

# MATH 595 (Group Cohomology) Notes

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## 1 AUG 21, 2023: INTRODUCTION

Group cohomology works over different settings of groups, like finite groups, profinite groups, and topological groups. The course will develop towards

- duality in  $H^*(G, -)$ , and
- focus on computations, e.g., spectral sequences.

We first establish some notations.

- Let  $G$  be a group. If  $G$  has a topology, that would also be part of the information of  $G$ .
- A (left)  $G$ -module is an abelian group  $M$  with an action map

$$\begin{aligned} G \times M &\rightarrow M \\ (g, m) &\mapsto g \cdot m = gm \end{aligned}$$

satisfying

- $1 \cdot m = m$ ,
- $(gh) \cdot m = g \cdot (hm)$ ,
- $g(m + m') = gm + gm'$ .

**Remark 1.1.** If  $G$  is a finite group, then the associated (non-commutative) group ring  $\mathbb{Z}[G] := \bigoplus_{g \in G} \mathbb{Z}e_g$ , where the multiplication is determined by  $e_g e_h = e_{gh}$ . Therefore, a  $G$ -module is just a  $\mathbb{Z}[G]$ -module.

**Example 1.2.** • Trivial module  $\mathbb{Z}$ , or any abelian group with the trivial action  $g \cdot a = a$ .

- $C_2$ , or any group with  $f : G \twoheadrightarrow C_2$ , then  $G$  with  $C_2$  as a quotient gives the sign representation  $\mathbb{Z}_{\text{sgn}}$ , with  $g \cdot (a) = (-1)^{\rho(g)} a$ .
- $\mathbb{Z}[G]$  is a  $G$ -module via the left multiplication action, and/or the conjugation action.

**Definition 1.3** (Fixed points/Invariants). The set of fixed points of  $M$  over  $G$  is  $M^G = \{m \in M \mid gm = m \ \forall g \in G\}$ .

**Definition 1.4** (Orbits/Coinvariants). The set of orbits of  $M$  over  $G$  is  $M_G = M/(gm - m)$ .

**Example 1.5.** If  $M = \mathbb{Z}_{\text{sgn}}$ , then everything gets multiplied by  $-1$ , so there are no fixed points. The orbits of  $M$  over  $G$  would be  $\mathbb{Z}_{\text{sgn}}/(-2) \cong \mathbb{Z}/2\mathbb{Z}$ .

**Example 1.6.** If  $M = \mathbb{Z}[G]$ , then the fixed points are  $\mathbb{Z} \left\{ \sum_{g \in G} e_g \right\}$ .

Thinking in a categorical setting, we have a trivial action function  $\mathbb{Z}\text{-Mod} \rightarrow G\text{-Mod}$ , sending  $ga \mapsto a$  for all  $g \in G$  and  $a \in A$ . This gives an exact functor from  $\mathbf{Ab}$  to  $G\text{-Mod}$ . Then this functor has a right adjoint  $( )^G : G\text{-Mod} \rightarrow \mathbf{Ab}$ , and a left adjoint  $( )_G : \mathbf{Ab} \rightarrow G\text{-Mod}$ . More specifically,  $M^G$  becomes the maximal trivial action submodule of  $M$ , namely  $\text{Hom}_G(\mathbb{Z}, M)$ ;  $M_G$  becomes the largest quotient of  $M$  with trivial action, namely  $\mathbb{Z} \otimes_{\mathbb{Z}[G]} M$ . This simplifies to the tensor-hom adjunction in some sense. For a more detailed derivation of this, see Chapter 6.1 of Weibel.

**Remark 1.7.** In general, as in the category of  $G$ -sets, we have the orbit functor  $X \mapsto X/G$  and the fixed point functor  $X \mapsto X^G$ . The orbit functor is left adjoint to the free  $G$ -set functor, and the fixed point functor is the right adjoint of the trivial  $G$ -set functor.

**Remark 1.8.** Read more about the setting in profinite groups with their topologies in Neukirch-Schmidt-Wingberg.

**Definition 1.9** (Profinite Group). A profinite group of a collection of groups is  $G = \varprojlim_i G_i$  as an inverse limit, where each  $G_i$  is a finite group of the form  $G/U_i$  for some open  $U_i$ . This gives a topology to the profinite group.

**Remark 1.10.** The groups rings  $\mathbb{Z}[[G]] = \varprojlim_i \mathbb{Z}[G_i]$ . For instance, let  $G = \hat{\mathbb{Z}}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ , then  $\mathbb{Z}_p[[G]] = \varprojlim_n \mathbb{Z}_p[\mathbb{Z}/p^n\mathbb{Z}]$ , where each  $\mathbb{Z}[\mathbb{Z}/p^n\mathbb{Z}] \cong \bigoplus_{i=0}^{n-1} \mathbb{Z}\{e_i\}$  where  $e_i \cdot e_j = e_{ij}$ . Therefore,  $\mathbb{Z}_p[[G]]$  is now equivalent to  $\varprojlim_n \mathbb{Z}_p[t]/(t^{p^n} - 1_e)$ , and hence becomes a power series.

**Remark 1.11.** By a change of variables, this becomes  $\varprojlim_n \mathbb{Z}_p[x]/(x^{p^n})$ , but this only works in the finite group  $\mathbb{Z}_p$  case, and not in general for  $\mathbb{Z}$ .

**Example 1.12.**  $\mathbb{Z}[C_n] \cong \mathbb{Z}\{e\} \oplus \mathbb{Z}\{g\} \oplus \mathbb{Z}\{g^2\} \oplus \cdots \oplus \mathbb{Z}\{g^{n-1}\} \cong \mathbb{Z}[g]/(g^n - 1_e)$ .

## 2 AUG 23, 2023: COHOMOLOGY OF GROUPS

**Definition 2.1.** Let  $G$  be a group, then we have a diagram

$$EG : \cdots \rightrightarrows G \times G \rightrightarrows G$$

where the arrows are given by

$$EG^n = G^{n+1} \xrightarrow{d_i} G^n$$

for all  $0 \leq i \leq n$ . In the sense of simplicial sets, we have  $d_i(g_0, \dots, g_n) = (g_0, \dots, \hat{g}_i, \dots, g_n)$ .

Now let  $M$  be a  $G$ -module, then we define  $X^n = X^n(G, M) = \text{Map}_{\text{Set}}(G^{n+1}, M)$ .  $G$  now has an action on this set, given by

$$(g \circ f)(g_0, \dots, g_n) = gf(g^{-1}g_0, \dots, g^{-1}g_n).$$

The action on  $d^i$ 's are contravariant, namely we obtain  $d_i^* : X_n \rightarrow X^{n+1}$  with an inherited structure. Note that  $M$  sits inside  $X^0$ , therefore we have a complex  $(*)$ :

$$0 \longrightarrow M \xleftarrow{\partial_0} X^0 \xrightarrow{\partial_1} X^1 \xrightarrow{\partial_2} X^2 \xrightarrow{\partial_3} \cdots$$

Here  $\partial_0$  includes  $M$  as the constant functions into  $X$ , namely  $\partial_0(m) = f$  for  $f(g) = m$ , and so on. In general, for  $n > 0$ , we have

$$\partial_n = \sum_{i=0}^n (-1)^i d_i^*.$$

**Lemma 2.2.** The complex  $(*) : M \rightarrow X^\cdot$  is an exact complex of  $G$ -modules, i.e.,  $\partial^2 = 0$  and  $\ker(\partial_{n+1}) = \text{im}(\partial_n)$ , and the  $\partial_i$ 's preserves the  $G$ -action. This is called the standard resolution of  $M$  as a  $G$ -module.

*Proof.* Exercise. □

**Definition 2.3.** The  $G$ -fixed points of the  $X^n$ 's are defined by  $C^n(G, M) = (X^n(G, M))^G$ , called the homogeneous  $n$ -cochains of  $G$  with coefficients in  $M$ . Because the complex preserves  $G$ -actions, then we obtain a complex of  $C^n(G, M)$ 's, given by

$$0 \longrightarrow C^0(G, M) \xrightarrow{\partial_0} C^1(G, M) \xrightarrow{\partial_1} \dots$$

**Remark 2.4.** To see what the induced mapping is, suppose  $A \rightarrow B$  is a  $G$ -module map, then there is an induced map of fixed points  $A^G \rightarrow B^G$  by the restriction. In particular, let  $a \in A$  be fixed with  $ga = a$  for all  $g \in G$ , then  $f(a) = f(ga) = gf(a)$ .

**Remark 2.5.** In the complex of Definition 2.3,  $\partial^2 = 0$  as well, but in general this is not an exact sequence.

**Definition 2.6** (Group Cohomology). The group cohomology of  $G$  with coefficients in  $M$  is the collection

$$\{H^n(G, M)\}_{n \geq 0},$$

where  $H^n(G, M) := H^n(C^\bullet(G, M)) = \ker(\partial : C^n \rightarrow C^{n+1}) / \text{im}(\partial : C^{n-1} \rightarrow C^n)$ . We usually use the notion of cocycles  $Z^n(G, M) = \ker(\partial : C^n \rightarrow C^{n+1})$  and coboundaries  $B^n(G, M) = \text{im}(\partial : C^{n-1} \rightarrow C^n)$ .

**Exercise 2.7.** Show that  $H^0(G, M)$  is isomorphic to  $M^G$ .

**Definition 2.8.** The inhomogeneous cochains  $C_i^n(G, M)$  are given by

- $C_i^0 = M$ , and
- for  $n > 0$ ,  $C_i^n = \text{Map}(G^n, M)$ ,

with coboundary maps  $\partial^{n+1} : C_i^n \rightarrow C_i^{n+1}$ , given by

- $\partial^1(m)(g) = gm - m$ ,
- $\partial^2(f)(g_1, g_2) = g_1 f(g_2) - f(g_1 g_2) + f(g_1)$ , and so on, with
- $\partial^{n+1}(f)(g_1, \dots, g_{n+1}) = g_1 f(g_2, \dots, g_{n+1}) + \sum_{i=1}^n (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) + (-1)^{n+1} f(g_1, \dots, g_n)$ .

This gives the inhomogeneous setting of this cochain.

**Lemma 2.9.** The maps

$$\begin{aligned} C^n(G, M) &\rightarrow C_i^n(G, M) \\ (\varphi : G^{n+1} \rightarrow M) &\mapsto (f : G^n \rightarrow M) \\ f(g_1, \dots, g_n) &:= \varphi(1, g_1, g_1 g_2, \dots, g_1 g_2 \cdots g_n) \end{aligned}$$

give a cochain homotopy equivalence  $C^\bullet(G, M) \xrightarrow{\sim} C_i^\bullet(G, M)$ , and hence this is a quasi-isomorphism.

**Corollary 2.10.** The cohomology  $H^*(C_i^\bullet(G, M)) \cong H^*(G, M)$ .

**Remark 2.11.** Any cohomology class can be represented by a normalized inhomogeneous cocycle  $f : G^n \rightarrow M$ , i.e.,  $f(g_1, \dots, g_n) = 0$  where  $g_i = 1$  for some  $i$ .

**Remark 2.12.** Even for  $G = C_2$ ,  $C_i^n$  or  $C^n$  get large as  $n$  grows.

**Remark 2.13.** • Using homological algebra, we can find other cochain complexes which computes group cohomology  $H^*(G, M)$ .

- We would also understand  $H^*(G, M)$  as the failure of exactness of  $( )^G : G\text{-Mod} \rightarrow \mathbf{Ab}$ . Therefore, when taking the fixed points, the exact sequence may not be mapped to another exact sequence. In particular, if we take an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of  $G$ -modules, the induced sequence

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G$$

do not give a surjection at  $B^G \rightarrow C^G$ . One needs to take higher cohomology to obtain a long exact sequence. Hence,  $( )^G : G\text{-Mod} \rightarrow \mathbf{Ab}$  is a left exact functor, but not necessarily right exact.

## 3 AUG 25, 2023: COHOMOLOGY OF GROUPS, CONTINUED

**Example 3.1.** Let  $G$  be  $C_2$ , or any group with a surjection  $p$  onto  $C_2$ , then it has an action on  $\mathbb{Z}_{\text{sgn}}$  given by  $g \cdot a = (-1)^{p(g)}a$ , therefore we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}_{\text{sgn}} \xrightarrow{\times 2} \mathbb{Z}_{\text{sgn}} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

and taking the fixed point functor we have

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

**Remark 3.2.** Higher homologies measure the failure of exactness.

**Remark 3.3.** The collection  $\{H^n(G, -)\}_{n \in \mathbb{Z}}$  satisfies

- $H^n(G, -) = 0$  for  $n < 0$ ;
- for short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  in  $G\text{-Mod}$ , we have a long exact sequence

$$0 \longrightarrow H^0(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, C) \xrightarrow{\delta} H^1(G, A) \longrightarrow \cdots$$

where  $\delta$  is the connecting homomorphism.

- the connecting homomorphisms  $\delta$  are natural, i.e., given a commuting diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \longrightarrow 0 \end{array}$$

the induced diagram

$$\begin{array}{ccc} H^n(G, C) & \xrightarrow{\delta} & H^{n+1}(G, A) \\ \downarrow & & \downarrow \\ H^n(G, C') & \xrightarrow{\delta} & H^{n+1}(G, A') \end{array}$$

also commutes, and  $\{H^n(G, -)\}_{n \in \mathbb{Z}}$  is a cohomological  $\delta$ -functor. Note that a  $\delta$ -functor is additive, and usually occurs in abelian categories.

**Definition 3.4** ( $\delta$ -functor). A map of  $\delta$ -functors  $T^* \rightarrow F^*$  is a collection of natural transformations  $T^n \rightarrow F^n$ , commuting with the  $\delta$ 's, i.e.,

$$\begin{array}{ccc} T^n & \longrightarrow & F^n \\ \delta_T \downarrow & & \downarrow \delta_F \\ T^{n+1} & \longrightarrow & F^{n+1} \end{array}$$

A  $\delta$ -functor  $T^*$  is universal if, given any other  $\delta$ -functor  $F^*$ , a map  $T^* \rightarrow F^*$  is uniquely determined by  $T^0 \rightarrow F^0$ .

**Proposition 3.5.**  $H^*(G, -) : G\text{-Mod} \rightarrow \mathbf{Ab}$  is a  $\delta$ -functor.

*Proof.* We need to show:

- each  $H^n(G, -)$  is a well-defined functor,
- the connecting homomorphisms  $\delta$ 's gives a long exact sequence,
- the naturality of  $\delta$ .

First, let  $f : A \rightarrow B$  be in  $G\text{-Mod}$ , then  $C^*(G, A) \rightarrow C^*(G, B)$  is equivalent to  $\text{Map}(G^{*+1}, A)^G \rightarrow \text{Map}(G^{*+1}, B)^G$  by composition with  $f$ . One can show that this is equivariant, i.e., respects the  $G$ -action, so it is well-defined to take the fixed points, and thus commutes with  $\partial$ 's.

Second, we need to apply the snake lemma. Given a short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ , we claim:

**Claim 3.6.**  $0 \rightarrow C^*(G, A) \rightarrow C^*(G, B) \rightarrow C^*(G, C) \rightarrow 0$  is a short exact sequence of cochain complexes, i.e.,  $C^*(G, -) : G\text{-Mod} \rightarrow \mathbf{coCh}$  is an exact functor.

*Subproof.* Exercise. ■

Now take the complex

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^n(G, A) & \longrightarrow & C^n(G, B) & \longrightarrow & C^n(G, C) \longrightarrow 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\ 0 & \longrightarrow & C^{n+1}(G, A) & \longrightarrow & C^{n+1}(G, B) & \longrightarrow & C^{n+1}(G, C) \longrightarrow 0 \end{array}$$

and quotient the boundaries everywhere (and thus lose the injectivity/surjectivity when applicable)

$$\begin{array}{ccccccc} C^n(G, A)/B^n(G, A) & \longrightarrow & C^n(G, B)/B^n(G, B) & \longrightarrow & C^n(G, C)/B^n(G, C) & \longrightarrow & 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\ 0 & \longrightarrow & Z^{n+1}(G, A) & \longrightarrow & Z^{n+1}(G, B) & \longrightarrow & Z^{n+1}(G, C) \end{array}$$

Taking the kernels and cokernels on  $\partial$ 's, we obtain a complex

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & H^n(G, A) & \longrightarrow & H^n(G, B) & \longrightarrow & H^n(G, C) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & C^n(G, A)/B^n(G, A) & \longrightarrow & C^n(G, B)/B^n(G, B) & \longrightarrow & C^n(G, C)/B^n(G, C) \longrightarrow 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\ 0 & \longrightarrow & Z^{n+1}(G, A) & \longrightarrow & Z^{n+1}(G, B) & \longrightarrow & Z^{n+1}(G, C) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & H^{n+1}(G, A) & \longrightarrow & H^{n+1}(G, B) & \longrightarrow & H^{n+1}(G, C) \end{array}$$

By the snake lemma, we obtain the long exact sequence. □

**Proposition 3.7.** If  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is a short exact sequence such that  $H^*(G, B) = 0$  for  $* > 0$  (or at least  $H^n(G, B) = 0 = H^{n+1}(G, B)$ ), then  $\delta : H^n(G, C) \rightarrow H^{n+1}(G, A)$  is an isomorphism.

**Definition 3.8** (Acyclic, Cohomologically Trivial). A  $G$ -module  $M$  is

- acyclic if  $H^*(G, M) = 0$  for  $* > 0$ ,
- cohomologically trivial if  $H^*(H, M) = 0$  for  $* > 0$  and any (closed) subgroup  $H \subseteq G$ .

**Definition 3.9** (Induced Module). Given any  $G$ -module  $M$ , the induced module  $\text{ind}_G(M) = \text{Map}(G, M) = X^0(G, M)$ .

**Example 3.10.**  $M$  could have the trivial action.

**Exercise 3.11.** For any  $M$ , the induced module of  $M$  over  $G$  is isomorphic (under the  $G$ -action) to the induced module of module given by forgetful action over  $G$ .

**Remark 3.12.** •  $\text{Ind}_G(-) : G\text{-Mod} \rightarrow G\text{-Mod}$  is exact.

- We say  $A$  is an induced module if  $A \cong \text{Ind}_G(M)$  for some module  $M$ . If  $A$  is an induced  $G$ -module, then  $A$  is induced as an  $H$ -module for any subgroup  $H \subseteq G$ .

**Lemma 3.13.** Induced modules are cohomologically trivial.

*Proof.* There is an isomorphism

$$C^*(G, \text{Ind}_G(M)) \cong X^*(G, M).$$

□

**Remark 3.14.** We have an equivariant inclusion of fixed points

$$M \hookrightarrow \text{Ind}_G(M)$$

which is an embedding, and we take  $Q \cong \text{Ind}_G(M)/M$ , then this extends to a short exact sequence

$$0 \longrightarrow M \hookrightarrow \text{Ind}_G(M) \longrightarrow Q \longrightarrow 0$$

then  $H^{n+1}(G, M) \cong H^n(G, Q)$ . One say that  $H^*(G, -)$  is effaceable. By Tohoku, an effaceable is universal.

#### 4 AUG 28, 2023: FIRST COHOMOLOGY OF GROUPS

There are three ways to think about  $H^1(G, M)$ .

##### 4.1 CROSSED HOMOMORPHISMS

Recall that  $H^1(G, M) = Z_i^1(G, M)/B_i^1(G, M)$  as inhomogeneous cochains, where

- $Z_i^1(G, M) = \ker(\text{Map}(G, M) \rightarrow \text{Map}(G \times G, M))$  where the map sends  $f \mapsto (g, h) \mapsto gf(h) - f(gh) + f(g)$ . The kernel of this is exactly the maps  $f$  such that  $f(gh) = gf(h) + f(g)$ , and note that this is not a group homomorphism.
- $B_i^1(G, M) = \text{im}(M \rightarrow \text{Map}(G, M))$  given by  $m \mapsto (g \mapsto gm - m)$ , where the image is called a principal crossed homomorphism.

**Exercise 4.1.**  $B_i^1(G, M) \cong M/M^G$  as an isomorphism of  $\mathbb{Z}[G]$ -modules.

**Remark 4.2.** If the  $G$ -action is trivial, then  $H^1(G, M) = \text{Hom}_{\text{Grp}}(G, M)$ .

**Corollary 4.3.** If  $G$  is a finite group with trivial action, then  $H^1(G, \mathbb{Z}) = 0$ .

**Theorem 4.4** (Hilbert's Theorem 90). Let  $L/K$  be a Galois extension with (finite or profinite) Galois group  $G$ , then  $H^1(G, L^\times) = 0$ .

*Proof.* Let  $f : G \rightarrow L^\times$  be a crossed homomorphism. We know the addition is given by  $f(gh) = gf(h) + f(g)$ , and the multiplication is given by  $f(gh) = (g \cdot f(h))f(g)$ , where  $\cdot$  represents the group action. Now for any  $l \in L^\times$ , the multiplication with respect to  $l$  is given by  $m_l = \sum_{h \in G} f(h)(h \cdot l)$ . We can first choose  $l$  so that  $m_l \neq 0$ , since the Galois conjugates  $h \cdot l$  over  $l \in L$  are linearly independent. For  $g \in G$ , we have

$$\begin{aligned} g \cdot m_l &= \sum_{h \in G} (g \cdot f(h))(gh \cdot l) \\ &= \sum_{h \in G} \frac{f(gh)}{f(g)} (gh \cdot l) \\ &= \frac{1}{f(g)} \sum_{h \in G} f(gh)(gh \cdot l) \\ &= \frac{1}{f(g)} m_l. \end{aligned}$$

Therefore,  $f(g) = \frac{m_l}{g \cdot m_l}$ . For any crossed homomorphism, there exists  $m \in L^\times$  such that  $f(g) = \frac{gm}{m}$ , so every crossed homomorphism is principal. □

**Exercise 4.5.** Let  $G$  acts over a commutative ring  $R$ , then  $H^1(G, R^\times)$  classifies invariant  $R$ -modules with a compatible  $G$ -action.

4.2 NON-ABELIAN  $H^1$  AND TORSORS

Let  $A$  be a group with  $G$ -action, so let the action  $g \cdot a = {}^g a$ . Hence,  $g \cdot (ab) = {}^g a {}^g b$ . Define the  $G$ -cocycles to be  $f : G \rightarrow A$  such that  $f(gh) = f(g) {}^g f(h)$ . Two cocycles  $f$  and  $f'$  are said to be cohomologous as  $f \sim f'$  if there exists  $a \in A$  such that for all  $g \in G$ ,  $f'(g) = a^{-1} f(g) {}^g a$ . This becomes an equivalence relation on the set of  $G$ -cocycles with coefficients in  $A$ , then  $H^1(G, A)$  is the set of equivalence classes of  $G$ -cocycles. Now the first cohomology  $H^1(G, A)$  has only a pointed set structure with distinguished point  $f \equiv 1$ , the constant function at 1.

**Exercise 4.6.** This definition is equivalent to the inhomogeneous cochain definition in the abelian case.

**Definition 4.7.** An  $A$ -torsor is a  $G$ -set  $X$  with action

$$\begin{aligned} X \times A &\rightarrow A \\ (x, a) &\mapsto xa \end{aligned}$$

that is free and transitive, i.e., for any  $x, y \in X$ , there exists a unique  $a \in A$  such that  $y = xa$ . Moreover, the action  $X \times A \rightarrow X$  respects the  $G$ -action, i.e.,  ${}^g(xa) = {}^g x {}^g a$ .

**Remark 4.8.** •  $A$  is an  $A$ -torsor.

- An isomorphism of  $A$ -torsors is a bijection that respects the  $G$ - and  $A$ - action.
- If  $A \subseteq B$  is a sub- $G$ -group, then  $bA$  is an  $A$ -torsor.
- An  $A$ -torsor is a principal  $A$ -bundle on the classifying space  $BG$ .

**Theorem 4.9.** There is a canonical bijection of pointed sets

$$H^1(G, A) \cong \text{Torsor}(G, A)$$

*Proof.* • The backwards map  $\lambda : \text{Torsor}(G, A) \rightarrow H^1(G, A)$  is defined as follows: for  $x \in \text{Torsor}(G, A)$ , we want to define a cocycle  $f(X) : G \rightarrow A$ . For arbitrary  $x \in X$ , note that for any  $g \in G$ , there exists a unique  $f_x(g) \in A$  such that  ${}^g x = x f_x(g)$  by the simple transitivity of the  $A$ -action on  $X$ . To see this is well-defined, if we have another  $y \in X$ , then  $y = xb$  for some  $b \in A$ , then  $f_y(g) = b^{-1} f_x(g) {}^g b$ , so  $f_x$  and  $f_y$  are cohomologous and define the same class in  $H^1(G, A)$ , which is defined to be the image  $\lambda(X)$ .

- To define  $\mu : H^1(G, A) \rightarrow \text{Torsor}(G, A)$ , given a cocycle  $f : G \rightarrow A$ , let  $X_f$  be the group  $A$ , then the action of  $A$  on  $X_f$  is by multiplication on the right, and one can twist the  $G$ -action on it using cocycle  $f : G \rightarrow A$  with  ${}^g x = f(g)gx$ , which defines an  $A$ -torsor. This is well-defined.

□

**Remark 4.10.** Suppose

$$1 \longrightarrow A \longrightarrow B \xrightarrow{p} C \longrightarrow 1$$

is a short exact sequence of  $G$ -groups, i.e.,  $A$  is a sub- $G$ -group and  $C \cong B/A$ , then there is a long exact sequence

$$1 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \xrightarrow{\delta} H^1(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, C)$$

where  $\delta$  is given by  $\delta(c) = p^{-1}(c)$ . For the exactness in the sense of pointed sets to work, the kernel is the subset mapping to the distinguished element.

## 4.3 EXTENSION SPLITTING

Consider the a split extension

$$1 \longrightarrow A \longrightarrow E \xrightarrow{p} G \longrightarrow 1$$

That is,  $E$  is the direct product  $A \times G$  with group action  $(a, g)(a', g') = (a {}^g a', gg')$ , and by definition  $E$  is the semidirect product  $A \rtimes G$ . Equivalently, there exists a section (as group homomorphism)  $s : G \rightarrow E$ .

There is an equivalence relation on the set of sections to the projection  $p : E \rightarrow G$ , where the sections  $s, s' : G \rightarrow E$  are conjugates if there exists  $a \in A$  such that  $s'(g) = a^{-1} s(g) a$ . We denote  $\text{sec}(E \rightarrow G)$  to be the conjugacy class of sections of  $p$ . Note that the class of trivial section  $s : g \mapsto (1, g) \in E$  is the distinguished element.

**Proposition 4.11.** The pointed set  $H^1(G, A)$  is isomorphic to  $\text{sec}(E \rightarrow G)$ .

*Proof.* Take  $\varphi \in \text{sec}(E \rightarrow G)$ , then the composition  $G \xrightarrow{\varphi} E \xrightarrow{\pi_1} A$ , where  $\pi_1$  is the set-theoretic projection to the first component, defines a cocycle  $G \rightarrow A$ . Conversely, given a cocycle  $f : G \rightarrow A$ , the section is given by  $g \mapsto (f(g), g)$ .  $\square$

**Exercise 4.12.** Expand the proof above.

**Exercise 4.13.** Describe  $\mathbb{Z} \rtimes C_2$  where  $C_2$  acts on  $\mathbb{Z}$  by inversion. How many sections are there of  $\mathbb{Z} \rtimes C_2 \rightarrow C_2$ ?

**Exercise 4.14.** How many sections are there to the projection  $D_{2n} \rightarrow C_2$ ?

## 5 AUG 30, 2023: $H^2$ , ABELIAN EXTENSIONS, AND BRAUER GROUP

Suppose we have an abelian extension, that is, let  $A$  be abelian, the short exact sequence of group extensions

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{p} G \longrightarrow 1$$

is such that  $E/i(A) \cong G$ . Note that  $A$  can be regarded as a normal subgroup in  $E$  given this notation.

Note that two extensions are equivalent if there exists a group isomorphism  $\varphi : E \rightarrow E'$  such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G \longrightarrow 1 \\ & & \parallel & & \downarrow \varphi & & \parallel \\ 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G \longrightarrow 1 \end{array}$$

commutes.

Consider the continuous functions

$$\varphi : G \times G \rightarrow A$$

such that  $\varphi(g_1g_2, g_3) + \varphi(g_1, g_2) = \varphi(g_1, g_2g_3) + g_1\varphi(g_2, g_3)$ . We know  $H^2(G, M)$  is the quotient of all such functions over the coboundaries, i.e., the functions  $\varphi$  such that  $\varphi(g_1, g_2) = f(g_1) - f(g_1g_2) + g_1f(g_2)$ .

Now  $E \cong A \times G$  can be considered as a bijection, so we pick a set-theoretic section  $s : G \rightarrow E$  with  $s(1) = 1$ , and now every element in  $E$  is written as  $as(g)$  uniquely for some  $a \in A$  and  $g \in G$ , we have

$$s(g)a = s(g)as(g)^{-1}s(g) = {}^g as(g).$$

Note that  $s$  may not be a homomorphism, but we have  $s(g)s(h) = f(g, h)s(gh)$  since  $s(g)s(h)$  and  $s(gh)$  are both lifts of  $gh$ .

As a consequence, we have

$$(s(g_1)s(g_2))s(g_3) = f(g_1, g_2)s(g_1g_2)s(g_3) = f(g_1, g_2)f(g_1g_2, g_3)s(g_1g_2g_3)$$

and

$$s(g_1)(s(g_2)s(g_3)) = s(g_1)f(g_2, g_3)s(g_2, g_3) = {}^{g_1}f(g_2, g_3)s(g_1)s(g_2g_3) = {}^{g_1}f(g_2, g_3)f(g_1, g_2g_3)s(g_1g_2g_3).$$

In additive notation, we have

$$f(g_1, g_2) + f(g_1g_2, g_3) = g_1f(g_2, g_3) + f(g_1, g_2g_3).$$

Therefore,  $f$  becomes an inhomogeneous 2-cocycle.

**Proposition 5.1.** The induced map  $\lambda : \text{ext}(G, A) \rightarrow H^2(G, A)$  is a well-defined bijection between the set of equivalence classes of extensions and  $H^2(G, A)$ .



**Example 5.2.** The two elements in  $H^2(C_2, \mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$  are given by non-split extension of  $Q_8$

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow Q_8 \longrightarrow C_2 \longrightarrow 1$$

and the identity element given by  $D_8 \cong \mathbb{Z}/4\mathbb{Z} \rtimes C_2$

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow D_8 \longrightarrow C_2 \longrightarrow 1$$

where  $D_8$  has the action of  $C_2$  over  $\mathbb{Z}/4\mathbb{Z}$ .

**Proposition 5.3.** An associative finite-dimensional  $K$ -algebra  $A$  is a CSA if and only if one of the following equivalent conditions hold:

1. Base-changed to the separable closure  $\bar{K}$  of  $K$  via  $\bar{K} \otimes_K A$ ,  $A \cong M_n(\bar{K})$  for some integer  $n \geq 1$ .
2. there exists a finite Galois extension  $L/K$  such that base-changed to  $L$  via  $L \otimes_K A$ ,  $A$  becomes isomorphic to a matrix algebra  $M_n(L)$  for some integer  $n \geq 1$ .
3.  $A \cong M_n(D)$  matrix algebra for some  $m \geq 1$  and some finite division algebra  $D$  over  $K$ .

A CSA  $A$  over  $K$  is said to be split over  $L$  if the above holds, i.e.,  $A \otimes_K L \cong M_n(L)$ . One can define an equivalence class on CSAs, such that  $A \sim B$  if and only if  $A \otimes_K M_n(K) \cong B \otimes_K M_m(K)$ . Now the Brauer group of  $K$  is the abelian group of equivalence classes of CSAs over  $K$  equipped with tensor product.

Suppose  $L/K$  is an extension, then there exists a homomorphism of base-change of algebras  $\text{Br}(K) \rightarrow \text{Br}(L)$ . We say the kernel  $\text{Br}(L | K)$  is the relative Brauer group of  $K$ -CSAs that split over  $L$ . The absolute Brauer group is  $\text{Br}(\bar{K} | K) = \text{Br}(K)$ , then

$$\text{Br}(K) = \bigcup_{L/K \text{ finite}} \text{Br}(L | K).$$

Now let  $L/K$  be a finite Galois extension with Galois group  $G$ , and we pick a normalized inhomogeneous 2-cycle  $\varphi : G \times G \rightarrow L^\times$  as the representative of its class, and we can construct  $A_\varphi$  as a  $K$ -CSA, then  $A_\varphi = \bigoplus_{g \in G} L e_g$  has dimension  $|G|^2$ , where  $e_g$ 's are the generators, with a multiplication operation  $(l e_g)(m e_h) = l(g \cdot m) \varphi(g, h) e_{gh}$  which can be extended via distribution.  $A_\varphi$  is said to be the crossed product of  $L$  and  $G$  via  $\varphi$ .

**Theorem 5.4.** 1.  $A_\varphi$  is a split algebra over  $L$ .

2. If  $\varphi, \varphi'$  are two normalized inhomogeneous 2-cocycles, then  $A_\varphi \sim A_{\varphi'}$  if and only if  $\varphi \sim \varphi'$ .
3.  $A_{\varphi\varphi'} \sim A_\varphi \otimes_K A_{\varphi'}$ .
4. Any  $K$ -CSA which is split over  $L$  is similar to a crossed product  $A_\varphi$  for some  $\varphi : G \times G \rightarrow L^\times$ .

**Corollary 5.5.**  $H^2(G, L^\times)$  is isomorphic to  $\text{Br}(L | K)$ , and  $H^2(\text{Gal}(\bar{K}/K), \bar{K}^\times)$  is isomorphic to  $\text{Br}(K)$ .