MATH 540 Notes

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1 Abstract Measure Theory

Definition 1.1. Let X be an (non-empty) underlying space we are working over. We denote $\mathcal{P}(X)$ to be the power set of X, i.e., the set of all subsets of X.

Example 1.2. Let $X = \{1, 2\}$, then $\mathcal{P}(X) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$.

Remark 1.3. If X is a finite set of size n, then $\mathcal{P}(X)$ is a finite set of size 2^n.

We will consider a subcollection $\mathcal{A}$ of subsets of X, i.e., a subset of the power set. We will try to define this as an algebra. Note that an algebra is just a ring with a module structure with respect to some other ring.

Definition 1.4. $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra on X if it is

a. closed under finite union, i.e., given $E_1, E_2 \in \mathcal{A}$, then $E_1 \cup E_2 \in \mathcal{A}$, and

b. closed under complements, i.e., if $E \in \mathcal{A}$, then the complement $E^c \in \mathcal{A}$ as well.

Remark 1.5. An algebra $\mathcal{A}$ would be closed under finite intersection. Indeed, for any $E_1, E_2 \in \mathcal{A}$, we have $E_1 \cap E_2 \in \mathcal{A}$ if and only if $(E_1 \cap E_2)^c \in \mathcal{A}$, if and only if $E_1^c \cup E_2^c \in \mathcal{A}$, which is true by definition.

Lemma 1.6. If $\mathcal{A}$ is an non-empty algebra on X, then $\varnothing \in \mathcal{A}$ and $X \in \mathcal{A}$.

Proof. Since $\mathcal{A}$ is non-empty, take $E \in \mathcal{A}$, then $\varnothing = E \cap E^c \in \mathcal{A}$ as well. Also, $X = E \cup E^c \in \mathcal{A}$. □

Example 1.7. Let X be a set, and let $\mathcal{A} = \{\varnothing, X\} \subseteq \mathcal{P}(X)$. It is easy to verify that $\mathcal{A}$ is an algebra.

Definition 1.8. Let $\varnothing \neq \mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra, then we say $\mathcal{A}$ is a σ-algebra on X if

a. closed under countable union, i.e., if $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, then $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$;

b. if $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$.

Lemma 1.9. If $\mathcal{A} \neq \varnothing$ is a σ-algebra on X, then $\{\varnothing, X\} \subseteq \mathcal{A}$ is a σ-algebra.

Example 1.10. Let X be an uncountable set, let $\mathcal{A} = \{E \subseteq X : E \text{ is countable or } E^c \text{ is countable}\}$, then $\mathcal{A}$ is a σ-algebra on X.

Theorem 1.11. Suppose a non-empty algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ such that,

- if $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, and E_j's are pairwise disjoint, then $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$,

then $\mathcal{A}$ is a σ-algebra on X.

Proof. Take $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, we will show that $\bigcup\limits_{j=1}^{\infty} E_j \in \mathcal{A}$. To do this, we will rearrange the sets. Let $F_1 = E_1$, let $F_2 = E_2 \backslash E_1$, let $F_3 = E_3 \backslash (E_1 \cup E_2)$, and so on, such that let $F_k = E_k \backslash \bigcup\limits_{i=1}^{k-1} E_i$. We note

$$\begin{aligned} F_k &= E_k \cap \left(\bigcup_{j=1}^{k-1} E_j \right)^c \\ &= E_k \cap \left(\bigcap_{j=1}^{k-1} E_j^c \right) \in \mathcal{A}. \end{aligned}$$

One can also verify that $\bigcup\limits_{j=1}^{\infty} E_j = \bigcup\limits_{k=1}^{\infty} F_k$, and that F_k's are disjoint from the definition. $\square$