

MATH 526 Notes

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Let X be a topological space with basepoint $x_0 \in X$. We already know two invariants,

- the fundamental group $\pi_1(X, x_0)$, and
- the homology groups $H_n(X)$ for $n \geq 0$, which are abelian groups.

We will look at two more invariants,

- the cohomology groups $H^n(X)$ for $n \geq 0$, and
- the higher homotopy groups $\pi_n(X, x_0)$ for $n \geq 0$.

In particular, $\pi_*(X, x_0)$ is a very good invariant in the following sense:

Theorem 1.1 (Whitehead). If $f : (X, x_0) \rightarrow (Y, y_0)$ is a map of CW-complexes, then f is a homotopy equivalence if and only if $\pi_*(f) : \pi_*(X, x_0) \rightarrow \pi_*(Y, y_0)$ is an isomorphism.

However, π_* is very hard to compute. On the other hand, $H^*(X)$ is relatively easy to compute, but this is not a complete invariant. For instance, $\mathbb{C}P^2$ and $S^2 \vee S^4$ have isomorphic cohomology groups, but they are not equivalent. $H^*(X)$ is closely related to $H_*(X)$, but $H^*(X)$ is a graded ring structure with cup product. It is contravariant in X , where $H_*(X)$ is covariant. The cup product is defined by the composition of induced diagonal map with an external product:

$$H^i(X) \times H^j(X) \xrightarrow{\times} H^{i+j}(X \times X) \xrightarrow{\Delta^*} H^{i+j}(X)$$

Other things we will talk about include:

- Natural transformations $H^i(-) \rightarrow H^j(-)$ encoded by Steenrod operations.
- $H^n(-)$ becomes a representable functor, i.e., $H^n(X) = [X, K(\mathbb{Z}, n)]$, where $K(\mathbb{Z}, n)$ is the Eilenberg-MacLane space, and the bracket indicates the homotopy classes of maps.
- Poincaré duality in $H^*(M)$ for compact manifold M , namely the cup product gives

$$H^i(M) \otimes H^{\dim(M)-i}(M) \xrightarrow{\sim} H^{\dim(M)}(M).$$

- Characteristic classes in $H^*(X)$ associated to vector bundles over X .

Recall for a topological space X , we obtain a collection of (singular) homology groups $H_n(X)$, with $H_*(X) = \bigoplus_{n \geq 0} H_n(X)$. The functoriality of morphisms says that $X \xrightarrow{f} Y \xrightarrow{g} Z$ induces $f_*g_* = (fg)_* : H_*(X) \xrightarrow{f_*} H_*(Y) \xrightarrow{g_*} H_*(Z)$. So

$$H_*(-) : \mathbf{Top} \rightarrow \mathbf{Ab}$$

is a well-defined functor. This factors into

$$\begin{array}{ccc} \mathbf{Top} & \xrightarrow{H_*(-)} & \mathbf{Ab} \\ & \searrow C_*(-) & \nearrow H_*(-) \\ & \mathbf{Ch} & \end{array}$$

Here $C_*(-)$ is usually the singular chain, given by $\partial : C_n(X) \rightarrow C_{n-1}(X)$, where $C_n(X)$ is the free abelian group generated by $\text{Hom}_{\mathbf{Top}}(\Delta^n, X) \cong \bigoplus \mathbb{Z}\sigma$. $\Delta^n \subseteq \mathbb{R}^{n+1}$ is the set of tuples (t_0, \dots, t_n) such that the coordinates sum to 1. The boundary is $\partial\sigma = \sum_{0 \leq i \leq n} (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$.

We say $C_*(-)$ is homotopy invariant, i.e., if $f : X \rightarrow Y$ is a homotopy equivalence, then the induced map $C_*(X) \rightarrow C_*(Y)$ on chain complexes is a chain equivalence.

Remark 1.2. $C_*^\Delta(X)$ and $C_*^{\text{CW}}(X)$ are both chain equivalent to $C_*(X)$.

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Here is a list of properties of $C_*(-) : \mathbf{Top} \rightarrow \mathbf{Ch}$:

- Functoriality: given a continuous map $f : X \rightarrow Y$, there is an induced map

$$\begin{aligned} f_* : C_*(X) &\rightarrow C_*(Y) \\ (\sigma : \Delta^n \rightarrow X) &\mapsto (f\sigma : \Delta^n \rightarrow Y) \end{aligned}$$

- Homotopy invariance: given $f, g : X \rightarrow Y$ such that $f \simeq g$, i.e., there is $H : X \times [0, 1] \rightarrow Y$ such that $H|_0 = f$ and $H|_1 = g$, then $f_* \simeq g_*$ as a chain homotopy equivalence, i.e., there exists maps $h_n : C_n(X) \rightarrow C_{n+1}(Y)$ making a diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{n+1}(X) & \longrightarrow & C_n(X) & \longrightarrow & C_{n-1}(X) \longrightarrow \cdots \\ & & \swarrow h & \downarrow g \parallel f & \swarrow h & \downarrow g \parallel f & \swarrow h \downarrow g \parallel f \\ \cdots & \longrightarrow & C_{n+1}(Y) & \longrightarrow & C_n(Y) & \longrightarrow & C_{n-1}(Y) \longrightarrow \cdots \end{array}$$

such that $f - g = \partial h + h\partial$. Therefore $f_* = g_* : H_*(X) \rightarrow H_*(Y)$.

Remark 2.1. $f : A_* \rightarrow B_*$ is a chain equivalence if there exists $g : B_* \rightarrow A_*$ and $fg \simeq \text{id}_B$ and $gf \simeq \text{id}_A$, then $f_* : H_*(A_*) \rightarrow H_*(B_*)$ is an isomorphism, i.e., f is a quasi-isomorphism.

Example 2.2. The complexes $A : 0 \rightarrow \mathbb{Z} \xrightarrow{-2} \mathbb{Z} \rightarrow 0$ and $B : 0 \rightarrow 0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$ gives a quasi-isomorphism $f : A \rightarrow B$ in the canonical way, but this is not a chain equivalence, since the backwards map has to be zero.

- Additivity: $C_*(\coprod_\alpha X_\alpha) \cong \bigoplus_\alpha C_*(X_\alpha)$.
- Excision: given a pair (X, A) with $Z \subseteq A$ such that $\bar{Z} \subseteq \text{int}(A)$, then we have $C_*(X \setminus Z, A \setminus Z) \cong C_*(X, A)$.
- Mayer-Vietoris: given $A, B \subseteq X$, with $X = \text{int}(A) \cup \text{int}(B)$, then we have a short exact sequence

$$0 \longrightarrow C_*(A \cap B) \longrightarrow C_*(A) \oplus C_*(B) \xrightarrow{*} C_*(X) \longrightarrow 0$$

The cochain complex is obtained via inverting the indices and maps δ from a chain complex. This induces a cohomology $H^*(C^*) = \ker(\delta)/\text{im}(\delta)$ as the quotient of cocycles over coboundaries. Now $f : A^* \rightarrow B^*$ is a quasi-isomorphism if $f^* : H^*(A^*) \rightarrow H^*(B^*)$ is an isomorphism. Similarly, one can define the cochain homotopy equivalence.

Example 2.3. If $C_* \in \mathbf{Ch}$, and $k \in \mathbf{Ab}$, then we can form cochain complex $C_k^* := \text{Hom}(C_*, k)$, where $C_k^n = \text{Hom}_{\mathbf{Ab}}(C_n, k) \xrightarrow{\delta} C_k^{n+1}$ by sending $f : C_n \rightarrow k$ to $f\partial : C_{n+1} \rightarrow C_n \rightarrow k$.

- $\text{Hom}(-, k) : \mathbf{Ch} \rightarrow \mathbf{coCh}$ is a functor.
- The functor preserves quasi-isomorphisms between chain complexes of free abelian groups.

Definition 2.4. For $k \in \mathbf{Ab}$, the singular cochains with coefficients in k is

$$\begin{array}{ccc} C^*(-, k) : \mathbf{Top} & \xrightarrow{\quad} & \mathbf{coCh} \\ & \searrow C_*(-) & \nearrow \text{Hom}(-, k) \\ & \mathbf{Ch} & \end{array}$$

The cohomology of X with coefficients in k is defined by $H^*(X; k) = H^*(C^*X, k)$. We have the convention $C^*(X) = C^*(X, \mathbb{Z})$.

Alternatively, we take the opposite categories \mathbf{Top}^* and \mathbf{Ch}^* so that the functors are viewed as covariant.

The corresponding map $\delta : C^n(X; k) \rightarrow C^{n+1}(X; k)$ is given by δf that maps $\sigma \in C_{n+1}(X)$ to $(-1)^{n+1}f(\partial\sigma)$. Although the cochains are in general the dual of chains, the cohomology is not going to be the dual of the homology.

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Recall:

$$\begin{array}{ccccc} & & H^*(-, k) & & \\ & \searrow & \curvearrowright & \searrow & \\ \mathbf{Top}^{\text{op}} & \xrightarrow{C_*} & \mathbf{Ch}^{\text{op}} & \xrightarrow{\text{Hom}(-, k)} & \mathbf{coCh} & \xrightarrow{H^*} & \mathbf{GrAb} \end{array}$$

Properties of $H^*(-, k) : \mathbf{Top} \rightarrow \mathbf{GrAb}$:

- Dimension:

Claim 3.1. $H^i(\{*\}, k) = \begin{cases} 0, & i \neq 0 \\ k, & i = 0 \end{cases}$

Proof. Note that each degree of cohomology is given the free abelian group generated by $\text{Hom}(\Delta^n, \{*\})$, but the singleton set is the terminal object in the category of topological spaces, so there is always a unique generator, thus the chain complex is given by \mathbb{Z} 's on each degree $n \geq 0$.

Now the generating map at degree n is $\sigma_n : \Delta^n \rightarrow \{*\}$, and see Homework 1 where we proved the homology. Now looking at $C^*(\{*\}, k)$, we have

$$k \xrightarrow{0} k \xrightarrow{\cong} k \xrightarrow{0} k \longrightarrow \dots$$

and this gives the cohomology. □

- Homotopy: if $f \simeq g : X \rightarrow Y$, then $f^* = g^* : H^*(Y, k) \rightarrow H^*(X, k)$.

Proof. We have $f_* = g_* : C_*X \rightarrow C_*Y$, and then $\text{Hom}(f_*, k) \cong \text{Hom}(g_*, k)$, so $H^*(-)$ is invariant under cochain homotopies. □

- Additivity: $H^*(\coprod_{\alpha} X_{\alpha}, k) \cong \prod_{\alpha} H^*(X_{\alpha}, k)$.

Proof. We know that for chains there is $C_*(\coprod_{\alpha} X_{\alpha}) = \bigoplus_{\alpha} C_*(X_{\alpha})$, so the cochain version says that $C^*(\coprod_{\alpha} X_{\alpha}, k) \cong \text{Hom}(\bigoplus_{\alpha} C_*(X_{\alpha}), k) \cong \prod_{\alpha} \text{Hom}(C_*(X_{\alpha}), k) \cong \prod_{\alpha} C^*(X_{\alpha})$ and $H^* : \mathbf{coCh} \rightarrow \mathbf{GrAb}$ commutes with the product. □

- Exactness: for a pair (X, A) , there is a natural long exact sequence

$$\cdots \longrightarrow H^n(X, A; k) \longrightarrow H^n(X; k) \longrightarrow H^n(A; k) \longrightarrow \cdots$$

Proof. We have a short exact sequence

$$0 \longrightarrow C_*A \longrightarrow C_*X \longrightarrow C_*(X, A) \longrightarrow 0$$

where $C_*A \rightarrow C_*X$ is an inclusion of summands. Therefore, the quotient $C_*(X, A)$ is also a chain complex of free abelian groups. Therefore, taking the cochains also gives a short exact sequence. We then obtain a short exact sequence of cochain complexes

$$0 \longrightarrow C^*(X, A; k) \longrightarrow C^*(X; k) \longrightarrow C^*(A; k) \longrightarrow 0$$

and can then apply cohomology functor. □

- Excision: given a pair (X, A) and Z such that $\bar{Z} \subseteq \text{int}(A)$, we have $H^*(X, A; k) \cong H^*(X \setminus Z, A \setminus Z; k)$.
- Mayer-Vietoris: given $A, B \subseteq X$ such that $\text{int}(A) \cup \text{int}(B) = X$, then we have a natural long exact sequence

$$\cdots \longrightarrow H^n(X; k) \longrightarrow H^n(A; k) \oplus H^n(B; k) \longrightarrow H^n(A \cap B; k) \longrightarrow \cdots$$

Definition 3.2. A functor $E^* : \mathbf{Top}^{\text{op}} \rightarrow \mathbf{GrAb}$ is called a generalized cohomology theory if it satisfies the four middle property (except the dimension property and Mayer-Vietoris).

Remark 3.3. If E^* also satisfies the dimension property, then E^* is naturally isomorphic to the cohomology $H^*(-; k)$. There are also other generalized cohomology theories like K -theory, cobordism, etc.

The Mayer-Vietoris becomes a consequence of the first five properties.

We will now try to use homological algebra to relate $H_*(X) = H_*(CX)$ and $H^*(X; k) = H^*(\text{Hom}(C_*X, k))$.

Definition 3.4. We say $C_*(X; k) \cong C_*(X) \otimes_{\mathbb{Z}} k$ and $H_*(X; k) \cong H_*(C_*X \otimes k)$ gives the singular homology of X with coefficients in k .

Lemma 3.5. $-\otimes k : \mathbf{Ab} \rightarrow \mathbf{Ab}$ is a right exact functor. $\text{Hom}(-, k) : \mathbf{Ab}^{\text{op}} \rightarrow \mathbf{Ab}$ is left exact.

Proof. Exercise. □

Remark 3.6. The covariant hom functor is also left exact.

Remark 3.7. The left adjoint is right exact, the right adjoint is left exact. In particular, we have the hom-tensor adjunction

$$\text{Hom}(A, \text{Hom}(B, C)) \cong \text{Hom}(A \otimes B, C).$$

Note that

$$\text{Hom}(A, \text{Hom}(B, C)) \cong \text{Hom}(A \otimes B, C) \cong \text{Hom}(B \otimes A, C) \cong \text{Hom}(B, \text{Hom}(A, C))$$

Example 3.8. Consider

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

Tensoring with $\mathbb{Z}/n\mathbb{Z}$, we do not have exactness.

Example 3.9.

$$0 \longrightarrow A \longrightarrow A \oplus C \longrightarrow C \longrightarrow 0$$

is always exact after tensoring $-\otimes k$ or applying the hom functor $\text{Hom}(-, k)$.

Definition 3.10. A short exact sequence $0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0$ is split if any of the following equivalence conditions hold:

- (i) p has a section $s : C \rightarrow B$ such that $ps = 1$;
- (ii) i has a retraction $r : B \rightarrow A$ such that $ri = 1$;
- (iii) $B \cong A \oplus C$, i.e.,

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C \longrightarrow 0 \\ & & \parallel & & \downarrow & & \parallel \\ 0 & \longrightarrow & A & \xrightarrow{i} & A \oplus C & \xrightarrow{p} & C \longrightarrow 0 \end{array}$$

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We will prove that (ii) implies (iii).

Suppose $b \in B$, then $b = (b - irb) + irb$, which is a decomposition of elements in $\ker(r)$ and in $\text{im}(i)$, respectively. Also, $\ker(r) \cap \text{im}(i) = 0$, therefore $B = \ker(r) \oplus \text{im}(i)$. Since i is an inclusion, then $\text{im}(i) \cong A$. Now $p : B \rightarrow C$ factors through the projection onto $\ker(r)$ since $ri = 0$. By restricting p onto $\ker(r)$, we see p is also injective, thereby an isomorphism.

Lemma 4.1. If we have a split exact sequence

$$0 \longrightarrow A \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{r} \end{array} B \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{s} \end{array} C \longrightarrow 0$$

then $- \otimes k$ and $\text{Hom}(-, k)$ preserves the split exactness, i.e.,

$$0 \longrightarrow A \otimes k \longrightarrow B \otimes k \longrightarrow C \otimes k \longrightarrow 0$$

and

$$0 \longrightarrow \text{Hom}(C, k) \longrightarrow \text{Hom}(B, k) \longrightarrow \text{Hom}(A, k) \longrightarrow 0$$

The point is tensors and homs preserve retracts.

Proof. • $(r \otimes \text{id}_k)(i \otimes \text{id}_k) = ri \otimes \text{id}_k = \text{id}_{A \otimes k}$, so $i \otimes \text{id}_k$ is split injective.

- Similarly, $\text{Hom}(i, \text{id})$ is split surjective.

□

Example 4.2. Given a surjection $B \rightarrow C \rightarrow 0$ such that C is free abelian, then there is always a section $s : C \rightarrow B$ making the exact sequence split. (That is, C is projective.) That is, if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence where C is free, then the sequence is split exact.

Definition 4.3. Let $C \in \mathbf{Ab}$. A free resolution of C is a chain complex of free objects

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

and an augmentation $F_0 \rightarrow C$, so that

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0$$

is acyclic, i.e., exact everywhere.

Example 4.4.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow 0$$

is a free resolution of $\mathbb{Z}/n\mathbb{Z}$. So is

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

as well as

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\text{id} \oplus (\times n)} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0$$

Lemma 4.5. Any $C \in \mathbf{Ab}$ admits a free resolution, and moreover, it admits a resolution of length 1; given by

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0$$

Proof. Choose a surjection $p : F_0 \rightarrow C$ from a free abelian group F_0 to C . Let $F_1 = \ker(p)$, then F_1 is free, so we are done. \square

Lemma 4.6. Free resolutions are essentially unique, i.e., if $F \rightarrow C$ and $F' \rightarrow C$ are free resolutions, then there is a quasi-isomorphism $F \xrightarrow{\sim} F'$ which commutes with the augmentations to C .

Definition 4.7. Let $C \in \mathbf{Ab}$ and let $F \rightarrow C$ be a free resolution, then we define the torsion groups to be $\mathrm{Tor}_n^{\mathbb{Z}}(C, k) = H_n(F \otimes k)$, and the ext groups to be $\mathrm{Ext}_{\mathbb{Z}}^n(C, k) = H^n(\mathrm{Hom}_{\mathbb{Z}}(F, k))$.

Remark 4.8. • Tor and Ext are independent of the choice of resolutions.

- $\mathrm{Tor}_n^{\mathbb{Z}}$ and $\mathrm{Ext}_{\mathbb{Z}}^n$ are zero for $n > 1$.
- $\mathrm{Tor}_n^{\mathbb{Z}}(C, k) \cong \mathrm{Tor}_n^{\mathbb{Z}}(k, C)$.
- $\mathrm{Tor}_0^{\mathbb{Z}}(C, k) \cong C \otimes k$.
- $\mathrm{Ext}_{\mathbb{Z}}^0(C, k) \cong \mathrm{Hom}(C, k)$.

Example 4.9. • If C is free, then $\mathrm{Tor}_1(C, k) = \mathrm{Ext}^1(C, k) = 0$.

- $\mathrm{Tor}_1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.
- $\mathrm{Tor}_1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = 0$.
- $\mathrm{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.
- $\mathrm{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$.
- $\mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = 0$.

Proof. Look at

$$0 \longrightarrow F_1 = \mathbb{Z} \longrightarrow F_0 = \mathbb{Z} \longrightarrow C = \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

then $\mathrm{Tor}_*(\mathbb{Z}/p\mathbb{Z}, k) = H_*(F_1 \otimes k = k \xrightarrow{\times p} F_0 \otimes k = k) = \begin{cases} k[p], * = 0 \\ k/pk, * = 1 \end{cases}$. Here $k[p]$ denotes p -torsion subgroup

of k . Moreover, $\mathrm{Ext}^*(\mathbb{Z}/p\mathbb{Z}, k) = H^*(\mathrm{Hom}(F_1, k) = k \xleftarrow{\times p} \mathrm{Hom}(F_0, k) = k) = \begin{cases} k[p], * = 0 \\ k/pk, * = 1 \end{cases}$. \square

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Recall that cohomology are basically the dual of homology, where the difference originates from the failure of exactness of the hom functor.

Theorem 5.1 (Universal Coefficient Theorem). Let C_* be a chain of free abelian groups and $k \in \mathbf{Ab}$, then there exists a natural short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(H_{n-1}(C_*), k) \longrightarrow H^n(\mathrm{Hom}(C_*, k)) \xrightarrow{h} \mathrm{Hom}(H_n(C_*), k) \longrightarrow 0$$

that splits in an unnatural sense.

Here we define $h \in \mathrm{Hom}(H^n(\mathrm{Hom}(C_*, k)), \mathrm{Hom}(H_n(C_*), k))$. Note that this hom set is isomorphic to the hom set $\mathrm{Hom}(H^n(\mathrm{Hom}(C_*, k)) \otimes H_n(C_*), k)$ via the tensor-hom adjunction. That is, h is given by a bilinear pairing $H^n(\mathrm{Hom}(C_*, k)) \times H_n(C_*) \rightarrow k$. We use the Kronecker pairing $([f], [x]) \mapsto f(x)$. To see this is well-defined, let $f \in \mathrm{Hom}(C_n, k)$ with $\delta f = 0$, for $x \in C_n$, we have $\partial x = 0$. Now replace x by $x + \partial y$, then $f(x + \partial y) = f(x) = f(\partial y) = f(x) \pm (\delta f)(y) = f(x)$. Also, replace f by $f + \delta(g)$ gives $(f + \delta(g))(x) = f(x) + (\delta g)(x) = f(x) + g(\delta x) = f(x)$.

Lemma 5.2. h is a split surjection.

Proof. Write $C_k^* = \text{Hom}(C_*, k)$. Now $h : \ker(\delta, C_k^n \rightarrow C_k^{n+1}) \rightarrow \text{Hom}(H_n(C_*), k)$ via $h : f \mapsto (x \mapsto f(x))$, then we will construct a section of h via $\varphi \mapsto \tilde{\varphi}$. Let $Z_n = \ker(\partial)$ and $B_n = \text{im}(\partial)$, then $H_n(C_*) = Z_n/B_n$, and the short exact sequence of free abelian groups

$$0 \longrightarrow Z_n \xrightarrow{i} C_n \longrightarrow B_{n-1} \longrightarrow 0$$

and this splits so $C_n \cong Z_n \oplus B_{n-1}$. Given $\varphi : H_n(C_*) \rightarrow k$, we have

$$C_n \xrightarrow{r} Z_n \longrightarrow Z_n/B_n \xrightarrow{\varphi} k$$

where r is the retraction to i , and we define the composition to be $\tilde{\varphi}$. Now the composition

$$C_{n+1} \xrightarrow{\partial} C_n \longrightarrow Z_n \longrightarrow Z_n/B_n \longrightarrow k$$

is still zero since $C_{n+1} \rightarrow Z_n$ is zero, but that means $\delta\tilde{\varphi}$ is also zero. \square

We will now prove the universal coefficient theorem.

Proof. Since h is a split surjection, then we know this extends to a short exact sequence, hence we just need to identify the kernel of h , i.e., to show that $\ker(h) \cong \text{Ext}^1(H_{n-1}(C_*), k)$. Given the split short exact sequence

$$0 \longrightarrow Z_n \xrightarrow{i} C_n \xrightarrow{\partial} B_{n-1} \longrightarrow 0$$

we have a diagram

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow 0 & & \downarrow \delta & & \downarrow 0 \\ 0 & \longrightarrow & \text{Hom}(B_{n-1}, k) & \longrightarrow & \text{Hom}(C_n, k) & \longrightarrow & \text{Hom}(Z_n, k) \longrightarrow 0 \\ & & \downarrow 0 & & \downarrow \delta & & \downarrow 0 \\ 0 & \longrightarrow & \text{Hom}(B_n, k) & \longrightarrow & \text{Hom}(C_{n+1}, k) & \longrightarrow & \text{Hom}(Z_{n+1}, k) \longrightarrow 0 \\ & & \downarrow 0 & & \downarrow \delta & & \downarrow 0 \\ & & \vdots & & \vdots & & \vdots \end{array}$$

which is a short exact sequence of complexes. By the snake lemma, we have the long exact sequence of cohomology $\cdots \rightarrow H^n(B_k^{*-1}) \rightarrow H^n(C_k^*) \rightarrow H^n(Z_k^*) \rightarrow H^{n+1}(B_k^{*-1}) \rightarrow \cdots$. We claim that the connecting homomorphism $H^n(Z_k^*) \rightarrow H^{n+1}(B_k^{*-1})$ is $\text{Hom}(B_n \subseteq Z_n, k)$. But $0 \rightarrow B^n \rightarrow Z^n \rightarrow H_n(C_*) \rightarrow 0$ is a free resolution of $H_n(C_*)$ of length 1. Then $H^*(\beta : \text{Hom}(Z_n, k) \rightarrow \text{Hom}(B_n, k)) = \text{Ext}^*(H_n(C_*), k)$ where β has kernel $\text{Hom}(H_n(C_*), k)$ and cokernel $\text{Ext}^1(H_n(C_*), k)$. Therefore, the long exact sequence of cohomology is the splicing (as epi-mono factorization) of

$$0 \longrightarrow \text{coker}(\beta_{n-1}) \longrightarrow H_n(C_k^*) \longrightarrow \ker(\beta_n) \longrightarrow 0$$

and by identification we are done. \square

Corollary 5.3. If $C_* \rightarrow C'_*$ is a quasi-isomorphism, then $\text{Hom}(C'_*, k) \rightarrow \text{Hom}(C_*, k)$ is a quasi-isomorphism.

Corollary 5.4. Let $X \in \mathbf{Top}$ and $A \subseteq X$, then there exists a short exact sequence

$$0 \longrightarrow \text{Ext}^1(H_{n-1}(X, A), k) \longrightarrow H^n(X, A; k) \longrightarrow \text{Hom}(H_n(X, A); k) \longrightarrow 0$$

which is natural in (X, A) . This also splits in (X, A) in an unnatural way.

Theorem 5.5. If C_* is a chain complex of free abelian groups, then there is a short exact sequence

$$0 \longrightarrow H_n(C_*) \otimes k \longrightarrow H_n(C_* \otimes k) \longrightarrow \mathbf{Tor}_1(H_{n-1}(C_*), k) \longrightarrow 0$$

which is natural. It splits unnaturally.

Corollary 5.6. For any pair (X, A) , there is a natural short exact sequence

$$0 \longrightarrow H_n(X, A) \otimes k \longrightarrow H_n(X, A; k) \longrightarrow \mathbf{Tor}_1(H_{n-1}(X, A), k) \longrightarrow 0$$

which splits in an unnatural way.