MATH 214B Notes

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1 Lecture 1

Globally, every divisor (codimension-1 subvariety) of $\mathbb{P}^n$ is defined by a single polynomial hom in $n+1$ variables.

Definition 1.1 (Complete Intersection)**.** A *complete intersection* in $\mathbb{P}^n_k$ is a subvariety of codimension-m that is defined by m equations.

A subscheme in $\mathbb{P}^n_k$ of codimension m is said to be set-theoretic a complete intersection if there is an ideal $I \subseteq K[x_0, \dots, x_n]$ generated by m elements whose vanishing locus is exactly the subvariety.

Remark 1.2 (Hartshorne's Conjecture)**.** Every closed curve in $\mathbb{P}^3$ is a set-theoretic complete intersection.

Example 1.3 (Normal Rational Curve of Degree n)**.**

$$\begin{aligned} \varphi_n : \mathbb{P}^1_k &\to \mathbb{P}^n_k \\ [s:t] &\mapsto [s^n : s^{n-1}t : \cdots : t^n] \end{aligned}$$

This is an embedding with $C_n := \varphi_n(\mathbb{P}^1_k)$.

Theorem 1.4 (Perron, 1941)**.** If $\frac{2^{n-1}}{n} \in \mathbb{Z}$, then C_n is a set-theoretic complete intersection of $n-1$ quadrics.

Theorem 1.5 (Gallarati-Rollero, 1988)**.** C_n is the set-theoretic complete intersection of $s-1$ quadrics and $n-s$ forms of degree $s+1, \dots, n$ where $s = \max\{k \in \mathbb{N} \mid 2^k \leq n\}$.

Therefore, although the case of codimension-1 is easy, it becomes increasingly difficult for higher codimensions.

Locally, every codimension-1 subvariety of a smooth variety is locally cut out by a single equation.

Example 1.6. The cone over $C_3 \hookrightarrow \mathbb{P}^3$ is given by

$$\mathrm{Cone}(C_3) \hookrightarrow \mathbb{A}^4,$$

where $\mathrm{Cone}(C_3)$ is a surface.

Remark 1.7. There is a correspondence (equivalence of categories) between projective varieties and germs of a singularity: given projective variety $X \hookrightarrow \mathbb{P}^n$, we obtain the germ $\bar{X} \hookrightarrow \mathbb{A}^{n+1}$ by looking at the cones; given a germ $\bar{X} \hookrightarrow \mathbb{A}^{n+1}$, we recover the projective variety by considering the collection of tangent directions.

The automorphisms (bijectives) of projective n-space (i.e., $\mathrm{Aut}(\mathbb{P}^n_k)$) is well-studied, and is known to be $\mathrm{PGL}(n+1)$. However, it is hard to study that for affine $\mathbb{A}^n_k$ spaces.

Theorem 1.8 (Lefschetz Principle, 1960s)**.** A first order logic proposition is true on $\mathbb{C}$ if it is true for $\mathbb{F}_p$ for infinitely many p's.

Theorem 1.9 (Ax-Grothendieck)**.** If $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ is a injective morphism, then it is surjective.

To do projective geometry, we have the following strategy: given a projective variety X, understand all its codimension-1 subvarieties.

Definition 1.10 (Irreducible Divisor)**.** Let X be an integral Noetherian scheme. An *irreducible divisor* on X is a closed irreducible subvariety of codimension 1.

Remark 1.11. If η_Y is the generic point of an irreducible subvariety $Y \hookrightarrow X$, then $\dim(\mathcal{O}_{X,Y}) = 1$ is a DVR with $v_Y : k(X)^* \to \mathbb{Z}$ such that $v_Y(f)$ is the order of vanishing of f at Y.

Example 1.12. $f = \frac{x}{y} \in k(\mathbb{A}^2_k)$, then the order of f at $x = 0$ is 1, and the order of f at $y = 0$ is -1.

Lemma 1.13. If X is an integral regular Northerian scheme with $f \in k(X)^*$, then $\mathrm{ord}_Y(f) = 0$ for all but finitely $Y \subseteq X$.

The *group of divisors* of X is the collection of finite formal sums $\sum\limits_{i \in I} \alpha_i D_i$ for $\alpha_i \in \mathbb{Z}$ and D_i are irreducible divisors. For $f \in k(X)^*$, $\mathrm{div}(f) = (f) = \sum\limits_{Y \subseteq X} \mathrm{ord}_Y(f) Y$. The group of divisors is denoted by $\mathrm{Div}(X)$ or Weil divisors.

Definition 1.14 (Principal Divisor)**.** A *principal divisor* is a divisor of the form $\operatorname{div}(f)$ for $f \in k(X)^*$. This forms a subgroup $\operatorname{Prin}(X)$ in the group of divisors, since $\operatorname{div}(fg) = \operatorname{div}(f) + \operatorname{div}(g)$.

Proof of Lemma. For $f \in k(X)^*$, there exists open subset $U \subseteq X$ such that $f\mid_U$ is regular. If $Y \cap U \neq \varnothing$, then $\operatorname{ord}_Y(f) \geq 0$, but $X \setminus U$ contains finitely many divisors. For the other part, consider f^{-1}. □

Remark 1.15. The order is well-defined from the condition.

Definition 1.16 (Divisor Class Group)**.** The *divisor class group* of X is defined to be $\operatorname{Div}(X)/\operatorname{Prin}(X)$.

Definition 1.17 (Linearly Equivalent)**.** Two divisors D_1 and D_2 on X are said to be *linearly equivalent* (denoted by $D_1 \sim D_2$) if $D_1 - D_2 = \operatorname{div}(f)$ for some rational function f.

We denote the class group of X by $\operatorname{Cl}(X)$.

Example 1.18. $\operatorname{Cl}(\mathbb{P}^n_k) = \mathbb{Z}$. Let $H_d \subseteq \mathbb{P}^n$ be the vanishing locus of f_d polynomial hom of degree d. At x_0 (the first coordinate), the function $\frac{f_d}{x_0^d}$ is rational, and $\operatorname{div}\left(\frac{f_d}{x_0^d}\right) = H_d - dH_0$.

2 Lecture 2

Proposition 2.1. Let A be a normal Noetherian domain. Let $U = \operatorname{Spec}(A)$. Then $\operatorname{Cl}(U) = 0$ if and only if A is a UFD.

Proof. First suppose A is a UFD. Let $Y \subseteq U$ be a codimension-1 subvariety. Y corresponds to a prime ideal of $\mathfrak{p} \subseteq A$ with height 1, hence $\mathfrak{p} = (f)$ for some $f \in A$. Thus, $Y = (f)$ in $\operatorname{Div}(U)$. □

Exercise 2.2. Describe the class group of $U_n = \operatorname{Spec}(k[x,y,z]/\langle xy - z^n\rangle)$. Compute $\operatorname{Cl}(U_n)$. How does it depend on n?

In fact, $\operatorname{Cl}(U_n) = \mathbb{Z}_n$.

Example 2.3. For any field k and $n \geq 1$, $\operatorname{Cl}(\mathbb{A}^n_k) = 0$.

Definition 2.4 (Factorial Variety)**.** A variety is called *factorial* if every divisor D is locally cut out by a single equation. In other words, for $p \in X$, there exists a neighborhood $U \ni p$ for which $[D|_U] = 0 \in \operatorname{Cl}(U)$.

A variety is called $\mathbb{Q}$-*factorial* if for every divisor D and $p \in X$, there exists a neighborhood U of p in X and an integer $m \neq 0$, for which $[mD|_U] = 0$ in $\operatorname{Cl}(U)$.

Remark 2.5. If $\mathrm{Cl}(X)$ is finitely-generated, then $\mathrm{Cl}(X) \simeq \mathbb{Z}^k \oplus T$ where T is finite, abelian, and torsion.

To construct a pullback for a map $f : X \to Y$ with respect to $D = \{g = 0\}$,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left] & Y \arrow[r, "g"] & \mathbb{P}^1 \end{tikzcd}$$

we define it by defining it locally with respect to the definition of D.

Remark 2.6. Suppose Y is $\mathbb{Q}$-factorial and D is on Y, and mD is *Cartier* (locally cut out by a single equation), then

$$f * D = f^*(mD)/m.$$

Theorem 2.7 (Algebraic Hartog Lemma)**.** Let R be a normal Noetherian domain, then

$$R = \bigcap \{R_p \mid p \subseteq R \text{ prime ideal of codimension-1}\} \subseteq \mathrm{Frac}(R).$$

Corollary 2.8. Let X be an affine normal integral Noetherian scheme, then a rational function f on X is regular if and only if $\mathrm{ord}_Y(f) \geq 0$ for each $Y \subseteq X$ is an irreducible divisor.

Corollary 2.9. Let X be a normal scheme and $Y \subseteq X$ be a closed subset of codimension ≥ 2. Then the restriction $\mathcal{O}(X) \to \mathcal{O}(X \backslash Y)$ is an isomorphism. In particular, $\mathcal{O}(\mathbb{A}_k^n \backslash \{0\}) \cong \mathcal{O}(\mathbb{A}_k^n) = k[x_1, \ldots, x_n]$.

Remark 2.10. Normal singularities are the "worst" class of singularities for which Hartog's lemma holds.

Example 2.11.

$$\{x = y = 0\} \cup \{z = w = 0\} \subseteq \mathbb{A}_k^4$$

is not normal.

Theorem 2.12. Let k be a field and $n \geq 1$, then $\mathrm{Cl}(\mathbb{P}_k^n) \cong \mathbb{Z}$ and is generated by the class of a hyperplane.

Proof. Consider

$$\begin{aligned} \psi : \mathbb{Z} &\to \mathrm{Cl}(\mathbb{P}_k^n) \\ 1 &\mapsto [H] \end{aligned}$$

We first show surjectivity. Let $Y \subset \mathbb{P}_k^n$ be a codimension-1 closed subvariety that corresponds to a homogeneous prime ideal $p \subseteq k[x_0, \ldots, x_n]$, then $p = (f)$ because we are in a

UFD, so $\deg(f) = d$ and $Y = \{f = 0\} \subseteq \mathbb{P}^n_k$. Let $H = \{x_0 = 0\} \subseteq \mathbb{P}^n_j$, then $\frac{f}{x_0^d}$ is rational on $\mathbb{P}^n_k$ and $\left(\frac{f}{x_0^d}\right) = Y \setminus dH$, thus $Y = dH$.

We now show injectivity. Assume $dH \sim 0$ and $d > 0$, then $dH = (f)$. Since all the orders of vanishing of the rational function f are ≥ 0, by corollary of the algebraic Hartog Lemma, f must be regular. Therefore, f must be a non-zero constant, contradiction. $\square$

Theorem 2.13 (Auslander–Buchsbaum, 1950s)**.** A regular local ring is a UFD.

Definition 2.14 (Sheaf associated to a Divisor)**.** Let D be a divisor on a normal Noetherian scheme X. Consider open set $U \subseteq X$, then

$$\mathcal{O}_X(D)(U) = \{f \in k(X) \mid (\mathrm{div}(f) + D)|_U \geq 0\}$$

gives a coherent sheaf of $\mathcal{O}_X$-modules. Furthermore, $\mathcal{O}_X(0) = \mathcal{O}_X$. The *sheaf associated to* D is denoted $\mathcal{O}_X(D)$.

Proposition 2.15. If divisors $D_1 \sim D_2$, then $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$ as $\mathcal{O}_X$-modules.

Proof. $D_1 \setminus D_2 = (\varphi)$ for $\varphi \in k(X)^*$, then there is an isomorphism

$$\begin{gathered} \mathcal{O}_X(D_1) \xrightarrow{\cong} \mathcal{O}_X(D_2) \\ f \mapsto fg \end{gathered}$$

. $\square$

Proposition 2.16. If X is a regular scheme, then $\mathcal{O}_X(D)$ is a line bundle for every D.

Proof. Fix $p \in X$, and write $D = \sum a_i D_i$, so $D_i = (f_i)$ near p. Then $D \sim 0$ near p, hence $\mathcal{O}_X(D) \cong \mathcal{O}_X$ near p, so $D = \mathrm{div}(f_i^{a_i})$. $\square$

Theorem 2.17. Let X be a normal Noetherian integral scheme, then there is an injective homomorphism (first Chern class)

$$c_1 : \mathrm{Pic}(X) \to \mathrm{Cl}(X).$$

Moreover, if X is regular, then c_1 is an isomorphism.

Remark 2.18. To construct c_1, consider $\mathcal{L} \to X$ where $\mathcal{L}$ is a line bundle and let s be a rational section of $\mathcal{L}$. We think of s as an element $\mathcal{L}_{\eta_X}$ defined by $(s) = \sum_{Y \subseteq X} \mathrm{ord}_Y(s) Y \in \mathrm{Div}(X)$. Note that $\mathcal{L}_{\eta_Y}$ is free of rank-1 over $\mathcal{O}_{X,\eta_Y} = \mathcal{P}_{X,Y}$, then $\mathrm{ord}_Y(s) = -\min\{r \in \mathbb{Z} \mid t^r s \in \mathcal{L}_Y\}$, then t is a local uniformizer of $\mathcal{O}_{X,Y}$. Now $(fs) = (s) + (f)$ for any $f \in k(X)^*$, so this is a homomorphism.

3 Lecture 3

Proof. Let X be normal and $\mathcal{L}$ be a line bundle over X, then $c_1(\mathcal{L}) = 0$. Let s be a rational section of $\mathcal{L}$, then $(s) = (f)$ for some $f \in k(X)$. Define $t = \frac{s}{f}$ of $\mathcal{L}$, so $(t) = 0$. Since $(t) \geq 0$, we have a morphism

$$\begin{tikzcd} \mathcal{L}(\mathcal{O}_X) \arrow[rr, "t"] \arrow[dr] & & \mathcal{L} \arrow[dl] \\ & X & \end{tikzcd}$$

Since $(t) \leq 0$, we have a morphism

$$\begin{tikzcd} \mathcal{L} \arrow[rr, "t^{-1}"] \arrow[dr] & & \mathcal{L}(\mathcal{O}_X) \arrow[dl] \\ & X & \end{tikzcd}$$

This proves injectivity. Suppose X is regular furthermore, then for every divisor $Y \subseteq X$, $\mathcal{O}(Y)$ is a line bundle, then $c_1(\mathcal{O}(Y)) = Y$. $\square$

Proposition 3.1. Let X be a smooth proper curve over k. Let $f : X \to Y$ be a morphism over k. Then, either

1. $f(X) = p$, or

2. $f : X \to Y$ is a surjective finite morphism, $k(X) \supseteq k(Y)$ is a finite field extension, and Y is proper over k.

Lemma 3.2. Let $f : X \to Y$ be a dominant rational map of varieties of the same dimension over k, then $k(X) \supseteq k(Y)$ is finite.

Lemma 3.3 (Hartshorne, 4.4)**.** If $f : X \to Y$ is a morphism over k, where X and Y are over k. If X is proper over k and Y is separated over k, then $f(X)$ is proper over k.

Proof of Proposition. Observe that $f(X)$ is closed and irreducible. Suppose $f : X \to Y$ is surjective, and $V = \mathrm{Spec}(B)$ is open in Y. Here, $B \subseteq K(Y)$ and in fact $\mathrm{Frac}(B) = k(Y)$. Let A be an integral closure of B in $k(X)$. Since $k(X) \supseteq k(Y)$ is finite, we conclude that A is a finitely-generated k-algebra and a domain. Denote $U = \mathrm{Spec}(A)$. We obtain a finite morphism $f : U \to V$ as open sets, where U is a smooth curve over k and X is its unique proper model. If $f^{-1}(V) = U$, then we are done. If this was true, then for any point $x \in X$, there is a neighborhood on which f is finite, which is a local property. Let U_0 be adding a point onto U (i.e., supposing there is some point not defined on U that cannot attain image in V). The valuative criteria for properness implies that $U_0 \dashrightarrow U$ (inclusion map) extends to a rational morphism $U_0 \to U$, which implies $f^{-1}(V) = U$. $\square$

Proposition 3.4 (Algebraic Version of Liouville's Theorem)**.** Let X be a projective variety over k, and $f \in k(X)^*$ for which $(f) \geq 0$, then f is constant.

Proof. Consider $X \hookrightarrow \mathbb{P}^N_k$. There is a map $f : X \dashrightarrow \mathbb{P}^1_k$ that is defined at codimension-1 points. Let Z be the loci where f f is not defined and $n = \dim(X)$. Let $H_1, \ldots, H_{n-1}$ be general hyperplanes on $\mathbb{P}^N_k$ with $\operatorname{codim}(Z, X) \geq 2$. Then $C := (H_1 \cap \cdots \cap H_{n-1} \cap X) \cap Z = \varnothing$. We obtain a restricted map $f|_C : C \to \mathbb{P}^1_k$, but since $(f) \geq 0$ so $(f|_C) \geq 0$. Therefore, f is constant. $\square$

Definition 3.5 (Degree)**.** Let $f : X \dashrightarrow Y$ be a dominant rational map of varieties of the same dimension, then $\deg(f) = [k(X) : k(Y)] < \infty$. If X is a curve and $p \in X$ is a point, then the *degree* of the point is $[k(p) : k] < \infty$. If k is algebraically closed and $p \in X$ is closed, then $\deg(p) = 1$.

Suppose $D = \sum_{i \in I} \alpha_i p_i$, where $\alpha_i \in \mathbb{Z}$ and $p_i \in C$be closed points in C over k. Then the *degree* of the divisor is defined by $\deg(D) = \sum_{i \in I} \alpha_i \deg(p_i)$.

Definition 3.6 (Pullback)**.** Let $f : X \to Y$ be a finite morphism of smooth curves over k, then there is a map $f^* : \operatorname{Div}(Y) \to \operatorname{Div}(X)$ defined by $f^*(p) = \sum_{q \in f^{-1}(p)} \alpha_q \cdot q$. For $t \in \mathcal{O}_{X,p}$ a uniformizer (i.e., $\operatorname{ord}_p(f) = 1$), then $f^*(t) = v^{\alpha_q}$ in $\mathcal{O}_{Y,q}$ where v is a uniformizer of $\mathcal{O}_{Y,q}$.

Example 3.7. Consider $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ defined by $[s : t] \mapsto [s^2 : t^2]$. The pullback satisfies $f^*[1 : 0] = 2[1 : 0]$ and $f^*[1 : 1] = [1 : -1] + [1 : 1]$.

Theorem 3.8 (Degree of Pullback)**.** Let $f : X \to Y$ be a finite morphism between smooth curves over k. Let $D \in \operatorname{Div}(Y)$, then $\deg(f^*D) = \deg(f) \cdot \deg(D)$.

Proof. First suppose $D = p$ is closed in Y. (Note that the linearity of both sides takes care of the general case.) Suppose $V = \operatorname{Spec}(B)$ and $f^{-1}(V) = \operatorname{Spec}(A) = U$, and A is finite over B. Consider $A \otimes_B \mathcal{O}_{Y,p}$ be a module over $\mathcal{O}_{Y,p}$ is a finitely-generated torsion-free $\mathcal{O}_{Y,p}$-module, free of rank d. So $A \otimes_B k(Y)$ is free of rank d over $k(Y)$ and $d = \deg(f)$. Thus, $A \otimes_B k(p)$ is free of rank d over $k(p)$. This implies that $f^*(p) = (\deg(f))(\deg(p))$. $\square$

Corollary 3.9. Let X be a smooth proper curve over k and $f \in k(X)^*$, then $\deg((f)) = 0$.

Proof. Consider $f : X \dashrightarrow \mathbb{P}^1_k$. Since X is smooth and $\mathbb{P}^1_k$ is proper, then f extends to a morphism $f : X \to \mathbb{P}^1_k$. If f is constant, then $(f) = 0$. If f is not constant, it must be a finite surjective morphism of degree d by Proposition 3.1. Moreover, by Theorem 3.8, $\deg(f^*(0)) = \deg(f) = \deg(f^*(\infty))$, and $(f) = f^*(0) - f^*(\infty)$, so $\deg((f)) = 0$. $\square$

Remark 3.10. This defines a degree map $\deg : \operatorname{Cl}(X) \cong \operatorname{Pic}(X) \to \mathbb{Z}$ for any smooth curve X over k, which agrees with the concept in complex geometry.

4 Lecture 4

Consider the projective morphisms, usually given by

$$X \to \mathbb{P}^n_A = \mathrm{Proj}(A[x_0, \dots, x_n])$$

which is basically equivalent to $\mathcal{L}$ on X and global sections $s_1, \dots, s_n \in \Gamma(X, \mathcal{L})$.

Let A be a ring, consider $\mathbb{P}^n_A$ projective, and a scheme X over A. There is a map $\varphi : X \to \mathbb{P}^n_A$ as sending line bundles $\varphi^*(\mathcal{O}(1))$ (such that $\varphi^*(x_i) = s_i \in \Gamma(X, \mathcal{L})$ as described) to line bundles $\mathcal{O}(1)$ generated by global sections $x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$.

Theorem 4.1. Let A be a ring and X be a scheme over A. If $\mathcal{L}$ describes line bundles on X and $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ generating $\mathcal{L}$. Then there exists a unique $\varphi : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong \varphi^*(\mathcal{O}(1))$ and $s_i = \varphi^*(x_i)$.

Proof. Consider $X_i = \{P \in X \mid (s_i)_p \notin m_p \mathcal{L}_p\}$ which are open in X. Since s_i's generate $\mathcal{L}$, then $X = \bigcup X_i$. Define $U_i = \{x_i \neq 0\}$, then $U = \mathrm{Spec}(A[y_0, \dots, \tilde{y}_i, \dots, y_n] = \mathrm{Spec}\left(A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]\right)$. Therefore, a map $X_i \to U_i$ is equivalent to a morphism $A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \to \Gamma(X_i, \mathcal{O}_i)$ global sections, where the map is defined by $\frac{x_j}{x_i} \mapsto \frac{s_j}{s_i}$ and thus gives a morphism ψ in the global section. In particular, these maps glue together. To see uniqueness, the morphism has to be exactly ψ locally on X_i. Therefore, φ is unique. $\square$

Remark 4.2 (Automorphisms of $\mathbb{P}^n_k$)**.** Let k be a field, consider the $n \times n$-matrices in $\mathrm{GL}_k(n+1)$. This gives $\mathrm{Aut}(k[x_0, \dots, x_n])$ and therefore gives $\mathrm{Aut}(\mathbb{P}(k[x_0, \dots, x_n])$. Note that $M \sim N$ gives the same automorphism if and only if $M = \lambda N$ for $\lambda \in K \backslash \{0\}$. Therefore, if $M = \lambda N$ then they have the same automorphism group of rings; if $M \neq \lambda N$, then use the coordinates and look at points $(1 : 0 : \cdots : 0), (0 : 1 : 0 : \cdots : 0), \dots, (0 : \cdots : 0 : 1)$. Hence, $\mathrm{PGL}_k(n+1) = \mathrm{GL}_k(n+1)/\lambda < \mathrm{Aut}(\mathbb{P}^n_k)$. On the other hand, let $\varphi : \mathbb{P}^n_k \to \mathbb{P}^n_k$ be an automorphism of projective space, then note that $\mathrm{Pic}(\mathbb{P}^n_k) \cong \mathbb{Z}$ with generators $\mathcal{O}(1)$. Thus, φ induces $\varphi^* : \mathrm{Pic}(\mathbb{P}^n_k) \to \mathrm{Pic}(\mathbb{P}^n_k)$ defined by $\mathcal{O}(1) \mapsto \varphi^*(\mathcal{O}(1))$. Now $\varphi^*(\mathcal{O}(1))$ is either $\mathcal{O}(1)$ or $\mathcal{O}(-1)$, i.e., has no global sections. Therefore, $\varphi^*(\mathcal{O}(1)) = \mathcal{O}(1)$. Therefore, the global sections of the sheaf $\Gamma(\mathbb{P}^n_k, \mathcal{O}(1))$ is a k-vector space with basis $x_0, \dots, x_n$. The morphisms to $\mathbb{P}^n_k$ corresponds to the global sections s_i's. Each $\varphi^*(\lambda_0) = s_i = \sum a_i x_i$. Hence, the only choice that corresponds to picking the morphism is this matrix, and corresponds to the original construction.

Proposition 4.3. Let A be a ring and let $\varphi : X \to \mathbb{P}^n_A$ corresponds to $\mathcal{L}$, with $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$. Then φ is a closed immersion if

1. $X_i = \{p \in X \mid (s_i)_p \notin m_p \mathcal{L}_p\}$ is affine, and

2. $A[y_0, \ldots, y_n] \to \Gamma(x_i, \mathcal{O}_i)$ are surjective maps.

Proof. If φ is a closed immersion, with $X \subseteq \mathbb{P}^N_A$. Let $X_i = X \cap U_i$ be closed subschemes of U_i, an affine scheme. In particular, the closed subschemes of $\operatorname{Spec}(B)$, i.e., $\operatorname{Spec}(B/b) \hookrightarrow \operatorname{Spec}(B)$, should correspond to $B \twoheadrightarrow B/b$, hence 1) and 2). Also, given 1) and 2), we know that $X_i = \operatorname{Spec}(C)$, then $\Gamma(X_i, \mathcal{O}_i) = C$, therefore corresponds to a map $A[x_0, \ldots, x_n] \twoheadrightarrow C = A[x_0, \ldots, x_n]/b$, therefore corresponds to a closed subscheme of $U_i \subseteq \mathbb{P}^N_A$, hence gives a closed subscheme $X \subseteq \mathbb{P}^N_A$, as the inclusion glues to one inclusion. $\square$

Proposition 4.4. Let $k = \bar{k}$ and X be a projective scheme over k. Define $\varphi : X \to \mathbb{P}^n_k$, and consider $\mathcal{L}$ with $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$. Let $V \subseteq \Gamma(X, \mathcal{L})$ be generated by $s_0, \ldots, s_n$. Now φ is a closed immersion if

1. V separates points, i.e., for P, Q closed, there exists $s \in V$ such that $s \in m_P \mathcal{L}_P$ but $s \notin m_Q \mathcal{L}_Q$ (and vice versa), and
2. V separates tangent vectors, i.e., for P closed, $\{s \in V \mid s_p \in m_p \mathcal{L}_p\}$ spans $m_p \mathcal{L}_p / m_p^2 \mathcal{L}_p$.

Remark 4.5. Geometrically, this comes from hyperplanes in $\mathbb{P}^n$ separating points on tangent directions. On the other hand, φ as a closed map gives a surjection $\mathcal{O}_{\mathbb{P}^n} \twoheadrightarrow \varphi_* \mathcal{O}_X$, which comes from an algebraic lemma, and the closedness comes from properness since X is projective.

5 Lecture 5

Proof. ($\Rightarrow$): Separate by hyperplanes on $\mathbb{P}^n_k$.

($\Leftarrow$): Since X is projective over k, then it is proper over k, so $\varphi(X) \subseteq \mathbb{P}^n$ is closed and so φ is proper. Therefore, φ is a closed map. It now suffices to show that $\mathcal{O}_{\mathbb{P}^n} \twoheadrightarrow \varphi_* \mathcal{O}_X$, which can be shown on stalks, by using the following lemma on the immersion case. $\square$

Lemma 5.1. If $f : A \to B$ is a local homomorphism of locally Noetherian rings. If

1. $A/m_A \to B/m_B$ is an isomorphism,
2. $m_A \to m_B/m_B^2$ is a surjection, and
3. B is a finitely-generated A-module,

then $f : A \twoheadrightarrow B$ is a surjection.

Definition 5.2. Let $\mathcal{L}$ be a line bundle on X. We say $\mathcal{L}$ is *very ample* with respect to Y if there exists an immersion $i : X \to \mathbb{P}^n_Y$ such that $\mathcal{L} \cong i^*(\mathcal{O}(1))$.

Remark 5.3. We say examples of very ample line bundles and criterions over $\mathrm{Spec}(k)$.

Proposition 5.4. Let X be a projective space over A, and $\mathcal{L}$ a very ample line bundle over $\mathrm{Spec}(A)$. For all coherent sheaves $\mathcal{F}$ on X, there exists $n_0 > 0$ such that $n \geq 0$ and that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections.

Definition 5.5. Let $\mathcal{L}$ be a line bundle on X. We say $\mathcal{L}$ is *ample* if for all coherent sheaves $\mathcal{F}$, there exists $n_0 > 0$ such that $n \geq n_0$ and $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections.

Proposition 5.6. Let $\mathcal{L}$ be a line bundle on X. The following are equivalent:

1. $\mathcal{L}$ is ample,
2. $\mathcal{L}^m$ is ample for any $m > 0$,
3. $\mathcal{L}^m$ is ample for some $m > 0$.

Proof. (2) $\Rightarrow$ (3): pick $m = 1$.

(1) $\Rightarrow$ (2): from definition.

(3) $\Rightarrow$ (1): Suppose $\mathcal{L}^m$ is ample for some coherent sheaf $\mathcal{F}$. Then there exists n_0 such that whenever $n > n_0$, $\mathcal{F} \otimes (\mathcal{L}^m)^n = \mathcal{F} \otimes \mathcal{L}^{mn}$ is generated by global sections. But $\mathcal{F} \otimes \mathcal{L}$ is coherent since $\mathcal{L}$ is a line bundle, so there exists n_1 such that for any $n > n_1$, $(\mathcal{F} \otimes \mathcal{L}^n) \otimes \mathcal{L}^{mn}$ is generated by global sections. Continuing for all $0 \leq i < n$, we have $\mathcal{F} \otimes \mathcal{L}^i$ and can pick some n_i correspondingly, such that $\mathcal{F} \otimes \mathcal{L}^{mn+i}$ is generated by global sections. Therefore, for $\mathcal{F}$ and $\mathcal{L}$, pick $(\max_i n_i) \cdot m =: N$, so $n \geq N$, therefore $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections, by some case in the argument above. □

Theorem 5.7. Let X be a scheme of finite type over a Noetherian ring A, and $\mathcal{L}$ is a line bundle on X. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^m$ is very ample over $\mathrm{Spec}(A)$ for some $m > 0$.

Proof. Suppose $\mathcal{L}^m$ is very ample, then it is ample, and so $\mathcal{L}$ is ample.

Suppose $\mathcal{L}$ is ample, then for all $P \in X$, let there is an open affine neighborhood $P \in U \subseteq X$ such that $\mathcal{L}|_U$ is free. Let $\bar{Y} := X \backslash U$, and let $\mathcal{I}_Y$ be a sheaf of ideals of Y. Since $\mathcal{L}$ is ample, then for some n we know $\mathcal{I}_Y \otimes \mathcal{L}^n$ is generated by global sections. Therefore, there exists $s \in \Gamma(X, \mathcal{I}_Y \otimes \mathcal{L}^n)$ such that $s_p \notin m_p(\mathcal{I} \times \mathcal{L}^n)_p$. We can think of $s \in \Gamma(X, \mathcal{L}^n)$ since the global section $\Gamma(X, \mathcal{I}_Y \otimes \mathcal{L}^n)$ is a subsheaf of $\mathcal{O}_X$. Define $X_s = \{Q \in X \mid s_Q \notin m_Q(\mathcal{I}_Y \otimes \mathcal{L}^n)_Q\}$, then $P \in X_s \subseteq U$ such that $\mathcal{L}_U$ is trivial, and s induces a section $f \in \Gamma(U, \mathcal{O}_U)$. So we have $X_s = U_{\mathcal{F}}$ is also affine. Therefore, for all $P \in X$, there exists $n > 0$ and $s \in \Gamma(X, \mathcal{L}^n)$ such that $P \in X_s$ affine is of finite type over Noetherian ring, so there exists $N > 0$ such that for all $P \in X$ and affine $P \in X_s$ with $s \in \Gamma(X, \mathcal{L}^N = \mathcal{L}^1)$, and $X_1, \ldots, X_k$ over X_s.

Let $B_i = \Gamma(X_i, \mathcal{O}_i)$, all B_i are finitely-generated A-algebras with generators $\{b_{ij}\}$ over B_i. By definition of X_i, for all $b_{ij} \in \Gamma(X_i, \mathcal{O}_i)$, there exists some m such that $s_i^m \cdot b_{ij} \in \Gamma(X, \mathcal{L}_1^m)$. Pick M such that for all i, j, $s_i^M b_{ij} \in \Gamma(X, \mathcal{L}_1^M)$, so $\mathcal{L}_1^M$ are line bundles and sections $\{\{s_1^M, \ldots, s_n^M\}, \{s_i^M b_{ij}\}\}$, which defines a morphism to $\mathbb{P}_A$ as an immersion. $\square$

6 Lecture 6

Let X be projective and non-singular, and k is algebraically closed. The line bundles $\mathcal{L}$ are in one-to-one correspondence with the classes of of divisors.

Let $s \in \Gamma(X, \mathcal{L})$. Then $D = (s)_0$ is the divisor of zeros of s. For all $U \subseteq X$, where $\mathcal{L}$ is trivial, there is a map $\varphi : \mathcal{L}|_U \cong \mathcal{O}(U)$ where $\varphi(s) \in \Gamma(U, \mathcal{O}_U)$. We call $\{U, \varphi(s)\}$ an effective Cartier divisor.

Proposition 6.1. Let D_0 be a divisor and $\mathcal{L} = \mathcal{L}(D_0)$.

(a) For all $s \in \Gamma(X, \mathcal{L})$, $(s)_0 \sim D_0$.

(b) For all D effective such that $D \sim D_0$, there exists $s \in \Gamma(X, \mathcal{L})$ such that $(s)_0 = D$.

(c) For any $s, s' \in \Gamma(X, \mathcal{L})$, $(s)_0 = (s')_0$ if and only if there exists $\lambda \in k^*$ such that $s' = \lambda s$.

Definition 6.2 (Complete Linear System)**.** For D_0, all the effective divisors $D \geq 0$ such that $D \sim D_0$ form a collection called *complete linear system* $|D_0|$. There is a one-to-one correspondence between $|D_0|$ and $(\Gamma(X, \mathcal{L}) \setminus \{0\})/k^*$.

Linear systems are k-linear subsets of complete linear systems.

Example 6.3. For a line bundle $\mathcal{L}$, if we take some $s_1, \ldots, s_n$, they form a linear system.

The conics of $\mathbb{P}^2$ form a complete linear system.

Definition 6.4 (Basepoint)**.** Let $|D_0|$ be a complete linear system and ∂ be a linear system in it. The *basepoint* ∂_0 is a linear system of $P \in X$ such that $P \in \operatorname{supp}(D)$ for all $D \in \partial$. We say a linear system is *basepoint-free* if it has no basepoints.

We say $\mathcal{L}$ is *basepoint-free* if the complete linear system given by $s_1, \ldots, s_n$ is basepoint free. The line bundle being basepoint-free implies that there is a morphism to the projective space.

Definition 6.5. We say $\mathcal{L}$ is *semi-ample* if $\mathcal{L}^{\otimes n}$ is generated by global sections for some $n \geq 0$.

$\mathcal{L}$ is *nef* if for all curves $C \subseteq X$, $\deg(\mathcal{L}|_C) \geq 0$.

Remark 6.6. Very ample implies ample and globally generated. Ample implies semi-ample. Globally generated implies semiample and nef.

Definition 6.7 (Proj Bundle)**.** Let $\mathscr{J}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Let λ be the sheaf, then $\mathscr{J}$ is the sheaf of graded $\mathcal{O}_X$-algebras, so $\mathscr{J} \equiv \bigoplus_{d \geq 0} \mathscr{J}_d$. Therefore, $\mathscr{J}_0 = \mathcal{O}_X$, $\mathscr{J}_1$ is coherent, and $\mathscr{J}$ is locally generated by $(\mathscr{J}_1)$ as $\mathcal{O}_X$-algebra. For all $U = \operatorname{Spec}(A) \subseteq X$, we define $\mathscr{J}(U) = \Gamma(U, \mathscr{J}|_U)$, which defines a proj of graded ring and morphism $\operatorname{Proj}(\mathscr{J}(U)) \to U$. By gluing the morphisms, we obtain $\pi : \operatorname{Proj}(\varphi) \to X$. For all $\operatorname{Proj}(\varphi(U))$ we have $\mathcal{O}(1)$ and they glue to $\mathcal{O}(1)$ on $\operatorname{Proj}(\varphi)$.

Proposition 6.8. Let X be a space and $\mathscr{J}$ be a sheaf of graded $\mathcal{O}_X$-algebras.

(a) π is proper,

(b) if there exists $\mathcal{L}$ ample line bundle on X and π is projective, then $\mathcal{O}_{\operatorname{Proj}(\varphi)} \otimes \pi^* \mathcal{L}^n$ is very ample for some $n > 0$.

Definition 6.9 (Symmetric Algebra)**.** Let X be a Noetherian scheme and $\mathcal{E}$ be a locally free coherent sheaf. Then $\varphi = S(\mathcal{E})$ is a *symmetric algebra* of $\mathcal{E}$ and $\mathbb{P}(\mathcal{E}) = \operatorname{Proj}(\varphi)$.

Definition 6.10 (Blow Up)**.** Let X be a space and $\mathscr{J}$ a coherent sheaf of ideals. Denote $\varphi = \bigoplus_{d \geq 0} \mathscr{J}^d$ with $\mathscr{J}^0 = \mathcal{O}_X$. Then $\tilde{X} = \operatorname{Proj}(\varphi)$ is the *blow up* with respect to $\mathscr{J}$.

Remark 6.11. Classically blow up is defined at a point or at a closed subvariety $Y \subseteq X$, denoted $\operatorname{Bl}_Y(X)$. This corresponds to $\tilde{X} = \operatorname{Proj}(\varphi)$ when $\mathscr{J}$ corresponds to Y.

7 Lecture 7

Definition 7.1. Let X be a topological space and $\mathcal{A}$ is an abelian group. We denote by $\mathcal{A}_x$ the constant sheaf on X with values on $\mathcal{A}$, defining $\mathcal{A}_X(U)$ to be the locally constant functions $f : U \to \mathcal{A}$. We just write $H^i_{\text{Sheaf}}(X, \mathcal{A})$ instead of $H^i_{\text{Sheaf}}(X, \mathcal{A}_X)$ for the cohomology groups.

Remark 7.2. Whenever X is a topological manifold or it has the structure of a CW complex, we have that

$$H^i_{\text{Sheaf}}(X, \mathcal{A}) \cong H^i_{\text{Sing}}(X, \mathcal{A}).$$

Note that $H^0_{\text{Sheaf}}(X, \mathcal{A})$ is the set of locally constant functions $f : X \to \mathcal{A}$, and $H^0_{\text{Sing}}(X, \mathcal{A})$ is the set of functions from X to $\mathcal{A}$.

Example 7.3. Consider the cantor set $\mathcal{A} = \{0, 1\}^{\mathbb{N}}$, so

$$H^0_{\text{Sing}}(\mathcal{C}, \mathbb{Z})$$

is the set of functions $\mathcal{C}$ to $\mathbb{Z}$, whereas

$$H^0_{\text{Sing}}(\mathcal{C}, \mathbb{Z})$$

is the set of locally constant functions $\mathcal{C} \to \mathbb{Z}$. Therefore, the first set has cardinality $2^{2^{\mathbb{N}}}$, and the second set has cardinality $2^{\mathbb{N}}$.

Definition 7.4. The *sheaf cohomology* is the right derived functor of the functor from the sheaves of Abelian groups on X to the abelian groups, given by $\mathcal{E} \mapsto \mathcal{E}(X)$.

Definition 7.5. An *abelian category* $\mathcal{Q}$ is a category such that for any objects $A, B \in \mathbb{Q}$, $\mathbf{Hom}(A, B)$ is given an abliean group structure with bilinear map $(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$.

Remark 7.6. Finite direct sums, coproducts, kernels, and cokernels all exist in abelian categories.

Every monomorphsim is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel.

Any morphism can be factored as an epimorphism and then a monomorphism.

Example 7.7.

- R-modules over a left Noetherian ring R.
- $\text{Shv}(X)$, the sheaves of abelian groups on a topologicla space X.
- $\text{Shv}(\mathcal{O}_X)$, the $\mathcal{O}_X$-modules over ringed spaces.
- $\text{Qcho}(X)$, the quasi-coherent sheaves on schemes.
- $\text{Coh}(X)$, the coherent sheaves on Noetherian schemes.

Definition 7.8. Let $\mathcal{Q}$ be an abelian category. A *cochain complex* on $\mathcal{Q}$ is a sequence of maps

$$A := \cdots \to A^i \xrightarrow{d} A^{i+1} \xrightarrow{d} A^{i+2} \xrightarrow{d} \cdots$$

where $A^i \in \mathbb{Q}$ and the d's are morphisms in the category such that $d^2 = 0$.

To the cochain complex A, we can associate cohomological objects as

$$H^i(A) = \ker(d : A^i \to A^{i+1}) / \operatorname{im}(d : A^{i-1} \to A^i).$$

We say that a cochain complex in $\mathcal{Q}$ is exact if $H^*(A) = 0$.

A *chain map* of maps of complexes is a commutative diagram is a series of commutative squares between two chain complexes.

If $f : A \to B$ is a chain map, we get an induced homomorphism of cohomology groups $f_* : H^i(A) \to H^i(B)$ for each $i \in \mathbb{Z}$. A homotopy between maps $f_*, g_* : H^i(A) \to H^i(B)$ is a sequence of morphisms F making the square commutes:

$$\begin{tikzcd}
A^{i-1} \arrow[r, "d"] \arrow[d] & A^i \arrow[r, "d"] \arrow[d] \arrow[dl] & A^{i+1} \arrow[d] \arrow[dl] \\
B^{i-1} \arrow[r, "d"] & B^i \arrow[r, "d"] & B^{i+1}
\end{tikzcd}$$

such that $dF + Fd = f - g$.

We say that f is homotopic to g, written as $f \sim g$, if there is a homotopy from f to g.

Remark 7.9. If $f \sim g$, then $f_* = g_* : H^i(A) \to H^i(B)$ for every $i \in \mathbb{Z}$.

Exercise 7.10. Find two maps f, g not homotopic with $f_* = g_*$ for every $i \in \mathbb{Z}$.

Lemma 7.11. A short exact sequence of cochain complexes induce a long exact sequence of cohomology groups: given $0 \to A \to B \to C \to 0$ a short exact sequence, there is a long exact sequence

$$\cdots \to H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \cdots$$

Lemma 7.12. Let X be an integral Noetherian scheme and let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of cohomology sheaves on X. Then we have a long exact sequence

$$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^0(X, \mathcal{H}) \to H^1(X, \mathcal{F}) \to \cdots$$

Definition 7.13. Let $X \subseteq \mathbb{P}^n$ be a smooth projective variety, and let $\omega_X = \mathcal{O}_X(K_X) = \bigwedge^n \Omega_x$ where $\Omega_X = T_X^v$. The *canonical ring* is denoted

$$\bigoplus_{m \in \mathbb{Z}} H^0(X, \omega_X^{\otimes m}).$$

Lemma 7.14. Let S be a smooth divisor on X and $\omega_S = \omega_X(S)$. Then there is a short sequence

$$0 \to \omega_X \to \omega_X \otimes S \to \omega_X \otimes S|_S \to 0$$

where $\omega_X \otimes S|_S \cong \omega_S$.

8 Lecture 8

Throughout this lecture we consider abelian categories.

Definition 8.1 (Exact Functor)**.** A functor $F : \mathcal{A} \to \mathcal{B}$ is said to be *additive* if the induced morphism $F : \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{B}}(FX, FY)$ is a homomorphism of abelian groups for all X, Y.

An additive functor is *left exact* if $0 \to FA \to FB \to FC$ is exact whenever $0 \to A \to B \to C \to 0$ is exact. Likewise, we say the functor is right exact if $FA \to FB \to FC \to 0$ is exact whenever $0 \to A \to B \to C \to 0$ is exact.

An additive functor is exact if it is both left and right exact.

Example 8.2. For abelian category $\mathcal{A}$ and $X \in \mathcal{A}$, then

$$\begin{aligned} \mathcal{A} &\to \mathbf{Ab} \\ Y &\mapsto \mathbf{Hom}(X, Y) \end{aligned}$$

is left exact, and

$$\begin{aligned} \mathcal{A}^{\mathrm{op}} &\to \mathbf{Ab} \\ Y &\mapsto \mathbf{Hom}(Y, X) \end{aligned}$$

is left exact.

Example 8.3. For $\mathcal{A} = \mathbf{Shv}(X)$, the category of abelian sheaves on a topological space X, then $\mathcal{E} \to \mathcal{E}(X)$ is a left exact functor and $\mathcal{E}(X) = \mathbf{Hom}_{\mathbf{Shv}(X)}(\mathbb{Z}_X, \mathcal{E})$.

Definition 8.4 (Injective Object)**.** An object I in an abelian category $\mathcal{A}$ is *injective* if $\mathbf{Hom}(-, I)$ is exact. Equivalently, for any monomorphism $A \hookrightarrow B$ in $\mathcal{A}$, every $A \to I$ extends to a map $B \to I$.

Analogously, P is projective if $\mathbf{Hom}(P, -)$ is exact. Equivalently, $B \twoheadrightarrow C$ gives a surjection $\mathbf{Hom}(P, B) \to \mathbf{Hom}(P, C)$.

Definition 8.5 (Resolution)**.** An injective *resolution* of an object $A \in \mathcal{A}$ is a complex

$$0 \to I^0 \to I^1 \to \cdots$$

of injective objects with a map $A \to I^0$ such that the induced complex

$$0 \to A \to I^0 \to I^1 \to \cdots$$

is exact. That is,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & A \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots
\end{tikzcd}$$

is a quasi-isomorphism, i.e., induces an isomorphism of cohomology groups.

Definition 8.6. An abelian category $\mathcal{A}$ has *enough injectives* if for all $A \in \mathcal{A}$ there is a monomorphism $A \to I$ where I is injective.

Lemma 8.7. If $\mathcal{A}$ has enough injectives, then every element admits an injective resolution.

Proof. $0 \to A \to I^0$ by definition of enough injectives, $I^0/A \hookrightarrow I^1$ by definition of enough injectives, where I^1 is injective, so this induces a map $I^0 \to I^1$, and we continue the resolution inductively. $\square$

Definition 8.8. Two complexes A^* and B^* in $\mathcal{A}$ are *homotopic* or homotopy equivalent if there are $f : A^* \to B^*$ and $g : B^* \to A^*$ such that $fg \sim \mathrm{id}_{B^*}$ and $gf \sim \mathrm{id}_{A^*}$

Remark 8.9. This implies $f_* : H^i(A^*) \to H^i(B^*)$ are isomorphisms for all $i \in \mathbb{Z}$.

Lemma 8.10. Two injective resolutions of the same element $A \in \mathcal{A}$ are homotopic.

Proof. Consider I^* and J^* to be

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "1_A"] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \\
0 \arrow[r] & A \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & \cdots
\end{tikzcd}$$

Since I^0 is injective, then there exists $f_0 : J^0 \to I^0$ via A, so analogously I^0/A induces a map $f_1 : I^1 \to J^1$, and analogously we have all the maps we need. Similarly, there are maps g_i's from J^i to I^i. Now $fg - 1$ is a map $I^0/A \to I^0$, since I^0 is injective, then this induces $F : I^1 \to I^0$ that satisfies $Fd = fg - 1$ as maps between $I^0 \to I^0$. We proceed inductively to get all such F's.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "d"] \arrow[d, "0"] & I^0 \arrow[r, "d"] \arrow[d, "fg-1"] & I^1 \arrow[r, "d"] \arrow[d, "fg-1"] \arrow[dl, "F"] & \cdots \\
0 \arrow[r] & A \arrow[r, "d"] & I^0 \arrow[r, "d"] & I^1 \arrow[r, "d"] & \cdots
\end{tikzcd}$$

$\square$

Corollary 8.11. I^* and J^* are injective resolutions of A, then $H^k(I^*) \cong H^k(J^*)$ for every $k \in \mathbb{Z}$.

Definition 8.12. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Suppose $\mathcal{A}$ has enough injective, then the *right derived functor* of F are functors $R^iF : \mathcal{A} \to \mathcal{B}$ defined as follows: given injective resolution I^* as $0 \to A \to I^0 \to I^1 \to \cdots$, we apply F, and obtain a resolution FI^* as $0 \to FA \to FI^0 \to FI^1 \to \cdots$. Let $(R^iF)(A) = H^i(FI^*)$ in $\mathcal{B}$. In particular, $(R^iF)(A) = 0$ for all $i < 0$.

Remark 8.13. $(R^0F)(A) \cong F(A)$.

Theorem 8.14. If $0 \to A \to B \to C \to 0$ is exact in $\mathcal{A}$ and $F : \mathcal{A} \to \mathcal{B}$ is left exact, then we have a long exact sequence

$$0 \to FA \to FB \to FC \to (R^1F)(A) \to (R^1F)(B) \to (R^1F)(C) \to (R^2F)(A) \to \cdots$$

Proof. Look at their injective resolutions I^*, J^*, K^* and produce a complex with vertical maps. We want the vertical sequences between the resolutions to be split. Choose $J^0 \supseteq (I^0 \oplus B)/A$, then this gives a vertical diagram after applying the functor F. Again, we want this to be exact on vertical sequences, so we apply the theorem from last time. $\square$

Definition 8.15. Given $\mathbf{Shv}(X) \to \mathbf{Ab}$ given by $\mathcal{E} \to \mathcal{E}(X)$, then the right derived functors are called *sheaf cohomologies*, denoted by $H^i(X, \mathcal{E})$.

Theorem 8.16. If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of sheaves on X, then there is a long exact sequence

$$0 \to H^0(\mathcal{F}) \to H^0(\mathcal{G}) \to H^0(\mathcal{H}) \to H^1(\mathcal{F}) \to \cdots$$

9 Lecture 9

Let $\mathbb{P}^1_k$ be a field, $S = \{0, \infty\}$ in $\mathbb{P}^1_k$. Then there is

$$0 \to I_{S/\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^1} \to i_*(\mathcal{O}_S) \to 0.$$

We then get a long exact sequence

$$\cdots \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \to H^0(\mathbb{P}^1, i_*(\mathcal{O}_S) \to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2) \to \cdots$$

where $I_{S/\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ and $H^0(\mathbb{P}^1, i_*(\mathcal{O}_S) \cong H^0(S, \mathcal{O}_S) \cong H^0(p, \mathcal{O}_p) \oplus H^0(q, \mathcal{O}_q)$. The diagonal map $k \to k \otimes k$ given by $1 \mapsto (1, 1)$ induces the fact that $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \neq 0$.

Now there is a complex

$$\begin{tikzcd}
0 \arrow[r] & \omega^v_{\mathbb{P}^1} \otimes I_{S/\mathfrak{P}^1} \arrow[r] \arrow[d, "\cong"] & \omega^v_{\mathbb{P}^1} \arrow[r] \arrow[d, "\cong"] & \omega^v_{\mathbb{P}^1} \otimes i_*(\mathcal{O}_S) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \omega^v_{\mathbb{P}^1}(-2) \arrow[r] \arrow[d, "\cong"] & \omega^v_{\mathbb{P}^1} \arrow[r] & i_*(\mathcal{O}_S) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1} \arrow[r] & \omega^v_{\mathbb{P}^1} \arrow[r] & i_*(\mathcal{O}_S) \arrow[r] & 0
\end{tikzcd}$$

In particular, we have

$$H^0(\mathbb{P}^1, \omega^v_{\mathbb{P}}) \to k \otimes k \to H^1(\mathcal{O}_{\mathbb{P}^1}) \cong H^0(\omega_{\mathbb{P}^1})$$

where $H^1(\mathcal{O}_{\mathbb{P}^1}) = 0$, and the final isomorphism is given by Serre duality.

Remark 9.1. For every two values $\alpha_0, \alpha_\infty \in k$, there is a section $\Gamma \in H^0(\mathbb{P}^1, \omega^v_{\mathbb{P}})$ taking value α_0 at $\{0\}$ and taking value α_∞ and $\{\infty\}$.

The question is, does $\mathbf{Shv}(X)$ have enough injectives? Let R be a ring, the abelian groups $\mathbf{Ext}^i_R(M, N)$ for R-modules M and N and $i \in \mathbb{Z}$ are the right derived functors of the left exact functor $N \mapsto \mathbf{Hom}_R(M, N)$. There is an isomorphism $H^i(X, \mathcal{E}) \cong \mathbf{Ext}^i_{\mathbf{Shv}(X)}(\mathbb{Z}_X, \mathcal{E})$. In particualr, $H^0(X, \mathcal{E}) \cong \mathbf{Hom}_{\mathbf{Shv}(X)}(\mathbb{Z}_X, \mathcal{E})$.

Lemma 9.2. A $\mathbb{Z}$-module is injective if and only if M is divisible.

Proof. ($\Rightarrow$): Let M be an injective $\mathbb{Z}$-module, pick $m \in M$, $n \in \mathbb{Z} > 0$. Then

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\times n"] \arrow[d, "1 \mapsto M"] & \mathbb{Z} \arrow[dl, dashed] \\
M &
\end{tikzcd}$$

($\Leftarrow$): Let M be an abelian group, and let $A \subseteq B$ be an inclusion of abelian groups with a homomorphism $f : A \to M$. Consider the poset of abelian groups $A \subseteq S \subseteq B$ and homomorphisms $g_S : S \to M$ making

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "f"] & S \arrow[r, hook] \arrow[dl, "g_S"] & B \\
M & &
\end{tikzcd}$$

commutative.

Now $\{(S_\alpha, g_\alpha)\}$ is a totally ordered subset of this poset, then $\{(U_\alpha S_\alpha, U_\alpha g_\alpha)\}$ is an element of this poset, meaning that every totally ordered subset has an upper bound $S_\alpha \leq U_\alpha S_\alpha$. Zorn's Lemma implies that this poset has a maximal element

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "f"] & H \arrow[r, hook] \arrow[dl, "h"] & B \\ M & & \end{tikzcd}$$

Claim 9.3. $H = B$.

Subproof. Pick $b \in B$ not in H, $\bar{H} = \langle H, b \rangle$. Consider $\varphi : H \otimes \mathbb{Z} \twoheadrightarrow \bar{H}$ by $(\alpha, 1) \mapsto \alpha + b$. Let $K = \ker(\varphi)$, then K injects into $\mathbb{Z}$ by projection. If $K = 0$, then we can extend $h : H \to M$ to $\bar{h} : \bar{H} \to M$, contradiction. ■

Therefore, $K \cong \langle n \rangle$ for some n, so $nb = s$ in the group H. Since M is divisible, then $h(b)$ is divisible in M, so there exists $nm = h(b)$ in M. Hence, we can extend h to $\bar{H}$ by mapping b to m, contradiction. □

Remark 9.4. Any product of an injective is an injective. Moreover,

$$\mathbf{Hom}_{\mathcal{A}}(A, \prod_{\alpha \in S} B_\alpha) \cong \prod_{\alpha \in S} \mathbf{Hom}_{\mathcal{A}}(A, B_\alpha).$$

Example 9.5. $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective.

Lemma 9.6. The category of $\mathbb{Z}$-modules has enough injective.

Proof. Let M be an abelian group. Pick $m \in M$, if m is torsion, set $I_m = \mathbb{Q}/\mathbb{Z}$, otherwise set $I_m = \mathbb{Q}$. Consider

$$\begin{tikzcd} \langle m \rangle \arrow[r, hook] \arrow[d, hook] & I_m \\ M \arrow[ur, dashed] & \end{tikzcd}$$

so we have $M \to I_m$ an abelian group homomorphism. Then we have a homomorphism $M \to \prod_{m \in M} I_m$ where the group in the right is injective. This is a group monomorphism. Indeed, pick $m_0 \in M$, then $M \to \prod_{m \in M} I_m \to I_{m_0}$, the image of m_0 in the composition is non-zero as soon as $m_0 \neq 0$. Therefore, $M \to \prod_{m \in M} I_m$ is injective. □

Exercise 9.7. A $\mathbb{Z}$-module is projective if and only if it is free. What is the analogous of Remark 9.4? Can we prove this without axiom of choice?

Lemma 9.8. If $\mathbb{T}$ is an injective $\mathbb{Z}$-module, then $\mathbf{Hom}_{\mathbb{Z}}(R, T)$ is an injective R-module.

Proof. Let $X \hookrightarrow X_2$ be an injection of R-modules. We want to prove that

$$\mathbf{Hom}_R(X_2, \mathbf{Hom}_{\mathbb{Z}}(R,T)) \to \mathbf{Hom}_R(X_1, \mathbf{Hom}_{\mathbb{Z}}(R,T))$$

is surjective.

Exercise 9.9. There is an isomorphism $\mathbf{Hom}_R(X_2, \mathbf{Hom}_{\mathbb{Z}}(R,T)) \cong \mathbf{Hom}_{\mathbb{Z}}(X_2, T)$.

Therefore, we have a commutative diagram

$$\begin{array}{ccc}
\mathbf{Hom}_R(X_2, \mathbf{Hom}_{\mathbb{Z}}(R,T)) & \longrightarrow & \mathbf{Hom}_R(X_1, \mathbf{Hom}_{\mathbb{Z}}(R,T)) \\
\downarrow \cong & & \downarrow \cong \\
\mathbf{Hom}_{\mathbb{Z}}(X_2, T) & \longrightarrow & \mathbf{Hom}_{\mathbb{Z}}(X < T)
\end{array}$$

$\square$

Theorem 9.10. Let R be a ring, then the category of R-modules has enough injectives.

Proof. Let M be an R-module. Let $M \hookrightarrow T$ be an injection into an injective $\mathbb{Z}$-module. By the lemma, $\mathbf{Hom}_{\mathbb{Z}}(R,T)$ is an injective R-module. Define $m \mapsto f_m$, with $f_m(a) = f(am) \in T$ for all $a \in R$. This gives an injection of R-modules $M \hookrightarrow \mathbf{Hom}_{\mathbb{Z}}(T, R)$. $\square$

Theorem 9.11. Let X be a locally ringed space, then $\mathbf{Shv}(\mathcal{O}_X)$ has enough injectives.

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Pick $x \in X$, then $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module. There is now an injection $\mathcal{F}_x \hookrightarrow \mathcal{J}_x$ where $\mathcal{J}_x$ is an injective $\mathcal{O}_{X,x}$-module. Define $j_x : \{x\} \hookrightarrow X$, then we define the $\mathcal{O}_X$-sheaf. Therefore, $\mathcal{J} = \prod\limits_{x \in X} (j_x)_* J_x$. We have a map of sheaves $\mathcal{F} \to \mathcal{J}$ that on open subsets $U \subseteq X$, there is

$$\mathcal{F}(U) \hookrightarrow \prod_{x \in X} \mathcal{F}_x \to \prod_{x \in U} \mathcal{J}_x = \mathcal{J}(U).$$

Note that for an $\mathcal{O}_X$-sheaf $\mathcal{G}$, we have $\mathbf{Hom}_{\mathcal{O}_X}(\mathcal{G}, (j_x)_* \mathcal{J}_X) \cong \mathbf{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, \mathcal{J}_x)$. Let $\mathcal{G}_1 \subseteq \mathcal{G}_2$ be an injection of $\mathcal{O}_X$-modules, then we have a diagram

$$\begin{array}{ccc}
\mathbf{Hom}_{\mathbf{Shv}(\mathcal{O}_X)}(\mathcal{G}_2, (j_x)_* \mathcal{J}_x) & \longrightarrow & \mathbf{Hom}_{\mathbf{Shv}(\mathcal{O}_X)}(\mathcal{G}_1, (j_x)_* \mathcal{J}_X) \\
\downarrow \cong & & \downarrow \cong \\
\mathbf{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_{2,x}, \mathcal{J}_x) & \longrightarrow & \mathbf{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_{1,x}, \mathcal{J}_x)
\end{array}$$

which is an surjection of $\mathcal{O}_{X,x}$-modules, so $\mathbf{Hom}_{\mathbf{Shv}(\mathcal{O}_X)}(\mathcal{G}_2, \mathcal{J}) \twoheadrightarrow \mathbf{Hom}_{\mathbf{Shv}(\mathcal{O}_X)}(\mathcal{G}_1, \mathcal{J})$. $\square$

10 Lecture 10

Lemma 10.1. Let X be a topological space. Assume X contains a point $x \in X$ such that for every open neighborhood U of $x \in X$ there exists a proper connected open neighborhood $x \in V \subseteq U \subseteq X$, then $\mathrm{Shv}(X)$ does not have enough projectives.

Proof. Let $i : \{x\} \hookrightarrow X$ be the inclusion, consider $i_*\mathbb{Z}_x$. We will prove that $i_*\mathbb{Z}_x$ is not the image of a projective sheaf, which is evaluated as $\mathbb{Z}$ on U whenever $x \in U$, and is 0 on U whenever $x \notin U$.

For $j : V \hookrightarrow X$ and a sheaf $\mathcal{E}$ on V, the extension by zeros, denoted by $j!\mathcal{E}$, is the sheafification of the presheaf defined by $W \mapsto \mathcal{E}(W)$ if $W \subseteq X$, and is 0 otherwise. The stalks of $j!\mathcal{E}$ are $\mathcal{E}x$ if $x \in V$ and 0 otherwise. Hence, for any open $W \subseteq X$, $(j!\mathcal{E})(W)$ is the set $\{s \in \mathcal{E}(V \cap W) \mid s = 0 \in V \cap N, N \text{ is a neighborhood of } W \setminus V\}$. Note that

$$\mathbf{Hom}_X(j!\mathcal{E}, \mathcal{F}) = \mathbf{Hom}_V(\mathcal{E}, j^*\mathcal{F}).$$

Let U be a connected neighborhood of $x \in X$. Let $V \subsetneq U$ be a connected neighborhood of $x \in X$. Write $\mathbb{Z}_V$ for the extension by zero of the sheaf $\mathbb{Z}_V$ on V ($j : V \hookrightarrow X$, $\mathbb{Z}_V := j!\mathbb{Z}_V$), so $\mathbb{Z}_V(W)$ is the set of constant functions $W \to \mathbb{Z}$ that are zero on an open neighborhood of $W \setminus V$. We have that $\mathbb{Z}_V \to i_*\mathbb{Z}_x \leftarrow P$, then by the projective assumption there is a map $P \to \mathbb{Z}_V$, then applying this to U we get

$$\begin{tikzcd} \mathbb{Z}_V(U) \arrow[r] & i_*\mathbb{Z}_x(U) \\ & P(U) \arrow[ul] \arrow[u] \end{tikzcd}$$

but $\mathbb{Z}_V(U) = 0$ by definition, therefore this lift is the zero map for every U connected neighborhood of $x \in X$. From that, any neighborhood of $x \in X$ contains a connected neighborhood of $x \in X$, we conclude that this is true for every neighborhood of $U \ni x \in X$, then taking the inverse limit, we conclude that $P_x \to (i_*\mathbb{Z}_x)_x \cong \mathbb{Z}$ is zero. Therefore, $P \to i_*\mathbb{Z}_x$ is not surjective, otherwise it is surjective on stalks. □

Definition 10.2. A sheaf $\mathcal{E}$ on X is *flasque* (or flabby) if for every open $U \subseteq X$, $\mathcal{E}(X) \to \mathcal{E}(U)$ is onto.

Lemma 10.3. Let $(X, \mathcal{O}_X)$ be a locally ringed space, then every injective $\mathcal{O}_X$-module is flasque.

Proof. Let $V \subseteq X$ be open, $\mathcal{I}$ is an injective $\mathcal{O}_X$-modules. For $j : V \hookrightarrow X$ open, write $\mathcal{O}_V$ for $j!\mathcal{O}_V$ on X. Observe that for any open $V \subseteq X$, we have an injection of sheaves $0 \to$

$\mathcal{O}_V \to \mathcal{O}_X$. Since $\mathcal{I}$ is injective, we have $\mathbf{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{I}) \to \mathbf{Hom}_{\mathcal{O}_X}(\mathcal{O}_V, I)$ a surjection. Note $\mathbf{Hom}_{\mathcal{O}_X}(j!\mathcal{O}_V, \mathcal{I}) \cong \mathbf{Hom}_{\mathcal{O}_V}(\mathcal{O}_V, j^*\mathcal{I}) \cong \mathcal{I}(V)$, so we have a commutative diagram

$$\begin{tikzcd}
\mathbf{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{I}) \arrow[r] \arrow[d, "\sim"] & \mathbf{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{I}) \arrow[d, "\sim"] \\
\mathcal{I}(X) \arrow[r] & \mathcal{I}(V)
\end{tikzcd}$$

□

Example 10.4. Let $X = \mathbb{R}^1$ and $x = \{0\}$, with $i : X \hookrightarrow \mathbb{R}$ and A an abelian group. Then i_*A is a flasque sheaf. Not injective unless A is an injective $\mathbb{Z}$-module.

Definition 10.5. A sheaf $\mathcal{F}$ is called *acyclic* if $H^i(X, \mathcal{F}) = 0$ for $i > 0$.

Example 10.6. Consider $X \hookrightarrow \mathbb{P}^n$ projective variety and $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^N}(H)|_X$ and $\mathcal{O}_X(m) = \mathcal{O}_X(1)^{\otimes m}$, then $\mathcal{O}_X(m)$ is acyclic for $m \gg 0$.

Proposition 10.7. Let $\mathcal{F}$ be a flasque sheaf on a space X, then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$ (therefore $H^i(U, \mathcal{F}) = 0$ for all $U \subseteq X$ open and $i > 0$, which implies $H^i(U, \mathcal{F}|_U = 0$ is flasque).

Proof. $\mathcal{F} \hookrightarrow \mathcal{I}$ where $\mathcal{I}$ is injective hence flasque, and $0 \to \mathcal{F} \to \mathcal{I} \to \mathbb{Q} \to 0$ is exact.

Claim 10.8. $\mathbb{Q}$ is also flasque.

Subproof. $U \subseteq X$ and $s \in \mathbb{Q}(U)$, we first lift s to a section of $\mathcal{I}(U)$. Consider the poset (V, t) with $V \subseteq U$, t a section of $\mathbb{Q}(V)$ lifting to $\mathcal{I}(V)$. Every totally ordered subset has an upper bound (gluing axiom on sheaves). By Zorn's lemma, there exists a maximal with respect to W and $W \subseteq U$. Assume by contradiction, $W \neq U$, since $I \to \mathbb{Q}$ is surjective, there is an open cover U_α of U such that $I(U_\alpha) \twoheadrightarrow \mathbb{Q}(U_\alpha)$ for all α. We can choose $t_\alpha \in I(U_\alpha)$ mapping to $s_\alpha = s|_{U_\alpha}$ for every α. Since $W \subsetneq U$, then there exists $U_\alpha \not\subseteq W$. Note $t_W - t_\alpha \in \mathcal{F}(W \cap U_\alpha)$. Since $\mathcal{F}$ is flasque, $t_W - t_\alpha$ extends to $b \in \mathcal{F}(U_\alpha)$. Replace t_W with $t_\alpha + b$ so now t_W and t_α agree on the intersection $W \cap U_\alpha$. This gives a section on $I(W \cup U_\alpha)$ that maps to $w|_{W|cupU_\alpha} \in \mathbb{Q}(W \cup U_\alpha)$, contradiction, so $W = U$. ■

We note that there is now a commutative diagram

$$\begin{tikzcd}
I(X) \arrow[r] \arrow[d, two heads] & \mathbb{Q}(X) \arrow[d] \\
I(U) \arrow[r, two heads] & \mathbb{Q}(U)
\end{tikzcd}$$

then in the claim above we proved $0 \to H^0(X, \mathcal{F}) \to H^0(X, I) \to H^0(X, \mathbb{Q}) \to 0$. Hence $H^i(X, \mathcal{F}) = 0$. We proved that the first cohomology of a flasque is zero. Therefore, looking at the long exact sequence, all cohomology of the flasque is zero. $\square$

Remark 10.9. Injective implies flasque implies acyclic.

Proposition 10.10. Consider a long exact sequence $0 \to \mathcal{E} \to \mathcal{A}_0 \to \mathcal{A}_1 \to \cdots$ where $\mathcal{A}_i$'s are acyclic, then $H^*(X, \mathcal{E}) \cong H^*(0 \to H^0(X, \mathcal{A}_0) \to H^0(X, \mathcal{A}^1) \to \cdots)$.

11 Lecture 11

Let R be a commutative ring and M be an R-module, then we have an associated $\mathcal{O}_X$-module $\tilde{M}$ on $X = \operatorname{Spec}(R)$. There is a natural isomorphism $H^0(X, \tilde{M}) \cong M$ as $\mathcal{O}_X$-modules, and as R-modules.

For every $f \in R$, $V_f = \{f \neq 0\} \subseteq X$ is affine and $H^0(V_f, M) \cong M[\frac{1}{f}]$. So $\tilde{M}_p = M_p$ for every $p \in \operatorname{Spec}(R)$.

The question is, in the setting with $\mathcal{E}$ quasi-compact and X affine, what can we say about $H^i(X, \mathcal{E})$?

Theorem 11.1. For an affine scheme X, there is an equivalence of categories between $\mathcal{O}(X)$-modules and quasi-coherent sheaves on X, given by $M \mapsto \tilde{M}$ and $H^0(X, \mathcal{E}) \hookleftarrow \mathcal{E}$.

Corollary 11.2. Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be a short exact sequence of quasi-coherent sheaves on affine scheme X. Then the sequence $0 \to H^0(X, \mathcal{A}) \to H^0(X, \mathcal{B}) \to H^0(X, \mathcal{C}) \to 0$ is exact.

Theorem 11.3 (Serre, 1955)**.** For a quasi-coherent sheaf $\mathcal{E}$ on X an affine scheme, we have that $H^i(X, \mathcal{E}) = 0$ for $i > 0$.

Remark 11.4. There are three major approaches:

- Hartshorne (in 1990s) shows that

  Theorem 11.5. For X a Noetherian scheme, the following are equivalent:

  1. X is affine,
  2. $H^i(X, \mathcal{F}) = 0$ for $i > 0$ and $\mathcal{F}$ quasi-coherent,
  3. $H^1(X, \mathcal{I}) = 0$ for $\mathcal{I}$ coherent.

  Exercise 11.6. Write down the proof and point out where Noetherian is used.

- In EGA (in 1960s) and the Stack Project, one shows that: Čech cohomology and sheaf cohomology ar e the same thing and use it to prove Serre vanishing.

- We will look at the proof of Kempf in *Algebraic Varieties* in 1970s.

Proof. Let X be a topological space, $\mathcal{E}$ be a sheaf on X, and $j : U \hookrightarrow X$ be an open embedding. Define $u\mathcal{E} := j_* j^* \mathcal{E}$, that is for any $V \subseteq X$ we have $u\mathcal{E}(V) = \mathcal{E}(U \cap V)$. There is a natural map of sheaves $\mathcal{E} \to u\mathcal{E}$ that gives a homology

$$H^i(X, \mathcal{E}) \to H^i(X, u\mathcal{E}) \to H^i(U, j^*\mathcal{E}) := H^i(U, \mathcal{E}).$$

Proposition 11.7. Let $\mathcal{U}$ be a basis of opens of X a topological space. Assume $\mathcal{U}$ is closed under finite intersections. Let $i \in \mathbb{Z}_{>0}$. Let $\mathcal{F}$ be a sheaf of abelian groups on X. Suppose that $H^j(U, \mathcal{F}) = 0$ for all $0 < j < i$ and $U \in \mathcal{U}$. Then for any element $\alpha \in H^i(X, \mathcal{F})$, there is an open covering $X = U_\sigma W_\sigma$ with $W_\sigma \in \mathcal{U}$ such that $\alpha \mapsto 0 \in H^I(X, W_\sigma \mathcal{F})$ induced by the sequence above.

Subproof. We proceed by induciton on i. First, we prove the case $i = 1$. We embed $\mathcal{F} \hookrightarrow \mathcal{L}$ flasque that vies a short exact sequence, then for any open $W \subseteq X$ we have

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{L} \arrow[r] \arrow[d] & \mathcal{L}/\mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W\mathcal{F} \arrow[r] & W\mathcal{I} \arrow[r] & W\mathcal{I}/W\mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

and this gives $H^0(\mathcal{I}/\mathcal{F}) \to H^1(\mathcal{F}) \to H^1(\mathcal{I}) = 0$. Let $\alpha \in H^1(X, \mathcal{F})$, so $\alpha = d\beta$ for some $\beta \in H^0(X, \mathcal{I}/\mathcal{F})$. We take an open cover $\{W_\sigma\}$ where $W_\sigma \in \mathcal{U}$, then β lifts to $\mathcal{L}(W_\sigma)$ for each σ and gives $\mathcal{L} \twoheadrightarrow \mathcal{L}/\mathcal{F}$. The image of β in $H^0(X, W_\sigma \mathcal{I}/W_\sigma \mathcal{F})$ lifts to $H^0(X, W_\sigma \mathcal{I})$ for each σ. Therefore, $d\beta = \alpha$ is zero in $H^1(X, W_\sigma \mathcal{F})$ for every σ. This proves the case for $i = 1$.

Claim 11.8.

- There is a short exact sequence of sheaves

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{L} \arrow[r] \arrow[d] & \mathcal{L}/\mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W\mathcal{F} \arrow[r] & W\mathcal{L} \arrow[r] & W(\mathcal{L}/\mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

- The sheaf $\mathcal{L}/\mathcal{F}$ satisfies all the assumptions of the proposition for $i = 1$. More precisely, $H^j(U, \mathcal{F}/\mathcal{L}) = 0$ for all $0 < j < i - 1$ and $U \in \mathcal{U}$.

Proof of Claim. If $V \in \mathcal{U}$ is an open different from W, using the corollary and the assumption on $\mathcal{U}$, we get that

$$0 \to H^0(W \cap V, \mathcal{F}) \to H^0(W \cap V, \mathcal{I}) \to H^0(W \cap V, \mathcal{L}/\mathcal{F}) \to 0$$

which is exact. Note that the sequence above is just equivalent to

$$0 \to H^0(V, W\mathcal{F}) \to H^0(V, W\mathcal{I}) \to H^0(V, W(\mathcal{L}/\mathcal{F})) \to 0$$

This proves the first part. Observe that $H^j(V, \mathcal{L}/\mathcal{F}) \cong H^{j+1}(V, \mathcal{F})$ for all $j \geq 1$, by restricting from

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{I}/\mathcal{F} \to 0$$

and the fact that $\mathcal{F}$ is flasque (restricting it to V). ■

Now we prove that the claim implies the proposition. Consider the complex

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{L}/\mathcal{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & W\mathcal{F} & \longrightarrow & W\mathcal{I} & \longrightarrow & W(\mathcal{I}/\mathcal{F}) & \longrightarrow & 0 \end{array}$$

with $W\mathcal{I}$ flasque, so by the first part of the claim, we have a commutative diagram

$$\begin{array}{ccc} H^{i-1}(X, \mathcal{I}/\mathcal{F}) & \xrightarrow{\cong} & H^i(X, \mathcal{F}) \\ \downarrow & & \downarrow \\ H^{i-1}(X, W(\mathcal{I}/\mathcal{F})) & \xrightarrow{\cong} & H^i(X, W\mathcal{F}) \end{array}$$

Note that the right map gives $\alpha \mapsto 0$, then we lift it back to β on the top-left corner, therefore this forces the map on the left to be $\beta \mapsto 0$, which proves the proposition. ■

We will now prove the main theorem. Consider $V_f = \{f \neq 0\}$ for $f \in \mathcal{O}(X)$ and $W_\sigma = \{f_\sigma \neq 0\}$ is a finite basis of X quasi-compact. We may assume $\alpha \in H^i(X, \mathcal{F})$ goes to zero in $H^i(X, W_\sigma\mathcal{F})$ by the proposition. Observe $W_\sigma\mathcal{F}$ is quasi-coherent on X as $W_\sigma\mathcal{F} = (M[\widetilde{\frac{1}{f_\sigma}}])$. Consider the sequence

$$0 \to \mathcal{F} \to \bigoplus_\sigma W_\sigma\mathcal{F} \to \mathfrak{g} \to 0$$

where $\mathfrak{g}$ is quasi-coherent. Take $\alpha \in H^i(X, \mathcal{F})$ and take the long exact sequence in homology, we obtain

$$H^{i-1}(X, \mathfrak{g}) \to^i (X, \mathcal{F}) \to H^i(X, \bigoplus_{W_\sigma} \mathcal{F}) \to H^i(X, \mathfrak{g}) \to \cdots$$

Take $\alpha \in H^i(X, \mathcal{F})$ which maps to 0, then there is a lifting to $\beta \mapsto \alpha$ in the homology. Since $H^{i-1}(X, \mathcal{G}) = 0$ by induction on i, we get $\alpha = 0$ so $H^i(X, \mathcal{F}) = 0$. The induction works provided $H^1 = 0$. To show this, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(X, \mathcal{F}) & \longrightarrow & H^0(X, \bigoplus_\alpha W_\alpha\mathcal{F}) & \longrightarrow & H^0(X, \mathcal{G}) & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & H^1(X, \mathcal{F}) & & \end{array}$$

Take $\beta \in H^0(X, \mathcal{G})$ mapping to $\alpha = H^1(X, \mathcal{F})$, but the mapping to β is surjective, so as $\beta = d\gamma$, this forces $\alpha = d^2\gamma = 0$, therefore the homology there is zero as desired. □

12 Lecture 12

Recall the following results:

Theorem. Category of quasi-coherent sheaves admits enough injectives and cannot be used to define sheaf cohomology $H^i(X, \mathcal{F})$.

Theorem. Sheaf $\mathcal{F}$ being injective implies being flabby implies being acyclic.

Theorem (Serre's Vanishing)**.** If $\mathcal{F}$ is quasi-coherent on an affine scheme X, then $H^i(X, \mathcal{F}) = 0$ for $i > 0$.

The question is, what about cohomology of projective schemes?

Theorem 12.1 (Mayer-Vietoris)**.** Let $\mathcal{E}$ be a sheaf of abelian groups on X, let $X = U \cup V$ be an open cover, then there is a long exact sequence

$$\cdots \to H^i(X, \mathcal{E}) \to H^i(U, \mathcal{E}) \oplus H^i(V, \mathcal{E}) \to H^i(U \cap V, \mathcal{E}) \to H^{i+1}(X, \mathcal{E})$$

Proof. Consider an (flabby) injective resolution

$$0 \to \mathcal{E} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \cdots$$

and restricts to a flabby resolution

$$0 \to \mathcal{E}|_V \to \mathcal{I}_0|_V \to \mathcal{I}_1|_V \to \cdots$$

Observe that restricting on open sets and intersection of open sets preserves the flabby property for all I_i. Therefore, we have a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{I}_0(X) \arrow[r] \arrow[d] & \mathcal{I}_1(X) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{I}_0(U) \oplus \mathcal{I}_0(V) \arrow[r] \arrow[d] & \mathcal{I}_1(U) \oplus \mathcal{I}_1(V) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{I}_0(U \cap V) \arrow[r] & \mathcal{I}_1(U \cap V) \arrow[r] & \cdots
\end{tikzcd}$$

The only non-trivial part of the diagram is $\mathcal{I}_j(U) \oplus \mathcal{I}_j(V) \to \mathcal{I}_j(U \cap V)$ being surjective. This follows from the flasque condition, so we take LES in cohomology associated to the short exact sequence of complexes. □

Example 12.2. Compute $H^*(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$, $U = \mathbb{P}^1 \setminus [0:1] \simeq \mathbb{A}^1$ and $V = \mathbb{P}^1 \setminus [1:0] \simeq \mathbb{A}^1$, then $U \cap V \simeq \mathbb{G}_m := \mathbb{A}^1 \setminus \{0\}$. From Theorem 12.1, we have a long exact sequence

$$0 \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1} \to H^0(U, \mathcal{O}_{\mathbb{P}^1}|_U) \oplus H^0(V, \mathcal{O}_V) \to H^0(U \cap V, \mathcal{O}_{U\cap V}) \to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \to 0$$

Indeed, we have

- $H^i(U, \mathcal{O}_U) = 0$ for all $i \geq 1$,
- $H^i(V, \mathcal{O}_V) = 0$ for all $i \geq 1$, and
- $H^i(U \cap V, \mathcal{O}_{U\cap V}) = 0$ for all $i \geq 1$.

Therefore, $H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$ for $i \geq 2$.

Moreover, we have a short exact sequence

$$0 \to k \to k[x^{-1}] \oplus k[x] \to k[x^{\pm 1}] \to 0$$

defined by $1 \mapsto (1,1)$ and $(a,b) \mapsto a+b$ and therefore $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$.

Theorem 12.3 (Grothendieck, 1975)**.** If X is a noetherian topological space of finite Krull dimension n, then $H^i(X, \mathcal{E}) = 0$ if $i > n$ and $\mathcal{E}$ is any sheaf of abelian groups.

Theorem 12.4. If X is a noetherian projective scheme and $\mathcal{E}$ is a quasi-coherent sheaf on X, then $H^i(X, \mathcal{E}) = 0$ for $i > \dim(X)$.

Remark 12.5 (Čech Cohomology)**.** Let X be a topological space and let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open covering of X. Fix an order in I. For any sequence $i_0, \ldots, i_p \in I$, write $U_{i_0,\ldots,i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$. Let $\mathcal{F}$ be a sheaf of abelian groups on X, define the Čech complex to be $\mathcal{C}^*(X, \mathcal{F})$: for $p \geq 0$, let $\mathcal{C}^p(\mathcal{U}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0,\ldots,i_p})$. This induces a long sequence

$$0 \to \mathcal{C}^0 \to \mathcal{C}^1 \to \mathcal{C}^2 \to \cdots$$

also known as

$$\mathcal{F}(X) \to \bigoplus_{i \in I} \mathcal{F}(U_i) \to \bigoplus_{i<j} \mathcal{F}(U_{i,j}) \to \cdots$$

The differentials are $d : \mathcal{C}^p \to \mathcal{C}^{p+1}$ given by

$$(d\alpha)_{i_0,\ldots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\ldots,\tilde{i}_k,\ldots,i_{p+1}}|_{U_{i_0,\ldots,i_{p+1}}}.$$

For $X = U_0 \cup U_1$ and $\mathcal{U} = \{U_0, U_1\}$, we have a sequence

$$0 \to \mathcal{F}(U_0) \oplus \mathcal{F}(U_1) \to \mathcal{F}(U_0 \cap U_1) \to \cdots$$

defined by $(s_0, s_1) \mapsto (s_0 - s_1)|_{U_0 \cap U_1}$.

Definition 12.6 (Čech Cohomology)**.** The *Čech cohomology* of $\mathcal{F}$ in X with respect to the cover $\mathcal{U} = \{U_i \mid i \in I\}$ is $H^*(\mathcal{U}, \mathcal{F}) = H^*(\mathcal{C}(X, \mathcal{F}))$.

Remark 12.7.

1. There is a natural homomorphism $H^i(X, \mathcal{F}) \to H^i(\mathcal{U}, \mathcal{F})$, from sheaf cohomology to Čech cohomology.
2. If $\mathcal{F}$ is acyclic on $U_{i_0,\dots,i_p}$ for all $i_0, \dots, i_p$, then the homomorphism is an isomorphism.

Theorem 12.8. Let X be a noetherian separated scheme. Let $\mathcal{U}$ be an affine open cover of X. Let $\mathcal{F}$ be a quasi-coherent sheaf on X, then $H^i(X, \mathcal{F}) \simeq H^i(\mathcal{C}(X, \mathcal{F}))$ with respect to $\mathcal{U}$.

Proof. If suffices to show that $H^i(U_{i_0,\dots,i_p}, \mathcal{F}) = 0$ for all $i_0, \dots, i_p$ (by Remark 12.7). This would be true if finite intersections of affine is affine, which uses Lemma 12.9. □

Lemma 12.9. Let X be a separated scheme and let U and V be affine on X. Then $U \cap V$ is affine.

Proof. The diagram functor $\Delta : X \to X \times X$ gives a closed embedding and a commutative diagram

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \times V \arrow[d] & \\ X \arrow[r, "\Delta"] & X \times X \arrow[r] & X \end{tikzcd}$$

Since the bottom map is a closed immersion, then so is the top one. Thus, since $U \times V$ is affine, we conclude that the image in the first $\mathrm{Proj}(X)$ is affine. □

Note that intersections of affines in general is not affine.

Example 12.10.

- Consider $X = \mathbb{A}^2_k \cup_{\mathbb{A}^2 \setminus \{0\}} \mathbb{A}^2_k$ with $U = \mathbb{A}^2_k$ and $V = \mathbb{A}^2_k$, then $U \cap V = \mathbb{A}^2_k \setminus \{0\}$ is not affine.
- $\mathbb{Z} = \bigcap_{\lambda \in \mathbb{C} \setminus \mathbb{Z}} V(x - \lambda) \subseteq \mathbb{A}^1_{\mathbb{C}}$ is not affine. Every countable affine variety over $\mathbb{C}$ is finite.

Lemma 12.11. Let X be a projective scheme with $X \hookrightarrow \mathbb{P}^n$. Let H be a hyperplane such that $X \not\subseteq H$, then $X \setminus H$ (set-theoretically) is an affine variety.

Proof. $X \setminus H \hookrightarrow \mathbb{P}^n \setminus H = \mathbb{A}^n$. Assume $H = \{x_0 = 0\}$, let $I(X)$ be the hom ideal defining X, and let the dehomogenization of $I(X)$ with respect to x_0 be the ideal $k[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}]$ defining $X \setminus H$. □

Theorem 12.12. Let X be a projective scheme (separated) of dimension n. Let $\mathcal{F}$ be a quasi-coherent sheaf on X, then $H^i(X, \mathcal{F}) = 0$ for $i \geq n$.

Proof. $X \hookrightarrow \mathbb{P}^n$, and consider $H_1, \ldots, H_{n+1}$ be general hyperplanes, so $H_1 \cap \cdots \cap H_{n+1} \cap X = \varnothing$. This gives an affine open cover $U_1, \ldots, U_{n+1}$ of X by Lemma 12.11. Set $\mathcal{U} = \{U_1, \ldots, U_{n+1}\}$, then since X is separated, we have $H^i(X, \mathcal{F}) \simeq H^i(\mathcal{C}(X, \mathcal{F}))$ with respect to $\mathcal{U}$, then the Čech complex stabilize at $n+1$, so $H^i(\mathcal{C}(X, \mathcal{F})) = 0$ for $i \geq n+1$. □

Proposition 12.13. Let C be a smooth projective curve, $c \in C$ a closed point, then $C \setminus \{c\}$ is affine.

Remark 12.14. Suppose C has genus at least 1, then there is an embedding $C \hookrightarrow \mathbb{P}^2$. Therefore, let d be such that $g = \frac{d(d+1)}{2}$, there are d intersections on $\mathbb{P}^2$. However, by Proposition 12.13, if we embed P as $C \hookrightarrow \mathbb{P}^N$ for some large enough N, then there exists a hyperplane H of $\mathbb{P}^N$ such that $H \cap C = p$ intersects at a point.

Example 12.15. Let $L \subseteq \mathbb{P}^2$ and $P \in E$ for E elliptic, $3P \sim 0$, then $E \setminus \{p\}$ is affine.

Exercise 12.16. Let E be an elliptic curve and $p \in E$, prove that $E \setminus \{p\}$ is affine.

13 Lecture 13

Theorem 13.1. Let k be a field and n be a positive integer. Then for $r \in \mathbb{Z}$,

$$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = \begin{cases} \bigoplus_k x_0^{a_0} \cdots x_n^{a_n}, a_k \geq 0, \sum a_i = r, & \text{if } i = 0 \\ 0 & \text{if } i \neq \{0, n\} \\ \bigoplus_k x_0^{a_0} \cdots x_n^{a_n}, a_k < 0, \sum a_i = r, & \text{if } i = n \end{cases}$$

Remark 13.2. We will see that for any projective scheme X over k and any coherent sheaf $\mathcal{E}$ on X, the cohomology group $H^i(X, \mathcal{E})$ is a finite-dimensional vector space.

If X over k is proper and $\mathcal{E}$ is coherent, we write $h^i(X, \mathcal{E}) = \dim_k(H^i(X, \mathcal{E}))$.

Corollary 13.3. We have

$$h^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = \begin{cases} \binom{n+r}{n}, & \text{if } i = 0 \text{ and } r \geq 0 \\ 0, & \text{otherwise} \\ \binom{-r-1}{n}, & \text{if } i = n, r < 0 \end{cases}$$

Remark 13.4. If $-n \leq r \leq -1$, then $\mathcal{O}(r)$ has no cohomologies in any degree in $\mathbb{P}^n$.

Remark 13.5. Recall $\omega_{\mathbb{P}^n}$. Hence, for every line bundle $\mathcal{L}$ on $\mathbb{P}^n$, we have an equality of dimensions $h^i(\mathbb{P}^n, \mathcal{L}) = h^{n-i}(\mathbb{P}^n, \omega_{\mathbb{P}^n} \otimes \mathcal{L}^\vee)$.

Example 13.6. On $\mathbb{P}^n$, we have the following table:

	$\mathcal{O}(-3)$	$\mathcal{O}(-2)$	$\mathcal{O}(-1)$	$\mathcal{O}(0)$	$\mathcal{O}(1)$	$\mathcal{O}(2)$	$\mathcal{O}(3)$
h^0	0	0	0	1	2	3	4
h^1	2	1	0	0	0	0	0

Definition 13.7 (Tensor Product)**.** Let A and B be two complex of abelian groups, then the *tensor product* of A and B is defined by $(A \otimes B)^j = \bigoplus_{i \in \mathbb{Z}} A^i \otimes B^{j-i}$, such that $d(a \otimes b) = (da) \otimes b + (-1)^{|a|} a \otimes (db)$.

Exercise 13.8. Show that this is indeed a complex.

Exercise 13.9. Prove the Kunneth formula, that $H^i(A \otimes B) = H^i(A) \otimes_k H^{j-i}(B)$.

Proof of Theorem. We use Čech cohomology for affine open cover of $\mathbb{P}^n$ given by $U_i = \{x_i \neq 0\}$ for $0 \leq i \leq n$. Since $\mathbb{P}^n$ is separated, we have

$$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = \check{H}^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r))$$

So we have a complex

$$\cdots \to \bigoplus_{i=1}^{n} \mathcal{O}(r)(U_i) \to \bigoplus_{0 \leq i_0 < i_1 \leq n} \mathcal{O}(r)(U_{i_0,i_1}) \to \cdots$$

Now observe that

$$\bigoplus_{i \in \mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) \cong H^i(\mathbb{P}^n, \bigoplus_{r \in \mathbb{Z}} \mathcal{O}(r)).$$

Set $\mathcal{F} = \bigoplus_{r \in \mathbb{Z}} \mathcal{O}(r)$, and set $U = U_{i_0,\ldots,i_p}$. Consider the map

$$\begin{aligned} \pi : \mathbb{A}^{n+1} \setminus \{0\} &\to \mathbb{P}^n \\ (x_0, \ldots, x_n) &\mapsto [x_0 : \ldots : x_n] \end{aligned}$$

and denote $\mathcal{F}(U) = \mathcal{F}(\pi^{-1}(U)) = k[x_{i_0}, \ldots, x_{i_n}, x_{i_0}^{-1}, \ldots, x_{i_p}^{-1}]$. This is a k-vector space with a basis $k\{x_0^{a_0} \cdots x_n^{a_n} \mid a_k \in \mathbb{Z}, a_k \geq 0, \text{ if } k \notin \{i_0, \ldots, i_p\}\}$. Therefore,

$$H^*(\mathbb{P}^n, \mathcal{F}) = H^*(0 \to \bigoplus_{i=0}^{n} k[x_0, \ldots, x_n, x_i^{-1}] \to \bigoplus_{0 \leq i < j \leq n} k[x_0, \ldots, x_n, x_i^{-1}, x_j^{-1}] \to \cdots)$$

This complex looks similar to the following complex T:

$$(0 \to k[x_0] \to k[x_0, x_0^{-1}] \to 0) \otimes_k (0 \to k[x_0] \to k[x_1, x_1^{-1}] \to 0) \otimes_k \cdots \otimes_k (0 \to k[x_0] \to k[x_n, x_n^{-1}] \to 0)$$

The difference is that we have to remove the group in degree) and shift by 1. By Kunneth formula,

$$H^i(T) = \begin{cases} k[x_0, x_0^{-1}]/k[x_0] \otimes_k \cdots \otimes_k k[x_n, x_n^{-1}]/k[x_n], & \text{if } j = n-1 \\ 0 & \text{if } j \neq n-1 \end{cases}$$

Therefore,

$$H^j(\mathbb{P}^n, \mathcal{F}) = \begin{cases} k[x_0, \ldots, x_n] & \text{if } j = 0 \\ k[x_0, x_0^{-1}]/k[x_0] \otimes_k \cdots \otimes_k k[x_n, x_n^{-1}]/k[x_n] & \text{if } j = n \\ 0 & \text{if } j \neq \{0, n\} \end{cases}$$

□

Remark 13.10. Most Riemann surfaces do not embed into $\mathbb{P}^2_{\mathbb{C}}$, not even analytically.

Remark 13.11. Every smooth projective curve C admits an embedding into $\mathbb{P}^3_{\mathbb{C}}$.

Lemma 13.12. Let $i : S \hookrightarrow X$ be a closed set of a topological space. Then

1. the functor $i_* : \mathrm{Shv}(S) \to \mathrm{Shv}(X)$, where $i_*(\mathcal{E}(U)) = \mathcal{E}(U \cap S)$, is exact;
2. for a sheaf $\mathcal{E}$ of abelian groups on S, we have

$$H^j(S, \mathcal{E}) \cong H^j(X, i_*(\mathcal{E})).$$

Proof.

1. Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be exact on S. It suffices to show $0 \to (i_*\mathcal{A})_x \to (i_*\mathcal{B})_x \to (i_*\mathcal{C})_x \to 0$ is exact on X for every $x \in X$. If $x \in S$, then $(i_*\mathcal{A})_x \cong \mathcal{A}_x$. If $x \notin S$, then since S is closed, we get $(i_*\mathcal{A})_x = (i_*\mathcal{B})_x = (i_*\mathcal{C})_x = 0$.
2. Consider an injective resolution $0 \to \mathcal{E} \to I^0 \to I^1 \to \cdots$ where each I^j is flabby. Now

$$H^*(S, \mathcal{E}) = H^*(0 \to I^0(S) \to I^1(S) \to \cdots)$$

then the sheaves i_*I^j are flabby, now

$$H^*(X, i_*\mathcal{E}) = H^*(0 \to i_*I^0(X) \to i_*I^1(X) \to \cdots)$$

and the two cohomologies agree degreewise.

□

Definition 13.13 (Genus)**.** Let C be a smooth projective curve over k, then the *genus* of C is $g(C) = \dim_k(H^1(C, \mathcal{O}_C))$.

Proposition 13.14. Let $C_j \subseteq \mathbb{P}^2$ be a smooth projective curve of degree d, then $g(C_d) = \frac{(d-1)(d-2)}{2}$.

Proof. We have

$$0 \to I_{\mathcal{C}/\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2} \to i_*\mathcal{O}_C \to 0$$

where $I_{\mathcal{C}/\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-d)$. Taking the first cohomology, we have

$$\cdots \to H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \to H^1(C, \mathcal{O}_C) \to H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d)) \to H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2} \to \cdots$$

where $H^1(C, \mathcal{O}_C) \cong H^1(\mathbb{P}^2, i_*\mathcal{O}_C)$, and by the previous result, the first and last term are zero, so the middle two terms agree, therefore $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-d))$ has dimension $\binom{n-1}{2}$. $\square$

14 Lecture 14

Theorem 14.1. Consider $\mathbb{P}^n_k$ for some $n \geq 0$ and k a field. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_k$, then $\mathcal{F}(m) = \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n_k}(m)$ is globally generated for some $m > 0$. That is, there is a surjection $\mathcal{O}^{\oplus r} \twoheadrightarrow \mathcal{F}(m)$ over $\mathbb{P}^n$.

Remark 14.2. For simplicity , we have $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^n_k}(H)$ for H hyperplane and $\mathcal{O}(m) = \mathcal{O}(1)^{\oplus m}$. Then $\mathcal{F}(k)$ is globally generated for $k \geq m$.

Proof. Observe that only coherent sheaf on affine scheme is globally generated. For $\mathcal{E}$ coherent sheaf on X affine, there is an equivalence of categories between quasi-coherent sheaves on X and $\mathcal{O}(X)$-modules. Every finitely-generated $\mathcal{O}(X)$-module admits a surjection $\mathcal{O}(X)^{\oplus r} \twoheadrightarrow H^0(X, \mathcal{E})$. Sheafifying this surjection of modules gives $\mathcal{O}^{\oplus r} \cong \tilde{\mathcal{O}(X)}^{\oplus r} \twoheadrightarrow H^0(\tilde{X}, \mathcal{E}) \cong \mathcal{E}$. In conclusion, $\mathcal{F}|_{U_i}$ spanned by sections $s_1, \ldots, s_j$ in $H^0(U_i, \mathcal{F}|_{U_i})$ where $U_i = \{x_i \neq 0\}$. Note that for $\mathcal{E}$ quasi-coherent on X affine and $f \in \mathcal{O}(X)$, we have $H^0(\{f \neq 0\}, \mathcal{E}) \cong H^0(X, \mathcal{E})\left[\frac{1}{f}\right]$. Then $(\frac{x_1}{x_j})^m s_1 \cdots (\frac{x_r}{x_j})^m s_r$ extends to sections of $\mathcal{F}$ in U_j. Equivalently, $x_i^m s_1 \cdots x_i^m s_r$ extends to sections of $\mathcal{F}(m)$ on $U_i \cup U_j$. To extend to $\mathbb{P}^n_k$, we need to agree on overlaps $U_i \cap U_j$. If $s \in H^0(X, \mathcal{E})[\frac{1}{f}] \cong H^0(\{f \neq 0\}, \mathcal{E})$, then there exists $m \geq 0$ such that $f^m \in H^0(X, \mathcal{E})$. Furthermore, if $s \in \frac{a}{f^m} = \frac{b}{f^m} \in M[\frac{1}{f}]$ where $M = H^0(X, \mathcal{E})$, there exists $j \geq 0$ for which $f^j a = f^j b$. Hence, if m is large enough, the sections $x_i^m s_r \cdots x_i^m s_r$ extend to sections of $\mathcal{F}(m)$ on $\mathbb{P}^n$. $\square$

Theorem 14.3 (Serre)**.** Let $i : X \hookrightarrow \mathbb{P}^n_k$ a projective variety and let $\mathcal{F}$ be a coherent sheaf on X. Then

1. $H^i(X, \mathcal{F})$ is a finite-dimensional k-vector space.

2. there exists $m_0 = m(\mathcal{F})$ such that $H^0(X, \mathcal{F}(m)) = 0$ for $m \geq m_0$.

Proof. We know $H^i(X, \mathcal{F}) \cong H^i(\mathbb{P}^n, i_*\mathcal{F})$ and $i_*\mathcal{F} \otimes \mathcal{O}(m) \cong i_*(\mathcal{F} \otimes i^*\mathcal{O}(m)) = i_*(\mathcal{F}(m))$, therefore $X = \mathbb{P}^n_k$ and $\mathcal{F}$ is coherent in $\mathbb{P}^n_k$.

Also, since $\mathcal{F}(m)$ is globally generated for $m \gg 0$, then there is a surjection $\mathcal{O}^{\oplus r} \twoheadrightarrow \mathcal{F}(m)$, and we have a short exact sequence

$$0 \to K \to \mathcal{O}^{\oplus r} \to \mathcal{F}(m) \to 0$$

and by Serre twisting we have

$$0 \to K(-m) \to \mathcal{O}(-m)^{\oplus r} \to \mathcal{F} \to 0$$

This gives a long exact sequence in cohomology:

$$\cdots \to H^j(\mathbb{P}^n, \mathcal{O}(-m)^{\oplus r} \to H^j(\mathbb{P}^n, \mathcal{F}) \to H^{j+1}(\mathbb{P}^n, K(-m)) \to \cdots$$

We know $H^i(\mathbb{P}^n, \mathcal{E}) = 0$ for $j > n$ and $\mathcal{E}$ coherent. We conclude that $H^j(\mathbb{P}^n, \mathcal{F})$ is a finite-dimensional k-vector space by descending induction. This proves the first part.

Fix s and let $m \gg s$. We know $\mathcal{F}(m)$ is globally generated for $s \gg 0$, then we have $\mathcal{O}^{\oplus r} \twoheadrightarrow \mathcal{F}(s)$ and therefore $\mathcal{O}(-s)^{\oplus r} \twoheadrightarrow \mathcal{F}$. Similar as above by Serre twisting we have

$$0 \to K(m) \to \mathcal{O}(m-s)^{\oplus r} \to \mathcal{F}(m) \to 0$$

We get a long exact sequence of finite-dimensional k-vector spaces, such that for $m \gg 0$,

$$0 = H^j(\mathbb{P}^n, \mathcal{O}(m-s)^{\oplus r}) \to H^i(\mathbb{P}^n, \mathcal{F}(m)) \to H^{j+1}(\mathbb{P}^n, K(m)) \to \cdots$$

We know $H^{n+1}(\mathbb{P}^n, K(m)) = 0$ and so $H^n(\mathbb{P}^n, \mathcal{F}(m)) = 0$. By descending induction, we are done. □

Corollary 14.4 (Serre)**.** Let $\mathcal{L}$ be ample on X, then for every coherent sheaf on X we have a constant $m(\mathcal{F}) = m_0$ for which

1. $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ is globally generated and

2. $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$ for $i > 0$.

Proof. Because $\mathcal{O}^{\otimes a}$ is very ample, then $i : X \hookrightarrow \mathbb{P}^N$ such that $i^*(\mathcal{O}(1)) \cong \mathcal{L}^{\otimes a}$. Think of $\mathcal{O}(1) = \mathcal{L}^{\otimes a}$, now apply the previous theorem to this $\mathcal{O}(1)$ for $\mathcal{F}, \mathcal{F} \otimes \mathcal{L}, \ldots, \mathcal{F} \otimes \mathcal{L}^{\otimes(a-1)}$. Note that $\mathcal{F} \otimes \mathcal{O}(m)$ is globally generated for $m \gg 0$, so $\mathcal{F} \otimes \mathcal{L}^{\otimes m_0}$ is globally generated. Therefore, there exists m_0 such that for all $0 \leq b \leq a-1$, $(\mathcal{F} \otimes \mathcal{L}^b) \otimes \mathcal{L}^{\otimes m_0}$ satisfies the two properties. □

Theorem 14.5 (Cartan-Serre-Grothendieck)**.** Let $\mathcal{L}$ be a line bundle on a complete scheme X. The following are equivalent:

1. $\mathcal{L}$ is ample,
2. $\mathcal{L}^{\otimes m}$ is very ample for some $m \geq 0$,
3. for every coherent sheaf $\mathcal{F}$, there is $m_0 = m_0(\mathcal{F})$ such that $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ is globally generated for $m \geq m_0$,
4. for every coherent sheaf $\mathcal{F}$, there is $m_1 = m_1(\mathcal{F})$ such that $H^i(\mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$ for $i > 0$ and $m \geq m_1$.

Corollary 14.6. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are ample, then $\mathcal{L}_1^{\otimes a} \otimes \mathcal{L}_2^{\otimes b}$ is ample for $a > 0$ and $b > 0$.

Proof. Since $\mathcal{L}_1$ is ample, then $\mathcal{L}_1^{\otimes am}$ is globally generated for $m \gg 0$. Similarly, $\mathcal{L}_2^{\otimes bm} \otimes \mathcal{F}$ is globally generated for $m \gg 0$. Therefore, $\mathcal{F} \otimes (\mathcal{L}_1^{\otimes a} \otimes \mathcal{L}_2^{\otimes b})^{\otimes m}$ is globally generated for $m \gg 0$. $\square$

Remark 14.7 (Where is the cone?)**.** Let X be a smooth projective scheme, then we have

$$0 \to 2\pi i\mathbb{Z}|to\mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

then in cohomology we have

$$\cdots \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to \cdots$$

where

$$\begin{array}{ccc} H^1(X, \mathcal{O}_X^) & \longrightarrow & H^2(X, \mathbb{Z}) \\ \big\downarrow \cong & & \big\uparrow \\ \operatorname{Pic}(X) & \xrightarrow{\mathrm{ch}_1} & \mathbb{Z}^\rho \end{array}$$

and extends to a short exact sequence

$$0 \longrightarrow \operatorname{Pic}^0(X) \cong H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}) \longrightarrow \operatorname{Pic}(X) \longrightarrow \mathbb{Z}^\rho \cong \operatorname{NS}(X) \longrightarrow 0$$

where $\operatorname{Pic}^0(X)$ is the complex tori.

Theorem 14.8. $\mathcal{L} \in \operatorname{Pic}^0(X)$ if and only if $\mathcal{L}$ deforms to $\mathcal{O}_X$ if and only if $\mathcal{L}.C = 0$ for all curves $C \subseteq X$.

Remark 14.9. The space of divisors $N_1(X) = \operatorname{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Theorem 14.10. Ampleness is a numerical condition. That is, the ample property forms a cone on the plane, and further inside the cone there is no higher cohomology.

15 Lecture 15

Proposition 15.1. Let X be a separated quasi-compact scheme over a field k, and let $\mathcal{E}$ be a quasi-coherent sheaf on X. Let F/k be a field extension, and write $X_F = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(F)$. Therefore we have

$$\begin{array}{ccc} X_F & \longrightarrow & \mathrm{Spec}(F) \\ \downarrow{\scriptstyle\pi} & & \downarrow \\ X & \longrightarrow & \mathrm{Spec}(k) \end{array}$$

Then $H^i(X_F, \pi^*\mathcal{E}) \cong H^i(X, \mathcal{E}) \otimes_k F$.

Proof. $\pi^*(\mathcal{E}) = (\pi^{-1}(\mathcal{E})) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_F}$ sheafified. If X is affine, then $H^0(X_F, \mathcal{E}) \cong H^0(X, \mathcal{E}) \otimes_k F$. Since X is quasi-compact, we write $X = \bigcup_{i=1}^{n} U_i$ such that U_i is an affine open variety of X for all i. Now $H^*(X, \mathcal{E}) = H^*(0 \to \prod_i H^0(U_i, \mathcal{E}) \to \prod_{i<j} H^0(U_{ij}, \mathcal{E}) \to \cdots)$ by Cech homology. Here, we are using $\mathcal{E}$ quasi-compact and X separable. We get $H^*(X_F, \mathcal{E} := \pi^*(\mathcal{E})) = H^*(0 \to \prod_i [H^0(U_i, \mathcal{E}) \otimes_k F] \to \cdots)$. It now suffices to tensor with F over k, which is an exact functor on k-vector spaces and commutes with finite products. □

Remark 15.2. If we drop the quasi-compact condition, this would not work. Say $X = \coprod_{n \geq 0} \mathrm{Spec}(k)$ then $\mathcal{O}(X) = \prod_{n \geq 0} k$ and $\mathcal{O}(X_F) = \prod_{n \geq 0} F$, then $(\prod k) \otimes_k F \ncong \prod F$ if the product is not finite.

Remark 15.3. A property of schemes that is preserved under taking field extensions is called "geometric". For instance, $\dim_k(H^1(X, \mathcal{O}_X))$ is geometric for quasi-coherent sheaves.

Remark 15.4. Suppose X is a projective scheme defined over an algebraically closed field k, then $\mathcal{O}(X) \cong k$.

Proposition 15.5. Let X be a projective variety defined over a field k, then $\mathcal{O}(X)$ is a finite field extension of k.

Proof. It suffices to check that the field extension is finite: $\mathcal{O}(X)$ is a domain and we can invert elements. Note that $\mathcal{O}(X) = H^0(X, \mathcal{O}_X)$ is finite-dimensional over k. □

Definition 15.6 (Geometrically Integral)**.** Let X be a scheme defined over k. X is *geometrically integral* if $X_{\bar{k}}$ is integral where $\bar{k}$ is the algebraic closure of k.

Remark 15.7. Given a property P that is not geometric, we say that that X is "geometrically P" if $X_{\bar{k}}$ satisfies P.

Proposition 15.8. Let X be a smooth projective variety over k, then the following are equivalent:

1. X is geometrically integral,
2. X is geometrically connected,
3. The map $k \to \mathcal{O}(X)$ is an isomorphism.

Proof. If X is smooth over k, then $X_{\bar{k}}$ is smooth over $|bark$. Therefore, $X_{\bar{k}}$ is connected with local rings that are domains, which means $X_{\bar{k}}$ is a domain. This proves (2) $\Rightarrow$ (1). (1) $\Rightarrow$ (2) is trivial.

We now show (1) $\Rightarrow$ (3). The extension $\bar{k} \to \mathcal{O}(X_{\bar{k}})$ is an isomorphism. Observe that $\bar{k} = k \otimes_k \bar{k}$ and $\mathcal{O}(X_{\bar{k}}) = \mathcal{O}(X) \otimes_k \bar{k}$, therefore $k \to \mathcal{O}(X)$ is an isomorphism. To see (3) $\Rightarrow$ (2), if $k \to \mathcal{O}(X)$ is an isomorphism, then $\bar{k} \to \mathcal{O}(X_{\bar{k}})$ is an isomorphism, then X is geometrically connected. □

Proposition 15.9. If X is a smooth projective variety over k and $X(k) \neq \varnothing$, then X is geometrically integral and $k \to \mathcal{O}(X)$ is an isomorphism.

Proof. Consider $X \to \mathrm{Spec}(k)$ and a section $s : \mathrm{Spec}(k) \to X$, this corresponds to a ring homomorphism $\mathcal{O}(X) \to k$ that is injective, so $k \cong \mathcal{O}(X)$. Hence, X is geometrically integral by the previous proposition. □

Let X be a smooth projective complex variety over $\mathbb{C}$, we will view it as a scheme over $\mathbb{R}$. We have $\mathcal{O}(X) \cong \mathbb{C}$, then X is not geometrically integral as a scheme over $\mathbb{R}$. To calculate $X_{\mathbb{C}}$, we have $\mathrm{Spec}(\mathbb{C}) \otimes_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C}) = \mathrm{Spec}(\mathbb{C} \otimes \mathbb{R}\mathbb{C}) \cong \mathrm{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}[x]/(x^2+1)) \cong \mathrm{Spec}(\mathbb{C}[x]/(x^2+1)) \cong \mathrm{Spec}(\mathbb{C}) \coprod \mathrm{Spec}(\mathbb{C})$. Therefore, $X_{\mathbb{C}} = X \coprod X^{\mathrm{conj}}$, where X^{conj} is defined by conjugating equations of X.

Definition 15.10 (Genus)**.** Let X be a smooth projective geometrically integral curve, then the genus of X is $\dim_k(H^1(X, \mathcal{O}_X))$.

Theorem 15.11. Let X be a smooth projective curve over a field k, assume $g(C) = 0$, then X is isomorphic to a conic in $\mathbb{P}^2_k$.

Theorem 15.12 (Diophantus, 200 AD)**.** Let X be a smooth projective curve over a field k. Assume $g(C) = 0$, then $X \cong \mathbb{P}^1_k$ if and only if $X(k) \neq \varnothing$.

Example 15.13. $\{x^2 + y^2 + z^2 = 0\} \subseteq \mathbb{P}^2_{\mathbb{R}}$ as an $\mathbb{R}$-scheme.

Remark 15.14. If $k = \bar{k}$, then every genus-0 smooth projective curve is isomorphic to $\mathbb{P}^1_k$.

Proof. Let $p \in X$ be a k-point, then we have a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}(p) \to i_* i^* \mathcal{O}(p) \to 0$$

where $i : \{p\} \hookrightarrow X$. We have an induced long exact sequence of homology

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}(p)) \to H^0(X, i_* i^* \mathcal{O}(p)) \to H^1(X, \mathcal{O}_X) \to \cdots$$

where $H^0(X, i_* i^* \mathcal{O}(p)) \cong H^0(p, i^* \mathcal{O}(p)) \cong k$, $H^0(X, \mathcal{O}_X) \cong k$, and $H^1(X, \mathcal{O}_X) \cong 0$. We conclude that $H^0(X, \mathcal{O}(p))$ has a non-trivial element $f \in H^0(X, \mathcal{O}(p))$ has a simple pole at p. We consider $f : X \to \mathbb{P}^1$ to be $x \mapsto [g_1(x), g_2(x)]$ where $f = \frac{g_1}{g_2}$. But $\mathbb{P}^1$ is projective, and X is smooth, so X is a morphism. But the pullback $f^*([\infty]) = [p]$ as divisors, so $\deg(f) = 1$, and $\deg[k(X); k(\mathbb{P}^1)] = 1$, so $k(\mathbb{P}^1) \cong k(X)$, therefore f is birational, and so f is an isomorphism since X and $\mathbb{P}^1$ are smooth. $\square$

Index