lemma 5.34 ([MVW06], Lemma 9.10)**.** If $f, g : \mathcal{F} \to \mathcal{G}$ in $\mathrm{Sh}(S)$ are $\mathbb{A}^1$-homotopic, then $f = g$ in $\mathrm{DM}^{\mathrm{eff},-}(S)$.

Proof. We have
$$\mathbb{Z}(*) \overset{i_0}{\underset{i_1}{\rightrightarrows}} \mathbb{Z}(\mathbb{A}^1) \xrightarrow{\cong} \mathbb{Z}(*)$$
where both compositions are identity, so this forces $i_0 = i_1$ in DM. □

Corollary 5.35. An $\mathbb{A}^1$-homotopy equivalence is an $\mathbb{A}^1$-weak equivalence.

The next goal is to show that the natural map $K \to C_*K = \mathrm{Tot}((K^i)^{\Delta^j})$ is an $\mathbb{A}^1$-weak equivalence.

Lemma 5.36 ([MVW06], Lemma 9.12)**.** Let $f : B \to B'$ be a map of double complexes which are vertically bounded above in the sense that there is an N such that $B^{*,n} = B'^{*,n} = 0$ for $n \geqslant N$. Suppose that the restriction of f to each row is an $\mathbb{A}^1$-weak equivalence and that $\mathrm{Tot}(B)$ and $\mathrm{Tot}(B')$ are bounded above, then $\mathrm{Tot}(f)$ is an $\mathbb{A}^1$-weak equivalence.

Proof. Let $S(n)$ be the double complex of B consisting of $B^{p,q}$ for $q \geqslant n$, then there is an exact sequence

$$0 \longrightarrow \mathrm{Tot}(S(n+1)) \longrightarrow \mathrm{Tot}(S(n)) \longrightarrow B^{n,*}[-n] \longrightarrow 0$$

Similarly, for $S'(n)$ of B' we have a similar result. By induction on n, i.e., taking the commutative diagram of short exact sequences, each $\mathrm{Tot}(S(n)) \to \mathrm{Tot}(S'(n))$ is an $\mathbb{A}^1$-weak equivalence. So the statement follows from the observation that $\mathrm{Tot}(B) = \varinjlim_n S(n)$ and $\mathrm{Tot}(B') = \varinjlim_n S'(n)$. $\square$

Corollary 5.37 ([MVW06], Corollary 9.13)**.** If $f : C \to C'$ is a morphism in $C^-(S)$ and $f_n : C_n \to C'_n$ is an $\mathbb{A}^1$-weak equivalence for every n, then f is an $\mathbb{A}^1$-weak equivalence.

Lemma 5.38 ([MVW06], Lemma 9.14)**.** For every $\mathcal{F}$ and $n \in \mathbb{N}$, the map $s : \mathcal{F} \to \mathcal{F}^{\Delta^n}$ is an $\mathbb{A}^1$-weak equivalence.

Proof. Since $\Delta^n \cong \mathbb{A}^n$ as schemes, we know for internal homs that $\mathcal{F}^{\Delta^n} = (\mathcal{F}^{\Delta^{n-1}})^{\Delta^1}$, so by induction we may suppose that $n = 1$. We define a map $m : \mathcal{F}^{\Delta^1} \to \mathcal{F}^{\Delta^2}$ induced by the multiplication

$$\begin{aligned} \Delta^2 = \mathbb{A}^1 \times \mathbb{A}^1 &\to \mathbb{A}^1 = \Delta^1 \\ (x, y) &\mapsto xy. \end{aligned}$$

Since $\mathcal{F}^{\Delta^2} = \underline{\mathrm{Hom}}(\mathbb{Z}(\Delta^1), \mathcal{F}^{\Delta^1})$, then the adjunction gives a map $h : \mathcal{F}^{\Delta^1} \otimes \mathbb{Z}(\Delta^1) \to \mathcal{F}^{\Delta^1}$, which is an $\mathbb{A}^1$-homotopy between the composite $\mathcal{F}^{\Delta^1} \xrightarrow{d_0} \mathcal{F} \xrightarrow{s} \mathcal{F}^{\Delta^1}$ and $\mathrm{id}_{\mathcal{F}^{\Delta^1}}$. In particular, they are the same map over DM, so $sd_0 = \mathrm{id}_{\mathcal{F}^{\Delta^1}}$. Also, we have $d_0 s = \mathrm{id}_{\mathcal{F}}$, therefore s is an $\mathbb{A}^1$-weak equivalence. $\square$

Proposition 5.39 ([MVW06], Lemma 9.15)**.** For every $K \in C^-(S)$, the map $K \to C_*K = \mathrm{Tot}(C_*K)$ is an $\mathbb{A}^1$-weak equivalence. Hence $K \cong C_*K$ in $\mathrm{DM}^{\mathrm{eff},-}(S)$, i.e., K is the same as the Suslin complex of itself.

Proof. For every n, we know $(K \to C_*(K))_n = (K_n \to C_*(K_n))$, so by Lemma 5.36 we may assume that K is a sheaf. We have a diagram of complexes

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & K \arrow[d, equal] \\ \cdots \arrow[r, "0"] & K \arrow[r, equal] \arrow[d] & K \arrow[r, "0"] \arrow[d] & K \arrow[d, equal] \\ \cdots \arrow[r] & K^{\Delta^2} \arrow[r] & K^{\Delta^1} \arrow[r] & K \end{tikzcd}$$

where the upper morphism is a quasi-isomorphism, and the lower morphism is an $\mathbb{A}^1$-weak equivalence by Lemma 5.38 and Corollary 5.37, therefore the composition of the two morphisms gives the isomorphism we want. $\square$

5.3 Étale $\mathbb{A}^1$-locality

Definition 5.40. An object L in $D^-(S)$ is called $\mathbb{A}^1$-local if for all $\mathbb{A}^1$-weak equivalences $K' \to K$, the induced map $\mathrm{Hom}_{D^-(S)}(K, L) \to \mathrm{Hom}_{D^-(S)}(K', L)$ is an isomorphism on the derived category. We denote $\mathscr{L}$ to be the full subcategory of $\mathbb{A}^1$-local objects of $D^-(S)$.

Remark 5.41. The notion of local objects occurs whenever we have weak equivalences. With local objects, we can reduce homotopy categories to model categories.

Remark 5.42. $\mathscr{L}$ is a thick triangulated subcategory of $D^-(S)$.

Proposition 5.43 ([MVW06], Lemma 9.19)**.** If L is $\mathbb{A}^1$-local, then for every $K \in D^-(S)$, we have

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}(S)}(K, L) \cong \mathrm{Hom}_{D^-(S)}(K, L).$$

Hence, the natural functor $\mathscr{L} \to \mathrm{DM}^{\mathrm{eff},-}(S)$ is fully faithful.

Proof. We know there is a natural map $\mathrm{Hom}_{D^-(S)}(K, L) \to \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}(S)}(K, L)$ induced by the localization. Using calculus of fractions, c.f., [Wei94], Theorem 10.3.7, we know every morphism in $\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}(S)}(K, L)$ is a roof $K \leftarrow M \to L$ in $D^-(S)$, where $M \to K$ is an $\mathbb{A}^1$-weak equivalence. Since L is $\mathbb{A}^1$-local, then $M \to K$ induces an isomorphism $\mathrm{Hom}_{D^-(S)}(K, L) \cong \mathrm{Hom}_{D^-(S)}(M, L)$, therefore we can regard the roof as a map in $\mathrm{Hom}_{D^-(S)}(K, L)$, thus the natural map is surjective. Moreover, if $K \to L$ is zero in $\mathrm{DM}^{\mathrm{eff},-}(S)$, then there exists an $\mathbb{A}^1$-weak equivalence $M \to K$ such that $M \to K \to L$ is zero in $D^-(S)$. Therefore, the map $K \to L$ is zero by $\mathbb{A}^1$-locality again. ☐

Proposition 5.44 ([MVW06], Lemma 9.20)**.** An object $L \in D^-(S)$ is $\mathbb{A}^1$-local if and only if

$$\mathbb{H}^{-n}(X, L) \cong \mathrm{Hom}_{D^-(S)}(\mathbb{Z}(X)[n], L) \to \mathrm{Hom}_{D^-(S)}(\mathbb{Z}(X \times \mathbb{A}^1)[n], L) \cong \mathbb{H}^{-n}(X \times \mathbb{A}^1, L), \tag{5.45}$$

induced by the projection $X \times \mathbb{A}^1 \to X$, is an isomorphism for all X and $n \in \mathbb{Z}$.

Proof. If L is $\mathbb{A}^1$-local, then Equation (5.45) is obvious from Proposition 5.43. Conversely, let $\mathscr{K}$ be the full subcategory of $D^-(S)$, whose objects are all K's for which $\mathrm{Hom}_{D^-(S)}(K[n], L) = 0$ for all n. Now Equation (5.45) is an isomorphism for all n if and only if the cone $C(f)$ of $f : \mathbb{Z}(X \times \mathbb{A}^1) \to \mathbb{Z}(X)$ satisfies $\mathrm{Hom}_{D^-(S)}(C(f)[n], L) = 0$, which reduces to proving that $\mathrm{Hom}_{D^-(S)}(K[n], L) = 0$. Note that $\mathscr{K}$ is a thick subcategory of $D^-(S)$, and is closed under direct sums and shifts. By construction, $\mathscr{K}$ contains the cone $C(f)$ of each homotopy relation $f : \mathbb{Z}(X \times \mathbb{A}^1) \to \mathbb{Z}(X)$, so all maps that are $\mathbb{A}^1$-equivalent to 0 are contained in $\mathscr{K}$, i.e., $\mathscr{E}_{\mathbb{A}} \subseteq \mathscr{K}$. Therefore, for every map that is $\mathbb{A}^1$-equivalent to 0, we know shifting by n and taking $\mathrm{Hom}_{D^-(S)}(-, L)$ gives 0. Hence, L is $\mathbb{A}^1$-local. ☐

Definition 5.46. An étale sheaf with transfers $\mathcal{F}$ is strictly $\mathbb{A}^1$-homotopy invariant if

$$H^n_{\acute{e}t}(X, \mathcal{F}) \to H^n_{\acute{e}t}(X \times \mathbb{A}^1, \mathcal{F})$$

is an isomorphism for all smooth X and every $n \in \mathbb{Z}$.

Remark 5.47. For $n = 0$, being strictly $\mathbb{A}^1$-homotopy invariant implies $\mathcal{F}$ is homotopy invariant.

We now give a sufficient condition for $\mathbb{A}^1$-locality.

Proposition 5.48. $L \in D^-(S)$ is $\mathbb{A}^1$-local if $H^n_{\acute{e}t}(L)$ is strictly $\mathbb{A}^1$-homotopy invariant for all $n \in \mathbb{Z}$.

Proof. Since $H^n(L)$'s are strictly $\mathbb{A}^1$-homotopy invariant, then $H^n_{\acute{e}t}(X, H^n(L)) \cong H^n_{\acute{e}t}(X \times \mathbb{A}^1, H^n(L))$ for all n. Moreover, the hypercohomology spectral sequence gives

$$H^p_{\acute{e}t}(X, H^q(L)) \Rightarrow \mathbb{H}^{p+q}(X, L) = \mathrm{Hom}_{D^-(S)}(X, L[p+q])$$

and

$$H^p_{\acute{e}t}(X \times \mathbb{A}^1, H^q(L)) \Rightarrow \mathbb{H}^{p+q}(X \times \mathbb{A}^1, L) = \mathrm{Hom}_{D^-(S)}(X \times \mathbb{A}^1, L[p+q]).$$

We know $H^p_{\acute{e}t}(X, H^q(L)) \cong H^p_{\acute{e}t}(X \times \mathbb{A}^1, H^q(L))$ already from the projection map $X \times \mathbb{A}^1 \to X$ by strict $\mathbb{A}^1$-homotopy invariance, so they converge to the same place, so $\mathbb{H}^*(X, L) \cong \mathbb{H}^*(X \times \mathbb{A}^1, L)$, i.e.,

$$\mathrm{Hom}_{D^-(S)}(X, L[n]) \cong \mathrm{Hom}_{D^-(S)}(X \times \mathbb{A}^1, L[n])$$

for all $n \in \mathbb{Z}$. Finally, apply Proposition 5.44. ☐

The converse of Proposition 5.48 is also true under some circumstances. For instance,

Proposition 5.49. Let $S = \operatorname{Spec}(k)$ for a perfect field k, and suppose $L \in D^-(S)$ is $\mathbb{A}^1$-local, then $H^n_{\text{ét}}(L)$ is strictly $\mathbb{A}^1$-homotopy invariant.

Remark 5.50. Therefore, if the underlying field k is perfect, then the complex is $\mathbb{A}^1$-local if and only if the cohomology sheaves are strictly $\mathbb{A}^1$-homotopy invariant, if and only if the cohomology sheaves are $\mathbb{A}^1$-homotopy invariant.

The proof makes use of the following theorem.

Theorem 5.51 ([MVW06], Theorem 13.8)**.** Assume that k is a perfect field and $\mathcal{F} \in \operatorname{PSh}(k)$ is homotopy invariant, then the Nisnevich sheafification $\mathcal{F}^+$ is strictly $\mathbb{A}^1$-homotopy invariant.

Proof of Proposition 5.49. We want to prove that the two presheaves defined by $X \mapsto \operatorname{Hom}_{D^-(k)}(\mathbb{Z}(X), L[n])$ and $X \mapsto H^n(L(X))$ have the same sheafification, namely $H^n_{\text{ét}}(L)$ for any $L \in D^-(k)$.

Define $\beta_i(L)$ to be the truncation of L

$$0 \longrightarrow L_i \longrightarrow L_{i+1} \longrightarrow \cdots$$

Since L is bounded above, then $\beta_i(L)$ is a bounded complex, so $L = \varinjlim_i \beta_i(L)$. Let us take an injective resolution $\beta_i \to I^i$ for each β_i, then

$$\begin{aligned}
\operatorname{Hom}_{D^-(S)}(\mathbb{Z}(X), \beta_i(L)[n]) &\cong \operatorname{Hom}_{D^-(S)}(\mathbb{Z}(X), I^i[n]) \\
&\cong \operatorname{Hom}_{K^-(S)}(\mathbb{Z}(X), I^i[n]) \\
&\cong H^n(I^i(X)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\operatorname{Hom}_{D^-(k)}(\mathbb{Z}(X), L[n]) &= \varinjlim_i \operatorname{Hom}_{D^-(k)}(\mathbb{Z}(X), \beta_i(L)[n]) \\
&= \varinjlim_i H^n(I^i(X)).
\end{aligned} \tag{5.52}$$

We know the sheafification of $\varinjlim_i H^n(I^i(X))$ is the direct limit $\varinjlim_i H^n(\beta_i(L))$, which is just $H^n(L)$. Finally, the sheafification of $\varinjlim_i H^n(I^i(X))$ is just the sheafification of $X \mapsto H^n(L(X))$, so by Equation (5.52) we know the two sheafifications agree.

Now suppose that L is $\mathbb{A}^1$-local. We know the presheaf $X \mapsto \operatorname{Hom}_{D^-(k)}(\mathbb{Z}(X), L[n])$ is $\mathbb{A}^1$-homotopy invariant by Proposition 5.44, therefore the sheafification is strictly $\mathbb{A}^1$-homotopy invariant by Theorem 5.51. Since the two sheafifications agree, then $H^n(L)$ is also strictly $\mathbb{A}^1$-homotopy invariant. □

In fact, one can prove a stronger statement in some other cases.

Lemma 5.53 ([MVW06], Lemma 9.24)**.** Let $\mathcal{F}$ be an étale sheaf of R-modules with transfers, then $\mathcal{F}$ is $\mathbb{A}^1$-local if and only if $\mathcal{F}$ is strictly $\mathbb{A}^1$-homotopy invariant.

Proposition 5.54. Assume that k is a perfect field, then the Suslin complex $C_*(K)$ is $\mathbb{A}^1$-local for any complex in $C^-(k)$.

Proof. By Proposition 5.39, we know the map $K \mapsto C_*(K)$ is an $\mathbb{A}^1$-weak equivalence, therefore we can replace any complex K by an $\mathbb{A}^1$-local object. We know the presheaf $H_n(C_*(K))$ is $\mathbb{A}^1$-homotopy invariant by Proposition 5.22, then its sheafification is strictly $\mathbb{A}^1$-homotopy invariant by Theorem 5.51. We know for each sheaf that $H_n(C_*(K))$ is homotopy invariant, therefore the total complex of the bicomplex is homotopy invariant by taking the spectral sequence associated to the double complex. Therefore, we transform the result onto the morphism $K \mapsto C_*(K)$ of double complexes, and now by Proposition 5.48, $C_*(K)$ is $\mathbb{A}^1$-local. □

Remark 5.55.

1. Every complex has an $\mathbb{A}^1$-local resolution given by the Suslin complex.

2. We have

$$\begin{aligned} H^{p,q}(X,\mathbb{Z}) &= \mathbb{H}^p_{\mathrm{Nis}}(X,\mathbb{Z}(q)) \text{ by definition} \\ &= \mathrm{Hom}_{D^-(k)}(\mathbb{Z}(X), C_*\mathbb{Z}(\mathbb{G}_m^{\wedge q})[-q]) \\ &= \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}(k)}(\mathbb{Z}(X), (\mathbb{Z}(\mathbb{G}_m^{\wedge 1})[-1])^{\otimes q}) \text{ since } C_*\mathbb{Z}[\mathbb{G}_m^{\wedge q}[-q] \text{ is } \mathbb{A}^1\text{-local} \end{aligned}$$

Therefore, conventionally we denote $\mathbb{Z}(\mathbb{G}_m^{\wedge 1}) = \mathbb{Z}(1)[1]$. Equivalently, $\mathbb{Z}(\mathbb{G}_m^{\wedge 1})[-1] = \mathbb{Z}(1)$ is the Tate twist.

Let $\mathrm{Sh}_{\acute{e}t}(\mathrm{Cor}_k, R)$ be the category of étale sheaves of R-modules with transfers. If $\frac{1}{m} \in k$, then let $\mathscr{L}$ be the corresponding full subcategory of $\mathbb{A}^1$-local complexes in $D^-(\mathrm{Sh}_{\acute{e}t}(\mathrm{Cor}_k, \mathbb{Z}/m\mathbb{Z}))$. One can equip the $\mathscr{L}$ with a tensor-triangulated category structure.

Definition 5.56. For $E, F \in \mathscr{L}$, we define $E \otimes_{\mathscr{L}} F = \mathrm{Tot}(C_*((E\otimes F)^+_{\acute{e}t}))$, where $(E\otimes F)^+_{\acute{e}t}$ is the étale sheafification of the tensor product on the sheaves with transfers. In particular, the tensor product is $\mathbb{A}^1$-local, hence $\otimes_{\mathscr{L}}$ is a bifunctor.

Theorem 5.57 ([MVW06], Theorem 9.35)**.** Suppose $\frac{1}{m} \in k$, then $(\mathscr{L}, \otimes_{\mathscr{L}})$ is a tensor-triangulated category, and $\mathscr{L}$ and $\mathrm{DM}^{\mathrm{eff},-}_{\acute{e}t}(\mathbb{Z}/m\mathbb{Z})$ are equivalent as tensor-triangulated categories.

6 Cancellation Theorem

We want to understand the suspension by $\mathbb{G}_m$, that is, the morphism

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}(k)}(K,L) \xrightarrow{\otimes\mathbb{Z}(1)} \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}(k)}(K(1),L(1))$$

where $K(1)=K\otimes\mathbb{Z}(1)$ and $L(1)=L\otimes\mathbb{Z}(1)$. In fact, we can show that this induces an isomorphism over $\mathrm{DM}^{\mathrm{eff},-}(k)$ on a perfect field k, c.f., [Voe10], Corollary 4.10.

In Section 6, the notation $A\times B$ indicates the fiber product $A\times_S B$ over a Noetherian base scheme S. Note that this is the tensor product structure on $\mathrm{Cor}(S)$ as well.

Let $X,Y\in\mathrm{Sm}/k$, we define f_1,f_2 to be the projections

$$\begin{tikzcd} \mathbb{G}_m\times X\times\mathbb{G}_m\times Y \arrow[r, "f_1"] \arrow[d, "f_2"'] & \mathbb{G}_m \\ \mathbb{G}_m & \end{tikzcd}$$

For any $n\in\mathbb{N}$, we can define a rational function $g_n=\frac{f_1^{n+1}-1}{f_1^{n+1}-f_2}\in K(\mathbb{G}_m\times X\times\mathbb{G}_m\times Y)$.

Proposition 6.1 ([Voe10], Lemma 4.1)**.** Suppose $Z\in\mathrm{Cor}_k(\mathbb{G}_m\times X,\mathbb{G}_m\times Y)$, then there is some $N\in\mathbb{N}$ such that for any $n>N$, the principal divisor $\mathrm{div}(g_n)$ intersects Z properly, and $\mathrm{supp}(Z\cdot\mathrm{div}(g_n))$ is finite over X.

Proposition 6.1 shows that a correspondence $Z\in\mathrm{Cor}_k(\mathbb{G}_m\times X,\mathbb{G}_m\times Y)$ induces a finite correspondence in $\mathrm{Cor}_k(X,Y)$, which motivates the cancellation theorem.

Proof. Without loss of generality, we may assume that $\mathrm{supp}(Z)$ is integral, since the general case can be proven componentwise. Since $\mathrm{supp}(Z)$ is closed in $\mathbb{G}_m\times X\times\mathbb{G}_m\times Y$, we have a natural map, then we have a projection onto $\mathbb{G}_m\times X$, which is included as an open subset in $\mathbb{P}^1\times X$. Collecting all of this, we define the composite φ to be

$$\begin{tikzcd} \mathrm{supp}(Z) \arrow[r] \arrow[rrr, bend left, "\varphi"] & \mathbb{G}_m\times X\times\mathbb{G}_m\times Y \arrow[r] & \mathbb{G}_m\times X \arrow[r] & \mathbb{P}^1\times X \end{tikzcd}$$

Since Z is a finite correspondence, then $\mathrm{supp}(Z)$ is finite over $\mathbb{G}_m\times X$, therefore φ has finite fibers. By Zariski's main theorem, c.f., [Gro66], Theorem 8.12.6, φ can be factorized as

$$\mathrm{supp}(Z)\xrightarrow{i} C\xrightarrow{\pi}\mathbb{P}^1\times X$$

where i is an open immersion and π is finite. We can now compactify $\mathrm{supp}(Z)$. Here we may assume C to be integral as well, otherwise we can consider $\mathrm{im}(i)$ instead. To compute the principal divisor, we should take the normalization, and get maps

$$\widetilde{\mathrm{supp}(Z)}\xrightarrow{\bar{i}}\tilde{C}\xrightarrow{\bar{\pi}:=(\bar{f}_1,q)}\mathbb{P}^1\times X$$

Now $\bar{f}_1$ extends f_1 to the compactification $\tilde{C}$, and is a rational function on $\tilde{C}$. Also, we know f_2 is a function that can be restricted to $\mathrm{supp}(Z)$, which is an open subset of C, so f_2 is a rational function on C and therefore on $\tilde{C}$ as well. Since $\bar{f}_1$ is a projection on $\mathbb{P}^1$, then it has positive order at the divisor over 0, that is, there exists natural number N such that $\frac{\bar{f}_1^N}{f_2}$ is regular at a dense open subset of $\bar{f}_1^{-1}(0)$. Indeed, since $\bar{f}_1$ has positive order at the divisor $\bar{f}_1^{-1}(0)$, then for large enough N, the morphism $\frac{\bar{f}_1^N}{f_2}$ should also have positive order at the divisor.[26] Similarly, the morphism $\frac{f_2}{\bar{f}_1^N}$ is regular at

[26] This does not imply f_2 also has positive order at the divisor, since we extended the morphism.

a dense open subset of $\bar{f}_1^{-1}(\infty)$. Now define $\bar{g}_n = g_n|_{\widetilde{\operatorname{supp}(Z)}}$, then by direct computation we know $\bar{g}_n f_2 = \frac{\bar{f}_1^{n+1}-1}{\frac{\bar{f}_1^{n+1}}{f_2}-1}$. As rational functions on $\tilde{C}$, we know $\bar{g}_n f_2|_{\bar{f}_1^{-1}(0)} = 1$: if $\bar{f}_1$ valuates as 0, then $\bar{g}_n f_2$ valuates as 1. Since it is a regular function evaluated as 1 at 0, then

$$\operatorname{supp}(\operatorname{div}(\bar{g}_n f_2)) \cap \bar{f}_1^{-1}(0) = \varnothing. \tag{6.2}$$

Similarly, $\bar{g}_n|_{\bar{f}_1^{-1}(\infty)} = 1$ as well, so

$$\operatorname{supp}(\operatorname{div}(\bar{g}_n)) \cap \bar{f}_1^{-1}(\infty) = \varnothing. \tag{6.3}$$

But f_2 is an invertible function on $\mathbb{G}_m \times X \times \mathbb{G}_m \times Y$, so it remains invertible when restricted to $\operatorname{supp}(Z)$, therefore $\operatorname{supp}(\operatorname{div}(f_2)) \cap \widetilde{\operatorname{supp}(Z)} = \varnothing$. We know $\widetilde{\operatorname{supp}(Z)}$ is an open subset of $\tilde{C}$ by construction. Since $\operatorname{supp}(\operatorname{div}(fg)) \subseteq \operatorname{supp}(\operatorname{div}(f)) \cup \operatorname{supp}(g)$, then

$$\overline{\operatorname{supp}(\operatorname{div}(\bar{g}_n)) \cap \widetilde{\operatorname{supp}(Z)}} \cap \bar{f}_1^{-1}(0) \subseteq \overline{\operatorname{supp}(\operatorname{div}(\bar{g}_n f_2)) \cap \widetilde{\operatorname{supp}(Z)}} \cap \bar{f}_1^{-1}(0) = \varnothing$$

by Equation (6.2). Combining with Equation (6.3), we know $\overline{\operatorname{supp}(\operatorname{div}(\bar{g}_n)) \cap \widetilde{\operatorname{supp}(Z)}}$ has no intersection at both 0 and ∞, thus it is contained in $\bar{f}_1^{-1}(\mathbb{G}_m \times X)$. Pushing forward the inclusion, i.e., taking the image, along the normalization map $\tilde{C} \to C$, since $\pi : C \to \mathbb{P}^1 \times X$ is finite, then

$$\pi(\overline{\operatorname{supp}(\operatorname{div}(g_n) \cap \operatorname{supp}(Z))}) \subseteq \mathbb{G}_m \times X. \tag{6.4}$$

Since Z is a finite correspondence, then it is finite over $\mathbb{G}_m \times X$, hence $\operatorname{supp}(Z \cdot \operatorname{div}(g_n))$ is also finite over $\mathbb{G}_m \times X$. However, Equation (6.4) shows that its closure is contained in $\mathbb{G}_m \times X$, so $\operatorname{supp}(Z \cdot \operatorname{div}(g_n))$ is finite over $\mathbb{P}^1 \times X$, hence it is proper over X. But Z has finite fibers over X, i.e., no fiber contains $\mathbb{P}^1 \times \{x\}$ for any $x \in X$, therefore this proper map has finite fibers over X as well, i.e., no fiber contains $\mathbb{P}^1 \times \{x\}$ for any $x \in X$. $\square$

Given a finite correspondence Z over $\mathbb{G}_m \times X \to \mathbb{G}_m \times Y$, once we know $Z \cdot \operatorname{div}(g_n)$ is finite over X, we define $\rho_n(Z) \in \operatorname{Cor}_k(X, Y)$ to be the pushforward of $Z \cdot \operatorname{div}(g_n)$ along projection $\mathbb{G}_m \times X \times \mathbb{G}_m \times Y \to X \times Y$.

Proposition 6.5 ([Voe10], Lemma 4.3)**.**

a. For any $W \in \operatorname{Cor}_k(X, Y)$ and $n \geqslant 1$, we have $\rho_n(\operatorname{id}_{\mathbb{G}_m} \times W) = W$.

b. Let e_x be the composition

$$\mathbb{G}_m \times X \xrightarrow{\operatorname{pr}_2} X \xrightarrow{(1,\operatorname{id}_X)} \mathbb{G}_m \times X$$

then $\rho_n(e_x) = 0$ for any $n \geqslant 0$.

Proof.

a. The cycle on $\mathbb{G}_m \times X \times \mathbb{G}_m \times Y$ over $\mathbb{G}_m \times X$ which represents $\operatorname{id}_{\mathbb{G}_m} \times W$ is $\Delta_*(\mathbb{G}_m \times W)$[27], where Δ is the diagonal embedding $\mathbb{G}_m \times X \times Y \to \mathbb{G}_m \times X \times \mathbb{G}_m \times Y$. We know the cycle $\Delta_*(\mathbb{G}_m \times W) \cdot \operatorname{div}(g_n)$ is the same as $\rho_n(\operatorname{id}_{\mathbb{G}_m} \times W)$ after pushing forward. Recall that $g_n = \frac{f_1^{n+1}-1}{f_1^{n+1}-f_2}$, then applying the pullback Δ^* yields $f_1 = f_2$. Therefore, by the projection formula, we have

$$\begin{aligned} \Delta_*(\mathbb{G}_m \times W) \cdot \operatorname{div}(g_n) &= \Delta_*((\mathbb{G}_m \times W) \cdot \Delta^*(\operatorname{div}(g_n))) \\ &= \Delta_*((\mathbb{G}_m \times W) \cdot \operatorname{div}(\Delta^*(g_n))) \end{aligned}$$

[27] Since W is a closed subset of $X \times Y$, then the pushforward of $\mathbb{G}_m \times W$ is well-defined.

$$= \Delta_* \left(\operatorname{div}\left(\frac{t^{n+1}-1}{t^{n+1}-t}\right) \times W\right)$$

Denote $p : \mathbb{G}_m \times X \times \mathbb{G}_m \times Y \to X \times Y$ to be the projection. Since t is invertible in $\mathbb{G}_m$, then the rational function $\frac{t^{n+1}-1}{t^{n+1}-t}$ has degree 1 in $\mathbb{G}_m$, hence by the projection formula and the base-change formula,

$$\begin{aligned}
\rho_n(\mathrm{id}_{\mathbb{G}_m \times W}) &= p_*\Delta_* \left(\operatorname{div}\left(\frac{t^{n+1}-1}{t^{n+1}-t}\right) \times W\right) \\
&= \deg\left(\frac{t^{n+1}-1}{t^{n+1}-t}\right) \cdot W \\
&= W.
\end{aligned}$$

b. The cycle Z on $\mathbb{G}_m \times X \times \mathbb{G}_m \times Y$ representing e_X is the image of the diagonal embedding on X

$$\begin{aligned}
\mathbb{G}_m \times X &\to \mathbb{G}_m \times X \times \mathbb{G}_m \times X \\
(t, x) &\mapsto (t, x, 1, x)
\end{aligned}$$

on $\mathbb{G}_m$. Pulling back g_n along the morphism, we know the restriction of g_n to $\operatorname{supp}(Z)$ is 1 just as in part a., therefore $Z \cdot \operatorname{div}(g_n) = 0$.

$\square$

Proposition 6.6 ([Voe10], Lemma 4.4)**.** Let $Z : \mathbb{G}_m \times X \to \mathbb{G}_m \times Y$ be a finite correspondence such that ρ_n is defined, then for any finite correspondence $W : X' \to X$, $\rho_n(Z \circ (\mathrm{id}_{\mathbb{G}_m} \times W))$ is defined, and

$$\rho_n(Z \circ (\mathrm{id}_{\mathbb{G}_m} \times W)) = \rho_n(Z) \circ W.$$

Proof. By definition, we can write $\rho_n(Z) \circ W$ as the composition

$$X' \xrightarrow{W} X \xrightarrow{Z \cdot \operatorname{div}(g_n)} \mathbb{G}_m \times \mathbb{G}_m \times Y \xrightarrow{\text{projection}} Y$$

where $Z \cdot \operatorname{div}(g_n)$ is well-defined by Proposition 6.1, and $\rho_n(Z \circ (\mathrm{id}_{\mathbb{G}_m} \times W))$ is the composition

$$X' \xrightarrow{Z \circ (\mathrm{id}_{\mathbb{G}_m} \times W) \cdot \operatorname{div}(g_n)} \mathbb{G}_m \times \mathbb{G}_m \times Y \longrightarrow Y$$

Hence, we need to prove that these two compositions are the same. Consider the diagram

$$\begin{tikzcd}
\mathbb{G}_m \times X' \times \mathbb{G}_m \times Y & \mathbb{G}_m \times X' \times X \times \mathbb{G}_m \times Y \arrow[l, "p_1"'] \arrow[r, "r"] \arrow[d, "\pi"] & \mathbb{G}_m \times X \times \mathbb{G}_m \times Y \\
& X' \times X &
\end{tikzcd}$$

where arrows are projections. Then by the projection formula,

$$\begin{aligned}
(Z \cdot \operatorname{div}(g_n)) \circ W &= p_{1*}(r^*(Z \cdot \operatorname{div}(g_n)) \cdot \pi^*(W)) \\
&= p_{1*}(r^*(Z) \cdot p_1^*(\operatorname{div}(g_n)) \cdot \pi^*(W)) \\
&= p_{1*}(r^*(Z) \cdot \pi^*(W)) \cdot \operatorname{div}(g_n) \\
&= (Z \circ (\mathrm{id}_{\mathbb{G}_m} \times W)) \cdot \operatorname{div}(g_n)
\end{aligned}$$

$\square$

Proposition 6.7 ([Voe10], Lemma 4.5)**.** Let $Z \in \mathrm{Cor}_k(\mathbb{G}_m \times X, \mathbb{G}_m \times Y)$ be a finite correspondence such that $\rho_n(Z)$ is defined, then for any $f : X' \to Y'$ in Sm/k, $\rho_n(Z \times f)$ is well-defined, and

$$\rho_n(Z \times f) = \rho_n(Z) \times f.$$

Proof. Consider the diagram

$$\begin{tikzcd}
 & \mathbb{G}_m \times \mathbb{G}_m & \\
\mathbb{G}_m \times X \times \mathbb{G}_m \times Y \arrow[ur, "u"] \arrow[d, "a"'] & \mathbb{G}_m \times X \times X' \times \mathbb{G}_m \times Y \times Y' \arrow[l, "p"'] \arrow[u, "r"'] \arrow[r, "s"] \arrow[d, "q"] & X' \times Y' \\
X \times Y & X \times X' \times Y \times Y' \arrow[l, "b"] \arrow[ur, "t"'] &
\end{tikzcd}$$

where $a \circ p = b \circ q$ gives a Cartesian square, then by the projection formula,

$$\begin{aligned}
\rho_n(Z \times f) &= q_*(p^*Z \cdot s^*\Gamma_f \cdot r^* \operatorname{div}(g_n)) \\
&= q_*(p^*Z \cdot r^* \operatorname{div}(g_n)) \cdot t^*(\Gamma_f) \\
&= q_*(p^*(Z \cdot u^* \operatorname{div}(g_n))) \cdot t^*(\Gamma_f) \\
&= b^*a_*(Z \cdot u^* \operatorname{div}(g_n)) \cdot t^*(\Gamma_f) \\
&= \rho_n(Z) \times f
\end{aligned}$$

where Γ_f is the graph of f. $\square$

Proposition 6.8 ([Voe10], Theorem 4.6)**.** Let $\mathcal{F} \in \mathrm{Sh}(k)$ such that there exists an epimorphism $\mathbb{Z}(X) \to \mathcal{F}$ for some $X \in \mathrm{Sm}/k$. Let $\varphi : \mathbb{G}_m^{\wedge 1} \otimes \mathcal{F} \to \mathbb{G}_m^{\wedge 1} \otimes \mathbb{Z}(Y)$ be a map of sheaves, then there exists a unique (up to $\mathbb{A}^1$-homotopy[28]) morphism $\rho(\varphi) : \mathcal{F} \to \mathbb{Z}(Y)$ such that $\mathbb{G}_m^{\wedge 1} \otimes \rho(\varphi)$ is $\mathbb{A}^1$-homotopic to φ.

Proof. Fix an epimorphism $p : \mathbb{Z}(X) \to \mathcal{F}$, and note $\mathbb{G}_m \cong \mathbb{G}_m^{\wedge 1} \oplus \mathbb{Z}$. We construct $\varphi_+ : \mathbb{G}_m \otimes \mathcal{F} \to \mathbb{G}_m \otimes \mathbb{Z}(Y)$ be the pointed map $\varphi_+ = \varphi \coprod \mathrm{id}_{\{*\}}$, then the map $\varphi_+ \circ (\mathrm{id}_{\mathbb{G}_m} \times p)$ is a map from $\mathbb{G}_m \times X$ to $\mathbb{G}_m \times Y$, so by Yoneda lemma it induces $Z \in \mathrm{Cor}_k(\mathbb{G}_m \times X, \mathbb{G}_m \times Y)$. Moreover, for sufficiently large n, we consider $\rho_n(Z) : X \to Y$, defined by $\pi_*(Z \cdot \operatorname{div}(g_n))$ where $g_n = \frac{f_1^{n+1}-1}{f_1^{n+1}-f_2}$ and π_* is the pushforward of $\pi : \mathbb{G}_m \times X \times \mathbb{G}_m \times Y \to X \times Y$. Suppose $f : W \to X$ is a finite correspondence such that $p \circ f = 0$, then by Proposition 6.6, we have

$$\begin{aligned}
\rho_n(Z) \circ f &= \rho_n(Z \circ (\mathrm{id}_{\mathbb{G}_m} \times f)) \\
&= \rho_n(\varphi_+ \circ (\mathrm{id}_{\mathbb{G}_m} \times p) \circ (\mathrm{id}_{\mathbb{G}_m} \times f)) \\
&= 0.
\end{aligned}$$

Hence, $\rho_n(Z)|_{\ker(p)} = 0$, so we get a map $\rho_n(\varphi) : \mathcal{F} \to \mathbb{Z}(Y)$.

We may now show that for large enough n one has $\mathbb{G}_m^{\wedge 1} \otimes \rho_n(\varphi) \sim_{\mathbb{A}^1} \varphi$. We define φ' by the commutative diagram

$$\begin{tikzcd}
\mathbb{G}_m^{\wedge 1} \otimes \mathcal{F} \arrow[r, "\varphi"] \arrow[d, "\cong"'] & \mathbb{G}_m^{\wedge 1} \otimes \mathbb{Z}(Y) \arrow[d, "\cong"] \\
\mathcal{F} \otimes \mathbb{G}_m^{\wedge 1} \arrow[r, "\varphi'"'] & \mathbb{Z}(Y) \otimes \mathbb{G}_m^{\wedge 1}
\end{tikzcd}$$

We claim that

[28] By $\mathbb{A}^1$-homotopy, we mean by $\sim_{\mathbb{A}^1}$: for any two sheaves $\mathcal{F}$ and $\mathcal{G}$, the hom set is $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) / \langle \mathbb{A}^1\text{-homotopy} \rangle$. Note that the $\mathbb{A}^1$-homotopy relation $X \times \mathbb{A}^1 \to Y$ is not transitive, so we need to consider the subgroup generated.

Claim 6.9 ([Voe10], Lemma 4.7)**.** The maps

$$\begin{aligned}\varphi \otimes \mathrm{id}_{\mathbb{G}_m^{\wedge 1}} : \mathbb{G}_m^{\wedge 1} \otimes \mathcal{F} \otimes \mathbb{G}_m^{\wedge 1} &\to \mathbb{G}_m^{\wedge 1} \otimes \mathbb{Z}(Y) \otimes \mathbb{G}_m^{\wedge 1} \\ (t_1, s, t_2) &\mapsto (\varphi(t_1, s), t_2)\end{aligned}$$

and

$$\begin{aligned}\mathrm{id}_{\mathbb{G}_m^{\wedge 1}} \otimes \varphi' : \mathbb{G}_m^{\wedge 1} \otimes \mathcal{F} \otimes \mathbb{G}_m^{\wedge 1} &\to \mathbb{G}_m^{\wedge 1} \otimes \mathbb{Z}(Y) \otimes \mathbb{G}_m^{\wedge 1} \\ (t_1, s, t_2) &\mapsto (t_1, \varphi'(s, t_2))\end{aligned}$$

are $\mathbb{A}^1$-homotopic.

First note that the two morphisms differ by a conjugation of swapping map $\tau : \mathbb{G}_m^{\wedge 2} \to \mathbb{G}_m^{\wedge 2}$. However,

Claim 6.10 ([Voe10], Lemma 4.8)**.** The swapping map τ is $\mathbb{A}^1$-homotopic to the map

$$\begin{aligned}\mathbb{G}_m^{\wedge 2} &\to \mathbb{G}_m^{\wedge 2} \\ (x, y) &\mapsto (x, y^{-1}).\end{aligned}$$

Therefore, to prove Claim 6.9, it suffices to prove Claim 6.10.

Proof of Claim 6.10. For any $f_1, \dots, f_n : X \to \mathbb{G}_m$, we write $\tilde{f}_1, \dots, \tilde{f}_n : X \to \mathbb{G}_m^{\wedge 1}$ to be the maps defined by composing with the projection $\mathbb{G}_m \to \mathbb{G}_m^{\wedge 1}$.[29] We denote $\tilde{f}_1 \otimes \cdots \otimes \tilde{f}_n : X \to \mathbb{G}_m^{\wedge n}$ by $[f_1] \cdots [f_n]$. Suppose $f_1, f_2, g : X \to \mathbb{G}_m$ are morphisms, then we define $Z \in \mathrm{Cor}_k(X \times \mathbb{A}^1, \mathbb{G}_m)$ by

$$y^2 - (t(\tilde{f}_1(x) + \tilde{f}_2(x)) + (1-t)(1 + \tilde{f}_1(x)\tilde{f}_2(x)))y + \tilde{f}_1(x)\tilde{f}_2(x) = 0$$

where $(x, t, y) \in X \times \mathbb{A}^1 \times \mathbb{G}_m$. We have

$$t = \frac{(y-1)(\tilde{f}_1\tilde{f}_2 - y)}{y(\tilde{f}_1 - 1)(\tilde{f}_2 - 1)},$$

therefore $Z \cong X \times \mathbb{G}_m$ are isomorphic as schemes, thus Z is integral. The projection $Z \to X \times \mathbb{A}^1$ is finite by locality. We have

$$Z|_{t=0} = [1] + [f_1 f_2] = [f_1 f_2]$$

and

$$Z|_{t=1} = [f_1] + [f_2].$$

Since $Z \in \mathrm{Cor}_k(X \times \mathbb{A}^1, \mathbb{G}_m)$, then

$$[f_1 f_2] \sim_{\mathbb{A}^1} [f_1] + [f_2].$$

Therefore,

$$[f_1 f_2][g] \sim_{\mathbb{A}^1} [f_1][g] + [f_2][g].$$

Now consider some $Z \in \mathrm{Cor}_k(X \times \mathbb{A}^1, \mathbb{G}_m \times \mathbb{G}_m)$ given by

$$\begin{cases} y_1^2 - (t(\tilde{f}(x) + \tilde{g}(x))) + (1-t)(1 + \tilde{f}(x)\tilde{g}(x))y_1 + \tilde{f}(x)\tilde{g}(x) = 0 \\ y_1 = y_2 \end{cases}$$

where $(x, t, y_1, y_2) \in X \times \mathbb{A}^1 \times \mathbb{G}_m \times \mathbb{G}_m$ and $f, g : X \to \mathbb{G}_m$. Restricting at $t = 0$ and $t = 1$ now gives the relation $[fg][fg] \sim_{\mathbb{A}^1} [f][f] + [g][g]$. But $[fg][fg] \sim_{\mathbb{A}^1} [f][f] + [f][g] + [g][f] + [g][g]$, then $[f][g] + [g][f] = 0$.

[29] Note that $\tilde{f}_i \not\equiv 1$ everywhere since $\mathbb{G}_m^{\wedge 1}$ has the point 1 killed.

Therefore, $[g][f]+[g][f^{-1}]\sim_{\mathbb{A}^1}[g][1]=0$, hence $[f][g]\sim_{\mathbb{A}^1}[g][f^{-1}]$. Now if we have $f,g:X\to\mathbb{G}_m^{\wedge 1}$, then we obtain $(f,g),(g,f^{-1}):X\to\mathbb{G}_m^{\wedge 2}$. In particular, they are maps $\mathrm{id}_{\mathbb{G}_m^{\wedge 2}}$ and $(x,y)\mapsto(y,x^{-1})$, and by our observation above they must be $\mathbb{A}^1$-homotopic. Applying the swapping map τ on both maps, we know $(x,y)\mapsto(y,x)$ and $(x,y)\mapsto(x,y^{-1})$ are $\mathbb{A}^1$-homotopic as well. This is exactly what we want to show. ■

Now we know Claim 6.9 holds. For sufficiently large n, we have $\rho_n(\varphi\otimes\mathrm{id}_{\mathbb{G}_m^{\wedge 1}})=\rho_n(\varphi)\otimes\mathrm{id}_{\mathbb{G}_m^{\wedge 1}}$ by Proposition 6.7. Moreover, we know $\rho_n(\mathrm{id}_{\mathbb{G}_m^{\wedge 1}}\otimes\varphi')=\varphi'$ by Proposition 6.5. Hence,

$$\varphi'=\rho_n(\mathrm{id}_{\mathbb{G}_m^{\wedge 1}}\otimes\varphi')\sim_{\mathbb{A}^1}\rho_n(\varphi\otimes\mathrm{id}_{\mathbb{G}_m^{\wedge 1}})=\rho_n(\varphi)\otimes\mathrm{id}_{\mathbb{G}_m^{\wedge 1}}$$

by Claim 6.9, therefore $\mathrm{id}_{\mathbb{G}_m^{\wedge 1}}\otimes\rho_n(\varphi)\sim_{\mathbb{A}^1}\varphi$. This proves the existence.

To prove the uniqueness up to $\mathbb{A}^1$-homotopy, consider a morphism of the form $\varphi=\mathrm{id}_{\mathbb{G}_m^{\wedge 1}}\otimes\psi$, then $Z\in\mathrm{Cor}_k(\mathbb{G}_m\times X,\mathbb{G}_m\times Y)$ defined above is of the form $\mathrm{id}_{\mathbb{G}_m^{\wedge 1}}\otimes W$ where $W\in\mathrm{Cor}_k(X,Y)$ corresponds to ψ. By Proposition 6.5, we have $\rho_n(Z)=W$. If ρ and ρ' satisfy $\mathrm{id}_{\mathbb{G}_m^{\wedge 1}}\otimes\rho\sim_{\mathbb{A}^1}\varphi$ and $\mathrm{id}_{\mathbb{G}_m^{\wedge 1}}\otimes\rho'\sim_{\mathbb{A}^1}\varphi$, then applying ρ_n for large n gives $\rho\sim_{\mathbb{A}^1}\rho'$. □

Proposition 6.11 ([Voe10], Corollary 4.9)**.** Let $\mathcal{F}_Y$ be the presheaf defined by $X\mapsto\mathrm{Hom}(\mathbb{G}_m^{\wedge 1}\otimes\mathbb{Z}(X),\mathbb{G}_m^{\wedge 1}\otimes\mathbb{Z}(Y))$, and define the map

$$\begin{aligned}\mathbb{G}_m^{\wedge 1}\otimes - :\mathbb{Z}(Y)&\to\mathcal{F}_Y\\ f&\mapsto\mathbb{G}_m^{\wedge 1}\otimes f.\end{aligned}$$

For any $X\in\mathrm{Sm}/k$, the map

$$\theta:C_*(\mathbb{Z}(Y))(X)\to C_*(\mathcal{F}_Y)(X)$$

between complexes is a quasi-isomorphism.

Proof. Consider each term in the Suslin complex. The map $C_p(\mathbb{Z}(Y))(X)\to C_p(\mathcal{F}_Y)(X)$, by definition, is

$$\mathrm{Hom}(\mathbb{Z}(X\otimes\Delta^p),\mathbb{Z}(Y))\xrightarrow{\mathbb{G}_m^{\wedge 1}\otimes -}\mathrm{Hom}(\mathbb{G}_m^{\wedge 1}\otimes\mathbb{Z}(X\otimes\Delta^p),\mathbb{G}_m^{\wedge 1}\otimes\mathbb{Z}(Y)).$$

For any sheaf $\mathcal{G}$, there is a notion of $C_*(\mathcal{G})(X)$ with sections on X. Taking the pth cycle $Z_p(C_*(\mathcal{G})(X))$, i.e., elements with zero differential at degree p, we know

$$Z_p(C_*(\mathcal{G})(X))=\mathrm{Hom}(\mathbb{Z}(X)\otimes\mathbb{Z}(\Delta^p/d\Delta^p),\mathcal{G}).$$

[30]By Proposition 6.8, for every $f\in Z_p(C_*(\mathcal{F}_Y)(X))$, there exists $g\in Z_p(C_*(\mathbb{Z}(Y))(X))$ such that $\theta(g)\sim_{\mathbb{A}^1}f$. By the prism decomposition technique as in Lemma 5.21, we have a chain homotopy $s_n:\mathcal{G}^{\Delta^n}(X\times\mathbb{A}^1)\to\mathcal{G}^{\Delta^{n+1}}(X)$, therefore $\theta(g)-f\in B_p(C_*(\mathcal{F}_Y)(X))$ lives in the boundary. Therefore, $H_*(\theta)$ is a surjective map between homologies.

Moreover, $\theta(g)=\partial(f)$ is in the boundary, then it is easy to check that $\partial(\rho_n(f))=g$ for large enough n, hence g is in the boundary as well, therefore θ induces an injection on homology, thus θ is a quasi-isomorphism. □

Theorem 6.12. Let $\mathcal{F}\in\mathrm{PSh}(k)$ be homotopy invariant and $\mathcal{F}(\mathrm{Spec}(E))=0$ for every field E/k, then $\mathcal{F}_{\mathrm{Zar}}=0$, i.e., $\mathcal{F}=0$ after Zariski sheafification.

In particular, Theorem 6.12 implies that if $f:\mathcal{F}_1\to\mathcal{F}_2$ is a map of sheaves such that $f(E)$ is an isomorphism for any field E/k, then f is an isomorphism.

[30]Here is an alternative argument. For any sheaf $\mathcal{F}$, there is a normalized complex $C_*^{\mathrm{DK}}(\mathcal{F})\subseteq C_*(\mathcal{F})$, defined by $C_*^{\mathrm{DK}}(\mathcal{F})_n=\{s\in\mathcal{F}^{\Delta^n}:s|_{i\text{-th face }\Delta^{n-1}}=0,0\leqslant i<n\}$. Proposition 7.8 states that $C_*^{\mathrm{DK}}(\mathcal{F})(X)\to C_*(\mathcal{F})(X)$ is a quasi-isomorphism for any $X\in\mathrm{Sm}/k$, so we may as well replace C_* by C_*^{DK}. For any $\mathcal{G}$, we have $Z_p(C_*^{\mathrm{DK}}(\mathcal{G})(X))=\{s\in\mathcal{G}(X\times\Delta^p):s|_{i\text{-th face}}=0,0\leqslant i\leqslant p\}=\mathrm{Hom}(X\times\Delta^p/\partial\Delta^p,\mathcal{G})$.

Proof. See [MVW06], Corollary 11.2. □

Proposition 6.13. Suppose $\mathcal{F} \in \mathrm{Sh}(k)$ is a sheaf with transfer over a perfect field k and is homotopy invariant then $R^i p_* \mathcal{F} = 0$ for all $i > 0$. Recall that $p : \mathbb{G}_m \to \mathrm{Spec}(k)$ is the structure map of $\mathbb{G}_m$ and $p_*(\mathcal{F}) = \mathcal{F}^{\mathbb{G}_m}$.

Proof. We know $R^i p_* \mathcal{F}$ is the sheaf associated to the presheaf defined by $X \mapsto H^i(X \times \mathbb{G}_m, \mathcal{F})$.[31] Note that it suffices to show this where $X = \mathrm{Spec}(E)$ for some field E, then there is an exact sequence of localizations

$$0 \longrightarrow H^i(\mathbb{A}^1_E, \mathcal{F}) \longrightarrow H^i((\mathbb{G}_m)_E, \mathcal{F}) \longrightarrow H^{i+1}_0(\mathbb{A}^1_E, \mathcal{F}) \longrightarrow 0$$

for $i > 0$. Since $\mathcal{F}$ is homotopy invariant, then $H^i(\mathbb{A}^1_E, \mathcal{F}) = H^i(\mathrm{Spec}(E), \mathcal{F}) = 0$ since $\mathbb{A}^1_E$ has cohomological dimension 1. Since $\mathcal{F}$ is $\mathbb{A}^1$-homotopy invariant, then it is strictly $\mathbb{A}^1$-homotopy invariant, thus it induces an isomorphism between cohomologies. Since $i > 0$, then $i+1 \geqslant 2$, now by the cohomological dimension again, we have $H^{i+1}_0(\mathbb{A}^1_E, \mathcal{F}) = 0$. Therefore, this forces $H^i((\mathbb{G}_m)_E, \mathcal{F}) = 0$. Note that the presheaf defined above is Nisnevich with transfers, so it is homotopy invariant as well. By Theorem 6.12, we know $R^i p_* \mathcal{F} = 0$ whenever $i > 0$. □

Theorem 6.14 (Cancellation Theorem, [Voe10], Corollary 4.10)**.** Suppose k is a perfect field. For any $K, L \in \mathrm{DM}^{\mathrm{eff},-}(k)$, the map

$$\mathrm{Hom}_{\mathrm{DM}}(K, L) \xrightarrow{-\otimes \mathbb{Z}(1)} \mathrm{Hom}_{\mathrm{DM}}(K(1), L(1))$$

is an isomorphism.

Proof. By Lemma 5.29, it suffices to show that for any $X, Y \in \mathrm{Sm}/k$ and $n \in \mathbb{Z}$, we have

$$\mathrm{Hom}_{\mathrm{DM}}(\mathbb{Z}(X), \mathbb{Z}(Y)[n]) = \mathrm{Hom}_{\mathrm{DM}}(\mathbb{Z}(X) \otimes \mathbb{G}_m^{\wedge 1}, \mathbb{Z}(Y) \otimes \mathbb{G}_m^{\wedge 1}[n]).$$

Recall that

$$\mathrm{Hom}_{\mathrm{DM}}(\mathbb{Z}(X), \mathbb{Z}(Y)[n]) \cong \mathbb{H}^n(X, C_* \mathbb{Z}(Y))$$

since $C_* \mathbb{Z}(Y)$ is $\mathbb{A}^1$-local and $\mathbb{Z}(Y)$ is $\mathbb{A}^1$-equivalent to $C_*(\mathbb{Z}(Y))$. Similarly,

$$\mathrm{Hom}_{\mathrm{DM}}(\mathbb{Z}(X) \otimes \mathbb{G}_m^{\wedge 1}, \mathbb{Z}(Y) \otimes \mathbb{G}_m^{\wedge 1}) \cong \mathbb{H}^n(X \otimes \mathbb{G}_m^{\wedge 1}, C_*(\mathbb{Z}(Y) \otimes \mathbb{G}_m^{\wedge 1})).$$

By Proposition 6.11, the map

$$- \otimes \mathbb{G}_m^{\wedge 1} : C_* \mathbb{Z}(Y) \to C_* \underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1}, \mathbb{Z}(Y) \otimes \mathbb{G}_m^{\wedge 1}) \tag{6.15}$$

is a quasi-isomorphism since it is a quasi-isomorphism on every section. By definition, we know

$$C_* \underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1}, \mathbb{Z}(Y) \otimes \mathbb{G}_m^{\wedge 1}) \cong \underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1}, C_*(\mathbb{Z}(Y) \otimes \mathbb{G}_m^{\wedge 1})).$$

Therefore, we have a Grothendieck/Leray spectral sequence[32]

$$H^j(X, R^i p_* C_*(\mathbb{Z}(Y) \otimes \mathbb{G}_m^{\wedge 1})) \Rightarrow \mathbb{H}^{i+j}(X \times \mathbb{G}_m, C_*(\mathbb{Z}(Y) \otimes \mathbb{G}_m^{\wedge 1})) \tag{6.16}$$

where $p : \mathbb{G}_m \to \mathrm{Spec}(k)$.[33] Moreover, we have a hypercohomology spectral sequence

$$R^i p_*(H^j(C_*(\mathbb{Z}(Y) \otimes \mathbb{G}_m^{\wedge 1}))) \Rightarrow R^{i+j} p_* C_*(\mathbb{Z}(Y) \otimes \mathbb{G}_m^{\wedge 1}) \tag{6.17}$$

[31] In particular, when $i = 0$, we have $\mathcal{F}^{\mathbb{G}_m}$.
[32] Taking sections on X after applying p_* is equivalent to taking sections on $X \times \mathbb{G}_m$.
[33] This might be $p : X \times \mathbb{G}_m \to X$ instead.

by Lemma 1.26. By Proposition 5.49 and Proposition 6.13, we have

$$R^i p_*(C_*(\mathbb{Z}(Y) \otimes \mathbb{G}_m^{\wedge 1})) = H^i(C_*(\mathbb{Z}(Y) \otimes \mathbb{G}_m^{\wedge 1})^{\mathbb{G}_m^{\wedge 1}}) = 0 \tag{6.18}$$

if $i > 0$. Indeed, considering Equation (6.17), if $i + j > 0$, then either $i > 0$ or $j > 0$. If $i > 0$, then we retrieve Equation (6.18) by Proposition 6.13; if $j > 0$, we know from the construction of C_* that $H^j(C_*\mathcal{F}) = 0$ for any sheaf $\mathcal{F}$, therefore $H^j(C_*(\mathbb{Z}(Y) \otimes \mathbb{G}_m^{\wedge 1})) = 0$ and we obtain Equation (6.18) again. By the quasi-isomorphism in Equation (6.15), we know

$$\begin{aligned}\mathbb{H}^n(X, C_*(\mathbb{Z}(Y))) &= \mathbb{H}^n(X, C_*(\mathbb{Z}(Y) \otimes \mathbb{G}_m^{\wedge 1})^{\mathbb{G}_m^{\wedge 1}}) \\ &= \mathbb{H}^n(X, \underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1}, C_*(\mathbb{Z}(Y) \otimes \mathbb{G}_m^{\wedge 1}))).\end{aligned}$$

By Equation (6.18) and Equation (6.16), this forces $i = 0$, so we get

$$\begin{aligned}\mathbb{H}^n(X, C_*\mathbb{Z}(Y)) &= \mathbb{H}^n(X, \underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1}, C_*(\mathbb{Z}(Y) \otimes \mathbb{G}_m^{\wedge 1}))) \\ &= \mathbb{H}^n(X \otimes \mathbb{G}_m^{\wedge 1}, C_*(\mathbb{Z}(Y) \otimes \mathbb{G}_m^{\wedge 1}))\end{aligned}$$

as desired. □

Remark 6.19. The assumption that k is perfect can be dropped, c.f., [CD15], Proposition 8.1, which states that the functor $\mathrm{DM}^{\mathrm{eff},-}(k) \to \mathrm{DM}^{\mathrm{eff},-}(k^{\mathrm{perf}})$ is fully faithful, where k^{perf} is the perfect closure.

7 Comparison Theorem for Weight-1 Motivic Cohomology

In this section, we want to identify weight-1 motivic cohomology $H^{p,1}(X,\mathbb{Z})$.

Definition 7.1. Let $X\in\mathrm{Sm}/k$, we define a presheaf with transfer[34]

$$\mathcal{M}^*(\mathbb{P}^1;0,\infty)(X)=\{f\in K(X\times\mathbb{P}^1): f|_{X\times\{0,\infty\}}=1\}=\{f\in K(X\times\mathbb{P}^1):\bar{f}\equiv 1 \text{ on } k(x) \text{ for } x\in X\times\{0,\infty\}\}$$

in the function field of $X\times\mathbb{P}^1$.

Proposition 7.2. Suppose $X\in\mathrm{Sm}/k$, and let $C\in\mathrm{Cor}_k(X,\mathbb{G}_m)$ be an (integral) finite correspondence, then C is a principal divisor on $X\times\mathbb{G}_m$. Conversely, suppose $C=\mathrm{div}(f)|_{X\times\mathbb{G}_m}$, i.e., C is the divisor of some rational function on $X\times\mathbb{P}^1$ restricted to $X\times\mathbb{G}_m$, where $f\in K(X\times\mathbb{P}^1)$, then f can be chosen to be the form $t^n+a_1t^{n-1}+\cdots+a_{n-1}t+a_n$ where t is the parameter of $\mathbb{G}_m$, with regular functions $a_1,\dots,a_n\in\mathcal{O}_X(X)$, and $n=[K(C):K(X)]$ is the extension degree of the rational function of C over X. In this case, $\mathrm{supp}(\mathrm{div}(f))\cap\{X\times\{0\}\}=\varnothing$ and $a_n\in\mathcal{O}_X^*(X)$.

Proof. For every affine open subset $U\subseteq X$, we define $f_0\in\mathcal{O}(U)[t]$ to be the minimal polynomial of $t|_C$ over $K(U)$, i.e., coefficients are regular. Since it is integral and closed, this is a regular function, with the degree of the function is just the field extension degree $[K(C):K(X)]$. These coefficients glue together so we obtain $f=t^n+a_1t^{n-1}+\cdots+a_0$. Here $a_n\in\mathcal{O}^*$ because it is invertible locally, therefore $\mathrm{div}(f)=C$ since f is the minimal polynomial of t and $f|_C=0$. Also $\mathrm{supp}(\mathrm{div}(f))\cap\{X\times\{0\}\}=\varnothing$ because $a_n\in\mathcal{O}^*(X)$. □

Proposition 7.3. Suppose $X\in\mathrm{Sm}/k$ and let $f\in K(X\times\mathbb{P}^1)$. If $\mathrm{supp}(\mathrm{div}(f))\subseteq X\times\mathbb{G}_m$ and $\mathrm{div}(f)\in\mathrm{Cor}_k(X,\mathbb{G}_m)$, then $\mathcal{O}^*(\mathrm{div}(f))(t)=\frac{f|_{t=0}}{f|_{t=\infty}}\in\mathcal{O}^*(X)$, where $t\in\mathcal{O}^*(\mathbb{G}_m)$ is the parameter.

Proof. We can check this equation locally, so without loss of generality, we may assume X is affine. Suppose $\mathrm{div}(f)=\sum\limits_a n_aC_a$ and let $g_a\in\mathcal{O}(X)[t]$ be the polynomial corresponding to C_a as in Proposition 7.2, then $\mathcal{O}^*(\mathrm{div}(f))(t)=\prod\limits_a((-1)^{\deg(g_a)}g_a(0))^{n_a}$ by properties of norms. Since $\mathrm{div}(g_a)=C_a$, then $\frac{f}{\prod\limits_a g_a^{n_a}}\in\mathcal{O}^*(X\times\mathbb{G}_m)$ is invertible.[35] Therefore we can write $\frac{f}{\prod\limits_a g_a^{n_a}}=u\cdot t^m$ for some $u\in\mathcal{O}^*(X)$ and $m\in\mathbb{Z}$. Since $\mathrm{supp}(\mathrm{div}(f))\cap\{X\times\{0\}\}=\varnothing$, therefore $m=0$, so $f=u\cdot\prod\limits_a g_a^{n_a}=u\cdot t^{\sum\limits_a n_a\deg(g_a)}\prod\limits_a\left(\frac{g_a}{t^{\deg(g_a)}}\right)^{n_a}$. We know the product is regular at ∞, and to make the power of t to be regular at ∞ as well, we need $\mathrm{ord}_{X\times\{\infty\}}(f)=-\sum\limits_a n_a\deg(g_a)=0$ because $\mathrm{supp}(\mathrm{div}(f))\cap\{X\times\{\infty\}\}=\varnothing$. Therefore, $f|_{t=0}=u\cdot\prod\limits_a g_a^{n_a}(0)$ and $f|_{t=\infty}=u$ by direct computation. Hence

$$\mathcal{O}^*(\mathrm{div}(f))(t)=\prod_a g_a(0)^{n_a}=\frac{f|_{t=0}}{f|_{t=\infty}}.$$

□

Proposition 7.4 ([MVW06], Lemma 4.4)**.** For any $X\in\mathrm{Sm}/k$, there is an exact sequence of abelian groups

$$0\longrightarrow\mathcal{M}^*(\mathbb{P}^1;0,\infty)(X)\xrightarrow{\mathrm{div}}\mathrm{Cor}_k(X,\mathbb{G}_m)\xrightarrow{\lambda}\mathrm{Cor}_k(X,\mathrm{Spec}(k))\oplus\mathcal{O}^*(X)\longrightarrow 0 \tag{7.5}$$

where $\lambda(C)=(\pi\circ C,\mathcal{O}^*(C)(t))$ and $\pi:\mathbb{G}_m\to\mathrm{Spec}(k)$.

Proof. We first show that div is injective. Suppose $\mathrm{div}(f)=0$, then $f\in\mathcal{O}^*(X\times\mathbb{P}^1)\cong\mathcal{O}^*(X)$, hence f comes from $\mathcal{O}^*(X)$. But $f|_{t=0}=1$, therefore $f\equiv 1$ as well.

[34] This is proven in [MVW06], Lemma 4.5. We will see later that this is actually a sheaf with transfer.
[35] Given a ring A, note that the invertible functions are $(A[t]_t)^*=A^*\oplus\mathbb{Z}$.

We then show that $\lambda \circ \operatorname{div} = 0$. Consider the commutative diagram

$$\begin{tikzcd} X \times \mathbb{G}_m \arrow[r, "a"] \arrow[d, hook] & X \\ X \times \mathbb{P}^1 \arrow[ur, "b"'] & \end{tikzcd}$$

then $\pi \circ (\operatorname{div}(f)) = a_*(\operatorname{div}(f)) = b_*(\operatorname{div}(f)) \in \mathrm{CH}^0(X)$. By changing the base to $K(X)$, we find

$$b_*(\operatorname{div}(f)) = \deg_{\mathbb{P}^1_{K(X)}}(\operatorname{div}(f_{K(X)})) = 0.$$

Moreover, $\mathcal{O}^*(\operatorname{div}(f))(t) = \frac{f|_{t=0}}{f|_{t=\infty}} = 1$ by Proposition 7.3.

Moreover, we show that $\ker(\lambda) \subseteq \operatorname{im}(\operatorname{div})$. Suppose we have $\sum\limits_a n_a C_a \in \mathrm{Cor}_k(X, \mathbb{G}_m)$ satisfying $\sum\limits_a n_a \pi \circ C_a = p_{1*}(\sum\limits_a n_a C_a) = 0$ where $p_1 : X \times \mathbb{G}_m \to X$, and $\mathcal{O}^*(\sum\limits_a n_a C_a)(t) = 1$. For any C_a, we pick $f_a \in \mathcal{O}(X)[t]$ as constructed in Proposition 7.2, then $\operatorname{supp}(\operatorname{div}(\prod\limits_a f_a^{n_a})) \cap \{X \times \{0\}\} = \varnothing$. Moreover,

$$\begin{aligned} 0 &= p_{1*}(\sum_a n_a C_a) \\ &= \sum_a n_a \deg(f_a) p_1(C_a). \end{aligned}$$

In particular, if we take $p_1(C_a) = 0$, then $\sum\limits_a n_a \deg(f_a) = 0$. Now let us write $f_a = t^d + a_1 t^{d-1} + \cdots + a_d = t^d(1 + \frac{a_1}{t} + \cdots + \frac{a_d}{t^d})$, then note that $(1 + \frac{a_1}{t} + \cdots + \frac{a_d}{t^d})$ is regular at ∞. Since $\sum\limits_a n_a \deg(f_a) = 0$, then $\prod\limits_a f_a^{n_a} = \prod\limits_a h_a^{n_a}$ where $h_a \in \mathcal{O}(X \times (\mathbb{P}^1 \backslash \{0\}))$, and $h_a|_{t=\infty} = 1$. Hence, $\left.\prod\limits_a f_a^{n_a}\right|_{t=\infty} = 1$. Therefore, we know $\operatorname{supp}(\operatorname{div}(\prod\limits_a f_a^{n_a})) \cap \{X \times \{\infty\}\} = \varnothing$. By Proposition 7.3, we have

$$\begin{aligned} 1 &= \mathcal{O}^*(\sum_a n_a C_a)(t) \\ &= \mathcal{O}^*(\operatorname{div}(\prod_a f_a^{n_a}))(t) \\ &= \frac{\left.\prod_a f_a^{n_a}\right|_{t=0}}{\left.\prod_a f_a^{n_a}\right|_{t=\infty}} \\ &= \left.\prod_a f_a^{n_a}\right|_{t=0}. \end{aligned}$$

Hence, $\operatorname{supp}(\operatorname{div}(\prod\limits_a f_a^{n_a})) \in \mathcal{M}^*(\mathbb{P}^1; 0, \infty)$, and $\operatorname{div}(\prod\limits_a f_a^{n_a}) = \sum\limits_a n_a C_a$.

Finally, we prove that λ is surjective. Let $\beta : \operatorname{Spec}(k) \to \mathbb{G}_m$ be the constant map $\beta \equiv 1$, then for every $C \in \mathrm{Cor}_k(X, \operatorname{Spec}(k))$, $\pi \circ \beta \circ C = C$ and $\mathcal{O}^*(\beta \circ C)(t) = 1$. Therefore, $(C, 1) = \lambda(\beta \circ C)$. For any $u \in \mathcal{O}^*(X)$, it corresponds to some $\varphi : X \to \mathbb{G}_m$, hence $\lambda(\varphi) = (\pi \circ \varphi, u)$. Therefore, λ is a surjection. $\square$

Remark 7.6. Note that $\mathrm{Cor}_k(X, \mathbb{G}_m)$ and $\mathrm{Cor}_k(X, \operatorname{Spec}(k)) \oplus \mathcal{O}^*(X)$ are sheaves with transfers, therefore Proposition 7.4 implies that $\mathcal{M}^*(\mathbb{P}^1; 0, \infty)$ is a sheaf with transfer as well.

Definition 7.7. Let A be a simplicial object in Ab, namely a functor $A : \mathrm{Sim}^{\mathrm{op}} \to \mathrm{Ab}$, then we define the (Dodd-Kan) normalized complex $C_*^{\mathrm{DK}}(A) \subseteq C_*(A)$ by $(C_*^{\mathrm{DK}}(A))_n = \{X \in A_n : \partial_i(X) = 0 \, \forall i < n\}$.

Proposition 7.8. In the context above, $C_*^{\mathrm{DK}}(A)$ is quasi-isomorphic to $C_*(A)$.[36]

Proof. See [GJ09], Theorem 2.4. ☐

Proposition 7.9 ([MVW06], Lemma 4.6)**.** For any $X \in \mathrm{Sm}/k$, $(C_*\mathcal{M}^*(\mathbb{P}^1;0,\infty))(X)$ is an acyclic complex.

Proof. Define

$$\begin{aligned} i_j : X &\to X \times \mathbb{A}^1 \\ x &\mapsto (x,j) \end{aligned}$$

for $j = 0,1$. We know the two maps $i_0^*, i_1^* : (C_*\mathcal{M}^*(\mathbb{P}^1;0,\infty))(X \times \mathbb{A}^1) \to (C_*\mathcal{M}^*(\mathbb{P}^1;0,\infty))(X)$ are pullbacks between complexes, and by Lemma 5.21 they are chain homotopic, hence $H_*(i_0^*) \cong H_*(i_1^*)$ on the level of Suslin complexes. Moreover, by Proposition 7.8, $H_*(i_0^*) \cong H_*(i_1^*)$ holds on the level of (Dodd-Kan) normalized complexes. Suppose

$$f \in Z_n(C_*^{\mathrm{DK}}\mathcal{M}^*(\mathbb{P}^1;0,\infty)(X)) \subseteq \mathcal{M}^*(\mathbb{P}^1;0,\infty)(X \times \Delta^n) \subseteq K(X \times \Delta^n \times \mathbb{P}^1),$$

and define $g = 1 - t(1-f) \in K(X \times \mathbb{A}^1 \times \Delta^n \times \mathbb{P}^1)$, where t is the parameter in $\mathbb{A}^1$, then $g|_{X \times \mathbb{A}^1 \times \Delta^n \times \{0,\infty\}} \equiv 1$. Therefore, $g \in \mathcal{M}^*(\mathbb{P}^1;0,\infty)(X \times \mathbb{A}^1 \times \Delta^n) = (C_n\mathcal{M}^*(\mathbb{P}^1;0,\infty))(X \times \mathbb{A}^1)$, by definition is the nth term of the Suslin complex. Similarly, $g|_{X \times \mathbb{A}^1 \times \Delta^{n-1} \times \mathbb{P}^1} \equiv 1$ for any face $\Delta^{n-1} \subseteq \Delta^n$ because f is a cycle in the Dodd-Kan complex that satisfies the same condition. Moreover, $g|_{X \times \{0\} \times \Delta^n \times \mathbb{P}^1} \equiv 0$ by plugging in $t = 0$, and $g|_{X \times \{1\} \times \Delta^n \times \mathbb{P}^1} \equiv f$ by plugging in $t = 1$. Therefore, 1 differs from f by a boundary, but 1 is a boundary itself, so f is also a boundary. ☐

Theorem 7.10 ([MVW06], Theorem 4.1)**.** Note that λ from Proposition 7.4 induces

$$\bar{\lambda} : \mathbb{Z}(\mathbb{G}_m^{\wedge 1}) \to \mathcal{O}^*$$

by taking the quotient over a rational point on the finite correspondence on both sides of λ. In fact, $\bar{\lambda}$ is an $\mathbb{A}^1$-weak equivalence in DM.

Proof. From Equation (7.5), we obtain an exact sequence of complexes

$$0 \longrightarrow C_*(\mathcal{M}^*(\mathbb{P}^1;0,\infty)) \longrightarrow C_*\mathbb{Z}(\mathbb{G}_m^{\wedge 1}) \xrightarrow{C_*\bar{\lambda}} C_*\mathcal{O}^* \longrightarrow 0$$

since it is exact on each level. By Proposition 7.9, we know $C_*\bar{\lambda}$ is a quasi-isomorphism. Hence, we have a commutative diagram

$$\begin{tikzcd} \mathbb{Z}(\mathbb{G}_m^{\wedge 1}) \arrow[r, "\bar{\lambda}"] \arrow[d] & \mathcal{O}^* \arrow[d] \\ C_*\mathbb{Z}(\mathbb{G}_m^{\wedge 1}) \arrow[r, "C_*\bar{\lambda}"'] & C_*\mathcal{O}^* \end{tikzcd}$$

where the vertical morphisms are $\mathbb{A}^1$-weak equivalences by Proposition 5.39. Therefore, $\bar{\lambda}$ is also an $\mathbb{A}^1$-weak equivalence as well. ☐

Remark 7.11.

- $\mathbb{Z}(\mathbb{G}_m^{\wedge 1}) = \mathbb{Z}(1)[1]$ and $\mathbb{Z}(1) \cong \mathcal{O}^*[-1]$.
- For any $(l, \mathrm{char}(k)) = 1$, we have $\mathbb{Z}/l\mathbb{Z}(1)_{\text{ét}} = \mu_l$, and $\mathbb{H}^p(X, \mathbb{Z}/l\mathbb{Z}(q)) = H^p_{\text{ét}}(X, \mu_l^{\otimes q})$.

[36] In fact, they are chain homotopic.

Proposition 7.12 ([MVW06], Proposition 13.9; [V+00], Theorem 5.7)**.** Let k be a perfect field. If $\mathcal{F} \in \mathrm{Sh}(k)$ is homotopy invariant (as a Nisnevich sheaf with transfers), then

$$H^i_{\mathrm{Zar}}(X, \mathcal{F}) \cong H^i_{\mathrm{Nis}}(X, \mathcal{F})$$

for every $i \in \mathbb{N}$ and $X \in \mathrm{Sm}/k$.

Proof. We have a (forgetful) functor $\pi : \mathrm{Sh}_{\mathrm{Nis}} \to \mathrm{Sh}_{\mathrm{Zar}}$ from Nisnevich sites to Zariski sites, since every Nisnevich sheaf is a Zariski sheaf. Moreover, we have a Leray spectral sequence

$$H^p_{\mathrm{Zar}}(X, R^q(\pi_* \mathcal{F})) \Rightarrow H^{p+q}_{\mathrm{Nis}}(X, \mathcal{F})$$

between sites. It suffices to show that $R^q(\pi_* \mathcal{F}) = 0$ if $q > 0$, then the statement follows from the spectral sequence. We know $R^q \pi_* \mathcal{F}$ is the Zariski sheafification of the prsheaf $X \mapsto H^q_{\mathrm{Nis}}(X, \mathcal{F})$ from higher direct image, but it is a presheaf with transfers, homotopy invariant, and whose sections at fields vanish by the cohomological dimension argument since $q > 0$. We conclude the statement by Theorem 6.12, which states that $R^q \pi_* \mathcal{F} = 0$. $\square$

Corollary 7.13 ([MVW06], Proposition 13.10)**.** If k is perfect, then

$$H^{p,q}(X, \mathbb{Z}) \cong \mathbb{H}^p_{\mathrm{Zar}}(X, \mathbb{Z}(q)).$$

Therefore, the motivic cohomology, defined by the hypercohomology with respect to Nisnevich topology, agrees with the hypercohomology with respect to Zariski topology.

Proof. By Theorem 5.51, the cohomology sheaves $H^*(\mathbb{Z}(q))$ are homotopy invariant. Therefore, by Proposition 7.12, we know the cohomology with respect to Nisnevich topology agrees with the cohomology with respect to Zariski topology. The statement now follows from the hypercohomology spectral sequence. $\square$

Proposition 7.14 ([MVW06], Corollary 4.2)**.** We have

$$H^{p,1}(X, \mathbb{Z}) = \begin{cases} \mathcal{O}^*(X), & p = 1 \\ \mathrm{Pic}(X) \cong \mathrm{CH}^1(X), & p = 2 \\ 0, & \text{otherwise} \end{cases}$$

Remark 7.15. As a comparison, recall that

$$H^{p,0}(X, \mathbb{Z}) = \begin{cases} \mathbb{Z}(X), & p = 0 \\ 0, & \text{otherwise} \end{cases}$$

Proof. By Theorem 7.10 and Proposition 7.12, we know $H^{p,1}(X, \mathbb{Z}) = H^{p-1}_{\mathrm{Nis}}(X, \mathcal{O}^*) \cong H^{p-1}_{\mathrm{Zar}}(X, \mathcal{O}^*)$ since $\mathbb{Z}(1)[1] \cong \mathbb{Z}(\mathbb{G}_m^{\wedge 1}) \cong \mathcal{O}^*$. Note that $\mathcal{O}^*$ is homotopy invariant, therefore the statement for $p \leqslant 2$ follows from a description of Zariski cohomology. In other cases, we have a sequence

$$0 \longrightarrow \mathcal{O}^* \longrightarrow K^* \xrightarrow{\mathrm{div}} \bigoplus_{x \in X^{(1)}} \mathbb{Z} \longrightarrow 0$$

Since X is smooth and thus normal, so if a rational function has no valuation at every divisor, then it is regular. Therefore, this sequence is exact. But this is a flasque resolution of $\mathcal{O}^*$, hence $H^i(X, \mathcal{O}^*) = 0$ if $i > 1$. $\square$

8 Comparison Theorem for Large-weight Motivic Cohomology

We now want to compute $H^{p,q}(X,\mathbb{Z})$ if $p \geqslant 2q-1$. Let us denote $A^p(X;K_n^M) = H^p(C^*(X;K_n^M))$, where $C^*(X;K_n^M)_m = \bigoplus_{x\in X^{(m)}} K_{n-m}^M(k(X))$ to be the cohomology of the Rost complex. Most of the results follow from [Ros96].

Example 8.1. Note that $A^0(X;K_n^M) = K_n^M(X)$ and $A^n(X;K_n^M) = \mathrm{CH}^n(X)$.

Recall Proposition 3.30, which indicates for any flat morphism $f: X \to Y$, we have a pullback $f^*: A^p(Y;K_n^M) \to A^p(X;K_n^M)$. Moreover, every proper morphism $g: X \to Y$ gives rise to a pushforward $g_*: A^p(X;K_n^M) \to A^{p+d_Y-d_X}(Y;K^M_{n+d_Y-d_X})$.

8.1 Gabber's Representation Theorem

Proposition 8.2. Define $\pi: X\times\mathbb{A}^1 \to X$ to be the projection, then the pullback $\pi^*: A^p(X;K_n^M) \to A^p(X\times\mathbb{A}^1;K_n^M)$ is an isomorphism.

Therefore, the cohomology is invariant under homotopy conditions.

Proof. Define
$$C^{i,p}(\pi) = \bigoplus_{\substack{x\in(X\times\mathbb{A}^1)^{(p)}\\ \mathrm{codim}(\widetilde{\pi(X)})\geqslant i}} K_{n-p}^M(k(X)) \subseteq C^p(X\times\mathbb{A}^1;K_n^M),$$
then we have a finite filtration
$$\cdots \subseteq C^{i,p}(\pi) \subseteq C^{i-1,p}(\pi) \subseteq \cdots$$
of $C^p(X\times\mathbb{A}^1;K_n^M)$, where
$$C^{i,p}(\pi)/C^{i+1,p}(\pi) = \bigoplus_{u\in X^{(i)}} C^{p-i}(\mathbb{A}^1_{k(u)};K_n^M).$$
This induces a spectral sequence
$$E_1^{a,b}(\pi) = \bigoplus_{u\in X^{(a)}} A^b(\mathbb{A}^1_{k(u)};K_n^M) \Rightarrow A^{a+b}(X\times\mathbb{A}^1;K_n^M)$$
where
$$\begin{aligned} A^b(\mathbb{A}^1_{k(u)};K_n^M) &= \begin{cases} 0, & b\neq 0 \\ K_n^M(k(u)), & b=0 \end{cases} \\ &= A^b(k(u);K_n^M) \end{aligned}$$
by Theorem 3.5. Therefore the spectral sequence degenerates to the complex $E_1^{*,0}(\pi) \cong C^*(X;K_n^M)$. Hence the statement follows. □

Corollary 8.3 ([Ros96], Proposition 8.6)**.** Suppose $\pi: V \to X$ is an $\mathbb{A}^n$-bundle, then $\pi^*: A^p(X;K_n^M) \to A^p(V;K_n^M)$ is an isomorphism.

Theorem 8.4 (Gabber's Representation Theorem)**.** Let F be a field and X be the localization of a smooth F-scheme of dimension n. Suppose $Y \subseteq X$ is a closed subscheme such that $\mathrm{codim}(Y) \geqslant 1$, then there exists a closed point $t \in \mathbb{A}_F^{n-1}$ and an étale morphism $\pi: X \to \mathbb{A}^1 \times S$, where $S = \mathrm{Spec}(\mathcal{O}_{\mathbb{A}_F^{n-1}}, t)$, such that $\pi|_Y$ is a closed immersion, Y is finite over S, and that $Y = \pi^{-1}(\pi(Y))$.

Proof. See [CTHK97], Theorem 3.1.1, for the case where F is infinite; for finite field F, see [HK20]. □

Theorem 8.5. Suppose $X = \operatorname{Spec}(\mathcal{O}_{Y,k})$ where $Y \in \operatorname{Sm}/k$, then $A^p(X; K_n^M) = 0$ for any $p > 0$.

Proof. We prove by induction on $\dim(X)$. If $\dim(X) = 0$, this is trivial. Therefore, let $r \in Z^i(C^*(X; K_n^M))$ for $i > 0$. Let $r = \alpha_1 + \cdots + \alpha_\ell$ where $\alpha_j \in K_{n-i}^M(k(y_j))$ for $y_j \in X^{(i)}$. By Theorem 8.4, one may choose an étale morphism $f : X \to \mathbb{A}^1 \times S$ where $S = \operatorname{Spec}(\mathcal{O}_{\mathbb{A}^{d_X - 1}})$ for closed point t, and we have $\bar{y}_j \subseteq X \to \mathbb{A}^1 \times S$ are closed immersions for any j. Therefore, the residue field is maintained. Since $\pi^{-1}(\pi(y_j)) = y_j$ for each j, we have unique pullback choices locally, by which we can find $r' = \alpha_1' + \cdots + \alpha_\ell' \in Z^i(C^*(\mathbb{A}^1_S; K_n^M))$[37] such that $f^*(\alpha_1' + \cdots + \alpha_\ell') = r$, where $\alpha_j' \in K_{n-i}^M(k(f(y_j))) = K_{n-i}^M(k(y_j))$ by closed immersion. It suffices to show that r' is a boundary, since then by uniquness of pullback we know r is also a boundary. By Proposition 8.2, $H^*(C^*(\mathbb{A}^1_S; K_n^M)) = H^*(C^*(S; K_n^M))$ and the latter is zero when $p > 0$ since $\dim(S) < \dim(X)$. $\square$

8.2 Gysin Map

Now for every closed embedding $i : Y \to X$ of smooth schemes, we want to define the Gysin map $i^* : A^p(X; K_n^M) \to A^p(Y; K_n^M)$, as a pullback, by performing deformation to normal bundle. This would be an important improvement since we were only able to construct pullback for flat morphisms.

Definition 8.6. The deformation space $D(X, Y)$ is defined as the exclusion of blow-ups

$$D(X, Y) = \operatorname{Bl}_{Y \times \{0\}}(X \times \mathbb{A}^1) \backslash \operatorname{Bl}_{Y \times \{0\}}(X \times \{0\}),$$

Remark 8.7. First note that $D(X, Y)$ is smooth with a closed embedding $j : Y \times \mathbb{A}^1 \to D(X, Y)$, which is a strict transformation. Moreover, there is a flat morphism $\rho : D(X, Y) \to \mathbb{A}^1$ derived from the blow-down diagram

$$\begin{tikzcd} D(X,Y) \arrow[r, "\rho"] \arrow[d] & \mathbb{A}^1 \\ X \times \mathbb{A}^1 & \end{tikzcd}$$

such that the diagram

$$\begin{tikzcd} Y \times \mathbb{A}^1 \arrow[r, "j"] \arrow[dr, hook, "i \times \mathbb{A}^1"'] & D(X,Y) \arrow[d] \arrow[dr, "\rho"] & \\ & X \times \mathbb{A}^1 \arrow[r, "\pi_{\mathbb{A}^1}"'] & \mathbb{A}^1 \end{tikzcd}$$

In addition, we have

1. $\rho^{-1}(\mathbb{A}^1 \backslash \{0\}) = X \times (\mathbb{A}^1 \backslash \{0\})$, and after restricting onto j, it is the embedding $i \times \mathrm{id} : Y \times (\mathbb{A}^1 \backslash \{0\}) \to X \times (\mathbb{A}^1 \backslash \{0\})$;
2. $\rho^{-1}(0) = \mathcal{N}_{Y/X}$ is the normal bundle of Y over X, and its restriction onto j is the zero section $s_0 : Y \to \mathcal{N}_{Y/X}$.

This process is known as deformation to normal bundle in intersection theory.

For this closed immersion, there exists a localization sequence.

Definition 8.8. Let X be of finite type over a field k, and let $i : Y \to X$ be a closed immersion, and let $j : U = X \backslash Y \hookrightarrow X$ be the inclusion. We define a map $\partial : C^p(U; K_n^M) \to C^{p+1-c}(Y; K_{n-c}^M)$ on the Rost complexes, where $c = d_X - d_Y$ is the difference of dimensions of X and Y, as the following: for any $x \in U^{(p)}$ and $y \in Y^{(p+1-c)}$, we define it to be the partial map in Rost complex if $y \in \bar{x}$ and 0 otherwise. That is,

$$\partial_y^x = \begin{cases} \partial_y^x, & y \in \bar{x} \\ 0, & \text{otherwise} \end{cases}$$

[37] To see this is a cycle, by Theorem 8.4, $\pi^{-1}(\pi(y_j)) = y_j$ for each j, so it is maintained as cycle locally.

By Theorem 3.31, one can check that ∂ induces a map

$$\partial : A^p(U; K_n^M) \to A^{p+1-c}(Y; K_{n-c}^M).$$

By definition, we know there is a decomposition

$$C^p(X; K_n^M) = C^{p-c}(Y; K_{n-c}^M) \oplus C^p(U; K_n^M),$$

which induces the boundary map on the level of cohomology, which is the same as ∂ defined above, thus by snake lemma, we have a long exact sequence

$$\cdots \longrightarrow A^{p+c}(U; K_{n+c}^M) \xrightarrow{\partial} A^{p+1}(Y; K_n^M) \xrightarrow{i_*} A^{p+1}(X; K_{n+c}^M) \xrightarrow{j^*} A^{p+1+c}(U; K_{n+c}^M) \longrightarrow \cdots$$

of localization. Similarly, for any $f \in \mathcal{O}^*(X) = H^{1,1}(X, \mathbb{Z})$, multiplication by $[t]$ induces a map

$$[t] : A^p(X; K_n^M) \to A^p(X; K_{n+1}^M).$$

Now we define the Gysin map $i^* : A^p(X; K_n^M) \to A^p(Y; K_n^M)$ by the composition

$$A^p(X; K_n^M) \longrightarrow A^p(X \times (\mathbb{A}^1 \backslash \{0\}); K_n^M) \xrightarrow{[t]} A^p(X \times (\mathbb{A}^1 \backslash \{0\}); K_{n+1}^M) \xrightarrow{\partial} A^p(\mathcal{N}_{Y/X}; K_n^M) \xrightarrow{\cong} A^p(Y; K_n^M)$$

where t is the parameter of $\mathbb{A}^1$, ∂ is the boundary map by consider including $A^p(X \times (\mathbb{A}^1 \backslash \{0\}); K_{n+1}^M)$ into $D(X, Y)$, and the last isomorphism is given by Corollary 8.3.

Definition 8.9. Suppose X and Y are of finite type over a field k. For every $x \in X^{(n)}$, $y \in Y^{(m)}$ and $z \in (\bar{x} \times \bar{y})^{(0)}$, we define

$$\begin{aligned} \times : K_a^M(k(x)) \times K_b^M(k(y)) &\to K_{a+b}^M(k(z)) \\ (u, v) &\mapsto \pi_X^*(u)_z \cdot \pi_Y^*(v)_z \end{aligned}$$

where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are projections. This induces an exterior product

$$\times : C^p(X : K_n^M) \times C^q(X; K_m^M) \to C^{p+q}(X \times Y; K_{n+m}^M).$$

One can show that this exterior product descends to the exterior product on the level of cohomology.

Proposition 8.10 (Lebniz Rule)**.** For any $\rho \in C^p(X; K_n^M)$ and $\mu \in C^q(Y; K_m^M)$, we have

$$\partial_{X \times Y}(\rho \times \mu) = \partial_X(\rho) \times \mu + (-1)^p \rho \times \partial_Y(\mu).$$

Proof. Suppose $x \in X^{(a)}$, $y \in Y^{(b)}$, and z is a generic point of $\bar{x} \times \bar{y}$, i.e., $z \in (\bar{x} \times \bar{y})^{(0)}$. Consider the divisors on $\bar{z}$. For instance, take $w \in \bar{z}^{(1)}$, then it corresponds to the projection pair $(w_x, w_y) \in \bar{x} \times \bar{y}$. By considering a dimension argument on the transcendental degree, we know either

- $w_x \in \bar{x}^{(1)}$ is a divisor and $w_y \in \bar{y}^{(0)}$ is a generic point, or
- $w_x \in \bar{x}^{(0)}$ is a generic point and $w_y \in \bar{y}^{(1)}$ is a divisor.

By symmetry, it suffices to discuss the first case. Suppose $\rho \in K_{n-p}^M(k(x))$ and $\mu \in K_{m-q}^M(k(y))$, we have a diagram

$$\begin{tikzcd}
& w \arrow[d] & & w_i \arrow[ll] \arrow[d] \\
& z \arrow[d, hook] & & \tilde{z} \arrow[ll, "\pi"'] \arrow[dd, "\pi_Y"] \\
& \bar{x} \times \bar{y} \arrow[dl] \arrow[dr] & & \\
\bar{x} & & \bar{y} & \tilde{y} \arrow[l]
\end{tikzcd}$$

where $\tilde{y}$ and z are the normalization of $\bar{y}$ and $\bar{z}$, respectively. For $w \in \bar{z}$, we get to pick points in the fiber with $\{w_i\}_i = \pi^{-1}(w)$, which gives a pullback square in the diagram above.

Since $\tilde{z}$ is normal, then its local ring is a DVR, and therefore each w_i is a divisor of $\tilde{z}$. For any $f \in K_*^M(k(y))$, we know $\pi_Y^*(f)$ has zero valuation on each w_i, otherwise we must have $\pi_Y^*(f) = 0$ generically on each w_i, but that means $f = 0$ on $K(Y)$ generically since $w \to \bar{y}$ is dominant. By construction of ∂ in Proposition 3.4, let ∂_i be the partial map with respect to valuation at w_i, and let $a, b \in K_*^M(K(z))$, and π be a uniformizer of this valuation, then

$$\partial_i(a \cdot b) = \partial_i(a) \cdot s^\pi(b) + (-1)^{\deg(a)} s^\pi(a) \cdot \partial_i(b) + \partial_i(a)\partial_i(b)[-1].$$

Since $\pi_Y^*(f)$ has zero valuation on each w_i, then $\partial_i(\pi_Y^*(\mu)) = 0$, and therefore $\partial_i(\pi_X^*(\rho)\pi_Y^*(\mu)) = \partial_i(\pi_X^*(\rho))s^\pi(\pi_Y^*(\mu))$. Hence,

$$\begin{aligned}
\partial_{X\times Y}(\rho \times \mu)_w &= \sum_i N_{k(w_i)/k(w)}(\partial_i(\pi_X^*(\rho)\pi_Y^*(\mu))) \\
&= \sum_i N_{k(w_i)/k(w)}(\partial_i(\pi_X^*(\rho))s^\pi(\pi_Y^*(\mu))) \\
&= \sum_i N_{k(w_i)/k(w)}(\partial_i(\pi_X^*(\rho))) \cdot s^\pi(\pi_Y^*(\mu)) \\
&= (\partial_X(\rho) \times \mu)_w + (-1)^p \rho \times \partial_Y(\mu) \\
&= (\partial_X(\rho) \times \mu)_w.
\end{aligned}$$

by the projection formula since $s^\pi(\pi_Y^*(\mu))$ lands in $k(w)$. □

Corollary 8.11. There is an exterior product

$$\times : A^p(X; K_n^M) \times A^q(Y; K_m^M) \to A^{p+q}(X \times Y; K_{n+m}^M)$$

for any schemes X, Y of finite type over k.

Proof. If one of ρ and μ is a boundary, then the exterior product must also be a boundary by Proposition 8.10. □

Now for every (separated) $f : X \to Y$, we can decompose it as a composition

$$X \xrightarrow{\Gamma_f} X \times Y \xrightarrow{\pi_Y} Y$$

of the graph morphism, as a closed immersion, and the projection, as a flat morphism. Therefore, we get to define $f^* = \Gamma_f^* \pi_Y^* : A^p(Y; K_n^M) \to A^p(X; K_n^M)$, which is functorial, c.f., [Ros96], Theorem 12.1. Conversely, for any $X \in \mathrm{Sm}/k$, we obtain an intersection product by the composition

$$A^p(X; K_n^M) \times A^q(X; K_n^M) \xrightarrow{\times} A^{p+q}(X \times X; K_{m+n}^M) \xrightarrow{\Delta^*} A^{p+q}(X; K_{m+n}^M)$$

This product is associative and graded-commutative, c.f., [Ros96], Properties 14.2 and 14.3. In this case, graded-commutative indicates $x \cdot y = (-1)^{(n-p)(m-q)} y \cdot x$.

Proposition 8.12. The functor $A^p(-; K_n^M)$ is a homotopy invariant presheaf with transfers.

Proof. Note that we already have products and pullbacks on this functor structure. For any irreducible correspondence C of $X \to Y$, we have a diagram

$$\begin{tikzcd}
C \arrow[r, "\subseteq"] \arrow[dr, "\text{finite surjective}"'] & X \times Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"] & Y \\
& X &
\end{tikzcd}$$

then we define

$$\begin{aligned} A^p(C; K_n^M) : A^p(Y; K_n^M) &\to A^p(X; K_n^M) \\ \alpha &\mapsto \pi_{X*}(\pi_Y^*(\alpha) \cdot C) \end{aligned}$$

where $C \in A_C^{d_Y}(X \times Y; K_{d_Y}^M)$, supported on C itself. The fact that it is homotopy invariant just follows from Proposition 8.2. □

Proposition 8.13. $A^0(-; K_n^M) = K_n^M(-)$ is a homotopy invariant Nisnevich sheaf with transfers.

Proof. Clearly $A^0(-; K_n^M)$ is a Zariski sheaf. To show that it is Nisnevich, consider a Nisnevich covering $p : U \to X$ over, without loss of generality, an integral scheme X. If U is connected, then $K_n^M(X) = K_n^M(U)$: note that there is a pullback morphism of $K_n^M(X)$ to $K_n^M(U)$ along the immersion, and both are subgroups of $K_n^M(K(X))$; in addition, they have the same function field, also, given a point on X, there exists a point on U such that they have the same residue field, and therefore having zero residue on X is equivalent to having zero residue on U by the Nisnevich property. Suppose U is not connected, then we have an injection $K_n^M(X) \subseteq K_n^M(U) \subseteq K_n^M(K(X))$. Therefore we have a separable presheaf, and it suffices to prove the existence of gluing. If we have $\alpha \in K_n^M(U)$ such that $\alpha|_{U \times_X U} = 0$, then for any connected component U_i of U, we get $\alpha_i \in K_n^M(p(U_i)) \supseteq K_n^M(X)$. Therefore, $\alpha|_{U \times_X U} = 0$, which indicates these α_i's are equal on the intersections. □

Proposition 8.14. We have $H^p_{\text{Nis}}(X; H_n^M) = A^p(X; K_n^M) = H^p_{\text{Zar}}(X; K_n^M)$ for $X \in \mathrm{Sm}/k$.

Proof. The Rost complex gives rise to a complex of sheaves on X:

$$0 \longrightarrow K_n^M \longrightarrow \bigoplus_{x \in X^{(0)}} K_n^M(k(x)) \longrightarrow \bigoplus_{x \in X^{(1)}} K_{n-1}^M(k(x)) \longrightarrow \cdots$$

Now Theorem 8.5 shows that this is an exact complex of Zariski sheaves. Note that each term is a skyscrapper sheaf, so $K_n^M(k(x))$ is flasque as Zariski sheaf, and therefore $H^p_{\text{Zar}}(X, K_n^M) = A^p(X; K_n^M)$. The second equality comes from Proposition 7.12 and Proposition 8.13. □

The method we adopted, for example, Milnor K-theory sheaves and its Rost complex, can be generalized to the notion of cycle modules as described in [Ros96], which describes the zeroth homotopy group of the spectrum. In particular, this is equivalent to the heart on the t-structure of DM. We give a basic sketch for this theory.

8.3 Cycle Modules

Fix a base field k, and let $\mathscr{F}(k)$ be the collection of all fileds that are finitely-generated over k. We can now axiomatize the theory.

Definition 8.15. A cycle premodule $\mathcal{M}$ consists of a function $\mathcal{M} : \mathscr{F}(k) \to \mathrm{Ab}$, with a $\mathbb{Z}$-grading $\mathcal{M} = \bigoplus_{n \in \mathbb{Z}} \mathcal{M}_n$, as well as the following data and rules:

(D1) for each field extension $\varphi : F \to E$, there exists a degree-0 map

$$\varphi_* : \mathcal{M}(F) \to \mathcal{M}(E);$$

(D2) for each finite field extension $\varphi : F \to E$, there exists a norm map

$$N_{E/F} : \mathcal{M}(E) \to \mathcal{M}(F)$$

of degree 0;

(D3) for each F, the group $\mathcal{M}(F)$ is equipped with a left $K_*^M(F)$-module structure, denoted by $X \cdot \rho$ for $X \in K_*^M(F)$ and $\rho \in \mathcal{M}(F)$, with $K_n^M(F) \cdot \mathcal{M}_m(F) \subseteq \mathcal{M}_{n+m}(F)$;

(D4) for a discrete valuation ν on F, there exists $\partial_\nu : \mathcal{M}(F) \to \mathcal{M}(k(\nu))$ of degree -1. For a uniformizer π of ν of F, we define a map

$$\begin{aligned} s_\nu^\pi : \mathcal{M}(F) &\to \mathcal{M}(k(\nu)) \\ \rho &\mapsto \partial_\nu([-\pi] \cdot \rho) \end{aligned}$$

of degree 0;

(R1a) for $F \xrightarrow{\varphi} E \xrightarrow{\psi} L$, one has $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$;

(R1b) for finite extensions $F \to E \to L$, we have $N_{L/F} = N_{E/F} \circ N_{L/E}$;

(R1c) for finite field extension E/F and field extension L/F, there is a commutative diagram

$$\begin{tikzcd} \mathcal{M}(E) \arrow[r, "\ell((L\otimes_F E)_{\mathfrak{p}_i})"] \arrow[d, "N_{E/F}"'] & \bigoplus\limits_{\mathfrak{p}_i \in \operatorname{Spec}(L\otimes_F E)} \mathcal{M}(k(\mathfrak{p}_i)) \arrow[d, "\sum N_{k(\mathfrak{p}_i)/L}"] \\ \mathcal{M}(F) \arrow[r] & \mathcal{M}(L) \end{tikzcd}$$

(R2) for field extension $\varphi : F \to E$, suppose $X \in K_*^M(F)$, $Y \in K_*^M(E)$, $\rho \in \mathcal{M}(F)$ and $\mu \in \mathcal{M}(E)$, one has

(R2a) $\varphi_*(X \cdot \rho) = X \cdot \varphi_*(\rho)$;

(R2b) if E/F is a finite field extension, then $N_{E/F}(\varphi_*(X) \cdot \mu) = X \cdot N_{E/F}(\mu)$;

(R2c) if E/F is a finite field extension, then $N_{E/F}(y \cdot \varphi_*(\rho)) = N_{E/F}(y) \cdot \rho$.

(R3) finally, we have compatibility of valuations with other maps:

(R3a) for field extension $\varphi : E \to F$, let ν be a discrete valuation on F with ramification index e of F. Denote $\bar{\nu}$ to be the induced discrete valuation on E and let $\bar{\varphi}$ be the induced map on function fields, then there is a commutative diagram

$$\begin{tikzcd} \mathcal{M}(E) \arrow[r, "\partial_{\bar{\nu}}"] \arrow[d, "\varphi_*"'] & \mathcal{M}(k(\bar{\nu})) \arrow[d, "\bar{\varphi}_* \cdot e"] \\ \mathcal{M}(F) \arrow[r, "\partial_\nu"'] & \mathcal{M}(k(\nu)) \end{tikzcd}$$

(R3b) suppose E/F is a finite field extension and $\nu \in \mathrm{DV}(F)$, then there is a commutative diagram

$$\begin{tikzcd} \mathcal{M}(E) \arrow[r, "(\partial_w)"] \arrow[d, "N_{E/F}"'] & \bigoplus\limits_{w/\nu} \mathcal{M}(k(w)) \arrow[d, "N_{k(w)/k(\nu)}"] \\ \mathcal{M}(F) \arrow[r, "\partial_\nu"'] & \mathcal{M}(k(\nu)) \end{tikzcd}$$

(R3c) let $\varphi : E \to F$ be a field extension, and suppose $\nu \in \mathrm{DV}(F)$ is such that $\nu|_E = 0$, then $\partial_\nu \circ \varphi_* = 0$;

(R3d) let $\varphi : E \to F$ be a field extension, and suppose $\nu \in \mathrm{DV}(F)$ is such that $\nu|_E = 0$, then there is a commutative diagram

$$\begin{tikzcd} \mathcal{M}(E) \arrow[rr, "\varphi_*"] \arrow[dr, "\bar{\varphi}_*"'] & & \mathcal{M}(F) \arrow[dl, "s_\nu^\pi"] \\ & \mathcal{M}(k(\nu)) & \end{tikzcd}$$

(R3e) for $\nu \in \mathrm{DV}(F)$, $u \in \mathcal{O}_\nu^*$, and $\rho \in \mathcal{M}(F)$, one has $\partial_\nu([u] \cdot \rho) = -[\bar{u}] \cdot \partial_\nu(\rho)$.

Definition 8.16. A pairing $\mathcal{M} \times \mathcal{M}' \to \mathcal{M}''$ of cycle premodules is given by bilinear maps

$$\begin{aligned} \mathcal{M}(F) \times \mathcal{M}'(F) &\to \mathcal{M}''(F) \\ (\rho, \mu) &\mapsto \rho \cdot \mu \end{aligned}$$

for each F in $\mathscr{F}(k)$, satisfying the following conditions.

(P1) for any $X \in K_*^M(F)$, $\rho \in \mathcal{M}(F)$, and $\mu \in \mathcal{M}'(F)$, one has

(P1a) $(X \cdot \rho) \cdot \mu = X \cdot (\rho \cdot \mu)$;

(P1b) $(-1)^{\deg(X)\deg(\rho)} \rho \cdot (X \cdot \mu) = X \cdot (\rho \cdot \mu)$.

(P2) for any field extension $\varphi : F \to E$, $\eta \in \mathcal{M}(F)$, $\nu \in \mathcal{M}(F)$, $\rho \in \mathcal{M}'(F)$, and $\mu \in \mathcal{M}'(E)$, one has

(P2a) $\varphi_*(\eta \cdot \rho) = \varphi_*(\eta) \cdot \varphi_*(\rho)$;

(P2b) for any finite field extension E/F, $N_{E/F}(\varphi_*(\eta) \cdot \mu) = \eta \cdot N_{E/F}(\mu)$;

(P2c) for any finite field extension E/F, $N_{E/F}(\nu \cdot \varphi_*(\rho)) = N_{E/F}(\nu) \cdot \rho$.

(P3) for $\nu \in \mathrm{DV}(F)$, $\eta \in \mathcal{M}_n(F)$, $\rho \in \mathcal{M}(F)$, and a uniformizer π of ν, one has

$$\partial_\nu(\eta \cdot \rho) = \partial_\nu(\eta) \cdot s_\nu^\pi(\rho) + (-1)^n s_\nu^\pi(\eta) \cdot \partial_\nu(\rho) + [-1] \cdot \partial_\nu(\eta)\partial_\nu(\rho).$$

A ring structure on $\mathcal{M}$ is a pairing $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ which is associative and graded-commutative.

To define cycle modules, we require a notion of ∂ that mimicks the construction given in Rost complexes.

Definition 8.17. For any scheme X over k, let $x \in X$, pick divisor $y \in \bar{X}^{(1)}$, and let $p : \tilde{Z} \to Z = \bar{X}$ be the normalization of the closure of X. We define

$$\begin{aligned} \partial_y^x : \mathcal{M}(k(x)) &\to \mathcal{M}(k(y)) \\ \rho &\mapsto \sum_{p(y_i)=y} N_{k(y_i)/k(y)}(\partial_i(\rho)) \end{aligned}$$

by summation of norm over fibers y_i of y.

A cycle module $\mathcal{M}$ is a cycle premodule satisfying

(FD) let X be a normal scheme and $\rho \in \mathcal{M}(K(X))$, then $\partial_x^{\xi_X}(\rho) = 0$ for all but finitely many divisors $x \in X^{(1)}$;

(C) let X be an integral schemes and local of dimension 2, then $\sum\limits_{x \in X^{(1)}} \partial_{\mathfrak{m}_x}^x \circ \partial_x^{\xi_X} = 0$.

Here $\mathfrak{m}_x$ is the maximal ideal at $x \in X$, and ξ_X is the generic point of X.

Example 8.18. Milnor K-theory, étale cohomology with finite coefficients, Quillen K-theory, and Galois cohomology are all cycle modules.

We now see that cycle modules have properties similar to those of Milnor K-theory.

Proposition 8.19. Suppose that $\mathcal{M}$ is a cycle module, then

(H) there is a split exact sequence

$$0 \longrightarrow \mathcal{M}(F) \longrightarrow \mathcal{M}(F(t)) \longrightarrow \bigoplus_{x \in \mathbb{A}^1_F} \mathcal{M}(k(x)) \longrightarrow 0$$

where ∂_x passes through all monic irreducible polynomial with coefficeints in F;

(RC) let X be a proper curve over F, then the composite

$$\mathcal{M}(\xi_X) \xrightarrow{\partial_X^{\xi_X}} \bigoplus_{x \in X^{(1)}} \mathcal{M}(k(x)) \xrightarrow{N_{k(x)/F}} \mathcal{M}(F)$$

is zero. That is, $\sum_{x \in X^{(1)}} N_{k(x)/F} \circ \partial_x^{\xi_X} = 0$.

Proof.

(H) for $x \in (\mathbb{A}^1_F)^{(1)}$, we have an embedding $i_x : k(x) \to k(x)(t)$. Therefore, $(i_x)_*(\mu) = \mathcal{M}_{k(x)(t)}$, then we define

$$\begin{aligned} \tau_x : \mathcal{M}(k(x)) &\to \mathcal{M}(F(t)) \\ \mu &\mapsto N_{k(x)(t)/F(t)}((t - t(x))\mathcal{M}_{k(x)(t)}) \end{aligned}$$

where $t(x)$ is the canonical generator of $k(x)/F$. Therefore, one can check that τ_x's form the section of the exact sequence $s^0 : \mathcal{M}(F(t)) \to \mathcal{M}(F)$ which is the evaluation of valuation at 0. Therefore one can prove (H) directly;

(RC) we find a finite map $X \to \mathbb{P}^1$, then we can proceed as in the proof of Theorem 3.23.

□

For every integral scheme X over k, we now define $\mathcal{M}(X)$ to be the kernel in the exact sequence

$$0 \longrightarrow \mathcal{M}(X) \longrightarrow \mathcal{M}(K(X)) \xrightarrow{\partial_x^{\xi_X}} \bigoplus_{x \in X^{(1)}} \mathcal{M}(k(x))$$

Also, we have a Rost complex

$$C^n(X; \mathcal{M}) = \bigoplus_{x \in X^{(n)}} \mathcal{M}(k(x))$$

which is a well-defined complex of the form

$$\bigoplus_{x \in X^{(0)}} \mathcal{M}(k(x)) \xrightarrow{\partial_y^x} \bigoplus_{x \in X^{(1)}} \mathcal{M}(k(x)) \xrightarrow{\partial_y^x} \bigoplus_{x \in X^{(2)}} \mathcal{M}(k(x)) \xrightarrow{\partial_y^x} \cdots$$

and therefore we can define $A^p(X; \mathcal{M}) = H^p(C^*(X; \mathcal{M}))$.

Proposition 8.20. Suppose $\mathcal{M}$ is a cycle module, then

1. $A^p(-; \mathcal{M})$ is a presheaf with transfers, and $\mathcal{M}(-)$ is a Nisnevich sheaf;
2. $A^p(-; \mathcal{M})$ is homotopy invariant;
3. $A^p(X; \mathcal{M}) \cong H^p_{\mathrm{Nis}}(X; \mathcal{M}) \cong H^p_{\mathrm{Zar}}(X; \mathcal{M})$.

Proof. For every flat morphism $f : X \to Y$, one can construct a flat pullback $f^* : C^*(Y; \mathcal{M}) \to C^*(X; \mathcal{M})$ as in Definition 3.29 (with coefficient with respect to the fiber), which induces a pullback $f^* : A^p(Y; \mathcal{M}) \to A^p(X; \mathcal{M})$. For every proper morphism $f : X \to Y$, one can construct a proper pushforward $f_* : A^p(X; \mathcal{M}) \to A^{p+d_Y-d_X}(Y; \mathcal{M})$ as in Definition 3.28. Therefore, part 2 follows from the spectral sequence as described in the proof of Proposition 8.2. The deformation to normal bundle gives a Gysin pullback along the closed immersions, and moreover it gives a notion of general pullback by the graph decomposition we saw before. Now part 1 follows from the same proof as in Proposition 8.12 and Proposition 8.13. Finally, part 3 follows from Theorem 8.5 and Proposition 8.14. □

Collecting results from Chapter 3, we conclude the following.

Theorem 8.21. $\mathcal{M}(F) = \bigoplus_{n\in\mathbb{N}} K_n^M(F)$ is a cycle module.

Definition 8.22. Suppose $\mathcal{F} \in \mathrm{PSh}(k)$ is homotopy invariant, then we define $\mathcal{F}_{-1}(X)$ for any $X \in \mathrm{Sm}/k$ from the exact sequence

$$\mathcal{F}(X \times \mathbb{A}^1) \longrightarrow \mathcal{F}(X \times \mathbb{G}_m) \longrightarrow \mathcal{F}_{-1}(X) \longrightarrow 0$$

Since $\mathcal{F}(X) = \mathcal{F}(X \times \mathbb{A}^1)$, the composition $\mathcal{F}(X) \to \mathcal{F}(X \times \mathbb{G}_m)$ has a section $i_1^* : \mathcal{F}(X \times \mathbb{G}_m) \to \mathcal{F}(X)$, i.e., pullback along evaluation at 1, then we have split-exactness, which means $\mathcal{F}(X \times \mathbb{G}_m) = \mathcal{F}(X) \oplus \mathcal{F}_{-1}(X)$. That is, we can define $\mathcal{F}_{-1}(X) = \ker(i_1^*)$. Therefore, $\mathcal{F}_{-1}(-)$ acts as a contraction.

Proposition 8.23. Suppose $\mathcal{M}$ is a cycle module, then we have $(\mathcal{M}_{n+1})_{-1} = \mathcal{M}_n$.

Proof. We have an exact (localization) sequence

$$A^0(X \times \mathbb{A}^1; \mathcal{M}_{n+1}) \longrightarrow A^0(X \times \mathbb{G}_m; \mathcal{M}_{n+1}) \xrightarrow{\partial} A^0(X; \mathcal{M}_n) \xrightarrow{i_{0*}} A^1(X \times \mathbb{A}^1; \mathcal{M}_{n+1})$$

where ∂ is with respect to the zero section $X \to X \times \mathbb{A}^1$, and i_{0*} is the pushforward along the zero section. Therefore, it induces a map $(\mathcal{M}_{n+1})_{-1} \to \mathcal{M}_n$.[38] When $X = \mathrm{Spec}(F)$ of a field F, since F has dimension 0, we have $A^1(\mathbb{A}^1_F; \mathcal{M}_{n+1}) = A^1(F; \mathcal{M}_{n+1}) = H^1(F; \mathcal{M}_{n+1}) = 0$. Therefore, $A^1(X \times \mathbb{A}^1; \mathcal{M}_{n+1}) = 0$, and so ∂ is a surjection. Hence, $(\mathcal{M}_{n+1})_{-1}(F) = \mathcal{M}_n(F)$. In particular, the map $(\mathcal{M}_{n+1})_{-1} \to \mathcal{M}_n$ is an isomorphism on fields, since they are both homotopy invariant Nisnevich sheaves with transfers. By Theorem 6.12, we have an isomorphism. □

What we have shown is the following procedure. If we start with a cycle module $\mathcal{M}$, it is a functor $\mathcal{M} : \mathscr{F}(k) \to \mathrm{Ab}$, then we want to construct homotopy invariant Nisnevich sheaves with transfers $\mathcal{M}_n$ for each n. These $\mathcal{M}_n$'s satisfy $(\mathcal{M}_{n+1})_{-1} = \mathcal{M}_n$.

With formalism, we get to construct two categories, 1) the category of cycle modules $\mathcal{M}(-)$, and 2) the category of sequences of homotopy invariant sheaves with transfers $\{\mathcal{M}_n\}$ such that $(\mathcal{M}_{n+1})_{-1} = \mathcal{M}_n$. The procedure above and Proposition 8.23 have given us a functor from 1) to 2). Moreover, Déglise proved a more complicated result in his thesis, which states that the two categories are equivalent (if k is perfect).

Theorem 8.24 (Déglise, [Dég02])**.** Let k be a perfect field. The category of sequences of homotopy invariant sheaves $\{\mathcal{M}_n\} \subseteq \mathrm{Sh}(k)$ with transfers and $(\mathcal{M}_{n+1})_{-1} = \mathcal{M}_n$ defined by Voevodsky and the category of cycle modules defined by Rost are closely related. In particular, the two categories stated above are equivalent. Moreover, the category of cycle modules over k is a Grothendieck abelian category equipped with a monoidal structure for which Milnor K-theory is the unit.

To set the stage for finding the explicit equivalence, we need to define a suitable functor from 2) to 1). Suppose k is perfect, and we are given a series of homotopy invariant Nisnevich sheaves with transfers $\mathcal{M}_n \in \mathrm{Sh}(k)$, such that $(\mathcal{M}_{n+1})_{-1} = \mathcal{M}_n$. We hope to construct a cycle module $\mathcal{M}$.

Lemma 8.25. If k is a perfect field, then every finitely-generated field extension E/k is a filtered direct limit of smooth k-algebras $A_i \subseteq E$, such that the function fields $K(A_i) = E$.

Proof. The existence follows from [Dég02], Lemma 2.1.32. To see that A_i's form a directed set, consider all smooth finitely-generated k-algebra $A \subseteq E$. If $A, B \subseteq E$ are smooth such that $K(A) = E$, then we want to construct a smooth ring contained in E, containing both A and B. To do this, consider the smallest k-algebra generated by A and B, denoted by $k(A, B)$. Note that $k(A, B)$ is generically smooth: $K(k(A, B)) = E$, then it is smooth at ξ_A. Therefore, we can choose a non-empty open subset with respect to a polynomial f, so that A, B are contained in the smooth algebra $\subseteq k(A, B)_f$ for some $f \neq 0$. □

[38] Indeed, note that $A^0(X \times \mathbb{A}^1; \mathcal{M}_{n+1}) \to A^0(X \times \mathbb{G}_m; \mathcal{M}_{n+1})$ is just $\mathcal{F}(X \times \mathbb{A}^1) \to \mathcal{F}(X \times \mathbb{G}_m)$, and $(\mathcal{M}_{n+1})_{-1}$ is its cokernel.

Therefore, we can define $\mathcal{M}(E)=\varinjlim_i \mathcal{M}(\operatorname{Spec}(A_i)) := \varinjlim_i \bigoplus_n \mathcal{M}_n(\operatorname{Spec}(A_i))$.

Lemma 8.26. $\mathcal{M}(E)$ satisfies (D1) - (D4) as specified in Definition 8.15.

Proof.

(D1) suppose E/F is an extension in $\mathscr{F}(k)$. By Lemma 8.25, we can set $E=\varinjlim A_i$ and $F=\varinjlim B_j$, where A_i's and B_j's are smooth over k. For every j, the composite $B_j\to F\to E=\varinjlim A_i$ factors through some A_{i_j} since B_j is finitely-generated. Therefore, we define $\mathcal{M}(F)\to\mathcal{M}(E)$ by the commutative diagram

$$\begin{array}{ccc} \mathcal{M}(\operatorname{Spec}(B_j)) & \longrightarrow & \mathcal{M}(\operatorname{Spec}(A_{i_j})) \\ \downarrow & & \downarrow \\ \mathcal{M}(F) & \longrightarrow & \mathcal{M}(E) \end{array}$$

(D2) suppose E/F is a finite extension and again by Lemma 8.25 we have $F=\varinjlim A_i$. For any A_i, let $\tilde{A}_i$ be its normalization in E. Each $\tilde{A}_i$ is a finite A_i-module, and is generically smooth over k.[39] Suppose $\varnothing\neq U\subseteq\operatorname{Spec}(\tilde{A}_i)$ is a non-empty dense open subset which is smooth over k, and let $f:\operatorname{Spec}(\tilde{A}_i)\to\operatorname{Spec}(A_i)$ be the natural map, then $f|_{f(U^c)^c}$ is finite,[40] and $f^{-1}(f(U^c)^c)$ is non-empty, contained in U, and must be smooth over k. Therefore, we can find a finite dominant map $f:X\to Y$ in Sm/k such that $K(X)=E$ and $K(Y)=F$, and $X\times_Y\operatorname{Spec}(K(Y))=\operatorname{Spec}(K(X))$. In particular, $\mathcal{M}(E)=\mathcal{M}(K(X))=\mathcal{M}(X\times_Y\operatorname{Spec}(K(Y)))$. Now the generic fiber $X\times_Y\operatorname{Spec}(K(Y))$ has a projection to $\operatorname{Spec}(K(Y))$. Since the map from X to Y is finite, then the projection is finite and surjective. However, $\mathcal{M}(-)$ is a Nisnevich sheaves with transfers, then the finite surjection has an inverse with transfer. By the finite correspondence, this constructs a morphism $\operatorname{Spec}(K(Y))\to X\times_Y\operatorname{Spec}(K(Y))$. By (D1), we can define $\mathcal{M}(E)\to\mathcal{M}(F)$ by the composite

$$\mathcal{M}(E)=\mathcal{M}(K(X))=\mathcal{M}(X\times_Y\operatorname{Spec}(K(Y)))\xrightarrow{\mathcal{M}(f^T)}\mathcal{M}(K(Y))=\mathcal{M}(F)$$

where f^T is the transpose of the finite surjective morphism f.

(D3) Again, by Lemma 8.25, we can assume $E=\varinjlim A_i$ is a direct limit of smooth k-algebras. For every smooth k-algebra A, we can define a pairing

$$\begin{aligned} \mathbb{Z}(\mathbb{G}_m^{\wedge 1})(\operatorname{Spec}(A))\times\mathcal{M}_n(\operatorname{Spec}(A)) &\to \mathcal{M}_{n+1}(\operatorname{Spec}(A)) \\ (a,S) &\mapsto \mathcal{M}_{n+1}(((\mathrm{id},a):\operatorname{Spec}(A)\to\operatorname{Spec}(A)\times\mathbb{G}_m))(S) \end{aligned}$$

where $\mathcal{M}_n(\operatorname{Spec}(A))\subseteq\mathcal{M}_{n+1}(\operatorname{Spec}(A)\times\mathbb{G}_m)$.[41] We claim that this pairing descends to the Milnor K-groups. Suppose a is in the image of $\partial_0-\partial_1:\mathbb{Z}(\mathbb{G}_m^{\wedge 1})(\operatorname{Spec}(A)\times\mathbb{A}^1)\to\mathbb{Z}(\mathbb{G}_m^{\wedge 1})(\operatorname{Spec}(A))$, i.e., as the zeroth boundary in the Suslin complex. Since $\mathcal{M}_{n+1}$ is homotopy invariant, then $\mathcal{M}_{n+1}(i_0^*)=\mathcal{M}_{n+1}(i_1^*)$ where $i_0,i_1:\operatorname{Spec}(A)\to\operatorname{Spec}(A)\times\mathbb{A}^1$ are constant embeddings. Therefore, the image of the pairing $a\cdot S=0$. This proves that the pairing descends to the pairing on quotient

$$\operatorname{coker}(\partial_0-\partial_1)\times\mathcal{M}_n(\operatorname{Spec}(A))\to\mathcal{M}_{n+1}(\operatorname{Spec}(A)).$$

where $\partial_0-\partial_1:\mathbb{Z}(\mathbb{G}_m^{\wedge 1})(\operatorname{Spec}(A)\times\mathbb{A}^1)\to\mathbb{Z}(\mathbb{G}_m^{\wedge 1})(\operatorname{Spec}(A))$. Since $\mathbb{Z}(\mathbb{G}_m^{\wedge n})\otimes\mathbb{Z}(\mathbb{G}_m^{\wedge m})=\mathbb{Z}(\mathbb{G}_m^{\wedge(n+m)})$, we also have pairings

$$H_0(C_*\mathbb{Z}(\mathbb{G}_m^{\wedge k})(\operatorname{Spec}(A)))\times\mathcal{M}_n(\operatorname{Spec}(A))\to\mathcal{M}_{n+k}(\operatorname{Spec}(A))$$

[39] Since k is perfect, then every finitely-generated algebra is generically smooth.

[40] Finiteness is preserved through base-change.

[41] That is, we take the sections of $\mathcal{M}_{n+1}((\mathrm{id},a))$ on S.

for arbitrary n and k. Recall that $E = \varinjlim A_i$, then by taking limits with respect to i, we obtain a pairing

$$K_k^M(E) \times \mathcal{M}_n(E) \to \mathcal{M}_{n+k}(E)$$

since $K_k^M(E) = H_0(C_*\mathbb{Z}(\mathbb{G}_m^{\wedge k})(E))$ by Theorem 4.10.

(D4) Establishing (D4) requires a result called homotopy purity, which is described in Theorem 8.27. Let $\nu \in \mathrm{DV}(E/k)$, then by [Dég02], Lemma 2.1.32, $\mathcal{O}_\nu = \varinjlim A_i$ where A_i's are smooth over k, and are contained in $\mathcal{O}_\nu$, and that $\mathcal{O}_\nu$ is a localization of A_i. For every A_i, since $\mathcal{O}_\nu$ has codimension 1, we regard it as a divisor over A_i, therefore we obtain a closed immersion $Z_i \subseteq \operatorname{Spec}(A_i)$ from a divisor Z_i, which gives the valuation ν. By repeatedly running localization over perfect field k, we get to assume that Z_i is smooth, and therefore $\mathcal{N}_{Z_i/\operatorname{Spec}(A_i)}$ is the trivial bundle. Therefore, we can define a map via localization sequence

$$\mathcal{M}_{n+1}(\operatorname{Spec}(A_i)\backslash Z_i) \xrightarrow{\partial} H^1_{Z_i}(\operatorname{Spec}(A_i)\backslash \mathcal{M}_{n+1})$$

But by the supported cohomology and Theorem 8.27, we have

$$H^1_{Z_i}(\operatorname{Spec}(A_i)\backslash \mathcal{M}_{n+1}) \cong H^1_{Z_i}(Z_i \times \mathbb{A}^1, \mathcal{M}_{n+1}).$$

To calculate the latter, we have an exact sequence of localization

$$\mathcal{M}_{n+1}(Z_i \times \mathbb{A}^1) \to \mathcal{M}_{n+1}(Z_i \times \mathbb{G}_m) \to H^1_{Z_i}(Z_i \times \mathbb{A}^1, \mathcal{A}_{n+1}) \to H^1(Z_i \times \mathbb{A}^1, \mathcal{M}_{n+1}) \to H^1(Z_i \times \mathbb{G}_m, \mathcal{M}_{n+1})$$

By homotopy invariance of $\mathcal{M}_{n+1}$, we know $H^1(Z_i \times \mathbb{A}^1, \mathcal{M}_{n+1}) \cong H^1(Z_i, \mathcal{M}_{n+1})$, and the last morphism $H^1(Z_i \times \mathbb{A}^1, \mathcal{M}_{n+1}) \to H^1(Z_i \times \mathbb{G}_m, \mathcal{M}_{n+1})$ admits a section and is therefore injective. Hence,

$$H^1_{Z_i}(Z_i \times \mathbb{A}^1, \mathcal{M}_{n+1}) \cong (\mathcal{M}_{n+1})_{-1}(Z_i) = \mathcal{M}_n(Z_i).$$

Therefore, we have a morphism $\partial_i : \mathcal{M}_{n+1}(\operatorname{Spec}(A_i)\backslash Z_i) \to \mathcal{M}_n(Z_i)$. Taking limits over i, we get the residue map $\partial_\nu : \mathcal{M}_{n+1}(E) \to \mathcal{M}_n(k(\nu))$.

□

Let us try to introduce the technique used in the proof of (D4) above. Suppose that $\mathcal{F} \in \operatorname{Sh}(k)$ is homotopy invariant and $Y \hookrightarrow X$ is a closed embedding (immersion) in Sm/k. We want to understand the supported cohomology $H^*_Y(X, \mathcal{F})$. Recall from Proposition 2.32 that for any étale morphism $\varphi : Y \to X$ such that there exists a closed subset Z in X with $\varphi^{-1}(Z) = Z$, then $H^*_Z(Y, \mathcal{F}) \cong H^*_Z(Z, \mathcal{F})$. Since $\mathcal{F}$ is homotopy invariant, then it is $\mathbb{A}^1$-local, therefore we have

$$H^n_Y(X, \mathcal{F}) = \operatorname{Hom}_{D^-(k)}(\mathbb{Z}(X)/\mathbb{Z}(X\backslash Y), \mathcal{F}[n]) = \operatorname{Hom}_{\mathrm{DM}^{\mathrm{eff},-}(k)}(\mathbb{Z}(X)/\mathbb{Z}(X\backslash Y), \mathcal{F}[n]).$$

Hence, it reduces to identifying $\mathbb{Z}(X)/\mathbb{Z}(X\backslash Y)$ in $\mathrm{DM}^{\mathrm{eff},-}(k)$. This is where the Gysin map comes into play.

Theorem 8.27 (Homotopy Purity)**.** For any closed subset $Y \subseteq X$, we have an isomorphism

$$\mathbb{Z}(X)/\mathbb{Z}(X\backslash Y) \cong \mathbb{Z}(\mathcal{N}_{Y/X})/\mathbb{Z}(\mathcal{N}^\times_{Y/X}) =: \operatorname{Th}(\mathcal{N}_{Y/X})$$

in $\mathrm{DM}^{\mathrm{eff},-}(k)$, where $\mathcal{N}_{Y/X}$ is the normal bundle of Y over X. We often define this to be the Thom space of the normal bundle $\mathcal{N}_{Y/X}$.

That is, we identify $\mathbb{Z}(X)/\mathbb{Z}(X\backslash Y)$ to be the quotient of the normal bundle over its non-zero sections. Therefore, given a closed immersion $Y \hookrightarrow X$, we can deform it into the zero section of the normal bundle $Y \mapsto \mathcal{N}_{Y/X}$. This result can be generalized to unstable $\mathbb{A}^1$-homotopy category.

Proof. Recall that the deformation space $D(X,Y) = \mathrm{Bl}_{Y\times\{0\}}(X\times\mathbb{A}^1)\backslash\mathrm{Bl}_{Y\times\{0\}}(X\times\{0\})$ admits a projection $\rho : D(X,Y)\to X\times\mathbb{A}^1$ and a closed embedding $i : Y\times\mathbb{A}^1\to D(X,Y)$ which can be further mapped into $\mathbb{A}^1$. Now $D(X,Y)$ satisfies that taking the non-zero fiber of ρ recovers the inclusion $Y\hookrightarrow X$ on embedding i, and taking the zero fiber of ρ recovers the zero section of the normal bundle on embedding i. Therefore, $i|_{p^{-1}(1)}$ is the inclusion $Y\hookrightarrow X$, and we obtain a fiber-inclusion map

$$g_{X,Y} : \mathbb{Z}(X)/\mathbb{Z}(X\backslash Y)\to\mathbb{Z}(D(X,Y))/\mathbb{Z}(D(X,Y)\backslash(Y\times\mathbb{A}^1)).$$

This is maintained on the complement by using properties of the closed immersion. Similarly, $i|_{p^{-1}(0)}$ is the zero section of $Y\to\mathcal{N}_{Y/X}$, therefore we obtain a map

$$\alpha_{X,Y} : \mathbb{Z}(\mathcal{N}_{Y/X})/\mathbb{Z}(\mathcal{N}_{Y/X}^{\times})\to\mathbb{Z}(D(X,Y))/\mathbb{Z}(D(X,Y)\backslash(Y\times\mathbb{A}^1)).$$

It suffices to show that $g_{X,Y}$ and $\alpha_{X,Y}$ are $\mathbb{A}^1$-weak equivalences, as we obtain an isomorphism $\alpha_{X,Y}^{-1}\circ g_{X,Y}$.

Step 1: we first consider the case for the embedding $0 : Y\hookrightarrow\mathbb{A}^n\times Y$ by the zero sections. We have $\mathrm{Bl}_{Y\times\{0\}}(Y\times\mathbb{A}^n)\subseteq\mathbb{A}^n_Y\times_Y\mathbb{P}^{n-1}_Y$, then note that

Remark 8.28.

- the fiber of the point $\infty = (0:\cdots:0:1)\in\mathbb{P}^{n-1}$ is the projection $\mathbb{A}^1\times Y\to Y$. Indeed, this follows from the definition of blow-up;
- given the exceptional divisor E, the composition given by inclusions $E\hookrightarrow\mathrm{Bl}_{Y\times\{0\}}(Y\times\mathbb{A}^1)\hookrightarrow\mathbb{A}^n_Y\times_Y\mathbb{P}^{n-1}\to\mathbb{P}^{n-1}_Y$ is an isomorphism. This follows from the definition of the blow-up with respect to a point in the affine space.

By étale excision, c.f., Proposition 2.32, we know $(Y\cap\mathbb{A}^1)\cap\mathrm{Bl}_{Y\times\{0\}}(X\times\{0\}) = \varnothing$, therefore we have

$$\mathbb{Z}(D(X,Y))/\mathbb{Z}(D(X,Y)\backslash(Y\times\mathbb{A}^1))\cong\mathbb{Z}(\mathrm{Bl}_{Y\times\{0\}}(X\times\mathbb{A}^1))/\mathbb{Z}(\mathrm{Bl}_{Y\times\{0\}}(X\times\mathbb{A}^1)\backslash(Y\times\mathbb{A}^1))$$

in $D^-(S)$ since they differ by a closed subset. Therefore, we can replace the deformation space by the blow-up via étale excision. Therefore, we just have to consider the map

$$g_{Y\times\mathbb{A}^n,Y} : \mathbb{Z}(Y\times\mathbb{A}^n)/\mathbb{Z}((Y\times\mathbb{A}^n)\backslash(Y\times\{0\}))\to\mathbb{Z}(\mathrm{Bl}_{Y\times\{0\}}(Y\times\mathbb{A}^n\times\mathbb{A}^1))/\mathbb{Z}(\mathrm{Bl}_{Y\times\{0\}}(Y\times\mathbb{A}^n\times\mathbb{A}^1)\backslash(Y\times\mathbb{A}^1)),$$

then we obtain a Cartesian square

$$\begin{tikzcd}
\mathrm{Bl}_{Y\times\{0\}}(Y\times\mathbb{A}^n\times\mathbb{A}^1)\backslash(Y\times\mathbb{A}^1) \arrow[r] \arrow[d, hook] & Y\times(\mathbb{P}^n\backslash\{\infty\}) \arrow[d] \\
\mathrm{Bl}_{Y\times\{0\}}(Y\times\mathbb{A}^n\times\mathbb{A}^1) \arrow[r, two heads] & Y\times\mathbb{P}^n
\end{tikzcd}$$

from our first observation in Remark 8.28. To see this, note that the bottom row is the blow-down map, and $Y\times(\mathbb{P}^n\backslash\{\infty\})$ is an open subest of $Y\times\mathbb{P}^n$, then by the observation the fiber is $Y\times\mathbb{A}^1$, so we record its complement. Moreover, one can check that the horizontal maps are structure morphisms of vector bundles, and therefore the horizontal maps are isomorphisms in $\mathrm{DM}^{\mathrm{eff},-}(k)$, then they have the same quotient in $\mathrm{DM}^{\mathrm{eff},-}(k)$, i.e., they have $\mathbb{A}^1$-weak equivalent mapping cones, namely

$$\mathbb{Z}(\mathrm{Bl}_{Y\times\{0\}}(Y\times\mathbb{A}^n\times\mathbb{A}^1))/\mathbb{Z}(\mathrm{Bl}_{Y\times\{0\}}(Y\times\mathbb{A}^n\times\mathbb{A}^1)\backslash(Y\times\mathbb{A}^1))\cong\mathbb{Z}(Y\times\mathbb{P}^n)/\mathbb{Z}(Y\times(\mathbb{P}^n\backslash\{\infty\}))$$

in $\mathrm{DM}^{\mathrm{eff},-}(k)$. Moreover, we know

$$\mathbb{Z}(Y\times\mathbb{P}^n)/\mathbb{Z}(Y\times(\mathbb{P}^n\backslash\{\infty\}))\cong\mathbb{Z}(Y\times\mathbb{A}^n)/\mathbb{Z}(Y\times(\mathbb{A}^n\backslash\{0\}))$$

by étale excision. Therefore, $g_{Y\times\mathbb{A}^n,Y}$ is an $\mathbb{A}^1$-weak equivalence. Moreover, we have Cartesian squares

$$\begin{tikzcd}
\mathcal{N}^{\times}_{Y/(Y\times\mathbb{A}^n)} \arrow[r] \arrow[d] & Y\times(\mathbb{P}^n\backslash\{\infty\}) \arrow[r] \arrow[d] & \mathrm{Bl}_{Y\times\{0\}}(Y\times\mathbb{A}^n\times\mathbb{A}^1)\backslash(Y\times\mathbb{A}^1) \arrow[d] \\
\mathcal{N}_{Y/(Y\times\mathbb{A}^n)} \arrow[r] & Y\times\mathbb{P}^n \arrow[r] & \mathrm{Bl}_{Y\times\{0\}}(Y\times\mathbb{A}^n\times\mathbb{A}^1)
\end{tikzcd}$$

Since $Y\times\mathbb{P}^n$ is the exceptional divisor of the blow-up $\mathrm{Bl}_{Y\times\{0\}}(Y\times\mathbb{A}^n\times\mathbb{A}^1)$. Therefore, $\mathcal{N}_{Y/Y\times\mathbb{P}^n}$ identifies with $Y\times\mathbb{A}^n$, which gives the natural embedding into $Y\times\mathbb{P}^n$, as described above. Moreover, we know $Y\times\mathbb{P}^n$ corresponds to the normal bundle of Y over $Y\times\mathbb{A}^n$, then deleting the ∞ point just gives non-zero divisor of the normal bundle. We then have an open immersion $\mathrm{Bl}_{Y\times\{0\}}(Y\times\mathbb{A}^n\times\mathbb{A}^1)(Y\times\mathbb{A}^1)\to\mathrm{Bl}_{Y\times\{0\}}(Y\times\mathbb{A}^n\times\mathbb{A}^1)$, then we can pullback towards $Y\times(\mathbb{P}^n\backslash\{\infty\})$, with fiber as non-zero sections. Note that we can now combine the two diagrams and obtain

$$\begin{tikzcd}
\mathcal{N}^{\times}_{Y/(Y\times\mathbb{A}^n)} \arrow[r] \arrow[d] & Y\times(\mathbb{P}^n\backslash\{\infty\}) \arrow[r] \arrow[d] \arrow[rr, bend left, "\mathrm{id}"] & \mathrm{Bl}_{Y\times\{0\}}(Y\times\mathbb{A}^n\times\mathbb{A}^1)\backslash(Y\times\mathbb{A}^1) \arrow[r] \arrow[d] & Y\times(\mathbb{P}^n\backslash\{\infty\}) \arrow[d] \\
\mathcal{N}_{Y/(Y\times\mathbb{A}^n)} \arrow[r] & Y\times\mathbb{P}^n \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \mathrm{Bl}_{Y\times\{0\}}(Y\times\mathbb{A}^n\times\mathbb{A}^1) \arrow[r] & Y\times\mathbb{P}^n
\end{tikzcd}$$

such that the compositions denoted above are identities. Since $\mathrm{Bl}_{Y\times\{0\}}(Y\times\mathbb{A}^n\times\mathbb{A}^1)(Y\times\mathbb{A}^1)\to Y\times(\mathbb{P}^n\backslash\{\infty\})$ and $\mathrm{Bl}_{Y\times\{0\}}(Y\times\mathbb{A}^n\times\mathbb{A}^1)\to Y\times\mathbb{P}^n$ are vector bundles, then $Y\times(\mathbb{P}^n\backslash\{\infty\})\to\mathrm{Bl}_{Y\times\{0\}}(Y\times\mathbb{A}^n\times\mathbb{A}^1)(Y\times\mathbb{A}^1)$ and $Y\times\mathbb{P}^n\to\mathrm{Bl}_{Y\times\{0\}}(Y\times\mathbb{A}^n\times\mathbb{A}^1)$ are isomorphisms, according to the second observation in Remark 8.28. This means we identify $Y\times(\mathbb{P}^n\backslash\{\infty\})\to Y\times\mathbb{P}^n$ with $\mathrm{Bl}_{Y\times\{0\}}(Y\times\mathbb{A}^n\times\mathbb{A}^1)\backslash(Y\times\mathbb{A}^1)\to\mathrm{Bl}_{Y\times\{0\}}(Y\times\mathbb{A}^n\times\mathbb{A}^1)$. Moreover, we can apply étale excision to the square

$$\begin{tikzcd}
\mathcal{N}^{\times}_{Y/(Y\times\mathbb{A}^n)} \arrow[r] \arrow[d] & Y\times(\mathbb{P}^n\backslash\{\infty\}) \arrow[d] \\
\mathcal{N}_{Y/(Y\times\mathbb{A}^n)} \arrow[r] & Y\times\mathbb{P}^n
\end{tikzcd}$$

on the left, then we see that $\alpha_{X,Y}$ is also an $\mathbb{A}^1$-weak equivalence.

Step 2: now suppose we have an étale morphism $\varphi: U\to X$, closed subset $Y\subseteq X$, such that $\varphi^{-1}(Y)=Y$, then $\pi:\mathrm{Bl}_{Y\times\{0\}}(U\times\mathbb{A}^1)\to\mathrm{Bl}_{Y\times\{0\}}(X\times\mathbb{A}^1)$ and $\pi':\mathcal{N}_{Y/U}\to\mathcal{N}_{Y/X}$ are both étale, with $\pi^{-1}(Y\times\mathbb{A}^1)=Y\times\mathbb{A}^1$, and $\pi'^{-1}(Y)=Y$. Therefore, the statement for the pairs (U,Y) and (X,Y) are equivalent: they both satisfy the requirement for étale excision, therefore their quotients are isomorphic.

Step 3: by [GR02], Proposition II.4.9, there is a finite Zariski covering $X=\bigcup_i U_i$ such that for any i, the embedding $Y\cap U_i\hookrightarrow U_i$ admits a Cartesian square

$$\begin{tikzcd}
Y\cap U_i \arrow[r] \arrow[d] & U_i \arrow[d, "q"] \\
\mathbb{A}^{d_Y} \arrow[r, "0"'] & \mathbb{A}^{d_X}
\end{tikzcd}$$

with an embedding of zero sections $0:\mathbb{A}^{d_Y}\to\mathbb{A}^{d_X}$ and étale vertical morphisms. We want to prove the statement for the pair $(U_i, Y\cap U_i)$ for arbitrary i. Consider the fiber product $U_i\times_{\mathbb{A}^{d_X}}((Y\cap U_i)\times\mathbb{A}^{d_X-d_Y})$. Here the structure maps are given by $q : U_i \to \mathbb{A}^{d_X}$ and $q \times 0 : (Y \cap U_i) \times \mathbb{A}^{d_X - d_Y} \to \mathbb{A}^{d_X}$. This fiber of this fiber product over $\mathbb{A}^{d_Y} \subseteq \mathbb{A}^{d_X}$ is given by zero section map, therefore it is just $(Y \cap U_i) \times_{\mathbb{A}^{d_Y}} (Y \cap U_i)$. Since the morphism $Y \cap U_i \to \mathbb{A}^{d_Y}$ is étale, then the diagonal $\Delta : Y \cap U_i \to (Y \cap U_i) \times_{\mathbb{A}^{d_Y}} (Y \cap U_i)$ is both an open and a closed immersion, therefore it induces a decomposition (of connected component) $(Y \cap U_i) \coprod R = (Y \cap U_i) \times_{\mathbb{A}^{d_Y}} (Y \cap U_i)$ via the self-intersection into closed and open subsets, i.e., one component through the diagonal and the other components through R. Set $V = (U_i \times_{\mathbb{A}^{d_X}} ((Y \cap U_i) \times \mathbb{A}^{d_X - d_Y}))\backslash R$, since R is included in the self-intersection, which is included in the space. Then we have two étale morphisms $p_1 : V \to U_i$ and $p_2 : V \to (Y \cap U_i) \times \mathbb{A}^{d_X - d_Y}$ given by projections (since we have open subsets), such that $p_1^{-1}(Y \cap U_i) = Y \cap U_i$, and $p_2^{-1}((Y \cap U_i) \times \{0\}) = Y \cap U_i$, as we have removed R from the space. Therefore, the two preimages are isomorphic, and we have

$$\begin{aligned}\mathbb{Z}(U_i)/\mathbb{Z}(U_i\backslash(Y \cap U_i)) &\cong \mathbb{Z}(U)/\mathbb{Z}(U\backslash(Y \cap U_i)) \\ &\cong \mathbb{Z}((Y \cap U_i) \times \mathbb{A}^{d_X - d_Y})/\mathbb{Z}(((Y \cap U_i) \times \mathbb{A}^{d_X - d_Y})\backslash((Y \cap U_i) \times \{0\}))\end{aligned}$$

by étale excision for p_1 and p_2. However, by Step 2, the statements regarding both sides should be equivalent. Now the statement concerning the quotient $\mathbb{Z}((Y \cap U_i) \times \mathbb{A}^{d_X - d_Y})/\mathbb{Z}(((Y \cap U_i) \times \mathbb{A}^{d_X - d_Y})\backslash((Y \cap U_i) \times \{0\}))$ can be reduced to Step 1, which proves the statement.

Step 4: by the Mayer-Vietoris sequence in $\mathrm{DM}^{\mathrm{eff},-}(k)$, we can write down a sequence of distinguished triangles such that two of the maps are isomorphic in $\mathrm{DM}^{\mathrm{eff},-}(k)$, then by the axioms of triangulated categories, the third map is also an isomorphism, which conclude the proof.

□

Finally, one can verify the remaining axioms of cycle modules.

Proof of Theorem 8.24. See [Dég02], Theorem 6.1.1. □

Corollary 8.29. Suppose k is a perfect field, and let $\{\mathcal{M}_n\}$ be a sequence of sheaves as specified in Theorem 8.24, then we have $H^p(X, \mathcal{M}_n) = A^p(X, \mathcal{M}_n)$ for every $X \in \mathrm{Sm}/k$. Therefore, the cohomology agrees with the one defined by Rost complex.

Theorem 8.30. For any $X \in \mathrm{Sm}/k$, we have $H^{p,q}(X, \mathbb{Z}) \cong H^{p-q}(X, K_q^M)$ if $p \geqslant 2q - 1$.

Proof. If k is not perfect, then we can perform base-change via a fully faithful functor, so that it lands in the perfect closure, which maintains the cohomology, c.f., Remark 6.19. Therefore, without loss of generality, we assume k is a perfect field. By the discussion in Proposition 5.49, we have two presheaves defined by $X \mapsto \mathrm{Hom}_{D^-(k)}(\mathbb{Z}(X), C_*\mathbb{Z}(q)[p])$ and $X \mapsto H^p(C_*\mathbb{Z}(q)(X))$, and they have the same sheafification, denoted by $H_M^{p,q}$. Recall by definition that $H^p(C_*\mathbb{Z}(q)) = H_{q-p}(C_*\mathbb{Z}(\mathbb{G}_m^{\wedge q}))$, then $H_M^{p,q} = 0$ if $p > q$. If $p = q$, then $H_M^{p,q}(E) = K_q^M(E)$ for every field E by Theorem 4.10, hence $H_M^{p,q} = K_q^M$ (as homotopy invariant presheaves with transfers) by Theorem 6.12. For any homotopy invariant $\mathcal{F} \in \mathrm{PSh}(k)$, it is a Zariski sheaf on $\mathbb{A}^1$, c.f., [MVW06], Lemma 22.4. Therefore, $\mathcal{F}$ is a Nisnevich sheaf on $\mathbb{A}^1$: a regular birational map between smooth curves is just an open immersion, which means their topologies agree. Hence, $\mathcal{F}((\mathbb{G}_m)_E) = \mathcal{F}^+((\mathbb{G}_m)_E)$ for every field E, which means $(\mathcal{F}_{-1})^+ \to (\mathcal{F}^+)_{-1}$, a map between homotopy invariant sheaves with transfers, induces an isomorphism on every field E. Hence, it is an isomorphism by Theorem 6.12.[42] Together with the cancellation theorem in Theorem 6.14, this shows that $(H_M^{p,q})_{-1} = H_M^{p-1,q-1}$. If $q \leqslant 0$, then $H_M^{p,q} = 0$ if $p \neq 0$: recall

$$H_M^{p,0} = \begin{cases} \mathbb{Z}, & p = 0 \\ 0, & p \neq 0 \end{cases}$$

[42] Also see [MVW06], Proposition 23.5.

We have a sequence

$$H^n(X, H_M^{p,q}) = H^n(\bigoplus_{x\in X^{(0)}} H^{p,q}(k(x),\mathbb{Z})) \longrightarrow \bigoplus_{x\in X^{(1)}} H^{p-1,q-1}(k(x),\mathbb{Z}) \longrightarrow \cdots$$

by Corollary 8.29. Hence $H^n(X, H_M^{p,q}) = 0$ if either $n > q$, or $n = q \neq p$, or $p > q$. We have the hypercohomology spectral sequence

$$H^n(X, H_M^{p,q}) \Rightarrow H^{n+p,q}(X,\mathbb{Z}),$$

and so if $p \geqslant 2q-1$, we know $H^a(X, H_M^{p-a,q}) = 0$ if $a \neq p-q$ by the vanishing conditions above, as desired. □

Corollary 8.31. We have $H^{p,q}(X,\mathbb{Z}) = 0$ if $p > 2q$.

Proof. Note that $H^n(X, K_n^M) = 0$ if $n > m$, since $(K_m^M)_{-n} = 0$. □

Corollary 8.32. We have $H^{2n,n}(X,\mathbb{Z}) \cong \mathrm{CH}^n(X) \cong H^n(X, K_n^M)$.

9 Orientation and Decomposition

9.1 Projective Bundle Theorem and Gysin Isomorphisms

Definition 9.1. Let $X \in \mathrm{Sm}/S$ and E be a vector bundle over X. We define the Thom space of E to be $\mathrm{Th}_S(E) = \mathbb{Z}_S(E)/\mathbb{Z}_S(E^\times)$.

Proposition 9.2.

1. Suppose that E_1 and E_2 are vector bundles over $X \in \mathrm{Sm}/k$, then $\mathrm{Th}_X(E_1) \otimes_X \mathrm{Th}_X(E_2) \cong \mathrm{Th}_X(E_1 \oplus E_2)$ in $\mathrm{DM}^{\mathrm{eff},-}(X)$.
2. Suppose $f : S \to T$ is a morphism, and $E \to X$ is a vector bundle for $X \in \mathrm{Sm}/T$, then $f^* \mathrm{Th}_T(E) \cong \mathrm{Th}_S(f^* E)$.
3. Suppose $f : S \to T$ is a smooth morphism, and $E \to X$ is a vector bundle for $X \in \mathrm{Sm}/T$, then $f_\# \mathrm{Th}_S(E) \cong \mathrm{Th}_T(E)$.

Proof. It suffices to prove the first part. The total space of $E_1 \oplus E_2$ is $E_1 \times_X E_2$, so by definition, $\mathrm{Th}_X(E)$ is quasi-isomorphic to the complex $\mathbb{Z}_X(E \backslash X) \to \mathbb{Z}(E)$. Hence, $\mathrm{Th}_X(E_1) \otimes_X \mathrm{Th}_X(E_2)$ is the total complex

$$\mathbb{Z}_X((E_1\backslash X) \times_X (E_2 \backslash X)) \longrightarrow \mathbb{Z}_X((E_1\backslash X) \times_X E_2) \oplus \mathbb{Z}_X(E_1 \times_X (E_2\backslash X)) \longrightarrow \mathbb{Z}_X(E_1 \times_X E_2)$$

By Theorem 2.35, the complex

$$\mathbb{Z}_X((E_1\backslash X) \times_X (E_2\backslash X)) \longrightarrow \mathbb{Z}_X(E_1^\times \times_X E_2) \oplus \mathbb{Z}_X(E_1 \times_X E_2^\times)$$

is quasi-isomorphic to

$$0 \longrightarrow \mathbb{Z}_X((E_1 \oplus E_2)^\times)$$

since $(E_1 \oplus E_2)^\times = (E_1^\times \times_X E_2) \cup (E_1 \times_X E_2^\times)$. Hence, we have a quasi-isomorphism

$$\begin{tikzcd}
\mathbb{Z}_X(E_1^\times \times_X E_2^\times) \arrow[r] \arrow[d] & \mathbb{Z}_X(E_1^\times \times_X E_2) \oplus \mathbb{Z}_X(E_1 \times_X E_2^\times) \arrow[r] \arrow[d] & \mathbb{Z}_X(E_1 \oplus E_2) \arrow[d, equal] \\
0 \arrow[r] & \mathbb{Z}_X((E_1 \oplus E_2)^\times) \arrow[r] & \mathbb{Z}_X(E_1 \oplus E_2)
\end{tikzcd}$$

□

Proposition 9.3. If E is a trivial bundle of rank n over $X \in \mathrm{Sm}/S$, then $\mathrm{Th}_S(E) \cong \mathbb{Z}_S(X)(n)[2n]$ in $\mathrm{DM}^{\mathrm{eff},-}(S)$.

Proof. If $n = 1$, then $\mathrm{Th}_X(E) \cong \mathbb{Z}_X(X \times \mathbb{A}^1)/\mathbb{Z}_X(X \times \mathbb{G}_m) \cong \mathbb{Z}_X(1)[2]$.[43] Therefore, for general n, $\mathrm{Th}_X(E) \cong (\mathbb{Z}_X(1)[2])^{\otimes n} \cong \mathbb{Z}_X(n)[2n]$ by Proposition 9.2. Now the statement follows by applying $f_\#$ where $f : X \to S$. □

Proposition 9.4. We have a decomposition

$$\mathbb{Z}(\mathbb{P}^n) \cong \bigoplus_{i=0}^{n} \mathbb{Z}(i)[2i]$$

in $\mathrm{DM}^{\mathrm{eff},-}(k)$.

[43] Note $\mathbb{Z}_X(X \times \mathbb{A}^1) \cong \mathbb{Z}_X(\mathbb{A}^1_X) \cong \mathbb{Z}_X$, and $\mathbb{Z}_X(X \times \mathbb{G}_m) \cong \mathbb{Z}_X({\mathbb{G}_m}_X) \cong \mathbb{Z}_X \oplus \mathbb{Z}_X(1)[1]$.

Proof. We proceed by induction on n. For $n = 0$, this is trivial since $\mathbb{P}^0$ is a point. For general n, we have a distinguished triangle

$$\mathbb{Z}(\mathbb{P}^n\backslash\{\infty\}) \longrightarrow \mathbb{Z}(\mathbb{P}^n) \longrightarrow \mathbb{Z}(\mathbb{P}^n)/\mathbb{Z}(\mathbb{P}^n\backslash\{\infty\}) \longrightarrow \mathbb{Z}(\mathbb{P}^n\backslash\{\infty\})[1]$$

Moreover, there is a Cartesian square

$$\begin{tikzcd} \mathbb{A}^n\backslash\{0\} \arrow[r] \arrow[d] & \mathbb{P}^n\backslash\{\infty\} \arrow[d] \\ \mathbb{A}^n \arrow[r] & \mathbb{P}^n \end{tikzcd}$$

By étale excision, we know $\mathbb{Z}(\mathbb{P}^n)/\mathbb{Z}(\mathbb{P}^n\backslash\{\infty\}) \cong \mathbb{Z}(\mathbb{A}^n)/\mathbb{Z}(\mathbb{A}^n\backslash\{0\})$, where the latter term is $\mathrm{Th}(k^{\oplus n}) \cong \mathbb{Z}(n)[2n]$ by Proposition 9.3. Moreover, with

$$\mathbb{P}^n\backslash\{\infty\} \to \mathbb{P}^{n-1}$$
$$(x_0 : \cdots : x_n) \mapsto (x_0 : \cdots : x_{n-1})$$

we get to identify $\mathbb{P}^n\backslash\{\infty\} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(1)$. Therefore, each $(x_0 : \cdots : x_n)$ gives a morphism $(x_0 : \cdots : x_{n-1}) \mapsto x_n$. Therefore, we have a distinguished triangle

$$\mathbb{Z}(\mathbb{P}^{n-1}) \longrightarrow \mathbb{Z}(\mathbb{P}^n) \longrightarrow \mathbb{Z}(n)[2n] \longrightarrow \mathbb{Z}(\mathbb{P}^{n-1})[1]$$

By induction, $\mathbb{Z}(\mathbb{P}^{n-1}) \cong \bigoplus_{i=1}^{n-1} \mathbb{Z}(i)[2i]$ by induction. Therefore, it suffices to compute

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}(k)}(\mathbb{Z}(n)[2n], \mathbb{Z}(i)[2i+1]),$$

which vanishes since $\mathbb{Z}(n)[2n]$ is a direct summand of $(\mathbb{P}^1)^{\times n}$ and $H^{2i+1,i}((\mathbb{P}^1)^{\times n}, \mathbb{Z}) = 0$ by Corollary 8.31. Therefore, the triangle splits. $\square$

Proposition 9.5. Suppose $\mathbb{Z}(X)$ can be written as $\bigoplus_i \mathbb{Z}(n_i)[2n_i]$ in $\mathrm{DM}^{\mathrm{eff},-}(k)$ for $X \in \mathrm{Sm}/k$. Given a map

$$\varphi = (\varphi_i) : \mathbb{Z}(X) \to \mathbb{Z}(X) \cong \bigoplus_i \mathbb{Z}(n_i)[2n_i],$$

the following are equivalent.

a. For every $k \in \mathbb{N}$, we have $\mathrm{CH}^k(X) = \bigoplus_{n_i=k} \mathbb{Z} \cdot p_i$.

b. φ is an isomorphism in $\mathrm{DM}^{\mathrm{eff},-}(k)$.

Proof.

a. $\Rightarrow$ *b.*: it suffices to show that $\mathrm{Hom}_{\mathrm{DM}}(\varphi, \mathbb{Z}(k)[2k])$ is an isomorphism. We now compute $\mathrm{Hom}_{\mathrm{DM}}(\mathbb{Z}(i)[2i], \mathbb{Z}(j)[2j])$ for $i, j \in \mathbb{N}$. If $i = j$, then it is just $\mathbb{Z}$ by cancellation. If $i < j$, by the vanishing of hypercohomology in Proposition 2.46, it is zero. If $i > j$, then $\mathbb{Z}(i-j)[2i-2j]$ is a direct summand of $((\mathbb{P}^1)^{\times(i-j)}, \{*\})$, so the group vanishes by the fact that $\mathrm{CH}^0((\mathbb{P}^1)^{\times(i-j)}/\{*\}) = 0$ and the cancellation theorem. Therefore, $\mathrm{Hom}_{\mathrm{DM}}(\mathbb{Z}(i)[2i], \mathbb{Z}(j)[2j])$ is the same as the map

$$\bigoplus_{n_i=k} \mathbb{Z} \to \mathrm{CH}^k(X)$$
$$e_i \mapsto \varphi_i$$

Hence the statement follows.

$b. \Rightarrow a.$: this is obvious from the discussion above. □

Corollary 9.6. The map $c_1(\mathcal{O}(1))^i : \mathbb{Z}(\mathbb{P}^n) \to \bigoplus_{i=0}^{n} \mathbb{Z}(i)[2i]$ is an isomorphism in DM.

Proposition 9.7. Let $\{U_i\}$ be an open covering of $X \in \mathrm{Sm}/k$ and f be a morphism in $\mathrm{DM}^{\mathrm{eff},-}(X)$. If $f|_{U_i}$ is an isomorphism for all i, then f is an isomorphism.

Proof. One can assume that we have a finite covering. By the condition, we know the restriction on mapping cone $C(f)|_{U_i} = 0$ in $\mathrm{DM}^{\mathrm{eff},-}(U_i)$ for all i. It suffices to show that the mapping cone $C(f) = 0$ in $\mathrm{DM}^{\mathrm{eff},-}(X)$.

For any $Y \in \mathrm{Sm}/U_i$ and any $n \in \mathbb{Z}$, we have

$$0 = \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}(U_i)}(\mathbb{Z}_{U_i}(Y)[n], C(f)|_{U_i}) = \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}(X)}(\mathbb{Z}_X(Y)[n], C(f))$$

by the adjunction $f_\# \dashv f^*$. Therefore, for any $Y \in \mathrm{Sm}/X$ we have an open cover $Y = \bigcup_i Y_i$ where $Y_i \in \mathrm{Sm}/U_i$. Therefore, by the Mayer-Vietoris sequence, we have

$$\mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}(X)}(\mathbb{Z}_X(Y)[n], C(f)) = 0$$

for any $n \in \mathbb{Z}$ and arbitrary choice of Y. Therefore, by Lemma 5.29, we have $C(f) = 0$. □

For any two maps $f_i : \mathbb{Z}(X) \to C_i$ for $i = 1, 2$, we define $f_1 \boxtimes f_2$ as the composite

$$\mathbb{Z}(X) \xrightarrow{\Delta} \mathbb{Z}(X) \otimes \mathbb{Z}(X) \xrightarrow{f_1 \otimes f_2} C_1 \otimes C_2$$

Theorem 9.8 (Projective Bundle Theorem)**.** Let E be a vector bundle of rank n over $X \in \mathrm{Sm}/S$. Given a structure map $p : \mathbb{P}(E) \to X$, the map

$$p \boxtimes c_1(\mathcal{O}_E(1))^i : \mathbb{Z}_S(\mathbb{P}(E)) \to \bigoplus_{i=0}^{n-1} \mathbb{Z}_S(X)(i)[2i]$$

is an isomorphism in $\mathrm{DM}^{\mathrm{eff},-}(S)$.

Proof. By Corollary 9.6, we have

$$c_1(\mathcal{O}(1))^i : \mathbb{Z}_S(\mathbb{P}^n \times S) \xrightarrow{\cong} \bigoplus_{i=0}^{n-1} \mathbb{Z}_S(i)[2i]$$

after pullback along the structure map $S \to \mathrm{Spec}(k)$. Now we take a trivialization $\{U_i\}$ of E, that is, $E|_{U_i} = \mathcal{O}_{U_i}^{\oplus n}$, then the map

$$c_1(\mathcal{O}_E(1))^i : \mathbb{Z}_X(\mathbb{P}(E)) \to \bigoplus_{i=0}^{n-1} \mathbb{Z}_X(i)[2i]$$

is an isomorphism on every U_i, hence it is an isomorphism on X by Proposition 9.7. Finally, apply $f_\#$ for $f : X \to S$ to pass it from X to S. □

Corollary 9.9. We have $H^{*,*}(\mathbb{P}(E), \mathbb{Z}) \cong \bigoplus_{i=0}^{n-1} H^{*-2i,*-i}(X, \mathbb{Z}) \cdot c_1(\mathcal{O}_E(1))^i$.

Proof. Apply the cancellation theorem Theorem 6.14 to Theorem 9.8. □

Definition 9.10. By Theorem 9.8, the Chern class $c_1(\mathcal{O}_E(1))^n$ can be written as a summation $\sum_{i=0}^{n-1} a_{n-i} \cdot c_1(\mathcal{O}_E(1))^i$ uniquely, where $a_{n-i} \in \mathrm{CH}^{n-i}(X)$. We get to define the ith Chern class of E by $c_i(E) = (-1)^{i-1}a_i \in H^{2i,i}(X,\mathbb{Z}) \cong \mathrm{CH}^i(X)$ for $i = 1, \cdots, n$.

From Theorem 9.8, we obtain split injections

$$\ell_r(E) : \mathbb{Z}_S(X)(r)[2r] \to \bigoplus_{i=0}^{n-1} \mathbb{Z}_S(X)(i)[2i] \cong \mathbb{Z}(\mathbb{P}(E))$$

for $r = 0, \dots, n-1$.

For every vector bundle E, we can consider its projective completion $\mathbb{P}(E^\vee \oplus \mathcal{O}_X)$ to characterize the quotient line bundle. We have a natural embedding $\mathbb{P}(E^\vee) \to \mathbb{P}(E^\vee \oplus \mathcal{O}_X)$ by the inclusion, and we know $\mathbb{P}(E^\vee \oplus \mathcal{O}_X)$ has a trivial section with a mapping $\mathbb{P}(E^\vee \oplus \mathcal{O}_X)\backslash X \to \mathbb{P}(E^\vee \oplus \mathcal{O}_X)$ defined by the complement. Locally, this section X we removed is just the point ∞, which means deleting this point gives a projective space of dimension one less. Therefore, we have an inclusion $\mathbb{P}(E^\vee) \to \mathbb{P}(E^\vee \oplus \mathcal{O}_X)\backslash X$, thus the diagram

$$\begin{tikzcd} \mathbb{P}(E^\vee) \arrow[r] \arrow[dr, "i"'] & \mathbb{P}(E^\vee \oplus \mathcal{O}_X) \\ & \mathbb{P}(E^\vee \oplus \mathcal{O}_X)\backslash X \arrow[u] \end{tikzcd}$$

commutes. Now $P := \mathbb{P}(E^\vee \oplus \mathcal{O}_X)\backslash\mathbb{P}(E^\vee)$ can be identified with the structure map $p : E \to X$,[44] given by the quotient $p^*E^\vee \oplus \mathcal{O}_E \to p^*E^\vee \to \mathcal{O}_E$ where the second map is induced by id_E.[45] There is an $\mathbb{A}^1$-bundle

$$\mathbb{P}(E^\vee \oplus \mathcal{O}_X)\backslash X \to \mathbb{P}(E^\vee)$$
$$(s,t) \mapsto s$$

with zero section i above, therefore it is an isomorphism over DM, and hence so is its zero section i. Therefore, $\mathbb{Z}_S(i)$ is an isomorphism in $\mathrm{DM}^{\mathrm{eff},-}(S)$.

Theorem 9.11. Suppose $Z \subseteq X$ is a closed immersion in Sm/S and $n = d_X - d_Z$, then there exists a unique family of (Gysin) isomorphisms of the form

$$p_{(X,Z)} : M_Z(X) \to \mathbb{Z}(Z)(n)[2n]$$

where $M_Z(X) := \mathbb{Z}(X)/\mathbb{Z}(X\backslash Z)$ which parametrizes the cohomology of X supported in Z, such that

1. for every Cartesian diagram

$$\begin{tikzcd} T \arrow[r, "g"] \arrow[d] & Z \arrow[d] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

between closed pairs (X,Z) and (Y,T) of relative dimension (or codimension) n, the diagram

$$\begin{tikzcd} M_T(Y) \arrow[r, "(f,g)"] \arrow[d, "p_{(Y,T)}"'] & M_Z(X) \arrow[d, "p_{(X,Z)}"] \\ \mathbb{Z}(T)(n)[2n] \arrow[r, "g(n)[2n]"'] & \mathbb{Z}(Z)(n)[2n] \end{tikzcd}$$

commutes;

[44] We try to identify it as E as X-schemes.
[45] This is induced by $E \to \mathbb{P}(E^\vee \oplus \mathcal{O}_X)$.

2. let $X \in \mathrm{Sm}/k$ and E be a vector bundle of rank n on X. Consider the pair (P, X) where $P = \mathbb{P}(E^\vee \oplus \mathcal{O}_X)$, so X is a zero section of P (as the point ∞), then the Gysin morphism $p_{(P,X)}$ is the inverse of the composed morphism

$$\mathbb{Z}(X)(n)[2n] \xrightarrow{\ell_n} \mathbb{Z}(P) \longrightarrow M_X(P)$$

which is an isomorphism by Theorem 9.8.

Proof. By Theorem 9.8, we can write down the first two isomorphisms between the two rows of split distinguished triangles. In particular, there is an inclusion between the direct sums.

$$\begin{tikzcd}
\mathbb{Z}(\mathbb{P}(E^\vee)) \cong \mathbb{Z}(P\backslash X) \arrow[r] & \mathbb{Z}(P) \arrow[r] & M_X(P) \arrow[r] & \mathbb{Z}(\mathbb{P}(E^\vee))[1] \\
\bigoplus_{i=0}^{n-1} \mathbb{Z}(X)(i)[2i] \arrow[u, "\cong"] \arrow[r, hook] \arrow[dr, equal] & \bigoplus_{i=0}^{n} \mathbb{Z}(X)(i)[2i] \arrow[u, "\cong"] \arrow[r] \arrow[d] & \mathbb{Z}(X)(n)[2n] \arrow[u, dashed] \arrow[r] & \left(\bigoplus_{i=0}^{n-1} \mathbb{Z}(X)(i)[2i]\right)[1] \arrow[u, "\cong"'] \\
& \bigoplus_{i=0}^{n-1} \mathbb{Z}(X)(i)[2i] & &
\end{tikzcd}$$

Now the projection $\bigoplus_{i=0}^{n} \mathbb{Z}(X)(i)[2i] \to \bigoplus_{i=0}^{n-1} \mathbb{Z}(X)(i)[2i]$ gives an identity when composed with the inclusion. However, if we look at the projection $\bigoplus_{i=0}^{n} \mathbb{Z}(X)(i)[2i] \to \mathbb{Z}(X)(n)[2n]$, then the composition

$$\bigoplus_{i=0}^{n} \mathbb{Z}(X)(i)[2i] \to \mathbb{Z}(X)(n)[2n]$$

is given by the Chern classes. Thus, the mapping cone of the second row is exactly $\mathbb{Z}(X)(n)[2n]$, and we have a morphism between two split distinguished triangles, which gives a uniquely determined natural map on the mapping cone, namely it is the composition of ℓ_n and the quotient map

$$\mathbb{Z}(X)(n)[2n] \xrightarrow{\ell_n} \mathbb{Z}(P) \longrightarrow M_X(P)$$

In particular, this is also an isomorphism. This proves the existence of this family of maps of part 2. For a general closed pair (X, Z), we define the Gysin morphism $p_{(X,Z)}$ to be the composite of isomorphisms

$$M_Z(X) \cong \mathrm{Th}_S(\mathcal{N}_{Z/X}) \cong M_Z(\mathbb{P}(\mathcal{N}_{Z/X}^\vee \oplus \mathcal{O}_Z)) \xrightarrow[\cong]{p_{(P,Z)}} \mathbb{Z}(Z)(n)[2n]$$

where the first isomorphism is given by homotopy purity in Theorem 8.27, the second isomorphism is given by étale excision in Proposition 2.32, and the last isomorphism is induced via $P = \mathbb{P}(\mathcal{N}_{Z/X}^\vee \oplus \mathcal{O}_Z)$. This finishes the proof for existence.

To show uniqueness, recall that the deformation space's section at 1 recovers the closed pair, so we adopt the following Cartesian diagram

$$\begin{tikzcd}
M_Z(X) \arrow[r] \arrow[d] & M_{Z\times\mathbb{A}^1}(D(X,Z)) \arrow[d] \\
\mathbb{Z}(Z)(n)[2n] \arrow[r] & \mathbb{Z}(Z\times\mathbb{A}^1)(n)[2n]
\end{tikzcd}$$

Moreover, recall that the fiber of the deformation space at 0 is the zero section of the normal bundle $\mathcal{N}_{Z/X}$, then we have another commutative diagram

$$\begin{tikzcd}
\mathrm{Th}_S(\mathcal{N}_{Z/X}) \arrow[r] \arrow[d] & M_{Z\times\mathbb{A}^1}(D(X,Z)) \arrow[d] \\
\mathbb{Z}(Z)(n)[2n] \arrow[r] & \mathbb{Z}(Z\times\mathbb{A}^1)(n)[2n]
\end{tikzcd}$$

Finally, by étale excision, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Th}_S(\mathcal{N}_{Z/X}) \arrow[r] \arrow[d] & M_Z(P) \arrow[dl] \\
\mathbb{Z}(Z)(n)[2n] &
\end{tikzcd}$$

Combining all of this, we have a diagram

$$\begin{tikzcd}
M_Z(X) \arrow[r] \arrow[d] & M_{Z\times\mathbb{A}^1}(D(X,Z)) \arrow[d] & \mathrm{Th}_S(\mathcal{N}_{Z/X}) \arrow[l] \arrow[r] \arrow[d] & M_Z(P) \arrow[dl] \\
\mathbb{Z}(Z)(n)[2n] \arrow[r] & \mathbb{Z}(Z\times\mathbb{A}^1)(n)[2n] & \mathbb{Z}(Z)(n)[2n] \arrow[l] &
\end{tikzcd}$$

where all morphisms are isomorphisms. In particular, this shows that $M_Z(X) \to \mathbb{Z}(Z)(n)[2n]$ is uniquely determined by $M_Z(P) \to \mathbb{Z}(Z)(n)[2n]$, which is determined by the property in part 2, hence it is uniquely determined by this diagram. □

Remark 9.12. In particular, Theorem 9.11 shows that $\mathrm{Th}_S(E) \cong \mathbb{Z}_S(X)(n)[2n]$ for any vector bundle E over X of rank n.

Corollary 9.13 (Gysin Triangle)**.** In the context of Theorem 9.11, we have a distinguished triangle (as a localization sequence)

$$\mathbb{Z}(X\backslash Z) \longrightarrow \mathbb{Z}(X) \xrightarrow{p_{(X,Z)}} \mathbb{Z}(Z)(n)[2n] \longrightarrow \mathbb{Z}(X\backslash Z)[1]$$

in $\mathrm{DM}^{\mathrm{eff},-}(S)$ (of motivic cohomology).

9.2 Białynicki-Birula Decomposition

We now introduce a common situation where the motive $\mathbb{Z}(X)$ of X can be written as a direct sum of the form $\mathbb{Z}(i)[2i]$ in $\mathrm{DM}^{\mathrm{eff},-}(k)$.

Theorem 9.14 (Białynicki-Birula Decomposition)**.** Let X be a smooth projective variety over a field k equipped with a $\mathbb{G}_m$-action, then

1. the fixed point locus $X^{\mathbb{G}_m}$ is a smooth closed subscheme of X;
2. there exists a numbering upon the connected components Z_i's of $X^{\mathbb{G}_m} = \coprod_{i=1}^n Z_i$ such that there is a filtration
$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_0 \supseteq X_{-1} = \varnothing$$
of closed subsets and affine bundles $\varphi_i : X_i\backslash X_{i-1} \to Z_i$. In particular, they are homotopic;
3. the relative dimension (or rank) a_i of the affine bundle φ_i is the dimension of the positive eigenspace of the $\mathbb{G}_m$-action on the tangent space $T_{X,z}$, where z is an arbitrary closed point of Z_i. The dimension of Z_i is the dimension of $(T_{X,z})^{\mathbb{G}_m}$.

Proof. The original proof assumed $k = \mathbb{C}$, and was later generalized so that it applies to arbitrary field k. See [Bro05], Theorem 3.2. □

Theorem 9.15. In the context of Theorem 9.14, there exists a decomposition

$$\mathbb{Z}(X) \cong \bigoplus_{i=0}^n \mathbb{Z}(Z_i)(b_i)[2b_i]$$

in $\mathrm{DM}^{\mathrm{eff},-}(k)$, where $b_i = \dim(T^-_{X,z})$ is the dimension of the negative-weight eigenspace at an arbitrary closed point z of Z_i.

Proof. By the Gysin triangle in Corollary 9.13, we have a distinguished triangle

$$\mathbb{Z}(X\backslash X_i) \longrightarrow \mathbb{Z}(X\backslash X_{i-1}) \longrightarrow \mathbb{Z}(X_i\backslash X_{i-1})(c_i)[2c_i] \cong \mathbb{Z}(Z_i)(c_i)[2c_i] \longrightarrow \mathbb{Z}(X\backslash X_i)[1]$$

where the identification of motives is given by the affine bundle φ_i. We need to calculate c_i's. Based on the eigenvalue of each eigenspace[46], we know

$$\begin{aligned} c_i &= d_X - d_{X_i} \\ &= \dim(T^+_{X,z}) + \dim(T^-_{X,z}) + \dim(T^0_{X,z}) - (\dim(T^+_{X,z}) + \dim(T^0_{X,z})) \\ &= \dim(T^-_{X,z}) \\ &= b_i \end{aligned}$$

for closed point $z \in Z_i$. We prove by induction that

$$\mathbb{Z}(X\backslash X_i) \cong \bigoplus_{j=i+1}^{n} \mathbb{Z}(Z_j)(b_j)[2b_j].$$

If $i = n$, we have an empty set and the statement is trivial. For $i < n$, by inductive hypothesis, we know $\mathbb{Z}(X\backslash X_i)[1]$ splits into the form of $\bigoplus_{j=i+1}^{n} \mathbb{Z}(Z_j)(b_j)[2b_j]$, where Z_j's are projective, therefore it suffices to prove

$$\mathrm{Hom}_{\mathrm{DM}}(\mathbb{Z}(U)(n)[2n], \mathbb{Z}(V)[m](2m+1)) = 0$$

for any $m, n \in \mathbb{N}$, $U, V \in \mathrm{Sm}/k$, and V projective. We postpone the proof of this fact until we obtain a result on duality, that is, the hom group above is isomorphic to

$$\mathrm{Hom}_{\mathrm{DM}}(\mathbb{Z}(U \times V)(n - \dim(V))[2n - 2\dim(V)], \mathbb{Z}(m)[2m+1]),$$

which is zero by Corollary 8.31. □

Corollary 9.16. In particular, in the context of Theorem 9.15, if $\dim(X^{\mathbb{G}_m}) = 0$, e.g., say it is the set of rational isolated points for instance, then

$$\mathbb{Z}(X) \cong \bigoplus_{i=0}^{n} \mathbb{Z}(b_i)[2b_i].$$

Here we get to introduce a situation where the projective scheme admits a $\mathbb{G}_m$-action such that the fixed points are isolated.

Proposition 9.17. Let G be a connected reductive linear algebraic group which splits over k. Suppose P is a parabolic subgroup[47], and T is a maximal torus contained in a Borel subgroup B contained in P. Then G/P is smooth and projective with a $\mathbb{G}_m$-action such that $(G/P)^{\mathbb{G}_m}$ has dimension zero.

Proof. Since the G-action is transitive on a homogeneous variety,[48] then G/P is smooth. Its projectivity is a classical result, for example see [Hum12], Corollary B in Chapter 21.3. Therefore, G/P has a natural T-action since $T \subseteq P$, and there is a T-equivariant map $\pi : G/B \to G/P$. Now the fixed point locus $(G/B)^T$ is isomorphic to the Weyl group $N_G(T)/Z_G(T)$, i.e., the normalizer quotient over the centralizer, which is finite. Note that given any T-fixed point xP of $(G/P)^T$ as a coset, the preimage $\pi^{-1}(xP) = P/B$ is a projective T-variety. Moreover, it admits a T-action,

[46] We know X_i has an affine bundle on Z_i, so its relative dimension is a_i, which is the dimension of the positive-weight eigenspace; similarly, the dimension of Z_i is the dimension of the zero-weight eigenspace.

[47] By definition, this is a closed subgroup of G containing a Borel subgroup of G. A Borel subgroup of G is a maximal connected closed solvable subgroup of G.

[48] It is isomorphic at every point.

so by Borel's fixed point theorem, the T-action admits a T-fixed point, c.f., [Hum12], Chapter 21.2. Therefore, the map $(G/B)^T \to (G/P)^T$ is a surjection, since every fixed point on G/P admits a fixed point in G/B, hence $(G/P)^T$ is also a finite group. Now the tangent space $T_1(G/B)$ at unit $e = 1$ can be identified with the vector space Φ^+ spanned by the positive roots. Choose a cocharacter $f : \mathbb{G}_m \to T$ (as a group homomorphism) such that the inner proudct $\langle f, \chi\rangle > 0$ for every $\chi \in \Phi^+$,[49] then the $\mathbb{G}_m$-action of G/B, given by the restriction from the T-action and is induced from f, satisfies $(G/B)^{\mathbb{G}_m} \cong (G/B)^T$, which is also a finite set as well. Using the same argument, we know that $(G/P)^{\mathbb{G}_m}$ is also finite by applying the surjection $(G/B)^T \to (G/P)^T$. $\square$

Let G be a connected reductive linear algebraic group that is split over k, then we get to classify all parabolic subgroups that contains some fixed Borel subgroup. In fact, we know the set of parabolic subgroups containing a fixed Borel subgroup is isomorphic to subsets of simple roots. For any subset J of simple roots, let P_J be the correcponding set of parabolic subgroups, then we have a decomposition

$$\mathbb{Z}(G/P_J) \cong \bigoplus_{\bar{w}\in W/\langle J\rangle} \mathbb{Z}(d_{G/P_J} - \ell(\bar{w}))[2d_{G/P_J} - 2\ell(\bar{w})]$$

induced by Theorem 9.14, where W is the Weyl group, and

$$\ell(\bar{w}) = \min\{\ell(v) : \bar{v} = \bar{w} \in W/\langle J\rangle\}.$$

Remark 9.18. The identification of the index $\bar{w} \in W/\langle J\rangle$ with fixed points follows from the fact that $W/\langle J\rangle \cong (G/P_J)^T \subseteq (G/P_J)^{\mathbb{G}_m}$, where T is a maximal torus contained in a Borel subgroup B, and we have a commutative diagram

$$\begin{tikzcd}
(G/B)^T \arrow[r, two heads] \arrow[d, hook, "\cong"'] & (G/P_J)^T \arrow[d, hook] \\
(G/B)^{\mathbb{G}_m} \arrow[r, two heads] & (G/P_J)^{\mathbb{G}_m}
\end{tikzcd}$$

where all exhibited properties are due to the proof of Proposition 9.17. In particular, this means $(G/P_J)^T \to (G/P_J)^{\mathbb{G}_m}$ is a surjection, therefore we have an isomorphism $(G/P_J)^T \cong (G/P_J)^{\mathbb{G}_m}$. This is actually equivalent to the Bruhat decomposition.

Remark 9.19. In particular, this gives a decomposition of the generalized flag varieties.

Example 9.20. We can take the Grassmannians as an example. Consider $G = \mathrm{GL}_n$ and suppose P is given by block matrices

$$P = \begin{pmatrix} M_{d\times d} & * \\ 0 & M_{(n-d)\times(n-d)} \end{pmatrix}$$

then P is a parabolic subgroup containing the Borel subgroup of upper triangular matrices. Therefore, $G/P = \mathrm{Gr}(d, n) := \{V \subseteq k^{\oplus n} : \dim(V) = d\}$ is the Grassmannian of d-dimensional subspaces in n-dimensional space, then the Białynicki-Birula Decomposition is the morphism given by

$$\mathbb{Z}(\mathrm{Gr}(d, n)) \xrightarrow[\cong]{c_\Gamma} \bigoplus \mathbb{Z}(|\Gamma|)[2|\Gamma|]$$

where Γ is a Young tableau of size $d \times (n-d)$, and $|\Gamma|$ is the number of boxes in Γ. Let $c_i = c_i(U^\perp)$ be the ith Chern class of the complement of the tautological bundle of rank $n - d$ (of the Grassmannian $\mathrm{Gr}(n, d)$). More explicitly, given a Young tableau Γ of row lengths $(a_1, \ldots, a_d)$ for $n - d \geqslant a_1 \geqslant \cdots \geqslant a_d$, we get to interpret $|\Gamma| = \sum_{i=1}^{d} a_i$. Under this

[49] With $\mathbb{G}_m$-action, the coordinates are given by the exponents as integers, therefore we have a notion of inner product.

setting, we define c_Γ to be the $d \times d$ matrix of the form

$$\begin{pmatrix} c_{a_1} & c_{a_1+1} & \cdots & c_{a_1+(d-1)} \\ c_{a_2-1} & c_{a_2} & \ddots & c_{a_2+(d-2)} \\ \vdots & \ddots & \ddots & \vdots \\ c_{a_d-(d-1)} & c_{a_d-(d-2)} & \cdots & c_{a_d} \end{pmatrix}$$

which illustrates the Giambelli's formula. In particular, c_Γ gives rise to an isomorphism between the two motives.

10 Category of Stabilized Motives

Recall that we have the following formula

$$H^{i+1}(X \wedge S^1, \mathbb{Z}) \cong H^i(X, \mathbb{Z})$$

for singular cohomology of any pointed space X. Let $K(\mathbb{Z}, i)$ be the Eilenberg-Maclane space of $H^i(-, \mathbb{Z})$, i.e., we have

$$[X, K(\mathbb{Z}, i)] \cong H^i(X, \mathbb{Z}).$$

Such spaces are characterized by

$$\pi_n(K(\mathbb{Z}, i)) \cong \begin{cases} \mathbb{Z}, & n = i \\ *, & n \neq i \end{cases}.$$

Then by the adjunction, we have

$$\begin{aligned}
[X, \mathrm{Map}(S^1, K(\mathbb{Z}, i+1))] &\cong [X \wedge S^1, K(\mathbb{Z}, i+1)] \\
&\cong H^{i+1}(X \wedge S^1, \mathbb{Z}) \\
&\cong H^i(X, \mathbb{Z}) \\
&\cong [X, K(\mathbb{Z}, i)].
\end{aligned}$$

Therefore, by Yoneda Lemma, $\mathrm{Map}(S^1, K(\mathbb{Z}, i+1))$ is weak homotopy equivalent to $K(\mathbb{Z}, i)$, i.e., they have the same homotopy groups. In particular, we have maps $S^1 \wedge K(\mathbb{Z}, i) \to K(\mathbb{Z}, i+1)$. This suggests that we should study sequences $E = \{E_i\}$ of spaces of structure maps $S^1 \wedge E_i \to E_{i+1}$, which is equivalent to $E_i \to \mathrm{Map}(S^1, E_{i+1})$. Such sequences are called spectra in topology.

We can define a naive smash product for spectra E and F, with $(E \wedge F)_{2n} = E_n \wedge F_n$ and $(E \wedge F)_{2n+1} = E_n \wedge F_{n+1}$, where F_{n+1} exhibits a S^1-action. However, this smash product is not commutative, nor homotopy commutative.

Example 10.1. Let $E = F = (S^0, S^1, \cdots)$. To define $E \wedge F \to F \wedge E$ along with $E_i \wedge F_j \simeq F_j \wedge E_i$, we need a commutative diagram

$$\begin{array}{ccc}
S^1 \wedge S^p \wedge S^q & \longrightarrow & S^{p+1} \wedge S^q \\
\downarrow & & \downarrow \\
S^1 \wedge S^q \wedge S^p & \longrightarrow & S^{q+1} \wedge S^p
\end{array}$$

where we identify $S^{p+q} \wedge S^q \cong S^{p+q+1} \cong S^{q+1} \wedge S^p$. In this diagram, the two compositions are different up to a smashing $S^1 \wedge S^1$. However, the swapping of $S^1 \wedge S^1$ is not identity in the homotopy category, so the diagram does not commute.

Therefore, the construction of smash product requires a notion of symmetric spectra Ω^∞, which is usually hard to define.

We want to consider the same idea in the motivic context. Suppose $K^{p,q}$ is the Eilenberg-Maclane space of the (p,q)th cohomology of some theory realized in DM,[50] then we have

$$\underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1}, K^{p+1,q+1}) \cong K^{p,q}$$

in DM by the cancellation theorem, c.f., Theorem 6.14. Therefore, it is natural to study the notion of $\mathbb{G}_m$-spectra, which are spaces $\{E_i\}$ with maps $\mathbb{G}_m \wedge E_i \to E_{i+1}$.

[50]For instance, consider $\mathrm{Hom}_{\mathrm{DM}}(X, K^{p,q}) \cong H^{p,q}(X)$.

It turns out that it is difficult to define smash products between spectra of derived motives, but it is easy to define the infinity loopspace Ω^∞. On the other hand, we have an alternative choice, namely the symmetric spectra, where it is easy to define a notion of tensor product, but hard to define Ω^∞. We are mostly interested in the construction of symmetric spectra.[51]

10.1 Symmetric Spectra

For references, see [Sch12].

Definition 10.2. Let $\mathscr{A}$ be a symmetric closed monoidal abelian category with arbitrary products, i.e., $\mathscr{A}$ admits a tensor product $\otimes$ which is commutative, associative, with a unit $\mathbb{1}$,[52] and a right adjoint given by the inner hom $\underline{\mathrm{Hom}}$.

A symmetric sequence of $\mathscr{A}$ is a sequence $(A_n)_{n\in\mathbb{N}}$ of $\mathscr{A}$ such that every A_n has an S_n-action. A morphism of symmetric sequences $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ is a collection of S_n-equivariant morphisms $f_n : A_n \to B_n$. Therefore, there is a notion of the category of symmetric sequences $\mathscr{A}^S$ over $\mathscr{A}$.[53]

We now define a tensor product of symmetric sequences.

Definition 10.3. Suppose $A, B \in \mathscr{A}^S$, then we define the tensor product $A \otimes^S B$ of A and B to be a symmetric sequence such that

$$(A \otimes^S B)_n = \bigoplus_{p=0}^{n} S_n \times_{S_p \times S_{n-p}} (A_p \otimes B_{n-p}).$$

Proposition 10.4. $\mathscr{A}^S$ is a symmetric closed monoidal abelian category.

Proof. Note that the kernel and cokernel are defined termwise, therefore it is easy to see it is an abelian category. For any $A, B, C \in \mathscr{A}^S$, we know $(A \otimes^S B) \otimes^S C$ and $A \otimes^S (B \otimes^S C)$ are both isomorphic to

$$\bigoplus_{i+j+k=n} S_n \times_{S_i \times S_j \times S_k} (A_i \otimes B_j \otimes C_k)$$

(which does not depend on associativity!), therefore the product is associative. For any $A, B \in \mathscr{A}^S$, we define $\tau : A \otimes^S B \to B \otimes^S A$ via the universal property of the diagram

$$\begin{tikzcd}
A_i \otimes B_j \arrow[r] \arrow[d] & B_j \otimes A_i \arrow[r] & S_n \times_{S_j \times S_i} (B_j \otimes A_i) \arrow[d, "\theta_{j,i}"] \\
S_n \times_{S_i \times S_j} (A_i \otimes B_j) \arrow[rr, dashed, "\tau_{i,j}"'] & & S_n \times_{S_j \times S_i} (B_j \otimes A_i)
\end{tikzcd}$$

for any $i + j = n$, where $\theta_{i,j} : S_n \to S_n$ is an automorphism given by swapping the first i elements and the last j elements. Therefore, one can check that $\mathscr{A}^S$ is symmetric monoidal. To show that it is closed, we define the inner hom set via

$$\underline{\mathrm{Hom}}^S(A, B)_n = \prod_{p\in\mathbb{N}} \underline{\mathrm{Hom}}_{S_p}(A_p, B_{n+p}),$$

where $\underline{\mathrm{Hom}}_{S_p}(A_p, B_{n+p})$ is the kernel of the map (or, the equalizer of two maps separately)

$$\sigma^* - (\mathrm{id}_{S_n} \times \sigma)_* : \underline{\mathrm{Hom}}(A_p, B_{n+p}) \to \prod_{\sigma\in S_p} \underline{\mathrm{Hom}}(A_p, B_{n+p}).$$

Therefore, giving a morphism $A \otimes^S B \to C$ is equivalent to giving $(S_p \times S_q)$-equivariant maps

$$f_{p,q} : A_p \otimes B_q \to C_{p+q},$$

[51] However, the two constructions eventually give the same homotopy category.

[52] These notions should be with respect to some equivalence.

[53] To consider the usual spectra (for non-symmetric case), we should forget about the S_n-action.

which is equivalent to giving S_p-equivariant maps

$$g_{p,q} : A_p \to \underline{\mathrm{Hom}}(B_q, C_{p+q})$$

such that for any $\sigma \in S_q$, the diagram

$$\begin{tikzcd}
A_p \arrow[r, "g_{p,q}"] \arrow[d, "g_{p,q}"'] & \underline{\mathrm{Hom}}(B_q, C_{p+q}) \arrow[d, "\sigma^*"] \\
\underline{\mathrm{Hom}}(B_q, C_{p+q}) \arrow[r, "(\mathrm{id}_{S_p} \times \sigma)_*"'] & \underline{\mathrm{Hom}}(B_q, C_{p+q})
\end{tikzcd}$$

commutes, which is equivalent to saying that $g_{p,q}$ factors through $\underline{\mathrm{Hom}}(B_q, C_{p+q})$. □

Remark 10.5. We have an adjunction

$$\begin{tikzcd}
\mathscr{A} \arrow[d, shift right, "i_0"'] \\
\mathscr{A}^S \arrow[u, shift right, "\mathrm{ev}_0"']
\end{tikzcd}$$

where $i_0(A) = (A, 0, 0, \cdots)$ for any $A \in \mathscr{A}$, and ev_0 is the evaluation at 0 defined by $\mathrm{ev}_0(\{A_n\}_{n\in\mathbb{N}}) = A_0$.

Definition 10.6. For any symmetric sequence $A \in \mathscr{A}^S$ and $n \in \mathbb{N}$, we can define its shifting via induction

$$(A\{-n\})_m = \begin{cases} S_m \times_{S_{m-n}} A_{m-n}, & m \geqslant n \\ 0, & m < n \end{cases}$$

and restriction

$$(A\{n\})_m = \mathrm{Res}^{S_{n+m}}_{S_m}(A_{n+m}).$$

Remark 10.7. This has a non-symmetric analogue.

Remark 10.8. This gives an adjunction

$$\begin{tikzcd}
\mathscr{A}^S \arrow[d, shift right, "\{-i\}"'] \\
\mathscr{A}^S \arrow[u, shift right, "\{i\}"']
\end{tikzcd}$$

for any $i \in \mathbb{N}$.

Now suppose $A \in \mathscr{A}$, we can define $\mathrm{Sym}(A) = (\mathbb{1}, A, A \otimes A, A^{\otimes 3}, \cdots) \in \mathscr{A}^S$, where each $A^{\otimes n}$ has an S_n-action given by permutation of factors.

Proposition 10.9. For any $A \in \mathscr{A}$, $\mathrm{Sym}(A)$ is a commutative monoid object in $\mathscr{A}^S$, i.e., equipped with a unit map and a notion of multiplication that is commutative. Therefore, $\mathrm{Sym}(A)$ has a structure analogous to that of a commutative ring.

Proof. There is an obvious unit map given by

$$(\mathbb{1}, 0, \cdots, 0) \to \mathrm{Sym}(A).$$

We define the multiplication operation as

$$\begin{aligned} \mu : \mathrm{Sym}(A) \otimes^S \mathrm{Sym}(A) &\to \mathrm{Sym}(A) \\ (A^{\otimes a}, A^{\otimes b}) &\mapsto A^{\otimes(a+b)} \end{aligned}$$

To see that this multiplication is commutative, we want to show that the diagram

$$\begin{tikzcd} \mathrm{Sym}(A) \otimes^S \mathrm{Sym}(A) \arrow[rr, "\tau"] \arrow[dr, "\mu"'] & & \mathrm{Sym}(A) \otimes^S \mathrm{Sym}(A) \arrow[dl, "\mu"] \\ & \mathrm{Sym}(A) & \end{tikzcd}$$

commutes, and it is sufficient to show it commutes termwise, that is, the diagram

$$\begin{tikzcd} S_n \times_{S_i \times S_j} (A^{\otimes i} \otimes A^{\otimes j}) \arrow[rr, "\tau_{i,j}"] \arrow[dr, "\mu"'] & & S_n \times_{S_i \times S_j} (A^{\otimes j} \otimes A^{\otimes i}) \arrow[dl, "\mu"] \\ & A^{\otimes n} & \end{tikzcd}$$

commutes, where $n = i + j$. This follows from the commutative diagram

$$\begin{tikzcd} A^{\otimes i} \otimes A^{\otimes j} \arrow[r] \arrow[d, equal] & A^{\otimes j} \otimes A^{\otimes i} \arrow[r] \arrow[dr, equal] & S_n \times_{S_j \times S_i} (A^{\otimes j} \otimes A^{\otimes i}) \arrow[r, "\theta_{j,i}"] \arrow[d] & S_n \times_{S_j \times S_i} (A^{\otimes j} \otimes A^{\otimes i}) \arrow[d] \\ A^{\otimes n} \arrow[rr, "\theta_{i,j}"'] \arrow[rrr, equal, bend right] & & A^{\otimes n} \arrow[r, "\theta_{j,i}"'] & A^{\otimes n} \end{tikzcd}$$

□

Remark 10.10. This is not true in the non-symmetric case.

If we think of symmetric sequences as abelian groups, then symmetric R-spectra are the abelian groups endowed with $\mathrm{Sym}(R)$-module strucutre.

Definition 10.11. Fix $R \in \mathscr{A}$. We define the category of symmetric R-spectra, denoted by $\mathrm{Sp}_R(\mathscr{A})$, to be the category of $\mathrm{Sym}(R)$-modules in $\mathscr{A}^S$.[54]

Also equivalently, given a symmetric sequence E, there are $(S_p \times S_q)$-equivariant maps $f_{p,q} : R^{\otimes p} \otimes E_q \to E_{p+q}$ such that $f_{o,q} = \mathrm{id}_{E_q}$, and the diagram

$$\begin{tikzcd} R^{\otimes p} \otimes R^{\otimes q} \otimes E_r \arrow[r, "f_{q,r}"] \arrow[dr, "f_{p+q,r}"'] & R^{\otimes p} \otimes E_{q+r} \arrow[d, "f_{p,q+r}"] \\ & E_{p+q+r} \end{tikzcd}$$

commutes.

This is to say that, equivalently, given a symmetric sequence E, there are maps $R \otimes E_p \to E_{p+1}$ such that the composite

$$R^{\otimes p} \otimes E_q \longrightarrow R^{\otimes (p-1)} \otimes E_{q+1} \longrightarrow \cdots \longrightarrow E_{p+q}$$

is $(S_p \times S_q)$-equivariant.

Remark 10.12. To obtain a non-symmetric analogue, we just need to give a structure map $R \otimes E_p \to E_{p+1}$ without an equivariant condition.

Proposition 10.13. $\mathrm{Sp}_R(\mathscr{A})$ is a symmetric closed monoidal abelian category.

This mimics the idea over abelian groups and R-modules.

[54] By $\mathrm{Sym}(R)$-modules, we mean each object $E \in \mathrm{Sp}_R(\mathscr{A})$ admits a map $\mathrm{Sym}(R) \otimes^S E \to E$ with associativity and unity laws.

Proof. Suppose $M, N \in \mathrm{Sp}_R(\mathscr{A})$, then we define $M \otimes N \in \mathrm{Sp}_R(\mathscr{A})$ by the exact sequence

$$M \otimes^S \mathrm{Sym}(R) \otimes^S N \xrightarrow{m\otimes \mathrm{id}_N - \mathrm{id}_M \otimes m} M \otimes^S N \longrightarrow M \otimes N \longrightarrow 0$$

where m is the structure map on M and on N. Define $\underline{\mathrm{Hom}}(M, N) \in \mathrm{Sp}_R(\mathscr{A})$ by the exact sequence

$$0 \longrightarrow \underline{\mathrm{Hom}}(M, N) \longrightarrow \underline{\mathrm{Hom}}^S(M, N) \xrightarrow{m^*-m_*} \underline{\mathrm{Hom}}^S(\mathrm{Sym}(R) \otimes^S M, N)$$

i.e., as a coequalizer, where m^* is given by the module structure, and m_* is given by the composition

$$\underline{\mathrm{Hom}}^S(M, N) \xrightarrow{\mathrm{Sym}(R)\otimes^S -} \underline{\mathrm{Hom}}^S(\mathrm{Sym}(R) \otimes^S M, \mathrm{Sym}(R) \otimes^S N) \longrightarrow \underline{\mathrm{Hom}}^S(\mathrm{Sym}(R) \otimes^S M, N)$$

One can now check the statement. □

Remark 10.14. There is now an adjunction

$$\begin{tikzcd} \mathscr{A}^S \arrow[d, shift right, "\mathrm{Sym}(R)\otimes^S -"'] \\ \mathrm{Sp}_R(\mathscr{A}) \arrow[u, shift right, "U"'] \end{tikzcd}$$

where U is the forgetful functor. Moreover, this gives another adjunction

$$\begin{tikzcd} \mathscr{A} \arrow[d, shift right, "\Sigma^\infty"'] \\ \mathrm{Sp}_R(\mathscr{A}) \arrow[u, shift right, "\Omega^\infty"'] \end{tikzcd}$$

where the left adjoint is the infinite suspension functor $\Sigma^\infty = (\mathrm{Sym}(R) \otimes^S -) \circ i_0$, and the right adjoint is the infinite loopspace functor $\Omega^\infty = \mathrm{ev}_0 \circ U$, using notations in Remark 10.5. More explicitly, we have $\Sigma^\infty(A) = (A, R \otimes A, R^{\otimes 2} \otimes A, \cdots)$ and $\Omega^\infty(E) = E_0$.

Remark 10.15. For symmetric sequences, we also have natural identification $A \otimes^S (B\{-i\}) = (A \otimes^S B)\{-i\}$ for $i \in \mathbb{N}$, given by the identity $S_{p+q} \times_{S_p \times S_q} (A_p \otimes (S_q \times_{S_{q-i}} B_{q-i})) = S_{p+q} \times_{S_p \times S_{q-i}} (A_p \otimes B_{q-i})$. Moreover, we have a natural map $A \otimes^S (B\{i\}) \to (A \otimes^S B)\{i\}$ defined by the composite

$$A \otimes^S (B\{i\}) \longrightarrow (A \otimes^S B\{i\})\{-i\}\{i\} = (A \otimes^S B\{i\}\{-i\})\{i\} \longrightarrow (A \otimes^S B)\{i\}$$

which is induced by the counit and the unit of the adjunction for any $i \in \mathbb{N}$. Restricting the functors $\{-i\}$ and $\{i\}$ to $\mathrm{Sp}_R(\mathscr{A})$, we obtain an adjunction

$$\begin{tikzcd} \mathrm{Sp}_R(\mathscr{A}) \arrow[d, shift right, "\{-i\}"'] \\ \mathrm{Sp}_R(\mathscr{A}) \arrow[u, shift right, "\{i\}"'] \end{tikzcd}$$

with the same property, that is, we have $A \otimes (B\{-i\}) = (A \otimes B)\{-i\}$ and a morphism $A \otimes (B\{i\}) \to (A \otimes B)\{i\}$ for any $i \in \mathbb{N}$ and $A, B \in \mathrm{Sp}_R(\mathscr{A})$.

10.2 Applications in Sheaves with Transfers

We now apply these general constructions to our category of sheaves with transfers.

Definition 10.16. Fix some $S \in \mathrm{Sm}/k$, then we can define the categories $\mathrm{Sp}(S) = \mathrm{Sp}_{\mathbb{Z}(\mathbb{G}_m^{\wedge 1})}(\mathrm{Sh}(S))$ and $\mathrm{Sp}'(S) = \mathrm{Sp}_{\mathbb{Z}(\mathbb{G}_m^{\wedge 1})}(\mathrm{PSh}(S))$ on sheaves and presheaves, respectively. By termwise definition, we obtain a sheafification-forgetful adjunction

$$\begin{tikzcd} \mathrm{Sp}'(S) \arrow[d, shift right, "+"'] \\ \mathrm{Sp}(S) \arrow[u, shift right, "U"'] \end{tikzcd}$$

a pullback-pushforward adjunction

$$\begin{tikzcd} \mathrm{Sp}(T) \arrow[d, shift right, "f^*"'] \\ \mathrm{Sp}(S) \arrow[u, shift right, "f_*"'] \end{tikzcd}$$

induced from $f : S \to T$, and a direct-image-pullback adunction

$$\begin{tikzcd} \mathrm{Sp}(T) \arrow[d, shift right, "f^\#"'] \\ \mathrm{Sp}(S) \arrow[u, shift right, "f^*"'] \end{tikzcd}$$

induced from smooth morphism $f : S \to T$.

Remark 10.17. We have $f^*(A \otimes B) = f^*(A) \otimes f^*(B)$ and $f_\#(A \otimes_S f^*(B)) = (f_\#(A)) \otimes_T B$.

Remark 10.18. For any natural number i and $\mathcal{F} \in \mathrm{Sh}(S)$, we have

$$(\Sigma^\infty \mathcal{F})\{i\} \cong \Sigma^\infty(\mathbb{G}_m^{\wedge i} \otimes \mathcal{F})$$

by construction. Moreover, for any $X \in \mathrm{Sm}/S$ and $A, B \in \mathrm{Sp}(S)$, we have

$$\mathrm{Hom}_{\mathrm{Sp}(S)}((\Sigma^\infty \mathbb{Z}_S(X))\{-i\}, A) \cong A_i(X)$$

according to the adjunctions, therefore every spectrum $A \in \mathrm{Sp}(S)$ has a resolution $L^* \to A$, where each L^i is given by a direct sum of terms of the form $(\Sigma^\infty \mathbb{Z}(X))\{-n\}$.

Define $D^-_{\mathrm{Sp}}(S)$ to be the derived category of bounded-above complexes of spectra in $\mathrm{Sp}(S)$.

Proposition 10.19. Let $X, U \in \mathrm{Sm}/S$, and let $p : U \to X$ be a Nisnevich covering. For any natural number $i \in \mathbb{N}$, the Čech complex, defined by

$$(\Sigma^\infty \check{C}(U/X))\{-i\} : \cdots \longrightarrow \Sigma^\infty \mathbb{Z}(U \times_X U)\{-i\} \longrightarrow (\Sigma^\infty \mathbb{Z}(U))\{-i\} \longrightarrow (\Sigma^\infty \mathbb{Z}(X))\{-i\} \longrightarrow 0$$

is exact.

Proof. We have $((\Sigma^\infty \mathcal{F})\{-i\})_m = S_m \times_{S_{m-i}} (\mathcal{F} \otimes \mathbb{G}_m^{\wedge m-i})$. The functor $S_m \times_{S_{m-i}} -$ is exact and one can apply Theorem 2.35. $\square$

Definition 10.20. We say a spectrum $A \in \mathrm{Sp}'(S)$ is free if it is a direct sum of spectra of the form $(\Sigma^\infty \mathbb{Z}_S(X))\{-i\}$, which are precisely the generators of $\mathrm{Sp}'(S)$. We say A is projective if it is as direct summand of a free spectrum in $\mathrm{Sp}'(S)$.

Similarly, a spectrum $A \in \mathrm{Sp}(S)$ is free (respectively, projective) if it is the sheafification of a free (respectively, projective) spectrum in $\mathrm{Sp}'(S)$.

A bounded-above complex of spectra is free (respectively, projective) if the term on each degree is free (respectively, projective).

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