MATH 595 (Group Cohomology) Notes

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1 Aug 21, 2023: Introduction

Group cohomology works over different settings of groups, like finite groups, profinite groups, and topological groups. The course will develop towards

- duality in $H^*(G, -)$, and
- focus on computations, e.g., spectral sequences.

We first establish some notations.

- Let G be a group. If G has a topology, that would also be part of the information of G.
- A (left) G-module is an abelian group M with an action map

$$\begin{aligned} G \times M &\to M \\ (g, m) &\mapsto g \cdot m = gm \end{aligned}$$

  satisfying

  - $1 \cdot m = m$,
  - $(gh) \cdot m = g \cdot (hm)$,
  - $g(m + m') = gm + gm'$.

Remark 1.1. If G is a finite group, then the associated (non-commutative) group ring $\mathbb{Z}[G] := \bigoplus_{g \in G} \mathbb{Z}e_g$, where the multiplication is determined by $e_g e_h = e_{gh}$. Therefore, a G-module is just a $\mathbb{Z}[G]$-module.

Example 1.2.
- Trivial module $\mathbb{Z}$, or any abelian group with the trivial action $g \cdot a = a$.
- C_2, or any group with $f : G \twoheadrightarrow C_2$, then G with C_2 as a quotient gives the sign representation $\mathbb{Z}_{\text{sgn}}$, with $g \cdot (a) = (-1)^{\rho(g)} a$.
- $\mathbb{Z}[G]$ is a G-module via the left multiplication action, and/or the conjugation action.

Definition 1.3 (Fixed points/Invariants)**.** The set of fixed points of M over G is $M^G = \{m \in M \mid gm = m \ \forall g \in G\}$.

Definition 1.4 (Orbits/Coinvariants)**.** The set of orbits of M over G is $M_G = M/(gm - m)$.

Example 1.5. If $M = \mathbb{Z}_{\text{sgn}}$, then everything gets multiplied by -1, so there are no fixed points. The orbits of M over G would be $\mathbb{Z}_{\text{sgn}}/(-2) \cong \mathbb{Z}/2\mathbb{Z}$.

Example 1.6. If $M = \mathbb{Z}[G]$, then the fixed points are $\mathbb{Z}\left\{\sum_{g \in G} e_g\right\}$.

Thinking in a categorical setting, we have a trivial action function $\mathbb{Z}$-**Mod** $\to G$-**Mod**, sending $ga \mapsto a$ for all $g \in G$ and $a \in A$. This gives an exact functor from **Ab** to G-**Mod**. Then this functor has a right adjoint $(\ )^G : G$-**Mod** $\to$ **Ab**, and a left adjoint $(\ )_G :$ **Ab** $\to G$-**Mod**. More specifically, M^G becomes the maximal trivial action submodule of M, namely $\operatorname{Hom}_G(\mathbb{Z}, M)$; M_G becomes the largest quotient of M with trivial action, namely $\mathbb{Z} \otimes_{\mathbb{Z}[G]} M$. This simplifies to the tensor-hom adjunction in some sense. For a more detailed derivation of this, see Chapter 6.1 of Weibel.

Remark 1.7. In general, as in the category of G-sets, we have the orbit functor $X \mapsto X/G$ and the fixed point functor $X \mapsto X^G$. The orbit functor is left adjoint to the free G-set functor, and the fixed point functor is the right adjoint of the trivial G-set functor.

Remark 1.8. Read more about the setting in profinite groups with their topologies in Neukirch-Schmidt-Wingberg.

Definition 1.9 (Profinite Group)**.** A profinite group of a collection of groups is $G = \varprojlim_i G_i$ as an inverse limit, where each G_i is a finite group of the form G/U_i for some open U_i. This gives a topology to the profinite group.

Remark 1.10. The groups rings $\mathbb{Z}\left[[G]\right] = \varprojlim_i \mathbb{Z}[G_i]$. For instance, let $G = \hat{\mathbb{Z}_p} = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$, then $\mathbb{Z}_p\left[[G]\right] = \varprojlim_n \mathbb{Z}_p\left[\mathbb{Z}/p^n\mathbb{Z}\right]$, where each $\mathbb{Z}[\mathbb{Z}/p^n\mathbb{Z}] \cong \bigoplus_{i=0}^{n-1} \mathbb{Z}\{e_i\}$ where $e_i \cdot e_j = e_{ij}$. Therefore, $\mathbb{Z}_p\left[[G]\right]$ is now equivalent to $\varprojlim_n \mathbb{Z}_p[t]/(t^{p^n} - 1_e)$, and hence becomes a power series.

Remark 1.11. By a change of variables, this becomes $\varprojlim_n \mathbb{Z}_p[x]/(x^{p^n})$, but this only works in the finite group $\mathbb{Z}_p$ case, and not in general for $\mathbb{Z}$.

Example 1.12. $\mathbb{Z}[C_n] \cong \mathbb{Z}\{e\} \oplus \mathbb{Z}\{g\} \oplus \mathbb{Z}\{g^2\} \oplus \cdots \oplus \mathbb{Z}\{g^{n-1}\} \cong \mathbb{Z}[g]/(g^n - 1_e)$.

2 Aug 23, 2021: Cohomology of groups

Definition 2.1. Let G be a group, then we have a diagram

$$EG^{\cdot} : \cdots \Rrightarrow G \times G \rightrightarrows G$$

where the arrows are given by

$$EG^n = G^{n+1} \xrightarrow{d_i} G^n$$

for all $0 \leqslant i \leqslant n$. In the sense of simplicial sets, we have $d_i(g_0, \ldots, g_n) = (g_0, \ldots, \hat{g}_i, \ldots, g_n)$.

Now let M be a G-module, then we define $X^n = X^n(G, M) = \operatorname{Map}_{\mathbf{Set}}(G^{n+1}, M)$. G now has an action on this set, given by

$$(g \circ f)(g_0, \ldots, g_n) = gf(g^{-1}g_0, \ldots, g^{-1}g_n).$$

The action on d^i's are contravariant, namely we obtain $d_i^* : X_n \to X^{n+1}$ with an inherited structure. Note that M sits inside X^0, therefore we have a complex $(*)$:

$$0 \longrightarrow M \xhookrightarrow{\partial_0} X^0 \xrightarrow{\partial_1} X^1 \xrightarrow{\partial_2} X^2 \xrightarrow{\partial_3} \cdots$$

Here ∂_0 includes M as the constant functions into X, namely $\partial_0(m) = f$ for $f(g) = m$, and so on. In general, for $n > 0$, we have

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i^*.$$

Lemma 2.2. The complex $(*) : M \to X^{\cdot}$ is an exact complex of G-modules, i.e., $\partial^2 = 0$ and $\ker(\partial_{n+1}) = \operatorname{im}(\partial_n)$, and the ∂_i's preserves the G-action. This is called the standard resolution of M as a G-module.

Proof. Exercise. $\square$

Definition 2.3. The G-fixed points of the X^n's are defined by $C^n(G,M) = (X^n(G,M))^G$, called the homogeneous n-cochains of G with coefficients in M. Because the complex preserves G-actions, then we obtain a complex of $C^n(G,M)$'s, given by

$$0 \longrightarrow C^0(G,M) \xrightarrow{\partial_0} C^1(G,M) \xrightarrow{\partial_1} \cdots$$

Remark 2.4. To see what the induced mapping is, suppose $A \to B$ is a G-module map, then there is an induced map of fixed points $A^G \to B^G$ by the restriction. In particular, let $a \in A$ be fixed with $ga = a$ for all $g \in G$, then $f(a) = f(ga) = gf(a)$.

Remark 2.5. In the complex of Definition 2.3, $\partial^2 = 0$ as well, but in general this is not an exact sequence.

Definition 2.6 (Group Cohomology)**.** The group cohomology of G with coefficients in M is the collection

$$\{H^n(G,M)\}_{n\geqslant 0},$$

where $H^n(G,M) := H^n(C^\cdot(G,M)) = \ker(\partial : C^n \to C^{n+1})/\operatorname{im}(\partial : C^{n-1} \to C^n)$. We usually use the notion of cocycles $Z^n(G,M) = \ker(\partial : C^n \to C^{n+1})$ and coboundaries $B^n(G,M) = \operatorname{im}(\partial : C^{n-1} \to C^n)$.

Exercise 2.7. Show that $H^0(G,M)$ is isomorphic to M^G.

Definition 2.8. The inhomogeneous cochains $C_i^\cdot(G,M)$ are given by

- $C_i^0 = M$, and
- for $n > 0$, $C_i^n = \operatorname{Map}(G^n, M)$,

with coboundary maps $\partial^{n+1} : C_i^n \to C_i^{n+1}$, given by

- $\partial^1(m)(g) = gm - m$,
- $\partial^2(f)(g_1,g_2) = g_1 f(g_2) - f(g_1g_2) + f(g_1)$, and so on, with
- $\partial^{n+1}(f)(g_1,\ldots,g_{n+1}) = g_1 f(g_2,\ldots,g_{n+1}) + \sum_{i=1}^{n}(-1)^i f(g_1,\ldots,g_ig_{i+1},\ldots,g_{n+1}) + (-1)^{n+1} f(g_1,\ldots,g_n)$.

This gives the inhomogeneous setting of this cochain.

Lemma 2.9. The maps

$$\begin{aligned} C^n(G,M) &\to C_i^n(G,M) \\ (\varphi : G^{n+1} \to M) &\mapsto (f : G^n \to M) \\ &\qquad f(g_1,\ldots,g_n) := \varphi(1, g_1, g_1g_2, \ldots, g_1g_2\cdots g_n) \end{aligned}$$

give a cochain homotopy equivalence $C^\cdot(G,M) \xrightarrow{\sim} C_i^\cdot(G,M)$, and hence this is a quasi-isomorphism.

Corollary 2.10. The cohomology $H^*(C_i^\cdot(G,M)) \cong H^*(G,M)$.

Remark 2.11. Any cohomology class can be represented by a normalized inhomogeneous cocycle $f : G^n \to M$, i.e., $f(g_1,\ldots,g_n) = 0$ where $g_i = 1$ for some i.

Remark 2.12. Even for $G = C_2$, C_i^n or C^n get large as n grows.

Remark 2.13.

- Using homological algebra, we can find other cochain complexes which computes group cohomology $H^*(G,M)$.
- We would also understand $H^*(G,M)$ as the failure of exactness of $(\ )^G : G\text{-}\mathbf{Mod} \to \mathbf{Ab}$. Therefore, when taking the fixed points, the exact sequence may not be mapped to another exact sequence. In particular, if we take an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of G-modules, the induced sequence

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G$$

do not give a surjection at $B^G \to C^G$. One needs to take higher cohomology to obtain a long exact sequence. Hence, $(\ )^G : G\text{-}\mathbf{Mod} \to \mathrm{Ab}$ is a left exact functor, but not necessarily right exact.

3 Aug 25, 2021: Cohomology of groups, continued

Example 3.1. Let G be C_2, or any group with a surjection p onto C_2, then it has an action on $\mathbb{Z}_{\text{sgn}}$ given by $g \cdot a = (-1)^{p(g)}a$, therefore we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}_{\text{sgn}} \xrightarrow{\times 2} \mathbb{Z}_{\text{sgn}} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

and taking the fixed point functor we have

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

Remark 3.2. Higher homologies measure the failure of exactness.

Remark 3.3. The collection $\{H^n(G, -)\}_{n \in \mathbb{Z}}$ satisfies

- $H^n(G, -) = 0$ for $n < 0$;
- for short exact sequence $0 \to A \to B \to C \to 0$ in G-**Mod**, we have a long exact sequence

  $$0 \longrightarrow H^0(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, C) \xrightarrow{\delta} H^1(G, A) \longrightarrow \cdots$$

  where δ is the connecting homomorphism.
- the connecting homomorphisms δ are natural, i.e., given a commutating diagram

  $$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

  the induced diagram

  $$\begin{tikzcd} H^n(G, C) \arrow[r, "\delta"] \arrow[d] & H^{n+1}(G, A) \arrow[d] \\ H^n(G, C') \arrow[r, "\delta"'] & H^{n+1}(G, A') \end{tikzcd}$$

  also commutes, and $\{H^n(G, -)\}_{n \in \mathbb{Z}}$ is a cohomological δ-functor. Note that a δ-functor is additive, and usually occurs in abelian categories.

Definition 3.4 (δ-functor)**.** A map of δ-functors $T^* \to F^*$ is a collection of natural transformations $T^n \to F^n$, commuting with the δ's, i.e.,

$$\begin{tikzcd} T^n \arrow[r] \arrow[d, "\delta_T"'] & F^n \arrow[d, "\delta_F"] \\ T^{n+1} \arrow[r] & F^{n+1} \end{tikzcd}$$

A δ-functor T^* is universal if, given any other δ-functor F^*, a map $T^* \to F^*$ is uniquely determined by $T^0 \to F^0$.

Proposition 3.5. $H^*(G, -) : G$-**Mod** $\to$ **Ab** is a δ-functor.

Proof. We need to show:

- each $H^n(G, -)$ is a well-defined functor,
- the connecting homomorphisms δ's gives a long exact sequence,
- the naturality of δ.

First, let $f : A \to B$ be in G-**Mod**, then $C^*(G,A) \to C^*(G,B)$ is equivalent to $\mathrm{Map}(G^{*+1}, A)^G \to \mathrm{Map}(G^{*+1}, B)^G$ by composition with f. One can show that this is equivariant, i.e., respects the G-action, so it is well-defined to take the fixed points, and thus commutes with ∂'s.

Second, we need to apply the snake lemma. Given a short exact sequence $0 \to A \to B \to C \to 0$, we claim:

Claim 3.6. $0 \longrightarrow C^*(G,A) \longrightarrow C^*(G,B) \longrightarrow C^*(G,C) \longrightarrow 0$ is a short exact sequence of cochain complexes, i.e., $C^*(G,-) : G\text{-}\mathbf{Mod} \to \mathbf{coCh}$ is an exact functor.

Subproof. Exercise. ■

Now take the complex

$$\begin{tikzcd}
0 \arrow[r] & C^n(G,A) \arrow[r] \arrow[d, "\partial"] & C^n(G,B) \arrow[r] \arrow[d, "\partial"] & C^n(G,C) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C^{n+1}(G,A) \arrow[r] & C^{n+1}(G,B) \arrow[r] & C^{n+1}(G,C) \arrow[r] & 0
\end{tikzcd}$$

and quotient the boundaries everywhere (and thus lose the injectivity/surjectivity when applicable)

$$\begin{tikzcd}
 & C^n(G,A)/B^n(G,A) \arrow[r] \arrow[d, "\partial"] & C^n(G,B)/B^n(G,B) \arrow[r] \arrow[d, "\partial"] & C^n(G,C)/B^n(G,C) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & Z^{n+1}(G,A) \arrow[r] & Z^{n+1}(G,B) \arrow[r] & Z^{n+1}(G,C) &
\end{tikzcd}$$

Taking the kernels and cokernels on ∂'s, we obtain a complex

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & H^n(G,A) \arrow[r] \arrow[d] & H^n(G,B) \arrow[r] \arrow[d] & H^n(G,C) \arrow[d] & \\
 & C^n(G,A)/B^n(G,A) \arrow[r] \arrow[d, "\partial"] & C^n(G,B)/B^n(G,B) \arrow[r] \arrow[d, "\partial"] & C^n(G,C)/B^n(G,C) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & Z^{n+1}(G,A) \arrow[r] \arrow[d] & Z^{n+1}(G,B) \arrow[r] \arrow[d] & Z^{n+1}(G,C) \arrow[d] & \\
 & H^{n+1}(G,A) \arrow[r] & H^{n+1}(G,B) \arrow[r] & H^{n+1}(G,C) &
\end{tikzcd}$$

By the snake lemma, we obtain the long exact sequence. □

Proposition 3.7. If $0 \to A \to B \to C \to 0$ is a short exact sequence such that $H^*(G,B) = 0$ for $* > 0$ (or at least $H^n(G,B) = 0 = H^{n+1}(G,B)$), then $\delta : H^n(G,C) \to H^{n+1}(G,A)$ is an isomorphism.

Definition 3.8 (Acyclic, Cohomologically Trivial)**.** A G-module M is

- acyclic if $H^*(G,M) = 0$ for $* > 0$,
- cohomologically trivial if $H^*(H,M) = 0$ for $* > 0$ and any (closed) subgroup $H \subseteq G$.

Definition 3.9 (Induced Module)**.** Given any G-module M, the induced module $\mathrm{ind}_G(M) = \mathrm{Map}(G,M) = X^0(G,M)$.

Example 3.10. M could have the trivial action.

Exercise 3.11. For any M, the induced module of M over G is isomorphic (under the G-action) to the induced module of module given by forgetful action over G.

Remark 3.12.
- $\mathrm{Ind}_G(-) : G\text{-}\mathbf{Mod} \to G\text{-}\mathbf{Mod}$ is exact.

- We say A is an induced module if $A \cong \mathrm{Ind}_G(M)$ for some module M. If A is an induced G-module, then A is induced as an H-module for any subgroup $H \subseteq G$.

Lemma 3.13. Induced modules are cohomologically trivial.

Proof. There is an isomorphism
$$C^*(G, \mathrm{Ind}_G(M)) \cong X^*(G, M).$$
□

Remark 3.14. We have an equivariant inclusion of fixed points
$$M \hookrightarrow \mathrm{Ind}_G(M)$$
which is an embedding, and we take $Q \cong \mathrm{Ind}_G(M)/M$, then this extends to a short exact sequence
$$0 \longrightarrow M \hookrightarrow \mathrm{Ind}_G(M) \longrightarrow Q \longrightarrow 0$$
then $H^{n+1}(G, M) \cong H^n(G, Q)$. One say that $H^*(G, -)$ is effaceable. By Tohoku, an effaceable is universal.