MATH 526 Notes

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Let X be a topological space with basepoint $x_0 \in X$. We already know two invariants,

- the fundamental group $\pi_1(X, x_0)$, and
- the homology groups $H_n(X)$ for $n \geqslant 0$, which are abelian groups.

We will look at two more invariants,

- the cohomology groups $H^n(X)$ for $n \geqslant 0$, and
- the higher homotopy groups $\pi_n(X, x_0)$ for $n \geqslant 0$.

In particular, $\pi_*(X, x_0)$ is a very good invariant in the following sense:

Theorem 1.1 (Whitehead)**.** If $f : (X, x_0) \to (Y, y_0)$ is a map of CW-complexes, then f is a homotopy equivalence if and only if $\pi_*(f) : \pi_*(X, x_0) \to \pi_*(Y, y_0)$ is an isomorphism.

However, π_* is very hard to compute. On the other hand, $H^*(X)$ is relatively easy to compute, but this is not a complete invariant. For instance, $\mathbb{C}P^2$ and $S^2 \vee S^4$ have isomorphic cohomology groups, but they are not equivalent. $H^*(X)$ is closely related to $H_*(X)$, but $H^*(X)$ is a graded ring structure with cup product. It is contravariant in X, where $H_*(X)$ is covariant. The cup product is defined by the composition of induced diagonal map with an external product:

$$H^i(X) \times H^j(X) \xrightarrow{\times} H^{i+j}(X \times X) \xrightarrow{\Delta^*} H^{i+j}(X)$$

Other things we will talk about include:

- Natural transformations $H^i(-) \to H^j(-)$ encoded by Steenrod operations.
- $H^n(-)$ becomes a representable functor, i.e., $H^n(X) = [X, K(\mathbb{Z}, n)]$, where $K(\mathbb{Z}, n)$ is the Eilenberg-Maclane space, and the bracket indicates the homotopy classes of maps.
- Poincaré duality in $H^*(M)$ for compact manifold M, namely the cup product gives
$$H^i(M) \otimes H^{\dim(M)-i}(M) \xrightarrow{\smile} H^{\dim(M)}(M).$$
- Characteristic classes in $H^*(X)$ associated to vector bundles over X.

Recall for a topological space X, we obtain a collection of (singular) homology groups $H_n(X)$, with $H_*(X) = \bigoplus_{n \geqslant 0} H_n(X)$. The functoriality of morphisms says that $X \xrightarrow{f} Y \xrightarrow{g} Z$ induces $f_* g_* = (fg)_* : H_*(X) \xrightarrow{f_*} H_*(Y) \xrightarrow{g_*} H_*(Z)$. So

$$H_*(-) : \mathbf{Top} \to \mathbf{Ab}$$

is a well-defined functor. This factors into

$$\begin{tikzcd} \mathbf{Top} \arrow[rr, "H_*(-)"] \arrow[dr, "C_*(-)"'] & & \mathbf{Ab} \\ & \mathbf{Ch} \arrow[ur, "H_*(-)"'] & \end{tikzcd}$$

Here $C_*(-)$ is usually the singular chain, given by $\partial : C_n(X) \to C_{n-1}(X)$, where $C_n(X)$ is the free abelian group generated by $\mathrm{Hom}_{\mathbf{Top}}(\Delta^n, X) \cong \bigoplus_{\sigma:\Delta^n\to X} \mathbb{Z}\sigma$. $\Delta^n \subseteq \mathbb{R}^{n+1}$ is the set of tuples $(t_0, \ldots, t_n)$ such that the coordinates sum to 1. The boundary is $\partial\sigma = \sum_{0\leqslant i\leqslant n} (-1)^i \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$.

We say $C_*(-)$ is homotopy invariant, i.e., if $f : X \to Y$ is a homotopy equivalence, then the induced map $C_*(X) \to C_*(Y)$ on chain complexes is a chain equivalence.

Remark 1.2. $C_*^\Delta(X)$ and $C_*^{\mathrm{CW}}(X)$ are both chain equivalent to $C_*(X)$.

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Here is a list of properties of $C_*(-) : \mathbf{Top} \to \mathbf{Ch}$:

- Functoriality: given a continuous map $f : X \to Y$, there is an induced map
$$\begin{aligned} f_* : C_*(X) &\to C_*(Y) \\ (\sigma : \Delta^n \to X) &\mapsto (f\sigma : \Delta^n \to Y) \end{aligned}$$

- Homotopy invariance: given $f, g : X \to Y$ such that $f \simeq g$, i.e., there is $H : X \times [0,1] \to Y$ such that $H|_0 = f$ and $H|_1 = g$, then $f_* \simeq g_*$ as a chain homotopy equivalence, i.e., there exists maps $h_n : C_n(X) \to C_{n+1}(Y)$ making a diagram
$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(X) \arrow[r] \arrow[dl, "h"'] \arrow[d, shift right, "g"'] \arrow[d, shift left, "f"] & C_n(X) \arrow[r] \arrow[dl, "h"'] \arrow[d, shift right, "g"'] \arrow[d, shift left, "f"] & C_{n-1}(X) \arrow[r] \arrow[dl, "h"'] \arrow[d, shift right, "g"'] \arrow[d, shift left, "f"] & \cdots \\ \cdots \arrow[r] & C_{n+1}(Y) \arrow[r] & C_n(Y) \arrow[r] & C_{n-1}(Y) \arrow[r] & \cdots \end{tikzcd}$$
such that $f - g = \partial h + h\partial$. Therefore $f_* = g_* : H_*(X) \to H_*(Y)$.

  Remark 2.1. $f : A_* \to B_*$ is a chain equivalence if there exists $g : B_* \to A_*$ and $fg \simeq \mathrm{id}_B$ and $gf \simeq \mathrm{id}_A$, then $f_* : H_*(A_*) \to H_*(B_*)$ is an isomorphism, i.e., f is a quasi-isomorphism.

  Example 2.2. The complexes $A : 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to 0$ and $B : 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0$ gives a quasi-isomorphism $f : A \to B$ in the canonical way, but this is not a chain equivalence, since the backwards map has to be zero.

- Additivity: $C_*(\coprod_\alpha X_\alpha) \cong \bigoplus_\alpha C_*(X_\alpha)$.

- Excision: given a pair (X, A) with $Z \subseteq A$ such that $\bar{Z} \subseteq \mathrm{int}(A)$, then we have $C_*(X\backslash Z, A\backslash Z) \cong C_*(X, A)$.

- Mayer-Vietoris: given $A, B \subseteq X$, with $X = \mathrm{int}(A) \cup \mathrm{int}(B)$, then we have a short exact sequence
$$0 \longrightarrow C_*(A \cap B) \longrightarrow C_*(A) \oplus C_*(B) * C_*(X) \longrightarrow 0$$

The cochain complex is obtained via inverting the indices and maps δ from a chain complex. This induces a cohomology $H^*(C^*) = \ker(\delta)/\mathrm{im}(\delta)$ as the quotient of cocycles over coboundaries. Now $f : A^* \to B^*$ is a quasi-isomorphism if $f^* : H^*(A^*) \to H^*(B^*)$ is an isomorphism. Similarly, one can define the cochain homotopy equivalence.

Example 2.3. If $C_* \in \mathbf{Ch}$, and $k \in \mathbf{Ab}$, then we can form cochain complex $C_k^* := \mathrm{Hom}(C_*, k)$, where $C_k^n = \mathrm{Hom}_{\mathbf{Ab}}(C_n, k) \xrightarrow{\delta} C_k^{n+1}$ by sending $f : C_n \to k$ to $f\partial : C_{n+1} \to C_n \to k$.

- $\mathrm{Hom}(-,k) : \mathbf{Ch} \to \mathbf{coCh}$ is a functor.
- The functor preserves quasi-isomorphisms between chain complexes of free abelian groups.

Definition 2.4. For $k \in \mathbf{Ab}$, the singular cochains with coefficients in k is

$$\begin{tikzcd} C^*(-,k) : \mathbf{Top} \arrow[rr] \arrow[dr, "C_*(-)"'] & & \mathbf{coCh} \\ & \mathbf{Ch} \arrow[ur, "\mathrm{Hom}(-,k)"'] & \end{tikzcd}$$

The cohomology of X with coefficients in k is defined by $H^*(X;k) = H^*(C^*X, k)$. We have the convention $C^*(X) = C^*(X, \mathbb{Z})$.

Alternatively, we take the opposite categories $\mathbf{Top}^*$ and $\mathbf{Ch}^*$ so that the functors are viewed as covariant.

The corresponding map $\delta : C^n(X;k) \to C^{n+1}(X;k)$ is given by δf that maps $\sigma \in C_{n+1}(X)$ to $(-1)^{n+1} f(\partial\sigma)$. Although the cochains are in general the dual of chains, the cohomology is not going to be the dual of the homology.

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Recall:

$$\begin{tikzcd} \mathbf{Top}^{\mathrm{op}} \arrow[r, "C_*"] \arrow[rrr, bend left, "H^*(-,k)"] & \mathbf{Ch}^{\mathrm{op}} \arrow[r, "\mathrm{Hom}(-,k)"] & \mathbf{coCh} \arrow[r, "H^*"] & \mathbf{GrAb} \end{tikzcd}$$

Properties of $H^*(-,k) : \mathbf{Top} \to \mathbf{GrAb}$:

- Dimension:

  Claim 3.1. $H^i(\{*\}, k) = \begin{cases} 0, & i \neq 0 \\ k, & i = 0 \end{cases}$

  Proof. Note that each degree of cohomology is given the free abelian group generated by $\mathrm{Hom}(\Delta^n, \{*\})$, but the singleton set is the terminal object in the category of topological spaces, so there is always a unique generator, thus the chain complex is given by $\mathbb{Z}$'s on each degree $n \geqslant 0$.

  Now the generating map at degree n is $\sigma_n : \Delta^n \to \{*\}$, and see Homework 1 where we proved the homology. Now looking at $C^*(\{*\}, k)$, we have

  $$k \xrightarrow{0} k \xrightarrow{\cong} k \xrightarrow{0} k \longrightarrow \cdots$$

  and this gives the cohomology. ☐

- Homotopy: if $f \simeq g : X \to Y$, then $f^* = g^* : H^*(Y,k) \to H^*(X,k)$.

  Proof. We have $f_* = g_* : C_*X \to C_*Y$, and then $\mathrm{Hom}(f_*, k) \cong \mathrm{Hom}(g_*, k)$, so $H^*(-)$ is invariant under cochain homotopies. ☐

- Additivity: $H^*(\coprod_\alpha X_\alpha, k) \cong \prod_\alpha H^*(X_\alpha, k)$.

  Proof. We know that for chains there is $C_*(\coprod_\alpha X_\alpha) = \bigoplus_\alpha C_*(X_\alpha)$, so the cochain version says that $C^*(\coprod_\alpha, X_\alpha, k) \cong \mathrm{Hom}(\bigoplus_\alpha C_*(X_\alpha), k) \cong \prod_\alpha \mathrm{Hom}(C_*(X_\alpha), k) \cong \prod_\alpha C^*(X_\alpha)$ and $H^* : \mathbf{coCh} \to \mathbf{GrAb}$ commutes with the product. ☐

- Exactness: for a pair (X, A), there is a natural long exact sequence
$$\cdots \longrightarrow H^n(X, A; k) \longrightarrow H^n(X; k) \longrightarrow H^n(A; k) \longrightarrow \cdots$$

  Proof. We have a short exact sequence
$$0 \longrightarrow C_*A \longrightarrow C_*X \longrightarrow C_*(X, A) \longrightarrow 0$$
  where $C_*A \to C_*X$ is an inclusion of summands. Therefore, the quotient $C_*(X, A)$ is also a chain complex of free abelian groups. Therefore, taking the cochains also gives a short exact sequence. We then obtain a short exact sequence of cochain complexes
$$0 \longrightarrow C^*(X, A; k) \longrightarrow C^*(X; k) \longrightarrow C^*(A; k) \longrightarrow 0$$
  and can then apply cohomology functor. □

- Excision: given a pair (X, A) and Z such that $\bar{Z} \subseteq \text{int}(A)$, we have $H^*(X, A; k) \cong H^*(X \backslash Z, A \backslash Z; k)$.
- Mayer-Vietoris: given $A, B \subseteq X$ such that $\text{int}(A) \cup \text{int}(B) = X$, then we have a natural long exact sequence
$$\cdots \longrightarrow H^n(X; k) \longrightarrow H^n(A; k) \oplus H^n(B; k) \longrightarrow H^n(A \cap B; k) \longrightarrow \cdots$$

Definition 3.2. A functor $E^* : \textbf{Top}^{\text{op}} \to \textbf{GrAb}$ is called a generalized cohomology theory if it satisfies the four middle property (except the dimension property and Mayer-Vietoris).

Remark 3.3. If E^* also satisfies the dimension property, then E^* is naturally isomorphic to the cohomology $H^*(-; k)$. There are also other generalized cohomology theories like K-theory, cobordism, etc.

The Mayer-Vietoris becomes a consequence of the first five properties.

We will now try to use homological algebra to relate $H_*(X) = H_*(CX)$ and $H^*(X; k) = H^*(\text{Hom}(C_*X, k))$.

Definition 3.4. We say $C_*(X; k) \cong C_*(X) \otimes_{\mathbb{Z}} k$ and $H_*(X; k) \cong H_*(C_*X \otimes k)$ gives the singular homology of X with coefficients in k.

Lemma 3.5. $- \otimes k : \textbf{Ab} \to \textbf{Ab}$ is a right exact functor. $\text{Hom}(-, k) : \textbf{Ab}^{\text{op}} \to \textbf{Ab}$ is left exact.

Proof. Exercise. □

Remark 3.6. The covariant hom functor is also left exact.

Remark 3.7. The left adjoint is right exact, the right adjoint is left exact. In particular, we have the hom-tensor adjunction
$$\text{Hom}(A, \text{Hom}(B, C)) \cong \text{Hom}(A \otimes B, C).$$
Note that
$$\text{Hom}(A, \text{Hom}(B, C)) \cong \text{Hom}(A \otimes B, C) \cong \text{Hom}(B \otimes A, C) \cong \text{Hom}(B, \text{Hom}(A, C))$$

Example 3.8. Consider
$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$
Tensoring with $\mathbb{Z}/n\mathbb{Z}$, we do not have exactness.

Example 3.9.
$$0 \longrightarrow A \longrightarrow A \oplus C \longrightarrow C \longrightarrow 0$$
is always exact after tensoring $- \otimes k$ or applying the hom functor $\text{Hom}(-, k)$.

Definition 3.10. A short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ is split if any of the following equivalence conditions hold:

(i) p has a section $s : C \to B$ such that $ps = 1$;

(ii) i has a retraction $r : B \to A$ such that $ri = 1$;

(iii) $B \cong A \oplus C$, i.e.,

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, equal] & B \arrow[r, "p"] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, hook, "i"] & A \oplus C \arrow[r, two heads, "p"] & C \arrow[r] & 0 \end{tikzcd}$$

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We will prove that (ii) implies (iii).

Suppose $b \in B$, then $b = (b - irb) + irb$, which is a decomposition of elements in $\ker(r)$ and in $\operatorname{im}(i)$, respectively. Also, $\ker(r) \cap \operatorname{im}(i) = 0$, therefore $B = \ker(r) \cap \operatorname{im}(i)$. Since i is an inclusion, then $\operatorname{im}(i) \cong A$. Now $p : B \to C$ factors through the projection onto $\ker(r)$ since $ri = 0$. By restricting p onto $\ker(r)$, we see p is also injective, thereby an isomorphism.

Lemma 4.1. If we have a split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, bend left, "i"] & B \arrow[l, bend left, "r"] \arrow[r, bend left, "p"] & C \arrow[l, bend left, "s"] \arrow[r] & 0 \end{tikzcd}$$

then $- \otimes k$ and $\operatorname{Hom}(-, k)$ preserves the split exactness, i.e.,

$$0 \longrightarrow A \otimes k \longrightarrow B \otimes k \longrightarrow C \otimes k \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Hom}(C, k) \longrightarrow \operatorname{Hom}(B, k) \longrightarrow \operatorname{Hom}(A, k) \longrightarrow 0$$

The point is tensors and homs preserve retracts.

Proof.
- $(r \otimes \mathrm{id}_k)(i \otimes \mathrm{id}_k) = ri \otimes \mathrm{id}_k = \mathrm{id}_{A \otimes k}$, so $i \otimes \mathrm{id}_k$ is split injective.
- Similarly, $\operatorname{Hom}(i, \mathrm{id})$ is split surjective.

□

Example 4.2. Given a surjection $B \to C \to 0$ such that C is free abelian, then there is always a section $s : C \to B$ making the exact sequence split. (That is, C is projective.) That is, if $0 \to A \to B \to C \to 0$ is an exact sequence where C is free, then the sequence is split exact.

Definition 4.3. Let $C \in \mathbf{Ab}$. A free resolution of C is a chain complex of free objects

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

and an augmentation $F_0 \to C$, so that

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0$$

is acyclic, i.e., exact everywhere.

Example 4.4.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow 0$$

is a free resolution of $\mathbb{Z}/n\mathbb{Z}$. So is

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

as well as

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\mathrm{id} \oplus (\times n)} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0$$

Lemma 4.5. Any $C \in \mathbf{Ab}$ admits a free resolution, and moreover, it admits a resolution of length 1¡ given by

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0$$

Proof. Choose a surjection $p : F_0 \to C$ from a free abelian group F_0 to C. Let $F_1 = \ker(p)$, then F_1 is free, so we are done. □

Lemma 4.6. Free resolutions are essentially unique, i.e., if $F_. \to C$ and $F'_. \to C$ are free resolutions, then there is a quasi-isomorphism $F_. \xrightarrow{\sim} F'_.$ which commutes with the augmentations to C.

Definition 4.7. Let $C \in \mathbf{Ab}$ and let $F_. \to C$ be a free resolution, then we define the torsion groups to be $\operatorname{Tor}_n^{\mathbb{Z}}(C, k) = H_n(F_. \otimes k)$, and the ext groups to be $\operatorname{Ext}_{\mathbb{Z}}^n(C, k) = H^n(\operatorname{Hom}_{\mathbb{Z}}(F_., k))$.

Remark 4.8.

- Tor and Ext are independent of the choice of resolutions.
- $\operatorname{Tor}_n^{\mathbb{Z}}$ and $\operatorname{Ext}_{\mathbb{Z}}^n$ are zero for $n > 1$.
- $\operatorname{Tor}_n^{\mathbb{Z}}(C, k) \cong \operatorname{Tor}_n^{\mathbb{Z}}(k, C)$.
- $\operatorname{Tor}_0^{\mathbb{Z}}(C, k) \cong C \otimes k$.
- $\operatorname{Ext}_{\mathbb{Z}}^0(C, k) \cong \operatorname{Hom}(C, k)$.

Example 4.9.

- If C is free, then $\operatorname{Tor}_1(C, k) = \operatorname{Ext}^1(C, k) = 0$.
- $\operatorname{Tor}_1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.
- $\operatorname{Tor}_1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = 0$.
- $\operatorname{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.
- $\operatorname{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.
- $\operatorname{Ext}^1(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = 0$.

Proof. Look at

$$0 \longrightarrow F_1 = \mathbb{Z} \longrightarrow F_0 = \mathbb{Z} \longrightarrow C = \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

then $\operatorname{Tor}_*(\mathbb{Z}/p\mathbb{Z}, k) = H_*(F_1 \otimes k = k \xrightarrow{\times p} F_0 \otimes k = k) = \begin{cases} k[p], * = 0 \\ k/pk, * = 1 \end{cases}$. Here $k[p]$ denotes p-torsion subgroup of k. Moreover, $\operatorname{Ext}^*(\mathbb{Z}/p\mathbb{Z}, k) = H^*(\operatorname{Hom}(F_1, k) = k \xleftarrow{\times p} \operatorname{Hom}(F_0, k) = k) = \begin{cases} k[p], * = 0 \\ k/pk, * = 1 \end{cases}$. □

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Recall that cohomology are basically the dual of homology, where the difference originates from the failure of exactness of the hom functor.

Theorem 5.1 (Universal Coefficient Theorem)**.** Let C_* be a chain of free abelian groups and $k \in \mathbf{Ab}$, then there exists a natural short exact sequence

$$0 \longrightarrow \operatorname{Ext}^1(H_{n-1}(C_*), k) \longrightarrow H^n(\operatorname{Hom}(C_*, k)) \xrightarrow{h} \operatorname{Hom}(H_n(C_*), k) \longrightarrow 0$$

that splits in an unnatural sense.

Here we define $h \in \operatorname{Hom}(H^n(\operatorname{Hom}(C_*, k)), \operatorname{Hom}(H_n(C_*), k))$. Note that this hom set is isomorphic to the hom set $\operatorname{Hom}(H^n(\operatorname{Hom}(C_*, k)) \otimes H_n(C_*), k)$ via the tensor-hom adjunction. That is, h is given by a bilinear pairing $H^n(\operatorname{Hom}(C_*, k)) \times H_n(C_*) \to k$. We use the Kronecker pairing $([f], [x]) \mapsto f(x)$. To see this is well-defined, let $f \in \operatorname{Hom}(C_n, k)$ with $\delta f = 0$, for $x \in C_n$, we have $\partial x = 0$. Now replace x by $x + \partial y$, then $f(x + \partial y) = f(x) = f(\partial y) = f(x) \pm (\delta f)(y) = f(x)$. Also, replace f by $f + \delta(g)$ gives $(f + \partial g)(x) = f(x) + (\delta g)(x) = f(x) + g(\delta x) = f(x)$.

Lemma 5.2. h is a split surjection.

Proof. Write $C_k^* = \mathrm{Hom}(C_*, k)$. Now $h : \ker(\delta, C_k^n \to C_k^{n+1}) \to \mathrm{Hom}(H_n(C_*), k)$ via $h : f \mapsto (x \mapsto f(x))$, then we will construct a section of h via $\varphi \mapsto \tilde{\varphi}$. Let $Z_n = \ker(\partial)$ and $B_n = \mathrm{im}(\partial)$, then $H_n(C_*) = Z_n/B_n$, and the short exact sequence of free abelian groups

$$0 \longrightarrow Z_n \xrightarrow{i} C_n \longrightarrow B_{n-1} \longrightarrow 0$$

and this splits so $C_n \cong Z_n \oplus B_{n-1}$. Given $\varphi : H_n(C_*) \to k$, we have

$$C_n \xrightarrow{r} Z_n \longrightarrow Z_n/B_n \xrightarrow{\varphi} k$$

where r is the retraction to i, and we define the composition to be $\tilde{\varphi}$. Now the composition

$$C_{n+1} \xrightarrow{\partial} C_n \longrightarrow Z_n \longrightarrow Z_n/B_n \longrightarrow k$$

is still zero since $C_{n+1} \to Z_n$ is zero, but that means $\delta\tilde{\varphi}$ is also zero. □

We will now prove the universal coefficient theorem.

Proof. Since h is a split surjection, then we know this extends to a short exact sequence, hence we just need to identify the kernel of h, i.e., to show that $\ker(h) \cong \mathrm{Ext}^1(H_{n-1}(C_*), k)$. Given the split short exact sequence

$$0 \longrightarrow Z_n \xrightarrow{i} C_n \xrightarrow{\partial} B_{n-1} \longrightarrow 0$$

we have a diagram

$$\begin{tikzcd}
 & \vdots \arrow[d, "0"] & \vdots \arrow[d, "\delta"] & \vdots \arrow[d, "0"] & \\
0 \arrow[r] & \mathrm{Hom}(B_{n-1}, k) \arrow[r] \arrow[d, "0"] & \mathrm{Hom}(C_n, k) \arrow[r] \arrow[d, "\delta"] & \mathrm{Hom}(Z_n, k) \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \mathrm{Hom}(B_n, k) \arrow[r] \arrow[d, "0"] & \mathrm{Hom}(C_{n+1}, k) \arrow[r] \arrow[d, "\delta"] & \mathrm{Hom}(Z_{n+1}, k) \arrow[r] \arrow[d, "0"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

which is a short exact sequence of complexes. By the snake lemma, we have the long exact sequence of cohomology$\cdots \to H^n(B_k^{*-1}) \to H^n(C_k^*) \to H^n(Z_k^*) \to H^{n+1}(B_k^{*-1}) \to \cdots$. We claim that the connecting homomorphism $H^n(Z_k^*) \to H^{n+1}(B_k^{*-1})$ is $\mathrm{Hom}(B_n \subseteq Z_n, k)$. But $0 \to B^n \to Z^n \to H_n(C_*) \to 0$ is a free resolution of $H_n(C_*)$ of length 1. Then $H^*(\beta : \mathrm{Hom}(Z_n, k) \to \mathrm{Hom}(B_n, k)) = \mathrm{Ext}^*(H_n(C_*), k)$ where β has kernel $\mathrm{Hom}(H_n(C_*), k)$ and cokernel $\mathrm{Ext}^1(H_n(C_*), k)$. Therefore, the long exact sequence of cohomomology is the splicing (as epi-mono factorization) of

$$0 \longrightarrow \mathrm{coker}(\beta_{n-1}) \longrightarrow H_n(C_k^*) \longrightarrow \ker(\beta_n) \longrightarrow 0$$

and by identification we are done. □

Corollary 5.3. If $C_* \to C'_*$ is a quasi-isomorphism, then $\mathrm{Hom}(C'_*, k) \to \mathrm{Hom}(C_*, k)$ is a quasi-isomorphism.

Corollary 5.4. Let $X \in \mathbf{Top}$ and $A \subseteq X$, then there exists a short exact sequence

$$0 \longrightarrow \mathbf{Ext}^1(H_{n-1}(X, A), k) \longrightarrow H^n(X, A; k) \longrightarrow \mathbf{Hom}(H_n(X, A); k) \longrightarrow 0$$

which is natural in (X, A). This also splits in (X, A) in an unnatural way.

Theorem 5.5. If C_* is a chain complex of free abelian groups, then there is a short exact sequence

$$0 \longrightarrow H_n(C_*) \otimes k \longrightarrow H_n(C_* \otimes k) \longrightarrow \mathbf{Tor}_1(H_{n-1}(C_*, k) \longrightarrow 0$$

which is natural. It splits unnaturally.

Corollary 5.6. For any pair (X, A), there is a natural short exact sequence

$$0 \longrightarrow H_n(X, A) \otimes k \longrightarrow H_n(X, A; k) \longrightarrow \mathbf{Tor}_1(H_{n-1}(X, A), k) \longrightarrow 0$$

which splits in an unnatural way.

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Example 6.1. Take $X = \mathbb{C}P^2$, then the Tor and Ext terms go away, so the cohomology is equivalent to the homology.

Example 6.2. Take $X = \mathbb{R}P^2$, the Tor term gives $\mathrm{Tor}_1(\mathbb{Z}/2\mathbb{Z}, k) = k/2 \cong k[2]$, as the 2-torsion of k, i.e., the set of $a \in k$ such that $2a = 0$. Also, $\mathrm{Ext}^1(\mathbb{Z}/2\mathbb{Z}, k) = k/2k$.

Indeed, the Tor is given by the homology on multiplication by 2 map over k via tensor, and the Ext is given by the cohomology on multiplication by 2 map over k via hom.

Tor stands for torsion and Ext stands for extension.

Went on to talk about the limits and colimits.

Remark 6.3. In many abelian categories (and in particular, the category of abelian groups), we find a short exact sequence

$$0 \longleftarrow \mathrm{colim}_I \longleftarrow \bigoplus_{i \geqslant 0} X_i \longleftarrow \bigoplus_{i \geqslant 0} X_i \longleftarrow 0$$

and note that taking the dual version in the opposite category, we should obtain a sequence in the covariant sense. However, there is an asymmetry given by

$$0 \longrightarrow \lim_{I^{\mathrm{op}}} X \longrightarrow \prod_{i \geqslant 0} X_i \longrightarrow \prod_{i \geqslant 0} X_i \longrightarrow \lim_{I^{\mathrm{op}}}{}^1 X \longrightarrow 0$$

which is not short anymore. This is called a Milnor sequence.

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The colimit of the empty diagram is the initial object; dually, the limit of the empty diagram is the terminal object.

Definition 7.1. We say $X : I \to \mathscr{C}$ is a filtered diagram if

- $\mathrm{Ob}(\mathscr{C}) \neq \varnothing$,
- for all $i, j \in I$, there exists $k \in I$ and morphisms $i \to k$ and $j \to k$, and
- for parallel morphisms $a, b : i \to j$ in I, then there exists coequalizers.

Example 7.2. A poset (as a category) P is a directed set if for any $i, j \in P$, there exists $k \in P$ such that $i \leqslant k$ and $j \leqslant k$.

For a filtered diagram $X : I \to \mathbf{Set}$, the colimit $\mathrm{colim}_I X$ exists and is isomorphic to $\coprod_{i \in I} X_i/\sim$, where $x_i \in X_i$ and $x_j \in X_j$ are equivalent if for some $k \in I$, we have $a : i \to k$ and $b : j \to k$ and that $a(x_i) = b(x_j)$

For concrete categories, we forget the additional structure to the category of sets, and find the colimits there, and give it the additional structure we want.

Lemma 7.3. If I is a directed set, then

$$0 \longrightarrow \bigoplus_{i\in I} A_i \longrightarrow \bigoplus_{i\in I} A_i \longrightarrow \operatorname{colim}_{i\in I} A_i \longrightarrow 0$$

$$(a_i)_{i\in I} \longrightarrow (a_j - f_{ij}(a_i))$$

where $f_{ij} : i \to j$.

Example 7.4. The colimit of a sequence given by $A \xrightarrow{\times n} A$ is $A\left[\frac{1}{n}\right]$.

Lemma 7.5. Colimit functor is exact in category of abelian groups.