

# Motivic Cohomology Notes

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## 0 INTRODUCTION

Let  $X \in \mathbf{Sm}/k$  be a smooth separated scheme over a field  $k$ . The study of motivic cohomology started with the hope that

**Conjecture 0.1** (Beilinson and Lichtenbaum, 1982-1987). There exists some complexes  $\mathbb{Z}(n)$  for  $n \in \mathbb{N}$  of sheaves in Zariski topology on  $\mathbf{Sm}/k$  such that

1.  $\mathbb{Z}(0)$  is (quasi-isomorphic to) the constant sheaf  $\mathbb{Z}$ , i.e., the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$

concentrated at degree 0;

2.  $\mathbb{Z}(1)$  is the complex  $\mathcal{O}^*[-1]$ , i.e., the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{O}^* \longrightarrow 0 \longrightarrow \cdots$$

concentrated at degree 1;

3. for every field  $F/k$ , the hypercohomology over Zariski topology satisfies<sup>1</sup>

$$\mathbb{H}_{\mathrm{Zar}}^n(F, \mathbb{Z}(n)) = H^n(\mathbb{Z}(n)(\mathrm{Spec}(F))) = K_n^M(F),$$

where  $K_n^M(F)$  is the  $n$ th Milnor K-theory of a field  $F$ , given by the quotient of the tensor algebra  $T(F^*)/\{x \otimes (1-x) : x \in F^*\}$  over  $\mathbb{Z}$ ; (lecture 5 of [\[MVW06\]](#), page 29)

**Example 0.2.**

- a.  $K_0^M(F) = K_0(F) = \mathbb{Z}$ ;
  - b.  $K_1^M(F) = K_1(F) = F^\times$ ;
  - c.  $K_2^M(F) = K_2(F)$ .
4.  $\mathbb{H}_{\mathrm{Zar}}^{2n}(X, \mathbb{Z}(n)) = \mathrm{CH}^n(X)$  (lecture 17 of [\[MVW06\]](#), page 135), where the  $n$ th classical Chow group  $\mathrm{CH}^n(X)$  is the free group given by

$$\mathrm{CH}^n(X) = \mathbb{Z}\{\text{cycles of codimension } n\}/\text{rational equivalences};$$

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<sup>1</sup>Here we use the convention that the (hyper)cohomology of  $F$  should be interpreted as of  $\mathrm{Spec}(F)$ , the corresponding space.

5. there is a natural Atiyah–Hirzebruch spectral sequence

$$E_2^{p,q} = \mathbb{H}_{\text{Zar}}^p(X, \mathbb{Z}(q)) \Rightarrow K_{2q-p}(X).$$

Moreover, tensoring with  $\mathbb{Q}$ , the spectral sequence degenerates and one has

$$\mathbb{H}_{\text{Zar}}^i(X, \mathbb{Z}(n))_{\mathbb{Q}} = \text{gr}_{\gamma}^n(K_{2n-i}(X)_{\mathbb{Q}})$$

where  $\text{gr}_{\gamma}^n$ 's are the quotients (graded pieces) of  $\gamma$ -filtration. ([Lev94]; [Lev99], Theorem 11.7)

**Remark 0.3.** Such choice of complexes  $\mathbb{Z}(q)$  exists, and is called the motivic complex. For a clear definition of these complexes, see Lecture 3 of [MVW06]. Moreover, by convention  $\mathbb{Z}(q) = 0$  for  $q < 0$ .

**Definition 0.4.** The motivic cohomology of  $X$  is defined by  $H^{p,q}(X, \mathbb{Z}) = \mathbb{H}_{\text{Zar}}^p(X, \mathbb{Z}(q))$ , the hypercohomology of the motivic complexes with respect to the Zariski topology.

**Remark 0.5.** In general, a motivic cohomology with coefficient in an abelian group  $A$  is a family of contravariant functors  $H^{p,q}(-, A) : \text{Sm}/k \rightarrow \text{Ab}$ .

**Remark 0.6.** The motivic cohomology of  $X$  satisfies the cancellation property: set  $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$ , then

$$H^{p,q}(X \times \mathbb{G}_m, \mathbb{Z}) = H^{p,q}(X, \mathbb{Z}) \oplus H^{p-1, q-1}(X, \mathbb{Z}).$$

**Remark 0.7.** It turns out that the group remains unchanged if we replace the Zariski topology by Nisnevich topology.<sup>2</sup> If one uses étale topology instead, we retrieve Lichtenbaum motivic cohomology  $H_L^{p,q}(X, \mathbb{Z})$ . If  $\text{char}(k) \nmid n$ , it admits the comparison

$$H_L^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) = H_{\text{étale}}^p(X, \mathbb{Z}/n\mathbb{Z}(q)),$$

where  $\mathbb{Z}/n\mathbb{Z}(q)$  is the  $q$ -twist  $\mu_n^{\otimes q}$ .

We may compare Lichtenbaum motivic cohomology with motivic cohomology by the following theorem, formerly known as Beilinson-Lichtenbaum Conjecture<sup>3</sup>:

**Theorem 0.8** ([Voe11]). The natural map

$$H^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) \rightarrow H_L^{p,q}(X, \mathbb{Z}/n\mathbb{Z})$$

is an isomorphism if  $p \leq q$ , is a monomorphism if  $p = q + 1$ , and gives a spectral sequence for any pair of  $p, q$ .

**Corollary 0.9.** For  $p \leq q$ , we have

$$H^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) = H_{\text{étale}}^p(X, \mathbb{Z}/n\mathbb{Z}(q)).$$

In particular, for  $X = \text{Spec}(k)$  as a point, this is the theorem formerly known as Milnor conjecture:

**Corollary 0.10** ([Voe97], [Voe03a], [Voe03b]).

- $H^{p,p}(k, \mathbb{Z}/n\mathbb{Z}) = K_p^M(k)/n = H_{\text{étale}}^p(X, \mathbb{Z}/n\mathbb{Z}(p))$  as the Galois cohomology;
- in general,

$$H^{p,q}(k, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} 0, & p > q \\ H^{p,p}(k, \mathbb{Z}/n\mathbb{Z}) \cdot \tau^{q-p}, & p \leq q \end{cases}$$

where  $\tau \in \mu_n(k) = H^{0,1}(k, \mathbb{Z})$  is a primitive  $n$ th root of unity.

**Remark 0.11.** Unlike the case with finite coefficients,  $H^{p,q}(k, \mathbb{Z})$  is quite hard to compute for small  $p < q$ ; for  $p \geq q$ , this is 0.

<sup>2</sup>Recall that the Nisnevich topology is a Grothendieck topology on the category of schemes that is finer than the Zariski topology but coarser than the étale topology.

<sup>3</sup>This is also known as the norm residue isomorphism theorem, or (formerly) Bloch-Kato conjecture.

A current long-standing conjecture is

**Conjecture 0.12** (Beilinson-Soulé Vanishing Conjecture, [Lev93]).  $H^{p,q}(k, \mathbb{Z}) = 0$  if  $p < 0$ .

**Remark 0.13.** Here are a few known cases:

- for  $\text{char}(k) = 0$ , this is known for number fields ([Bor74]), function fields of genus 0 ([Dég08]), curves over number fields, and their inductive limits (more precise references required); ([DG05])
- for  $\text{char}(k) > 0$ , this is known for finite fields ([Qui72]) and global fields ([Har77]).

**Remark 0.14.** The motivic cohomology could be realized in a tensor triangulated category, namely the category of effective motives  $\text{DM}(k)$ . For any pair of  $p, q$ , we can find an Eilenberg-MacLane space and a corresponding representable functor so that

$$H^{p,q}(X, \mathbb{Z}) = \text{Hom}_{DM}(\mathbb{Z}(X), \mathbb{Z}(q)[p])$$

where  $\mathbb{Z}(X)$  is the motive of  $X$  and  $\mathbb{Z}(q)[p] = \mathbb{G}_m^{\wedge q}[p - q]$ .<sup>4</sup>

**Remark 0.15.** Dually, we can define the motivic homology by

$$H_{p,q}(X, \mathbb{Z}) = \text{Hom}_{DM}(\mathbb{Z}(q)[p], \mathbb{Z}(X)).$$

**Remark 0.16** ([MVW06] Properties 14.5, page 110). By taking the hom functor from the aspect of motives, we can derive theorems for all (co)homologies which can be represented in  $DM$ . The main derives are the following:

1. If  $E \rightarrow X$  is an  $\mathbb{A}^n$ -bundle, then motives  $\mathbb{Z}(E) = \mathbb{Z}(X)$  in  $DM$ .
2. If  $\{U, V\}$  is a Zariski open covering of  $X$ , we have a Mayer-Vietoris sequence

$$\mathbb{Z}(U \cap V) \longrightarrow \mathbb{Z}(U) \oplus \mathbb{Z}(V) \longrightarrow \mathbb{Z}(X) \longrightarrow \mathbb{Z}(U \cap V)[1]$$

in the form of a distinguished triangle in  $DM$ .

3. If  $Y \subseteq X$  is a closed embedding of codimension  $c$  in  $\text{Sm}/k$ , then we have a Gysin triangle

$$\mathbb{Z}(X \setminus Y) \longrightarrow \mathbb{Z}(X) \longrightarrow \mathbb{Z}(Y)(c)[2c] \longrightarrow \mathbb{Z}(X \setminus Y)[1]$$

which is a distinguished triangle where  $\mathbb{Z}(Y)(c)[2c] := \mathbb{Z}(Y) \otimes \mathbb{Z}(c)[2c]$ .

4. For any vector bundle of rank  $n$  on  $X$ , we have the projective bundle formula

$$\mathbb{Z}(\mathbb{P}(E)) = \bigoplus_{i=0}^n \mathbb{Z}(X)(i)[2i]$$

which defines the Chern class of  $E$ .

5. Let  $X$  be a proper smooth scheme and let  $d_X$  be its dimension, then  $\mathbb{Z}(X)$  has a strong dual  $\mathbb{Z}(X)(-d_X)[-2d_X]$  in  $DM$  by stabilization. This gives a Poincaré duality<sup>5</sup>

$$H^{p,q}(X, \mathbb{Z}) \cong H_{2d_X - p, d_X - q}(X, \mathbb{Z}).$$

<sup>4</sup>Again, this notation goes back to the concise definition of the motivic complexes: see Lecture 3 from [MVW06] as well as the concept of presheaves with transfers.

<sup>5</sup>We can use cohomology with compact support for this.

## 1 INTERSECTION THEORY

**Definition 1.1.** Let  $X$  be a scheme of finite type over  $k$ . We define the  $i$ th cycle on the scheme  $X$  to be a free abelian group

$$Z_i(X) = \bigoplus_{\substack{\text{irreducible closed } c \subseteq X \\ \dim(c)=i}} \mathbb{Z} \cdot c$$

and set  $Z(X) = \bigoplus_i Z_i(X)$ . Define a set  $K_i(X)$  to be the set of coherent sheaves  $\mathcal{F}$  on  $X$  with  $\dim(\text{supp}(\mathcal{F})) \leq i$ .<sup>6</sup>

**Remark 1.2.** Let  $(A, \mathfrak{m})$  be a Noetherian local ring and  $M$  be an  $A$ -module, then by the dimension theorem, we know  $\dim(M) = d(M) = \dim(\text{supp}(M))$ , where  $d(M)$  is the degree of the Hilbert-Samuel polynomial  $P_{\mathfrak{m}}(M, n)$ .

**Definition 1.3.** Let  $X \in \text{Sm}/k$  and let  $U, V \subseteq X$  be irreducible and closed. Suppose  $W \subseteq U \cap V$  is a irreducible and closed component. If  $\dim(W) = \dim(U) + \dim(V) - \dim(X)$ , i.e.,  $\text{codim}(W) = \text{codim}(U) + \text{codim}(V)$ , we say that  $U$  and  $V$  intersect properly at  $W$ .

**Remark 1.4.** This condition is weaker than saying they intersect transversely: we do not require information about tangent spaces.

**Theorem 1.5.** Let  $A \supseteq k$  be a Noetherian regular ring,  $M, N$  be finitely-generated  $A$ -modules, and suppose  $\ell(M \otimes_A N) < \infty$ , then

1.  $\ell(\text{Tor}_i^A(M, N)) < \infty$  for all  $i \geq 0$ ;
2. the Euler-Poincaré characteristic  $\chi(M, N) := \sum_{i=0}^{\dim(A)} (-1)^i \ell(\text{Tor}_i^A(M, N)) \geq 0$ ;
3. by Remark 1.2, we have  $\dim(M) + \dim(N) \leq \dim(A)$ ;
4. in particular, we have  $\dim(M) + \dim(N) < \dim(A)$  if and only if  $\chi(M, N) = 0$ .

*Proof.* See [Ser12], page 106. □

**Remark 1.6.** Part 3. from Theorem 1.5 implies that  $\dim(W) \geq \dim(U) + \dim(V) - \dim(X)$ , i.e.,  $\text{codim}(W) \leq \text{codim}(U) + \text{codim}(V)$  in the notation of Definition 1.3.

**Definition 1.7.** Let  $X, U, V, W$  be as in Definition 1.3, then we define the intersection multiplicity  $m_W(U, V)$  of  $U$  and  $V$  at  $W$  by

$$m_W(U, V) = \chi^{\mathcal{O}_{X,W}}(\mathcal{O}_{X,W}/P_U, \mathcal{O}_{X,W}/P_V)$$

where  $P_U$  and  $P_V$  are prime ideals defining  $U$  and  $V$ , respectively.

**Remark 1.8.** By Theorem 1.5, we know  $m_W(U, V) \geq 0$ , and  $m_W(U, V) = 0$  if and only if  $U$  and  $V$  do not intersect properly at  $W$ .

**Definition 1.9.** Let  $X \in \text{Sm}/k$ , and let  $U \in Z_a(X)$  and  $V \in Z_b(X)$ . If  $U$  and  $V$  intersect properly at every component, then we define the intersection product to be the cycle

$$U \cdot V = \sum_{\substack{W \subseteq U \cap V \\ \dim(W)=a+b-d_X}} m_W(U, V) \cdot W \in Z_{a+b-d_X}(X).$$

**Example 1.10.** Let  $X$  be a smooth projective surface, and let  $C$  and  $D$  be divisors on  $X$ . For any point  $x \in C \cap D$ , locally we think of  $C = \{f = 0\}$  and  $D = \{g = 0\}$  around  $x$ , then  $m_x(C, D) = \ell_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(f, g))$ .

**Definition 1.11.** Suppose  $X$  is a scheme of finite type over  $k$ , and  $\mathcal{F} \in K_n(X)$  is a coherent sheaf, then we define  $Z_a(\mathcal{F}) = \sum_{\dim(\bar{\eta})=a} (\mathcal{O}_{X,\eta}(\mathcal{F}_{\bar{\eta}}) \cdot \bar{\eta}) \in Z_a(X)$ .

<sup>6</sup>Despite the notation, this has nothing to do with a K-theory.

Therefore, we define the cycle of  $F$  as an element of the cycle of  $X$ .

**Definition 1.12** ([Har13], Exercise III.6.9). Every coherent sheaf  $\mathcal{F}$  on  $X \in \mathbf{Sm}/k$  has a resolution

$$0 \longrightarrow E_k \longrightarrow E_{k-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

where  $E_i$ 's are locally free of finite rank. Therefore, for any coherent sheaf  $\mathcal{G}$ , we can define the Tor functor<sup>7</sup> of coherent sheaves by

$$\mathrm{Tor}_i(\mathcal{F}, \mathcal{G}) = H_i(E_* \otimes_{\mathcal{O}_X} \mathcal{G}).$$

**Proposition 1.13.** Let  $X \in \mathbf{Sm}/k$ . Suppose  $\mathcal{F} \in K_a(X)$  and  $\mathcal{G} \in K_b(X)$  intersect properly, then

$$Z_a(\mathcal{F}) \cdot Z_b(\mathcal{G}) = \sum_{i=0}^{d_X} (-1)^i \cdot Z_{a+b-d_X}(\mathrm{Tor}_i(\mathcal{F}, \mathcal{G})).$$

*Proof.* We only have to do it locally, so we can assume  $X$  to be affine, and count the coefficients of  $\bar{\xi}$  where  $\dim(\xi) = a + b - d_X$ . It suffices to show that the stalks at  $\xi$  satisfies

$$\chi(F_\xi, G_\xi) = \sum_{\substack{\dim(\bar{\lambda})=a \\ \dim(\bar{\eta})=b \\ \xi \in \bar{\lambda} \cap \bar{\eta}}} \ell(\mathcal{F}_\lambda) \cdot \ell(\mathcal{G}_\eta) \cdot m_{\bar{\xi}}(\bar{\lambda}, \bar{\eta}).$$

Because our ring is Noetherian, then  $\mathcal{F}$  admits a filtration

$$0 = M_0 \subseteq \cdots \subseteq M_d = \mathcal{F}$$

such that  $M_i/M_{i-1} \cong \mathcal{O}_X/\mathcal{I}$  is coherent for prime ideal  $\mathcal{I}$ . By the additivity of both sides of the isomorphism, we may assume  $\mathcal{F} = \mathcal{O}_X/\mathfrak{p}$  with dimension at most  $a$ , where  $\mathfrak{p} \sim \lambda \in X$ . Similarly, we may assume  $\mathcal{G} = \mathcal{O}_X/\mathfrak{q}$  with dimension at most  $b$ , where  $\mathfrak{q} \sim \eta \in X$ . Moreover, set  $\xi \in \bar{\lambda} \cap \bar{\eta}$ . By definition, we now have  $\chi(\mathcal{F}_\xi, \mathcal{G}_\xi) = m_{\bar{\xi}}(\bar{\lambda}, \bar{\eta})$ .

- If  $\dim(\bar{\lambda}) = a$  and  $\dim(\bar{\eta}) = b$ , then the equality follows from the fact that  $\ell(\mathcal{F}_\lambda) = \ell(\mathcal{G}_\eta) = 1$ .
- If not, then either  $\dim(\bar{\lambda}) < a$  or  $\dim(\bar{\eta}) < b$ , then  $\bar{\lambda}$  and  $\bar{\eta}$  do not intersect properly at  $\bar{\xi}$ , so both the left-hand side and the right-hand side become 0.

□

**Proposition 1.14.** The intersection product is commutative.

*Proof.* This is obvious since the Tor functor is commutative.

□

**Proposition 1.15.** The intersection product is associative.

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<sup>7</sup>Since we are working over sheaves of  $\mathcal{O}_X$ -modules, using the same argument on the level of modules shows that the Tor functor is independent from the choice of resolution.

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