MATH 205A Notes

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1 Lecture 1: September 23, 2022

1.1 Motivation of the Subject

Example 1.1 (Motivating Example)**.** • Fermat's Last Theorem. For any $n \geq 3$, the equation $x^n + y^n = z^n$ has no integer solutions. This was stated by Fermat in 1637, who solved the case for $n = 4$, and was eventually proven by Wiles in 1995.

Kummer (approximately 1850) proved the case for prime $n = p \geq 3$, and $\gcd(x, y, z) = 1$, where $p \nmid xyz$. This is called the first case of Fermat's Last Theorem. Take $\xi_p = e^{\frac{2\pi i}{p}}$, we then study $\mathbb{Z}[\xi_p] = \{\sum_{i=0}^{p-1} a_i \xi_p^i \mid a_i \in \mathbb{Z}\}$. Suppose $\mathbb{Z}[\xi_p]$ is a UFD ($p \leq 19$). Note that $x^p + y^p = \prod_{i=0}^{p-1}(x + \xi_p^i y)$. By our assumption, the $x + \xi_p^i y$ are all relatively prime. Their product is z^p, so each $x + \xi_p^i y$ is a pth power times a unit. They are also all congruent modulo $(1 - \xi_p)$, the unique prime of $\mathbb{Z}[\xi_p]$ over (p). One obtains a contradiction using

1. the structure of $\mathbb{Z}[\xi_p]^\times$,
2. properties of pth powers in $\mathbb{Z}[\xi_p]$ modulo (p).

Note that for any p, $\mathbb{Z}[\xi_p]$ has unique factorization of nonzero ideals into prime ideals: Dedekind domain. It is in fact enough that no non-principal ideal has principal pth power. We say p is regular. This includes all $p < 100$ except $37, 59, 67$. Also, Kummer did not require $p \nmid xyz$.

- Power residue. When is 2 a cube modulo p? (c.f. reciprocity) If $p = 3$ or $p \equiv 2 \pmod 3$, the answer is always. If $p \equiv 1 \pmod 3$, then 2 is a cube modulo p if and only if $p = a^2 + 27b^2$ with $a, b \in \mathbb{Z}$. Note that 2 is a cube modulo p if and only if 2 is a cube modulo π.[1] The cubic reciprocity result by Eisenstein says that 2 is a cube modulo π if and only if π is a cube modulo 2. But when is π a cube modulo 2? Note $(\mathbb{Z}[\xi_3]/(2))^\times \cong \mathbb{F}_2[\xi_3]^\times = \mathbb{F}_4^\times$. So π is a cube modulo 2 if and only if $\pi \equiv 1 \pmod 2$. We can choose $\pi \equiv 1 \pmod 3$, so this is when $\pi \equiv 1 \pmod 6$, and with $\xi_3^2 + \xi_3 + 1 = 0$, this is true if and only if $\pi = a + 6b\xi_3$ with $a \equiv 1 \pmod 3$ and $b \in \mathbb{Z}$, if and only if $p = \pi\bar{\pi} = a^2 - 6ab + 36b^2 = (a - 3b)^2 + 27b^2$.

[1] When we say modulo π, we consider $p = \pi\bar{\pi}$ in $\mathbb{Z}[\xi_3]$ for π irreducible.

1.2 Integrality

Definition 1.2 (Number Field)**.** A number field is a finite extension of $\mathbb{Q}$. Being a number field implies it is algebraic. An algebraic number is algebraic over $\mathbb{Q}$, but inside $\mathbb{C}$, i.e. $\bar{\mathbb{Q}} \subseteq \mathbb{C}$. We like to think of $\bar{\mathbb{Q}}$ as an algebraic closure itself.[2]

Definition 1.3 (Ring of Integers)**.** The ring of integers $\mathcal{O}_F$ of a number field F is the set of all roots of monic polynomials in $\mathbb{Z}[x]$ in F. We will see later that this is indeed a ring because it is the integral closure of F.[3]

Let B/A be an extension of commutative rings.

Definition 1.4 (Integral Element)**.** An element of B is integral over A if it is the root of some monic $f \in A[x]$.

Proposition 1.5. Let $\beta \in B$. The following are equivalent:

(i) β is integral over A.

(ii) There exists $n \geq 0$ such that $A[\beta] = \bigoplus_{i=0}^{n} A \cdot \beta^i$, i.e. $\{1, \beta, \cdots, \beta^n\}$ generates $A[\beta]$ as an A-module.

(iii) $A[\beta]$ is finitely-generated as an A-module.

(iv) There exists a finitely-generated A-submodule M of B such that $\beta M \subseteq M$ and M is faithful as an $A[\beta]$-module.

Proof. The proof from (i) to (ii) to (iii) to (iv) is fairly simple. We now prove (iv) implies (i). Suppose $M = \sum_{i=1}^{n} A \cdot \gamma_i \subseteq B$ has the properties in (iv), then $\beta\gamma_i = \sum_{j=1}^{n} a_{ij}\gamma_j$, where (a_{ij}) is defining $T : A^n \to A^n$, which is B-linear. Now the characteristic polynomial $c_T(x) = \det(x \cdot \mathbf{id} - T)$, so $c_T(\beta) \cdot M = 0$, and so $c_T(\beta) = 0$ as M is faithful over $A[\beta]$. □

Definition 1.6 (Integral Extension)**.** An extension B/A is integral if every $\beta \in B$ is integral over A.

Proposition 1.7. Suppose $B = A[\beta_1, \cdots, \beta_k]$ is finitely-generated over A. The following are equivalent:

(i) B/A is integral.

(ii) Each β_i is integral over A.

(iii) B is finitely-generated as an A-module.

[2] In the notes, we defined the ring of algebraic integers to be the integral closure $\bar{\mathbb{Z}}$ of $\mathbb{Z}$ inside $\mathbb{C}$, and an algebraic integer is an element of $\bar{\mathbb{Z}}$

[3] We can define the ring of integers of a number field to be the integral closure of $\mathbb{Z}$ over F.

Proof. Easy if one assumes that we proved "if C/B is an extension and C is a finitely-generated B-module and B is a finitely-generated A-module, then C is a finitely-generated A-module". □

Corollary 1.8. If C/B and B/A are integral extensions, then so is C/A.

Proof. Suppose $\gamma \in C$. It is the root of some monic polynomial $f \in B[x]$. Let B' be an A-algebra (subring) generated by the coefficients of f. Then γ is integral over B' and B' is integral over A, and so $B'[\gamma]$ is integral over A, and so γ is integral over A. □

Definition 1.9 (Integral Closure)**.** The integral closure of A in B is the set of elements of B integral over A.

Proposition 1.10. The integral closure of A in B is a ring.

Proof. Suppose α, β are in the integral closure of A in B. Consider the ring $A[\alpha, \beta]$, then it is integral over A, but it also contains $-\alpha, \alpha + \beta, \alpha \cdot \beta$, and so we have closure. □

Corollary 1.11. If F is a number field, then $\mathcal{O}_F$ is a ring.

Note that we can define $\bar{\mathbb{Z}}$ to be the ring of algebraic integers, i.e. the integral closure of $\mathbb{Z}$ in $\mathbb{Q} \subseteq \mathbb{C}$.

Definition 1.12 (Integrally Closed)**.** We say A is integrally closed in B if the integral closure of A in B is A.

Definition 1.13 (Integrally Closed/Normal)**.** We say a domain A is integrally closed if it is integrally closed in its quotient field $Q(A)$. We use normal and integrally closed interchangably.

This gives an absolute notion of closure.

Example 1.14. $\mathbb{Z}$ is not integrally closed. For example, suppose $\frac{c}{d} \in \mathbb{Q}$ is a reduced fraction, then $\mathbb{Z}[\frac{c}{d}]$ is not finitely generated over $\mathbb{Z}$ if $d > 1$.

Proposition 1.15. Suppose A is integrally closed domain, and $K = Q(A)$, and L/K is a field extension. If $\beta \in L$ is integral over A with minimal polynomial $f \in K[x]$, then $f \in A[x]$.

Proof. See notes. □

Corollary 1.16. Suppose B is an integrally closed domain, then the integral closure of A in B is integrally closed.

2 Lecture 2: September 26, 2022

Recall the following proposition from last time.

Proposition 2.1. Suppose A is integrally closed domain, and $K = Q(A)$, and L/K is a field extension. If $\beta \in L$ is integral over A with minimal polynomial $f \in K[x]$, then $f \in A[x]$.

Proof. There exists a monic polynomial $g \in A[x]$ such that $g(\beta) = 0$. Now f as a minimal polynomial divides g in $K[x]$. However, all roots of g are integral over A, so all roots of f are. But f being a monic polynomial has the form $f = \prod_{i=1}^{n}(x - \alpha_i)$, where α_i's are integral over A, so sums and products of α_i's are also integral over A, and so all coefficients of f are integral over A, and therefore in K, so it is in A as A is normal. □

Proposition 2.2. UFDs are normal, i.e. integrally closed.

Proof. See notes. □

Proposition 2.3. Let B/A be an integral extension of domains. Then B is a field if and only if A is a field.

Proposition 2.4. Suppose B/A is a normal domain. Then the integral closure of A in B is normal.

Proof. Let $\bar{A}$ be the integral closure of A in B, let $\beta \in Q(\bar{A})$ be integral over $\bar{A}$, then $\bar{A}[\beta]$ is integral over $\bar{A}$ and $\bar{A}$ is integral over A, so $\bar{A}[\beta]$ is integral over A, then β is integral over A. Therefore, $\beta \in \bar{A}$. □

Corollary 2.5. If F is a number field, then $\mathcal{O}_F$ is normal.

Proposition 2.6. Let A be normal and $K = Q(A)$, let L/K be an algebraic extension, and B be the integral closure of A in L, then $Q(B) = L$, and in fact, any $\beta \in L$ has the form $\frac{b}{d}$ where $b \in B$ and $d \in A\backslash\{0\}$.

Proof. Let $\beta \in L$ be the root of some monic $f = \sum_{i=0}^{n} a_i x^i \in K[x]$. There exists $d \in A\backslash\{0\}$ such that $df \in A[x]$. Now $d^n f(d^{-1}x) = \sum_{i=0}^{n} a_i d^{n-i} x^i \in A[x]$ monic, and it has $d\beta$ as a root. Now $d\beta \in B$ since it is the root of a monic polynomial in $A[x]$. □

Corollary 2.7. $Q(\mathcal{O}_F) = F$.

We now give a different interpretation of the proposition we just proved.

Remark 2.8. The proposition tells us that $B \otimes_A K \twoheadrightarrow L$ is a surjection given by $b \otimes \frac{1}{d} \mapsto \frac{b}{d}$. In fact, this is an isomorphism. (Left as an exercise.) Then the rank of B over A is just $\dim_K(B \otimes_A K) = [L : K]$.

In general, it is not obvious that this implies that B is a finitely-generated A-module, but we do get $\mathcal{O}_F$ as a finitely-generated Abelian group.

Definition 2.9 (Square-free Integer)**.** A square-free integer is an integer which is divisible by no square number other than 1. That is, its prime factorization has exactly one factor for each prime that appears in it.

Theorem 2.10. Let d be a square-free integer that is not 1. Then we know $\mathcal{O}_{\mathbb{Q}(\sqrt{d})} = \begin{cases} \mathbb{Z}[\frac{1+\sqrt{d}}{2}], & d \equiv 1 \pmod 4 \\ \mathbb{Z}[\sqrt{d}], & d \equiv 2, 3 \pmod 4 \end{cases}$.

Proof. Note $\mathbb{Z}[\sqrt{d}] \subseteq \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. If $\alpha = a + b\sqrt{d}$ with $a \in \mathbb{Q}$ and $b \in \mathbb{Q}^\times$ that are integral over $\mathbb{Z}$, then $f = x^2 - 2ax + a^2 - b^2 d$ is its minimal polynomial in $\mathbb{Z}[x]$, then $a \in \frac{1}{2}\mathbb{Z}$. If $a \in \mathbb{Z}$, then $b^2 d \in \mathbb{Z}$ and d is square-free, so $b \in \mathbb{Z}$. If $a \notin \mathbb{Z}$, $a' = 2a \in \mathbb{Z}$ and $b' = 2b \in \mathbb{Z}$ are odd. And $(a')^2 \equiv (b')^2 d \pmod 4$. Since $(a')^2, (b')^2 \equiv 1 \pmod 4$, $d \equiv 1 \pmod 4$. Since all elements $\frac{a'+b'\sqrt{d}}{2} \in \mathbb{Z}[\frac{1+\sqrt{d}}{2}]$, we are done. □

2.1 Dedekind Domains

Definition 2.11 (Dedekind Domain)**.** A Dedekind domain is a Noetherian, normal domain of Krull dimension at most 1.

Remark 2.12. Krull dimension at most 1 means all nonzero prime ideals are maximal.

Example 2.13.
- Fields.
- PIDs. A PID is Noetherian, and it is a UFD, so it is integrally closed. Its nonzero prime ideals are maximal, generated by its irreducible elements.

Lemma 2.14. Suppose B/A is integral. If $\mathfrak{b} \in B$ is an ideal containing a nonzero element that is not a zero divisor, then $\mathfrak{b} \cap A \neq (0)$.

Proof. Let $\beta \in B \backslash \{0\}$ not be a zero divisor. Let $f \in A[x]$ be a minimal polynomial of β, so $f(0) \neq 0$. Suppose $\beta \in \mathfrak{b}$, then $f(\beta) - f(0) \in \mathfrak{b}$, but $f(\beta) = 0$, then $f(0) \in \mathfrak{b}$, and so $f(0) \in \mathfrak{b} \cap A$. □

Proposition 2.15. If $\dim(A) \leq 1$, and B/A is an integral extension of domains, then $\dim(B) \leq 1$.

Proof. Let P be a nonzero prime ideal of B and $\mathfrak{p} = P \cap A$ prime. Then $\mathfrak{p} \neq 0$ by the lemma, and so $F = A/\mathfrak{p}$ is a field since $\dim(A) = 1$. For $\beta \in B$, let $f \in A[x]$ be monic with $f(\beta) = 0$. Let $\bar{f} \in F[x]$ be its image under the reduction modulo $\mathfrak{p}$ map, $\bar{\beta} \in B/P$ be the image of β, then $\bar{f}(\bar{\beta}) = 0$. Then $\bar{\beta}$ is algebraic over F, so $B/P = F[\bar{\beta} \mid \bar{\beta} \in B]$ is a field since all of them are algebraic elements. Therefore, P is maximal. □

We want to show the following theorem.

Theorem. Let A be a Dedekind domain, $K = Q(A)$, L/K is a finite extension, B is the integral closure of A in L, then B is a Dedekind domain.

This will help us prove the corollary.

Corollary. $\mathcal{O}_F$ is a Dedekind domain.

2.2 Norm and Trace

Definition 2.16 (Trace Map, Norm Map)**.** Let L/K be a finite extension of fields. For $\alpha \in L$, let $m_\alpha : L \to L$ denote the linear transformation of K-vector spaces defined by left multiplication by α. Then

- The trace map $Tr_{L/K}$ is defined by sending $\alpha \in L$ to the trace of m_α.
- The norm map $N_{L/K}$ is defined by sending $\alpha \in L$ to the determinant of m_α.

Proposition 2.17. Let L/K be a finite extension of fields, and let $\alpha \in L$. Let $f \in K[x]$ be the minimal polynomial of α over K, let $d = [K(\alpha) : K]$ and $s = [L : K(\alpha)]$. Suppose f factors in $\bar{K}[x]$ as $f = \prod_{i=1}^{d}(x - \alpha_i)$ for some $\alpha_1, \cdots, \alpha_d \in \bar{K}$. Then the characteristic polynomial of m_α is f^s, and we have

$$N_{L/K}(\alpha) = \prod_{i=1}^{d} \alpha_i^s$$

and

$$Tr_{L/K}(\alpha) = s \sum_{i=1}^{d} \alpha_i.$$

Proof. See notes. □

Proposition 2.18. Let L/K be a finite extension of fields, and let $m = [L : K]_i$ be its degree of inseparability. Let $\mathfrak{S}$ denote the set of embeddings of L fixing K in a given algebraic closure of K, i.e. $K \hookrightarrow L$. Then, for $\alpha \in L$, we have

$$N_{L/K}(\alpha) = \prod_{\sigma \in \mathfrak{S}} \sigma\alpha^m$$

and

$$Tr_{L/K}(\alpha) = m \sum_{\sigma \in \mathfrak{S}} \sigma\alpha.$$

Remark 2.19. Note that the distinct conjugates of α in a fixed algebraic closure $\bar{K}$ of K are exactly the $\tau\alpha$ for τ in the set of distinct embeddings of $K(\alpha)$ in K, and these $\tau\alpha$'s are the distinct roots of the minimal polynomial of α over K.

Proof. See notes. □

Corollary 2.20. Let L/K be a finite separable extension of fields. Let S denote the set of embeddings of L fixing K in a given algebraic closure of K. Then, for $\alpha \in L$, we have

$$N_{L/K}(\alpha) = \prod_{\sigma \in \mathfrak{S}} \sigma\alpha$$

and

$$Tr_{L/K}(\alpha) = \sum_{\sigma \in \mathfrak{S}} \sigma\alpha.$$

Proposition 2.21. Let M/K be a finite field extension and L be an intermediate field in the extension. Then we have

$$N_{M/K} = N_{L/K} \circ N_{M/L}$$

and

$$Tr_{M/K} = Tr_{L/K} \circ Tr_{M/L}.$$

2.3 Discriminant

Definition 2.22 (Symmetric Bilinear Form)**.** Let V be a K-vector space. A symmetric bilinear form is a bilinear form $\psi : V \times V \to K$ which is K-linear in each variable, with symmetric if $\psi(w, v) = \psi(v, w)$ for all $v, w \in V$.

Example 2.23. $V = K^n$, $Q \in M_n(F)$, $\psi(v, w) = v^T Q w$ bilinear. It is symmetric if and only if Q is.

Another example of symmetric bilinear form is the trace form.

Example 2.24. If L/K is a finite extension of fields, then $\psi : L \times L \to K$ defined by $\psi(\alpha, \beta) = Tr_{L/K}(\alpha\beta)$ for $\alpha, \beta \in L$ is a symmetric K-bilinear form on L.

Definition 2.25. The discriminant of $\psi : V \times V \to K$ with respect to (ordered) basis $(v_1, \cdots, v_n)$ of V/K is $\det(\psi(v_i, v_j))_{i,j}$.

Lemma 2.26. If $T : V \to V$ is K-linear, then $\det(\psi(Tv_i, Tv_j)) = \det(T)^2 \det(\psi(v_i, v_j))$.

Proof. See notes. □

Definition 2.27. The discriminant of a finite field extension L/K related to a basis of L as a K-vector space is the discriminant of the trace form related to that basis $\beta_1, \cdots, \beta_n \in L$: $D(\beta_1, \cdots, \beta_n) = \det(Tr_{L/K}(\beta_i \beta_j)_{i,j})$.

Remark 2.28. This depends on the basis you choose.

3 Lecture 3: September 28, 2022

Exercise 3.1. If L/K is inseparable, then $D(\beta_1, \cdots, \beta_n) = 0$.

Suppose L/K is separable and let $\sigma_1, \cdots, \sigma_n : L \hookrightarrow \bar{K}$ be the distinct embeddings of L in an algebraic closure of K that fix K.

Proposition 3.2. Then $D(\beta_1, \cdots, \beta_n) = \det((\sigma_i(\beta_j))_{i,j})^2$.

Proof. Note $Tr_{L/K}((\beta_i\beta_j)_{i,j}) = \sum_{k=1}^{n} \sigma_k(\beta_i)\sigma_k(\beta_j)$, and so $(Tr_{L/K}(\beta_i\beta_j))_{i,j} = Q^TQ$, where $Q = (\sigma_i(\beta_j))_{i,j}$. □

Definition 3.3. Let $\alpha_1, \cdots, \alpha_n \in L$. The Vandermonde matrix $Q(\alpha_1, \cdots, \alpha_n)$ with respect to those coefficients is

$$(\alpha_i^{j-1})_{i,j} = \begin{pmatrix} 1 & \alpha_1 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \cdots & \alpha_2^{n-1} \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \alpha_n & \cdots & \alpha_n^{n-1} \end{pmatrix}$$

Lemma 3.4. $\det(Q(\alpha_1, \cdots, \alpha_n)) = \prod_{1 \le i < j < n} (\alpha_j - \alpha_i)$.

Proof. Prove by induction. See notes. □

Proposition 3.5. Suppose $L = K(\alpha)$, then $D(1, \alpha, \cdots, \alpha^{n-1}) = \prod_{1 \le i < j \le n} (\alpha_j - \alpha_i)^2 \neq 0$.

Proof. Let $\alpha_i = \sigma_i(\alpha)$ for all i. Then $D(1, \alpha, \cdots, \alpha^{n-1}) = \det((\alpha_i^{j-1})_{i,j}) = \prod_{i<j}(\alpha_i - \alpha_i)^2$ by the lemma. □

Example 3.6. Suppose d is square-free and not 1, and consider $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. Now $D(1, \sqrt{d}) = (\sqrt{d} - (-\sqrt{d}))^2 = 4d$.

Corollary 3.7. Suppose f is a minimal polynomial of α, then the discriminant can be expressed as $D(1, \alpha, \cdots, \alpha^{n-1}) = (-1)^{\frac{n(n-1)}{2}} N_{L/K}(f'(\alpha))$, where f' is the derivative of f.

Proof. Left as an exercise using $f'(\alpha_j) = \prod_{i \neq j}(\alpha_j - \alpha_i)$. □

Corollary 3.8. $D(\beta_1, \cdots, \beta_n) \neq 0$ for any ordered basis $(\beta_1, \cdots, \beta_n)$ of L/K.

Now let A be a normal domain and suppose B/A is integral.

Definition 3.9. Suppose B is free of rank n over A, i.e, $B \cong A^n$ as an A-module. Let $(\beta_1, \cdots, \beta_n) \in B^n$ be an ordered basis of B over A. The discriminant of B/A relative to $(\beta_1, \cdots, \beta_n)$ is $D(\beta_1, \cdots, \beta_n)$.

Remark 3.10. This discriminant is well-defined up to multiplication up to an element of $(A^\times)^2$, i.e. square of a unit. Therefore, if $A = \mathbb{Z}$, the discriminant is well-defined, i.e. independence of choice.

In particular, we can define:

Definition 3.11. The discriminant $disc(K)$ of a number field K is the discriminant of $\mathcal{O}_k/\mathbb{Z}$ relative to some basis (but does not matter what choice we make).

Example 3.12. Suppose d is square-free and not 1 and $K = \mathbb{Q}(\sqrt{d})$, then $disc(K) = \begin{cases} d, & d \equiv 1 \pmod 4 \\ 4d, & d \equiv 2, 3 \pmod 4 \end{cases}$.

Suppose $K = Q(A)$ and L/K is finite separable, and let B be the integral closure of A in L, with $n = [L : K]$.

Lemma 3.13. Let $(\alpha_1, \cdots, \alpha_n) \in B^n$ be an ordered basis of L as a K-vector space. (Note that it exists.) Let $\beta \in L$ be such that $Tr_{L/K}(\alpha\beta) \in A$ for all $\alpha \in B$, then $disc(\alpha_1, \cdots, \alpha_n)\beta \in \sum_{i=1}^{n} A \cdot \alpha_i$.

Proof. We write $\beta = \sum_{i=1}^{n} a_i\alpha_i$ for some $a_i \in K$. Then $Tr_{L/K}(\alpha_i\beta) = \sum_{j=1}^{n} a_i Tr_{L/K}(\alpha_i\alpha_j) =: c_i$. Now let $Q = (Tr_{L/K}(\alpha_i\alpha_j))_{i,j}$, so $Q \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in A^n$. If we left multiply it by Q^*, the adjoint of Q, then $Q^*Q = dI_n$ for some $d \in A$. Note that by our definition we have $d = D(\alpha_1, \cdots, \alpha_n)$. Therefore, $A^n \ni Q^*Q \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = d \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$, and so $da_i \in A$ for all i, which means $d\beta \in B$. □

Corollary 3.14. Let $(\alpha_1, \cdots, \alpha_n) \in B^n$ be an ordered basis of L as a K-vector space. Then $\sum_{i=1}^{n} A\alpha_i \subseteq B \subseteq \sum_{i=1}^{n} Ad^{-1}\alpha_i$, with $d = D(\alpha_1, \cdots, \alpha_n)$.

Remark 3.15. We squeeze B between two free A-modules of rank n.

Definition 3.16. The rank of a module M over a domain A is $rank_A(M) = \dim_K(K \otimes_A M)$.

Corollary 3.17. Suppose in addition that A is Noetherian. Then B is a finitely-generated torsion-free A-module of rank $[L : K]$.

Proof. B is now a submodule of a free A-module, so it is finitely-generated. □

3.1 Fractional Ideal

Definition 3.18 (Fractional Ideal)**.** A fractional ideal of a Noetherian domain R is a non-zero finitely-generated R-submodule of $Q(R)$.

Proposition 3.19. Suppose in addition that A is Neotherian. Any fractional ideal of B is a finitely-generated A-module of rank n.

Proof. Suppose $\mathfrak{A} \subseteq Q(B)$ is a fractional ideal of B. If $\beta \in L^\times$, then $\beta : B \xrightarrow{\sim} B \cdot \beta$ that sends $x \mapsto \beta x$, so $B\beta$ has A-rank n. Take $\beta \in \mathfrak{A}$, then the rank of $\mathfrak{A}$ over A is bounded below by the rank of B over A, which is n. By assumption, $\mathfrak{A}$ is B-finitely generated in L, so there exists $\alpha \in A$ such that $\alpha\mathfrak{A} \subseteq B$. Now $\alpha : \mathfrak{A} \xrightarrow{\sim} \alpha\mathfrak{A} \subseteq B$, so the rank of $\mathfrak{A}$ over A is bounded above by the rank of B over A, which is n. □

Corollary 3.20. In a number field F, any fractional ideal of $\mathcal{O}_F$ is $\mathbb{Z}$-free of rank $[F : \mathbb{Q}]$.

Theorem 3.21. Suppose A is a Dedekind domain, and B is the integral closure of A in a finite separable extension of $Q(A)$. Then B is a Dedekind domain.

Proof. By corollary, B is a finitely-generated A-module, so any ideal $\mathfrak{b} \subseteq B$ is finitely-generated. Therefore, B is Noetherian as A is. Recall that $\dim(A) \leq 1$ indicates $\dim(B) \leq 1$, and we already know that A normal implies B normal, so we are done. □

Corollary 3.22. $\mathcal{O}_F$ is a Dedekind domain for any number field F.

Definition 3.23. A fractional ideal $\mathfrak{A}$ of a domain R is a non-zero R-submodule of $Q(R)$ such that there exists $d \in R\backslash\{0\}$ with $d\mathfrak{A} \subseteq R$.

Lemma 3.24. If R is a Noetherian domain, then a R-submodule $\mathfrak{A} \subseteq Q(R)$ is a fractional ideal if and only if it is R-finitely-generated.

Proof. Left as an exercise. □

Definition 3.25. $\mathfrak{A}^{-1} = \{b \in Q(R) \mid ab \in R \ \ \forall a \in \mathfrak{A}\}$.

Exercise 3.26. This is a fractional ideal if $\mathfrak{A}$ is.

Now for $b \in Q(R)$, we denote $(b) = Rb$ to be the principal fractional ideal. Then $(b)^{-1} = (b^{-1})$. Moreover, $\mathfrak{A}\mathfrak{B} = R \cdot (ab \mid a \in \mathfrak{A}, b \in \mathfrak{B})$ is also a fractional ideal. The intersection of two fractional ideals is also a fractional ideal. But in general, $\mathfrak{A} \cdot \mathfrak{A}^{-1} \neq R$.

Example 3.27. Note $(x, y) \subsetneq \mathbb{Q}[x, y]$, with $(x, y)^{-1} = \mathbb{Q}[x, y]$. But $(x, y)\cdot(x, y)^{-1} = (x, y) \neq \mathbb{Q}[x, y]$.

4 Lecture 4: September 30, 2022

Lemma 4.1. Let A be a Noetherian domain and $\mathfrak{A} \subseteq A$ is a nonzero ideal. Then

(a) There exists $k \geq 0$ and nonzero prime ideals $\mathfrak{P}_1, \cdots, \mathfrak{P}_k$ of A such that $\mathfrak{P}_1 \cdots \mathfrak{P}_k \subseteq \mathfrak{A}$.

(b) Suppose $\dim(A) \leq 1$. If $\mathfrak{P}_1, \cdots, \mathfrak{P}_k$ are as in (a) and $\mathfrak{P}$ is prime with $\mathfrak{A} \subseteq \mathfrak{P}$, then $\mathfrak{P} = \mathfrak{P}_i$ for some i.

Proof. (a) Let X be the set of non zero ideals $\mathfrak{B}$ of A such that there does not exist primes $\mathfrak{P}'_1, \cdots, \mathfrak{P}'_l$ with $\mathfrak{P}'_1 \cdots \mathfrak{P}'_l \subseteq \mathfrak{B}$. Suppose $X \neq \varnothing$. Order X by the partial relation $\subseteq$. Any chain in X has a maximal element since A is Noetherian. Therefore, X has a maximal element $\mathfrak{A}$ by Zorn's Lemma. In particular, $\mathfrak{A}$ is not a prime ideal. Therefore, there exists $a, b \in A \backslash \mathfrak{A}$ such that $ab \in \mathfrak{A}$. Consider $\mathfrak{A} + (a)$ and $\mathfrak{A} + (b)$ which contain $\mathfrak{A}$. So both ideals are not in X, which means there exists $\mathfrak{P}_1, \cdots, \mathfrak{P}_m$ and $\mathfrak{Q}_1, \cdots, \mathfrak{Q}_n$ such that $\mathfrak{P}_1 \cdots \mathfrak{P}_m \subseteq \mathfrak{A} + (a)$ and $\mathfrak{Q}_1 \cdots \mathfrak{Q}_n \subseteq \mathfrak{A} + (b)$. Then $\mathfrak{P}_1 \cdots \mathfrak{P}_m \mathfrak{Q}_1 \cdots \mathfrak{Q}_n \subseteq (\mathfrak{A} + (a))(\mathfrak{A} + (b)) \subseteq \mathfrak{A}$, contradiction.

(b) Consider $\mathfrak{P}_1 \cdots \mathfrak{P}_k \subseteq \mathfrak{A} \subseteq \mathfrak{P}$. If $\mathfrak{P} \neq \mathfrak{P}_i$, since $\mathfrak{P}_i$ is maximal, then there exists $b_i \in \mathfrak{P}_i$ with $b_i \notin \mathfrak{P}$. If $\mathfrak{P} \neq \mathfrak{P}_i$ for all i, then $b_1 \cdots b_k \notin \mathfrak{P}$ as $\mathfrak{P}$ is prime. But $b_1 \cdots b_k \in \mathfrak{P}_1 \cdots \mathfrak{P}_k \subseteq \mathfrak{P}$, contradiction.

□

Lemma 4.2. Let A be a Dedekind domain and $\mathfrak{P} \subseteq A$ be a nonzero prime ideal. Then $\mathfrak{P} \cdot \mathfrak{P}^{-1} = A$.

Proof. Let $a \in \mathfrak{P} \backslash \{0\}$. By Lemma 4.1, we take $k \geq 1$ minimal such that $\mathfrak{P}_1 \cdots \mathfrak{P}_k \subseteq (a)$, and without loss of generality we take $\mathfrak{P}_k = \mathfrak{P}$. Let $b \in \mathfrak{P}_1 \cdots \mathfrak{P}_{k-1}$, $b \notin (a)$. Then $a^{-1}b \notin A$. But $a^{-1}b\mathfrak{P} \subseteq a^{-1}\mathfrak{P}_1 \cdots \mathfrak{P}_k \subseteq A$, so $a^{-1}b \in \mathfrak{P}^{-1}$. If $\mathfrak{P}^{-1}\mathfrak{P} = \mathfrak{P}$, then $a^{-1}b\mathfrak{P} \subseteq \mathfrak{P}$. Since $\mathfrak{P}$ is a finitely-generated faithful A-module, then $a^{-1}b$ is integral over A. But A is integrally closed, so $a^{-1}b \in A$, contradiction, so $\mathfrak{P}^{-1}\mathfrak{P} \neq \mathfrak{P}$. Now this is an ideal bigger than $\mathfrak{P}$, so it has to be the whole ring since $\mathfrak{P}$ is maximal, i.e. $\mathfrak{P}^{-1}\mathfrak{P} = A$. □

Theorem 4.3. Let A be a Dedekind domain and $\mathfrak{A}$ is a fractional ideal of A. Then there exists $k \geq 0$ and nonzero prime ideals $\mathfrak{P}_1, \cdots, \mathfrak{P}_k$, and integers $r_1, \cdots, r_k \neq 0$ such that $\mathfrak{A} = \mathfrak{P}_1^{r_1} \cdots \mathfrak{P}_k^{r_k}$. Moreover, this factorization is unique up to reordering. If $\mathfrak{A} \subseteq \mathfrak{A}$ as an ideal, then $r_i \geq 1$ for all i.

Proof. Suppose $\mathfrak{A} \subseteq A$ is a nonzero ideal. If $\mathfrak{A} \neq A$ ($m \neq 0$), there exists $m \geq 1$ such that there exists nonzero ideals $\mathfrak{Q}_1, \cdots, \mathfrak{Q}_m$ of A with $\mathfrak{Q}_1 \cdots \mathfrak{Q}_m \subseteq \mathfrak{A}$, according to Lemma 4.1. Without loss of generality, $\mathfrak{Q}_m \supseteq \mathfrak{A}$. Then $\mathfrak{Q}_1 \cdots \mathfrak{Q}_{m-1} = \mathfrak{Q}_1 \cdots \mathfrak{Q}_m \mathfrak{Q}_m^{-1} \subseteq \mathfrak{A}\mathfrak{Q}_m^{-1} \subseteq A$. By induction on m, there exists primes $\mathfrak{Q}'_1, \cdots, \mathfrak{Q}'_l$ of A such that $\mathfrak{Q}'_1 \cdots \mathfrak{Q}'_l = \mathfrak{A} = \mathfrak{Q}_m^{-1}$. So $\mathfrak{A}$ has a factorization into primes.

In general, suppose $\mathfrak{A}$ is a fractional ideal. Let $d \in A \backslash \{0\}$ such that $d\mathfrak{A} \subseteq A$. Then $d\mathfrak{A} = \mathfrak{P}_1 \mathfrak{P}_k$ with some primes $\mathfrak{P}_i$ and $(d) = \mathfrak{P}'_1 \cdots \mathfrak{P}'_l$, so $\mathfrak{A} = \mathfrak{P}_1 \cdots \mathfrak{P}_k (\mathfrak{P}'_1)^{-1} \cdots (\mathfrak{P}'_l)^{-1}$. For uniqueness, if $\mathfrak{P}_1^{r_1} \cdots \mathfrak{P}_k^{r_k} = \mathfrak{Q}_1^{s_1} \cdots \mathfrak{Q}_l^{s_l}$ with $r_i, s_j \geq 1$ for all i, j, then the right-hand-side contains $\mathfrak{P}_k$, so there exists $\mathfrak{Q}_i$ (say $i = l$ without loss of generality) such that $\mathfrak{P}_k = \mathfrak{Q}_i$ by Lemma 4.1. Then $\mathfrak{P}_1^{r_1} \cdots \mathfrak{P}_{k-1}^{r_{k-1}} \mathfrak{P}_k^{r_k - 1} = \mathfrak{Q}_1^{s_1} \cdots \mathfrak{Q}_{l-1}^{s_{l-1}} \mathfrak{Q}_l^{s_l - 1}$,. By induction on the sum of r_i's ($\sum_{i=1}^{r} s_i$), there are the same factorizations up to the reordering of primes. □

Definition 4.4 (Divides)**.** A nonzero ideal $\mathfrak{b}$ of a commutative ring divides an ideal $\mathfrak{a}$ if there exists an ideal $\mathfrak{c}$ such that $\mathfrak{bc} = \mathfrak{a}$.

Let A be a Dedekind domain.

Corollary 4.5. Suppose $\mathfrak{A}, \mathfrak{B}$ are nonzero ideals of A.

(a) $\mathfrak{A}$ and $\mathfrak{B}$ have no common divisors if and only if $\mathfrak{A} + \mathfrak{B} = A$, i.e. $\gcd(\mathfrak{A}, \mathfrak{B}) = A$.

(b) $\mathfrak{A} \subseteq \mathfrak{B}$ if and only if $\mathfrak{B} \mid \mathfrak{A}$.

Definition 4.6 (Ideal Group)**.** The ideal group $I(A)$ of A is the group of fractional ideals of A under $\cdot$.

By the theorem, $I(A)$ is a free Abelian group on the nonzero prime ideals of A.

Definition 4.7 (Principal Ideal Group, Ideal Class Group)**.** The principal ideal group $P(A)$ is the subgroup of $I(A)$ of principal fractional ideals.

The class group $Cl(A)$ of A is $I(A)/P(A)$.

Exercise 4.8. The class group is trivial if and only only if A is a PID.

Proposition 4.9. A Dedekind domain A is a PID if and only if it is a UFD.

Proof. Let A be a Dedekind UFD. Let $P \in I(A)$ be prime. If $a \in P \backslash \{0\}$, there exists irreducible element π in A such that $\pi \mid a$ and $\pi \in P$ since P is prime. But (π) is maximal as $\dim(A) \leq 1$, so $P = (\pi)$. Then the unqiue factorization of ideals implies A is a PID. □

Definition 4.10 (Class Group)**.** The class group Cl_F of a number field F is $Cl(\mathcal{O}_F)$. (Set $I_F = I(\mathcal{O}_F), P_F = P(\mathcal{O}_F)$). Then there is a map from $\mathfrak{A} \in I(A)$ to $[\mathfrak{A}] \in Cl(A)$.

Example 4.11. $F = \mathbb{Q}(\sqrt{-5})$ and $\mathcal{O}_F = \mathbb{Z}[\sqrt{-5}]$. Then $Cl_{\mathbb{Q}(\sqrt{-5})} \neq 0$. In fact, $[\mathfrak{A}] \neq 0$ for $\mathfrak{A} = (2, 1 + \sqrt{-5})$.

Here $N_{F/\mathbb{Q}}(2) = 4$ and $N_{F/\mathbb{Q}}(1 + \sqrt{-5}) = 6$, so if $\mathfrak{A} = (x)$, then $N_{F/\mathbb{Q}}(x) \in \{\pm 1, \pm 2\}$. But $N_{F/\mathbb{Q}}(a + b\sqrt{-5}) = a^2 + 5b^2$ forces $x = \pm 1$. Therefore, A is the whole ring. This is a contradiction, because

$$\mathbb{Z}[\sqrt{-5}]/(2, 1 + \sqrt{-5}) \cong \mathbb{Z}[x]/(x^2 + 5, 2, 1 + x) \cong \mathbb{Z}[x]/(2, 1 + x) \cong \mathbb{Z}/2\mathbb{Z} = \mathbb{F}_2 \neq 0.$$

Hence, $x \neq \pm 1$, and so $(2, 1 + \sqrt{-5})$ is not principal.

Exercise 4.12. In a Dedekind domain, every ideal can be generated by two elements.

5 Lecture 5: October 3, 2022

5.1 Discrete Valuation Ring

Proposition 5.1. Any localization of a Dedekind domain is Dedekind.

Definition 5.2 (Discrete Valuation Ring)**.** A discrete valuation ring (DVR) is a PID with exactly on non-zero prime ideal. The prime ideal therefore has a generator. A generator of this ideal is therefore called a uniformizer.

Proposition 5.3. Let A be a domain, then A is a DVR if and only if it is a local Dedekind domain which is not a field.

Proof. ($\Rightarrow$): PID implies Dedekind.

($\Leftarrow$): Let $\mathfrak{p} \neq 0$ be the unique prime ideal of A. Choose $\pi \in \mathfrak{p} - \mathfrak{p}^2$, then $(\pi) = \mathfrak{p}^n$ for some n, so $n = 1$, then $\mathfrak{p}^n = (\pi^n)$, so A is a PID. □

Theorem 5.4. A Noetherian domain is Dedekind if and only if its localization at every nonzero prime ideal is a DVR.

Proof. ($\Rightarrow$): By the proposition, it is trivial.

($\Leftarrow$): Consider A where $A_\mathfrak{p}$ is a DVR for all $\mathfrak{p} \neq 0$. Let B be the intersection of $A_\mathfrak{p}$ for nonzero prime $\mathfrak{p}$. Let $\frac{c}{d} \in B$, $c \in A$ and $d \in A\backslash\{0\}$. Set $\mathfrak{A} = \{a \in A \mid ac \in (d)\}$. We have $\frac{c}{d} = \frac{r}{s}$ with $r \in A$ and $s \in A\backslash\mathfrak{p}$. Therefore, $sc = rd \in (d)$, then by definition $s \in \mathfrak{A}$. Then $\mathfrak{A} \not\subseteq \mathfrak{p}$ for all $\mathfrak{p}$, so $\mathfrak{A} = A$. But that means $1 \in \mathfrak{A}$, so $c \in (d)$, and $\frac{c}{d} \in A$. Therefore, $B = A$. Now each $A_\mathfrak{p}$ is normal, so $B = A$ is normal. Suppose $\mathfrak{q} \neq 0$ is a prime ideal in A. Let $\mathfrak{m} \supseteq \mathfrak{q}$ be a maximal ideal. Then $\mathfrak{q}A_\mathfrak{m}$ is a nonzero prime ideal of the DVR $A_\mathfrak{m}$, but then $\mathfrak{q}A_\mathfrak{m} = \mathfrak{m}A_\mathfrak{m}$. Note $\mathfrak{q} = A \cap \mathfrak{q}A_\mathfrak{m}$ (exercise) as $\mathfrak{q} \subseteq \mathfrak{m}$. So $\mathfrak{q} = A \cap \mathfrak{q}A_m = A \cap \mathfrak{m}A_\mathfrak{m} = \mathfrak{m}$. Therefore, $\dim(A) \leq 1$. □

Definition 5.5 (Discrete Valuation)**.** A discrete valuation v on a field K is a surjective function $v : K \to \mathbb{Z} \cup \{\infty\}$ such that

1. $v(a) = \infty$ if and only if $a = 0$, and
2. $v(ab) = v(a) + v(b)$, and
3. $v(a + b) \geq \min(v(a), v(b))$ for all $a, b \in K$.

We call $v(a)$ the valuation of a. (K, v) is called a discrete valuation field.

Remark 5.6. $v(a + b) = \min(v(a), v(b))$ if $v(a) \neq v(b)$.
$v(1) = 0$.
$v(-a) = v(a)$.

Definition 5.7 (Valuation Ring)**.** The valuation ring of v is $\mathcal{O}_v = \{a \in K \mid v(a) \geq 0\}$.

Lemma 5.8. $\mathcal{O}_v$ is a DVR with maximal ideal $\mathfrak{m}_v = \{a \in K \mid v(a) \geq 1\}$.

Proof. Take $\pi \in \mathcal{O}_v$ with $v(\pi) = 1$. Any $a \in \mathcal{O}_v$ with $v(a) = n$ has $v(a\pi^{-n}) = 0$. So $u = a\pi^{-n} \in \mathcal{O}_v$ and this is a unit. Then $a = u\pi^n$. Thus, $\mathcal{O}_v$ is a DVR with uniformizer π. □

Definition 5.9 (p-adic Valuation)**.** Let A be Dedekind with $Q(A) = K$ and p is a prime in A. The p-adic valuation of A is $v_p : K \to \mathbb{Z} \cup \{\infty\}$ given by $(a) = p^{v_p(a)}\mathfrak{b}\mathfrak{c}^{-1}$ where $p \nmid \mathfrak{b}\mathfrak{c}$, for $a \in K^\times$.

Remark 5.10 (Why is this a valuation?)**.** It suffices to check the last property. Note that for $a, b \in K^\times$, $(a+b) = p^{v_p(a+b)} \subseteq (a) + (b) = p^{v_p(a)}\frac{\mathfrak{b}}{\mathfrak{c}} + p^{v_p(b)}\frac{\mathfrak{b}'}{\mathfrak{c}'} = p^{\min(v_p(a),v_p(b))}\frac{\mathfrak{b}''}{\mathfrak{c}''}$. Therefore, $v_p(a+b) \geq \min(v_p(a), v_p(b))$.

Remark 5.11. Valuation ring of v_p is A_p.

Example 5.12. Let p be a prime. Then $v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ with $v_p = v_{(p)}$ is a p-adic valuation. Now $\mathcal{P}_{v_p} = \mathbb{Z}_{(p)} = \{\frac{c}{d} \mid c, d \in \mathbb{Z}, p \nmid d\}$.

Example 5.13. Let K be a field. $v_\infty : K(t) \to \mathbb{Z} \cup \{\infty\}$ is given by $v_\infty(\frac{f}{g}) = \deg(g) - \deg(f)$ for $f.g \in K[t]$ and $g \neq 0$.[4] Now consider $A = K[t^{-1}]$, then $v_\infty = v_{(t^{-1})}$. In particular, $\mathcal{O}_v = A_{(t^{-1})} = K[t^{-1}]_{(t^{-1})}$.

5.2 Orders

Definition 5.14 (Order)**.** An order R in a normal domain $A \subseteq Q(R)$ is a Noetherian subring of Krull dimension at most 1 with integral closure A.

Lemma 5.15. An integral extension B of an order R that is a domain and finitely-generated as an R-algebra is also an order.

Theorem 5.16 (Krull-Akizuki)**.** Let A be a Noetherian domain with $\dim(A) \leq 1$ and $K = Q(A)$. Let L/K be a finite extension and B is any subring of L containing A. Then B is Noetherian and $\dim(B) \leq 1$.

Corollary 5.17. Let A be an order and $K = Q(A)$ and L/K is a finite extension and B is the integral closure of A in L. Then B is a Dedekind domain.

In particular, for a number field F, we know that any subring of F is finitely-generated over $\mathbb{Z}$ if and only if it is contained in $\mathcal{O}_F$. So an order in $\mathcal{O}_F$ is exactly a subring that is finitely-generated over $\mathbb{Z}$ and has rank $[F : \mathbb{Q}]$.

Example 5.18. Let F be a number field and $F = \mathbb{Q}(\alpha)$ where $\alpha \in \mathcal{O}_F$. Then $\mathbb{Z}[\alpha] \subseteq \mathcal{O}_F$ is an order.

Definition 5.19 (Discriminant)**.** The discriminant $disc(R)$ of an order R in $\mathcal{O}_F$ is its discriminant relative to a $\mathbb{Z}$-basis.

Remark 5.20. $disc(R) = [\mathcal{O}_F : R]^2 disc(\mathcal{O}_F)$. So if $disc(R)$ is square-free, then $R = \mathcal{O}_F$.

[4] Here we assume $\deg(0) = -\infty$.

Definition 5.21 (Conductor)**.** Let R be an order with integral closure A. The conductor f_R of R is $f_R = \{a \in A \mid aA \subseteq R\}$.

Remark 5.22. f_R is the largest ideal of A contained in R, so it is also an ideal of R.

Lemma 5.23. $f_R \neq 0$ if and only if A is a finitely-generated R-module.

Proof. ($\Leftarrow$): Let A be finitely-generated as an R-module, so $A = \sum_{i=1}^{m} Ra_i$, then there exists $r_i \in R\backslash\{0\}$ such that $r_i a_i \in R$ (as $A \subseteq Q(R)$). Now $r_1 \cdots r_m \in f_R$, which is nonzero, and we are done.

($\Rightarrow$): Consider $r \in f_R\backslash\{0\}$ and $r : AA \hookrightarrow R$ is the map $x \mapsto rx$ and $rA \cong A$ (as R-modules), so R is Notherian implies rA is finitely generated over R (since it is an ideal of R). $\square$

6 Lecture 6: October 5, 2022

Example 6.1. Suppose $d \neq 1$ is square-free, then $f_{\mathbb{Z}[\sqrt{d}]} = \begin{cases} \mathbb{Z}[\sqrt{d}], & d \equiv 2,3 \pmod 4 \\ 2\mathbb{Z}[\sqrt{d}], & d \equiv 1 \pmod 4 \end{cases}$.

Lemma 6.2. Let A be Dedekind and $K = Q(A)$, and L/K is a finite extension and B is the integral closure of A in L. Suppose $L = K(\alpha)$ with $\alpha \in B$, then $D(1, \alpha, \cdots, \alpha^{n-1}) \in f_{A[\alpha]}$.

Proposition 6.3. Let R be an order and $\mathfrak{p} \subseteq R$ is a nonzero prime ideal and A is the integral closure of R in $Q(R)$. Suppose $f_R \neq 0$, then $\mathfrak{p} \not\supseteq f_R$ if and only if $R_\mathfrak{p}$ is a DVR.

Example 6.4. Consider $\mathbb{Z}[\sqrt{5}]$ with $\mathfrak{p} = (2, 1 - \sqrt{5})$. Then

$$\mathbb{Z}[\sqrt{5}]/\mathfrak{p} \cong \mathbb{Z}[x]/(x^2 - 5, 2, 1 - x) \cong \mathbb{F}_2[x]/(x - 1) \cong \mathbb{F}_2,$$

so $\mathfrak{p}$ is prime. Now $\mathfrak{p} \supset (2) = f_{\mathbb{Z}[\sqrt{5}]}$, and $\mathcal{O}_{\mathbb{Q}(\sqrt{5})} = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$, and the ideal $\mathfrak{p}A = (2)$ is prime:

$$\mathbb{Z}[\frac{\sqrt{5}-1}{2}]/(2) \cong \mathbb{Z}[x]/(x^2 + x - 1, 2) \cong \mathbb{F}_2[x]/(x^2 + x + 1) \cong \mathbb{F}_4[x].$$

Therefore, we have an embedding $\mathbb{Z}[\sqrt{5}]/\mathfrak{p} \hookrightarrow A_{\mathfrak{p}A}/\mathfrak{p}A$, but their isomorphism fields give $\mathbb{F}_2 \hookrightarrow \mathbb{F}_4$ is not an isomorphism, and so $\mathbb{Z}[\sqrt{5}]/\mathfrak{p} \ncong A_{\mathfrak{p}A}$ with $A = \mathcal{O}_{\mathbb{Q}(\sqrt{5})}$. Hence, $\mathfrak{p}\mathbb{Z}[\sqrt{5}]_\mathfrak{p}$ is not principal: for $v_{(2)} : \mathbb{Q}(\sqrt{5}) \to \mathbb{Z} \cup \{\infty\}$, we have $v_{(2)}(2) = 1$ and $v_{(2)}(\sqrt{5} - 1) = 1$. So if $2a + (\sqrt{5} - 1)b$ generates $p\mathbb{Z}[\sqrt{5}]_\mathfrak{p}$, it has 2-adic valuation 1, and then it is associated to 2 (say $b \notin \mathfrak{p}\mathbb{Z}[\sqrt{5}]_\mathfrak{p}$), for example

$$\frac{2a + b(\sqrt{5} - 1)}{2} \in \mathbb{Z}[\sqrt{5}]_\mathfrak{p}^\times$$

which means $b\frac{\sqrt{5}-1}{2} \in \mathbb{Z}[\sqrt{5}]_\mathfrak{p}$, contradiction.

This shows that the order is not a DVR, and therefore the proposition fails.

Proof. Suppose $f_R \not\subseteq \mathfrak{p}$. Let $x \in f_R$ and $x \notin \mathfrak{p}$. Then $xA \subseteq R$ and $x \in R_\mathfrak{p}^\times$. Thus, $A \subseteq R_\mathfrak{p}$. Let $\mathfrak{q} = A \cap \mathfrak{p}R_\mathfrak{p}$ be a prime ideal of A. containing $\mathfrak{p}$. As $\mathfrak{q} \cap R$ is prime in R, $\mathfrak{p} = \mathfrak{q} \cap R$ as $\dim(R) \leq 1$. Note $R_\mathfrak{p} \subseteq A_\mathfrak{q}$. If $\frac{a}{s} \in A_\mathfrak{q}$ with $a \in A$ and $s \in A \backslash \mathfrak{q}$, then $xa \in R$ and $xs \in R \backslash \mathfrak{p}$, and so $\frac{a}{s} = \frac{xa}{xs} \in R_\mathfrak{p}$. Therefore, $R_\mathfrak{p} = A_\mathfrak{q}$.

Claim 6.5. $\mathfrak{q} = \mathfrak{p}A$.

Subproof. Note $\mathfrak{q} \mid \mathfrak{p}A$ by definition. If $\mathfrak{q}'$ prime with $\mathfrak{q}' \mid \mathfrak{p}A$, then $A_{\mathfrak{q}'} \supseteq R_\mathfrak{p} = A_\mathfrak{q}$. Since $\mathfrak{q}'$ is maximal, then $A_{\mathfrak{q}'} = A_\mathfrak{q}$, and so $\mathfrak{q}' = \mathfrak{q}$. Thus, $\mathfrak{p}A = \mathfrak{q}^e$ for some $e \geq 1$. Therefore, $\mathfrak{p}R_\mathfrak{p} = \mathfrak{p}A_\mathfrak{q} = \mathfrak{q}^e A_\mathfrak{q}$, which is maximal in $R_\mathfrak{p} = A_\mathfrak{q}$, so $e = 1$. Thus, $\mathfrak{p}A = \mathfrak{q}$. ■

Conversely, suppose $R_\mathfrak{p}$ is a DVR. Then $R_\mathfrak{p}$ is normal. Since A is integrally closed in R, then $A \subseteq R_\mathfrak{p}$, then $\mathfrak{p} = R \cap \mathfrak{p}R_\mathfrak{p} \supseteq R \cap \mathfrak{p}A$, so $\mathfrak{p} = R \cap \mathfrak{p}A$. Write $A = \sum_{i=1}^{n} Ra_i$, where $a_i \in A$ for $1 \leq i \leq n$ (since $f_R \neq 0$). Then $a_i = \frac{y_i}{s_i}$ for $y_i \in R$ and $s_i \in R \backslash \mathfrak{p}$. Take $s = s_1 \cdots s_n$, now $sa_i \in R$ for all i, and so $s \in f_R$, and $s \notin \mathfrak{p}$. So $\mathfrak{p} \not\supseteq f_R$. □

Lemma 6.6. Let F be a number field and $R \subseteq \mathcal{O}_F$ be an order. The prime numbers dividing $[\mathcal{O}_F : R]$ are exactly those dividing a generator of $f_R \cap \mathbb{Z}$.

Proof. $f_R \cap \mathbb{Z} = (f)$. Now $f\mathcal{O}_F \subseteq R$, so f is a multiple of exponent of $\mathcal{O}_{F/R}$. If $p \mid f$ but not $[\mathcal{O}_F : R]$, there exists prime $\mathfrak{p}$ of $\mathcal{O}_F$ with $\mathfrak{p} \cap \mathbb{Z} = (p)$ and dividing f_R. Set $\mathfrak{g} = \mathfrak{p}^{-1} f_R$ as an ideal of $\mathcal{O}_F$.

Now there is $p : \mathcal{O}_F/R \xrightarrow{\sim} \mathcal{O}_F/R$, so $\mathfrak{g}\mathcal{O}_F/R = \mathfrak{p}\mathfrak{g}\mathcal{O}_F/R \subseteq f_R\mathcal{O}_F/R = 0$. Therefore, $\mathfrak{g}$ is contained in the conductor, but it is not by definition, contradiction. □

6.1 Ramification

Let A be Dedekind and $K = Q(A)$ and L/K is a finite extension and B is integral closure of A in L.

Because $\mathfrak{p}$ is prime in A, then $\mathfrak{p}B = \prod_{i=1}^{g} P_i^{e_i}$ where P_i's are distinct primes and $e_i \geq 1$ for all i, and for some $g \geq 1$.

Definition 6.7. (a) $\mathfrak{p}$ ramifies in B/A (or L/K) if $e_i \geq 2$ for some i. We then say P_i is ramified in B/A.

(b) $\mathfrak{p}$ is inert in B/A if $\mathfrak{p}B$ remains prime.

(c) $\mathfrak{p}$ splits in B/A if $g \geq 2$.

Example 6.8. Let $A = \mathbb{Z}$, and $L = \mathbb{Q}(\sqrt{-5})$ and $\mathcal{O}_L = \mathbb{Z}[\sqrt{-5}]$. Now (2) ramifies $(2, 1 - \sqrt{-5})^2 = (2)$, and (5) ramifies $(\sqrt{-5})^2 = (5)$. (3) splits: $\mathbb{Z}[x]/(3, x^2 + 5) \cong \mathbb{F}_3[x]/(x^2 - 1) \cong \mathbb{F}_3 \times \mathbb{F}_3$, and $(3) = (3, \sqrt{-5} - 1)(3, \sqrt{5} + 1)$.

(7) also splits, and (11) is inert: $-5 \notin \mathbb{F}_{11}^{\times 2}$, $\mathbb{Z}[\sqrt{-5}]/(11) \cong \mathbb{F}_{121}$.

Definition 6.9 (Residue Field)**.** The residue field of $\mathfrak{p}$ is $A/\mathfrak{p}$.

Remark 6.10. If $P \mid \mathfrak{p}$, then $(B/P)/(A/\mathfrak{p})$ becomes a field extension, an extension of residue field.[5]

Definition 6.11. $e_{P/\mathfrak{p}}$, called the ramification index, is the largest e such that $P^e \mid \mathfrak{p}$.

$f_{R/\mathfrak{p}}$, called the residue degree, is $[B/P : A/\mathfrak{p}]$.

Definition 6.12 (Lying Over, Lying Under)**.** If $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals of A and B, respectively, such that $\mathfrak{q} \cap A = \mathfrak{p}$, (note that $\mathfrak{q} \cap A$ is automatically a prime ideal of A,) then we say that $\mathfrak{p}$ lies under $\mathfrak{q}$ and that $\mathfrak{q}$ lies over $\mathfrak{p}$.

A ring extension $A \subseteq B$ of commutative rings is said to satisfy the lying over property if every prime ideal $\mathfrak{p}$ of A lies under some prime ideal $\mathfrak{q}$ of B.

[5] We usually say P is a prime of B lying over $\mathfrak{p}$.

7 Lecture 7: October 7, 2022

As usual, let A be a Dedekind domain, $K = Q(A)$, L/K is a finite extension, and B is the integral closure of A in L.

Theorem 7.1. Write $\mathfrak{p}R = P_1^{e_1} \cdots P_g^{e_g}$ with P_i distinct and $e_i \geq 1$, then $e_i = e_{P_i}/\mathfrak{p}$ and set $f_i = f_{P_i}/\mathfrak{p}$, then $\sum_{i=1}^{g} e_i f_i = [L : K]$.

We will use the folloing lemma to prove the theorem.

Lemma 7.2. Suppose $S \subseteq A$ is a multiplicatively closed set. Let P be a set of primes of A such that $S \cap \mathfrak{q} = \varnothing$ for all $\mathfrak{q} \in P$. Let $\mathfrak{a}$ be a nonzero ideal of A is divisible only by primes in P, then

$$A/\mathfrak{a} \xrightarrow{\sim} S^{-1}A/S^{-1}\mathfrak{a}.$$

Proof. Injective: Let $b \in S^{-1}\mathfrak{a} \cap A$, then $b = \frac{a}{s}$ for $a \in \mathfrak{a}$ and $s \in S$. Therefore, $(s) + \mathfrak{a} = A$. Then $b \in \mathfrak{a}$ by the unique factorization into primes.

Surjective: For $c \in A$ and $t \in S$, we have $(t) + \mathfrak{a} = A$, so there exists $u \in A$ such that $ut - 1 \in A$. Then we have $cu + \mathfrak{a} \mapsto \frac{c}{t} + S^{-1}\mathfrak{a}$. □

Proof of Theorem. When considering them as $A/\mathfrak{p}$-algebras, we have

$$B/\mathfrak{p}B = \prod_{i=1}^{g} B/P_i^{e_i}$$

by the Chinese Remainder Theorem. Now $\dim_{A/\mathfrak{p}A} B/\mathfrak{p}B = \sum_{i=1}^{g} \sum_{j=r}^{e_i-1} \dim_{A/P_i}(P_i^j/P_i^{j+1})$. This equals to $\sum_{i=1}^{g} e_i f_i$ because

$$P_i^j/P_i^{j+1} \cong P_i^j B_{P_i}/P_i^{j+1} B_{P_i}$$

is one-dimensional over $B_{P_i}/P_i \cong B/P_i$. Consider $S_{\mathfrak{p}} = A \backslash \mathfrak{p}$, then $S_{\mathfrak{p}}^{-1}B$ is the integral closure of A_p in L, so $S_{\mathfrak{p}}^{-1}B$ is free of rank $[L : K]$ over $A_{\mathfrak{p}}$. Then $B/\mathfrak{p}B$ is isomorphic to $S_{\mathfrak{p}}^{-1}B/\mathfrak{p}S_{\mathfrak{p}}^{-1}B$ by lemma, and so it is $[L : K]$-dimensional over $A/\mathfrak{p}$, i.e. $\dim_{A/\mathfrak{p}}(B/\mathfrak{p}B) = [L : K]$. □

Example 7.3. Let $[L : K] = 2$. Now $\mathfrak{p}B = P_1P_2$ splits where P_1 and P_2 have residue degree 1, and $\mathfrak{p} = P_1^2$ ramified has residue degree 1, and $\mathfrak{p} = P_1$ inert has residue degree 2.

Let $[L : K] = 3$. Now $\mathfrak{p}B = P_1P_2P_3$ is completely split and each P_i has residue degree 1. The possibilities are $P_1^2P_2$, where each has residue degree 1, and P_1P_2, where P_1 has degree 2 and P_2 has degree 1, and P_1^3 which is totally ramified with degree 1, and P_1 which is inert with degree 3.

Theorem 7.4 (Kummer-Dedekind)**.** Let $h \in A[x]$ be the minimal polynomial of α, and $\bar{h} \in A/\mathfrak{p}[x]$ is its reduction modulo $\mathfrak{p}$. Suppose $\mathfrak{p}B + f_{A[\alpha]} + B$. Write $\bar{h} = \bar{h}_1^{e_1} \cdots \bar{h}_g^{e_g}$, with $\bar{h}_i$ distinct irreducible with $e_i \geq 1$. Let $h_i \in A[x]$ be a lift of $\bar{h}_i$. Set $P_i = \mathfrak{p}B + (h_i(\alpha))$. Then P_i's are distinct primes over $\mathfrak{p}$, and $\mathfrak{p}B = \prod_{i=1^g} P_i^{e_i}$, and $f_{P_i/\mathfrak{p}} = \deg(\bar{h}_i)$.

Proof. Set $F = A/\mathfrak{p}$, then

$$\begin{aligned} A[\alpha]/\mathfrak{p}A[\alpha] &\cong A[x]/(\mathfrak{p}A[x] + (h)) \\ &\cong F[x]/(\bar{h}) \\ &\cong \prod_{i=1}^{g} F[x]/(\bar{h}_i^{e_i}). \end{aligned}$$

Let $Q_i = \mathfrak{p}A[\alpha] + (h_i(\alpha)) \subseteq A[\alpha]$ and let $\varphi_i : A[\alpha] \twoheadrightarrow F[x]/(\bar{h}_i^{e_i})$.

Claim 7.5. $\ker(\varphi_i) = Q_i^{e_i}$.

Subproof. Since the $\bar{h}_i$'s are relatively prime, so are the Q_i's, and $A[\alpha]/Q_i \cong F[x]/(\bar{h}_i)$ so Q_i's are prime. Therefore, $[A[\alpha]/Q_i : F] = f_i := \deg(\bar{h}_i)$. Then $A[\alpha]/Q^{e_i} \cong A[\alpha]_{Q_i}/Q_i^{e_i}A[\alpha]_{Q_i}$. Since $Q_i = P_i \cap A[\alpha]$, $f_{A[\alpha}$'s are prime, and the ring $A[\alpha]_{Q_i}$ is a DVR, so only ideals of $A[x]/Q_i^{e_i}$ are $Q_i^j/Q_i^{e_i}$ for $0 \leq j \leq e_i$. Therefore,

$$\ker(A[\alpha]/Q_i^{e_i} \twoheadrightarrow F[x]/(\bar{h}_i^{e_i})) = 0,$$

which means $\ker(\varphi_i) = Q_i^{e_i}$. ■

Now we know

$$\prod_{i=1}^{g} A[\alpha]/Q_i^{e_i} \cong \prod_{i=1}^{g} F[x]/(\bar{h}_i^{e_i})$$

and so $\mathfrak{p}A[\alpha] = \prod_{i=1}^{g} Q_i^{e_i}$, and so $\mathfrak{p}B = \prod_{i=1}^{g} P_i^{e_i}$.

Now $P_i = Q_iB$ is prime and the residue fields $B_{P_i} \cong A[\alpha]_{Q_i}$, so P_i are distinct and $f_{P_i/\mathfrak{p}} = \deg(\bar{h}_i)$. □

Example 7.6. Let $h(x) = x^3 + x + 1$, then it is irreducible in $\mathbb{Q}[x]$. Let $L = Q(\alpha)$ and $h(\alpha) = 0$. Exercise: the discriminant of $\mathbb{Z}[\alpha] = -31$. Therefore, the discriminant is square-free, so $\mathcal{O}_L = \mathbb{Z}[\alpha]$. Now $h(x)$ is irreducible modulo 2, so (2) is inert in L. Also, $h(x) = (x-1)(x^2+x-1)$ modulo 3, so $3\mathbb{Z}[\alpha] = P_1P_2$ with residue degree 1 and 2 respectively, where $P_1 = (3, \alpha - 1)$ and $P_2 = (3, \alpha^2 + \alpha - 1)$.

Corollary 7.7. Let p be an odd prime and $a \in \mathbb{Z}$ is square-free with $p \nmid a$. Then $a \in \mathbb{F}_p^{\times 2}$ if and only if (p) splits in $\mathbb{Q}(\sqrt{a})$.

Proof. Note $f_{\mathbb{Z}[a]} \mid 2$. We can determine $p\mathcal{O}_{\mathbb{Q}(\sqrt{a})}$ by factoring $x^2 - a$ modulo p. Because $p \nmid a$, then $x^2 - a$ is not a square modulo p. So (p) splits if and only if $x^2 - a$ splits over $\mathbb{F}_p$, if and only if $a \in \mathbb{F}_p^{\times 2}$. □

Proposition 7.8. If $\mathfrak{p}$ ramifies in B, then $\mathfrak{p} \mid D(1, \alpha, \cdots, \alpha^{[L:K]-1}) =: d(\alpha)$.

Proof. Let h be the minimal polynomial of α. Suppose $\mathfrak{p} + f_{A[\alpha]} = (1)$, then by Theorem 7.4, $\mathfrak{p}$ ramifies in B if and only if $\bar{h}$ is divisible by a square, i.e. $\bar{h}$ has a multiple root, and that is true if and only if $d(\alpha) \equiv 0 \pmod{\mathfrak{p}}$.

Note that $f_{A[\alpha]} \mid (d(\alpha))$, and $(d(\alpha))$ is an ideal of A and $f_{A[\alpha]}$ is an ideal of $A[\alpha]$. So if $p + f_{A[\alpha]} \neq (1)$, then $\mathfrak{p} \mid (d(\alpha))$. □

Corollary 7.9. Only finitely-many primes are ramified in L/K.

Lemma 7.10. Let $b \in B$. Every prime of B dividing (b) lies over a prime of A dividing $N_{L/K}A$. Every prime of Adividing $N_{L/K}(b)$ lies below some prime of B dividing (b).

Corollary 7.11. $b \in B^\times$ if and only if $N_{L/K}(b) \in A^\times$.

8 Lecture 8: October 10, 2022

Example 8.1. $\mathbb{Z}[\sqrt{5}] \subseteq \mathcal{O}_{\mathbb{Q}(\sqrt{5})}$. Notice that 2 is inert in $\mathbb{Q}(\sqrt{5})$, $(2) = f_{\mathbb{Z}[\sqrt{5}]}$, where $x^2 - 5 \equiv (x-1)^2 \pmod 2$. However, $2\mathbb{Z}[\sqrt{5}] \neq (2, \sqrt{5}-1)^2 = (4, 2(\sqrt{5}-1))$.

Exercise 8.2. Let A be a domain, $K = Q(A)$, L/K is separable, and B is integral closure of A in L. Given $\mathfrak{A} \subseteq B$ is a nonzero ideal, there exists $\alpha \in B$ such that $L = K(\alpha)$ and $\mathfrak{A} + f_{A[\alpha]} = B$.

In particular, we can always apply Kummer-Dedekind Theorem to get factorization of pB for p prime of A.

8.1 Decomposition Groups

Suppose we take the notation in the exercise above, but now L/K is a Galois extension, and $G = \mathbf{Gal}(L/K)$. Let $p \subseteq A$ be prime and P be prime of B over p.

Definition 8.3 (Galois Conjugate)**.** For $\sigma \in G$, $\sigma(P)$ is called a Galois conjugate of P. This is essentially an orbit.

Proposition 8.4. All primes of B over p are conjugate, i.e. G acts transitively on the set of primes over p.

Proof. Let Q be a prime that is not $\sigma(P)$ for all $\sigma \in G$. By the Chinese Remainder Theorem, there exists $b \in Q$ such that $b \equiv 1 \pmod{\sigma(P)}$ for all $\sigma \in G$. Then $N/_{L/K}(b) \in Q \cap A$ and $N_{L/K}(b) \equiv 1 \pmod p$, so $Q \cap A \neq p$. □

Definition 8.5. The decomposition group G_P of P is the stabilizer of P under the action of G on primes.

By the orbit-stabilizer theorem, there is a bijection from set of cosets G/G_P to the set of primes of B over p (prime of A), given by $\sigma \mapsto \sigma \cdot P$.

Proposition 8.6. $f_{P/p}$ and $e_{P/p}$ are independent of choice of P/p.

Proof. Let S be the set of coset representatives of G/G_P. Now $pB = \prod_{\sigma \in S} (\sigma P)^{e_{\sigma P/p}}$. If $\tau \in G$, then $pB = \tau pB = \prod_{\sigma \in S} (\tau\sigma P)^{e_{\sigma P/p}}$. Therefore, $e_{P/p} = e_{\tau P/p}$ for all τ by uniqueness of factorization. Note that $\tau : B/P \xrightarrow{\sim} B/\tau P$ is an isomorphism of A/p-vector spaces, so they have the same dimension, i.e. $f_{P/p} = f_{\tau P/p}$. □

Corollary 8.7. Suppose $\sigma, \cdots, \sigma_g$ are coset representatives of G/G_P, then $pB = \prod_{i=1}^{g} (\sigma_i P)^e$ with $e = e_{P/p}$. Setting $f = f_{P/p}$, we have $efg = [L : K]$.

Remark 8.8. $G_{\sigma(P)} = \sigma G_P \sigma^{-1}$ for $\sigma \in G$.

So, if L/K is Abelian, then $G_{\sigma(P)} = G_P$, so we can speak of "G_p").

Lemma 8.9. Consider the usual L/K extension. Let $E = L^{G_P}$ be the fixed field and C be the integral closure of A in E. Then P is the only prime of E lying over $\mathfrak{P} = P \cap C$, and the ramification index and residue degree $e_{\mathfrak{P}/p} = f_{\mathfrak{P}/p} = 1$. Therefore, p splits completely in E/K and $G_P = \mathbf{Gal}(L/E)_P = \mathbf{Gal}(L/E)$.

Proof. $\mathbf{Gal}(L/E)_P = \mathbf{Gal}(L/E)$, so P is the only prime over $\mathfrak{P}$ by transitivity. Now $e_{P/\mathfrak{P}} f_{P/\mathfrak{P}} = [L : E]$. There are $g = [E : K]$ primes dividing p in L. Therefore, $e_{P/\mathfrak{P}} f_{P/\mathfrak{P}} g = [L : K]$, but $e_{P/p} f_{P/p} g = [L : K]$. Since $e_{P/\mathfrak{P}} \mid e_{P/p}$ and $f_{P/\mathfrak{P}} \mid f_{P/p}$, we have $e_{P/\mathfrak{P}} = e_{P/p}$ and $f_{P/\mathfrak{P}} = f_{P/p}$, so $e_{\mathfrak{P}/p} = f_{\mathfrak{P}/p} = 1$. □

Proposition 8.10. Set $K_P = B/P$ and $K_P = A/P$, the extension K_P/K_p is normal, and $\pi_P : G_P \to \mathbf{Gal}(K_P/K_p)$ given by $\sigma \mapsto (b + P \mapsto \sigma b + P)$ is a surjective homomorphism.

Proof. Let $\alpha \in B$. Let $f \in A[x]$ be its minimal polynomial. Consider $f \mapsto \bar{f} \in A_p[x] = K_p[x]$. Then $\alpha \mapsto \bar{\alpha} \in K_P$ and $\bar{\alpha}$ is a root of $\bar{f}$. Since f splits completely in L, with roots in B, $\bar{f}$ splits completely in the residue field K_P. Therefore, the minimal polynomial of $\bar{\alpha}$ under K_p splits completely as it divides $\bar{f}$. This is saying that the extension is normal.

π_P is obviously a well-defined homomorphism. We now prove surjectivity. Let $\bar{\sigma} \in \mathbf{Gal}(K_P/K_p)$. Let $E = L^{G_P}$ and C be the integral closure of A in E and $\mathfrak{P} = P \cap C$. Let $\bar{\theta} \in K_P$ generate the maximal separable subextension of K_P/K_p (note that $K_p = C/\mathfrak{P}$). Let $\theta \in B$ lift $\bar{\theta}$. Let $g \in C[x]$ be the minimal polynomial of θ over E. Let $\bar{g} \in K_p[x]$ be its residue modulo $\mathfrak{P}$, so $\bar{g}(\bar{\theta}) = 0$. Let $\bar{h} \in K_p[x]$ be the minimal polynomial of $\bar{\theta}$, so $\bar{h} \mid \bar{g}$. Then $\bar{g}(\bar{\sigma}(\bar{\theta})) = 0$ as well, so there exists a root of g, say $\theta' \in B$ such that $\theta' \mapsto \bar{\sigma}(\bar{\theta})$, then there exists $\sigma \in G$ such that $\sigma(\theta) = \theta'$. Then the reduction at $\sigma(\theta)$, $\pi_P(\sigma)(\bar{\theta}) = \bar{\sigma}(\bar{\theta})$. This forces $\pi_P(\sigma) = \bar{\sigma}$, as $\bar{\theta}$ generates a maximal separable subextension of K_P/K_p. This proves the surjectivity. □

Definition 8.11. The inertia group I_P of I over p is $\ker(\pi_P)$.

This gives an exact sequence

$$1 \to I_P \to G_P \xrightarrow{\pi_P} \mathbf{Gal}(K_P/K_p) \to 1$$

If K_P/K_p is separable, then $|\mathbf{Gal}(K_P/K_p))| = f_{P/p}$, so $|I_P| = e_{P/p}$.

Example 8.12. Let $L = \mathbb{Q}(\zeta_3, \sqrt[3]{2})$ over $K = \mathbb{Q}(\zeta_3)$. Now $G = \mathbf{Gal}(L/\mathbb{Q}) \triangleright N = \mathbf{Gal}(L/K)$. We know that $\mathcal{O}_{\mathbb{Q}(\sqrt[3]{2})} = \mathbb{Z}[\sqrt[3]{2}]$.

Now, (2) is inert in $K/\mathbb{Q}$ as $x^2 + x + 1$ is irreducible modulo 2 and ramifies in L/K since $(2) = (\sqrt[3]{2})^3$, here $f_{(\sqrt[3]{2})/(2)} = 2$ and $I_{(\sqrt[3]{2})} = N$ and $G_{(\sqrt[3]{2})} = G$.

Moreover, (3) is totally ramified: $I_P = G$ and $3\mathcal{O}_L = P^6$.

We also know (5) is inert in K and splits in L/K: $x^3 - 2$ has a single root (3) modulo 5 and splits over $\mathbb{F}_{25}$. So $5\mathcal{O}_L = Q_1 Q_2 Q_3$, and $G_{Q_i} = \mathbf{Gal}(L/E_i)$ where $E_i = \mathbb{Q}(\zeta_3^{i-1}\sqrt[3]{2})$. Here we have $E_1 = \mathbb{Q}(\sqrt[3]{2})$, $5\mathcal{O}_{E_1} = \mathfrak{q}_1\mathfrak{q}_2$, then $Q_1 \mid \mathfrak{q}_1$ and $Q_2, Q_3 \mid \mathfrak{q}_2$. Here $f_{\mathfrak{q}_1|(5)} = (1)$ and $f_{\mathfrak{q}_2/(5)} = f_{\mathfrak{q}_3/(5)=2}$. Therefore, $I_{Q_i} = 1$ for all i, and GQ_1 permutes Q_2 and Q_3.

9 Lecture 9: October 12, 2022

Definition 9.1 (Absolute Norm)**.** Let L/K be a Galois extension of number fields, let $G = \mathrm{Gal}(L/K)$. Consider the extension P/p of $P \subseteq \mathcal{O}_L$ and $p \subseteq \mathcal{O}_K$. Then the absolute norm of P is $N(P) = [\mathcal{O}_L : P]$.

Definition 9.2 (Frobenius Element)**.** A Frobenius element at P for L/K is $\varphi_P \in G$ such that $\varphi_P(x) = x^{N_p} \pmod P$.

For the map $\pi_P : G_P \twoheadrightarrow \mathrm{Gal}(\mathcal{O}_{L/P}/\mathcal{O}_{K/p})$, we have $\pi_P(\sigma)(y) = y^{N_p}$. Note that the Galois group has a generator.

There are two types of global fields.

1. Number fields.
2. Function fields: finite extension of $\mathbb{F}_p(x)$ for some p in $\mathbb{F}_p(x)$, there are many Dedekind subrings, e.g. $\mathbb{F}_p[x]$.

In both cases, residue fields are finite.

Note that the Galois group is an finite extension, so it makes sense to talk about a Frobenius element.

9.1 Cyclotominc Fields

Let K be a field and $n \geq 1$. We denote $\mu_n(K)$ to the the nth roots of unity of K. Now $\mu_n(\bar{K})$ has order N if and only if $\mathrm{char}(K) \nmid n$.

Definition 9.3 (Cyclotomic Field)**.** The field $\mathbb{Q}(\mu_n)$ is the nth cyclotomic field.

The field $\mathbb{Q}(\mu_n)$ is Galois over $\mathbb{Q}$, as it is the splitting field of $x^n - 1$. All nth roots of unity are powers of any primitive nth root of unity ζ_n, so $\mathbb{Q}(\mu_n) = \mathbb{Q}(\zeta_n)$.

Definition 9.4 (Cyclotomic Polynomial)**.** The nth cyclotomic polynomial $\Phi_n \in \mathbb{Z}[x]$ is the polynomial which has as its roots the primitive nth roots of unity. Note that $x^n - 1 = \prod_{d|n} \Phi_d(x)$.

Definition 9.5 (Mobius Function)**.** The Mobius function $\mu : \mathbb{Z}_{\geq 1} \to \mathbb{Z}$ sends an integer n to $(-1)^k$ when $n = p_1 \cdots p_k$ where p_i's are distinct, and 0 otherwise.

Proposition 9.6 (Mobius Inversion Formula)**.** Let $f, G : \mathbb{Z}_{\geq 1} \to A$ where A be an Abelian group, and such that $F(n) = \sum_{d|n} f(A)$, then $f(n) = \sum_{d|n} \mu(\frac{n}{d}) F(d)$.

Lemma 9.7. For all $n \geq 1$, we have $\Phi_n = \prod_{d|n} (x^d - 1)^{\mu(\frac{n}{d})}$.

Proof. Use Mobius Inversion Formula. □

Example 9.8. $\Phi_{15} = \frac{(x^{15}-1)(x-1)}{(x^5-1)(x^3-1)} \cdots$.

$\Phi_{p^n} = x^{p^{n-1}(p-1)} + \cdots + x^{p^{n-1}} + 1$.

Lemma 9.9. If $i, j \geq 1$ are relatively prime to n, then $\frac{1-\zeta_n^i}{1-\zeta_n^j} \in \mathbb{Z}[\mu_n]^\times = \mathcal{O}_{\mathbb{Q}(\mu_n)}^\times$.

Proof. Take $k \in \mathbb{Z}$ such that $jk \equiv 1 \pmod{n}$, then $1 - \zeta_n^i = 1 - \zeta_n^{ijk}$, and as $1 - x^j$ divides $1 - x^{ijk}$, then $\frac{1-\zeta_n^i}{1-\zeta_n^j} \in \mathbb{Z}[\mu_n]$, so this is a unit. □

Lemma 9.10. Let p be a prime number and $r \geq 1$. Then the absolute value of the discriminant of $\mathbb{Z}[\mu_{p^r}]$ is a power of p, and (p) is the only prime of $\mathbb{Z}$ that ramifies in $\mathbb{Q}(\mu_{p^r})$. It is totally ramified and lies below $(1 - \zeta_{p^r})$. Moreover, $[\mathbb{Q}(\mu_{p^r}) : \mathbb{Q}] = p^{r-1}(p-1)$.

Proof. Note that $[\mathbb{Q}(\mu_{p^r}) : \mathbb{Q}] \mid \deg(\Phi_{p^r}) = p^{r-1}(p-1)$. By the lemma, we have

$$\prod_{i=1, p\nmid i}^{p^r-1} (1 - \zeta_{p^r}^i) = \Phi_{p^r}(1) = p.$$

Therefore, $p.Z[\mu_{p^r}] = (1 - \zeta_{p^r})^{p^{r-1}(p-1)}$, which is the same in $\mathcal{O}_{\mathbb{Q}(\mu_{p^r})}$.

Now $efg = [\mathbb{Q}(\mu_{p^r}) : \mathbb{Q}]$, so all claims about ramifications of p holds because $e = p^{r-1}(p-1)$.

Then $\operatorname{disc}(\mathbb{Z}[\mu_{p^r}] = \prod_{1 \leq i < j \leq p-1} (\zeta_{p^j} - \zeta_{p^i})^2$, but they are primes dividing p, so the result on discriminant holds. □

Proposition 9.11. The nth cyclotomic polynomial is irreducible for all $n \geq 1$. In other words, $[\mathbb{Q}(\mu_n) : \mathbb{Q}] = \varphi(n)$, where φ is Euler's phi-function. Moreover, the prime ideals of $\mathbb{Z}$ that ramify in $\mathcal{O}_{Q(\mu_n)}$ are those generated by the odd primes dividing n and, if n is a multiple of 4, the prime 2.

Proof. Note that $\mathbb{Q}(\mu_n) = \mathbb{Q}(\mu_{2n})$ if n is odd, so we may ask the case when n is odd or $4 \mid n$, let $n = p_1^{r_1} \cdots p_k^{r_k}$. Now $\mathbb{Q}(\mu_n) = \prod_i \mathbb{Q}(\mu_{p_i}^{r_i})$, here p_i's are ramified in $\mathbb{Q}(\mu_{p_i}^{r_i})$, so it is in $\mathbb{Q}(\mu_n)$. Also, if no other primes ramified in any $\mathbb{Q}(\mu_{p_i}^{r_i})$, then it is not in $\mathbb{Q}(\mu_n)$.

Since p_i is totally ramified in $\mathbb{Q}(\mu_{p_i}^{r_i})$ but unramified in $\mathbb{Q}(\mu_{p_i}^{r_i})$, these two fields have intersection $\mathbb{Q}$. So $[\mathbb{Q}(\mu_n) : \mathbb{Q}] = \prod_{i=1}^k [\mathbb{Q}(\mu_{p_i}^{r_i} : \mathbb{Q}] = \varphi(n)$ by induction. □

Proposition 9.12. $\mathcal{O}_{\mathbb{Q}(\mu_n)} = \mathbb{Z}[\mu_n]$.

Proof. We first consider $n = p^r$ for prime p. Now $f_{\mathbb{Z}[\mu_p^r]} \mid (D(1, \zeta_{p^r}, \cdots, \zeta_{p^r}^{p^{r-1}(p-1)-1}) = (\operatorname{disc}(\mathbb{Z}[\mu_{p^r}])) = (p^m)$ for some $m \geq 1$.

Now let $\lambda_r = 1 - \zeta_{p^r}$, which generates the unqiue primes over (p) in $\mathbb{Q}(\mu_{p^r})$. Since (p) is totally ramified in $\mathbb{Q}(\mu_{p^r})/\mathbb{Q}$, we have that $\mathcal{O}_{\mathbb{Q}(\mu_{p^r})})/(\lambda_r) \cong \mathbb{Z}/p\mathbb{Z}$. In particular,

$$\begin{aligned}
\mathcal{O}_{\mathbb{Q}(\mu_{p^r})} &= \mathbb{Z}[\mu_{p^r}] + \lambda_r \mathcal{O}_{\mathbb{Q}(\mu_{p^r})} \\
&= \mathbb{Z}[\mu_{p^r}] + \lambda_r(\mathbb{Z}[\mu_r] + \lambda_r \mathcal{O}_{\mathbb{Q}(\mu_{p^r})}) \\
&= \mathbb{Z}[\mu_{p^r}] + \lambda_r^2 \mathcal{O}_{\mathbb{Q}(\mu_{p^r})} \\
&= \cdots \\
&= \mathbb{Z}[\mu_{p^r}] + p^m \mathcal{O}_{\mathbb{Q}(\mu_{p^r})} = \mathbb{Z}[\mu_p^r].
\end{aligned}$$

In the general case, we write $n = p_1^{r_1} \cdots p_k^{r_k}$. We have a basis $\zeta_{p_1^{r_1}}^{i_1} \cdots \zeta_{p_k^{r_k}}^{i_k}$ with $0 \leq i_j \leq \varphi(p_j^{r_j}) - 1$ of $\mathbb{Z}[\mu_n]$ over $\mathbb{Z}$.

We need the following useful result (1.4.28):

Proposition 9.13. Let A be a normal domain, $K = Q(A)$, and L and L' are linearly disjoint and are finite separable extensions of K. Suppose B and B' are integral closures of A in L and L', respectively. Suppose B is A-free with basis $\beta_1, \cdots, \beta_n$ and B' is A-free with basis $\gamma_1, \cdots, \gamma_m$. Set $d = D(\beta_1, \cdots, \beta_n)$, $d' = D(\gamma_1, \cdots, \gamma_m)$, then $\{\beta_i \gamma_j\}$ has discriminant $d^m (d')^n$. If C is the integral closure of A in LL' and C' is the A-span of $\{\beta_i \gamma_j\}$, then $(d, d')C \subseteq C'$.

Here take $\mathbb{Z}[\mu_{p_k^{r_k}}]$ and $\mathbb{Z}[\mu_n / p_k^{r_k}]$ by induction on k. The discriminants of these rings, d and d', are relatively prime. Therefore, $(d, d')\mathcal{O}_{\mathbb{Q}(\mu_n)} \subseteq \mathbb{Z}[\mu_n]$ by the proposition. Now (d, d') is the unit ideal (1), and we are done. □

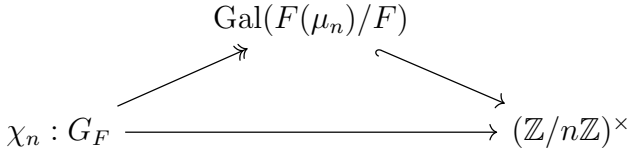
We revise linear disjoint for a bit.

Definition 9.14 (Linear Disjoint)**.** Let $L, L' \subseteq \Omega$ be extensions of K. We say L and L' are linearly disjoint over K if every K-linear independent subset of L is linearly independent over L'. Equivalently, $L \otimes_K L' \hookrightarrow LL'$ sends $x \otimes_y \mapsto xy$. If $\Omega = \bar{K}$, this is equivalent to saying $L \otimes_K L'$. If L and L' are finite over K, then this is equivalent to $[LL' : K] = [L : K][L' : K]$. Finally, if L be finite Galois over K, then this is equivalent to $L \cap L' = K$.

10 Lecture 10: October 14, 2022

Proposition 10.1. Let $n \geq 1$, p be a prime, $r \geq 0$ such that $p^r \mid\mid n$, $m = \frac{n}{p^r}$, f is the order of p in $(\mathbb{Z}/m\mathbb{Z})^\times$, and $g = \frac{\varphi(m)}{f}$. Then $p\mathbb{Z}[\mu_n] = (p_1 \cdots p_g)^{\varphi(p^r)}$, where $p_1, \cdots, p_g$ are distinct primes.

Remark 10.2. The nth cyclotomic character χ_n of a field K of character prime to n is $\chi_n : G_F \to (\mathbb{Z}/n\mathbb{Z})^\times$ determined by $\sigma(\zeta_n) = \zeta_n^{\chi_n(\sigma)}$ for $\sigma \in G_F$, where $\mu_n = \langle \zeta_n \rangle$. χ_n is injective on $\mathrm{Gal}(F(\mu_n)/F)$. For $F = \mathbb{Q}$, it is an isomorphism from $\mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})$ to $(\mathbb{Z}/n\mathbb{Z})^\times$.



Proof. Denote $\Phi_n = \prod_{i=1,\gcd(i,m)=1}^{m-1} \prod_{j=1,p\nmid j}^{p^r} (X - \zeta_m^i \zeta_{p^r}^j)$, since $X^{p^r} - 1 \equiv (x-1)^{p^r} \pmod{p}$, then $(\zeta_{p^r} - 1)^{p^r} \equiv 0 \pmod{p\mathbb{Z}[\mu_n]}$. If $\mathfrak{p} \subseteq \mathbb{Z}[\mu_n]$ lies over p, then $\zeta_{p^r} \equiv 1 \pmod{\mathfrak{p}}$. Then

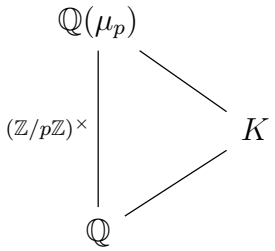
$$\Phi_n \equiv \prod_{i=1,\gcd(i,m)=1}^{m-1} (X - \zeta_m^i)^{\varphi(p^r)} \pmod{\mathfrak{p}},$$

so $\Phi_n = \Phi_m^{\varphi(p^n)} \pmod{p}$, which is equivalent to $\mathfrak{p} \cap \mathbb{Z}$. As $x^m - 1$ is separable in $F = \mathbb{Z}[\mu_m]/(\mathfrak{p} \cap \mathbb{Z}[\mu_m]$ as $p \nmid m$, so $|\mu_m(F)| = m$. Let $F \cong \mathbb{F}_{p^f}$ for f minimal such that $m \mid p^f - 1$, i.e. f is the order of p in $(\mathbb{Z}/m\mathbb{Z})^\times$. Then $e_{\mathfrak{p}/p} = \varphi(p^r)$, $f_{\mathfrak{p}/p} = f$, and degree formula gives $g = \frac{\varphi(n)}{ef} = \frac{\varphi(m)}{f}$, the number of primes. □

Corollary 10.3. p splits completely in $\mathbb{Q}(\mu_n/\mathbb{Q})$ if and only if $p \equiv 1 \pmod{n}$.

10.1 Quadratic Reciprocity

For odd p, let $p^* = (-1)^{\frac{p-1}{2}} p \equiv 1 \pmod{4}$. Now $\mathcal{O}_{\mathbb{Q}(\sqrt{p^*})} = \mathbb{Z}[\frac{1+\sqrt{p^*}}{2}]$ which is unramified at 2, and $\mathcal{O}_{\mathbb{Q}(\sqrt{-p^*})} = \mathbb{Z}[\sqrt{-p^*}]$, which is ramified at 2. The unique quadratic field $K/\mathbb{Q}$ is unramified outside p and totally ramified at p, so $K = \mathbb{Q}(\sqrt{p^*})$.



Proposition 10.4. Let q be a prime. q splits in $\mathbb{Q}(\sqrt{p^*})$ if and only if q splits into an even number of primes in $\mathbb{Q}(\mu_p)$.

Proof. Let g be the number of primes dividing q in $\mathbb{Q}(\mu_p) = [G : G_q]$, where we have $G = \mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$ and G_q as the decomposition group. So $2 \mid g$ (since G acyclic) if and only if G_q fixes $\mathbb{Q}(\sqrt{p^*})$ if and only if q splits in $\mathbb{Q}(\sqrt{p^*})/\mathbb{Q}$. □

Definition 10.5 (Legendre Symbol)**.** Suppose p is an odd prime that does not divide $a \in \mathbb{Z}$, we denote $\left(\frac{a}{p}\right) \in \{\pm 1\}$, where

$$\left(\frac{a}{p}\right) = a^{\frac{p-1}{2}} \pmod p = \begin{cases} 1, & a \in \mathbb{F}_p^{\times 2} \\ -1, & a \notin \mathbb{F}_p^{\times 2} \end{cases}$$

Theorem 10.6 (Quadratic Reciprocity)**.** Let p, q be odd primes, then $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}$. Also, $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$ and $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$.

Proof.

$$\begin{aligned} p^{\frac{q-1}{2}} \equiv (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}} \pmod q \equiv &\iff (p^*)^{\frac{q-1}{2}} \equiv 1 \pmod q \\ &\iff \left(\frac{p^*}{q}\right) = 1 \\ &\iff x^2 - p^* \text{ factors in } \mathbb{F}_q \\ &\iff q \text{ splits in } \mathbb{Q}(\sqrt{p^*}), \end{aligned}$$

and by proposition, this is equivalent to saying q splits into an even number of primes in $\mathbb{Q}(\mu_p)$. Recall (q) splits into $\frac{p-1}{f}$ primes, where f is the order of q modulo p. Also, note that the equivalent statement above is equivalent to $\frac{p-1}{f}$ is even, if and only if $f \mid \frac{p-1}{2}$, if and only if $q^{\frac{p-1}{2}} \equiv 1 \pmod p$, i.e. $\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}$ if and only if $\left(\frac{q}{p}\right) = 1$. □

10.2 Lattice

Definition 10.7. A lattice Λ in a finite dimensional $\mathbb{R}$-vector space V is an Abelian subgroup generated by a finite set of $\mathbb{R}$-linearly independent vectors in Λ. The lattice Λ is complete if it has a basis of $\mathbb{R}$-linearly indepednent vectors spanning V.

Definition 10.8. The fundamental domain D of a complete lattice Λ in V relative to a $\mathbb{Z}$-basis $\{v_1, \cdots, v_n\}$ of Λ is $D = \{\sum_{i=1}^{n} c_i v_i \mid c_i \in [0, 1)\}$.

Remark 10.9. Every $v \in V$ can be written uniquely as $v = \lambda + d$ with $\lambda \in \Lambda$ and $d \in D$.

Definition 10.10. A subgroup A of V is discrete if it is discrete with respect to the subspace topology for the Euclidean topology on V.

Proposition 10.11. A subgroup Λ of V is discrete if and only if Λ is a lattice in V.

Proof. If Λ is a lattice, $\Lambda = \sum_{i=1}^{m} \mathbb{Z} v_i$ with v_i's are $\mathbb{R}$-linearly independent, and extend to basis $v_1, \cdots, v_n$ of V. If $v = \sum_{i=1}^{m} a_i v_i \in \Lambda$, then $U = \{v + \sum_{i=1}^{n} c_i v_i \mid c_i \in (-1, 1)\ \forall i\}$ open with $U \cap \Lambda = \{v\}$. Therefore, Λ is discrete.

Now let Λ be discrete, $W = \mathbb{R}$-span of Λ in V, with $v_1, \cdots, v_m \in \Lambda$ forming an $\mathbb{R}$-basis of W. Now $\Sigma = \sum_{i=1} \mathbb{Z} v_i \leq \Lambda$ Now Σ is a complete lattice in W. Let D be its fundamental domain. Let $S \subseteq D$ be a set of coset representations of Λ/Σ, but $S \subseteq \Lambda \cap D$ is finite since Λ is discrete. Let $d = |S|$, then $\Lambda \subseteq \frac{1}{d}\Sigma \cong \Sigma$, so Λ is free of rank $\leq m$, but $\Sigma \leq \Lambda$, so it is equal to m. So Λ contains m $\mathbb{R}$-linearly independent vectors, so it has a basis of $\mathbb{R}$-linear vectors. Hence, Λ is a lattice. $\square$

Lemma 10.12. A lattice $\Lambda \subseteq V$ is complete if and only if there exists a bounded subset B of V such that $V = B + \Lambda$.

Proof. Suppose Λ is complete, then B as a fundamental domain works.

Suppose there exists B, then there is W as a $\mathbb{R}$-span of Λ. Now for $v \in V$, for any $k \geq 1$, we can write $kv = b_k + \lambda_k$ for $b_k \in B$ and $\lambda_k \in \Lambda$. Now B is bounded, with $\frac{1}{k} b_k \to 0$, and so $\frac{1}{k}\lambda_k \to v$. As $\frac{1}{k}\lambda_k \in W$ for all k, and $W \subseteq V$ closed, this forces $v \in W$, and so $V = W$. $\square$

11 Lecture 11: October 17, 2022

Let V be a finite-dimensional vector space with symmetric, positive definite inner product $\langle\ ,\ \rangle$. Let Λ be a complete lattice in V. Let μ_V be a Lebesgue measure in V, which gives the notion of volumes.

Definition 11.1 (Volume)**.** The volume $\mathrm{Vol}(\Lambda)$ of Λ is $\mu_V(D)$ for any fundamental domain D of Λ.

Exercise 11.2. If $e_1, \cdots, e_n$ is an orthonormal basis of the inner product space V and $v_1, \cdots, v_n$ is a $\mathbb{Z}$-basis of Λ, we can write $v_i = \sum_{j=1}^{n} a_{ij} e_j$ and set $A = (a_{ij})$, then $\mathrm{vol}(\Lambda) = \det(A) = \det((\langle v_i, v_j\rangle)_{i,j})^{\frac{1}{2}}$.

Definition 11.3. Let $T \subseteq V$ be a subset.

T is convex if for all $v, w \in T$, $rv + (1-r)w \in T$ for all $r \in [0,1]$.

T is symmetric about the origin 0 if $T = -T$.

Theorem 11.4 (Minkowski)**.** Let X be a convex, measureable subset of V that is symmetric about 0. Let n be the dimension of V. Suppose that $\mu_V(X) > 2^n \mathrm{Vol}(\Lambda)$. Then $X \cap \Lambda \neq \{0\}$.

Proof. Let $Y = \frac{1}{2}X = \{\frac{1}{2}x \mid x \in X\}$.

Claim 11.5. $Y - Y \subseteq X$.

Subproof. Let $y, y' \in V$, then $y' - y = \frac{1}{2}(2y') + \frac{1}{2}(-2y) \in X$ since X is symmetric about 0 and convex. ■

Now $\mu_V(Y) = \frac{1}{2^n}\mu_V(X) > \mathrm{Vol}(\Lambda)$. Let D be a fundamental domain of Λ. If all $v + Y$ for $v \in \Lambda$ are disjoint, then $\mathrm{Vol}(\Lambda) = \sum_{v\in\Lambda} \mu_V(D \cap (v+Y)) = \sum_{v\in\Lambda} \mu_V((D-v) \cap Y) = \mu_V(Y)$, contradiction. Therefore, $\exists v, v' \in \Lambda$ and $v \neq v'$ such that $(v+Y) \cap (v'+Y) \neq \varnothing$. Then $0 \neq v - v' \subseteq (Y - Y) \cap \Lambda \subseteq X \cap \Lambda$. □

11.1 Real and Complex Embeddings

Let $[F : \mathbb{Q}] = n$, then there is an embedding $F \hookrightarrow F \otimes_{\mathbb{Q}} \mathbb{C} \cong \prod_{\sigma: F \hookrightarrow \mathbb{C}} \mathbb{C}$ given by $x \mapsto x \otimes 1$. This is true because $F = \mathbb{Q}[x]/(f)$ and $f = \prod_{i=1}^{n}(x - \alpha_i)$ for distinct α_i's. Now $F \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{Q}[x]/(f) \otimes_{\mathbb{Q}} \mathbb{C} \cong \prod_{i=1}^{n} \mathbb{C}[x]/(x - \alpha_i) \cong \mathbb{C}^n$. The α_i's are the images under field embeddings of a roof of f in F.

Definition 11.6. Let $\sigma : F \hookrightarrow \mathbb{C}$ be a field embedding.

We say σ is a real embedding if $\sigma(F) \subseteq \mathbb{R}$, otherwise we say it is a complex embedding.

A real prime is a real embedding. A complex prime is a pair of complex embeddings $(\sigma, \bar{\sigma})$ such that $\bar{\sigma}(\alpha) = \overline{\sigma(\alpha)}$ for $\alpha \in \bar{F}$, where $\bar{z}$ is the complex conjugation of $z \in \mathbb{C}$.

$r_1(F)$ is the number of real primes of F and $r_2(F)$ is the number of complex primes of F.

Remark 11.7. $F \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{\text{real primes}} \mathbb{R} \times \prod_{\text{complex primes}} \mathbb{C} \cong \mathbb{R}^{r_1(F)} \times \mathbb{R}^{r_2(F)}$, and $[F : \mathbb{Q}] = r_1(F) + 2r_2(F)$.

Remark 11.8. If $F/\mathbb{Q}$ is Galois, then given an (Archemedian) embedding $\sigma : F \hookrightarrow \mathbb{C}$, all others are $\sigma \circ \tau$ with $\tau \in \mathrm{Gal}(F/\mathbb{Q})$, so either $r_1(F) = 0$ or $r_2(F) = 0$.

Example 11.9.

F	r_1	r_2
$\mathbb{Q}(\sqrt{2})$	2	0
$\mathbb{Q}(i)$	0	1
$\mathbb{Q}(\sqrt[3]{2})$	1	1

11.2 Finiteness of the Class Group

Now consider $V = F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \cong \mathbb{R}^n$ is given by $(x_1, \cdots, x_{r_1}, z_1, \cdots, z_{r_2}) \mapsto (x_1, \cdots, x_{r_1}, \mathrm{Re}(z_1), \mathrm{Im}(z_1), \cdots, \mathrm{Re}(z_{r_2}), \mathrm{Im}(z_{r_2}))$. The usual inner product on $\mathbb{R}^n$ gives an inner product on V. A Lebesgue measure μ can be defined on the structure.

We denote $v_F : F \hookrightarrow F \otimes_{\mathbb{Q}} \mathbb{R}$, and real embeddings $\sigma_i : F \hookrightarrow \mathbb{R}$ and non-conjugate complex embeddings $\tau_j : F \hookrightarrow \mathbb{C}$ where $1 \le i \le r_1$ and $1 \le j \le r_2$.

Proposition 11.10. Let $\mathfrak{a}$ be a nonzero ideal of $\mathcal{O}_F$. The $v_F(\mathfrak{a})$ is a complex lattice in V and $\mathrm{Vol}(v_F(\mathfrak{a})) = 2^{-r_2} N_{\mathfrak{a}} |\mathrm{disc}(F)|^{\frac{1}{2}}$.

Proof. Let $\alpha_1, \cdots, \alpha_n$ be a $\mathbb{Z}$-basis of $\mathfrak{a}$.

Let $A \in M_n(\mathbb{C})$ have ith row

$$(\sigma_1(\alpha_i), \cdots, \sigma_{r_1}(\alpha_i), \tau_1(\alpha_i), \bar{\tau}_1(\alpha_i), \cdots, \tau_{r_2}(\alpha_i), \bar{\tau}_{r_2}(\alpha_i)).$$

Let $B \in M_n(\mathbb{R})$ have ith row

$$(\sigma_1(\alpha_i), \cdots, \sigma_{r_1}(\alpha_i), \mathrm{Re}(\tau_1(\alpha_i)), \mathrm{Im}(\tau_1(\alpha_i)), \cdots, \mathrm{Re}(\tau_{r_2}(\alpha_i)), \mathrm{Im}(\tau_{r_2}(\alpha_i))).$$

Then $\det(A) = (-2i)^{r_2} \det(B)$ and $|\det(A)| = |D(\alpha_1, \cdots, \alpha_n)|^{\frac{1}{2}} = N_{\mathfrak{a}} |\mathrm{disc}(F)|^{\frac{1}{2}}$. So $\mathrm{Vol}(v_F(\mathfrak{a})) = |\det(B)| = 2^{r_2} |\det(A)| = 2^{-r_2} N_{\mathfrak{a}} |\mathrm{disc}(F)|^{\frac{1}{2}}$. □

Norm on V is given by

$$||(x_1, \cdots, x_{r_1}, z_1, \cdots, z_{r_2})|| = \sum_{i=1}^{r_1} |x_i| + 2 \sum_{j=1}^{r_2} |z_j|.$$

Set $D_t = D_t^{(r_1, r_2)} = \{v \in V \mid ||v|| < t\}$, the open ball of radius t.

Lemma 11.11. $\mu_V(D_t) = 2^{r_1 - r_2} \pi^{r_2} \frac{t^n}{n!}$.

Proof. Induction on r_1 with $r_2 = 0$ and then induction on r_2 with $r_1 = 0$. Integrate over the reals and do polar coordinates. □

Proposition 11.12. For any non-zero ideal $\mathfrak{a}$ of $\mathcal{O}_F$, there exists $\alpha \in \mathfrak{a} \backslash \{0\}$ such that $|N_{F/\mathbb{Q}}(\alpha)| \le (\frac{4}{\pi})^{r_2} \frac{n!}{n^n} N_{\mathfrak{a}} |\mathrm{disc}(F)|^{\frac{1}{2}}$.

Proof. We denote the right hand side as C. Let t be such that $\mu_V(D_t) > 2^n \mathrm{Vol}(v_F(\mathfrak{a}))$. By Minkowski's Theorem, as $v_F(\mathfrak{a})$ is a complex lattice, there exists $\alpha \in \mathfrak{a} \backslash \{0\}$ such that $\alpha \in D_t$. Note that

$$|N_{F/\mathbb{Q}}(\alpha)| = |\sigma_1(\alpha)| \cdots |\sigma_{r_1}(\alpha)||\tau_1(\alpha)|^2 \cdots |\tau_{r_2}(\alpha)|^2.$$

Because the geometric mean is bounded above by the arithmetic mean, then

$$|N_{F/\mathbb{Q}}(\alpha)|^{\frac{1}{n}} \leq \frac{1}{n}(\sum_{i=1}^{r_1} |\sigma_i(\alpha)| + 2\sum_{j=1}^{r_2} |\tau_j(\alpha)|) < \frac{t}{n}$$

since $\alpha \in D_t$. We can rewrite $\mu_V(D_t) > 2^n \mathrm{Vol}(v_F(\mathfrak{a}))$ by

$$2^{r_1 - r_2} \pi^{r_2} \frac{t^n}{n!} > 2^n 2^{-r_2} N_{\mathfrak{a}} |\mathrm{disc}(F)|^{\frac{1}{2}}$$

and this is equivalent to

$$\frac{t^n}{n!} > (\frac{4}{\pi})^{r_2} N_{\mathfrak{a}} |\mathrm{disc}(F)|^{\frac{1}{2}}$$

which is equivalent to $(\frac{t}{n})^n > C$.

Choose t such that $(\frac{t}{n})^n$ is less than the smallest integer greater than C. Then $|N_{F/\mathbb{Q}}(\alpha)| \leq (\frac{t}{n})^n$ and $N_{F/\mathbb{Q}}(\alpha) \in \mathbb{Z}$, so $|N_{F/\mathbb{Q}}| \leq C$. □

Definition 11.13. The Minkowski bound for F is

$$B_F = \frac{n!}{n^n}(\frac{4}{\pi})^{r_2} |\mathrm{disc}(F)|^{\frac{1}{2}}.$$

Corollary 11.14. $|\mathrm{disc}(F)|^{\frac{1}{2}} \geq (\frac{\pi}{4})^{r_2} \frac{n^n}{n!}$.

Proof. Let $\mathfrak{a} = \mathcal{O}_F$ and note that $|N_{F/\mathbb{Q}}\alpha| \geq 1$. □

12 Lecture 12: October 19, 2022

Theorem 12.1 (Minkowski)**.** There exists a set of representatives of Cl_F consisting of ideals $\mathfrak{c}$ such that $N\mathfrak{c} \leq B_F$.

Proof. Let $\mathfrak{a} \in I_F$ be a fractional ideal and let $d \in F^\times$ such that $\mathfrak{b} = d\mathfrak{a}^{-1} \subseteq \mathcal{O}_F$. Now, there exists $\beta \in \mathfrak{b}\backslash\{0\}$ such that $|N_{F/\mathbb{Q}}(\beta)| \leq N\mathfrak{b} \cdot B_F$. Therefore, $\beta\mathcal{O}_F = \mathfrak{b}\mathfrak{c}$ with $\mathfrak{c} \subseteq \mathcal{O}_K$ as an ideal. The class $[\mathfrak{c}]$ of $\mathfrak{c}$ is $[\mathfrak{b}^{-1}]$, which is the same as the class $[\mathfrak{a}]$. The norm $|N_{F/\mathbb{Q}}(\beta))| = N\mathfrak{b} \cdot N\mathfrak{c}$, and so $N\mathfrak{c} \leq B_F$. □

Theorem 12.2. Cl_F is finite.

Proof. It is enough to show that $\{\mathfrak{a} \subseteq \mathcal{O}_F \text{ ideal } \mid N\mathfrak{a} \leq B_F\}$ is finite. We factor $\mathfrak{a} = \mathfrak{p}_1^{r_1} \cdots \mathfrak{p}_k^{r_k}$ with distinct $\mathfrak{p}_i$ and $r_i \geq 1$. If $k >> 0$, then at least one of the $\mathfrak{p}_i$'s satisfies $N\mathfrak{p}_I > B_F$. For example, $k = n \cdot B_F$ works. Then $N\mathfrak{a} \leq N\mathfrak{p}_i > B_F$. Also, for $r_i >> 0$, $N\mathfrak{p}_i > B_k$ for any $\mathfrak{p}_i$. This leaves only finitely many choices of $\mathfrak{p}_i$ and r_i such that $N\mathfrak{a} \leq B_F$. □

Definition 12.3. The class number h_F is $|\mathrm{Cl}_F|$.

Example 12.4. For $F = \mathbb{Q}(\sqrt{-5})$, the discriminant $\mathrm{disc}(F) = -20$, and $B_F = \frac{2}{\pi}\sqrt{20} < 3$. As $\mathbb{Z}[\sqrt{-5}]$ is not a PID, so $h_F = 2$.

Example 12.5. For $F = \mathbb{Q}(\sqrt{17})$, $\mathrm{disc}(F) = 17$, and $B_F = \frac{1}{2}\sqrt{17} < 3$. The ring of integers is generated by $\alpha = \frac{\sqrt{17}+1}{2}$ with $\mathcal{O}_F = \mathbb{Z}[\alpha]$, then the minimal polynomial of α is $x^2 - x - 4$, which splits modulo 2, i.e. $2\mathbb{Z}[\alpha] = \mathfrak{p}_1\mathfrak{p}_2$. Now Cl_F has representatives with norm ≤ 2, then $\mathrm{Cl}_F = \{0, [\mathfrak{p}_1], [\mathfrak{p}_2]\}$, but $[\mathfrak{p}_1] = -[\mathfrak{p}_2]$. Note that $N(\frac{\sqrt{17}+5}{2}) = 2$, so $[\mathfrak{p}_1] = [\mathfrak{p}_2] = 0$, i.e. $h_F = 1$ and $\mathcal{O}_F$ is a PID.

12.1 Dirichlet's Unit Theorem

Lemma 12.6. Let $m, N \geq 1$. The set of algebraic integers α such that $[\mathbb{Q}(\alpha) : \mathbb{Q}] \leq m$ and $|\sigma(\alpha)| \leq 1$ for any embedding $\sigma : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$ is finite.

Proof. Let α be an algebraic integer with $f = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ as its minimal polynomial. Then in $\mathbb{C}[x]$, $f = \prod_{\sigma:\mathbb{Q}(\alpha)\hookrightarrow\mathbb{C}} (x - \sigma(\alpha))$. Then $|a_i| \leq N^{n-i}\binom{n}{i}$, which is bounded in terms of N and n. Therefore, the number of f's is finite, and the number of α's is finite. □

Corollary 12.7. $\mu(F)$ is finite, which is the group of roots of unity in F. Moreover, $\mu(F) = \{\alpha \in \mathcal{O}_F \mid |\sigma(\alpha)| = 1 \forall \sigma : F \hookrightarrow \mathbb{C}\}$ for Archimedean embeddings σ.

Proof. If $\alpha \in \mathcal{O}_F$ and $|\sigma(\alpha)| = 1$ for all σ, then $|\sigma(\alpha^n)| = 1$ for all σ, n. But there are only finitely many $\beta \in \mathcal{O}_F$ such that $|\sigma(\beta)| \leq 1$ for all σ. So α has finite order, i.e. $\alpha \in \mu(F)$. □

Definition 12.8 (Unit Group)**.** The unit group of F is $\mathcal{O}_F^\times$.

Set $l_F : F^\times \to \mathbb{R}^{r_1+r_2}$ as $l_F(\alpha) = (\log|\sigma_1(\alpha)|, \cdots, \log|\sigma_{r_1}(\alpha)|, \log|\tau_1(\alpha)|, \cdots, \log|\tau_{r_2}(\alpha)|)$ where σ_i are the real embeddings of F and τ_j's are one of each complex conjugate pair of complex embeddings.

Proposition 12.9. $l_F(\mathcal{O}_F^\times)$ is a lattice in $\mathbb{R}^{r_1+r_2}$ that is contained in the hyperplane

$$H = \{(x_1, \cdots, x_{r_1+r_2}) \mid \sum_{i=1}^{r_1} x_i + 2\sum_{j=1}^{r_2} x_{j+r_1} = 0\}$$

and $\ker(l_F) = \mu(F)$. Therefore, there is an exact sequence

$$1 \to \mu(F) \to \mathcal{O}_F^\times \xrightarrow{l_F} l_F(\mathcal{O}_F^\times) \to 1$$

Proof. By the corollary, the kernel is just $\mu(F)$. For $\alpha \in \mathcal{O}_F^\times$, we have

$$\sum_{i=0}^{r_1} \log|\sigma_i(\alpha)| + 2\sum_{j=0}^{r_2} \log|\tau_j(\alpha)| = \log|N_{F/\mathbb{Q}}(\alpha)| = \log(1) = 0,$$

so $l_F(\alpha) \in H$. For $N \geq 0$, let $D_N = \{(x_1, \cdots, x_{r_1+r_2}) \in H \mid |x_i| \leq N \ \forall i\}$. Now $l_F(\mathcal{O}_F^\times) \cap D_N$ is finite by the lemma, so there exists $U \subseteq H$ an open neighborhood of 0 such that $l_F(\mathcal{O}_F^\times) \cap U = \{0\}$. Therefore, $l_F(\mathcal{O}_F^\times) \subseteq H$ is discrete, and therefore is a lattice. □

Lemma 12.10. Let $A = (a_{ij})_{i,j} \in M_k(\mathbb{R})$ for $k \geq 1$ be such that $a_{ij} < 0$ for all $i \neq j$ and $\sum_{j=1}^{k} a_{ij} > 0$ for all i, then $\det(A) \neq 0$.

Proof. Pick $v = (v_i) \in \mathbb{R}^k$ with $Av = 0$ such that $v_j = 1$ for some j and $|v_i$
$le1$ for all i. Then

$$0 = \sum_{i=1}^{k} a_{ji}v_i = a_{jj} + \sum_{i \neq j} a_{ji}v_i$$

and $a_{ji}v_i \geq a_{ji}$, so the right-hand side is greater than $\sum_{i=1}^{k} a_{ji} > 0$, contradiction. □

Lemma 12.11. Set $D = \{v \in F \otimes_{\mathbb{Q}} \mathbb{R} \mid \frac{1}{2} \leq [v] \leq 1\}$. Let X be a bounded convex subset of $F \otimes_{\mathbb{Q}} \mathbb{R}$ that is symmetric about 0 and has $\mu_V(X) > 2^m \mathrm{vol}(v\iota_F(\mathcal{O}_F))$, then there exists some $\alpha_1, \cdots, \alpha_s \in \mathcal{O}_F$ for some s such that for each $w \in D$, there exists $\varepsilon \in \mathcal{O}_F^\times$ such that $w\iota_F(\varepsilon) \in \iota_F(\alpha_i^{-1})X$ for some $1 \leq i \leq s$.

Proof. Consider $\iota_F : F \hookrightarrow F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, with $[\ ,\ ] : F \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{R}$ defined by $[(x_1, \cdots, x_{r_1}, z_1, \cdots, z_{r_2})] = |x_1| \cdots |x_{r_1}||z_1|^2 \cdots |z_{r_2}|^2$.

For $v \in F \otimes_{\mathbb{Q}} \mathbb{R}$, set $v \cdot \iota_F(\mathcal{O}_F) = \{v\iota_F(\alpha) \mid \alpha \in \mathcal{O}_F\}$. Set $D = \{v \in F \otimes_{\mathbb{Q}} \mathbb{R} \mid \frac{1}{2} \leq |v| \leq 1\}$. Let $X \subseteq F \otimes_{\mathbb{Q}} \mathbb{R}$ be bounded, convex, and symmetric about 0 such that the volume $\mu_V(X) > 2^n \mathrm{Vol}(v \cdot \iota_F(\mathcal{O}_F))$. By previous result, the right-hand side is bounded below by $2^{r_1+r_2-1}|\mathrm{disc}(F)|^{\frac{1}{2}}$. By Minkowski's theorem, there exists $\alpha \in \mathcal{O}_F \backslash \{v\}$ such that $v\iota_F(\alpha) \in X$. Since X is bounded, there exists $M > 0$ such that $[x] < M$ for all $x \in X$. Then $[v\iota_F(\alpha)] \leq M$. But $[v\iota_F(\alpha)] = [v]|N_{F/\mathbb{Q}}(\alpha)| \leq M$, so $|N_{F/\mathbb{Q}}(\alpha)| \leq 2M$.

Since there are only finitely many ideals $\subseteq \mathcal{O}_F$ with $N\mathfrak{a} \leq 2M$, there are only infinitely many $\beta\mathcal{O}_F$ such that $w\iota_F(\beta\mathcal{O}_F) \cap X \neq \{0\}$ for some $w \in D$. Let $\alpha_1\mathcal{O}_F, \cdots, \alpha_s\mathcal{O}_F$ be these finitely many ideals. Set $Y = \bigcup_{i=1}^{s} \iota_F(\alpha_i^{-1})X$. For $w \in D$, let $\beta \in \mathcal{O}_F \backslash \{0\}$ with $w\iota_F(\beta) \in X$. Then $\beta\mathcal{O}_F = \alpha_i\mathcal{O}_F$ for some i, and $\varepsilon = \beta\alpha_i^{-1} \in \mathcal{O}_F^\times$, and $w\iota_F(\varepsilon)\iota_F(\alpha_i)X \subseteq Y$. □

13 Lecture 13: October 21, 2022

Theorem 13.1 (Dirichlet's Unit Theorem)**.** $\mathcal{O}_F^\times \cong \mathbb{Z}^{r_1+r_2-1} \times \mu(F)$. In particular, $l_F(\mathcal{O}_F^\times)$ gives a complete lattice in H.

Proof. Set

$$l_F : F^\times \to \mathbb{R}^{r_1+r_2}$$

by defining

$$l_F(\alpha) = (\log|\sigma_1(\alpha)|, \cdots, \log|\sigma_{r_1}(\alpha)|, \log|\tau_1(\alpha)|, \cdots, \log|\tau_{r_2}(\alpha)|)$$

where σ_i are the real embeddings of F and τ_j's are one of each complex conjugate pair of complex embeddings. We have shown that $l_F(\mathcal{O}_F^\times) \subseteq H = \{(x_i)_{i=1}^{r_1+r_2} \mid \sum_{i=1}^{r_i} x_i + \sum_{j=1}^{r_2} x_{j+r_1} = 0\}$ is a lattice and $\ker(l_F) = \mu(F)$.

It suffices to show $l_F(\mathcal{O}_F^\times)$ is complete in H. We now use the notation as in Lemma 12.11.

Set $Y = \bigcup_{i=1}^{s} \iota_F(\alpha_i^{-1})X$. Since Y is bounded, there exists N such that $(x_i)_{i=1}^{r_1+r_2} \in Y$, then $|\lambda_i| \leq N$ for all i. For each $1 \leq i \leq r_1 + r_2$, let $v^{(i)} = (v_j^{(i)})_j \in F \otimes_{\mathbb{Q}} \mathbb{R}$ be such that $|v_j^{(i)}| > N$ for $j \neq i$ and $[v^{(i)}] = 1$. By the lemma, $v^{(i)} \in D$, so there exists $\varepsilon^{(i)} \in \mathcal{O}_F^\times$ such that $v^{(i)}\iota_F(\varepsilon^{(i)}) \in Y$. Note that $\iota_F(\varepsilon^{(i)} = (\varepsilon_j^{(i)})_j = (\sigma_1(\varepsilon^{(i)}), \cdots, \tau_{r_2}(\varepsilon^{(i)}))$. Since $v^{(i)}\iota_F(\varepsilon^{(i)}) \in Y$, then $|v_j^{(i)}\varepsilon_j^{(i)}| \leq N$. For $j \neq 1$, we then have $|\varepsilon_j^{(i)} < 1|$, then $l_F(\varepsilon^{(i)})$ has negative coordinates aside from the ith one.

Without loss of generality say $r_1+r_2 > 1$. Set $a_{ij} = \begin{cases} \log|\sigma_j(\varepsilon^{(i)})|, & 1 \leq j \leq r_1 \\ 2\log|\tau_{j-r_1}(\varepsilon^{(i)})|, & r_1 < j \leq r_1 + r_2 \end{cases}$.

Set $A = (a_{ij})_{i,j=1}^{r_1+r_2-1} \in M_{r_1+r_2-1}(\mathbb{R})$, then $l_F(\mathcal{O}_F^\times) \subseteq H$ implies $\sum_{j=1}^{r_1+r_2-1} a_{ij}a_{ij} = -a_{r_1+r_2} > 0$ for $i < r_1 + r_2$. Moreover, $a_{ij} < 0$ for $i \neq j$. By a lemma last time, A is invertible, so the $l_F(\varepsilon^{(i)})$ with $1 \leq i < r_1 + r_2$ are $\mathbb{R}$-linearly independent, and thus $l_F(\mathcal{O}_F^\times) \subseteq H \cong \mathbb{R}^{r_1+r_2-1}$ is a complete lattice. □

Example 13.2. Consider $F = \mathbb{Q}(\sqrt{d})$ where $d \neq 1$ is square-free integer. Suppose $d > 0$, then $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}^\times \cong \mathbb{Z} \times \langle -1 \rangle$. Suppose $d < 0$, then $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}^\times \cong \mu(F) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & d \neq -1, -3 \\ \mathbb{Z}/4\mathbb{Z}, & d = -1 \\ \mathbb{Z}/6\mathbb{Z}, & d = -3 \end{cases}$.

Example 13.3. Let $F = \mathbb{Q}(\sqrt{2})$ and let $\varepsilon \in \mathcal{O}_F^\times$ have infinite order. We have $\varepsilon = a + b\sqrt{d}$ where $a, b \in \mathbb{Z}$, and $\{\pm\varepsilon, \pm\varepsilon^{-1}\} = \{\pm a \pm b\sqrt{d}\}$. We may ask $a, b > 0$, then ε is called the fundamental unit of F, and $N_{F/\mathbb{Q}}(\varepsilon) = a^2 - db^2 = \pm 1$. Then $\varepsilon^n = a_n + b_n\sqrt{d}$, and $a_n > a$, $b_n > b$. For $d = 2$, $a = b = a$, then $\varepsilon = 1 + \sqrt{2}$.

Example 13.4. Let $F = \mathbb{Q}(\mu_n)$ for $n \geq 3$ and $r_1 = 0$, $r_2 = \frac{\varphi(n)}{2}$, and let $\mathbb{Z}[\mu_n]^\times \cong \mathbb{Z}^{\frac{\varphi(n)}{2}-1} \times \mu_m$, where m is $2n$ if n is odd, and n if n is even.

Definition 13.5. A number field F is totally real if $r_1 = n$ (and therefore $r_2 = 0$).

A number field F is purely imaginary if $r_2 = \frac{n}{2}$ (and therefore $r_1 = 0$).

A number field is CM if it is a purely imaginary quadratic extension of a totally real field. For example, $\mathbb{Q}(\mu_n)$ is CM for $n \geq 3$.

Example 13.6. For $\mu_n = \langle \zeta_n \rangle$, the extension $\mathbb{Q}(\mu_n) = \mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_n + \zeta_n^{-1}) = \mathbb{Q}(\mu_n)^+$ has degree 2, and therefore $\mathbb{Q}(\mu_n)^+$ is the maximal totally real subfield.

Remark 13.7 (Global Fields in Characteristic p)**.** Suppose $F/\mathbb{F}_p(t)$ is finite, we set $\mathcal{O}_F$ to be the integral closure of $\mathbb{F}_p[t]$ in F.

The primes of $\mathcal{O}_F$ are called finite primes of F. Let $\mathcal{O}'_F$ be the integral closure of $\mathbb{F}_p[t^{-1}]$ in F, and the primes of $\mathcal{O}'_F$ over (t^{-1}) are called infinite primes.

For a number field F, a finite prime is a nonzero prime ideal of $\mathcal{O}_F$, and an infinite prime is a real or complex prime of F.

13.1 Multiplicative Valuations

Definition 13.8. A multiplicative valuation $|\cdot|$ on a field K is a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ such that

(i) $|a| = 0$ if and only if $a = 0$.

(ii) $|ab| = |a||b|$.

(iii) $|a + b| \leq |a| + |b|$ for all $a, b \in K$.

We say the multiplicative valuation is trivial if $|a| = 1$ for all $a \neq 0$.

Remark 13.9. $|\mu(K)| = 1$.

The multiplicative valuation gives a topology on K defined by the metric $d(a, b) = |a - b|$. The trivial valuation gives the discrete topology.

Definition 13.10. We say that two valuations $|\cdot|_1$ and $|\cdot|_2$ on K are equivalent if they define the same topology on K.

Proposition 13.11. Let $|\cdot|_1$ and $|\cdot|_2$ be two valuations on K. The following are equivalent:

(i) $|\cdot|_1$ and $|\cdot|_2$ are equivalent.

(ii) $a \in K$ satisfies $|a|_1 < 1$ if and only if $|a|_2 < 1$.

(iii) There exists $s > 0$ such that $|a|_2 = |a|_1^s$ for all $a \in K$.

Remark 13.12. Let $|\cdot|$ be a valuation and let $s > 1$, then $|\cdot|^s$ need not be a valuation.

Proof. $|\cdot|_1$ is nontrivial, then there exists $b \in K$ with $|b|_1 > 1$ and $b^{-n} \to 0$ so the topology on K is not discrete: trivial valuation is its own equivalent class. Therefore, we may assume the two valuations are non-trivial.

$(i) \Rightarrow (ii)$: $a^n \to 0$ if and only if $|a|_i^n \to 0$ if and only if $|a|_i < 1$.

$(iii) \Rightarrow (i)$: $B_i(a,\varepsilon) = \{b \in K \mid |a-b|_i < \varepsilon\}$, then $B_1(a,\varepsilon) = B_2(a,\varepsilon)$ are the same as open balls.

$(ii) \Rightarrow (iii)$: Let $b \in K$ be such that $|b|_1 > 1$. Set $s = \frac{\log|b|_2}{\log|b|_1} > 0$, then $|b|_2 = |b|_1^s$. For $a \in K^\times$, let $t = \frac{\log|a|_1}{\log|b|_1} \in \mathbb{R}$, so $|a|_1 = |b|_1^t$. Then let $m, n \in \mathbb{Z}$, $n \neq 0$ be such that $q = \frac{m}{n} > t$, then $|a|_1 < |b|_1^q$, so $\left|\frac{a^n}{b^m}\right| < 1$, and in particular $|a|_2 < |b|_2^q$. Let $q \searrow t$ to see $|a|_2 \leq |b|_2^t$. Now do the same with $q < t$ to get $|a|_2 \geq |b|_2^t$, so $|a|_2 = |b|_2^t$. Then $|a|_2 = |b|_2^t = |b|_1^{st} = |a|_1^s$. $\square$

14 Lecture 14: October 24, 2022

Definition 14.1 (Nonarchimedean Valuation)**.** A nonarchimedean valuation on K is a multiplicative valuation such that $|a+b| \leq \max(|a|,|b|)$ for all $a, b \in K$.

Lemma 14.2. A valuation $|\cdot|$ on K is nonarchimedean if and only if $|n| \leq 1$ for all $n \geq 2$.

Proof. ($\Rightarrow$) $|m| \leq \max(|1|, \cdots, |n|) = 1$.

($\Leftarrow$): $|a+b|^k \leq \sum_{i=0}^{k} |\binom{k}{i}||a|^i|b|^{k-i} \leq \max(|a|,|b|)^k = (k+1)\max(|a|,|b|)^k$. Then $|a+b| \leq \lim_{k\to\infty}(k+1)^{\frac{1}{k}} \max(|a|,|b|) = \max(|a|,|b|)$. □

Definition 14.3 (Additive Valuation)**.** An additive ($(\mathbb{R})$-valued) valuation K is a function $v : K \cup \{\infty\}$ such that

(i) $v(a) = \infty$ if and only if $a = 0$,

(ii) $v(ab) = v(a) + v(b)$,

(iii) $v(a+b) \geq \min(v(a), v(b))$.

Lemma 14.4. Let $v : K \to \mathbb{R} \cup \{\infty\}$ and $|\cdot| : K \to \mathbb{R} \cup \{\infty\}$ be functions such that there exists $c > 1$ such that $|a| = c^{-v(a)}$ for all $a \in K$. Then v is an additive valuation if and only if $|\cdot|$ is a multiplicative valuation.

Definition 14.5. The value group of $|\cdot|$ is $|K^\times| \leq \mathbb{R}_{>0}$. We say $|\cdot|$ is discrete if and only if $|K^\times|$ is discrete.

Lemma 14.6. A nonarchimedean valuation $|\cdot|$ is discrete if and only if there exists $c \in \mathbb{R}_{>1}$ such that $v : K \to \mathbb{Z} \cup \{\infty\}$ with $v(a) = -\log_c |a|$ $\forall a \in K$ is a discrete valuation.

A nonarchimedean valuation $|\cdot|$ has a valuation ring $\mathcal{O} = \{a \in K \mid |a| \leq 1\}$ (because of the correspondence), which is local with maximal ideal $\mathfrak{m} = \{a \in K \mid |a| < 1\}$. Now $|\cdot|$ is discrete if and only if $\mathcal{O}$ is a DVR, in which case $\mathfrak{m}^n = \{a \in K \mid |a| \leq r^n\}$ for $r = \max(|K^\times| \cap \mathbb{R}_{<1}) < 1$.

Let F be a global field and p is a finite prime, or an infinite prime if $\mathrm{char}(F) > 0$. The p-adic valuation $|\cdot|_p$ on F is defined as $|a|_p = p^{f_p v_p(a)}$. Note that p^{f_p} is the order of the residue field of p. Therefore, when F is a number field, f_p is the residue degree over $\mathbb{Z}$. When F is a functional field, f_p is either the residue degree over $\mathbb{F}_p[t]$ for finite p, or is the residue degree of $\mathbb{F}_p[t^{-1}]$ for infinite p.[6]

Example 14.7. (a) $|a|_p = p^{-v_p(a)}$ on $\mathbb{Q}$.

(b) $|\frac{f}{g}|_\infty = q^{\deg(g)-\deg(f)}$ on $\mathbb{F}_q(t)$.

[6]Here we think of p lying over $(t^{-1}) \subseteq \mathbb{F}_p[t^{-1}]$.

These are all discrete (nonarchimedean) valuations. We now think of Archimedean valuations on a number field.

Let $\sigma : F \hookrightarrow \mathbb{C}$ be an embedding. Then $|\cdot|_\sigma : F \to \mathbb{R}_{\geq 0}$ given by $|a|_\sigma = |\sigma(a)|$ for all $a \in F$ is called the absolute value of F with respect to σ. Then F gets the subspace topology from $\mathbb{C}$ (or $\mathbb{R}$), so we say that $|\cdot|_\sigma$ is an Archimedean absolute value.

There are no valuations on local fields that are neither Archimedean nor nonarchimedean.

Definition 14.8 (Place)**.** A place of a global field is an equivalence class of non-trivial valuations on it.

Theorem 14.9 (Ostrowski)**.** The places of $\mathbb{Q}$ are exactly the equivalence classes of p-adic absolute values and the real absolute values.

Proof. Let $|\cdot|$ be a non-trivial valuation on $\mathbb{Q}$. Let $m, n \geq 2$ be integers. Write $m = \sum_{i=0}^{k} a_i n^i$ with $0 \leq a_i \leq n$, $k \geq 1$, and $a_k \neq 0$. Then $n^k \leq m$, so $k \leq \frac{\log(m)}{\log(n)}$. Let $N = \max(1, |n|)$. Then $|a_1| < n$ as $|1| = 1$, so $|m| < \sum_{i=0}^{k} n|n|^i \leq (1+k)nN^k \leq (1 + \frac{\log(m)}{\log(n)})nN^{\frac{\log(m)}{\log(n)}}$. Replacing m by m^t, $t > 0$ gives $|m| \leq (1 + t\frac{\log(m)}{\log(n)})^{\frac{1}{t}} n^{\frac{1}{t}} N^{\frac{\log(m)}{\log(n)}}$, so letting $t \to \infty$, we get $|m| \leq N^{\frac{\log(m)}{\log(n)}}$.

If $|n| \leq 1$ for some $n \geq 2$, then $N = 1$, so $|m| \leq 1$ for all $m \geq 2$, hence for all $m \in \mathbb{Z}$. Since the valuation $|\cdot|$ is non-trivial, then there exists p prime such that $|p| < 1$. Set $\mathfrak{m} = \{a \in \mathbb{Z} \mid |a| < 1\}$. Since $|\cdot|$ is now nonarchimedean, $\mathfrak{m}$ is not a proper ideal of $\mathbb{Z}$ and since $p \in \mathfrak{m}$, then $\mathfrak{m} = (p)$. Let $s > 0$ such that $|p| = p^{-s}$. Given $q = p^{v_p(q)\frac{m}{n}}$ with $m, n \in \mathbb{Z}$, $p \nmid mn$, then $|q| = p^{-sv_p(q)}$, so $|\cdot| \sim |\cdot|_p$.

If $|n| \geq 1$ for all $n \geq 2$, then $N = |n|$. Then $m^{\frac{1}{\log(m)}} \leq |n|^{\frac{1}{\log(n)}}$ for all $m \geq 2$. But m and n are arbitrary, so we must have $m^{\frac{1}{\log(m)}} = |n|^{\frac{1}{\log(n)}}$ for all $m, n \geq 2$, say the constant is $s > 1$. Then $|n| = s^{\log(n)} = n^{\log(s)}$ for all $n \geq 2$, so $|q| = |q|_\infty^{\log(s)}$ for all $q \in \mathbb{Q}$, so $|\cdot| \sim |\cdot|_\infty$.

Note that the valuations are different by Lemma 14.2. □

Exercise 14.10. The places of $\mathbb{F}_p$ are $|\cdot|_f$ for $f \in \mathbb{F}_q[t]$ irreducible and $|\cdot|_\infty$.

Let V_F be the set of places of F. The product formula for $\mathbb{Q}$ is given by $\prod_{v \in V_\mathbb{Q}} |a|_v = 1$, where $|a|_v$ is a representative of v. Indeed, by multiplicity, we can check this on a prime p. $|p|_p = p^{-1}$ and $|p|_l = 1$ for $l \neq p$, then $|p|_\infty = p$.

Note that this also gives a product formula for $\mathbb{F}_q(t)$.

15 Lecture 15: October 26, 2022

Lemma 15.1. Let $|\cdot|_1, \cdots, |\cdot|_k$ nontrivial inequivalent valuations on K, then there exists $a \in K$ such that $|a|_1 < 1$ and $|a|_j > 1$ for all $1 < j \leq k$.

Proof. Suppose $k = 2$. Consider $\alpha, \beta \in K$ such that $|\alpha|_1 < 1$ and $|\alpha|_2 \geq 1$, and $|\beta|_1 \geq 1$ and $|\beta|_2 < 1$. Set $c = \frac{\alpha}{\beta}$, then $|c|_1 < 1$ and $|c|_2 > 1$.

Suppose $k \geq 2$. By induction, there exists some $\alpha \in K$ such that $|\alpha|_1 < 1$ and $|\alpha|_j > 1$ for all $2 \leq j \leq k-1$, and there exists some $\beta \in K$ such that $|\beta|_1 < 1$ and $|\beta|_k > 1$.

If $|\alpha|_k > 1$, then we are done. If $|\alpha|_k = 1$, then choose $s >> 0$ such that $|\alpha|_j^s > |\beta|_j^{-1}$ for all $2 \leq j \leq k-1$ and $|\alpha|_1^s < |\beta|_1^{-1}$. Then take $a = \alpha^s\beta$. If $|\alpha|_k < 1$, let $c_m = \beta^{-1}(1+\alpha^m)^{-1}$ be under the topology $|\cdot|_i$. Note that this term converges to 1 under $|\cdot|_i$ if $|x|_i < 1$ and converges to 0 under $|\cdot|_i$ if $|x|_i > 1$. Therefore, $|c_m|_1 \to |\beta|_1^{-1}$ and $|c_m|_j \to 0$ for $2 \leq j \leq k-1$, and $|c_m|_k \to |\beta|_k^{-1}$. Take $a = c_m^{-1}$. □

Theorem 15.2 (Weak Approximation Theorem)**.** Consider the setup in Lemma 15.1 above: let $|\cdot|_1, |\cdot|_2, \cdots, |\cdot|_k$ be k inequivalent nontrivial valuations on K and consider $a_1, \cdots, a_k \in K$. For every $\varepsilon > 0$, there exists $b \in K$ such that $|b - a_i|_i < \varepsilon$ for all $1 \leq i \leq k$.

Remark 15.3. This is a generalization of Chinese Remainder Theorem.

Proof. By Lemma 15.1, there exists $a_i \in K$ such that $|a_i|_i < 1$ and $|a_i|_j > 1$ if $j \neq i$. Given $\delta > 0$, let $\beta_i = (1+\alpha_i^m)^{-1}$ with $m >> 0$ such that $|\beta_i - 1|_i < \delta$ and $|\beta_i|_j < \delta$ for $j \neq i$. Then set $b = \sum_{j=1}^{k} a_j\beta_j$. We see that $|b - a_i|_i \leq |a_i|_i|\beta_i - 1|_i + \sum_{j=1, j\neq i}^{k} |a_j|_i|\beta_j|_i < \delta \sum_{j=1}^{k} |a_j|_i < \varepsilon$ if we choose δ sufficiently small. □

15.1 Completions

Definition 15.4 (Valued Field)**.** A pair consisting of a field K and a valuation $|\cdot|$ on K is called a valued field.

A valued field has a topology given by $|\cdot|$ and is a topological field with respect to it, that is, the maps $K \times K \to K$ defined by the addition, subtraction, multiplication, and inverse mappings are all continuous.

Remark 15.5. If a field has a topology, it becomes a topological ring, but it is not necessarily a topological field, which requires the mappings to be continuous.

Definition 15.6 (Complete)**.** A valued field is complete if it is complete with respect to the metric defined by its valuation.

Given two valued fields $(L, |\cdot|)$ and $(L, |\cdot|')$ and a field embedding $\iota : K \hookrightarrow L$, then it is an embedding of valued fields if $|\iota(a)|' = |a|$ for all $a \in K$.

Theorem 15.7. Let K be a valued field, then there exists a complete valued field $(\tilde{K}, |\cdot|)$ and an embedding $\iota : K \hookrightarrow \tilde{K}$ of valued fields such that $\iota(K)$ dense in $\tilde{K}$.

Proof. Let R be the set of Cauchy sequences. By definition, if $(a_n)_{n\geq 1} \in R$, then for any ε, there exists $N \geq 1$ such that $|a_n - a_m|_n < \varepsilon$ for all $n, m \geq N$. But $\big||a_n| - |a_m|\big| < |a_n - a_m|$, so $(|a_n|)_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$, so converges. Then we can define $||\cdot|| : R \to \mathbb{R}_{\geq 0}$ by $||(a_n)_n|| = \lim_{n\to\infty} |a_n|$.

Claim 15.8. R is a ring.

Subproof. We prove its addition closure. Take Cauchy sequences $(a_n)_{n\geq 1}, (b_n)_{n\geq 1} \in R$. Then $|a_n b_n - a_m b_m| \leq |a_n||b_n - b_m| + |b_m||a_n - a_m|$. Take $M > 0$ such that $|a_n|, |b_m| \leq M$ for all large enough $n, m > 0$, then $|b_n - b_m|, |a_n - a_m| < \frac{\varepsilon}{2<}$ for all $n, m >> 0$, then we get $|a_n b_n - a_m b_m| < \varepsilon$ for all $n, m >> 0$, so $(a_n b_n)_n \in R$ as desired. ■

Claim 15.9. $\mathfrak{M} = \{(a_n)_{n\geq 1} \in R \mid a_n \to 0\}$ is a maximal ideal of R.

Subproof. We check the maximal property. If $(a_n)_{n\geq 1} \in R\backslash\mathfrak{M}$, then there exists $(b_n)_{n\geq 1} \in \mathfrak{M}$ such that $a_n + b_n \neq 0$ for all n. Therefore, we may assume $a_n \neq 0$ for all n, then $(a_n^{-1})_{n\geq 1} \in R$: $|a_n^{-1} - a_m^{-1}| = \frac{|a_m - a_n|}{|a_m||a_n|}$ small for $m, n >> 0$, so $(a_n)_{n\geq 1} \in R^\times$. ■

Define $\tilde{K} = R/\mathfrak{M}$. Note that the embedding $K \hookrightarrow \tilde{K}$ maps $a \mapsto (a_n)_{n\geq 1}$ and $||\cdot||$ defines a valuation of $|tildeK$ extending $|\cdot|$.

Claim 15.10. $(\tilde{K}, ||\cdot||)$ is complete.

Subproof. Let $c_m = (c_{m,n})_{n\geq 1}$ and $m \geq 1$ give a Cauchy sequence in R: $||c_m - c_k|| = \lim_{n\to\infty} |c_{m,n} - c_{k,n}| < \varepsilon$ for $k, m \geq N$ for some N. Then there exists $N' \geq N$ such that $|c_{m,n} - c_{k,n}| < \varepsilon$ for all $k, m, n \geq N'$. As each c_m is Cauchy, then there exists an increasing sequence $(l_m)_{m\geq 1}$ starting from $l_1 \geq N'$, such that $|c_{m,n} - c_{m,k}| < \varepsilon$ for $k, n \geq l_m$.

Set $a_n = c_n l_n$, then $|a_n - a_m| \leq |c_{n,l_n} - c_{m,l_n}| + |c_{m,l_n} - c_{m,l_m}| < 2\varepsilon$ (which is good enough). So $(a_n)_{n\geq 1} \in R$. Also, $||c_m - (a_n)_{n\geq 1}|| = \lim_{n\to\infty} |c_{m,n} - c_{n,l_n}|$ and $|c_{m,n} - c_{n,l_n}| \leq |c_{m,n} - c_{m,l_n}| + |c_{m,l_n} - c_{n,l_n}| < 2\varepsilon$ for $m \geq N'$ and $n \geq l_m$, so $||c_m - (a_n)_{n\geq 1})|| \leq 2\varepsilon$ for $m \geq N$;, then $(c_m)_{m\geq 1} \to (a_n)_{n\geq 1} \in R$. ■

Claim 15.11. K is dense in $\tilde{K}$.

Subproof. For $(a_n)_{n\geq 1} \in R$, there is $||(a_m)_n - (a_n)_n|| = \lim_{n\to\infty} |a_m - a_n| < \varepsilon$ for $m \geq N$ with some given N. Therefore, $(\iota(a_m))_{,\geq 1} \to (a_n)_{n\geq 1}$ as $m \to \infty$. ■

□

Proposition 15.12. Let K be a valued field and $\tilde{K}$ be the complete valued field of Theorem 15.7, with an embedding of valued fields $\iota : K \to \tilde{K}$. If L is another complete valued field with a field embedding $\sigma : K \to L$, then there exists a unique extension of σ to $\tilde{\sigma} : \tilde{K} \hookrightarrow L$ such that $\sigma = \tilde{\sigma} \circ \iota$.

Proof. See lecture notes, proposition 5.3.13. □

Definition 15.13 (Completion)**.** Therefore, $\tilde{K}$ is unique up to unique isomorphism of valued fields. Therefore, we call it the completion of K.

16 Lecture 16: October 28, 2022

Theorem 16.1 (Ostrowski)**.** Let K be a complete valued field with respect to a valuation which is archimedean. Then K is isomorphic to $(\mathbb{R}, |\cdot|^s)$ or $(\mathbb{C}, |\cdot|^s)$ for some $s \leq 1$.

Remark 16.2. $|\cdot|^s$ is a valuation on $\mathbb{C}$ (or $\mathbb{R}$) if and only if $s \in (0, 1]$.

Lemma 16.3. $(K, |\cdot|)$ is a nonarchimedean valued field with completion $(\hat{K}, |\cdot|)$. Then $|\cdot|$ on $\hat{K}$ is nonarchimedean. Letting $\mathcal{O}$ be the valuation ring and let $\mathfrak{m}$ be a maximal ideal for K, and $\hat{\mathcal{O}}$ and $\hat{\mathfrak{m}}$ for $\hat{K}$ is defined similarly, the canonical map $\hat{\imath} : \mathcal{O}/\mathfrak{m} \to \hat{\mathcal{O}}/\hat{\Updownarrow}$ is an isomorphism. Moreover, if $|\cdot|$ is discrete on K, then $|K^\times| = |\hat{K}^\times|$ and $\hat{\imath}_n : \mathcal{O}/\mathfrak{m}^n \to \hat{\mathcal{I}}/\mathfrak{m}^n$ is an isomorphism.

Proof. $|\cdot|$ is nonarchimedean by density of K in $\hat{K}$ and if $|\cdot|$ is discrete, then $|K^\times| = |\hat{K}^\times|$ for similar reasons. $\mathfrak{m} = \mathcal{O} \cap \hat{\mathfrak{m}}$, so $\hat{\imath}$ is injective. If $a \in \hat{\mathcal{O}}$, then there exists $b \in K$ such that $|b - a| < 1$. Then $b - a \in \hat{\mathfrak{m}}$, so $b \in \hat{\mathcal{O}} \cap K = \mathcal{O}$ and $\bar{\imath}(b + \mathfrak{m}) = a + \hat{\mathfrak{m}}$. Then $\bar{\imath}_n$ is an isomorphism, left as an exercise. □

Proposition 16.4. Let K be a complete discrete valuation field. Let $\mathcal{O}$ be its valuation ring. Let T be a set of representations in $\mathcal{O}$ of $\mathcal{O}/\mathfrak{m}$ with $0 \in T$. Every $a \in K^\times$ equals $\sum_{k=m}^{\infty} c_k \pi^k$ for uniformizer π, and $m \in \mathbb{Z}$ with $c_m \neq 0$ and $c_k \in T$ for all $m \leq k$. This expression is unique, and $v(a) = m$, where v corresponds to $|\cdot|$.

Proof. If $v(a) = m$, then $a - c_m \pi^m \in \mathfrak{m}^{m+1}$ for some unique $c_m \in T \backslash \{0\}$. If $a - \sum_{k=m}^{n} c_k \pi^k \in \mathfrak{m}^{n+1}$, then there exists a unique $c_{n+1} \in T$ such that $a - \sum_{k=m}^{n+1} c_m \pi^m \in \mathfrak{m}^{n+2}$, then take the limit. □

Corollary 16.5. $K(t)$ with t-adic valuation has completion $K((t))$, the Laurent series in t, with valuation ring $K[[t]]$, the power series in K.

Definition 16.6. The p-adic numbers (or field) $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$. The valuation ring is $\mathbb{Z}_p$, the p-adic integers.

Example 16.7. Consider $T = \{0, 1, \cdots, p-1\}$. Now $1 + p + p^2 + \cdots = (1-p)^{-1}$, so $\sum_{i=0}^{\infty} (p-1)p^i = -1 \in \mathbb{Z}_p$.

Proposition 16.8. Let K be a complete discrete valuation field with valuation ring $\mathcal{O}$ and maximal ideal $\mathfrak{m}$. Then there is a canonical map $\mathcal{O} \to \varprojlim_n \mathcal{O}/\mathfrak{m}^n \subseteq \prod_n \mathcal{O}/\mathfrak{m}^n$ which is an isomorphism of (topological) rings.

Proof. The map sends $\sum_{m=0}^{\infty} c_k \pi^k$ to $\left(\sum_{m=0}^{\infty} c_k \pi^k \right)_n$. □

Definition 16.9. We say a DVR A with maximal ideal $\mathfrak{p}$ is complete if the map $A \xrightarrow{\sim} \varprojlim_n A/\mathfrak{p}^n$ is an isomorphism.

Lemma 16.10. Let A be a DVR and $K = Q(A)$. Then K is complete if and only if A is complete.

Proof. Left as an exercise. □

Theorem 16.11 (Hensel's Lemma, Weak Form)**.** Let K be a complete nonarchimedean valuation field and define $\mathcal{O}$ and $\mathfrak{m}$ to be its valuation ring and a maximal ideal, respectively. Let $f \in \mathcal{O}[x]$ and $\bar{f}$ be the image of f in $\mathcal{O}/\mathfrak{m}[x]$. Let $\bar{\alpha} \in \mathcal{O}/\mathfrak{m}$ be a simple roof of $\bar{f}$. Then there exists a unique $\alpha \in \mathcal{O}$ as a roof of f such that $\alpha \mapsto \bar{\alpha} \pmod{\mathfrak{m}}$.

Proof. Let $\alpha_0 \in \mathcal{O}$ be any lift of $\bar{\alpha}$ and $\pi = f(\alpha_0) \in \mathfrak{m}$. We argue by induction to build α_k that is congruent to some power of π. Therefore, we suppose we have $\alpha_k \in \mathcal{O}$ for some $0 \leq k \leq n$ such that $\alpha_n \equiv \alpha_k \pmod{\mathfrak{m}^{2^k}}$ for all $0 \leq k \leq n$, and $f(\alpha_n) \equiv 0 \pmod{\pi^{2^n}}$. Define $f' \in \mathcal{O}[x]$ formally, i.e. as the algebraic derivative.

Consider $f(\alpha_n + x) - f(\alpha_n) - f'(\alpha_n)x \in (x^2)$, then $f(\alpha_n + \beta\pi^{2^n}) \equiv f(\alpha_n) + f'(\alpha_n)\beta\pi^{2^n} \pmod{\pi^{2^{n+1}}}$ for all $\beta \in \mathcal{O}$. Now $f'(\alpha_n) \notin \mathfrak{m}$ because $\bar{\alpha}$ is a simple root, so $f'(\alpha_n) \in \mathcal{O}^\times$, and $f(\alpha_n) \in (\pi^{2^n})$, so $f(\alpha_n + \beta\pi^{2^n})$ for some unique $\beta \pmod{\pi^{2^n}}$. Set $\alpha_{n+1} = \alpha_n + \beta\pi^{2^n} \equiv \alpha_n \pmod{\pi^{2^n}}$. Now α_{n+1} is unique and $\alpha = \lim_{n\to\infty} \alpha_n \in \mathcal{O}$ works (and is unique). □

Example 16.12. Note that $f = x^2 - 5$ has two simple roots in $\mathbb{F}_{11}$, namely 4 and 7. Take $\alpha_0 = 4$. $f(4) = 11$ and $f'(4) = 8$. By formula, $\beta = -\frac{f(\alpha_0)}{f'(\alpha_0)\pi^{2n}}$ and $\alpha_1 = 4 - \frac{11}{8} = 4 + 4 \cdot 11 \pmod{121}$ as $\alpha_1 = \alpha_0 - \frac{f(\alpha_0)}{f'(\alpha_0)}$. Similarly, $\alpha_2 = (4 + 4 \cdot 11) - \frac{r^2+32\cdot 11+16\cdot 11^2-5}{2(4+4\cdot 11)} \equiv 4 + 4 \cdot 11 + 10 \cdot 11^2 + 4 \cdot 11^3 \equiv 6582 \pmod{11^4}$.

Lemma 16.13. Let p be an odd prime. $|\mu(\mathbb{Q}_p)| = p - 1$. If $p = 2$, the order is 2.

Proof. $x^{p-1} - 1$ splits over $\mathbb{F}_p$ with simple roots. These roots lift to distinct roots in $\mathbb{Z}_p$. Therefore, $|\mu_{p-1}(\mathbb{Q}_p)| = p-1$. Suppose $\zeta_n \in \mathbb{Z}_p$ prime of order n. Suppose $m \mid n$ and $\zeta_n^m \equiv 1 \pmod{p}$, but then if $m \neq n$, there exists $l \mid n$ such that $\zeta_k = \zeta_n^{\frac{n}{l}} \equiv 1 \pmod{p}$. If $l \neq p$, then $\zeta_l - 1 \mid l = \prod_{i=0}^{l-1}(1 - \zeta_l^i)$, now as an ideal $(l) = \left(\prod_{i=0}^{l-1}(1 - \zeta_l^i)\right) = (1 - \zeta_l)^{l-1}$ in $\mathbb{Z}_p$, and $p \mid (1 - \zeta_l)^{l-1} = (l)$, contradiction. If $l = p$, we have a similar contradiction as $p \mid (1 - \zeta_p)$. Therefore, $\mu(\mathbb{Q}_p) \cong \mu(\mathbb{F}_p)$ with order $p - 1$. □

Remark 16.14. There is a unique (injective) homomorphism $\mathbb{F}_p^\times \hookrightarrow \mathbb{Z}_p^\times$ which matches the lift.

Theorem 16.15 (Hansel's Lemma, Strong Form)**.** Let K be a complete nonarchimedean valuation field with its valuation ring $\mathcal{O}$ and maximal ideal $\mathfrak{m}$. If $f \in \mathcal{O}[x]$ is primitive with image $\bar{f} \in \mathcal{O}/\mathfrak{m}[x]$ which factors as $\bar{f} = \bar{g}\bar{h}$, where $\bar{g}$ and $\bar{h}$ are relatively prime, then $f = gh$ factors where $g, h \in \mathcal{O}[x]$ with reductions $g \mapsto \bar{g}$ and $h \mapsto \bar{h}$ such that $\deg(g) = \deg(\bar{g})$. Moreover, if $g', h' \in \mathcal{O}[x]$ with $\deg g' = \deg \bar{g}$ satisfy $f \equiv g'h' \pmod{\mathfrak{b}}$ for ideal $\mathfrak{b}$ in $\mathcal{O}$, and $g' \mapsto \bar{g}$ and $h' \mapsto \bar{h}$, then g and h can be chosen so that $g \equiv g' \pmod{\mathfrak{b}}$ and $h \equiv h' \pmod{\mathfrak{b}}$.

Remark 16.16. A valuation ring satisfying the weak form of Hensel's Lemma (but without uniqueness property) is called Henselian. Henselian rings also satisfy the strong form.

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Proof. Note that $f \in \mathcal{O}[x]$ is primitive if and only if $f \not\equiv 0 \pmod{\mathfrak{m}}$. Set $k = \deg(\bar{g})$ and $d = \deg(f)$. Let $g_0, h_0 \in \mathcal{O}[x]$ lift $\bar{g}$ and $\bar{h}$ respectively such that $\deg(g_0) = k$ and $\deg(h_0) \leq d - k$, then $f \equiv g_0 h_0 \pmod{\mathfrak{m}}$.

Let $a, b \in \mathcal{O}$ be such that $ag_0 + bh_0 \equiv 1 \pmod{\mathfrak{m}}$. Note that $\mathfrak{m}$ is not necessarily generated by the uniformizer. Let $\mathfrak{a} \subseteq \mathfrak{m}$ be the ideal generated by the coefficients of $ag_0 + bh_0 - 1$, then $\mathfrak{a} = (\pi)$ for π with $|\pi|$ being the maximal of valuations of coefficients.

Suppose that for $n \geq 1$ (using the induction argument), there exists $g_m, h_m \in \mathcal{O}[x]$ such that $\deg(g_m - g_0) < k$ and $\deg(h_m) \leq d - k$, and $f \equiv g_m h_m \pmod{\mathfrak{a}^{m+1}}$ and $g_m \equiv g_{m-1} \pmod{\mathfrak{a}^m}$ and $h_m \equiv h_{m-1} \pmod{\mathfrak{a}^m}$. Set $f_n = \pi^{-n}(f - g_{n-1}h_{n-1}) \in \mathcal{O}[x]$. Note that the leading coefficient of g_0 is a unit since $\deg(g_0) = \deg(\bar{g})$, then by the division algorithm $bf_n = q_n g_0 + r_n$ with $q_n, r_n \in \mathcal{O}[x]$, with $\deg(r_n) < k$. Then

$$(*)\ (af_n + q_n h_0)g_0 + r_n h_0 = af_n g_0 + bf_n h_9 \equiv f_n \pmod{\mathfrak{m}}.$$

Let $s_n \in \mathcal{O}[x]$ have coefficients agreeing with those of $af_n + q_n h_0$ which are not equivalent to 0 modulo $\mathfrak{m}$, and which are 0 otherwise. Set $g_n = g_{n-1} + \pi^n r_n$ and $h_n = h_{n-1} + \pi^n s_n$. Then

$$\begin{aligned} g_n h_n &\equiv g_{n-1}h_{n-1} + \pi^n(r_n h_{n-1} + s_n g_{n-1}) \\ &\equiv g_{n-1}h_{n-1} + \pi^n(r_n h_0 + s_n g_0) \\ &\equiv g_{n-1}h_{n-1} + \pi^n f_n \text{ by } (*) \\ &\equiv f \pmod{\pi^{n+1}}. \end{aligned}$$

Since $\deg(g_{n-1} - g_0) < k$ and $\deg(r_n) < k$, then $\deg(g_n - g_0) < k$. Since $\deg(r_n h_0) < d$ and $\deg(f_n) \leq d$, reduction of $(af_n + q_n h_0)g_0$ has degree at most d by $(*)$. As the nonzero coefficients of s_n are units, then $\deg(s_n g_0) \leq d$, so $\deg(s_n) \leq d - k$, so $\deg(h_n) \leq d - k$. □

17.1 Extension of Valuation

Definition 17.1 (Extension of Valuation)**.** Let L/K be a field extension. An extension of a valuation on K is a valuation on L which restrict to the valuation on K.

Remark 17.2. Suppose L/K is an extension of global fields and $\mathfrak{p}$ is a nonarchimedean prime of K, $\mathfrak{P}$ is a prime of L over $\mathfrak{p}$. Fix $\alpha \in K$, then

$$\begin{aligned} |\alpha|_{\mathfrak{P}} &= p^{-f_P v_P(\alpha)} \\ &= p^{-f_{\mathfrak{P}} e_{\mathfrak{P}/\mathfrak{p}} v_{\mathfrak{p}}(\alpha)} \\ &= p^{-f_{\mathfrak{p}} e_{\mathfrak{P}/\mathfrak{p}} e_{\mathfrak{P}/\mathfrak{p}} v_{\mathfrak{p}}(\alpha)} \\ &= |\alpha|_{\mathfrak{p}}^{e_{\mathfrak{P}/\mathfrak{p}} f_{\mathfrak{P}/\mathfrak{p}}} \end{aligned}$$

Therefore, in general, $|\cdot|_{\mathfrak{P}}$ is not an extension of $|\cdot|_{\mathfrak{p}}$.

Proposition 17.3. Suppose $(V, |\cdot|)$ is a finite-dimensional normed vector space over K such that $|\alpha||v| = |\alpha v|$ for all $\alpha \in K$ and $v \in V$. Then V is complete with respect to $|\cdot|$ and if $(v_1, \cdots, v_n)$ is an ordered basis of V, then the linear isomorphism $\varphi : K^n \to V$ that maps $e_i \mapsto v_i$ is a homeomorphism.

Proof. $||(a_1, \cdots, a_n)|| = \max(|a_i|)$ gives the product topology on K^n, therefore induces norm $||\sum_{i=1}^{n} a_i v_i|| = \max(|a_i|)$ on V. It is easy to see there exists $c_1, c_2 > 0$ such that $c_1||v|| < |v| < c_2||v||$ for all $v \in V$. For instance, we can pick $c_2 = \sum_{i=1}^{n} |v_i|$ and c_1 can be found by induction on n. Take $c_1 = |v_1|$ and in general set $W_j = \sum_{i \neq j} K \cdot v_i$ for all j, then W_i is complete with respect to $|\cdot|$, so W_j is closed in V. Let $B = B(0, \varepsilon)$ for some $\varepsilon > 0$ such that $B \cap (v_i + W_i) = \varnothing$ for all i. Let $v = \sum_i a_i v_i \neq 0$. If $a_j \neq 0$, then $a_j^{-1} v \in v_j + W_j$, so $|a_j^{-1} v| \geq \varepsilon$, so $|v| \geq \varepsilon ||(a_1, \cdots, a_n)|| = \varepsilon ||v||$. Take $c_1 = \varepsilon$ in these cases. □

Lemma 17.4. Suppose K is nonarchimedean. Let $f = \sum_{i=0}^{n} a_i x^i \in K[x]$ be irreducible with $a_0, a_n \neq 0$. Then either $|a_0|$ or $|a_n|$ is maximal among the $|a_i|$.

Proof. We may assume $f \in \mathcal{O}[x]$ and has a unit coefficient by scaling. Let j be minimal such that $a_j \in \mathcal{O}^\times$. Then $f \equiv x^j g \pmod{m}$, where $g = a_n x^{n-j} + \cdots + a_j$. Therefore, we can factor f unless $j = 0$ or $j = n$ by strong Hensel's lemma. □

Corollary 17.5. Suppose f is monic and irreducible in $K[x]$ with $f(0) \in \mathcal{O}$, then $f \in \mathcal{O}[x]$.

Corollary 17.6. Suppose L/K is finite, then the integral closure of $\mathcal{O}$ of L is $\{\beta \in L \mid N_{L/K}(\beta) \in \mathcal{O}\}$.

Theorem 17.7. Suppose L/K is algebraic. There exists a unique extension of $|\cdot|_K$ to a valuation $|\cdot|_L$ on L. It is nonarchimedean if and only if $|\cdot|$ is. If L/K is finite, then L is complete with respect to $|\cdot|_K$ and $|B|_L = |N_{L/K}(\beta)|_K^{\frac{1}{[L:K]}}$ for all $\beta \in K$.

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