MATH 214A Notes

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1 Lecture 1

Algebraic geometry is about shapes defined by polynomial equations. One may realize it is especially easier to understand algebraic sets over $\mathbb{C}$.

Example 1.1. $\{(x,y)\in\mathbb{C}^2 : x^2+y^2=1\}\cong\mathbb{C}\backslash\{0\}$.

Algebraic geometry studies algebraic curves over $\mathbb{C}$, i.e., structure of dimension 1. Because the field $\mathbb{C}$ is algebraically closed, then every polynomial $f\in\mathbb{C}[x]$ can be factored into degree 1 polynomials, i.e., $f(x)=a(x-b_1)\cdots(x-b_n)$ for some $a\in\mathbb{C}$, $n\geq 0$, and $b_1,\ldots,b_n\in\mathbb{C}$. This would not happen over $\mathbb{R}$, for instance.

Algebraic geometry looks at equations with more variables, in general.

Example 1.2. Consider $\{x\in\mathbb{R} : x^3+ax^2+bx+c=0\}$ for some $a,b,c\in\mathbb{R}$. Typically, the equation has 1 root or 3 roots, depending on the shape of the diagram. However, if we substitute $\mathbb{R}$ with $\mathbb{C}$, then we essentially always have 3 roots in this equation, even though sometimes there exists a double root.

To classify algebraic varieties, one key step for varieties over $\mathbb{C}$ is to look at them just as topological spaces.

Example 1.3. Consider $\{(x,y)\in\mathbb{C}^2 : x^d+y^d=1\}$. This is a complex curve homeomorphic to a real 2-manifold of genus g minus a finite set. In this case, we have $g=\frac{(d-1)(d-2)}{2}$.

Theorem 1.4 (Faltings)**.** If an algebraic curve X over $\mathbb{Q}$ has genus $g\geq 2$, then the set of rational points $X(\mathbb{Q})$ is finite.

In some sense, complexity in algebra and topology are related.

Sometimes people also look at the connection between algebraic geometry and number theory.

Example 1.5. What is $\{(x, y, z) \in \mathbb{Z}^3 : x^5 + y^5 = z^5\}$? The only solution is $(0, 0, 0)$. Note that this set is equivalent to $\{(x, y) \in \mathbb{Q}^2 : x^5 + y^5 = 1\}$.

Number theory allows us to study numbers in finite fields. We can define numbers like the genus and topology even in finite characteristics.

Definition 1.6 (Affine Space)**.** Let k be an algebraically closed field. The *affine n-space over k* is

$$\mathbb{A}_k^n = k^n = \{(a_1, \ldots, a_n) : a_1, \ldots, a_n \in k\}.$$

Let $R = k[x_1, \ldots, x_n]$. An element $f \in R$ determines a function $\mathbb{A}_k^n \to k$. For an element $f \in R$, its *zero set* is $\{f = 0\} \subseteq \mathbb{A}_k^n$, often defined by

$$Z(f) = \{f = 0\} := \{(a_1, \ldots, a_n) \in \mathbb{A}_k^n : f(a_1, \ldots, a_n) = 0\}.$$

Similarly, for a set T, its zero set is

$$Z(T) = \{a \in \mathbb{A}_k^n : f(a) = 0 \ \forall f \in T\}.$$

An *affine algebraic set* over k is a subset of $\mathbb{A}_k^n$ for some $n \geq 0$ of the form $Z(T)$ for some subset $T \subseteq R = k[x_1, \ldots, x_n]$.

Remark 1.7. Given a subset $T \subseteq R$, let $I \subseteq R$ be the ideal generated by T, then $Z(T) = Z(I)$.

Example 1.8. What is the algebraic set of the affine line $\mathbb{A}_k^1$? We want to find all subsets of $\mathbb{A}_k^1 \cong k$ defined by some ideal $I \subseteq k[x]$. If $I = \{0\}$, then $Z(I) = \mathbb{A}_k^1$. If not, then pick $f \neq 0$ in I, then $Z(I) \subseteq Z(f)$, and $f = a(x - b_1) \cdots (x - b_n)$, so $Z(f) = \{b_1, \ldots, b_n\}$.

We conclude that an affine set in $\mathbb{A}_k^1$ is either all of $\mathbb{A}_k^1$ or a finite set of points.

2 Lecture 2

Definition 2.1 (Zariski Topology)**.** Let k be an algebraically closed field and let $n \geq 0$. The *Zariski Topology* on $\mathbb{A}_k^n \cong k^n$ is defined by closed sets, which is defined as follows: a subset $S \subseteq \mathbb{A}_k^n$ is closed if and only if it is of the form $S = Z(I)$ for some ideal $I \subseteq R$ where $R = k[x_1, \ldots, x_n]$.

Example 2.2. The twisted cubic curve in $\mathbb{A}_k^3$ is defined as

$$\{(\mathcal{A}, \mathcal{A}^2, \mathcal{A}^3) : \mathcal{A} \in k\} \subseteq \mathbb{A}_k^3.$$

This is Zariski-closed in $\mathbb{A}_k^3$ since

$$S = \{y = x^2, z = x^3\} \subseteq \mathbb{A}_k^3$$

is equivalent to $Z(\{y - x^2, z - x^3\}$, which is just $Z(I)$ where $I \subseteq k[x, y, z]$ is just the ideal $(y - x^2, z - x^3)$.

Remark 2.3. If $k = \mathbb{C}$, then we also have the classical topology on $\mathbb{A}_{\mathbb{C}}^n = \mathbb{C}^n$, based on the usual metric on $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

It is easy to see that Zariski-closed in $\mathbb{A}_{\mathbb{C}}^n$ implies closure in the classical topology. The converse is obviously not true, for example consider the closed balls in $\mathbb{C}^3$.

Lemma 2.4. The Zariski topology in $\mathbb{A}_k^n$ is a well-defined topology.

Proof. By definition, a topological space is a set with a colletion of subsets called "the open subsets of X", such that

1. $\varnothing$ and X are open in X,

2. union of any collection of open sets is open,

3. intersection of finitely many open sets is open.

Equivalently, the closed subsets of X satisfy

1. $\varnothing$ and X are closed in X,

2. intersection of any collection of closed sets is closed,

3. union of finitely many closed sets is closed.

Indeed,

1. $\mathbb{A}_k^n = Z(0)$ and $\varnothing = Z(R)$.

2. Given a collection S_α of closed subsets of $X = \mathbb{A}_k^n$ where $\alpha \in I$ set, which could be infinite, the intersection of the collection is just the union of the zero sets.

   By definition, for each $\alpha \in I$, we can choose an ideal $I_\alpha \subseteq R$ with $S_\alpha = Z(I_\alpha) \subseteq \mathbb{A}_k^n$.

   Define $I = \sum_{\alpha \in I} I_\alpha \subseteq R$ (i.e., the set of all possible finite sums), then $Z(I) = \bigcap_{\alpha \in I} Z(I_\alpha) = \bigcap_{\alpha \in I} S_\alpha$, so it is closed.

3. Given closed sets $S, T \subseteq \mathbb{A}_k^n$, we want to show that $S \cup T$ is closed. By definition, choose I and J such that $S = Z(I)$ and $T = Z(J)$. Take $K = I \cap J$ or $J = IJ$ (i.e., finite sum of elements ab with $a \in I$ and $b \in J$), then it suffices to show that $Z(I \cap J) = Z(IJ) = Z(I) \cup (J)$.

   Example 2.5. Note that the two structures may not be equivalent. Let $R = k[x]$ and let $I = J = (x)$. Now $Z(I) = Z(J) = \{0\}$, then $I \cap J = (x)$, but $IJ = (x^2)$.

   Remark 2.6. Essentially, if $I = (f_1, \ldots, f_r)$ and $J = (g_1, \ldots, g_s)$, then $IJ = (f_i g_j : \forall i, j)$.

   However, things look better if we look at their radicals.

   Exercise 2.7. Show that for any commutative R and ideals I and J, the radicals satisfy $\operatorname{rad}(I \cap J) = \operatorname{rad}(IJ)$.

   To finish the proof, we show that $Z(IJ) = Z(I) \cup Z(J)$. Indeed, we have $IJ \subseteq I$ and $IJ \subseteq J$, so $Z(IJ) \supseteq Z(I)$ and $Z(IJ) \supseteq Z(J)$, so $Z(I) \cup Z(J) \subseteq Z(IJ)$.

   Conversely, we want to show $Z(IJ) \subseteq Z(I) \cup Z(J) \subseteq \mathbb{A}_k^n$.

   Let $a = (a_1, \ldots, a_n) \in k^n$ be a point in $Z(IJ)$. Suppose $a \notin Z(I)$ and $a \notin Z(J)$, so there exists $f \in I$ such that $f(a) \neq 0$, and there exists $g \in J$ such that $g(a) \neq 0$, then $(fg)(a) = f(a)g(a) = 0$, but $fg \in IJ$, $(fg)(a) \neq 0$, contradiction.

$\square$

Remark 2.8. Note that $\mathbb{A}_k^n$ is not Hausdorff for $n > 1$. In fact, the intersection of any two non-empty open subsets is non-empty.

For $\mathbb{A}_k^1$, an open subset of $\mathbb{A}_k^1$ is either $\varnothing$ or a $\mathbb{A}_k^1$-finite set. Note that k is infinite since it is algebraically closed, so the intersection of two intervals on $\mathbb{A}_k^1$ (with finitely many isolated points excluded) should not be empty.

Definition 2.9 (Connected, Irreducible)**.** A topological space X is *connected* if $X \neq \varnothing$, and you cannot write X as the disjoint union of two non-empty closed subsets.

A topological space X is *irreducible* if $X \neq \varnothing$, and you cannot write X as the union of two proper closed subsets.

Example 2.10. For example, the set defined by two parallel lines is not connected; the set defined by the union of a circle and a line passing through the circle is connected, but not irreducible.

Remark 2.11. A Hausdorff space with at least 2 points is never irreducible.

Example 2.12. $[0,1]$ is not irreducible since $[0,1] = [0,\frac{1}{2}] \cup [\frac{1}{2},1]$, but $\mathbb{A}_k^n$ is irreducible.

Theorem 2.13 (Hilbert's Nullstellensatz)**.** For an algebraically closed field k and $n \geq 0$, there is a one-to-one correspondence between radical ideals in $R = k[x_1,\ldots,x_n]$ and the Zariski closed subsets of $\mathbb{A}_k^n$. More precisely, this correspondence is given by the mapping $I \mapsto Z(I)$ for radical ideals I and the mapping $S \mapsto I(S) = \{f \in R : f(a) = 0 \ \forall a \in S\}$ for closed subset $S \subseteq \mathbb{A}_k^n$.

Definition 2.14 (Reduced Ring, Radical Ideal)**.** A commutative ring R is *reduced* if every nilpotent element is 0, i.e., if $a \in R$ such that $a^m = 0$ for some $m > 0$, then $a = 0$.

An ideal I in a commutative ring R is *radical* if the ring R/I is radical. In particular, $I \subseteq R$ is radical if and only if for any $a \in R$ with $a^m \in I$ for some $m > 0$, we know $a \in I$.

For any ideal I, $\mathrm{rad}(I) = \{a \in R : a^m \in I \text{ for some } m > 0\}$.

Lemma 2.15. An affine algebraic set $X \subseteq \mathbb{A}_k^n$ is irreducible if and only if $I(Y) \subseteq R$ is prime.

Proof. ($\Longrightarrow$): Let $Y \subseteq \mathbb{A}_k^n$ be an irreducible algebraic set.

We define the subspace topology on Y as follows: a subset of Y is closed in Y if it is the intersection of some closed subset (of X) and Y.

Therefore, since $Y \neq \varnothing$, so $I(Y) \neq R$ as $1 \in R$ is not in $I(Y)$.

Suppose $f, g \in R$ with $fg \in I(Y)$. We want to show that f or g is in $I(Y)$. Since $fg \in I(Y)$, $Y = (Y \cap \{f = 0\}) \cup (Y \cap \{g = 0\})$ is the union of two closed sets in Y. Therefore, since Y is irreducible, then either $Y = Y \cap \{f = 0\}$, or $Y = Y \cap \{g = 0\}$. That is, $f \in I(Y)$ or $g \in I(Y)$, as desired.

($\Longleftarrow$): Given an affine algebraic set $X \subseteq \mathbb{A}_k^n$ such that the ideal $I(X) \subseteq R$ is prime. That means $1 \notin I(X)$, and, if $f, g \in R$ such that $fg \in I(X)$, then $f \in I(X)$ or $g \in I(X)$. Note that if $X = \varnothing$, then $I(X)$ would be R, which is not prime. Therefore, $X \neq \varnothing$. Suppose $X = S_1 \cup S_2$ for closed subsets $S_1, S_2 \subsetneq X$. We pick $p \in S_2 \backslash S_1$ and $q \in S_1 \backslash S_2$. Since S_1 and S_2 are closed in $\mathbb{A}_k^n$, there is a polynomial $f \in I(S_1)$ and $f(q) \neq 0 \in k$. Similarly, there is a polynomial $g \in I(S_2)$ but with $g(p) \neq 0$. Then $fg \in I(X)$. Since $I(X)$ is prime, $f \in I(X)$ or $g \in I(X)$, contradiction. $\square$

3 Lecture 3

Remark 3.1. For any subset $X \subseteq \mathbb{A}_k^n$, $I(X) \subseteq R$ is radical.

Proof. If $f \in R$ has $f^m \in I(X)$ for some $m > 0$, then $f \in I(X)$. Therefore, at any $p \in X$, $f(p)^m = 0 \in k$. Hence, $f(p) = 0 \in k$. $\square$

Remark 3.2. $Z(I) = Z(\text{rad}(I))$ for ideal $I \subseteq R = k[x_1, \ldots, x_n]$.

Example 3.3. Affine n-space $\mathbb{A}_k^n$ is irreducible.

Proof. Think of $\mathbb{A}_k^n$ as a closed set in itself, then $I(\mathbb{A}_k^n) = 0$, and so $\mathbb{A}_k^n$ is irreducible if and only if $0 \subseteq k[x_1, \ldots, x_n]$ is prime, if and only if $k[x_1, \ldots, x_n]$ is a domain. $\square$

Remark 3.4. For any irreducible topological space, the intersection of any two non-empty open subsets is non-empty. (So this holds in $\mathbb{A}_k$ per se.)

Definition 3.5 (Affine Variety)**.** An *affine variety* over k is an irreducible affine algebraic set in some $\mathbb{A}_k^n$.

Definition 3.6 (Irreducible)**.** Let R be a domain. Any element $f \in R$ is *irreducible* if $f \neq 0$ and for any $g, h \in R$ such that $f = gh$, either g or h must be a unit.

Remark 3.7. This concept is useless unless R is a UFD, where R admits a unique factorization.

Proposition 3.8. If R is a UFD, and $f \in R$ is irreducible, then (f) is a prime ideal. In particular, for any field k, the polynomial ring $k[x_1, \ldots, x_n]$ is a UFD.

We now have the notion of an irreducible polynomial $f \in k[x_1, \ldots, x_n]$ over k. In particular, the units in the polynomial ring $k[x_1, \ldots, x_n]$ is just k^*, i.e., the units in k.

Remark 3.9. The proposition implies that for any irreducible polynomial f over a field k, the ideal $(f) \subseteq R$ is prime.

Corollary 3.10. For an irreducible polynomial $f \in k[x_1, \ldots, x_n]$ over an algebraically closed field k, $\{f = 0\} \subseteq \mathbb{A}_k^n$ is an affine variety over k. This is called an *irreducible hypersurface* in $\mathbb{A}_k^n$.

For $n = 1$, an irreducible polynomial in $k[x]$ (with k algebraically closed) is of the form $c(x - a)$ for $a, c \in k$.

Recall the following exercise in homework:

Exercise 3.11. Let $g \in k[x_1, \ldots, x_{n-1}]$. Then $x_n^2 - g(x_1, \ldots, x_{n-1})$ is irreducible over k if and only if g is not a square in $k[x_1, \ldots, x_{n-1}]$.

For example, $x^2 - y^{17}$ is irreducible over $\mathbb{C}$, i.e., $\{x^2 = y^{17}\} \subseteq \mathbb{A}_{\mathbb{C}}^2$ is a variety.

Example 3.12. Over $\mathbb{R}$, x^2+y^2 is irreducible since $-y^2$ is not a square in $\mathbb{R}[y]$. Geometrically, we see that the set $\{(x,y) \in \mathbb{R}^2 : x^2+y^2=0\} = \{(0,0)\}$.

Over $\mathbb{C}$, as $x^2+y^2 = (x+iy)(x-iy)$, then geometrically we see $\{(x,y) \in \mathbb{C}^2 : x^2+y^2 = 0\} = \{(x=iy)\} \cup \{(x=-iy)\}$.

Note that for $n \geq 3$, $x_1^2 + \cdot + x_n^2$ is irreducible over $\mathbb{C}$.

Definition 3.13 (Coordinate Ring)**.** For an affine algebraic set $X \subseteq \mathbb{A}_k^n$, the *coordinate ring* of X (or *ring of regular functions* on X) is $\mathcal{O}(X) := k[x_1, \ldots, x_n]/I(X)$. This is isomorphic to the image of mapping from $k[x_1, \ldots, x_n]$ to the ring of all functions $X \to k$.

Example 3.14. Consider $X = \{x^2 = y^3\} \subseteq \mathbb{A}_{\mathbb{C}}^2$. Then $x^5 - 7y$ is a regular function on X, and is equal to $x^5 - 7x + 8(x^2 - y^3)$ on X.

Remark 3.15. For an affine algebraic set X, $\mathcal{O}(X)$ is a finitely-generated commutative k-algebra. Also, for an affine variety $X \subseteq \mathbb{A}_k^n$, $\mathcal{O}(X)$ is a domain as well.

Conversely, for any finitely-generated commutative k-algebra R (which is a domain), $R \cong \mathcal{O}(X)$ for some affine variety $X \subseteq \mathbb{A}_k^n$ for some $n \geq 0$. Similar classification holds for general affine algebraic sets.

Proof. Let R be a finitely-generated k-algebra which is a domain, then $R = k[x_1, \ldots, x_n]/I$ for some $n \geq 0$ and some ideal I. Since R is a domain, I is prime. So $Z(I) \subseteq \mathbb{A}_k^n$ is an affine variety X.

We want to show that $R \cong \mathcal{O}(X)$ as k-algebras. Here $\mathcal{O}(X) \cong k[x_1, \ldots, x_n]/I(X)$, where we can denote $I(X) = I(Z(I))$. By Nullstellensatz, $I(Z(I))$ is just I if it is radical. Now since I is prime, then it is radical indeed, and we are done. $\square$

Example 3.16. $\mathbb{A}_k^1$ and $X = \{y = x^2\} \subseteq \mathbb{A}_k^2$ have isomorphic coordinate rings (as k-algebras).

Proof. One would realize that $\mathcal{O}(\mathbb{A}_k^1) = k[x]$ and $\mathcal{O}(X) = k[x,y]/I(X)$. Note that $y - x^2$ is irreducible, so $(y - x^2) \subseteq k[x,y]$ is prime, then $I(X) = I(Z(y-x^2)) = (y-x^2)$. Therefore, $\mathcal{O}(X) = k[x,y]/I(X) \cong k[x,y]/(y-x^2) \cong k[x]$.

Geometrically, the two structures are just a horizontal line and a quadratic curve, respectively. The isomorphic is given by the projection of the quadratic curve onto the horizontal axis. $\square$

4 Lecture 4

Definition 4.1 (Noetherian)**.** A topological space X is *Noetherian* if every descending sequence of closed subsets $X \supset Y_1 \supset Y_2 \supset \cdots$, there is some $N \in \mathbb{Z}^+$ such that $Y_N = Y_{N+1} =$

$\cdots$. This is essentially a DCC on X.

Remark 4.2. Note that $\mathbb{R}$ and $[0,1]$ are not Noetherian with the classical topology.

Lemma 4.3. Every affine algebraic set over an algebraically closed field k is Noetherian (as a topological space).

Proof. We are given a closed subset $X \subseteq \mathbb{A}_k^n$ for some $n \geq 0$. Here $\mathcal{O}(X)$ is a finitely-generated (commutative) k-algebra (and a reduced ring). By the Nullstellensatz, we have a one-to-one correspondence between closed subsets of X and radical ideals of $\mathcal{O}(X)$. To see this, we know a one-to-one correspondence between closed subsets of $\mathbb{A}_k^n$ and radical ideals in $k(x_1, \ldots, x_n)$, then $\mathcal{O}(X) = k[x_1, \ldots, x_n]/I(X)$. By Hilbert's basis theorem, $\mathcal{O}(X)$ is a Noetherian ring, i.e., every ideal in $\mathcal{O}(X)$ is finitely-generated as an ideal, or equivalently, the ACC condition. Therefore, every decreasing sequence of closed subsets of X terminates, i.e., X is Noetherian as a topological space. $\square$

Theorem 4.4. Every Noetherian topological space X can be written as a finite union of irreducible closed subsets, i.e., $X = Y_1 \cup \cdots \cup Y_n$ for some $n \geq 0$ and irreducible closed subsets Y_i of X.

Moreover, if we also require that Y_i is not contained in Y_j for all $i \neq j$, then this decomposition is unique up to reordering.

Remark 4.5. We call the Y_i's (with all the conditions above) the *irreducible component* of X.

Definition 4.6 (Dimension)**.** The *dimension* of a topological space X is $\dim(X) = \sup\{n \geq 0 :$ there is a chain of length n of irreducible closed subsets of $X,\ Y_0 \subsetneq Y_1 \subsetneq Y_2 \subsetneq \cdots \subsetneq Y_n\}$.

Exercise 4.7. Show that $\dim(\mathbb{R}^3) = 0$ for $\mathbb{R}^3$ with the classical topological space.

Example 4.8. $\dim(\mathbb{A}_k^1) = 1$ with the Zariski topology. Recall that any closed set on this space is either itself or a set of finitely many points. Therefore, the largest chain of irreducible closed subsets has length $\{a\} \subsetneq \mathbb{A}_k^1$ for any $a \in k$.

Definition 4.9 (Krull Dimension)**.** The *(Krull) dimension* of a commutative ring R is $\sup\{n \geq 0 :$ there is a chain of length n of prime ideals in $R : \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n\}$.

Lemma 4.10. Let X be an affine algebraic set over k. Then $\dim(X) = \dim(\mathcal{O}(X))$, i.e., the dimension of the topological space equals the (Krull) dimension of the ring.

Proof. We have a one-to-one correspondence between prime ideals in $\mathcal{O}(X)$ and irreducible closed subsets of X (containing whatever $I(X)$ we quotient out), reversing the directions of the inclusions. $\square$

Definition 4.11 (transcendence degree)**.** Let $k \subseteq E$ be a field extension (not necessarily finite, or even algebraic). There is a set I and a set of elements $x_i \in E$ for $i \in I$ such that $k \subseteq k(x_i : i \in I) \subseteq$, where $k(x_i : i \in I) = \mathrm{Frac}(k[x_i : i \in I])$ is the rational function field on a set of variables, such that E is algebraic over $k(x_i : i \in I)$. The *transcendence degree* of E over k is the cardinality $|I|$. This is well-defined.

Theorem 4.12. Let k be any field and let A be a domain which is also a finitely-generated (commutative) k-algebra. Then $\dim(A)$ is the transcendence degree of $\mathrm{Frac}(A)/k$, i.e., $\dim(A) = \operatorname{tr}\deg(\mathrm{Frac}(A)/k)$.

Corollary 4.13. For any $n \geq 0$ and algebraically closed field k, $\dim(\mathbb{A}_k^n) = n$.

Proof. We have $\dim(\mathbb{A}_k^n) = \dim(k[x_1, \ldots, x_n]) = \dim(\mathcal{O}(\mathbb{A}_k^n)) = \operatorname{tr}\deg(k(x_1, \ldots, x_n)/k) = n$. $\square$

Proposition 4.14 (Krull's Principal Ideal Theorem)**.** Let A be a Noetherian ring, and let $f \in A$ be an element which is neither zero divisor nor a unit, then every minimal prime ideal $\mathfrak{p}$ containing f has height 1.

Corollary 4.15. A variety in $\mathbb{A}_k^n$ has dimension $n-1$ if and only if it is the zero set $Z(f)$ of a single non-constant irreducible polynomial in $A = k[x_1, \ldots, x_n]$.

Proof. See Hartshorne Section I.1 Proposition 1.13. $\square$

In the classical topology, $\mathbb{C}P^n$ is a compact complex manifold, containing $\mathbb{C}^n$ as an open subset; note that $\mathbb{C}^n$ is not compact for $n \geq 1$.

Example 4.16. The 2-sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is compact in the classical topology in $\mathbb{R}^3$. However, $S_{\mathbb{C}}^2 = \{(x, y, z) \in \mathbb{A}_{\mathbb{C}}^3 : x^2 + y^2 + z^2 = 1\}$ is not compact in the classical topology in $\mathbb{C}^3$.

Indeed, consider the function z desecending in $\mathbb{C}$. So we have an unbounded compact function on $S_{\mathbb{C}}^2$ with values decreasing in $\mathbb{C}$, so $S_{\mathbb{C}}^2$ is not compact.

Definition 4.17 (Projective Space)**.** For $n \geq 0$ and k algebraically closed, the *projective n-space over k P_k^n* is the set of one-dimensional k-linear subspaces of the k-vector space k^{n+1}.

Example 4.18. P_k^0 is just a point.

Definition 4.19 (Homogeneous Coordinates)**.** For $a_0, \dots, a_n \in k$, not all zeros, we write $[a_0, \dots, a_n] \in P_k^n$ to mean the line $k(a_0, \dots, a_n) \subseteq k^{n+1}$.

Remark 4.20. Note that $[0, \dots, 0]$ is not defined in P_k^n.

Clearly, $[a_0, \dots, a_n] = [b_0, \dots, b_n]$ if and only if there exists $c \in k^*$ such that $b_i = ca_i$ for all $0 \leq i \leq n$.

Example 4.21. We can define a bijection $P_k^1 \cong \mathbb{A}_k^1 \cup \{\infty\}$ by the following correspondence: every point in P_k^1, $[a_0, a_1]$ with coordinates not both 0, is either equal to $[0, 1]$ or to $[1, b]$ for some $b \in k$, and that is a unique way of writing the point.

Remark 4.22. By adding a point of infinity, we make sure parallel lines intersect at infinity.

5 Lecture 5

Remark 5.1. In fact, we can make a generalization: $P_k^1 := \mathbb{A}_k^1 \cup \{\infty\}$. Let k be an algebraically closed field and let $n \geq 0$, let $0 \leq i \leq n$, then $[x_0, \dots, x_n] \in P^n(k)$. Note that there exists a bijective correspondence between $\{x_i \neq 0\}$ $(\subseteq P_k^n)$ and $\mathbb{A}_k^n$, via $[x_0, \dots, x_i, \dots, x_n] \mapsto (\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i})$. Clearly P_k^n is covered by these $n+1$ "coordinate charts", as in $P_k^1 \cong (P_k^1 \backslash \{\infty\}) \cup (P_k^1 \backslash \{0\}) \cong \mathbb{A}_k^1 \cup \mathbb{A}_k^1$.

We can also see that $P_k^2 = \{x_0 \neq 0\} \cup P_k^1 \cong \mathbb{A}_k^2 \cup P_k^1 = \mathbb{A}_k^2 \cup \mathbb{A}_k^1 \cup \{*\}$, where $* = [0, x_1, x_2] \in P_k^2$.

Definition 5.2 (Homogeneous Polynomial)**.** A polynomial $f \in k[x_0, \dots, x_n]$ is *homogeneous* of degree $d \geq 0$ if $f = \sum\limits_{\text{finite sum}} a_{i_0, \dots, i_n} x_0^{i_0} \dots x_i^{i_n}$ with $a_I \in k$ and $i_0 + \dots + i_n = d$.

Remark 5.3. Note that a polynomial f (homogeneous or not) does not give a well-defined function $f : P^n(k) \to k$: for a point $[b_0, \dots, b_n] \in P^n(k)$, if there is another point in the same class (off by a scaling), the polynomial then produces a different value.

But, if f is homogeneous of degree d, then $f(ca_0, \dots, ca_n) = c^d f(a_0, \dots, a_n)$ for any $c \in k$.

Therefore, the zero set of a homogeneous polynomial f is a well-defined subset of P_k^n, $Z(f) = \{f = 0\} \subseteq P_k^n$, called a *hypersurface* in P_k^n.

Definition 5.4 (Projective Algebraic Set)**.** A *projective algebraic set* over k is a subset $X \subseteq P_k^n$ (for some $n \geq 0$) that equal to $Z(T) := \bigcap\limits_{f \in T} Z(f)$ for some set T of homogeneous polynomials in $k[x_0, \dots, x_n]$.

Remark 5.5. We will see later that this set T is defined as $T = Z(I)$ for a homogeneous ideal in $k[x_0, \dots, x_n]$.

Definition 5.6 (Zariski Topology)**.** The *Zariski topology on* P_k^n (for $n \geq 0$) is the topology whose closed subsets are the projective algebraic sets in P_k^n.

Remark 5.7. This is a topology.

There is a correspondence $\mathbb{A}_k^{n+1} \backslash \{0\} \to P^n$ given by sending $(x_0, \ldots, x_n)$ to $[x_0, \ldots, x_n]$.

Definition 5.8 (Cone)**.** A *cone* in $\mathbb{A}_k^{n+1}$ is a closed subset that is a union of lines through 0.

Remark 5.9. The zero set of a homogeneous polynomial in $\mathbb{A}_k^{n+1}$ is a cone.

Definition 5.10 (Graded Ring)**.** A *graded ring* is a (commutative ring) $R = \bigoplus_{i \geq 0} R_i$ such that $R_i R_j \subseteq R_{i+j}$ for all i, j.

Example 5.11. $k[x_0, \ldots, x_n]$ is graded with $|x_i| = 1$ for each i.

Definition 5.12 (Homogeneous Ideal)**.** An ideal I in a graded ring R is *homogeneous* if

$$I = \sum_{d \geq 0} (I \cap R_d).$$

In particular, this implies that

$$I = \bigoplus_{d \geq 0} (I \cap R_d).$$

Definition 5.13 (Zero Set)**.** For a homogeneous ideal $I \subseteq k[x_0, \ldots, x_n]$, its zero set in P_k^n is $Z(I) = \bigcap_{f \in I \text{ homogeneous}} Z(f)$.

Remark 5.14. If $I = (g_1, \ldots, g_r)$ with $g_1, \ldots, g_r$ homogeneous, then $Z(I) = Z(g_1) \cap \cdots \cap Z(g_r)$.

Definition 5.15 (Projective Algebraic Variety)**.** A *projective algebraic variety* is an irreducible projective algebraic set $X \subseteq P_k^n$ for some $n \geq 0$.

Remark 5.16. A projective algebraic set over k is a Noetherian topological space. So it is a finite union of its irreducible components.

Remark 5.17. Given an affine algebraic set $X \subseteq \mathbb{A}_k^n$, we can think of $\mathbb{A}_k^n$ as an open subset of P_k^n, and therefore produces a bijective correspondence between $\{x_0 \neq 0\} (\subseteq CP^n) \Leftrightarrow \mathbb{A}_k^n$.

Note that

1. The bijection above is a homeomorphis.

2. $\{x_0 \neq 0\} \subseteq P_k^n$ is open.

We can then consider its *projective closure*, i.e., its closure in P_k^n.

Remark 5.18. How would we usually calculate that closure?

Given as set of polynomials with $X = \{f(x_1, \ldots, x_n) = 0, \ldots\} \subseteq \mathbb{A}_k^n$, then say that f_i has degree at most d, then we can write down an "associated" homogeneous polynomial $g_i(x_1, \ldots, x_n)$ with degree d by $x_1^{i_1} \ldots x_n^{i_n} \mapsto x_0^{d-i_1-\ldots-i_n} x_1^{i_1} \ldots x_n^{i_n}$.

The correspondence is now given by

$$[1, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] \in P_k^n \Longleftrightarrow (x_1, \ldots, x_n) \in \mathbb{A}_k^n.$$

Therefore,

$$\{g_1 = 0, \ldots, g_r = 0\}(\subseteq P_k^n) \cap \{x_0 \neq 0\}(\cong \mathbb{A}_k^n) = \{f_1 = 0, \ldots, f_r = 0\} \subseteq \mathbb{A}_k^n.$$

The subtlety is that the set on the left might be bigger than the precise closure in P_k^n of the set in the right. (That is, the calculation from right to left may not be well-defined.)

Definition 5.19 (Regular Function)**.** Let X be an affine algebraic set over algebraically closed field k. (That is, $X \subseteq \mathbb{A}_k^n$ is closed.) Let $U \subseteq X$ be an open subset, then a function $f : U \to K$ is called *regular* if for every $x \in U$ there exists an open neighborhood $x \in V \subseteq U$ on which we can write $f = \frac{g}{h}$ where g and h are polynomials in $k[x_1, \ldots, x_n]$ such that $h \neq 0$ at all points of V.

Remark 5.20. This is a locally defined class of functions. That is, the expression may not be the same in different neighborhoods.

Example 5.21. $\frac{1}{x}$ is a regular function on $\mathbb{A}_k^1 \backslash \{0\}$. In fact, as we will see, the ring of all regular functions $\mathcal{O}(\mathbb{A}_k^1 \backslash \{0\}) \cong k[x][\frac{1}{x}]$, i.e., the ring of Laurent polynomials.

Remark 5.22. Note that for a function to be regular on the entire affine variety, this is equivalent to the following: a function is *regular* on the entire affine variety if it can globally be written as a polynomial.

Therefore, it is not so interesting to define regularity on an affine algebraic set with the same definition: one can just take the definition on the entire affine variety and restrict its domain. Our alternative definition essentially looks for the localization on open subsets.

6 Lecture 6

Definition 6.1 (Quasi-affine Algebraic Set)**.** A *quasi-affine algebraic set* over k an algebraically closed field is an open subset of an affine algebraic (closed) set $X \subseteq \mathbb{A}_k^n$. That is, $X \cap U$ where U is open in $\mathbb{A}_k^n$, i.e., $X - Y$ where Y is closed in $\mathbb{A}_k^n$, i.e., $X - Y$ where Y is a closed in X. This describes the idea of "a solution set minus another solution set".

Lemma 6.2. A regular function $f : U \to k$ on a quasi-affine algebraic set U is continuous as a mapping $f : U \to \mathbb{A}^1_k$ (with the Zariski topology).

Proof. We have to show that for every closed $S \subseteq \mathbb{A}^k_1$, $f^{-1}(U)$ is closed in U. By our knowledge of the closed subset of $\mathbb{A}^1_k$, it suffices to prove this for $S = \{a\}$ for some $a \in k$. By assumption, U is covered by open set $V \subseteq U$, on which $f = \frac{g}{h}$ with $g, h \in x[k_1 \dots, x_n]$ with $h \mid_V \neq 0$ everywhere on V.

Lemma 6.3. For a topological space X with an open covering by open V_α, a subset S is closed in X if and only if $S \cap V_\alpha$ is closed in V_α for all α, and likewise for open subsets.

Subproof. Left as an exercise. ■

So it suffices to show that $f^{-1}(a) \cap V$, for each open $V \subseteq U$ as above. Now $f^{-1}(a) \cap V = \{x \in V : f(x) = a\} = \{x \in V : \frac{g(x)}{h(x)} = a\} = \{x \in V : g(x) - ah(x) = 0\}$, but this is a polynomial function on $\mathbb{A}^n_k$, restricted to V, and therefore this is a closed subset of V. □

Definition 6.4 (Quasi-projective Algebraic Set)**.** A *quasi-projective algebraic set* V over k is an open subset V of some projective algebraic set $X \subseteq P^n_k$ for some $n \geq 0$.

Remark 6.5. A quasi-affine algebraic set can be viewed as a quasi-projective algebraic set in P^n_k by the inclusion $\mathbb{A}^n_k \subseteq P^n_k$ as $\mathbb{A}^n_k = \{x_i \neq 0\} \subseteq P^n_k$ for any $0 \leq i \leq n$.

Definition 6.6 (Morphism of Quasi-projective Algebraic Set)**.** Let X and Y be quasi-projective algebraic sets over k. A *morphism* $f : X \to Y$ is a continuous function such that for every open $U \subseteq Y$ and every regular function g on U, the composition $g \circ f : f^{-1}(U) \to k$ is a regular function open in X.

Definition 6.7 (Regular functions on Quasi-projective Algebraic Set)**.** Let U be a quasi-projective algebraic set over k. A function $f : U \to k$ is *regular* if and only if for every point $x \in U$, there is an open $x \in V \subseteq U$ and $g, h \in k[x_0, \dots, x_n]$ homogeneous of the same degree d such that

1. $h \neq 0$ at every point of V, and
2. $f = \frac{g}{h}$ on V.

Remark 6.8. Note that for homogeneous polynomial g, h of the same degree d,

$$\frac{g(ca_0, \dots, ca_n)}{h(ca_0, \dots, ca_n)} = \frac{c^d g(a_0, \dots, a_n)}{c^d h(a_0, \dots, a_n)} = \frac{g(a_0, \dots, a_n)}{h(a_0, \dots, a_n)}.$$

Remark 6.9. In defining a morphism, it is not enough to take $U = Y$ in the definition.

Example 6.10. The ring of regular functions on P_k^1 is just k, i.e., the constant functions.

Remark 6.11. Note that $P_k^1 \backslash \{\infty\} \cong P_k^1 \backslash \{0\} \cong \mathbb{A}_k^1$.

Proof Sketch. We will see that $\mathcal{O}(\mathbb{A}_k^1) = k[x]$, even by our new definition. So a regular function$f : P_k^1 \to k$ would restrict to regular functions on $V_0 = \{x_0 \neq 0\} \cong \mathbb{A}_k^1$ but also in $V_1 = \{x_1 \neq 0\} \cong \mathbb{A}_k^1$, and as $[x_0, x_1] \in P_k^1$, therefore f would be in $k[x]$ and also $k[y]$. But $[1, a] = [\frac{1}{a}, 1]$, so f is both a polynomial in x and in $\frac{1}{x}$, which forces f to be a constant. □

Example 6.12. For a quasi-projective algebraic set X, a morphism $f : X \to \mathbb{A}_k^n$ is of the form $f(x) = (f_1(x), \ldots, f_n(x))$ where $f_1, \ldots, f_n$ are regular functions on X, and the converse is true.

Corollary 6.13. If X is a quasi-projective variety (meaning that it is irreducible), and f is a regular function on X that is not identically zero, then every irreducible component of the closed subset $\{f = 0\} \subseteq X$ has dimension $\dim(X) - 1$.

Proof. This is a corollary of Krull's Principal Ideal Theorem. □

Theorem 6.14. Let $X \subseteq \mathbb{A}_k^n$ be a closed subset (i.e. an affine algebraic set), then the definition of the ring $\mathcal{O}(X)$ of regular functions agrees with our old definition $k[x_1, \ldots, x_n]/I(X)$.

Proof.

Definition 6.15. For an affine algebraic set $X \subseteq \mathbb{A}_k^n$, a *standard open subset* of X is a subset of the form $\{g \neq 0\} \subseteq X$, where $g \in k[x_1, \ldots, x_n]$.

Lemma 6.16. The standard open subsets of X form a basis for the topology of X, for X an affine algebraic set.

Subproof. We have to show that every open subset of X is a union of standard ones. By definition, an open set $U \subseteq X$ is $X - \{g_1 = 0, \ldots, g_r = 0\}$ for some $g_1, \ldots, g_r \in k[x_1, \ldots, x_n]$, and this is just the set $\bigcup\limits_{1 \leq i \leq r} \{g_i \neq 0\}$. ■

Write $\mathcal{O}(X)$ for our new descriptions of regular functions. Clearly there is a homomorphism of k-algebras

$$\varphi : k[x_1, \ldots, x_n]/I(X) \to \mathcal{O}(X),$$

and clearly φ is injective. We now show that it is surjective. Let $f \in \mathcal{O}(X)$, we know we can cover X by open sets $U_\alpha \subseteq X$ on which $f = \frac{g_\alpha}{h_\alpha}$ with g_α, h_α as polynomials in $k[x_1, \ldots, x_n]$, and $h_\alpha \neq 0$ everywhere on U_α. By Lemma 6.16, we can assume that each U_α is a standard open subset in X, i.e., $U_\alpha = \{k_\alpha \neq 0\} \subseteq X$ for some $k_\alpha \in k[x_1, \dots, x_n]$. Note that on U_α,

$$f = \frac{g_\alpha}{h_\alpha} = \frac{g_\alpha k_\alpha}{h_\alpha k_\alpha},$$

and this is still well-defined. Note that $\{k_\alpha \neq 0\} = \{h_\alpha k_\alpha \neq 0\} \subseteq X$. Therefore, we can replace h_α and k_α by $h_\alpha k_\alpha$ in our discussion. We now have polynomials g_α and h_α such that

$$X = \bigcup_\alpha \{h_\alpha \neq 0\}$$

and, on $\{h_\alpha \neq 0\}$, $f = \frac{g_\alpha}{h_\alpha}$. Note that $h_\alpha^2 \cdot f = g_\alpha h_\alpha$ on $\{h_\alpha \neq 0\} \subseteq X$, and also on $\{h_\alpha = 0\} \subseteq X$. Therefore, the equation is true on all of X.

Because $X = \bigcup_\alpha \{h_\alpha \neq 0\}$, we have $Z(h_\alpha^2 : \alpha \in \zeta) \subseteq X$ as the empty set $\varnothing$. By the Nullstellensatz, let $I = (h_\alpha : \alpha \in \zeta) \subseteq k[x_1, \dots, x_n]/I(X) = R/I(X)$, then it has $\operatorname{rad}(I) = R$. In particular, $I = R$. Therefore, 1 can be expressed as some finite sum of the forms $r_\alpha h_\alpha^2$ for some $r_\alpha \in R$. Hence, on all of X, $1 \cdot f = (\sum r_\alpha h_\alpha^2) \cdot f = \sum r_\alpha h_\alpha^2 f = \sum r_\alpha g_\alpha h_\alpha \in R = k[x_1, \dots, x_n]/I(X)$. $\square$

7 Lecture 7

Lemma 7.1. Let X be a quasi-projective algebraic set over k algebraically closed. $\mathcal{O}(X)$ is a ring, in fact a commutative reduced k-algebra.

Proof. The main point is to show that the sum and product of regular functions are still regular. Call our set U, then given functions $f_1, f_2 : U \to k$ that locally are of the form $\frac{g}{h}$ with $g, h \in k[x_1, \dots, x_n]$, both homogeneous of same degree d, with $h \neq 0$ of the given point p. Then say $f_1 = \frac{g_1}{h_1}$ near p and $f_2 = \frac{g_2}{h_2}$ near p. Obviously, $f_1 f_2 = \frac{g_1 g_2}{h_1 h_2}$ where the numerator and the denominator are homogeneous of the same degree, and the denominator is still non-zero at this point. The sum is similar: $\frac{g_1}{h_1} + \frac{g_2}{h_2} = \frac{g_1 h_2 + h_1 g_2}{h_1 h_2}$, and therefore we have the same argument. $\square$

Lemma 7.2. For a quasi-projective algebraic set X over k, a morphism $f : X \to \mathbb{A}_k^n$ is equivalent to a list of n regular functions $f_1, \dots, f_n$ on X.

Proof. Clearly, a function $U \to \mathbb{A}_k^n = k^n$ is equivalent to a list of n functions $U \to k$, i.e., $f(x) = (f_1(x), \dots, f_n(x))$. If f is a morphism, then the pullbacks of the n regular functions, $x_1, \dots, x_n \in \mathcal{O}(\mathbb{A}_k^n) = k[x_1, \dots, x_n]$, so $f_1, \dots, f_n$ are regular functions on X.

Conversely, suppose $f_1, \dots, f_n$ are regular functions on $X = U$. To show that $f(x) = (f(x_1), \dots, f(x_n))$ is a morphism $U \to \mathbb{A}_k^n$ over k, let $V \subseteq \mathbb{A}_k^n$ be open and let $g \in \mathcal{O}(U)$.

(One can check that f is indeed continuous.) To show that $i(g) = g \circ f$ is regular on $f^{-1}(V)$, here g can be written locally as $\frac{h}{k}$, with h, k polynomials near each point $p \in U$ with $k(p) \neq 0$. We want to show that $\frac{h(f_1,\ldots,f_n)}{k(f_1,\ldots,f_n)}$ is regular on $f^{-1}(V)$, so one has to write this as a ratio of homogeneous polynomials of the same degree, using that each function is of that form (near p). $\square$

Remark 7.3. For a quasi-affine algebraic set $Y \subseteq \mathbb{A}_k^n$ and X a quasi-projective algebraic set X over k, a morphism $f : X \to Y$ is equal to n regular functions $f_1, \ldots, f_n \in \mathcal{O}(X)$ such that $(f_1(x), \ldots, f_n(x)) \in Y$ for every $x \in X$.

Remark 7.4. The morphisms of quasi-projective algebraic sets over k form a category.

Definition 7.5 (Isomorphism)**.** An *isomorphism* $f : X \to Y$ of quasi-projective algebraic set over k is a morphism $f : X \to Y$ that has a two-sided inverse.

Example 7.6. $X = \mathbb{A}_k^1 \backslash \{0\} \cong \{xy = 1\} \subseteq \mathbb{A}_k^2 = Y$. Note that X is quasi-affine and Y is affine.

Proof. Use the morphism $Y \to X$ by $(x, y) \mapsto x$ and $X \to Y$ by $x \mapsto (x, x^{-1})$, and this is well-defined since $x^{-1} \in \mathcal{O}(\mathbb{A}_k^1 \backslash \{0\})$. $\square$

Remark 7.7. Sometimes we say that a quasi-projective algebraic set is affine if it is isomorphic to an affine algebraic set, i.e., a closed subset of some $\mathbb{A}_k^n$.

Example 7.8. The hypersurface $\{x_n = f(x_1, \ldots, x_{n-1})\} \subseteq \mathbb{A}_k^n$ is isomorphic to $\mathbb{A}_k^{n-1}$, where f is any polynomial in $k[x_1, \ldots, x_{n-1}]$.

Example 7.9. Let $X \subseteq \mathbb{A}_k^n$ be an affine algebraic set over k (i.e., a closed subset of $\mathbb{A}_k^n$). Let $g \in \mathcal{O}(X)$, then the standard open subset $\{g \neq 0\}$ is affine, in fact it is isomorphic to $\{(x_1, \ldots, x_n, x_{n+1}) : x_{n+1}g(x_1, \ldots, x_n) = 1\} \subseteq \mathbb{A}_k^{n+1}$.

Proof. Map $U = \{g \neq 0\} \subseteq X$ by $(a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n, g(a_1, \ldots, a_n)^{-1}) \in Y$, then this is a morphism. The inverse morphism is given by $(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n) \in U = \{g \neq 0\}$. $\square$

Example 7.10. $\mathbb{A}_k^2 \backslash \{0\} = \{x_1 = 0\} \cup \{x_2 = 0\}$ is a quasi-affine algebraic set which is not affine.

Corollary 7.11. Let $X \subseteq \mathbb{A}_k^n$ be an affine algebraic set (i.e., closed in $\mathbb{A}_k^n$), and let $g \in \mathcal{O}(X)$, then $\mathcal{O}(\{g \neq 0\}) = \mathcal{O}(X)[\frac{1}{g}]$.

Proof. A morphism $f : X \to Y$ of quasi-projective algebraic sets induces a k-algebraic homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. Therefore, an isomorphism $f : X \to Y$ induces an isomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ of k-algebras. Therefore,

$$\begin{aligned}\mathcal{O}(\{g \neq 0\}) &= \mathcal{O}(\{x_{n+1}g(x_1, \ldots, x_n) = 1\} \subseteq \mathbb{A}_k^{n+1}) \\ &= k[x_1, \ldots, x_{n+1}]/(f_1(x_1, \ldots, x_n), \ldots, f_r(x_1, \ldots, x_n), x_{n+1}g(x_1, \ldots, x_n) - 1) \\ &= \mathcal{O}(X)[x_{n+1}]/(x_{n+1}g(x_1, \ldots, x_n) - 1) \\ &\cong \mathcal{O}(X)[\frac{1}{g}].\end{aligned}$$

□

Theorem 7.12. The correspondence mentioned in the proof above can be formalized. Let $f : X \to Y$ be a morphism of quasi-projective algebraic sets over an algebraically closed field k. f determines a pullback homomorphism of k-algebras $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. Moreover, if Y is affine (i.e., isomorphic to a closed subset of some $\mathbb{A}_k^n$), then this construction gives a one-to-one correspondence between morphisms $X \to Y$ and k-algebra homomorphisms $\mathcal{O}(Y) \to \mathcal{O}(X)$. It follows that if both X and Y are affine, then X and Y are isomorphic if and only if the k-algebras $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are isomorphic.

8 Lecture 8

Lemma 8.1. Let $X \subseteq \mathbb{A}_k^{n+1}$ be a cone (that is, X is closed and is a union of lines through 0), then the ideal $I(X) \subseteq k[x_0, \ldots, x_n]$ is homogeneous.

Proof. We have to say: for any $f \in I(X)$, if we write $f = f_0 + \ldots + f_d$ with f_i homogeneous of degree i, then f_i should be in $I(X)$.

Let $(a_0, \ldots, a_n)$ be a point in X, then we know that (because X is a cone and $f \in I(X)$) $f(ca_0, \ldots, ca_n) = 0$ for all $c \in k$. In particular, $f_0(a_0, \ldots, a_n) + cf_1(a_0, \ldots, a_n) + \cdots + c^d f_d(a_0, \ldots, a_n)$. Note that every term is in k, but as polynomial in c, this polynomial $g(c) \in k[c]$ such that $g(c) = 0$ for all $c \in k$. Hence, all its coefficients are 0.

Since k is algebraically closed, it is infinite. So $g = 0 \in k[c]$, that is, $f_i(a_0, \ldots, a_n) = 0$ for each $0 \leq i \leq d$. Since $(a_0, \ldots, a_n) \in X$ are arbitrary, $f_i \in I(X)$, so the ideal $I(X)$ is homogeneous. □

Remark 8.2. Note that the zero set in P^n of the ideal $(x_0, \ldots, x_n)$ in $k[x_0, \ldots, x_n]$ since $[0, \ldots, 0]$ is not a point in P^n. We get a one-to-one correspondence between homogenous prime ideals that are not $(x_0, \ldots, x_n)$ (called the *irrelevant ideal*), and irreducible closed subsets of P_k^n.

Definition 8.3 (Local Ring)**.** Let X be a quasi-projective algebraic set over k algebraically closed. Then for a point $p \in X$, the *local ring* of X at p is

1. an equivalence class of pairs (U, f) with open $p \in U \subseteq X$ and $f \in \mathcal{O}(U)$, with $(U, f) \sim (V, g)$ if there is an open neighborhood $p \in W \subseteq U \cap V$ such that $f\mid_W = g\mid_W$. (That is, an element of $\mathcal{O}_{X,p}$ is a germ of regular functions at p.)

2. The direct limit $\varinjlim_{p \in U \subseteq X} \mathcal{O}(U)$, i.e., with $p \in U \subseteq V \subseteq X$, there is a restriction map $\mathcal{O}(V) \to \mathcal{O}(U)$.

Lemma 8.4. $\mathcal{O}_{X,p}$ is a local ring.

Proof. That is, we want to show that $\mathcal{O}_{X,p}$ has exactly one maximal ideal. Equivalently, $\mathcal{O}_{X,p}$ has a maximal ideal $\mathfrak{m}$ such that for all $f \in \mathcal{O}_{X,p} \backslash \mathfrak{m}_{X,p}$, then $f \in \mathcal{O}^*_{X,p}$. Let $\mathfrak{m} = \ker(\mathcal{O}_{X,p} \to k)$, i.e., the kernel of the evaluation at p. One can see this is surjective (using constant functions), then let $f \in \mathcal{O}_{X,p} \backslash \mathfrak{m}$, then we can view $f \in \mathcal{O}(U)$ for some open set $p \in U \subseteq X$. Then $\{f \neq 0\} \subseteq U$ is an open subset of X containing p, so $\frac{1}{f} \in \mathcal{O}(V)$, hence $\frac{1}{f} \in \mathcal{O}_{X,p}$. □

Lemma 8.5. Let X be an affine algebraic set over k, then for a point $p \in X$ with $\mathfrak{m} = \ker(\mathcal{O}_{X,p} \to k)$ as the evaluation map at p, then $\mathcal{O}_{X,p} = \mathcal{O}(X)_\mathfrak{m}$ as the localization.

Proof. For a commutative ring R and prime ideal $\mathfrak{p} \subseteq R$, an element of the localization $R_\mathfrak{p}$ can be written as $\frac{a}{b}$ with $a \in R$ and $b \in R \backslash \mathfrak{p}$. So an element of $\mathcal{O}(X)_\mathfrak{m}$ is a fraction $\frac{a}{b}$ with $a \in \mathcal{O}(X)$ and $b \in \mathcal{O}(X)$ with $b(p) \neq 0$. Therefore $\frac{a}{b} \in \mathcal{O}(\{b \neq 0\})$ hence is contained in $\mathcal{O}_{X,p}$. □

Remark 8.6. Recall that $\mathcal{O}(\{g \neq 0\}) = \mathcal{O}[x][\frac{1}{g}]$.

Remark 8.7. An isomorphism $f : X \to Y$ of quasi-projective algebraic sets over k induces an isomorphism of local rings $\mathcal{O}_{Y,f(p)} \cong \mathcal{O}_{X,p}$.

Definition 8.8 (Dimension Near a Point)**.** Let $X \subseteq \mathbb{A}^n_k$ be a closed subset, write $I(X) = (f_1, \dots, f_r) \in k[x_1, \dots, x_n]$, and let $p \in X$. Let m be the dimension of X near p, i.e., the dimension of U for all small enough open neighborhoods of p.

Remark 8.9. If X is irreducible, then it has the same dimension near every point. Note that we can define derivatives of polynomials manually:

$$\frac{\partial}{\partial x_j}(x_1^{i_1}, \dots, x_n^{i_n}) = i_j x_1^{i_1} \dots x_j^{i_j - 1} \dots x_n^{i_n}$$

Note that we have a unique ring homomorphism $\mathbb{Z} \to k$, and can be viewed as a polynomial in $k[x_1, \ldots, x_n]$.

We have

$$\frac{\partial}{\partial x}(fg) = f\frac{\partial g}{\partial x} + \frac{\partial f}{\partial x}g$$

and etc.

Remark 8.10. If k has characteristic $p > 0$, then $p = 0 \in k$, so $\frac{\partial}{\partial x}(x^p) = px^{p-1} = 0 \in k[x]$. We now get a $n \times r$ matrix in k, of the form $\left(\frac{\partial f_i}{\partial x_j} \mid_p\right)$, and therefore a map $A^n \to A^r$.

Definition 8.11 (Smooth)**.** $X \subseteq \mathbb{A}^n_k$ is *smooth* over k at $p \in X(k)$ if the matrix $D_p = \left(\frac{\partial f_i}{\partial x_j} \mid_p\right)$ has rank $n - m$ where m is the dimension of X near p.

Definition 8.12 (Zariski Tangent Space)**.** The *Zariski tangent space* is defined to be $T_{X,p} = \ker(D_p : k^n \to k^r)$. The smoothness of X at p means that (X, p) has dimension $\dim(X)$ near p. Note that we always have a $\geq$ relation.

Example 8.13. Let $X = \{xy = 0\} \subseteq \mathbb{A}^2_k$. Where is X smooth? Let $(a, b) \in X(k)$, then the matrix $D_p = \left(\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y}\right) \mid_{(a,b)} = (y\ x) \mid_{a,b} = (b\ a) \in M_{1\times 2}(k)$. Therefore, X is smooth if and only if this matrix has rank 1 (note that it always has rank at most 1), if and only if $a \neq 0$ or $b \neq 0$.

Thus, X is smooth (of dimension 1) everywhere except $(0, 0)$.

Example 8.14. Where is the curve $X = \{xy = 1\} \subseteq \mathbb{A}^2_K$ smooth?

The matrix of derivatives is (write $f = xy - 1$) $(y\ x)$, and so X is smooth at (x, y) if and only if $(x, y) \neq (0, 0)$. But $(0, 0)$ is not on the curve, so X is smooth everywhere.

9 Lecture 9

Remark 9.1.

1. Smoothness does not depend on the choice of generators $g_1, \ldots, g_r$.
2. This "commutes with localization".
3. Smoothness is preserved by isomorphisms.

Example 9.2 (Zariski Tangent Space)**.** Consider $X = \{xy = 0\} \subseteq \mathbb{A}^2_k$, then at every point $x \in X$, we define a vector space $T_pX \subseteq k^n$ for $X \subseteq \mathbb{A}^n_k$. The tangent space is two-dimensional at the origin, and is one-dimensional everywhere else.

Definition 9.3 (Presheaf)**.** Let X be a topological space. A *presheaf* of Abelian groups on X is an Abelian group $A(U)$ for every open set $U \subseteq X$, together with restriction homomorphisms $r^V_U : A(V) \to A(U)$ for every open $U \subseteq V \subseteq X$, such that

- $r_U^U = 1_{A(U)}$ for every $U \subseteq X$,
- $r_U^W = r_U^V r_V^W$ for open $U \subseteq V \subseteq W \subseteq X$ as homomorphism $A(W) \to A(U)$.

Example 9.4. Let X be a topological space. Let $C(U)$ be the presheaf of continuous $\mathbb{R}$-valued functions.

Example 9.5. Let X be C^∞-manifold, then we have the presheaf of C^∞ (smooth) $\mathbb{R}$-valued functions.

Example 9.6. Let X be a complex manifold. We have the presheaf $\mathcal{O}_{an}$ of $\mathbb{C}$-analytic functions (on open subsets of X). For instance, if $X = \mathbb{C}P^1$, then $\mathcal{O}_{an}(X) = \mathbb{C}$.

Example 9.7. Let X be a quasi-projective algebraic set over k algebraically closed, then we have the presheaf $\mathcal{O}_X$ of regular functions.

Remark 9.8. We may call $A(U)$ the Abelian group of section of A on U.

Remark 9.9. Let X be a topological space. Define a category $\mathbf{Top}(X)$ with objects the open subsets of X, and $\mathbf{Hom}_{\mathbf{Top}(X)}(U, V) = \begin{cases} *, & \text{if } U \subseteq V \\ \varnothing, & \text{if } U \nsubseteq V \end{cases}$. A presheaf of Abelian groups on X is exactly a contravariant functor $\mathbf{Top}(X) \to \mathbf{Ab}$.

Definition 9.10 (Sheaf)**.** A *sheaf* of Abelian groups on a topological space X is a presheaf A of Abelian groups such that

- for every open set $U \subseteq X$ and every open cover $\{U_\alpha\}_{\alpha \in I}$ of U if $a, b \in A(U)$ such that $a \mid_{U_\alpha} = b \mid_{U_\alpha}$ for every $\alpha \in I$, then $a = b \in A(U)$,
- for every open set $U \subseteq X$ and every open cover $\{U_\alpha\}_{\alpha \in I}$ of U, for any collection of $a_\alpha \in A(U_\alpha)$ for all $\alpha \in I$, if $a_\alpha \mid_{U_\alpha \cap U_\beta} = a_\beta \mid_{U_\alpha \cap U_\beta}$ for all $\alpha, \beta \in I$, then there is an $a \in A(U)$ such that $a \mid_{U_\alpha} = a_\alpha$ for all $\alpha \in I$.

Remark 9.11. If A is a sheaf, then the $a \in A(U)$ described in the second property is unique, given by the first property.

Example 9.12. The presheaves described above are sheaves.

Remark 9.13. If A is a sheaf, then $A(\varnothing) = 0$ is the trivial Abelian group.

Proof. Take $U = \varnothing$, notice that U is covered by no open subsets. □

Example 9.14. Let A be an Abelian group and X be a topological space. The constant presheaf T_A on X is defined by $T_A(U) = A$ for every open $U \subseteq X$. This is not a sheaf if $A \neq 0$, since $T_A(\varnothing) = A$, not 0.

Example 9.15. Let A be an Abelian group on a space X. Define a presheaf S_A on X by $S_A(U) = \begin{cases} 0, & \text{if } V = \varnothing \\ A, & \text{otherwise} \end{cases}$. This is not a sheaf, for many spaces X, e.g., $X = \mathbb{R}$ with classical topology. Take the real line $\mathbb{R}$, and two disjoint open subsets U_1 and U_2, then let $U = U_1 \cup U_2 \subseteq \mathbb{R}$. Now $7 \in S_{\mathbb{Z}}(U_1)$ and $8 \in S_{\mathbb{Z}}(U_2)$, then the sections agree on the intersection, but there is not $a \in S_{\mathbb{Z}}(U_1 \cup U_2) = \mathbb{Z}$ that restricts to both 7 and 8.

Example 9.16. For a topological space X and Abelian group A, the sheaf A_X of locally constant A-valued functions on X is $A_X(U)$, the set of functions $f : U \to A$ for $U \subseteq X$ open that are locally constant, i.e., for every $p \in U$, there exists $p \in V \subseteq U$ such that $f\mid_V$ is constant.

Definition 9.17 (Stalk)**.** Let A be a presheaf on a space X. The *stalk* of A at a point $p \in X$ is $A_p = \varinjlim_{p \in U \subseteq X} A(U)$ for any open U of X containing p. That is, an element A_p is a germ of section of A at p.

Example 9.18. For a quasi-projective algebraic set X over k, the stalk $\mathcal{O}_{X,p}$ is exactly the local ring of X at p.

Definition 9.19 (homomorphism of presheaves)**.** A *homomorphism of presheaves* of Abelian groups A and B on a space X is a natural transformation $A \to B$ (as contravariant functors on $\mathbf{Top}(X)$): for every open $U \subseteq X$ we are given a homomorphism $f_U : A(U) \to B(U)$ of Abelian groups such that for every open inclusion $U \subseteq V$, the diagram

$$\begin{tikzcd} A(V) \arrow[r] \arrow[d] & B(V) \arrow[d] \\ A(U) \arrow[r] & B(U) \end{tikzcd}$$

commutes.

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