## MATH 526 Notes

Jiantong Liu

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Let X be a topological space with basepoint  $x_0 \in X$ . We already know two invariants,

- the fundamental group  $\pi_1(X, x_0)$ , and
- the homology groups  $H_n(X)$  for  $n \ge 0$ , which are abelian groups.

We will look at two more invariants,

- the cohomology groups  $H^n(X)$  for  $n \ge 0$ , and
- the higher homotopy groups  $\pi_n(X, x_0)$  for  $n \ge 0$ .

In particular,  $\pi_*(X, x_0)$  is a very good invariant in the following sense:

**Theorem 1.1** (Whitehead). If  $f:(X,x_0)\to (Y,y_0)$  is a map of CW-complexes, then f is a homotopy equivalence if and only if  $\pi_*(f):\pi_*(X,x_0)\to\pi_*(Y,y_0)$  is an isomorphism.

However,  $\pi_*$  is very hard to compute. On the other hand,  $H^*(X)$  is relatively easy to compute, but this is not a complete invariant. For instance,  $\mathbb{C}P^2$  and  $S^2\vee S^4$  have isomorphic cohomology groups, but they are not equivalent.  $H^*(X)$  is closely related to  $H_*(X)$ , but  $H^*(X)$  is a graded ring structure with cup product. It is contravariant in X, where  $H_*(X)$  is covariant. The cup product is defined by the composition of induced diagonal map with an external product:

$$H^{i}(X) \times H^{j}(X) \xrightarrow{\times} H^{i+j}(X \times X) \xrightarrow{\Delta^{*}} H^{i+j}(X)$$

Other things we will talk about include:

- Natural transformations  $H^i(-) \to H^j(-)$  encoded by Steenrod operations.
- $H^n(-)$  becomes a representable functor, i.e.,  $H^n(X) = [X, K(\mathbb{Z}, n)]$ , where  $K(\mathbb{Z}, n)$  is the Eilenberg-Maclane space, and the bracket indicates the homotopy classes of maps.
- Poincaré duality in  $H^*(M)$  for compact manifold M, namely the cup product gives

$$H^i(M)\otimes H^{\dim(M)-i}(M) \xrightarrow{\smile} H^{\dim(M)}(M).$$

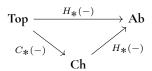
• Characteristic classes in  $H^*(X)$  associated to vector bundles over X.

Recall for a topological space X, we obtain a collection of (singular) homology groups  $H_n(X)$ , with  $H_*(X) = \bigoplus_{n \geq 0} H_n(X)$ . The functoriality of morphisms says that  $X \xrightarrow{f} Y \xrightarrow{g} Z$  induces  $f_*g_* = (fg)_* : H_*(X) \xrightarrow{f_*} H_*(Y) \xrightarrow{g_*} H_*(Z)$ . So

$$H_*(-): \text{Top} \to \text{Ab}$$

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is a well-defined functor. This factors into



Here  $C_*(-)$  is usually the singular chain, given by  $\partial: C_n(X) \to C_{n-1}(X)$ , where  $C_n(X)$  is the free abelian group generated by  $\operatorname{Hom}_{\operatorname{Top}}(\Delta^n,X) \cong \bigoplus_{\sigma:\Delta^n \to X} \mathbb{Z}\sigma$ .  $\Delta^n \subseteq \mathbb{R}^{n+1}$  is the set of tuples  $(t_0,\ldots,t_n)$  such that the coordinates sum to 1. The boundary is  $\partial \sigma = \sum_{0 \leqslant i \leqslant n} (-1)^i \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$ .

We say  $C_*(-)$  is homotopy invariant, i.e., if  $f: X \to Y$  is a homotopy equivalence, then the induced map  $C_*(X) \to C_*(Y)$  on chain complexes is a chain equivalence.

**Remark 1.2.**  $C_*^{\Delta}(X)$  and  $C_*^{\text{CW}}(X)$  are both chain equivalent to  $C_*(X)$ .