MATH 540 Notes

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Contents

1 Abstract Measure Theory

1.1 Introduction

Definition 1.1. Let X be an (non-empty) underlying space we are working over. We denote $\mathcal{P}(X)$ to be the power set of X, i.e., the set of all subsets of X.

Example 1.2. Let $X = \{1, 2\}$, then $\mathcal{P}(X) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$.

Remark 1.3. If X is a finite set of size n, then $\mathcal{P}(X)$ is a finite set of size 2^n.

We will consider a subcollection $\mathcal{A}$ of subsets of X, i.e., a subset of the power set. We will try to define this as an algebra. Note that an algebra is just a ring with a module structure with respect to some other ring.

Definition 1.4. $\mathcal{A} \subseteq \mathcal{P}(X)$ is an algebra on X if it is

a. closed under finite union, i.e., given $E_1, E_2 \in \mathcal{A}$, then $E_1 \cup E_2 \in \mathcal{A}$, and

b. closed under complements, i.e., if $E \in \mathcal{A}$, then the complement $E^c \in \mathcal{A}$ as well.

Remark 1.5. An algebra $\mathcal{A}$ would be closed under finite intersection. Indeed, for any $E_1, E_2 \in \mathcal{A}$, we have $E_1 \cap E_2 \in \mathcal{A}$ if and only if $(E_1 \cap E_2)^c \in \mathcal{A}$, if and only if $E_1^c \cup E_2^c \in \mathcal{A}$, which is true by definition.

Lemma 1.6. If $\mathcal{A}$ is an non-empty algebra on X, then $\varnothing \in \mathcal{A}$ and $X \in \mathcal{A}$.

Proof. Since $\mathcal{A}$ is non-empty, take $E \in \mathcal{A}$, then $\varnothing = E \cap E^c \in \mathcal{A}$ as well. Also, $X = E \cup E^c \in \mathcal{A}$. □

Example 1.7. Let X be a set, and let $\mathcal{A} = \{\varnothing, X\} \subseteq \mathcal{P}(X)$. It is easy to verify that $\mathcal{A}$ is an algebra.

Definition 1.8. Let $\varnothing \neq \mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra, then we say $\mathcal{A}$ is a σ-algebra on X if

a. closed under countable union, i.e., if $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, then $\displaystyle\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$;

b. if $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$.

Lemma 1.9. If $\mathcal{A} \neq \varnothing$ is a σ-algebra on X, then $\{\varnothing, X\} \subseteq \mathcal{A}$ is a σ-algebra.

Example 1.10. Let X be an uncountable set, let $\mathcal{A} = \{E \subseteq X : E \text{ is countable or } E^c \text{ is countable}\}$, then $\mathcal{A}$ is a σ-algebra on X.

Theorem 1.11. Suppose a non-empty algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ such that,

- if $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, and E_j's are pairwise disjoint, then $\displaystyle\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$,

then $\mathcal{A}$ is a σ-algebra on X.

Proof. Take $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, we will show that $\displaystyle\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$. To do this, we will rearrange the sets. Let $F_1 = E_1$, let $F_2 = E_2 \backslash E_1$, let $F_3 = E_3 \backslash (E_1 \cup E_2)$, and so on, such that let $F_k = E_k \backslash \displaystyle\bigcup_{i=1}^{k-1} E_i$. We note

$$\begin{aligned} F_k &= E_k \cap \left(\bigcup_{j=1}^{k-1} E_j \right)^c \\ &= E_k \cap \left(\bigcap_{j=1}^{k-1} E_j^c \right) \in \mathcal{A}. \end{aligned}$$

One can also verify that $\displaystyle\bigcup_{j=1}^{\infty} E_j = \bigcup_{k=1}^{\infty} F_k$, and that F_k's are disjoint from the definition. □

Definition 1.12. Let X be a non-empty space. A topology on X is a family $\mathcal{F}$ of subsets of X satisfying the following conditions:

i. $\varnothing, X \in \mathcal{F}$;

ii. $\mathcal{F}$ is closed under arbitrary union;

iii. $\mathcal{F}$ is closed under finite intersection.

Every member of $\mathcal{F}$ is now called an open subset of X. A complement of an open subset of X is called a closed subset.

Definition 1.13. Let $\mathcal{A}_1, \mathcal{A}_2$ be σ-algebras. We say $\mathcal{A}_1$ is smaller than $\mathcal{A}_2$ if $\mathcal{A}_1 \subseteq \mathcal{A}_2$, and equivalently $\mathcal{A}_2$ is larger than $\mathcal{A}_1$.

Definition 1.14. Let $\mathcal{F}$ be a family of subsets of X, the smallest σ-algebra containing $\mathcal{F}$ is called the σ-algebra generated by $\mathcal{F}$. This is denoted by $\mathcal{M}(\mathcal{F})$.

Lemma 1.15. Let $\mathcal{F}$ be a family of subsets of X. Suppose $\mathcal{F} \subseteq \mathcal{A}$ where $\mathcal{A}$ is a σ-algebra, then $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{A}$.

Proof. Obvious. $\square$

Definition 1.16. Let $\mathcal{F}$ be a topology on X, then we say $(X, \mathcal{F})$ is a topological space. We say $\mathcal{M}(\mathcal{F})$ is the Borel σ-algebra on X, denoted by $\mathcal{B}_X = \mathcal{B}_{X,\mathcal{F}}$. Any member of $\mathcal{B}_X$ is called a Borel set.

Example 1.17. Let $X = \mathbb{R}$, we denote the corresponding Borel σ-algebra to be $\mathcal{B}_{\mathbb{R}}$.

Definition 1.18. A G_δ-set is a countable intersection of open subsets of X. A F_σ-set is a countable union of closed subsets of X.

Theorem 1.19. Both G_δ-sets and F_σ-sets are Borel sets, that is, $G_\delta, F_\sigma \subseteq \mathcal{B}_X$.

Proof. We will prove that any G_δ-set E is a Borel set, and similarly any F_σ-set is a Borel set. By definition $E = \bigcap\limits_{j=1}^{\infty} O_j$, where each O_j is an open subset. To show $E \in \mathcal{B}_X$, we show that $E^c \in \mathcal{B}_X$. Note that $E^c = \left(\bigcap\limits_{j=1}^{\infty} O_j\right)^c = \bigcup\limits_{j=1}^{\infty} O_j^c$. Since $O_j \in \mathcal{B}_X$ for all j, then $O_j^c \in \mathcal{B}_X$ as well. Therefore, $E^c \in \mathcal{B}_X$ since a σ-algebra $\mathcal{B}_X$ is closed under countable unions. $\square$

Definition 1.20. Let $X_1, \ldots, X_n$ be non-empty spaces. The product space is $\prod\limits_{j=1}^{n} X_j$. Define $\pi_j : \prod\limits_{i=1}^{n} X_i \to X_j$ by $\pi_j(x_1, \ldots, x_n) = x_j$. Let $\mathcal{A}_j$ be a σ-algebra on X_j, the product σ-algebra on $\prod\limits_{i=1}^{n} X_j$ is the σ-algebra generated by $\{\pi_j^{-1}(E_j) : E_j \in \mathcal{A}_j \ \forall j \in \{1, \ldots, n\}\}$. The product σ-algebra is denoted by $\bigotimes\limits_{j=1}^{n} \mathcal{A}_j = \prod\limits_{j=1}^{n} \mathcal{A}_j$.

Example 1.21. $\mathcal{B}_{\mathbb{R}^n} = \bigotimes\limits_{j=1}^{n} \mathcal{B}_{\mathbb{R}}$.

1.2 Measures

Definition 1.22. Let $\mathcal{A}$ be a σ-algebra on X. A measure μ on X and $\mathcal{A}$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that

a. $\mu(\varnothing) = 0$;

b. if $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$ and E_j's are disjoint, then $\mu\left(\bigcup\limits_{j=1}^{\infty} E_j\right) = \sum\limits_{j=1}^{\infty} \mu(E_j)$.

We then say $(X, \mathcal{A})$ is a measureable space. A measureable space is a triple $(X, \mathcal{A}, \mu)$ with measure μ specified.

Definition 1.23. Let μ be a measure on $(X, \mathcal{A})$.

1. If $\mu(X) < \infty$, then we say μ is a finite measure. In particular, if $\mu(X) = 1$, this is a probability measure.
2. If $X = \bigcup_{j=1}^{\infty} E_j$ such that $\mu(E_j) < \infty$ for all $j \in \mathbb{N}$, then we say μ is σ-finite.
3. If for all $E \in \mathcal{A}$ with $\mu(E) = \infty$, there is $F \in \mathcal{A}$ such that $F \subseteq E$ and $0 < \mu(F) < \infty$, then we say μ is semi-finite.

Remark 1.24. A σ-finite measure is semi-finite. However, the converse is not true.

Example 1.25. Let $f : X \to [0, \infty]$ be a function. For any $E \subseteq \mathcal{P}(E)$, we can define a measure $\mu(E) = \sum_{x \in E} f(x)$. Note that the summation makes sense only when E is finite. In case E is infinite, we should define $\sum_{x \in E} f(x) = \sup\{\sum_{x \in F} f(x) : F \subseteq E \text{ for finite } F\}$. Let μ be a measure on $\mathcal{P}(X)$.

- If $f(x) \equiv 1$ for all $x \in X$, then $\mu(E) = \sum_{x \in E} 1 = \mathrm{Card}(E)$. In this case, μ is called a counting measure.
- Suppose $x_0 \in X$ is fixed. Define

  $$f(x) = \begin{cases} 1, & \text{if } x = x_0 \\ 0, & \text{if } x \neq x_0 \end{cases}$$

  then for any $E \in \mathcal{P}(X)$,

  $$\mu(E) = \begin{cases} 1, & \text{if } x_0 \in E \\ 0, & \text{if } x_0 \notin E \end{cases}$$

  This is called the Dirac measure of x_0.

Definition 1.26. Let $(X, \mathcal{A}, \mu)$ be a measure space. A set $E \subseteq \mathcal{A}$ is called a null set if $\mu(E) = 0$.

If a statement about points $x \in X$ is true except for null sets, then we say the statement is true almost everywhere.

Example 1.27. Suppose $f(x) \leqslant 1$ for all $x \in X$, then we say f is bounded above by 1 everywhere. If we want to weaken this statement, we can say $f(x) \leqslant 1$ almost everywhere $x \in X$, which is true if and only if $\mu(\{x \in X : f(x) > 1\} = 0$.

Theorem 1.28. Let $E, F \in \mathcal{A}$ be such that $E \subseteq F$, then $\mu(E) \leqslant \mu(F)$.

Proof. We can write $F = E \cup (E \backslash F)$, then

$$\begin{aligned} \mu(F) &= \mu(E) + \mu(F \backslash E) \\ &\geqslant \mu(E) \end{aligned}$$

since $\mu(F \backslash E) \geqslant 0$. $\square$

Theorem 1.29 (Sub-additivity)**.** Let $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, then $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) \leqslant \sum_{j=1}^{\infty} \mu(E_j)$.

Proof. Set $F_1 = E_1$ and let $F_k = E_k \backslash \left(\bigcup_{j=1}^{k-1} E_j\right)$ be defined inductively, then $\bigcup_{k \in \mathbb{N}} F_k = \bigcup_{j \in \mathbb{N}} E_j$. Since F_k's are disjoint, we have

$$\begin{aligned} \mu\left(\bigcup_{j \in \mathbb{N}} E_j\right) &= \mu\left(\bigcup_{k \in \mathbb{N}} F_k\right) \\ &= \sum_{k=1}^{\infty} \mu(F_k) \\ &= \sum_{k=1}^{\infty} \mu(E_k) \end{aligned}$$

$$= \sum_{j=1}^{\infty} \mu(E_j)$$

by Theorem 1.28. □

Theorem 1.30. Let $E_j \in \mathcal{A}$ for all $j \in \mathbb{N}$.

a. (Continuity from below): If $E_1 \subseteq E_2 \subseteq \cdots E_j \subseteq \cdots$ for all j, then $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j)$.

b. (Continuity from above): If $E_1 \supseteq E_2 \supseteq \cdots E_j \supseteq \cdots$ for all $j \in \mathbb{N}$, then $\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{j\to\infty} \mu(E_j)$ if $\mu(E_1) < \infty$.

In particular, the limits on the right exist on $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.

Example 1.31. Let μ be the counting measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. For each $j \in \mathbb{N}$, we define $E_j = \{n \in \mathbb{N} : n > j\}$. Therefore $E_1 \supseteq E_2 \supseteq \cdots$ is a decreasing sequence of sets. Note that $\mu(E_1) = \mu(\{n \in \mathbb{N}\}) = \mathbb{N} = \infty$, and $\lim_{j\to\infty} \mu(E_j) = \lim_{j\to\infty} \infty = \infty$, but $\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \mu(\varnothing) = 0$.

Proof.

a. Set $E_0 = \varnothing$. Now

$$\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} (E_j \backslash E_{j-1})$$

and therefore

$$\begin{aligned}
\mu\left(\bigcup_{j=1}^{\infty} E_j\right) &= \mu\left(\bigcup_{j=1}^{\infty} (E_j \backslash E_{j-1})\right) \\
&= \sum_{j=1}^{\infty} \mu(E_j \backslash E_{j-1}) \\
&= \lim_{k\to\infty} \sum_{j=1}^{k} \mu(E_j \backslash E_{j-1}) \\
&= \lim_{k\to\infty} \mu\left(\bigcup_{j=1}^{k} E_j \backslash E_{j-1}\right) \\
&= \lim_{k\to\infty} \mu(E_k) \\
&= \lim_{j\to\infty} \mu(E_j).
\end{aligned}$$

b. For any $j \in \mathbb{N}$, set $F_j = E_1 \backslash E_j$. Note that $F_j \subseteq F_{j+1}$ since $E_j \supseteq E_{j-1}$. This is now an increasing sequence as in part a. By part a., we know $\mu\left(\bigcup_{j=1}^{\infty} F_j\right) = \lim_{j\to\infty} \mu(F_j)$. Now note that

$$\begin{aligned}
\bigcup_{j=1}^{\infty} F_j &= \bigcup_{j=1}^{\infty} (E_1 \backslash E_j) \\
&= \bigcup_{j=1}^{\infty} (E_1 \cap E_j^c)
\end{aligned}$$

$$
\begin{aligned}
&= E_1 \cap \bigcup_{j=1}^{\infty} E_j^c \\
&= E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j \right)^c \\
&= \left(\left(E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j \right)^c \right) \cup \left(\bigcap_{j=1}^{\infty} E_j \right) \right) \cap \left(\bigcap_{j=1}^{\infty} E_j \right)^c \\
&= \left(E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j \right)^c \right) \cup \left(\bigcap_{j=1}^{\infty} E_j \right).
\end{aligned}
$$

Note that $E_1 \cap \left(\bigcap\limits_{j=1}^{\infty} E_j \right)^c$ and $\bigcap\limits_{j=1}^{\infty} E_j$ are disjoint, therefore by property of measure we have

$$
\begin{aligned}
\mu(E_1) &= \mu\left(E_1 \cap \left(\bigcap_{j=1}^{\infty} E_j \right)^c \right) + \mu\left(\bigcap_{j=1}^{\infty} E_j \right) \\
&= \mu\left(\bigcup_{j=1}^{\infty} F_j \right) + \mu\left(\bigcap_{j=1}^{\infty} E_j \right) \\
&= \lim_{j \to \infty} \mu(F_j) + \mu\left(\bigcap_{j=1}^{\infty} E_j \right).
\end{aligned}
$$

Recall that $F_j = E_1 \backslash E_j$ for all j, therefore $E_1 = F_j \cup F_j^c = F_j \cup E_j$, where F_j and E_j are disjoint, therefore $\mu(E_1) = \mu(F_j) + \mu(E_j)$. Since $\mu(E_1) < \infty$, and F_j is a subset of E_1 and hence also a real number, then $\mu(E_1)$ is a sum of two real numbers. Therefore, we have $\mu(E_1) - \mu(E_j) = \mu(F_j)$. With this, we have

$$
\begin{aligned}
\mu(E_1) &= \lim_{j \to \infty} (\mu(E_1) - \mu(E_j)) + \mu\left(\bigcap_{j=1}^{\infty} E_j \right) \\
&= \mu(E_1) - \lim_{j \to \infty} (\mu(E_j)) + \mu\left(\bigcap_{j=1}^{\infty} E_j \right).
\end{aligned}
$$

In particular, we get

$$
\lim_{j \to \infty} (\mu(E_j)) = \mu\left(\bigcap_{j=1}^{\infty} E_j \right).
$$

□

1.3 Outer Measure

Definition 1.32. An outer measure μ^* on X (or $\mathcal{P}(X)$) is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that

i. $\mu^*(\varnothing) = 0$,

ii. $\mu^*(A) \leqslant \mu^*(B)$ for all $A \subseteq B \subseteq X$,

iii. σ-subaddivity: $\mu^*\left(\bigcup\limits_{j=1}^{\infty} A_j \right) \leqslant \sum\limits_{j=1}^{\infty} \mu^*(A_j)$.

Example 1.33. Let $\rho : \mathcal{A} \to [0, \infty]$ be such that $\rho(\varnothing) = 0$, where $\mathcal{A} \subseteq \mathcal{P}(X)$ is a subcollection (but not necessarily an algebra) such that $\varnothing, X \in \mathcal{A}$.

For all $A \in \mathcal{P}(X)$, i.e., $A \subseteq X$, we define

$$
\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \rho(E_j) : E_j \in \mathcal{A} \text{ and } A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}.
$$

Theorem 1.34. μ^* defined in Example 1.33 is an outer measure.

Proof.

i. Let $E_j = \varnothing$ for all $j \in \mathbb{N}$, then $\varnothing \subseteq \bigcup_{j=1}^{\infty} E_j$, and so

$$\sum_{j=1}^{\infty} \rho(E_j) = \sum_{j=1}^{\infty} \rho(\varnothing) = \sum_{j=1}^{\infty} 0 = 0$$

and therefore $\mu^*(\varnothing) = 0$.

ii. Let $A \subseteq B \subseteq X$. If $B \subseteq \bigcup_{j=1}^{\infty} E_j$, we have $A \subseteq \bigcup_{j=1}^{\infty} E_j$, then

$$\left\{ \sum_{j=1}^{\infty} \rho(E_j) : E_j \in \mathcal{A}, B \subseteq \bigcup_{j=1}^{\infty} E_j \right\} \subseteq \left\{ \sum_{j=1}^{\infty} \rho(E_j) : E_j \in \mathcal{A}, A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}.$$

In particular, given subsets $S_1 \subseteq S_2$, then $\inf S_2 \leqslant \inf S_1$ and $\sup S_1 \leqslant \sup S_2$. This implies $\mu^*(A) \leqslant \mu^*(B)$.

iii. We want to show $\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leqslant \sum_{j=1}^{\infty} \mu^*(A_j)$. Now for any $j \in \mathbb{N}$, we have

$$\mu^*(A_j) = \inf \left\{ \sum_{k=1}^{\infty} \rho(E_k) : E_k \in \mathcal{A} \text{ and } A_j \subseteq \bigcup_{k=1}^{\infty} E_k \right\}.$$

For any $\varepsilon > 0$, we note that $\mu^*(A_j) + \varepsilon \cdot 2^{-j}$ is not a lower bound of $\left\{ \sum_{k=1}^{\infty} \rho(E_k) : E_k \in \mathcal{A} \text{ and } A_j \subseteq \bigcup_{k=1}^{\infty} E_k \right\}$. Then there exists $E_k^{(j)} \in \mathcal{A}$ for $k \in \mathbb{N}$ such that $A_j \subseteq \bigcup_{k=1}^{\infty} E_k^{(j)}$ and $\sum_{k=1}^{\infty} \rho(E_k^{(j)}) \leqslant \mu^*(A_j) + \varepsilon \cdot 2^{-j}$. Summing with respec to j, we get

$$\begin{aligned}\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \rho(E_k^{(j)}) &\leqslant \sum_{j=1}^{\infty} \mu^*(A_j) + \sum_{j=1}^{\infty} \varepsilon \cdot 2^{-j} \\ &= \sum_{j=1}^{\infty} \mu^*(A_j) + \varepsilon.\end{aligned}$$

Note that

$$\bigcup_{j=1}^{\infty} A_j \subseteq \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} E_k^{(j)}$$

is a countable union of subsets of $\mathcal{A}$. We will calculate the value over μ^*. By definition of μ^*, we have

$$\begin{aligned}\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) &\leqslant \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \rho(E_k^{(j)}) \\ &\leqslant \sum_{j=1}^{\infty} \mu^*(A_j) + \varepsilon.\end{aligned}$$

Since this is true for all $\varepsilon > 0$, then take $\varepsilon \to 0$, we are done.

□

Definition 1.35. Let μ^* be an outer measure on $(X, \mathcal{P}(X))$. A set $A \subseteq X$ is called μ^*-measurable if $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for all $E \subseteq X$.

Remark 1.36. First note that $\mu^*(E) = \mu^*((E \cap A) \cup (E \cap A^c))$, therefore $\mu^*(E) \leqslant \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for all $E \subseteq X$.

Theorem 1.37 (Fundamental Theorem of Measure Theory)**.** Let μ^* be an outer measure on X. Let $\mathcal{A}$ be the collection of all μ^*-measurable set, then $\mathcal{A}$ is a σ-algebra, and $\mu^*|_{\mathcal{A}}$ is a measure on $\mathcal{A}$, i.e., $(X, \mathcal{A}, \mu^*)$ is a measure space.

Proof. We first prove that $\mathcal{A}$ is an algebra. To see $\mathcal{A}$ is closed under complement, we have $A \in \mathcal{A}$ if and only if $A^c \in \mathcal{A}$. by the definition of measurable set. To show $\mathcal{A}$ is closed under finite union, suppose $A, B \in \mathcal{A}$, and we want to show $A \cup B \in \mathcal{A}$, which is true if and only if $\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$ for all $E \subseteq X$, hence it suffices to show that $\mu^*(E) \geqslant \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$. We have

$$\begin{aligned}\mu^*(E \cap (A \cup B)) &= \mu^*(E \cap (A \cup B) \cap A) + \mu^*(E \cap (A \cup B) \cap A^c) \\ &= \mu^*(E \cap A) + \mu^*(E \cap B \cap A^c)\end{aligned}$$

and

$$\begin{aligned}\mu^*(E \cap (A \cup B)^c) &= \mu^*(E \cap (A \cup B)^c \cap A) + \mu^*(E \cap (A \cup B)^c \cap A^c) \\ &= \mu^*(\varnothing) + \mu^*(E \cap A^c \cap B^c) \\ &= \mu^*(E \cap A^c \cap B^c).\end{aligned}$$

Therefore

$$\begin{aligned}\mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c) &= \mu^*(E \cap A) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\ &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E)\end{aligned}$$

where the last two steps follow from the fact that $A, B \in \mathcal{A}$ are μ^*-measurable. Therefore, $\mathcal{A}$ is an algebra. We now want to show that it is a σ-algebra. It suffices to prove that $\mathcal{A}$ is closed under disjoint σ-unions. Let $A_j \in \mathcal{A}$ for all $j \in \mathbb{N}$ where they are pairwise disjoint, and we want to show that $\bigcup\limits_{j=1}^{\infty} A_j \in \mathcal{A}$. That is,

$$\mu^*(E) = \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)\right) + \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)^c\right)$$

for all $E \subseteq X$.

Lemma 1.38. For a pairwise disjoint family $A_1, \dots, A_n \in \mathcal{A}$,

$$\mu^*\left(E \cap \bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu^*(E \cap A_j).$$

Subproof. We proceed by induction. For $n = 1$, this is obviously true. Now suppose $n > 1$. To simplify the notation, let $B_n = \bigcup\limits_{j=1}^{n} A_j$, and use the convention that $B_0 = \varnothing$. Now

$$\begin{aligned}\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) \\ &= \sum_{i=1}^{n}(E \cap A_i) + \mu^*(E \cap B_0) \\ &= \sum_{i=1}^{n}(E \cap A_i) \\ &= \sum_{i=1}^{n}(E \cap A_i)\end{aligned}$$

for all $n \in \mathbb{N}$. This finishes the proof. ■

Now for any $E \subseteq X$, we have

$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \\
&= \sum_{j=1}^{n} \mu^*(E \cap A_j) + \mu^*(E \cap B_n^c) \\
&\geqslant \sum_{j=1}^{n} \mu^*(E \cap A_j) + \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)^c\right)
\end{aligned}$$

since $B_n = \bigcup_{j=1}^{n} A_j \subseteq \bigcup_{j=1}^{\infty} A_j$. Now take $n \to \infty$, we get

$$\begin{aligned}
\mu^*(E) &\geqslant \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*\left(\left(\bigcup_{j=1}^{\infty} A_j\right)^c\right) \\
&\geqslant \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)\right) + \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)^c\right) \\
&\geqslant \mu^*(E).
\end{aligned}$$

This forces all inequalities here to be equality, therefore

$$\mu^*(E) = \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)\right) + \mu^*\left(E \cap \left(\bigcup_{j=1}^{\infty} A_j\right)^c\right)$$

as desired. Finally, we need to show that the restriction still gives a measure. We already know

$$\mu^*(E) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*\left(\left(\bigcup_{j=1}^{\infty} A_j\right)^c\right)$$

for any $E \subseteq X$, then in particular take $E = \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$ to be the disjoint union, then this forces

$$\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*(\varnothing) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j).$$

Therefore $\mu^*|_{\mathcal{A}}$ is a measure. □

Definition 1.39. A measure μ is said to be complete if its domain contains all subsets of null sets.

Example 1.40. Let $X = \{a, b\}$, $\mathcal{A} = \{\varnothing, \{a, b\}\}$. Define $\mu : \mathcal{A} \to [0, \infty]$ by setting $\mu^*(X) = 0$, $\mu^*(\varnothing) = 0$. This is not a complete measure because $\{a\} \notin \mathcal{A}$.

Theorem 1.41. Let $\mathcal{A}$ be the collection of all μ^*-measurable sets, then the measure $\mu^*|_{\mathcal{A}}$ is complete.

Proof. Let N be any null set in $\mathcal{A}$, i.e., $\mu^*(N) = 0$. Take an arbitrary subset $A \subseteq N$, we need to show $A \in \mathcal{A}$. Since $\mu^*(N) = 0$, then $\mu^*(A) = 0$ as well. For any $E \subseteq X$, we prove $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$. It is clear that

$$\begin{aligned}
\mu^*(E) &\leqslant \mu^*(E \cap A) + \mu^*(E \cap A^c) \\
&\leqslant \mu^*(A) + \mu^*(E \cap A^c) \\
&\leqslant \mu^*(N) + \mu^*(E \cap A^c) \\
&= \mu^*(E \cap A^c) \\
&= \mu^*(E).
\end{aligned}$$

by the subadditivity of μ^*. □

Definition 1.42. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra. A function $\mu_0 : \mathcal{A} \to [0, \infty]$ is a pre-measure if

i. $\mu_0(\varnothing) = 0$,

ii. if $A_j \in \mathcal{A}$ for all $j \in \mathbb{N}$ with $\bigcup\limits_{j=1}^{\infty} A_j \in \mathcal{A}$, and they are pairwise disjoint, then $\mu_0\left(\bigcup\limits_{j=1}^{\infty} A_j\right) = \sum\limits_{j=1}^{\infty} \mu_0(A_j)$.

Therefore, the difference of a pre-measure from a measure is that a pre-measure is not defined on a σ-algebra.

Theorem 1.43. Let μ_0 be a pre-measure, then $\mu_0(A) \leqslant \mu_0(B)$ if $A, B \in \mathcal{A}$ are such that $A \subseteq B$.

Proof. We write $B = (B\backslash A) \cup A$, where $B\backslash A = B \cap A^c \in \mathcal{A}$, therefore

$$\begin{aligned} \mu_0(B) &= \mu_0(B\backslash A) + \mu_0(A) \\ &\geqslant \mu_0(A). \end{aligned}$$

□

Definition 1.44. Given a pre-measure μ_0, we extend it to an outer measure as follows: for any $E \subseteq X$, define $\mu^*(E) = \inf\{\sum\limits_{j=1}^{\infty} \mu_0(A_j) : E \subseteq \bigcup\limits_{j=1}^{\infty} A_j, A_j \in \mathcal{A}\}$.

Theorem 1.45 (Carathéodory's Extension Theorem)**.** Let μ^* be the outer measure induced by μ_0 specified in Definition 1.44, then

i. $\mu^*|_{\mathcal{A}} = \mu_0$, or equivalently, for any $A \in \mathcal{A}$, we have $\mu^*(A) = \mu_0(A)$;

ii. if $A \in \mathcal{A}$, then A is μ^*-measurable.

Proof.

i. We want to show that for any $E \in \mathcal{A}$, $\mu^*(E) = \mu_0(E)$. To show $\mu^*(E) \leqslant \mu_0(E)$, we choose $A_1 = E \in \mathcal{A}$, and $A_j = \varnothing$ for all $j \geqslant 2$, then $E \subseteq \bigcup\limits_{j=1}^{\infty} A_j$, therefore

$$\begin{aligned} \mu^*(E) &\leqslant \sum_{j=1}^{\infty} \mu_0(A_j) \\ &= \mu_0(E). \end{aligned}$$

It now suffices to show that $\mu_0(E)$ is a lower bound of $\{\sum\limits_{j=1}^{\infty} \mu_0(A_j) : E \subseteq \bigcup\limits_{j=1}^{\infty}, A_j \in \mathcal{A}\}$. Let $A_j \in \mathcal{A}$ and $\bigcup\limits_{j=1}^{\infty} A_j \supseteq E$. We prove that $\mu_0(E) \leqslant \sum\limits_{j=1}^{\infty} \mu_0(A_j)$. For any $n \in \mathbb{N}$, define $B_n = E \cap \left(A_n \backslash \bigcup\limits_{j=1}^{n-1} A_j\right)$, therefore $\bigcup\limits_{n=1}^{\infty} B_n = E \cap \left(\bigcup\limits_{j=1}^{\infty} A_j\right) = E$ where B_n's are disjoint. We have

$$\begin{aligned} \mu_0(E) &= \mu_0\left(\bigcup_{n=1}^{\infty} B_n\right) \\ &= \sum_{n=1}^{\infty} \mu_0(B_n) \\ &\leqslant \sum_{n=1}^{\infty} \mu_0(A_n) \\ &= \sum_{j=1}^{\infty} \mu_0(A_j). \end{aligned}$$

ii. For any $A \in \mathcal{A}$, we want to prove that $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for all $E \subseteq X$. It suffices to show that for any $E \subseteq X$, we have $\mu^*(E) \geqslant \mu^*(E \cap A) + \mu^*(E \cap A^c)$.

Pick arbitrary $\varepsilon > 0$, then $\mu^*(E) + \varepsilon$ is not a lower bound of $\{\sum_{j=1}^{\infty} \mu_0(A_j) : E \subseteq \bigcup_{j=1}^{\infty}, A_j \in \mathcal{A}\}$. Therefore, there exists some $A_j \in \mathcal{A}$ such that $E \subseteq \bigcup_{j=1}^{\infty} A_j$ and $\sum_{j=1}^{\infty} \mu_0(A_j) \leqslant \mu^*(E) + \varepsilon$. Since $\mu_0(A_j) = \mu_0(A_j \cap A) + \mu_0(A_j \cap A^c)$, then

$$\begin{aligned}
\sum_{j=1}^{\infty} \mu_0(A_j) &= \sum_{j=1}^{\infty} \mu_0(A_j \cap A) + \sum_{j=1}^{\infty} \mu_0(A_j \cap A^c) \\
&= \sum_{j=1}^{\infty} \mu^*(A_j \cap A) + \sum_{j=1}^{\infty} \mu^*(A_j \cap A^c) \\
&\geqslant \mu^*\left(\left(\bigcup_{j=1}^{\infty} A_j\right) \cap A\right) + \mu^*\left(\left(\bigcup_{j=1}^{\infty} A_j\right) \cap A^c\right) \\
&\geqslant \mu^*(E \cap A) + \mu^*(E \cap A^c).
\end{aligned}$$

Let $\varepsilon \to 0$, then $\mu^*(E) \geqslant \mu^*(E \cap A) + \mu^*(E \cap A^c)$, as desired.

$\square$

Theorem 1.46. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra, and let μ_0 be a pre-measure on $\mathcal{A}$. Define $\mathcal{M}(\mathcal{A})$ to be the σ-algebra generated by $\mathcal{A}$.

a. The outer measure μ^* induced by μ_0 defines a measure function on $\mathcal{M}(\mathcal{A})$, and $\mu^*|_{\mathcal{A}} = \mu_0$.

b. If $\tilde{\mu}$ is another measure on $\mathcal{M}(\mathcal{A})$ that extends μ_0, then $\tilde{\mu}(E) \leqslant \mu^*(E)$ for all $E \subseteq \mathcal{M}(\mathcal{A})$, with equality if and only if $\mu^*(E) < \infty$.

c. If μ_0 is σ-finite, i.e., $X = \bigcup_{j=1}^{\infty} A_j$ with $A_j \in \mathcal{A}$ and $\mu_0(A_j) < \infty$ for all j, then $\mu^*|_{\mathcal{M}(\mathcal{A})}$ is the unique extension of μ_0 to a measure on $\mathcal{M}(\mathcal{A})$.

Proof.

a. Let $\mathcal{B}$ be the set of all μ^*-measurable sets, then $\mu^*|_{\mathcal{B}}$ is a measure on $\mathcal{B}$ that extends μ_0. By the fundamental theorem of measure theory, we know $\mathcal{B}$ is a σ-algebra. In particular, $\mathcal{B} \supseteq \mathcal{A}$, therefore $\mathcal{B} \supseteq \mathcal{M}(\mathcal{A})$. That means $\mu^*|_{\mathcal{M}(\mathcal{A})}$ is a measure as well.

b. Let $\tilde{\mu}$ be any measure on $\mathcal{M}(\mathcal{A})$ that extends μ_0. We first show that for all $E \in \mathcal{M}(\mathcal{A})$, then $\tilde{\mu}(E) \leqslant \mu^*(E)$. Recall that $\mu^*(E) = \inf\{\sum_{j=1}^{\infty} \mu_0(A_j) : E \subseteq \bigcup_{j=1}^{\infty} A_j, A_j \in \mathcal{A}\}$. Given a cover $E \subseteq \bigcup_{j=1}^{\infty} A_j$ and fix $A_j \in \mathcal{A}$. Therefore,

$$\begin{aligned}
\tilde{\mu}(E) &\leqslant \tilde{\mu}\left(\bigcup_{j=1}^{\infty} A_j\right) \\
&\leqslant \sum_{j=1}^{\infty} \tilde{\mu}(A_j) \\
&= \sum_{j=1}^{\infty} \mu_0(A_j),
\end{aligned}$$

therefore $\tilde{\mu}(E) \leqslant \mu^*(E)$. Assume we have $\mu^*(E) < \infty$, and we want to show that $\tilde{\mu}(E) = \mu^*(E)$. It suffices to show $\mu^*(E) \leqslant \tilde{\mu}(E)$.

Claim 1.47. Let $A_j \in \mathcal{A}$ for all $j \in \mathbb{N}$, then $\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) = \tilde{\mu}\left(\bigcup_{j=1}^{\infty} A_j\right)$.

Subproof. Note that $\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}(\mathcal{A})$, then we can just work on $\mathcal{M}(\mathcal{A})$. Consider $\mu^*|_{\mathcal{M}(\mathcal{A})}$ and $\tilde{\mu}$ are measures on $\mathcal{M}(\mathcal{A})$. Let $E_n = \bigcup_{j=1}^{n} A_j$ for all $n \in \mathbb{N}$, then we have a nested increasing sequence of E_n's. In particular, we know $\bigcup_{n=1}^{\infty} E_n = \bigcup_{j=1}^{\infty} A_j$. Therefore

$$\begin{aligned}
\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) &= \mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \\
&= \lim_{n\to\infty} \mu^*(E_n) \\
&= \lim_{n\to\infty} \mu^*\left(\bigcup_{j=1}^{n} A_j\right) \\
&= \lim_{n\to\infty} \mu_0\left(\bigcup_{j=1}^{n} A_j\right) \\
&= \lim_{n\to\infty} \tilde{\mu}\left(\bigcup_{j=1}^{n} A_j\right) \\
&= \tilde{\mu}\left(\bigcup_{j=1}^{\infty} A_j\right)
\end{aligned}$$

by continuity from below and closure of finite union. ■

We know from the claim that

$$\begin{aligned}
\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) &= \lim_{n\to\infty} \mu_0\left(\bigcup_{j=1}^{n} A_j\right) \\
&\leqslant \lim_{n\to\infty} \sum_{j=1}^{n} \mu_0(A_j) \\
&= \sum_{j=1}^{\infty} \mu_0(A_j).
\end{aligned}$$

Take arbitrary $\varepsilon > 0$, then consider $\mu^*(E) + \varepsilon$, which is not a lower bound of the set anymore. Therefore, there exists $A_j \in \mathcal{A}$ for each $j \in \mathbb{N}$ such that $E \subseteq \bigcup_{j=1}^{\infty} A_j$ and that $\sum_{j=1}^{\infty} \mu_0(A_j) \leqslant \mu^*(E) + \varepsilon$. In particular, this means $\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leqslant \mu^*(E) + \varepsilon$. Since $\mu^*(E) < \infty$, then

$$\begin{aligned}
\mu^*\left(\bigcup_{j=1}^{\infty} A_j \backslash E\right) &= \mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) - \mu^*(E) \\
&< \varepsilon.
\end{aligned}$$

Now that

$$\mu^*(E) \leqslant \mu^*\left(\bigcup_{j=1}^{\infty} A_j\right)$$

$$
\begin{aligned}
&= \tilde{\mu}\left(\bigcup_{j=1}^{\infty} A_j\right) \\
&= \tilde{\mu}(E) + \tilde{\mu}\left(\bigcup_{j=1}^{\infty} A_j \backslash E\right) \\
&< \tilde{\mu}(E) + \varepsilon
\end{aligned}
$$

by the claim. Therefore, for any $\varepsilon > 0$, we have $\mu^*(E) \leqslant \tilde{\mu}(E) + \varepsilon$ whenever $\mu^*(E) < \infty$. Take $\varepsilon \to 0$, we get $\mu^*(E) \leqslant \tilde{\mu}(E)$.

c. Since μ_0 is σ-finite, then there exists a decomposition $X = \bigcup_{j=1}^{\infty} A_j$ for $A_j \in \mathcal{A}$ and that $\mu_0(A_j) < \infty$. For any $E \in \mathcal{M}(\mathcal{A})$, then

$$
\begin{aligned}
E &= E \cap X \\
&= E \cap \left(\bigcup_{j=1}^{\infty} A_j\right) \\
&= \bigcup_{j=1}^{\infty} (E \cap A_j)
\end{aligned}
$$

and

$$
\begin{aligned}
\mu^*(E) &= \mu^*\left(\bigcup_{j=1}^{\infty} (E \cap A_j)\right) \\
&= \sum_{j=1}^{\infty} \mu^*(E \cap A_j) \\
&= \sum_{j=1}^{\infty} \tilde{\mu}(E \cap A_j) \\
&= \tilde{\mu}\left(\bigcup_{j=1}^{\infty} (E \cap A_j)\right) \\
&= \tilde{\mu}(E)
\end{aligned}
$$

since $\mu^*(E \cap A_j) \leqslant \mu^*(A_j) = \mu_0(A_j) < \infty$.

□

1.4 Borel Measure

Recall that the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$ is the smallest σ-algebra containing all open sets. Let $\mathcal{G}$ be the set of all open sets in $\mathbb{R}$ with respect to the standard topology. Therefore $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{G})$. We can in fact use something smaller than $\mathcal{G}$.

Theorem 1.48. $\mathcal{B}_{\mathbb{R}}$ is a σ-algebra generated by

a. $\mathcal{A}_0 = \{(a, b) : a, b \in \mathbb{R}, a < b\}$, or by

b. $\mathcal{A}_1 = \{(a, b] : a, b \in \mathbb{R}, -\infty \leqslant a < b < \infty\} \cup \{(a, \infty) : -\infty \leqslant a < \infty\} \cup \{\varnothing\}$.

Any member in $\mathcal{A}_1$ is called an h-interval.

Proof.

a. We want to show that $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{A})$. Obviously $\mathcal{A}_0 \subseteq \mathcal{G}$, then $\mathcal{M}(\mathcal{G})$ is a σ-algebra containing $\mathcal{A}_0$, then $\mathcal{M}(\mathcal{A}_0) \subseteq \mathcal{M}(\mathcal{G}) = \mathcal{B}_{\mathbb{R}}$. Conversely, recall that any open subset in $\mathbb{R}$ is a σ-union of open intervals, therefore $\mathcal{G} \subseteq \mathcal{M}(\mathcal{A})$, so $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{G}) \subseteq \mathcal{M}(\mathcal{A}_0)$, therefore $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{A}_0)$.

b. We first show that $\mathcal{M}(\mathcal{A}_1) \subseteq \mathcal{B}_{\mathbb{R}}$. Since $\mathcal{M}(\mathcal{A}_1)$ is the smallest σ-algebra containing $\mathcal{A}_1$, then it suffices to show that $\mathcal{A}_1 \subseteq \mathcal{B}_{\mathbb{R}}$. It is easy to see that $(a,b] = \bigcap\limits_{n=1}^{\infty}(a, b+\frac{1}{n}) \in \mathcal{B}_{\mathbb{R}}$, and $(a,\infty) = \bigcup\limits_{n=1}^{\infty}(a,n) \in \mathcal{B}_{\mathbb{R}}$.

We now verify that $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{A}_1)$. By a. we know $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{A}_0)$, so it suffices to show that $\mathcal{A}_0 \subseteq \mathcal{M}(\mathcal{A}_1)$. For $a < b$, we have $(a,b) = \bigcup\limits_{n=1}^{\infty}(a, b-\frac{1}{n}]$, therefore the right-hand side is a σ-union of intervals, hence belongs to $\mathcal{M}(\mathcal{A}_1)$, and we are done.

$\square$

Definition 1.49. We define $\mathcal{A}_2$ to be the collection of finite disjoint unions of h-intervals, e.g., $\bigcup\limits_{j=1}^{n}(a_j, b_j]$, then $\mathcal{A}_2$ is an algebra.

Definition 1.50. A function on $\mathbb{R}$ is said to be right continuous if $\lim\limits_{x\to x_0^+} F(x) = F(x_0)$.

Theorem 1.51. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous. Let $I_j = (a_j, b_j]$ for $j = 1, \ldots, n$ be disjoint h-intervals. We define the pre-measure μ_0 on $\mathcal{A}_2$ by $\mu_0(\varnothing) = 0$ and $\mu_0\left(\bigcup\limits_{j=1}^{n}(a_j, b_j]\right) = \sum\limits_{j=1}^{n}[F(b_j) - F(a_j)]$.

Proof. First one cancheck that μ_0 is well-defined, that is, given any partition of h-interval, the μ_0-measurements on the interval are the same.

Second, we need to show that μ_0 satisfies σ-additivity, that is, if $\bigcup\limits_{j=1}^{\infty} I_j \in \mathcal{A}_2$ such that I_j's are disjoint, then $\mu_0\left(\bigcup\limits_{j=1}^{\infty} I_j\right) = \sum\limits_{j=1}^{\infty}\mu_0(I_j)$. It is easy to verify finite additivity, so we now assume

$$\bigcup_{j=1}^{\infty} I_j = I = (a,b] \in \mathcal{A}_2$$

for $-\infty \leqslant a < b < \infty$, then we will show that

$$F(b) - F(a) = \mu_0(I) = \sum_{j=1}^{\infty}\mu_0(I_j)$$

for $I_j = (a_j, b_j]$.

To show $\mu_0(I) \geqslant \sum\limits_{j=1}^{\infty}\mu(I_j)$, we know $F(b) - F(a) \geqslant \sum\limits_{j=1}^{n}[F(b_j) - F(a_j)]$, therefore taking the limit of $n \to \infty$ gives $F(b) - F(a) \geqslant \sum\limits_{j=1}^{\infty}\mu_0(I_j)$.

To show $\mu_0(I) \leqslant \sum\limits_{j=1}^{\infty}\mu(I_j)$, since F is right continuous, then for all $\varepsilon > 0$, there exist $\delta > 0$ such that $F(a+\delta) - F(a) < \varepsilon$. Therefore, for every $j > 0$, there exists $\delta_j > 0$ such that $F(b_j + \delta_j) - F(b_j) < 2^{-j}\varepsilon$, then

$$\begin{aligned}[a+\delta, b] &\subseteq (a,b] \\ &= \bigcup_{j=1}^{\infty}(a_j, b_j] \\ &= \bigcup_{j=1}^{\infty}(a_j, b_j + \delta_j).\end{aligned}$$

By compactness, there exists some $N \in \mathbb{N}$ such that $[a+\delta, b] \subseteq \bigcup\limits_{j=1}^{N}(a_j, b_j + \delta_j)$. Assume $b_j + \delta_j \in (a_{j+1}, b_{j+1}]$, then

$$\mu_0(I) = \mu_0((a,b])$$

$$\begin{aligned}
&= F(b) - F(a)\\
&\leqslant F(b) - F(a+\delta) + \varepsilon\\
&\leqslant F(b_N + \delta_N) - F(a+\delta) + \varepsilon\\
&= F(b_N + \delta_N) - F(a_N) + F(a_N) - F(a+\delta) + \varepsilon\\
&= F(b_N + \delta_N) - F(a_N) + \sum_{j=1}^{N-1} [F(a_{j-1}) - F(a_j)] + \varepsilon\\
&\leqslant F(b_N + \delta_N) - F(a_N) + \sum_{j=1}^{N-1} [F(b_j + \delta_j) - F(a_j)] + \varepsilon\\
&= \sum_{j=1}^{N} [F(b_j + \delta_j) - F(a_j)] + \varepsilon\\
&= \sum_{j=1}^{N} [F(b_j + \delta_j) - F(b_j) + F(b_j) - F(a_j)] + \varepsilon\\
&= \sum_{j=1}^{N} [F(b_j + \delta_j) - F(b_j)] + \sum_{j=1}^{N} [F(b_j) - F(a_j)] + \varepsilon\\
&\leqslant \sum_{j=1}^{N} 2^{-j}\varepsilon + \sum_{j=1}^{N} \mu_0(I_j) + \varepsilon\\
&\leqslant 2\varepsilon + \sum_{j=1}^{\infty} \mu_0(I_j)
\end{aligned}$$

since F is increasing. Let $\varepsilon \to 0$ and we are done. □

Theorem 1.52. Let F be increasing and right-continuous, then

a. there is a unique measure μ_F on $\mathbb{R}$ such that $\mu_F((a,b]) = F(b) - F(a)$ for all $a, b \in \mathbb{R}$;

b. if G is another increasing and right-continuous function, then $\mu_F = \mu_F$ if and only if $F - G$ is a constant function;

c. if μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets, i.e., a set $S \subseteq \mathbb{R}$ contained in $[-M, M]$ for some $M \in \mathbb{R}$, then

$$F(x) = \begin{cases} \mu((0,x]), & x > 0\\ 0, & x = 0\\ -\mu((x,0]), & x < 0 \end{cases}$$

is an increasing function and right-continuous, and $\mu_F = \mu$.

Proof.

a. Consider $\mathbb{R} = \bigcup_{j=-\infty}^{\infty} (j, j+1]$, then the pre-measure $\mu_0((j,j+1]) = F(j+1) - F(j) < \infty$ defined on h-intervals is σ-finite. Therefore there exists a unique extension of measure μ of μ_0 on $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{A}_2)$ such that $\mu|_{\mathcal{A}_2} = \mu_0$.

b. We have $\mu_F((a,b]) = F(b) - F(a)$ and $\mu_G((a,b]) = G(b) - G(a)$, then

$$\begin{aligned}
\mu_F((a,b]) = \mu_G((a,b]) &\iff F(b) - F(a) = G(b) - G(a)\\
&\iff F(b) - G(b) = G(a) - F(a)\\
&\iff F - G \text{ is constant.}
\end{aligned}$$

c. First note that F is an increasing function since the measure function is increasing. Take any $x_0 \in \mathbb{R}$, we want to show that $\lim\limits_{x \to x_0^+} F(x) = F(x_0)$. We prove this by cases, either $x_0 = 0$, $x_0 > 0$, or $x_0 < 0$. We will only prove the first case, but the two other cases are analogous. Suppose $x_0 = 0$, take a nested sequence of intervals $E_n = (0, \frac{1}{n}]$, with $E_n \supseteq E_{n+1}$ for all $n \in \mathbb{N}$, then

$$\begin{aligned}
\lim_{x \to 0^+} F(x) &= \lim_{x \to 0^+} \mu((0, x]) \\
&= \lim_{n \to 0} \mu((0, \frac{1}{n}]) \\
&= \lim_{n \to \infty} \mu(E_n) \\
&= \mu\left(\bigcap_{n=1}^{\infty} E_n\right) \\
&= \mu(\varnothing) \\
&= 0 \\
&= F(0)
\end{aligned}$$

since $\mu(E_1) < \infty$.

□

Definition 1.53. Suppose F is increasing and right-continuous, then we can use F to create μ_0 on $\mathcal{A}_2$, and get an outer measure μ^* induced by μ_0. Let $\mathcal{A}$ be the collection of all μ^*-measurable sets, then $\mu^*|_{\mathcal{A}}$ is a measure. Note that $\mathcal{A} \supseteq \mathcal{B}_{\mathbb{R}}$: since μ_F is only defined on $\mathcal{B}_{\mathbb{R}}$, then $\mu^*|_{\mathcal{A}}$ becomes the extension of μ_F on $\mathcal{A}$. We denote this measure to be $\bar{\mu}_F$, as the extension of μ_F, called the Lebesgue-Stieltjes measure.

Remark 1.54. In particular, if $F(x) = x$ for all $x \in \mathbb{R}$, then $\bar{\mu}_F$ is called a Lebesgue measure, denoted by $\mathfrak{m}$, with $\mathfrak{m}((a, b]) = F(b) - F(a) = b - a$.

Definition 1.55. Let μ be a Lebesgue-Stieltjes measure associated to an increasing and right-continuous function F. Let $\mathcal{M}_\mu$ be the domain of the measure μ, which gives the collection of measurable sets. For any measurable set $E \in \mathcal{M}_\mu$, we have

$$\begin{aligned}
\mu(E) &= \inf\left\{\sum_{j=1}^{\infty} [F(b_j) - F(a_j)] : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j]\right\} \\
&= \inf\left\{\sum_{j=1}^{\infty} \mu((a_j, b_j]) : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j]\right\}.
\end{aligned}$$

Theorem 1.56. For all $E \in \mathcal{M}_\mu$, we have

$$\mu^*(E) = \inf\left\{\sum_{j=1}^{\infty} \mu((a_j, b_j)) : E \subseteq \bigcup_{j=1}^{\infty} (a_j, b_j)\right\}.$$

Proof. Let $\tilde{\mu}(E)$ be the right-hand side of this equation, so we will show that $\mu(E) = \tilde{\mu}(E)$. Note that we have a partition

$$(a_j, b_j) = \bigcup_{k=1}^{\infty} I_k^{(j)},$$

where $I_k^{(j)} = (b_j - \frac{1}{2^k}(b_j - a_j), b_j - \frac{1}{2^{k+1}}(b_j - a_j)]$. Now $E \subseteq \bigcup\limits_{j=1}^{\infty} (a_j, b_j)$, so $E \subseteq \bigcup\limits_{j=1}^{\infty} \bigcup\limits_{k=1}^{\infty} I_k^{(j)}$, and thus

$$\sum_{j=1}^{\infty} \mu((a_j, b_j)) = \sum_{j=1}^{\infty} \mu\left(\bigcup_{k=1}^{\infty} I_k^{(j)}\right)$$

$$= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\mu(I_k^{(j)}).$$

Because $\left\{\sum_{j=1}^{\infty}\mu((a_j,b_j)) : E\subseteq\bigcup_{j=1}^{\infty}(a_j,b_j)\right\}\subseteq\left\{\sum_{j=1}^{\infty}\mu((a_j,b_j]) : E\subseteq\bigcup_{j=1}^{\infty}(a_j,b_j]\right\}$, then $\tilde{\mu}(E)\geqslant\mu(E)$.

We now show that $\mu(E)\geqslant\tilde{\mu}(E)$. Pick arbitrary $\varepsilon>0$, then we know $\mu(E)+\varepsilon$ is not a lower bound of the set $\left\{\sum_{j=1}^{\infty}\mu((a_j,b_j]) : E\subseteq\bigcup_{j=1}^{\infty}(a_j,b_j]\right\}$, hence there exists $(a_j,b_j]$ for $j\geqslant 1$ such that $E\subseteq\bigcup_{j\geqslant 1}(a_j,b_j]$. Therefore $\sum_{j=1}^{\infty}\mu((a_j,b_j])\leqslant\mu(E)+\varepsilon$. By the right continuity of F, for $\varepsilon\cdot 2^{-j}>0$, there exists $\delta_j>0$ such that $F(b_j+\delta_j)-F(b_j)<\varepsilon\cdot 2^{-j}$, then $E\subseteq\bigcup_{j=1}^{\infty}(a_j,b_j+\delta_j)$. We know

$$\begin{aligned}
\tilde{\mu}(E)&\leqslant\sum_{j=1}^{\infty}\mu((a_j,b_j+\delta_j))\\
&=\sum_{j=1}^{\infty}[F(b_j+\delta_j)-F(a_j)]\\
&=\sum_{j=1}^{\infty}[F(b_j+\delta_j)-F(b_j)+F(b_j)-F(a_j)]\\
&\leqslant\sum_{j=1}^{\infty}[F(b_j+\delta_j)-F(b_j)]+\sum_{j=1}^{\infty}[F(b_j)-F(a_j)]\\
&<\sum_{j=1}^{\infty}\varepsilon\cdot 2^{-j}+\sum_{j=1}^{\infty}\mu((a_j,b_j])\\
&<\varepsilon+\mu(E)+\varepsilon\\
&=\mu(E)+2\varepsilon.
\end{aligned}$$

Taking small enough ε finishes the proof. $\square$

Remark 1.57. The union of h-intervals may not be open, so often times we use the characterization in Theorem 1.56 instead.

Theorem 1.58. For any $E\subseteq\mathcal{M}_\mu$, we have

$$\mu(E)=\inf\{\mu(U):\ \text{open } U\supseteq E\}=\sup\{\mu(K):\ \text{compact } K\subseteq E\}.$$

Proof. Let $\tilde{\mu}(E)=\inf\{\mu(U):\ \text{open } U\supseteq E\}$. First, $\mu(E)\leqslant\tilde{\mu}(E)$: since $E\subseteq U$, then $\mu(E)\leqslant\mu(U)$, therefore $\mu(E)\leqslant\tilde{\mu}(E)$. To see $\tilde{\mu}(E)\leqslant\mu(E)$, we have $\mu(E)+\varepsilon$ is not a lower bound of $\left\{\sum_{j=1}^{\infty}\mu((a_j,b_j]) : E\subseteq\bigcup_{j=1}^{\infty}(a_j,b_j)\right\}$, then there exists (a_j,b_j) for each $j\in\mathbb{N}$ such that $E\subseteq\bigcup_{j=1}^{\infty}(a_j,b_j)$, and that $\sum_{j=1}^{\infty}\mu((a_j,b_j))\leqslant\mu(E)+\varepsilon$. Therefore, take U to be the open set $\bigcup_{j=1}^{\infty}(a_j,b_j)$, then

$$\tilde{\mu}(E)\leqslant\mu(U)\leqslant\sum_{j=1}^{\infty}\mu((a_j,b_j))\leqslant\mu(E)+\varepsilon$$

as desired.

Now let $\nu(E)=\sup\{\mu(K):\ \text{compact } K\subseteq E\}$. We note that if $K\subseteq E$, then $\mu(K)\leqslant\mu(E)$, therefore $\nu(E)\leqslant\mu(E)$. To prove the reverse inequality, we consider the following cases:

- E is bounded.
  - E is closed. Since E is bounded and closed, it is compact over $\mathbb{R}$, thus $\mu(E) \leqslant \nu(E)$.
  - E is bounded but not closed. We have $\mu(\bar{E}\backslash E) = \inf\{\mu(U) : \text{ open } U \supseteq \bar{E}\backslash E\}$. For any $\varepsilon > 0$, there exists an open set U such that $U \supseteq \bar{E}\backslash E$ and $\mu(U) \leqslant \mu(\bar{E}\backslash E) + \varepsilon$. Set $K = \bar{E}\backslash U$, then K is compact. Since all measures here are finite, we have

$$\begin{aligned}
\mu(K) &= \mu(E) - \mu(E \cap U) \\
&= \mu(E) - [\mu(U) - \mu(U\backslash E)] \\
&\geqslant \mu(E) - \mu(U) + \mu(\bar{E}\backslash E) \\
&\geqslant \mu(E) - \varepsilon.
\end{aligned}$$

    Therefore $\nu(E) \geqslant \mu(E) - \varepsilon$, and we are done by taking $\varepsilon \to 0$.

- E is not bounded. Suppose $E = \bigcup\limits_{j=-\infty}^{\infty} ((j, j+1] \cap E)$, then denote $E_j = E \cap (j, j+1]$, which is bounded. Therefore, we know the statement is true for each E_j for $j \geqslant 1$, thus $\mu(E_j) = \sup\{\mu(K) : \text{ compact } K \subseteq E_j\}$. Take arbitrary $\varepsilon > 0$, then $\mu(E_j) - \frac{1}{3}\varepsilon \cdot 2^{-|j|}$ is not the upper bound of $\{\mu(K) : \text{ compact } K \subseteq E_j\}$, then there exists a compact set $K_j \subseteq E_j$ such that $\mu(K_j) \geqslant \mu(E_j) - \frac{1}{3}\varepsilon \cdot 2^{-|j|}$. Since $K_j \subseteq E_j$ and E_j's are disjoint, then K_j's are disjoint. Therefore, for $n \in \mathbb{N}$, set $H_n = \bigcup\limits_{j=-n}^{n} K_j$, which is a finite disjoint union of compact sets, so this is a compact set. But $H_n \subseteq E$, then

$$\begin{aligned}
\mu(H_n) &= \mu\left(\bigcup_{j=-n}^{n} K_j\right) \\
&= \sum_{j=-n}^{n} \mu(K_j) \\
&\geqslant \sum_{j=-n}^{n} \mu(E_j) - \frac{\varepsilon}{3}\sum_{j=-n}^{n} 2^{-|j|} \\
&\geqslant \sum_{j=-n}^{n} \mu(E_j) - \frac{\varepsilon}{3}\sum_{j=-\infty}^{\infty} 2^{-|j|} \\
&\geqslant \sum_{j=-n}^{n} \mu(E_j) - \varepsilon.
\end{aligned}$$

  Note that H_n still depends on n, so we should not take $n \to \infty$ here. Since $\nu(E)$ is the upper bound of $\mu(K)$'s for compact $K \subseteq E$, then $\nu(E) \geqslant \mu(H_n)$, therefore

$$\begin{aligned}
\nu(E) &\geqslant \sum_{j=-n}^{n} \mu(E_j) - \varepsilon \\
&= \mu\left(\bigcup_{j=-n}^{n} E_j\right) - \varepsilon.
\end{aligned}$$

  Take $n \to \infty$, then

$$\begin{aligned}
\nu(E) &\geqslant \lim_{n\to\infty} \mu\left(\bigcup_{j=-n}^{n} E_j\right) - \varepsilon \\
&= \mu\left(\bigcup_{j=-\infty}^{\infty} E_j\right) - \varepsilon
\end{aligned}$$

$$= \mu(E) - \varepsilon.$$

Let $\varepsilon \to 0$, we are done.

□

Theorem 1.59. Let $E \subseteq \mathbb{R}$, then the following are equivalent:

a. $E \in \mathcal{M}_\mu$;

b. $E = V \backslash N_1$, where V is a G_δ-set and $\mu(N_1) = 0$;

c. $E = H \cup N_2$, where H is a F_σ-set and $\mu(N_2) = 0$.

Proof.

- *b.* $\Rightarrow$ *a.*: note that $\mathcal{M}_\mu \supseteq \mathcal{B}_{\mathbb{R}}$, then both V and N_1 are measurable, therefore E is measurable, i.e., $E \in \mathcal{M}_\mu$.
- *c.* $\Rightarrow$ *a.*: similar to the case above.
- *a.* $\Rightarrow$ *b.*:
  - If $\mu(E) < \infty$, recall $\mu(E) = \inf\{\mu(U) : \text{ open } U \supseteq E\}$. For any $k \in \mathbb{N}$, consider $2^{-k} > 0$, then there exists open subset $U_k \supseteq E$ such that $\mu(U_k) \leqslant \mu(E) + 2^{-k}$. Let $V = \bigcap\limits_{k=1}^{\infty} U_k$ be a G_δ-set, then $V \supseteq E$ as well. It suffices to show that $V \backslash E$ is a null set. We know

$$\begin{aligned} \mu(V) &= \mu\left(\bigcap_{k=1}^{\infty} U_k\right) \\ &\leqslant \mu(U_k) \\ &\leqslant \mu(E) + 2^{-k} \end{aligned}$$

    for all $k \in \mathbb{N}$. Since $\mu(V)$ and $\mu(E)$ are independent of k, then take $k \to \infty$, therefore $\mu(V) \leqslant \mu(E)$. But since $E \subseteq V$, then $\mu(E) \leqslant \mu(V)$, therefore this gives equality. Since $\mu(E) < \infty$, then $\mu(V) - \mu(E) = 0$, then $\mu(V \backslash E) = 0$ by additivity.
  - If $\mu(E) = \infty$, then the proof can be done using the previous case.
- *a.* $\Rightarrow$ *c.*: the proof is similar to the case above.

□

Theorem 1.60. Let $E \in \mathcal{M}_\mu$, and suppose $\mu(E) < \infty$. For any $\varepsilon > 0$, there exists some set A that is a finite union of open intervals such that $\mu(E \Delta A) = \mu((E \backslash A) \cup (A \backslash E)) < \varepsilon$.

Proof. Note that $\mu(E) = \sup\{\mu(K) : \text{ compact } K \subseteq E\}$. For any $\varepsilon > 0$, there exists compact $K \subseteq E$ such that $\mu(E) - \frac{\varepsilon}{2} < \mu(K)$, which is equivalent to having $\mu(E \backslash K) < \frac{\varepsilon}{2}$. Similarly, recall that $\mu(E) = \inf\{\mu(U) : \text{ open } U \supseteq E\}$, but open set U on $\mathbb{R}$ is characterized as a union of open intervals, therefore this is just $\mu(E) = \inf\{\sum\limits_{j=1}^{\infty} \mu((a_j, b_j)) : \bigcup\limits_{j=1}^{\infty} (a_j, b_j) \supseteq E\}$. Therefore, there exists $\bigcup\limits_{j=1}^{\infty} I_j \supseteq E$, where I_j is open interval for each j, such that $\mu\left(\bigcup\limits_{j=1}^{\infty} I_j\right) < \mu(E) + \frac{\varepsilon}{2}$. Since $\mu(E)$ is finite, then $\mu\left(\bigcup\limits_{j=1}^{\infty} I_j \backslash E\right) < \frac{\varepsilon}{2}$. Now $K \subseteq E \subseteq \bigcup\limits_{j=1}^{\infty} I_j$, but K is compact, so there exists $I_1, \dots, I_n$ such that their union cover K. Set $A = \bigcup\limits_{j=1}^{m} I_j$, and we are done. □

Definition 1.61. Let $F(x) = x$ be a function for all $x \in \mathbb{R}$, then μ_F is called the Lebesgue measure defined by $\mathfrak{m}((a, b]) = b - a$. The domain of m is $\mathcal{L}$.

For $E \subseteq \mathbb{R}$ and $s, r \in \mathbb{R}$, we denote $E + s = \{x + s : x \in E\}$ and $rE = \{rx : x \in E\}$.

Theorem 1.62. If $E \in \mathcal{L}$, then $\mathfrak{m}(E+s) = \mathfrak{m}(E)$ and $\mathfrak{m}(rE) = |r|\mathfrak{m}(E)$.

Proof. We prove the first claim. For any $E \in \mathcal{L}$ and $s \in \mathbb{R}$, define $m_s = \mathfrak{m}(E+s)$, then this is a measure.

Claim 1.63. For any $E \in \mathcal{L}$, $m_s(E) = \mathfrak{m}(E)$.

Subproof. First note that this is true if E is a finite (disjoint) union of h-intervals of m_s, as $\mathfrak{m}$ extends the pre-measure μ_0. On $\mathcal{B}_{\mathbb{R}}$, the extension is unique, so $m_s(E) = \mathfrak{m}(E)$ if $E \in \mathcal{B}_{\mathbb{R}}$. Moreover, recall $E \in \mathcal{L}$ if and only if $E = V \setminus N_1$ for $V \in \mathcal{B}_{\mathbb{R}}$. Therefore this is true for all $E \in \mathcal{L}$. ■

□

Definition 1.64. The Cantor set $\mathscr{C}$ is constructed iteratively from the interval $[0,1]$, that for any remaining connected interval $[m,n]$, we delete the subinterval $(m + \frac{1}{3}(n-m), m + \frac{2}{3}(n-m))$ from $[m,n]$.

Remark 1.65. Note that

$$\begin{aligned}\mathfrak{m}(\mathscr{C}) &= \mathfrak{m}([0,1]) - \frac{1}{3} - \frac{2}{3^2} - \frac{2^2}{3^3} - \cdots \\ &= 1 - \sum_{j=0}^{\infty} \frac{2^j}{3^{j+1}} \\ &= 1 - 1 \\ &= 0.\end{aligned}$$

Remark 1.66. If E is countable, then

$$\begin{aligned}\mathfrak{m}(E) &= \sum_{j=1}^{\infty} \mathfrak{m}(\{a_j\}) \\ &= 0.\end{aligned}$$

Theorem 1.67. The Cantor set $\mathscr{C}$ is uncountable.

Proof. Alternatively, the Cantor set $\mathscr{C}$ can be represented as

$$\mathscr{C} = \{x \in [0,1] : x = \sum_{j=1}^{\infty} a_j 3^{-j}, a_j \in \{0,2\}\}.$$

To prove that $\mathscr{C}$ is uncountable, it suffices to build a surjection $f : \mathscr{C} \to [0,1]$. For $x \in \mathscr{C}$, we have $x = \sum_{j=1}^{\infty} a_j 3^{-j}, a_j \in \{0,2\}$. Set $f(x) = \sum_{j=1}^{\infty} \frac{a_j}{2} 2^{-j}$ for $\frac{a_j}{2} \in \{0,1\}$, therefore this gives a decimal representation with base 2, so any real number in $[0,1]$ can be represented in this form, therefore we have a surjection. □

Theorem 1.68. Let $F \subseteq \mathbb{R}$ be such that every subset of F is Lebesgue measurable, then $\mathfrak{m}(F) = 0$.

Corollary 1.69. If $\mathfrak{m}(F) > 0$, then there exists a subset S of F such that $S \notin \mathcal{L}$.

Remark 1.70 (Banach-Tarski Paradox)**.** Given a ball $B = S^2$, then there exists some $m \in \mathbb{N}$ such that $B = V_1 \cup \cdots \cup V_m$ is a union of subsets V_i that are not Lebesgue measurable and $\mathfrak{m}(B) \neq \mathfrak{m}(V_1 \cup \cdots \cup V_m)$.

Definition 1.71. For any $x \in \mathbb{R}$, we defined the cosets over $\mathbb{Q}$ to be $\mathbb{Q} + x = \{r + x : r \in \mathbb{Q}\}$ for any x. This is called the coset of an additive group $\mathbb{R}$.

Let E be the set that contains exactly one point from each coset of $\mathbb{Q}$ as representations, which requires the axiom of choice. Now E allows us make a partition on $\mathbb{R}$.

Lemma 1.72.

1. $(E+r_1)\cap(E+r_2)=\varnothing$ if $r_1\neq r_2$ and $r_1,r_2\in\mathbb{Q}$.
2. $\mathbb{R}=\bigcup\limits_{r\in\mathbb{Q}}(E+r)$.

Proof.

1. Suppose $x\in(E+r_1)\cap(E+r_2)$, then $x=e_1+r_1=e_2+r_2$ for some $e_1,e_2\in E$. Therefore $e_1-e_2=r_2-r_1$, which is a non-zero rational number, therefore $0\neq e_1-e_2\in\mathbb{Q}$. Therefore e_1 and e_2 are in the same coset, so $e_1=e_2$, contradiction.
2. Obviously $\mathbb{R}\supseteq\bigcup\limits_{r\in\mathbb{Q}}(E+r)$. Take any $x\in\mathbb{R}$, then E contains a point y from the coset $\mathbb{Q}+x$, therefore $y-x\in\mathbb{Q}$, so take $r=y-x$, then $x\in E+r$.

□

Proof of Theorem 1.68. We have

$$\begin{aligned}F&=F\cap\mathbb{R}\\&=F\cap\bigcup_{r\in\mathbb{Q}}(E+r)\\&=\bigcup_{r\in\mathbb{Q}}(F\cap(E+r)).\end{aligned}$$

Now let $F_r=F\cap(E+r)$ for all $r\in\mathbb{Q}$, then $F=\bigcup\limits_{r\in\mathbb{Q}}F_r$ for $F_r\in\mathcal{L}$ by Lemma 1.72. It remains to verify that $\mathfrak{m}(F_r)=0$ for all $r\in\mathbb{Q}$. Recall

$$\mathfrak{m}(F_r)=\sup\{\mathfrak{m}(K):\ \text{compact } K\subseteq F_r\},$$

then it suffices to show that

Claim 1.73. For any compact set $K\subseteq F_r$, $\mathfrak{m}(K)=0$.

Indeed, take the supremum over all compact subsets and we are done.

Subproof. Let $K_r=K+r$ for all $r\in\mathbb{Q}$.

First, we show that $K_{r_1}\cap K_{r_2}=\varnothing$ if $r_1\neq r_2$ for $r_1,r_2\in\mathbb{Q}$. Assume there exists $x\in K_{r_1}\cap K_{r_2}$, then $K\subseteq F_r\subseteq E+r$, so we know $K_{r_1}=K+r_1\subseteq E+r+r_1$ and $K_{r_2}=K+r_2\subseteq E+r+r_2$. Therefore, $x\in(E+r+r_1)\cap(E+r+r_2)$, but by Lemma 1.72 we know $(E+r+r_1)\cap(E+r+r_2)=\varnothing$, contradiction.

Set $H=\bigcup\limits_{r\in\mathbb{Q}}K_r$ be a disjoint union. Since the right-hand side is a Borel set, then it is Lebesgue measurable, so by σ-additivity, we have

$$\begin{aligned}\mathfrak{m}(H)&=\mathfrak{m}\left(\bigcup_{r\in\mathbb{Q}}K_r\right)\\&=\sum_{r\in\mathbb{Q}}\mathfrak{m}(K_r)\\&=\sum_{r\in\mathbb{Q}}\mathfrak{m}(K)\\&=\mathfrak{m}(K)\sum_{r\in\mathbb{Q}}1.\end{aligned}$$

We need to bound the set, so instead of summation over $\mathbb{Q}$, we will sum over $\mathbb{Q}\cap[0,1]$ instead, so for $H=\bigcup\limits_{r\in\mathbb{Q}\cap[0,1]}K_r$ we get

$$\mathfrak{m}(H)=\mathfrak{m}(K)\sum_{r\in\mathbb{Q}\cap[0,1]}1.$$

That is, $\mathfrak{m}(H)$ is just $\mathfrak{m}(K)$ times the number of rational numbers in $[0,1]$, which are countably many, therefore $\mathfrak{m}(H) = \mathfrak{m}(K) \cdot \mathbb{N}$.

Assume, towards contradiction, that $\mathfrak{m}(K) \neq 0$, then we have $\mathfrak{m}(K) > 0$, so $\mathfrak{m}(H) = \infty$. But we know H is bounded by $[0,1]$ already, therefore $\mathfrak{m}(H)$ is finite, contradiction. ■

□

Remark 1.74. Not every set is Lebesgue measurable.

2 Integration

2.1 Measurable Functions

Definition 2.1. Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. A function $f : X \to Y$ is called $(\mathcal{A}, \mathcal{B})$-measurable if $f^{-1}(E) \in \mathcal{A}$ for any $E \in \mathcal{B}$. That is, the preimage of a measurable set is measurable.

Definition 2.2. Let $(X, \mathcal{A})$ be a measurable space.

a. If $f : X \to \mathbb{R}$ is $(\mathcal{A}, \mathcal{B}_{\mathbb{R}})$-measurable, then we say the function f is $\mathcal{A}$-measurable.

b. A complex-valued function $f : X \to \mathbb{C}$ is $\mathcal{A}$-measurable if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are $\mathcal{A}$-measurable.

Definition 2.3. A function $f : \mathbb{R} \to \mathbb{C}$ is called Lebesgue measurable if it is $\mathcal{L}$-measurable (on both the real part and the imaginary part).

Lemma 2.4. Let $\mathcal{B}$ be a σ-algebra generated by $\mathcal{B}_0$, then $f : X \to Y$ is $(\mathcal{A}, \mathcal{B})$-measurable if and only if $f^{-1}(E) \in \mathcal{A}$ for all $E \in \mathcal{B}_0$.

Proof.

($\Rightarrow$): this is obvious by Definition 2.1.

($\Leftarrow$): let $M = \{E \subseteq Y : f^{-1}(E) \in \mathcal{A}\}$. Note that $\mathcal{M} \supseteq \mathcal{B}_0$ is a σ-algebra, and since $\mathcal{B}$ is the σ-algebra generated by $\mathcal{B}_0$, then $\mathcal{M} \supseteq \mathcal{B}$. Therefore, for all $E \in \mathcal{B}$, we have $f^{-1}(E) \in \mathcal{A}$.

□

Theorem 2.5. Let X and Y be topological spaces, then every continuous function $f : X \to Y$ is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.

Proof. Note that f is continuous if and only if $f^{-1}(U)$ is open in X for any open subset U in Y, and since $\mathcal{B}_Y$ is the σ-algebra generated by all open subsets of Y, therefore by Lemma 2.4 we know f is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable. □

Theorem 2.6. Let $f : X \to \mathbb{R}$ be a function, then the following are equivalent:

a. f is $\mathcal{A}$-measurable;

b. $f^{-1}((a, \infty)) \in \mathcal{A}$ for all $a \in \mathbb{R}$;

c. $f^{-1}([a, \infty)) \in \mathcal{A}$ for all $a \in \mathbb{R}$;

d. $f^{-1}((-\infty, a)) \in \mathcal{A}$ for all $a \in \mathbb{R}$;

e. $f^{-1}((-\infty, a]) \in \mathcal{A}$ for all $a \in \mathbb{R}$;

Proof. Since the proofs will be analogous to one another, it suffices to show the equivalence between a. and b.

a. $\Rightarrow$ *b.*: since $(a, \infty) \in \mathcal{B}_{\mathbb{R}}$ is a Borel set, then $f^{-1}((a, \infty)) \in \mathcal{A}$ since f is $\mathcal{A}$-measurable.

b. $\Rightarrow$ *a.*: let $\mathcal{B}_0 = \{(a, \infty) : a \in \mathbb{R}\}$, then $\mathcal{B}_{\mathbb{R}}$ is a σ-algebra generated by $\mathcal{B}_0$. The statement then follows from Lemma 2.4. □

Theorem 2.7. If $f, g : X \to \mathbb{C}$ are $\mathcal{A}$-measurable, then so are $f + g$ and $f \cdot g$.

Proof. Assume, without loss of generality, that f and g are $\mathbb{R}$-valued functions.

First, we show that $f + g$ is $\mathcal{A}$-measurable. By Theorem 2.6, it suffices to show that $(f + g)^{-1}((-\infty, a)) \in \mathcal{A}$ for all $a \in \mathbb{R}$. Fix $a \in \mathbb{R}$, this is the set of elements $x \in X$ such that $(f + g)(x) < a$. Note that $x \in X$ satisfies $(f + g)(x) = f(x) + g(x) < a$ if and only if $f(x) < a - g(x)$, where both expressions are real numbers. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists some $r \in \mathbb{Q}$ such that $f(x) < r < a - g(x)$. Therefore,

$$\begin{aligned}
\{x \in X : f(x) + g(x) < a\} &= \bigcup_{r \in \mathbb{Q}} (\{x \in X : f(x) < r\} \cap \{x \in X : r < a - g(x)\}) \\
&= \bigcup_{r \in \mathbb{Q}} \left(f^{-1}((-\infty, r)) \cap g^{-1}((-\infty, a - r))\right) \in \mathcal{A}
\end{aligned}$$

since $f^{-1}((-\infty, r)) \in \mathcal{A}$ and $g^{-1}((-\infty, a - r)) \in \mathcal{A}$.

Remark 2.8. Note that if f is $\mathcal{A}$-measurable, then $-f$ is $\mathcal{A}$-measurable. Therefore, the sum and the difference of two $\mathcal{A}$-measurable functions is still $\mathcal{A}$-measurable.

We now show that $f \cdot g$ is also $\mathcal{A}$-measurable.

Claim 2.9. If $f : X \to \mathbb{R}$ is $\mathcal{A}$-measurable, then f^2 is $\mathcal{A}$-measurable as well.

Subproof. By Theorem 2.6, it suffices to show $\{x \in X : f^2(x) > \alpha\} \in \mathcal{A}$ for all $\alpha \in \mathbb{R}$.

- If $\alpha < 0$, then $\{x \in X : f^2(x) > \alpha\} = X \in \mathcal{A}$.
- If $\alpha \geqslant 0$, then $\{x \in X : f^2(x) > \alpha\} = \{x \in X : f(x) > \sqrt{\alpha}\} \cup \{x \in X : f(x) < -\sqrt{\alpha}\}$. Since f is $\mathcal{A}$-measurable, then this is a union of two $\mathcal{A}$-measurable sets, which is still $\mathcal{A}$-measurable.

■

Now $fg = \frac{1}{2}\left((f+g)^2 - f^2 - g^2\right)$ which is $\mathcal{A}$-measurable. □

Definition 2.10. The extended real line is $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, and correspondingly $\mathcal{B}_{\bar{\mathbb{R}}} = \{E \subseteq \bar{\mathbb{R}} : E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}$. Any member in $\mathcal{B}_{\bar{\mathbb{R}}}$ is called a Borel set in $\bar{\mathbb{R}}$.

A function $f : X \to \bar{\mathbb{R}}$ is called $\mathcal{A}$-measurable if $f^{-1}(E) \in \mathcal{A}$ for all $E \in \mathcal{B}_{\bar{\mathbb{R}}}$.

We deduce results analogous to Theorem 2.6.

Theorem 2.11. Let $f : X \to \bar{\mathbb{R}}$ be a function, then the following are equivalent:

a. f is $\mathcal{A}$-measurable;

b. $f^{-1}((a, \infty]) \in \mathcal{A}$ for all $a \in \mathbb{R}$;

c. $f^{-1}([a, \infty]) \in \mathcal{A}$ for all $a \in \mathbb{R}$;

d. $f^{-1}([-\infty, a)) \in \mathcal{A}$ for all $a \in \mathbb{R}$;

e. $f^{-1}([-\infty, a]) \in \mathcal{A}$ for all $a \in \mathbb{R}$;

Theorem 2.12. Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of $\bar{\mathbb{R}}$-valued measurable functions on $(X, \mathcal{A})$, then the functions

- $g_1(x) = \sup\limits_{j \in \mathbb{N}} f_j(x) = \sup\{f_j(x) : j \in \mathbb{N}\}$;
- $g_2(x) = \inf\limits_{j \in \mathbb{N}} f_j(x) = \inf\{f_j(x) : j \in \mathbb{N}\}$;
- $g_3(x) = \limsup\limits_{j \in \mathbb{N}} f_j(x) = \limsup\{f_j(x) : j \in \mathbb{N}\}$;
- $g_4(x) = \liminf\limits_{j \in \mathbb{N}} f_j(x) = \liminf\{f_j(x) : j \in \mathbb{N}\}$

are measurable.

Proof. We prove $g_1^{-1}((a, \infty]) \in \mathcal{A}$ for all $a \in \mathbb{R}$. Recall that $g_1^{-1}((a, \infty]) = \{x \in X : \infty \geqslant \sup\limits_j f_j(x) > a\} = \bigcup\limits_{j=1}^{\infty} \{x \in X : \infty \geqslant f_j(x) > a\}$. Since each f_j is $\mathcal{A}$-measurable, then each set is measurable, and so is the countable union of such functions. Therefore $g_1(x)$ is measurable. Similarly, we can show that $g_2(x)$ is measurable.

We also prove that g_3 is measurable. Recall that $\limsup\limits_{j \to \infty} f_j(x) = \inf\limits_{j \in \mathbb{N}} \sup\limits_{k > j} f_k(x)$, then it is measurable since supremum and infimum are measurable as functions. Similarly, we can show that $g_4(x)$ is measurable. □

Definition 2.13. Let $f : X \to \bar{\mathbb{R}}$ be a function, then define $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$.

Remark 2.14.

- $f^+ \geqslant 0$;

- $f^- \geqslant 0$;
- $f = f^+ - f^-$;
- $|f| = f^+ + f^-$;
- If f is measurable, then so are f^+, f^-, $|f|$.

Definition 2.15. Let $E \subseteq X$. The characteristic function or the indicator function is

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases}$$

Remark 2.16. If $E \in \mathcal{A}$, then χ_E is ($\mathcal{A}$-)measurable.

Definition 2.17. A simple function on X is a function that can be written as a finite $\mathbb{C}$-linear combination of characteristic functions of sets in $\mathcal{A}$.

Theorem 2.18. Any simple function f can be represented as a standard representation of the form

$$f(x) = \sum_{j=1}^{n} a_j \chi_{E_j}$$

where E_j's are disjoint, $a_j \in \mathbb{C}$ and $\bigcup_{j=1}^{n} E_j = X$.

Proof. We can write $f(x) = \sum_{k=1}^{m} a_k \chi_{E_k}(X)$ for some measurable sets $E_k \in \mathcal{A}$. Since each characteristic function takes only two values, then f takes finitely many valuers, say $z_1, \dots, z_m$. Now we can write $f(x) = \sum_{j=1}^{m} z_j \chi_{E_j}(x)$ where $E_j = \{x \in X : f(x) = z_j\} = f^{-1}(\{z_j\})$. In particular, E_j's are disjoint. However, these sets may not cover X. Let $E_{m+1} = X \setminus \bigcup_{j=1}^{m} E_j$, then $\bigcup_{j=1}^{m+1} E_j = X$, hence

$$f(x) = \sum_{j=1}^{m+1} z_j \chi_{E_j}(x)$$

where $z_{m+1} = 0$. $\square$

Remark 2.19. Equivalently, a function $f : X \to \mathbb{C}$ is simple if and only if f is measurable and the range of f is a finite subset of $\mathbb{C}$.

Theorem 2.20. Let $(X, \mathcal{A})$ be a measurable space.

a. If $f : X \to [0, \infty]$ is measurable, then there exists a sequence $\{\varphi_n\}_{n \geqslant 1}$ of simple functions such that
   - $0 \leqslant \varphi_1 \leqslant \varphi_2 \leqslant \cdots \leqslant f$,
   - $\lim_{n\to\infty} \varphi_n(x) = f(x)$ for all $x \in X$, and
   - $\varphi_n \rightrightarrows f$ converges uniformly on A, i.e., $\lim_{n\to\infty} \sup_{x\in A} |\varphi_n(x) - f(x)| = 0$, for any set A on which f is bounded.

b. If $f : X \to \mathbb{C}$ is measurable, then there exists a sequence $\{\varphi_n\}_{n \geqslant 1}$ of simple functions such that
   - $0 \leqslant |\varphi_1| \leqslant |\varphi_2| \leqslant \cdots \leqslant |f|$.
   - $\lim_{n\to\infty} \varphi_n(x) = f(x)$ for all $x \in X$.
   - $\varphi_n \rightrightarrows f$ converges uniformly on any set on which f is bounded.

Proof.

a. Take arbitrary $n \in \mathbb{N} \cup \{0\}$ and arbitrary $k \in \mathbb{Z}$. We define a dyadic interval to be

$$I_{k,n} = (k2^{-n}, (k+1)2^{-n}],$$

then let $\mathcal{I} = \{I_{k,n} : k, n\}$. For any $I, J \in \mathcal{I}$, we either have $I \subseteq J$, $J \subseteq I$, or $I \cap J = \varnothing$. That is, we have a graded structure on $\mathcal{I}$. Now define $E_{k,n} = \{x \in X : f(x) \in I_{k,n}\} = f^{-1}(I_{k,n})$ and $F_n = f^{-1}((2^n, \infty))$. Therefore, for a fixed n, the $I_{k,n}$'s give a partition of $(0, 2^n)$ on the y-axis, and $f(F_n)$ covers the rest of the y-axis. We define a simple function

$$\varphi_n(x) = \sum_{k=1}^{2^{2n}-1} k2^{-n}\chi_{E_{k,n}}(x) + 2^n\chi_{F_n}(x).$$

Claim 2.21. For any $n \in \mathbb{N}$, $\varphi_n(x) \leqslant \varphi_{n+1}(x)$.

Subproof. This follows from the definition. ■

Claim 2.22. We have $0 \leqslant f(x) - \varphi_n(x) \leqslant 2^{-n}$ for all $x \in F_n^c = \{x \in X : f(x) \leqslant 2^n\}$.

Subproof. We have

$$f(x) = \sum_{k=0}^{2^{2n}-1} f(x)\chi_{E_{k,n}}(x) + f(x)\chi_{F_n}(x)$$

which partitions $(0, \infty)$ to $\bigcup_{k=0}^{2^{2n}-1} I_{k,n}$ and $(2^n, \infty)$. Therefore

$$\begin{aligned} f(x) - \varphi_n(x) &= \sum_{k=0}^{2^{2n}-1} [f(x) - k \cdot 2^{-n}]\chi_{E_{k,n}}(x) + (f(x) - 2^n)\chi_{F_n}(x) \\ &= \sum_{k=0}^{2^{2n}-1} [f(x) - k \cdot 2^{-n}]\chi_{E_{k,n}}(x) \\ &\geqslant 0 \end{aligned}$$

if $x \in F_n^c$. We now bound the difference from above by enlarging it, and since $E_{k,n}$'s are disjoint, then

$$\begin{aligned} \sum_{k=0}^{2^{2n}-1} [f(x) - k \cdot 2^{-n}]\chi_{E_{k,n}}(x) &\leqslant \sum_{k=0}^{2^{2n}-1} [(k+1)2^{-n} - k2^{-n}]\chi_{E_{k,n}}(x) \\ &= \sum_{k=0}^{2^{2n}-1} 2^{-n}\chi_{E_{k,n}}(x) \\ &= 2^{-n}\sum_{k=0}^{2^{2n}-1} \chi_{E_{k,n}}(x) \\ &\leqslant 2^{-n} \end{aligned}$$

as desired. ■

Claim 2.23. $\lim_{n\to\infty} \varphi_n(x) = f(x)$ for all $x \in X$.

Subproof.

- Suppose $f(x) = \infty$, then recall $\varphi_n(x) = 2^n\chi_{F_n}(x) = 2^n$, so obviously both values equal to ∞.

- Suppose $0 \leqslant f(x) < \infty$, then for large enough n, we have $2^n > f(x)$, therefore $x \in F_n^c$ in this case. By Claim 2.22, $0 \leqslant f(x) - \varphi_n(x) \leqslant 2^{-n}$ for n large enough, so when we let $n \to \infty$, then

$$0 \leqslant \lim_{n\to\infty} [f(x) - \varphi_n(x)] \leqslant 0$$

and therefore by squeeze theorem the limit exists and must equal to 0, i.e., $\lim\limits_{n\to\infty} \varphi_n(x) = f(x)$.

■

Claim 2.24. $\varphi_n \rightrightarrows f$ converges uniformly on any set on which f is bounded.

Subproof. Let A be a set on which f is bounded. For any $x \in A$, there exists some large enough n such that $0 \leqslant f(x) - \varphi_n(x) \leqslant 2^{-n}$ by Claim 2.22, so

$$0 \leqslant \sup_{x\in A} |f(x) - \varphi_n(x)| \leqslant 2^{-n},$$

so taking $n \to \infty$ gives

$$\lim_{n\to\infty} \sup_{x\in A} |f(x) - \varphi_n(x)| = 0,$$

i..e, $\varphi_n \rightrightarrows f$ on A. ■

b. Write $f = \mathrm{Re}(f) + i\,\mathrm{Im}(f)$, then both $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are measurable. Now write $\mathrm{Re}(f) = (\mathrm{Re}(f))^+ - (\mathrm{Re}(f))^-$ and $\mathrm{Im}(f) = (\mathrm{Im}(f))^+ - (\mathrm{Im}(f))^-$. By part a., we find a desirable sequence for each of these four parts of the function, then taking the sum/difference gives the desired sequence for f.

□

2.2 Integration of Non-negative Functions

Definition 2.25. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let L^+ be the collection of all non-negative measurable functions on X, i.e., $f \in L^+$ if and only if $f : X \to [0, \infty]$.

Let $\varphi \in L^+$ be a simple function, then we can represent φ as

$$\varphi(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x)$$

for disjoint $E_j \in \mathcal{A}$ such that $\bigcup\limits_{j=1}^{n} = X$.

We first define the integral for simple functions to be

$$\int_X \varphi d\mu = \sum_{j=1}^{n} a_j \mu(E_j).$$

Here we set $0 \cdot \infty = 0$. For any $A \subseteq X$, we define the integral to be

$$\int_A \varphi d\mu = \int_X \varphi\chi - A d\mu.$$

To extend our definition to general non-negative functions, we need to define the following. For any $f \in L^+$, set

$$\int_X f d\mu = \sup\left\{ \int_X \varphi d\mu : 0 \leqslant \varphi \leqslant f \text{ for simple function } \varphi \right\}.$$

Since any non-negative measurable function is a limit of simple functions, then such simple functions exist, hence the supremum exists, which is either a real number or ∞.

Proposition 2.26. Let φ and ψ be simple functions in L^+, then

a. if $c \geqslant 0$, $\int_X c\varphi d\mu = c\int_X \varphi d\mu$;

b. $\int_X \varphi d\mu + \int_X \psi d\mu = \int_X (\varphi+\psi) d\mu$;

c. if $\varphi \leqslant \psi$ pointwise, then $\int_X \varphi d\mu \leqslant \int_X \psi d\mu$;

d. for any $A \in \mathcal{A}$, define $\nu : A \to \int_A \varphi d\mu$, then ν is a measure on $\mathcal{A}$.

Proof.

a. This follows from the definition.

b. Set $\varphi(X) = \sum_{j=1}^{n} a_j \chi_{E_j}(X)$ and $\psi(x) = \sum_{k=1}^{m} b_k \chi_{F_k}(x)$ as standard representations. To add the functions together, we need to refine the partition. Recall $X = \bigcup_{j=1}^{m} E_j = \bigcup_{k=1}^{m} F_k$, then we write

$$E_j = E_j \cap X = E_j \cap \left(\bigcup_{k=1}^{m} F_k\right) = \bigcup_{k=1}^{m} (E_j \cap F_k)$$

and similarly

$$F_k = F_k \cap X = F_k \cap \left(\bigcup_{j=1}^{n} E_j\right) = \bigcup_{j=1}^{n} (F_k \cap E_j).$$

Therefore

$$\begin{aligned}\varphi(x) &= \sum_{j=1}^{n} a_j \chi_{E_j} \\ &= \sum_{j=1}^{n} a_j \sum_{k=1}^{m} \chi_{E_j \cap F_k} \\ &= \sum_{j=1}^{n}\sum_{k=1}^{m} a_j \chi_{E_j \cap F_k}\end{aligned}$$

and similarly

$$\psi(x) = \sum_{j=1}^{n}\sum_{k=1}^{m} b_k \chi_{E_j \cap F_k}.$$

Therefore

$$\begin{aligned}(\varphi+\psi)(x) &= \varphi(x) + \psi(x) \\ &= \sum_{j,k} (a_j + b_k) \chi_{E_j \cap F_k}.\end{aligned}$$

Finally,

$$\begin{aligned}\int_X (\varphi+\psi) d\mu &= \sum_{j,k} (a_j + b_k) \mu(E_j \cap F_k) \\ &= \sum_{j,k} a_j \mu(E_j \cap F_k) + \sum_{j,k} b_k \mu(E_j \cap F_k) \\ &= \int_X \varphi d\mu + \int_X \psi d\mu.\end{aligned}$$

c. Using the same partition trick, since $\varphi \leqslant \psi$, then $a_j \leqslant b_k$ whenever $E_j \cap F_k \neq \varnothing$. Therefore,

$$\begin{aligned}\int_X \varphi d\mu &= \sum_{j,k} a_j \mu(E_j \cap F_k) \\ &\leqslant \sum_{j,k} b_k \mu(E_j \cap F_k) \\ &= \int_X \psi d\mu.\end{aligned}$$

d. It is easy to verify that

$$\nu(\varnothing) = \int_{\varnothing} \varphi d\mu = 0.$$

It remains to show that ν satisfies σ-additivity. Take a sequence $\{A_k\}_{k \geqslant 1} \subseteq \mathcal{A}$, such that A_k's are disjoint. Given a standard representation $\varphi = \sum\limits_{j=1}^{n} a_j \chi_{E_j}$, and we have

$$\begin{aligned}\nu\left(\bigcup_{k=1}^{\infty} A_k\right) &= \int_{\bigcup\limits_{k=1}^{\infty} A_k} \varphi d\mu \\ &= \int_X \varphi \chi_{\bigcup\limits_{k=1}^{\infty} A_k} d\mu \\ &= \int_X \sum_{j=1}^{n} a_j \chi_{E_j} \chi_{\bigcup\limits_{k=1}^{\infty} A_k} d\mu \\ &= \int_X \sum_{j=1}^{n} a_j \chi_{E_j \cap \left(\bigcup\limits_{k=1}^{\infty} A_k\right)} d\mu \\ &= \sum_{j=1}^{n} a_j \mu\left(E_j \cap \bigcup_{k=1}^{\infty} A_k\right) \\ &= \sum_{j=1}^{n} a_j \mu\left(\bigcup_{k=1}^{\infty} (E_j \cap A_k)\right) \\ &= \sum_{j=1}^{n} a_j \sum_{k=1}^{\infty} \mu(E_j \cap A_k) \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^{n} a_j \mu(E_j \cap A_k) \\ &= \sum_{k=1}^{\infty} \int_{A_k} \varphi d\mu \\ &= \sum_{k=1}^{\infty} \nu(A_k).\end{aligned}$$

Note that we can only switch the summation because one of them is infinite while the other one is finite.

□

Remark 2.27. Let φ, ψ be simple functions such that $\varphi \leqslant \psi$, then $\int\limits_X \varphi \leqslant \int\limits_X \psi$. Therefore, this is true for any functions $f, g \in L^+$ as well.

Theorem 2.28 (Monotone Convergence)**.** Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in L^+ such that $f_j \leqslant f_{j+1}$ for all $j \in \mathbb{N}$, then

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X \lim_{n\to\infty} f_n d\mu$$

Remark 2.29. By Remark 2.27, the limit on the left-hand side exists.

Proof. Since the sequence $\{f_n\}_{n\in\mathbb{N}}$ is monotonely increasing, then $\lim\limits_{n\to\infty} f_n$ exists in $\bar{\mathbb{R}}$. Set $f = \lim\limits_{n\to\infty} f_n$, then $f \in L^+$ as well. In particular, $f = \sup\limits_{n\in\mathbb{N}} f_n$ as well, so $f_n \leqslant f$ for all $n \in \mathbb{N}$. Therefore,

$$\int_X f_n d\mu \leqslant \int_X f d\mu$$

for all $n \in \mathbb{N}$. Since $\{\int_X f_n d\mu\}_{n\geqslant 1}$ is a monotone sequence, the limit exists, therefore taking the limit $n \to \infty$ gives

$$\lim_{n\to\infty} \int_X f_n d\mu \leqslant \int_X \lim_{n\to\infty} f_n d\mu.$$

It remains to show

$$\lim_{n\to\infty} \int_X f_n d\mu \geqslant \int_X \lim_{n\to\infty} f_n d\mu.$$

Claim 2.30. Let φ be any simple function such that $0 \leqslant \varphi \leqslant f$. For any fixed $\alpha \in (0,1)$, let $E_n = \{x \in X : f_n(x) \geqslant \alpha\varphi(x)\}$, then

a. $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$, and $X = \bigcup\limits_{n=1}^{\infty} E_n$;

b. $\int_X \varphi d\mu = \lim\limits_{n\to\infty} \int_{E_n} \varphi d\mu$.

Subproof.

a. Since $f_{n+1} \geqslant f_n$, then $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$. To show $X = \bigcup\limits_{n=1}^{\infty} E_n$, we note that $E_n \subseteq X$ for all n implies $\bigcup\limits_{n=1}^{\infty} E_n \subseteq X$, and we claim that $X \subseteq \bigcup\limits_{n=1}^{\infty} E_n$. Take arbitrary $x \in X$,
   - if $\varphi(x) = 0$, then $f_n(x) \geqslant 0 = \varphi(x)$, so $x \in E_n$ for all n by definition;
   - if $\varphi(x) > 0$, recall $f(x) = \lim\limits_{n\to\infty} f_n(x)$, then there exists large enough $N \in \mathbb{N}$ such that $0 \leqslant f(x) - f_N(x) < (1-\alpha)\varphi(x)$, but $\varphi(x) \leqslant f(x)$, then $0 \leqslant f(x) - \varphi(x) < f_N(x) - \alpha\varphi(x)$. In particular, $x \in E_N$.

b. Recall from Proposition 2.26 that $\nu(A) = \int_A \varphi d\mu$ for all $A \in \mathcal{A}$ defines a measure. By the continuity from below for ν and part a., we know

$$\begin{aligned}\lim_{n\to\infty} \int_{E_n} \varphi d\mu &= \lim_{n\to\infty} \nu(E_n)\\ &= \nu\left(\bigcup_{n=1}^{\infty} E_n\right)\\ &= \nu(X)\\ &= \int_X \varphi d\mu.\end{aligned}$$

■

By Claim 2.30, we now have

$$\begin{aligned}\int_X f_n d\mu &= \int_X f_n \chi_{E_n} d\mu \\ &= \int_X \alpha\varphi\chi_{E_n} d\mu \\ &= \alpha \int_X \varphi\chi_{E_n} d\mu.\end{aligned}$$

Since this is true for all n, then taking $n \to \infty$ gives

$$\lim_{n\to\infty} \int_X f_n d\mu \geqslant \alpha \lim_{n\to\infty} \int_X \varphi\chi_{E_n} d\mu = \alpha \int_X \varphi d\mu$$

for any $\alpha \in (0,1)$. Taking $\alpha \to 1$, we get

$$\lim_{n\to\infty} \int_X f_n d\mu \geqslant \int_X \varphi d\mu$$

for any function φ bounded by 0 and f. Taking the supremum over all such φ gives

$$\lim_{n\to\infty} \int_X f_n d\mu \geqslant \int_X f d\mu.$$

□

Theorem 2.31. Let $f_n \in L^+$ for all $n \in \mathbb{N}$, then

$$\int_X \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu.$$

Proof.

Claim 2.32. Given any $f_1, f_2 \in L^+$,

$$\int_X (f_1 + f_2) d\mu = \int_X f_1 d\mu + \int_X f_2 d\mu.$$

Subproof. Since $f_1 \geqslant 0$, there exists simple functions φ_j's such that $0 \leqslant \varphi_j \leqslant f_1$ for all $\in \mathbb{N}$, $\varphi_j \leqslant \varphi_{j+1}$ for all j, and $\lim\limits_{j\to\infty} \varphi_j = f_1$. Similarly, there are simple functions $0 \leqslant \psi_j \leqslant f_2$ for all $j \in \mathbb{N}$ with $\psi_j \leqslant \psi_{j+1}$ for all j, and that $\lim\limits_{j\to\infty} \psi_j = f_2$. Therefore

$$\begin{aligned}\int_X (f_1 + f_2) d\mu &= \int_X \lim_{j\to\infty} \varphi_j + \lim_{j\to\infty} \psi_j d\mu \\ &= \int_X \lim_{j\to\infty} (\varphi_j + \psi_j) d\mu.\end{aligned}$$

Since $\varphi_j + \psi_j$ increases monotonically, so by Theorem 2.28, we have

$$\begin{aligned}\int_X (f_1 + f_2) d\mu &= \int_X \lim_{j\to\infty} (\varphi_j + \psi_j) d\mu \\ &= \lim_{j\to\infty} \int_X \varphi_j + \psi_j d\mu\end{aligned}$$

$$
\begin{aligned}
&= \lim_{j\to\infty}\left(\int_X \varphi_j d\mu + \int_X \psi_j d\mu\right)\\
&= \lim_{j\to\infty}\int_X \varphi_j d\mu + \lim_{j\to\infty}\int_X \psi_j d\mu\\
&= \int_X \lim_{j\to\infty}\varphi_j d\mu + \int_X \lim_{j\to\infty}\psi_j d\mu\\
&= \int_X f_1 d\mu + \int_X f_2 d\mu
\end{aligned}
$$

where we apply Theorem 2.28 at the last steps. ∎

By Claim 2.32,

$$\int_X \sum_{n=1}^{N} f_n d\mu = \sum_{n=1}^{N}\int_X f_n d\mu$$

for all $n \in \mathbb{N}$. By Theorem 2.28,

$$
\begin{aligned}
\int_X \sum_{n=1}^{\infty} f_n d\mu &= \lim_{N\to\infty}\int_X \sum_{n=1}^{N} f_n d\mu\\
&= \lim_{N\to\infty}\sum_{n=1}^{N}\int_X f_n d\mu\\
&= \sum_{n=1}^{N}\int_X f_n d\mu.
\end{aligned}
$$

□

Theorem 2.33. Let $f \in L^+$, then $\int_X f d\mu = 0$ if and only if $f \equiv 0$ almost everywhere.

Proof.

(⇐): Suppose $f \equiv 0$ almost everywhere, then for every choice of simple function φ such that $0 \leqslant \varphi \leqslant f$, $\varphi \equiv 0$ almost everywhere. Take the standard representation $\varphi = \sum_{j=1}^{n} a_j \chi_{E_j}$, then either $a_j = 0$ or $\mu(E_j) = 0$. Therefore,

$$
\begin{aligned}
\int_X \varphi d\mu &= \sum_{j=1}^{n} a_j \mu(E_j)\\
&= 0
\end{aligned}
$$

according to the convention that $0 \cdot \infty = 0$.

(⇒): We claim that $\mu(\{x \in X : f(x) > 0\}) = 0$. To see this, note that

$$\{x \in X : f(x) > 0\} = \bigcup_{n=1}^{\infty}\{x \in X : f(x) > \frac{1}{n}\}.$$

Denote $E_n = \{x \in X : f(x) > \frac{1}{n}\}$, then we just need to show that $\mu(E_n) = 0$ for all $n \in \mathbb{N}$. Note that

$$0 = \int_X f d\mu$$

$$
\begin{aligned}
&\geqslant \int_{E_n} f d\mu \\
&\geqslant \int_{E_n} \frac{1}{n} d\mu \\
&= \frac{1}{n} \times \mu(E_n),
\end{aligned}
$$

so $0 \leqslant \mu(E_n) \leqslant n \cdot 0 = 0$, hence $\mu(E_n) = 0$ for all $n \in \mathbb{N}$.

□

Corollary 2.34. If $f \in L^+$ and $\mu(E) = 0$, then

$$\int_E f d\mu = 0.$$

Proof. Note that

$$\int_E f d\mu = \int_X f \chi_E d\mu,$$

but $f\chi_E = 0$ almost everywhere since $\mu(E) = 0$, so by Theorem 2.33 we are done. □

Theorem 2.35. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in L^+. Suppose that $f_n \leqslant f_{n+1}$ for all $n \in \mathbb{N}$, and that $\lim_{n\to\infty} f_n(x) = f(x)$ almost everywhere $x \in X$, then

$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu.$$

Proof. Let $E = \{x \in X : \lim_{n\to\infty} f_n(x) = f(x)\}$, so E^c is a null set. Extend the function f to

$$f_{\text{ext}}(x) = \begin{cases} f(x), & \text{if } x \in E \\ 0, & \text{if } x \in E^c \end{cases}$$

then by Theorem 2.28 we have

$$
\begin{aligned}
\int_X f d\mu &= \int_E d\mu + \int_{E^c} 0 d\mu \\
&= \int_E f d\mu \\
&= \int_E \lim_{n\to\infty} f_n d\mu \\
&= \int_X \lim_{n\to\infty} f_n \chi_E d\mu \\
&= \lim_{n\to\infty} \int_X f_n \chi_E d\mu \\
&= \lim_{n\to\infty} \left(\int_E f_n d\mu + \int_{E^c} f_n d\mu \right) \\
&= \lim_{n\to\infty} \int_X f_n d\mu.
\end{aligned}
$$

□

Theorem 2.36 (Fatou's Lemma)**.** Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in L^+, then

$$\int_X \liminf_{n\to\infty} f_n d\mu \leqslant \liminf_{n\to\infty} \int_X f_n d\mu.$$

Remark 2.37. Note that Theorem 2.36 does not require Theorem 2.28, but we will use it to give a quick proof.

Proof. Note that for all $j \geqslant n$, we have

$$\inf_{k\geqslant n} f_k(x) \leqslant f_j(x).$$

Taking the integral, we have

$$\int_X \inf_{k\geqslant n} f_k d\mu \leqslant \int_X f_j d\mu$$

for all $j \geqslant n$. Therefore,

$$\int_X \inf_{k\geqslant n} f_k d\mu \leqslant \inf_{j\geqslant n} \int_X f_j d\mu$$

for all $n \in \mathbb{N}$. By definition,

$$\liminf_{n\to\infty} f_n(x) = \lim_{n\to\infty} \inf_{k\geqslant n} f_k(x).$$

By Theorem 2.28, taking the limit gives

$$\begin{aligned}\int_X \liminf_{n\to\infty} f_n d\mu &= \lim_{n\to\infty} \int_X \inf_{k\geqslant n} f_k d\mu \\ &\leqslant \lim_{n\to\infty} \inf_{j\geqslant n} \int_X f_j d\mu \\ &= \liminf_{n\to\infty} \int_X f_n d\mu.\end{aligned}$$

□

Remark 2.38. There is a different version of Theorem 2.36 concerning $\limsup$ instead.

Corollary 2.39. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in L^+ and $\lim\limits_{n\to\infty} f_n(x) = f(x)$ almost everywhere in $x \in X$, then

$$\int_X f d\mu \leqslant \liminf_{n\to\infty} \int_X f_n d\mu.$$

Theorem 2.40. Let $f \in L^+$ and $\int_X f d\mu < \infty$, then $\{x \in X : f(x) = \infty\}$ is a null set, and $\{x \in X : f(x) > 0\}$ is σ-finite.

Proof. We know that

$$\infty > \int_X f d\mu \geqslant \int_{\{x\in X: f(x)=\infty\}} f d\mu = \infty\mu(\{x \in X : f(x) = \infty\})$$

which forces $\mu(\{x \in X : f(x) = \infty\} = 0$. Also note that the level set

$$\{x \in X : f(x) > 0\} = \bigcup_{n=1}^{\infty} \{x \in X : f(x) > \frac{1}{n}\},$$

so we define $E_n = \{x \in X : f(x) > \frac{1}{n}\}$, so it remains to verify that $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$. To see this,

$$\infty > \int_X f d\mu > \int_{E_n} f d\mu > \frac{1}{n}\mu(E_n),$$

therefore $\mu(E_n) < \infty$. □

2.3 Integration of Complex-Valued Functions

If f is a real-valued measurable function, we know $f = f^+ - f^-$ for $f^+, f^- \in L^+$. We know how to define $\int_X f^+ d\mu$ and $\int_X f^- d\mu$. To find the integral of f, we define

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu$$

if one of the two terms is not ∞. We need to resolve the issue when both of them are ∞.

Definition 2.41. Let f be a complex-valued measurable function, we say f is integrable if

$$\int_X |f| d\mu < \infty,$$

that is, the L^1-norm $||f||_1 = \int_X |f| d\mu$ is finite. We define

$$L^1(X) = \left\{ f : \int_X |f| d\mu < \infty \right\}.$$

to be the set of L^1-integrable functions.

The following properties are obvious.

Theorem 2.42. Let $f, g \in L^1(X)$, then

a. $\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu$ for all $\alpha, \beta \in \mathbb{C}$;

b. if $|f| \leqslant |g|$ almost everywhere, then $\int_X |f| d\mu \leqslant \int_X |g| d\mu$;

c. let $\lambda(A) = \int_A |f| d\mu$ for all $A \in \mathcal{A}$, then λ is a measure on $\mathcal{A}$.

Theorem 2.43 (Triangle Inequality)**.** Let $f \in L^1(X)$, then

$$\left| \int_X f d\mu \right| \leqslant \int_X |f| d\mu.$$

Proof.

- If f is real-valued, then

$$\left.\int_X f d\mu \right| = \left| \int_X f^+ d\mu - \int_X f^- d\mu \right| \leqslant \int_X f^+ d\mu + \int_X f^- d\mu = \int_X f^+ + f^- d\mu.$$

- If f is complex-valued, now we can just assume $\int_X f d\mu \neq 0$. Set

$$\alpha = \frac{\overline{\int_X f d\mu}}{\left| \int_X f d\mu \right|},$$

then we have $|\alpha| = 1$, and

$$\left|\int_X f d\mu\right| = \frac{\overline{\int_X f d\mu} \int_X f d\mu}{\left|\int_X f d\mu\right|} = \alpha \int_X f d\mu.$$

In particular, $\alpha \int_X f d\mu \in \mathbb{R}$. We know

$$\begin{aligned}\left|\int_X f d\mu\right| &= \operatorname{Re}\left(\alpha \int_X f d\mu\right) \\ &= \operatorname{Re}\left(\int_X \alpha f d\mu\right) \\ &= \int_X \operatorname{Re}(\alpha f) d\mu \\ &\leqslant \int_X |\operatorname{Re}(\alpha f)| d\mu \\ &\leqslant \int_X |\alpha f| d\mu \\ &= |\alpha| \int_X |f| d\mu \\ &= \int_X |f| d\mu.\end{aligned}$$

□

Theorem 2.44. Let $f, g \in L^1(X)$, then

a. $\int_X |f - g| d\mu = 0$ if and only if $f = g$ almost everywhere;

b. $\int_E f d\mu = \int_E g d\mu$ for all $E \in \mathcal{A}$ if and only if $f = g$ almost everywhere.

Proof.

a. We know $\int_X |f - g| d\mu = 0$ if and only if $|f - g| = 0$ almost everywhere, if and only if $f = g$ almost everywhere.

b. If $f = g$ almost everywhere, then obviously $\int_E f d\mu = \int_E g d\mu$ for all $E \in \mathcal{A}$. The other direction is left as an exercise.

□

By Theorem 2.44, we know if $f = g$ almost everywhere, then $\int_X f d\mu = \int_X g d\mu$.

Example 2.45. Let $X = [0, 1]$, set $f \equiv 1$ on X and

$$g(x) = \begin{cases} 1, & x \in [0,1] \backslash \mathbb{Q} \\ 0, & x \in \mathbb{Q} \cap [0,1] \end{cases}$$

on X, then $f = g$ almost everywhere. Therefore, in $L^1(X, \mathcal{A}, \mathcal{M})$, we say $f = g$. Note that in the sense of Riemann, they do not agree in terms of Riemann integrability, which is designed only for continuous functions in general.

Theorem 2.46 (Dominated Convergence Theorem)**.** Let $\{f_n\}_{n\geqslant 1}$ be a sequence in $L^1(X)$ such that

a. $\lim\limits_{n\to\infty} f_n = f$ almost everywhere,

b. there exists integrable function $g \in L^1$ such that $|f_n| \leqslant g$ for all $n \in \mathbb{N}$,

then $\int\limits_X \lim\limits_{n\to\infty} f_n d\mu = \lim\limits_{n\to\infty} \int\limits_X f_n d\mu$.

Proof. First, note that $f \in L^1$: since $|f| = \lim\limits_{n\to\infty} |f_n| \leqslant g \in L^1$, so $\int\limits_X |f| d\mu \leqslant \int\limits_X |g| d\mu < \infty$, hence $f \in L^1(X)$ by definition. Now note that $|f_n| \leqslant g$ if and only if $-g \leqslant f_n \leqslant g$ almost everywhere, then $f_n + g \in L^+$ for all $n \in \mathbb{N}$. By Theorem 2.36, we know

$$\begin{aligned}
\int\limits_X \liminf_{n\to\infty} f_n d\mu + \int\limits_X g d\mu &= \int\limits_X \left(\liminf_{n\to\infty} f_n d\mu\right) + g \\
&= \int \liminf_{n\to\infty}(f_n + g) d\mu \\
&\leqslant \liminf_{n\to\infty} \int\limits_X (f_n + g) d\mu \\
&= \liminf_{n\to\infty} \left(\int\limits_X f_n d\mu + \int\limits_X g d\mu\right) \\
&= \left(\liminf_{n\to\infty} \int\limits_X f_n d\mu\right) + \int\limits_X g d\mu,
\end{aligned}$$

therefore $\int\limits_X \liminf\limits_{n\to\infty} f_n d\mu \leqslant \liminf\limits_{n\to\infty} \int\limits_X f_n d\mu$. Since $g - f_n \in L^+$, then by Theorem 2.36 again, we know

$$\begin{aligned}
\int\limits_X g d\mu - \int\limits_X \limsup_{n\to\infty} f_n d\mu &= \int\limits_X (g - \limsup_{n\to\infty} f_n) d\mu \\
&= \int\limits_X (g + \liminf_{n\to\infty}(-f_n)) d\mu \\
&= \int\limits_X \liminf_{n\to\infty}(g - f_n) d\mu \\
&\leqslant \liminf_{n\to\infty} \int\limits_X (g - f_n) d\mu \\
&= \liminf_{n\to\infty} (\int\limits_X g d\mu - \int\limits_X f_n d\mu) \\
&= \int\limits_X g d\mu - \limsup_{n\to\infty} \int\limits_X f_n d\mu,
\end{aligned}$$

hence $\int\limits_X \limsup\limits_{n\to\infty} f_n d\mu \geqslant \limsup\limits_{n\to\infty} \int\limits_X f_n d\mu$. This gives

$$\int\limits_X f d\mu = \int\limits_X \limsup_{n\to\infty} f_n d\mu \geqslant \limsup_{n\to\infty} \int\limits_X f_n d\mu \geqslant \liminf_{n\to\infty} \int\limits_X f_n d\mu \geqslant \int\limits_X \liminf_{n\to\infty} f_n d\mu = \int\limits_X f d\mu$$

and forces

$$\limsup_{n\to\infty} \int\limits_X f_n d\mu = \liminf_{n\to\infty} \int\limits_X f_n d\mu = \int\limits_X f d\mu.$$

In particular, the limit exists, hence

$$\int_X \lim_{n\to\infty} f_n d\mu = \lim_{n\to\infty}\int_X f_n d\mu.$$

□

Theorem 2.47. Suppose that $\{f_j\}_{j\geqslant 1}$ is a sequence in L^1 such that $\sum_{j=1}^{\infty}\int_X |f_j| d\mu < \infty$, then $\sum_{j=1}^{\infty} f_j$ converges almost everywhere to a function in L^1 such that

$$\int_X \sum_{j=1}^{\infty} f_j d\mu = \sum_{j=1}^{\infty}\int_X f_j d\mu.$$

Proof. Let $g(x) = \sum_{j=1}^{\infty} |f_j(x)|$ for all $x \in X$, then

$$\int_X g d\mu = \int_X \sum_{j=1}^{\infty} |f_j| d\mu = \sum_{j=1}^{\infty}\int_X |f_j| d\mu < \infty.$$

Therefore $g \in L^1$. For all $n \in \mathbb{N}$, we set $g_n = \sum_{j=1}^{n} f_j$ and therefore $|g_n| \leqslant g$for all $n \in \mathbb{N}$. Now by Theorem 2.46 we know

$$\begin{aligned}\int_X \sum_{j=1}^{\infty} f_j d\mu &= \int_X \lim_{n\to\infty} g_n d\mu \\ &= \lim_{n\to\infty}\int_X g_n d\mu \\ &= \lim_{n\to\infty}\int_X \sum_{j=1}^{n} f_j d\mu \\ &= \lim_{n\to\infty}\sum_{j=1}^{n}\int_X f_j d\mu \\ &= \sum_{j=1}^{\infty}\int_X f_j d\mu.\end{aligned}$$

□

Theorem 2.48. Let $f \in L^1$. For any $\varepsilon > 0$, there exists a simple function $\varphi \in L^1$ such that $||f - \varphi||_1 < \varepsilon$.

Proof. Note that $|f| \in L^+$, therefore there exists a sequence $\{\varphi_n\}_{n\geqslant 1}$ of simple functions such that $0 \leqslant |\varphi_1| \leqslant \cdots \leqslant |\varphi_n| \leqslant \cdots \leqslant |f|$ with $\lim_{n\to\infty} \varphi_n = f$. Therefore

$$|f - \varphi_n| \leqslant |f| + |\varphi_n| \leqslant 2|f| \in L^1.$$

By Theorem 2.46, we have

$$0 = \int_X \lim_{n\to\infty} |f - \varphi_n| d\mu = \lim_{n\to\infty}\int_X |f - \varphi_n| d\mu,$$

hence $\lim_{n\to\infty}\int_X |f - \varphi_n| d\mu = 0$. Now for any $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that $\int_X |f - \varphi_N| < \varepsilon$. Take $\varphi = \varphi_N$, we have $||f - \varphi||_1 < \varepsilon$ as desired. □

Theorem 2.49. Let $f : [a, b] \to \mathbb{R}$ be a bounded function where $a, b \in \mathbb{R}$, then f is Riemann integrable if and only if the Lebesgue measure $\mathfrak{m}(\{x \in [a, b] : f \text{ is discontinuous}\}) = 0$.

Example 2.50. $\chi_{\mathbb{Q}}$ is not Riemann integrable on $[0,1]$ because it is discontinuous everywhere.

Example 2.51. Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$, then χ_S is Riemann integrable on $[0,1]$ because

$$\mathfrak{m}(\{x \in [0,1] : \chi_S \text{ is discontinuous at } x\} = \mathfrak{m}(S) = 0.$$

Example 2.52. Let $\mathscr{C}$ be the Cantor set, c.f., Definition 1.64, then $\chi_{\mathscr{C}}$ is Riemann integrable on $[0,1]$.

Proof. Given a partition $\mathcal{P}$ of $[a,b]$

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b,$$

recall that $||\mathcal{P}|| = \max\{|x_j - x_{j-1}| : 1 \leqslant j \leqslant n\}$, then we have two simple functions

$$U_{\mathcal{P}}(x) = \sum_{j=1}^{n} \sup_{x \in [x_{j-1}, x_j)} f(x) \cdot \chi_{[x_{j-1}, x_j)}(x)$$

and

$$L_{\mathcal{P}}(x) = \sum_{j=1}^{n} \inf_{x \in [x_{j-1}, x_j)} f(x) \cdot \chi_{[x_{j-1}, x_j)}(x).$$

We try to create a Riemann sum with respect to these two functions. We have

$$\begin{aligned} \int_{[a,b]} U_{\mathcal{P}} d\mathfrak{m} &= \sum_{j=1}^{n} \sup_{x \in [x_{j-1}, x_j)} f(x)(x_j - x_{j-1}) \\ &:= U(f, \mathcal{P}) \end{aligned}$$

and

$$\begin{aligned} \int_{[a,b]} L_{\mathcal{P}} d\mathfrak{m} &= \sum_{j=1}^{n} \inf_{x \in [x_{j-1}, x_j)} f(x)(x_j - x_{j-1}) \\ &:= L(f, \mathcal{P}). \end{aligned}$$

Let us take a sequence of partitions $\{\mathcal{P}_n\}_{n \geqslant 1}$ such that

$$\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \cdots \subseteq \mathcal{P}_n \subseteq \cdots$$

and $\lim\limits_{n\to\infty} ||\mathcal{P}_n|| = 0$. Recall that f is Riemann integrable if and only if $L(f) =: \lim\limits_{n\to\infty} L(f, \mathcal{P}_n) = \lim\limits_{n\to\infty} U(f, \mathcal{P}_n) := U(f)$. We can bound f by the simple functions

$$L_{\mathcal{P}_1} \leqslant \cdots \leqslant L_{\mathcal{P}_n} \leqslant \cdots \leqslant f \leqslant \cdots \leqslant U_{\mathcal{P}_n} \leqslant \cdots \leqslant U_{\mathcal{P}_1}.$$

Therefore we get a monotone sequence and take the limit $n \to \infty$ since it exists in $\bar{\mathbb{R}}$, then $L := \lim\limits_{n\to\infty} L_{\mathcal{P}_n}$ and $U = \lim\limits_{n\to\infty} U_{\mathcal{P}_n}$ are $\bar{\mathbb{R}}$-valued functions, and are measurable. Since the limit preserves the order, we know that $L \leqslant f \leqslant U$. In particular, there exists some constant C such that

$$|U_{\mathcal{P}_n}| \leqslant \sup_{x \in [a,b]} |f(x)| \leqslant C$$

and

$$|L_{\mathcal{P}_n}| \leqslant \inf_{x \in [a,b]} |f(x)| \leqslant C$$

for all $n \in \mathbb{N}$. Therefore we get $|U| \leqslant C$ and $|L| \leqslant C$, where $C \in L^1([a,b])$. By Theorem 2.46, we have that

$$\int_{[a,b]} U d\mathfrak{m} = \int_{[a,b]} \lim_{n\to\infty} U_{\mathcal{P}_n} d\mathfrak{m}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \int\limits_{[a,b]} U_{\mathcal{P}_n} d\mathfrak{m} \\
&= \lim_{n\to\infty} U(f, \mathcal{P}_n) \\
&= U(f)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\int\limits_{[a,b]} L d\mathfrak{m} &= \int\limits_{[a,b]} \lim_{n\to\infty} L_{\mathcal{P}_n} d\mathfrak{m} \\
&= \lim_{n\to\infty} \int\limits_{[a,b]} L_{\mathcal{P}_n} d\mathfrak{m} \\
&= \lim_{n\to\infty} L(f, \mathcal{P}_n) \\
&= L(f).
\end{aligned}
$$

Therefore, we know

$$
\begin{aligned}
f \text{ is Riemann integrable } &\iff U(f) = L(f) = \int\limits_a^b f dx \text{ in the Riemann sense} \\
&\iff \int\limits_{[a,b]} U d\mathfrak{m} = \int\limits_{[a,b]} L d\mathfrak{m} \\
&\iff \int\limits_{[a,b]} (U - L) d\mathfrak{m} = 0 \\
&\iff \mathfrak{m}(\{x \in [a,b] : U(x) > L(x)\}) = 0.
\end{aligned}
$$

Claim 2.53. If $f : [a,b] \to \mathbb{R}$ is a bounded Riemann integrable function, then f is Lebesgue integrable. Moreover,

$$
\int\limits_{[a,b]} f d\mathfrak{m} = \int\limits_a^b f dx.
$$

Subproof. We have

$$
\begin{aligned}
\{x \in [a,b] : f(x) \neq U(x)\} &\subseteq \{x \in [a,b] : L(x) \neq U(x)\} \\
&= \{x \in [a,b] : U(x) > L(x)\}
\end{aligned}
$$

and therefore

$$
\mathfrak{m}(\{x \in [a,b] : f(x) \neq U(x)\}) = 0.
$$

Hence,

$$
\begin{aligned}
\int\limits_{[a,b]} f d\mathfrak{m} &= \int\limits_{[a,b]} U d\mathfrak{m} \\
&= U(f) \\
&= \int\limits_a^b f dx.
\end{aligned}
$$

■

It now suffices to prove the following claim.

Claim 2.54. $\mathfrak{m}(\{x \in [a,b] : U(x) > L(x)\}) = 0$ if and only if $\mathfrak{m}(\{x \in [a,b] : f \text{ is discontinuous at } x\}) = 0$.

Subproof. For any $A \subseteq [a,b]$, we define the oscillation of f to be $\omega_f(A) = \sup_{x\in A} f(x) - \inf_{x\in A} f(x)$. Now f is continuous at x_0 if and only if the oscillation of f at x_0 is $\Omega_f(x_0) := \lim_{\delta\to 0} \omega_f((x_0-\delta, x_0+\delta)) = 0$. Note that the function is monotone with respect to δ, therefore the limit exists. Let $x \in [a,b] \setminus \bigcup_{n=1}^{\infty} \mathcal{P}_n$ with a zero-measure subset removed. Denote the subinterval in $\mathcal{P}_n$ containing x by I_n, then

$$\begin{aligned}\Omega_f(x) &= \lim_{n\to\infty} \omega_f(I_n)\\ &= \lim_{n\to\infty} [U_{\mathcal{P}_n}(x) - L_{\mathcal{P}_n}(x)]\\ &= U(x) - L(x).\end{aligned}$$

Therefore,

$$\begin{aligned} f \text{ is continuous at } x &\iff \Omega_f(x) = 0\\ &\iff U(x) = L(x)\\ &\iff U(x) = L(x),\end{aligned}$$

and we conclude that

$$\mathfrak{m}(\{x \in [a,b] : f \text{ is discontinuous at } x\} = \mathfrak{m}(\{x \in [a,b] : U(x) > L(x)\}$$

as desired. ■

□

2.4 Modes of Convergences

Definition 2.55. We say $\{f_n\}_{n\geqslant 1}$ converges to f uniformly on E if $\lim_{n\to\infty} \sup_{x\in E} |f_n(x) - f(x)| = 0$, and write $f_n \rightrightarrows f$ on E as $n \to \infty$.

Remark 2.56. If $f_n \rightrightarrows f$ on E, then $f_n \to f$ on E.

Definition 2.57. We say $\{f_n\}_{n\geqslant 1}$ converges to f in L^1 if $\lim_{n\to\infty} ||f_n - f||_1 = 0$, and write $f_n \xrightarrow{L^1} f$ as $n \to \infty$.

Definition 2.58. We say that $\{f_n\}_{n\geqslant 1}$ converges to f in measure μ if for all $\varepsilon > 0$, $\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\} = 0$. We write $f_n \xrightarrow{\mu} f$ as $n \to \infty$.

We now study the relations between different types of convergence.

Theorem 2.59. If $f_n \xrightarrow{L^1} f$, then $f_n \xrightarrow{\mu} f$.

Proof. Pick $\varepsilon > 0$, and let $E_n = \{x \in X : |f_n(x) - f(x)| > \varepsilon\}$. Now

$$\begin{aligned}\varepsilon\mu(E_n) &= \int_{E_n} \varepsilon d\mu\\ &\leqslant \int_{E_n} |f_n - f| d\mu\\ &\leqslant \int_X |f_n - f| d\mu\end{aligned}$$

$$= ||f_n - f||_1,$$

therefore $0 \leqslant \mu(E_n) \leqslant \frac{1}{\varepsilon}||f_n - f||_1$. Let $n \to \infty$, then $0 \leqslant \lim\limits_{n\to\infty} \mu(E_n) \leqslant 0$ so by squeeze theorem we have $\lim\limits_{n\to\infty} \mu(E_n) = 0$. By definition, $f_n \xrightarrow{u} f$. ☐

Example 2.60. Let $f_n = \frac{\chi_{(0,n)}}{n}$ be a function on $\mathbb{R}$, then $f_n \rightrightarrows 0$ on $\mathbb{R}$. Thus, $f_n \to 0$ on $\mathbb{R}$ pointwise. Moreover, $f_n \xrightarrow{u} 0$, but $f_n \overset{L^1}{\not\to} 0$, thus the converse of Theorem 2.59 is not true:

$$\begin{aligned}
\lim_{n\to\infty} \int_X |f_n - 0| d\mathfrak{m} &= \lim_{n\to 0} \int_X |f_n| d\mathfrak{m} \\
&= \frac{1}{n} \int_X \chi_{(0,n)} d\mathfrak{m} \\
&= \frac{n}{n} \\
&= 1.
\end{aligned}$$

Example 2.61. Let $f_n = \chi_{(n,n+1)}$ be a function on $\mathbb{R}$, then $f_n \to 0$ on $\mathbb{R}$ pointwise, but $f_n \overset{\mathfrak{m}}{\not\to} 0$ does not converge to 0 on measure $\mathfrak{m}$: for any $\varepsilon > 0$,

$$\mathfrak{m}(\{x \in X : |\chi_{(n,n+1)}(x) > \varepsilon|\} = \mathfrak{m}(\{x \in (n, n+1) : \varepsilon < 1\},$$

so for any $1 > \varepsilon > 0$, taking the limit $n \to 0$ gives

$$\lim_{n\to\infty} \mathfrak{m}(\{x \in X : |\chi_{(n,n+1)}(x) > \varepsilon|\} = 1.$$

Definition 2.62. Let $\{f_n\}_{n\geqslant 1}$ be a sequence of measurable functions. We say the sequence is Cauchy in measure if for all $\sigma > 0$, for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that $\mu(\{x \in X : |f_n(x) - f_m(x)| > \varepsilon\} < \sigma$ for all $m, n \geqslant N$.

Equivalently, the sequence is Cauchy in measure if for any $\varepsilon > 0$,

$$\lim_{m,n\to\infty} \mu(\{x \in X : |f_n(x) - f_m(x)| > \varepsilon\} = 0.$$

Theorem 2.63. Suppose $\{f_n\}_{n\geqslant 1}$ is Cauchy in measure, then there exists a subsequence $\{f_{n_j}\}_{j\geqslant 1}$ such that $f_{n_j} \to f$ almost everywhere as $j \to \infty$.

Proof. Let $\sigma = \varepsilon = 2^{-j}$ for all $j \in \mathbb{N}$, then there exists $n_j \in \mathbb{N}$ such that $\mu(\{x \in X : |f_{n_{j+1}}(x) - f_{n_j}(x)| > 2^{-j}\} < 2^{-j}$, therefore we have choices $n_j < n_{j+1}$ for all J. Now we know $\{f_{n_j}\}_{j\geqslant 1}$ is a subsequence, so let $g_j = f_{n_j}$ for all $j \in \mathbb{N}$. Therefore,

$$\mu(\{x \in X : |g_{j+1}(x) - g_j(x)| > 2^{-j}\} \leqslant 2^{-j}$$

for all j. Let $E_j = \{x \in X : |g_{j+1}(x) - g_j(x)| > 2^{-j}\}$, then $\mu(E_j) \leqslant 2^{-j}$.

Claim 2.64. For all $k \in \mathbb{N}$ and $F_k = \bigcup\limits_{j=k}^{\infty} E_j$, then $\{g_j\}_{j\geqslant 1}$ is pointwise Cauchy on F_k^c.

Subproof. We show that for $x \in F_k^c$, we have $\lim\limits_{m,n\to\infty} |g_m(x) - g_n(x)| = 0$, which is equivalent to saying for all $\varepsilon > 0$, for all $x \in F_k^c$, there exists $N \in \mathbb{N}$ such that $|g_m(x) - g_n(x)| < \varepsilon$ for all $m, n \geqslant N$. Since $x \in F_k^c$, then $x \in \left(\bigcup\limits_{j=k}^{\infty} E_j\right)^c = \bigcap\limits_{j=k}^{\infty} E_j^c$, so for all $j \geqslant k$ we know $x \in E_j^c$, which is equivalent to saying that for all $j \geqslant k$, $|g_{j+1}(x) - g_j(x)| < 2^{-j}$. Without loss of generality, take arbitrary $m > n \geqslant k$, we get

$$|g_m(x) - g_n(x)| = \left|\sum_{j=n}^{m-1} [g_{j+1}(x) - g_j(x)]\right|$$

$$\begin{aligned}
&\leqslant \sum_{j=n}^{m+1} |g_{j+1}(x) - g_j(x)| \\
&\leqslant \sum_{j=n}^{m+1} 2^{-j} \\
&\leqslant 2^{1-n}.
\end{aligned}$$

Taking $n \to \infty$, we forces $\lim_{m,n\to\infty} |g_m(x) - g_n(x)| = 0$, as desired. ■

Claim 2.65. Let $F = \bigcap_{k=1}^{\infty} F_k$, then $\mu(F) = 0$.

Subproof. We know that for all $n \in \mathbb{N}$,

$$\begin{aligned}
\mu(F) &\leqslant \mu(F_n) \\
&= \mu\left(\bigcup_{j=n}^{\infty} F_j\right) \\
&\leqslant \sum_{j=n}^{\infty} \mu(E_j) \\
&\leqslant \sum_{j=n}^{\infty} 2^{-j} \\
&\leqslant 2^{1-n},
\end{aligned}$$

so for $n \to \infty$, we forces $\mu(F) = 0$. ■

Claim 2.66. If $x \in F^c$, then $\{g_j(x)\}_{j\geqslant 1}$ is a pointwise Cauchy sequence.

Subproof. For any $x \in F^c$, we know $x \in (\bigcap_{k=1}^{\infty} F_k)^c = \bigcup_{k=1}^{\infty} F_k^c$, therefore $x \in F_k^c$ for some $k \in \mathbb{N}$. By Claim 2.64, we conclude that $\{g_j(x)\}_{j\geqslant 1}$ is a pointwise Cauchy sequence. ■

Therefore, for any $x \in F^c$, we know $\{g_j(x)\}$ is Cauchy, so $\lim_{j\to\infty} g_j(x)$ exists in $\mathbb{R}$. Let f be given by

$$f(x) = \begin{cases} \lim_{j\to\infty} g_j(x), & x \in F^c \\ 0, & x \in F \end{cases}$$

then $\{g_j\}$ converges to f almost everywhere. Consider $\{g_j\}_{j\geqslant 1}$ as the said subsequence $\{f_{n_j}\}_{j\geqslant 1}$ of $\{f_n\}_{n\geqslant 1}$, then we are done. □

Theorem 2.67 (Cauchy Criterion)**.** The sequence $\{f_n\}_{n\geqslant 1}$ is Cauchy in measure if and only if there is a measurable function f such that $f_n \xrightarrow{\mu} f$.

Proof.

($\Leftarrow$): Suppose $f_n \xrightarrow{\mu} f$, and set $\varepsilon > 0$, then we want to show that $\lim_{m,n\to 0} \mu(\{x \in X : |f_m(x) - f_n(x)| > \varepsilon\}) = 0$. We know, for any $x \in X$ that lies in the given subset, that

$$\begin{aligned}
\varepsilon &< |f_m(x) - f_n(x)| \\
&= |(f_m(x) - f(x)) + (f(x) - f_n(x))| \\
&\leqslant |f_m(x) - f(x)| + |f_n(x) - f(x)|,
\end{aligned}$$

therefore either $|f_m(x) - f(x)| > \frac{\varepsilon}{2}$ or $|f_n(x) - f(x)| > \frac{\varepsilon}{2}$. Therefore,

$$\{x \in X : |f_m(x) - f_n(x)| > \varepsilon\} \subseteq \{x \in X : |f_m(x) - f(x)| > \frac{\varepsilon}{2}\} \cup \{x \in X : |f_n(x) - f(x)| > \frac{\varepsilon}{2}\}.$$

Hence,

$$\mu(\{x \in X : |f_m(x) - f_n(x)| > \varepsilon\}) \leqslant \mu(\{x \in X : |f_m(x) - f(x)| > \frac{\varepsilon}{2}\}) + \mu(\{x \in X : |f_n(x) - f(x)| > \frac{\varepsilon}{2}\}),$$

but as $m, n \to \infty$, the two measures of the right-hand side converges to 0, which forces the measure on the left also converges to 0.

($\Rightarrow$): Since $\{f_n\}_{n\geqslant 1}$ is Cauchy in measure, then there exists a subsequence $\{g_j\}_{j\geqslant 1} = \{f_{n_j}\}_{j\geqslant 1}$ such that $\lim\limits_{j\to\infty} f_{n_j} = \lim\limits_{j\to\infty} g_j = f$ almost everywhere.

Claim 2.68. $g_j \xrightarrow{\mu} f$.

Subproof. Again, let $E_j = \{x \in X : |g_{j+1}(x) - g_j(x)| > 2^{-j}\}$, and set $F_k = \bigcup\limits_{j=k}^{\infty} E_j$ as in Theorem 2.63, then we know for all $x \in F_k^c$, we have

$$|g_m(x) - g_j(x)| \leqslant 2^{1-j}$$

for all $m, j \geqslant k$. Now let $m \to \infty$, then

$$|f(x) - g_j(x)| \leqslant 2^{1-j}$$

for any $j \geqslant k$ and $x \in F_k^c$. Fix $\varepsilon > 0$. For large enough j, we know $2^{1-j} < \varepsilon$ and therefore satisfies

$$\{x \in X : |g_j(x) - f(x)| > \varepsilon|\} = \{x \in F_j : |g_j(x) - f(x)| > \varepsilon\} \cup \{x \in F_j^c : |g_j(x) - f(x)| > \varepsilon\}.$$

But note that for any $x \in F_j^c$, $|g_j(x) - f(x)| \leqslant 2^{1-j} < \varepsilon$, which forces the second set to be empty, therefore we have

$$\{x \in X : |g_j(x) - f(x)| > \varepsilon|\} = \{x \in F_j : |g_j(x) - f(x)| > \varepsilon\} \subseteq F_j.$$

Taking the measure, we have

$$\begin{aligned} \mu(\{x \in X : |g_j(x) - f(x)| > \varepsilon|\}) &\leqslant \mu(F_j) \\ &\leqslant 2^{1-j} \\ &\to 0 \end{aligned}$$

as $j \to \infty$. Therefore, $g_j \xrightarrow{\mu} f$. ■

Claim 2.69. $f_n \xrightarrow{\mu} f$.

Subproof. We know that

$$\begin{aligned} \varepsilon &< |f_n(x) - f(x)| \\ &< |f_n(x) - g_j(x)| + |g_j(x) - f(x)| \\ &\leqslant |f_n(x) - g_j(x)| + |g_j(x) - f(x)| \end{aligned}$$

and therefore either $|f_n(x) - g_j(x)| > \frac{\varepsilon}{2}$ or $|g_j(x) - f(x)| > \frac{\varepsilon}{2}$. Therefore,

$$\{x \in X : |f_n(x) - f(x)| > \varepsilon\} \subseteq \{x \in X : |f_n(x) - g_j(x)| > \frac{\varepsilon}{2}\} \cup \{x \in X : |g_j(x) - f(x) > \frac{\varepsilon}{2}|\}.$$

Taking the measure, we know that

$$\mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) \leqslant \mu(\{x \in X : |f_n(x) - g_j(x)| > \frac{\varepsilon}{2}\}) + \mu(\{x \in X : |g_j(x) - f(x) > \frac{\varepsilon}{2}|\}).$$

Let $j, n \to \infty$, then $\mu(\{x \in X : |g_j(x) - f(x) > \frac{\varepsilon}{2}|\}) \to 0$ since $g_j \xrightarrow{\mu} f$, and $\mu(\{x \in X : |f_n(x) - g_j(x)| > \frac{\varepsilon}{2}\}) \to 0$ since $\{f_n\}_{n\geqslant 1}$ is Cauchy in measure. Therefore, $\mu(\{x \in X : |f_n(x) - f(x)|\}) \to 0$ as $j, n \to \infty$. In particular, that means

$$\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)|\}) = 0.$$

■

□

Theorem 2.70. Suppose $f_n \xrightarrow{\mu} f$ in measure, then there exists a subsequence $\{f_{n_j}\}_{j\geqslant 1}$ such that $f_{n_j} \to f$ almost everywhere.

Proof. Since $f_n \xrightarrow{\mu} f$, then $\{f_n\}_{n\geqslant 1}$ is Cauchy in measure, therefore by Theorem 2.63 there exists a subsequence $\{f_{n_j}\}_{j\geqslant 1}$ such that $f_{n_j} \to f$ almost everwhere. □

Corollary 2.71. If $\{f_n\}_{n\geqslant 1}$ converges to f in L^1, i.e., $||f_n - f||_1 \to 0$, then there exists a subsequence $\{f_{n_j}\}_{j\geqslant 1}$ such that $f_{n_j} \to f$ almost everywhere.

Proof. This is obvious from Theorem 2.70. □

Definition 2.72. We say $\{f_n\}_{n\geqslant 1}$ converges to f almost uniformly on X if for any $\varepsilon > 0$, there exists a subset $E \subseteq X$ such that $\mu(E) < \varepsilon$ and $f_n \rightrightarrows f$ on E^c.

Theorem 2.73 (Egoroff)**.** Suppose that $\mu(X) < \infty$ and $f_n \to f$ almost everywhere on X, then $\{f_n\}_{n\geqslant 1}$ converges to f almost uniformly.

Proof. Without loss of generality, suppose $f_n \to f$ for all $x \in X$. For any $k \in \mathbb{N}$ and $n \in \mathbb{N}$, we define

$$E_n(k) = \bigcup_{m=n}^{\infty} \{x \in X : |f_m(x) - f(x)| > \frac{1}{k}\}.$$

Claim 2.74. Given any k, $E_n(k) \supseteq E_{n+1}(k)$ for all $n \in \mathbb{N}$.

Subproof. This follows from the definition of $E_n(k)$. ■

Claim 2.75. $\bigcap_{n\geqslant 1} E_n(k) = \varnothing$.

Subproof. Suppose not, then there exists $x \in \bigcap_{n\geqslant 1} E_n(k)$, hence $x \in E_n(k)$ for all $n \in \mathbb{N}$. By definition, we know there is a subsequence $\{f_{n_j}\}_{j\geqslant 1}$ of $\{f_n\}_{n\geqslant 1}$ such that $|f_{n_j}(x) - f(x)| > \frac{1}{k}$ for any $j \in \mathbb{N}$. Let $j \to \infty$, we know $0 = \lim_{j\to\infty} |f_{n_j}(x) - f(x)| \geqslant \frac{1}{k}$, contradiction. ■

Since $\mu(X) < \infty$, then

$$\begin{aligned}\lim_{n\to\infty} \mu(E_n(k)) &= \mu\left(\bigcap_{n=1}^{\infty} E_n(k)\right) \\ &= \mu(\varnothing) \\ &= 0.\end{aligned}$$

For arbitrary $\varepsilon > 0$, there exists some $n_k \in \mathbb{N}$ such that $\mu(E_{n_k}(k)) < \varepsilon \cdot 2^{-k}$. Take $E = \bigcup_{k\geqslant 1} E_{n_k}(k)$, then

$$\mu(E) \leqslant \sum_{k\geqslant 1} \mu(E_{n_k}(k)) < \sum_{k\geqslant 1} \varepsilon \cdot 2^{-k} \leqslant \varepsilon.$$

Finally, we need to show that $f_n \rightrightarrows f$ on E^c. Take $x \in E^c$, then $x \in \bigcap_{k\geqslant 1} [E_{n_k}(k)]^c$, therefore $x \in (E_{n_k}(k)^c$ for all $k \in \mathbb{N}$. Recall that

$$(E_{n_k}(k))^c = \bigcap_{m\geqslant n_k} \{x \in X : |f_m(x) - f(x)| \leqslant \frac{1}{k}\},$$

Thus, if $x \in E^c$, we know $|f_n(x) - f(x)| \leqslant \frac{1}{k}$ for all $k \in \mathbb{N}$ and $n \geqslant n_k$, hence $\sup_{x \in E^c} |f_n(x) - f(x)| \leqslant \frac{1}{k}$ for all $k \in \mathbb{N}$ and $n \geqslant n_k$, therefore

$$0 \leqslant \lim_{n\to\infty} \sup_{x\in E^c} |f_n(x) - f(x)| \leqslant \frac{1}{k}.$$

In particular, this limits tends to 0 when $k \to \infty$. This shows that $\lim_{n\to\infty} \sup_{x\in E^c} |f_n(x) - f(x)| = 0$, in other words $f_n \rightrightarrows f$ on E^c. Therefore, f_n converges almost uniformly to f on E^c. $\square$

Remark 2.76. If f_n converges to f almost uniformly on X, then $f_n \to f$ almost everywhere on X and $f_n \xrightarrow{\mu} f$ on X.

Remark 2.77. The condition $\mu(X) < \infty$ in Theorem 2.73 is necessary. To see this, consider the measure space $(\mathbb{R}, \mathcal{L}, \mathfrak{m})$, and consider $f_n = \chi_{[n,\infty)}$ for all $n \in \mathbb{N}$. Now $f_n \to 0$ converges, but f_n does not converge to 0 in measure $\mathfrak{m}$. Indeed,

$$\begin{aligned}\mathfrak{m}(\{x \in \mathbb{R} : |f_n(x)| > \frac{1}{2}\}) &= \mathfrak{m}(\{x \in [n,\infty)\}) \\ &= \infty \not\to 0.\end{aligned}$$

By Remark 2.76, $\{f_n\}_{n\geqslant 1}$ does not converge to 0 almost uniformly on $\mathbb{R}$.

Remark 2.78. The hypothesis $\mu(X) < \infty$ in Theorem 2.73 can be replaced by $|f_n| \leqslant g$ for all $n \in \mathbb{N}$ and $g \in L^1(X)$.

Theorem 2.79. Let f be any complex-valued measurable function on E with $\mu(E) < \infty$. Then for any $\varepsilon > 0$, there exist a simple function φ and a measurable set $F \subseteq E$ such that

1. $\mu(E \backslash F) < \varepsilon$, and
2. $|f(x) - \varphi(x)| < \varepsilon$ for all $x \in F$.

Proof. Without loss of generality, assume $f \in L^+$. Let $\varphi_n(x) = \sum_{k=0}^{2^{2n}-1} k2^{-n}\chi_{E_{n,k}}(x) + 2^n\chi_{F_n}(k)$, where $E_{n,k} = \{x \in E : f(x) \in (k2^{-n}, (k+1)2^{-n}]\}$ and $F_n = \{x \in E : f(x) > 2^n\}$. Therefore, $F_n \supseteq F_{n+1}$ and $\mu(F_n) \leqslant \mu(E) < \infty$ for all $n \in \mathbb{N}$, so by continuity from above we have

$$\begin{aligned}\lim_{n\to\infty} \mu(F_n) &= \mu\left(\bigcap_{n\geqslant 1} F_n\right) \\ &= \mu(\varnothing) \\ &= 0.\end{aligned}$$

For any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that $\mu(F_n) < \varepsilon$ for all $n \geqslant N_1$. Recall that $|\varphi_n(x) - f(x)| \leqslant 2^{-n}$ for all $x \notin F_n$, then $\sup_{x\in F_n^c} |\varphi_n(x) - f(x)| \leqslant 2^{-n}$, then by squeeze theorem we have $\lim_{n\to\infty} \sup_{x\in F_n^c} |\varphi_n(x) - f(x)| = 0$. Hence, for any $\varepsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that $\sup_{x\in F_n^c} |\varphi_n(x) - f(x)| < \varepsilon$ for all $n \geqslant N_2$. Let $N = \max\{N_1, N_2\}$, then $|\varphi_N(x) - f(x)| < \varepsilon$ for all $x \notin F_N$, and $\mu(F_N) < \varepsilon$. Define $\varphi = \varphi_N$ to be the said simple function, and let $F = E\backslash F_N$. $\square$

Theorem 2.80. Let $\mu(X) < \infty$ and f be a complex-valued measurable function on X. For any $\varepsilon > 0$, there exists $0 < M \in \mathbb{R}$ and a measurable set $E \subseteq X$ such that $|f(x)| < M$ for all $x \in E$ and $\mu(E^c) < \varepsilon$.

Proof. By Theorem 2.79, for any $\varepsilon > 0$, there exists a simple function φ and a measurable set $E \subseteq X$ such that $\mu(E^c) < \varepsilon$ and $|f(x) - \varphi(x)| < \varepsilon$ for all $x \in E$. Using the triangle inequality and the fact that φ is a simple function on E, we know for any $x \in E$ that

$$\begin{aligned}|f(x)| &\leqslant |f(x) - \varphi(x)| + |\varphi(x)| \\ &< \varepsilon + |\varphi(x)| \\ &< \varepsilon + \sup_{x\in E} |\varphi(x)| \\ &=: M \in \mathbb{R}.\end{aligned}$$

$\square$

Theorem 2.81. For any $f \in L^1(\mathbb{R}, \mathcal{A}, \mu)$ where μ is a Lebesgue-Stieltjes measure, then for any $\varepsilon > 0$, there exists a continuous function g on $\mathbb{R}$ such that $||f - g||_1 < \varepsilon$.

Proof. For any $\varepsilon > 0$, there exists a simple function $\varphi \in L^1$ such that $||f - \varphi||_1 < \varepsilon$. Let us write $\varphi(x) = \sum\limits_{j=1}^{n} a_j \chi_{E_j}$, where each $a_j \neq 0$, and each $\mu(E_j) < \infty$ for all j. We can replace E_j by a finite union of disjoint open intervals $I_k^{(j)}$ for each j, then $\mu\left(E_j \Delta \left(\bigcup\limits_{k=1}^{K} I_k^{(j)}\right)\right) < \frac{\varepsilon}{2^j |a_j|}$. Therefore, χ_{E_j} can be replaced by $\chi_{\bigcup\limits_{j=1}^{K} I_k^{(j)}}$, which can then be replaced by continuous functions g_j, where we replace the function upon intervals on $I_k^{(j)}$ for each k, such that $g = \sum\limits_{i=1}^{n} g_j$. This gives the desired function g. ☐

Theorem 2.82 (Lusin)**.** Let μ be a Lebesgue-Stieltjes measure on $\mathbb{R}$, and let f be any complex-valued function measurable function on E with $\mu(E) < \infty$, then f is almost a continuous function on E in the following sense: for any $\varepsilon > 0$, there exists a function g on E and a measurable set $F \subseteq E$ such that

1. g is continuous on E,
2. $\mu(E \backslash F) < \varepsilon$, and
3. $|f(x) - g(x)| < \varepsilon$ for all $x \in F$.

Proof Sketch.

- By Theorem 2.79, we know any complex-valued function is "almost simple", i.e., close to a simple function $\varphi \in L^1$ on E.
- Since φ is integrable, then by Theorem 2.81, we know continuous functions are dense in L^1, i.e., there exists a sequence $\{g_j\}_{j \geqslant 1}$ of continuous functions such that $||g_j - \varphi||_1 \to 0$ as $j \to \infty$. Here we can replace $|| \cdot ||_1$ by $|| \cdot ||_{L^1(E)}$.
- We can now find a subsequence $\{g_{n_j}\}_{j \geqslant 1}$ of $\{g_j\}_{j \geqslant 1}$ such that $g_{n_j} \to \varphi$ almost everywhere as $j \to \infty$.
- Note that limit of continuous functions may not be continuous, but the limit of uniform continuous functions is continuous, so we can find the continuous function g after applying Theorem 2.73 to $\{g_{n_j}\}_{j \geqslant 1}$.

☐

Remark 2.83 (Littlewood's Three Principles on $\mathbb{R}$)**.**

- Every (finite) measurable set in $\mathbb{R}$ is nearly a finite union of intervals.
- Every measurable (complex-valued) function on $\mathbb{R}$ is nearly continuous, c.f., Theorem 2.82.
- Every convergent sequence of measurable functions on a finite measure set is nearly uniformly convergent, c.f., Theorem 2.73.