

MATH 540 Notes

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1 ABSTRACT MEASURE THEORY

1.1 INTRODUCTION

Definition 1.1. Let X be an (non-empty) underlying space we are working over. We denote $\mathcal{P}(X)$ to be the power set of X , i.e., the set of all subsets of X .

Example 1.2. Let $X = \{1, 2\}$, then $\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

Remark 1.3. If X is a finite set of size n , then $\mathcal{P}(X)$ is a finite set of size 2^n .

We will consider a subcollection \mathcal{A} of subsets of X , i.e., a subset of the power set. We will try to define this as an algebra. Note that an algebra is just a ring with a module structure with respect to some other ring.

Definition 1.4. $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra on X if it is

- a. closed under finite union, i.e., given $E_1, E_2 \in \mathcal{A}$, then $E_1 \cup E_2 \in \mathcal{A}$, and
- b. closed under complements, i.e., if $E \in \mathcal{A}$, then the complement $E^c \in \mathcal{A}$ as well.

Remark 1.5. An algebra \mathcal{A} would be closed under finite intersection. Indeed, for any $E_1, E_2 \in \mathcal{A}$, we have $E_1 \cap E_2 \in \mathcal{A}$ if and only if $(E_1 \cap E_2)^c \in \mathcal{A}$, if and only if $E_1^c \cup E_2^c \in \mathcal{A}$, which is true by definition.

Lemma 1.6. If \mathcal{A} is a non-empty algebra on X , then $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$.

Proof. Since \mathcal{A} is non-empty, take $E \in \mathcal{A}$, then $\emptyset = E \cap E^c \in \mathcal{A}$ as well. Also, $X = E \cup E^c \in \mathcal{A}$. □

Example 1.7. Let X be a set, and let $\mathcal{A} = \{\emptyset, X\} \subseteq \mathcal{P}(X)$. It is easy to verify that \mathcal{A} is an algebra.

Definition 1.8. Let $\emptyset \neq \mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra, then we say \mathcal{A} is a σ -algebra on X if

- a. closed under countable union, i.e., if $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, then $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$;
- b. if $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$.

Lemma 1.9. If $\mathcal{A} \neq \emptyset$ is a σ -algebra on X , then $\{\emptyset, X\} \subseteq \mathcal{A}$ is a σ -algebra.

Example 1.10. Let X be an uncountable set, let $\mathcal{A} = \{E \subseteq X : E \text{ is countable or } E^c \text{ is countable}\}$, then \mathcal{A} is a σ -algebra on X .

Theorem 1.11. Suppose a non-empty algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ such that,

- if $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, and E_j 's are pairwise disjoint, then $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$,

then \mathcal{A} is a σ -algebra on X .

Proof. Take $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, we will show that $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$. To do this, we will rearrange the sets. Let $F_1 = E_1$, let

$F_2 = E_2 \setminus E_1$, let $F_3 = E_3 \setminus (E_1 \cup E_2)$, and so on, such that let $F_k = E_k \setminus \bigcup_{i=1}^{k-1} E_i$. We note

$$\begin{aligned} F_k &= E_k \cap \left(\bigcup_{j=1}^{k-1} E_j \right)^c \\ &= E_k \cap \left(\bigcap_{j=1}^{k-1} E_j^c \right) \in \mathcal{A}. \end{aligned}$$

One can also verify that $\bigcup_{j=1}^{\infty} E_j = \bigcup_{k=1}^{\infty} F_k$, and that F_k 's are disjoint from the definition. □

Definition 1.12. Let X be a non-empty space. A topology on X is a family \mathcal{F} of subsets of X satisfying the following conditions:

- i. $\emptyset, X \in \mathcal{F}$;
- ii. \mathcal{F} is closed under arbitrary union;
- iii. \mathcal{F} is closed under finite intersection.

Every member of \mathcal{F} is now called an open subset of X . A complement of an open subset of X is called a closed subset.

Definition 1.13. Let $\mathcal{A}_1, \mathcal{A}_2$ be σ -algebras. We say \mathcal{A}_1 is smaller than \mathcal{A}_2 if $\mathcal{A}_1 \subseteq \mathcal{A}_2$, and equivalently \mathcal{A}_2 is larger than \mathcal{A}_1 .

Definition 1.14. Let \mathcal{F} be a family of subsets of X , the smallest σ -algebra containing \mathcal{F} is called the σ -algebra generated by \mathcal{F} . This is denoted by $\mathcal{M}(\mathcal{F})$.

Lemma 1.15. Let \mathcal{F} be a family of subsets of X . Suppose $\mathcal{F} \subseteq \mathcal{A}$ where \mathcal{A} is a σ -algebra, then $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{A}$.

Proof. Obvious. □

Definition 1.16. Let \mathcal{F} be a topology on X , then we say (X, \mathcal{F}) is a topological space. We say $\mathcal{M}(\mathcal{F})$ is the Borel σ -algebra on X , denoted by $\mathcal{B}_X = \mathcal{B}_{X, \mathcal{F}}$. Any member of \mathcal{B}_X is called a Borel set.

Example 1.17. Let $X = \mathbb{R}$, we denote the corresponding Borel σ -algebra to be $\mathcal{B}_{\mathbb{R}}$.

Definition 1.18. A G_δ -set is a countable intersection of open subsets of X . A F_σ -set is a countable union of closed subsets of X .

Theorem 1.19. Both G_δ -sets and F_σ -sets are Borel sets, that is, $G_\delta, F_\sigma \subseteq \mathcal{B}_X$.

Proof. We will prove that any G_δ -set E is a Borel set, and similarly any F_σ -set is a Borel set. By definition $E = \bigcap_{j=1}^{\infty} O_j$, where each O_j is an open subset. To show $E \in \mathcal{B}_X$, we show that $E^c \in \mathcal{B}_X$. Note that $E^c = \left(\bigcap_{j=1}^{\infty} O_j \right)^c = \bigcup_{j=1}^{\infty} O_j^c$. Since $O_j \in \mathcal{B}_X$ for all j , then $O_j^c \in \mathcal{B}_X$ as well. Therefore, $E^c \in \mathcal{B}_X$ since a σ -algebra \mathcal{B}_X is closed under countable unions. □

Definition 1.20. Let X_1, \dots, X_n be non-empty spaces. The product space is $\prod_{j=1}^n X_j$. Define $\pi_j : \prod_{i=1}^n X_i \rightarrow X_j$ by $\pi_j(x_1, \dots, x_n) = x_j$. Let \mathcal{A}_j be a σ -algebra on X_j , the product σ -algebra on $\prod_{i=1}^n X_j$ is the σ -algebra generated by $\{\pi_j^{-1}(E_j) : E_j \in \mathcal{A}_j \forall j \in \{1, \dots, n\}\}$. The product σ -algebra is denoted by $\bigotimes_{j=1}^n \mathcal{A}_j = \prod_{j=1}^n \mathcal{A}_j$.

Example 1.21. $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_{j=1}^n \mathcal{B}_{\mathbb{R}}$.

1.2 MEASURES

Definition 1.22. Let \mathcal{A} be a σ -algebra on X . A measure μ on X and \mathcal{A} is a function $\mu : \mathcal{A} \rightarrow [0, \infty]$ such that

- a. $\mu(\emptyset) = 0$;
- b. if $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$ and E_j 's are disjoint, then $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$.

We then say (X, \mathcal{A}) is a measureable space. A measureable space is a triple (X, \mathcal{A}, μ) with measure μ specified.

Definition 1.23. Let μ be a measure on (X, \mathcal{A}) .

1. If $\mu(X) < \infty$, then we say μ is a finite measure. In particular, if $\mu(X) = 1$, this is a probability measure.
2. If $X = \bigcup_{j=1}^{\infty} E_j$ such that $\mu(E_j) < \infty$ for all $j \in \mathbb{N}$, then we say μ is σ -finite.
3. If for all $E \in \mathcal{A}$ with $\mu(E) = \infty$, there is $F \in \mathcal{A}$ such that $F \subseteq E$ and $0 < \mu(F) < \infty$, then we say μ is semi-finite.

Remark 1.24. A σ -finite measure is semi-finite. However, the converse is not true.

Example 1.25. Let $f : X \rightarrow [0, \infty]$ be a function. For any $E \subseteq \mathcal{P}(E)$, we can define a measure $\mu(E) = \sum_{x \in E} f(x)$. Note that the summation makes sense only when E is finite. In case E is infinite, we should define $\sum_{x \in E} f(x) = \sup \left\{ \sum_{x \in F} f(x) : F \subseteq E \text{ for finite } F \right\}$. Let μ be a measure on $\mathcal{P}(X)$.

- If $f(x) \equiv 1$ for all $x \in X$, then $\mu(E) = \sum_{x \in E} 1 = \text{Card}(E)$. In this case, μ is called a counting measure.
- Suppose $x_0 \in X$ is fixed. Define

$$f(x) = \begin{cases} 1, & \text{if } x = x_0 \\ 0, & \text{if } x \neq x_0 \end{cases}$$

then for any $E \in \mathcal{P}(X)$,

$$\mu(E) = \begin{cases} 1, & \text{if } x_0 \in E \\ 0, & \text{if } x_0 \notin E \end{cases}$$

This is called the Dirac measure of x_0 .

Definition 1.26. Let (X, \mathcal{A}, μ) be a measure space. A set $E \subseteq \mathcal{A}$ is called a null set if $\mu(E) = 0$.

If a statement about points $x \in X$ is true except for null sets, then we say the statement is true almost everywhere.

Example 1.27. Suppose $f(x) \leq 1$ for all $x \in X$, then we say f is bounded above by 1 everywhere. If we want to weaken this statement, we can say $f(x) \leq 1$ almost everywhere $x \in X$, which is true if and only if $\mu(\{x \in X : f(x) > 1\}) = 0$.

Theorem 1.28. Let $E, F \in \mathcal{A}$ be such that $E \subseteq F$, then $\mu(E) \leq \mu(F)$.

Proof. We can write $F = E \cup (E \setminus F)$, then

$$\begin{aligned} \mu(F) &= \mu(E) + \mu(F \setminus E) \\ &\geq \mu(E) \end{aligned}$$

since $\mu(F \setminus E) \geq 0$. □

Theorem 1.29 (Sub-additivity). Let $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, then $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu(E_j)$.

Proof. Set $F_1 = E_1$ and let $F_k = E_k \setminus \left(\bigcup_{j=1}^{k-1} E_j\right)$ be defined inductively, then $\bigcup_{k \in \mathbb{N}} F_k = \bigcup_{j \in \mathbb{N}} E_j$. Since F_k 's are disjoint, we have

$$\begin{aligned} \mu\left(\bigcup_{j \in \mathbb{N}} E_j\right) &= \mu\left(\bigcup_{k \in \mathbb{N}} F_k\right) \\ &= \sum_{k=1}^{\infty} \mu(F_k) \\ &= \sum_{k=1}^{\infty} \mu(E_k) \end{aligned}$$

$$= \sum_{j=1}^{\infty} \mu(E_j)$$

by [Theorem 1.28](#). □

Theorem 1.30. Let $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$.

- a. (Continuity from below): If $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_j \subseteq \cdots$ for all j , then $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j \rightarrow \infty} \mu(E_j)$.
- b. (Continuity from above): If $E_1 \supseteq E_2 \supseteq \cdots \supseteq E_j \supseteq \cdots$ for all $j \in \mathbb{N}$, then $\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j \rightarrow \infty} \mu(E_j)$ if $\mu(E_1) < \infty$.

In particular, the limits on the right exist on $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.

Example 1.31. Let μ be the counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. For each $j \in \mathbb{N}$, we define $E_j = \{n \in \mathbb{N} : n > j\}$. Therefore $E_1 \supseteq E_2 \supseteq \cdots$ is a decreasing sequence of sets. Note that $\mu(E_1) = \mu(\{n \in \mathbb{N}\}) = \mathbb{N} = \infty$, and $\lim_{j \rightarrow \infty} \mu(E_j) =$

$$\lim_{j \rightarrow \infty} \infty = \infty, \text{ but } \mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \mu(\emptyset) = 0.$$

Proof.

- a. Set $E_0 = \emptyset$. Now

$$\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} (E_j \setminus E_{j-1})$$

and therefore

$$\begin{aligned} \mu\left(\bigcup_{j=1}^{\infty} E_j\right) &= \mu\left(\bigcup_{j=1}^{\infty} (E_j \setminus E_{j-1})\right) \\ &= \sum_{j=1}^{\infty} \mu(E_j \setminus E_{j-1}) \\ &= \lim_{k \rightarrow \infty} \sum_{j=1}^k \mu(E_j \setminus E_{j-1}) \\ &= \lim_{k \rightarrow \infty} \mu\left(\bigcup_{j=1}^k E_j \setminus E_{j-1}\right) \\ &= \lim_{k \rightarrow \infty} \mu(E_k) \\ &= \lim_{j \rightarrow \infty} \mu(E_j). \end{aligned}$$

- b. For any $j \in \mathbb{N}$, set $F_j = E_1 \setminus E_j$. Note that $F_j \subseteq F_{j+1}$ since $E_j \supseteq E_{j+1}$. This is now an increasing sequence as in part a. By part a., we know $\mu\left(\bigcup_{j=1}^{\infty} F_j\right) = \lim_{j \rightarrow \infty} \mu(F_j)$. Now note that

$$\begin{aligned} \bigcup_{j=1}^{\infty} F_j &= \bigcup_{j=1}^{\infty} (E_1 \setminus E_j) \\ &= \bigcup_{j=1}^{\infty} (E_1 \cap E_j^c) \end{aligned}$$

$$\begin{aligned}
&= E_1 \cap \bigcup_{j=1}^{\infty} E_j^c \\
&= E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j \right)^c \\
&= \left(\left(E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j \right)^c \right) \cup \left(\bigcap_{j=1}^{\infty} E_j \right) \right) \cap \left(\bigcap_{j=1}^{\infty} E_j \right)^c \\
&= \left(E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j \right)^c \right) \cup \left(\bigcap_{j=1}^{\infty} E_j \right).
\end{aligned}$$

Note that $E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j \right)^c$ and $\bigcap_{j=1}^{\infty} E_j$ are disjoint, therefore by property of measure we have

$$\begin{aligned}
\mu(E_1) &= \mu \left(E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j \right)^c \right) + \mu \left(\bigcap_{j=1}^{\infty} E_j \right) \\
&= \mu \left(\bigcup_{j=1}^{\infty} F_j \right) + \mu \left(\bigcap_{j=1}^{\infty} E_j \right) \\
&= \lim_{j \rightarrow \infty} \mu(F_j) + \mu \left(\bigcap_{j=1}^{\infty} E_j \right).
\end{aligned}$$

Recall that $F_j = E_1 \setminus E_j$ for all j , therefore $E_1 = F_j \cup F_j^c = F_j \cup E_j$, where F_j and E_j are disjoint, therefore $\mu(E_1) = \mu(F_j) + \mu(E_j)$. Since $\mu(E_1) < \infty$, and F_j is a subset of E_1 and hence also a real number, then $\mu(E_1)$ is a sum of two real numbers. Therefore, we have $\mu(E_1) - \mu(E_j) = \mu(F_j)$. With this, we have

$$\begin{aligned}
\mu(E_1) &= \lim_{j \rightarrow \infty} (\mu(E_1) - \mu(E_j)) + \mu \left(\bigcap_{j=1}^{\infty} E_j \right) \\
&= \mu(E_1) - \lim_{j \rightarrow \infty} (\mu(E_j)) + \mu \left(\bigcap_{j=1}^{\infty} E_j \right).
\end{aligned}$$

In particular, we get

$$\lim_{j \rightarrow \infty} (\mu(E_j)) = \mu \left(\bigcap_{j=1}^{\infty} E_j \right).$$

□

1.3 OUTER MEASURE

Definition 1.32. An outer measure μ^* on X (or $\mathcal{P}(X)$) is a function $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ such that

- i. $\mu^*(\emptyset) = 0$,
- ii. $\mu^*(A) \leq \mu^*(B)$ for all $A \subseteq B \subseteq X$,
- iii. σ -subadditivity: $\mu^* \left(\bigcup_{j=1}^{\infty} A_j \right) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$.

Example 1.33. Let $\rho : \mathcal{A} \rightarrow [0, \infty]$ be such that $\rho(\emptyset) = 0$, where $\mathcal{A} \subseteq \mathcal{P}(X)$ is a subcollection (but not necessarily an algebra) such that $\emptyset, X \in \mathcal{A}$.

For all $A \in \mathcal{P}(X)$, i.e., $A \subseteq X$, we define

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \rho(E_j) : E_j \in \mathcal{A} \text{ and } A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}.$$

Theorem 1.34. μ^* defined in [Example 1.33](#) is an outer measure.

Proof.

- i. Let $E_j = \emptyset$ for all $j \in \mathbb{N}$, then $\emptyset \subseteq \bigcup_{j=1}^{\infty} E_j$, and so

$$\sum_{j=1}^{\infty} \rho(E_j) = \sum_{j=1}^{\infty} \rho(\emptyset) = \sum_{j=1}^{\infty} 0 = 0$$

and therefore $\mu^*(\emptyset) = 0$.

- ii. Let $A \subseteq B \subseteq X$. If $B \subseteq \bigcup_{j=1}^{\infty} E_j$, we have $A \subseteq \bigcup_{j=1}^{\infty} E_j$, then

$$\left\{ \sum_{j=1}^{\infty} \rho(E_j) : E_j \in \mathcal{A}, B \subseteq \bigcup_{j=1}^{\infty} E_j \right\} \subseteq \left\{ \sum_{j=1}^{\infty} \rho(E_j) : E_j \in \mathcal{A}, A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}.$$

In particular, given subsets $S_1 \subseteq S_2$, then $\inf S_2 \leq \inf S_1$ and $\sup S_1 \leq \sup S_2$. This implies $\mu^*(A) \leq \mu^*(B)$.

- iii. We want to show $\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$. Now for any $j \in \mathbb{N}$, we have

$$\mu^*(A_j) = \inf \left\{ \sum_{k=1}^{\infty} \rho(E_k) : E_k \in \mathcal{A} \text{ and } A_j \subseteq \bigcup_{k=1}^{\infty} E_k \right\}.$$

For any $\varepsilon > 0$, we note that $\mu^*(A_j) + \varepsilon \cdot 2^{-j}$ is not a lower bound of $\left\{ \sum_{k=1}^{\infty} \rho(E_k) : E_k \in \mathcal{A} \text{ and } A_j \subseteq \bigcup_{k=1}^{\infty} E_k \right\}$.

Then there exists $E_k^{(j)} \in \mathcal{A}$ for $k \in \mathbb{N}$ such that $A_j \subseteq \bigcup_{k=1}^{\infty} E_k^{(j)}$ and $\sum_{k=1}^{\infty} \rho(E_k^{(j)}) \leq \mu^*(A_j) + \varepsilon \cdot 2^{-j}$. Summing with respect to j , we get

$$\begin{aligned} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \rho(E_k^{(j)}) &\leq \sum_{j=1}^{\infty} \mu^*(A_j) + \sum_{j=1}^{\infty} \varepsilon \cdot 2^{-j} \\ &= \sum_{j=1}^{\infty} \mu^*(A_j) + \varepsilon. \end{aligned}$$

Note that

$$\bigcup_{j=1}^{\infty} A_j \subseteq \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} E_k^{(j)}$$

is a countable union of subsets of \mathcal{A} . We will calculate the value over μ^* . By definition of μ^* , we have

$$\begin{aligned} \mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) &\leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \rho(E_k^{(j)}) \\ &\leq \sum_{j=1}^{\infty} \mu^*(A_j) + \varepsilon. \end{aligned}$$

Since this is true for all $\varepsilon > 0$, then take $\varepsilon \rightarrow 0$, we are done. □

Definition 1.35. Let μ^* be an outer measure on $(X, \mathcal{P}(X))$. A set $A \subseteq X$ is called μ^* -measurable if $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for all $E \subseteq X$.

Remark 1.36. First note that $\mu^*(E) = \mu^*((E \cap A) \cup (E \cap A^c))$, therefore $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for all $E \subseteq X$.

Theorem 1.37 (Fundamental Theorem of Measure Theory). Let μ^* be an outer measure on X . Let \mathcal{A} be the collection of all μ^* -measurable set, then \mathcal{A} is a σ -algebra, and $\mu^*|_{\mathcal{A}}$ is a measure on \mathcal{A} , i.e., (X, \mathcal{A}, μ^*) is a measure space.

Proof. We first prove that \mathcal{A} is an algebra. To see \mathcal{A} is closed under complement, we have $A \in \mathcal{A}$ if and only if $A^c \in \mathcal{A}$ by the definition of measurable set. To show \mathcal{A} is closed under finite union, suppose $A, B \in \mathcal{A}$, and we want to show $A \cup B \in \mathcal{A}$, which is true if and only if $\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$ for all $E \subseteq X$, hence it suffices to show that $\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$. We have

$$\begin{aligned} \mu^*(E \cap (A \cup B)) &= \mu^*(E \cap (A \cup B) \cap A) + \mu^*(E \cap (A \cup B) \cap A^c) \\ &= \mu^*(E \cap A) + \mu^*(E \cap B \cap A^c) \end{aligned}$$

and

$$\begin{aligned} \mu^*(E \cap (A \cup B)^c) &= \mu^*(E \cap (A \cup B)^c \cap A) + \mu^*(E \cap (A \cup B)^c \cap A^c) \\ &= \mu^*(\emptyset) + \mu^*(E \cap A^c \cap B^c) \\ &= \mu^*(E \cap A^c \cap B^c). \end{aligned}$$

Therefore

$$\begin{aligned} \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c) &= \mu^*(E \cap A) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\ &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E) \end{aligned}$$

where the last two steps follow from the fact that $A, B \in \mathcal{A}$ are μ^* -measurable. Therefore, \mathcal{A} is an algebra. We now want to show that it is a σ -algebra. It suffices to prove that \mathcal{A} is closed under disjoint σ -unions. Let $A_j \in \mathcal{A}$ for all $j \in \mathbb{N}$ where they are pairwise disjoint, and we want to show that $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$. That is,

$$\mu^*(E) = \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)\right) + \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)^c\right)$$

for all $E \subseteq X$.

Lemma 1.38. For a pairwise disjoint family $A_1, \dots, A_n \in \mathcal{A}$,

$$\mu^*\left(E \cap \bigcup_{j=1}^n A_j\right) = \sum_{j=1}^n \mu^*(E \cap A_j).$$

Subproof. We proceed by induction. For $n = 1$, this is obviously true. Now suppose $n > 1$. To simplify the notation, let $B_n = \bigcup_{j=1}^n A_j$, and use the convention that $B_0 = \emptyset$. Now

$$\begin{aligned} \mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) \\ &= \sum_{i=1}^n \mu^*(E \cap A_i) + \mu^*(E \cap B_0) \\ &= \sum_{i=1}^n \mu^*(E \cap A_i) \\ &= \sum_{i=1}^n \mu^*(E \cap A_i) \end{aligned}$$

for all $n \in \mathbb{N}$. This finishes the proof. ■

Now for any $E \subseteq X$, we have

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \\ &= \sum_{j=1}^n \mu^*(E \cap A_j) + \mu^*(E \cap B_n^c) \\ &\geq \sum_{j=1}^n \mu^*(E \cap A_j) + \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)^c\right)\end{aligned}$$

since $B_n = \bigcup_{j=1}^n A_j \subseteq \bigcup_{j=1}^{\infty} A_j$. Now take $n \rightarrow \infty$, we get

$$\begin{aligned}\mu^*(E) &\geq \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*\left(\left(\bigcup_{j=1}^{\infty} A_j\right)^c\right) \\ &\geq \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)\right) + \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)^c\right) \\ &\geq \mu^*(E).\end{aligned}$$

This forces all inequalities here to be equality, therefore

$$\mu^*(E) = \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)\right) + \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)^c\right)$$

as desired. Finally, we need to show that the restriction still gives a measure. We already know

$$\mu^*(E) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*\left(\left(\bigcup_{j=1}^{\infty} A_j\right)^c\right)$$

for any $E \subseteq X$, then in particular take $E = \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$ to be the disjoint union, then this forces

$$\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*(\emptyset) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j).$$

Therefore $\mu^*|_{\mathcal{A}}$ is a measure. □

Definition 1.39. A measure μ is said to be complete if its domain contains all subsets of null sets.

Example 1.40. Let $X = \{a, b\}$, $\mathcal{A} = \{\emptyset, \{a, b\}\}$. Define $\mu : \mathcal{A} \rightarrow [0, \infty]$ by setting $\mu^*(X) = 0$, $\mu^*(\emptyset) = 0$. This is not a complete measure because $\{a\} \notin \mathcal{A}$.

Theorem 1.41. Let \mathcal{A} be the collection of all μ^* -measurable sets, then the measure $\mu^*|_{\mathcal{A}}$ is complete.

Proof. Let N be any null set in \mathcal{A} , i.e., $\mu^*(N) = 0$. Take an arbitrary subset $A \subseteq N$, we need to show $A \in \mathcal{A}$. Since $\mu^*(N) = 0$, then $\mu^*(A) = 0$ as well. For any $E \subseteq X$, we prove $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$. It is clear that

$$\begin{aligned}\mu^*(E) &\leq \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &\leq \mu^*(A) + \mu^*(E \cap A^c) \\ &\leq \mu^*(N) + \mu^*(E \cap A^c) \\ &= \mu^*(E \cap A^c) \\ &= \mu^*(E).\end{aligned}$$

by the subadditivity of μ^* . □

Definition 1.42. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra. A function $\mu_0 : \mathcal{A} \rightarrow [0, \infty]$ is a pre-measure if

- i. $\mu_0(\emptyset) = 0$,
- ii. if $A_j \in \mathcal{A}$ for all $j \in \mathbb{N}$ with $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$, and they are pairwise disjoint, then $\mu_0\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu_0(A_j)$.

Therefore, the difference of a pre-measure from a measure is that a pre-measure is not defined on a σ -algebra.

Theorem 1.43. Let μ_0 be a pre-measure, then $\mu_0(A) \leq \mu_0(B)$ if $A, B \in \mathcal{A}$ are such that $A \subseteq B$.

Proof. We write $B = (B \setminus A) \cup A$, where $B \setminus A = B \cap A^c \in \mathcal{A}$, therefore

$$\begin{aligned} \mu_0(B) &= \mu_0(B \setminus A) + \mu_0(A) \\ &\geq \mu_0(A). \end{aligned}$$

□

Definition 1.44. Given a pre-measure μ_0 , we extend it to an outer measure as follows: for any $E \subseteq X$, define $\mu^*(E) = \inf\left\{\sum_{j=1}^{\infty} \mu_0(A_j) : E \subseteq \bigcup_{j=1}^{\infty} A_j, A_j \in \mathcal{A}\right\}$.

Theorem 1.45 (Carathéodory's Extension Theorem). Let μ^* be the outer measure induced by μ_0 specified in Definition 1.44, then

- i. $\mu^*|_{\mathcal{A}} = \mu_0$, or equivalently, for any $A \in \mathcal{A}$, we have $\mu^*(A) = \mu_0(A)$;
- ii. if $A \in \mathcal{A}$, then A is μ^* -measurable.

Proof.

- i. We want to show that for any $E \in \mathcal{A}$, $\mu^*(E) = \mu_0(E)$. To show $\mu^*(E) \leq \mu_0(E)$, we choose $A_1 = E \in \mathcal{A}$, and $A_j = \emptyset$ for all $j \geq 2$, then $E \subseteq \bigcup_{j=1}^{\infty} A_j$, therefore

$$\begin{aligned} \mu^*(E) &\leq \sum_{j=1}^{\infty} \mu_0(A_j) \\ &= \mu_0(E). \end{aligned}$$

It now suffices to show that $\mu_0(E)$ is a lower bound of $\left\{\sum_{j=1}^{\infty} \mu_0(A_j) : E \subseteq \bigcup_{j=1}^{\infty} A_j, A_j \in \mathcal{A}\right\}$. Let $A_j \in \mathcal{A}$ and

$\bigcup_{j=1}^{\infty} A_j \supseteq E$. We prove that $\mu_0(E) \leq \sum_{j=1}^{\infty} \mu_0(A_j)$. For any $n \in \mathbb{N}$, define $B_n = E \cap \left(A_n \setminus \bigcup_{j=1}^{n-1} A_j\right)$, therefore

$\bigcup_{n=1}^{\infty} B_n = E \cap \left(\bigcup_{j=1}^{\infty} A_j\right) = E$ where B_n 's are disjoint. We have

$$\begin{aligned} \mu_0(E) &= \mu_0\left(\bigcup_{n=1}^{\infty} B_n\right) \\ &= \sum_{n=1}^{\infty} \mu_0(B_n) \\ &\leq \sum_{n=1}^{\infty} \mu_0(A_n) \\ &= \sum_{j=1}^{\infty} \mu_0(A_j). \end{aligned}$$

- ii. For any $A \in \mathcal{A}$, we want to prove that $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for all $E \subseteq X$. It suffices to show that for any $E \subseteq X$, we have $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$.

Pick arbitrary $\varepsilon > 0$, then $\mu^*(E) + \varepsilon$ is not a lower bound of $\{\sum_{j=1}^{\infty} \mu_0(A_j) : E \subseteq \bigcup_{j=1}^{\infty} A_j, A_j \in \mathcal{A}\}$. Therefore, there exists some $A_j \in \mathcal{A}$ such that $E \subseteq \bigcup_{j=1}^{\infty} A_j$ and $\sum_{j=1}^{\infty} \mu_0(A_j) \leq \mu^*(E) + \varepsilon$. Since $\mu_0(A_j) = \mu_0(A_j \cap A) + \mu_0(A_j \cap A^c)$, then

$$\begin{aligned} \sum_{j=1}^{\infty} \mu_0(A_j) &= \sum_{j=1}^{\infty} \mu_0(A_j \cap A) + \sum_{j=1}^{\infty} \mu_0(A_j \cap A^c) \\ &= \sum_{j=1}^{\infty} \mu^*(A_j \cap A) + \sum_{j=1}^{\infty} \mu^*(A_j \cap A^c) \\ &\geq \mu^*\left(\left(\bigcup_{j=1}^{\infty} A_j\right) \cap A\right) + \mu^*\left(\left(\bigcup_{j=1}^{\infty} A_j\right) \cap A^c\right) \\ &\geq \mu^*(E \cap A) + \mu^*(E \cap A^c). \end{aligned}$$

Let $\varepsilon \rightarrow 0$, then $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$, as desired. □

Theorem 1.46. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra, and let μ_0 be a pre-measure on \mathcal{A} . Define $\mathcal{M}(\mathcal{A})$ to be the σ -algebra generated by \mathcal{A} .

- The outer measure μ^* induced by μ_0 defines a measure function on $\mathcal{M}(\mathcal{A})$, and $\mu^*|_{\mathcal{A}} = \mu_0$.
- If $\tilde{\mu}$ is another measure on $\mathcal{M}(\mathcal{A})$ that extends μ_0 , then $\tilde{\mu}(E) \leq \mu^*(E)$ for all $E \subseteq \mathcal{M}(\mathcal{A})$, with equality if and only if $\mu^*(E) < \infty$.
- If μ_0 is σ -finite, i.e., $X = \bigcup_{j=1}^{\infty} A_j$ with $A_j \in \mathcal{A}$ and $\mu_0(A_j) < \infty$ for all j , then $\mu^*|_{\mathcal{M}(\mathcal{A})}$ is the unique extension of μ_0 to a measure on $\mathcal{M}(\mathcal{A})$.

Proof.

- Let \mathcal{B} be the set of all μ^* -measurable sets, then $\mu^*|_{\mathcal{B}}$ is a measure on \mathcal{B} that extends μ_0 . By the fundamental theorem of measure theory, we know \mathcal{B} is a σ -algebra. In particular, $\mathcal{B} \supseteq \mathcal{A}$, therefore $\mathcal{B} \supseteq \mathcal{M}(\mathcal{A})$. That means $\mu^*|_{\mathcal{M}(\mathcal{A})}$ is a measure as well.
- Let $\tilde{\mu}$ be any measure on $\mathcal{M}(\mathcal{A})$ that extends μ_0 . We first show that for all $E \in \mathcal{M}(\mathcal{A})$, then $\tilde{\mu}(E) \leq \mu^*(E)$. Recall that $\mu^*(E) = \inf\{\sum_{j=1}^{\infty} \mu_0(A_j) : E \subseteq \bigcup_{j=1}^{\infty} A_j, A_j \in \mathcal{A}\}$. Given a cover $E \subseteq \bigcup_{j=1}^{\infty} A_j$ and fix $A_j \in \mathcal{A}$. Therefore,

$$\begin{aligned} \tilde{\mu}(E) &\leq \tilde{\mu}\left(\bigcup_{j=1}^{\infty} A_j\right) \\ &\leq \sum_{j=1}^{\infty} \tilde{\mu}(A_j) \\ &= \sum_{j=1}^{\infty} \mu_0(A_j), \end{aligned}$$

therefore $\tilde{\mu}(E) \leq \mu^*(E)$. Assume we have $\mu^*(E) < \infty$, and we want to show that $\tilde{\mu}(E) = \mu^*(E)$. It suffices to show $\mu^*(E) \leq \tilde{\mu}(E)$.

Claim 1.47. Let $A_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, then $\mu^* \left(\bigcup_{j=1}^{\infty} A_j \right) = \tilde{\mu} \left(\bigcup_{j=1}^{\infty} A_j \right)$.

Subproof. Note that $\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}(\mathcal{A})$, then we can just work on $\mathcal{M}(\mathcal{A})$. Consider $\mu^*|_{\mathcal{M}(\mathcal{A})}$ and $\tilde{\mu}$ are measures on $\mathcal{M}(\mathcal{A})$. Let $E_n = \bigcup_{j=1}^n A_j$ for all $n \in \mathbb{N}$, then we have a nested increasing sequence of E_n 's. In particular, we know $\bigcup_{n=1}^{\infty} E_n = \bigcup_{j=1}^{\infty} A_j$. Therefore

$$\begin{aligned} \mu^* \left(\bigcup_{n=1}^{\infty} E_n \right) &= \mu^* \left(\bigcup_{j=1}^{\infty} A_j \right) \\ &= \lim_{n \rightarrow \infty} \mu^*(E_n) \\ &= \lim_{n \rightarrow \infty} \mu^* \left(\bigcup_{j=1}^n A_j \right) \\ &= \lim_{n \rightarrow \infty} \mu_0 \left(\bigcup_{j=1}^n A_j \right) \\ &= \lim_{n \rightarrow \infty} \tilde{\mu} \left(\bigcup_{j=1}^n A_j \right) \\ &= \tilde{\mu} \left(\bigcup_{j=1}^{\infty} A_j \right) \end{aligned}$$

by continuity from below and closure of finite union. ■

We know from the claim that

$$\begin{aligned} \mu^* \left(\bigcup_{j=1}^{\infty} A_j \right) &= \lim_{n \rightarrow \infty} \mu_0 \left(\bigcup_{j=1}^n A_j \right) \\ &\leq \lim_{n \rightarrow \infty} \sum_{j=1}^n \mu_0(A_j) \\ &= \sum_{j=1}^{\infty} \mu_0(A_j). \end{aligned}$$

Take arbitrary $\varepsilon > 0$, then consider $\mu^*(E) + \varepsilon$, which is not a lower bound of the set anymore. Therefore, there exists $A_j \in \mathcal{A}$ for each $j \in \mathbb{N}$ such that $E \subseteq \bigcup_{j=1}^{\infty} A_j$ and that $\sum_{j=1}^{\infty} \mu_0(A_j) \leq \mu^*(E) + \varepsilon$. In particular, this means

$$\mu^* \left(\bigcup_{j=1}^{\infty} A_j \right) \leq \mu^*(E) + \varepsilon. \text{ Since } \mu^*(E) < \infty, \text{ then}$$

$$\begin{aligned} \mu^* \left(\bigcup_{j=1}^{\infty} A_j \setminus E \right) &= \mu^* \left(\bigcup_{j=1}^{\infty} A_j \right) - \mu^*(E) \\ &< \varepsilon. \end{aligned}$$

Now that

$$\mu^*(E) \leq \mu^* \left(\bigcup_{j=1}^{\infty} A_j \right)$$

$$\begin{aligned}
&= \tilde{\mu} \left(\bigcup_{j=1}^{\infty} A_j \right) \\
&= \tilde{\mu}(E) + \tilde{\mu} \left(\bigcup_{j=1}^{\infty} A_j \setminus E \right) \\
&< \tilde{\mu}(E) + \varepsilon
\end{aligned}$$

by the claim. Therefore, for any $\varepsilon > 0$, we have $\mu^*(E) \leq \tilde{\mu}(E) + \varepsilon$ whenever $\mu^*(E) < \infty$. Take $\varepsilon \rightarrow 0$, we get $\mu^*(E) \leq \tilde{\mu}(E)$.

- c. Since μ_0 is σ -finite, then there exists a decomposition $X = \bigcup_{j=1}^{\infty} A_j$ for $A_j \in \mathcal{A}$ and that $\mu_0(A_j) < \infty$. For any $E \in \mathcal{M}(\mathcal{A})$, then

$$\begin{aligned}
E &= E \cap X \\
&= E \cap \left(\bigcup_{j=1}^{\infty} A_j \right) \\
&= \bigcup_{j=1}^{\infty} (E \cap A_j)
\end{aligned}$$

and

$$\begin{aligned}
\mu^*(E) &= \mu^* \left(\bigcup_{j=1}^{\infty} (E \cap A_j) \right) \\
&= \sum_{j=1}^{\infty} \mu^*(E \cap A_j) \\
&= \sum_{j=1}^{\infty} \tilde{\mu}(E \cap A_j) \\
&= \tilde{\mu} \left(\bigcup_{j=1}^{\infty} (E \cap A_j) \right) \\
&= \tilde{\mu}(E)
\end{aligned}$$

since $\mu^*(E \cap A_j) \leq \mu^*(A_j) = \mu_0(A_j) < \infty$.

□

1.4 BOREL MEASURE

Recall that the Borel σ -algebra $\mathcal{B}_{\mathbb{R}}$ is the smallest σ -algebra containing all open sets. Let \mathcal{G} be the set of all open sets in \mathbb{R} with respect to the standard topology. Therefore $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{G})$. We can in fact use something smaller than \mathcal{G} .

Theorem 1.48. $\mathcal{B}_{\mathbb{R}}$ is a σ -algebra generated by

- $\mathcal{A}_0 = \{(a, b) : a, b \in \mathbb{R}, a < b\}$, or by
- $\mathcal{A}_1 = \{(a, b] : a, b \in \mathbb{R}, -\infty \leq a < b < \infty\} \cup \{(a, \infty) : -\infty \leq a < \infty\} \cup \{\emptyset\}$.

Any member in \mathcal{A}_1 is called an h -interval.

Proof.

- We want to show that $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{A})$. Obviously $\mathcal{A}_0 \subseteq \mathcal{G}$, then $\mathcal{M}(\mathcal{G})$ is a σ -algebra containing \mathcal{A}_0 , then $\mathcal{M}(\mathcal{A}_0) \subseteq \mathcal{M}(\mathcal{G}) = \mathcal{B}_{\mathbb{R}}$. Conversely, recall that any open subset in \mathbb{R} is a σ -union of open intervals, therefore $\mathcal{G} \subseteq \mathcal{M}(\mathcal{A})$, so $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{G}) \subseteq \mathcal{M}(\mathcal{A}_0)$, therefore $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{A}_0)$.

b. We first show that $\mathcal{M}(\mathcal{A}_1) \subseteq \mathcal{B}_{\mathbb{R}}$. Since $\mathcal{M}(\mathcal{A}_1)$ is the smallest σ -algebra containing \mathcal{A}_1 , then it suffices to show that $\mathcal{A}_1 \subseteq \mathcal{B}_{\mathbb{R}}$. It is easy to see that $(a, b] = \bigcap_{n=1}^{\infty} (a, b + \frac{1}{n}) \in \mathcal{B}_{\mathbb{R}}$, and $(a, \infty) = \bigcup_{n=1}^{\infty} (a, n) \in \mathcal{B}_{\mathbb{R}}$.

We now verify that $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{A}_1)$. By a. we know $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{A}_0)$, so it suffices to show that $\mathcal{A}_0 \subseteq \mathcal{M}(\mathcal{A}_1)$. For $a < b$, we have $(a, b) = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}]$, therefore the right-hand side is a σ -union of intervals, hence belongs to $\mathcal{M}(\mathcal{A}_1)$, and we are done. \square

Definition 1.49. We define \mathcal{A}_2 to be the collection of finite disjoint unions of h -intervals, e.g., $\bigcup_{j=1}^n (a_j, b_j]$, then \mathcal{A}_2 is an algebra.

Definition 1.50. A function on \mathbb{R} is said to be right continuous if $\lim_{x \rightarrow x_0^+} F(x) = F(x_0)$.

Theorem 1.51. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be increasing and right continuous. Let $I_j = (a_j, b_j]$ for $j = 1, \dots, n$ be disjoint h -intervals. We define the pre-measure μ_0 on \mathcal{A}_2 by $\mu_0(\emptyset) = 0$ and $\mu_0\left(\bigcup_{j=1}^n (a_j, b_j]\right) = \sum_{j=1}^n [F(b_j) - F(a_j)]$.

Proof. First one can check that μ_0 is well-defined, that is, given any partition of h -interval, the μ_0 -measurements on the interval are the same.

Second, we need to show that μ_0 satisfies σ -additivity, that is, if $\bigcup_{j=1}^{\infty} I_j \in \mathcal{A}_2$ such that I_j 's are disjoint, then

$$\mu_0\left(\bigcup_{j=1}^{\infty} I_j\right) = \sum_{j=1}^{\infty} \mu_0(I_j). \text{ It is easy to verify finite additivity, so we now assume}$$

$$\bigcup_{j=1}^{\infty} I_j = I = (a, b] \in \mathcal{A}_2$$

for $-\infty \leq a < b < \infty$, then we will show that

$$F(b) - F(a) = \mu_0(I) = \sum_{j=1}^{\infty} \mu_0(I_j)$$

for $I_j = (a_j, b_j]$.

To show $\mu_0(I) \geq \sum_{j=1}^{\infty} \mu_0(I_j)$, we know $F(b) - F(a) \geq \sum_{j=1}^n [F(b_j) - F(a_j)]$, therefore taking the limit of $n \rightarrow \infty$ gives $F(b) - F(a) \geq \sum_{j=1}^{\infty} \mu_0(I_j)$.

To show $\mu_0(I) \leq \sum_{j=1}^{\infty} \mu_0(I_j)$, since F is right continuous, then for all $\varepsilon > 0$, there exist $\delta > 0$ such that $F(a + \delta) - F(a) < \varepsilon$. Therefore, for every $j > 0$, there exists $\delta_j > 0$ such that $F(b_j + \delta_j) - F(b_j) < 2^{-j}\varepsilon$, then

$$\begin{aligned} [a + \delta, b] &\subseteq (a, b] \\ &= \bigcup_{j=1}^{\infty} (a_j, b_j] \\ &= \bigcup_{j=1}^{\infty} (a_j, b_j + \delta_j). \end{aligned}$$

By compactness, there exists some $N \in \mathbb{N}$ such that $[a + \delta, b] \subseteq \bigcup_{j=1}^N (a_j, b_j + \delta_j)$. Assume $b_j + \delta_j \in (a_{j+1}, b_{j+1}]$, then

$$\mu_0(I) = \mu_0((a, b])$$

$$\begin{aligned}
&= F(b) - F(a) \\
&\leq F(b) - F(a + \delta) + \varepsilon \\
&\leq F(b_N + \delta_N) - F(a + \delta) + \varepsilon \\
&= F(b_N + \delta_N) - F(a_N) + F(a_N) - F(a + \delta) + \varepsilon \\
&= F(b_N + \delta_N) - F(a_N) + \sum_{j=1}^{N-1} [F(a_{j-1}) - F(a_j)] + \varepsilon \\
&\leq F(b_N + \delta_N) - F(a_N) + \sum_{j=1}^{N-1} [F(b_j + \delta_j) - F(a_j)] + \varepsilon \\
&= \sum_{j=1}^N [F(b_j + \delta_j) - F(a_j)] + \varepsilon \\
&= \sum_{j=1}^N [F(b_j + \delta_j) - F(b_j) + F(b_j) - F(a_j)] + \varepsilon \\
&= \sum_{j=1}^N [F(b_j + \delta_j) - F(b_j)] + \sum_{j=1}^N [F(b_j) - F(a_j)] + \varepsilon \\
&\leq \sum_{j=1}^N 2^{-j} \varepsilon + \sum_{j=1}^N \mu_0(I_j) + \varepsilon \\
&\leq 2\varepsilon + \sum_{j=1}^{\infty} \mu_0(I_j)
\end{aligned}$$

since F is increasing. Let $\varepsilon \rightarrow 0$ and we are done. \square

Theorem 1.52. Let F be increasing and right-continuous, then

- there is a unique measure μ_F on \mathbb{R} such that $\mu_F((a, b]) = F(b) - F(a)$ for all $a, b \in \mathbb{R}$;
- if G is another increasing and right-continuous function, then $\mu_F = \mu_G$ if and only if $F - G$ is a constant function;
- if μ is a Borel measure on \mathbb{R} that is finite on all bounded Borel sets, i.e., a set $S \subseteq \mathbb{R}$ contained in $[-M, M]$ for some $M \in \mathbb{R}$, then

$$F(x) = \begin{cases} \mu((0, x]), & x > 0 \\ 0, & x = 0 \\ -\mu((x, 0]), & x < 0 \end{cases}$$

is an increasing function and right-continuous, and $\mu_F = \mu$.

Proof.

- Consider $\mathbb{R} = \bigcup_{j=-\infty}^{\infty} (j, j+1]$, then the pre-measure $\mu_0((j, j+1]) = F(j+1) - F(j) < \infty$ defined on h -intervals is σ -finite. Therefore there exists a unique extension of measure μ of μ_0 on $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{A}_2)$ such that $\mu|_{\mathcal{A}_2} = \mu_0$.
- We have $\mu_F((a, b]) = F(b) - F(a)$ and $\mu_G((a, b]) = G(b) - G(a)$, then

$$\begin{aligned}
\mu_F((a, b]) = \mu_G((a, b]) &\iff F(b) - F(a) = G(b) - G(a) \\
&\iff F(b) - G(b) = G(a) - F(a) \\
&\iff F - G \text{ is constant.}
\end{aligned}$$

- c. First note that F is an increasing function since the measure function is increasing. Take any $x_0 \in \mathbb{R}$, we want to show that $\lim_{x \rightarrow x_0^+} F(x) = F(x_0)$. We prove this by cases, either $x_0 = 0$, $x_0 > 0$, or $x_0 < 0$. We will only prove the first case, but the two other cases are analogous. Suppose $x_0 = 0$, take a nested sequence of intervals $E_n = (0, \frac{1}{n}]$, with $E_n \supseteq E_{n+1}$ for all $n \in \mathbb{N}$, then

$$\begin{aligned}
 \lim_{x \rightarrow 0^+} F(x) &= \lim_{x \rightarrow 0^+} \mu((0, x]) \\
 &= \lim_{n \rightarrow \infty} \mu((0, \frac{1}{n}]) \\
 &= \lim_{n \rightarrow \infty} \mu(E_n) \\
 &= \mu\left(\bigcap_{n=1}^{\infty} E_n\right) \\
 &= \mu(\emptyset) \\
 &= 0 \\
 &= F(0)
 \end{aligned}$$

since $\mu(E_1) < \infty$.

□

Definition 1.53. Suppose F is increasing and right-continuous, then we can use F to create μ_0 on \mathcal{A}_2 , and get an outer measure μ^* induced by μ_0 . Let \mathcal{A} be the collection of all μ^* -measurable sets, then $\mu^*|_{\mathcal{A}}$ is a measure. Note that $\mathcal{A} \supseteq \mathcal{B}_{\mathbb{R}}$: since μ_F is only defined on $\mathcal{B}_{\mathbb{R}}$, then $\mu^*|_{\mathcal{A}}$ becomes the extension of μ_F on \mathcal{A} . We denote this measure to be $\bar{\mu}_F$, as the extension of μ_F , called the Lebesgue-Stieltjes measure.

Remark 1.54. In particular, if $F(x) = x$ for all $x \in \mathbb{R}$, then $\bar{\mu}_F$ is called a Lebesgue measure, denoted by \mathbf{m} , with $\mathbf{m}((a, b]) = F(b) - F(a) = b - a$.

Definition 1.55. Let μ be a Lebesgue-Stieltjes measure associated to an increasing and right-continuous function F . Let \mathcal{M}_{μ} be the domain of the measure μ , which gives the collection of measurable sets. For any measurable set $E \in \mathcal{M}_{\mu}$, we have

$$\begin{aligned}
 \mu(E) &= \inf \left\{ \sum_{j=1}^{\infty} [F(b_j) - F(a_j)] : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j] \right\} \\
 &= \inf \left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j]) : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j] \right\}.
 \end{aligned}$$

Theorem 1.56. For all $E \in \mathcal{M}_{\mu}$, we have

$$\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j)) : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j) \right\}.$$

Proof. Let $\tilde{\mu}(E)$ be the right-hand side of this equation, so we will show that $\mu(E) = \tilde{\mu}(E)$. Note that we have a partition

$$(a_j, b_j) = \bigcup_{k=1}^{\infty} I_k^{(j)},$$

where $I_k^{(j)} = (b_j - \frac{1}{2^k}(b_j - a_j), b_j - \frac{1}{2^{k+1}}(b_j - a_j)]$. Now $E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j)$, so $E \subseteq \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} I_k^{(j)}$, and thus

$$\sum_{j=1}^{\infty} \mu((a_j, b_j)) = \sum_{j=1}^{\infty} \mu\left(\bigcup_{k=1}^{\infty} I_k^{(j)}\right)$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mu(I_k^{(j)}).$$

Because $\left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j)) : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j) \right\} \subseteq \left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j]) : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j] \right\}$, then $\tilde{\mu}(E) \geq \mu(E)$.

We now show that $\mu(E) \geq \tilde{\mu}(E)$. Pick arbitrary $\varepsilon > 0$, then we know $\mu(E) + \varepsilon$ is not a lower bound of the set $\left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j]) : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j] \right\}$, hence there exists $(a_j, b_j]$ for $j \geq 1$ such that $E \subseteq \bigcup_{j \geq 1} (a_j, b_j]$. Therefore $\sum_{j=1}^{\infty} \mu((a_j, b_j]) \leq \mu(E) + \varepsilon$. By the right continuity of F , for $\varepsilon \cdot 2^{-j} > 0$, there exists $\delta_j > 0$ such that $F(b_j + \delta_j) - F(b_j) < \varepsilon \cdot 2^{-j}$, then $E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j + \delta_j)$. We know

$$\begin{aligned} \tilde{\mu}(E) &\leq \sum_{j=1}^{\infty} \mu((a_j, b_j + \delta_j)) \\ &= \sum_{j=1}^{\infty} [F(b_j + \delta_j) - F(a_j)] \\ &= \sum_{j=1}^{\infty} [F(b_j + \delta_j) - F(b_j) + F(b_j) - F(a_j)] \\ &\leq \sum_{j=1}^{\infty} [F(b_j + \delta_j) - F(b_j)] + \sum_{j=1}^{\infty} [F(b_j) - F(a_j)] \\ &< \sum_{j=1}^{\infty} \varepsilon \cdot 2^{-j} + \sum_{j=1}^{\infty} \mu((a_j, b_j]) \\ &< \varepsilon + \mu(E) + \varepsilon \\ &= \mu(E) + 2\varepsilon. \end{aligned}$$

Taking small enough ε finishes the proof. \square

Remark 1.57. The union of h -intervals may not be open, so often times we use the characterization in [Theorem 1.56](#) instead.

Theorem 1.58. For any $E \subseteq \mathcal{M}_{\mu}$, we have

$$\mu(E) = \inf\{\mu(U) : \text{open } U \supseteq E\} = \sup\{\mu(K) : \text{compact } K \subseteq E\}.$$

Proof. Let $\tilde{\mu}(E) = \inf\{\mu(U) : \text{open } U \supseteq E\}$. First, $\mu(E) \leq \tilde{\mu}(E)$: since $E \subseteq U$, then $\mu(E) \leq \mu(U)$, therefore $\mu(E) \leq \tilde{\mu}(E)$. To see $\tilde{\mu}(E) \leq \mu(E)$, we have $\mu(E) + \varepsilon$ is not a lower bound of $\left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j]) : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j] \right\}$, then there exists $(a_j, b_j]$ for each $j \in \mathbb{N}$ such that $E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j]$, and that $\sum_{j=1}^{\infty} \mu((a_j, b_j]) \leq \mu(E) + \varepsilon$. Therefore, take

U to be the open set $\bigcup_{j=1}^{\infty} (a_j, b_j)$, then

$$\tilde{\mu}(E) \leq \mu(U) \leq \sum_{j=1}^{\infty} \mu((a_j, b_j)) \leq \mu(E) + \varepsilon$$

as desired.

Now let $\nu(E) = \sup\{\mu(K) : \text{compact } K \subseteq E\}$. We note that if $K \subseteq E$, then $\mu(K) \leq \mu(E)$, therefore $\nu(E) \leq \mu(E)$. To prove the reverse inequality, we consider the following cases:

- E is bounded.
 - E is closed. Since E is bounded and closed, it is compact over \mathbb{R} , thus $\mu(E) \leq \nu(E)$.
 - E is bounded but not closed. We have $\mu(\bar{E} \setminus E) = \inf\{\mu(U) : \text{open } U \supseteq \bar{E} \setminus E\}$. For any $\varepsilon > 0$, there exists an open set U such that $U \supseteq \bar{E} \setminus E$ and $\mu(U) \leq \mu(\bar{E} \setminus E) + \varepsilon$. Set $K = \bar{E} \setminus U$, then K is compact. Since all measures here are finite, we have

$$\begin{aligned}
 \mu(K) &= \mu(E) - \mu(E \cap U) \\
 &= \mu(E) - [\mu(U) - \mu(U \setminus E)] \\
 &\geq \mu(E) - \mu(U) + \mu(\bar{E} \setminus E) \\
 &\geq \mu(E) - \varepsilon.
 \end{aligned}$$

Therefore $\nu(E) \geq \mu(E) - \varepsilon$, and we are done by taking $\varepsilon \rightarrow 0$.

- E is not bounded. Suppose $E = \bigcup_{j=-\infty}^{\infty} ((j, j+1] \cap E)$, then denote $E_j = E \cap (j, j+1]$, which is bounded. Therefore, we know the statement is true for each E_j for $j \geq 1$, thus $\mu(E_j) = \sup\{\mu(K) : \text{compact } K \subseteq E_j\}$. Take arbitrary $\varepsilon > 0$, then $\mu(E_j) - \frac{1}{3}\varepsilon \cdot 2^{-|j|}$ is not the upper bound of $\{\mu(K) : \text{compact } K \subseteq E_j\}$, then there exists a compact set $K_j \subseteq E_j$ such that $\mu(K_j) \geq \mu(E_j) - \frac{1}{3}\varepsilon \cdot 2^{-|j|}$. Since $K_j \subseteq E_j$ and E_j 's are disjoint, then K_j 's are disjoint. Therefore, for $n \in \mathbb{N}$, set $H_n = \bigcup_{j=-n}^n K_j$, which is a finite disjoint union of compact sets, so this is a compact set. But $H_n \subseteq E$, then

$$\begin{aligned}
 \mu(H_n) &= \mu\left(\bigcup_{j=-n}^n K_j\right) \\
 &= \sum_{j=-n}^n \mu(K_j) \\
 &\geq \sum_{j=-n}^n \mu(E_j) - \frac{\varepsilon}{3} \sum_{j=-n}^n 2^{-|j|} \\
 &\geq \sum_{j=-n}^n \mu(E_j) - \frac{\varepsilon}{3} \sum_{j=-\infty}^{\infty} 2^{-|j|} \\
 &\geq \sum_{j=-n}^n \mu(E_j) - \varepsilon.
 \end{aligned}$$

Note that H_n still depends on n , so we should not take $n \rightarrow \infty$ here. Since $\nu(E)$ is the upper bound of $\mu(K)$'s for compact $K \subseteq E$, then $\nu(E) \geq \mu(H_n)$, therefore

$$\begin{aligned}
 \nu(E) &\geq \sum_{j=-n}^n \mu(E_j) - \varepsilon \\
 &= \mu\left(\bigcup_{j=-n}^n E_j\right) - \varepsilon.
 \end{aligned}$$

Take $n \rightarrow \infty$, then

$$\begin{aligned}
 \nu(E) &\geq \lim_{n \rightarrow \infty} \mu\left(\bigcup_{j=-n}^n E_j\right) - \varepsilon \\
 &= \mu\left(\bigcup_{j=-\infty}^{\infty} E_j\right) - \varepsilon
 \end{aligned}$$

$$= \mu(E) - \varepsilon.$$

Let $\varepsilon \rightarrow 0$, we are done. □

Theorem 1.59. Let $E \subseteq \mathbb{R}$, then the following are equivalent:

- a. $E \in \mathcal{M}_\mu$;
- b. $E = V \setminus N_1$, where V is a G_δ -set and $\mu(N_1) = 0$;
- c. $E = H \cup N_2$, where H is a F_σ -set and $\mu(N_2) = 0$.

Proof.

- $b. \Rightarrow a.$: note that $\mathcal{M}_\mu \supseteq \mathcal{B}_\mathbb{R}$, then both V and N_1 are measurable, therefore E is measurable, i.e., $E \in \mathcal{M}_\mu$.
- $c. \Rightarrow a.$: similar to the case above.
- $a. \Rightarrow b.$:
 - If $\mu(E) < \infty$, recall $\mu(E) = \inf\{\mu(U) : \text{open } U \supseteq E\}$. For any $k \in \mathbb{N}$, consider $2^{-k} > 0$, then there exists open subset $U_k \supseteq E$ such that $\mu(U_k) \leq \mu(E) + 2^{-k}$. Let $V = \bigcap_{k=1}^{\infty} U_k$ be a G_δ -set, then $V \supseteq E$ as well. It suffices to show that $V \setminus E$ is a null set. We know

$$\begin{aligned} \mu(V) &= \mu\left(\bigcap_{k=1}^{\infty} U_k\right) \\ &\leq \mu(U_k) \\ &\leq \mu(E) + 2^{-k} \end{aligned}$$

for all $k \in \mathbb{N}$. Since $\mu(V)$ and $\mu(E)$ are independent of k , then take $k \rightarrow \infty$, therefore $\mu(V) \leq \mu(E)$. But since $E \subseteq V$, then $\mu(E) \leq \mu(V)$, therefore this gives equality. Since $\mu(E) < \infty$, then $\mu(V) - \mu(E) = 0$, then $\mu(V \setminus E) = 0$ by additivity.

- If $\mu(E) = \infty$, then the proof can be done using the previous case.

- $a. \Rightarrow c.$: the proof is similar to the case above. □

Theorem 1.60. Let $E \in \mathcal{M}_\mu$, and suppose $\mu(E) < \infty$. For any $\varepsilon > 0$, there exists some set A that is a finite union of open intervals such that $\mu(E \Delta A) = \mu((E \setminus A) \cup (A \setminus E)) < \varepsilon$.

Proof. Note that $\mu(E) = \sup\{\mu(K) : \text{compact } K \subseteq E\}$. For any $\varepsilon > 0$, there exists compact $K \subseteq E$ such that $\mu(E) - \frac{\varepsilon}{2} < \mu(K)$, which is equivalent to having $\mu(E \setminus K) < \frac{\varepsilon}{2}$. Similarly, recall that $\mu(E) = \inf\{\mu(U) : \text{open } U \supseteq E\}$, but open set U on \mathbb{R} is characterized as a union of open intervals, therefore this is just $\mu(E) = \inf\left\{\sum_{j=1}^{\infty} \mu((a_j, b_j)) : \bigcup_{j=1}^{\infty} (a_j, b_j) \supseteq E\right\}$.

Therefore, there exists $\bigcup_{j=1}^{\infty} I_j \supseteq E$, where I_j is open interval for each j , such that $\mu\left(\bigcup_{j=1}^{\infty} I_j\right) < \mu(E) + \frac{\varepsilon}{2}$. Since $\mu(E)$ is finite, then $\mu\left(\bigcup_{j=1}^{\infty} I_j \setminus E\right) < \frac{\varepsilon}{2}$. Now $K \subseteq E \subseteq \bigcup_{j=1}^{\infty} I_j$, but K is compact, so there exists I_1, \dots, I_n such that their union cover K . Set $A = \bigcup_{j=1}^n I_j$, and we are done. □

Definition 1.61. Let $F(x) = x$ be a function for all $x \in \mathbb{R}$, then μ_F is called the Lebesgue measure defined by $\mathbf{m}((a, b]) = b - a$. The domain of m is \mathcal{L} .

For $E \subseteq \mathbb{R}$ and $s, r \in \mathbb{R}$, we denote $E + s = \{x + s : x \in E\}$ and $rE = \{rx : x \in E\}$.

Theorem 1.62. If $E \in \mathcal{L}$, then $\mathbf{m}(E + s) = \mathbf{m}(E)$ and $\mathbf{m}(rE) = |r|\mathbf{m}(E)$.

Proof. We prove the first claim. For any $E \in \mathcal{L}$ and $s \in \mathbb{R}$, define $m_s = \mathbf{m}(E + s)$, then this is a measure.

Claim 1.63. For any $E \in \mathcal{L}$, $m_s(E) = \mathbf{m}(E)$.

Subproof. First note that this is true if E is a finite (disjoint) union of h -intervals of m_s , as \mathbf{m} extends the pre-measure μ_0 . On $\mathcal{B}_{\mathbb{R}}$, the extension is unique, so $m_s(E) = \mathbf{m}(E)$ if $E \in \mathcal{B}_{\mathbb{R}}$. Moreover, recall $E \in \mathcal{L}$ if and only if $E = V \setminus N_1$ for $V \in \mathcal{B}_{\mathbb{R}}$. Therefore this is true for all $E \in \mathcal{L}$. ■

□

Definition 1.64. The Cantor set \mathcal{C} is constructed iteratively from the interval $[0, 1]$, that for any remaining connected interval $[m, n]$, we delete the subinterval $(m + \frac{1}{3}(n - m), m + \frac{2}{3}(n - m))$ from $[m, n]$.

Remark 1.65. Note that

$$\begin{aligned} \mathbf{m}(\mathcal{C}) &= \mathbf{m}([0, 1]) - \frac{1}{3} - \frac{2}{3^2} - \frac{2^2}{3^3} - \cdots \\ &= 1 - \sum_{j=0}^{\infty} \frac{2^j}{3^{j+1}} \\ &= 1 - 1 \\ &= 0. \end{aligned}$$

Remark 1.66. If E is countable, then

$$\begin{aligned} \mathbf{m}(E) &= \sum_{j=1}^{\infty} \mathbf{m}(\{a_j\}) \\ &= 0. \end{aligned}$$

Theorem 1.67. The Cantor set \mathcal{C} is uncountable.

Proof. Alternatively, the Cantor set \mathcal{C} can be represented as

$$\mathcal{C} = \{x \in [0, 1] : x = \sum_{j=1}^{\infty} a_j 3^{-j}, a_j \in \{0, 2\}\}.$$

To prove that \mathcal{C} is uncountable, it suffices to build a surjection $f : \mathcal{C} \rightarrow [0, 1]$. For $x \in \mathcal{C}$, we have $x = \sum_{j=1}^{\infty} a_j 3^{-j}$, $a_j \in \{0, 2\}$.

Set $f(x) = \sum_{j=1}^{\infty} \frac{a_j}{2} 2^{-j}$ for $\frac{a_j}{2} \in \{0, 1\}$, therefore this gives a decimal representation with base 2, so any real number in $[0, 1]$ can be represented in this form, therefore we have a surjection. □

Theorem 1.68. Let $F \subseteq \mathbb{R}$ be such that every subset of F is Lebesgue measurable, then $\mathbf{m}(F) = 0$.

Corollary 1.69. If $\mathbf{m}(F) > 0$, then there exists a subset S of F such that $S \notin \mathcal{L}$.

Remark 1.70 (Banach-Tarski Paradox). Given a ball $B = S^2$, then there exists some $m \in \mathbb{N}$ such that $B = V_1 \cup \cdots \cup V_m$ is a union of subsets V_i that are not Lebesgue measurable and $\mathbf{m}(B) \neq \mathbf{m}(V_1 \cup \cdots \cup V_m)$.

Definition 1.71. For any $x \in \mathbb{R}$, we defined the cosets over \mathbb{Q} to be $\mathbb{Q} + x = \{r + x : r \in \mathbb{Q}\}$ for any x . This is called the coset of an additive group \mathbb{R} .

Let E be the set that contains exactly one point from each coset of \mathbb{Q} as representations, which requires the axiom of choice. Now E allows us make a partition on \mathbb{R} .

Lemma 1.72.

1. $(E + r_1) \cap (E + r_2) = \emptyset$ if $r_1 \neq r_2$ and $r_1, r_2 \in \mathbb{Q}$.
2. $\mathbb{R} = \bigcup_{r \in \mathbb{Q}} (E + r)$.

Proof.

1. Suppose $x \in (E + r_1) \cap (E + r_2)$, then $x = e_1 + r_1 = e_2 + r_2$ for some $e_1, e_2 \in E$. Therefore $e_1 - e_2 = r_2 - r_1$, which is a non-zero rational number, therefore $0 \neq e_1 - e_2 \in \mathbb{Q}$. Therefore e_1 and e_2 are in the same coset, so $e_1 = e_2$, contradiction.
2. Obviously $\mathbb{R} \supseteq \bigcup_{r \in \mathbb{Q}} (E + r)$. Take any $x \in \mathbb{R}$, then E contains a point y from the coset $\mathbb{Q} + x$, therefore $y - x \in \mathbb{Q}$, so take $r = y - x$, then $x \in E + r$.

□

Proof of Theorem 1.68. We have

$$\begin{aligned}
 F &= F \cap \mathbb{R} \\
 &= F \cap \bigcup_{r \in \mathbb{Q}} (E + r) \\
 &= \bigcup_{r \in \mathbb{Q}} (F \cap (E + r)).
 \end{aligned}$$

Now let $F_r = F \cap (E + r)$ for all $r \in \mathbb{Q}$, then $F = \bigcup_{r \in \mathbb{Q}} F_r$ for $F_r \in \mathcal{L}$ by Lemma 1.72. It remains to verify that $\mathbf{m}(F_r) = 0$ for all $r \in \mathbb{Q}$. Recall

$$\mathbf{m}(F_r) = \sup\{\mathbf{m}(K) : \text{compact } K \subseteq F_r\},$$

then it suffices to show that

Claim 1.73. For any compact set $K \subseteq F_r$, $\mathbf{m}(K) = 0$.

Indeed, take the supremum over all compact subsets and we are done.

Subproof. Let $K_r = K + r$ for all $r \in \mathbb{Q}$.

First, we show that $K_{r_1} \cap K_{r_2} = \emptyset$ if $r_1 \neq r_2$ for $r_1, r_2 \in \mathbb{Q}$. Assume there exists $x \in K_{r_1} \cap K_{r_2}$, then $K \subseteq F_r \subseteq E + r$, so we know $K_{r_1} = K + r_1 \subseteq E + r + r_1$ and $K_{r_2} = K + r_2 \subseteq E + r + r_2$. Therefore, $x \in (E + r + r_1) \cap (E + r + r_2)$, but by Lemma 1.72 we know $(E + r + r_1) \cap (E + r + r_2) = \emptyset$, contradiction.

Set $H = \bigcup_{r \in \mathbb{Q}} K_r$ be a disjoint union. Since the right-hand side is a Borel set, then it is Lebesgue measurable, so by σ -additivity, we have

$$\begin{aligned}
 \mathbf{m}(H) &= \mathbf{m}\left(\bigcup_{r \in \mathbb{Q}} K_r\right) \\
 &= \sum_{r \in \mathbb{Q}} \mathbf{m}(K_r) \\
 &= \sum_{r \in \mathbb{Q}} \mathbf{m}(K) \\
 &= \mathbf{m}(K) \sum_{r \in \mathbb{Q}} 1.
 \end{aligned}$$

We need to bound the set, so instead of summation over \mathbb{Q} , we will sum over $\mathbb{Q} \cap [0, 1]$ instead, so for $H = \bigcup_{r \in \mathbb{Q} \cap [0, 1]} K_r$ we get

$$\mathbf{m}(H) = \mathbf{m}(K) \sum_{r \in \mathbb{Q} \cap [0, 1]} 1.$$

That is, $\mathbf{m}(H)$ is just $\mathbf{m}(K)$ times the number of rational numbers in $[0, 1]$, which are countably many, therefore $\mathbf{m}(H) = \mathbf{m}(K) \cdot \mathbb{N}$.

Assume, towards contradiction, that $\mathbf{m}(K) \neq 0$, then we have $\mathbf{m}(K) > 0$, so $\mathbf{m}(H) = \infty$. But we know H is bounded by $[0, 1]$ already, therefore $\mathbf{m}(H)$ is finite, contradiction. ■

□

Remark 1.74. Not every set is Lebesgue measurable.

2 INTEGRATION

2.1 MEASURABLE FUNCTIONS

Definition 2.1. Let (X, \mathcal{A}) and (Y, \mathcal{B}) be measurable spaces. A function $f : X \rightarrow Y$ is called $(\mathcal{A}, \mathcal{B})$ -measurable if $f^{-1}(E) \in \mathcal{A}$ for any $E \in \mathcal{B}$. That is, the preimage of a measurable set is measurable.

Definition 2.2. Let (X, \mathcal{A}) be a measurable space.

- a. If $f : X \rightarrow \mathbb{R}$ is $(\mathcal{A}, \mathcal{B}_{\mathbb{R}})$ -measurable, then we say the function f is \mathcal{A} -measurable.
- b. A complex-valued function $f : X \rightarrow \mathbb{C}$ is \mathcal{A} -measurable if $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ are \mathcal{A} -measurable.

Definition 2.3. A function $f : \mathbb{R} \rightarrow \mathbb{C}$ is called Lebesgue measurable if it is \mathcal{L} -measurable (on both the real part and the imaginary part).

Lemma 2.4. Let \mathcal{B} be a σ -algebra generated by \mathcal{B}_0 , then $f : X \rightarrow Y$ is $(\mathcal{A}, \mathcal{B})$ -measurable if and only if $f^{-1}(E) \in \mathcal{A}$ for all $E \in \mathcal{B}_0$.

Proof.

(\Rightarrow): this is obvious by [Definition 2.1](#).

(\Leftarrow): let $M = \{E \subseteq Y : f^{-1}(E) \in \mathcal{A}\}$. Note that $\mathcal{M} \supseteq \mathcal{B}_0$ is a σ -algebra, and since \mathcal{B} is the σ -algebra generated by \mathcal{B}_0 , then $\mathcal{M} \supseteq \mathcal{B}$. Therefore, for all $E \in \mathcal{B}$, we have $f^{-1}(E) \in \mathcal{A}$. □

Theorem 2.5. Let X and Y be topological spaces, then every continuous function $f : X \rightarrow Y$ is $(\mathcal{B}_X, \mathcal{B}_Y)$ -measurable.

Proof. Note that f is continuous if and only if $f^{-1}(U)$ is open in X for any open subset U in Y , and since \mathcal{B}_Y is the σ -algebra generated by all open subsets of Y , therefore by [Lemma 2.4](#) we know f is $(\mathcal{B}_X, \mathcal{B}_Y)$ -measurable. □

Theorem 2.6. Let $f : X \rightarrow \mathbb{R}$ be a function, then the following are equivalent:

- a. f is \mathcal{A} -measurable;
- b. $f^{-1}((a, \infty)) \in \mathcal{A}$ for all $a \in \mathbb{R}$;
- c. $f^{-1}([a, \infty)) \in \mathcal{A}$ for all $a \in \mathbb{R}$;
- d. $f^{-1}((-\infty, a)) \in \mathcal{A}$ for all $a \in \mathbb{R}$;
- e. $f^{-1}((-\infty, a]) \in \mathcal{A}$ for all $a \in \mathbb{R}$;

Proof. Since the proofs will be analogous to one another, it suffices to show the equivalence between a. and b.

a. \Rightarrow b.: since $(a, \infty) \in \mathcal{B}_{\mathbb{R}}$ is a Borel set, then $f^{-1}((a, \infty)) \in \mathcal{A}$ since f is \mathcal{A} -measurable.

b. \Rightarrow a.: let $\mathcal{B}_0 = \{(a, \infty) : a \in \mathbb{R}\}$, then $\mathcal{B}_{\mathbb{R}}$ is a σ -algebra generated by \mathcal{B}_0 . The statement then follows from [Lemma 2.4](#). □

Theorem 2.7. If $f, g : X \rightarrow \mathbb{C}$ are \mathcal{A} -measurable, then so are $f + g$ and $f \cdot g$.

Proof. Assume, without loss of generality, that f and g are \mathbb{R} -valued functions.

First, we show that $f + g$ is \mathcal{A} -measurable. By [Theorem 2.6](#), it suffices to show that $(f + g)^{-1}((-\infty, a)) \in \mathcal{A}$ for all $a \in \mathbb{R}$. Fix $a \in \mathbb{R}$, this is the set of elements $x \in X$ such that $(f + g)(x) < a$. Note that $x \in X$ satisfies $(f + g)(x) = f(x) + g(x) < a$ if and only if $f(x) < a - g(x)$, where both expressions are real numbers. Since \mathbb{Q} is dense in \mathbb{R} , there exists some $r \in \mathbb{Q}$ such that $f(x) < r < a - g(x)$. Therefore,

$$\begin{aligned} \{x \in X : f(x) + g(x) < a\} &= \bigcup_{r \in \mathbb{Q}} (\{x \in X : f(x) < r\} \cap \{x \in X : r < a - g(x)\}) \\ &= \bigcup_{r \in \mathbb{Q}} (f^{-1}((-\infty, r)) \cap g^{-1}((-\infty, a - r))) \in \mathcal{A} \end{aligned}$$

since $f^{-1}((-\infty, r)) \in \mathcal{A}$ and $g^{-1}((-\infty, a - r)) \in \mathcal{A}$.

Remark 2.8. Note that if f is \mathcal{A} -measurable, then $-f$ is \mathcal{A} -measurable. Therefore, the sum and the difference of two \mathcal{A} -measurable functions is still \mathcal{A} -measurable.

We now show that $f \cdot g$ is also \mathcal{A} -measurable.

Claim 2.9. If $f : X \rightarrow \mathbb{R}$ is \mathcal{A} -measurable, then f^2 is \mathcal{A} -measurable as well.

Subproof. By [Theorem 2.6](#), it suffices to show $\{x \in X : f^2(x) > \alpha\} \in \mathcal{A}$ for all $\alpha \in \mathbb{R}$.

- If $\alpha < 0$, then $\{x \in X : f^2(x) > \alpha\} = X \in \mathcal{A}$.
- If $\alpha \geq 0$, then $\{x \in X : f^2(x) > \alpha\} = \{x \in X : f(x) > \sqrt{\alpha}\} \cup \{x \in X : f(x) < -\sqrt{\alpha}\}$. Since f is \mathcal{A} -measurable, then this is a union of two \mathcal{A} -measurable sets, which is still \mathcal{A} -measurable. ■

Now $fg = \frac{1}{2}((f+g)^2 - f^2 - g^2)$ which is \mathcal{A} -measurable. □

Definition 2.10. The extended real line is $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, and correspondingly $\mathcal{B}_{\bar{\mathbb{R}}} = \{E \subseteq \bar{\mathbb{R}} : E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}$. Any member in $\mathcal{B}_{\bar{\mathbb{R}}}$ is called a Borel set in $\bar{\mathbb{R}}$.

A function $f : X \rightarrow \bar{\mathbb{R}}$ is called \mathcal{A} -measurable if $f^{-1}(E) \in \mathcal{A}$ for all $E \in \mathcal{B}_{\bar{\mathbb{R}}}$.

We deduce results analogous to [Theorem 2.6](#).

Theorem 2.11. Let $f : X \rightarrow \bar{\mathbb{R}}$ be a function, then the following are equivalent:

- a. f is \mathcal{A} -measurable;
- b. $f^{-1}((a, \infty]) \in \mathcal{A}$ for all $a \in \mathbb{R}$;
- c. $f^{-1}([a, \infty]) \in \mathcal{A}$ for all $a \in \mathbb{R}$;
- d. $f^{-1}([-\infty, a)) \in \mathcal{A}$ for all $a \in \mathbb{R}$;
- e. $f^{-1}([-\infty, a]) \in \mathcal{A}$ for all $a \in \mathbb{R}$;

Theorem 2.12. Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of $\bar{\mathbb{R}}$ -valued measurable functions on (X, \mathcal{A}) , then the functions

- $g_1(x) = \sup_{j \in \mathbb{N}} f_j(x) = \sup\{f_j(x) : j \in \mathbb{N}\}$;
- $g_2(x) = \inf_{j \in \mathbb{N}} f_j(x) = \inf\{f_j(x) : j \in \mathbb{N}\}$;
- $g_3(x) = \limsup_{j \in \mathbb{N}} f_j(x) = \limsup\{f_j(x) : j \in \mathbb{N}\}$;
- $g_4(x) = \liminf_{j \in \mathbb{N}} f_j(x) = \liminf\{f_j(x) : j \in \mathbb{N}\}$

are measurable.

Proof. We prove $g_1^{-1}((a, \infty]) \in \mathcal{A}$ for all $a \in \mathbb{R}$. Recall that $g_1^{-1}((a, \infty]) = \{x \in X : \infty \geq \sup_j f_j(x) > a\} = \bigcup_{j=1}^{\infty} \{x \in X : \infty \geq f_j(x) > a\}$. Since each f_j is \mathcal{A} -measurable, then each set is measurable, and so is the countable union of such functions. Therefore $g_1(x)$ is measurable. Similarly, we can show that $g_2(x)$ is measurable.

We also prove that g_3 is measurable. Recall that $\limsup_{j \rightarrow \infty} f_j(x) = \inf_{j \in \mathbb{N}} \sup_{k > j} f_k(x)$, then it is measurable since supremum and infimum are measurable as functions. Similarly, we can show that $g_4(x)$ is measurable. □

Definition 2.13. Let $f : X \rightarrow \bar{\mathbb{R}}$ be a function, then define $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$.

Remark 2.14.

- $f^+ \geq 0$;

- $f^- \geq 0$;
- $f = f^+ - f^-$;
- $|f| = f^+ + f^-$;
- If f is measurable, then so are f^+ , f^- , $|f|$.

Definition 2.15. Let $E \subseteq X$. The characteristic function or the indicator function is

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases}$$

Remark 2.16. If $E \in \mathcal{A}$, then χ_E is (\mathcal{A}) -measurable.

Definition 2.17. A simple function on X is a function that can be written as a finite \mathbb{C} -linear combination of characteristic functions of sets in \mathcal{A} .

Theorem 2.18. Any simple function f can be represented as a standard representation of the form

$$f(x) = \sum_{j=1}^n a_j \chi_{E_j}$$

where E_j 's are disjoint, $a_j \in \mathbb{C}$ and $\bigcup_{j=1}^n E_j = X$.

Proof. We can write $f(x) = \sum_{k=1}^m a_k \chi_{E_k}(x)$ for some measurable sets $E_k \in \mathcal{A}$. Since each characteristic function takes only two values, then f takes finitely many values, say z_1, \dots, z_m . Now we can write $f(x) = \sum_{j=1}^m z_j \chi_{E_j}(x)$ where $E_j = \{x \in X : f(x) = z_j\} = f^{-1}(\{z_j\})$. In particular, E_j 's are disjoint. However, these sets may not cover X . Let $E_{m+1} = X \setminus \bigcup_{j=1}^m E_j$, then $\bigcup_{j=1}^{m+1} E_j = X$, hence

$$f(x) = \sum_{j=1}^{m+1} z_j \chi_{E_j}(x)$$

where $z_{m+1} = 0$. □

Remark 2.19. Equivalently, a function $f : X \rightarrow \mathbb{C}$ is simple if and only if f is measurable and the range of f is a finite subset of \mathbb{C} .

Theorem 2.20. Let (X, \mathcal{A}) be a measurable space.

- If $f : X \rightarrow [0, \infty]$ is measurable, then there exists a sequence $\{\varphi_n\}_{n \geq 1}$ of simple functions such that
 - $0 \leq \varphi_1 \leq \varphi_2 \leq \dots \leq f$,
 - $\lim_{n \rightarrow \infty} \varphi_n(x) = f(x)$ for all $x \in X$, and
 - $\varphi_n \Rightarrow f$ converges uniformly on A , i.e., $\lim_{n \rightarrow \infty} \sup_{x \in A} |\varphi_n(x) - f(x)| = 0$, for any set A on which f is bounded.
- If $f : X \rightarrow \mathbb{C}$ is measurable, then there exists a sequence $\{\varphi_n\}_{n \geq 1}$ of simple functions such that
 - $0 \leq |\varphi_1| \leq |\varphi_2| \leq \dots \leq |f|$.
 - $\lim_{n \rightarrow \infty} \varphi_n(x) = f(x)$ for all $x \in X$.
 - $\varphi_n \Rightarrow f$ converges uniformly on any set on which f is bounded.

Proof.

a. Take arbitrary $n \in \mathbb{N} \cup \{0\}$ and arbitrary $k \in \mathbb{Z}$. We define a dyadic interval to be

$$I_{k,n} = (k2^{-n}, (k+1)2^{-n}],$$

then let $\mathcal{I} = \{I_{k,n} : k, n\}$. For any $I, J \in \mathcal{I}$, we either have $I \subseteq J$, $J \subseteq I$, or $I \cap J = \emptyset$. That is, we have a graded structure on \mathcal{I} . Now define $E_{k,n} = \{x \in X : f(x) \in I_{k,n}\} = f^{-1}(I_{k,n})$ and $F_n = f^{-1}((2^n, \infty))$. Therefore, for a fixed n , the $I_{k,n}$'s give a partition of $(0, 2^n)$ on the y -axis, and $f(F_n)$ covers the rest of the y -axis. We define a simple function

$$\varphi_n(x) = \sum_{k=1}^{2^{2n}-1} k2^{-n} \chi_{E_{k,n}}(x) + 2^n \chi_{F_n}(x).$$

Claim 2.21. For any $n \in \mathbb{N}$, $\varphi_n(x) \leq \varphi_{n+1}(x)$.

Subproof. This follows from the definition. ■

Claim 2.22. We have $0 \leq f(x) - \varphi_n(x) \leq 2^{-n}$ for all $x \in F_n^c = \{x \in X : f(x) \leq 2^n\}$.

Subproof. We have

$$f(x) = \sum_{k=0}^{2^{2n}-1} f(x) \chi_{E_{k,n}}(x) + f(x) \chi_{F_n}(x)$$

which partitions $(0, \infty)$ to $\bigcup_{k=0}^{2^{2n}-1} I_{k,n}$ and $(2^n, \infty)$. Therefore

$$\begin{aligned} f(x) - \varphi_n(x) &= \sum_{k=0}^{2^{2n}-1} [f(x) - k \cdot 2^{-n}] \chi_{E_{k,n}}(x) + (f(x) - 2^n) \chi_{F_n}(x) \\ &= \sum_{k=0}^{2^{2n}-1} [f(x) - k \cdot 2^{-n}] \chi_{E_{k,n}}(x) \\ &\geq 0 \end{aligned}$$

if $x \in F_n^c$. We now bound the difference from above by enlarging it, and since $E_{k,n}$'s are disjoint, then

$$\begin{aligned} \sum_{k=0}^{2^{2n}-1} [f(x) - k \cdot 2^{-n}] \chi_{E_{k,n}}(x) &\leq \sum_{k=0}^{2^{2n}-1} [(k+1)2^{-n} - k2^{-n}] \chi_{E_{k,n}}(x) \\ &= \sum_{k=0}^{2^{2n}-1} 2^{-n} \chi_{E_{k,n}}(x) \\ &= 2^{-n} \sum_{k=0}^{2^{2n}-1} \chi_{E_{k,n}}(x) \\ &\leq 2^{-n} \end{aligned}$$

as desired. ■

Claim 2.23. $\lim_{n \rightarrow \infty} \varphi_n(x) = f(x)$ for all $x \in X$.

Subproof.

- Suppose $f(x) = \infty$, then recall $\varphi_n(x) = 2^n \chi_{F_n}(x) = 2^n$, so obviously both values equal to ∞ .

- Suppose $0 \leq f(x) < \infty$, then for large enough n , we have $2^n > f(x)$, therefore $x \in F_n^c$ in this case. By [Claim 2.22](#), $0 \leq f(x) - \varphi_n(x) \leq 2^{-n}$ for n large enough, so when we let $n \rightarrow \infty$, then

$$0 \leq \lim_{n \rightarrow \infty} [f(x) - \varphi_n(x)] \leq 0$$

and therefore by squeeze theorem the limit exists and must equal to 0, i.e., $\lim_{n \rightarrow \infty} \varphi_n(x) = f(x)$. ■

Claim 2.24. $\varphi_n \rightrightarrows f$ converges uniformly on any set on which f is bounded.

Subproof. Let A be a set on which f is bounded. For any $x \in A$, there exists some large enough n such that $0 \leq f(x) - \varphi_n(x) \leq 2^{-n}$ by [Claim 2.22](#), so

$$0 \leq \sup_{x \in A} |f(x) - \varphi_n(x)| \leq 2^{-n},$$

so taking $n \rightarrow \infty$ gives

$$\lim_{n \rightarrow \infty} \sup_{x \in A} |f(x) - \varphi_n(x)| = 0,$$

i.e., $\varphi_n \rightrightarrows f$ on A . ■

- b. Write $f = \operatorname{Re}(f) + i \operatorname{Im}(f)$, then both $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ are measurable. Now write $\operatorname{Re}(f) = (\operatorname{Re}(f))^+ - (\operatorname{Re}(f))^-$ and $\operatorname{Im}(f) = (\operatorname{Im}(f))^+ - (\operatorname{Im}(f))^-$. By part a., we find a desirable sequence for each of these four parts of the function, then taking the sum/difference gives the desired sequence for f . □

2.2 INTEGRATION OF NON-NEGATIVE FUNCTIONS

Definition 2.25. Let (X, \mathcal{A}, μ) be a measure space, and let L^+ be the collection of all non-negative measurable functions on X , i.e., $f \in L^+$ if and only if $f : X \rightarrow [0, \infty]$.

Let $\varphi \in L^+$ be a simple function, then we can represent φ as

$$\varphi(x) = \sum_{j=1}^n a_j \chi_{E_j}(x)$$

for disjoint $E_j \in \mathcal{A}$ such that $\bigcup_{j=1}^n E_j = X$.

We first define the integral for simple functions to be

$$\int_X \varphi d\mu = \sum_{j=1}^n a_j \mu(E_j).$$

Here we set $0 \cdot \infty = 0$. For any $A \subseteq X$, we define the integral to be

$$\int_A \varphi d\mu = \int_X \varphi \chi_A d\mu.$$

To extend our definition to general non-negative functions, we need to define the following. For any $f \in L^+$, set

$$\int_X f d\mu = \sup \left\{ \int_X \varphi d\mu : 0 \leq \varphi \leq f \text{ for simple function } \varphi \right\}.$$

Since any non-negative measurable function is a limit of simple functions, then such simple functions exist, hence the supremum exists, which is either a real number or ∞ .

Proposition 2.26. Let φ and ψ be simple functions in L^+ , then

- a. if $c \geq 0$, $\int_X c\varphi d\mu = c \int_X \varphi d\mu$;
- b. $\int_X \varphi d\mu + \int_X \psi d\mu = \int_X (\varphi + \psi) d\mu$;
- c. if $\varphi \leq \psi$ pointwise, then $\int_X \varphi d\mu \leq \int_X \psi d\mu$;
- d. for any $A \in \mathcal{A}$, define $\nu : A \rightarrow \int_A \varphi d\mu$, then ν is a measure on \mathcal{A} .

Proof.

- a. This follows from the definition.
- b. Set $\varphi(X) = \sum_{j=1}^n a_j \chi_{E_j}(X)$ and $\psi(x) = \sum_{k=1}^m b_k \chi_{F_k}(x)$ as standard representations. To add the functions together, we need to refine the partition. Recall $X = \bigcup_{j=1}^n E_j = \bigcup_{k=1}^m F_k$, then we write

$$E_j = E_j \cap X = E_j \cap \left(\bigcup_{k=1}^m F_k \right) = \bigcup_{k=1}^m (E_j \cap F_k)$$

and similarly

$$F_k = F_k \cap X = F_k \cap \left(\bigcup_{j=1}^n E_j \right) = \bigcup_{j=1}^n (F_k \cap E_j).$$

Therefore

$$\begin{aligned} \varphi(x) &= \sum_{j=1}^n a_j \chi_{E_j} \\ &= \sum_{j=1}^n a_j \sum_{k=1}^m \chi_{E_j \cap F_k} \\ &= \sum_{j=1}^n \sum_{k=1}^m a_j \chi_{E_j \cap F_k} \end{aligned}$$

and similarly

$$\psi(x) = \sum_{j=1}^n \sum_{k=1}^m b_k \chi_{E_j \cap F_k}.$$

Therefore

$$\begin{aligned} (\varphi + \psi)(x) &= \varphi(x) + \psi(x) \\ &= \sum_{j,k} (a_j + b_k) \chi_{E_j \cap F_k}. \end{aligned}$$

Finally,

$$\begin{aligned} \int_X (\varphi + \psi) d\mu &= \sum_{j,k} (a_j + b_k) \mu(E_j \cap F_k) \\ &= \sum_{j,k} a_j \mu(E_j \cap F_k) + \sum_{j,k} b_k \mu(E_j \cap F_k) \\ &= \int_X \varphi d\mu + \int_X \psi d\mu. \end{aligned}$$

c. Using the same partition trick, since $\varphi \leq \psi$, then $a_j \leq b_k$ whenever $E_j \cap F_k \neq \emptyset$. Therefore,

$$\begin{aligned} \int_X \varphi d\mu &= \sum_{j,k} a_j \mu(E_j \cap F_k) \\ &\leq \sum_{j,k} b_k \mu(E_j \cap F_k) \\ &= \int_X \psi d\mu. \end{aligned}$$

d. It is easy to verify that

$$\nu(\emptyset) = \int_{\emptyset} \varphi d\mu = 0.$$

It remains to show that ν satisfies σ -additivity. Take a sequence $\{A_k\}_{k \geq 1} \subseteq \mathcal{A}$, such that A_k 's are disjoint. Given a standard representation $\varphi = \sum_{j=1}^n a_j \chi_{E_j}$, and we have

$$\begin{aligned} \nu\left(\bigcup_{k=1}^{\infty} A_k\right) &= \int_{\bigcup_{k=1}^{\infty} A_k} \varphi d\mu \\ &= \int_X \varphi \chi_{\bigcup_{k=1}^{\infty} A_k} d\mu \\ &= \int_X \sum_{j=1}^n a_j \chi_{E_j} \chi_{\bigcup_{k=1}^{\infty} A_k} d\mu \\ &= \int_X \sum_{j=1}^n a_j \chi_{E_j \cap \left(\bigcup_{k=1}^{\infty} A_k\right)} d\mu \\ &= \sum_{j=1}^n a_j \mu\left(E_j \cap \bigcup_{k=1}^{\infty} A_k\right) \\ &= \sum_{j=1}^n a_j \mu\left(\bigcup_{k=1}^{\infty} (E_j \cap A_k)\right) \\ &= \sum_{j=1}^n a_j \sum_{k=1}^{\infty} \mu(E_j \cap A_k) \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^n a_j \mu(E_j \cap A_k) \\ &= \sum_{k=1}^{\infty} \int_{A_k} \varphi d\mu \\ &= \sum_{k=1}^{\infty} \nu(A_k). \end{aligned}$$

Note that we can only switch the summation because one of them is infinite while the other one is finite. □

Remark 2.27. Let φ, ψ be simple functions such that $\varphi \leq \psi$, then $\int_X \varphi \leq \int_X \psi$. Therefore, this is true for any functions $f, g \in L^+$ as well.

Theorem 2.28 (Monotone Convergence). Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of functions in L^+ such that $f_j \leq f_{j+1}$ for all $j \in \mathbb{N}$, then

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X \lim_{n \rightarrow \infty} f_n d\mu$$

Remark 2.29. By [Remark 2.27](#), the limit on the left-hand side exists.

Proof. Since the sequence $\{f_n\}_{n \in \mathbb{N}}$ is monotonely increasing, then $\lim_{n \rightarrow \infty} f_n$ exists in $\bar{\mathbb{R}}$. Set $f = \lim_{n \rightarrow \infty} f_n$, then $f \in L^+$ as well. In particular, $f = \sup_{n \in \mathbb{N}} f_n$ as well, so $f_n \leq f$ for all $n \in \mathbb{N}$. Therefore,

$$\int_X f_n d\mu \leq \int_X f d\mu$$

for all $n \in \mathbb{N}$. Since $\{\int_X f_n d\mu\}_{n \geq 1}$ is a monotone sequence, the limit exists, therefore taking the limit $n \rightarrow \infty$ gives

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu \leq \int_X \lim_{n \rightarrow \infty} f_n d\mu.$$

It remains to show

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu \geq \int_X \lim_{n \rightarrow \infty} f_n d\mu.$$

Claim 2.30. Let φ be any simple function such that $0 \leq \varphi \leq f$. For any fixed $\alpha \in (0, 1)$, let $E_n = \{x \in X : f_n(x) \geq \alpha \varphi(x)\}$, then

- a. $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$, and $X = \bigcup_{n=1}^{\infty} E_n$;
- b. $\int_X \varphi d\mu = \lim_{n \rightarrow \infty} \int_{E_n} \varphi d\mu$.

Subproof.

- a. Since $f_{n+1} \geq f_n$, then $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$. To show $X = \bigcup_{n=1}^{\infty} E_n$, we note that $E_n \subseteq X$ for all n implies $\bigcup_{n=1}^{\infty} E_n \subseteq X$, and we claim that $X \subseteq \bigcup_{n=1}^{\infty} E_n$. Take arbitrary $x \in X$,
 - if $\varphi(x) = 0$, then $f_n(x) \geq 0 = \varphi(x)$, so $x \in E_n$ for all n by definition;
 - if $\varphi(x) > 0$, recall $f(x) = \lim_{n \rightarrow \infty} f_n(x)$, then there exists large enough $N \in \mathbb{N}$ such that $0 \leq f(x) - f_N(x) < (1 - \alpha)\varphi(x)$, but $\varphi(x) \leq f(x)$, then $0 \leq f(x) - \varphi(x) < f_N(x) - \alpha\varphi(x)$. In particular, $x \in E_N$.
- b. Recall from [Proposition 2.26](#) that $\nu(A) = \int_A \varphi d\mu$ for all $A \in \mathcal{A}$ defines a measure. By the continuity from below for ν and part a., we know

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{E_n} \varphi d\mu &= \lim_{n \rightarrow \infty} \nu(E_n) \\ &= \nu\left(\bigcup_{n=1}^{\infty} E_n\right) \\ &= \nu(X) \\ &= \int_X \varphi d\mu. \end{aligned}$$

■

By [Claim 2.30](#), we now have

$$\begin{aligned}\int_X f_n d\mu &= \int_X f_n \chi_{E_n} d\mu \\ &= \int_X \alpha \varphi \chi_{E_n} d\mu \\ &= \alpha \int_X \varphi \chi_{E_n} d\mu.\end{aligned}$$

Since this is true for all n , then taking $n \rightarrow \infty$ gives

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu \geq \alpha \lim_{n \rightarrow \infty} \int_X \varphi \chi_{E_n} d\mu = \alpha \int_X \varphi d\mu$$

for any $\alpha \in (0, 1)$. Taking $\alpha \rightarrow 1$, we get

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu \geq \int_X \varphi d\mu$$

for any function φ bounded by 0 and f . Taking the supremum over all such φ gives

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu \geq \int_X f d\mu.$$

□

Theorem 2.31. Let $f_n \in L^+$ for all $n \in \mathbb{N}$, then

$$\int_X \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu.$$

Proof.

Claim 2.32. Given any $f_1, f_2 \in L^+$,

$$\int_X (f_1 + f_2) d\mu = \int_X f_1 d\mu + \int_X f_2 d\mu.$$

Subproof. Since $f_1 \geq 0$, there exists simple functions φ_j 's such that $0 \leq \varphi_j \leq f_1$ for all $j \in \mathbb{N}$, $\varphi_j \leq \varphi_{j+1}$ for all j , and $\lim_{j \rightarrow \infty} \varphi_j = f_1$. Similarly, there are simple functions $0 \leq \psi_j \leq f_2$ for all $j \in \mathbb{N}$ with $\psi_j \leq \psi_{j+1}$ for all j , and that $\lim_{j \rightarrow \infty} \psi_j = f_2$. Therefore

$$\begin{aligned}\int_X (f_1 + f_2) d\mu &= \int_X \lim_{j \rightarrow \infty} \varphi_j + \lim_{j \rightarrow \infty} \psi_j d\mu \\ &= \int_X \lim_{j \rightarrow \infty} (\varphi_j + \psi_j) d\mu.\end{aligned}$$

Since $\varphi_j + \psi_j$ increases monotonically, so by [Theorem 2.28](#), we have

$$\begin{aligned}\int_X (f_1 + f_2) d\mu &= \int_X \lim_{j \rightarrow \infty} (\varphi_j + \psi_j) d\mu \\ &= \lim_{j \rightarrow \infty} \int_X \varphi_j + \psi_j d\mu\end{aligned}$$

$$\begin{aligned}
&= \lim_{j \rightarrow \infty} \left(\int_X \varphi_j d\mu + \int_X \psi_j d\mu \right) \\
&= \lim_{j \rightarrow \infty} \int_X \varphi_j d\mu + \lim_{j \rightarrow \infty} \int_X \psi_j d\mu \\
&= \int_X \lim_{j \rightarrow \infty} \varphi_j d\mu + \int_X \lim_{j \rightarrow \infty} \psi_j d\mu \\
&= \int_X f_1 d\mu + \int_X f_2 d\mu
\end{aligned}$$

where we apply [Theorem 2.28](#) at the last steps. ■

By [Claim 2.32](#),

$$\int_X \sum_{n=1}^N f_n d\mu = \sum_{n=1}^N \int_X f_n d\mu$$

for all $n \in \mathbb{N}$. By [Theorem 2.28](#),

$$\begin{aligned}
\int_X \sum_{n=1}^{\infty} f_n d\mu &= \lim_{N \rightarrow \infty} \int_X \sum_{n=1}^N f_n d\mu \\
&= \lim_{N \rightarrow \infty} \sum_{n=1}^N \int_X f_n d\mu \\
&= \sum_{n=1}^{\infty} \int_X f_n d\mu.
\end{aligned}$$

□

Theorem 2.33. Let $f \in L^+$, then $\int_X f d\mu = 0$ if and only if $f \equiv 0$ almost everywhere.

Proof.

(\Leftarrow): Suppose $f \equiv 0$ almost everywhere, then for every choice of simple function φ such that $0 \leq \varphi \leq f$, $\varphi \equiv 0$ almost everywhere. Take the standard representation $\varphi = \sum_{j=1}^n a_j \chi_{E_j}$, then either $a_j = 0$ or $\mu(E_j) = 0$. Therefore,

$$\begin{aligned}
\int_X \varphi d\mu &= \sum_{j=1}^n a_j \mu(E_j) \\
&= 0
\end{aligned}$$

according to the convention that $0 \cdot \infty = 0$.

(\Rightarrow): We claim that $\mu(\{x \in X : f(x) > 0\}) = 0$. To see this, note that

$$\{x \in X : f(x) > 0\} = \bigcup_{n=1}^{\infty} \{x \in X : f(x) > \frac{1}{n}\}.$$

Denote $E_n = \{x \in X : f(x) > \frac{1}{n}\}$, then we just need to show that $\mu(E_n) = 0$ for all $n \in \mathbb{N}$. Note that

$$0 = \int_X f d\mu$$

$$\begin{aligned}
&\geq \int_{E_n} f d\mu \\
&\geq \int_{E_n} \frac{1}{n} d\mu \\
&= \frac{1}{n} \times \mu(E_n),
\end{aligned}$$

so $0 \leq \mu(E_n) \leq n \cdot 0 = 0$, hence $\mu(E_n) = 0$ for all $n \in \mathbb{N}$. □

Corollary 2.34. If $f \in L^+$ and $\mu(E) = 0$, then

$$\int_E f d\mu = 0.$$

Proof. Note that

$$\int_E f d\mu = \int_X f \chi_E d\mu,$$

but $f \chi_E = 0$ almost everywhere since $\mu(E) = 0$, so by [Theorem 2.33](#) we are done. □

Theorem 2.35. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence in L^+ . Suppose that $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$, and that $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ almost everywhere $x \in X$, then

$$\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu.$$

Proof. Let $E = \{x \in X : \lim_{n \rightarrow \infty} f_n(x) = f(x)\}$, so E^c is a null set. Extend the function f to

$$f_{\text{ext}}(x) = \begin{cases} f(x), & \text{if } x \in E \\ 0, & \text{if } x \in E^c \end{cases}$$

then by [Theorem 2.28](#) we have

$$\begin{aligned}
\int_X f d\mu &= \int_E f d\mu + \int_{E^c} 0 d\mu \\
&= \int_E f d\mu \\
&= \int_E \lim_{n \rightarrow \infty} f_n d\mu \\
&= \int_X \lim_{n \rightarrow \infty} f_n \chi_E d\mu \\
&= \lim_{n \rightarrow \infty} \int_X f_n \chi_E d\mu \\
&= \lim_{n \rightarrow \infty} \left(\int_E f_n d\mu + \int_{E^c} f_n d\mu \right) \\
&= \lim_{n \rightarrow \infty} \int_X f_n d\mu.
\end{aligned}$$
□

Theorem 2.36 (Fatou's Lemma). Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence in L^+ , then

$$\int_X \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int_X f_n d\mu.$$

Remark 2.37. Note that [Theorem 2.36](#) does not require [Theorem 2.28](#), but we will use it to give a quick proof.

Proof. Note that for all $j \geq n$, we have

$$\inf_{k \geq n} f_k(x) \leq f_j(x).$$

Taking the integral, we have

$$\int_X \inf_{k \geq n} f_k d\mu \leq \int_X f_j d\mu$$

for all $j \geq n$. Therefore,

$$\int_X \inf_{k \geq n} f_k d\mu \leq \inf_{j \geq n} \int_X f_j d\mu$$

for all $n \in \mathbb{N}$. By definition,

$$\liminf_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \inf_{k \geq n} f_k(x).$$

By [Theorem 2.28](#), taking the limit gives

$$\begin{aligned} \int_X \liminf_{n \rightarrow \infty} f_n d\mu &= \lim_{n \rightarrow \infty} \int_X \inf_{k \geq n} f_k d\mu \\ &\leq \lim_{n \rightarrow \infty} \inf_{j \geq n} \int_X f_j d\mu \\ &= \liminf_{n \rightarrow \infty} \int_X f_n d\mu. \end{aligned}$$

□

Remark 2.38. There is a different version of [Theorem 2.36](#) concerning \limsup instead.

Corollary 2.39. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence in L^+ and $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ almost everywhere in $x \in X$, then

$$\int_X f d\mu \leq \liminf_{n \rightarrow \infty} \int_X f_n d\mu.$$

Theorem 2.40. Let $f \in L^+$ and $\int_X f d\mu < \infty$, then $\{x \in X : f(x) = \infty\}$ is a null set, and $\{x \in X : f(x) > 0\}$ is σ -finite.

Proof. We know that

$$\infty > \int_X f d\mu \geq \int_{\{x \in X : f(x) = \infty\}} f d\mu = \infty \mu(\{x \in X : f(x) = \infty\})$$

which forces $\mu(\{x \in X : f(x) = \infty\}) = 0$. Also note that the level set

$$\{x \in X : f(x) > 0\} = \bigcup_{n=1}^{\infty} \{x \in X : f(x) > \frac{1}{n}\},$$

so we define $E_n = \{x \in X : f(x) > \frac{1}{n}\}$, so it remains to verify that $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$. To see this,

$$\infty > \int_X f d\mu > \int_{E_n} f d\mu > \frac{1}{n} \mu(E_n),$$

therefore $\mu(E_n) < \infty$.

□

2.3 INTEGRATION OF COMPLEX-VALUED FUNCTIONS

If f is a real-valued measurable function, we know $f = f^+ - f^-$ for $f^+, f^- \in L^+$. We know how to define $\int_X f^+ d\mu$ and $\int_X f^- d\mu$. To find the integral of f , we define

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu$$

if one of the two terms is not ∞ . We need to resolve the issue when both of them are ∞ .

Definition 2.41. Let f be a complex-valued measurable function, we say f is integrable if

$$\int_X |f| d\mu < \infty,$$

that is, the L^1 -norm $\|f\|_1 = \int_X |f| d\mu$ is finite. We define

$$L^1(X) = \left\{ f : \int_X |f| d\mu < \infty \right\}.$$

to be the set of L^1 -integrable functions.

The following properties are obvious.

Theorem 2.42. Let $f, g \in L^1(X)$, then

- a. $\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu$ for all $\alpha, \beta \in \mathbb{C}$;
- b. if $|f| \leq |g|$ almost everywhere, then $\int_X |f| d\mu \leq \int_X |g| d\mu$;
- c. let $\lambda(A) = \int_A |f| d\mu$ for all $A \in \mathcal{A}$, then λ is a measure on \mathcal{A} .

Theorem 2.43 (Triangle Inequality). Let $f \in L^1(X)$, then

$$\left| \int_X f d\mu \right| \leq \int_X |f| d\mu.$$

Proof.

- If f is real-valued, then

$$\left| \int_X f d\mu \right| = \left| \int_X f^+ d\mu - \int_X f^- d\mu \right| \leq \int_X f^+ d\mu + \int_X f^- d\mu = \int_X f^+ + f^- d\mu.$$

- If f is complex-valued, now we can just assume $\int_X f d\mu \neq 0$. Set

$$\alpha = \frac{\overline{\int_X f d\mu}}{\left| \int_X f d\mu \right|},$$

then we have $|\alpha| = 1$, and

$$\left| \int_X f d\mu \right| = \frac{\overline{\int_X f d\mu} \int_X f d\mu}{\left| \int_X f d\mu \right|} = \alpha \int_X f d\mu.$$

In particular, $\alpha \int_X f d\mu \in \mathbb{R}$. We know

$$\begin{aligned} \left| \int_X f d\mu \right| &= \operatorname{Re} \left(\alpha \int_X f d\mu \right) \\ &= \operatorname{Re} \left(\int_X \alpha f d\mu \right) \\ &= \int_X \operatorname{Re}(\alpha f) d\mu \\ &\leq \int_X |\operatorname{Re}(\alpha f)| d\mu \\ &\leq \int_X |\alpha f| d\mu \\ &= |\alpha| \int_X |f| d\mu \\ &= \int_X |f| d\mu. \end{aligned}$$

□

Theorem 2.44. Let $f, g \in L^1(X)$, then

- a. $\int_X |f - g| d\mu = 0$ if and only if $f = g$ almost everywhere;
- b. $\int_E f d\mu = \int_E g d\mu$ for all $E \in \mathcal{A}$ if and only if $f = g$ almost everywhere.

Proof.

- a. We know $\int_X |f - g| d\mu = 0$ if and only if $|f - g| = 0$ almost everywhere, if and only if $f = g$ almost everywhere.
- b. If $f = g$ almost everywhere, then obviously $\int_E f d\mu = \int_E g d\mu$ for all $E \in \mathcal{A}$. The other direction is left as an exercise.

□

By [Theorem 2.44](#), we know if $f = g$ almost everywhere, then $\int_X f d\mu = \int_X g d\mu$.

Example 2.45. Let $X = [0, 1]$, set $f \equiv 1$ on X and

$$g(x) = \begin{cases} 1, & x \in [0, 1] \setminus \mathbb{Q} \\ 0, & x \in \mathbb{Q} \cap [0, 1] \end{cases}$$

on X , then $f = g$ almost everywhere. Therefore, in $L^1(X, \mathcal{A}, \mathcal{M})$, we say $f = g$. Note that in the sense of Riemann, they do not agree in terms of Riemann integrability, which is designed only for continuous functions in general.

Theorem 2.46 (Dominated Convergence Theorem). Let $\{f_n\}_{n \geq 1}$ be a sequence in $L^1(X)$ such that

- a. $\lim_{n \rightarrow \infty} f_n = f$ almost everywhere,
- b. there exists integrable function $g \in L^1$ such that $|f_n| \leq g$ for all $n \in \mathbb{N}$,

then $\int_X \lim_{n \rightarrow \infty} f_n d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu$.

Proof. First, note that $f \in L^1$: since $|f| = \lim_{n \rightarrow \infty} |f_n| \leq g \in L^1$, so $\int_X |f| d\mu \leq \int_X |g| d\mu < \infty$, hence $f \in L^1(X)$ by definition. Now note that $|f_n| \leq g$ if and only if $-g \leq f_n \leq g$ almost everywhere, then $f_n + g \in L^+$ for all $n \in \mathbb{N}$. By [Theorem 2.36](#), we know

$$\begin{aligned} \int_X \liminf_{n \rightarrow \infty} f_n d\mu + \int_X g d\mu &= \int_X \left(\liminf_{n \rightarrow \infty} f_n d\mu \right) + g \\ &= \int_X \liminf_{n \rightarrow \infty} (f_n + g) d\mu \\ &\leq \liminf_{n \rightarrow \infty} \int_X (f_n + g) d\mu \\ &= \liminf_{n \rightarrow \infty} \left(\int_X f_n d\mu + \int_X g d\mu \right) \\ &= \left(\liminf_{n \rightarrow \infty} \int_X f_n d\mu \right) + \int_X g d\mu, \end{aligned}$$

therefore $\int_X \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int_X f_n d\mu$. Since $g - f_n \in L^+$, then by [Theorem 2.36](#) again, we know

$$\begin{aligned} \int_X g d\mu - \int_X \limsup_{n \rightarrow \infty} f_n d\mu &= \int_X (g - \limsup_{n \rightarrow \infty} f_n) d\mu \\ &= \int_X (g + \liminf_{n \rightarrow \infty} (-f_n)) d\mu \\ &= \int_X \liminf_{n \rightarrow \infty} (g - f_n) d\mu \\ &\leq \liminf_{n \rightarrow \infty} \int_X (g - f_n) d\mu \\ &= \liminf_{n \rightarrow \infty} \left(\int_X g d\mu - \int_X f_n d\mu \right) \\ &= \int_X g d\mu - \limsup_{n \rightarrow \infty} \int_X f_n d\mu, \end{aligned}$$

hence $\int_X \limsup_{n \rightarrow \infty} f_n d\mu \geq \limsup_{n \rightarrow \infty} \int_X f_n d\mu$. This gives

$$\int_X f d\mu = \int_X \limsup_{n \rightarrow \infty} f_n d\mu \geq \limsup_{n \rightarrow \infty} \int_X f_n d\mu \geq \liminf_{n \rightarrow \infty} \int_X f_n d\mu \geq \int_X \liminf_{n \rightarrow \infty} f_n d\mu = \int_X f d\mu$$

and forces

$$\limsup_{n \rightarrow \infty} \int_X f_n d\mu = \liminf_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu.$$

In particular, the limit exists, hence

$$\int_X \lim_{n \rightarrow \infty} f_n d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu.$$

□

Theorem 2.47. Suppose that $\{f_j\}_{j \geq 1}$ is a sequence in L^1 such that $\sum_{j=1}^{\infty} \int_X |f_j| d\mu < \infty$, then $\sum_{j=1}^{\infty} f_j$ converges almost everywhere to a function in L^1 such that

$$\int_X \sum_{j=1}^{\infty} f_j d\mu = \sum_{j=1}^{\infty} \int_X f_j d\mu.$$

Proof. Let $g(x) = \sum_{j=1}^{\infty} |f_j(x)|$ for all $x \in X$, then

$$\int_X g d\mu = \int_X \sum_{j=1}^{\infty} |f_j| d\mu = \sum_{j=1}^{\infty} \int_X |f_j| d\mu < \infty.$$

Therefore $g \in L^1$. For all $n \in \mathbb{N}$, we set $g_n = \sum_{j=1}^n f_j$ and therefore $|g_n| \leq g$ for all $n \in \mathbb{N}$. Now by [Theorem 2.46](#) we know

$$\begin{aligned} \int_X \sum_{j=1}^{\infty} f_j d\mu &= \int_X \lim_{n \rightarrow \infty} g_n d\mu \\ &= \lim_{n \rightarrow \infty} \int_X g_n d\mu \\ &= \lim_{n \rightarrow \infty} \int_X \sum_{j=1}^n f_j d\mu \\ &= \lim_{n \rightarrow \infty} \sum_{j=1}^n \int_X f_j d\mu \\ &= \sum_{j=1}^{\infty} \int_X f_j d\mu. \end{aligned}$$

□

Theorem 2.48. Let $f \in L^1$. For any $\varepsilon > 0$, there exists a simple function $\varphi \in L^1$ such that $\|f - \varphi\|_1 < \varepsilon$.

Proof. Note that $|f| \in L^+$, therefore there exists a sequence $\{\varphi_n\}_{n \geq 1}$ of simple functions such that $0 \leq |\varphi_1| \leq \dots \leq |\varphi_n| \leq \dots \leq |f|$ with $\lim_{n \rightarrow \infty} \varphi_n = f$. Therefore

$$|f - \varphi_n| \leq |f| + |\varphi_n| \leq 2|f| \in L^1.$$

By [Theorem 2.46](#), we have

$$0 = \int_X \lim_{n \rightarrow \infty} |f - \varphi_n| d\mu = \lim_{n \rightarrow \infty} \int_X |f - \varphi_n| d\mu,$$

hence $\lim_{n \rightarrow \infty} \int_X |f - \varphi_n| d\mu = 0$. Now for any $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that $\int_X |f - \varphi_N| < \varepsilon$. Take $\varphi = \varphi_N$, we have $\|f - \varphi\|_1 < \varepsilon$ as desired. □

Theorem 2.49. Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function where $a, b \in \mathbb{R}$, then f is Riemann integrable if and only if the Lebesgue measure $\mathbf{m}(\{x \in [a, b] : f \text{ is discontinuous}\}) = 0$.

Example 2.50. $\chi_{\mathbb{Q}}$ is not Riemann integrable on $[0, 1]$ because it is discontinuous everywhere.

Example 2.51. Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$, then χ_S is Riemann integrable on $[0, 1]$ because

$$\mathbf{m}(\{x \in [0, 1] : \chi_S \text{ is discontinuous at } x\}) = \mathbf{m}(S) = 0.$$

Example 2.52. Let \mathcal{C} be the Cantor set, c.f., [Definition 1.64](#), then $\chi_{\mathcal{C}}$ is Riemann integrable on $[0, 1]$.

Proof. Given a partition \mathcal{P} of $[a, b]$

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b,$$

recall that $||\mathcal{P}|| = \max\{|x_j - x_{j-1}| : 1 \leq j \leq n\}$, then we have two simple functions

$$U_{\mathcal{P}}(x) = \sum_{j=1}^n \sup_{x \in [x_{j-1}, x_j)} f(x) \cdot \chi_{[x_{j-1}, x_j)}(x)$$

and

$$L_{\mathcal{P}}(x) = \sum_{j=1}^n \inf_{x \in [x_{j-1}, x_j)} f(x) \cdot \chi_{[x_{j-1}, x_j)}(x).$$

We try to create a Riemann sum with respect to these two functions. We have

$$\begin{aligned} \int_{[a,b]} U_{\mathcal{P}} d\mathbf{m} &= \sum_{j=1}^n \sup_{x \in [x_{j-1}, x_j)} f(x) (x_j - x_{j-1}) \\ &:= U(f, \mathcal{P}) \end{aligned}$$

and

$$\begin{aligned} \int_{[a,b]} L_{\mathcal{P}} d\mathbf{m} &= \sum_{j=1}^n \inf_{x \in [x_{j-1}, x_j)} f(x) (x_j - x_{j-1}) \\ &:= L(f, \mathcal{P}). \end{aligned}$$

Let us take a sequence of partitions $\{\mathcal{P}_n\}_{n \geq 1}$ such that

$$\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \cdots \subseteq \mathcal{P}_n \subseteq \cdots$$

and $\lim_{n \rightarrow \infty} ||\mathcal{P}_n|| = 0$. Recall that f is Riemann integrable if and only if $L(f) =: \lim_{n \rightarrow \infty} L(f, \mathcal{P}_n) = \lim_{n \rightarrow \infty} U(f, \mathcal{P}_n) := U(f)$. We can bound f by the simple functions

$$L_{\mathcal{P}_1} \leq \cdots \leq L_{\mathcal{P}_n} \leq \cdots \leq f \leq \cdots \leq U_{\mathcal{P}_n} \leq \cdots \leq U_{\mathcal{P}_1}.$$

Therefore we get a monotone sequence and take the limit $n \rightarrow \infty$ since it exists in $\bar{\mathbb{R}}$, then $L := \lim_{n \rightarrow \infty} L_{\mathcal{P}_n}$ and $U = \lim_{n \rightarrow \infty} U_{\mathcal{P}_n}$ are $\bar{\mathbb{R}}$ -valued functions, and are measurable. Since the limit preserves the order, we know that $L \leq f \leq U$. In particular, there exists some constant C such that

$$|U_{\mathcal{P}_n}| \leq \sup_{x \in [a,b]} |f(x)| \leq C$$

and

$$|L_{\mathcal{P}_n}| \leq \inf_{x \in [a,b]} |f(x)| \leq C$$

for all $n \in \mathbb{N}$. Therefore we get $|U| \leq C$ and $|L| \leq C$, where $C \in L^1([a, b])$. By [Theorem 2.46](#), we have that

$$\int_{[a,b]} U d\mathbf{m} = \int_{[a,b]} \lim_{n \rightarrow \infty} U_{\mathcal{P}_n} d\mathbf{m}$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \int_{[a,b]} U_{\mathcal{P}_n} d\mathbf{m} \\
&= \lim_{n \rightarrow \infty} U(f, \mathcal{P}_n) \\
&= U(f)
\end{aligned}$$

and similarly

$$\begin{aligned}
\int_{[a,b]} L d\mathbf{m} &= \int_{[a,b]} \lim_{n \rightarrow \infty} L_{\mathcal{P}_n} d\mathbf{m} \\
&= \lim_{n \rightarrow \infty} \int_{[a,b]} L_{\mathcal{P}_n} d\mathbf{m} \\
&= \lim_{n \rightarrow \infty} L(f, \mathcal{P}_n) \\
&= L(f).
\end{aligned}$$

Therefore, we know

$$\begin{aligned}
f \text{ is Riemann integrable} &\iff U(f) = L(f) = \int_a^b f dx \text{ in the Riemann sense} \\
&\iff \int_{[a,b]} U d\mathbf{m} = \int_{[a,b]} L d\mathbf{m} \\
&\iff \int_{[a,b]} (U - L) d\mathbf{m} = 0 \\
&\iff \mathbf{m}(\{x \in [a, b] : U(x) > L(x)\}) = 0.
\end{aligned}$$

Claim 2.53. If $f : [a, b] \rightarrow \mathbb{R}$ is a bounded Riemann integrable function, then f is Lebesgue integrable. Moreover,

$$\int_{[a,b]} f d\mathbf{m} = \int_a^b f dx.$$

Subproof. We have

$$\begin{aligned}
\{x \in [a, b] : f(x) \neq U(x)\} &\subseteq \{x \in [a, b] : L(x) \neq U(x)\} \\
&= \{x \in [a, b] : U(x) > L(x)\}
\end{aligned}$$

and therefore

$$\mathbf{m}(\{x \in [a, b] : f(x) \neq U(x)\}) = 0.$$

Hence,

$$\begin{aligned}
\int_{[a,b]} f d\mathbf{m} &= \int_{[a,b]} U d\mathbf{m} \\
&= U(f) \\
&= \int_a^b f dx.
\end{aligned}$$

■

It now suffices to prove the following claim.

Claim 2.54. $\mathbf{m}(\{x \in [a, b] : U(x) > L(x)\}) = 0$ if and only if $\mathbf{m}(\{x \in [a, b] : f \text{ is discontinuous at } x\}) = 0$.

Subproof. For any $A \subseteq [a, b]$, we define the oscillation of f to be $\omega_f(A) = \sup_{x \in A} f(x) - \inf_{x \in A} f(x)$. Now f is continuous at x_0 if and only if the oscillation of f at x_0 is $\Omega_f(x_0) := \lim_{\delta \rightarrow 0} \omega_f((x_0 - \delta, x_0 + \delta)) = 0$. Note that the function is monotone with respect to δ , therefore the limit exists. Let $x \in [a, b] \setminus \bigcup_{n=1}^{\infty} \mathcal{P}_n$ with a zero-measure subset removed. Denote the subinterval in \mathcal{P}_n containing x by I_n , then

$$\begin{aligned} \Omega_f(x) &= \lim_{n \rightarrow \infty} \omega_f(I_n) \\ &= \lim_{n \rightarrow \infty} [U_{\mathcal{P}_n}(x) - L_{\mathcal{P}_n}(x)] \\ &= U(x) - L(x). \end{aligned}$$

Therefore,

$$\begin{aligned} f \text{ is continuous at } x &\iff \Omega_f(x) = 0 \\ &\iff U(x) = L(x) \\ &\iff U(x) = L(x), \end{aligned}$$

and we conclude that

$$\mathbf{m}(\{x \in [a, b] : f \text{ is discontinuous at } x\}) = \mathbf{m}(\{x \in [a, b] : U(x) > L(x)\})$$

as desired. ■

□

2.4 MODES OF CONVERGENCES

Definition 2.55. We say $\{f_n\}_{n \geq 1}$ converges to f uniformly on E if $\lim_{n \rightarrow \infty} \sup_{x \in E} |f_n(x) - f(x)| = 0$, and write $f_n \rightrightarrows f$ on E as $n \rightarrow \infty$.

Remark 2.56. If $f_n \rightrightarrows f$ on E , then $f_n \rightarrow f$ on E .

Definition 2.57. We say $\{f_n\}_{n \geq 1}$ converges to f in L^1 if $\lim_{n \rightarrow \infty} \|f_n - f\|_1 = 0$, and write $f_n \xrightarrow{L^1} f$ as $n \rightarrow \infty$.

Definition 2.58. We say that $\{f_n\}_{n \geq 1}$ converges to f in measure μ if for all $\varepsilon > 0$, $\lim_{n \rightarrow \infty} \mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) = 0$. We write $f_n \xrightarrow{\mu} f$ as $n \rightarrow \infty$.

We now study the relations between different types of convergence.

Theorem 2.59. If $f_n \xrightarrow{L^1} f$, then $f_n \xrightarrow{\mu} f$.

Proof. Pick $\varepsilon > 0$, and let $E_n = \{x \in X : |f_n(x) - f(x)| > \varepsilon\}$. Now

$$\begin{aligned} \varepsilon \mu(E_n) &= \int_{E_n} \varepsilon d\mu \\ &\leq \int_{E_n} |f_n - f| d\mu \\ &\leq \int_X |f_n - f| d\mu \end{aligned}$$

$$= \|f_n - f\|_1,$$

therefore $0 \leq \mu(E_n) \leq \frac{1}{\varepsilon} \|f_n - f\|_1$. Let $n \rightarrow \infty$, then $0 \leq \lim_{n \rightarrow \infty} \mu(E_n) \leq 0$ so by squeeze theorem we have $\lim_{n \rightarrow \infty} \mu(E_n) = 0$. By definition, $f_n \xrightarrow{\mu} f$. \square

Example 2.60. Let $f_n = \frac{\chi_{(0,n)}}{n}$ be a function on \mathbb{R} , then $f_n \rightarrow 0$ on \mathbb{R} pointwise. Thus, $f_n \rightarrow 0$ on \mathbb{R} pointwise. Moreover, $f_n \xrightarrow{u} 0$, but $f_n \not\xrightarrow{L^1} 0$, thus the converse of [Theorem 2.59](#) is not true:

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_X |f_n - 0| d\mathbf{m} &= \lim_{n \rightarrow \infty} \int_X |f_n| d\mathbf{m} \\ &= \frac{1}{n} \int_X \chi_{(0,n)} d\mathbf{m} \\ &= \frac{n}{n} \\ &= 1. \end{aligned}$$

Example 2.61. Let $f_n = \chi_{(n,n+1)}$ be a function on \mathbb{R} , then $f_n \rightarrow 0$ on \mathbb{R} pointwise, but $f_n \not\xrightarrow{\mathbf{m}} 0$ does not converge to 0 on measure \mathbf{m} : for any $\varepsilon > 0$,

$$\mathbf{m}(\{x \in X : |\chi_{(n,n+1)}(x) > \varepsilon|\}) = \mathbf{m}(\{x \in (n, n+1) : \varepsilon < 1\}),$$

so for any $1 > \varepsilon > 0$, taking the limit $n \rightarrow \infty$ gives

$$\lim_{n \rightarrow \infty} \mathbf{m}(\{x \in X : |\chi_{(n,n+1)}(x) > \varepsilon|\}) = 1.$$

Definition 2.62. Let $\{f_n\}_{n \geq 1}$ be a sequence of measurable functions. We say the sequence is Cauchy in measure if for all $\sigma > 0$, for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that $\mu(\{x \in X : |f_n(x) - f_m(x)| > \varepsilon\}) < \sigma$ for all $m, n \geq N$.

Equivalently, the sequence is Cauchy in measure if for any $\varepsilon > 0$,

$$\lim_{m, n \rightarrow \infty} \mu(\{x \in X : |f_n(x) - f_m(x)| > \varepsilon\}) = 0.$$

Theorem 2.63. Suppose $\{f_n\}_{n \geq 1}$ is Cauchy in measure, then there exists a subsequence $\{f_{n_j}\}_{j \geq 1}$ such that $f_{n_j} \rightarrow f$ almost everywhere as $j \rightarrow \infty$.

Proof. Let $\sigma = \varepsilon = 2^{-j}$ for all $j \in \mathbb{N}$, then there exists $n_j \in \mathbb{N}$ such that $\mu(\{x \in X : |f_{n_{j+1}}(x) - f_{n_j}(x)| > 2^{-j}\}) < 2^{-j}$, therefore we have choices $n_j < n_{j+1}$ for all J . Now we know $\{f_{n_j}\}_{j \geq 1}$ is a subsequence, so let $g_j = f_{n_j}$ for all $j \in \mathbb{N}$. Therefore,

$$\mu(\{x \in X : |g_{j+1}(x) - g_j(x)| > 2^{-j}\}) \leq 2^{-j}$$

for all j . Let $E_j = \{x \in X : |g_{j+1}(x) - g_j(x)| > 2^{-j}\}$, then $\mu(E_j) \leq 2^{-j}$.

Claim 2.64. For all $k \in \mathbb{N}$ and $F_k = \bigcup_{j=k}^{\infty} E_j$, then $\{g_j\}_{j \geq 1}$ is pointwise Cauchy on F_k^c .

Subproof. We show that for $x \in F_k^c$, we have $\lim_{m, n \rightarrow \infty} |g_m(x) - g_n(x)| = 0$, which is equivalent to saying for all $\varepsilon > 0$, for

all $x \in F_k^c$, there exists $N \in \mathbb{N}$ such that $|g_m(x) - g_n(x)| < \varepsilon$ for all $m, n \geq N$. Since $x \in F_k^c$, then $x \in \left(\bigcup_{j=k}^{\infty} E_j\right)^c =$

$\bigcap_{j=k}^{\infty} E_j^c$, so for all $j \geq k$ we know $x \in E_j^c$, which is equivalent to saying that for all $j \geq k$, $|g_{j+1}(x) - g_j(x)| < 2^{-j}$.

Without loss of generality, take arbitrary $m > n \geq k$, we get

$$|g_m(x) - g_n(x)| = \left| \sum_{j=n}^{m-1} [g_{j+1}(x) - g_j(x)] \right|$$

$$\begin{aligned}
&\leq \sum_{j=n}^{m+1} |g_{j+1}(x) - g_j(x)| \\
&\leq \sum_{j=n}^{m+1} 2^{-j} \\
&\leq 2^{1-n}.
\end{aligned}$$

Taking $n \rightarrow \infty$, we forces $\lim_{m,n \rightarrow \infty} |g_m(x) - g_n(x)| = 0$, as desired. ■

Claim 2.65. Let $F = \bigcap_{k=1}^{\infty} F_k$, then $\mu(F) = 0$.

Subproof. We know that for all $n \in \mathbb{N}$,

$$\begin{aligned}
\mu(F) &\leq \mu(F_n) \\
&= \mu\left(\bigcup_{j=n}^{\infty} F_j\right) \\
&\leq \sum_{j=n}^{\infty} \mu(E_j) \\
&\leq \sum_{j=n}^{\infty} 2^{-j} \\
&\leq 2^{1-n},
\end{aligned}$$

so for $n \rightarrow \infty$, we forces $\mu(F) = 0$. ■

Claim 2.66. If $x \in F^c$, then $\{g_j(x)\}_{j \geq 1}$ is a pointwise Cauchy sequence.

Subproof. For any $x \in F^c$, we know $x \in (\bigcap_{k=1}^{\infty} F_k)^c = \bigcup_{k=1}^{\infty} F_k^c$, therefore $x \in F_k^c$ for some $k \in \mathbb{N}$. By [Claim 2.64](#), we conclude that $\{g_j(x)\}_{j \geq 1}$ is a pointwise Cauchy sequence. ■

Therefore, for any $x \in F^c$, we know $\{g_j(x)\}$ is Cauchy, so $\lim_{j \rightarrow \infty} g_j(x)$ exists in \mathbb{R} . Let f be given by

$$f(x) = \begin{cases} \lim_{j \rightarrow \infty} g_j(x), & x \in F^c \\ 0, & x \in F \end{cases}$$

then $\{g_j\}$ converges to f almost everywhere. Consider $\{g_j\}_{j \geq 1}$ as the said subsequence $\{f_{n_j}\}_{j \geq 1}$ of $\{f_n\}_{n \geq 1}$, then we are done. □

Theorem 2.67 (Cauchy Criterion). The sequence $\{f_n\}_{n \geq 1}$ is Cauchy in measure if and only if there is a measurable function f such that $f_n \xrightarrow{\mu} f$.

Proof.

(\Leftarrow): Suppose $f_n \xrightarrow{\mu} f$, and set $\varepsilon > 0$, then we want to show that $\lim_{m,n \rightarrow \infty} \mu(\{x \in X : |f_m(x) - f_n(x)| > \varepsilon\}) = 0$. We know, for any $x \in X$ that lies in the given subset, that

$$\begin{aligned}
\varepsilon &< |f_m(x) - f_n(x)| \\
&= |(f_m(x) - f(x)) + (f(x) - f_n(x))| \\
&\leq |f_m(x) - f(x)| + |f_n(x) - f(x)|,
\end{aligned}$$

therefore either $|f_m(x) - f(x)| > \frac{\varepsilon}{2}$ or $|f_n(x) - f(x)| > \frac{\varepsilon}{2}$. Therefore,

$$\{x \in X : |f_m(x) - f_n(x)| > \varepsilon\} \subseteq \{x \in X : |f_m(x) - f(x)| > \frac{\varepsilon}{2}\} \cup \{x \in X : |f_n(x) - f(x)| > \frac{\varepsilon}{2}\}.$$

Hence,

$$\mu(\{x \in X : |f_m(x) - f_n(x)| > \varepsilon\}) \leq \mu(\{x \in X : |f_m(x) - f(x)| > \frac{\varepsilon}{2}\}) + \mu(\{x \in X : |f_n(x) - f(x)| > \frac{\varepsilon}{2}\}),$$

but as $m, n \rightarrow \infty$, the two measures of the right-hand side converges to 0, which forces the measure on the left also converges to 0.

(\Rightarrow): Since $\{f_n\}_{n \geq 1}$ is Cauchy in measure, then there exists a subsequence $\{g_j\}_{j \geq 1} = \{f_{n_j}\}_{j \geq 1}$ such that $\lim_{j \rightarrow \infty} f_{n_j} = \lim_{j \rightarrow \infty} g_j = f$ almost everywhere.

Claim 2.68. $g_j \xrightarrow{\mu} f$.

Subproof. Again, let $E_j = \{x \in X : |g_{j+1}(x) - g_j(x)| > 2^{-j}\}$, and set $F_k = \bigcup_{j=k}^{\infty} E_j$ as in [Theorem 2.63](#), then we know for all $x \in F_k^c$, we have

$$|g_m(x) - g_j(x)| \leq 2^{1-j}$$

for all $m, j \geq k$. Now let $m \rightarrow \infty$, then

$$|f(x) - g_j(x)| \leq 2^{1-j}$$

for any $j \geq k$ and $x \in F_k^c$. Fix $\varepsilon > 0$. For large enough j , we know $2^{1-j} < \varepsilon$ and therefore satisfies

$$\{x \in X : |g_j(x) - f(x)| > \varepsilon\} = \{x \in F_j : |g_j(x) - f(x)| > \varepsilon\} \cup \{x \in F_j^c : |g_j(x) - f(x)| > \varepsilon\}.$$

But note that for any $x \in F_j^c$, $|g_j(x) - f(x)| \leq 2^{1-j} < \varepsilon$, which forces the second set to be empty, therefore we have

$$\{x \in X : |g_j(x) - f(x)| > \varepsilon\} = \{x \in F_j : |g_j(x) - f(x)| > \varepsilon\} \subseteq F_j.$$

Taking the measure, we have

$$\begin{aligned} \mu(\{x \in X : |g_j(x) - f(x)| > \varepsilon\}) &\leq \mu(F_j) \\ &\leq 2^{1-j} \\ &\rightarrow 0 \end{aligned}$$

as $j \rightarrow \infty$. Therefore, $g_j \xrightarrow{\mu} f$. ■

Claim 2.69. $f_n \xrightarrow{\mu} f$.

Subproof. We know that

$$\begin{aligned} \varepsilon &< |f_n(x) - f(x)| \\ &< |f_n(x) - g_j(x)| + |g_j(x) - f(x)| \\ &\leq |f_n(x) - g_j(x)| + |g_j(x) - f(x)| \end{aligned}$$

and therefore either $|f_n(x) - g_j(x)| > \frac{\varepsilon}{2}$ or $|g_j(x) - f(x)| > \frac{\varepsilon}{2}$. Therefore,

$$\{x \in X : |f_n(x) - f(x)| > \varepsilon\} \subseteq \{x \in X : |f_n(x) - g_j(x)| > \frac{\varepsilon}{2}\} \cup \{x \in X : |g_j(x) - f(x)| > \frac{\varepsilon}{2}\}.$$

Taking the measure, we know that

$$\mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) \leq \mu(\{x \in X : |f_n(x) - g_j(x)| > \frac{\varepsilon}{2}\}) + \mu(\{x \in X : |g_j(x) - f(x)| > \frac{\varepsilon}{2}\}).$$

Let $j, n \rightarrow \infty$, then $\mu(\{x \in X : |g_j(x) - f(x)| > \frac{\varepsilon}{2}\}) \rightarrow 0$ since $g_j \xrightarrow{\mu} f$, and $\mu(\{x \in X : |f_n(x) - g_j(x)| > \frac{\varepsilon}{2}\}) \rightarrow 0$ since $\{f_n\}_{n \geq 1}$ is Cauchy in measure. Therefore, $\mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) \rightarrow 0$ as $j, n \rightarrow \infty$. In particular, that means

$$\lim_{n \rightarrow \infty} \mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) = 0.$$

■

□

Theorem 2.70. Suppose $f_n \xrightarrow{\mu} f$ in measure, then there exists a subsequence $\{f_{n_j}\}_{j \geq 1}$ such that $f_{n_j} \rightarrow f$ almost everywhere.

Proof. Since $f_n \xrightarrow{\mu} f$, then $\{f_n\}_{n \geq 1}$ is Cauchy in measure, therefore by [Theorem 2.63](#) there exists a subsequence $\{f_{n_j}\}_{j \geq 1}$ such that $f_{n_j} \rightarrow f$ almost everywhere. □

Corollary 2.71. If $\{f_n\}_{n \geq 1}$ converges to f in L^1 , i.e., $\|f_n - f\|_1 \rightarrow 0$, then there exists a subsequence $\{f_{n_j}\}_{j \geq 1}$ such that $f_{n_j} \rightarrow f$ almost everywhere.

Proof. This is obvious from [Theorem 2.70](#). □

Definition 2.72. We say $\{f_n\}_{n \geq 1}$ converges to f almost uniformly on X if for any $\varepsilon > 0$, there exists a subset $E \subseteq X$ such that $\mu(E) < \varepsilon$ and $f_n \rightrightarrows f$ on E^c .

Theorem 2.73 (Egoroff). Suppose that $\mu(X) < \infty$ and $f_n \rightarrow f$ almost everywhere on X , then $\{f_n\}_{n \geq 1}$ converges to f almost uniformly.

Proof. Without loss of generality, suppose $f_n \rightarrow f$ for all $x \in X$. For any $k \in \mathbb{N}$ and $n \in \mathbb{N}$, we define

$$E_n(k) = \bigcup_{m=n}^{\infty} \{x \in X : |f_m(x) - f(x)| > \frac{1}{k}\}.$$

Claim 2.74. Given any k , $E_n(k) \supseteq E_{n+1}(k)$ for all $n \in \mathbb{N}$.

Subproof. This follows from the definition of $E_n(k)$. ■

Claim 2.75. $\bigcap_{n \geq 1} E_n(k) = \emptyset$.

Subproof. Suppose not, then there exists $x \in \bigcap_{n \geq 1} E_n(k)$, hence $x \in E_n(k)$ for all $n \in \mathbb{N}$. By definition, we know there is a subsequence $\{f_{n_j}\}_{j \geq 1}$ of $\{f_n\}_{n \geq 1}$ such that $|f_{n_j}(x) - f(x)| > \frac{1}{k}$ for any $j \in \mathbb{N}$. Let $j \rightarrow \infty$, we know $0 = \lim_{j \rightarrow \infty} |f_{n_j}(x) - f(x)| \geq \frac{1}{k}$, contradiction. ■

Since $\mu(X) < \infty$, then

$$\begin{aligned} \lim_{n \rightarrow \infty} \mu(E_n(k)) &= \mu\left(\bigcap_{n=1}^{\infty} E_n(k)\right) \\ &= \mu(\emptyset) \\ &= 0. \end{aligned}$$

For arbitrary $\varepsilon > 0$, there exists some $n_k \in \mathbb{N}$ such that $\mu(E_{n_k}(k)) < \varepsilon \cdot 2^{-k}$. Take $E = \bigcup_{k \geq 1} E_{n_k}(k)$, then

$$\mu(E) \leq \sum_{k \geq 1} \mu(E_{n_k}(k)) < \sum_{k \geq 1} \varepsilon \cdot 2^{-k} \leq \varepsilon.$$

Finally, we need to show that $f_n \rightrightarrows f$ on E^c . Take $x \in E^c$, then $x \in \bigcap_{k \geq 1} [E_{n_k}(k)]^c$, therefore $x \in (E_{n_k}(k))^c$ for all $k \in \mathbb{N}$. Recall that

$$(E_{n_k}(k))^c = \bigcap_{m \geq n_k} \{x \in X : |f_m(x) - f(x)| \leq \frac{1}{k}\},$$

Thus, if $x \in E^c$, we know $|f_n(x) - f(x)| \leq \frac{1}{k}$ for all $k \in \mathbb{N}$ and $n \geq n_k$, hence $\sup_{x \in E^c} |f_n(x) - f(x)| \leq \frac{1}{k}$ for all $k \in \mathbb{N}$ and $n \geq n_k$, therefore

$$0 \leq \lim_{n \rightarrow \infty} \sup_{x \in E^c} |f_n(x) - f(x)| \leq \frac{1}{k}.$$

In particular, this limit tends to 0 when $k \rightarrow \infty$. This shows that $\lim_{n \rightarrow \infty} \sup_{x \in E^c} |f_n(x) - f(x)| = 0$, in other words $f_n \rightrightarrows f$ on E^c . Therefore, f_n converges almost uniformly to f on E^c . \square

Remark 2.76. If f_n converges to f almost uniformly on X , then $f_n \rightarrow f$ almost everywhere on X and $f_n \xrightarrow{\mu} f$ on X .

Remark 2.77. The condition $\mu(X) < \infty$ in [Theorem 2.73](#) is necessary. To see this, consider the measure space $(\mathbb{R}, \mathcal{L}, \mathbf{m})$, and consider $f_n = \chi_{[n, \infty)}$ for all $n \in \mathbb{N}$. Now $f_n \rightarrow 0$ converges, but f_n does not converge to 0 in measure \mathbf{m} . Indeed,

$$\begin{aligned} \mathbf{m}(\{x \in \mathbb{R} : |f_n(x)| > \frac{1}{2}\}) &= \mathbf{m}(\{x \in [n, \infty)\}) \\ &= \infty \nrightarrow 0. \end{aligned}$$

By [Remark 2.76](#), $\{f_n\}_{n \geq 1}$ does not converge to 0 almost uniformly on \mathbb{R} .

Remark 2.78. The hypothesis $\mu(X) < \infty$ in [Theorem 2.73](#) can be replaced by $|f_n| \leq g$ for all $n \in \mathbb{N}$ and $g \in L^1(X)$.

Theorem 2.79. Let f be any complex-valued measurable function on E with $\mu(E) < \infty$. Then for any $\varepsilon > 0$, there exist a simple function φ and a measurable set $F \subseteq E$ such that

1. $\mu(E \setminus F) < \varepsilon$, and
2. $|f(x) - \varphi(x)| < \varepsilon$ for all $x \in F$.

Proof. Without loss of generality, assume $f \in L^+$. Let $\varphi_n(x) = \sum_{k=0}^{2^{2n}-1} k2^{-n}\chi_{E_{n,k}}(x) + 2^n\chi_{F_n}(x)$, where $E_{n,k} = \{x \in E : f(x) \in (k2^{-n}, (k+1)2^{-n}]\}$ and $F_n = \{x \in E : f(x) > 2^n\}$. Therefore, $F_n \supseteq F_{n+1}$ and $\mu(F_n) \leq \mu(E) < \infty$ for all $n \in \mathbb{N}$, so by continuity from above we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \mu(F_n) &= \mu\left(\bigcap_{n \geq 1} F_n\right) \\ &= \mu(\emptyset) \\ &= 0. \end{aligned}$$

For any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that $\mu(F_n) < \varepsilon$ for all $n \geq N_1$. Recall that $|\varphi_n(x) - f(x)| \leq 2^{-n}$ for all $x \notin F_n$, then $\sup_{x \in F_n^c} |\varphi_n(x) - f(x)| \leq 2^{-n}$, then by squeeze theorem we have $\lim_{n \rightarrow \infty} \sup_{x \in F_n^c} |\varphi_n(x) - f(x)| = 0$. Hence, for any $\varepsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that $\sup_{x \in F_n^c} |\varphi_n(x) - f(x)| < \varepsilon$ for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$, then $|\varphi_N(x) - f(x)| < \varepsilon$ for all $x \notin F_N$, and $\mu(F_N) < \varepsilon$. Define $\varphi = \varphi_N$ to be the said simple function, and let $F = E \setminus F_N$. \square

Theorem 2.80. Let $\mu(X) < \infty$ and f be a complex-valued measurable function on X . For any $\varepsilon > 0$, there exists $0 < M \in \mathbb{R}$ and a measurable set $E \subseteq X$ such that $|f(x)| < M$ for all $x \in E$ and $\mu(E^c) < \varepsilon$.

Proof. By [Theorem 2.79](#), for any $\varepsilon > 0$, there exists a simple function φ and a measurable set $E \subseteq X$ such that $\mu(E^c) < \varepsilon$ and $|f(x) - \varphi(x)| < \varepsilon$ for all $x \in E$. Using the triangle inequality and the fact that φ is a simple function on E , we know for any $x \in E$ that

$$\begin{aligned} |f(x)| &\leq |f(x) - \varphi(x)| + |\varphi(x)| \\ &< \varepsilon + |\varphi(x)| \\ &< \varepsilon + \sup_{x \in E} |\varphi(x)| \\ &=: M \in \mathbb{R}. \end{aligned}$$

\square

Theorem 2.81. For any $f \in L^1(\mathbb{R}, \mathcal{A}, \mu)$ where μ is a Lebesgue-Stieltjes measure, then for any $\varepsilon > 0$, there exists a continuous function g on \mathbb{R} such that $\|f - g\|_1 < \varepsilon$.

Proof. For any $\varepsilon > 0$, there exists a simple function $\varphi \in L^1$ such that $\|f - \varphi\|_1 < \varepsilon$. Let us write $\varphi(x) = \sum_{j=1}^n a_j \chi_{E_j}$, where each $a_j \neq 0$, and each $\mu(E_j) < \infty$ for all j . We can replace E_j by a finite union of disjoint open intervals $I_k^{(j)}$ for each j , then $\mu\left(E_j \Delta \left(\bigcup_{k=1}^K I_k^{(j)}\right)\right) < \frac{\varepsilon}{2^j |a_j|}$. Therefore, χ_{E_j} can be replaced by $\chi_{\bigcup_{k=1}^K I_k^{(j)}}$, which can then be replaced by continuous functions g_j , where we replace the function upon intervals on $I_k^{(j)}$ for each k , such that $g = \sum_{j=1}^n g_j$. This gives the desired function g . \square

Theorem 2.82 (Lusin). Let μ be a Lebesgue-Stieltjes measure on \mathbb{R} , and let f be any complex-valued function measurable function on E with $\mu(E) < \infty$, then f is almost a continuous function on E in the following sense: for any $\varepsilon > 0$, there exists a function g on E and a measurable set $F \subseteq E$ such that

1. g is continuous on E ,
2. $\mu(E \setminus F) < \varepsilon$, and
3. $|f(x) - g(x)| < \varepsilon$ for all $x \in F$.

Proof Sketch.

- By [Theorem 2.79](#), we know any complex-valued function is “almost simple”, i.e., close to a simple function $\varphi \in L^1$ on E .
- Since φ is integrable, then by [Theorem 2.81](#), we know continuous functions are dense in L^1 , i.e., there exists a sequence $\{g_j\}_{j \geq 1}$ of continuous functions such that $\|g_j - \varphi\|_1 \rightarrow 0$ as $j \rightarrow \infty$. Here we can replace $\|\cdot\|_1$ by $\|\cdot\|_{L^1(E)}$.
- We can now find a subsequence $\{g_{n_j}\}_{j \geq 1}$ of $\{g_j\}_{j \geq 1}$ such that $g_{n_j} \rightarrow \varphi$ almost everywhere as $j \rightarrow \infty$.
- Note that limit of continuous functions may not be continuous, but the limit of uniform continuous functions is continuous, so we can find the continuous function g after applying [Theorem 2.73](#) to $\{g_{n_j}\}_{j \geq 1}$. \square

Remark 2.83 (Littlewood’s Three Principles on \mathbb{R}).

- Every (finite) measurable set in \mathbb{R} is nearly a finite union of intervals.
- Every measurable (complex-valued) function on \mathbb{R} is nearly continuous, c.f., [Theorem 2.82](#).
- Every convergent sequence of measurable functions on a finite measure set is nearly uniformly convergent, c.f., [Theorem 2.73](#).