MATH 526 Notes

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Let X be a topological space with basepoint $x_0 \in X$. We already know two invariants,

- the fundamental group $\pi_1(X, x_0)$, and
- the homology groups $H_n(X)$ for $n \geqslant 0$, which are abelian groups.

We will look at two more invariants,

- the cohomology groups $H^n(X)$ for $n \geqslant 0$, and
- the higher homotopy groups $\pi_n(X, x_0)$ for $n \geqslant 0$.

In particular, $\pi_*(X, x_0)$ is a very good invariant in the following sense:

Theorem 1.1 (Whitehead)**.** If $f : (X, x_0) \to (Y, y_0)$ is a map of CW-complexes, then f is a homotopy equivalence if and only if $\pi_*(f) : \pi_*(X, x_0) \to \pi_*(Y, y_0)$ is an isomorphism.

However, π_* is very hard to compute. On the other hand, $H^*(X)$ is relatively easy to compute, but this is not a complete invariant. For instance, $\mathbb{C}P^2$ and $S^2 \vee S^4$ have isomorphic cohomology groups, but they are not equivalent. $H^*(X)$ is closely related to $H_*(X)$, but $H^*(X)$ is a graded ring structure with cup product. It is contravariant in X, where $H_*(X)$ is covariant. The cup product is defined by the composition of induced diagonal map with an external product:

$$H^i(X) \times H^j(X) \xrightarrow{\times} H^{i+j}(X \times X) \xrightarrow{\Delta^*} H^{i+j}(X)$$

Other things we will talk about include:

- Natural transformations $H^i(-) \to H^j(-)$ encoded by Steenrod operations.
- $H^n(-)$ becomes a representable functor, i.e., $H^n(X) = [X, K(\mathbb{Z}, n)]$, where $K(\mathbb{Z}, n)$ is the Eilenberg-Maclane space, and the bracket indicates the homotopy classes of maps.
- Poincaré duality in $H^*(M)$ for compact manifold M, namely the cup product gives

$$H^i(M) \otimes H^{\dim(M)-i}(M) \xrightarrow{\smile} H^{\dim(M)}(M).$$

- Characteristic classes in $H^*(X)$ associated to vector bundles over X.

Recall for a topological space X, we obtain a collection of (singular) homology groups $H_n(X)$, with $H_*(X) = \bigoplus_{n\geqslant 0} H_n(X)$. The functoriality of morphisms says that $X \xrightarrow{f} Y \xrightarrow{g} Z$ induces $f_* g_* = (fg)_* : H_*(X) \xrightarrow{f_*} H_*(Y) \xrightarrow{g_*} H_*(Z)$. So

$$H_*(-) : \textbf{Top} \to \textbf{Ab}$$

is a well-defined functor. This factors into

$$\begin{tikzcd} \mathbf{Top} \arrow[rr, "H_*(-)"] \arrow[dr, "C_*(-)"'] & & \mathbf{Ab} \\ & \mathbf{Ch} \arrow[ur, "H_*(-)"'] & \end{tikzcd}$$

Here $C_*(-)$ is usually the singular chain, given by $\partial : C_n(X) \to C_{n-1}(X)$, where $C_n(X)$ is the free abelian group generated by $\mathrm{Hom}_{\mathbf{Top}}(\Delta^n, X) \cong \bigoplus_{\sigma:\Delta^n\to X} \mathbb{Z}\sigma$. $\Delta^n \subseteq \mathbb{R}^{n+1}$ is the set of tuples $(t_0, \dots, t_n)$ such that the coordinates sum to 1. The boundary is $\partial\sigma = \sum_{0\leqslant i\leqslant n} (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$.

We say $C_*(-)$ is homotopy invariant, i.e., if $f : X \to Y$ is a homotopy equivalence, then the induced map $C_*(X) \to C_*(Y)$ on chain complexes is a chain equivalence.

Remark 1.2. $C_*^{\Delta}(X)$ and $C_*^{\mathrm{CW}}(X)$ are both chain equivalent to $C_*(X)$.

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Here is a list of properties of $C_*(-) : \mathbf{Top} \to \mathbf{Ch}$:

- Functoriality: given a continuous map $f : X \to Y$, there is an induced map

$$\begin{aligned} f_* : C_*(X) &\to C_*(Y) \\ (\sigma : \Delta^n \to X) &\mapsto (f\sigma : \Delta^n \to Y) \end{aligned}$$

- Homotopy invariance: given $f, g : X \to Y$ such that $f \simeq g$, i.e., there is $H : X \times [0, 1] \to Y$ such that $H|_0 = f$ and $H|_1 = g$, then $f_* \simeq g_*$ as a chain homotopy equivalence, i.e., there exists maps $h_n : C_n(X) \to C_{n+1}(Y)$ making a diagram

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(X) \arrow[r] \arrow[d, shift right, "g"'] \arrow[d, shift left, "f"] \arrow[dl, "h"] & C_n(X) \arrow[r] \arrow[d, shift right, "g"'] \arrow[d, shift left, "f"] \arrow[dl, "h"] & C_{n-1}(X) \arrow[r] \arrow[d, shift right, "g"'] \arrow[d, shift left, "f"] \arrow[dl, "h"] & \cdots \\ \cdots \arrow[r] & C_{n+1}(Y) \arrow[r] & C_n(Y) \arrow[r] & C_{n-1}(Y) \arrow[r] & \cdots \end{tikzcd}$$

  such that $f - g = \partial h + h\partial$. Therefore $f_* = g_* : H_*(X) \to H_*(Y)$.

  Remark 2.1. $f : A_* \to B_*$ is a chain equivalence if there exists $g : B_* \to A_*$ and $fg \simeq \mathrm{id}_B$ and $gf \simeq \mathrm{id}_A$, then $f_* : H_*(A_*) \to H_*(B_*)$ is an isomorphism, i.e., f is a quasi-isomorphism.

  Example 2.2. The complexes $A : 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to 0$ and $B : 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0$ gives a quasi-isomorphism $f : A \to B$ in the canonical way, but this is not a chain equivalence, since the backwards map has to be zero.

- Additivity: $C_*(\coprod_\alpha X_\alpha) \cong \bigoplus_\alpha C_*(X_\alpha)$.
- Excision: given a pair (X, A) with $Z \subseteq A$ such that $\bar{Z} \subseteq \mathrm{int}(A)$, then we have $C_*(X\backslash Z, A\backslash Z) \cong C_*(X, A)$.
- Mayer-Vietoris: given $A, B \subseteq X$, with $X = \mathrm{int}(A) \cup \mathrm{int}(B)$, then we have a short exact sequence

$$0 \longrightarrow C_*(A \cap B) \longrightarrow C_*(A) \oplus C_*(B) * C_*(X) \longrightarrow 0$$

The cochain complex is obtained via inverting the indices and maps δ from a chain complex. This induces a cohomology $H^*(C^*) = \ker(\delta)/\mathrm{im}(\delta)$ as the quotient of cocycles over coboundaries. Now $f : A^* \to B^*$ is a quasi-isomorphism if $f^* : H^*(A^*) \to H^*(B^*)$ is an isomorphism. Similarly, one can define the cochain homotopy equivalence.

Example 2.3. If $C_* \in \mathbf{Ch}$, and $k \in \mathbf{Ab}$, then we can form cochain complex $C_k^* := \mathrm{Hom}(C_*, k)$, where $C_k^n = \mathrm{Hom}_{\mathbf{Ab}}(C_n, k) \xrightarrow{\delta} C_k^{n+1}$ by sending $f : C_n \to k$ to $f\partial : C_{n+1} \to C_n \to k$.

- $\operatorname{Hom}(-, k) : \mathbf{Ch} \to \mathbf{coCh}$ is a functor.
- The functor preserves quasi-isomorphisms between chain complexes of free abelian groups.

Definition 2.4. For $k \in \mathbf{Ab}$, the singular cochains with coefficients in k is

$$\begin{tikzcd} C^*(-,k) : \mathbf{Top} \arrow[rr] \arrow[dr, "C_*(-)"'] & & \mathbf{coCh} \\ & \mathbf{Ch} \arrow[ur, "\operatorname{Hom}(-,k)"'] & \end{tikzcd}$$

The cohomology of X with coefficients in k is defined by $H^*(X; k) = H^*(C^*X, k)$. We have the convention $C^*(X) = C^*(X, \mathbb{Z})$.

Alternatively, we take the opposite categories $\mathbf{Top}^*$ and $\mathbf{Ch}^*$ so that the functors are viewed as covariant.

The corresponding map $\delta : C^n(X; k) \to C^{n+1}(X; k)$ is given by δf that maps $\sigma \in C_{n+1}(X)$ to $(-1)^{n+1} f(\partial\sigma)$. Although the cochains are in general the dual of chains, the cohomology is not going to be the dual of the homology.

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Recall:

$$\begin{tikzcd} \mathbf{Top}^{\mathrm{op}} \arrow[r, "C_*"] \arrow[rrr, bend left, "H^*(-,k)"] & \mathbf{Ch}^{\mathrm{op}} \arrow[r, "\operatorname{Hom}(-,k)"] & \mathbf{coCh} \arrow[r, "H^*"] & \mathbf{GrAb} \end{tikzcd}$$

Properties of $H^*(-, k) : \mathbf{Top} \to \mathbf{GrAb}$:

- Dimension:

  Claim 3.1. $H^i(\{*\}, k) = \begin{cases} 0, & i \neq 0 \\ k, & i = 0 \end{cases}$

  Proof. Note that each degree of cohomology is given the free abelian group generated by $\operatorname{Hom}(\Delta^n, \{*\})$, but the singleton set is the terminal object in the category of topological spaces, so there is always a unique generator, thus the chain complex is given by $\mathbb{Z}$'s on each degree $n \geqslant 0$.

  Now the generating map at degree n is $\sigma_n : \Delta^n \to \{*\}$, and see Homework 1 where we proved the homology. Now looking at $C^*(\{*\}, k)$, we have

  $$k \xrightarrow{0} k \xrightarrow{\cong} k \xrightarrow{0} k \longrightarrow \cdots$$

  and this gives the cohomology. $\square$

- Homotopy: if $f \simeq g : X \to Y$, then $f^* = g^* : H^*(Y, k) \to H^*(X, k)$.

  Proof. We have $f_* = g_* : C_*X \to C_*Y$, and then $\operatorname{Hom}(f_*, k) \cong \operatorname{Hom}(g_*, k)$, so $H^*(-)$ is invariant under cochain homotopies. $\square$

- Additivity: $H^*(\coprod\limits_\alpha X_\alpha, k) \cong \prod\limits_\alpha H^*(X_\alpha, k)$.

  Proof. We know that for chains there is $C_*(\coprod\limits_\alpha X_\alpha) = \bigoplus\limits_\alpha C_*(X_\alpha)$, so the cochain version says that $C^*(\coprod\limits_\alpha, X_\alpha, k) \cong \operatorname{Hom}(\bigoplus\limits_\alpha C_*(X_\alpha), k) \cong \prod\limits_\alpha \operatorname{Hom}(C_*(X_\alpha), k) \cong \prod\limits_\alpha C^*(X_\alpha)$ and $H^* : \mathbf{coCh} \to \mathbf{GrAb}$ commutes with the product. $\square$

- Exactness: for a pair (X, A), there is a natural long exact sequence

$$\cdots \longrightarrow H^n(X,A;k) \longrightarrow H^n(X;k) \longrightarrow H^n(A;k) \longrightarrow \cdots$$

Proof. We have a short exact sequence

$$0 \longrightarrow C_*A \longrightarrow C_*X \longrightarrow C_*(X,A) \longrightarrow 0$$

where $C_*A \to C_*X$ is an inclusion of summands. Therefore, the quotient $C_*(X,A)$ is also a chain complex of free abelian groups. Therefore, taking the cochains also gives a short exact sequence. We then obtain a short exact sequence of cochain complexes

$$0 \longrightarrow C^*(X,A;k) \longrightarrow C^*(X;k) \longrightarrow C^*(A;k) \longrightarrow 0$$

and can then apply cohomology functor. □

- Excision: given a pair (X, A) and Z such that $\bar{Z} \subseteq \mathrm{int}(A)$, we have $H^*(X, A; k) \cong H^*(X \backslash Z, A \backslash Z; k)$.
- Mayer-Vietoris: given $A, B \subseteq X$ such that $\mathrm{int}(A) \cup \mathrm{int}(B) = X$, then we have a natural long exact sequence

$$\cdots \longrightarrow H^n(X;k) \longrightarrow H^n(A;k) \oplus H^n(B;k) \longrightarrow H^n(A \cap B;k) \longrightarrow \cdots$$

Definition 3.2. A functor $E^* : \mathbf{Top}^{\mathrm{op}} \to \mathbf{GrAb}$ is called a generalized cohomology theory if it satisfies the four middle property (except the dimension property and Mayer-Vietoris).

Remark 3.3. If E^* also satisfies the dimension property, then E^* is naturally isomorphic to the cohomology $H^*(-;k)$. There are also other generalized cohomology theories like K-theory, cobordism, etc.

The Mayer-Vietoris becomes a consequence of the first five properties.

We will now try to use homological algebra to relate $H_*(X) = H_*(CX)$ and $H^*(X;k) = H^*(\mathrm{Hom}(C_*X, k))$.

Definition 3.4. We say $C_*(X;k) \cong C_*(X) \otimes_{\mathbb{Z}} k$ and $H_*(X;k) \cong H_*(C_*X \otimes k)$ gives the singular homology of X with coefficients in k.

Lemma 3.5. $- \otimes k : \mathbf{Ab} \to \mathbf{Ab}$ is a right exact functor. $\mathrm{Hom}(-,k) : \mathbf{Ab}^{\mathrm{op}} \to \mathbf{Ab}$ is left exact.

Proof. Exercise. □

Remark 3.6. The covariant hom functor is also left exact.

Remark 3.7. The left adjoint is right exact, the right adjoint is left exact. In particular, we have the hom-tensor adjunction

$$\mathrm{Hom}(A, \mathrm{Hom}(B,C)) \cong \mathrm{Hom}(A \otimes B, C).$$

Note that

$$\mathrm{Hom}(A, \mathrm{Hom}(B,C)) \cong \mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(B \otimes A, C) \cong \mathrm{Hom}(B, \mathrm{Hom}(A,C))$$

Example 3.8. Consider

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

Tensoring with $\mathbb{Z}/n\mathbb{Z}$, we do not have exactness.

Example 3.9.

$$0 \longrightarrow A \longrightarrow A \oplus C \longrightarrow C \longrightarrow 0$$

is always exact after tensoring $- \otimes k$ or applying the hom functor $\mathrm{Hom}(-,k)$.

Definition 3.10. A short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ is split if any of the following equivalence conditions hold:

(i) p has a section $s : C \to B$ such that $ps = 1$;

(ii) i has a retraction $r : B \to A$ such that $ri = 1$;

(iii) $B \cong A \oplus C$, i.e.,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & B \arrow[r, "p"] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, hook, "i"] & A \oplus C \arrow[r, two heads, "p"] & C \arrow[r] & 0
\end{tikzcd}$$

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We will prove that (ii) implies (iii).

Suppose $b \in B$, then $b = (b - irb) + irb$, which is a decomposition of elements in $\ker(r)$ and in $\operatorname{im}(i)$, respectively. Also, $\ker(r) \cap \operatorname{im}(i) = 0$, therefore $B = \ker(r) \cap \operatorname{im}(i)$. Since i is an inclusion, then $\operatorname{im}(i) \cong A$. Now $p : B \to C$ factors through the projection onto $\ker(r)$ since $ri = 0$. By restricting p onto $\ker(r)$, we see p is also injective, thereby an isomorphism.

Lemma 4.1. If we have a split exact sequence

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, bend left, "i"] & B \arrow[l, bend left, "r"] \arrow[r, bend left, "p"] & C \arrow[l, bend left, "s"] \arrow[r] & 0
\end{tikzcd}$$

then $- \otimes k$ and $\operatorname{Hom}(-, k)$ preserves the split exactness, i.e.,

$$0 \longrightarrow A \otimes k \longrightarrow B \otimes k \longrightarrow C \otimes k \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Hom}(C, k) \longrightarrow \operatorname{Hom}(B, k) \longrightarrow \operatorname{Hom}(A, k) \longrightarrow 0$$

The point is tensors and homs preserve retracts.

Proof.
- $(r \otimes \mathrm{id}_k)(i \otimes \mathrm{id}_k) = ri \otimes \mathrm{id}_k = \mathrm{id}_{A \otimes k}$, so $i \otimes \mathrm{id}_k$ is split injective.
- Similarly, $\operatorname{Hom}(i, \mathrm{id})$ is split surjective.

□

Example 4.2. Given a surjection $B \to C \to 0$ such that C is free abelian, then there is always a section $s : C \to B$ making the exact sequence split. (That is, C is projective.) That is, if $0 \to A \to B \to C \to 0$ is an exact sequence where C is free, then the sequence is split exact.

Definition 4.3. Let $C \in \mathbf{Ab}$. A free resolution of C is a chain complex of free objects

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

and an augmentation $F_0 \to C$, so that

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0$$

is acyclic, i.e., exact everywhere.

Example 4.4.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow 0$$

is a free resolution of $\mathbb{Z}/n\mathbb{Z}$. So is

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

as well as

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\mathrm{id} \oplus (\times n)} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0$$

Lemma 4.5. Any $C \in \mathbf{Ab}$ admits a free resolution, and moreover, it admits a resolution of length 1¡ given by

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0$$

Proof. Choose a surjection $p : F_0 \to C$ from a free abelian group F_0 to C. Let $F_1 = \ker(p)$, then F_1 is free, so we are done. $\square$

Lemma 4.6. Free resolutions are essentially unique, i.e., if $F_. \to C$ and $F'_. \to C$ are free resolutions, then there is a quasi-isomorphism $F_. \xrightarrow{\sim} F'_.$ which commutes with the augmentations to C.

Definition 4.7. Let $C \in \mathbf{Ab}$ and let $F_. \to C$ be a free resolution, then we define the torsion groups to be $\mathrm{Tor}_n^{\mathbb{Z}}(C,k) = H_n(F_. \otimes k)$, and the ext groups to be $\mathrm{Ext}^n_{\mathbb{Z}}(C,k) = H^n(\mathrm{Hom}_{\mathbb{Z}}(F_.,k))$.

Remark 4.8.
- Tor and Ext are independent of the choice of resolutions.
- $\mathrm{Tor}_n^{\mathbb{Z}}$ and $\mathrm{Ext}^n_{\mathbb{Z}}$ are zero for $n > 1$.
- $\mathrm{Tor}_n^{\mathbb{Z}}(C,k) \cong \mathrm{Tor}_n^{\mathbb{Z}}(k,C)$.
- $\mathrm{Tor}_0^{\mathbb{Z}}(C,k) \cong C \otimes k$.
- $\mathrm{Ext}^0_{\mathbb{Z}}(C,k) \cong \mathrm{Hom}(C,k)$.

Example 4.9.
- If C is free, then $\mathrm{Tor}_1(C,k) = \mathrm{Ext}^1(C,k) = 0$.
- $\mathrm{Tor}_1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.
- $\mathrm{Tor}_1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = 0$.
- $\mathrm{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.
- $\mathrm{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.
- $\mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = 0$.

Proof. Look at

$$0 \longrightarrow F_1 = \mathbb{Z} \longrightarrow F_0 = \mathbb{Z} \longrightarrow C = \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

then $\mathrm{Tor}_*(\mathbb{Z}/p\mathbb{Z}, k) = H_*(F_1 \otimes k = k \xrightarrow{\times p} F_0 \otimes k = k) = \begin{cases} k[p], * = 0 \\ k/pk, * = 1 \end{cases}$. Here $k[p]$ denotes p-torsion subgroup of k. Moreover, $\mathrm{Ext}^*(\mathbb{Z}/p\mathbb{Z}, k) = H^*(\mathrm{Hom}(F_1,k) = k \xleftarrow{\times p} \mathrm{Hom}(F_0,k) = k) = \begin{cases} k[p], * = 0 \\ k/pk, * = 1 \end{cases}$. $\square$

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Recall that cohomology are basically the dual of homology, where the difference originates from the failure of exactness of the hom functor.

Theorem 5.1 (Universal Coefficient Theorem)**.** Let C_* be a chain of free abelian groups and $k \in \mathbf{Ab}$, then there exists a natural short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(H_{n-1}(C_*),k) \longrightarrow H^n(\mathrm{Hom}(C_*,k)) \xrightarrow{h} \mathrm{Hom}(H_n(C_*),k) \longrightarrow 0$$

that splits in an unnatural sense.

Here we define $h \in \mathrm{Hom}(H^n(\mathrm{Hom}(C_*,k)), \mathrm{Hom}(H_n(C_*),k))$. Note that this hom set is isomorphic to the hom set $\mathrm{Hom}(H^n(\mathrm{Hom}(C_*,k)) \otimes H_n(C_*),k)$ via the tensor-hom adjunction. That is, h is given by a bilinear pairing $H^n(\mathrm{Hom}(C_*,k)) \times H_n(C_*) \to k$. We use the Kronecker pairing $([f],[x]) \mapsto f(x)$. To see this is well-defined, let $f \in \mathrm{Hom}(C_n,k)$ with $\delta f = 0$, for $x \in C_n$, we have $\partial x = 0$. Now replace x by $x + \partial y$, then $f(x+\partial y) = f(x) = f(\partial y) = f(x) \pm (\delta f)(y) = f(x)$. Also, replace f by $f + \delta(g)$ gives $(f+\partial g)(x) = f(x) + (\delta g)(x) = f(x) + g(\delta x) = f(x)$.

Lemma 5.2. h is a split surjection.

Proof. Write $C_k^* = \mathrm{Hom}(C_*, k)$. Now $h : \ker(\delta, C_k^n \to C_k^{n+1}) \to \mathrm{Hom}(H_n(C_*), k)$ via $h : f \mapsto (x \mapsto f(x))$, then we will construct a section of h via $\varphi \mapsto \tilde{\varphi}$. Let $Z_n = \ker(\partial)$ and $B_n = \mathrm{im}(\partial)$, then $H_n(C_*) = Z_n/B_n$, and the short exact sequence of free abelian groups

$$0 \longrightarrow Z_n \xrightarrow{i} C_n \longrightarrow B_{n-1} \longrightarrow 0$$

and this splits so $C_n \cong Z_n \oplus B_{n-1}$. Given $\varphi : H_n(C_*) \to k$, we have

$$C_n \xrightarrow{r} Z_n \longrightarrow Z_n/B_n \xrightarrow{\varphi} k$$

where r is the retraction to i, and we define the composition to be $\tilde{\varphi}$. Now the composition

$$C_{n+1} \xrightarrow{\partial} C_n \longrightarrow Z_n \longrightarrow Z_n/B_n \longrightarrow k$$

is still zero since $C_{n+1} \to Z_n$ is zero, but that means $\delta\tilde{\varphi}$ is also zero. $\square$

We will now prove the universal coefficient theorem.

Proof. Since h is a split surjection, then we know this extends to a short exact sequence, hence we just need to identify the kernel of h, i.e., to show that $\ker(h) \cong \mathrm{Ext}^1(H_{n-1}(C_*), k)$. Given the split short exact sequence

$$0 \longrightarrow Z_n \xrightarrow{i} C_n \xrightarrow{\partial} B_{n-1} \longrightarrow 0$$

we have a diagram

$$\begin{tikzcd}
 & \vdots \arrow[d, "0"] & \vdots \arrow[d, "\delta"] & \vdots \arrow[d, "0"] & \\
0 \arrow[r] & \mathrm{Hom}(B_{n-1}, k) \arrow[r] \arrow[d, "0"] & \mathrm{Hom}(C_n, k) \arrow[r] \arrow[d, "\delta"] & \mathrm{Hom}(Z_n, k) \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \mathrm{Hom}(B_n, k) \arrow[r] \arrow[d, "0"] & \mathrm{Hom}(C_{n+1}, k) \arrow[r] \arrow[d, "\delta"] & \mathrm{Hom}(Z_{n+1}, k) \arrow[r] \arrow[d, "0"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

which is a short exact sequence of complexes. By the snake lemma, we have the long exact sequence of cohomology$\cdots \to H^n(B_k^{*-1}) \to H^n(C_k^*) \to H^n(Z_k^*) \to H^{n+1}(B_k^{*-1}) \to \cdots$. We claim that the connecting homomorphism $H^n(Z_k^*) \to H^{n+1}(B_k^{*-1})$ is $\mathrm{Hom}(B_n \subseteq Z_n, k)$. But $0 \to B^n \to Z^n \to H_n(C_*) \to 0$ is a free resolution of $H_n(C_*)$ of length 1. Then $H^*(\beta : \mathrm{Hom}(Z_n, k) \to \mathrm{Hom}(B_n, k)) = \mathrm{Ext}^*(H_n(C_*), k)$ where β has kernel $\mathrm{Hom}(H_n(C_*), k)$ and cokernel $\mathrm{Ext}^1(H_n(C_*), k)$. Therefore, the long exact sequence of cohomomology is the splicing (as epi-mono factorization) of

$$0 \longrightarrow \mathrm{coker}(\beta_{n-1}) \longrightarrow H_n(C_k^*) \longrightarrow \ker(\beta_n) \longrightarrow 0$$

and by identification we are done. $\square$

Corollary 5.3. If $C_* \to C'_*$ is a quasi-isomorphism, then $\mathrm{Hom}(C'_*, k) \to \mathrm{Hom}(C_*, k)$ is a quasi-isomorphism.

Corollary 5.4. Let $X \in$ **Top** and $A \subseteq X$, then there exists a short exact sequence

$$0 \longrightarrow \mathbf{Ext}^1(H_{n-1}(X, A), k) \longrightarrow H^n(X, A; k) \longrightarrow \mathbf{Hom}(H_n(X, A); k) \longrightarrow 0$$

which is natural in (X, A). This also splits in (X, A) in an unnatural way.

Theorem 5.5. If C_* is a chain complex of free abelian groups, then there is a short exact sequence

$$0 \longrightarrow H_n(C_*) \otimes k \longrightarrow H_n(C_* \otimes k) \longrightarrow \mathbf{Tor}_1(H_{n-1}(C_*, k) \longrightarrow 0$$

which is natural. It splits unnaturally.

Corollary 5.6. For any pair (X, A), there is a natural short exact sequence

$$0 \longrightarrow H_n(X, A) \otimes k \longrightarrow H_n(X, A; k) \longrightarrow \mathbf{Tor}_1(H_{n-1}(X, A), k) \longrightarrow 0$$

which splits in an unnatural way.

6 Sept 1, 2023

Example 6.1. Take $X = \mathbb{C}P^2$, then the Tor and Ext terms go away, so the cohomology is equivalent to the homology.

Example 6.2. Take $X = \mathbb{R}P^2$, the Tor term gives $\mathrm{Tor}_1(\mathbb{Z}/2\mathbb{Z}, k) = k/2 \cong k[2]$, as the 2-torsion of k, i.e., the set of $a \in k$ such that $2a = 0$. Also, $\mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z}, k) = k/2k$.

Indeed, the Tor is given by the homology on multiplication by 2 map over k via tensor, and the Ext is given by the cohomology on multiplication by 2 map over k via hom.

Tor stands for torsion and Ext stands for extension.

Went on to talk about the limits and colimits.

Remark 6.3. In many abelian categories (and in particular, the category of abelian groups), we find a short exact sequence

$$0 \longleftarrow \mathrm{colim}_I \longleftarrow \bigoplus_{i \geqslant 0} X_i \longleftarrow \bigoplus_{i \geqslant 0} X_i \longleftarrow 0$$

and note that taking the dual version in the opposite category, we should obtain a sequence in the covariant sense. However, there is an asymmetry given by

$$0 \longrightarrow \lim_{I^{\mathrm{op}}} X \longrightarrow \prod_{i \geqslant 0} X_i \longrightarrow \prod_{i \geqslant 0} X_i \longrightarrow \lim_{I^{\mathrm{op}}}^1 X \longrightarrow 0$$

which is not short anymore. This is called a Milnor sequence.

7 Sept 6, 2023

The colimit of the empty diagram is the initial object; dually, the limit of the empty diagram is the terminal object.

Definition 7.1. We say $X : I \to \mathscr{C}$ is a filtered diagram if

- $\mathrm{Ob}(\mathscr{C}) \neq \varnothing$,
- for all $i, j \in I$, there exists $k \in I$ and morphisms $i \to k$ and $j \to k$, and
- for parallel morphisms $a, b : i \to j$ in I, then there exists coequalizers.

Example 7.2. A poset (as a category) P is a directed set if for any $i, j \in P$, there exists $k \in P$ such that $i \leqslant k$ and $j \leqslant k$.

For a filtered diagram $X : I \to \mathbf{Set}$, the colimit $\mathrm{colim}_I X$ exists and is isomorphic to $\coprod_{i \in I} X_i / \sim$, where $x_i \in X_i$ and $x_j \in X_j$ are equivalent if for some $k \in I$, we have $a : i \to k$ and $b : j \to k$ and that $a(x_i) = b(x_j)$

For concrete categories, we forget the additional structure to the category of sets, and find the colimits there, and give it the additional structure we want.

Lemma 7.3. If I is a directed set, then

$$0 \longrightarrow \bigoplus_{i\in I} A_i \longrightarrow \bigoplus_{i\in I} A_i \longrightarrow \operatorname{colim}_{i\in I} A_i \longrightarrow 0$$

$$(a_i)_{i\in I} \longrightarrow (a_j - f_{ij}(a_i))$$

where $f_{ij} : i \to j$.

Example 7.4. The colimit of a sequence given by $A \xrightarrow{\times n} A$ is $A\left[\frac{1}{n}\right]$.

Lemma 7.5. Colimit functor is exact in category of abelian groups.

8 Sept 8, 2023

- For a sequential diagram
$$\cdots \longrightarrow A_2 \longrightarrow A_1 \longrightarrow A_0$$
the limit of A_i's is the terminal cone, and in fact is the kernel of
$$\prod_{i\geqslant 0} A_i \to \prod_{i\geqslant 0} A_i$$
$$(a_i) \mapsto (a_i - f_{i+1}(a_{i+1}))_i$$
However, this sequence is not exact, as we discussed before.

Lemma 8.1. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_i & \longrightarrow & B_i & \longrightarrow & C_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_{i-1} & \longrightarrow & B_{i-1} & \longrightarrow & C_{i-1} & \longrightarrow & 0 \end{array}$$

then we have a long exact sequence

$$0 \longrightarrow \lim A_i \longrightarrow \lim B_i \longrightarrow \lim C_i \longrightarrow \lim^1 A_i \longrightarrow \lim^1 B_i \longrightarrow \lim^1(C_1) \longrightarrow 0$$

Proof. Take the products to get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \prod_i A_i & \longrightarrow & \prod_i B_i & \longrightarrow & \prod_i C_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \prod_i A_{i-1} & \longrightarrow & \prod_i B_{i-1} & \longrightarrow & \prod_i C_{i-1} & \longrightarrow & 0 \end{array}$$

and now use the snake lemma. □

Example 8.2. The p-adic integers $\mathbb{Z}_p = \lim(\cdots \twoheadrightarrow \mathbb{Z}/p^k \twoheadrightarrow \mathbb{Z}/p^{k+1} \twoheadrightarrow \cdots)$ is a limit.

Theorem 8.3 (Mittag-Leffler Condition)**.** If $\{A_{i+1} \to A_i\}$ satisfies for each k, there is $i \geqslant k$ such that $\operatorname{im}(A_i \to A_k) \to \operatorname{im}(A_j \to A_k)$ for all $j \geqslant i \leqslant k$, then $\lim^1(A_i) = 0$.

Example 8.4.

1. This is true if all maps are surjections.
2. This is also true if all A_i's are finite.

Definition 8.5. Recall that a mapping cylinder is $M_f = (X \times s[0,1] \coprod Y)/((x,1) \sim f(x))$, so there is an inclusion $X \hookrightarrow M_f \cong Y$. Now given a sequence with $f_i : X_i \to X_{i+1}$, then the mapping telescope is

$$T = \mathrm{Tel}(X_*) = (\coprod_{n \geqslant 0} X_n \times [0,1])/((n,x,1) \sim (n+1, f_n(x), 0)),$$

with

$$\begin{aligned} i_n : X_n &\to T \\ x &\mapsto (n, x_n, 0) \end{aligned}$$

and homotopies $(i_n \circ f_{n-1}) \cong i_{n-1} : X_{n-1} \to T$. Therefore, the diagrams

$$\begin{tikzcd} H_*(X_{n-1}) \arrow[dr, "(i_{n-1})_*"] \arrow[dd, "(f_{n-1})_*"'] & \\ & H_*(T) \\ H_*(X_n) \arrow[ur, "(i_n)_*"'] & \end{tikzcd}$$

commute. This induces a map $\mathrm{colim}_n(H_*(X_n)) \to H_*(T)$. We claim that this is an isomorphism.

Proof. Indeed, consider the refinement

$$\begin{aligned} \lambda : T = \coprod_n X_n \times [0,1]/\sim &\to \mathbb{R}_{\geqslant 0} \\ (n,x,t) &\mapsto n+t \end{aligned}$$

Let $T_{\leqslant a} = \lambda^{-1}([0,a])$ or $T_{<a} = \lambda^{-1}([0,a])$. We observe that $T_{\leqslant n}$ has a homotopy equivalence via $X_n \hookrightarrow T_{\leqslant n}$ with a deformation retraction. But $T_{\leqslant n}$ is also homotopy equivalent to $T_{<n+1}$. The upshot is that it suffices to show that $\mathrm{colim}(H_*(T_{<n})) \to H_*(T)$ is an isomorphism. $\square$

Proposition 8.6. Let Y be a space and let $\mathcal{A}$ be a collection of subspaces forming a direct system under inclusion. Assume that $Y = \bigcup_{A \in \mathcal{A}} A$, and for any compact $K \subseteq Y$, $K \subseteq A$ for some $A \in \mathcal{A}$. Then the map $\mathrm{colim}_{A \in \mathcal{A}} C_*(A \to C_*(Y)$ is an isomorphism, hence induces an isomorphism on the level of homology: $\mathrm{colim}(H_*(A)) \cong H_*(Y)$.

9 Sept 11, 2023

Recall that $H_*(\mathrm{Tel}(X_n)) \cong \mathrm{colim}_n H_*(X_n)$, with the proof replying on $C_*(\mathrm{Tel}(X_n)) \cong \mathrm{colim}_n C_*(X_n)$.

Example 9.1. $\mathrm{Tel}(S^1 \xrightarrow{p} S^1 \xrightarrow{p} \cdots) = T = S^1\left[\frac{1}{p}\right]$. Correspondingly, we have $\mathrm{colim}(H_0(S^1) \cong \mathbb{Z} \xrightarrow{p_*} H_0(S^1) \cong \mathbb{Z} \xrightarrow{p_*} \cdots) = \mathbb{Z}$, where the induced maps are just identities. Also, $\mathrm{colim}(H_1(S^1) \cong \mathbb{Z} \xrightarrow{p_*} H_1(S^1) \cong \mathbb{Z} \xrightarrow{p_*} \cdots) = \mathbb{Z}\left[\frac{1}{p}\right] \cong H_1(T)$, where the induced maps are multiplications by p.

By the Universal Coefficient theorem, we can calculate the cohomology of T as follows:

$$0 \longrightarrow \mathrm{Ext}^1(H_n^1(S^1\left[\tfrac{1}{p}\right], \mathbb{Z}) \longrightarrow H^n(S^1\left[\tfrac{1}{p}\right]) \, \mathrm{Hom}(H_n(S^1\left[\tfrac{1}{p}\right]), \mathbb{Z}) \longrightarrow 0$$

Here

- $H^0 * (S^1\left[\frac{1}{p}\right]) = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$;
- $H^1(S^1\left[\frac{1}{p}\right]) \cong \mathrm{Hom}(\mathbb{Z}\left[\frac{1}{p}\right]) = 0$, since the Ext term is 0;

- Higher homologies are zero, so $H^2(S^1\left[\frac{1}{p}\right]) \cong \operatorname{Ext}(\mathbb{Z}\left[\frac{1}{p}\right], \mathbb{Z}) \cong \mathbb{Z}_p/\mathbb{Z}$, the p-adic integers over $\mathbb{Z}$.

We are interested in calculating $H^*(\mathrm{Tel})$ in terms of $H^*(X_i)$'s. Note that the chain complex $C_*(\mathrm{Tel}(X_i)) \cong \operatorname{colim}_i(C_* X_i)$, so

$$\begin{aligned} C^*(\mathrm{Tel}(X_i)) &= \operatorname{Hom}(\operatorname{colim}_i(C_* X_i), \mathbb{Z}) \\ &= \lim_i(C^*(X_i)). \end{aligned}$$

Therefore, the question becomes, what is $H^*(\lim_i(C_i^*)$?

Theorem 9.2 (Milnor Exact Sequence)**.** Suppose $\{C_i^*\}$ is an inverse system of cochain complexes, such that for each n, $\{C_i^n\}$ is an inverse system that satisfies Mittag-Leffler condition, i.e., we need $\lim^1 = 0$, then we have a short exact sequence

$$0 \longrightarrow \lim_i^1(H^{n-1}(C_i^*)) \longrightarrow H^n(\lim_i C_i^*) \longrightarrow \lim_i(H^n(C_n^*)) \longrightarrow 0$$

Proof. We set $B_i^n = \operatorname{im}(\delta : C_i^{n-1} \to C_i^n)$, and $Z_i^n = \ker(\delta : C_i^n \to C_i^{n+1})$. With this notation, we have a system of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & Z_i^n \arrow[r] \arrow[d] & C_i^n \arrow[r, "\delta"] \arrow[d] & B_i^{n+1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Z_{i-1}^n \arrow[r] & C_{i-1}^n \arrow[r, "\delta"'] & B_{i-1}^{n+1} \arrow[r] & 0 \end{tikzcd}$$

Therefore we have a long exact sequence

$$0 \longrightarrow \lim_i Z_i^n \longrightarrow \lim_i C_i^n \longrightarrow \lim_i B_i^{n+1} \longrightarrow \lim_i^1 Z_i^n \longrightarrow \lim_i^1 C_i^n \longrightarrow \lim_i^1 B_i^{n+1} \longrightarrow 0$$

By assumption, $\lim_i^1 C_i^n = 0$, so $\lim_i^1 B_i^{n+1} = 0$, and we have the sequence

$$0 \longrightarrow \lim_i Z_i^n \longrightarrow \lim_i C_i^n \longrightarrow \lim_i B_i^{n+1} \longrightarrow \lim_i^1 Z_i^n \longrightarrow 0$$

Denote $C^* = \lim_i C_i^*$, and $Z^n = \ker(C^n \xrightarrow{\delta} C^{n+1})$, and $B^n = \operatorname{im}(C^{n-1} \to C^n)$. This gives

$$\begin{tikzcd} 0 \arrow[r] & Z^n \arrow[r] & C^n \arrow[dr, two heads, "\delta"'] & & \lim_i B_i^{n+1} \arrow[r] & \lim_i^1 Z_i^n \arrow[r] & 0 \\ & & & B_{n+1} \arrow[ur, hook] & & & \end{tikzcd}$$

We know have $0 \subseteq B^{n+1} \subseteq \lim_i B_i^{n+1} \subseteq \lim_i Z_i^{n+1} = Z^{n+1}$, therefore this gives an exact sequence

$$0 \longrightarrow \lim_i B_i^{n+1}/B^{n+1} \longrightarrow Z^{n+1}/B^{n+1} \longrightarrow Z^{n+1}/\lim_i B_i^{n+1} \longrightarrow 0$$

so this is

$$0 \longrightarrow \lim_i^1 Z_i^n \longrightarrow H^{n+1}(C^*) \longrightarrow Z^{n+1}/\lim_i B_i^{n+1} \longrightarrow 0$$

From the canonical exact sequence

$$0 \longrightarrow B_i^n \longrightarrow Z_i^n \longrightarrow H^n(C_i^*) \longrightarrow 0$$

we induce

$$0 \longrightarrow \lim_i B_i^n \longrightarrow Z^n \longrightarrow \lim_i H^n(C_i^n) \longrightarrow \lim_i^1 B_i^n \longrightarrow \lim_I^1 Z_i^n \longrightarrow \lim_i^1 H^n(C_i^*) \longrightarrow 0$$

but we have $\lim_i^1 B_i^n = 0$, so $\lim_i^1 Z_i^n \cong \lim_i^1 H^n(C_i^*)$, therefore we identify $Z^{n+}/\lim_i B_i^{n+1} \cong \lim_i H^{n+1}(C_i^*)$. □

Corollary 9.3. Let $X \in$ **Top** and $X = \bigcup_i X_i$ such that if there is compact $K \subseteq X$, then there exists some i such that $K \subseteq X_i$. If this is the case, then we have a short exact sequence in cohomology given by

$$0 \longrightarrow \lim_i^1 H^{n-1}(X_i) \longrightarrow H^n(X) \longrightarrow \lim_i H^n(X_i) \longrightarrow 0$$

Proof. We have $C_*(X) \cong \operatorname{colim}(C_*(X_i))$, and $C^*(X) \cong \lim C^*(X_i)$.

Claim 9.4. $\lim_i^1(C^n(X_i)) = 0$ for all n.

Subproof. We want the open cover of X to be a direct system, i.e., nested in some sense, so that we have a telescope and by the Mittag-Leffler condition we win. For instance, if we have telescopes, then $T = \mathrm{Tel}(X_0 \to X_1 \to \cdots)$, then $\bigcup_n T_{\leqslant n}$ gives $T_{\leqslant 0} \subseteq T_{\leqslant 1} \subseteq \cdots \subseteq T = \bigcup T_{\leqslant i}$. The point being, now we have $T_{\leqslant i} \cong X_i$ by deformation retraction, so we have a Milnor exact sequence on the level of cohomology of T, and we are done. ■

□

Example 9.5.

$$0 \longrightarrow \lim{}^1 H^1(S^1) \xrightarrow{\cong} H^2(S^1\left[\tfrac{1}{p}\right]) \longrightarrow H^2(S^1) \longrightarrow 0$$

where $\lim^1 H^1(S^1)$ is $\lim^1(\cdots \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \cdots) \cong \mathbb{Z}_p/\mathbb{Z}$.

10 Sept 13, 2023

We now want to define a map on cohomology groups. Let R be a commutative ring, and let $\varphi_i \in C^{n_i}(X, R)$ be with $i = 1, 2$, then we can define the cup product on $\smile$ with

$$\begin{aligned} C^{n_1}(X,R) \times C^{n_2}(X,R) &\to C^{n_1+n_2}(X,R) \\ (\varphi_1 \smile \varphi_2)(\sigma) &= \varphi_1(\sigma|_{[v_0,\ldots,v_{n_1}]}\,\varphi_2(\sigma|_{[v_{n_1},\ldots,v_{n_2}]} \end{aligned}$$

and we extend it linearly. Note that if $n_1 = 0$, then the map sends σ to $\varphi_1(\sigma|_{v_0})\varphi_2(\sigma)$. Moreover, if $\varphi_1 = e$ is the constant mapping with image 1, then $e \smile \varphi = \varphi = \varphi \smile e$. By associativity, we know $C^*(X, R)$ is a graded ring.

Lemma 10.1. $\smile$ is functorial in X, that is, if $f : X \to Y$, then $f^* : C^*(Y, R) \to C^*(X, R)$ is a ring homomorphism.

Lemma 10.2. $\partial(\varphi_1 \smile \varphi_2) = \partial\varphi_1 \smile \varphi_2 + (-1)^{|\varphi_1|} \smile \partial\varphi_2$.

Corollary 10.3.

- If $\varphi_1, \varphi_2 \in Z^*$ are cocycles, then the cup product $\varphi_1 \smile \varphi_2 \in Z^*$.
- If $\varphi_i \in Z^*$, and one is in B^*, then $\varphi_1 \smile \varphi_2 \in B^*$.

Using these two facts, we know that $\smile$: $H^{n_1}(X, \mathbb{R}) \times H^{n_2}(X, \mathbb{R}) \to H^{n_1+n_2}(X, R)$ is an induced map. In particular, if X is connected, then $H^0(X, R) \cong R$, and the cup product becomes the product on R. This has a graded ring structure.

Theorem 10.4. The cohomology cup product satisfies:

1. naturality in X,

2. $1 \smile \alpha = \alpha = \alpha \smile 1$ for $\alpha \in H^*(X, R)$. This is given by $1 : C_0X \to R$ with $\sigma : \Delta^0 \to X$ sent to 1. Therefore, $1 = [1]$.
3. $\alpha \smile (\beta \smile \gamma) = (\alpha \smile \beta) \smile \gamma$.
4. $\alpha \smile \beta = (-1)^{|\alpha||\beta|}\beta \smile \alpha$.
5. For any pair (X, A) with $i : A \hookrightarrow X$ with $\delta : H^*(A; R) \to H^{*+1}(X, A; R)$, then for $\alpha \in H^*(A; R)$ and $\beta \in H^*(X; R)$, then $\delta(\alpha \smile i^*\beta) = \delta(\alpha) \smile \beta)$, and $\delta(i^*\beta \smile \alpha) = (-1)^{|\beta|}\beta \smile \delta(\alpha)$.

Remark 10.5. The cup product $\smile$ comes from $C^*(X) \otimes C^*(X) \to C^*(X)$, also regarded as $\mathrm{Hom}(C_*X, R) \otimes \mathrm{Hom}(C_*X, R) \to \mathrm{Hom}(C_*X, R)$, which is given by the factoring via $\mathrm{Hom}(C_*X \otimes C_*X, R)$. This gives a pairing on C^*X if we have a commutative diagram

$$\begin{tikzcd} C_*X \arrow[r] \arrow[d, "\sigma_n \mapsto 0"] & C_*X \otimes C_*X \arrow[d] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z} \end{tikzcd}$$

The map $C_*X \to C_*X \otimes C_*X$ is called the diagonal approximation. More generally, if we think of X and Y, then we have

$$\begin{tikzcd} C_*(X \times Y) \arrow[r] \arrow[d] & C_*X \otimes C_*Y \arrow[d] \\ \mathbb{Z} \arrow[r, equal] & \mathbb{Z} \end{tikzcd}$$

In particular, if $X = Y$, then we have a diagonal mapping $X \to X \times X$, therefore induces $C_*X \to C_*(X \times X)$.

Definition 10.6. The Alexander-Whitney map is given by

$$AW_{X,Y} : C_*(X \times Y) \to C_*X \otimes C_*Y$$

where $C_*X \otimes C_*Y$ is given by total complex of degree n, i.e., $\bigoplus\limits_{i+j=n} C_iX \otimes C_jY$, and differential $\partial(a \otimes b) = \partial a \otimes b + (-1)^{|a|}a \otimes \partial b$.

$$\begin{tikzcd} & X \\ \Delta^n \arrow[ur, "\sigma"] \arrow[r, "(\sigma,\tau)"] \arrow[dr, "\tau"'] & X \times Y \arrow[u, "\pi_X"'] \arrow[d, "\pi_Y"] \\ & Y \end{tikzcd}$$

The Alexander-Whitney map defines $AW(\sigma, \tau) = \sum\limits_{i+j=n} \sigma|_{[v_0,\ldots,v_i]} \otimes \tau|_{[v_i,\ldots,v_n]}$. On the level of cochains, the cup product is $\mathrm{Hom}(-, R)$ of composition of Alexander-Whitney map and the induced diagonal mapping.

Similarly, we can define the cochain version, with a pair (X, A), then

$$\begin{tikzcd} C_*(X \times Y, A \times Y) \arrow[r, dashed] & C_*(X, A) \otimes C_*Y \\ C_*(X \times Y) \arrow[u, two heads] \arrow[r, "AW_{X \times Y}"] & C_*X \otimes C_*Y \arrow[u, two heads] \\ C_*(A \times Y) \arrow[u, hook] \arrow[r, "AW_{A,Y}"'] & C_*A \otimes C_*Y \arrow[u, hook] \end{tikzcd}$$

We now want $(X, A) \times (Y, B) = (X \times Y, A \times Y \cup X \times B)$ to have the suitable mapping. Naturally, we get the Alexander-Whitney map

$$\begin{tikzcd} C_*(X \times Y)/(C_*(X \times B) + C_*(A \times Y)) \arrow[r] \arrow[d] & C_*(X, A) \otimes C_*(Y, B) \\ C_*(X \times Y)/C_*(A \times Y \cup X \times B) \arrow[ur, dashed] & \end{tikzcd}$$

The summation is not the direct sum but not summation in complex.

11 Sept 15, 2023

Recall that the Alexander-Whitney map is the natural transformation of functors **Top** $\times$ **Top** $\to$ **Ch** via $C_*(X \times Y) \to C_*(X) \to C_*(Y)$, where

$$AW(\sigma, \tau) = \sum_{i+j=n} \sigma|_{[v_0,\dots,v_i]} \otimes \tau|_{[v_i,\dots,v_n]}$$

for $\sigma, \tau : \Delta^n \to X \times Y$. We also note that the cross product is defined as the composition

$$\begin{tikzcd} H^*(\mathrm{Hom}(C_*X, R)) \otimes H^*(\mathrm{Hom}(C_*Y, R)) \arrow[r] \arrow[dr, dashed, "\times"'] & H^*(\mathrm{Hom}(C_*X \otimes C_*Y, R)) \arrow[d, "AW^*"] \\ & H^*(\mathrm{Hom}(C_*(X, Y), R)) \end{tikzcd}$$

where the horizontal map is induced by homological algebra. The cup product is composed by the diagonal inclusion and the cross product:

$$\begin{tikzcd} H^*(X) \otimes H^*(X) \arrow[r, "\times"] \arrow[rr, bend right, "\smile"'] & H^*(X \times X) \arrow[r, "\Delta^*"] & H^*(X) \end{tikzcd}$$

given by

$$(f \smile g)(\sigma) = f(\sigma|_{[v_0,\dots,v_i]}\, g(\sigma|_{[v_i,\dots,v_{i+j}]}$$

for $f \in H^i(X)$, $g \in H^j(X)$, $\sigma : \Delta^{i+j} \to X$.

Remark 11.1.
- If X is connected, then $H^0(X, R) = R$.
- The cup product gives the R-module structure on $H^n(X)$.

Example 11.2. Let $X = S^n$, then

$$H^*(X, R) = \begin{cases} R, & * = 0, n \\ 0, & \text{otherwise} \end{cases}$$

This says that the induced multiplication map $R \otimes R \to R$ on cohomology has the same behavior, i.e., $H^n(S^n; R) \otimes H^n(S^n; R) \to H^{2n}(S^n, R) = 0$. That is, we have $H^*(S^n; R) \cong R[e_n]/e_n^2$.

For the unit interval $I = [0, 1]$, then

$$\tilde{H}^*(S^1) \cong H^*(I, \partial I) = \begin{cases} \mathbb{Z}, & * = 1 \\ 0, & \text{otherwise} \end{cases}$$

Claim 11.3.

$$H^1(I, \partial I) \otimes H^n(Y) \xrightarrow{\times} H^{n+1}(I \times Y, \partial I \times Y)$$

is an isomorphism for any Y.

Corollary 11.4.

$$H^*(S^1) \otimes H^*(Y) \xrightarrow{\times} H^*(S^1 \times Y)$$

is an isomorphism for any space Y.

Example 11.5. Consider the Moore spaces. For any $m \in \mathbb{Z}$, we have $X_m = S^1 \cup_m e^2$, so we have

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, "m"'] & D^2 \arrow[d] \\ S^1 \arrow[r] & X_m \end{tikzcd}$$

We can give this a cell structure, so for instance $m = 2$, we have $X_m = \mathbb{R}P^2$. For general m, we have the cell structure with vertices x and y, $m+1$ edges $a, e_0, \ldots, e_m$, and m faces $C_0, \ldots, C_{m-1}$, then the boundary map is given by $\partial(a) = 0$, $\partial(e_i) = y - x$, and $\partial(C_i) = a - e_{i+1} + e_i$.

In the case $m = 2$, we have

$$\begin{tikzcd} x \arrow[r, "e_1"] \arrow[d, "e_0"'] & y \\ y \arrow[ur, "a"] & x \arrow[l, "e_1"] \arrow[u, "e_0"'] \end{tikzcd}$$

where the upper triangle is the face C_0 and the bottom triangle is the face C_1. We look at the chain equivalences

$$\begin{tikzcd} C_0X_2 & C_1X_2 \arrow[l] & C_0X_2 \arrow[l] & \cdots \arrow[l] \\ \mathbb{Z}\{x,y\} \arrow[u, hook] & \mathbb{Z}\{\alpha, e_0, e_1\} \arrow[l] \arrow[u, hook] & \mathbb{Z}\{C_0, C_1\} \arrow[l] \arrow[u, hook] & \cdots \arrow[l] \end{tikzcd}$$

The integral cohomology is just the cohomology of the above chain with respect to the dual basis, then checking the kernel and image, we know $\delta(x^\vee) = -e_0^\vee - e_1^\vee$, $\delta(y^\vee) = e_0^\vee + e_1^\vee$, $\delta(a^\vee) = C_0^\vee + C_1^\vee$, and $\delta(e_0^\vee) = C_0^\vee - C_1^\vee$, $\delta(e_1^\vee) = C_1^\vee - C_0^\vee$, therefore $x^\vee + y^\vee$ generates H^0. Similarly, we can show that $H^1 = 0$ and $H^2 = \mathbb{Z}/2$.

12 Sept 18, 2023

We need to prove that $\alpha \smile \beta = (-1)^{|\alpha||\beta|}\beta \smile \alpha$ in $H^*(X;k)$. Define $\rho : \Delta^n \to \Delta^n$ by sending $[v_0, \ldots, v_n]$ to $[v_n, \ldots, v_0]$. Using this, we can define a map

$$\begin{aligned} \rho : C_*X &\to C_*X \\ \sigma &\mapsto (-1)^{\varepsilon_n}\ \sigma|_{[v_n,\ldots,v_0]} \end{aligned}$$

where ε_n is the number of permutations required to permute $(0, \ldots, n)$ into $(n, \ldots, 0)$. This should just be $\binom{n+1}{2}$.

Exercise 12.1. ρ is a chain map.

This induces $\rho : C^*X \to C^*X$ with $\rho(\alpha)(\sigma) = (-1)^{\varepsilon_i}\alpha(\sigma)_{[v_n,\ldots,v_0]}$. Therefore,

$$\begin{aligned} \rho(\alpha \smile \beta)(\sigma) &= (-1)^{\varepsilon_n}(\alpha \smile \beta)(\sigma|_{[v_n,\ldots,v_0]}) \\ &= (-1)^{\varepsilon_n}\alpha(\sigma|_{[v_n,\ldots,v_j]})\beta(\sigma|_{[v_j,\ldots,v_0]}) \\ &= (-1)^{\varepsilon_n}(-1)^{\varepsilon_i}\rho(\alpha)(\sigma|_{[v_j,\ldots,v_n]}) \cdot (-1)^{\varepsilon_j}\rho(\beta)(\sigma|_{[v_0,\ldots,v_j]} \\ &= (-1)^{\varepsilon_n+\varepsilon_i+\varepsilon_j}\rho(\beta) \smile \rho(\alpha)(\sigma). \end{aligned}$$

Claim 12.2. $\varepsilon_i + \varepsilon_j - \varepsilon_{i+n} \equiv ij \pmod 2$.

In particular, this proves the claim. Moreover, ρ is a chain equivalence.

Proposition 12.3. If $f, g : C_*X \to C_*X$ are natural transformations of functors **Top** $\to$ **Ch**, such that f_0 and g_0 are naturally isomorphic (as components of the natural transformations), then f and g are naturally equivalent. Here f_0, g_0 : **Top** $\to$ **Ab**.

Theorem 12.4. Given a functor $F : \mathscr{C} \to$ **Ch**, there is an equivalence of categories **Func**$(\mathscr{C}$, **Ch**$) \cong$ **Ch**(**Func**$(\mathscr{C}$, **Ab**)).

To prove the theorem, we introduce acyclic models. Suppose we have a functor $F : \mathscr{C} \to$ **Ch**. We regard $\mathscr{C}$ as **Top**, or **Top** $\times$ **Top**.

Definition 12.5. A functor $F : \mathscr{C} \to$ **Ab** is called free on models M if

- there exists a set $M \subseteq \mathrm{Ob}(\mathscr{C})$ such that F is naturally isomorphic to the functor defined by the mapping $X \mapsto \bigoplus_{A \in M} \mathbb{Z}\{\mathrm{Hom}_{\mathscr{C}}(A, X)\}$.

Remark 12.6. Note that if $G : \mathscr{C} \to \mathbf{Set}$ is representable with respect to $A \in \mathscr{C}$, then the composition of the free functor $\mathbf{Set} \to \mathbf{Ab}$ and $G : \mathscr{C} \to \mathbf{Set}$ is free on model A.

- A functor $F_* : \mathscr{C} \to \mathbf{Ch}$ is free on models $\{M_n\}_{n\in\mathbb{Z}}$ if each $F_n : \mathscr{C} \to \mathbf{Ab}$ is free on M_n.
- Given $M \subseteq \mathrm{Ob}(\mathscr{C})$, a functor $F : \mathscr{C} \to \mathbf{Ch}$ is M-acyclic in positive degrees if for all $A \in M$, $H_q(F(A)) = 0$ for all $q > 0$.

Example 12.7. $C_* : \mathbf{Top} \to \mathbf{Ch}$ is acyclic in positive degrees on $\{\Delta^n\}_{n\in\mathbb{Z}}$.

Example 12.8. Consider $\mathbf{Top}^2 \to \mathbf{Ch}$.

1. If we have $(X, Y) \mapsto C_*(X \times Y)$, then $C_n(- \times -)$ is free on the model $\Delta^n \times \Delta^n$, and $C_*(- \times -)$ is acyclic on $\{\Delta^p \times \Delta^q\}_{p,q\geqslant 0}$.
2. If we have $(X, Y) \mapsto C_*(X) \otimes C_*(Y)$, then $(C_*(-) \otimes C_*(-))_n$ is free on the models $\{\Delta^p \times \Delta^{n-p}\}_p$, which is acyclic in positive degrees on $\{\Delta^p \times \Delta^q\}$.

13 Sept 20, 2023

Theorem 13.1 (Acyclic Models)**.** Suppose $F_*, G_* : \mathscr{C} \to \mathbf{Ch}$ are functors, and assume $F_n = 0 = G_n$ for $n < 0$. Assume

(a) each $F_n : \mathscr{C} \to \mathbf{Ab}$ is free on models $M_n \subseteq \mathrm{ob}(\mathscr{C})$, and

(b) G_* is acyclic in positive degrees on $\bigcup\limits_{n\geqslant 0} M_n$,

then

1. any natural transformation $H_0F_* \to H_0G_*$ of functors $\mathscr{C} \to \mathbf{Ab}$ is induced by a natural transformation $F_* \to G_*$, and
2. if $f, g : F_* \to G_*$ are natural transformations such that $H_0f = H_0g$, then there exists a natural chain homotopy $f \simeq g$, and
3. assume, in addition, that G_* is free on some model N, then if $f : F_* \to G_*$ is a natural transformation such that $H_0f : H_0F_* \to H_0G_*$ is a natural isomorphism, then f is a natural chain equivalence.

Claim 13.2. Any natural transformation $C_*X \to C_*X$ that induces an isomorphism $H_0X \to H_0X$ is a chain equivalence.

Example 13.3. Take $\rho : C_*X \to C_*X$ that inverts orientation, then ρ induces identity on cohomology, so

$$\alpha \smile \beta = \rho(\alpha \smile \beta) = (-1)^{|\beta|\times|\alpha|}\rho(\beta) \smile \rho(\alpha) = (-1)^{|\beta|\times|\alpha|}\beta \smile \alpha$$

Claim 13.4.

$$AW : C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$$

is a natural chain equivalence.

Proof. Apply acyclic models. □

Lemma 13.5 (Yoneda)**.** If $G : \mathscr{C} \to \mathbf{Set}$ is a functor, and let $C \in \mathrm{Ob}(\mathscr{C}($ be a representation of the functor, that is, $F_c(d) = \mathrm{Hom}_{\mathscr{C}}(c, d)$, then there is a natural bijection of sets $\mathrm{Nat}(F_c, G) \cong G(c)$ by $f : F_c \to G \mapsto f(\mathrm{id}_c)$.

Corollary 13.6. If $F : \mathscr{C} \to \mathbf{Ab}$ is free on models M, that is, $F(X) = \mathbb{Z}\left\{\coprod\limits_{A\in M} \mathrm{Hom}_{\mathscr{C}}(A, X)\right\} \cong \bigoplus\limits_{A\in M} \mathbb{Z}\{F_A(X)\}$, which induces

$$F : \mathscr{C} \xrightarrow{\coprod_{A\in M} F_A} \mathbf{Set} \xrightarrow{\text{Free}} \mathbf{Ab}$$

then for any $G : \mathscr{C} \to \mathbf{Ab}$, then we have a natural isomorphism $\mathrm{Nat}(F, G) \cong \prod\limits_{A\in M} G(A)$ given by $(f : F \to G) \mapsto (f(\mathrm{id}_A))_{A\in M}$.

We will now prove the acyclic models theorem.

Proof. 1. Take $F_* \to G_*$, then we are given a natural transformation $\bar{\varphi}_{-1} : H_0F_* \to H_0G_*$ with

$$\begin{tikzcd} 0 & H_0F_* \arrow[l] \arrow[d, "\bar{\varphi}_{-1}"] & F_0 \arrow[l] & F_1 \arrow[l] & \cdots \arrow[l] \\ 0 & H_0G_* \arrow[l] & G_0 \arrow[l] & G_1 \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

We want to lift $\varphi_0 \in \mathrm{Nat}(F_0, G_0)$, we take a look into the commutative diagram

$$\begin{tikzcd} \mathrm{Nat}(F_0, G_0) \arrow[r, "\cong"] \arrow[d, "\partial_G"'] & \prod\limits_{A \in M_0} G_0(A) \arrow[d, two heads] \\ \mathrm{Nat}(F_0, H_0G_0) \arrow[r, "\cong"'] & \prod\limits_{A \in M_0} H_0G_0(A) \end{tikzcd}$$

so we take $\varphi_{-1} \circ \partial_F \in \prod\limits_{A \in M_0} H_0G_0(A)$, then lift it to $\varphi_0 \in \prod\limits_{A \in M_0} G_0(A)$, then we obtain $\varphi_0 : F_0 \to G_0$ as desired. By construction, $\partial_G\varphi_0 = \varphi_{-1}\partial_F$. Proceeding inductively, we complete the diagram.

2. Now given $f, g : F_* \to G_*$, with $H_0f = H_0g$, we want $f \simeq g$. We want $h_i : F_i \to G_i$ to be such that $f_i - g_i = h_{i-1}\partial_F + \partial_F h$.

□

14 Sept 22, 2023

A complex C_* that is chain equivalent to 0 implies it is acyclic, i.e., $H_q(C_*) = 0$ for all q.

Proposition 14.1. If C_* is a complex of free abelian groups with $C_n = 0$ for $n \ll 0$, then C_* is acyclic if and only if it is chain equivalent to 0.

Proof. We can assume $C_n = 0$ for $n < 0$. Now consider $F : \mathscr{C} = \{*\} \to \mathbf{Ch}$ where $F(*) = C_*$. Now F is free and acyclic on models $\{*\}$, then the identity and zero map gives the same map on H_0, and by the acyclic model theorem we are done. □

Example 14.2. If $X \in \mathbf{Top}$, then X is acyclic if $H_*X = \begin{cases} \mathbb{Z}, * = 0 \\ 0, \text{otherwise} \end{cases}$, and so we extend the kernel and get a short exact sequence

$$0 \longrightarrow \tilde{C}_*X \longrightarrow C_*X \twoheadrightarrow C_*\{*\} \longrightarrow 0$$

Note that the last map admits a section with respect to a choice of a point $x_0 \in X$. Therefore, X is acyclic if and only if $\tilde{C}_*X$ is acyclic. Also, $\tilde{C}_*X$ is a complex of free abelian groups, so eX being acyclic implies $\tilde{C}_*X$ is chain equivalent to 0. Therefore, C_*X is chain homotopic to zero, as a complex concentrated at degree 0.

For instance, let $X = \Delta^p$ or $\Delta^p \times \Delta^q$.

Corollary 14.3 (Eilenberg-Zilber)**.** For any $X, Y \in \mathbf{Top}$, $C_*(X \times Y) \cong C_*X \otimes C_*Y$.

Claim 14.4. There is an anti-commutative diagram

$$\begin{tikzcd} H^p(X) \times H^q(Y) \arrow[r, "\times"] \arrow[d, "s"', "\cong"] & H^{p+q}(X \times Y) \arrow[d, "s^*"] \\ H^q(Y) \times H^p(X) \arrow[r, "\times"'] & H^{q+p}(Y \times X) \end{tikzcd}$$

with $\alpha \times \beta = (-1)^{|\alpha||\beta|} s^*(\beta \times \alpha)$.

This follows from

Lemma 14.5.

$$\begin{tikzcd} C_*(X \times Y) \arrow[r, "AW"] \arrow[d, "s_*"'] & C_*(X) \otimes C_*(Y) \\ C_*(Y \times X) \arrow[r, "AW"'] & C_*(Y) \otimes C_*(X) \arrow[u, "T"'] \end{tikzcd}$$

where T is a twist map via $T(y \otimes x) = (-1)^{|x||y|} x \otimes y$.

Theorem 14.6 (Kunneth)**.** Let $C_*, D_* \in$ **Ch**, say C_* is built out of free abelian groups, then

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(C_*) \otimes H_j(D_*) \longrightarrow H_n(C_* \otimes D_*) \longrightarrow \bigoplus_{i+j=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_iC_*, H_jD_*) \longrightarrow 0$$

splits unnaturally.

Remark 14.7. $\mathrm{Tor}(M, A) \cong \mathrm{Tor}(A, M)$.

Example 14.8. $\mathrm{Tor}(A, \mathbb{Z}) = 0 = \mathrm{Tor}(\mathbb{Z}, A)$, and $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$.

15 Sept 25, 2023

Remark 15.1. Similar results for chain complexes of R-modules for PID R holds. If R is not a PID, then there may be extra terms.

Example 15.2. If C_*, D_* are chain complexes of k-modules for a field k, then $H_*(C_* \otimes_k D_*) \cong H_*(C_*) \otimes_k H_*(D_*)$.

Theorem 15.3 (Kunneth)**.** If $X, Y \in$ **Top**, there is a short exact sequence

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \longrightarrow H_n(X \times Y) \longrightarrow \bigoplus_{i+j=n-1} \mathbf{Tor}(H_i(X), H_j(Y)) \longrightarrow 0$$

which splits unnaturally.

Proof. Identify $C_*(X \times Y) \cong C_*(X) \otimes C_*(Y)$ and use the previous Kunneth theorem. □

Remark 15.4. Let $(X, A) \times (Y, B) = (X \times Y, A \times Y \cup X \times B)$, then $H_*(X, A) \cong \tilde{H}_*(X) = \ker(H_*(X) \to H_*(*)) \cong \mathrm{coker}(H_*(A) \to H_*(X))$.

Definition 15.5 (Smash Product)**.** We denote $(X, *) \times (Y, *) \cong (X \times Y, C := X \times * \cup * \times Y)$, then $X \wedge Y = X \times Y/C$ is the smash product.

Theorem 15.6 (Kunneth)**.** We have an unnatural short exact sequence

$$0 \longrightarrow \bigoplus_{i+j=n} \tilde{H}_i(X) \otimes \tilde{H}_j(Y) \longrightarrow \tilde{H}_n(X \wedge Y) \longrightarrow \bigoplus_{i+j=n-1} \mathrm{Tor}(\tilde{H}_i(X), \tilde{H}_j(Y)) \longrightarrow 0$$

Example 15.7. Take $X = S^k$, then $S^k \wedge Y = \Sigma^k Y$ with $Y = S^m$, i.e., $S^k \wedge S^m \cong S^{k+m}$. Therefore, $\tilde{H}_n(\Sigma^k Y) \cong \tilde{H}_{n-k}(Y)$ as the suspension isomorphism.

Example 15.8. Let A be an abelian group, then there is a space $M(A, n)$ with the property that $\tilde{H}_* M(A, n) = A$ if $* = n$ and is 0 otherwise.

- If $A = \mathbb{Z}$, then $M(A, n) = S^n$;
- $A = \mathbb{Z}\left[\frac{1}{p}\right]$, then $M(A, n)$ is the mapping telescope of $S^n \xrightarrow{p} S^n \to \cdots$);
- if $A = \mathbb{Z}/k\mathbb{Z}$, then $M(A, n) \cong S^n \cup e^{n+1}$.

Therefore,

$$0 \longrightarrow \tilde{H}_r(Y) \otimes A \longrightarrow \tilde{H}_{n+r}(M(A,n) \wedge Y) \longrightarrow \mathrm{Tor}(A, \tilde{H}_{r-1}(Y)) \longrightarrow 0$$

One can compare this to the universal coefficient theorem for homology, i.e., $\tilde{H}_*(Y \wedge M(A,n)) = \tilde{H}_*(Y, A)$.

Definition 15.9. If A^* and B^* are cochain complexes, with a multiplication structure, then $A^* \otimes B^*$ also has a multiplicative structure by $(a \otimes b)(a' \otimes b') = (-1)^{|a| \times |b|}(aa' \otimes bb')$.

For instance, $H^*X \otimes H^*Y$ is a graded commutative ring.

Proposition 15.10. The cross product $\times : H^*X \otimes H^*Y \to H^*(X \times Y)$ is a map of graded rings via $(a \times b) \smile (a' \times b') = (-1)^{|a'||b|}(a \smile a') \times (b \smile b')$.

Proof. Consider the diagonal maps $\mathrm{diag}_{X \times Y} : X \times Y \to X \times Y \times X \times Y$ and $\mathrm{diag}_X \times \mathrm{diag}_Y : X \times Y \to X \times X \times Y \times Y$, then the left-hand side is just $\mathrm{diag}^*_{X \times Y}(a \times b \times a' \times b')$, and

$$\begin{aligned}(a \smile a') \times (b \smile b') &= \mathrm{diag}^*_X(a \times a') \times \mathrm{diag}^*_Y(b \times b') \\ &= (\mathrm{diag}_X \times \mathrm{diag}_Y)^*(a \times a' \times b \times b') \\ &= (\mathrm{diag}_{X \times Y})^*(1 \times \tau \times 1)^*(a \times a' \times b \times b') \\ &= (\mathrm{diag}_{X \times Y})^*(-1)^{|a'||b|} a \times b \times a' \times b'\end{aligned}$$

where τ swaps $X \times Y$ to $Y \times X$, therefore $1 \times \tau \times 1$ factors $\mathrm{diag}_X \times \mathrm{diag}_Y$ via $\mathrm{diag}_{X \times Y}$. ☐

16 Sept 27, 2023

Theorem 16.1 (Kunneth)**.** Let X and Y be topological spaces such that $H_n(Y)$ is finitely-generated as abelian groups, then we have a short exact sequence

$$0 \longrightarrow \bigoplus_{i+j=n} H^i(X) \otimes H^j(Y) \xrightarrow{\times} H^n(X \times Y) \longrightarrow \bigoplus_{i+j=n+1} \mathrm{Tor}(H^i(X), H^j(Y)) \longrightarrow 0$$

Remark 16.2.

1. One can think of this as

$$0 \longrightarrow H^*(X) \otimes H^*(Y) \xrightarrow{\times} H^*(X \times Y) \longrightarrow \mathrm{Tor}(H^*(X), H^{*+1}(Y)) \longrightarrow 0$$

2. same for coefficients in a PID;
3. If k is a field, then we have a Kunneth isomorphism $H^*(X, k) \otimes_k H^*(Y, k) \cong H^*(X \times Y)$.

Proof. Consider

$$\begin{tikzcd} C^*(X) \otimes C^*(Y) \arrow[rr] \arrow[dr] & & C^*(X \times Y) \\ & \mathrm{Hom}(C_*(X) \otimes C_*(Y), \mathbb{Z}) \arrow[ur, "\cong", "AW^*"'] & \end{tikzcd}$$

note that the first map in the splitting is not an equivalence in general. If $H_n(Y)$ is finitely-generated in each degree, then there is a complex D_* such that each D_n is finitely-generated, and $D_*D \cong C_*Y$, so

$$\begin{tikzcd} \mathrm{Hom}(C_*X \otimes D_*, \mathbb{Z}) & \mathrm{Hom}(C_*X \otimes C_*Y, \mathbb{Z}) \arrow[l, "\sim"'] \\ C^*X \otimes \mathrm{Hom}(D_*, \mathbb{Z}) \arrow[u, "\cong"] & C^*X \otimes C_*Y \arrow[l, "\sim"] \arrow[u, dashed] \end{tikzcd}$$

Note that in this case, the dashed map is an equivalence. By the algebraic Kunneth theorem, we are done. ☐

Example 16.3. 1. If $X = S^m$, then $\times : H^*(S^m) \otimes H^*(Y) \xrightarrow{\sim} H^*(S^m \times Y)$, so $H^*(S^m \times Y) \cong H^*(Y)[e_m]/e_m^2$.

2. $H^*(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{F}_2) = \mathbb{F}_2[x, y]/(x^3, y^3)$, with $|x| = |y| = 1$.

3. $H^*(\mathbb{C}P^2) = \mathbb{Z}[x]/x^3$, with $|x| = 2$. Note that $\mathbb{C}P^2$ and $S^2 \vee S^4$ have the same cohomology, but different cohomology rings. The first one is obtained by attaching a 3-cell on S^2 by the Hopf map, and the second one is obtained by attaching a 4-cell on S^2 by the trivial map.

Definition 16.4 (Bockstein Operation)**.** Consider a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of abelian groups, we then have two connecting homomorphisms $H_n(X, C) \to Hn - 1(X, A)$ and $H^n(X, C) \to H^{n+1}(X, A)$. For instance, consider

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

and therefore we have Bockstein maps $\beta : H_n(X, \mathbb{Z}/p\mathbb{Z}) \to H_{n-1}(X, \mathbb{Z}/p\mathbb{Z})$ as well as $\beta : H^n(X, \mathbb{Z}/p\mathbb{Z}) \to H^{n-1}(X, \mathbb{Z}/p\mathbb{Z})$. Consider another sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

where $\tilde{\beta} : H^n(X; \mathbb{Z}/p\mathbb{Z}) \to H^{n+1}(X, \mathbb{Z})$. This is also called a Bockstein map. In particular, they agree in the sense that

$$\begin{tikzcd} H^n(X;\mathbb{Z}/p\mathbb{Z}) \arrow[r, "\tilde{\beta}"] \arrow[dr, "\beta"'] & H^{n+1}(X,\mathbb{Z}) \arrow[d, "\bmod p"] \\ & H^{n+1}(X,\mathbb{Z}/p\mathbb{Z}) \end{tikzcd}$$

Considering

$$\cdots \longrightarrow H^n(X, \mathbb{Z}) \longrightarrow H^n(X, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\tilde{\beta}} H^{n+1}(X, \mathbb{Z}) \longrightarrow \cdots$$

then $\tilde{\beta}x = 0$ if and only if x lifts to an integral cohomology class.

Proposition 16.5. β is a derivation with respect to $\smile$ and $\times$, that is, $\beta(x \smile y) = (\beta x) \smile y + (-1)^{|x|} x \smile (\beta y)$, and similarly for $\times$.

Proof. We will prove this for $\smile$. Let $x = [f]$ and $y = [g]$ for $f \in C^n(X, \mathbb{Z}/p\mathbb{Z})$ and $g \in C^m(X, \mathbb{Z}/p\mathbb{Z})$ as cocycles. Given

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

this induces maps on the cochain level with respect to the connecting homomorphisms. Use the fact that $\delta(\tilde{f} \smile \tilde{g}). = \delta\tilde{f} \smile \tilde{g} + (-1)^{|\tilde{f}|} \tilde{f} \smile \delta\tilde{g}$. ☐

17 Sept 29, 2023

Lemma 17.1. $\beta^2 = 0$.

Proof. Identify the cycles as the cochains over the boundaries, then $\beta(x) = \frac{1}{n}\delta(\tilde{x})$ where δ is the connecting map between them, then $\beta^2(x)$ is identified to be $\frac{1}{n^2} \times \delta^2(\tilde{x}) = 0$. ☐

Example 17.2. We know $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z})$ is just $\mathbb{Z}/2\mathbb{Z}$ for all $* \geqslant 0$, so this is $\mathbb{Z}/2\mathbb{Z}[x]$. For every n, we know $H^*(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[x]/(x^{n+1})$ for $|x| = 1$, with $\beta(x^k) = x^{k+1}$ if k is odd and is zero otherwise.

Motivated by this, we will work on Steenrod operations with coefficients in $\mathbb{Z}/2\mathbb{Z}$, in particular the Steenrod squares. (The Steenrod powers are over $\mathbb{Z}/p\mathbb{Z}$ in general.)

A cohomology operation θ is a natural transformation $H^i(-, A) \to H^j(-, B)$. We usually want the functors **Top** $\to$ **Set** to be additive. For instance, the Bockstein map would be.

Definition 17.3. The Steenrod squares are additive cohomology operations $\operatorname{Sq}^i : H^n(X, A; \mathbb{Z}/2\mathbb{Z}) \to H^{n+i}(X, A; \mathbb{Z}/2\mathbb{Z})$ for $i \geqslant 0$, satisfying

1. $\operatorname{Sq}^0 = \mathrm{id}$;
2. if $|x| = i$, then $\operatorname{Sq}^i x = x^2$;
3. if $|x| < i$, then $\operatorname{Sq}^i x = 0$;
4. $\operatorname{Sq}^k(x \smile y) = \sum\limits_{k=i+j} \operatorname{Sq}^i x \smile \operatorname{Sq}^j y$. Alternatively, $\operatorname{Sq}^k(x \times y) = \sum\limits_{k=i+j} \operatorname{Sq}^i x \times \operatorname{Sq}^j y$.

Corollary 17.4.

1. $\beta = \operatorname{Sq}^1$;
2. Adem relation: if $0 < a < 2b$, then $\operatorname{Sq}^a \operatorname{Sq}^b = \binom{b-j-1}{a-2j} \sum\limits_{j=0}^{\lfloor \frac{a}{2} \rfloor} \operatorname{Sq}^{a+b-j} \operatorname{Sq}^j$. For example, if $a = b = 1$ and $j = 0$, then $\binom{0}{1} = 0$, so $\operatorname{Sq}^1 \operatorname{Sq}^1 = 0$.

Proposition 17.5. For any (X, A), the diagram

$$\begin{tikzcd}
H^s(A;\mathbb{Z}/2\mathbb{Z}) \arrow[r, "\delta"] \arrow[d, "\operatorname{Sq}^i"] & H^{s+1}(X,A;\mathbb{Z}/2\mathbb{Z}) \arrow[d, "\operatorname{Sq}^i"] \\
H^{s+i}(A;\mathbb{Z}/2\mathbb{Z}) \arrow[r, "\delta"'] & H^{s+i+1}(X,A;\mathbb{Z}/2\mathbb{Z})
\end{tikzcd}$$

commutes.

Proof. For $A \hookrightarrow X$, we have a mapping cylinder $M = X \smile_{A \times \{1\}} (A \times I)$. Then

$$(X, A) \xleftarrow{\sim} (M, A \times I) \xleftarrow{\sim} (M, A \times \{0\}) \longrightarrow (M, A \times \{0\}) \cup X \cup (A \times [0, \tfrac{1}{2}]) \longleftarrow (A \times I, A \times \partial I)$$

where the last map is the excision. Then

$$\begin{tikzcd}
H^s(A) \arrow[r, "\sim"] \arrow[d, "\delta"'] & H^s(A \times I) \arrow[r] \arrow[d, "\delta"] & H^s(A \times \{0\}) \arrow[d, "\delta"] & H^s(A \times \{0\} \sqcup Z) \arrow[l, two heads] \arrow[r, "\cong"] \arrow[d, "\delta"] & H^s(A \times \partial I) \arrow[d, "\delta"] \\
H^{s+1}(X, A) \arrow[r, "\sim"'] & H^{s+1}(M, A \times I) \arrow[r, equal] & H^{s+1}(M, A \times \{0\}) & H^{s+1}(M, A \times \{0\} \cup Z) \arrow[l] \arrow[r, "\cong"] & H^{s+1}(A \times I, A \times \partial I)
\end{tikzcd}$$

□

18 Oct 2, 2023

Let (X, A) be a pair, then we have a commutative diagram

$$\begin{tikzcd}
H^q(A;\mathbb{Z}/2\mathbb{Z}) \arrow[r, "\delta"] \arrow[d, "\operatorname{Sq}^i"'] & H^{q+1}(X,A) \arrow[d, "\operatorname{Sq}^i"] \\
H^{q+i}(A;\mathbb{Z}/2\mathbb{Z}) \arrow[r, "\delta"'] & H^{q+1+i}(X,A)
\end{tikzcd}$$

Corollary 18.1. Sq^i's are stable operations, i.e.,

$$\begin{tikzcd}
\tilde{H}^*(X;\mathbb{Z}/2\mathbb{Z}) \arrow[r, "\simeq"] \arrow[d, "\operatorname{Sq}^i"'] & \tilde{H}^{*+1}(\Sigma X;\mathbb{Z}/2\mathbb{Z}) \arrow[d, "\operatorname{Sq}^i"] \\
\tilde{H}^{*+i}(X;\mathbb{Z}/2\mathbb{Z}) \arrow[r, "\simeq"'] & \tilde{H}^{*+i+1}(\Sigma X;\mathbb{Z}/2\mathbb{Z})
\end{tikzcd}$$

Proof. Take (CX, X), then $H^*(CX, X) \cong \tilde{H}^*(\Sigma X)$. □

Example 18.2. Let $\eta : S^3 \to S^2$ be the Hopf map, then $\mathrm{cone}(\eta) = \mathbb{C}P^2 = S^2 \cup_\eta e^4$. This extends to

$$\begin{tikzcd} S^3 \arrow[r, "\eta"] \arrow[d, hook] & S^2 \arrow[d, "\simeq"] \\ \mathbb{C}^2\backslash\{0\} \arrow[r, "(z_1,z_2)\mapsto[z_1,z_2]"'] & \mathbb{C}P1. \end{tikzcd}$$

In fact, $H^*(\mathbb{C}P^2;\mathbb{Z}) = \mathbb{Z}[y]/(y^3)$, where $|y| = 2$, so the two structures do not have the same cohomology rings, therefore $\mathbb{C}P^2 \not\simeq S^2 \vee S^4$.

Example 18.3. Consider $\Sigma\eta : S^4 \to S^3$ with cone $\Sigma(\mathbb{C}P^2)$, and $H^*(\Sigma(\mathbb{C}P^2)) \cong H^*(S^3 \wedge S^5)$ as rings. Therefore $H^*(\mathbb{C}P^2;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[y]/(y^3) = \mathbb{Z}/2\mathbb{Z}\{y_0, y_2, y_4\}$. On the other hand, $H^*(\Sigma(\mathbb{C}P^2))$ is $\mathbb{Z}/2\mathbb{Z}\{1, z_3, z_5\}$, where $\mathrm{Sq}^2(z_3) = z_5$ (the Steenrod operation commutes with the suspension), and $\mathrm{Sq}^2(y_2) = y_2^2 = y_4$. By the same argument, $H^*(S^3 \wedge S^5) = \mathbb{Z}/2\mathbb{Z}\{1, w_3, w_5\}$. However, now we see $\mathrm{Sq}^2(w_3) = 0$.

Note that as a ring, we must map generators of degree n to generators of degree n as a ring isomorphism, but in this case, there are no mappings over the topological structure, as we see the transform to the topology structure does not preserve the Steenrod operation.

The punchline being, cohomology is a bad invariant in the sense that non-identical spaces can have the same cohomology. Moreover, if two spaces are homotopy equivalent, then they must have isomorphic cohomology rings. (Therefore, cohomology, cohomology rings, and Steenrod algebras, are all homotopy invariants.) However, even if they have the same cohomology rings (as algebraic structure), they can still have different cohomology modules over the Steenrod algebra, that is, the different topology structure over the Steenrod operation, so non-homotopy equivalent.

Corollary 18.4. $\Sigma^n\eta \not\simeq 0$ for any $n \geqslant 0$.

Example 18.5. $H^*(\mathbb{R}P^\infty;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}_2[x]$ for $|x| = 1$, then $\mathrm{Sq}^1(x) = x^2$ as dimension axiom says. More generally, let $x \in H^1(X;\mathbb{Z}/2\mathbb{Z})$, then $\mathbb{Z}/2\mathbb{Z}[x] \to H^*(X;\mathbb{Z}/2\mathbb{Z})$.

Lemma 18.6. For $x \in H^1(X;\mathbb{Z}/2\mathbb{Z})$, then $\mathrm{Sq}^i(x^k) = \binom{k}{i}x^{k+i}$.

Proof. This is true by induction on k and by the Cartan formula. On the inductive step, we have

$$\begin{aligned} \mathrm{Sq}^i(x^{k+1}) &= \sum_{a+b=i} \mathrm{Sq}^a(x)\,\mathrm{Sq}^b(x^k) \\ &= x\,\mathrm{Sq}^i(x^k) + x^2\,\mathrm{Sq}^{i-1}(x^k) \\ &= x\binom{k}{i}x^{k+i} + x^2\binom{k}{i-1}x^{k+i-1} \\ &= \binom{k+1}{i}x^{k+i+1} \end{aligned}$$

since $\mathrm{Sq}^a(x)$ is x if $a = 0$, is x^2 if $a = 1$, and is 0 otherwise. $\square$

Lemma 18.7. If $y \in H^2(X;\mathbb{Z}/2\mathbb{Z})$ such that $\beta y = \mathrm{Sq}^1(y) = 0$, then $\mathrm{Sq}^{2i}(y^k) = \binom{k}{i}y^{k+i}$, and $\mathrm{Sq}^{2i+1}(y^k) = 0$.

Remark 18.8. Let a and b be 2-adic, then $\binom{a}{b} \equiv \prod_i \binom{a_i}{b_i} \pmod 2$.

For $x \in H^1(X;\mathbb{Z}/2\mathbb{Z})$, we know $\mathrm{Sq}^i(x^k) = \binom{k}{i}x^{k+i}$, for instance $\mathrm{Sq}^i(x^{2^k})$ is x^{2^k} if $i = 0$, is $x^{2^{k+1}}$ if $i = 2^k$, and is 0 otherwise.

Theorem 18.9. A minimal set of algebraic generators of Steenrod algebra A^* is given by Sq^{2^i}. That is, for any Sq^i where $i \neq 2^k$, it is deceomposable as a sum of products of Sq^j's for $j < i$.

Proof. By Adem relations, for $0 < a < 2b$, then $\binom{b-1}{a}\mathrm{Sq}^{a+b} = \mathrm{Sq}^a\,\mathrm{Sq}^b = \sum_{j>0}\binom{b-1-j}{a-2j}\mathrm{Sq}^{a+b-j}\,\mathrm{Sq}^j$.

- If $\binom{b-1}{a} \equiv 1$, then Sq^{a+b} is decomposable.
- If $i \neq 2^k$, then $i = a = b$ where $b = 2^l$ for some l, now $\binom{b-1}{a} \equiv 1 \pmod 2$.

$\square$

19 Oct 4, 2023

$$\begin{tikzcd}
C^*(X;\mathbb{F}_2)\otimes C^*(X,\mathbb{F}_2) \arrow[r] \arrow[d, "\tau"'] \arrow[rr, bend left, "\smile"] & C^*(X\times X,\mathbb{F}_2) \arrow[r] & C^*(X;\mathbb{F}_2) \\
C^*(X;\mathbb{F}_2)\otimes C^*(X;\mathbb{F}_2) \arrow[urr, "\smile"'] & &
\end{tikzcd}$$

is not (graded) commutative, but it commutes in cohomology, therefore it is up to homotopy. In fact, it commutes up to "coherent homotopy", i.e., Steenrod operations.

Let k be a commutative ring and G be a finite group. The group ring $k[G]$ is free as module over k, therefore $k[G] \cong \bigoplus_{g_i\in G} k[g_i]$. The multiplication is given by $[g][h]=[gh]$ and extended k-linearly.

Example 19.1. Let $G=\Sigma_2\langle t\rangle$ be symmetric group on two letters. We write $k[\Sigma_2]=k\cdot 1\oplus k\cdot t = k[t]/(t^2-1)$.

Recall that a k-module with a G-action corresponds to a $k[G]$-module, where the action is given by

$$\begin{aligned} G\times M &\to M \\ (g,m) &\mapsto g\cdot m \end{aligned}$$

where $1\cdot m=m$ and $g\cdot(h\cdot m)=(gh)\cdot m$. The invariants/fixed points are $M^G\subseteq M$, i.e., $\{m \mid g\cdot m=m\ \forall g\in G\}$. Note that this corresponds to $\mathrm{Hom}_{k[G]}(k,M)$. The dual construction is the coinvariants/orbits, as $M\otimes_{k[G]}k\cong M_G = M/\langle m-gm \mid g\in G\rangle$.

In the example of $G=\Sigma_2$, then $M=k[\Sigma_2]=k[t]/(t^2-1)$. Then $M^{\Sigma_2}=k$ and $M_{\Sigma_2}=k$.

Moreover, (co)chain complexes of k-modules with G-actions correspond to (co)chain complexes of $k[G]$-modules. Therefore the construction $(C^*)^G$ and $(C^*)_G$ are well-defined. Therefore, we can build a free $k[G]$-resolution of k, where P_i is free, i.e., $P_i\cong\bigoplus k[G]$ and such that P_* is an acyclic complex.

Example 19.2. For $k[\Sigma_2]k[t]/(t^2-1)$, we have a resolution

$$k \xleftarrow[1\mapsfrom t]{} k[\Sigma_2] \xleftarrow[1-t]{} k[\Sigma_2] \xleftarrow[1+t]{} k[\Sigma_2] \xleftarrow[1-t]{} k[\Sigma_2] \longleftarrow \cdots$$

Let $EG_*\in \mathbf{Ch}_{k[G]}$ be any acyclic free resolution of k. We define

Definition 19.3. The homotopy fixed point is $M^{hG}=\mathrm{Hom}(EG_*,M)$. The homotopy orbit is $M_{hG}=M\otimes_{k[G]}EG_*$.

Note that $\mathrm{Hom}_{k[G]}(A,B)=\mathrm{Hom}_k(A,B)^G$ with the action $(g\cdot f)(a)=g\cdot f(g^{-1}\cdot a)$. Also, if $M=k[G]$, then $M_{hG}=EG_*$.

Example 19.4. There is a Σ_2-action on $C^*(X;\mathbb{F}_2)\otimes C^*(X;\mathbb{F}_2)$, given by $t\cdot(x\otimes y)=\tau(x\otimes y)=y\otimes x$. More generally, we have a Σ_2-action on $V^*\otimes V^*$.

We will now denote $D_2(V^*)=(V^*\otimes V^*)_{h\Sigma}=(V^*\otimes V^*)\otimes_{\mathbb{F}_2[\Sigma_2]}E\Sigma_2$. More generally, we can write $D_n(V^*)=((V^*)^{\otimes n})_{h\Sigma_n}$. If we have a (associative) multiplication $m: V^*\otimes V^*\to V^*$, then m is commutative if and only if

$$\begin{tikzcd}
V^*\otimes V^* \arrow[rr, "m"] \arrow[dr] & & V^* \\
& (V^*\otimes V^*)_{\Sigma_2} \arrow[ur, dashed] &
\end{tikzcd}$$

commutes.

We say m is symmetric if we can factor through $D_2(V^*)$, that is, commutative up to homotopy. Moreover, we can ask for multiplications $m_n: D_n(V^*)\to V^*$.

Proposition 19.5. There is a natural map of $\mathbb{Z}[\Sigma_2]$-chain complexes $C_*(X) \otimes E\Sigma_2 \to C_*(X) \otimes C_*(X)$ where Σ_2 acts by the twisting τ on $C_*(X) \otimes C_*(X)$, such that

$$\begin{tikzcd} C_*(X) \otimes E\Sigma_2 \arrow[r] \arrow[d] & C_*(X) \otimes C_*(X) \\ C_*(X) \arrow[ur, "AW_X"'] & \end{tikzcd}$$

commutes up to homotopy.

Proof. Acyclic models for functors $F, G : \mathbf{Top} \to \mathbf{Ch}_{\mathbb{Z}[\Sigma_2]}$ then $F(X) = C_*(X) \otimes E\Sigma_2$ is free over models, and $G(X) = C_*(X) \otimes C_*(X)$ are acyclic on those models, all with respect to $\mathbb{Z}[\Sigma_2]$. Extending this into tensoring with $E\Sigma_2$ makes sure this is free and acyclic, and we can apply the theorem. $\square$

20 Oct 6, 2023

Recall there is a Σ_n-action on $V^{\otimes n}$, so we have a commutative diagram

$$\begin{tikzcd} V \otimes V \arrow[rr, "m"] \arrow[dr] & & V \\ & (V \otimes V)_{\Sigma_2} \arrow[ur] & \end{tikzcd}$$

Proposition 20.1. There is a natural map of $\mathbb{Z}[\Sigma_2]$-cochain complexes $C_*(X) \otimes E\Sigma_2 \to C_*X \otimes C_*X$ which refines the Alexander-Whitney map.

Example 20.2. Consider the free resolution $EG_* \to \mathbb{Z}$. For instance, we have

$$\mathbb{Z}[\Sigma_2] \xrightarrow{1+T} \mathbb{Z}[\Sigma_2] \xrightarrow{1-T} \mathbb{Z}[\Sigma_2] \longrightarrow \mathbb{Z}$$

and therefore $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], \mathbb{Z}] \cong \mathbb{Z}[G]$ as G-modules. This implies $\mathrm{Hom}_{\mathbb{Z}}(EG_*, \mathbb{Z}) \cong EG_*$ gives $\mathbb{Z} \to EG_*$.

Proof. Apply acyclic models for functors $\mathbf{Top} \to \mathbf{Ch}_{\mathbb{Z}[\Sigma_2]}$. $\square$

Proposition 20.3. There is a symmetric multiplication

$$\begin{tikzcd} D_2(C^*X) = (C^*(X) \otimes C^*(X) \otimes E\Sigma_2)_{\Sigma_2} \arrow[r, "m_2"] & C^*(X) \\ C^*(X) \otimes C^*(X) \arrow[u] \arrow[ur, "\smile"'] & \end{tikzcd}$$

that commutes up to homotopy.

Proof.

$$\begin{tikzcd} C^*(X) \otimes C^*(X) \arrow[r] \arrow[d, "\smile"'] & \mathrm{Hom}(C_*(X) \otimes C_*(X), \mathbb{Z}) \arrow[d] \\ \mathrm{Hom}(C_*(X), \mathbb{Z}) & \mathrm{Hom}(C_*(X) \otimes E\Sigma_2, \mathbb{Z}) \cong \mathrm{Hom}(E\Sigma_2, C^*(X)) \arrow[l] \end{tikzcd}$$

The map $C^*(X) \otimes C^*(X) \to \mathrm{Hom}(E\Sigma_2, C^*(X))$ is equivalent to $C^*(X) \otimes C^*(X) \otimes E\Sigma_2 \to C^*(X)$ where $C^*(X)$ has the trivial action, and $C^*(X) \otimes C^*(X) \otimes E\Sigma_2$ has action T given by $T(x \otimes y \otimes a) = (y \otimes x \otimes Ta)$. This makes the map Σ_2-equivariant. Take $(-)_{\Sigma_2}$, we get $D_2(C^*(X)) \to C^*(X)$. $\square$

Remark 20.4. Let A_* be a chain complex. Then $A_* \otimes E_{\Sigma_2}$ is the same as having A_* tensoring $\cdots \to \mathbb{Z}[\Sigma_2] \xrightarrow{1+T} \mathbb{Z}[\Sigma_2] \xrightarrow{1-T} \mathbb{Z}[\Sigma_2])$ in non-negative degrees, so $(A_* \otimes E\Sigma_2)_n \cong (A_n \otimes \mathbb{Z}[\Sigma_2]) \oplus (A_{n-1} \otimes \mathbb{Z}[\Sigma_2]) \oplus \cdots \oplus (A_0 \otimes \mathbb{Z}[\Sigma_2])$.

What does an equivariant chain map out of $A_* \otimes E\Sigma_2$ look like? Note that $\mathrm{Hom}_{\Sigma_2}(A \otimes \mathbb{Z}[\Sigma_2], B) = \mathrm{Hom}(A, B)$, i.e, there is an adjunction between $\mathbf{Mod}_R$ and $\mathbf{Mod}_{R[G]}$, as $- \otimes R[G] \dashv U$, where U is forgetful.

To give a chain map of $\mathbb{F}_2[\Sigma_2]$-complexes $C_*(X; \mathbb{F}_2) \otimes E\Sigma_2 \to C_*(X; \mathbb{F}_2) \otimes C_*(X; \mathbb{F}_2)$, we want a collection of maps $\{d_j : C_*(X) \to C_*(X) \otimes C_*(X)\}$ such that d_j is a chain map of degree $-j$ such that $(1+T)d_{j-1} = \partial d_j + d_j \partial$, and $d_0 = AW$.

21 Oct 9, 2023

Let $A_* = C_*X \otimes E\Sigma_2$ and $B_* = C_*X \otimes C_*X$. Therefore, there is an Σ_2-equivalence between $A_* \otimes E\Sigma_2$ and B_* such that the diagram

$$\begin{tikzcd} A_* \otimes E\Sigma_2 \arrow[r, "\simeq"] \arrow[d] & B_* \\ C_*X \arrow[ur, "AX"'] & \end{tikzcd}$$

Let $E\Sigma_2$ be $\mathbb{F}_2[\Sigma_2] \xleftarrow{1+T} \mathbb{F}_2[\Sigma_2] \xleftarrow{1+T} \cdots$, then $(A_* \otimes E\Sigma_2)_n$ is just a direct sum $\bigoplus_{i=0}^{n} A_i \otimes \mathbb{F}_2[\Sigma_2]$ such that $f_* : (A_* \otimes E\Sigma_2)_n \to (B_*)_n$ corresponds to $f_n : \bigoplus_{i=0}^{n} A_i \otimes \mathbb{F}_2[\Sigma_2] \to B_n$, which corresponds to $f_n^{n-i} : A_i \to B_i$ for $0 \leqslant i \leqslant n$.

Lemma 21.1. Let B_* be a Σ_2-equivalence of chain complexes, given a map $\varphi : A_* \to B_*$, to give an extension of φ to a map

$$\begin{tikzcd} A_* \otimes E\Sigma_2 \arrow[r, "f"] & B_* \\ A_* \arrow[ur, "\varphi"'] & \end{tikzcd}$$

is equivalent to giving a collection of $f^j : A_* \to B_{*+j}$ such that $f^0 = \varphi$, and $(1+T)f^{j-1} = \partial_B f^j + f^j \partial_A$.

Therefore, all of this gives a degree j map $f^j C_*X \to C_*X \otimes C_*X$. On cochains, we have

$$\begin{tikzcd} C^*X \otimes C^*X \arrow[r] \arrow[dr, "h^j"'] & \mathrm{Hom}(C_*X \otimes C_*X, \mathbb{F}_2) \arrow[d, "\mathrm{Hom}(f^j,\mathbb{F}_2)"] \\ & C^*X \end{tikzcd}$$

where h^j has degree $-j$. Alternatively, we can write $h^j(\alpha \otimes \beta) = \alpha \smile_j \beta$.

Therefore, h^j's satisfy $h^j(1+\tau) = h^{j+1}\delta + \delta h^{j+1}$.

Remark 21.2.
- $h^{j+1}(\delta a \otimes \delta a) = \delta h^{j+1}(a \otimes \delta a) + \delta h^j(a \otimes a)$.
- $h^j((a+b) \otimes (a+b)) = h^j(a \otimes a) + h^j(b \otimes b) + h^{j+1}\delta(a \otimes b) + \delta h^{j+1}(a \otimes b)$.

Theorem 21.3. The map h^{q-n} induces a natural homomorphism $\mathrm{Sq}^n : H^q(X) \to H^{q+n}(X)$ by $\mathrm{Sq}^n([a]) = [h^{q-n}(a \otimes a)]$.

Proposition 21.4. If $a \in H^q(X)$, then $\mathrm{Sq}^q(a) = a^2$, and if $n > q$, then $\mathrm{Sq}^n(a) = 0$.

Theorem 21.5. The operations Sq^n are independent of the choice of a chain map $C_*X \otimes E\Sigma_2 \to C_*X \otimes C_*X$.

Recall if V^* is cochain complex, then $D_2(V^*) = (V \otimes V \otimes E\Sigma_2)/\Sigma_2$. The enhanced AW map gave us a symmetric multiplication from $m_2 : D_2(C^*X) \to C^*(X)$ to $H^*(D_2C^*X) \to H^*X$.

We should study $H^*(D_2\mathbb{F}_2[n])$. So given θ in this cohomology, and $x \in H^*V$, we get $\theta(x) \in H^*V$ by $m_2 \circ D_2x \circ \theta : \mathbb{F}_2[m] \to V$.

22 Oct 11, 2023

We saw that the enhanced AW map

$$\begin{tikzcd} C_*X \otimes E\Sigma_2 \arrow[r, "\Sigma_2 \cong"] & C_*X \otimes C_*X \\ C_*X \arrow[u] \arrow[ur, "AW"'] & \end{tikzcd}$$

and this gives rise to a symmetric multiplication $m_2 : D_2(C^*X) \cong (C^*X \otimes C^*X \otimes E\Sigma_2)_{\Sigma_2} \to C^*X$. Here $E\Sigma_2 = (\cdots \xrightarrow{1+T} \mathbb{F}_2[\Sigma_2] \xrightarrow{1+T} \mathbb{F}_2[\Sigma_2])$ is a chain complex in non-negative degree, i.e., cochain complex in non-positive degree. On cohomology, m_2 gives $H^*(D_2C^*X) \to H^*X$, and to get an operation on $H^*(X)$, we need natural transformations $H^k(X) \to H^m(D_2C^*X)$. We observe that if V^* is a cochain complex, then $H^n(V^*) = \mathrm{Hom}_{\mathbf{coCh}}(\mathbb{F}_2[n], V^*)/\sim$, quotient by chain homotopy.

Proposition 22.1. Natural transformations $H^n(V^*) \to H^m(D_2V^*)$ are in correspondence with elements of $H^m(D_2\mathbb{F}_2[n])$.

Proof. Given $\theta \in H^m(D_2\mathbb{F}_2[n])$ represented by $\theta : \mathbb{F}_2[m] \to D_2\mathbb{F}_2[n]$, let $x \in H^n(V^*)$, then x is represented by $x : \mathbb{F}+2[n] \to V^*$. Apply D_2 to get $D_2x : D_2\mathbb{F}_2[n] \to D_2V^*$, by precomposing with θ, we get $\mathbb{F}_2[m] \to D_2V^*$.

For the converse, given $\varphi : H^n(-) \to H^m(D_2-)$, let $x \in H^n(V^*)$, $x : \mathbb{F}_2[n] \to V^*$, so

$$\begin{tikzcd} H^n(\mathbb{F}_2[n]) \arrow[r, "\varphi"] \arrow[d] & H^m(D_2\mathbb{F}_2[n]) \arrow[d] \\ H^n(V^*) \arrow[r, "\varphi"'] & H^m(D_2V^*) \end{tikzcd}$$

this traces a generator in $\mathbb{F}_2$ to $x \in H^n(V^*)$ and to $\theta \in H^m(D_2\mathbb{F}_2[n])$. ☐

Therefore, to compute the cohomology of $H^*(D_2\mathbb{F}_2[n])$, note that $\mathbb{F}_2[n] \otimes_{\mathbb{F}_2} \mathbb{F}_2[n] \cong \mathbb{F}_2[2n]$, so $D_2\mathbb{F}_2[n] = (\mathbb{F}_2[2n] \otimes E\Sigma_2)_{\Sigma_2}$, but note that $\mathbb{F}_2[2n]$ now has a trivial action, so this is just $\mathbb{F}_2[2n] \otimes (E\Sigma_2/\Sigma_2)$, where $E\Sigma_2/\Sigma_2$ is the cochain complex in non-positive degrees

$$\cdots \longrightarrow \mathbb{F}_2 \xrightarrow{\times 2} \mathbb{F}_2 \xrightarrow{\times 2} \mathbb{F}_2 \longrightarrow 0 \longrightarrow \cdots$$

but over $\mathbb{F}_2$ they are just zero. Therefore, tensoring $\mathbb{F}_2[2n]$ with this complex is just identifying the degree 0 in the complex by the twisting, i.e., as degree $2n$, so the thing we want is

$$H^m(D_2\mathbb{F}_2[n]) = \begin{cases} \mathbb{F}_2, & m \leqslant 2n \\ 0, & \text{otherwise} \end{cases}$$

This means the natural transformations $\mathrm{Nat}(H^n(-) \to H^m(D_2-))$ is just $\mathbb{F}_2$ if $m \leqslant 2n$ and is 0 otherwise. Relabel $i = m-n$, and denote $\overline{\mathrm{Sq}}^i$ to be the non-zero transformation $H^n(-) \to H^{n+i}(D_2-)$. If V^* has a symmetric multiplication, then we get $\mathrm{Sq}^i : H^n(V^*) \xrightarrow{\overline{\mathrm{Sq}}^i} H^{n+i}(D_2V^*) \xrightarrow{m^*} H^{n+i}(V^*)$.

Example 22.2.

1. $V^* = C^*X$;
2. Suppose $X \in \mathbf{Top}$ has a homotopy commutative multiplication $X \times X \to X$, i.e., X is an H-space, for instance consider S^1, S^3, S^7, or BG where G is abelian, then $C_*(X)$, as a cochain complex in non-positive degrees, has a symmetric multiplication.

Proposition 22.3. If V^* has a symmetric multiplication, then for $x \in H^n(V^*)$, $\mathrm{Sq}^n(x) = x62$.

Proof. Note that Sq^n corresponds to the generator of $H^{2n}(D_2\mathbb{F}_2[n])$, which is $i_n \otimes i_n$, where i_n is the generator of $H^n(\mathbb{F}_2[n])$. Therefore $\overline{\mathrm{Sq}}^n(x) = [x \otimes x] \in H^{2n}(D_2V^*)$. ☐

Proposition 22.4. $\overline{\mathrm{Sq}}^i$ are additive, i.e., given $v, v' \in H^n(V^*)$, then $\overline{\mathrm{Sq}}^i(v + v') = \overline{\mathrm{Sq}}^i(v) + \overline{\mathrm{Sq}}^i(v') \in H^{n+i}(D_2V^*)$.

Proof. Let $v : \mathbb{F}_2[n] \to V^*$ and $v' : \mathbb{F}_2[n] \to V^*$, then $v + v' : \mathbb{F}_2[n] \oplus \mathbb{F}_2[n] \to V^* \oplus V^* \to V^*$, so it suffices to show additivity on the direct sum. Let $W = \mathbb{F}_2[n] \oplus \mathbb{F}_2[n]$, then it suffices to show that the diagram

$$\begin{tikzcd} H^n(W) \arrow[r, "\overline{\mathrm{Sq}}^i"] \arrow[d] & H^{n+i}(D_2W) \arrow[d] \\ H^n(\mathbb{F}_2[n]) \oplus H^n(\mathbb{F}_2[n]) \arrow[r, "(\overline{\mathrm{Sq}}^i, \overline{\mathrm{Sq}}^i)"'] & H^{n+i}(D_2\mathbb{F}_2[n]) \oplus H^{n+i}(D_2\mathbb{F}_2[n]) \end{tikzcd}$$

commutes. The vertical mappings are given by $D_2(A \oplus B) \cong D_2(A) \oplus D_2(B) \oplus \cdots$ and one can check this componentwise. ☐

23 Oct 13, 2023

We want to have Sq^i commutes with suspension. Note that there is a correspondence between suspension and shift as $(V^*[k])^m = V^{m-k}$ and $V^*[k] = V^* \otimes_{\mathbb{F}_2} \mathbb{F}_2[k]$.

Proposition 23.1. The diagram

$$\begin{tikzcd} H^*(V^*[1]) \arrow[r, "\cong", "\sigma"'] \arrow[d, "\overline{Sq}^i"'] & H^{*-1}(V^*) \arrow[d, "\overline{Sq}^i"] \\ H^{*+i}(D_2(V^*[1])) \arrow[r] & H^{*+i-1}(D_2V^*) \cong H^{*+i}((D_2V^*)[1]) \end{tikzcd}$$

where the map $H^{*+i}(D_2(V^*[1])) \to H^{*+i-1}(D_2V^*) \cong H^{*+i}((D_2V^*)[1])$ is $H^{*+i}(D_2(V[1]) \to (D_2V)[1])$, where $D_2(V[1])$ is $(V[1] \otimes V[1] \otimes E\Sigma_2)/\Sigma_2 \cong (V \otimes V \otimes E\Sigma_2[2])/\Sigma_2$, therefore the map becomes $(V \otimes V \otimes E\Sigma_2[2])/\Sigma_2 \to (V \otimes V \otimes E\Sigma_2[1])/\Sigma_2$, i.e., induced by $E\Sigma_2[2] \to E\Sigma_2[1]$, as a sort of inclusion.

Corollary 23.2. Steenrod operations on $\mathbb{F}_2$ commutes with suspensions.

We now want $\mathrm{Sq}^0 = \mathrm{id}$ on $H^*(X; \mathbb{F}_2)$.

Example 23.3. $\tilde{H}^*(S^n)$ is $\mathbb{F}_2\{e_n\}$ if $* = n$, and is 0 otherwise. Therefore, $\mathrm{Sq}^i = 0$ for $i \neq 0$.

To understand Sq^0, use suspension isomorphism to reduce to $n = 0$. We have $\mathrm{Sq}^0\{e_0\} = e_0$ by $\mathrm{Sq}^i(x_i) = x_i^2$, therefore $\mathrm{Sq}^0 = \mathrm{id}$ on $H^*(S^n)$.

Proposition 23.4. Suppose X is equivalent to a CW complex, then $\mathrm{Sq}^0 = \mathrm{id}$ on $H^*(X)$, and $\mathrm{Sq}^i = 0$ for $i < 0$.

Proof. Note that this is true for S^n. Now consider the good pair $(X, X^{(n)})$ with n-skeleton $X^{(n)}$. We trace the diagram

$$\cdots \longrightarrow H^n(X, X^{(n)}) \longrightarrow H^n(X) \hookrightarrow H^n(X^{(n)}) \longrightarrow \cdots$$

and note that $H^n(X, X^{(n)})$ is 0 because there are no cells of dimension at most n. Here $i : X^{(n)} \to X$ induces the inclusion map.

Theorem 23.5 (Hopf's Classification Theorem)**.** If Y is a CW-complex with all of its cells in dimension at most n, then $H^n(Y, \mathbb{Z}) \cong [Y, S^n]$ where $[Y, S^n]$ is the homotopy classes of maps $Y \to S^n$. That is, for $\gamma \in H^n(Y, \mathbb{Z})$ corresponds to a map $f_\gamma : Y \to S^n$ (we can pullback f_γ back to $\gamma = f_\gamma^*(e_n)$).

If we send $\gamma \in H^n(X)$ to $i^*\gamma \in H^n(X^{(n)})$, then since we have $i^*\gamma = f^*(e_n)$, we have $i^* \mathrm{Sq}^0(\gamma) = \mathrm{Sq}^0(i^*\gamma) = \mathrm{Sq}^0(f^*(e_n)) = f^*(\mathrm{Sq}^0(e_n)) = f^*(e_n) = i^*\gamma$. ☐

Theorem 23.6 (Cartan Formula)**.** Let $V^*, W^* \in \mathbf{coCh}$, then for $x \in H^n(V^*)$, $y \in H^m(W^*)$, we have

$$f\overline{Sq}^k(x \otimes y) = \sum_{i+j=k} \overline{Sq}^i(x) \otimes \overline{Sq}^j(y) \in H^{n+m+k}(D_2V^* \otimes D_2W)$$

where

$$\begin{tikzcd} D_2(V \otimes W) = (V \otimes W \otimes V \otimes W \otimes E\Sigma_2)_{\Sigma_2} \arrow[rr] \arrow[dr] & & D_2V \otimes D_2W = (V \otimes V \otimes E\Sigma_2)_{\Sigma_2} \\ & (V \otimes W \otimes V \otimes W \otimes E\Sigma_2 \otimes E\Sigma_2)_{\Sigma_2 \times \Sigma_2} \arrow[ur] & \end{tikzcd}$$

To define this, note that there is a diagonal map $\Sigma_2 \to \Sigma_2 \times \Sigma_2$, sending a projective resolution $E\Sigma_2$ to the tensor product over itself, which gives the resolution $E\Sigma_2 \otimes E\Sigma_2 \cong E(\Sigma_2 \times \Sigma_2)$.

Proof. Consider $x : \mathbb{F}_2[n] \to V^*$ and $y : \mathbb{F}_2[m] \to W^*$, wwe assume $V = \mathbb{F}_2[n]$ and $W = \mathbb{F}_2[m]$, then $\overline{Sq}^i x \in H^{n+i}(D_2\mathbb{F}_2[n])$ is the generator, and so

$$f : D_2(\mathbb{F}_2[n] \otimes \mathbb{F}_2[m]) \cong \mathbb{F}_2[n+m] \to D_2\mathbb{F}_2[n] \otimes D_2\mathbb{F}_2[m]$$

on cohomology, we have $f(e_k^{n+m}) = \sum_{i+j=k} e_i^n \otimes e_j^m$. ☐

24 Oct 16, 2023

Let M be a topological manifold, then every $m \in M$ has an open neighborhood U such that $U \cong \mathbb{R}^n$, Examples include $\mathbb{R}^n$ and $S^n \subseteq \mathbb{R}^{n+1}$. We say M is closed if M is compact, i.e., $\partial M = \varnothing$.

We use the following notation: let A be a subset of X, we write $(X \mid A) = (X, X \backslash A)$. For $A \subseteq B \subseteq X$, we write $j_A^B : (X \mid B) \hookrightarrow X(\mid A)$, so this induces $H^*(X \mid A) \to H^*(X \mid B)$. In particular, if $B = X$, then $H^*(X \mid A) \to H^*(X)$.

Example 24.1. Let $x \in U \subseteq M$, then $H_i(M \mid x) \cong H_i(U \mid x)$ by excision on $M \backslash U$, which is then isomorphic to $H_i(\mathbb{R}^n \mid 0) = \begin{cases} \mathbb{Z}, & i = n \\ 0, & i \neq n \end{cases}$.

Definition 24.2 (Orientation)**.** An orientation of M at a point $x \in M$ is a choice of a generator of $H_n(M \mid x) \cong \mathbb{Z}$. Note that there are exactly two generators of $\mathbb{Z}$, so the set of orientations of M at x is $\mathrm{Or}(M \mid x) \subseteq H_n(M \mid x)$, with two elements.

More generally, let R be a commutative ring, an R-orientation is just a generator of $H_n(M \mid x; R) \cong R$ as an R-module. Then $\mathrm{Or}(M \mid x; R) \cong R^\times$, the units of R.

Example 24.3. Let $R = \mathbb{F}_2$, then the set of orientation is a singleton.

Let $\pi : \mathrm{Or}(M) = \coprod_{x \in M} \mathrm{Or}(M \mid x) \to M$ be the natural projection, with $\pi^{-1}(x) = \mathrm{Or}(M \mid x)$. We will topologize this map. Also, we have $\pi : M_{\mathbb{Z}} = \coprod_{x \in M} H_n(M \mid x) \to M$ where $\mathrm{Or}(M) \subseteq M_{\mathbb{Z}}$.

Definition 24.4. We say $U \subseteq M$ is a small Euclidean neighborhood if there is $U \subseteq V$ where V is open and there is a homeomorphism $\varphi : V \cong \mathbb{R}^n$ such that $U = \varphi^{-1}(\mathrm{int}(D^n))$.

Example 24.5. $S^n \backslash \{*\} \subseteq S^n$ is not a small Euclidean neighborhood.

Remark 24.6. If $U \subseteq M$ is a small Euclidean neighborhood, then $\bar{U} \cong D^n$ is contained in some Euclidean neighborhood of M. Therefore, the small Euclidean neighborhoods form a basis for the topology on M.

Lemma 24.7. Let $x \in U \subseteq M$ be a small Euclidean neighborhood of $x \in M$, then we have an isomorphism $j_x^U : H_n(M \mid U) \cong H_n(M \mid x)$.

Proof. Choose $V \supseteq V$, then $\varphi : V \to \mathbb{R}^n$ is an isomorphism, so $U = \varphi^{-1}(\mathrm{int}(D^n))$, and $\varphi^{-1}(x) = 0$. We have

$$\begin{tikzcd}
(M \mid U) \arrow[d] & (V \mid U) \arrow[l, "H_* \cong"'] \arrow[r, "\cong"] \arrow[d] & (\mathbb{R}^n \mid \mathrm{int}(D^n)) \arrow[d, "\cong"] \\
(M \mid x) & (V \mid x) \arrow[l, "H_* \cong"] \arrow[r, "\cong"'] & (\mathbb{R}^n \mid 0)
\end{tikzcd}$$

so all maps are H_*-isomorphisms. ∎

For a small Euclidean neighborhood $U \subseteq M$ and any $\alpha \in H_n(M \mid U)$ with any $x \in U$ gives $j_x^U(\alpha) \in H_n(M \mid x)$.

Definition 24.8. Define $U_\alpha = \{j_x^U(\alpha) \mid x \in U\} \subseteq M_{\mathbb{Z}} = \coprod_{x \in M} H_n(M \mid x)$, then $U_\alpha \subseteq \pi^{-1}(U)$.

Claim 24.9. $\{U_\alpha\}$ is a basis of a topology on $M_{\mathbb{Z}}$.

Proof.

- Each point of $M_{\mathbb{Z}}$ is in some U_α;
- If $(x, \alpha_x) \in U'_{\alpha'} \cap U''_{\alpha''}$, then $j_x^{U'}(\alpha') = \alpha_x = j_x^{U''}(\alpha'')$. Let $V \ni x$ be a small Euclidean neighborhood in $U' \cap U''$, and let $\beta = (j_x^V)^{-1}(\alpha_x)$, then $\beta = j_V^{U'}(\alpha') = j_V^{U''}(\alpha'')$.

∎

Proposition 24.10. $\pi : M_{\mathbb{Z}} \to M$ is a covering with fiber $\mathbb{Z}$.

Proof. For a small Euclidean neighborhood $U \subseteq M$, we have

$$\begin{tikzcd} \pi^{-1}(U) \arrow[d, "\pi"'] & U \times H_n(M \mid U) \arrow[l, "\cong"'] \arrow[dl, "\text{proj}"] \\ U & \end{tikzcd}$$

where the isomorphism sends $(x, \alpha) \mapsto j_x^U(\alpha)$. □

This gives $\mathrm{Or}(M) \subseteq M$ a subspace topology, so $\pi : \mathrm{Or}(M) \to M$ is a 2-fold covering map.

Definition 24.11. An orientation of M is a section $s : M \to \mathrm{Or}(M)$ of π. (We say M is orientable if such a section exists.)

Example 24.12. A section $M \to M_{\mathbb{Z}}$ amounts to giving $\alpha_x \in H_n(M \mid x)$ which varies in a continuous way.

25 Oct 18, 2023

Lemma 25.1. Suppose M is connected, then M is orientable if and only if $\mathrm{Or}(M) = M \coprod M$, and M is not orientable if and only if $\mathrm{Or}(M)$ is connected, that is, for all $x \in M$, there exists a continuous path between (x, α) to $(x, -\alpha)$ where α is a generator; that path obstructs the existence of a section.

Remark 25.2. If M is orientable, the same argument says that $M_{\mathbb{Z}} \to M$ gives a splitting $M_{\mathbb{Z}} \cong \coprod_{k \in \mathbb{Z}} M = M \times \mathbb{Z}$. If M is not orientable, we have $M_{\mathbb{Z}} \cong M \coprod \coprod_{m \geqslant 1} \mathrm{Or}(M)$.

Example 25.3. Let M be the Möbius band, i.e., as $S^1 \times \mathbb{R}/\sim$ where $(x, y) \sim (-x, -y)$, then $M \to S^1$ corresponds to a 2-fold covering space $\mathrm{Or}(M) \cong S^1 \times \mathbb{R}$ to the choice of $(x, y) \sim (-x, -y)$.

Example 25.4. Suppose $M = \mathbb{R}P^2 \cong S^2/x \sim -x$, then there is a mapping $S^2 = \mathrm{Or}(\mathbb{R}P^2) \to M = \mathbb{R}P^2$.

Remark 25.5. For any M, $\mathrm{Or}(M)$ is an orientable manifold.

We know $\mathrm{Or}(M, x; R)$ is the set of generators of $H_n(M \mid x; R)$, then for $\pi : M_R \to M$ we have preimage $\pi^{-1}(x) = H_n(M \mid x; R) \cong H_n(M \mid x) \otimes R$. Therefore $\mathrm{Or}(M; R) \subseteq M_R$ is just $\pi^{-1} \cong R^\times$. Hence, $\mathrm{Or}(M; \mathbb{Z}/2\mathbb{Z}) = M$. Therefore, $M_{\mathbb{Z}/2\mathbb{Z}} \cong M \coprod M$, so R-orientations of M correspond to sections of $\mathrm{Or}(M; R) \to M$.

Let $\Gamma(M, M_R)$ be the set of continuous sections of $\pi : M_R \to M$, then this is an R-module. This gives a mapping

$$\begin{aligned} H_n(M; R) &\to \Gamma(M, M_R) \\ \alpha &\mapsto s_\alpha(x) = j_x^M(\alpha) \in H_n(M \mid x; R). \end{aligned}$$

More generally, if $A \subseteq M$, then the sections $\Gamma(A, M_R)$ is the set of sections $s : A \to M_R$ such that the diagram

$$\begin{tikzcd} & M_R \arrow[d] \\ A \arrow[ur, "s"] \arrow[r, hook] & M \end{tikzcd}$$

commutes. Then we have

$$H_n(M \mid A; R) \to \Gamma(A, M_R)$$

with the same formula, restricting elements to A. This corresponds to $(M, M \backslash A) \to (M, M \backslash \{x\})$ where $x \in A$.

Theorem 25.6 (Orientation)**.** Assuming $A \subseteq M$ is compact, then the map

$$f_A : H_n(M \mid A; R) \to \Gamma(A, M_R)$$

is an isomorphism.

Corollary 25.7. if M is a compact n-manifold, then

$$H_n(M;R) = \begin{cases} R, & M \text{ orientable, i.e., } M_R \cong \coprod_R M \\ R[2], & M \text{ not orientable} \end{cases}$$

where $R[2]$ means the 2-torsion in r, i.e., $r = -r$.

Remark 25.8. Fiberwise, we have

$$\begin{aligned} M_{\mathbb{Z}} &\to M_R \\ k &\mapsto k \otimes 1. \end{aligned}$$

In particular, given $r \in R$,

$$\begin{tikzcd} H_n(M \mid A) \arrow[r] \arrow[d, equal] & H_n(M \mid A, R) \\ H_n(M \mid A) \otimes \mathbb{Z} \arrow[r, "\mathrm{id} \otimes r"'] & H_n(M \mid A) \otimes R \end{tikzcd}$$

Therefore this defines $r : M_{\mathbb{Z}} \to M_R$. If M is orientable, then $M_{\mathbb{Z}} \cong \coprod_{\mathbb{Z}} M$, and if M is not orientable, then $M_{\mathbb{Z}} = M \coprod \coprod_{\mathbb{Z}} \mathrm{Or}(M)$. In particular, if $2r \neq 0$, then $r(\mathrm{Or}(M)) \simeq \mathrm{Or}(M)$; if $2r = 0$, then $r : \mathrm{Or}(M) \xrightarrow{\pi} M \hookrightarrow M_R$.

26 Oct 23, 2023

Recall that if M is an n-dimensional manifold, $M_{\mathbb{Z}} = \coprod_{x \in M} H_n(M \mid x) \cong \coprod_{x \in M} \mathbb{Z} \to M$. Note that $M_{\mathbb{Z}} \supseteq \mathrm{Or}(M)$.

Recall that if $U \subseteq M$ is a small Euclidean neighborhood, then for $x \in U$, we have $j_x^U : H_n(M \mid U) \cong H_n(M \mid x)$. We can topologize $M_{\mathbb{Z}}$ by $U_\alpha = \{j_x^U(\alpha) \mid x \in U\}$ open subsets which form a basis for a topology on $M_{\mathbb{Z}}$.

Proposition 26.1. $\pi : M_{\mathbb{Z}} \to M$ is a covering.

Proof Idea. If U is a small Euclidean neighborhood, then

$$\begin{tikzcd} \pi^{-1}(U) \arrow[d] & U \times H_n(\mathbb{M} \mid U) \cong U \times \mathbb{Z} \arrow[l, "j_x^U(\alpha) \leftarrow (x,\alpha)"'] \arrow[dl, two heads] \\ U & \end{tikzcd}$$

Therefore M is orientable, i.e., there exists a section $s : M \to \mathrm{Or}(M)$ if and only if $\mathrm{Or}(M) \cong M \coprod M$. Similarly, M being connected but not orientable if and only if $\mathrm{Or}(M)$ is connected. ☐

We have $M_R \cong \coprod_{x \in M} H_n(M \mid x; R)$. For any closed subset $A \subseteq M$, we know $\Gamma(A, M_R)$, the sections $A \to M_R$ of

$$\begin{tikzcd} & M_R \arrow[d] \\ A \arrow[r, hook] \arrow[ur, dashed] & M \end{tikzcd}$$

contains $\Gamma_c(A, M_R)$, the sections with compact support, i.e.,

$$\begin{aligned} s : A &\to M_R \\ a &\mapsto (a, \alpha(a) \in H_n(M \mid a; R)) \end{aligned}$$

where $\alpha(a) = 0$ outside of a compact subset of A.

This induces a map

$$\begin{aligned} J_A : H_n(M \mid A; R) &\to \Gamma(A, M_R) \\ \alpha &\mapsto J_A(\alpha)(x) = j_x^A(\alpha) \in H_n(M \mid x). \end{aligned}$$

Claim 26.2. $\operatorname{im}(J_A) \subseteq \Gamma_c(A, M_R)$.

Proof. Let α be represented by $a \in C_n(M; R)$, then a is a finite sum of $\lambda_i\sigma_i$'s where $\sigma_i : \Delta^n \to M$ and $\lambda_\in R$. Therefore a is a chain on a compact subset of M, i.e., compact subset $B = \bigcup_i \operatorname{im}(\sigma_i)$. If $x \notin B$, then we need $J_A(\alpha)(x) = (x, 0)$, i.e., $j_x^A(\alpha) = 0$. Since $x \notin B$, then $B \subseteq M \backslash \{x\}$, thus $\alpha \mapsto 0$ in $H_n(M \mid x; R)$. □

Theorem 26.3 (Orientation)**.** Let M be an n-manifold, and let $A \subseteq M$ be closed, then

1. $H_i(M \mid A; R) = 0$ for $i > n$.
2. $J_A : H_n(M \mid A; R) \xrightarrow{\cong} \Gamma_c(A, M_R)$.

Remark 26.4. Compare this to the fact that $H_n(M \mid U; R) \times U \cong \pi^{-1}(M)$ for all small Euclidean neighborhood U, so $\Gamma(U, M_R) \cong H_n(M \mid U; R)$.

Lemma 26.5. If $A, B \subseteq B$ are closed, and Theorem 26.3 holds for $A, B, A \cap B$, then it also holds for $A \cup B$.

Proof. By commutative diagrams, we have

$$\begin{tikzcd}
0 \arrow[r] & H_n(M \mid A \cup B) \arrow[r] \arrow[d, "J_{A\cup B}"] & H_n * M \mid A) \oplus H_n(M \mid B) \arrow[r] \arrow[d, "J_A \oplus J_B"', "\cong"] & H_n(M \mid A \cap B) \arrow[r] \arrow[d, "\cong"] & \cdots \\
0 \arrow[r] & \Gamma_c(A \cup B) \arrow[r] & \Gamma_c(A) \oplus \Gamma_c(B) \arrow[r] & \Gamma_c(A \cap B) \arrow[r] & \cdots
\end{tikzcd}$$

(since $H_{n+1}(M \mid A \cap B) = 0$) and apply five lemma. □

Proposition 26.6. Let $A_1 \supseteq A_2 \supseteq \cdots$ be a decreasing sequence of compact subsets of M and $A = \bigcap_{i \geqslant 1} A_i$; if the theorem holds for each of the A_i's, then it holds for A.

Proof. Consider $(M \mid A_1) \subseteq (M \mid A_2) \subseteq \cdots$ and we get a commutative diagram

$$\begin{tikzcd}
\operatorname{colim}_i H_n(M \mid A_i) \arrow[r, "\cong"] \arrow[d, "\cong"'] & H_n(M \mid A) \arrow[d] \\
\operatorname{colim}_i \Gamma_c(A_i) \arrow[r, "\cong"'] & \Gamma_c(A)
\end{tikzcd}$$

□

27 Oct 25, 2023

Recall that $J_A(\alpha)(x) = j_x^A(\alpha) \in H_n(M \mid x)$. One can show that $J_A(\alpha)$ has compact support and $J_A(\alpha)$ is continuous.

To see that it is continuous, let $a \in C_n(M; R)$ for $[a] = \alpha$ and $\sum \lambda_i\sigma_i\partial a \in C_n(M\backslash A; R)$, where $\sigma_i : \Delta^n \to M\backslash A$. Let $B = \bigcup \operatorname{im}(\sigma_1) \subseteq M\backslash A$ as a union of compact sets. Let $x \in A$, then it has a small Euclidean neighborhood U such that $B \subseteq M\backslash\bar{U}$, then $\partial a \in C_n(M\backslash\bar{U})$, therefore $\beta = [a] \in H_n(M \mid \bar{U})$. The map $U \to \pi^{-1}(U) \cong U \times H_n(M \mid U)$ sends x to (x, β), this implies continuity.

We will now prove the Orientation theorem through the following steps:

1. for $M = \mathbb{R}^n$, A compact and convex;
2. for $M = \mathbb{R}^n$, A is a finite union of compact and convex;
3. for $M = \mathbb{R}^n$, A compact;
4. for any M, A a finite union of compact subsets contained in the Euclidean neighborhood;
5. for any M, A compact;
6. for any M, A closed.

To prove the first part, we rescale and translate the area, so we can assume $A \subseteq D^n \subseteq \mathbb{R}^n = M$. Then $S^{n-1} = \partial D^{n-1} \subseteq \mathbb{R}^n \backslash A$ via $\frac{x}{||x||} \leftarrow\!\shortmid x$.

$$\begin{tikzcd} H_n(\mathbb{R}^n \mid 0) \cong H_n(\mathbb{R}^n \mid D^n) \arrow[r, "\cong"] \arrow[d, "J_{D^n}"', "\cong"] & H_n(\mathbb{R}^n \mid A) \arrow[d, "J_A"] \\ \Gamma_c(D^n, M_R) \arrow[r, "\cong"'] & \Gamma_c(A, M_R) \end{tikzcd}$$

where $\pi^{-1}(D^n) \cong D^n \times R$ gives J_{D^n}, therefore J_A is an isomorphism as well.

To prove the second part, consider Lemma 26.5 and

Lemma 27.1. Let $A_1 \supseteq A_2 \supseteq \cdots$ be a decreasing sequence of compact subsets of X (Hausdorff). If U is open and contains $A = \bigcap_{i \geqslant 1} A_i$, then $A_i \subseteq U$ for some i.

Proof. Since A is compact, and $\bigcap_{i\geqslant 1}(A_i \backslash (A_i \cap U)) = \varnothing$, then we have a sequence

$$A_1 \backslash (A_1 \cap U) \supseteq A_2 \backslash (A_2 \cap U) \supseteq \cdots$$

and therefore $A_i \subseteq U$. $\square$

Proposition 27.2. Let $A_1 \supseteq A_2 \supseteq \cdots$ be a decreasing sequence of compact subsets of M such that the orientation theorem holds for each A_i, then it holds for $A = \bigcap A_i$.

Proof. Recall $(M \mid A) = \bigcup (M \mid A_i)$, so we have

$$\begin{tikzcd} \operatorname{colim}_i H_n(M \mid A_i) \arrow[r, "\cong"] \arrow[d, "\operatorname{colim} J_{A_i}"', "\cong"] & H_n(M \mid A) \arrow[d, "J_A"] \\ \operatorname{colim} \Gamma_c(A_i, M_R) \arrow[r, "\cong"'] & \Gamma_c(A, M) \end{tikzcd}$$

To see the bottom map is an isomorphism, let $s \in \Gamma_c(A, M_R)$, cover A by finitely many small Euclidean neighborhoods. Suppose their union is U, then s extends to U as well (uniquely). By lemma, we have $A_i \subseteq U$, so s is in the image from $\Gamma_c(A_i)$, so the map is onto. Injectivity follows from a similar argument. $\square$

This means the first step of the proof implies the second step of the proof.

Let A be compact and $M = \mathbb{R}^n$, set the closed balls $B_{\frac{1}{j}}(x)$ over x, then $A \subseteq \bigcup_{x \in C_j} B_{\frac{1}{j}}(x)$ where $C_j \subseteq A$ are finite subsets. Let $A_k = \bigcap_{i=1}^{k} \left(\bigcup B_{\frac{1}{j}}(x) \right)$, then we can build A using finite union of compact subsets A_k's.

28 Oct 27, 2023

Recall that we have proved the first three steps of the theorem. Now let M be arbitrary and let $A = \bigcup_{i=1}^{r} A_i$ where $A_i \subseteq U_i \cong \mathbb{R}^n$ for Euclidean neighborhoods U_i's of A_i's. To do this, we need excision to show that this holds for $\mathrm{Or}(M, A_i)$, then recall that if the statement holds for $\mathrm{Or}(M, A)$, $\mathrm{Or}(M, B)$, $\mathrm{Or}(A \cap B)$, then $\mathrm{Or}(A \cup B)$. This proves step 4. To prove step 5, we want to show this for arbitrary M and compact $A \subseteq M$. This is the same proof as step 3 (from step 2), based on our result of step 4. This uses Proposition 27.2. Finally, to prove step 6, we need to show A being compact implies A being closed, which is proven in Bredon.

Corollary 28.1. Let M be a compact manifold, then $H_n(M; R) \cong \Gamma(M; M_R) \cong \begin{cases} R, M \text{ orientable} \\ R[2], M \text{ not orientable} \end{cases}$.

Definition 28.2. Let X be a topological space and R be a commutative rings, any (co)chain or (co)homology is denoted with R-coefficients. There is a pairing $\frown$: $C^p(X) \otimes C_n(X) \to C_{n-p}(X)$ for $f \in C^p(X)$ and $C \in C_n(X)$, defined by

$$C_n(X) \xrightarrow{\text{diag}_*} C_n(X \times X) \xrightarrow{AW} \bigoplus_{i+j=n} C_i X \otimes C_j X \xrightarrow{\pi} C_{n-p}(X) \otimes C_p X \xrightarrow{\text{id} \otimes f} C_{n-p}(X) \cong C_{n-p}(X) \otimes_R R$$

This sends $c \in C_n(X)$ to a class $f \cap C \in C_{n-p}(X)$.

Explicitly, let $C = \sigma$ for $\sigma : \Delta^n \to X$, then

$$\begin{aligned} f \frown \sigma &= (1 \otimes f)(AW \circ \text{diag} \circ \sigma) \\ &= (1 \otimes f)(\sum \sigma|_{[v_0,\ldots,v_i]} \otimes \sigma|_{[v_i,\ldots,v_n]}) \\ &= (-1)^{(n-p)p} f(\sigma|_{[v_{n-p},\ldots,v_n]})\, \sigma|_{[v_0,\ldots,v_{n-p}]} \end{aligned}$$

where $f(\sigma|_{[v_{n-p},\ldots,v_n]})$ gives a Kronecker pairing

$$\begin{aligned} \langle -,- \rangle : C^p(X) \otimes C_p(X) &\to R \\ \langle f, \sigma \rangle &\mapsto f(\sigma) \end{aligned}$$

Lemma 28.3. For $C \in C_n$, $f \in C^p$, and $g \in C^{n-p}$, we have $\langle f \smile g, C \rangle = \langle f, g \frown c \rangle$.

Proof. Check explicitly for $C = \sigma$. □

Lemma 28.4. $\partial(f \frown C) = \delta f \frown C + (-1)^{|f|} f \frown \partial C$.

Corollary 28.5. We have

$$\begin{aligned} \frown: H^p(X) \otimes H_n(X) &\to H_{n-p}(X) \\ [f] \frown [c] &\mapsto [f \frown C] \end{aligned}$$

that is natural, i.e., for $\varphi : X \to Y$, we have

$$\varphi_*(\varphi^*([f]) \frown [c]) = [f] \frown \varphi_*[c].$$

Proof of Lemma. Let $C \in C_n$ and $f \in C^p$, then we have a diagram

$$\begin{array}{ccc} C_n \otimes C_p & \xrightarrow{\text{id} \otimes f} & C_n \\ {\scriptstyle \partial \otimes 1}\downarrow & & \downarrow{\scriptstyle \partial} \\ C_{n-1} \otimes C_p & \xrightarrow[\text{id} \otimes f]{} & C_{n-1} \end{array}$$

that commutes up to multiplication of $(-1)^p$. Therefore

$$\partial(\text{id} \otimes f)(c_n \otimes c_P) = \partial(C_n \otimes f(c_p)) = (-1)^{pn} f(c_p) \partial c_n$$

and so

$$\begin{aligned} (\text{id} \otimes f)(\partial \otimes \text{id}(c_n \otimes c_p) &= (1 \otimes f)(\partial c_n \otimes c_p) \\ &= (-1)^{(n-1)p} \partial c_n \otimes f(c_p) \\ &+ (-1)^{(n-1)p} f(c_p) \partial c_n. \end{aligned}$$

This gives

$$\begin{aligned} \partial(f \frown c) &= \partial((1 \otimes f)(AW \circ \text{diag} \circ c)) \\ &= (-1)^p (1 \otimes f)(\partial \otimes 1)(AW \circ \text{diag} \circ c) \\ &= (-1)^p ((1 \otimes f)\, \text{diag}(\partial c) + (-1)^p (1 + \delta f)\, \text{diag}(c)). \end{aligned}$$

□

29 Nov 1, 2023

Remark 29.1.
1. $(f \smile g) \frown c = f \smile g \frown c)$,
2. $1 \frown c = c$,
3. for any $\alpha \in H^*X$ and $\beta \in H^*Y$, and $a \in H_*X$ and $b \in H_*Y$, then $(\alpha \times \beta) \frown (a \times b) = (-1)^{|\beta||a|}(\alpha \frown a) \times (\beta \frown b)$, where $a \times b$ is given by the inverse of Alexander-Whitney map, i.e.,
$$\begin{aligned} C_*X \otimes C_*Y &\to C_*(X \times Y) \\ a \otimes b &\mapsto a \times b. \end{aligned}$$
4. Relative version for open/good pairs:
$$\frown : H^p(X, A) \otimes H_n(X, A \smile B) \to H_{n-p}(X, B).$$

Definition 29.2. Let M be an n-manifold, then a fundamental class for M is $[M] \in H_n(M, R)$ such that for any $x \in M$, $j_x^M[M] \in H_n(M \mid x; R) \cong R$ is an R-module generator.

Remark 29.3. If $[M]$ is a fundamental class, then we have a continuous section
$$\begin{aligned} s : M &\to \mathrm{Or}(M, R) \subseteq M_R \\ x &\mapsto [M]_x. \end{aligned}$$
Therefore M is orientable. By the orientation theorem, if M is a closed (compact) manifold, then $H_n(M; R) \cong \Gamma(M; M_R)$. If so, and suppose M is connected, then $H_n(M; R) \cong R$, therefore we get a fundamental class $[M]$.

Theorem 29.4 (Poincaré Duality)**.** Let M be a compact, closed, and oriented (i.e., orientable and choosing a continuous section of $\mathrm{Or}(M; R)$, which is therefore equivalent to choosing a fundamental class $[M]$) n-manifold, with a fundamental class $[M]$, then
$$- \frown [M] : H^p(M; R) \xrightarrow{\cong} H_{n-p}(M; R)$$
is an isomorphism for all p.

Definition 29.5. Denote $C^p = \mathrm{Hom}(C_p, R)$, then we say $f \in C^p(M)$ is supported on $K \subseteq M$ if for all $\sigma : \Delta^p \to M \backslash K$, we have $f(\sigma) = 0$, i.e., $f \in C^p(M, M \backslash K)$, since on the level of cochains there is the exact sequence
$$0 \longrightarrow C^p(M, M\backslash K) \longrightarrow C^p(M) \longrightarrow C^p(M\backslash K) \longrightarrow 0$$
We say f is compactly supported if it is supported on some compact $K \subseteq M$.

Definition 29.6. Denote $C_c^p(M; R) = \mathrm{colim}_{\text{compact } K} C^q(M, M\backslash K) := \bigcup_{\text{compact } K \subseteq M} C^p(M, M\backslash K) \subseteq C^p(M)$. This is a subcomplex of $C^*(M)$. We denote $H_c^*(M; R) = H^*(C_c^*(M; R))$ to be the cohomology of this complex.

Remark 29.7. If $\{K_\alpha\}$ is a collection of compact subsets of M such that any compact $K \subseteq K_\alpha$ for some α, then
$$C_c^*(M) \cong \mathrm{colim}_{K_\alpha} C^*(M, M\backslash K_\alpha),$$
and so $H_c^*(M) \cong \mathrm{colim}_{K_\alpha} H^*(M, M\backslash K_\alpha)$.

Example 29.8. Let $M = \mathbb{R}^n$, then $H_c^*(\mathbb{R}^n) \cong \mathrm{colim}_i H^i(\mathbb{R}^n, \mathbb{R}^n \backslash B_i(0)) \cong \mathrm{colim}_i H^*(\mathbb{R}^n, \mathbb{R}^n \backslash \{0\}) \cong \tilde{H}^*(S^{n-1})$, where $B_i(0)$ is the ball centered at 0 with radius i.

Example 29.9. If M is compact, then this is just the ordinary cohomology, i.e., the statement is true vacuously.

Remark 29.10. If $f : X \to Y$ is a continuous map, we do not have an induced map on H_c^*. However,
1. if f is proper, i.e., the preimage of a compact set is compact, then this gives a map $f^* : H_c^*(Y) \to H_c^*(X)$;
2. if $i : U \hookrightarrow X$ is an inclusion of an open subset where X is Hausdorff, then we get $i_! : H_c^*(U) \to H_c^*(X)$ as extension by zero, and $H_c^*(U) = \mathrm{colim}_{K \subseteq U} H^*(U, U\backslash K) \cong \mathrm{colim}_{K \subseteq U} H^*(X, X\backslash K)$ by excision $\overline{X\backslash U} \subseteq K\backslash K = \mathrm{int}(X\backslash K)$. This induces a map $i_! : \mathrm{colim}_{K \subseteq U} H^*(X, X\backslash K) \to \mathrm{colim}_{L \subseteq X} H^*(X, X\backslash L)$ from the colimit. Note that if $j : V \hookrightarrow U$ and $i : U \hookrightarrow X$ are open inclusions, then $(i \circ j)_! = i_! \circ j_!$.

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Proposition 30.1. Suppose $X = U \cup V$, then we have Mayer-Vietoris sequence

$$\cdots \longrightarrow H_c^p(U \cap V) \longrightarrow H_c^p(U) \oplus H_c^p(V) \longrightarrow H_c^p(X) \longrightarrow H_c^{p+1}(U \cap V) \longrightarrow \cdots$$

Proof. Let $K \subseteq U$ and $L \subseteq V$ be compact subsets, then we have an exact sequence

$$\cdots \longrightarrow H^p(X, X\backslash(K \cap L)) \longrightarrow H^p(X, X\backslash K) \oplus H^p(X, X\backslash L) \longrightarrow H^p(X, X\backslash(K \cup L)) \longrightarrow \cdots$$

Let $\mathcal{K}_U \times \mathcal{K}_V = \{(K, L) \mid \text{compact } K \subseteq U, \text{ compact } L \subseteq V\}$ as a directed system, take $\operatorname{colim}_{\mathcal{K}_U \times \mathcal{K}_V}(*)$ and we get an exact sequence

$$\cdots \to \operatorname{colim} H^*(X, X\backslash(K \cap L)) \to \operatorname{colim} H^*(X, X\backslash K) \oplus H^*(X, X\backslash L) \to \operatorname{colim} H^*(X, X\backslash(K \cup L)) \to \cdots$$

by taking the colimits over $\mathcal{K}_U \times \mathcal{K}_V$. Now colimits distribute over direct sum, so the middle term is just

$$\operatorname{colim}_{\mathcal{K}_U} H^*(X, X\backslash K) \oplus \operatorname{colim}_{\mathcal{K}_V} H^*(X, X\backslash L)$$

Definition 30.2. A map of directed systems $\varphi : \mathcal{D} \to \mathcal{C}$ is final if for all $C \in \mathcal{C}$, there exists $D \in \mathcal{D}$ such that $C \leqslant \varphi(d)$, e.g., φ is surjective.

Therefore if $\varphi : \mathcal{D} \to \mathcal{C}$ is final, then $\operatorname{colim}_{\mathcal{D}} F \circ \varphi \cong \operatorname{colim}_{\mathcal{C}} F$ for any diagram $F : \mathcal{C} \to \mathcal{A}$.

Remark 30.3. This gives, for example, $\varphi_U : \mathcal{K}_U \times \mathcal{K}_V \to \mathcal{K}_U$, $\varphi_V : \mathcal{K}_U \times \mathcal{K}_V \to \mathcal{K}_V$, $\varphi_{U\cap V} : \mathcal{K}_U \times \mathcal{K}_V \xrightarrow{\cap} \mathcal{K}_{U\cap V}$, and finally $\varphi_{U\cup V} : \mathcal{K}_U \times \mathcal{K}_V \xrightarrow{\cup} \mathcal{K}_{U\cup V}$ is surjective. Suppose K is compact in $U \cup V$, then $K = (K \cap U) \cup (K \cap V)$ where $K\backslash(K \cap U)$ and $K\backslash(K \cap V)$ are contained in a disjoint union of open neighborhoods (by some separation axioms). Therefore, there exists open $W \subseteq$ such that $K\backslash(K \cap U) \subseteq W \subseteq \bar{W} \subseteq K \cap V$, now let $K_V = \bar{W}$ and $K_U = K\backslash W$, then $K_V \cup K_U = K$ as a union of compact subset of U and compact subset of V.

Finally, we look at the Mayer-Vietoris for pairs, given by

$$\begin{tikzcd}
0 \arrow[r] & C_(A) \arrow[r] & C_*(X) \arrow[r] & C_*(X, A) \arrow[r] & 0 \\
0 \arrow[r] & C_(A) \arrow[r] \arrow[u] & C_*(X) \arrow[r] \arrow[u, equal] & C_*(X, A) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

then this gives the complex

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_*(A \cap B) \arrow[r] \arrow[d] & C_*(A) \oplus C_*(B) \arrow[r] \arrow[d] & C_*(A \cup B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*(X) \arrow[r] \arrow[d] & C_*(X) \oplus C_*(X) \arrow[r] \arrow[d] & C_*(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*(X, A \cap B) \arrow[r] \arrow[d] & C_*(X, A) \oplus C_*(X, B) \arrow[r] \arrow[d] & C_*(X, A \cup B) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

□

Let M be an n-manifold that is R-oriented, i.e., $s : M \to M_R$ is a section or gives a fundamental class.

Definition 30.4. Let $A \subseteq M$ be compact, then a fundamental class along A is $[M]_A \in H_n(M \mid A, R)$, such that $j_x^A([M]_A) \in H_n(M \mid x) \cong R$ is a generator of R-module structure, i.e., a local orientation at x.

Let $A \subseteq B$ be two compact subsets of M, with $i : (M \mid B) \hookrightarrow (M \mid A)$ and $i_* : H_n(M \mid B) \to H_n(M \mid A)$, then $i_*([M]_B)$ is a fundamental class along A. Therefore, we have a commutative diagram

$$\begin{tikzcd} H^p(M \mid A) \arrow[d, "i^*"'] \arrow[dr, "-\frown [M]_A"] & \\ H^p(M \mid B) \arrow[r, "-\frown [M]_B"'] & H_{n-p}(M) \end{tikzcd}$$

recalling that $\cap : H^p(X, Z) \otimes H_n(X, Z \cup W) \to H_{n-p}(X, W)$, so taking $W = \varnothing$ gives $\cap : H^p(X, Z) \otimes H_n(X, Z) \to H_{n-p}(X, \varnothing)$. Take the colimits over compact subsets of M, then we have a duality map

$$D_M : H_c^p(M) \to H_{n-p}(M).$$