MATH 595 (Group Cohomology) Notes

Jiantong Liu

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1 Aug 21, 2023: Introduction

Group cohomology works over different settings of groups, like finite groups, profinite groups, and topological groups. The course will develop towards

- duality in $H^*(G,-)$, and
- focus on computations, e.g., spectral sequences.

We first establish some notations.

- Let G be a group. If G has a topology, that would also be part of the information of G.
- A (left) G-module is an abelian group M with an action map

$$\begin{aligned} G \times M &\to M \\ (g,m) &\mapsto g \cdot m = gm \end{aligned}$$

  satisfying

  - $1 \cdot m = m$,
  - $(gh) \cdot m = g \cdot (hm)$,
  - $g(m+m') = gm + gm'$.

Remark 1.1. If G is a finite group, then the associated (non-commutative) group ring $\mathbb{Z}[G] := \bigoplus_{g \in G} \mathbb{Z}e_g$, where the multiplication is determined by $e_g e_h = e_{gh}$. Therefore, a G-module is just a $\mathbb{Z}[G]$-module.

Example 1.2.

- Trivial module $\mathbb{Z}$, or any abelian group with the trivial action $g \cdot a = a$.
- C_2, or any group with $f : G \twoheadrightarrow C_2$, then G with C_2 as a quotient gives the sign representation $\mathbb{Z}_{\text{sgn}}$, with $g \cdot (a) = (-1)^{\rho(g)} a$.
- $\mathbb{Z}[G]$ is a G-module via the left multiplication action, and/or the conjugation action.

Definition 1.3 (Fixed points/Invariants)**.** The set of fixed points of M over G is $M^G = \{m \in M \mid gm = m\ \forall g \in G\}$.

Definition 1.4 (Orbits/Coinvariants)**.** The set of orbits of M over G is $M_G = M/(gm - m)$.

Example 1.5. If $M = \mathbb{Z}_{\text{sgn}}$, then everything gets multiplied by -1, so there are no fixed points. The orbits of M over G would be $\mathbb{Z}_{\text{sgn}}/(-2) \cong \mathbb{Z}/2\mathbb{Z}$.

Example 1.6. If $M = \mathbb{Z}[G]$, then the fixed points are $\mathbb{Z}\left\{\sum_{g \in G} e_g\right\}$.

Thinking in a categorical setting, we have a trivial action function $\mathbb{Z}$-**Mod** $\to G$-**Mod**, sending $ga \mapsto a$ for all $g \in G$ and $a \in A$. This gives an exact functor from **Ab** to G-**Mod**. Then this functor has a right adjoint $(\ )^G : G$-**Mod** $\to$ **Ab**, and a left adjoint $(\ )_G :$ **Ab** $\to G$-**Mod**. More specifically, M^G becomes the maximal trivial action submodule of M, namely $\mathrm{Hom}_G(\mathbb{Z}, M)$; M_G becomes the largest quotient of M with trivial action, namely $\mathbb{Z} \otimes_{\mathbb{Z}[G]} M$. This simplifies to the tensor-hom adjunction in some sense. For a more detailed derivation of this, see Chapter 6.1 of Weibel.

Remark 1.7. In general, as in the category of G-sets, we have the orbit functor $X \mapsto X/G$ and the fixed point functor $X \mapsto X^G$. The orbit functor is left adjoint to the free G-set functor, and the fixed point functor is the right adjoint of the trivial G-set functor.

Remark 1.8. Read more about the setting in profinite groups with their topologies in Neukirch-Schmidt-Wingberg.

Definition 1.9 (Profinite Group)**.** A profinite group of a collection of groups is $G = \varprojlim_i G_i$ as an inverse limit, where each G_i is a finite group of the form G/U_i for some open U_i. This gives a topology to the profinite group.

Remark 1.10. The groups rings $\mathbb{Z}\left[[G]\right] = \varprojlim_i \mathbb{Z}[G_i]$. For instance, let $G = \hat{\mathbb{Z}_p} = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$, then $\mathbb{Z}_p\left[[G]\right] = \varprojlim_n \mathbb{Z}_p\left[\mathbb{Z}/p^n\mathbb{Z}\right]$, where each $\mathbb{Z}[\mathbb{Z}/p^n\mathbb{Z}] \cong \bigoplus_{i=0}^{n-1} \mathbb{Z}\{e_i\}$ where $e_i \cdot e_j = e_{ij}$. Therefore, $\mathbb{Z}_p\left[[G]\right]$ is now equivalent to $\varprojlim_n \mathbb{Z}_p[t]/(t^{p^n} - 1_e)$, and hence becomes a power series.

Remark 1.11. By a change of variables, this becomes $\varprojlim_n \mathbb{Z}_p[x]/(x^{p^n})$, but this only works in the finite group $\mathbb{Z}_p$ case, and not in general for $\mathbb{Z}$.

Example 1.12. $\mathbb{Z}[C_n] \cong \mathbb{Z}\{e\} \oplus \mathbb{Z}\{g\} \oplus \mathbb{Z}\{g^2\} \oplus \cdots \oplus \mathbb{Z}\{g^{n-1}\} \cong \mathbb{Z}[g]/(g^n - 1_e)$.

2 Aug 23, 2023: Cohomology of groups

Definition 2.1. Let G be a group, then we have a diagram

$$EG^{\cdot} : \cdots \Rrightarrow G \times G \rightrightarrows G$$

where the arrows are given by

$$EG^n = G^{n+1} \xrightarrow{d_i} G^n$$

for all $0 \leqslant i \leqslant n$. In the sense of simplicial sets, we have $d_i(g_0, \ldots, g_n) = (g_0, \ldots, \hat{g}_i, \ldots, g_n)$.

Now let M be a G-module, then we define $X^n = X^n(G, M) = \mathrm{Map}_{\mathbf{Set}}(G^{n+1}, M)$. G now has an action on this set, given by

$$(g \circ f)(g_0, \ldots, g_n) = gf(g^{-1}g_0, \ldots, g^{-1}g_n).$$

The action on d^i's are contravariant, namely we obtain $d_i^* : X_n \to X^{n+1}$ with an inherited structure. Note that M sits inside X^0, therefore we have a complex $(*)$:

$$0 \longrightarrow M \xhookrightarrow{\partial_0} X^0 \xrightarrow{\partial_1} X^1 \xrightarrow{\partial_2} X^2 \xrightarrow{\partial_3} \cdots$$

Here ∂_0 includes M as the constant functions into X, namely $\partial_0(m) = f$ for $f(g) = m$, and so on. In general, for $n > 0$, we have

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i^*.$$

Lemma 2.2. The complex $(*) : M \to X^{\cdot}$ is an exact complex of G-modules, i.e., $\partial^2 = 0$ and $\ker(\partial_{n+1}) = \mathrm{im}(\partial_n)$, and the ∂_i's preserves the G-action. This is called the standard resolution of M as a G-module.

Proof. Exercise. □

Definition 2.3. The G-fixed points of the X^n's are defined by $C^n(G,M) = (X^n(G,M))^G$, called the homogeneous n-cochains of G with coefficients in M. Because the complex preserves G-actions, then we obtain a complex of $C^n(G,M)$'s, given by

$$0 \longrightarrow C^0(G,M) \xrightarrow{\partial_0} C^1(G,M) \xrightarrow{\partial_1} \cdots$$

Remark 2.4. To see what the induced mapping is, suppose $A \to B$ is a G-module map, then there is an induced map of fixed points $A^G \to B^G$ by the restriction. In particular, let $a \in A$ be fixed with $ga = a$ for all $g \in G$, then $f(a) = f(ga) = gf(a)$.

Remark 2.5. In the complex of Definition 2.3, $\partial^2 = 0$ as well, but in general this is not an exact sequence.

Definition 2.6 (Group Cohomology)**.** The group cohomology of G with coefficients in M is the collection

$$\{H^n(G,M)\}_{n\geqslant 0},$$

where $H^n(G,M) := H^n(C^\cdot(G,M)) = \ker(\partial : C^n \to C^{n+1})/\operatorname{im}(\partial : C^{n-1} \to C^n)$. We usually use the notion of cocycles $Z^n(G,M) = \ker(\partial : C^n \to C^{n+1})$ and coboundaries $B^n(G,M) = \operatorname{im}(\partial : C^{n-1} \to C^n)$.

Exercise 2.7. Show that $H^0(G,M)$ is isomorphic to M^G.

Definition 2.8. The inhomogeneous cochains $C_i^\cdot(G,M)$ are given by

- $C_i^0 = M$, and
- for $n > 0$, $C_i^n = \operatorname{Map}(G^n, M)$,

with coboundary maps $\partial^{n+1} : C_i^n \to C_i^{n+1}$, given by

- $\partial^1(m)(g) = gm - m$,
- $\partial^2(f)(g_1,g_2) = g_1 f(g_2) - f(g_1g_2) + f(g_1)$, and so on, with
- $\partial^{n+1}(f)(g_1,\dots,g_{n+1}) = g_1 f(g_2,\dots,g_{n+1}) + \sum_{i=1}^{n}(-1)^i f(g_1,\dots,g_ig_{i+1},\dots,g_{n+1}) + (-1)^{n+1} f(g_1,\dots,g_n)$.

This gives the inhomogeneous setting of this cochain.

Lemma 2.9. The maps

$$\begin{aligned} C^n(G,M) &\to C_i^n(G,M) \\ (\varphi : G^{n+1} \to M) &\mapsto (f : G^n \to M) \\ & \qquad f(g_1,\dots,g_n) := \varphi(1, g_1, g_1g_2, \dots, g_1g_2\cdots g_n) \end{aligned}$$

give a cochain homotopy equivalence $C^\cdot(G,M) \xrightarrow{\sim} C_i^\cdot(G,M)$, and hence this is a quasi-isomorphism.

Corollary 2.10. The cohomology $H^*(C_i^\cdot(G,M)) \cong H^*(G,M)$.

Remark 2.11. Any cohomology class can be represented by a normalized inhomogeneous cocycle $f : G^n \to M$, i.e., $f(g_1,\dots,g_n) = 0$ where $g_i = 1$ for some i.

Remark 2.12. Even for $G = C_2$, C_i^n or C^n get large as n grows.

Remark 2.13.
- Using homological algebra, we can find other cochain complexes which computes group cohomology $H^*(G,M)$.
- We would also understand $H^*(G,M)$ as the failure of exactness of $(\ )^G : G$-**Mod** $\to$ **Ab**. Therefore, when taking the fixed points, the exact sequence may not be mapped to another exact sequence. In particular, if we take an exact sequence

  $$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

  of G-modules, the induced sequence

  $$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G$$

  do not give a surjection at $B^G \to C^G$. One needs to take higher cohomology to obtain a long exact sequence. Hence, $(\ )^G : G$-**Mod** $\to$ Ab is a left exact functor, but not necessarily right exact.

3 Aug 25, 2023: Cohomology of groups, continued

Example 3.1. Let G be C_2, or any group with a surjection p onto C_2, then it has an action on $\mathbb{Z}_{\text{sgn}}$ given by $g \cdot a = (-1)^{p(g)}a$, therefore we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}_{\text{sgn}} \xrightarrow{\times 2} \mathbb{Z}_{\text{sgn}} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

and taking the fixed point functor we have

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

Remark 3.2. Higher homologies measure the failure of exactness.

Remark 3.3. The collection $\{H^n(G,-)\}_{n\in\mathbb{Z}}$ satisfies

- $H^n(G,-) = 0$ for $n < 0$;
- for short exact sequence $0 \to A \to B \to C \to 0$ in G-**Mod**, we have a long exact sequence

$$0 \longrightarrow H^0(G,A) \longrightarrow H^1(G,B) \longrightarrow H^1(G,C) \xrightarrow{\delta} H^1(G,A) \longrightarrow \cdots$$

  where δ is the connecting homomorphism.
- the connecting homomorphisms δ are natural, i.e., given a commutating diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

  the induced diagram

$$\begin{tikzcd} H^n(G,C) \arrow[r, "\delta"] \arrow[d] & H^{n+1}(G,A) \arrow[d] \\ H^n(G,C') \arrow[r, "\delta"'] & H^{n+1}(G,A') \end{tikzcd}$$

  also commutes, and $\{H^n(G,-)\}_{n\in\mathbb{Z}}$ is a cohomological δ-functor. Note that a δ-functor is additive, and usually occurs in abelian categories.

Definition 3.4 (δ-functor)**.** A map of δ-functors $T^* \to F^*$ is a collection of natural transformations $T^n \to F^n$, commuting with the δ's, i.e.,

$$\begin{tikzcd} T^n \arrow[r] \arrow[d, "\delta_T"'] & F^n \arrow[d, "\delta_F"] \\ T^{n+1} \arrow[r] & F^{n+1} \end{tikzcd}$$

A δ-functor T^* is universal if, given any other δ-functor F^*, a map $T^* \to F^*$ is uniquely determined by $T^0 \to F^0$.

Proposition 3.5. $H^*(G,-) : G\text{-}\mathbf{Mod} \to \mathbf{Ab}$ is a δ-functor.

Proof. We need to show:

- each $H^n(G,-)$ is a well-defined functor,
- the connecting homomorphisms δ's gives a long exact sequence,
- the naturality of δ.

First, let $f : A \to B$ be in G-**Mod**, then $C^*(G,A) \to C^*(G,B)$ is equivalent to $\mathrm{Map}(G^{*+1}, A)^G \to \mathrm{Map}(G^{*+1}, B)^G$ by composition with f. One can show that this is equivariant, i.e., respects the G-action, so it is well-defined to take the fixed points, and thus commutes with ∂'s.

Second, we need to apply the snake lemma. Given a short exact sequence $0 \to A \to B \to C \to 0$, we claim:

Claim 3.6. $0 \longrightarrow C^*(G,A) \longrightarrow C^*(G,B) \longrightarrow C^*(G,C) \longrightarrow 0$ is a short exact sequence of cochain complexes, i.e., $C^*(G,-) : G\text{-}\mathbf{Mod} \to \mathbf{coCh}$ is an exact functor.

Subproof. Exercise. ■

Now take the complex

$$\begin{tikzcd}
0 \arrow[r] & C^n(G,A) \arrow[r] \arrow[d, "\partial"] & C^n(G,B) \arrow[r] \arrow[d, "\partial"] & C^n(G,C) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C^{n+1}(G,A) \arrow[r] & C^{n+1}(G,B) \arrow[r] & C^{n+1}(G,C) \arrow[r] & 0
\end{tikzcd}$$

and quotient the boundaries everywhere (and thus lose the injectivity/surjectivity when applicable)

$$\begin{tikzcd}
& C^n(G,A)/B^n(G,A) \arrow[r] \arrow[d, "\partial"] & C^n(G,B)/B^n(G,B) \arrow[r] \arrow[d, "\partial"] & C^n(G,C)/B^n(G,C) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & Z^{n+1}(G,A) \arrow[r] & Z^{n+1}(G,B) \arrow[r] & Z^{n+1}(G,C) &
\end{tikzcd}$$

Taking the kernels and cokernels on ∂'s, we obtain a complex

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
& H^n(G,A) \arrow[r] \arrow[d] & H^n(G,B) \arrow[r] \arrow[d] & H^n(G,C) \arrow[d] & \\
& C^n(G,A)/B^n(G,A) \arrow[r] \arrow[d, "\partial"] & C^n(G,B)/B^n(G,B) \arrow[r] \arrow[d, "\partial"] & C^n(G,C)/B^n(G,C) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & Z^{n+1}(G,A) \arrow[r] \arrow[d] & Z^{n+1}(G,B) \arrow[r] \arrow[d] & Z^{n+1}(G,C) \arrow[d] & \\
& H^{n+1}(G,A) \arrow[r] & H^{n+1}(G,B) \arrow[r] & H^{n+1}(G,C) &
\end{tikzcd}$$

By the snake lemma, we obtain the long exact sequence. □

Proposition 3.7. If $0 \to A \to B \to C \to 0$ is a short exact sequence such that $H^*(G,B) = 0$ for $* > 0$ (or at least $H^n(G,B) = 0 = H^{n+1}(G,B)$), then $\delta : H^n(G,C) \to H^{n+1}(G,A)$ is an isomorphism.

Definition 3.8 (Acyclic, Cohomologically Trivial)**.** A G-module M is

- acyclic if $H^*(G,M) = 0$ for $* > 0$,
- cohomologically trivial if $H^*(H,M) = 0$ for $* > 0$ and any (closed) subgroup $H \subseteq G$.

Definition 3.9 (Induced Module)**.** Given any G-module M, the induced module $\mathrm{ind}_G(M) = \mathrm{Map}(G,M) = X^0(G,M)$.

Example 3.10. M could have the trivial action.

Exercise 3.11. For any M, the induced module of M over G is isomorphic (under the G-action) to the induced module of module given by forgetful action over G.

Remark 3.12.
- $\mathrm{Ind}_G(-) : G\text{-}\mathbf{Mod} \to G\text{-}\mathbf{Mod}$ is exact.

- We say A is an induced module if $A \cong \mathrm{Ind}_G(M)$ for some module M. If A is an induced G-module, then A is induced as an H-module for any subgroup $H \subseteq G$.

Lemma 3.13. Induced modules are cohomologically trivial.

Proof. There is an isomorphism
$$C^*(G, \mathrm{Ind}_G(M)) \cong X^*(G, M).$$
$\square$

Remark 3.14. We have an equivariant inclusion of fixed points
$$M \hookrightarrow \mathrm{Ind}_G(M)$$
which is an embedding, and we take $Q \cong \mathrm{Ind}_G(M)/M$, then this extends to a short exact sequence
$$0 \longrightarrow M \hookrightarrow \mathrm{Ind}_G(M) \longrightarrow Q \longrightarrow 0$$
then $H^{n+1}(G, M) \cong H^n(G, Q)$. One say that $H^*(G, -)$ is effaceable. By Tohoku, an effaceable is universal.

4 Aug 28, 2023: First Cohomology of Groups

There are three ways to think about $H^1(G, M)$.

4.1 Crossed Homomorphims

Recall that $H^1(G, M) = Z_i^1(G, M)/B_i^1(G, M)$ as inhomogeneous cochains, where

- $Z_i^1(G, M) = \ker(\mathrm{Map}(G, M) \to \mathrm{Map}(G \times G, M)$ where the map sends $f \mapsto (g, h) \mapsto gf(h) - f(gh) + f(g)$. The kernel of this is exactly the maps f such that $f(gh) = gf(h) + f(g)$, and note that this is not a group homomorphism.
- $B_i(G, M) = \mathrm{im}(M \to \mathrm{Map}(G, M))$ given by $m \mapsto (g \mapsto gm - m)$, where the image is called a principal crossed homomorphism.

Exercise 4.1. $B_i^1(G, M) \cong M/M^G$ as an isomorphism of $\mathbb{Z}[G]$-modules.

Remark 4.2. If the G-action is trivial, then $H^1(G, M) = \mathrm{Hom}_{\mathrm{Grp}}(G, M)$.

Corollary 4.3. If G is a finite group with trivial action, then $H^1(G, \mathbb{Z}) = 0$.

Theorem 4.4 (Hilbert's Theorem 90)**.** Let L/K be a Galois extension with (finite or profinite) Galois group G, then $H^1(G, L^\times) = 0$.

Proof. Let $f : G \to L^\times$ be a crossed homomorphism. We know the addition is given by $f(gh) = gf(h) + f(g)$, and the multiplication is given by $f(gh) = (g \cdot f(h))f(g)$, where $\cdot$ represents the group action. Now for any $l \in L^\times$, the multiplication with respect to l is given by $m_l = \sum\limits_{h \in G} f(h)(h \cdot l)$. We can first choose l so that $m_l \neq 0$, since the Galois conjugates $h \cdot l$ over $l \in L$ are linearly independent. For $g \in G$, we have
$$\begin{aligned} g \cdot m_l &= \sum_{h \in G} (g \cdot f(h))(gh \cdot l) \\ &= \sum_{h \in G} \frac{f(gh)}{f(g)}(gh \cdot l) \\ &= \frac{1}{f(g)} \sum_{h \in G} f(gh)(gh \cdot l) \\ &= \frac{1}{f(g)} m_l. \end{aligned}$$
Therefore, $f(g) = \frac{m_l}{g \cdot m_l}$. For any crossed homomorphism, there exists $m \in L^\times$ such that $f(g) = \frac{gm}{m}$, so every crossed homomorphism is principal. $\square$

Exercise 4.5. Let G acts over a commutative ring R, then $H^1(G, R^\times)$ classifies invariant R-modules with a compatible G-action.

4.2 Non-abelian H^1 and Torsors

Let A be a group with G-action, so let the action $g \cdot a = {}^g a$. Hence, $g \cdot (ab) = {}^g a {}^g b$. Define the G-cocycles to be $f : G \to A$ such that $f(gh) = f(g) {}^g f(h)$. Two cocycles f and f' are said to be cohomologous as $f \sim f'$ if there exists $a \in A$ such that for all $g \in G$, $f'(g) = a^{-1} f(g) {}^g a$. This becomes an equivalence relation on the set of G-cocycles with coefficients in A, then $H^1(G, A)$ is the set of equivalence classes of G-cocycles. Now the first cohomology $H^1(G, A)$ has only a pointed set structure with distinguished point $f \equiv 1$, the constant function at 1.

Exercise 4.6. This definition is equivalent to the inhomogeneous cochain definition in the abelian case.

Definition 4.7. An A-torsor is a G-set X with action

$$
\begin{aligned}
X \times A &\to A \\
(x, a) &\mapsto xa
\end{aligned}
$$

that is free and transitive, i.e., for any $x, y \in G$, there exists a unique $a \in A$ such that $y = xa$. Moreover, the action $X \times A \to X$ respects the G-action, i.e., ${}^g(xa) = {}^g x {}^g a$.

Remark 4.8.
- A is an A-torsor.
- An isomorphism of A-torsors is a bijection that respects the G- and A- action.
- If $A \subseteq B$ is a sub-G-group, then bA is an A-torsor.
- An A-torsor is a principal A-bundle on the classifying space BG.

Theorem 4.9. There is a canonical bijection of pointed sets

$$
H^1(G, A) \cong \mathrm{Torsor}(G, A)
$$

Proof.
- The backwards map $\lambda : \mathrm{Torsor}(G, A) \to H^1(G, A)$ is defined as follows: for $x \in \mathrm{Torsor}(G, A)$, we want to define a cocycle $f(X) : G \to A$. For arbitrary $x \in X$, note that for any $g \in G$, there exists a unique $f_x(g) \in A$ such that ${}^g x = x f_x(g)$ by the simple transitivity of the A-action on X. To see this is well-defined, if we have another $y \in X$, then $y = xb$ for some $b \in A$, then $f_y(g) = b^{-1} f_x(g) {}^g b$, so f_x and f_y are cohomologous and define the same class in $H^1(G, A)$, which is defined to be the image $\lambda(X)$.
- To define $\mu : H^1(G, A) \to \mathrm{Torsor}(G, A)$, given a cocycle $f : G \to A$, let X_f be the group A, then the action of A on X_f is by multiplication on the right, and one can twist the G-action on it using cocycle $f : G \to A$ with ${}^{\bar{g}} x = f(g) g x$, which defines an A-torsor. This is well-defined.

□

Remark 4.10. Suppose

$$
1 \longrightarrow A \longrightarrow B \xrightarrow{p} C \longrightarrow 1
$$

is a short exact sequence of G-groups, i.e., A is a sub-G-group and $C \cong B/A$, then there is a long exact sequence

$$
1 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \xrightarrow{\delta} H^1(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, C)
$$

where δ is given by $\delta(c) = p^{-1}(c)$. For the exactness in the sense of pointed sets to work, the kernel is the subset mapping to the distinguished element.

4.3 Extension Splitting

Consider the a split extension

$$
1 \longrightarrow A \longrightarrow E \xrightarrow{p} G \longrightarrow 1
$$

That is, E is the direct product $A \times G$ with group action $(a, g)(a', g') = (a {}^g a', gg')$, and by definition E is the semidirect product $A \rtimes G$. Equivalently, there exists a section (as group homomorphism) $s : G \to E$.

There is an equivalence relation on the set of sections to the projection $p : E \to G$, where the sections $s, s' : G \to E$ are conjugates if there exists $a \in A$ such that $s'(g) = a^{-1} s(g) a$. We denote $\sec(E \to G)$ to be the conjugacy class of sections of p. Note that the class of trivial section $s : g \mapsto (1, g) \in E$ is the distinguished element.

Proposition 4.11. The pointed set $H^1(G, A)$ is isomorphic to $\sec(E \to G)$.

Proof. Take $\varphi \in \sec(E \to G)$, then the composition $G \xrightarrow{\varphi} E \xrightarrow{\pi_1} A$, where π_1 is the set-theoretic projection to the first component, defines a cocycle $G \to A$. Conversely, given a cocycle $f : G \to A$, the section is given by $g \mapsto (f(g), g)$. $\square$

Exercise 4.12. Expand the proof above.

Exercise 4.13. Describe $\mathbb{Z} \rtimes C_2$ where C_2 acts on $\mathbb{Z}$ by inversion. How many sections are there of $\mathbb{Z} \rtimes C_2 \to C_2$?

Exercise 4.14. How many sections are there to the projection $D_{2n} \to C_2$?

5 Aug 30, 2023: H^2, abelian extensions, and Brauer group

Suppose we have an abelian extension, that is, let A be abelian, the short exact sequence of group extensions

$$0 \longrightarrow A \xhookrightarrow{i} E \xrightarrow{p} G \longrightarrow 1$$

is such that $E/i(A) \cong G$. Note that A can be regarded as a normal subgroup in E given this notation.

Note that two extensions are equivalent if there exists a group isomorphism $\varphi : E \to E'$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\varphi"] & G \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & A \arrow[r] & E \arrow[r] & G \arrow[r] & 1
\end{tikzcd}$$

commutes.

Consider the continuous functions

$$\varphi : G \times G \to A$$

such that $\varphi(g_1g_2, g_3) + \varphi(g_{1,2}) = \varphi(g_1, g_2g_3) + g_1\varphi(g_2, g_3)$. We know $H^2(G, M)$ is the quotient of all such functions over the coboundaries, i.e., the functions φ such that $\varphi(g_1, g_2) = f(g_1) - f(g_1g_2) + g_1 f(g_2)$.

Now $E \cong A \times G$ can be considered as a bijection, so we pick a set-theoretic section $s : G \to E$ with $s(1) = 1$, and now every element in E is written as $as(g)$ uniquely for some $a \in A$ and $g \in G$, we have

$$s(g)a = s(g)as(g)^{-1}s(g) = {}^g as(g).$$

Note that s may not be a homomorphism, but we have $s(g)s(h) = f(g, h)s(gh)$ since $s(g)s(h)$ and $s(gh)$ are both lifts of gh.

As a consequence, we have

$$(s(g_1)s(g_2))s(g_3) = f(g_1, g_2)s(g_1g_2)s(g_3) = f(g_1, g_2)f(g_1g_2, g_3)s(g_1g_2g_3)$$

and

$$s(g_1)(s(g_2)s(g_3)) = s(g_1)f(g_2, g_3)s(g_2, g_3) = {}^{g_1}f(g_2, g_3)s(g_1)s(g_2g_3) = {}^{g_1}f(g_2, g_3)f(g_1, g_2g_3)s(g_1g_2g_3).$$

In additive notation, we have

$$f(g_1, g_2) + f(g_1g_2, g_3) = g_1 f(g_2, g_3) + f(g_1, g_2g_3).$$

Therefore, f becomes an inhomogeneous 2-cocycle.

Proposition 5.1. The induced map $\lambda : \mathrm{ext}(G, A) \to H^2(G, A)$ is a well-defined bijection between the set of equivalence classes of extensions and $H^2(G, A)$.

Example 5.2. The two elements in $H^2(C_2, \mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ are given by non-split extension of Q_8

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow Q_8 \longrightarrow C_2 \longrightarrow 1$$

and the identity element given by $D_8 \cong \mathbb{Z}/4\mathbb{Z} \rtimes C_2$

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow D_8 \longrightarrow C_2 \longrightarrow 1$$

where D_8 has the action of C_2 over $\mathbb{Z}/4\mathbb{Z}$.

Proposition 5.3. An associative finite-dimensional K-algebra A is a CSA if and only if one of the following equivlaent conditions hold:

1. Based-changed to the separable closure $\bar{K}$ of K via $\bar{K} \otimes_K A$, $A \cong M_n(\bar{K})$ for some integer $n \geqslant 1$.
2. there exists a finite Galois extension L/K such that base-changed to L via $L \otimes_K A$, A becomes isomorphic to a matrix algebra $M_n(L)$ for some integer $n \geqslant 1$.
3. $A \cong M_n(D)$ matrix algebra for some $m \geqslant 1$ and some finite division algebra D over K.

A CSA A over K is said to be split over L if the above holds, i.e., $A \otimes_K L \cong M_n(L)$. One can define an equivalence class on CSAs, such that $A \sim B$ if and only if $A \otimes_K M_n(K) \cong B \otimes_K M_m(K)$. Now the Brauer group of K is the abelian group of equivalence classes of CSAs over K equipped with tensor product.

Suppose L/K is an extension, then there exists a homomorphism of base-change of algebras $\mathrm{Br}(K) \to \mathrm{Br}(L)$. We say the kernel $\mathrm{Br}(L \mid K)$ is the relative Brauer group of K-CSAs that split over K. The absolute Brauer group is $\mathrm{Br}(\bar{K} \mid K) = \mathrm{Br}(K)$, then

$$\mathrm{Br}(K) = \bigcup_{L/K \text{ finite}} \mathrm{Br}(L \mid K).$$

Now let L/K be a finite Galois extension with Galois group G, and we pick a normalized inhomogeneous 2-cycle $\varphi : G \times G \to L^\times$ as the representative of its class, and we can construct A_φ as a K-CSA, then $A_\varphi = \bigoplus_{g \in G} Le_g$ has dimension $|G|^2$, where e_g's are the generators, with a multiplication operation $(le_g)(me_h) = l(g \cdot m)\varphi(g,h)e_{gh}$ which can be extended via distribution. A_φ is said to be the crossed product of L and G via φ.

Theorem 5.4.

1. A_φ is a split algebra over L.
2. If φ, φ' are two normalized inhomogeneous 2-cocycles, then $A_\varphi \sim A_{\varphi'}$ if and only if $\varphi \sim \varphi'$.
3. $A_{\varphi\varphi'} \sim A_\varphi \otimes_K A_{\varphi'}$.
4. Any K-CSA which is split over L is similar to a crossed product A_φ for some $\varphi : G \times G \to L^\times$.

Corollary 5.5. $H^2(G, L^\times)$ is isomorphic to $\mathrm{Br}(L \mid K)$, and $H^2(\mathrm{Gal}(\bar{K}/K), \bar{K}^\times)$ is isomorphic to $\mathrm{Br}(K)$.

6 Sept 1, 2023: Cohomology of Cyclic and Free Groups

Recall that we can compute $H^*(G, M)$ using any acyclic resolution of M. We want to describe $H^*(G, M)$ for specific G using nice resolutions.

We have

$$\cdots \to G^3 \xrightarrow{\delta} G^2 \xrightarrow{\delta} G$$

and to obtain $X^*(G, M)$ we map out of the resolution and into M, so $\mathrm{Map}(G, M) \cong \mathrm{Hom}(\mathbb{Z}[G], M)$ as G-modules, and in general we obtain

$$\mathrm{Map}(G^k, M) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G]^{\otimes k}, M)$$

as $\mathbb{Z}$-modules.

We denote F^{st} to be the standard free resolution given by

$$\mathbb{Z}[G]^{\otimes k} \xrightarrow{d} \mathbb{Z}[G]^{\otimes(k-1)} \to \cdots \to \mathbb{Z}[G]^{\otimes 2} \xrightarrow{d_1 - d_0} \mathbb{Z}[G]$$

To obtain $X^*(G,M)$, we can map this into M. Now the standard resolution becomes an augmentation of $\mathbb{Z}$ that makes $X^*(G,M)$ exact, free, and acyclic. The kernel of $\mathbb{Z}[G] \to \mathbb{Z}$ is the agumentation ideal of G as of $\mathbb{Z}[G]$. Since this is a G-equivariant map, then the augmentation ideal is a G-submodule of $\mathbb{Z}[G]$, as a free abelian group generated by the set $\{(g-1) \mid 1 \neq g \in G\}$.

Lemma 6.1. If $P_* \to \mathbb{Z}$ is any free resolution of $\mathbb{Z}$ as a G-module, then for a G-module M, we have $H^*(G,M) \cong H^*(\mathrm{Hom}(P_*, M))^G$.

Proof. Since each P_i is free, then $\mathrm{Hom}(P_i, M)$ is an acyclic module, so $M \to \mathrm{Hom}(P_*, M)$ is an acyclic resolution of M. Now apply Proposition 2.28 in the notes. □

Remark 6.2. $H^*(G,M) \cong \mathrm{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z}, M)$ as universal δ-functors.

Now let C_n be the cyclic group of order n, generated by element g, then $\mathbb{Z}[C_n] \cong \mathbb{Z}[g]/(g^n-1)$, so we have $0 = g^n - 1 = (g-1)N_g$ in $\mathbb{Z}[C_n]$ where N_g is the norm element $N_g = 1 + g + \cdots + g^{n-1}$, so we have a free resolution of $\mathbb{Z}$:

$$\cdots \longrightarrow \mathbb{Z}[C_n] \xrightarrow{1-g} \mathbb{Z}[C_n] \xrightarrow{N_g} \mathbb{Z}[C_n] \xrightarrow{1-g} \mathbb{Z}[C_n] \xrightarrow{\varepsilon} \mathbb{Z}$$

where augmentation ε sends g to 1. This allows us to compute the cohomology of any C_n-modules.

Proposition 6.3. Let M be an C_n-module, then

$$H^i(G,M) = \begin{cases} M^G, & i = 0 \\ \{m \in M \mid N_g m = 0\}/(1-g)M, & i > 0 \text{ odd} \\ M^G/N_g M, & i > 0 \text{ even} \end{cases}$$

Proof. Taking $\mathrm{Hom}(P_*, M)^G$ gives

$$\cdots \longleftarrow M \xleftarrow[1-g]{} M \xleftarrow[N_g]{} M \xleftarrow[1-g]{} M \longleftarrow \cdots$$

□

Remark 6.4. If M has trivial action, then

$$H^i(G,M) = \begin{cases} M, & i = 0 \\ M[n], & i > 0 \text{ odd} \\ M/n, & i > 0 \text{ even} \end{cases}$$

where $M[n]$ is the n-torsion in M.

Now if $T = \mathbb{Z}$ be with generator t, then $\mathbb{Z}[T]$ is isomorphic to the Laurent polynomials, so we have a resolution

$$0 \longrightarrow \mathbb{Z}[T] \xrightarrow{1-t} \mathbb{Z}[T] \longrightarrow \mathbb{Z}$$

since $(1-t)$ is not a zero-divisor of $\mathbb{Z}[T]$. Therefore, taking $\mathrm{Hom}(P_*, M)^T$ gives

$$0 \longleftarrow M \xleftarrow[1-t]{} M$$

$$H^i(T,M) = \begin{cases} M^T, & i = 0 \\ M_T, & i = 1 \\ 0, & \text{otherwise} \end{cases}$$

Now let X be a set, and let G_X be the free group on X.

Proposition 6.5. The augmentation ideal I_X is a free $\mathbb{Z}[G_X]$-module, generated by the set $\{(x-1) \mid x \in X\}$, and so the exact sequence

$$0 \longrightarrow I_X \longrightarrow \mathbb{Z}[G_X] \longrightarrow \mathbb{Z} \longrightarrow 0$$

is a free resolution of $\mathbb{Z}$ as a G_X-module.

Proof. As $\mathbb{Z}$-bases of I_X, we have $\{(g-1) \mid g \in G_X\}$, but $\{h(x-1) \mid h \in G, x \in X\}$ is also a $\mathbb{Z}$-linear basis for I_X. □

Remark 6.6. Groups are free if and only if they have cohomological dimension 1.

7 Sept 6, 2023: Cup Product

Remark 7.1.

1. A crossed homomorphism would be a group homomorphism when G has trivial action on M.
2. If X is an A-torsor, then there is a given G-action and a right A-action so that $X \times A \to X$ is given by a diagonal action compatible to the G-action. Therefore, ${}^g(x \cdot a) = {}^g x \cdot {}^g a$.

Definition 7.2. Let A and B be G-modules, then there is a notion of tensor product $A \otimes_G B$ as a G-module via the diagonal action $g(a \otimes b) = ga \otimes gb$. On the level of cochain, we have a cup product

$$\begin{aligned} C^p(G,A) \otimes C^q(G,B) &\xrightarrow{\smile} C^{p+q}(G, A \otimes B) \\ (\alpha : G^{p+1} \to A) \otimes (\beta : G^{q+1} \to B) &\mapsto (\alpha \smile \beta) \\ &\qquad (g_0, \dots, g_{p+q}) \mapsto \alpha(g_0, \dots, g_p) \otimes \beta(g_p, \dots, g_{p+q}) \end{aligned}$$

Proposition 7.3. $\partial(\alpha \smile \beta) = (\partial\alpha) \cup \beta + (-1)^{|\alpha|}\alpha \smile \partial\beta$.

Corollary 7.4.

- If α and β are cocycles, then $\alpha \smile \beta$ is also a cocycle.
- If α is a cocycle β is a coboundary, or vice versa, then $\alpha \smile \beta$ is a coboundary. Indeed, if $\beta = \partial\gamma$, then $\partial(\alpha \smile \gamma) = (-1)^{|\alpha|}\alpha \smile \beta$.

Therefore, on the level of cohomology, we have a (bilinear) cup product as well:

$$H^p(G,A) \otimes H^q(G,B) \to H^{p+q}(G, A \otimes B)$$

Example 7.5.

- If $p = q = 0$, then

$$\begin{gathered} H^0(G,A) \otimes H^0(G,B) \cong A^G \otimes B^G \to H^0(G, A \otimes B) \cong (A \otimes B)^G \\ a \otimes b \mapsto a \otimes b \end{gathered}$$

- By extending this prioperty, we get a G-equivariant pairing $A \otimes B \to C$ and therefore

$$H^p(G,A) \otimes H^q(G,B) \xrightarrow{\smile} H^{p+q}(G, C).$$

Example 7.6. Let R be a commutative ring, and if there is a G-action on R, then the multiplication $m : R \otimes R \to R$ is G-equivariant, so we have a cup product

$$\smile : H^p(G,R) \otimes H^q(G,R) \to H^{p+q}(R)$$

This has the following properties:

1. This is natural in A, B, and C.
2. This is compatible with connecting homomorphism and exact sequences, that is,
   - Given short exact sequences

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

and

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

and pairing $A \otimes B \to C$, then this induces $A \otimes B \to C'$ and in the quotients we have $A'' \otimes B \to C''$, so $\delta(\alpha \smile \beta) = \delta\alpha \smile \beta$, so we have a commutative diagram[1]

$$\begin{tikzcd} & A' \otimes B \arrow[r] \arrow[d] & A \otimes B \arrow[r] \arrow[d] & A'' \otimes B \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C' \arrow[r] & C \arrow[r] & C'' \arrow[r] & 0 \end{tikzcd}$$

[1] This may require the assumption that the modules are flat.

and thus

$$\begin{tikzcd} H^o(G,A'')\otimes H^q(G,B) \arrow[r] \arrow[d, "\delta\otimes 1"] & H^{p+q}(G,A''\otimes B) \arrow[d, "\delta"] \\ H^{p+1}(G,A')\otimes H^q(G,B) \arrow[r] & H^{p+q+1}(G,A'\otimes B) \end{tikzcd}$$

- Given

$$0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$$

and

$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$

and pairings

$$\begin{tikzcd} & A\otimes B' \arrow[r] \arrow[d] & A\otimes B \arrow[r] \arrow[d] & A\otimes B'' \arrow[d] & \\ 0 \arrow[r] & C' \arrow[r] & C \arrow[r] & C'' \arrow[r] & 0 \end{tikzcd}$$

so $\delta(\alpha\smile\beta)=(-1)^{|\alpha|}\alpha\smile\delta\beta$.

Proof. Let $\alpha=[a]$ for $a:G^{p+1}\to A$ and $\beta=[b]$ for $b:G^{q+1}\to B''$, then there is a lift $b:G^{q+1}\xrightarrow{\tilde{b}}B\to B''$. Then we have

$$\begin{tikzcd} & C^q/B^q(B') \arrow[r] \arrow[d, "\partial"] & C^q/B^q(B) \arrow[r] \arrow[d, "\partial"] & C^q/B^q(B'') \arrow[r] \arrow[d, "\partial"] & 0 \\ 0 \arrow[r] & Z^q(B') \arrow[r] & Z^{q+1}(B) \arrow[r] & Z^{q+1}(B'') & \end{tikzcd}$$

and by the snake lemma we have a connecting homomorphism over group cohomologies. □

8 Sept 8, 2023: Restriction and Transfer

Recall that we have a chain-level cup product, and we extend it to the level of cohomology. The cup product has the following properties:

1. If $p=q=0$, then the cup product is the natural composition
$$A^G\otimes B^G\to(A\otimes B)^G\to C^G$$
2. Functoriality.
3. We have $\delta(\alpha\smile\beta)=\delta(\alpha)\smile\beta$, and incorporating this with the exact sequence, we have $\delta(\alpha\smile\beta)=(-1)^{|\alpha|}\alpha\smile\delta(\beta)$.

By the universal property of the tensor product, there exists a unique bilinear pairing that also satisfies these properties. To prove this, we use dimension-shifting.

Remark 8.1. Let M be a module, and map it into the induced module with an extended short exact sequence

$$0\longrightarrow M\longrightarrow \operatorname{Ind}^G(M)=\operatorname{Map}(G,M)\cong\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[G],M)\longrightarrow M_1\longrightarrow 0$$

Taking the fixed points, we have

$$0\longrightarrow M^G\longrightarrow(\operatorname{Ind}^G(M))^G\longrightarrow(M_1)^G\longrightarrow H^1(G,M)\longrightarrow 0\longrightarrow\cdots$$

$$\cdots\longrightarrow 0\longrightarrow H^k(G,M_1)\xrightarrow{\cong}H^{k+1}(G,M)$$

Here $(M_1)^G \to H^1(G, M)$ is a surjection. Now we know $\delta : H^i(G, M_1) \to H^{i+1}(G, M)$ is a surjection for $i = 0$, and is an isomorphism for $i > 0$.

Proceeding inductively, we define

$$0 \longrightarrow M_i \longrightarrow \mathrm{Ind}^G(M) \longrightarrow M_{i+1} \longrightarrow 0$$

If we start with $A \otimes B \to C$, then use property (3) repeatedly to the short exact sequence above, we get the uniqueness.

Example 8.2. Consider $G = C_2$, and consider the cohomology ring $H^*(C_2, \mathbb{F}_2)$. The action is obviously trivial. This induced the sequence with augmentation

$$0 \longrightarrow \mathbb{F}_2 \longrightarrow \mathbb{F}_2[C_2] \longrightarrow \mathbb{F}_2 \longrightarrow 0$$

The boundary map is $\delta : H^i(C_2, \mathbb{F}_2) \to H^{i+1}(C_2, \mathbb{F}_2)$ is an isomorphism for all i.

We know $H^i(C_2, \mathbb{F}_2) = \mathbb{F}_2\{x_i\}$, so we can write $x_{i+1} = \delta x_i$. The product $x_i \smile x_j = \delta^i x_0 \smile \delta^j x_0 = \delta^{i+j} x_0 \smile x_0 = \delta^{i+j} x_0 = x_{i+j}$. Hence, $H^*(C_2, \mathbb{F}_2) \cong \mathbb{F}_2[x]$ where $x = |x_1|$.

Note that

$$H^i(C_2, M) = \begin{cases} M^{C_2}, & i = 0 \\ \ker(N)/(\sim), & i \text{ odd} \\ M^{C_2}/N, & i > 0 \text{ even} \end{cases}$$

Remark 8.3. For odd prime p, we want to use the same method to calculate $H^i(C_p, \mathbb{F}_p)$ with trivial action, then this is $\{\mathbb{F}_p, i \geqslant 0\}$. For instance, if we look at $x_1 \smile x_1$, then this is $(-1)^{|x_1|} x_1 \smile x_1$, so this gives $2x_1 \smile x_1 = 0 \in H^2 = \mathbb{F}_p$, so this gives $x_1 \smile x_1 = 0$. Note that $H^*(C_p, \mathbb{F}_p) \cong \bigwedge(x_1) \otimes \mathbb{F}_p[y]$.

We now talk about the functoriality in G. Given G_1 acting on M_1 and G_2 acting on M_2, and say $\varphi : G_1 \to G_2$ is a group homomorphism, and a map of modules $f : M_2 \to M_1$, then we say φ and f is a compatible pair of morphisms if for any $g \in G_1$, the diagram

$$\begin{tikzcd} M_2 \arrow[r, "f"] \arrow[d, "\varphi(g)"'] & M_1 \arrow[d, "g"] \\ M_2 \arrow[r, "f"'] & M_1 \end{tikzcd}$$

This gives a map $C^*(G_2, M_2) \to C^*(G_1, M_1)$, and hence a map on cohomology $H^*(G_2, M_2) \to H^*(G_1, M_1)$. For instance, if $\varphi = \mathrm{id}$, we obtain the functoriality in M, as we previously saw. Similarly, if $f = \mathrm{id}$, and $M = M_2$ is a G_2-module, on which $g_1 \cdot m = \varphi(g_1) \cdot m$.

There are some special situations from the relations above.

1. Conjugation: let $H \subseteq G$ be a subgroup, and we consider A to be a G-module, then there is restriction of G-action on A to H, so A becomes a H-module. Let $B \subseteq A$ be a H-submodule in this sense. This is preserved by the action of A, but not necessarily by the action of G. For any $g \in G$, let the right conjugation be $h^g = g^{-1}hg$ on h, and let ${}^gH = gHg^{-1}$ on subgroup H. The compatible morphisms are now

$$\begin{aligned} {}^gH &\to H \\ h &\mapsto h^g \end{aligned}$$

and

$$\begin{aligned} B &\to gB \\ b &\mapsto gb \end{aligned}$$

Therefore, the induced maps on conjugation is given by $(g)_* = H^*(H, B) \to H^*({}^gH, gB)$. Therefore, $(g_1g_2)_* = (g_1)_*(g_2)_*$.

2. Inflation: suppose $H \lhd G$ is a normal subgroup. We have the canonical map $G \to G/H$. Let A be a G-module, then G/H acts on A^H, and we look at the inclusion $A^H \hookrightarrow A$. Now $\varphi : G \to G/H$ and $f : A^H \hookrightarrow A$ are compatible, so on the level of cohomology, we get an inflation map
$$\inf_G^{G/H} : H^*(G/H, A^H) \to H^*(G, A).$$
If we look at $H_1 \subseteq H_2 \lhd G$ where $H_i \lhd G$, we have $G \to G/H_1 \to G/H_2 \cong (G/H_1)/(H_2/H_1)$, then the inflation is
$$\inf_G^{G/H_1} \circ \inf_{G/H_1}^{G/H_2} = \inf_G^{G/H_2}.$$
3. Restriction: Let $\varphi : H \hookrightarrow G$ and consider A A as G-module and H-module respectively. There is now a restriction map
$$\operatorname{res}_H^G : H^*(G, A) \to H^*(H, A)$$
Now if $H_1 \subseteq H_2 \subseteq G$, then
$$\operatorname{res}_{H_1}^G = \operatorname{res}_{H_1}^{H_2} \circ \operatorname{res}_{H_2}^G$$
Inflation and restriction fit in a long exact sequence.

Finally, we discuss corestriction/transfer/norm. Let G be a finite group and let M be a G-module, then we have $M^G \hookrightarrow M$ as inclusion. On the other way around, we have
$$\begin{aligned} \operatorname{tr}/N : M &\to M^G \\ m &\mapsto \sum_{g \in G} gm. \end{aligned}$$

9 Sept 11, 2023:

Let $\varphi : G_1 \to G_2$ and $f : M_2 \to M_1$ be compatible, then we denote $(\varphi, f)^* = H^*(G_2, M_2) \to H^*(G_1, M_1)$, with
$$G_1^{\times(*+1)} \longrightarrow G_2^{\times(*+1)} \longrightarrow M_2 \xrightarrow{f} M_1$$
such that it follows composition, and $(\varphi, f)^*$ commutes with δ, i.e.,
$$\begin{tikzcd} 0 \arrow[r] & M_2' \arrow[r] \arrow[d, "f"] & M_2 \arrow[r] \arrow[d, "f"] & M_2'' \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & M_1' \arrow[r] & M_1 \arrow[r] & M_1'' \arrow[r] & 0 \end{tikzcd}$$
and therefore we have a commutative square
$$\begin{tikzcd} H^k(G, M_2'') \arrow[r, "\delta"] \arrow[d, "(\varphi,f)^*"'] & H^{k+1}(G_2, M_2') \arrow[d, "(\varphi,f)^*"] \\ H^k(G_1, M_1'') \arrow[r, "\delta"'] & H^{k+1}(G, M_1') \end{tikzcd}$$
For $\alpha \in C^k(M_2'')/B^k$, we trace it back to $\tilde{\alpha} \in C^k(M_2)/B_k$, and α is sent to $Z^{k+1}(M_2'')$, but now that means $\tilde{\alpha}$ lands in the kernel of $Z^{k+1}(M_2) \to Z^{k+1}(M_2'')$, so this is in $Z^{k+1}(M_2')$.
$$\begin{tikzcd} & C^k(M_2)/B_k \arrow[r] \arrow[d, "\partial"'] & C^k(M_2'')/B_k \arrow[r] \arrow[d, "\partial"] & 0 \\ 0 \arrow[r] & Z^{k+1}(M_2') \arrow[r] & Z^{k+1}(M_2) \arrow[r] & Z^{k+1}(M_2'') \end{tikzcd}$$
Moreover, we have $(\varphi, f)^*(\alpha \smile \beta) = (\varphi, f)^*\alpha \smile (\varphi, f)^*\beta$, whenever the modules are compatible.

For transfer/corestriction, if $H \subseteq G$ is a subgroup with finite index, and M is a G-module, then we have

$$\begin{aligned} \mathrm{tr}_G^H : M^H &\to M^G \\ m &\mapsto \sum_{g \in G/H} gm \end{aligned}$$

For instance, we have $\mathrm{tr} : \mathbb{Z}^H = \mathbb{Z} \to \mathbb{Z}^G = \mathbb{Z}$ is multiplication by $[G : H]$. Note that $H^*(X^*(G,M)^G) = H^*(G,M)$, but $H^*(X^*(G,M)^H) = H^*(H,M)$, and the latter maps to the former cohomology structure via the transfer mapping. Hence, we have $\mathrm{tr}_G^H : X^*(G,M)^H \to X^*(G,M)^G$ giving $\mathrm{tr}_G^H \equiv \mathrm{cores}_G^H : H^*(H,M) \to H^*(G,M)$. This is not a ring homomorphism.

Remark 9.1 (Properties).

1. tr commutes with δ, that is, for a short exact sequnece of G-modules (hence a short exact sequence of H-modules),

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

then we have

$$\begin{tikzcd} H^k(H,C) \arrow[r, "\delta"] \arrow[d, "\mathrm{tr}"'] & H^{k+1}(H,A) \arrow[d, "\mathrm{tr}"] \\ H^k(G,C) \arrow[r, "\delta"'] & H^{k+1}(G,A) \end{tikzcd}$$

2. If $H_1 \subseteq H_2 \subseteq G$ are subgroups with finite indices, then $\mathrm{tr}_G^{H_1} = \mathrm{tr}_G^{H_2} \mathrm{tr}_{H_2}^{H_1}$.

3. $\mathrm{tr}(\mathrm{res}(\alpha) \smile \beta) = \alpha \smile \mathrm{tr}(\beta)$. Now given a pairing $A \otimes B \to C$ of G-modules, with $H \subseteq G$, then

$$\begin{tikzcd} H^i(H,A) & \otimes & H^j(H,B) \arrow[r, "\smile"] \arrow[d, "\mathrm{tr}"] & H^{i+j}(H,C) \arrow[d, "\mathrm{tr}"] \\ H^i(G,A) \arrow[u, "\mathrm{res}"] & \otimes & H^j(G,B) \arrow[r, "\smile"'] & H^{i+j}(G,C) \end{tikzcd}$$

Proof Idea. By dimension shifting, we reduce the case H^0, in which we have an explicit description. We have $A^H \otimes B^H \to C^H$, so for $\alpha \in A^G$ and $\beta \in B^H$, we have $\mathrm{tr}(\alpha \otimes \beta) = \sum_{g \in G/H} g(\alpha \otimes \beta) = \sum g\alpha \otimes g\beta = \alpha \otimes \sum_{g \in G/H} g\beta$. □

Example 9.2. Let R be a commutative ring with a G-action, then the restriction $\mathrm{res} : H^*(G,R) \to H^*(H,R)$ is a ring homomorphism, so $H^*(H,R)$ is a $H^*(G,R)$-algebra. The opposite side has tr is a map of $H^(G,R)$-modules where the cohomology of H is given the module structure from the restriction. This induces the Frobennius reciprocity.

Remark 9.3 (Other compatibilities). Let $K \subseteq H \subseteq G$ be (normal) subgroups, then $G \to G/K \to G/H$ are quotient maps. The restrictions of inclusions correspond to inflations of surjections: if $K \lhd G$, then $G \to G/K$ and $H \to H/K$, so $\inf_H^{H/K} \circ \mathrm{res}_{H/K}^{G/K} = \mathrm{res}_H^G \circ \inf_G^{G/K}$. Note that the maps are contravariants. Moreover, we have $\inf_G^{G/K} \circ \mathrm{cores}_{G/K}^{H/K} = \mathrm{cores}_G^H \circ \inf_H^{H/K}$.

If $H \lhd G$, then $\mathrm{res}_H^G \circ \mathrm{cor}_G^H = N_{G/H}$; also, $\mathrm{cor}_G^H \circ \mathrm{res}_H^G = [G : H]$.

10 Sept 13, 2023: Spectral Sequence

Whenever G is not cyclic or Q_8, the group cohomology $H^*(G,M)$ would not have a small resolution. We know there is a pullback diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & \prod_p M_p^n \arrow[d] \\ M_{\mathbb{Q}} \arrow[r] & \prod_p (M_p^n)_{\mathbb{Q}} \end{tikzcd}$$

Here $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ is the base-change, and $M_p^n = \lim_i M/p^i$ is the completion. For finite group G, we have $H^*(G, M_{\mathbb{Q}}) = M_{\mathbb{Q}}^G$ if $* = 0$ and is trivial otherwise. Now we have the diagram

$$\begin{tikzcd} H^*(G,M) \arrow[r, "\mathrm{res}"] \arrow[dr, "|G|"'] & H^*(\{e\},M) \arrow[d, "\mathrm{tr}"] \\ & H^*(G,M) \end{tikzcd}$$

where $H^*(\{e\}, M)$ is M if $* = 0$ and is otherwise trivial. Note that if $* > 0$, then $H^*(G, M)$ is annihilated by $|G|$. Let $P \subseteq G$ be a Sylow p-subgroup, then if P is normal, then $H^*(G, M_p^n) \cong H^*(P, M_p^n)^{G/p}$. Therefore we have a normal series $\cdots \lhd P_2 \lhd P_1 \lhd P$ with simple enough quotients, e.g., as abelian series. Therefore, we need ways to reassemble the cohomology.

For $H \lhd G$ we know there is a G/H-action on $H^*(H, M)$ via conjugation, so we can calculate $H^*(G/H, H^*(H, M))$, hence calculate $H^*(G, M)$ using Lyndon-Hochschild-Serre spectral sequences.

We will first look at Bockstein spectral sequences. We start by looking at the sequence

$$\cdots \subseteq p^2\mathbb{Z} \subseteq p\mathbb{Z} \subseteq \mathbb{Z}$$

and factors each inclusion $p^k\mathbb{Z} \subseteq p^{k-1}\mathbb{Z}$ via $p^k(\mathbb{Z}/p\mathbb{Z})$, then we have cohomology $H^*(G, M/p)[p]$, thus calculate $H^*(G, M)$. (Here the attachment by p is given by tensoring $\mathbb{Z}[v_0]$ with grading p.) In general, we construct the abstract version as filtered cochain complex, with

$$\cdots \subseteq F^{p+1}C^* \subseteq F^pC^* \subseteq \cdots \subseteq C^*$$

so we can map each term to the graded version $\mathrm{gr}^p C^*$. We denote the inclusions by i and the projections to the graded versions by π. The goal is to understand $H^*(C^*)$ from the building blocks $H^*(\mathrm{gr}^* C^*)$. There exists the factoring

$$\begin{tikzcd} \cdots \arrow[r] & H^q(F^{p+2}) \arrow[r, "i"] & H^q(F^{p+1}) \arrow[r, "i"] \arrow[dl, "\pi"] & H^q(F^p) \arrow[r] \arrow[dl, "\pi"] & \cdots \\ & H^q(\mathrm{gr}^{p+1}) \arrow[u, "\delta"] & H^q(\mathrm{gr}^p) \arrow[u, "\delta"] & & \end{tikzcd}$$

This is the E_1-page of the spectral sequence, given by $E_1^{p,q} = H^q(\mathrm{gr}^p)$. We denote $d_1 : H^q(\mathrm{gr}^p) \to H^{q+1}(\mathrm{gr}^{p+1})$ as the composition. Obviously $d_1^2 = 0$.

Now the E_2-page is given by $H^*(E_1, d_1)$. For $a \in \ker(d_1)$, the map i induces $\tilde{\delta} \mapsto \delta a$ by lifting, so $\pi(\tilde{\delta a}) \in H^{q+1}(\mathrm{gr}^{p+2}) = E_1^{p+2,q+1}$, with $d_1(\pi(\tilde{\delta a})) = \pi\delta\pi(\tilde{\delta a}) = 0$. We then define $d_2([a]) = [\pi(\tilde{\delta a})] \in E_2$. We then proceed inductively and find higher pages. This is usually done by calculating derived pages.

11 Sept 15, 2023

Recall that: if H is a finite group, A is a finite H-module, then an extension of H by A is a group G such that

$$0 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 1$$

is exact, where the H-module structure on A is realized via conjugation $h \cdot a = hah^{-1} \in G$. We already know that the equivalence classes of extensions of H by A correspond to $H^2(H, A)$, where $A \rtimes H$ corresponds to $0 \in H^2(H, A)$.

Theorem 11.1. Let p be an odd prime, $|G| = p^{n+1}$, and G contains $\mathbb{Z}_q$ for $q = p^n$ as a subgroup. If this is the case, then G is either $\mathbb{Z}_{p^{n+1}}$, $\mathbb{Z}_q \times \mathbb{Z}_p$, or $\mathbb{Z}_q \rtimes \mathbb{Z}_p$, where the generator $e \in H$ acts on $1 \in \mathbb{Z}_q$ by $e1e^{-1} = 1 + p^{n-1}$. We denote $H = \mathbb{Z}_p$ in this case.

Proof. We want to look at the short exact sequence

$$0 \longrightarrow \mathbb{Z}_q \longrightarrow G \longrightarrow H \longrightarrow 1$$

where $H = \mathbb{Z}_p$.

Lemma 11.2. If p is an odd prime, and there exists integer a such that $a^p \equiv 1 \pmod{p^n}$ for $n \geqslant 2$, then $a \equiv 1 \pmod{p^{n-1}}$.

Subproof. This is trivial if $a = 1$. If $a \neq 1$, let $d(a)$ be the largest possible integer d such that $a \equiv 1 \pmod{p^d}$. It suffices to show that $d(a) \geqslant n-1$. By Fermat's Little theorem, we have $d(a) \geqslant 1$. We now want to show $d(a^p) = d(a)+1$. Indeed, let $a = 1 + p^d b$, then using the binomial theorem, we have $a^p = (1+p^d b)^p = 1 + p^{d+1}b + \cdots$ where the omitted terms have higher order of p^{d+2}. However, $d(a^p) \geqslant n$, so $d(a) \geqslant n-1$. ■

Now let

$$0 \longrightarrow \mathbb{Z}_q \longrightarrow G \longrightarrow H \longrightarrow 1$$

be the extension with $|H| = p$, then the H-module of $\mathbb{Z}_q$ is given by a map $\varphi : H \to \mathrm{Aut}(\mathbb{Z}_q) \cong \mathbb{Z}_q^\times$. Since $|H|$ is prime, then φ is either trivial or injective.

If φ is trivial, then $h1h^{-1} = 1$ for all $h \in H$, so G is an abelian group. By the fundamental theorem of abelian groups, we know G is either $\mathbb{Z}_{p^{n+1}}$ or $\mathbb{Z}_q \times \mathbb{Z}_p$.

If φ is injective, then $n \geqslant 2$, otherwise the size of H is larger than the size of the units. Given some element $h \in H$ such that $h1h^{-1} = k$, then $k^p \equiv 1 \pmod{p^n}$. By Lemma 11.2, $k = 1 + p^{n-1}b$ for some $b \in \mathbb{Z}_p$. Because φ is injective, then the image of φ has size p, but every element in the image has the form of k, therefore the image is just the set of such elements. Let $e \in H$ be a generator such that $e1e^{-1} = 1 + p^{n-1}$. Now let $A = \mathbb{Z}_q$ with this H-module structure, and it suffices to show that $H^2(H, A) = 0$, then we have the semidirect product only.

Since H and A are both cyclic groups, we write down the periodic resolution to be

$$A \xrightarrow{e-1} A \xrightarrow{N} A \xrightarrow{e-1} A \xrightarrow{N} A \longrightarrow \cdots$$

where N is the norm element $\sum_{h \in H} h$. We know the action via $e-1$ on 1 is $(e-1)\cdot 1 = (1+p^{n-1}) - 1 = p^{n-1}$, so $\ker(e-1) = p\mathbb{Z}/q\mathbb{Z}$; the action via N is $N \cdot 1 = \sum_{b \in \mathbb{Z}_p} (1 + p^{n-1}b) \equiv p \pmod{p^n}$, therefore the image of the norm map is $\mathrm{im}(\mathbb{Z}) = p\mathbb{Z}/q\mathbb{Z}$ as well. Therefore, $H^2(H, A) = 0$. □

Corollary 11.3. If we have a p-group G with $p \neq 2$, then there is a unique subgroup of order p and a unique subgroup of index p.

Let H be a normal subgroup of G, then we consider the free $\mathbb{Z}[H]$-resolution

$$\mathbb{Z} \longleftarrow C_H^0 \longleftarrow C_H^1 \longleftarrow C_H^2 \longleftarrow \cdots$$

and we can try turning it into a free G-resolution of $\mathbb{Z}[G/H]$ by taking the tensor via

$$\mathbb{Z} \otimes \mathbb{Z}[G/H] \cong \mathbb{Z}/[G/H] \longleftarrow C_H^* \otimes \mathbb{Z}[G/H]$$

Because $\mathbb{Z}[H] \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G] \cong \mathbb{Z}[G]$, then we have

$$\mathbb{Z}[G/H] \cong \mathbb{Z} \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G] \longleftarrow C_H^* \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G]$$

Now given an arbitrary free $\mathbb{Z}[G/H]$-resolution and we want to map the given resolution to it.

$$\begin{tikzcd}
\mathbb{Z} & D^0_{G/H} \cong \mathbb{Z}[G/H] \arrow[l] & D^1_{G/H} \cong \mathbb{Z}[G/H]^m \arrow[l] & \cdots \arrow[l] \\
& C_H^* \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G] \arrow[u] & (C_H^* \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G])^m \arrow[u] \arrow[l, dashed] &
\end{tikzcd}$$

The vertical maps are resolved as G-modules by using the resolution of $\mathbb{Z}[G/H]$. We claim that there are horizontal maps that gives a double complex whose total complex is a resolution of $\mathbb{Z}$ as a G-module.

Example 11.4. Consider the dihedral group $D_{2n} \rhd C_n$, so $D_{2n}/C_n \cong C_2$. In particular, say D_{2n} is generated by τ of order n and T of order 2, so C_n is generated by τ and C_2 is generated by T. Consider the resolutions

$$D^* : \mathbb{Z} \longleftarrow \mathbb{Z}[T]/(T^2-1) \xleftarrow[T-1]{} \mathbb{Z}[T]/(T^2-1) \xleftarrow[T+1]{} \mathbb{Z}[T]/(T^2-1) \xleftarrow[T-1]{} \mathbb{Z}[T]/(T^2-1) \xleftarrow[T+1]{} \cdots$$

and

$$C^* : \mathbb{Z} \longleftarrow \mathbb{Z}[\tau]/(\tau^n-1) \xleftarrow[\tau-1]{} \mathbb{Z}[\tau]/(\tau^n-1) \xleftarrow[N_\tau]{} \mathbb{Z}[\tau]/(\tau^n-1) \xleftarrow[\tau-1]{} \cdots$$

and so on. Therefore we have an induced resolution given by

$$\mathbb{Z}[T]/T^2 \longleftarrow \mathbb{Z}[D_{2n}] \xleftarrow[\tau-1]{} \mathbb{Z}[D_{2n}] \xleftarrow[N_\tau]{} \mathbb{Z}[D_{2n}] \xleftarrow[\tau-1]{} \mathbb{Z}[D_{2n}] \xleftarrow[N_\tau]{} \cdots$$

Now let the sequence of $D^n_{G/H}$'s be of

$$\begin{tikzcd}
\mathbb{Z} & \mathbb{Z}[T]/T^2 \arrow[l] & \mathbb{Z}[T]/T^2 \arrow[l, "T-1"'] & \mathbb{Z}[T]/T^2 \arrow[l, "T+1"'] & \mathbb{Z}[T]/T^2 \arrow[l, "T-1"'] & \cdots \arrow[l, "T+1"'] \\
\cdots & \mathbb{Z}[D_{2n}] \arrow[l] \arrow[u] & \mathbb{Z}[D_{2n}] \arrow[l, "T-1"'] \arrow[u] & \mathbb{Z}[D_{2n}] \arrow[l, "T+1"'] \arrow[u] & \mathbb{Z}[D_{2n}] \arrow[l, "T-1"'] \arrow[u] & \cdots \arrow[l] \\
\cdots & \mathbb{Z}[D_{2n}] \arrow[u, "\tau-1"] & \mathbb{Z}[D_{2n}] \arrow[u, "\tau-1"] & \mathbb{Z}[D_{2n}] \arrow[u, "\tau-1"] & \mathbb{Z}[D_{2n}] \arrow[u, "\tau-1"] & \cdots \arrow[l] \\
\cdots & \mathbb{Z}[D_{2n}] \arrow[u, "N_\tau"] & \mathbb{Z}[D_{2n}] \arrow[u, "N_\tau"] & \mathbb{Z}[D_{2n}] \arrow[u, "N_\tau"] & \mathbb{Z}[D_{2n}] \arrow[u, "N_\tau"] & \cdots \arrow[l] \\
\cdots & \cdots \arrow[u] & \cdots \arrow[u] & \cdots \arrow[u] & \cdots \arrow[u] & \cdots
\end{tikzcd}$$

The horizontal maps are hard to construct, they may look like $\tau - 1$, but we need to introduce signs at certain places.

12 Sept 18, 2023

We will build the resolution out of this diagram, using double complexes, where horizontal differential ∂^v and vertical differential ∂^h satisfies $\partial^v\partial^h + \partial^h\partial^v = 0$ between $C^{i,j}$'s. There now exists a total complex Tot with

$$(\mathrm{Tot}^{\oplus}(C^{*,*}))_n = \bigoplus_{i+j=n} C^{i,j}$$

and

$$(\mathrm{Tot}^{\prod}(C^{*,*}))_n = \prod_{i+j=n} C^{i,j}$$

so each degree of the total complex is given by a collection of terms with the same fixed total degree. From the above, we have

$$\begin{tikzcd}
\cdots & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \\
\cdots & \mathbb{Z}[D_{2n}] \arrow[l] \arrow[d, "T+1"] & \mathbb{Z}[D_{2n}] \arrow[l, "\tau-1"'] \arrow[d, dashed] & \mathbb{Z}[D_{2n}] \arrow[l, "N_\tau"'] \arrow[d, dashed] & \mathbb{Z}[D_{2n}] \arrow[l, "\tau-1"'] \arrow[d, dashed] & \cdots \arrow[l] \\
\cdots & \mathbb{Z}[D_{2n}] \arrow[l] \arrow[d, "T-1"] & \mathbb{Z}[D_{2n}] \arrow[l, "\tau-1"'] \arrow[d, dashed] & \mathbb{Z}[D_{2n}] \arrow[l, "N_\tau"'] \arrow[d, dashed] & \mathbb{Z}[D_{2n}] \arrow[l, "\tau-1"'] \arrow[d, dashed] & \cdots \arrow[l] \\
\cdots & \mathbb{Z}[D_{2n}] \arrow[l] \arrow[d, dashed] & \mathbb{Z}[D_{2n}] \arrow[l, "\tau-1"'] \arrow[d, dashed] & \mathbb{Z}[D_{2n}] \arrow[l, "N_\tau"'] \arrow[d, dashed] & \mathbb{Z}[D_{2n}] \arrow[l, "\tau-1"'] \arrow[d, dashed] & \cdots \arrow[l] \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{tikzcd}$$

One can fill in the diagram so that each square anticommutes, so that this becomes a double complex.

Example 12.1. If we calculate $H^*(D_{2n}, \mathbb{F}_2)$, we would find the differentials of the total complex to be zero, therefore the cohomology (after taking $\operatorname{Hom}(C^{*,*}, \mathbb{F}_2)$) is just determined by the number of copies in the total complex, enumerated on $\mathbb{F}_2$.

If we think of the quaternions Q_8 instead, with the presentation $\langle \tau, T \mid \tau^2 = T^2 = (\tau T)^2, \tau^4 = 1 \rangle$, then we obtain

$$\begin{tikzcd}
\cdots & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \\
\cdots & \mathbb{Z}[Q_8] \arrow[l] \arrow[d, "T+1"] & \mathbb{Z}[Q_8] \arrow[l, "\tau-1"'] \arrow[d, dashed] & \mathbb{Z}[Q_8] \arrow[l, "N_\tau"'] \arrow[d, dashed] & \mathbb{Z}[Q_8] \arrow[l, "\tau-1"'] \arrow[d, dashed] & \cdots \arrow[l] \\
\cdots & \mathbb{Z}[Q_8] \arrow[l] \arrow[d, "T-1"] & \mathbb{Z}[Q_8] \arrow[l, "\tau-1"'] \arrow[d, dashed] & \mathbb{Z}[Q_8] \arrow[l, "N_\tau"'] \arrow[d, dashed] & \mathbb{Z}[Q_8] \arrow[l, "\tau-1"'] \arrow[d, dashed] & \cdots \arrow[l] \\
\cdots & \mathbb{Z}[Q_8] \arrow[l] \arrow[d, dashed] & \mathbb{Z}[Q_8] \arrow[l, "\tau-1"'] \arrow[d, dashed] & \mathbb{Z}[Q_8] \arrow[l, "N_\tau"'] \arrow[d, dashed] & \mathbb{Z}[Q_8] \arrow[l, "\tau-1"'] \arrow[d, dashed] & \cdots \arrow[l] \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{tikzcd}$$

To make this a complex, we need to add notions of differentials, where we find a nullhomotopic map so that given a term in some degree and any term in the following degree, there exists a differential from the former to the latter.

13 Sept 20, 2023

We think of $H \lhd G$ with $G \twoheadrightarrow G/H$, then as we discussed before there are chains

$$\begin{tikzcd}
\mathbb{Z} & \mathbb{Z}[G/H] \arrow[l] & \cdots \arrow[l] \\
& \mathbb{Z}[G] \arrow[u] & \\
& \vdots \arrow[u] &
\end{tikzcd}$$

and therefore this gives an anti-commute square

$$\begin{tikzcd}
C_{i,j} & C_{i+1,j} \arrow[l, "\partial_h"'] \\
C_{i,j+1} \arrow[u, "\partial_v"] & C_{i+1,j+1} \arrow[l, "\partial_h"] \arrow[u, "\partial_v"']
\end{tikzcd}$$

where ∂_v and ∂_h are G-equivariant.

Theorem 13.1. In this situation, there are equivariant maps, where $d_0 = \partial_v : C_{i,j} \to C_{i,j-1}$, $d_2 : C_{i,j} \to C_{i-2,j+1}$, and so on, with $d_r : C_{i,j} \to C_{i-r,j+r-1}$, so that these differentials commute with the augmentation maps $\varepsilon_i : C_{i,0} \to B_i$, that is, $\varepsilon d_1^C = d_1^B \varepsilon$ and such that

$$\cdots \xrightarrow{\sum d_r} \bigoplus_{i+j=n} C_{i,j} \xrightarrow{\sum d_r} \bigoplus_{i+j=n-1} C_{i,j} \xrightarrow{\sum d_r} \cdots$$

is a free resolution of the trivial G-module $\mathbb{Z}$.

We will filter $C_{*,*}$ by $(F^p C_{*,*})_n = \bigoplus_{i+j=n, i \geqslant p} C_{i,j}$, then $\operatorname{gr}^p = F^p/F^{p+1}$, so the filtration (horizontally/vertically) gives a spectral sequence with page 2 as $E_2^{p,q} = H^p(G/H, H^q(H, M))$.

Example 13.2. Consider

$$0 \longrightarrow C_4 \longrightarrow Q_8 \longrightarrow C_2 \longrightarrow 0$$

with B_* given by $\mathbb{Z}[C_2]$'s, and $C_{i,j} = \mathbb{Z}[Q_8]$. The E_2-page is now $H^p(C_2, H^q(C_4, \mathbb{Z}/2\mathbb{Z}))$, and as τ acts trivially on the resolution, then $d_2 = \pm(\tau + 1)$ is the zero map on the spectral sequence. One can show that $d_3 = \pm T$. There will then be periodicity on the picture for d_4 and so on.

Now the spectral sequence gives us $H^p(G/H, H^q(H, M)) \Rightarrow H^{p+q}(G, M)$, and therefore the E_∞-page, with $\mathrm{gr}^* H^{p+q} \cong \bigoplus_{p+q} E_\infty^{p,q}$. In the example above we see $H^0(Q_8, \mathbb{Z}_2) \cong \mathbb{Z}_2$ since the filtration ends there; $\mathrm{gr}^* H^1(Q_8, \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$; $\mathrm{gr}^* H^2(Q_8, \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$; $H^3 = \mathbb{Z}/2\mathbb{Z}$. This describes a general picture of H^{4k+i}, and we can remove the graded version and yields the same result.

14 Sept 22, 2023

We think of how $H^p(G/H, H^q(H, M))$ turns into $H^{p+q}(G, M)$. We know $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$, and we consider total degree n.

- If $n = 0$, then $H^0(G/H, H^0(H, M)) \cong H^0(G, M)$.
- If $n = 1$, then we have a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^1(G/H, H^0(H, M)) \arrow[r, "inf"] & H^1(G, M) \arrow[r, "res"] \arrow[d] & H^0(G/H, H^1(G, M)) \arrow[r, "d_2"] & H^2(G/H, H^0(H, M)) \arrow[r, "inf"] \arrow[d] & H^2(G, M) \arrow[r, "\alpha"] & Q \arrow[r] & 0 \\ & & \ker(d_2) \arrow[ur] & & \mathrm{coker}(d_2) \arrow[ur] & & & \end{tikzcd}$$

More generally, we get a filtration on $H^n(G, M)$ with associated grading $E_\infty^{p,n-p} \cong E_R^{p,n-p}$ for some $R \gg 0$. In the exact sequence above, we obtain

$$0 \longrightarrow H^1(G/H, H^0(H, M)) \cong E_\infty^{1,0} \xrightarrow{inf} H^1(G, M) \longrightarrow \ker(d_2) \cong E_\infty^{0,1} \longrightarrow 0$$

and correspondingly $\mathrm{coker}(d_2) = E_\infty^{2,0}$ with Q given by

$$ker(d_2^{1,1}) \cong E_\infty^{1,1} \hookrightarrow Q \xrightarrow{\pi} \ker(d_3)^{0,2} \cong E_\infty^{0,2}$$

so that $res = \pi\alpha$. The edge maps are given by

$$\begin{tikzcd} E_\infty^{n,0} \arrow[r, hook] & H^n(G, M) \\ E_2^{n,0} \arrow[u, two heads] \arrow[r, equal] & H^n(G/H, H^0(H, M)) \arrow[u, "inf"'] \end{tikzcd}$$

and

$$\begin{tikzcd} H^n(G, M) \arrow[r, two heads] \arrow[dr, "res"'] & E_\infty^{0,n} \arrow[d, hook] \\ & H^0(G/H, H^n(H, M)) \end{tikzcd}$$

Example 14.1. Consider giving $H^p(C_2, H^q(C_2, \mathbb{Z}_2))$ to $H^{p+q}(C_4, \mathbb{Z}_2)$. The thing we want to calculate is the spectral sequence of

$$C^{p,q} = X^p(G/H, X^q(G, M)^H)^{G/H}.$$

Given $f_i \in C^{p_i,q_i}$, we take

$$C^{p_1,q_1} \times C^{p_2,q_2} \xrightarrow{\smile} X^{p_1+p_2}(G/H, X^{q_1}(G, M)^H \otimes X^{q_2}(G, M)^H)^{G/H} \xrightarrow{\smile} X^{p_1+p_2}(G/H, X^{q_1+q_2}(G, M)^H)^{G/H}$$

and so $d_r(x \smile y) = d_r(X) \smile y + (-1)^{|x|} x \smile d_r(y)$. Therefore this satisfies some kind of Leibniz's rule. We conclude that $E_2^{*,*} \cong \mathbb{F}_2[x, y]$. Therefore the arrows takes on grid other than ones of the form x^{2n} and $x^{2n}y$, which is given by the E_3-page and beyond. We conclude that $E_4 \cong E_\infty = \mathbb{F}_2[x^2] \otimes \bigwedge(y)$.

15 Sept 25, 2023

We will work over $\mathbb{F}_2$-coefficients today. We were trying to calculate the spectral sequence via

$$1 \longrightarrow C_2 \longrightarrow C_{2^n} \longrightarrow C_{2^{n-1}} \longrightarrow 0$$

Here $H^*(C_2) = \mathbb{F}_2[x]$ where $|x| = 1$.

Proposition 15.1. $H^*(C_{2^n}) \cong \mathbb{F}_2[x_n, y_n]/(x_n^2)$ for some $x_n \in H^1$ and $y_n \in H^2$ and $n > 1$.

On the E_2-page, we need to move $(0,1)$ to somewhere so that the total degree 1 would have only one piece of information, so we move $(0,1)$ to $(2,0)$, and similarly $(n,1)$ to $(n+2,0)$. In general, $E_\infty^{*,*} \cong E_3^{*,*} \cong \mathbb{F}_2[x^2] \otimes \mathbb{F}_2[x_{n-1}]/x_{n-1}^2$. We identify the column of $p = 1$ to be x_{n-1} and column of $p = 2$ to be y_{n-1} and we identify $y_{n-1} = x_{n-1}^2$. In general, $[f] \in E_\infty^{p,q}$ is equivalent to $F^pH^*(G)/F^{p+1}H^*(G)$, and given also $[f'] \in E_\infty^{p',q'}$ for, then $[f][f'] \in E_\infty^{p+p',q+q'}$, then $[ff'] = [f][f']$ modulo $F^{p+p'+1}H^*(G)$.

The edge maps are

$$H^k(G/H) \cong H^k(C_{2^{n-1}}) \xrightarrow{inf} H^k(G) \cong H^k(C_2) \xrightarrow{res} H^k(H) \cong H^k(C_2)$$

where inf is an isomorphism for $k = 0, 1$ and zero otherwise, and res is an isomorphism for even k, and is zero otherwise.

Note that if $G = \lim_i G_i$ for finite groups G_i's, then $H^*(G) \cong \operatorname{colim}_{i,inf} H^*(G_i)$.

Corollary 15.2. $H^*(\mathbb{Z}_2; \mathbb{F}_2) \cong \mathbb{F}_2[x]/x^2$ for $x \in H^1$.

If we think of $H^*(D_{2n})$, then we already have $C_{2^{n-1}} \to D_D2^n \to C_2$, so $H^p(C_2, H^q(C_{2^{n-1}})) \Rightarrow H^*(D_{2^n})$ already collapses. For $n = 1$, we have C_2; for $n = 2$, we have $C_2 \times C_2$ and resolve the cohomology by Kunneth isomorphism $H^*(C_2 \times C_2) \cong \mathbb{F}_2[x, y]$ for $x, y \in H^1$. For $n \geqslant 3$, $E_2^{*,*} \cong H^*(C_2) \otimes H^*(C_{2^{n-1}}) \cong \mathbb{F}_2[e] \otimes \mathbb{F}_2[x]/x^2 \otimes \mathbb{F}_2[y]$. Since higher pages vanishes, this is also $E_\infty^{*,*}$. Let $\mathcal{X} = [x] \in H^1(D_{2^n})$, and $\mathcal{Y} = [y]$ and $\mathcal{E} = [e]$, then $\mathcal{X}^2 \in \mathbb{F}_2\{\mathcal{E}\mathcal{X}, \mathcal{E}^2\}$. Eventually this would be hard to compute, so we would look at something different.

If we think of $D_8 \cong \langle T, \tau \mid T^2 = 1 = \tau^4, T\tau T = \tau' \rangle$, then we have $C_2 \cong \langle \tau^2 \rangle \to D_8 \to C_2 \times C_2$. Similarly, $E_2 \cong \mathbb{F}_2[e] \otimes \mathbb{F}_2[x, y]$, where e^i's are on position $(1, i+1)$ and $d_2(e) = \alpha x^2 + \beta y^2 + \gamma xy$, so we obtain maps of spectral sequences to our sequence $C_2 \cong \langle \tau^2 \rangle \to D_8 \to C_2 \times C_2$, including

$$C_2 \longrightarrow C_2 \times C_2 \longrightarrow C_2 = \langle \tau T \rangle$$

$$C_2 \cong \langle \tau^2 \rangle \longrightarrow C_4 \longrightarrow C_2 \cong \langle \tau \rangle$$

$$C_2 \longrightarrow C_2 \times C_2 \longrightarrow C_2 \cong \langle \tau \rangle$$

When we say a map of spectral sequences we mean $f^* : E_r^{*,*} \to \tilde{E}_r^{*,*}$ by sending $d_r(x)$ to $d_r(f^*x)$, as maps of differential graded algebras. From one of the sequence above, we obtain

$$H^*(C_2, H^*(C_2)) \Rightarrow H^*C_2 \times C_2)$$

with $d_2(e) = 0$. Take our original sequence with $H^*(C_2, H^*(C_2 \times C_2)) \Rightarrow H^*(D_8)$, we send this to above by $e \mapsto e$, $x \mapsto x$, and $y \mapsto 0$, then by naturality (as we compare with the sequence above), we note $d_2(e) = \alpha x^2 + \beta y^2 + \gamma xy$ where $\alpha = 0$; similarly we note $\beta = 0$ by comparing with another sequence. Therefore $d_2(e) = \gamma xy$.

16 Sept 27, 2023

The cohomology rings $H^*(G, F)$ we referred to today are with respect to $F = \mathbb{F}_p$ where p is a prime.

Theorem 16.1 (Evans-Venkov Theorem)**.** For any finite group G, the cohomology ring $H^*(G; \mathbb{F}_p)$ is Noetherian.

Proof. Suppose we know this holds for p-groups, then for an arbitrary group G, take its Sylow p-subgroup $P \subseteq G$. The cohomology rings give a restriction $\mathrm{res} : H^*(G) \to H^*(P)$ where $H^*(P)$ is Noetherian. By assumption, we know $\mathrm{tr} : H^*(P) \to H^*(G)$ is the backwards mapping, and that $\mathrm{tr} \circ \mathrm{res} = [G : P]$, therefore this is an isomorphism. The transfer is then surjective and the restriction is injective. Therefore, $H^*(G)$ is the subring of a Noetherian ring, then $H^*(G)$ is Noetherian, as the retraction tr is fully faithful. Alternatively, we can show that $I_1 \subseteq I_2 \subseteq \cdots \subseteq H^*(G)$ stabilizes: we note that

$$\mathrm{res}(I_1) \smile H^*(P) \subseteq \mathrm{res}(I_2) \smile H^*(P) \subseteq \cdots \subseteq H^*(P)$$

stabilizes. Let $x \in \mathrm{res}(I_k) \smile H^*(P)$, i.e., $x = \mathrm{res}(a_k) \smile b$ for some choices of a_k and b. Taking the transfer, we have $\mathrm{tr}(x) = \mathrm{tr}(\mathrm{res}(a_k) \smile b) = a_k \smile \mathrm{tr}(b)$. The point being I_k's and $(\mathrm{res}(I_k) \smile H^*(P)$ are now composes to be an isomorphism, therefore we identify them to be the same. In particular, if $a_j \in I_k \backslash I_{k-1}$, so taking the restriction we end up in $\mathrm{res}(I_{k-1}) \smile H^*(P)$, then sending it back via trace multiplies it by a unit, so it should end up in I_{k-1} again.

We now need to show that $H^*(P)$ is Noetherian for all finite p-groups P. By an induction on order of P, for $H^*(C_p) = \wedge(e) \otimes \mathbb{F}_p[y]$, and given a central extension $C_p \triangleleft P \twoheadrightarrow \bar{P}$, we need to show that the statement holds for P given it holds for $\bar{P}$. We consider the spectral sequence $E_2^{i,j} : H^i(\bar{P}, H^j(C_p)) \Rightarrow H^{i+j}(P)$, the $\bar{P}$-action on $H^j(C_p)$ is trivial since every action of p-group on $\mathbb{F}_p$ is always trivial, therefore the E_2-page decomposes as the tensor product of two cohomology rings, so $E_2^{*,*} = H^*(\bar{P}) \otimes_{\mathbb{F}_p} H^*(C_p) = H^*(P)[e, y]/e^2$. $E_2^{*,*}$ is Noetherian as a tensor product of two Noetherian rings. One can show that

- by induction, we can show that $E_r^{*,*}$ is Noetherian (the kernel of each d_r map will be finitely-generated over $E_r^{*,0}$ as an algebra), and
- moreover, there is $N \gg 0$ such that $E_N^{*,*} \cong E_\infty^{*,*}$.

It then allows us to conclude that E_∞ is Noetherian, hence $H^*P)$ is Noetherian as well. ☐

Suppose we have a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of G-modules, then we obtain $H^k(G, C) \to H^{k+1}(G, A)$ as a connecting homomorphism.

Example 16.2. Consider

$$0 \longrightarrow \mathbb{Z}_p \longrightarrow \mathbb{Z}_{p^2} \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

then we obtain Bockstein $\beta : H^k(G, \mathbb{Z}/p) \to H^{k+1}(G, \mathbb{Z}_p)$. So we have $\beta : H^*(G, \mathbb{F}_p) \to H^{*+1}(G, \mathbb{F}_p)$. This map is

- natural in G;
- a derivation, i.e., $\beta(x \smile y) = \beta x \smile y + (-1)^{|x|} x \smile \beta y$;
- $\beta^2 = 0$.

These are called the Steenrod operations, with $P^0 = \mathrm{id} : H^*(G) \to H^*(G)$, and $P^i : H^(G) \to H^{*+2(p-1)i}(G)$, satisfying

1. if $|x| = 2k$, then $P^k(x) = x^p$,
2. if $|x| < 2k$, then $P^k(x) = 0$, and
3. $P^k(x \smile y) = \sum\limits_{i=0}^{k} (P^i x) \smile (P^{k-i} y)$.

Example 16.3. For example, $H^*(C_p) \cong \wedge(e) \otimes \mathbb{F}_p[y]$, with $\beta(e) = y$, $\beta(y) = 0$, and $p^1(y) = y^p$.

17 Sept 29, 2023

Let p be odd, and all coefficients are over the field $\mathbb{F}_p$. The Steenrod operations P^i for $i \geqslant 0$ is given by

$$P^i : H^m(-) \to H^{m+2(p-1)i}(-)$$

satisfying

1. $P^2 = \mathrm{id}$;
2. if $|x| = 2n$, then $P^n x = x^p$;
3. if $|x| < 2n$, then $P^n x = 0$;
4. $P^n(x \smile y) = \sum_{i+j=n} P^i x \smile P^j y$,

as well as the algebraic relations, e.g., $P^1 P^1 = 2P^2$, as Adem relations.

Definition 17.1 (Steenrod Algebra)**.** The Steenrod algebra is $A^* = \mathbb{F}_p \left\langle \beta, P^i, i \geqslant 1 \right\rangle / \sim$, where $\sim$ is given by Adem relations.

Definition 17.2 (Milnor's Q_i-operations)**.** Denote $Q_0 = \beta$, $Q_i = [P^{p^{i-1}}, Q_{i-1}]$, e.g., $Q_1 = [P^1, \beta] = P^1\beta - \beta P^1$; $Q_2 = [P^p, P^1\beta - \beta P^1] = P^p P^1 \beta + \cdots$. The key fact is that $Q_i(x \smile y) = (Q_i x) \smile y + (-1)^{|Q_i||x|} x \smile Q_{i-1}$.

Example 17.3. $H^*(C_p)$ is the exterior algebra $\bigwedge(x) \otimes \mathbb{F}_p[y]$ where $|x| = 1$ and $|y| = 2$, with $\beta x = y$. Then $Q_1 x = (P^1\beta - \beta P^1)(x) = y^p$; $P^p y^p = y^{p^2} = Q_2 x$. In general, $Q_i x = y^{p^i}$.

Definition 17.4 (Fiber Bundle, Principal Bundle)**.** A fiber bundle is the diagram $F \to E \xrightarrow{\pi} B$, where B is the base space, E is the total space, and F is the fiber, such that for any $b \in B$, there exists a neighborhood U of b such that $\pi^{-1}(U) \simeq U \times F$, with certain compatibility.

A principal G-bundle is a fiber bundle with fiber G. In this case, E inherits a free G-action.

Remark 17.5. If G is a finite group, then this gives a finite covering.

For a nice enough group G, there is a classifying space BG characterized by the fact that if X is a CW complex, then homotopy classes of map from X to BG, denoted $[X, BG]$, correspond to the principal G-bundles over X, such that there is a universal principal G-bundle

$$\begin{array}{c} EG \\ \downarrow \\ BG \end{array}$$

where EG is contractible, with the universal property that given $f : X \to BG$, there is a pullback f^*EG with respect to these maps.

Remark 17.6.

- If G is a finite group, then $\pi_k(BG) = \begin{cases} G, k = 1 \\ 0, k \neq 1 \end{cases}$ and therefore $BG = K(G, 1)$.
- For a group A and integer $n \geqslant 0$, $K(A, n)$ is a space with

$$\pi_m(K(A, n)) = \begin{cases} A, m = n \\ 0, m \neq n \end{cases}$$

If $n \geqslant 2$, A needs to be abelian for these structures to exist.

Example 17.7.

1. $B(G \times H) = BG \times BH$.
2. If $G = H \rtimes K$, then the classifying space BG is isomorphic to the fiber product $BH \times_K EK = (BH \times EK)/\Delta$ with respect to the diagonal K-action Δ.

3. Let $H^n = \prod_n H$ be a product of n copies of H. Permuting these H's gives an action Σ_n on H^n, then there is the wreath product $H^n \rtimes \Sigma_n = H \wr \Sigma$. The classifying space $B(H \wr \Sigma_n) \simeq (BG)^n \times_{\Sigma_n} E\Sigma_n$. More generally, for a space X, we can permute the copies and get a fiber bundle

$$\begin{array}{c} X^n \times_{\Sigma_n} E\Sigma_n \\ \big\downarrow \\ B\Sigma_n \end{array}$$

where $F = X^n$. This bundle has a section

$$\begin{aligned} s: B\Sigma_n &\to X^n \times_{\Sigma_n} E\Sigma_n \\ s_x(y) &= (x, \dots, x, \tilde{y}). \end{aligned}$$

Definition 17.8 (Serre Spectral Sequence)**.** Given a fiber bundle $F \to E \to B$, there is a spectral sequence given by $H^i(B, H^j(F)) \Rightarrow H^{i+j}(E)$.

Example 17.9. For $H \lhd G$, the sequence $BH \to BG \to B(G/H)$ gives the Lyndon-Hochschild spectral sequences.

Example 17.10. Consider $X^p \to X^p \times_{C_p} EC_p \to BC_p$, it gives

$$H^i(BC_p, H^j(X^p)) \Rightarrow H^{i+j}(X^p \times_{C_p} EC_p).$$

We have

$$H^*(BC_p, H^*(X^p)) \Rightarrow H^*(X^p \times_{C_p} EC_p).$$

where $H^*(X^p) \cong H^*(X)^{\otimes p}$, which decomposes as a direct sum of free and trivial terms. Let $C_p = \langle T \rangle / (T^p - 1)$. The free terms are generated by the image of $1 + T + \cdots + T^{p-1}$, and the trivial terms are of the form $x \otimes \cdots \otimes x$, i.e., fixed by the permutation action on C_p.

18 Oct 2, 2023

Again, we work on cohomology with coefficients in $\mathbb{F}_p$.

Let Σ_n act on X^n for some space X. (Similarly, the action of C_n on X^n gives $X^n \times_{C_n} EC_n$) The space $X^n \times_{\Sigma n} E\Sigma_n$ has a free contractible Σ_n-space as Σ_n-fiber $X^n \times E\Sigma_n$. For instance, define $H2\Sigma_n = H^n \rtimes \Sigma_n$, then $B(H2\Sigma_n) = (BH)^n \times_{\Sigma_n} E\Sigma_n$. We will show that the spectral sequence for these collapses at E_2-page. Note that given a fibration $F \to E \to B$, there is a spectral sequence $H^i(F, H^j(B)) \Rightarrow H^{i+j}(E)$, for instance take $H \lhd G \to G/H$, then we have a fibration $BH \to BG \to B(G/H)$. For instance, take the fibration $X^n \to X^n \times_{\Sigma_n} E\Sigma_n \xrightarrow{\pi} B\Sigma_n$. This gives a spectral sequence $H^i(\Sigma_n, H^j(X)^{\otimes n}) \Rightarrow H^{i+j}(X^n \times_{\Sigma_n} E\Sigma_n$. Note that π has a section $s(y) = (x, \dots, x, \tilde{y})$. Looking at the edge homomorphisms $\pi^* : H^i(B\Sigma_n) \to E_\infty^{i,0} \to H^i(X^n \times_{\Sigma_n} E\Sigma_n)$, there is also a retraction hence $d_r : E_r^{*,*} \to E_r^{i,0}$'s are zero.

Let G be a finite group, then $BG = K(G, 1)$, so by definition $\pi_n(BG)$ is G if $n = 1$ and is zero otherwise. If A is abelian group, then there are (Eilenberg-Maclane) spaces $K(A, n)$ for all $n \geqslant 0$, with $\pi_k(K(A, n))$ being A if $n = k$ and is zero otherwise.

Remark 18.1.

- there is a fibration $K(A, n-1) \to E \to K(A, n)$ where E is contractible. Therefore, $K(A, n-1)$ is the loop space on $K(A, n)$.
- If X is a space and A is an abelian group, then $H^n(X; A)$, as a representable functor, is given by the homotopy classes $[X, K(A, n))]$ of maps of spaces.
- $K(A, n)$ is an ∞-loop space.
- $\tilde{H}^m(\mathbb{F}_p, j)$ is 0 if $m \leqslant j$, is $\mathbb{F}_p\{\iota_j\}$ if $m = j$.

Consider $X^p \to X^p \times_{C_p} EC_p \to BC_p$, so we have $H^i(BC_p, H^j(X)^{\otimes p}) \Rightarrow H^*(X^p \times_{C_p} EC_p)$.

Lemma 18.2. Let V be an $\mathbb{F}_p$-vector space, and let $V^{\otimes p}$ be a space with cyclic permutation acting upon it, then $V^{\otimes p}$ is isomorphic to a direct sum of free and trivial portions via action by C_p. The trivial portion is generated by the diagonal image $(v \otimes \cdots \otimes v)$ for some $v \in V$; the free portion is generated by the image of $(1 + T + \cdots + T^{p-1}) = N_T$, if we consider $C_p = \langle T \rangle$.

Remark 18.3. $H^*(X)^{\otimes p} = \bigoplus_{j_1+\cdots+j_p} H^{j_1}(X) \otimes H^{j_2}(X) \otimes \cdots \otimes H^{j_p}(X)$ and so $H^*(C_p, V^{\otimes p}) = H^0(C_p, V^{\otimes p}) \oplus \cdots \oplus H^*(C_p, \mathrm{diag})$, where the first terms are image of norm maps, and the last term is the portion representing the fixed points.

Exercise 18.4. Show that classes in $H^0(C_p, H^*(X^{\otimes p}))$ which are in the image of transfer are permanent cycles.

What about $H^0(C_p, \mathbb{F}_p\{w\otimes \cdots \otimes w\}) \subseteq H^*(X)^{\otimes C_p}$? Let $w \in H^j(X)$, so w is represented by $f_w : X \to K(\mathbb{F}_p, j)$, so the pullback $f_w^*(\iota_j) = w$. We have a fiber diagram

$$\begin{tikzcd}
X^p \arrow[r] \arrow[d, "f_w^p"'] & X^p \times_{C_p} EC_p \arrow[r] \arrow[d] & BC_p \arrow[d, equal] \\
K(\mathbb{F}_p, j) \arrow[r] & K(\mathbb{F}_p, j) \times_{C_p} EC_p \arrow[r] & BC_p
\end{tikzcd}$$

We interpret this as having the first few rows above the zeroth row as $K(\mathbb{F}_p, j)$, so all differentials vanishes in this class: in the reduced cohomology, we see the cohomology starts at $m = j$, everything below would be the image of transfer map, which gives as free summands and has no higher cohomology. Hence, the first non-zero differential would have been $\iota_j^{\otimes p}$ onto the zeroth row, but this is not allowed since it has no higher cohomology, so when we pullback w, we have $d_r(i_j^p) = 0$ and therefore $d_r(w^{\otimes p}) = 0$. By Leibniz rule, everything vanishes since this generates everything.

19 Oct 4, 2023

Theorem 19.1 (Evans-Venkos)**.** $H^*(G, \mathbb{F}_p)$ is Noetherian if G is a finite group.

Proof. We reduce the proof to p-groups and induct on orders of G. This works for C_p as a base case. We can also extend $C_p \triangleleft E \twoheadrightarrow G$ for some G with a smaller order than E, then there is a spectral sequence by $H^i(G, H^j(C_p)) \Rightarrow H^{i+j}(E)$. To run the induction, we need to know that

Proposition 19.2. The spectral sequence above collapses at a finite stage.

Subproof. Given $C_p \triangleleft E \twoheadrightarrow G$, we can write $E = \prod_{i=1}^{|G|} g_i C_p$ for some $g_i \in E$ as coset representatives of E/G. Note that this extension is central so the action on C_p is trivial, but not trivial on E. Now $h \in G$ will permute the $g_i C_p$'s, so there is a group homomorphism $G \to \Sigma_{|G|}$, hence $C_p^{|G|} \rtimes \Sigma_{|G|} = C_p \wr \Sigma_{|G|} \hookleftarrow E$, and

$$\begin{tikzcd}
C_p^{|G|} \arrow[r] & C_p \wr \Sigma_{|G|} \arrow[r] & \Sigma_{|G|} \\
C_p \arrow[r] \arrow[u, "\Delta"] & E \arrow[r] \arrow[u] & G \arrow[u]
\end{tikzcd}$$

Therefore this gives a mapping of spectral sequences, from $H^*(\Sigma_{|G|}, H^*(C_p^{|G|})) \Rightarrow H^*(C_p \wr \Sigma_{|G|})$ to $H^*(G, H^*(C_p)) \Rightarrow H^*(E)$. Now $H^*(G)$ is $\mathbb{F}_p[x]/(x^2) \otimes \mathbb{F}_p[y]$ where $|x| = 1$ and $|y| = 2$. Therefore, $H^*(G, H^*(G)) \cong H^*(G) \otimes \mathbb{F}_p[x, y]/(x^2)$. Recall that the first spectral sequence collapses at E_2, and we want to see the second spectral sequence collapses at finite stage. Also note that $H^*(G)$, the bottom row of the spectral sequence, is all zeros, so we need to find the action on $\mathbb{F}_p[x, y]/(x^2)$. This corresponds to the zeroth column of the spectral sequence. Since $y^{|G|} = f^*(y^{\otimes |G|})$, then $y^{|G|}$ is a permutation cycle in the spectral sequence $H^*(G, H^*(C_p)) \Rightarrow H^*(E)$. Hence, $E_\infty^{*,*} \cong \mathbb{F}_p[y^{|G}] \otimes \left(\bigoplus_{j<2|G|} E_\infty^{i,j} \right)$.

The rows are now $y^{|G|}$-cyclic, i.e., $1, x, y, xy, \ldots, y^{|G|}$, and arrows cannot cross this cycle anymore, since it is cyclic and would end up in the same class again. Therefore, the spectral sequence collapses at the $2|G|$-page. ■

□

Definition 19.3. An elementary abelian p-group is of the form $C_p^{\times r}$.

If G is a finite group, then we can approximate the spectral sequence over G by these elementary abelian p-groups.

Theorem 19.4 (Quillen)**.** If $w \in H^*(G)$ is such that the restriction $\operatorname{res}(w) \in H^*(V)$ for all elementary abelian subgroup V of G is nilpotent, then w is nilpotent.

Proof. It suffices to show that if $\operatorname{res}(w) = 0 \in H^*(V)$ for all V, then w is nilpotent. This is because $H^*(V) = \mathbb{F}_p[y_1, \ldots, y_r] \otimes \wedge(x_1, \ldots, x_r)$, so any nilpotent element in $H^(V)$ squares to zero.

We can reduce this to the case where G is a p-group. If $w \in H^*(G)$ is nilpotent, then the transfer $\operatorname{tr}(w) \in H^(P)$ into Sylow p-subgroup is nilpotent, and vice versa (invertible).

We have an extension $H \lhd G \to C_p$, so we assume inductively we know the result for H. Take $w \in H(G)$, then $\operatorname{res}(w)$ to elementary abelian groups is nilpotent, so by the inductive procedure we know $\operatorname{res}(w) \in H^*(H)$ is nilpotent, then take w to some power and the restriction in $H^*(H)$ would become zero. Therefore, we just need to show that if $w \in \ker(\operatorname{res}(H^*(G) \to H^*(H))$, then w is nilpotent.

If we regard $H^*(H)$ of C_p as the zeroth column in the spectral sequence, then for $w \in \ker(\operatorname{res}^G_H)$, $w \in F^1H^*(G)$, where F^i is the filtration on columns i and higher. □

20 Oct 6, 2023

Recall:

Theorem 20.1. Let G be a finite group, then if $w \in H^*(G)$ is such that w restricts to a nilpotent element in the cohomology of elementary abelian subgroups of G, then w is nilpotent. That is, $\operatorname{res}: H^*(G) \to \varprojlim_{V \subseteq G} H^*(V)$ where V's are elementary abelian, then kernel consists of nilpotent elements. That is, res is an f-isomorphism.

Proof. We reduced the proof to the case of p-groups, and we proceed inductively on $H \hookrightarrow G \to C_p$. If we consider the spectral sequence of $H^*(C_p, H^j(H)) \Rightarrow H^{i+j}(G)$, then the firs trow of the diagram would be $1, x, y, xy, y^2, \ldots$, and note that every term starting from 2 has a factor of y.

Note that for any Γ-module M, M an $\mathbb{F}_p$-vector space, then $H^*(\Gamma, M)$ is a module over $H^*(\Gamma, \mathbb{F}_p)$, i.e., $M \otimes_{\mathbb{F}_p} \mathbb{F}_p \cong M$, then $H^*(C_p, H^i(H))$ is a module over $H^*(C_p) \cong \bigwedge(x) \otimes \mathbb{F}_p[y]$, then

Claim 20.2. $E_2^{i \geqslant 2,*} = F^2(H^*(G)) \subseteq (y)$.

We need to show that if $w \in \ker(\operatorname{res}(H^*(G) \to H^*(H))$, then w is nilpotent. The kernel of the restriction would be $F^1(H^*(G))$, so whenever w is in the kernel of the restriction, $w^2 \in F^2H^*(G)$. Run an induction on r to show $\smile [y] : E_r^{i,j} \to E_r^{i+2,y}$ is surjective for all i, j. This means some power of w will be divisible by the image of some class in $H^1(G)$ over Bockstein β. Therefore, some power of w is divisible by all $\beta(H^1(G))$. (Note that $H^1(G) = \operatorname{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ where G is a p-group, so this is non-trivial.) Therefore, this power of w is a product of (βx_i)'s. To see this, we note $H_i \to G \xrightarrow{x_i} C_p$ has x_i's as generators of $H^1(G)$. Let $w \in H^(G)$, then we can assume inductively that some power of w restricts to 0 in every proper subgroup. From the spectral sequence for $H_i \lhd G \xrightarrow{x_i} C_p$, then this power of w is $(\beta x_i) \cdots$.

Lemma 20.3. Let G be a p-group. Then G is not elementary abelian if and only if there are non-zero classes $v_1, \ldots, v_k \in H^1(G)$ such that $\beta(v_1)\beta(v_k) = 0$.

Subproof. Consider $G' = [G, G]G^p \to G \xrightarrow{x_1, \ldots, x_r} C_p^{\times r}$ where $x_1, \ldots, x_r$ are generators of $H^1(G)$, and it suffices to check that the map $G \to C_p^{\times r}$ is an H_1-isomorphism. Eventually, finding such v_i's in $H^1(G)$ is equivalent to having $\beta(v_i)$ not linearly independent in $H^2(G)$. We have

$$H^1(C_p^{\times r}) \xrightarrow{\sim} H^1(G) \longrightarrow H^1(G') \xrightarrow{d_2} H_2(C_p^{\times r}) \longrightarrow H^2(G).$$

then the statement above is equivalent to $d_2 \neq 0$. This forces $H^1(G^1)$ is zero, so we have an H^1-isomorphism as required. ■

Therefore, this power of w has to be zero. □

21 Oct 9, 2023

Definition 21.1. Let G be a finite group, M be a G-module. The norm map $Nm_G : M \to M$ sends m to $\sum_{g\in G} gm$, so

$$\begin{tikzcd} M \arrow[r, "Nm_G"] \arrow[d] & M \\ M_G \arrow[r, "Nm_G"'] & M^G \arrow[u] \end{tikzcd}$$

Definition 21.2.

$$\hat{H}^*(G,M) = \begin{cases} H_{-*-1}(G,M), & * \leqslant -2 \\ \ker(Nm_G), & * = -1 \\ \operatorname{coker}(Nm_G), & * = 0 \\ H^*(G,M), & * \geqslant 1 \end{cases}$$

Example 21.3. Let $G = C_p$ and $M = \mathbb{Z}$, we have

$$\cdots \longrightarrow \mathbb{Z}[C_p] \xrightarrow{1-g} \mathbb{Z}[C_p] \xrightarrow{N_g} \mathbb{Z}[C_p] \xrightarrow{1-g} \mathbb{Z}[C_p] \xrightarrow{N_g} \mathbb{Z}[C_p] \xrightarrow{1-g} \mathbb{Z}[C_p] \xrightarrow{\varepsilon} \mathbb{Z}$$

where $\varepsilon \cdot g \mapsto 1$. We have

$$Nm_{C_p}(m) = \sum_{i=0}^{p-1} g^i m = \sum m = pm,$$

therefore $\operatorname{coker}(Nm) = \mathbb{Z}/p\mathbb{Z}$ and $\ker(Nm) = 0$. Therefore

$$\hat{H}^*(C_p,\mathbb{Z}) = \begin{cases} \mathbb{Z}/p\mathbb{Z}, * \text{ even} \\ 0, * \text{ odd} \end{cases}$$

More generally,

$$\hat{H}^*(C_p,M) = \begin{cases} M^G/N_gM, & * \text{ even} \\ \{m \in M : N_gM = 0\}/(1-g)M, & * \text{ odd} \end{cases}$$

Definition 21.4. A complete resolution F_* of G is an exact sequence

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \xrightarrow{d_0} F_{-1} \longrightarrow \cdots$$

of finitely-generated free $\mathbb{Z}[G]$-modules along with an element $e \in F_{-1}$ which is G-fixed and generates d_0.

To obtain a complete resolution, we get

$$\begin{tikzcd} \cdots \arrow[r] & F_1 \arrow[r] & F_0 \arrow[rr, "Nm_G"] \arrow[dr, "\varepsilon"'] & & \operatorname{Hom}(F_0,\mathbb{Z}) \arrow[r] & \cdots \\ & & & \mathbb{Z} \arrow[ur, "\varepsilon^*"'] & & \end{tikzcd}$$

where $e = \varepsilon^*(1)$. Conversely, given a complete resolution F, because e is G-fixed, F_{-1} is $\mathbb{Z}[G]$-free, e generates a copy of $\mathbb{Z} \subseteq F_{-1}$. Therefore we have

$$\begin{tikzcd} \cdots \arrow[r] & F_0 \arrow[rr, "d_0"] \arrow[dr, "\varepsilon"'] & & F_{-1} \arrow[r] & \cdots \\ & & \mathbb{Z} \arrow[ur, "\mu"'] & & \end{tikzcd}$$

for $\varepsilon : F_+ \to \mathbb{Z}$ and $\mu : \mathbb{Z} \to F$.

Definition 21.5. $\hat{H}^*(G, M) = H^*(\operatorname{Hom}_G(\hat{F}_*, M))$.

Intuitively, we can compare $F^* \otimes_G M$, so $\operatorname{Hom}(F, \mathbb{Z}) \otimes_G M \cong \operatorname{Hom}_G(F, M)$.

Lemma 21.6. Let F be a finitely-generated free $\mathbb{Z}[G]$-module, so $Nm_{\mathbb{Z}[G]}(F \otimes M)_G \to (F \otimes M)^G$ is an isomorphism.

To connect this definition with the previous one, we consider $\hat{F}_*, \operatorname{Hom}_G(\hat{F}_*, M)$ for $n < 0$, then $\operatorname{Hom}_G(F_n, M) \cong F^n \otimes M$. We can write F^+ as the complex $F_* \to \mathbb{Z}$ with augmentation $\varepsilon : F_0 \to \mathbb{Z}$, and $\operatorname{Hom}((F^-)^\times, \mathbb{Z})$ as $\mathbb{Z} \to F_{-1} \to F_{-2} \to \cdots$ where $G_\mu : \mathbb{Z} \to F_{-1}$. Therefore, $\hat{H}^n = H_{-n-1}(G, M)$ for $n \leqslant -2$ and is $H^n(G, M)$ for $n \geqslant -1$.

Lemma 21.7 (Shapiro)**.** $\hat{H}^*(H, M) \cong \hat{H}^*(G, \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M)$ where $H \subseteq G$ and M is an H-module.

For augmentation $\varepsilon : P_* \to \mathbb{Z}$, then let $\tilde{P}_*$ be the cone of ε.

Definition 21.8. The Tate complex is $T(G, M) = \tilde{P}_* \otimes \operatorname{Hom}(P_*, M)$.
In this sense, we can also define $\hat{H}^*(G, M) = H_{-*}(T_*(G, M)^G)$.

22 Oct 11, 2023

Let G be a finite group, a complete resolution would be

$$\begin{tikzcd} \cdots \arrow[r] & F_1 \arrow[r] & F_0 \arrow[rr] \arrow[dr, "\varepsilon"'] & & F_{-1} \arrow[r] & \cdots \\ & & & \mathbb{Z} \arrow[ur] & & \end{tikzcd}$$

so that $\hat{H}^*(G, M) = H^*(\operatorname{Hom}_G(F_*, M))$ and $\hat{H}_*(G, M) = H_*(F_* \otimes_G M)$. Observe that $\hat{H}^*(G, \mathbb{Z}[G]) = 0$. More generally, induced modules satisfy $\hat{H}^*(G, \operatorname{Ind}_G(M)) = 0$ and $\hat{H}^*(G, \operatorname{Ind}_G^H(M)) \cong \hat{H}^*(H, M)$.

Corollary 22.1 (Dimension Shifting)**.** For any finitely-generated module M, there are K and Q with

$$0 \longrightarrow M \longrightarrow \operatorname{Ind}_G(M) \longrightarrow Q \longrightarrow 0$$

and

$$0 \longrightarrow K \longrightarrow \operatorname{Ind}_G(M) \longrightarrow M \longrightarrow 0$$

such that $\hat{H}^i(G, M) \cong \hat{H}^{i+1}(G, K) \cong \hat{H}^{i-1}(G, Q)$. (Recall that if M is a G-module, then $\operatorname{Ind}_G(U(M)) \cong_G \mathbb{Z}[G] \otimes M$, where U is the forgetful functor and $\mathbb{Z}[G] \otimes M$ has the diagonal action.

Example 22.2. Let $G = C_n = \langle T \rangle$, with $y \in H^2(C_n, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ be the generator. The exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}[C_n] \arrow[rr, "1-T"] \arrow[dr] & & \mathbb{Z}[C_n] \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\ & & & I \arrow[ur] & & & \end{tikzcd}$$

where I is the augmentation ideal, as the kernel/cokernel of the sequences. Therefore $\hat{H}^{i-2}(C_n, \mathbb{Z}) \cong \hat{H}^i(C_n, \mathbb{Z}) \cong \hat{H}^{i+2}(C_n, \mathbb{Z})$.

Because the middle terms are free, this gives $H^0(-, \mathbb{Z}) \to H^1(-, I) \xrightarrow{\cong} H^2(-, \mathbb{Z})$.

Theorem 22.3. There is a unique product (i.e., for a pairing $A \otimes B \to C$ of G-modules, we get a pairing $\hat{H}^k(G, A) \otimes \hat{H}^m(G, M) \to \hat{H}^{k+m}(G, C)$) on $\hat{H}^*$ satisfying

- on $\hat{H}^0$, it is induced by $A^G \times B^G \to C^G$, and that
- the connecting homomorphism δ satisfies $\delta(a \smile b) = \delta a \smile b + (-1)^{|a|} a \smile \delta b$, and $\delta(a \smile b) = (-1)^{|a||b|}\delta(b \smile a)$.

Proof. Uniqueness is the direct result of dimension shifting. For existence, it suffices to construct a suitable pairing on standard Tate cochains. We build a standard resolution $X_* \to \mathbb{Z}$ where $X_i = \mathbb{Z}[G^{i+1}] \cong \mathbb{Z}[G]^{\otimes(i+1)}$ and so $\hat{X}_*$ is the diagram given by

$$\begin{tikzcd} X_* \arrow[rr] \arrow[dr] & & \mathrm{Hom}(X_*, \mathbb{Z}) \\ & \mathbb{Z} \arrow[ur] & \end{tikzcd}$$

For $i > 0$, $X_{-i} \cong \mathbb{Z}[G]^{\otimes i}$, so we need suitable maps $\varphi_{p,q} : X_{p+q} \to X_p \otimes X_q$ for all $p, q \in \mathbb{Z}$ because

$$\hat{C}^p(A) \otimes \hat{C}^q(B) = \mathrm{Hom}_G(X_p, A) \otimes \mathrm{Hom}_G(X_q, B) \longrightarrow \mathrm{Hom}_G(X_p \otimes X_q, C) \xrightarrow{\varphi_{p,q}^*} \mathrm{Hom}_G(X_{p+q}, C) = \hat{C}^{p+q}(C).$$

This allows us to write down what $\varphi_{p,q}$ is supposed to be. $\square$

Example 22.4. Consider

$$\hat{H}^p(G, \mathbb{Z}) \otimes \hat{H}^{-p}(G, \mathbb{Z}) \to \hat{H}^0(G, \mathbb{Z})$$

given by $f : G^{p+1} \to \mathbb{Z}$ and $g : G^p \to \mathbb{Z}$ in $\hat{H}^p(G, \mathbb{Z})$ and $\hat{H}^{-p}(G, \mathbb{Z})$ respectively, then

$$(f \smile g)(\sigma_0) = \sum_{\tau_i \in G} f(\sigma_0, \ldots, \sigma_p) \cdot g(\tau_p, \ldots, \tau_1)$$

but actually

$$\hat{H}^p(G, \mathbb{Z}) \otimes \hat{H}^{-p}(G, \mathbb{Z}) \to \hat{H}^0(G, \mathbb{Z}) = \mathbb{Z}/|G|$$

is a perfect pairing, i.e., $\hat{H}^{-p}(G, \mathbb{Z}) \cong \mathrm{Hom}(\hat{H}^p(G, \mathbb{Z}), \mathbb{Z}/|G|)$.

Remark 22.5. Let R be a ring with a G-action, then $H^*(G, R) \to \hat{H}^*(G, R)$ is a ring homomorphism.

For the case $G = C_n$, this gives $H^*(G, \mathbb{Z}) \cong \mathbb{Z}[y]/ny \to \hat{H}^*(G, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}[y^{\pm 1}]$.

More generally, for any C_n-module M, $H^*(C_n, M) \to \hat{H}^*(C_n, M)$ is a map between a module over $H^*(C_n, \mathbb{Z})$ and a module over $\hat{H}^*(C_n, \mathbb{Z})$. This map is therefore the inversion of y (due to the cup product structure). For instance, $\hat{H}^*(C_p, \mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z}[x, y/x^2)[y^{-1}]$.

For a general G, if we have an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow F_{k-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

where F_i's are G-free, then for $y_k \in \hat{H}^k(G, \mathbb{Z})$, then if we cup with y_k, we get an isomorphism $\hat{H}^n(G, M) \cong \hat{H}^{n+k}(G, M)$.

23 Oct 13, 2023

Recall that we have $\hat{H}^i(G, \mathbb{Z}) \otimes \hat{H}^{-i}(G, \mathbb{Z}) \to \hat{H}^0(G.\mathbb{Z})$. More generally,

Proposition 23.1. For a G-module M, $\hat{H}^i(G, M^\vee) \otimes \hat{H}^{-i-1}(G, M) \xrightarrow{\smile} \hat{H}^{-1}(G, \mathbb{Q}/\mathbb{Z})$ where we denote $M^\vee = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z}) = \frac{1}{|G|}\mathbb{Z}/\mathbb{Z}$ is a perfect pairing.

Proof. Use dimension shifting to reduce it to $i = 0$, then check explicitly. Recall for cyclic group G, we have

$$\hat{H}^n(G, M) \otimes \hat{H}^2(G, \mathbb{Z}) \xrightarrow{\sim} \hat{H}^{n+2}(G, M)$$

from

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[G] \xrightarrow{1-\sigma} \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

(When regarding $\mathbb{Z}[G]$'s as free modules, we have the second cohomology by noting the coboundary occurs twice.) $\square$

Definition 23.2 (Class Module)**.** C is called a class module if for all subgroups H of (finite group) G,

1. $H^1(H, C) = 0$;

2. $H^2(H, C) = \mathbb{Z}/|H|$, where the generator is called the fundamental class.

For any C and $\gamma \in H^2(G, C)$, i.e., $\gamma : G \times G \to C$ is an inhomogenous cocycle, we define $C(\gamma) = C \oplus \bigoplus_{1 \neq g \in G} \mathbb{Z} b_g$ where b_g is a formal basis element. The G-action is given by $g \cdot b_n = b_{gh} - g_g + \gamma(g, h)$ and $b_1 = \gamma(1, 1)$. The composition $\gamma : G \times G \to C \to C(\gamma)$ is a coboundary. ($\gamma = \delta\beta$, $\beta(g) = b_g$.) Therefore, $\gamma \in \ker(H^2(G, C) \to H^2(G, C(\gamma))$. We have an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] & C(\gamma) \arrow[rr] \arrow[dr, "b_g \mapsto g-1"'] & & \mathbb{Z}[G] \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\
& & & I_G \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

which gives $\hat{H}^0(G, \mathbb{Z}) = \mathbb{Z}/|G|\mathbb{Z} \xrightarrow{\cong} \hat{H}^1(G, I_G) \xrightarrow{\delta} \hat{H}^2(G, C)$.

Theorem 23.3. $\delta^2 : \hat{H}^n(H, \mathbb{Z}) \to \hat{H}^{n+2}(H, C)$ is $\delta^2(x) = x \smile \gamma_H$, where $\gamma_H = \operatorname{res}^G_H(\gamma)$. Moreover, the following are equivalent:

1. $C(\gamma)$ is cohomologically trivial.
2. C is a class module with fundamental class γ.
3. δ^2 is an isomorphism for all n and all H.

Proof. (1) $\Rightarrow$ (2): $\hat{H}^1(H, C) \cong \hat{H}^0(H, I_G) \cong \hat{H}^{-1}(H, \mathbb{Z}) = 0$ and $\hat{H}^2(H, C) = \hat{H}^0(H, \mathbb{Z}) = \mathbb{Z}/|H|\mathbb{Z}$.

(2) $\Rightarrow$ (1): We have

$$0 = \hat{H}^1(H, C) \longrightarrow \hat{H}^1(H, C(\gamma)) \longrightarrow \hat{H}^1(H, I_G) \longrightarrow \hat{H}^2(H, C) \longrightarrow \hat{H}^2(H, C(\gamma)) \longrightarrow \hat{H}^2(H, I_G)$$

By dimension shifting on $0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z}$, we have $\hat{H}^1(I_G) = \hat{H}^0(\mathbb{Z}) = \mathbb{Z}/|H|\mathbb{Z}$, and so $\hat{H}^2(H, C) = \mathbb{Z}/|H|\mathbb{Z}$, but it follows by a zero map to $\hat{H}^2(H, C(\gamma))$, therefore the map $\hat{H}^1(H, I_G) \to \hat{H}^2(H, C)$ is also the zero map. We then note that $\hat{H}^1(H, C(\gamma)) = 0 = \hat{H}^2(H, C(\gamma))$. This implies $C(\gamma)$ is cohomologically trivial. ☐

Theorem 23.4 (Nakayama-Tate)**.** If C is a class module with fundamental class γ, then

$$\hat{H}^i(G, \operatorname{Hom}(M, C)) \otimes \hat{H}^{2-i}(G, M) \xrightarrow{\smile} \hat{H}^2(G, C)$$

is a perfect pairing in the sense that $\operatorname{Hom}(\hat{H}^{2-i}(G, M), \mathbb{Q}/\mathbb{Z}) \cong \hat{H}^i(G, \operatorname{Hom}(M, C))$. Note $\operatorname{Hom}(\hat{H}^{2-i}(G, M), \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Hom}(\hat{H}^{2-i}(G, M, H^2(G, C)))$.

24 Oct 16, 2023

For a class module C, choose the generator γ of $\hat{H}^2(G, C)$, so γ is represented by $c : G \times C \to C$ and defines a map $G^{\text{ab}} \to C^G/N_G C = \hat{H}^0(G, C)$. Now we have $\hat{H}^2(G, \mathbb{Z}) \otimes \hat{H}^0(G, C) \to \mathbb{Z}/|G|$. Therefore, by connecting $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, we have $\hat{H}^2(G, \mathbb{Z}) \cong \hat{H}^1(G, \mathbb{Q}/\mathbb{Z}) = (G^{\text{ab}})^\vee$. Therefore $G^{\text{ab}} = \hat{H}^2(G, \mathbb{Z})^\vee \cong \hat{H}^0(G, C)$. Therefore, γ defines an isomorphism, with inverse extends to $C^G \to G^{\text{ab}}$.

Remark 24.1. If A is k-torsion, then $\operatorname{Hom}(A, \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Hom}(A, \mathbb{Z}/k\mathbb{Z})$.

Theorem 24.2. Let G be a profinite group, $G = \lim_u G/uG$ where G/uG is finite, then $H^*(G, M) \cong \operatorname{colim}_u H^*(G/uG.M^u)$.

By Tate cohomology, $\hat{H}^{>0}(G, M) = H^{>0}(G, M)$ and for $i \leqslant 0$ we have $\hat{H}^i(G, M) = \lim_{\text{deflation}} \hat{H}^i(G/u, M^u)$.

Let $P_* \to \mathbb{Z}$ be some projective/free G-resolution, so we obtain $H_*((P_* \otimes M)/G) = H^*(\operatorname{Hom}(P_*, M)^G) = H^*(G, M)$.

For $U \subseteq V \subseteq G$, we have $G/uG \twoheadrightarrow G/vG$, then we define the deflation to be the composition of norm and coinflation,

$$def : H_j(G/uG, M^u) \cong H_j(P_* \otimes M^u)/(G/uG) \xrightarrow{coinf} H_j(G/vG, (M^u)/v) \xrightarrow{norm} H_j(G/v, M^v).$$

25 Oct 18, 2023

Let k be a number field, then we may study $H^*(\mathrm{Gal}(\bar{k}/k), -)$. Over the localization k_p, we may want to study $\mathrm{Gal}(\bar{k}_p/k_p)$ in the same way as $\mathbb{C}/\mathbb{R}$ with absolute Galois group C_2. Note that $\mathrm{Gal}(\bar{k}_p/k_p)$ has finite cohomological dimension. To do this, we have patched Tate cohomology by putting duality in $\mathrm{Gal}(\bar{k}_p/k_p)$ and periodicity for C_2 together.

For finite groups, Tate cohomology gives $H^*(G, \mathbb{F}_p) \to \lim_{V \subseteq G} H^*(V, \mathbb{F}_p)$, where V is an elementary abelian subgroup, has nilpotent kernel and cokernel. This is based on $H^*(C_p, \mathbb{F}_p) = \mathbb{F}_p[y] \otimes \bigwedge(x)$ and $\hat{H}^*(C_p, \mathbb{F}_p) = \mathbb{F}_p[y^{\pm 1}] \otimes \bigwedge(x)$. Another idea is that if Γ is any group, then we have $H^*(\Gamma, \mathbb{F}_p) \to \lim_{G \subseteq \Gamma} H^*(G)$ where $G \subseteq \Gamma$ is a finite group. The question is how well does this approximate.

Farrell has the following version of Tate cohomology. We say Γ is of virtual cohomological dimension k, if there exists a finite index subgroup $U \subseteq \Gamma$ with codimension k. If the virtual cohomological dimension of Γ is finite, then

1. $\hat{H}^*(\Gamma, M) = H^*(\Gamma, M)$ for $* > k$,
2. if the cohomological dimension of Γ is finite, then $\hat{H}^*(\Gamma, M) = 0$.

When G is finite, we have complete resolutions

$$\begin{tikzcd} P_* \arrow[rr] \arrow[dr] & & \mathrm{Hom}(P_*, \mathbb{Z}) \\ & \mathbb{Z} \arrow[ur] & \end{tikzcd}$$

of free $\mathbb{Z}[G]$-modules since $\mathrm{Hom}(\mathbb{Z}[G], \mathbb{Z}] \cong \mathbb{Z}[G]$.

Definition 25.1. For any Γ, a complete resolution of Γ is an acyclic complex F_* of projective Γ-modules, as well as a projective resolution $P_* \to \mathbb{Z}$ such that $F_r \cong P_r$ for $r \gg 0$, then $\hat{H}^*(\Gamma, M) = H^*(\mathrm{Hom}_\Gamma(F_*, M))$.

Remark 25.2.

- There is a complete resolution such that $F_n \cong P_n$ for all n greater than the virtual cohomological dimension of Γ.
- Any two complete resolutions are chain equivalent.

Note that if $H^k(G, M) = 0$ for all $k > n$, then the cohomological dimension of G is n. This implies there is a projective resolution of $\mathbb{Z} \leftarrow P_0 \leftarrow \cdots \leftarrow P_n \leftarrow 0$ and vice versa.

Example 25.3. If G has finite cohomological dimension, $F_* = 0$, $P_* \to \mathbb{Z}$ has finite projective resolution. This is a complete resolution.

Lemma 25.4. If G has finite cohomological dimension, then any acyclic complex F_* of projectives is chain contractible.

Proof. Take $0 \to K \to F_k \to \cdots \to F_{k-n} \to B \to 0$, then $H^i(G, B) = H^{i+n}(G, K) = 0$, so B is projective therefore B as the kernel of differentials, which indicates we have a splitting on the image of differentials. We have chain nullhomotopy. ☐