Motivic Cohomology Notes

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0 Introduction

Let $X \in \mathrm{Sm}/k$ be a smooth separated scheme over a field k. The study of motivic cohomology started with the hope that

Conjecture 0.1 (Beilinson and Lichtenbaum, 1982-1987)**.** There exists certain complexes $\mathbb{Z}(n)$ for $n \in \mathbb{N}$ of sheaves in Zariski topology on Sm/k such that

1. $\mathbb{Z}(0)$ is (quasi-isomorphic to) the constant sheaf $\mathbb{Z}$, i.e., the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$

concentrated at degree 0;

2. $\mathbb{Z}(1)$ is the complex $\mathcal{O}^*[-1]$, i.e., the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{O}^* \longrightarrow 0 \longrightarrow \cdots$$

concentrated at degree 1;

3. for every field F/k, the hypercohomology[1]

$$\mathbb{H}^n_{\mathrm{Zar}}(F, \mathbb{Z}(n)) = H^n(\mathbb{Z}(n)(\mathrm{Spec}(F))) = K_n^M(F),$$

where $K_n^M(F)$ is the nth Milnor K-theory of a field F, given by the quotient of the tensor algebra $T(F^*)/\{x \otimes (1-x) : x \in F^*\}$ over $\mathbb{Z}$; (lecture 5 of [MVW06], page 29)

Example 0.2.

a. $K_0^M(F) = K_0(F) = \mathbb{Z}$;
b. $K_1^M(F) = K_1(F) = F^\times$;
c. $K_2^M(F) = K_2(F)$.

4. $\mathbb{H}^{2n}_{\mathrm{Zar}}(X, \mathbb{Z}(n)) = \mathrm{CH}^n(X)$ (lecture 17 of [MVW06], page 135), where the nth classical Chow group $\mathrm{CH}^n(X)$ is the free group given by

$$\mathrm{CH}^n(X) = \mathbb{Z}\{\text{cycles of codimension } n\}/\text{rational equivalences};$$

[1] Here we use the convention that the (hyper)cohomology of F should be interpreted as of $\mathrm{Spec}(F)$, the corresponding space.

5. there is a natural Atiyah–Hirzebruch spectral sequence

$$E_2^{p,q} = \mathbb{H}^p_{\text{Zar}}(X, \mathbb{Z}(q)) \Rightarrow K_{2q-p}(X).$$

Moreover, tensoring with $\mathbb{Q}$, the spectral sequence degenerates and one has

$$\mathbb{H}^i_{\text{Zar}}(X, \mathbb{Z}(n))_{\mathbb{Q}} = \text{gr}^n_\gamma(K_{2n-i}(X)_{\mathbb{Q}})$$

where gr^n_γ's are the quotients (graded pieces) of γ-filtration. ([Lev94]; [Lev99], Theorem 11.7)

Remark 0.3. Such choices of complexes $\mathbb{Z}(q)$ exists, and is called the motivic complex. For a clear definition of these complexes, see Lecture 3 of [MVW06]. Moreover, by convention $\mathbb{Z}(q) = 0$ for $q < 0$.

Definition 0.4. The motivic cohomology of X is defined by $H^{p,q}(X, \mathbb{Z}) = \mathbb{H}^p_{\text{Zar}}(X, \mathbb{Z}(q))$, the hypercohomology of the motivic complexes with respect to the Zariski topology.

Remark 0.5. In general, a motivic cohomology with coefficient in an abelian group A is a family of contravariant functors $H^{p,q}(-, A) : \text{Sm}/k \to \text{Ab}$.

Remark 0.6. The motivic cohomology of X satisfies the cancellation property: set $\mathbb{G}_m = \mathbb{A}^1 \backslash \{0\}$, then

$$H^{p,q}(X \times \mathbb{G}_m, \mathbb{Z}) = H^{p,q}(X, \mathbb{Z}) \oplus H^{p-1,q-1}(X, \mathbb{Z}).$$

Remark 0.7. It turns out that the group remains unchanged if we replace the Zariski topology by Nisnevich topology.[2] If one uses étale topology instead, we retrieve Lichtenbaum motivic cohomology $H^{p,q}_L(X, \mathbb{Z})$. If $\text{char}(k) \nmid n$, it admits the comparison

$$H^{p,q}_L(X, \mathbb{Z}/n\mathbb{Z}) = H_{\text{étale}}(X, \mathbb{Z}/n\mathbb{Z}(q)),$$

where $\mathbb{Z}/n\mathbb{Z}(q)$ is the q-twist $\mu_n^{\otimes q}$.

We may compare Lichtenbaum motivic cohomology with motivic cohomology by the following theorem, formerly known as Beilinson-Lichtenbaum Conjecture[3]:

Theorem 0.8 ([Voe11])**.** The natural map

$$H^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) \to H^{p,q}_L(X, \mathbb{Z}/n\mathbb{Z})$$

is an isomorphism if $p \leqslant q$, is a monomorphism if $p = q + 1$, and gives a spectral sequence for any pair of p, q.

Corollary 0.9. For $p \leqslant q$, we have

$$H^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) = H^p_{\text{étale}}(X, \mathbb{Z}/n\mathbb{Z}(q)).$$

In particular, for $X = \text{Spec}(k)$ as a point, this is the theorem formerly known as Milnor conjecture:

Corollary 0.10 ([Voe97], [Voe03a], [Voe03b])**.**

- $H^{p,p}(k, \mathbb{Z}/n\mathbb{Z}) = K^M_p(k)/n = H^p_{\text{étale}}(X, \mathbb{Z}/n\mathbb{Z}(p))$ as the Galois cohomology;
- in general,

$$H^{p,q}(k, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} 0, & p > q \\ H^{p,p}(k, \mathbb{Z}/n\mathbb{Z}) \cdot \tau^{q-p}, & p \leqslant q \end{cases}$$

where $\tau \in \mu_n(k) = H^{0,1}(k, \mathbb{Z})$ is a primitive nth root of unity.

Remark 0.11. Unlike finite coefficients, $H^{p,q}(k, \mathbb{Z})$ is quite hard to compute for small $p < q$; for $p \geqslant q$, this is 0.

A current long-standing conjecture is

[2] Recall that the Nisnevich topology is a Grothendieck topology on the category of schemes that is finer than the Zariski topology but coarser than the étale topology.

[3] This is also known as the norm residue isomorphism theorem, or (formerly) Bloch-Kato conjecture.

Conjecture 0.12 (Beilinson-Soulé Vanishing Conjecture, [Lev93])**.** $H^{p,q}(k,\mathbb{Z})=0$ if $p<0$.

Remark 0.13. Here are a few known cases:

- for $\operatorname{char}(k)=0$, this is known for number fields ([Bor74]), function fields of genus 0 ([Dég08]), curves over number fields, and their inductive limits (more precise references required); ([DG05])
- for $\operatorname{char}(k)>0$, this is known for finite fields ([Qui72]) and global fields ([Har77]).

Remark 0.14. The motivic cohomology could be realized in a tensor triangulated category, namely the category of effective motives $\mathrm{DM}(k)$. For any pair of p,q, we can find an Eilenberg-Maclane space and a corresponding representable functor so that

$$H^{p,q}(X,\mathbb{Z})=\operatorname{Hom}_{DM}(\mathbb{Z}(X),\mathbb{Z}(q)[p])$$

where $\mathbb{Z}(X)$ is the motive of X and $\mathbb{Z}(q)[p]=\mathbb{G}_m^{\wedge q}[p-q]$.[4]

Remark 0.15. Dually, we can define the motivic homology by

$$H_{p,q}(X,\mathbb{Z})=\operatorname{Hom}_{DM}(\mathbb{Z}(q)[p],\mathbb{Z}(X)).$$

Remark 0.16 ([MVW06] Properties 14.5, page 110)**.** By taking the hom functor from the aspect of motives, we can derive theorems for all (co)homologies which can be represented in DM. The main derives are the following:

1. If $E\to X$ is an $\mathbb{A}^n$-bundle, then motives $\mathbb{Z}(E)=\mathbb{Z}(X)$ in DM.
2. If $\{U,V\}$ is a Zariski open covering of X, we have a Mayer-Vietoris sequence

$$\mathbb{Z}(U\cap V)\longrightarrow\mathbb{Z}(U)\oplus\mathbb{Z}(V)\longrightarrow\mathbb{Z}(X)\longrightarrow\mathbb{Z}(U\cap V)[1]$$

   in the form of a distinguished triangle in DM.
3. If $Y\subseteq X$ is a closed embedding of codimension c in Sm/k, then we have a Gysin triangle

$$\mathbb{Z}(X\backslash Y)\longrightarrow\mathbb{Z}(X)\longrightarrow\mathbb{Z}(Y)(c)[2c]\longrightarrow\mathbb{Z}(X\backslash Y)[1]$$

   which is a distinguished triangle where $\mathbb{Z}(Y)(c)[2c]:=\mathbb{Z}(Y)\otimes\mathbb{Z}(c)[2c]$.
4. For any vector bundle of rank n on X, we have the projective bundle formula

$$\mathbb{Z}(\mathbb{P}(E))=\bigoplus_{i=0}^{n}\mathbb{Z}(X)(i)[2i]$$

   which defines the Chern class of E.
5. Let X be a proper smooth scheme and let d_X be its dimension, then $\mathbb{Z}(X)$ has a strong dual $\mathbb{Z}(X)(-d_X)[-2d_X]$ in DM by stabilization. This gives a Poincaré duality[5]

$$H^{p,q}(X,\mathbb{Z})\cong H_{2d_X-p,d_X-q}(X,\mathbb{Z})$$

1 Intersection Theory

Definition 1.1. Let X be a scheme of finite type over k. We define the ith cycle on the scheme X to be a free abelian group

$$Z_i(X)=\bigoplus_{\substack{\text{irreducible closed } c\subseteq X\\ \text{with }\dim(c)=i}}\mathbb{Z}\cdot c$$

and set $Z(X)=\bigoplus_i Z_i(X)$. Define a set $K_i(X)$ to be the set of coherent sheaves $\mathcal{F}$ on X with $\dim(\operatorname{supp}(F))\leqslant i$.[6]

[4] Again, this notation goes back to the concise definition of the motivic complexes: see Lecture 3 from [MVW06] as well as the concept of presheaves with transfers.

[5] We can use cohomology with compact support for this.

[6] Despite the notation, this has nothing to do with a K-theory.

Remark 1.2. Let $(A, \mathfrak{m})$ be a Noetherian local ring and M be an A-module, then by the dimension theorem, we know $\dim(M) = d(M) = \dim(\operatorname{supp}(M))$, where $d(M)$ is the degree of the Hilbert-Samuel polynomial $P_{\mathfrak{m}}(M, n)$.

Definition 1.3. Let $X \in \mathrm{Sm}/k$ and let $U, V \subseteq X$ be irreducible and closed. Suppose $W \subseteq U \cap V$ is a irreducible and closed component. If $\dim(W) = \dim(U) + \dim(V) - \dim(X)$, i.e., $\operatorname{codim}(W) = \operatorname{codim}(U) + \operatorname{codim}(V)$, we say that U and V intersect properly at W.

Remark 1.4. This condition is weaker than saying they intersect transversely: we do not require information about tangent spaces.

Theorem 1.5. Let $A \supseteq k$ be a Noetherian regular ring, M, N be finitely-generated A-modules, and suppose $\ell(M \otimes_A N) < \infty$, then

1. $\ell(\operatorname{Tor}_i^A(M, N)) < \infty$ for all $i \geqslant 0$;
2. the Euler-Poincaré characteristic $\chi(M, N) := \sum_{i=0}^{\dim(A)} (-1)^i \ell(\operatorname{Tor}_i^A(M, N)) \geqslant 0$;
3. by Remark 1.2, we have $\dim(M) + \dim(N) \leqslant \dim(A)$;
4. in particular, we have $\dim(M) + \dim(N) < \dim(A)$ if and only if $\chi(M, N) = 0$.

Proof. See [Ser12], page 106. □

Remark 1.6. Part 3. from Theorem 1.5 implies that $\dim(W) \geqslant \dim(U) + \dim(V) - \dim(X)$, i.e., $\operatorname{codim}(W) \leqslant \operatorname{codim}(U) + \operatorname{codim}(V)$ in the notation of Definition 1.3.

Definition 1.7. Let X, U, V, W be as in Definition 1.3, then we define the intersection multiplicity $m_W(U, V)$ of U and V at W by
$$m_W(U, V) = \chi^{\mathcal{O}_{X,W}}(\mathcal{O}_{X,W}/P_U, \mathcal{O}_{X,W}/P_V)$$
where P_U and P_V are prime ideals defining U and V, respectively.

Remark 1.8. By Theorem 1.5, we know $m_W(U, V) \geqslant 0$, and $m_W(U, V) = 0$ if and only if U and V do not intersect properly at W.

Definition 1.9. Let $X \in \mathrm{Sm}/k$, and let $U \in Z_a(X)$ and $V \in Z_b(X)$. If U and V intersect properly at every component, then we define the intersection product to be the cycle
$$U \cdot V = \sum_{\substack{W \subseteq U \cap V \\ \dim(W) = a+b-d_X}} m_W(U, V) \cdot W \in Z_{a+b-d_X}(X).$$

Example 1.10. Let X be a smooth projective surface, and let C and D be divisors on X. For any point $x \in C \cap D$, locally we think of $C = \{f = 0\}$ and $D = \{g = 0\}$ around x, then $m_x(C, D) = \ell_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(f, g))$.

Definition 1.11. Suppose X is a scheme of finite type over k, and $\mathcal{F} \in K_n(X)$ is a coherent sheaf, then we define $Z_a(\mathcal{F}) = \sum_{\dim(\overline{\eta})=a} (\mathcal{O}_{X,\eta}(\mathcal{F}_\eta) \cdot \overline{\eta}) \in Z_a(X)$.

Therefore, we define the cycle of F as an element of the cycle of X.

Definition 1.12 ([Har13], Exercise III.6.9). Every coherent sheaf $\mathcal{F}$ on $X \in \mathrm{Sm}/k$ has a resolution
$$0 \longrightarrow E_k \longrightarrow E_{k-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow \mathcal{F} \longrightarrow 0$$
where E_i's are locally free of finite rank. Therefore, for any coherent sheaf $\mathcal{G}$, we can define the Tor functor[7] of coherent sheaves by
$$\operatorname{Tor}_i(\mathcal{F}, \mathcal{G}) = H_i(E_* \otimes_{\mathcal{O}_X} \mathcal{G}).$$

[7] Since we are working over sheaves of $\mathcal{O}_X$-modules, using the same argument on the level of modules shows that the Tor functor is independent from the choice of resolution.

Proposition 1.13. Let $X \in \mathrm{Sm}/k$. Suppose $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(X)$ intersect properly, then

$$Z_a(\mathcal{F}) \cdot Z_b(G) = \sum_{i=0}^{d_X} (-1)^i \cdot Z_{a+b-d_X}(\mathrm{Tor}_i(\mathcal{F}, \mathcal{G})).$$

Proof. We only have to do it locally, so we can assume X to be affine, and count the coefficients of $\bar{\xi}$ where $\dim(\xi) = a + b - d_X$. It sufficess to show that the stalks at ξ satisfies

$$\chi(F_\xi, G_\xi) = \sum_{\substack{\dim(\bar{\lambda})=a \\ \dim(\beta\eta)=b \\ \xi \in \bar{\lambda} \cap \bar{\eta}}} \ell(\mathcal{F}_\lambda) \cdot \ell(G_\eta) \cdot m_{\bar{\xi}}(\bar{\lambda}, \bar{\eta}).$$

Because our ring is Noetherian, then $\mathcal{F}$ admits a filtration

$$0 = M_0 \subseteq \cdots \subseteq M_d = \mathcal{F}$$

such that $M_i/M_{i-1} \cong \mathcal{O}_X/\mathcal{I}$ is coherent for prime ideal $\mathcal{I}$. By the additivity of both sides of the isomorphism, we may assume $\mathcal{F} = \mathcal{O}_X/\mathfrak{p}$ with dimension at most a, where $\mathfrak{p} \sim \lambda \in X$. Similarly, we may assume $\mathcal{G} = \mathcal{O}_X/\mathfrak{q}$ with dimension at most b, where $\mathfrak{q} \sim \eta \in X$. Moreover, set $\xi \in \bar{\lambda} \cap \bar{\eta}$. By definition, we now have $\chi(\mathcal{F}_\xi, \mathcal{G}_\xi) = m_{\bar{\xi}}(\bar{\lambda}, \bar{\eta})$.

- If $\dim(\bar{\lambda}) = a$ and $\dim(\bar{\eta}) = b$, then the equality follows from the fact that $\ell(\mathcal{F}_\lambda) = \ell(\mathcal{G}_\eta) = 1$.
- If not, then either $\dim(\bar{\lambda}) < a$ or $\dim(\bar{\eta}) < b$, then $\bar{\lambda}$ and $\bar{\eta}$ do not intersect properly at $\bar{\xi}$, so both the left-hand side and the right-hand side become 0.

□

Proposition 1.14. The intersection product is commutative.

Proof. This is obvious since the Tor functor is commutative. □

Proposition 1.15. The intersection product is associative.

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