Descent Properties in Algebraic K-Theory

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These notes are meant to discuss Cisinski's paper [Cis13], and are reconstructed from a talk I gave in Spring 2025, deviated from the actual content I delivered.

1 Motivation

The main goal of the paper is to show that homotopy-invariant K-theory satisfies cdh descent, but we will recontextualize and give it more motivation, as discussed in [7].

1.1 Algebraic K-theory

Let us first discuss the notion of algebraic K-theory that we care about, i.e., in the context of algebraic geometry. This requires a brief overview of the history.

- For any scheme X, Quillen defined its algebraic K-theory to be, essentially, the algebraic K-theory of the exact category $\mathbf{Vect}(X)$ of vector bundles over S with exact sequences. Here we recall that the algebraic K-theory of an exact category $\mathcal{E}$ is just the homotopy groups of the algebraic K-theory space $\Omega BQ(\mathcal{E})$ where $Q(\mathcal{E})$ is the Quillen Q-construction of $\mathcal{E}$.[1]

- The Quillen Q-construction, being very helpful for producing K-theory spaces, eventually extended[2] to what we now know as Waldhausen S-construction, which is then used to define algebraic K-theory for Waldhausen categories, or stable $(\infty, 1)$-categories in general. Waldhausen K-theory is then the geometric realization of S-construction, c.f., [Wal06].

- Thomason-Trobaugh then noted that, the category of perfect complexes $\mathbf{Perf}(X)$ has a Waldhausen category structure (as a stable $(\infty, 1)$-category), therefore you can define the algebraic K-theory of schemes upon that, c.f., [TT90]. By [TT90, Proposition 3.10], this K-theory coincides with Quillen's K-theory whenever there exists an ample family of line bundles.

  Definition 1.1. Let X be a quasi-compact quasi-separated scheme, and set $\mathbf{Perf}(X)$ to be the category of perfect complexes on X. Suppose $\mathbf{Perf}(X)$ has globally finite Tor-amplitude, then $\mathbf{Perf}(X)$ has the structure of a Waldhausen category with cofibrations as degreewise split monomorphisms, and weak equivalences as quasi-isomorphisms.

  i. We define the K-theory $K(X)$ of X to be the K-theory of this Waldhausen category.

  ii. We define the K-theory $K(X \text{ on } Y)$ is the K-theory spectrum given by the Waldhausen subcategory of the perfect complexes on X which are acyclic on $X \backslash Y$ for some closed subspace Y of X. This stands in the place as "K-theory with support."

- We should comment that the same idea allowed people to define algebraic K-theory on ∞-categories, and characterize it by a universal property, c.f., [BGT13], but we digress.

[1] We should remark that for Noetherian schemes Quillen defined a different notion of algebraic K-theory, which coincides with our notion of algebraic K-theory when X is Noetherian.

[2] In the sense that, for any exact category, the two notions are equivalent.

An important I would make is that so far, all the K-theory groups K_n defined so far are for $n \geqslant 0$, therefore when interpreting the corresponding spectrum, they are connective. We will now introduce an extension of Thomason-Trobaugh K-theory to the negative K-groups. This involves Bass delooping which was originally studied for topological K-theory.

Definition 1.2. Let A be an ordinary ring. For $n > 0$, we define $K_{-n}(A)$ to be the cokernel of

$$K_{-n+1}(A[t]) \oplus K_{-n+1}(A[t^{-1}]) \to K_{-n+1}(A[t, t^{-1}]).$$

The defined groups $\{K_n(A)\}_{n \in \mathbb{Z}}$ is called Bass K-theory.

Note that for $K_n(A)$ with $n > 0$, this is part of the statement of the Fundamental theorem of Algebraic K-theory. Correspondingly, on the level of schemes X, we see producing the non-connective spectrum actually involves a delooping technique from the connective spectra, c.f., [TT90] for details.

The following result, c.f., [TT90, Proposition 6.8], allows us to recover Thomason-Trobaugh K-theory from Bass K-theory.

Proposition 1.3. Let X be a regular Noetherian scheme, then denote $X[T] = X \otimes_{\mathbb{Z}} \mathbb{Z}[T]$, then

a. the pullback $p^* : K(X) \simeq K(X[T])$ of the projection $p : K(X[T]) \to K(X)$ is a homotopy equivalence, and

b. $K(X) \simeq K^B(X)$ is a homotopy equivalence. In particular, $K_n^B(X) = 0$ for $n < 0$.

1.2 Representability

We will now ask a seemingly unrelated question, but one that I found more interesting than the main result:

Is the algebraic K-theory of $\mathbf{Sm}/S$ representable as an object in the stable motivic homotopy category $\mathbf{SH}(S)$?

We note that the representability of spectra corresponding to given K-theories is usually easy to produce, therefore the difficulty lies in understanding if these algebraic K-theories are actually in the stable homotopy category. Essentially, the proof involves showing three things are true: given a K-theory,

a. it satisfies Nisnevich descent;

b. it is $\mathbb{A}^1$-homotopy invariant;

c. it is $\mathbb{P}^1$-periodic, i.e., stabilized with respect to the suspension by $\mathbb{P}^1$.

Remark. *Let us make a few observations about Nisnevich descent. By definition, this is asking the presheaf on Grothendieck topology (in this case the Nisnevich topology) to satisfy (homotopy-coherent) sheaf condition. That is, we should have a sheaf in the sense of Grothendieck topology. A more useful but equivalent condition for Nisnevich topology (under quasi-compact quasi-separated assumptions of $\mathbf{Sm}/S$) is satisfying Nisnevich excision, c.f., [Hoy15, Appendix C].*

We will first understand this in the case where X is a regular Noetherian scheme for Bass K-theory K^B.

- From [TT90, Theorem 10.3, 10.8], we know this K-theory satisfies Zariski and Nisnevich descent for quasi-compact quasi-separated schemes.
- From Proposition 1.3, we know this K-theory satisfies $\mathbb{A}^1$-homotopy invariance for regular Noetherian schemes.
- From the projective bundle formula [TT90, Theorem 4.1], we know this K-theory is $\mathbb{P}^1$-periodic for quasi-compact quasi-separated schemes.

Therefore K^B satisfies Nisnevich descent. The representability is then recorded in [MV99, Theorem 4.3.13], given by $\mathbb{Z} \times \mathrm{BGL}_\infty$. Putting all this together, Bass K-theory has the right representability by the $\mathbb{P}^1$-spectrum given by the space $\mathbb{Z} \times \mathrm{BGL}_\infty$ levelwise, in the stable motivic homotopy category.

Remark. *An important remark we make here is that the Thomason-Trobaugh algebraic K-theory does not satisfy descent property on the level of spectra. Indeed, if you follow the same argument as the proof of Zariski descent in [TT90, Theorem 10.3], they have used the Localization Theorem [TT90, Theorem 7.4] in a crucial way.*

Theorem 1.4 (Localization)**.** Suppose X a quasi-compact quasi-separated scheme, suppose U a Zariski open in X such that U is also quasi-compact and quasi-separated, and suppose Z the closed complement. There exists a fiber sequence

$$K^B(X \text{ on } Z) \to K^B(X) \to K^B(U)$$

of spectra.

The localization theorem fails for Thomason-Trobaugh K-theory for the exact same reason as Bass delooping. This was highlighted in [TT90, Theorem 5.1] and known as proto-localization. The theorem would have worked in positive degrees, but is obstructed at degree 0 *by applying the connective cover functor. That is,* $K_0(X) \to K_0(U)$ *is not surjective in general: the obstruction to lifting* K_0*-classes from* U *to* X *is precisely* $K^B_{-1}(X \text{ on } Z)$*, i.e., the correction term, by the fundamental theorem of algebraic K-theory. ([6])*

However, we want to distinguish this from the fact that connective algebraic K-theory still satisfies Nisnevich descent property as a connective spectra. This is because $\mathbf{Sp} \to \mathbf{Sp}^{cn}$ *commutes with limits, so any descent property we show for non-connective K-theory will give a descent result for connective K-theory, but again this is only true as a presheaf of connective spectra.*

Now we may ask: what happens if we think about general (quasi-compact quasi-separated) schemes? This requires backtracking the things we talked about above, and we will see that K^B would no longer be $\mathbb{A}^1$-homotopy invariant, so the infinite Grassmannian $\mathbb{Z} \times \mathrm{BGL}_\infty$ is no longer $\mathbb{A}^1$-local, thereby we lost representability of K^B. (See [MV99, Proposition 4.3.14].) This motivates us to find a notion of "$\mathbb{A}^1$-homotopy invariant" K-theory, while maintaining Nisnevich descent and $\mathbb{P}^1$-periodicity, so that we have representability over general schemes by $\mathbb{Z} \times \mathrm{BGL}_\infty$. Under this motivation, the main result of [Cis13] becomes a byproduct that justifies our eventual choice of K-theory.

2 Building Homotopy-invariance

This is where we start talking about the actual content of [Cis13]. Unfortunately, the paper was written in the language of model categories, and instead of upgrading/polishing everything to discuss in the ∞-categorical framework, we will try to suppress the model-categorical language from this talk.

Let S be a (quasi-compact, quasi-separated) scheme. For the rest of the talk, unless stated otherwise, all model categories are equipped with the projective model structure (or induced from one). We define the Tate sphere to be $T \simeq S^1 \wedge \mathbb{G}_m$ in the pointed model category of simplicial sheaves $\mathcal{E}_*$ over S.

Whatever K-theory we decided to build, we do need it to be $\mathbb{P}^1$-stable. Recall that T and $\mathbb{P}^1$ agrees under $\mathbb{A}^1$-local conditions in Nisnevich topology, so it suffices to invert the Tate sphere and consider the spectra over it. But to get a stable category, we do need to invert S^1 first.

2.1 Building Over S^1-spectra

Let $\mathbf{Sp}_{S^1}$ be the model category of presheaves of symmetric S^1-spectra on the category of smooth S-schemes. (This is also the stable model category of symmetric S^1-spectra in $\mathcal{E}_*$.) The homotopy category $\mathbf{Ho}(\mathbf{Sp}_{S^1})$ has a triangulated structure. By representability, we have an object (and really a ring spectrum) $K \in \mathbf{Ho}(\mathbf{Sp}_{S^1})$ representing Thomason-Trobaugh K-theory, given by

$$\mathrm{Hom}_{\mathbf{Ho}(\mathbf{Sp}_{S^1})}(\Sigma^n \Sigma^\infty (X_+), K) \simeq K_n(X).$$

We can then ask for more. Inside $\mathbf{Ho}(\mathbf{Sp}_{S^1})$, there is a full subcategory of $\mathbb{A}^1$-homotopy invariant S^1-spectra, along with the inclusion functor. This inclusion functor has a left adjoint, known as $\mathbb{A}^1$-localization

$$R_{\mathbb{A}^1} : \mathbf{Ho}(\mathbf{Sp}_{S^1}) \to \mathbf{Ho}_{\mathbb{A}^1}(\mathbf{Sp}_{S^1}).$$

Writing down the formula would require using derived functors as well as internal hom in $\mathbf{Ho}(\mathbf{Sp}_{S^1})$ from the model structure, so we omit.

We will now build a T-action on K-theory spectrum K. Choosing a representation

$$\mathbb{G}_m = S \times \operatorname{Spec} \mathbb{Z}[t, t^{-1}],$$

the invertible section t corresponds to a class $b \in K_1(\mathbb{G}_m)$, therefore giving rise to a map in $\mathbf{Ho}(\mathcal{E}_*)$,

$$b : T = S^1 \wedge \mathbb{G}_m \to \mathbf{R}\Omega^\infty(K),$$

into the loopspace of K. This then gives rise to a cup product

$$b \smile - : T \wedge^{\mathbf{L}} K \xrightarrow{b \wedge^{\mathbf{L}} 1_K} K \wedge^{\mathbf{L}} K \xrightarrow{\mu} K$$

To understand this T-action, we really need to understand a general pair (E, w) for some S^1-spectrum $E \in \mathbf{Sp}_{S^1}$ and $w : T \wedge^{\mathbf{L}} E \to E$ in $\mathbf{Ho}(\mathbf{Sp}_{S^1})$. One first question we should ask being, does the map w actually depend on the choice of underlying map $\underline{w} : T \wedge E \to E$ in $\mathbf{Sp}_{S^1}$? The answer to this, after justification, is no. In short, thinking T-equivariantly,

- given a morphism $\underline{w} : T \wedge E \to E$, we can upgrade the morphism to $w : T \wedge^{\mathbf{L}} E \to E$ in $\mathbf{Ho}(\mathbf{Sp}_{S^1})$, defined using the canonical map $T \wedge^{\mathbf{L}} E \to T \wedge E$;
- if we are given $w : T \wedge^{\mathbf{L}} E \to E$ in $\mathbf{Ho}(\mathbf{Sp}_{S^1})$, then by replacement, we note that $T \wedge^{\mathbf{L}} E \to T \wedge E$ is an isomorphism in $\mathbf{Ho}(\mathbf{Sp}_{S^1})$, therefore w lifts to $\underline{w} : T \wedge E \to E$ in $\mathbf{Sp}_{S^1}$.

Therefore, to get any information in the homotopy category, it suffices to understand the information on the level of spectra.

Let us now try and build an $\mathbb{A}^1$-homotopy invariant K-theory.

Definition 2.1. We define the naive homotopy-invariant K-theory $\mathbb{K}$ to be the ring spectrum $\mathbb{K} = R_{\mathbb{A}^1}(K)$.

The whole story that I told before still holds: we have a cup product, and an identification between mappings in general. We will now move on to non-connective spectra. Given object E in $\mathbf{Sp}_{S^1}$ with morphism $w : T \wedge E \to E$, there are now two ways of producing new non-connective spectra.

- Recall that Bass delooping gives an assignment $K \mapsto K^B$ using the information (K, b). The point being, this process works in general for any pair (E, w). As the definition in [TT90] suggests, this is a very complicated construction. However, the construction preserves the representability in the universal way:

  Proposition 2.2. The spectrum K^B represents the Bass-Thomason-Trobaugh K-theory, i.e.,

  $$\mathrm{Hom}_{\mathbf{Ho}(\mathbf{Sp}_{S^1})}(\Sigma^n \Sigma^\infty (X_+), K^B) \simeq K_n^B(X).$$

- For every $n \geqslant 0$, we have a canonical map

  $$\mathbf{R}\,\mathrm{Hom}(T^{\wedge n}, E) \to \mathbf{R}\,\mathrm{Hom}(T^{\wedge (n+1)}, T \wedge^{\mathbf{L}} E) \xrightarrow{w_*} \mathbf{R}\,\mathrm{Hom}(T^{\wedge (n+1)}, E)$$

  where the first map is induced by $T \wedge -$, and the second map is induced by w. We thereby obtain a sequence

  $$E \to \mathbf{R}\,\mathrm{Hom}(T, E) \to \cdots \to \mathbf{R}\,\mathrm{Hom}(T^{\wedge n}, E) \to \mathbf{R}\,\mathrm{Hom}(T^{\wedge (n+1)}, E) \to \cdots$$

  We then set $E^{\#} = \mathbf{L} \varinjlim_{n \geqslant 0} \mathbf{R}\,\mathrm{Hom}(T^{\wedge n}, E)$.

  Remark. *This is analogous to taking suspensions and then loopspaces in the classical homotopy theory case, therefore $E^{\#}$ is a T-stabilization of E. Note that $E^{\#}$ is still not T-stable, mostly because $E^{\#}$ is not yet a T-spectra. In this case, we have a simple description of the delooping, but this was not done in a universal way, so we do not recover representability.*

We now have two non-connective spectra K^B and $K^{\#}$. Because of the lack of universality in $E^{\#}$, it does not quite make sense construct the $\mathbb{A}^1$-homotopy invariant counterpart, and we will only do this for K^B.

Definition 2.3. The spectrum of homotopy-invariant K-theory is $\mathrm{KH} = R_{\mathbb{A}^1}(K^B)$.

Again, the universality suggests the following representability result:

$$\mathrm{Hom}_{\mathbf{Ho}(\mathbf{Sp}_{S^1})}(\Sigma^n \Sigma^\infty (X_+), \mathrm{KH}) \simeq \mathrm{KH}_n(X),$$

where $\mathrm{KH}_n(X)$ is the nth homotopy-invariant K-group defined by Weibel, c.f., [Wei89].

We take a small detour into this notion of K-theory.

Definition 2.4. Here $\mathrm{KH}_n(X) = \pi_n(|K^B(\Delta^* \times X)|)$ is the geometric realization of the simplicial spectrum where

$$\Delta^* = \operatorname{Spec}\left(\mathbb{Z}[t_0, \dots, t_n]/\sum_i t_i - 1\right).$$

Remark. *The original definition of* KH *[Wei89] is defined for any ring* A *via* $K^B(\Delta A)$ *instead, where* ΔA *is the simplicial ring defined by the coordinate ring* $\Delta_n A = A[t_0, \dots, t_n]/(\sum_i t_i - 1)A$*.* KH *satisfies the following properties.*

- *For any set* X*, we have*
$$\mathrm{KH}(A) \cong \mathrm{KH}(A[X]) \cong \mathrm{KH}(A\{X\}).$$
Therefore KH *satisfies*
- *For all* $n \in \mathbb{Z}$*,*
$$\mathrm{KH}_n(A[x, x^{-1}]) \cong \mathrm{KH}_n(A) \oplus \mathrm{KH}_{n-1}(A),$$
and on the level of spectra we have
$$\mathrm{KH}(A[x, x^{-1}]) \cong \mathrm{KH}(A) \times \Omega^{-1}\mathrm{KH}(A).$$

Once we upgrade this to the K-theory of space using the definition, we note that

- KH *satisfies* $\mathbb{A}^1$*-homotopy invariance (just as we will see later), and*
- *if* X *is a regular scheme, then the canonical map* $K(X) \to \mathrm{KH}(X)$ *is an equivalence.*

These properties justify the fact that this is the "correct" homotopy-invariant K-theory.

We can now ask:

Being the "correct" version, how does this compare to $R_{\mathbb{A}^1}(K^{\#})$, as well as $\mathbb{K}^B$ and $\mathbb{K}^{\#}$?

This is partially answered by the following technical lemma.

Lemma 2.5. If $E \in \mathbf{Ho}_{\mathbb{A}^1}(\mathbf{Sp}_{S^1})$, then so are E^B and $E^{\#}$. Moreover, $E^B \simeq E^{\#}$.

Moreover, we want to control the behavior of T-equivariant $\mathbb{A}^1$-equivalence after taking $(-)^B$ and $(-)^{\#}$. The following proposition [Cis13, Proposition 2.9] is very useful.

Proposition 2.6. Consider $(E, w : T \wedge E \to E)$ and $(F, w' : T \wedge F \to F)$ given by objects in $\mathbf{Sp}_{S^1}$. Suppose there exists a map $\varphi : E \to F$ in $\mathbf{Ho}(\mathbf{Sp}_{S^1})$ that is

- T-equivariant, i.e., the diagram
$$\begin{tikzcd} T \wedge E \arrow[r, "w"] \arrow[d, "T\wedge\varphi"'] & E \arrow[d, "\varphi"] \\ T \wedge F \arrow[r, "w'"'] & F \end{tikzcd}$$
commutes;
- an $\mathbb{A}^1$-equivalence, i.e., image under $R_{\mathbb{A}^1}$ is an isomorphism,

then the induced maps
$$\varphi^B : E^B \to F^B, \quad \varphi^{\#} : E^{\#} \to F^{\#}$$
are also $\mathbb{A}^1$-equivalences.

Proof. Check that $\mathbf{R}\operatorname{Hom}(C, -) : \mathbf{Ho}(\mathbf{Sp}_{S^1}) \to \mathbf{Ho}(\mathbf{Sp}_{S^1})$ preserves $\mathbb{A}^1$-equivalences for any compact object C of $\mathbf{Ho}(\mathbf{Sp}_{S^1})$. In particular, for any presheaf E of S^1-spectra, we have an isomorphism

$$R_{\mathbb{A}^1}(\mathbf{R}\operatorname{Hom}(C, E)) \simeq \mathbf{R}\operatorname{Hom}(C, R_{\mathbb{A}^1}(E)).$$

□

Corollary 2.7. We have canonical isomorphisms

$$R_{\mathbb{A}^1}(E^B) \simeq R_{\mathbb{A}^1}(E)^B \simeq R_{\mathbb{A}^1}(E)^\# \simeq R_{\mathbb{A}^1}(E^\#)$$

in $\mathbf{Ho}(\mathbf{Sp}_{S^1})$.

Proof. Since $E \to R_{\mathbb{A}^1}(E)$ is universal, it is an $\mathbb{A}^1$-equivalence, i.e., image under $R_{\mathbb{A}^1}$ is automatically an isomorphism. By results above, we conclude that $E^B \to R_{\mathbb{A}^1}(E)^B$ is an $\mathbb{A}^1$-equivalences, therefore $R_{\mathbb{A}^1}(E)^B$ is $\mathbb{A}^1$-homotopy invariant. Applying $R_{\mathbb{A}^1}$ on $E^B \to R_{\mathbb{A}^1}(E)^B$ again, we get an isomorphism

$$R_{\mathbb{A}^1}(E^B) \cong R_{\mathbb{A}^1}(R_{\mathbb{A}^1}(E)^B) \cong R_{\mathbb{A}^1}(E)^B$$

by the universal property. Similarly,

$$R_{\mathbb{A}^1}(E^\#) \cong R_{\mathbb{A}^1}(R_{\mathbb{A}^1}(E)^\#) \cong R_{\mathbb{A}^1}(E)^\#.$$

We conclude by noting that since $R_{\mathbb{A}^1}(E)$ is $\mathbb{A}^1$-homotopy invariant, then $R_{\mathbb{A}^1}(E)^B \cong R_{\mathbb{A}^1}(E)^\#$. □

Corollary 2.8. We have isomorphisms

$$\mathrm{KH} \simeq \mathbb{K}^B \simeq \mathbb{K}^\#.$$

This is the story we have on S^1-spectra. Both K^B and $K^\#$ give some sort of delooping, but they exhibit very different properties.

- K^B follows the universal delooping done in the literature, therefore inherits the correct representability.
- $K^\#$ loses the said representability, but being stabilized already, producing a T-stable (and therefore S^1-stable spectrum) just requires a lifting into the category of T-spectra.

We see that both constructions have their unique advantage, and surprisingly they agree after $\mathbb{A}^1$-localization, producing KH. We will use this to our advantage to produce the right spectrum in $\mathbf{SH}(S)$. Let us prove that the $\mathbb{A}^1$-homotopy invariant spectrum KH is more powerful than it seems.

Remark. *If E satisfies Nisnevich descent, then so does $\mathbf{R}\operatorname{Hom}(C, E)$ for any presheaf C of S^1-spectra, and since the presheaves of S^1-spectra satisfying Nisnevich descent also form a localizing subcategory of $\mathbf{Ho}(\mathbf{Sp}_{S^1})$, then $R_{\mathbb{A}^1}(E)$ satisfies Nisnevich descent. We conclude that E^B and $E^\#$ also do.*

Corollary 2.9. $\mathrm{KH} \simeq \mathbb{K}^\#$ satisfies Nisnevich descent.

Proof. Since K^B satisfies Nisnevich descent, so does $\mathbb{K}^\# \simeq \mathrm{KH} = R_{\mathbb{A}^1}(K^B)$. □

The only thing we still require from homotopy-invariant K-theory are that

- we have not gotten a T-spectrum yet, and
- it needs to be $\mathbb{P}^1$-periodic, actually giving T-stable properties.

2.2 Lifting to $\mathbb{P}^1$-spectra

Let us now move on and localize $T \simeq S^1 \wedge \mathbb{G}_m$. We will then study the model category $\mathbf{Sp}_T\mathbf{Sp}_{S^1}$ of T-spectra in the category of presheaves of $\mathbf{Sp}_{S^1}$. Note analogous to the case of S^1-spectra, we have to again consider mappings T-equivariantly. Again, objects in this category are described by $E = (E_n, \sigma_n : T \wedge E_n \to E_{n+1})$. Our study of these pairs over S^1 has shown that our choice, again, does not matter. However, a few things have changed:

- the evaluation at zero functor $\Omega_T^\infty : \mathbf{Sp}_T\mathbf{Sp}_{S^1} \to \mathbf{Sp}_{S^1}$ is a right Quillen functor with left adjoint Σ_T^∞, and this upgrades to a derived adjunction

$$\mathbf{L}\Sigma_T^\infty : \mathbf{Ho}(\mathbf{Sp}_{S^1}) \rightleftarrows \mathbf{Ho}(\mathbf{Sp}_T\mathbf{Sp}_{S^1}) : \mathbf{R}\Omega_T^\infty$$

This gives us enough language to communicate between T-spectra and S^1-spectra.

- let us we repeat the same comparison between S^1-spectra and T-spectra. Suppose E be a presheaf of S^1-spectra over the category of smooth S-schemes, equipped with $w : T \wedge E \to E$, then this is associated to a T-spectrum
$$\underline{E} = (E_n, \sigma_n)_{n \geqslant 0}$$
by setting $E_n = E$ and $\sigma_n = w$ for all $n \geqslant 0$. Again, we get a morphism $\underline{w} : T \wedge^{\mathbf{L}} \underline{E} \to \underline{E}$ in $\mathbf{Ho}(\mathbf{Sp}_T\mathbf{Sp}_{S^1})$, but this time, the construction shows us that this is an isomorphism! This then induces a canonical isomorphism
$$E^{\#} \simeq \mathbf{R}\Omega_T^{\infty}(\underline{E})$$
in $\mathbf{Ho}(\mathbf{Sp}_{S^1})$. This tells us that, given a reason property $\mathcal{P}$ of objects in $\mathbf{Ho}(\mathbf{Sp}_{S^1})$, e.g., descent in a topology, or homotopy invariance, for $\underline{E}$ to satisfy $\mathcal{P}$ in $\mathbf{Ho}(\mathbf{Sp}_T\mathbf{Sp}_{S^1})$, i.e., for all n, the presheaf of S^1-spectra $\mathbf{R}\Omega_T^{\infty}(T^{\wedge n} \wedge^{\mathbf{L}} E)$ has property $\mathcal{P}$ in $\mathbf{Ho}(\mathbf{Sp}_{S^1})$, it is equivalent to show that $E^{\#}$ satisfies $\mathcal{P}$ in $\mathbf{Ho}(\mathbf{Sp}_{S^1})$.

This is important: we have liftings
$$K^{\#} \simeq \mathbf{R}\Omega_T^{\infty}(\underline{K}), \quad \mathrm{KH} \simeq \mathbf{R}\Omega_T^{\infty}(\underline{\mathbb{K}})$$
for spectra $\underline{K}, \underline{\mathbb{K}} \in \mathbf{Ho}(\mathbf{Sp}_T\mathbf{Sp}_{S^1})$. Again, we are at a situation where there are two things we can work with, but this time,

- $\underline{\mathbb{K}}$ is $\mathbb{A}^1$-homotopy invariant with the correct descent property, while
- it is unclear what $\underline{K}$ produces.

We will do something similar to the case of S^1-spectra. This time, we care about the full subcategory $\mathbf{SH}(S)$ of $\mathbf{Ho}(\mathbf{Sp}_T\mathbf{Sp}_{S^1})$ formed by $\mathbb{A}^1$-homotopy invariant objects satisfying Nisnevich descent. This inclusion functor has a left adjoint given by
$$\gamma : \mathbf{Ho}(\mathbf{Sp}_T\mathbf{Sp}_{S^1}) \to \mathbf{SH}(S).$$
We note that only in this category, i.e., under assumptions of being $\mathbb{A}^1$-local and satisfying Nisnevich descent, can we make local identification $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m \simeq T$.

Definition 2.10. The T-spectrum of K-theory KGL is defined by
$$\mathrm{KGL} = \gamma(\underline{K}).$$

We will again make an identification with $\underline{\mathbb{K}}$.

Proposition 2.11. The T-spectra KGL and $\underline{\mathbb{K}}$ are canonically isomorphic in $\mathbf{SH}(S)$.

Proof. Recall that $\underline{\mathbb{K}}$ is a homotopy-invariant presheaf satisfying Nisnevich descent, so $\gamma(\underline{\mathbb{K}}) \simeq \underline{\mathbb{K}}$. Now note that the map $\underline{K} \to \underline{\mathbb{K}}$ is a degreewise $\mathbb{A}^1$-equivalence, therefore after applying localization functor, we get
$$\mathrm{KGL} = \gamma(\underline{K}) \simeq \gamma(\underline{\mathbb{K}}) \simeq \underline{\mathbb{K}}.$$
□

So again, we conclude that the order of construction does not quite matter here. However, there is an advantage of working with KGL instead of $\underline{\mathbb{K}}$, which we will now talk about.

2.3 The $\mathbb{P}^1$-spectra of K-theory

Let K be the presheaf of K-theory, then working purely simplicially, we have an isomorphism
$$\mathbb{Z} \times \mathrm{BGL}_{\infty} \simeq \mathbf{R}\Omega^{\infty}(K)$$
in the unstable pointed homotopy category $\mathbf{H}_*(S)$. We will now build the $\mathbb{P}^1$-spectra of K-theory from this description, without the $\mathbf{Sp}_{S^1}$ as an intermediate layer.

Let $\beta = [\mathcal{O}_{\mathbb{P}^1}] - [\mathcal{O}_{\mathbb{P}^1}(-1)]$ be the Bott class in $K_0(\mathbb{P}^1) = \pi_0(\mathbf{R}\Omega^\infty(K)(\mathbb{P}^1))$, then this defines a morphism

$$\beta : \mathbb{P}^1 \to \mathbf{R}\Omega^\infty(K)$$

in $\mathbf{Ho}(\mathcal{E}_*)$, therefore by the $\mathbb{A}^1$-equivalence of $\mathbb{Z} \times \mathrm{BGL}_\infty \simeq \mathbf{R}\Omega^\infty(K)$, we have a morphism

$$\beta : \mathbb{P}^1 \to \mathbf{R}\Omega^\infty(K) \simeq \mathbb{Z} \times \mathrm{BGL}_\infty$$

in the pointed unstable homotopy category $\mathbf{H}_*(S)$.

Definition 2.12. We define the $\mathbb{P}^1$-spectrum of K-theory in the homotopy of schemes to be $\mathcal{K}$, given by the periodic $\mathbb{P}^1$-spectrum determined by $\beta \smile -$, that is, the collection of simplicial presheaves

$$(\mathbb{Z} \times \mathrm{BGL}_\infty, \mathbb{Z} \times \mathrm{BGL}_\infty, \mathbb{Z} \times \mathrm{BGL}_\infty, \ldots)$$

with structural morphism

$$\beta \smile - : \mathbb{P}^1 \wedge (\mathbb{Z} \times \mathrm{BGL}_\infty) \to \mathbb{Z} \times \mathrm{BGL}_\infty .$$

This is a description that we are fairly familiar with, being completely analogous to the case over S^1-spectra. Again, we find ourselves comparing two constructions that reach the same endproduct via different routes, namely the $\mathbb{P}^1$-spectra (of simplicial presheaves) with the T-spectra of S^1-presheaves. (Again, this uses the local identification $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m \simeq T$ mentioned before.) We then have a description of $\mathrm{SH}(S)$ via $\mathbb{P}^1$-spectra, which is done by comparing on the level of K-groups, c.f., [Cis13, Proposition 2.18].

Proposition 2.13. The comparison above gives a categorical equivalence when taking stable homotopy categories[3]. In particular, this assignment sends KGL to $\mathcal{K}$.

But we have seen that $\mathcal{K}$ has the simplest description among all three of them, namely it is a $\mathbb{P}^1$-periodic spectrum determined by $\mathbb{Z} \times \mathrm{BGL}_\infty$, so this gives $\underline{\mathbb{K}}$ the required property: it satisfies Nisnevich descent, being $\mathbb{A}^1$-homotopy invariant, and $\mathbb{P}^1$-periodic, and represented by $\mathbb{Z} \times \mathrm{BGL}_\infty$ levelwise.

Theorem 2.14. The T-spectrum KGL represents homotopy-invariant K-theory in $\mathbf{SH}(S)$: for any smooth S-scheme X and integer n, we have an isomorphism

$$\mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma^n \Sigma_T^\infty(X_+), \mathrm{KGL}) \simeq \mathrm{KH}_n(X).$$

Proof. Recall

$$\mathrm{Hom}_{\mathbf{Ho}(\mathbf{Sp}_{S^1})}(\Sigma^n \Sigma^\infty(X_+), \mathrm{KH}) \simeq \mathrm{KH}_n(X),$$

and we know $\mathrm{KH} \simeq \mathbb{K}^\# \simeq \mathbf{R}\Omega_T^\infty(\underline{\mathbb{K}})$ in $\mathbf{Ho}(\mathbf{Sp}_{S^1})$, and we identify KGL and $\underline{\mathbb{K}}$ in $\mathbf{SH}(S)$. □

You can find a streamlined illustration of the proof discussed so far from Figure 1.

Remark. *The key takeaway being, however we construct motivic spaces, i.e., elements in $\mathbf{SH}(S)$, using these methods, we end up with the same one.*

3 Extending Descent Property

For the rest of the talk, we will improve the descent property of homotopy-invariant K-theory from Nisnevich topology to cdh topology. We note that $\mathbf{SH}(S)$ satisfies the usual six-functor formalism, under the derived setting. For instance, given a morphism of schemes $f : S' \to S$, there is a pair of adjoint functors

$$\mathbf{L}f^* : \mathbf{SH}(S) \rightleftarrows \mathbf{SH}(S') : \mathbf{R}f_*.$$

Under this formalism, we have the usual properties like localization theorem, smooth base-change, proper base-change.

[3] This should be interpreted in the simplest fashion, namely the homotopy category with T being stable.

Definition 3.1. A morphism $p : X' \to X$ of schemes is an abstract blow-up at closed subscheme $Z \subseteq X$ if p is proper, and Z is such that
$$p^{-1}(X\backslash Z)_{\mathrm{red}} \to (X\backslash Z)_{\mathrm{red}}$$
is an isomorphism. The cdh topology is the Grothendieck topology on the category of schemes, generated by Nisnevich coverings and by coverings of the form $Z \coprod X' \to X$ for any abstract blow-up $X' \to X$ at Z.

So we can ask a question similar to the one we asked about Nisnevich descent: how do we characterize cdh descent without referring to the definition?

Definition 3.2. A presheaf of S^1-spectra E on the category of schemes satisfies cdh descent if and only if it satisfies Nisnevich descent, and if, for every abstract blow-up $p : X' \to X$ at Z, setting $Z' = p^{-1}(Z)$, we have a homotopy (co)Cartesian square
$$\begin{tikzcd} E(X) \arrow[r] \arrow[d] & E(X') \arrow[d] \\ E(Z) \arrow[r] & E(Z') \end{tikzcd}$$

Proposition 3.3. Let $p : X' \to X$ be an abstract blow-up at center Z. Suppose we have a Cartesian square of schemes
$$\begin{tikzcd} Z' \arrow[r, "k"] \arrow[d, "q"'] & X' \arrow[d, "p"] \\ Z \arrow[r, "i"'] & X \end{tikzcd}$$
with $r = pk = iq : Z' \to X$, then for any E of $\mathbf{SH}(X)$, the square
$$\begin{tikzcd} E \arrow[r] \arrow[d] & \mathbf{R}p_*\mathbf{L}p^*E \arrow[d] \\ \mathbf{R}i_*\mathbf{L}i^*E \arrow[r] & \mathbf{R}r_*\mathbf{L}r^*E \end{tikzcd}$$
is homotopy coCartesian.

This is proven purely using six functor yoga.

Proof. Let $j : U = X\backslash Z \to X$ be the open immersion. By localization and smooth base-change, we can do six functor yoga, then the image of the desired square under $\mathbf{L}j^*$ is
$$\begin{tikzcd} \mathbf{L}j^*E \arrow[r, equal] \arrow[d] & \mathbf{L}j^*E \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}$$
and similarly, its image under $\mathbf{L}i^*$ is
$$\begin{tikzcd} \mathbf{L}i^*E \arrow[r] \arrow[d, equal] & \mathbf{L}i^*E \arrow[d, equal] \\ \mathbf{L}i^*E \arrow[r] & \mathbf{R}q_*\mathbf{L}q^*\mathbf{L}i^*E \end{tikzcd}$$
which is also homotopy coCartesian. Now both $\mathbf{L}j^*$ and $\mathbf{L}i^*$ are conservative, therefore the square we want is also obviously coCartesian. $\square$

Proposition 3.4. For any morphism $f : S' \to S$ of schemes, the canonical morphism
$$\mathbf{L}f^*(\mathrm{KGL}) \to \mathrm{KGL}$$
is an isomorphism in $\mathbf{SH}(S')$.

Proof. By writing $\mathbb{Z} \times \mathrm{BGL}_\infty$ as a homotopy colimit of smooth schemes, we have a canonical isomorphism

$$\mathbf{L}f^*(\mathbb{Z} \times \mathrm{BGL}_\infty) \simeq \mathbb{Z} \times \mathrm{BGL}_\infty$$

in unstable homotopy category $\mathbf{H}(S')$. Since KGL is the $\mathbb{P}^1$-spectra corresponding to $\mathcal{K}$, which is described by spaces $\mathbb{Z} \times \mathrm{BGL}_\infty$, we are done. $\square$

Theorem 3.5. KH satisfies cdh descent.

Proof. It suffices to show that for every abstract blow-up $p : X' \to X$ at Z, setting $Z' = p^{-1}(Z)$, we have a homotopy (co)Cartesian square

$$\begin{array}{ccc} \mathrm{KGL}(X) & \longrightarrow & \mathrm{KGL}(X') \\ \downarrow & & \downarrow \\ \mathrm{KGL}(Z) & \longrightarrow & \mathrm{KGL}(Z') \end{array}$$

By Theorem 2.14 and Proposition 3.4, this corresponds to the square

$$\begin{array}{ccc} \mathrm{KGL}(X) & \longrightarrow & \mathbf{R}p_* \,\mathrm{KGL}(X) \\ \downarrow & & \downarrow \\ \mathbf{R}i_* \,\mathrm{KGL}(X) & \longrightarrow & \mathbf{R}r_* \,\mathrm{KGL}(X) \end{array}$$

But the latter is induced from the homotopy coCartesian square in Proposition 3.3, which has the desired property. $\square$

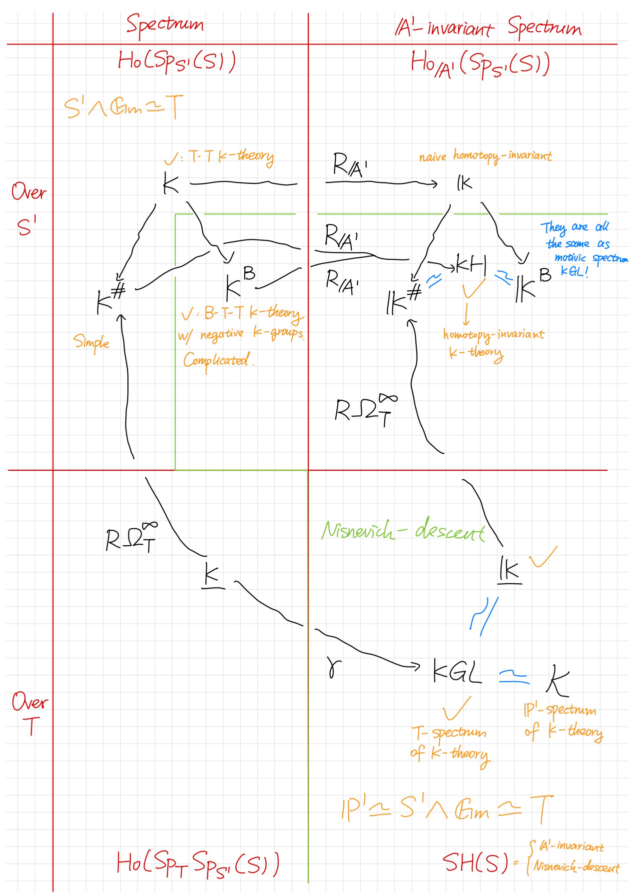

Figure 1: Streamlining the Proof

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