MATH 502 Notes

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References:

- Atiyah and MacDonald, *Commutative Algebra*.
- J.P. Serre, *Local Algebra*.
- Zariski and Samuel, *Commutative Algebra* Volume 1 and 2.
- Matsumura, *Commutative Algebra*.
- Bourbaki, *Commutative Algebra*.

We always assume a ring R has a multiplicative identity and is commutative.

0 Noetherian, Artinian, and Localization

Proposition 0.1. Let R be a (commutative) ring, and let M be an A-module, then the following are equivalent:

(i) Given an infinite increasing chain of submodules of M

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq M_{n+1} \subseteq \cdots$$

then there exists some $N \in \mathbb{N}$ such that $M_N = M_{N+1} = \cdots$, i.e., for all $n \geqslant N$, $M_n = M_{n+1}$.

(ii) Every non-empty family of submodules has a maximal element.

(iii) Every submodule of M is finitely-generated.

Proof. $(i) \Rightarrow (ii)$: This is a direct result of Zorn's lemma.

$(ii) \Rightarrow (i)$: Obvious.

$(i), (ii) \Rightarrow (iii)$: Take any submodule N of M and take $x_1 \in N$. If $(x_1) \neq N$, then there exists $x_2 \in N \backslash (x_1)$, so $(x_1, x_2) \subseteq N$, now we proceed inductively, but by the given property we know this stops in finite number of steps, hence we have $N = (x_1, \ldots, x_n)$ for some $n \in \mathbb{N}$, thus N is finitely-generated.

$(iii) \Rightarrow (i)$: Note that the property implies M is finitely-generated, but that means the chain of submodules must be finite. □

Definition 0.2 (Noetherian Module)**.** If any of the conditions in Proposition 0.1 holds, then M is said to be a Noetherian module. Alternatively, we say M satisfies the ascending chain condition.

Proposition 0.3. Let R be a (commutative) ring, and let M be an A-module, then the following are equivalent:

(i) Given an infinite decreasing chain of submodules of M

$$M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq M_{n+1} \supseteq \cdots$$

then there exists some $N \in \mathbb{N}$ such that $M_N = M_{N+1} = \cdots$, i.e., for all $n \geqslant N$, $M_n = M_{n+1}$.

(ii) Every non-empty family of submodules has a minimal element.

Proof. Again, Zorn's lemma. □

Definition 0.4 (Artinian Module)**.** If any of the conditions in Proposition 0.3 holds, then M is said to be a Artinian module. Alternatively, we say M satisfies the descending chain condition.

Example 0.5.

- $\mathbb{Z}$ is Noetherian.
- $\mathbb{Q}/\mathbb{Z}$ is not Noetherian.
- Let p be a prime. Let $\mathbb{Z}(p^\infty)$ be the union of chains (as direct limits)

$$\left\langle \frac{\bar{1}}{p} \right\rangle \subseteq \left\langle \frac{\bar{1}}{p^2} \right\rangle \subseteq \cdots \subseteq \left\langle \frac{\bar{1}}{p^n} \right\rangle \subseteq \cdots$$

  then there is an embedding $\mathbb{Z}(p^\infty) \subseteq \mathbb{Q}/\mathbb{Z}$, where $\bar{a}$ is the image of a in $\mathbb{Q}/\mathbb{Z}$. With this construction, $\mathbb{Z}(p^\infty)$ is Artinian.

Exercise 0.6. Show that $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}(p^\infty)$ where p traverses through all the primes.

Proposition 0.7. Let N be a submodule of M. Suppose M satisfies ascending (respectively, descending) chain condition, then N and M/N also satisfy ascending (respectively, descending) chain condition. If, for some submodule N of M, we know N and M/N satisfy ascending (respectively, descending) chain condition, then M also satisfies ascending (respectively, descending) chain condition.

Proof. Suppose M satisfies ascending (respectively, descending) chain condition, and let N be a submodule of M. Let $\{N_i\}$ be an increasing (respectively, decreasing) sequence of submodules of N, then they can be regarded as submodules of M, therefore by the Noetherian (respectively, Artinian) condition, we know N satisfies ascending (respectively, descending) chain condition. Now let $\bar{M} = M/N$, and take $\{\bar{M}_i\}$ be an increasing (respectively, decreasing) sequence of submodules of $\bar{M}$. Let $\pi : M \to M/N$ be the quotient map, then the preimages give an increasing (respectively, decreasing) sequence $\{M_i\}$ of submodules of M, where $M_i = \pi^{-1}(\bar{M}_i)$, but by the Notherian (respectively, Artinian) condition, we know the sequence stops in finite steps, therefore the original sequence stops in finite steps as well, hence $\bar{M}$ satisfies the ascending (respectively, descending) chain condition.

Suppose a submodule N of M is such that N and M/N both satisfy ascending chain condition. Take a submodule T of M, then we have a short exact sequence

$$0 \longrightarrow T \cap N \hookrightarrow T \longrightarrow T/(T \cap N) \longrightarrow 0$$

Now $T \cap N$ is finitely-generated as N is finitely-generated, therefore we have an embedding $T/T \cap N \hookrightarrow M/N$, thus $T/T \cap N$ is finitely-generated, therefore T is also finitely-generated by a vector space argument.

Suppose we have a decreasing sequence $\{M_n\}$ of M, then we have a decreasing sequence $\{N \cap M_n\}$. Let $\bar{M} = M/N$, then $\bar{M}_n := (M_n + N)/N$ defines a decreasing sequence of submodules in $\bar{M}$, but N satisfies the descending chain condition, so the sequence $\{N \cap M_n\}$ stops in finite number of steps, say n_0. Moreover, the sequence of $\bar{M}_n$'s also stops in finite number of steps, so by definition the sequence of $(M_n + N)/N$ stops in finite number of steps, say m_0, but by the isomorphism theorem this shows that the sequence of $M_n/(N \cap M_n)$ stops in m_0 steps. Therefore, whenever $n \geqslant m_0, n_0$, then $N \cap M_n = N \cap M_{n+1}$, hence $M_n = M_{n+1} = \cdots$ for such n. □

Remark 0.8. The final argument should also work in the Noetherian case.

Definition 0.9 (Simple Module)**.** An A-module M is simple if the submodules of M are either 0 or M.

Exercise 0.10. Let A be a commutative ring, and M is an A-module, then M is simple if and only if $M \cong A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of A.

Definition 0.11 (Jordan-Hölder Chain)**.** Let A be a commutative ring and M be an A-module. We say M has a Jordan-Hölder chain if there exists a decreasing chain of submodules $\{M_i\}$ such that

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_{n-1} \supsetneq M_n = 0$$

such that M_i/M_{i+1} is simple. In such a situation, we know n is the length of the Jordan-Hölder chain, and such n is unique. We say M is a module of finite length, and the length is $\ell_A(M) = n$.

Exercise 0.12. Let A be a commutative ring, and let M be an A-module, then M is of finite length if and only if M is both Noetherian and Artinian.

Theorem 0.13. Let A be a commutative ring, then A is Artinian if and only if A is Noetherian and every prime ideal of A is maximal.

Proof. ($\Leftarrow$):

Lemma 0.14. Let A be Noetherian, then every ideal of A contains a product of prime ideals.

Subproof. Suppose, towards contradiction, that there exists some ideal I of A that does not contain a product of prime ideals. Let $\mathcal{J}$ be the set of such ideals of A, then $\mathcal{J} \neq \varnothing$, and we can take a maximal element of $\mathcal{J}$, namely J. By definition, J is not prime, therefore there exists $a, b \in A$ such that $a \notin J$ and $b \notin J$, but $ab \in J$. Now $J \subsetneq J + Aa$ and $J \subsetneq J + Ab$, therefore $J + Aa, J + Ab \notin J$, therefore $J + Aa$ and $J + Ab$ both contain product of prime ideals. But now $(J + Aa)(J + Ab)$ should also contain products of prime ideals, but by distribution this is just $J^2 + Ja + Jb + Aab$, which is contained in J because every term is contained in J, so J contains a product of prime ideals as well, contradiction. ■

In particular, (0) contains a product of prime ideals, in particular (0) equals to this product, but every prime ideal is maximal, therefore $(0) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$ becomes the product of maximal ideals (which may not necessarily be distinct), hence we have a descending chain of ideals

$$A \supseteq \mathfrak{m}_1 \supseteq \mathfrak{m}_1\mathfrak{m}_2 \supseteq \cdots \supseteq \mathfrak{m}_1 \cdots \mathfrak{m}_n = (0),$$

and in particular $(\mathfrak{m}_1 \cdots \mathfrak{m}_{i-1})/(\mathfrak{m}_1 \cdots \mathfrak{m}_i)$ is a finite-dimensional since A is Noetherian, and it has a natural structure as a $A/\mathfrak{m}_i$-vector space. From the short exact sequence

$$0 \longrightarrow \mathfrak{m}_1 \cdots \mathfrak{m}_i \longrightarrow \mathfrak{m}_1 \cdots \mathfrak{m}_{i-1} \longrightarrow (\mathfrak{m}_1 \cdots \mathfrak{m}_{i-1})/(\mathfrak{m}_1 \cdots \mathfrak{m}_i) \longrightarrow 0$$

we know the two sides of the sequence are Artinian, hence the central term is Artinian. Proceeding inductively, we know that $\mathfrak{m}_1$ is Artinian, and $R/\mathfrak{m}_1$ would also be Artinian, hence A is Artinian.

($\Rightarrow$): Now suppose A is Artinian, and we want to show that every prime ideal is maximal, and (0) is a product of maximal ideals. The result then follows from the argument above.

Lemma 0.15. Every Artinian domain is a field.

Subproof. Let $0 \neq a \in A$, then consider the chain

$$(a) \supseteq (a^2) \supseteq \cdots \supseteq (a^n) \supseteq \cdots$$

and by the Artinian property, for some large enough n the descending chain stops. Hence, we have $a^n = \lambda a^{n+1}$ for some large enough n and some $\lambda \in A$. Hence, $a^n(1 - \lambda a) = 0$, by the cancellation property of a domain, since $a \neq 0$, we must have $\lambda a = 1$, therefore a is a unit, as desired. ■

Corollary 0.16. Let A be Artinian, then every prime ideal of A is maximal.

Finally, it suffices to show that $(0) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$. Let $\mathfrak{J}$ be the set of finite products of maximal ideals, then $\mathfrak{J}$ has a minimal element, and it suffices to show that this element is (0). Suppose not, let $I \neq (0)$ be a minimal element of R. For any two ideals α, β of A, let $(\alpha : \beta) = \{a \in A \mid a\beta \subseteq \alpha\}$. Note that this has a natural structure as an ideal of A. Let $J = ((0) : I)$, and suppose $J = A$, then $I = 0$, contradiction, so $J \neq A$ is a proper ideal of A, now consider A/J which is Artinian, then let $\mathfrak{G}$ be the set of all non-zero ideals of A/J, so $\mathfrak{G}$ has a minimal element as well, call it $\bar{H}$. Let $H = \pi^{-1}(\bar{H})$ where $\pi : A \to A/J$, so we have $J \subsetneq H$, thus let $P = (J : H)$.

Claim 0.17. P is a prime ideal.

Subproof. Given $c, d \notin P$, we want to show that $cd \notin P$. Indeed, consider $J \subsetneq J + cH \subseteq H$, then since H is minimal, then $J + cH = H$, and similarly we have that $J + dH = H$. Therefore, we have that $J + cdH = J + c(dH + J) = J + cH = H$, hence we know $cd \notin P$, as desired. ■

Now $P = (J : H)$ and $J = (0 : I)$, the by definition we have $PHI = (0)$. Since P is a prime ideal, then P is maximal, and now

$$(0 : PI) \supseteq H \supsetneq J = (0 : I)$$

Therefore $PI \subsetneq I$, where I is a minimal element, contradiction, hence (0) is a product of maximal ideals. □

Definition 0.18 (Short Exact Sequence)**.** Consider the sequence

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} T \longrightarrow 0$$

This is called a short exact sequence if $\ker(f) = 0$, $\operatorname{im}(g) = T$, and $\ker(g) = \operatorname{im}(f)$. In particular, one slot of the sequence is said to be exact if the kernel of the previous map equals to the image of the subsequent map.

Definition 0.19 (Flat Module)**.** Let M be an A-module, then we say M is a flat A-module if for every short exact sequence

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

the tensored sequence

$$0 \longrightarrow M \otimes_A N_1 \longrightarrow M \otimes_A N_2 \longrightarrow M \otimes_A N_3 \longrightarrow 0$$

remains exact.

Remark 0.20. We already know that the tensor functor is right exact, namely given the short exact sequence above, then

$$M \otimes_A N_1 \longrightarrow M \otimes_A N_2 \longrightarrow M \otimes_A N_3 \longrightarrow 0$$

is exact.

Exercise 0.21. Let M be an A-module, and if there exists a short exact sequence of A-modules

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

where N_1 and N_2 are finitely-generated as A-modules, and such that tensoring M preserves the short exact sequence, then M is flat.

Definition 0.22 (Multiplicatively Closed Subset)**.** Let A be a commutative ring and M be an A-module. Let $S \subseteq A$ be a subset. We say S is a multiplicatively closed subset of A if $1 \in S$, $0 \notin S$, and whenever $s_1, s_2 \in S$, then $s_1 s_2 \in S$.

Definition 0.23 (Localization)**.** Let $S \subseteq A$ be a multiplicatively closed subset, and let M be an A-module, then $S^{-1}M = (M \times S)/\sim$, where $\sim$ is an equivalence relation defined by the following: $(m_1, s_1) \sim (m_2, s_2)$ if and only if there exists $t \in S$ such that $t(m_1 s_2 - m_2 s_1) = 0$. $S^{-1}M$ is said to be the localization of M at S.

Given $(m, s) \in M \times S$, we write $\overline{(m, s)}$ to be the equivalence class in $S^{-1}M$ represented by (m, s).

Exercise 0.24. Similarly, one can define the localization $S^{-1}A$ of A at S. In fact, $S^{-1}A$ inherits a ring structure from A, namely

- $\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{a_1 s_2 + a_2 s_1}{s_1 s_2}$,
- $\frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{a_1 a_2}{s_1 s_2}$,
- $\frac{1}{s} \cdot \frac{s}{1} = \frac{1}{1} = 1$.

Remark 0.25. Note that a ring structure does not guarantee every element to have a multiplicative inverse. The localization of A at S ensures that every element of S now becomes invertible in the new ring $S^{-1}A$. In particular, this induces a ring homomorphism

$$f : A \to S^{-1}A$$
$$a \mapsto \frac{a}{1}$$

This homomorphism is injective if A is a domain.

Remark 0.26. Let I be an ideal of A.

- Consider the ring homomorphism $f : A \to S^{-1}A$ above, then
$$S^{-1}I = IS^{-1}A = f(I)S^{-1}A.$$
In particular, $f^{-1}(IS^{-1}A) \supseteq I$.
- If $I \cap S \neq \varnothing$, then $IS^{-1}A = S^{-1}A$.
- If P is a prime ideal of A such that $P \cap S = \varnothing$, then $f^{-1}(PS^{-1}A) = P$.
- Let M be an A-module, then if $N \subseteq M$ is a submodule, then $S^{-1}N \subseteq S^{-1}M$. That is, given an exact sequence
$$0 \longrightarrow N \longrightarrow M$$
then we obtain an exact sequence
$$0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M$$
Indeed, given $0 \to N \xrightarrow{f} M$, say we have it sending $\frac{n}{1} \mapsto \frac{f(n)}{1} = 0$, then there exists $s \in S$ such that $sf(n) = 0$, so $f(sn) = 0$, therefore $sn = 0$ by injection, hence $\frac{n}{1} = 0$ in $S^{-1}N$ as well.

Exercise 0.27. The localization functor is exact.

1 Primary Decomposition Theorem

2 Filtered Rings and Modules, Completions

3 Dimension Theory

4 Integral Extensions

5 Noether's Normalization Lemma

6 Homological Algebra