Bounds in Simple Hexagonal Lattice and Classification of 11-stick Knots

Jiantong Liu

University of California, Los Angeles

January 6, 2023

How to Classify Knots?

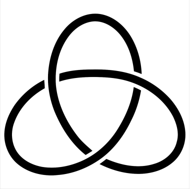
We usually think of a knot as an embedding of S^1 (1-sphere) on the Euclidean space $\mathbb{R}^3$.

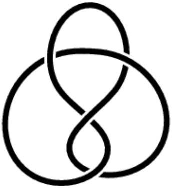
How to Classify Knots?

We usually think of a knot as an embedding of S^1 (1-sphere) on the Euclidean space $\mathbb{R}^3$.
In particular, we say two knots are *equivalent* if there exists an ambient isotopy that transforms one to another.

How to Classify Knots?

We usually think of a knot as an embedding of S^1 (1-sphere) on the Euclidean space $\mathbb{R}^3$.
In particular, we say two knots are *equivalent* if there exists an ambient isotopy that transforms one to another.
However, it is sometimes hard to tell one knot from another...





Knot Invariants

Instead of looking for ambient isotopies, we look for the properties of a knot that would be preserved by ambient isotopies. These are called *knot invariants*.

Knot Invariants

Instead of looking for ambient isotopies, we look for the properties of a knot that would be preserved by ambient isotopies. These are called *knot invariants*.

- Crossing Number
- Bridge Number
- ...

Knot Invariants

Instead of looking for ambient isotopies, we look for the properties of a knot that would be preserved by ambient isotopies. These are called *knot invariants*.

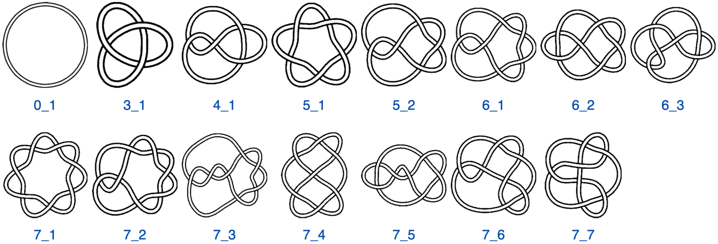
- Crossing Number
- Bridge Number
- ...

Definition

The *crossing number* of a knot type is the least number of crossings among all possible knots of this type.

Knot Types with Small Crossing Numbers

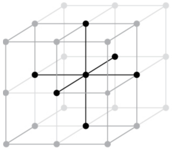
The crossing number gives us an idea of how simple/complex a knot really is.



Cubic Lattice

The *cubic lattice* is defined to be

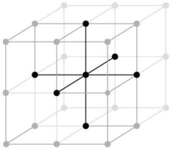
$$\mathbb{L}^3 = (\mathbb{R} \times \mathbb{Z} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{Z} \times \mathbb{R}).$$



Cubic Lattice

The *cubic lattice* is defined to be

$$\mathbb{L}^3 = (\mathbb{R} \times \mathbb{Z} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{Z} \times \mathbb{R}).$$

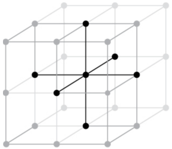


A *polygon* $\mathcal{P}$ in the cubic lattice is a continuous path consisting of line segments parallel to the x-, y-, and z-axes.

Cubic Lattice

The *cubic lattice* is defined to be

$$\mathbb{L}^3 = (\mathbb{R} \times \mathbb{Z} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{Z} \times \mathbb{R}).$$

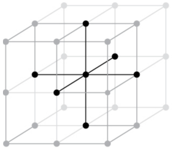


A *polygon* $\mathcal{P}$ in the cubic lattice is a continuous path consisting of line segments parallel to the x-, y-, and z-axes. A maximal line segment parallel to the x-axis is called an x*-stick*, and one can define y-stick and z-stick similarly.

Cubic Lattice

The *cubic lattice* is defined to be

$$\mathbb{L}^3 = (\mathbb{R} \times \mathbb{Z} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{Z} \times \mathbb{R}).$$



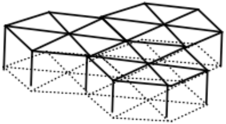
A *polygon* $\mathcal{P}$ in the cubic lattice is a continuous path consisting of line segments parallel to the x-, y-, and z-axes. A maximal line segment parallel to the x-axis is called an x-*stick*, and one can define y-stick and z-stick similarly. A *cubic lattice knot* is a non-intersecting closed polygon in the cubic lattice consisting of x-, y-, and z-sticks.

Simple Hexagonal Lattice

Let $x = \langle 1, 0, 0 \rangle$, $y = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}, 0 \right\rangle$, and $w = \langle 0, 0, 1 \rangle$. The *simple hexagonal lattice* (sh-lattice) is defined to be the set of $\mathbb{Z}$-combinations of x, y, w, i.e.,

$$sh = \{ax + by + cw \mid a, b, c \in \mathbb{Z}\}.$$

We define $z = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}, 0 \right\rangle$, i.e, $z = y - x$.



Mapping between Lattices

$$T : \mathbb{L}^3 \to \mathsf{sh}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

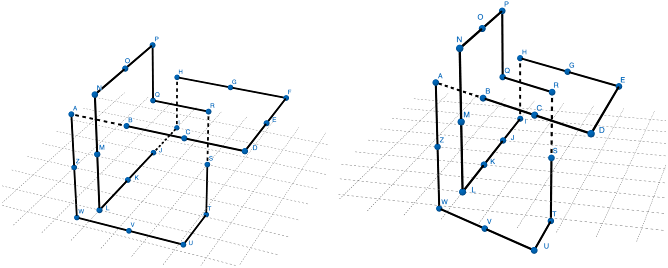


Figure: Effect of T on the Trefoil Knot

Mapping between Lattices

$$T : \mathbb{L}^3 \to \mathrm{sh}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Proposition

T is a well-defined linear transformation. Moreover, let $\mathcal{P}_L$ be a cubic lattice knot presentation and $\mathcal{P}_{sh}$ be its image over T, then T preserves

1. *the stick number of the lattice knot, i.e., $|\mathcal{P}_L| = |\mathcal{P}_{sh}|$.*
2. *the order and length of the sticks.*

Therefore, T preserves the overall structure and properties of lattice knots, only “squeezing” the knot a little.

Studying Knot Types

Definition

The *stick number of a knot type* $[K]$ is the least stick number among all knot conformations $\mathcal{P}$ of $[K]$ in a given lattice $\mathbb{A}$, i.e., $s_{\mathbb{A}}[K] = \min_{\mathcal{P} \in [K] \subset \mathbb{A}} |\mathcal{P}|$. We use $s_L[K]$ and $s_{\mathrm{sh}}[K]$ to denote the stick number of $[K]$ with respect to $\mathbb{L}^3$ and sh, respectively.

Studying Knot Types

Definition

The *stick number of a knot type* $[K]$ is the least stick number among all knot conformations $\mathcal{P}$ of $[K]$ in a given lattice $\mathbb{A}$, i.e., $s_{\mathbb{A}}[K] = \min_{\mathcal{P} \in [K] \subset \mathbb{A}} |\mathcal{P}|$. We use $s_L[K]$ and $s_{\text{sh}}[K]$ to denote the stick number of $[K]$ with respect to $\mathbb{L}^3$ and sh, respectively.

Proposition

For any knot type $[K]$, $s_{sh}[K] \leq s_L[K]$.

Studying Knot Types

Definition

The *stick number of a knot type* $[K]$ is the least stick number among all knot conformations $\mathcal{P}$ of $[K]$ in a given lattice $\mathbb{A}$, i.e., $s_{\mathbb{A}}[K] = \min_{\mathcal{P} \in [K] \subset \mathbb{A}} |\mathcal{P}|$. We use $s_L[K]$ and $s_{\text{sh}}[K]$ to denote the stick number of $[K]$ with respect to $\mathbb{L}^3$ and sh, respectively.

Proposition

For any knot type $[K]$, $s_{sh}[K] \leq s_L[K]$.

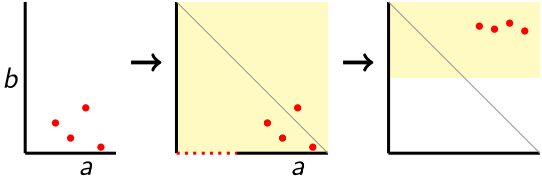
Theorem

Can we improve it to a strict bound, i.e., $s_{sh}[K] < s_L[K]$?

Proving the Strict Bound

Lemma

Project a polygon $\mathcal{P}$ in the cubic lattice down to the xy-plane. Suppose we have an x-stick named x and a y-stick named y of equal length, connected in the shape of an "L". If there are no z-sticks within the triangle with x and y as legs, then we can replace them with a z-stick in the sh-lattice after applying T.



By moving the z-sticks from the lattice knot in $\mathbb{L}^3$ out of the triangular region, the theorem is trivial.

Edge Length

Definition

An *edge* of a polygon in a lattice is a unit-length segment of the polygon between two points in the lattice. The *edge length* of a polygon in a lattice is the total number of edges in the polygon. We denote $e_L[K]$ and $e_{sh}[K]$ to be the (minimal) edge lengths of a knot type $[K]$ in $\mathbb{L}^3$ and sh, respectively.

Proposition

T preserves edge length.

Corollary

The theorem on stick numbers implies that we also have a strict bound on edge lengths, i.e., $e_{sh}[K] < e_L[K]$.

Lower Bounds

Proposition

For a non-trivial knot type $[K]$, $s_{sh}[K] \geq 2\sqrt{s_L[K] + \frac{9}{4}} - 3$.

Proposition

For a non-trivial knot type $[K]$, $e_{sh}[K] \geq \frac{3e_L[K]+30}{8}$.

Previous Classifications

Classification of a few knots with small stick numbers has been known as follows:

	3_1	4_1	5_1	5_2
$\mathbb{L}^3$	12	14	16	16
sh	11	?	?	?

Previous Classifications

Classification of a few knots with small stick numbers has been known as follows:

	3_1	4_1	5_1	5_2
$\mathbb{L}^3$	12	14	16	16
sh	11	?	?	?

We improve the classification by proving the following result:

Theorem

In the sh-lattice, the only non-trivial 11*-stick knots are* 3_1 *and* 4_1*.*

Stick Number of 4_1

Proposition

The stick number of a figure-eight knot in the sh-lattice is 11*, i.e.,* $s_{sh}(4_1) = 11$.

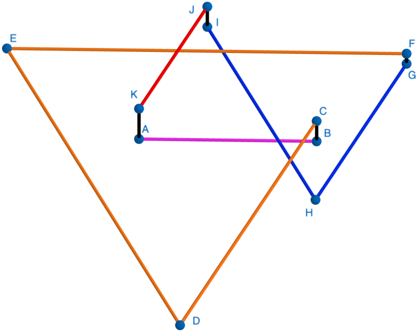


Figure: 4_1 knot in sh-lattice with 11 sticks

w-level Structure

When we say a "polygon" $\mathcal{P}$, we mean a knot presentation $\mathcal{P}$ of a knot type $[\mathcal{P}]$.

Definition

- A polygon $\mathcal{P}$ is *reducible* if its stick number is greater than the stick number of its knot type. Otherwise, $\mathcal{P}$ is *irreducible*.
- The plane formed by x-, y-, and z-sticks with w-coordinate k is called the w-level k.
- A polygon $\mathcal{P}$ is *properly leveled with respect to w-coordinate* if each w-level contains exactly two endpoints of w-sticks. In particular, the number of w-levels is equal to the number of w-sticks in the polygon.

Number of w-sticks in a 11-stick Polygon

Lemma

An 11-stick polygon with five w-sticks has to be trivial.

Proof.

We can determine the exact w-sticks in a knot, which is given by

$$w_{13}, w_{14}, w_{24}, w_{25}, w_{35}$$

where w_{ij} is a w-stick connecting w-level i and j. Based on the fact that exactly one of the w-levels has two sticks, every possible configuration then turns out to be trivial. □

Corollary

A non-trivial irreducible 11-stick polygon $\mathcal{P}$ has exactly four w-sticks.

Determine the Stick Number of Each Type

Lemma

A non-trivial 11*-stick polygon has at least three* x*-sticks, at least two* y*-sticks, and at least one* z*-stick, up to permutation of stick types.*

Corollary

A non-trivial 11*-stick polygon must have either*

1. $(4, 2, 1)$*: four* x*-sticks, two* y*-sticks, and one* z*-stick, or*
2. $(3, 3, 1)$*: three* x*-sticks, three* y*-sticks and one* z*-stick, or*
3. $(3, 2, 2)$*: three* x*-sticks, two* y*-sticks and two* z*-sticks.*

Square of Replacement

Imagine looking at the preimage of a sh-lattice knot with respect to T, i.e., a knot in the sh-lattice that is put into the cubic lattice. Note that the only sticks not embedded in the cubic lattice are the z-sticks. We call a square with a particular z-stick as diagonal a square of replacement. A stick is *within* the square of replacement if the stick intersects with the square at exactly one point.

Lemma

If there are no other z-sticks in the square of replacement, the z-stick can be reduced into x- and y-sticks with the addition of at most three sticks.

Theorem

In the sh-lattice, the only non-trivial 11-stick knots are 3_1 and 4_1.

Summary

	3_1	4_1	5_1	5_2
$\mathbb{L}^3$	12	14	16	16
sh	11	?	?	?

	3_1	4_1	5_1	5_2
$\mathbb{L}^3$	12	14	16	16
sh	11	11	$12 \sim 14$	$12 \sim 14$

Future Work

- Determine the stick number of 5_1 and 5_2 in sh-lattice.
- Determine the relationship between stick number and crossing number for knots with small stick numbers.
- For a properly leveled polygon $\mathcal{P}$ of type $[K]$, construct upper and lower bounds on the number of w-sticks, both in terms of stick number $s_{\text{sh}}[K]$ and in terms of crossing number $c[K]$.
- Improve the bounds of s_{sh} and e_{sh} in terms of s_L and e_L, respectively.

References

R. Bailey, et al., Stick numbers in the simple hexagonal lattice, *Involve, a Journal of Mathematics* **8**(3) (2015) 503–512.

Y. Huang and W. Yang, Lattice stick number of knots, *Journal of Physics A: Mathematical and Theoretical* **50**(50) (2017) p. 505204.

Y. Huh and S. Oh, Lattice stick numbers of small knots, *Journal of Knot Theory and Its Ramifications* **14**(07) (2005) 859–867.

C. E. Mann et al., The stick number for the simple hexagonal lattice, *Journal of Knot Theory and Its Ramifications* **21**(14) (2012) p. 1250120.