

MATH 540 Notes

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1 ABSTRACT MEASURE THEORY

1.1 INTRODUCTION

Definition 1.1. Let X be an (non-empty) underlying space we are working over. We denote $\mathcal{P}(X)$ to be the power set of X , i.e., the set of all subsets of X .

Example 1.2. Let $X = \{1, 2\}$, then $\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

Remark 1.3. If X is a finite set of size n , then $\mathcal{P}(X)$ is a finite set of size 2^n .

We will consider a subcollection \mathcal{A} of subsets of X , i.e., a subset of the power set. We will try to define this as an algebra. Note that an algebra is just a ring with a module structure with respect to some other ring.

Definition 1.4. $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra on X if it is

- a. closed under finite union, i.e., given $E_1, E_2 \in \mathcal{A}$, then $E_1 \cup E_2 \in \mathcal{A}$, and
- b. closed under complements, i.e., if $E \in \mathcal{A}$, then the complement $E^c \in \mathcal{A}$ as well.

Remark 1.5. An algebra \mathcal{A} would be closed under finite intersection. Indeed, for any $E_1, E_2 \in \mathcal{A}$, we have $E_1 \cap E_2 \in \mathcal{A}$ if and only if $(E_1 \cap E_2)^c \in \mathcal{A}$, if and only if $E_1^c \cup E_2^c \in \mathcal{A}$, which is true by definition.

Lemma 1.6. If \mathcal{A} is a non-empty algebra on X , then $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$.

Proof. Since \mathcal{A} is non-empty, take $E \in \mathcal{A}$, then $\emptyset = E \cap E^c \in \mathcal{A}$ as well. Also, $X = E \cup E^c \in \mathcal{A}$. □

Example 1.7. Let X be a set, and let $\mathcal{A} = \{\emptyset, X\} \subseteq \mathcal{P}(X)$. It is easy to verify that \mathcal{A} is an algebra.

Definition 1.8. Let $\emptyset \neq \mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra, then we say \mathcal{A} is a σ -algebra on X if

- a. closed under countable union, i.e., if $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, then $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$;
- b. if $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$.

Lemma 1.9. If $\mathcal{A} \neq \emptyset$ is a σ -algebra on X , then $\{\emptyset, X\} \subseteq \mathcal{A}$ is a σ -algebra.

Example 1.10. Let X be an uncountable set, let $\mathcal{A} = \{E \subseteq X : E \text{ is countable or } E^c \text{ is countable}\}$, then \mathcal{A} is a σ -algebra on X .

Theorem 1.11. Suppose a non-empty algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ such that,

- if $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, and E_j 's are pairwise disjoint, then $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$,

then \mathcal{A} is a σ -algebra on X .

Proof. Take $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, we will show that $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$. To do this, we will rearrange the sets. Let $F_1 = E_1$, let $F_2 = E_2 \setminus E_1$, let $F_3 = E_3 \setminus (E_1 \cup E_2)$, and so on, such that let $F_k = E_k \setminus \bigcup_{i=1}^{k-1} E_i$. We note

$$\begin{aligned} F_k &= E_k \cap \left(\bigcup_{j=1}^{k-1} E_j \right)^c \\ &= E_k \cap \left(\bigcap_{j=1}^{k-1} E_j^c \right) \in \mathcal{A}. \end{aligned}$$

One can also verify that $\bigcup_{j=1}^{\infty} E_j = \bigcup_{k=1}^{\infty} F_k$, and that F_k 's are disjoint from the definition. \square

Definition 1.12. Let X be a non-empty space. A topology on X is a family \mathcal{F} of subsets of X satisfying the following conditions:

- i. $\emptyset, X \in \mathcal{F}$;
- ii. \mathcal{F} is closed under arbitrary union;
- iii. \mathcal{F} is closed under finite intersection.

Every member of \mathcal{F} is now called an open subset of X . A complement of an open subset of X is called a closed subset.

Definition 1.13. Let $\mathcal{A}_1, \mathcal{A}_2$ be σ -algebras. We say \mathcal{A}_1 is smaller than \mathcal{A}_2 if $\mathcal{A}_1 \subseteq \mathcal{A}_2$, and equivalently \mathcal{A}_2 is larger than \mathcal{A}_1 .

Definition 1.14. Let \mathcal{F} be a family of subsets of X , the smallest σ -algebra containing \mathcal{F} is called the σ -algebra generated by \mathcal{F} . This is denoted by $\mathcal{M}(\mathcal{F})$.

Lemma 1.15. Let \mathcal{F} be a family of subsets of X . Suppose $\mathcal{F} \subseteq \mathcal{A}$ where \mathcal{A} is a σ -algebra, then $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{A}$.

Proof. Obvious. \square

Definition 1.16. Let \mathcal{F} be a topology on X , then we say (X, \mathcal{F}) is a topological space. We say $\mathcal{M}(\mathcal{F})$ is the Borel σ -algebra on X , denoted by $\mathcal{B}_X = \mathcal{B}_{X, \mathcal{F}}$. Any member of \mathcal{B}_X is called a Borel set.

Example 1.17. Let $X = \mathbb{R}$, we denote the corresponding Borel σ -algebra to be $\mathcal{B}_{\mathbb{R}}$.

Definition 1.18. A G_{δ} -set is a countable intersection of open subsets of X . A F_{σ} -set is a countable union of closed subsets of X .

Theorem 1.19. Both G_{δ} -sets and F_{σ} -sets are Borel sets, that is, $G_{\delta}, F_{\sigma} \subseteq \mathcal{B}_X$.

Proof. We will prove that any G_{δ} -set E is a Borel set, and similarly any F_{σ} -set is a Borel set. By definition $E = \bigcap_{j=1}^{\infty} O_j$, where each O_j is an open subset. To show $E \in \mathcal{B}_X$, we show that $E^c \in \mathcal{B}_X$. Note that $E^c = \left(\bigcap_{j=1}^{\infty} O_j \right)^c = \bigcup_{j=1}^{\infty} O_j^c$. Since $O_j \in \mathcal{B}_X$ for all j , then $O_j^c \in \mathcal{B}_X$ as well. Therefore, $E^c \in \mathcal{B}_X$ since a σ -algebra \mathcal{B}_X is closed under countable unions. \square

Definition 1.20. Let X_1, \dots, X_n be non-empty spaces. The product space is $\prod_{j=1}^n X_j$. Define $\pi_j : \prod_{i=1}^n X_i \rightarrow X_j$ by $\pi_j(x_1, \dots, x_n) = x_j$. Let \mathcal{A}_j be a σ -algebra on X_j , the product σ -algebra on $\prod_{i=1}^n X_i$ is the σ -algebra generated by $\{\pi_j^{-1}(E_j) : E_j \in \mathcal{A}_j \forall j \in \{1, \dots, n\}\}$. The product σ -algebra is denoted by $\bigotimes_{j=1}^n \mathcal{A}_j = \prod_{j=1}^n \mathcal{A}_j$.

Example 1.21. $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_{j=1}^n \mathcal{B}_{\mathbb{R}}$.

1.2 MEASURES

Definition 1.22. Let \mathcal{A} be a σ -algebra on X . A measure μ on X and \mathcal{A} is a function $\mu : \mathcal{A} \rightarrow [0, \infty]$ such that

- a. $\mu(\emptyset) = 0$;
- b. if $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$ and E_j 's are disjoint, then $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$.

We then say (X, \mathcal{A}) is a measureable space. A measureable space is a triple (X, \mathcal{A}, μ) with measure μ specified.

Definition 1.23. Let μ be a measure on (X, \mathcal{A}) .

1. If $\mu(X) < \infty$, then we say μ is a finite measure. In particular, if $\mu(X) = 1$, this is a probability measure.
2. If $X = \bigcup_{j=1}^{\infty} E_j$ such that $\mu(E_j) < \infty$ for all $j \in \mathbb{N}$, then we say μ is σ -finite.
3. If for all $E \in \mathcal{A}$ with $\mu(E) = \infty$, there is $F \in \mathcal{A}$ such that $F \subseteq E$ and $0 < \mu(F) < \infty$, then we say μ is semi-finite.

Remark 1.24. A σ -finite measure is semi-finite. However, the converse is not true.

Example 1.25. Let $f : X \rightarrow [0, \infty]$ be a function. For any $E \subseteq \mathcal{P}(E)$, we can define a measure $\mu(E) = \sum_{x \in E} f(x)$. Note that the summation makes sense only when E is finite. In case E is infinite, we should define $\sum_{x \in E} f(x) = \sup\{\sum_{x \in F} f(x) : F \subseteq E \text{ for finite } F\}$. Let μ be a measure on $\mathcal{P}(X)$.

- If $f(x) \equiv 1$ for all $x \in X$, then $\mu(E) = \sum_{x \in E} 1 = \text{Card}(E)$. In this case, μ is called a counting measure.
- Suppose $x_0 \in X$ is fixed. Define

$$f(x) = \begin{cases} 1, & \text{if } x = x_0 \\ 0, & \text{if } x \neq x_0 \end{cases}$$

then for any $E \in \mathcal{P}(X)$,

$$\mu(E) = \begin{cases} 1, & \text{if } x_0 \in E \\ 0, & \text{if } x_0 \notin E \end{cases}$$

This is called the Dirac measure of x_0 .

Definition 1.26. Let (X, \mathcal{A}, μ) be a measure space. A set $E \subseteq \mathcal{A}$ is called a null set if $\mu(E) = 0$.

If a statement about points $x \in X$ is true except for null sets, then we say the statement is true almost everywhere.

Example 1.27. Suppose $f(x) \leq 1$ for all $x \in X$, then we say f is bounded above by 1 everywhere. If we want to weaken this statement, we can say $f(x) \leq 1$ almost everywhere $x \in X$, which is true if and only if $\mu(\{x \in X : f(x) > 1\}) = 0$.

Theorem 1.28. Let $E, F \in \mathcal{A}$ be such that $E \subseteq F$, then $\mu(E) \leq \mu(F)$.

Proof. We can write $F = E \cup (E \setminus F)$, then

$$\begin{aligned} \mu(F) &= \mu(E) + \mu(F \setminus E) \\ &\geq \mu(E) \end{aligned}$$

since $\mu(F \setminus E) \geq 0$. □

Theorem 1.29 (Sub-additivity). Let $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, then $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu(E_j)$.

Proof. Set $F_1 = E_1$ and let $F_k = E_k \setminus \left(\bigcup_{j=1}^{k-1} E_j \right)$ be defined inductively, then $\bigcup_{k \in \mathbb{N}} F_k = \bigcup_{j \in \mathbb{N}} E_j$. Since F_k 's are disjoint, we have

$$\begin{aligned} \mu \left(\bigcup_{j \in \mathbb{N}} E_j \right) &= \mu \left(\bigcup_{k \in \mathbb{N}} F_k \right) \\ &= \sum_{k=1}^{\infty} \mu(F_k) \\ &= \sum_{k=1}^{\infty} \mu(E_k) \\ &= \sum_{j=1}^{\infty} \mu(E_j) \end{aligned}$$

by [Theorem 1.28](#). □

Theorem 1.30. Let $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$.

a. (Continuity from below): If $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_j \subseteq \cdots$ for all j , then $\mu \left(\bigcup_{j=1}^{\infty} E_j \right) = \lim_{j \rightarrow \infty} \mu(E_j)$.

b. (Continuity from above): If $E_1 \supseteq E_2 \supseteq \cdots \supseteq E_j \supseteq \cdots$ for all $j \in \mathbb{N}$, then $\mu \left(\bigcap_{j=1}^{\infty} E_j \right) = \lim_{j \rightarrow \infty} \mu(E_j)$ if $\mu(E_1) < \infty$.

In particular, the limits on the right exist on $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.

Example 1.31. Let μ be the counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. For each $j \in \mathbb{N}$, we define $E_j = \{n \in \mathbb{N} : n > j\}$. Therefore $E_1 \supseteq E_2 \supseteq \cdots$ is a decreasing sequence of sets. Note that $\mu(E_1) = \mu(\{n \in \mathbb{N}\}) = \mathbb{N} = \infty$, and $\lim_{j \rightarrow \infty} \mu(E_j) =$

$$\lim_{j \rightarrow \infty} \infty = \infty, \text{ but } \mu \left(\bigcap_{j=1}^{\infty} E_j \right) = \mu(\emptyset) = 0.$$

Proof. a. Set $E_0 = \emptyset$. Now

$$\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} (E_j \setminus E_{j-1})$$

and therefore

$$\begin{aligned} \mu \left(\bigcup_{j=1}^{\infty} E_j \right) &= \mu \left(\bigcup_{j=1}^{\infty} (E_j \setminus E_{j-1}) \right) \\ &= \sum_{j=1}^{\infty} \mu(E_j \setminus E_{j-1}) \\ &= \lim_{k \rightarrow \infty} \sum_{j=1}^k \mu(E_j \setminus E_{j-1}) \\ &= \lim_{k \rightarrow \infty} \mu \left(\bigcup_{j=1}^k E_j \setminus E_{j-1} \right) \\ &= \lim_{k \rightarrow \infty} \mu(E_k) \\ &= \lim_{j \rightarrow \infty} \mu(E_j). \end{aligned}$$

b. For any $j \in \mathbb{N}$, set $F_j = E_1 \setminus E_j$. Note that $F_j \subseteq F_{j+1}$ since $E_j \supseteq E_{j-1}$. This is now an increasing sequence as in part a. By part a., we know $\mu\left(\bigcup_{j=1}^{\infty} F_j\right) = \lim_{j \rightarrow \infty} \mu(F_j)$. Now note that

$$\begin{aligned}
 \bigcup_{j=1}^{\infty} F_j &= \bigcup_{j=1}^{\infty} (E_1 \setminus E_j) \\
 &= \bigcup_{j=1}^{\infty} (E_1 \cap E_j^c) \\
 &= E_1 \cap \bigcup_{j=1}^{\infty} E_j^c \\
 &= E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j\right)^c \\
 &= \left(\left(E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j\right)^c\right) \cup \left(\bigcap_{j=1}^{\infty} E_j\right)\right) \cap \left(\bigcap_{j=1}^{\infty} E_j\right)^c \\
 &= \left(E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j\right)^c\right) \cup \left(\bigcap_{j=1}^{\infty} E_j\right).
 \end{aligned}$$

Note that $E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j\right)^c$ and $\bigcap_{j=1}^{\infty} E_j$ are disjoint, therefore by property of measure we have

$$\begin{aligned}
 \mu(E_1) &= \mu\left(E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j\right)^c\right) + \mu\left(\bigcap_{j=1}^{\infty} E_j\right) \\
 &= \mu\left(\bigcup_{j=1}^{\infty} F_j\right) + \mu\left(\bigcap_{j=1}^{\infty} E_j\right) \\
 &= \lim_{j \rightarrow \infty} \mu(F_j) + \mu\left(\bigcap_{j=1}^{\infty} E_j\right).
 \end{aligned}$$

Recall that $F_j = E_1 \setminus E_j$ for all j , therefore $E_1 = F_j \cup F_j^c = F_j \cup E_j$, where F_j and E_j are disjoint, therefore $\mu(E_1) = \mu(F_j) + \mu(E_j)$. Since $\mu(E_1) < \infty$, and F_j is a subset of E_1 and hence also a real number, then $\mu(E_1)$ is a sum of two real numbers. Therefore, we have $\mu(E_1) - \mu(E_j) = \mu(F_j)$. With this, we have

$$\begin{aligned}
 \mu(E_1) &= \lim_{j \rightarrow \infty} (\mu(E_1) - \mu(E_j)) + \mu\left(\bigcap_{j=1}^{\infty} E_j\right) \\
 &= \mu(E_1) - \lim_{j \rightarrow \infty} (\mu(E_j)) + \mu\left(\bigcap_{j=1}^{\infty} E_j\right).
 \end{aligned}$$

In particular, we get

$$\lim_{j \rightarrow \infty} (\mu(E_j)) = \mu\left(\bigcap_{j=1}^{\infty} E_j\right).$$

□

1.3 OUTER MEASURE

Definition 1.32. An outer measure μ^* on X (or $\mathcal{P}(X)$) is a function $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ such that

- i. $\mu^*(\emptyset) = 0$,
- ii. $\mu^*(A) \leq \mu^*(B)$ for all $A \subseteq B \subseteq X$,
- iii. σ -subadditivity: $\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$.

Example 1.33. Let $\rho : \mathcal{A} \rightarrow [0, \infty]$ be such that $\rho(\emptyset) = 0$, where $\mathcal{A} \subseteq \mathcal{P}(X)$ is a subcollection (but not necessarily an algebra) such that $\emptyset, X \in \mathcal{A}$.

For all $A \in \mathcal{P}(X)$, i.e., $A \subseteq X$, we define

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \rho(E_j) : E_j \in \mathcal{A} \text{ and } A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}.$$

Theorem 1.34. μ^* defined in Example 1.33 is an outer measure.

Proof. i. Let $E_j = \emptyset$ for all $j \in \mathbb{N}$, then $\emptyset \subseteq \bigcup_{j=1}^{\infty} E_j$, and so

$$\sum_{j=1}^{\infty} \rho(E_j) = \sum_{j=1}^{\infty} \rho(\emptyset) = \sum_{j=1}^{\infty} 0 = 0$$

and therefore $\mu^*(\emptyset) = 0$.

ii. Let $A \subseteq B \subseteq X$. If $B \subseteq \bigcup_{j=1}^{\infty} E_j$, we have $A \subseteq \bigcup_{j=1}^{\infty} E_j$, then

$$\left\{ \sum_{j=1}^{\infty} \rho(E_j) : E_j \in \mathcal{A}, B \subseteq \bigcup_{j=1}^{\infty} E_j \right\} \subseteq \left\{ \sum_{j=1}^{\infty} \rho(E_j) : E_j \in \mathcal{A}, A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}.$$

In particular, given subsets $S_1 \subseteq S_2$, then $\inf S_2 \leq \inf S_1$ and $\sup S_1 \leq \sup S_2$. This implies $\mu^*(A) \leq \mu^*(B)$.

iii. We want to show $\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$. Now for any $j \in \mathbb{N}$, we have

$$\mu^*(A_j) = \inf \left\{ \sum_{k=1}^{\infty} \rho(E_k) : E_k \in \mathcal{A} \text{ and } A_j \subseteq \bigcup_{k=1}^{\infty} E_k \right\}.$$

For any $\varepsilon > 0$, we note that $\mu^*(A_j) + \varepsilon \cdot 2^{-j}$ is not a lower bound of $\left\{ \sum_{k=1}^{\infty} \rho(E_k) : E_k \in \mathcal{A} \text{ and } A_j \subseteq \bigcup_{k=1}^{\infty} E_k \right\}$.

Then there exists $E_k^{(j)} \in \mathcal{A}$ for $k \in \mathbb{N}$ such that $A_j \subseteq \bigcup_{k=1}^{\infty} E_k^{(j)}$ and $\sum_{k=1}^{\infty} \rho(E_k^{(j)}) \leq \mu^*(A_j) + \varepsilon \cdot 2^{-j}$. Summing with respect to j , we get

$$\begin{aligned} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \rho(E_k^{(j)}) &\leq \sum_{j=1}^{\infty} \mu^*(A_j) + \sum_{j=1}^{\infty} \varepsilon \cdot 2^{-j} \\ &= \sum_{j=1}^{\infty} \mu^*(A_j) + \varepsilon. \end{aligned}$$

Note that

$$\bigcup_{j=1}^{\infty} A_j \subseteq \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} E_k^{(j)}$$

is a countable union of subsets of \mathcal{A} . We will calculate the value over μ^* . By definition of μ^* , we have

$$\begin{aligned}\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) &\leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \rho(E_k^{(j)}) \\ &\leq \sum_{j=1}^{\infty} \mu^*(A_j) + \varepsilon.\end{aligned}$$

Since this is true for all $\varepsilon > 0$, then take $\varepsilon \rightarrow 0$, we are done. \square

Definition 1.35. Let μ^* be an outer measure on $(X, \mathcal{P}(X))$. A set $A \subseteq X$ is called μ^* -measurable if $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for all $E \subseteq X$.

Remark 1.36. First note that $\mu^*(E) = \mu^*((E \cap A) \cup (E \cap A^c))$, therefore $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for all $E \subseteq X$.

Theorem 1.37 (Fundamental Theorem of Measure Theory). Let μ^* be an outer measure on X . Let \mathcal{A} be the collection of all μ^* -measurable set, then \mathcal{A} is a σ -algebra, and $\mu^*|_{\mathcal{A}}$ is a measure on \mathcal{A} , i.e., (X, \mathcal{A}, μ^*) is a measure space.

Proof. We first prove that \mathcal{A} is an algebra. To see \mathcal{A} is closed under complement, we have $A \in \mathcal{A}$ if and only if $A^c \in \mathcal{A}$ by the definition of measurable set. To show \mathcal{A} is closed under finite union, suppose $A, B \in \mathcal{A}$, and we want to show $A \cup B \in \mathcal{A}$, which is true if and only if $\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$ for all $E \subseteq X$, hence it suffices to show that $\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$. We have

$$\begin{aligned}\mu^*(E \cap (A \cup B)) &= \mu^*(E \cap (A \cup B) \cap A) + \mu^*(E \cap (A \cup B) \cap A^c) \\ &= \mu^*(E \cap A) + \mu^*(E \cap B \cap A^c)\end{aligned}$$

and

$$\begin{aligned}\mu^*(E \cap (A \cup B)^c) &= \mu^*(E \cap (A \cup B)^c \cap A) + \mu^*(E \cap (A \cup B)^c \cap A^c) \\ &= \mu^*(\emptyset) + \mu^*(E \cap A^c \cap B^c) \\ &= \mu^*(E \cap A^c \cap B^c).\end{aligned}$$

Therefore

$$\begin{aligned}\mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c) &= \mu^*(E \cap A) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\ &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E)\end{aligned}$$

where the last two steps follow from the fact that $A, B \in \mathcal{A}$ are μ^* -measurable. Therefore, \mathcal{A} is an algebra. We now want to show that it is a σ -algebra. It suffices to prove that \mathcal{A} is closed under disjoint σ -unions. Let $A_j \in \mathcal{A}$ for all $j \in \mathbb{N}$ where they are pairwise disjoint, and we want to show that $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$. That is,

$$\mu^*(E) = \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)\right) + \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)^c\right)$$

for all $E \subseteq X$.

Lemma 1.38. For a pairwise disjoint family $A_1, \dots, A_n \in \mathcal{A}$,

$$\mu^*\left(E \cap \bigcup_{j=1}^n A_j\right) = \sum_{j=1}^n \mu^*(E \cap A_j).$$

Subproof. We proceed by induction. For $n = 1$, this is obviously true. Now suppose $n > 1$. To simplify the notation, let $B_n = \bigcup_{j=1}^n A_j$, and use the convention that $B_0 = \emptyset$. Now

$$\begin{aligned}\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) \\ &= \sum_{i=1}^n (E \cap A_i) + \mu^*(E \cap B_0) \\ &= \sum_{i=1}^n (E \cap A_i) \\ &= \sum_{i=1}^n (E \cap A_i)\end{aligned}$$

for all $n \in \mathbb{N}$. This finishes the proof. ■

Now for any $E \subseteq X$, we have

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \\ &= \sum_{j=1}^n \mu^*(E \cap A_j) + \mu^*(E \cap B_n^c) \\ &\geq \sum_{j=1}^n \mu^*(E \cap A_j) + \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)^c\right)\end{aligned}$$

since $B_n = \bigcup_{j=1}^n A_j \subseteq \bigcup_{j=1}^{\infty} A_j$. Now take $n \rightarrow \infty$, we get

$$\begin{aligned}\mu^*(E) &\geq \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*\left(\left(\bigcup_{j=1}^{\infty} A_j\right)^c\right) \\ &\geq \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)\right) + \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)^c\right) \\ &\geq \mu^*(E).\end{aligned}$$

This forces all inequalities here to be equality, therefore

$$\mu^*(E) = \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)\right) + \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)^c\right)$$

as desired. Finally, we need to show that the restriction still gives a measure. We already know

$$\mu^*(E) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*\left(\left(\bigcup_{j=1}^{\infty} A_j\right)^c\right)$$

for any $E \subseteq X$, then in particular take $E = \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$ to be the disjoint union, then this forces

$$\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*(\emptyset) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j).$$

Therefore $\mu^*|_{\mathcal{A}}$ is a measure. □

Definition 1.39. A measure μ is said to be complete if its domain contains all subsets of null sets.

Example 1.40. Let $X = \{a, b\}$, $\mathcal{A} = \{\emptyset, \{a, b\}\}$. Define $\mu : \mathcal{A} \rightarrow [0, \infty]$ by setting $\mu^*(X) = 0$, $\mu^*(\emptyset) = 0$. This is not a complete measure because $\{a\} \notin \mathcal{A}$.

Theorem 1.41. Let \mathcal{A} be the collection of all μ^* -measurable sets, then the measure $\mu^*|_{\mathcal{A}}$ is complete.

Proof. Let N be any null set in \mathcal{A} , i.e., $\mu^*(N) = 0$. Take an arbitrary subset $A \subseteq N$, we need to show $A \in \mathcal{A}$. Since $\mu^*(N) = 0$, then $\mu^*(A) = 0$ as well. For any $E \subseteq X$, we prove $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$. It is clear that

$$\begin{aligned} \mu^*(E) &\leq \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &\leq \mu^*(A) + \mu^*(E \cap A^c) \\ &\leq \mu^*(N) + \mu^*(E \cap A^c) \\ &= \mu^*(E \cap A^c) \\ &= \mu^*(E). \end{aligned}$$

by the subadditivity of μ^* . □

Definition 1.42. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra. A function $\mu_0 : \mathcal{A} \rightarrow [0, \infty]$ is a pre-measure if

i. $\mu_0(\emptyset) = 0$,

ii. if $A_j \in \mathcal{A}$ for all $j \in \mathbb{N}$ with $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$, and they are pairwise disjoint, then $\mu_0\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu_0(A_j)$.

Theorem 1.43. Let μ_0 be a pre-measure, then $\mu_0(A) \leq \mu_0(B)$ if $A, B \in \mathcal{A}$ are such that $A \subseteq B$.

Proof. We write $B = (B \setminus A) \cup A$, where $B \setminus A = B \cap A^c \in \mathcal{A}$, therefore

$$\begin{aligned} \mu_0(B) &= \mu_0(B \setminus A) + \mu_0(A) \\ &\geq \mu_0(A). \end{aligned}$$

□

Definition 1.44. Given a pre-measure μ_0 , we extend it to an outer measure as follows: for any $E \subseteq X$, define $\mu^*(E) = \inf\left\{\sum_{j=1}^{\infty} \mu_0(A_j) : E \subseteq \bigcup_{j=1}^{\infty} A_j, A_j \in \mathcal{A}\right\}$.

Theorem 1.45. Let μ^* be the outer measure induced by μ_0 specified in Definition 1.44, then

i. $\mu^*|_{\mathcal{A}} = \mu_0$, or equivalently, for any $A \in \mathcal{A}$, we have $\mu^*(A) = \mu_0(A)$;

ii. if $A \in \mathcal{A}$, then A is μ^* -measurable.

Proof. i. We want to show that for any $E \in \mathcal{A}$, $\mu^*(E) = \mu_0(E)$. To show $\mu^*(E) \leq \mu_0(E)$, we choose $A_1 = E \in \mathcal{A}$, and $A_j = \emptyset$ for all $j \geq 2$, then $E \subseteq \bigcup_{j=1}^{\infty} A_j$, therefore

$$\begin{aligned} \mu^*(E) &\leq \sum_{j=1}^{\infty} \mu_0(A_j) \\ &= \mu_0(E). \end{aligned}$$

It now suffices to show that $\mu_0(E)$ is a lower bound of $\left\{\sum_{j=1}^{\infty} \mu_0(A_j) : E \subseteq \bigcup_{j=1}^{\infty} A_j, A_j \in \mathcal{A}\right\}$. Let $A_j \in \mathcal{A}$ and

$\bigcup_{j=1}^{\infty} A_j \supseteq E$. We prove that $\mu_0(E) \leq \sum_{j=1}^{\infty} \mu_0(A_j)$. For any $n \in \mathbb{N}$, define $B_n = E \cap \left(A_n \setminus \bigcup_{j=1}^{n-1} A_j\right)$, therefore

$\bigcup_{n=1}^{\infty} B_n = E \cap \left(\bigcup_{j=1}^{\infty} A_j \right) = E$ where B_n 's are disjoint. We have

$$\begin{aligned} \mu_0(E) &= \mu_0 \left(\bigcup_{n=1}^{\infty} B_n \right) \\ &= \sum_{n=1}^{\infty} \mu_0(B_n) \\ &\leq \sum_{n=1}^{\infty} \mu_0(A_n) \\ &= \sum_{j=1}^{\infty} \mu_0(A_j). \end{aligned}$$

- ii. For any $A \in \mathcal{A}$, we want to prove that $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for all $E \subseteq X$. It suffices to show that for any $E \subseteq X$, we have $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$.

Pick arbitrary $\varepsilon > 0$, then $\mu^*(E) + \varepsilon$ is not a lower bound of $\left\{ \sum_{j=1}^{\infty} \mu_0(A_j) : E \subseteq \bigcup_{j=1}^{\infty} A_j, A_j \in \mathcal{A} \right\}$. Therefore, there exists some $A_j \in \mathcal{A}$ such that $E \subseteq \bigcup_{j=1}^{\infty} A_j$ and $\sum_{j=1}^{\infty} \mu_0(A_j) \leq \mu^*(E) + \varepsilon$. Since $\mu_0(A_j) = \mu_0(A_j \cap A) + \mu_0(A_j \cap A^c)$, then

$$\begin{aligned} \sum_{j=1}^{\infty} \mu_0(A_j) &= \sum_{j=1}^{\infty} \mu_0(A_j \cap A) + \sum_{j=1}^{\infty} \mu_0(A_j \cap A^c) \\ &= \sum_{j=1}^{\infty} \mu^*(A_j \cap A) + \sum_{j=1}^{\infty} \mu^*(A_j \cap A^c) \\ &\geq \mu^* \left(\left(\bigcup_{j=1}^{\infty} A_j \right) \cap A \right) + \mu^* \left(\left(\bigcup_{j=1}^{\infty} A_j \right) \cap A^c \right) \\ &\geq \mu^*(E \cap A) + \mu^*(E \cap A^c). \end{aligned}$$

Let $\varepsilon \rightarrow 0$, then $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$, as desired. □

Theorem 1.46. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra, and let μ_0 be a pre-measure on \mathcal{A} . Define $\mathcal{M}(\mathcal{A})$ to be the σ -algebra generated by \mathcal{A} .

- The outer measure μ^* induced by μ_0 defines a measure function on $\mathcal{M}(\mathcal{A})$, and $\mu^*|_{\mathcal{A}} = \mu_0$.
- If $\tilde{\mu}$ is another measure on $\mathcal{M}(\mathcal{A})$ that extends μ_0 , then $\tilde{\mu}(E) \leq \mu^*(E)$ for all $E \subseteq \mathcal{M}(\mathcal{A})$, with equality if and only if $\mu^*(E) < \infty$.
- If μ_0 is σ -finite, i.e., $X = \bigcup_{j=1}^{\infty} A_j$ with $A_j \in \mathcal{A}$ and $\mu_0(A_j) < \infty$ for all j , then $\mu^*|_{\mathcal{M}(\mathcal{A})}$ is the unique extension of μ_0 to a measure on $\mathcal{M}(\mathcal{A})$.

Proof. a. Let \mathcal{B} be the set of all μ^* -measurable sets, then $\mu^*|_{\mathcal{B}}$ is a measure on \mathcal{B} that extends μ_0 . By the fundamental theorem of measure theory, we know \mathcal{B} is a σ -algebra. In particular, $\mathcal{B} \supseteq \mathcal{A}$, therefore $\mathcal{B} \supseteq \mathcal{M}(\mathcal{A})$. That means $\mu^*|_{\mathcal{M}(\mathcal{A})}$ is a measure as well.

- Let $\tilde{\mu}$ be any measure on $\mathcal{M}(\mathcal{A})$ that extends μ_0 . We first show that for all $E \in \mathcal{M}(\mathcal{A})$, then $\tilde{\mu}(E) \leq \mu^*(E)$. Recall that $\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu_0(A_j) : E \subseteq \bigcup_{j=1}^{\infty} A_j, A_j \in \mathcal{A} \right\}$. Given a cover $E \subseteq \bigcup_{j=1}^{\infty} A_j$ and fix $A_j \in \mathcal{A}$.

Therefore,

$$\begin{aligned}\tilde{\mu}(E) &\leq \tilde{\mu}\left(\bigcup_{j=1}^{\infty} A_j\right) \\ &\leq \sum_{j=1}^{\infty} \tilde{\mu}(A_j) \\ &= \sum_{j=1}^{\infty} \mu_0(A_j),\end{aligned}$$

therefore $\tilde{\mu}(E) \leq \mu^*(E)$. Assume we have $\mu^*(E) < \infty$, and we want to show that $\tilde{\mu}(E) = \mu^*(E)$. It suffices to show $\mu^*(E) \leq \tilde{\mu}(E)$.

Claim 1.47. Let $A_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, then $\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) = \tilde{\mu}\left(\bigcup_{j=1}^{\infty} A_j\right)$.

Subproof. Note that $\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}(\mathcal{A})$, then we can just work on $\mathcal{M}(\mathcal{A})$. Consider $\mu^*|_{\mathcal{M}(\mathcal{A})}$ and $\tilde{\mu}$ are measures on $\mathcal{M}(\mathcal{A})$. Let $E_n = \bigcup_{j=1}^n A_j$ for all $n \in \mathbb{N}$, then we have a nested increasing sequence of E_n 's. In particular, we know $\bigcup_{n=1}^{\infty} E_n = \bigcup_{j=1}^{\infty} A_j$. Therefore

$$\begin{aligned}\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) &= \mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \\ &= \lim_{n \rightarrow \infty} \mu^*(E_n) \\ &= \lim_{n \rightarrow \infty} \mu^*\left(\bigcup_{j=1}^n A_j\right) \\ &= \lim_{n \rightarrow \infty} \mu_0\left(\bigcup_{j=1}^n A_j\right) \\ &= \lim_{n \rightarrow \infty} \tilde{\mu}\left(\bigcup_{j=1}^n A_j\right) \\ &= \tilde{\mu}\left(\bigcup_{j=1}^{\infty} A_j\right)\end{aligned}$$

by continuity from below and closure of finite union. ■

We know from the claim that

$$\begin{aligned}\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) &= \lim_{n \rightarrow \infty} \mu_0\left(\bigcup_{j=1}^n A_j\right) \\ &\leq \lim_{n \rightarrow \infty} \sum_{j=1}^n \mu_0(A_j) \\ &= \sum_{j=1}^{\infty} \mu_0(A_j).\end{aligned}$$

Take arbitrary $\varepsilon > 0$, then consider $\mu^*(E) + \varepsilon$, which is not a lower bound of the set anymore. Therefore, there exists $A_j \in \mathcal{A}$ for each $j \in \mathbb{N}$ such that $E \subseteq \bigcup_{j=1}^{\infty} A_j$ and that $\sum_{j=1}^{\infty} \mu_0(A_j) \leq \mu^*(E) + \varepsilon$. In particular, this means

$$\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \mu^*(E) + \varepsilon. \text{ Since } \mu^*(E) < \infty, \text{ then}$$

$$\begin{aligned} \mu^*\left(\bigcup_{j=1}^{\infty} A_j \setminus E\right) &= \mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) - \mu^*(E) \\ &< \varepsilon. \end{aligned}$$

Now that

$$\begin{aligned} \mu^*(E) &\leq \mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \\ &= \tilde{\mu}\left(\bigcup_{j=1}^{\infty} A_j\right) \\ &= \tilde{\mu}(E) + \tilde{\mu}\left(\bigcup_{j=1}^{\infty} A_j \setminus E\right) \\ &< \tilde{\mu}(E) + \varepsilon \end{aligned}$$

by the claim. Therefore, for any $\varepsilon > 0$, we have $\mu^*(E) \leq \tilde{\mu}(E) + \varepsilon$ whenever $\mu^*(E) < \infty$. Take $\varepsilon \rightarrow 0$, we get $\mu^*(E) \leq \tilde{\mu}(E)$.

- c. Since μ_0 is σ -finite, then there exists a decomposition $X = \bigcup_{j=1}^{\infty} A_j$ for $A_j \in \mathcal{A}$ and that $\mu_0(A_j) < \infty$. For any $E \in \mathcal{M}(\mathcal{A})$, then

$$\begin{aligned} E &= E \cap X \\ &= E \cap \left(\bigcup_{j=1}^{\infty} A_j\right) \\ &= \bigcup_{j=1}^{\infty} (E \cap A_j) \end{aligned}$$

and

$$\begin{aligned} \mu^*(E) &= \mu^*\left(\bigcup_{j=1}^{\infty} (E \cap A_j)\right) \\ &= \sum_{j=1}^{\infty} \mu^*(E \cap A_j) \\ &= \sum_{j=1}^{\infty} \tilde{\mu}(E \cap A_j) \\ &= \tilde{\mu}\left(\bigcup_{j=1}^{\infty} (E \cap A_j)\right) \\ &= \tilde{\mu}(E) \end{aligned}$$

since $\mu^*(E \cap A_j) \leq \mu^*(A_j) = \mu_0(A_j) < \infty$.

□

1.4 BOREL MEASURE

Recall that the Borel σ -algebra $\mathcal{B}_{\mathbb{R}}$ is the smallest σ -algebra containing all open sets. Let \mathcal{G} be the set of all open sets in \mathbb{R} with respect to the standard topology. Therefore $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{G})$. We can in fact use something smaller than \mathcal{G} .

Theorem 1.48. $\mathcal{B}_{\mathbb{R}}$ is a σ -algebra generated by

- a. $\mathcal{A}_0 = \{(a, b) : a, b \in \mathbb{R}, a < b\}$;
- b. $\mathcal{A}_1 = \{(a, b] : a, b \in \mathbb{R}, -\infty \leq a < b < \infty\} \cup \{(a, \infty) : -\infty \leq a < \infty\} \cup \{\emptyset\}$.

Any member in \mathcal{A}_1 is called an h -interval.

Proof. a. We want to show that $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{A})$. Obviously $\mathcal{A}_0 \subseteq \mathcal{G}$, then $\mathcal{M}(\mathcal{G})$ is a σ -algebra containing \mathcal{A}_0 , then $\mathcal{M}(\mathcal{A}_0) \subseteq \mathcal{M}(\mathcal{G}) = \mathcal{B}_{\mathbb{R}}$. Conversely, recall that any open subset in \mathbb{R} is a σ -union of open intervals, therefore $\mathcal{G} \subseteq \mathcal{M}(\mathcal{A})$, so $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{G}) \subseteq \mathcal{M}(\mathcal{A}_0)$, therefore $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{A}_0)$.

- b. We first show that $\mathcal{M}(\mathcal{A}_1) \subseteq \mathcal{B}_{\mathbb{R}}$. Since $\mathcal{M}(\mathcal{A}_1)$ is the smallest σ -algebra containing \mathcal{A}_1 , then it suffices to show that $\mathcal{A}_1 \subseteq \mathcal{B}_{\mathbb{R}}$. It is easy to see that $(a, b] = \bigcap_{n=1}^{\infty} (a, b + \frac{1}{n}) \in \mathcal{B}_{\mathbb{R}}$, and $(a, \infty) = \bigcup_{n=1}^{\infty} (a, n) \in \mathcal{B}_{\mathbb{R}}$.

We now verify that $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{A}_1)$. By a. we know $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{A}_0)$, so it suffices to show that $\mathcal{A}_0 \subseteq \mathcal{M}(\mathcal{A}_1)$. For $a < b$, we have $(a, b) = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}]$, therefore the right-hand side is a σ -union of intervals, hence belongs to $\mathcal{M}(\mathcal{A}_1)$, and we are done. □

Definition 1.49. We define \mathcal{A}_2 to be the collection of finite disjoint unions of h -intervals, e.g., $\bigcup_{j=1}^n (a_j, b_j]$, then \mathcal{A}_2 is an algebra.

Definition 1.50. A function on \mathbb{R} is said to be right continuous if $\lim_{x \rightarrow x_0^+} F(x) = F(x_0)$.

Theorem 1.51. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be increasing and right continuous. Let $I_j = (a_j, b_j]$ for $j = 1, \dots, n$ be disjoint h -intervals. We define the premeasure μ_0 on \mathcal{A}_2 by $\mu_0(\emptyset) = 0$ and $\mu_0\left(\bigcup_{j=1}^n (a_j, b_j]\right) = \sum_{j=1}^n [F(b_j) - F(a_j)]$.

Proof. First one can check that μ_0 is well-defined, that is, given any partition of h -interval, the μ_0 -measurements on the interval are the same.

Second, we need to show that μ_0 satisfies σ -additivity, that is, if $\bigcup_{j=1}^{\infty} I_j \in \mathcal{A}_2$ such that I_j 's are disjoint, then

$$\mu_0\left(\bigcup_{j=1}^{\infty} I_j\right) = \sum_{j=1}^{\infty} \mu_0(I_j). \text{ It is easy to verify finite additivity, so we now assume}$$

$$\bigcup_{j=1}^{\infty} I_j = I = (a, b] \in \mathcal{A}_2$$

for $-\infty \leq a < b < \infty$, then we will show that

$$F(b) - F(a) = \mu_0(I) = \sum_{j=1}^{\infty} \mu_0(I_j)$$

for $I_j = (a_j, b_j]$.

To show $\mu_0(I) \geq \sum_{j=1}^{\infty} \mu_0(I_j)$, we know $F(b) - F(a) \geq \sum_{j=1}^n [F(b_j) - F(a_j)]$, therefore taking the limit of $n \rightarrow \infty$ gives $F(b) - F(a) \geq \sum_{j=1}^{\infty} \mu_0(I_j)$.

To show $\mu_0(I) \leq \sum_{j=1}^{\infty} \mu(I_j)$, since F is right continuous, then for all $\varepsilon > 0$, there exist $\delta > 0$ such that $F(a + \delta) - F(a) < \varepsilon$. Therefore, for every $j > 0$, there exists $\delta_j > 0$ such that $F(b_j + \delta_j) - F(b_j) < 2^{-j}\varepsilon$, then

$$\begin{aligned} [a + \delta, b] &\subseteq (a, b] \\ &= \bigcup_{j=1}^{\infty} (a_j, b_j] \\ &= \bigcup_{j=1}^{\infty} (a_j, b_j + \delta_j). \end{aligned}$$

By compactness, there exists some $N \in \mathbb{N}$ such that $[a + \delta, b] \subseteq \bigcup_{j=1}^N (a_j, b_j + \delta_j)$. Assume $b_j + \delta_j \in (a_{j+1}, b_{j+1}]$, then

$$\begin{aligned} \mu_0(I) &= \mu_0((a, b]) \\ &= F(b) - F(a) \\ &\leq F(b) - F(a + \delta) + \varepsilon \\ &\leq F(b_N + \delta_N) - F(a + \delta) + \varepsilon \\ &= F(b_N + \delta_N) - F(a_N) + F(a_N) - F(a + \delta) + \varepsilon \\ &= F(b_N + \delta_N) - F(a_N) + \sum_{j=1}^{N-1} [F(a_{j-1}) - F(a_j)] + \varepsilon \\ &\leq F(b_N + \delta_N) - F(a_N) + \sum_{j=1}^{N-1} [F(b_j + \delta_j) - F(a_j)] + \varepsilon \\ &= \sum_{j=1}^N [F(b_j + \delta_j) - F(a_j)] + \varepsilon \\ &= \sum_{j=1}^N [F(b_j + \delta_j) - F(b_j) + F(b_j) - F(a_j)] + \varepsilon \\ &= \sum_{j=1}^N [F(b_j + \delta_j) - F(b_j)] + \sum_{j=1}^N [F(b_j) - F(a_j)] + \varepsilon \\ &\leq \sum_{j=1}^N 2^{-j}\varepsilon + \sum_{j=1}^N \mu_0(I_j) + \varepsilon \\ &\leq 2\varepsilon + \sum_{j=1}^{\infty} \mu_0(I_j) \end{aligned}$$

since F is increasing. Let $\varepsilon \rightarrow 0$ and we are done. \square

Theorem 1.52. Let F be increasing and right-continuous, then

- there is a unique measure μ_F on \mathbb{R} such that $\mu_F((a, b]) = F(b) - F(a)$ for all $a, b \in \mathbb{R}$;
- if G is another increasing and right-continuous function, then $\mu_F = \mu_G$ if and only if $F - G$ is a constant function;
- if μ is a Borel measure on \mathbb{R} that is finite on all bounded Borel sets, i.e., a set $S \subseteq \mathbb{R}$ contained in $[-M, M]$ for some $M \in \mathbb{R}$, then

$$F(x) = \begin{cases} \mu((0, x]), & x > 0 \\ 0, & x = 0 \\ -\mu(x, 0], & x < 0 \end{cases}$$

is an increasing function and right-continuous, and $\mu_F = \mu$.

Proof. a. Consider $\mathbb{R} = \bigcup_{j=-\infty}^{\infty} (j, j+1]$, then the pre-measure $\mu_0((j, j+1]) = F(j+1) - F(j) < \infty$ defined on h -intervals is σ -finite. Therefore there exists a unique extension of measure μ of μ_0 on $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{A}_2)$ such that $\mu|_{\mathcal{A}_2} = \mu_0$.

b. We have $\mu_F((a, b]) = F(b) - F(a)$ and $\mu_G((a, b]) = G(b) - G(a)$, then

$$\begin{aligned}\mu_F((a, b]) = \mu_G((a, b]) &\iff F(b) - F(a) = G(b) - G(a) \\ &\iff F(b) - G(b) = G(a) - F(a) \\ &\iff F - G \text{ is constant.}\end{aligned}$$

c. First note that F is an increasing function since the measure function is increasing. Take any $x_0 \in \mathbb{R}$, we want to show that $\lim_{x \rightarrow x_0^+} F(x) = F(x_0)$. We prove this by cases, either $x_0 = 0$, $x_0 > 0$, or $x_0 < 0$. We will only prove the first case, but the two other cases are analogous. Suppose $x_0 = 0$, take a nested sequence of intervals $E_n = (0, \frac{1}{n}]$, with $E_n \supseteq E_{n+1}$ for all $n \in \mathbb{N}$, then

$$\begin{aligned}\lim_{x \rightarrow 0^+} F(x) &= \lim_{x \rightarrow 0^+} \mu((0, x]) \\ &= \lim_{n \rightarrow \infty} \mu((0, \frac{1}{n}]) \\ &= \lim_{n \rightarrow \infty} \mu(E_n) \\ &= \mu\left(\bigcap_{n=1}^{\infty} E_n\right) \\ &= \mu(\emptyset) \\ &= 0 \\ &= F(0)\end{aligned}$$

since $\mu(E_1) < \infty$.

□

Definition 1.53. Suppose F is increasing and right-continuous, then we can use F to create μ_0 on \mathcal{A}_2 , and get an outer measure μ^* induced by μ_0 . Let \mathcal{A} be the collection of all μ^* -measurable sets, then $\mu^*|_{\mathcal{A}}$ is a measure. Note that $\mathcal{A} \supseteq \mathcal{B}_{\mathbb{R}}$: since μ_F is only defined on $\mathcal{B}_{\mathbb{R}}$, then $\mu^*|_{\mathcal{A}}$ becomes the extension of μ_F on \mathcal{A} . We denote this measure to be $\bar{\mu}_F$, as the extension of μ_F , called the Lebesgue-Stieltjes measure.

Remark 1.54. In particular, if $F(x) = x$ for all $x \in \mathbb{R}$, then $\bar{\mu}_F$ is called a Lebesgue measure, denoted by \mathbf{m} , with $\mathbf{m}((a, b]) = F(b) - F(a) = b - a$.

Definition 1.55. Let μ be a Lebesgue-Stieltjes measure associated to an increasing and right-continuous function F . Let \mathcal{M}_{μ} be the domain of the measure μ , which gives the collection of measurable sets. For any measurable set $E \in \mathcal{M}_{\mu}$, we have

$$\begin{aligned}\mu(E) &= \inf \left\{ \sum_{j=1}^{\infty} [F(b_j) - F(a_j)] : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j] \right\} \\ &= \inf \left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j]) : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j] \right\}.\end{aligned}$$

Theorem 1.56. For all $E \in \mathcal{M}_{\mu}$, we have

$$\mu^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j]) : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j] \right\}.$$

Proof. Let $\tilde{\mu}(E)$ be the right-hand side of this equation, so we will show that $\mu(E) = \tilde{\mu}(E)$. Note that we have a partition

$$(a_j, b_j) = \bigcup_{k=1}^{\infty} I_k^{(j)},$$

where $I_k^{(j)} = (b_j - \frac{1}{2^k}(b_j - a_j), b_j - \frac{1}{2^{k+1}}(b_j - a_j)]$. Now $E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j)$, so $E \subseteq \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} I_k^{(j)}$, and thus

$$\begin{aligned} \sum_{j=1}^{\infty} \mu((a_j, b_j)) &= \sum_{j=1}^{\infty} \mu\left(\bigcup_{k=1}^{\infty} I_k^{(j)}\right) \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \mu(I_k^{(j)}). \end{aligned}$$

Because $\left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j)) : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j) \right\} \subseteq \left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j]) : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j] \right\}$, then $\tilde{\mu}(E) \geq \mu(E)$.

We now show that $\mu(E) \geq \tilde{\mu}(E)$. Pick arbitrary $\varepsilon > 0$, then we know $\mu(E) + \varepsilon$ is not a lower bound of the set $\left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j]) : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j] \right\}$, hence there exists $(a_j, b_j]$ for $j \geq 1$ such that $E \subseteq \bigcup_{j \geq 1} (a_j, b_j]$. Therefore $\sum_{j=1}^{\infty} \mu((a_j, b_j]) \leq \mu(E) + \varepsilon$. By the right continuity of F , for $\varepsilon \cdot 2^{-j} > 0$, there exists $\delta_j > 0$ such that $F(b_j + \delta_j) - F(b_j) < \varepsilon \cdot 2^{-j}$, then $E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j + \delta_j)$. We know

$$\begin{aligned} \tilde{\mu}(E) &\leq \sum_{j=1}^{\infty} \mu((a_j, b_j + \delta_j)) \\ &= \sum_{j=1}^{\infty} [F(b_j + \delta_j) - F(a_j)] \\ &= \sum_{j=1}^{\infty} [F(b_j + \delta_j) - F(b_j) + F(b_j) - F(a_j)] \\ &\leq \sum_{j=1}^{\infty} [F(b_j + \delta_j) - F(b_j)] + \sum_{j=1}^{\infty} [F(b_j) - F(a_j)] \\ &< \sum_{j=1}^{\infty} \varepsilon \cdot 2^{-j} + \sum_{j=1}^{\infty} \mu((a_j, b_j]) \\ &< \varepsilon + \mu(E) + \varepsilon \\ &= \mu(E) + 2\varepsilon. \end{aligned}$$

Taking small enough ε finishes the proof. \square

Remark 1.57. The union of h -intervals may not be open, so often times we use the characterization in [Theorem 1.56](#) instead.

Theorem 1.58. For any $E \subseteq \mathcal{M}_{\mu}$, we have

$$\mu(E) = \inf\{\mu(U) : \text{open } U \supseteq E\} = \sup\{\mu(K) : \text{compact } K \subseteq E\}.$$

Proof. Let $\tilde{\mu}(E) = \inf\{\mu(U) : \text{open } U \supseteq E\}$. First, $\mu(E) \leq \tilde{\mu}(E)$: since $E \subseteq U$, then $\mu(E) \leq \mu(U)$, therefore $\mu(E) \leq \tilde{\mu}(E)$. To see $\tilde{\mu}(E) \leq \mu(E)$, we have $\mu(E) + \varepsilon$ is not a lower bound of $\left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j]) : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j] \right\}$,

then there exists (a_j, b_j) for each $j \in \mathbb{N}$ such that $E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j)$, and that $\sum_{j=1}^{\infty} \mu((a_j, b_j)) \leq \mu(E) + \varepsilon$. Therefore, take U to be the open set $\bigcup_{j=1}^{\infty} (a_j, b_j)$, then

$$\tilde{\mu}(E) \leq \mu(U) \leq \sum_{j=1}^{\infty} \mu((a_j, b_j)) \leq \mu(E) + \varepsilon$$

as desired.

Now let $\nu(E) = \sup\{\mu(K) : \text{compact } K \subseteq E\}$. We note that if $K \subseteq E$, then $\mu(K) \leq \mu(E)$, therefore $\nu(E) \leq \mu(E)$. To prove the reverse inequality, we consider the following cases:

- E is bounded.
 - E is closed. Since E is bounded and closed, it is compact over \mathbb{R} , thus $\mu(E) \leq \nu(E)$.
 - E is bounded but not closed. We have $\mu(\bar{E} \setminus E) = \inf\{\mu(U) : \text{open } U \supseteq \bar{E} \setminus E\}$. For any $\varepsilon > 0$, there exists an open set U such that $U \supseteq \bar{E} \setminus E$ and $\mu(U) \leq \mu(\bar{E} \setminus E) + \varepsilon$. Set $K = \bar{E} \setminus U$, then K is compact. Since all measures here are finite, we have

$$\begin{aligned} \mu(K) &= \mu(E) - \mu(E \cap U) \\ &= \mu(E) - [\mu(U) - \mu(U \setminus E)] \\ &\geq \mu(E) - \mu(U) + \mu(\bar{E} \setminus E) \\ &\geq \mu(E) - \varepsilon. \end{aligned}$$

Therefore $\nu(E) \geq \mu(E) - \varepsilon$, and we are done by taking $\varepsilon \rightarrow 0$.

- E is not bounded. Suppose $E = \bigcup_{j=-\infty}^{\infty} ((j, j+1] \cap E)$, then denote $E_j = E \cap (j, j+1]$, which is bounded. Therefore, we know the statement is true for each E_j for $j \geq 1$, thus $\mu(E_j) = \sup\{\mu(K) : \text{compact } K \subseteq E_j\}$. Take arbitrary $\varepsilon > 0$, then $\mu(E_j) - \frac{1}{3}\varepsilon \cdot 2^{-|j|}$ is not the upper bound of $\{\mu(K) : \text{compact } K \subseteq E_j\}$, then there exists a compact set $K_j \subseteq E_j$ such that $\mu(K_j) \geq \mu(E_j) - \frac{1}{3}\varepsilon \cdot 2^{-|j|}$. Since $K_j \subseteq E_j$ and E_j 's are disjoint, then K_j 's are disjoint. Therefore, for $n \in \mathbb{N}$, set $H_n = \bigcup_{j=-n}^n K_j$, which is a finite disjoint union of compact sets, so this is a compact set. But $H_n \subseteq E$, then

$$\begin{aligned} \mu(H_n) &= \mu\left(\bigcup_{j=-n}^n K_j\right) \\ &= \sum_{j=-n}^n \mu(K_j) \\ &\geq \sum_{j=-n}^n \mu(E_j) - \frac{\varepsilon}{3} \sum_{j=-n}^n 2^{-|j|} \\ &\geq \sum_{j=-n}^n \mu(E_j) - \frac{\varepsilon}{3} \sum_{j=-\infty}^{\infty} 2^{-|j|} \\ &\geq \sum_{j=-n}^n \mu(E_j) - \varepsilon. \end{aligned}$$

Note that H_n still depends on n , so we should not take $n \rightarrow \infty$ here. Since $\nu(E)$ is the upper bound of $\mu(K)$'s for compact $K \subseteq E$, then $\nu(E) \geq \mu(H_n)$, therefore

$$\nu(E) \geq \sum_{j=-n}^n \mu(E_j) - \varepsilon$$

$$= \mu \left(\bigcup_{j=-n}^n E_j \right) - \varepsilon.$$

Take $n \rightarrow \infty$, then

$$\begin{aligned} \nu(E) &\geq \lim_{n \rightarrow \infty} \mu \left(\bigcup_{j=-n}^n E_j \right) - \varepsilon \\ &= \mu \left(\bigcup_{j=-\infty}^{\infty} E_j \right) - \varepsilon \\ &= \mu(E) - \varepsilon. \end{aligned}$$

Let $\varepsilon \rightarrow 0$, we are done. □

Theorem 1.59. Let $E \subseteq \mathbb{R}$, then the following are equivalent:

- a. $E \in \mathcal{M}_\mu$;
- b. $E = V \setminus N_1$, where V is a G_δ -set and $\mu(N_1) = 0$;
- c. $E = H \cup N_2$, where H is a F_σ -set and $\mu(N_2) = 0$.

Proof. • $b. \Rightarrow a.$: note that $\mathcal{M}_\mu \supseteq \mathcal{B}_\mathbb{R}$, then both V and N_1 are measurable, therefore E is measurable, i.e., $E \in \mathcal{M}_\mu$.

• $c. \Rightarrow a.$: similar to the case above.

• $a. \Rightarrow b.$:

- If $\mu(E) < \infty$, recall $\mu(E) = \inf\{\mu(U) : \text{open } U \supseteq E\}$. For any $k \in \mathbb{N}$, consider $2^{-k} > 0$, then there exists open subset $U_k \supseteq E$ such that $\mu(U_k) \leq \mu(E) + 2^{-k}$. Let $V = \bigcap_{k=1}^{\infty} U_k$ be a G_δ -set, then $V \supseteq E$ as well. It suffices to show that $V \setminus E$ is a null set. We know

$$\begin{aligned} \mu(V) &= \mu \left(\bigcap_{k=1}^{\infty} U_k \right) \\ &\leq \mu(U_k) \\ &\leq \mu(E) + 2^{-k} \end{aligned}$$

for all $k \in \mathbb{N}$. Since $\mu(V)$ and $\mu(E)$ are independent of k , then take $k \rightarrow \infty$, therefore $\mu(V) \leq \mu(E)$. But since $E \subseteq V$, then $\mu(E) \leq \mu(V)$, therefore this gives equality. Since $\mu(E) < \infty$, then $\mu(V) - \mu(E) = 0$, then $\mu(V \setminus E) = 0$ by additivity.

- If $\mu(E) = \infty$, then the proof can be done using the previous case.

• $a. \Rightarrow c.$: the proof is similar to the case above. □

Theorem 1.60. Let $E \in \mathcal{M}_\mu$, and suppose $\mu(E) < \infty$. For any $\varepsilon > 0$, there exists some set A that is a finite union of open intervals such that $\mu(E \Delta A) = \mu((E \setminus A) \cup (A \setminus E)) < \varepsilon$.

Proof. Note that $\mu(E) = \sup\{\mu(K) : \text{compact } K \subseteq E\}$. For any $\varepsilon > 0$, there exists compact $K \subseteq E$ such that $\mu(E) - \frac{\varepsilon}{2} < \mu(K)$, which is equivalent to having $\mu(E \setminus K) < \frac{\varepsilon}{2}$. Similarly, recall that $\mu(E) = \inf\{\mu(U) : \text{open } U \supseteq E\}$, but open set U on \mathbb{R} is characterized as a union of open intervals, therefore this is just $\mu(E) = \inf\left\{ \sum_{j=1}^{\infty} \mu((a_j, b_j)) : \right.$

$\bigcup_{j=1}^{\infty} (a_j, b_j) \supseteq E$. Therefore, there exists $\bigcup_{j=1}^{\infty} I_j \supseteq E$, where I_j is open interval for each j , such that $\mu\left(\bigcup_{j=1}^{\infty} I_j\right) < \mu(E) + \frac{\varepsilon}{2}$. Since $\mu(E)$ is finite, then $\mu\left(\bigcup_{j=1}^{\infty} I_j \setminus E\right) < \frac{\varepsilon}{2}$. Now $K \subseteq E \subseteq \bigcup_{j=1}^{\infty} I_j$, but K is compact, so there exists I_1, \dots, I_n such that their union cover K . Set $A = \bigcup_{j=1}^m I_j$, and we are done. \square

Definition 1.61. Let $F(x) = x$ be a function for all $x \in \mathbb{R}$, then μ_F is called the Lebesgue measure defined by $m((a, b]) = b - a$. The domain of m is \mathcal{L} .

For $E \subseteq \mathbb{R}$ and $s, r \in \mathbb{R}$, we denote $E + s = \{x + s : x \in E\}$ and $rE = \{rx : x \in E\}$.

Theorem 1.62. If $E \in \mathcal{L}$, then $m(E + s) = m(E)$ and $m(rE) = |r|m(E)$.

Proof. We prove the first claim. For any $E \in \mathcal{L}$ and $s \in \mathbb{R}$, define $m_s = m(E + s)$, then this is a measure.

Claim 1.63. For any $E \in \mathcal{L}$, $m_s(E) = m(E)$.

Subproof. First note that this is true if E is a finite (disjoint) union of h -intervals of m_s , as m extends the pre-measure μ_0 . On $\mathcal{B}_{\mathbb{R}}$, the extension is unique, so $m_s(E) = m(E)$ if $E \in \mathcal{B}_{\mathbb{R}}$. Moreover, recall $E \in \mathcal{L}$ if and only if $E = V \setminus N_1$ for $V \in \mathcal{B}_{\mathbb{R}}$. Therefore this is true for all $E \in \mathcal{L}$. \blacksquare

\square

Definition 1.64. The Cantor set \mathcal{C} is constructed iteratively from the interval $[0, 1]$, that for any remaining connected interval $[m, n]$, we delete the subinterval $(m + \frac{1}{3}(n - m), m + \frac{2}{3}(n - m))$ from $[m, n]$.

Remark 1.65. Note that

$$\begin{aligned} m(\mathcal{C}) &= m([0, 1]) - \frac{1}{3} - \frac{2}{3^2} - \frac{2^2}{3^3} - \dots \\ &= 1 - \sum_{j=0}^{\infty} \frac{2^j}{3^{j+1}} \\ &= 1 - 1 \\ &= 0. \end{aligned}$$

Remark 1.66. If E is countable, then

$$\begin{aligned} m(E) &= \sum_{j=1}^{\infty} m(\{a_j\}) \\ &= 0. \end{aligned}$$

Theorem 1.67. The Cantor set \mathcal{C} is uncountable.

Proof. Alternatively, the Cantor set \mathcal{C} can be represented as

$$\mathcal{C} = \{x \in [0, 1] : x = \sum_{j=1}^{\infty} a_j 3^{-j}, a_j \in \{0, 2\}\}.$$

To prove that \mathcal{C} is uncountable, it suffices to build a surjection $f : \mathcal{C} \rightarrow [0, 1]$. For $x \in \mathcal{C}$, we have $x = \sum_{j=1}^{\infty} a_j 3^{-j}$, $a_j \in \{0, 2\}$. Set $f(x) = \sum_{j=1}^{\infty} \frac{a_j}{2} 2^{-j}$ for $\frac{a_j}{2} \in \{0, 1\}$, therefore this gives a decimal representation with base 2, so any real number in $[0, 1]$ can be represented in this form, therefore we have a surjection. \square

Theorem 1.68. Let $F \subseteq \mathbb{R}$ be such that every subset of F is Lebesgue measurable, then $m(F) = 0$.

Corollary 1.69. If $m(F) > 0$, then there exists a subset S of F such that $S \notin \mathcal{L}$.

Remark 1.70 (Banach-Tarski Paradox). Given a ball $B = S^2$, then there exists some $m \in \mathbb{N}$ such that $B = V_1 \cup \cdots \cup V_m$ is a union of subsets V_i that are not Lebesgue measurable and $m(B) \neq m(V_1 \cup \cdots \cup V_m)$.

Definition 1.71. For any $x \in \mathbb{R}$, we defined the cosets over \mathbb{Q} to be $\mathbb{Q} + x = \{r + x : r \in \mathbb{Q}\}$ for any x . This is called the coset of an additive group \mathbb{R} .

Let E be the set that contains exactly one point from each coset of \mathbb{Q} as representations, which requires the axiom of choice.

Now E allows us make a partition on \mathbb{R} .

Lemma 1.72. 1. $(E + r_1) \cap (E + r_2) = \emptyset$ if $r_1 \neq r_2$ and $r_1, r_2 \in \mathbb{Q}$.

2. $\mathbb{R} = \bigcup_{r \in \mathbb{Q}} (E + r)$.

Proof. 1. Suppose $x \in (E + r_1) \cap (E + r_2)$, then $x = e_1 + r_1 = e_2 + r_2$ for some $e_1, e_2 \in E$. Therefore $e_1 - e_2 = r_2 - r_1$, which is a non-zero rational number, therefore $0 \neq e_1 - e_2 \in \mathbb{Q}$. Therefore e_1 and e_2 are in the same coset, so $e_1 = e_2$, contradiction.

2. Obviously $\mathbb{R} \supseteq \bigcup_{r \in \mathbb{Q}} (E + r)$. Take any $x \in \mathbb{R}$, then E contains a point y from the coset $\mathbb{Q} + x$, therefore $y - x \in \mathbb{Q}$, so take $r = y - x$, then $x \in E + r$. □

Proof of Theorem 1.68. □