MATH 502 Notes

Jiantong Liu

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References:

- Atiyah and MacDonald, *Commutative Algebra*.
- J.P. Serre, *Local Algebra*.
- Zariski and Samuel, *Commutative Algebra* Volume 1 and 2.
- Matsumura, *Commutative Algebra*.
- Bourbaki, *Commutative Algebra*.

We always assume a ring R has a multiplicative identity and is commutative.

0 Noetherian and Artinian Property

Proposition 0.1. Let R be a (commutative) ring, and let M be an A-module, then the following are equivalent:

(i) Given an infinite increasing chain of submodules of M

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq M_{n+1} \subseteq \cdots$$

then there exists some $N \in \mathbb{N}$ such that $M_N = M_{N+1} = \cdots$, i.e., for all $n \geqslant N$, $M_n = M_{n+1}$.

(ii) Every non-empty family of submodules has a maximal element.

(iii) Every submodule of M is finitely-generated.

Proof. $(i) \Rightarrow (ii)$: This is a direct result of Zorn's lemma.

$(ii) \Rightarrow (i)$: Obvious.

$(i), (ii) \Rightarrow (iii)$: Take any submodule N of M and take $x_1 \in N$. If $(x_1) \neq N$, then there exists $x_2 \in N \backslash (x_1)$, so $(x_1, x_2) \subseteq N$, now we proceed inductively, but by the given property we know this stops in finite number of steps, hence we have $N = (x_1, \ldots, x_n)$ for some $n \in \mathbb{N}$, thus N is finitely-generated.

$(iii) \Rightarrow (i)$: Note that the property implies M is finitely-generated, but that means the chain of submodules must be finite. $\square$

Definition 0.2 (Noetherian Module)**.** If any of the conditions in Proposition 0.1 holds, then M is said to be a Noetherian module. Alternatively, we say M satisfies the ascending chain condition.

Proposition 0.3. Let R be a (commutative) ring, and let M be an A-module, then the following are equivalent:

(i) Given an infinite decreasing chain of submodules of M

$$M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq M_{n+1} \supseteq \cdots$$

then there exists some $N \in \mathbb{N}$ such that $M_N = M_{N+1} = \cdots$, i.e., for all $n \geqslant N$, $M_n = M_{n+1}$.

(ii) Every non-empty family of submodules has a minimal element.

Proof. Again, Zorn's lemma. □

Definition 0.4 (Artinian Module)**.** If any of the conditions in Proposition 0.3 holds, then M is said to be a Artinian module. Alternatively, we say M satisfies the descending chain condition.

Example 0.5.
- $\mathbb{Z}$ is Noetherian.
- $\mathbb{Q}/\mathbb{Z}$ is not Noetherian.
- Let p be a prime. Let $\mathbb{Z}(p^\infty)$ be the union of chains (as direct limits)
$$\left\langle \frac{\bar{1}}{p} \right\rangle \subseteq \left\langle \frac{\bar{1}}{p^2} \right\rangle \subseteq \cdots \subseteq \left\langle \frac{\bar{1}}{p^n} \right\rangle \subseteq \cdots$$
then there is an embedding $\mathbb{Z}(p^\infty) \subseteq \mathbb{Q}/\mathbb{Z}$, where $\bar{a}$ is the image of a in $\mathbb{Q}/\mathbb{Z}$. With this construction, $\mathbb{Z}(p^\infty)$ is Artinian.

Exercise 0.6. Show that $\mathbb{Q}/\mathbb{Z} \cong \bigoplus\limits_p \mathbb{Z}(p^\infty)$ where p traverses through all the primes.

Proposition 0.7. Let N be a submodule of M. Suppose M satisfies ascending (respectively, descending) chain condition, then N and M/N also satisfy ascending (respectively, descending) chain condition. If, for some submodule N of M, we know N and M/N satisfy ascending (respectively, descending) chain condition, then M also satisfies ascending (respectively, descending) chain condition.

Proof. Suppose M satisfies ascending (respectively, descending) chain condition, and let N be a submodule of M. Let $\{N_i\}$ be an increasing (respectively, decreasing) sequence of submodules of N, then they can be regarded as submodules of M, therefore by the Noetherian (respectively, Artinian) condition, we know N satisfies ascending (respectively, descending) chain condition. Now let $\bar{M} = M/N$, and take $\{\bar{M}_i\}$ be an increasing (respectively, decreasing) sequence of submodules of $\bar{M}$. Let $\pi : M \to M/N$ be the quotient map, then the preimages give an increasing (respectively, decreasing) sequence $\{M_i\}$ of submodules of M, where $M_i = \pi^{-1}(\bar{M}_i)$, but by the Notherian (respectively, Artinian) condition, we know the sequence stops in finite steps, therefore the original sequence stops in finite steps as well, hence $\bar{M}$ satisfies the ascending (respectively, descending) chain condition.

Suppose a submodule N of M is such that N and M/N both satisfy ascending chain condition. Take a submodule T of M, then we have a short exact sequence

$$0 \longrightarrow T \cap N \hookrightarrow T \longrightarrow T/(T \cap N) \longrightarrow 0$$

Now $T \cap N$ is finitely-generated as N is finitely-generated, therefore we have an embedding $T/T \cap N \hookrightarrow M/N$, thus $T/T \cap N$ is finitely-generated, therefore T is also finitely-generated by a vector space argument.

Suppose we have a decreasing sequence $\{M_n\}$ of M, then we have a decreasing sequence $\{N \cap M_n\}$. Let $\bar{M} = M/N$, then $\bar{M}_n := (M_n + N)/N$ defines a decreasing sequence of submodules in $\bar{M}$, but N satisfies the descending chain condition, so the sequence $\{N \cap M_n\}$ stops in finite number of steps, say n_0. Moreover, the sequence of $\bar{M}_n$'s also stops in finite number of steps, so by definition the sequence of $(M_n + N)/N$ stops in finite number of steps, say m_0, but by the isomorphism theorem this shows that the sequence of $M_n/(N \cap M_n)$ stops in m_0 steps. Therefore, whenever $n \geqslant m_0, n_0$, then $N \cap M_n = N \cap M_{n+1}$, hence $M_n = M_{n+1} = \cdots$ for such n. □

Remark 0.8. The final argument should also work in the Noetherian case.

Definition 0.9 (Simple Module)**.** An A-module M is simple if the submodules of M are either 0 or M.

Exercise 0.10. Let A be a commutative ring, and M is an A-module, then M is simple if and only if $M \cong A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of A.

Definition 0.11 (Jordan-Hölder Chain)**.** Let A be a commutative ring and M be an A-module. We say M has a Jordan-Hölder chain if there exists a decreasing chain of submodules $\{M_i\}$ such that

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_{n-1} \supsetneq M_n = 0$$

such that M_i/M_{i+1} is simple. In such a situation, we know n is the length of the Jordan-Hölder chain, and such n is unique. We say M is a module of finite length, and the length is $\ell_A(M) = n$.

Exercise 0.12. Let A be a commutative ring, and let M be an A-module, then M is of finite length if and only if M is both Noetherian and Artinian.

Theorem 0.13. Let A be a commutative ring, then A is Artinian if and only if A is Noetherian and every prime ideal of A is maximal.

1 Primary Decomposition Theorem

2 Filtered Rings and Modules, Completions

3 Dimension Theory

4 Integral Extensions

5 Noether's Normalization Lemma

6 Homological Algebra