# Power Operations and Global Algebra

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## 1 November 13, 2024

**Background.** In chromatic homotopy theory, we have a notion of height that measures complexity. In the case of height 1, we have a completion of complex K-theory

$$K \to K_p^\wedge = E_1$$

which then builds up to higher heights with $E_2, E_3$, and so on. When goes on to the height level of $\infty$, we have a map $\mathbb{S} \to H\mathbb{F}_p$. When valued in finite groups, this gives rise to objects in global algebra, which are the representative ring functor. This corresponds to the Burnside ring functor in terms of $K$-theory and $E$-cohomology of classifying spaces in terms of the spectrum $\{E_i\}_{i \geqslant 1}$.

### 1.1 The Complex Representation Ring

**Definition 1.1.** A $G$-representation is a finite-dimensional $\mathbb{C}$-vector space equipped with an action of $G$. A map $f : V \to W$ of $G$-representations is an equivariant linear map: $g \cdot f(v) = f \cdot g(v)$ for $g \in G$ and $v \in V$.

Given two $G$-representations $V$ and $W$, we may build $G$-representations $V \oplus W$ and $V \otimes W$ with respect to the $G$-diagonal action.

**Definition 1.2.** Let $[V]$ be the isomorphism class of $G$-representation $V$. We may define addition and multiplication of $G$-representations $V$ and $W$ as

$$[V] + [W] = [V + W] \qquad [V][W] = [V \otimes W].$$

This gives rise to a symmetric monoidal structure, only lacking the additive inverses.

Taking the Grothendieck construction, we may fill in the additive inverses. Let $\mathrm{RU}(G)$ be the Grothendieck ring of the isomorphism class of $G$-representations under addition and multiplication above.

**Lemma 1.3** (Schur)**.** If $V$ and $W$ are irreducible $G$-representations, i.e., no non-trivial $G$-subrepresentations, then

1. if $V \ncong W$ as $G$-representations, and $f : V \to W$ is a map of $G$-representations, then $f \equiv 0$;
2. if $V \cong W$, then any map $f : V \to V$ of $G$-representations must be defined by multiplication by a scalar.

**Fact 1.4.** Since every $G$-representation is a sum of irreducible $G$-representations in a unique way, then $\mathrm{RU}(G)$ is (additively) a free $\mathbb{Z}$-module with canonical basis given by the set of isomorphism classes of irreducible $G$-representations.

Therefore, $\mathrm{RU}(G)$ is quite simple with respect to the additive structure. However, it takes more effort to understand the ring multiplicatively.

**Example 1.5.** Let $e$ be the trivial group, then the isomorphism classes are given by $\mathbb{N}$, so taking the Grothendieck completion gives $\mathrm{RU}(e) \cong \mathbb{Z}$.

**Example 1.6.** Assume $A$ is an abelian group and $V$ is an irreducible $A$-representation. For $a \in A$, the action map $a : V \to V$ is a map of $A$-representations. Since $V$ is irreducible, then by Lemma 1.3, we know the map $a$ is described by $av = cv$ for some $c \in \mathbb{C}$. Therefore, the subspace $\langle v \rangle$ is a subrepresentation of $V$, hence $V = \langle v \rangle$. That is, $\dim(V) = 1$.

**Example 1.7.** Consider $A = C_n \subseteq S^1 \subseteq \mathbb{C}$, then $A$ inherits an $\mathbb{C}$-action. In particular, the action $\rho : C_n \times \mathbb{C} \to \mathbb{C}$ is such that $\rho^{\otimes n} = \mathrm{triv}$ and the tensor powers give $n$ irreducible representations. Therefore, $\mathrm{RU}(C_n) \cong \mathbb{Z}[x]/(x^n - 1)$ where $x = [\rho]$.

**Remark.** The spectrum $\mathrm{Spec}(\mathbb{Z}[x]/(x^n - 1)) \cong \mathbb{G}_m[n]$ is the $n$-torsion of the multiplicative group.

**Example 1.8.** Consider the free $\mathbb{C}$-vector space $\mathbb{C}\{C_n\}$ based on the cyclic group $C_n$ has a $C_n$-action. This is then called the regular representation. Since it can be written as a sum of irreducible representations, then one can show that

$$\mathbb{C}\{C_n\} \cong \bigoplus_{i=0}^{n-1} \rho^{\otimes i}.$$

Alternatively,

$$[\mathbb{C}\{C_n\}] = 1 + x + x^2 + \cdots + x^{n-1}$$

in the context of representation ring.

It is now natural to ask: how do representation rings interact as the group varies?

## 1.2 Restrictions and Transfers

Let $f : H \to G$ be a map of groups, then

- there is a (contravariant) restriction map $\mathrm{Res}_f : \mathrm{RU}(G) \to \mathrm{RU}(H)$: given $G$-representation $V$, we can send this to $H \xrightarrow{f} G$ acting on $V$, so thinking of $V$ as an $H$-representation. In particular, the restriction map above is a ring map;
- we can also define a (covariant) transfer map $\mathrm{Tr}_f : \mathrm{RU}(H) \to \mathrm{RU}(G)$: given $H$-representation $V$, we may notice that it is the same thing as a $\mathbb{C}[H]$-module over the group ring, then by base-change, we consider it as $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ as a $G$-representation. This map is not a ring map: it is additive but not multiplicative in general.

**Example 1.9.** Consider the trivial map $i : e \to G$, then this corresponds to a restriction map

$$\begin{aligned} \mathrm{Res}_i : \mathrm{RU}(G) &\to \mathbb{Z} \\ V &\mapsto \dim(V) \end{aligned}$$

that describes the dimension, and a transfer map

$$\begin{aligned} \mathrm{Tr}_i : \mathbb{Z} &\to \mathrm{RU}(G) \\ \mathbb{C} &\mapsto \mathbb{C}[G] \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}[G] \end{aligned}$$

as the regular representation.

The restriction and transfer map interacts via the Frobenius reciprocity and a double coset formula.

**Theorem 1.10** (Frobenius Reciprocity). Given $x \in \mathrm{RU}(G)$ and $y \in \mathrm{RU}(H)$, then $\mathrm{Tr}_f(\mathrm{Res}_f(x)y) = x\,\mathrm{Tr}_f(y)$. That is, the transfer map is a map of $\mathrm{RU}(G)$-modules for a module structure on $\mathrm{RU}(H)$ given by restriction along $f$.

**Theorem 1.11** (Double Coset Formula). Given subgroups $H, K \subseteq G$, then

$$\mathrm{Res}^G_K \mathrm{Tr}^G_H = \sum_{[g] \in K\backslash G/H} \mathrm{Tr}^K_{K \cap H^{g^{-1}}} c_g \mathrm{Res}^{K^g \cap H}_H$$

where $c_g$ is a conjugation action.

**Example 1.12.** Suppose $k \mid n$ and consider $f : C_k \to C_n$, then

$$\begin{aligned} \mathrm{Res}_f : \mathrm{RU}(C_n) \cong \mathbb{Z}[x]/(x^n - 1) &\to \mathrm{RU}(C_k) \cong \mathbb{Z}[x]/(x^k - 1) \\ x &\mapsto x \end{aligned}$$

is a surjection, and

$$\begin{aligned} \mathrm{Tr}_f : \mathrm{RU}(C_k) &\to \mathrm{RU}(C_n) \\ \mathbb{1} = [\mathbb{C}] &\mapsto [\mathbb{C}[C_n] \otimes_{\mathbb{C}[C_k]} \mathbb{C}] \cong [\mathbb{C}[C_n/C_k]] \end{aligned}$$

Since the restriction map is surjective and the transfer map is a map of modules, then the module structure implies that the transfer map is completely determined by the mapping of $\mathbb{1}$.

## 1.3 Character Theory

Let $G/\operatorname{conj}$ be the set of conjugacy classes of $G$. Let $\mathbb{Q}(\mu_\infty)$ be $\mathbb{Q}$ adjoining all roots of unity. Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be some $G$-representation, then the trace $\mathrm{Tr}(\rho(g))$ is a sum of roots of unity.

**Remark.** To see this, we note that every representation $\mathrm{GL}_n(\mathbb{C})$ can be conjugated to some representation of the unitary group, which can then be diagonalized. But $G$ has finite order, so the elements on the diagonal has to be some roots of unity. Alternatively, apply Jordan canonical form.

Furthermore, the trace function satisfies $\mathrm{Tr}(\rho(hgh^{-1})) = \mathrm{Tr}(\rho(g))$. So this process gives a map

$$\chi : \mathrm{RU}(G) \to \mathrm{Cl}(G, \mathbb{Q}(\mu_\infty)) = \mathrm{Fun}(G/\operatorname{conj}, \mathbb{Q}(\mu_\infty))$$

into the class functions.

**Fact 1.13.** $\chi$ is an injective ring map: we win by sending a complicated (multiplicative) structure into a much simpler structure, since the ring structure is defined pointwise. Moreover, the base-change

$$\mathbb{Q}(\mu_\infty) \otimes_{\mathbb{Z}} \mathrm{RU}(G) \to \mathrm{Cl}(G, \mathbb{Q}(\mu_\infty))$$

is an isomorphism. Even more: $\mathrm{Gal}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \cong \hat{\mathbb{Z}}^* := \varinjlim_n (\mathbb{Z}/n\mathbb{Z})^*$.

**Fact 1.14.** Here $\mathrm{Aut}(\hat{\mathbb{Z}}) \cong \hat{\mathbb{Z}}^*$ acts on $G/\operatorname{conj}$ naturally as $\mathrm{Hom}_{\mathrm{cts}}(\hat{\mathbb{Z}}, G)$.

Combining the two actions, we have an isomorphism

$$\mathbb{Q} \otimes \mathrm{RU}(G) \cong \mathrm{Cl}(G, \mathbb{Q}(\mu_\infty))^{\hat{\mathbb{Z}}^\times}.$$

**Example 1.15.** Let $G = \Sigma_m$, then we have a map

$$\mathrm{RU}(\Sigma_m) \to \mathrm{Cl}(\Sigma_m, \mathbb{Q}(\mu_\infty))^{\hat{\mathbb{Z}}^\times}.$$

A conjugacy class $[\sigma]$ of $\Sigma_m$ is determined completely by the cycle decomposition: given $\ell \in \hat{\mathbb{Z}}^*$ and $[\sigma] \in \Sigma_m/\operatorname{conj}$, we view $\ell \in (\mathbb{Z}/m!\mathbb{Z})^*$ and send $[\sigma]$ to $[\sigma^\ell]$ via $\ell$. In particular, $[\sigma] = [\sigma^\ell]$ have the same cycle decomposition. Therefore, the action of $\hat{\mathbb{Z}}^*$ on conjugacy classes must be trivial. Hence, the given map tells us that

$$\mathrm{Cl}(\Sigma_m, \mathbb{Q}(\mu_\infty))^{\hat{\mathbb{Z}}^\times} \cong \mathrm{Cl}(\Sigma, \mathbb{Q}).$$

Comparing this with $\mathrm{Cl}(\Sigma_m, \mathbb{Z})$, we notice that the trace map ensures the fractions of integers never appear in the image, therefore this map factors into $\mathrm{Cl}(\Sigma_m, \mathbb{Z})$.