MATH 595 (Group Cohomology) Notes

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1 Aug 21, 2023: Introduction

Group cohomology works over different settings of groups, like finite groups, profinite groups, and topological groups. The course will develop towards

- duality in $H^*(G, -)$, and
- focus on computations, e.g., spectral sequences.

We first establish some notations.

- Let G be a group. If G has a topology, that would also be part of the information of G.
- A (left) G-module is an abelian group M with an action map

$$\begin{aligned} G \times M &\to M \\ (g, m) &\mapsto g \cdot m = gm \end{aligned}$$

  satisfying

  - $1 \cdot m = m$,
  - $(gh) \cdot m = g \cdot (hm)$,
  - $g(m + m') = gm + gm'$.

Remark 1.1. If G is a finite group, then the associated (non-commutative) group ring $\mathbb{Z}[G] := \bigoplus_{g \in G} \mathbb{Z}e_g$, where the multiplication is determined by $e_g e_h = e_{gh}$. Therefore, a G-module is just a $\mathbb{Z}[G]$-module.

Example 1.2.

- Trivial module $\mathbb{Z}$, or any abelian group with the trivial action $g \cdot a = a$.
- C_2, or any group with $f : G \twoheadrightarrow C_2$, then G with C_2 as a quotient gives the sign representation $\mathbb{Z}_{\text{sgn}}$, with $g \cdot (a) = (-1)^{\rho(g)} a$.
- $\mathbb{Z}[G]$ is a G-module via the left multiplication action, and/or the conjugation action.

Definition 1.3 (Fixed points/Invariants)**.** The set of fixed points of M over G is $M^G = \{m \in M \mid gm = m \ \forall g \in G\}$.

Definition 1.4 (Orbits/Coinvariants)**.** The set of orbits of M over G is $M_G = M/(gm - m)$.

Example 1.5. If $M = \mathbb{Z}_{\text{sgn}}$, then everything gets multiplied by -1, so there are no fixed points. The orbits of M over G would be $\mathbb{Z}_{\text{sgn}}/(-2) \cong \mathbb{Z}/2\mathbb{Z}$.

Example 1.6. If $M = \mathbb{Z}[G]$, then the fixed points are $\mathbb{Z}\left\{\sum_{g \in G} e_g\right\}$.

Thinking in a categorical setting, we have a trivial action function $\mathbb{Z}$-**Mod** $\to$ G-**Mod**, sending $ga \mapsto a$ for all $g \in G$ and $a \in A$. This gives an exact functor from **Ab** to G-**Mod**. Then this functor has a right adjoint $(\ )^G : G\text{-}\mathbf{Mod} \to \mathbf{Ab}$, and a left adjoint $(\ )_G : \mathbf{Ab} \to G\text{-}\mathbf{Mod}$. More specifically, M^G becomes the maximal trivial action submodule of M, namely $\mathrm{Hom}_G(\mathbb{Z}, M)$; M_G becomes the largest quotient of M with trivial action, namely $\mathbb{Z} \otimes_{\mathbb{Z}[G]} M$. This simplifies to the tensor-hom adjunction in some sense. For a more detailed derivation of this, see Chapter 6.1 of Weibel.

Remark 1.7. In general, as in the category of G-sets, we have the orbit functor $X \mapsto X/G$ and the fixed point functor $X \mapsto X^G$. The orbit functor is left adjoint to the free G-set functor, and the fixed point functor is the right adjoint of the trivial G-set functor.

Remark 1.8. Read more about the setting in profinite groups with their topologies in Neukirch-Schmidt-Wingberg.

Definition 1.9 (Profinite Group)**.** A profinite group of a collection of groups is $G = \varprojlim_i G_i$ as an inverse limit, where each G_i is a finite group of the form G/U_i for some open U_i. This gives a topology to the profinite group.

Remark 1.10. The groups rings $\mathbb{Z}[[G]] = \varprojlim_i \mathbb{Z}[G_i]$. For instance, let $G = \hat{\mathbb{Z}_p} = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$, then $\mathbb{Z}_p[[G]] = \varprojlim_n \mathbb{Z}_p[\mathbb{Z}/p^n\mathbb{Z}]$, where each $\mathbb{Z}[\mathbb{Z}/p^n\mathbb{Z}] \cong \bigoplus_{i=0}^{n-1} \mathbb{Z}\{e_i\}$ where $e_i \cdot e_j = e_{ij}$. Therefore, $\mathbb{Z}_p[[G]]$ is now equivalent to $\varprojlim_n \mathbb{Z}_p[t]/(t^{p^n} - 1_e)$, and hence becomes a power series.

Remark 1.11. By a change of variables, this becomes $\varprojlim_n \mathbb{Z}_p[x]/(x^{p^n})$, but this only works in the finite group $\mathbb{Z}_p$ case, and not in general for $\mathbb{Z}$.

Example 1.12. $\mathbb{Z}[C_n] \cong \mathbb{Z}\{e\} \oplus \mathbb{Z}\{g\} \oplus \mathbb{Z}\{g^2\} \oplus \cdots \oplus \mathbb{Z}\{g^{n-1}\} \cong \mathbb{Z}[g]/(g^n - 1_e)$.

2 Aug 23, 2023: Cohomology of groups

Definition 2.1. Let G be a group, then we have a diagram

$$EG^{\cdot} : \cdots \Rrightarrow G \times G \rightrightarrows G$$

where the arrows are given by

$$EG^n = G^{n+1} \xrightarrow{d_i} G^n$$

for all $0 \leqslant i \leqslant n$. In the sense of simplicial sets, we have $d_i(g_0, \dots, g_n) = (g_0, \dots, \hat{g}_i, \dots, g_n)$.

Now let M be a G-module, then we define $X^n = X^n(G, M) = \mathrm{Map}_{\mathbf{Set}}(G^{n+1}, M)$. G now has an action on this set, given by

$$(g \circ f)(g_0, \dots, g_n) = gf(g^{-1}g_0, \dots, g^{-1}g_n).$$

The action on d^i's are contravariant, namely we obtain $d_i^* : X_n \to X^{n+1}$ with an inherited structure. Note that M sits inside X^0, therefore we have a complex $(*)$:

$$0 \longrightarrow M \xhookrightarrow{\partial_0} X^0 \xrightarrow{\partial_1} X^1 \xrightarrow{\partial_2} X^2 \xrightarrow{\partial_3} \cdots$$

Here ∂_0 includes M as the constant functions into X, namely $\partial_0(m) = f$ for $f(g) = m$, and so on. In general, for $n > 0$, we have

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i^*.$$

Lemma 2.2. The complex $(*) : M \to X^{\cdot}$ is an exact complex of G-modules, i.e., $\partial^2 = 0$ and $\ker(\partial_{n+1}) = \mathrm{im}(\partial_n)$, and the ∂_i's preserves the G-action. This is called the standard resolution of M as a G-module.

Proof. Exercise. $\square$

Definition 2.3. The G-fixed points of the X^n's are defined by $C^n(G,M)=(X^n(G,M))^G$, called the homogeneous n-cochains of G with coefficients in M. Because the complex preserves G-actions, then we obtain a complex of $C^n(G,M)$'s, given by

$$0 \longrightarrow C^0(G,M) \xrightarrow{\partial_0} C^1(G,M) \xrightarrow{\partial_1} \cdots$$

Remark 2.4. To see what the induced mapping is, suppose $A \to B$ is a G-module map, then there is an induced map of fixed points $A^G \to B^G$ by the restriction. In particular, let $a \in A$ be fixed with $ga = a$ for all $g \in G$, then $f(a) = f(ga) = gf(a)$.

Remark 2.5. In the complex of Definition 2.3, $\partial^2 = 0$ as well, but in general this is not an exact sequence.

Definition 2.6 (Group Cohomology)**.** The group cohomology of G with coefficients in M is the collection

$$\{H^n(G,M)\}_{n\geqslant 0},$$

where $H^n(G,M) := H^n(C^\cdot(G,M)) = \ker(\partial : C^n \to C^{n+1})/\operatorname{im}(\partial : C^{n-1} \to C^n)$. We usually use the notion of cocycles $Z^n(G,M) = \ker(\partial : C^n \to C^{n+1})$ and coboundaries $B^n(G,M) = \operatorname{im}(\partial : C^{n-1} \to C^n)$.

Exercise 2.7. Show that $H^0(G,M)$ is isomorphic to M^G.

Definition 2.8. The inhomogeneous cochains $C_i^\cdot(G,M)$ are given by

- $C_i^0 = M$, and
- for $n > 0$, $C_i^n = \operatorname{Map}(G^n, M)$,

with coboundary maps $\partial^{n+1} : C_i^n \to C_i^{n+1}$, given by

- $\partial^1(m)(g) = gm - m$,
- $\partial^2(f)(g_1,g_2) = g_1 f(g_2) - f(g_1g_2) + f(g_1)$, and so on, with
- $\partial^{n+1}(f)(g_1,\ldots,g_{n+1}) = g_1 f(g_2,\ldots,g_{n+1}) + \sum\limits_{i=1}^{n}(-1)^i f(g_1,\ldots,g_ig_{i+1},\ldots,g_{n+1}) + (-1)^{n+1} f(g_1,\ldots,g_n)$.

This gives the inhomogeneous setting of this cochain.

Lemma 2.9. The maps

$$\begin{aligned} C^n(G,M) &\to C_i^n(G,M) \\ (\varphi : G^{n+1} \to M) &\mapsto (f : G^n \to M) \\ & \qquad f(g_1,\ldots,g_n) := \varphi(1, g_1, g_1g_2, \ldots, g_1g_2\cdots g_n) \end{aligned}$$

give a cochain homotopy equivalence $C^\cdot(G,M) \xrightarrow{\sim} C_i^\cdot(G,M)$, and hence this is a quasi-isomorphism.

Corollary 2.10. The cohomology $H^*(C_i^\cdot(G,M)) \cong H^*(G,M)$.

Remark 2.11. Any cohomology class can be represented by a normalized inhomogeneous cocycle $f : G^n \to M$, i.e., $f(g_1,\ldots,g_n) = 0$ where $g_i = 1$ for some i.

Remark 2.12. Even for $G = C_2$, C_i^n or C^n get large as n grows.

Remark 2.13.

- Using homological algebra, we can find other cochain complexes which computes group cohomology $H^*(G,M)$.
- We would also understand $H^*(G,M)$ as the failure of exactness of $(\ )^G : G\text{-}\mathbf{Mod} \to \mathbf{Ab}$. Therefore, when taking the fixed points, the exact sequence may not be mapped to another exact sequence. In particular, if we take an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of G-modules, the induced sequence

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G$$

do not give a surjection at $B^G \to C^G$. One needs to take higher cohomology to obtain a long exact sequence. Hence, $(\ )^G : G\text{-}\mathbf{Mod} \to \mathrm{Ab}$ is a left exact functor, but not necessarily right exact.

3 Aug 25, 2023: Cohomology of groups, continued

Example 3.1. Let G be C_2, or any group with a surjection p onto C_2, then it has an action on $\mathbb{Z}_{\text{sgn}}$ given by $g \cdot a = (-1)^{p(g)} a$, therefore we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}_{\text{sgn}} \xrightarrow{\times 2} \mathbb{Z}_{\text{sgn}} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

and taking the fixed point functor we have

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

Remark 3.2. Higher homologies measure the failure of exactness.

Remark 3.3. The collection $\{H^n(G,-)\}_{n\in\mathbb{Z}}$ satisfies

- $H^n(G,-) = 0$ for $n < 0$;
- for short exact sequence $0 \to A \to B \to C \to 0$ in G-**Mod**, we have a long exact sequence

  $$0 \longrightarrow H^0(G,A) \longrightarrow H^1(G,B) \longrightarrow H^1(G,C) \xrightarrow{\delta} H^1(G,A) \longrightarrow \cdots$$

  where δ is the connecting homomorphism.
- the connecting homomorphisms δ are natural, i.e., given a commutating diagram

  $$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

  the induced diagram

  $$\begin{tikzcd} H^n(G,C) \arrow[r, "\delta"] \arrow[d] & H^{n+1}(G,A) \arrow[d] \\ H^n(G,C') \arrow[r, "\delta"'] & H^{n+1}(G,A') \end{tikzcd}$$

  also commutes, and $\{H^n(G,-)\}_{n\in\mathbb{Z}}$ is a cohomological δ-functor. Note that a δ-functor is additive, and usually occurs in abelian categories.

Definition 3.4 (δ-functor)**.** A map of δ-functors $T^* \to F^*$ is a collection of natural transformations $T^n \to F^n$, commuting with the δ's, i.e.,

$$\begin{tikzcd} T^n \arrow[r] \arrow[d, "\delta_T"'] & F^n \arrow[d, "\delta_F"] \\ T^{n+1} \arrow[r] & F^{n+1} \end{tikzcd}$$

A δ-functor T^* is universal if, given any other δ-functor F^*, a map $T^* \to F^*$ is uniquely determined by $T^0 \to F^0$.

Proposition 3.5. $H^*(G,-) : G$-**Mod** $\to$ **Ab** is a δ-functor.

Proof. We need to show:

- each $H^n(G,-)$ is a well-defined functor,
- the connecting homomorphisms δ's gives a long exact sequence,
- the naturality of δ.

First, let $f : A \to B$ be in G-**Mod**, then $C^*(G, A) \to C^*(G, B)$ is equivalent to $\mathrm{Map}(G^{*+1}, A)^G \to \mathrm{Map}(G^{*+1}, B)^G$ by composition with f. One can show that this is equivariant, i.e., respects the G-action, so it is well-defined to take the fixed points, and thus commutes with ∂'s.

Second, we need to apply the snake lemma. Given a short exact sequence $0 \to A \to B \to C \to 0$, we claim:

Claim 3.6. $0 \longrightarrow C^*(G, A) \longrightarrow C^*(G, B) \longrightarrow C^*(G, C) \longrightarrow 0$ is a short exact sequence of cochain complexes, i.e., $C^*(G, -) : G\text{-}\mathbf{Mod} \to \mathbf{coCh}$ is an exact functor.

Subproof. Exercise. ■

Now take the complex

$$\begin{tikzcd}
0 \arrow[r] & C^n(G,A) \arrow[r] \arrow[d, "\partial"] & C^n(G,B) \arrow[r] \arrow[d, "\partial"] & C^n(G,C) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C^{n+1}(G,A) \arrow[r] & C^{n+1}(G,B) \arrow[r] & C^{n+1}(G,C) \arrow[r] & 0
\end{tikzcd}$$

and quotient the boundaries everywhere (and thus lose the injectivity/surjectivity when applicable)

$$\begin{tikzcd}
& C^n(G,A)/B^n(G,A) \arrow[r] \arrow[d, "\partial"] & C^n(G,B)/B^n(G,B) \arrow[r] \arrow[d, "\partial"] & C^n(G,C)/B^n(G,C) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & Z^{n+1}(G,A) \arrow[r] & Z^{n+1}(G,B) \arrow[r] & Z^{n+1}(G,C) &
\end{tikzcd}$$

Taking the kernels and cokernels on ∂'s, we obtain a complex

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
& H^n(G,A) \arrow[r] \arrow[d] & H^n(G,B) \arrow[r] \arrow[d] & H^n(G,C) \arrow[d] & \\
& C^n(G,A)/B^n(G,A) \arrow[r] \arrow[d, "\partial"] & C^n(G,B)/B^n(G,B) \arrow[r] \arrow[d, "\partial"] & C^n(G,C)/B^n(G,C) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & Z^{n+1}(G,A) \arrow[r] \arrow[d] & Z^{n+1}(G,B) \arrow[r] \arrow[d] & Z^{n+1}(G,C) \arrow[d] & \\
& H^{n+1}(G,A) \arrow[r] & H^{n+1}(G,B) \arrow[r] & H^{n+1}(G,C) &
\end{tikzcd}$$

By the snake lemma, we obtain the long exact sequence. □

Proposition 3.7. If $0 \to A \to B \to C \to 0$ is a short exact sequence such that $H^*(G, B) = 0$ for $* > 0$ (or at least $H^n(G, B) = 0 = H^{n+1}(G, B)$), then $\delta : H^n(G, C) \to H^{n+1}(G, A)$ is an isomorphism.

Definition 3.8 (Acyclic, Cohomologically Trivial)**.** A G-module M is

- acyclic if $H^*(G, M) = 0$ for $* > 0$,
- cohomologically trivial if $H^*(H, M) = 0$ for $* > 0$ and any (closed) subgroup $H \subseteq G$.

Definition 3.9 (Induced Module)**.** Given any G-module M, the induced module $\mathrm{ind}_G(M) = \mathrm{Map}(G, M) = X^0(G, M)$.

Example 3.10. M could have the trivial action.

Exercise 3.11. For any M, the induced module of M over G is isomorphic (under the G-action) to the induced module of module given by forgetful action over G.

Remark 3.12.

- $\mathrm{Ind}_G(-) : G\text{-}\mathbf{Mod} \to G\text{-}\mathbf{Mod}$ is exact.

- We say A is an induced module if $A \cong \mathrm{Ind}_G(M)$ for some module M. If A is an induced G-module, then A is induced as an H-module for any subgroup $H \subseteq G$.

Lemma 3.13. Induced modules are cohomologically trivial.

Proof. There is an isomorphism
$$C^*(G, \mathrm{Ind}_G(M)) \cong X^*(G, M).$$
□

Remark 3.14. We have an equivariant inclusion of fixed points
$$M \hookrightarrow \mathrm{Ind}_G(M)$$
which is an embedding, and we take $Q \cong \mathrm{Ind}_G(M)/M$, then this extends to a short exact sequence
$$0 \longrightarrow M \hookrightarrow \mathrm{Ind}_G(M) \longrightarrow Q \longrightarrow 0$$
then $H^{n+1}(G, M) \cong H^n(G, Q)$. One say that $H^*(G, -)$ is effaceable. By Tohoku, an effaceable is universal.

4 Aug 28, 2023: First Cohomology of Groups

There are three ways to think about $H^1(G, M)$.

4.1 Crossed Homomorphims

Recall that $H^1(G, M) = Z_i^1(G, M)/B_i^1(G, M)$ as inhomogeneous cochains, where

- $Z_i^1(G, M) = \ker(\mathrm{Map}(G, M) \to \mathrm{Map}(G \times G, M)$ where the map sends $f \mapsto (g, h) \mapsto gf(h) - f(gh) + f(g)$. The kernel of this is exactly the maps f such that $f(gh) = gf(h) + f(g)$, and note that this is not a group homomorphism.
- $B_i(G, M) = \mathrm{im}(M \to \mathrm{Map}(G, M))$ given by $m \mapsto (g \mapsto gm - m)$, where the image is called a principal crossed homomorphism.

Exercise 4.1. $B_i^1(G, M) \cong M/M^G$ as an isomorphism of $\mathbb{Z}[G]$-modules.

Remark 4.2. If the G-action is trivial, then $H^1(G, M) = \mathrm{Hom}_{\mathrm{Grp}}(G, M)$.

Corollary 4.3. If G is a finite group with trivial action, then $H^1(G, \mathbb{Z}) = 0$.

Theorem 4.4 (Hilbert's Theorem 90)**.** Let L/K be a Galois extension with (finite or profinite) Galois group G, then $H^1(G, L^\times) = 0$.

Proof. Let $f : G \to L^\times$ be a crossed homomorphism. We know the addition is given by $f(gh) = gf(h) + f(g)$, and the multiplication is given by $f(gh) = (g \cdot f(h))f(g)$, where $\cdot$ represents the group action. Now for any $l \in L^\times$, the multiplication with respect to l is given by $m_l = \sum\limits_{h \in G} f(h)(h \cdot l)$. We can first choose l so that $m_l \neq 0$, since the Galois conjugates $h \cdot l$ over $l \in L$ are linearly independent. For $g \in G$, we have
$$\begin{aligned} g \cdot m_l &= \sum_{h \in G} (g \cdot f(h))(gh \cdot l) \\ &= \sum_{h \in G} \frac{f(gh)}{f(g)}(gh \cdot l) \\ &= \frac{1}{f(g)} \sum_{h \in G} f(gh)(gh \cdot l) \\ &= \frac{1}{f(g)} m_l. \end{aligned}$$
Therefore, $f(g) = \frac{m_l}{g \cdot m_l}$. For any crossed homomorphism, there exists $m \in L^\times$ such that $f(g) = \frac{gm}{m}$, so every crossed homomorphism is principal. □

Exercise 4.5. Let G acts over a commutative ring R, then $H^1(G, R^\times)$ classifies invariant R-modules with a compatible G-action.

4.2 Non-abelian H^1 and Torsors

Let A be a group with G-action, so let the action $g \cdot a = {}^g a$. Hence, $g \cdot (ab) = {}^g a {}^g b$. Define the G-cocycles to be $f : G \to A$ such that $f(gh) = f(g) {}^g f(h)$. Two cocycles f and f' are said to be cohomologous as $f \sim f'$ if there exists $a \in A$ such that for all $g \in G$, $f'(g) = a^{-1} f(g) {}^g a$. This becomes an equivalence relation on the set of G-cocycles with coefficients in A, then $H^1(G, A)$ is the set of equivalence classes of G-cocycles. Now the first cohomology $H^1(G, A)$ has only a pointed set structure with distinguished point $f \equiv 1$, the constant function at 1.

Exercise 4.6. This definition is equivalent to the inhomogeneous cochain definition in the abelian case.

Definition 4.7. An A-torsor is a G-set X with action

$$
\begin{aligned}
X \times A &\to A \\
(x, a) &\mapsto xa
\end{aligned}
$$

that is free and transitive, i.e., for any $x, y \in G$, there exists a unique $a \in A$ such that $y = xa$. Moreover, the action $X \times A \to X$ respects the G-action, i.e., ${}^g(xa) = {}^g x {}^g a$.

Remark 4.8.

- A is an A-torsor.
- An isomorphism of A-torsors is a bijection that respects the G- and A- action.
- If $A \subseteq B$ is a sub-G-group, then bA is an A-torsor.
- An A-torsor is a principal A-bundle on the classifying space BG.

Theorem 4.9. There is a canonical bijection of pointed sets

$$
H^1(G, A) \cong \operatorname{Torsor}(G, A)
$$

Proof.

- The backwards map $\lambda : \operatorname{Torsor}(G, A) \to H^1(G, A)$ is defined as follows: for $x \in \operatorname{Torsor}(G, A)$, we want to define a cocycle $f(X) : G \to A$. For arbitrary $x \in X$, note that for any $g \in G$, there exists a unique $f_x(g) \in A$ such that ${}^g x = x f_x(g)$ by the simple transitivity of the A-action on X. To see this is well-defined, if we have another $y \in X$, then $y = xb$ for some $b \in A$, then $f_y(g) = b^{-1} f_x(g) {}^g b$, so f_x and f_y are cohomologous and define the same class in $H^1(G, A)$, which is defined to be the image $\lambda(X)$.
- To define $\mu : H^1(G, A) \to \operatorname{Torsor}(G, A)$, given a cocycle $f : G \to A$, let X_f be the group A, then the action of A on X_f is by multiplication on the right, and one can twist the G-action on it using cocycle $f : G \to A$ with ${}^{\bar{g}} x = f(g) g x$, which defines an A-torsor. This is well-defined.

$\square$

Remark 4.10. Suppose

$$
1 \longrightarrow A \longrightarrow B \xrightarrow{p} C \longrightarrow 1
$$

is a short exact sequence of G-groups, i.e., A is a sub-G-group and $C \cong B/A$, then there is a long exact sequence

$$
1 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \xrightarrow{\delta} H^1(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, C)
$$

where δ is given by $\delta(c) = p^{-1}(c)$. For the exactness in the sense of pointed sets to work, the kernel is the subset mapping to the distinguished element.

4.3 Extension Splitting

Consider the a split extension

$$
1 \longrightarrow A \longrightarrow E \xrightarrow{p} G \longrightarrow 1
$$

That is, E is the direct product $A \times G$ with group action $(a, g)(a', g') = (a {}^g a', gg')$, and by definition E is the semidirect product $A \rtimes G$. Equivalently, there exists a section (as group homomorphism) $s : G \to E$.

There is an equivalence relation on the set of sections to the projection $p : E \to G$, where the sections $s, s' : G \to E$ are conjugates if there exists $a \in A$ such that $s'(g) = a^{-1} s(g) a$. We denote $\sec(E \to G)$ to be the conjugacy class of sections of p. Note that the class of trivial section $s : g \mapsto (1, g) \in E$ is the distinguished element.

Proposition 4.11. The pointed set $H^1(G, A)$ is isomorphic to $\sec(E \to G)$.

Proof. Take $\varphi \in \sec(E \to G)$, then the composition $G \xrightarrow{\varphi} E \xrightarrow{\pi_1} A$, where π_1 is the set-theoretic projection to the first component, defines a cocycle $G \to A$. Conversely, given a cocycle $f : G \to A$, the section is given by $g \mapsto (f(g), g)$. $\square$

Exercise 4.12. Expand the proof above.

Exercise 4.13. Describe $\mathbb{Z} \rtimes C_2$ where C_2 acts on $\mathbb{Z}$ by inversion. How many sections are there of $\mathbb{Z} \rtimes C_2 \to C_2$?

Exercise 4.14. How many sections are there to the projection $D_{2n} \to C_2$?

5 Aug 30, 2023: H^2, abelian extensions, and Brauer group

Suppose we have an abelian extension, that is, let A be abelian, the short exact sequence of group extensions

$$0 \longrightarrow A \xhookrightarrow{i} E \xrightarrow{p} G \longrightarrow 1$$

is such that $E/i(A) \cong G$. Note that A can be regarded as a normal subgroup in E given this notation.

Note that two extensions are equivalent if there exists a group isomorphism $\varphi : E \to E'$ such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\varphi"] & G \arrow[r] \arrow[d, equal] & 1 \\ 0 \arrow[r] & A \arrow[r] & E \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

commutes.

Consider the continuous functions

$$\varphi : G \times G \to A$$

such that $\varphi(g_1g_2, g_3) + \varphi(g_{1,2}) = \varphi(g_1, g_2g_3) + g_1\varphi(g_2, g_3)$. We know $H^2(G, M)$ is the quotient of all such functions over the coboundaries, i.e., the functions φ such that $\varphi(g_1, g_2) = f(g_1) - f(g_1g_2) + g_1f(g_2)$.

Now $E \cong A \times G$ can be considered as a bijection, so we pick a set-theoretic section $s : G \to E$ with $s(1) = 1$, and now every element in E is written as $as(g)$ uniquely for some $a \in A$ and $g \in G$, we have

$$s(g)a = s(g)as(g)^{-1}s(g) = {}^g as(g).$$

Note that s may not be a homomorphism, but we have $s(g)s(h) = f(g, h)s(gh)$ since $s(g)s(h)$ and $s(gh)$ are both lifts of gh.

As a consequence, we have

$$(s(g_1)s(g_2))s(g_3) = f(g_1, g_2)s(g_1g_2)s(g_3) = f(g_1, g_2)f(g_1g_2, g_3)s(g_1g_2g_3)$$

and

$$s(g_1)(s(g_2)s(g_3)) = s(g_1)f(g_2, g_3)s(g_2, g_3) = {}^{g_1}f(g_2, g_3)s(g_1)s(g_2g_3) = {}^{g_1}f(g_2, g_3)f(g_1, g_2g_3)s(g_1g_2g_3).$$

In additive notation, we have

$$f(g_1, g_2) + f(g_1g_2, g_3) = g_1f(g_2, g_3) + f(g_1, g_2g_3).$$

Therefore, f becomes an inhomogeneous 2-cocycle.

Proposition 5.1. The induced map $\lambda : \mathrm{ext}(G, A) \to H^2(G, A)$ is a well-defined bijection between the set of equivalence classes of extensions and $H^2(G, A)$.

Example 5.2. The two elements in $H^2(C_2, \mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ are given by non-split extension of Q_8

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow Q_8 \longrightarrow C_2 \longrightarrow 1$$

and the identity element given by $D_8 \cong \mathbb{Z}/4\mathbb{Z} \rtimes C_2$

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow D_8 \longrightarrow C_2 \longrightarrow 1$$

where D_8 has the action of C_2 over $\mathbb{Z}/4\mathbb{Z}$.

Proposition 5.3. An associative finite-dimensional K-algebra A is a CSA if and only if one of the following equivlaent conditions hold:

1. Based-changed to the separable closure $\bar{K}$ of K via $\bar{K} \otimes_K A$, $A \cong M_n(\bar{K})$ for some integer $n \geqslant 1$.
2. there exists a finite Galois extension L/K such that base-changed to L via $L \otimes_K A$, A becomes isomorphic to a matrix algebra $M_n(L)$ for some integer $n \geqslant 1$.
3. $A \cong M_n(D)$ matrix algebra for some $m \geqslant 1$ and some finite division algebra D over K.

A CSA A over K is said to be split over L if the above holds, i.e., $A \otimes_K L \cong M_n(L)$. One can define an equivalence class on CSAs, such that $A \sim B$ if and only if $A \otimes_K M_n(K) \cong B \otimes_K M_m(K)$. Now the Brauer group of K is the abelian group of equivalence classes of CSAs over K equipped with tensor product.

Suppose L/K is an extension, then there exists a homomorphism of base-change of algebras $\mathrm{Br}(K) \to \mathrm{Br}(L)$. We say the kernel $\mathrm{Br}(L \mid K)$ is the relative Brauer group of K-CSAs that split over K. The absolute Brauer group is $\mathrm{Br}(\bar{K} \mid K) = \mathrm{Br}(K)$, then

$$\mathrm{Br}(K) = \bigcup_{L/K \text{ finite}} \mathrm{Br}(L \mid K).$$

Now let L/K be a finite Galois extension with Galois group G, and we pick a normalized inhomogeneous 2-cycle $\varphi : G \times G \to L^\times$ as the representative of its class, and we can construct A_φ as a K-CSA, then $A_\varphi = \bigoplus_{g \in G} Le_g$ has dimension $|G|^2$, where e_g's are the generators, with a multiplication operation $(le_g)(me_h) = l(g \cdot m)\varphi(g,h)e_{gh}$ which can be extended via distribution. A_φ is said to be the crossed product of L and G via φ.

Theorem 5.4.
1. A_φ is a split algebra over L.
2. If φ, φ' are two normalized inhomogeneous 2-cocycles, then $A_\varphi \sim A_{\varphi'}$ if and only if $\varphi \sim \varphi'$.
3. $A_{\varphi\varphi'} \sim A_\varphi \otimes_K A_{\varphi'}$.
4. Any K-CSA which is split over L is similar to a crossed product A_φ for some $\varphi : G \times G \to L^\times$.

Corollary 5.5. $H^2(G, L^\times)$ is isomorphic to $\mathrm{Br}(L \mid K)$, and $H^2(\mathrm{Gal}(\bar{K}/K), \bar{K}^\times)$ is isomorphic to $\mathrm{Br}(K)$.

6 Sept 1, 2023: Cohomology of Cyclic and Free Groups

Recall that we can compute $H^*(G, M)$ using any acyclic resolution of M. We want to describe $H^*(G, M)$ for specific G using nice resolutions.

We have

$$\cdots \to G^3 \xrightarrow{\delta} G^2 \xrightarrow{\delta} G$$

and to obtain $X^*(G, M)$ we map out of the resolution and into M, so $\mathrm{Map}(G, M) \cong \mathrm{Hom}(\mathbb{Z}[G], M)$ as G-modules, and in general we obtain

$$\mathrm{Map}(G^k, M) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]^{\otimes k}, M)$$

as $\mathbb{Z}$-modules.

We denote F^{st} to be the standard free resolution given by

$$\mathbb{Z}[G]^{\otimes k} \xrightarrow{d} \mathbb{Z}[G]^{\otimes(k-1)} \to \cdots \to \mathbb{Z}[G]^{\otimes 2} \xrightarrow{d_1 - d_0} \mathbb{Z}[G]$$

To obtain $X^*(G, M)$, we can map this into M. Now the standard resolution becomes an augmentation of $\mathbb{Z}$ that makes $X^*(G, M)$ exact, free, and acyclic. The kernel of $\mathbb{Z}[G] \to \mathbb{Z}$ is the agumentation ideal of G as of $\mathbb{Z}[G]$. Since this is a G-equivariant map, then the augmentation ideal is a G-submodule of $\mathbb{Z}[G]$, as a free abelian group generated by the set $\{(g-1) \mid 1 \neq g \in G\}$.

Lemma 6.1. If $P_* \to \mathbb{Z}$ is any free resolution of $\mathbb{Z}$ as a G-module, then for a G-module M, we have $H^*(G, M) \cong H^*(\mathrm{Hom}(P_*, M))^G$.

Proof. Since each P_i is free, then $\mathrm{Hom}(P_i, M)$ is an acyclic module, so $M \to \mathrm{Hom}(P_*, M)$ is an acyclic resolution of M. Now apply Proposition 2.28 in the notes. □

Remark 6.2. $H^*(G, M) \cong \mathrm{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z}, M)$ as universal δ-functors.

Now let C_n be the cyclic group of order n, generated by element g, then $\mathbb{Z}[C_n] \cong \mathbb{Z}[g]/(g^n - 1)$, so we have $0 = g^n - 1 = (g-1)N_g$ in $\mathbb{Z}[C_n]$ where N_g is the norm element $N_g = 1 + g + \cdots + g^{n-1}$, so we have a free resolution of $\mathbb{Z}$:

$$\cdots \longrightarrow \mathbb{Z}[C_n] \xrightarrow{1-g} \mathbb{Z}[C_n] \xrightarrow{N_g} \mathbb{Z}[C_n] \xrightarrow{1-g} \mathbb{Z}[C_n] \xrightarrow{\varepsilon} \mathbb{Z}$$

where augmentation ε sends g to 1. This allows us to compute the cohomology of any C_n-modules.

Proposition 6.3. Let M be an C_n-module, then

$$H^i(G, M) = \begin{cases} M^G, & i = 0 \\ \{m \in M \mid N_g m = 0\}/(1-g)M, & i > 0 \text{ odd} \\ M^G/N_g M, & i > 0 \text{ even} \end{cases}$$

Proof. Taking $\mathrm{Hom}(P_*, M)^G$ gives

$$\cdots \longleftarrow M \xleftarrow[1-g]{} M \xleftarrow[N_g]{} M \xleftarrow[1-g]{} M \longleftarrow \cdots$$

□

Remark 6.4. If M has trivial action, then

$$H^i(G, M) = \begin{cases} M, & i = 0 \\ M[n], & i > 0 \text{ odd} \\ M/n, & i > 0 \text{ even} \end{cases}$$

where $M[n]$ is the n-torsion in M.

Now if $T = \mathbb{Z}$ be with generator t, then $\mathbb{Z}[T]$ is isomorphic to the Laurent polynomials, so we have a resolution

$$0 \longrightarrow \mathbb{Z}[T] \xrightarrow{1-t} \mathbb{Z}[T] \longrightarrow \mathbb{Z}$$

since $(1-t)$ is not a zero-divisor of $\mathbb{Z}[T]$. Therefore, taking $\mathrm{Hom}(P_*, M)^T$ gives

$$0 \longleftarrow M \xleftarrow[1-t]{} M$$

$$H^i(T, M) = \begin{cases} M^T, & i = 0 \\ M_T, & i = 1 \\ 0, & \text{otherwise} \end{cases}$$

Now let X be a set, and let G_X be the free group on X.

Proposition 6.5. The augmentation ideal I_X is a free $\mathbb{Z}[G_X]$-module, generated by the set $\{(x-1) \mid x \in X\}$, and so the exact sequence

$$0 \longrightarrow I_X \longrightarrow \mathbb{Z}[G_X] \longrightarrow \mathbb{Z} \longrightarrow 0$$

is a free resolution of $\mathbb{Z}$ as a G_X-module.

Proof. As $\mathbb{Z}$-bases of I_X, we have $\{(g-1) \mid g \in G_X\}$, but $\{h(x-1) \mid h \in G, x \in X\}$ is also a $\mathbb{Z}$-linear basis for I_X. □

Remark 6.6. Groups are free if and only if they have cohomological dimension 1.

7 Sept 6:

Remark 7.1.

1. A crossed homomorphism would be a group homomorphism when G has trivial action on M.
2. If X is an A-torsor, then there is a given G-action and a right A-action so that $X \times A \to X$ is given by a diagonal action compatible to the G-action. Therefore, ${}^g(x \cdot a) = {}^g x \cdot {}^g a$.

Definition 7.2. Let A and B be G-modules, then there is a notion of tensor product $A \otimes_G B$ as a G-module via the diagonal action $g(a \otimes b) = ga \otimes gb$. On the level of cochain, we have a cup product

$$\begin{aligned} C^p(G, A) \otimes C^q(G, B) &\xrightarrow{\smile} C^{p+q}(G, A \otimes B) \\ (\alpha : G^{p+1} \to A) \otimes (\beta : G^{q+1} \to B) &\mapsto (\alpha \smile \beta) \\ &\qquad (g_0, \dots, g_{p+q}) \mapsto \alpha(g_0, \dots, g_p) \otimes \beta(g_p, \dots, g_{p+q}) \end{aligned}$$

Proposition 7.3. $\partial(\alpha \smile \beta) = (\partial\alpha) \cup \beta + (-1)^{|\alpha|}\alpha \smile \partial\beta$.

Corollary 7.4.

- If α and β are cocycles, then $\alpha \smile \beta$ is also a cocycle.
- If α is a cocycle β is a coboundary, or vice versa, then $\alpha \smile \beta$ is a coboundary. Indeed, if $\beta = \partial\gamma$, then $\partial(\alpha \smile \gamma) = (-1)^{|\alpha|}\alpha \smile \beta$.

  Therefore, on the level of cohomology, we have a (bilinear) cup product as well:

$$H^p(G, A) \otimes H^q(G, B) \to H^{p+q}(G, A \otimes B)$$

Example 7.5.

- If $p = q = 0$, then

$$\begin{gathered} H^0(G, A) \otimes H^0(G, B) \cong A^G \otimes B^G \to H^0(G, A \otimes B) \cong (A \otimes B)^G \\ a \otimes b \mapsto a \otimes b \end{gathered}$$

- By extending this prioperty, we get a G-equivariant pairing $A \otimes B \to C$ and therefore

$$H^p(G, A) \otimes H^q(G, B) \xrightarrow{\smile} H^{p+q}(G, C).$$

Example 7.6. Let R be a commutative ring, and if there is a G-action on R, then the multiplication $m : R \otimes R \to R$ is G-equivariant, so we have a cup product

$$\smile : H^p(G, R) \otimes H^q(G, R) \to H^{p+q}(R)$$

This has the following properties:

1. This is natural in A, B, and C.
2. This is compatible with connecting homomorphism and exact sequences, that is,
   - Given short exact sequences

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

   and

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

   and pairing $A \otimes B \to C$, then this induces $A \otimes B \to C'$ and in the quotients we have $A'' \otimes B \to C''$, so $\delta(\alpha \smile \beta) = \delta\alpha \smile \beta$, so we have a commutative diagram[1]

$$\begin{tikzcd} & A' \otimes B \arrow[r] \arrow[d] & A \otimes B \arrow[r] \arrow[d] & A'' \otimes B \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C' \arrow[r] & C \arrow[r] & C'' \arrow[r] & 0 \end{tikzcd}$$

[1]This may require the assumption that the modules are flat.

and thus

$$\begin{array}{ccc} H^o(G,A'')\otimes H^q(G,B) & \longrightarrow & H^{p+q}(G,A''\otimes B) \\ \downarrow{\scriptstyle \delta\otimes 1} & & \downarrow{\scriptstyle \delta} \\ H^{p+1}(G,A')\otimes H^q(G,B) & \longrightarrow & H^{p+q+1}(G,A'\otimes B) \end{array}$$

- Given

  $$0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$$

  and

  $$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

  and pairings

  $$\begin{array}{ccccccccc} & & A\otimes B' & \longrightarrow & A\otimes B & \longrightarrow & A\otimes B'' & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0 \end{array}$$

  so $\delta(\alpha \smile \beta) = (-1)^{|\alpha|}\alpha \smile \delta\beta$.

Proof. Let $\alpha = [a]$ for $a : G^{p+1} \to A$ and $\beta = [b]$ for $b : G^{q+1} \to B''$, then there is a lift $b : G^{q+1} \xrightarrow{\tilde{b}} B \to B''$. Then we have

$$\begin{array}{ccccccccc} & & C^q/B^q(B') & \longrightarrow & C^q/B^q(B) & \longrightarrow & C^q/B^q(B'') & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\ 0 & \longrightarrow & Z^q(B') & \longrightarrow & Z^{q+1}(B) & \longrightarrow & Z^{q+1}(B'') & & \end{array}$$

and by the snake lemma we have a connecting homomorphism over group cohomologies. ◻