Triangulated Category's Christmas Wish List

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Abstract

We will follow Balmer's work (as summarized in his survey [Bal10b] in 2010) and discuss the notion and ramifications of tensor triangulated categories. First, we will give some motivations of studying tensor triangulated category, including a few examples in algebraic geometry. We will give definition to a tensor triangulated category as well as a general description of the corresponding Balmer spectrum. Finally, we will talk about the geometric aspects over a tensor triangulated category, which motivates the study of tensor triangulated geometry.

List of things I may omit due to time constraints:

- Example 1.5,
- some proofs in Section 2,
- Section 3 as a whole.

1 Introduction

The goal of this talk is to explain what tensor-triangulated category is and why we care about it (as in tensor-triangulated geometry, as well as algebraic geometry). We abbreviate the term "tensor-triangulated" as "tt", i.e., tt-geometry, tt-category. I will assume people have seen derived categories and triangulated categories. A handy reference would be Chapter 10 of Weibel's book [Wei94]. It would also help if people know a bit about symmetric monoidal categories, c.f., [Kel82].

1.1 Idea

From the name of this subject, you can probably tell tt-categories has something to do with equipping a tensor operation structure on a triangulated category. This is absolutely true: in fact, we will give the category a monoidal structure with respect to this mysterious $\otimes$-operation. Generally speaking, tt-categories allow us to study tt-geometry, and for the latter concept we have the following slogan:

tt-geometry is basically commutative algebra over a good enough triangulated category, but not really.

The natural question now would be: where do we find the geometry on a tt-category?

1.2 Motivation

We first talk about examples of tt-category, in particular in commutative algebra and algebraic geometry. These examples usually involve two categories, a small category $\mathcal{K}$, with the property of being "compact", and a large, ambient category $\mathcal{T}$, with the property of being "triangulated". We usually assume $\mathcal{T}$ contains all coproducts.

Definition 1.1. An element $x \in \mathcal{T}$ is compact if $\mathrm{Hom}_{\mathcal{T}}(x, -)$ commutes with coproducts. The subcategory $\mathcal{K}$ of $\mathcal{T}$ is the subcategory of all compact objects.

The takeaway from the examples below is that we often want to construct the compact subcategory of a triangulated category as a tt-category. However, I will note discuss the role of compactness in the construction in this talk.

Example 1.2. Let A be a commutative ring, $\mathcal{T} = D(A\text{-}\mathbf{Mod})$, the derived category of A-modules. Spelling the definition out, $\mathcal{T}$ is the derived category of a Grothendieck category, made up of chain complexes of A-modules, with quasi-isomorphisms inverted. Due to Neeman (see [NB92]), we know $\mathcal{K} = D_{\text{perf}}(A)$, the derived category of perfect complexes of A, also known as $K_b(A\text{-}\mathbf{proj})$, the homotopy category of bounded complexes of finitely-generated projective A-modules. The maps in this complex are up to homotopy simply because quasi-isomorphisms between such complexes have to be homotopy-equivalent. $\mathcal{K}$ is now a triangulated category.

Remark 1.3. We have not actually used the fact that A is commutative, which only helps us to construct the symmetrical monoidal structure. In particular, the tensor product $- \otimes_A -$ induces a tensor product $\otimes_A^L$ on the category, given by the left derived functor of the derived category. In this case, the unit object $\mathbb{1}$ is $\mathcal{O}_X$, as a complex concentrated in degree 0.

In general, in algebraic geometry, we have the following example.

Example 1.4. Let X be a quasi-compact and quasi-separated scheme, so the underlying space $|X|$ has a quasi-compact open basis. For example, take affine $X = \operatorname{Spec}(A)$ for commutative ring A. Let $\mathcal{T} = D(X)$, which happens to be the derived category of complexes of $\mathcal{O}_X$-modules with quasi-coherent homology. $\mathcal{K}$ is still equivalent to $D_{\text{perf}}(X)$, i.e., those that are in $D_{\text{perf}}(A)$ for all affine $\operatorname{Spec}(A)$. Then $\mathcal{K}$ is a tt-category.

Example 1.5. Consider $i : \operatorname{Spec}(k) \hookrightarrow \operatorname{Spec}(A)$, where k is a field obtained as some factor ring of A. For now, let $A = \mathbb{Z}$ and $k = \mathbb{Z}/p\mathbb{Z}$. Let V be a finite-dimensional k-vector space, then it is an A-module i_*V with respect to the pushforwards. Here A acts on V by projecting onto k. The k-dual of V has k-dimension $\dim_k(V^*) = \dim_k(V)$, but if we look at the A-homomorphisms, we have $\operatorname{Hom}_A(i_*V, A) = 0$ because the module i_*V is killed by p via the torsions, so every element lands in some element killed by p in A, but there is no such element. Hence, we note that the dual lost the information. Regardless, note that this gives $(i_*V)^* = (i_*V)[-1]$, shifted by -1.

If we take $A = k[x_1, \dots, x_n]$ and $k = A/\langle x_1, \dots, x_n\rangle$, then $(i_*V)^* = (i_*V)[-n]$, shifted by $-n$.

We see that the shifting is exactly the difference in codimensions of spectrum, which gives a sense of some geometric information over the suspension.

Other examples (which can be found in [Bal10b]) include

- modular representation theory,
- stable homotopy theory,
- vector bundles in K-theory,
- motivic theory, etc.

The whole point being, there is a notion of geometry that survives from the derived category $D(X)$, e.g., something about the dimension of a scheme. But how much?

Remark 1.6 (Mukai, [Muk81], 1981)**.** There exists non-isomorphic schemes X and X' such that $D(X)$ and $D(X')$ are equivalent as triangulated categories. Note that this is not necessarily maintaining the $\otimes$-structure. The example illustrated in [Muk81] is the abelian variety with its dual.

Theorem 1.7 (Balmer, [Bal05], 2005)**.** Let X be a quasi-compact and quasi-separated scheme, then we obtain an isomorphism $\operatorname{Spec}(D_{\text{perf}}(X)) \cong X$ of ringed spaces.

Corollary 1.8. Let X and Y be as quasi-separated (e.g., Noetherian) schemes, and suppose $D_{\text{perf}}(X) \cong D_{\text{perf}}(X')$ is an equivalence of tt-categories (that preserves the tensor structure), then X and X' are isomorphic as schemes.

Remark 1.9. From the theorem, we are able to recover a scheme X from a tt-category $D_{\text{perf}}(X)$. It is unclear what tt-category K we should pick to that it is equivalent to $D_{\text{perf}}(\operatorname{Spec}(K))$.

2 Basic Construction

We will now introduce the notion of commutative algebra on a tt-category.

2.1 Tensor-triangulated Category

Definition 2.1 (Tensor-triangulated Category)**.** A tensor-triangulated category $\mathcal{T}$ is a triangulated category together with a symmetric monoidal structure $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ given by $(a, b) \mapsto a \otimes b$, such that $a \otimes - : \mathcal{T} \to \mathcal{T}$ and $- \otimes b : \mathcal{T} \to \mathcal{T}$ are exact functors (with respect to the monoidal and exact triangle structure), with natural isomorphisms $(\Sigma a) \otimes b \cong \Sigma(a \otimes b) \cong a \otimes \Sigma b$, and (because of commutativity) such that

$$\begin{tikzcd}
(\Sigma a) \otimes (\Sigma b) \arrow[r, "\cong"] \arrow[d, "\cong"] & \Sigma((\Sigma a) \otimes b) \arrow[d, "\cong"] \\
\Sigma(a \otimes (\Sigma b)) \arrow[r, "\cong"] & \Sigma^2(a \otimes b)
\end{tikzcd}$$

In particular, $\mathbb{1}$ acts as the tensor unit. There are some trivial relations $\lambda : \mathbb{1} \otimes a \cong a$ and $\rho : a \otimes 1 \cong a$, therefore

$$\begin{tikzcd}
& \mathbb{1} \otimes a \arrow[dl, "\lambda"'] \arrow[dr, "\rho"] & \\
a \arrow[rr, equal] & & a
\end{tikzcd}$$

and we have $\alpha_{a,b,c} : (a \otimes b) \otimes c \cong a \otimes (b \otimes c)$ such that

$$\begin{tikzcd}
a \otimes (b \otimes (c \otimes d)) \arrow[rr, "\cong"] \arrow[d] & & a \otimes ((b \otimes c) \otimes d) \arrow[d, "\cong"] \\
(a \otimes b) \otimes (c \otimes d) \arrow[dr, "\cong"] & & (a \otimes (b \otimes c)) \otimes d \arrow[dl, "\cong"] \\
& ((a \otimes b) \otimes c) \otimes d &
\end{tikzcd}$$

Finally, there is a symmetric monoidal structure given by $a \otimes b \cong b \otimes a$.

Remark 2.2. This gives a good way of enriching triangulated categories over tt-categories, c.f., [Kel82].

Example 2.3.

- Category of bounded complexes of any tensor additive category, like $D_{\text{perf}}(A)$, $D^b(kG\text{-}\mathbf{Mod})$.
- Stable homotopy category of complexes.

Remark 2.4. In such a setting, we want to talk about functors that preserves the additional structures instead. Therefore, we will only work with exact functors that preserves the tensor structure and the suspension structure. They are called $\otimes$-exact functor.

Lemma 2.5. $\operatorname{End}_K(\mathbb{1})$ is a commutative ring.

Proof. Let $f, g : \mathbb{1} \to \mathbb{1}$ be morphisms, then we have

$$\begin{tikzcd}
& \mathbb{1} \arrow[r, "f"] \arrow[dl, equal] & \mathbb{1} \arrow[dr, equal] & \\
\mathbb{1} \arrow[d, "g"'] & \mathbb{1} \otimes \mathbb{1} \arrow[l, "\cong"] \arrow[u, "\rho", "\cong"'] \arrow[r, "f \otimes \mathbb{1}"] \arrow[d, "\mathbb{1} \otimes g"] \arrow[dr, "f \otimes g"] & \mathbb{1} \otimes \mathbb{1} \arrow[u, "\rho", "\cong"'] \arrow[r, "\cong", "\lambda"'] \arrow[d, "\mathbb{1} \otimes g"] & \mathbb{1} \arrow[d, "g"] \\
\mathbb{1} \arrow[dr, equal] & \mathbb{1} \otimes \mathbb{1} \arrow[l, "\cong"] \arrow[r, "f \otimes \mathbb{1}"] \arrow[d] & \mathbb{1} \otimes \mathbb{1} \arrow[r, "\cong", "\lambda"'] \arrow[d] & \mathbb{1} \arrow[dl, equal] \\
& \mathbb{1} \arrow[r, "f"] & \mathbb{1} &
\end{tikzcd}$$

Therefore $fg = gf$. □

We know have a basic setting of commutative algebra on this category, so the most important thing right now before we can talk about geometry becomes: what is a spectrum?

2.2 Balmer Spectrum

Let R be any commutative ring. What do you want from a spectrum of R? Here is a Christmas wish list:

- A topological space X, where for each $a \in R$ we have a closed subset $\{a = 0\}$.
- $Z(0) = X$.
- $Z(1) = \varnothing$.
- $Z(a \cdot b) = Z(a) \cup Z(b)$ for all $a, b \in R$.
- $Z(a + b) \supseteq Z(a) \cap Z(b)$ for all $a, b \in R$.

Remark 2.6. Given a topological space X and $(X, Z : R \to \mathrm{Closed}(X))$ and a continuous function $f : Y \to X$ we can build on Y a Z' by $Z'(a) = f^{-1}(Z(a))$ for all $a \in R$.

This makes a category of such "zero data" (X, Z) for R, and a trivial construction would be an initial object. The most interesting object would be the terminal object.

Theorem 2.7. There exists a terminal object in this category, namely the Zariski spectrum, i.e., $(\mathrm{Spec}(R), V)$, i.e., for any zero data (X, Z), there exists a unique continuous map $f : X \to \mathrm{Spec}(R)$ such that $Z(a) = f^{-1}(V(a))$ for all $a \in R$.

With this in mind, what can we say on a tt-category, with respect to the suspension structure?

Definition 2.8. Fix K. A support data is a pair (X, σ) where X is a topological space and σ is an assignment $\sigma : \mathrm{Ob}(K) \to \mathrm{Closed}(X)$, i.e., $\sigma(a) \subseteq X$ for all $a \in K$, such that

- $\sigma(0) = \varnothing$ and $\sigma(\mathbb{1}) = X$,
- $\sigma(\Sigma a) = \sigma(a)$ for all $a \in K$,
- $\sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$ for all $a, b \in K$,
- $\sigma(c) \subseteq \sigma(a) \cup \sigma(b)$ for all exact triangles $a \to b \to c \to \Sigma a$.

The morphism of support data is given by $(X, \sigma) \to (Y, \tau)$ by a continuous map $f : X \to Y$ such that $f^{-1}(\tau(a)) = \sigma(a)$ for all $a \in K$.

Theorem 2.9 (Definition/Theorem)**.** Let K be an essentially small tensor-triangulated category. There exists a terminal support data, called the Balmer spectrum of K, $(\mathrm{Spc}(K), \mathrm{supp})$.

Constructive Proof. The (prime) tt-ideals of K are (strict) subcategories of K that are

- triangulated, that is,
  - $0 \in P$,
  - closed under 2-out-of-3 in exact triangles,
  - thick, i.e., $a \oplus b \in P$ implies $a \in P$,
- $\otimes$-ideals, i.e., $P \otimes K \subseteq P$,
- prime, that is, $\mathbb{1} \notin P$, and if $a \otimes b \in P$, then either a or b is in P.

The topology is given by the support data $\mathrm{supp}(a) = \{P \mid P \not\ni a\}$ and general closed are $Z(E) = \{P \mid P \cap E = \varnothing\}$ for any $E \subseteq K$. The universality is as follows: given a support data (X, σ), we build

$$\begin{aligned} f : X &\to \mathrm{Spc}(K) \\ x &\mapsto \{a \in K \mid x \notin \sigma(a)\}. \end{aligned}$$

$\square$

Remark 2.10. The open subsets are therefore of the form $U(a) = \text{open}(a) = \text{Spc}(K)\backslash\{\text{supp}(a)\} = \{P \mid a \in P\}$. Now a general open set is given by $\bigcup_{a\in S} U(a) = \{P \mid P \cap S \neq \varnothing\}$ for every $S \subseteq \text{Ob}(K)$.

Lemma 2.11 (Existence Lemma)**.** Let $J \subsetneq K$ be a thick $\otimes$-ideal and $S \subseteq K$ is a $\otimes$-multiplicative set of objects (i.e., $1 \in S \supseteq S \otimes S$) such that $J \cap S = \varnothing$. Then there exists $P \in \text{Spc}(K)$ such that $P \supseteq J$ and $P \cap S = \varnothing$.

Proof. Let $\mathcal{F} = \{\mathcal{A} \subseteq K \mid \mathcal{A} \supseteq J$ and is tt-ideal, and $\mathcal{A} \cap S = \varnothing\}$. By Zorn's Lemma, we find $P \in \mathcal{F}$ maximal in terms of inclusion.

Claim 2.12. P is prime.

Subproof. Let $a, b \in K$ be such that $a \otimes b \in P$. Ab absurdo, suppose $a \notin P$ and $b \notin P$. Consider the tt-ideals $\langle P, a\rangle$ and $\langle P, b\rangle$. By assumption, these ideals are strictly greater than $P \supseteq J$, then these two ideals are not in the family, and therefore $\langle P, a\rangle$ and $\langle P, b\rangle$ are not in $\mathcal{F}$. Therefore, there exists $s, t \in S$ such that $s \in \langle P, a\rangle$ and $t \in \langle P, b\rangle$, then $s \otimes t \in \langle P, a \otimes b\rangle = P$. But $s \otimes t \in S$, we reach a contradiction. ■

□

Corollary 2.13. If $K \neq 0$, then $\text{Spc}(K) \neq \varnothing$.

Proof. Apply Lemma 2.11 to the case where $J = 0$ and $S = \{\mathbb{1}\}$. □

Proposition 2.14. For every $a \in K$, the open set $U(a)$ is quasi-compact. Conversely, any quasi-compact open is of the form $U(a)$ for some $a \in K$.

Proof. Suppose $U(a) \subseteq \bigcup_{s\in S} U(s) = \{P \mid P \cap S \neq \varnothing\}$. Let $S' = \{s_1 \otimes \ldots \otimes s_n \mid n \geqslant 0, s_i \in S\} \supseteq S$ be $\otimes$-multiplicative.

Claim 2.15. $\langle a\rangle \cap S' \neq \varnothing$.

Subproof. Suppose not, then by Lemma 2.11, there exists $P \in \text{Spc}(K)$ such that $a \in P$ (so $u(a) \ni P$), but $P \cap S' = \varnothing$, therefore $P \cap S = \varnothing$, hence $P \notin \bigcup_{s\in S} u(s)$, contradiction. ■

Therefore, there exists $n \geqslant 0$ and $s_1, \ldots, s_n \in S$ such that $s_1 \otimes \ldots \otimes s_n \in \langle a\rangle$, so

$$U(a) \subseteq U(s_1 \otimes \ldots \otimes s_n) = U(s_1) \cup \ldots \cup U(s_n).$$

Conversely, if U is quasi-compact, then $U = \bigcup_{s\in K: U(s)\subseteq U} U(s)$. Therefore, there exists $s_1, \ldots, s_n$ such that $U = U(s_1) \cup \ldots \cup U(s_n) = U(s_1 \otimes \ldots \otimes s_n)$. □

Proposition 2.16. For every $E \subseteq \text{Spc}(K)$, we have $\bar{E} = \bigcap_{\text{supp}(a)\supseteq E} \text{supp}(a)$. In particular, $\overline{\{P\}} = \{Q \in \text{Spc}(K) \mid Q \subseteq P\}$. Thus, $\overline{\{P\}} = \overline{\{Q\}}$ implies $P = Q$.

Proof. The first statement is formal. The second statement is true because (since $\text{supp}(a) \ni P$ if and only if $a \notin P$) $\overline{\{P\}} = \bigcap_{\text{supp}(a)\ni p} \text{supp}(a) = \{Q \mid \forall a \notin P, a \notin Q\}$, therefore this is the set of Q such that $Q \subseteq P$. The last statement is obvious. □

Proposition 2.17. In $\text{Spc}(K)$, every irreducible closed subset Z ($Z \neq \varnothing$, and $Z \subseteq Z_1 \cup Z_2$ closed implies $Z \subseteq Z_1$ or $Z \subseteq Z_2$) admits a unique generic point $Z = \overline{\{P\}}$. In case, we have $P = \{a \in K \mid Z \cap U(a) \neq \varnothing\}$.

Proof. Let Z be irreducible, so $Z \cap U(a) \cap U(b) = \varnothing$, and taking contrapositive of a previous statement gives: if $Z \cap U(a) \neq \varnothing$ and $Z \cap U(b) \neq \varnothing$, then $Z \cap U(a) \cap U(b) \neq \varnothing$, i.e., $Z \cap U(a \otimes b) \neq \varnothing$. In set-theoretic studies, we note $U(a \otimes b) = U(a) \cup U(b)$, so we note P is prime and $\otimes$-ideal for free. Therefore, P is triangulated because the contrapositive statement we just observed: $a \to b \to c \to \Sigma a$ then $U(c) \supseteq U(a) \cap U(b)$.

Therefore, $Z = \bigcap_{\text{supp}(a)\supseteq Z} \text{supp}(a) = \{Q \mid \forall a \notin P, a \notin Q\} = \{Q \mid Q \subseteq P\} = \overline{\{P\}}$ (note that $\text{supp}(a) \supseteq Z$ implies $Z \cap U(a) = \varnothing$, and $a \notin P$). □

Remark 2.18. $\mathrm{Spc}(K)$ is quasi-compact and quasi-separated (i.e., has a basis of quasi-compact open subsets), and every non-empty irreducible closed subset has a unique generic point. In particular, $\mathrm{Spc}(K)$ is T_0. These are the spectral topological spaces in the sense of Hochster.

Theorem 2.19 (Balmer, [Bal05])**.** Let

$$\begin{aligned} F : K &\to L \\ A &\mapsto F^{-1}A \end{aligned}$$

be a $\otimes$-exact functor, then this induces a continuous and spectral (i.e., preimage of a quasi-compact open subset is quasi-compact) map $\mathrm{Spc}(F) : \mathrm{Spc}(L) \to \mathrm{Spc}(K)$. Therefore, Spc is a contravariant functor, where we can carry the support over.

3 But where is the geometry?

Along with assuming K to be essentially small, we assume K is rigid and idempotent-complete. Finally, we will list a few results which indicates geometry survives on such tt-categories.

3.1 Further Assumptions

Definition 3.1 (Rigid)**.** A tt-category K is rigid if there exists an exact functor $D : K^{\mathrm{op}} \to K$ and a n atural isomorphism $\mathrm{Hom}_K(a \otimes b, c) \cong \mathrm{Hom}_K(b, Da \otimes c)$ for all $a, b, c \in K$, where $Da = a^\vee$ is the dual of a. In this sense, $(K, \otimes)$ becomes a closed symmetric monoidal category.

This is another indication that our category cannot be too big.

Definition 3.2 (Idempotent-complete)**.** A tt-category K is idempotent complete if every idempotent $e = e^2 : a \to a$ in K yields a decomposition $a = \mathrm{im}(e) \oplus \ker(e)$.

We will not worry about this assumption too much. The thing to note is the following:

Theorem 3.3 (Balmer, [Bal05])**.** The idempotent completion $\iota : K \to K^\natural$ induces a homeomorphism $\mathrm{Spc}(\iota) : \mathrm{Spc}(K^\natural) \to \mathrm{Spc}(K)$.

Therefore, the completion does not change the spectrum!

3.2 Geometry

For arbitrary quasi-compact open subset $U \subseteq \mathrm{Spc}(K)$, we denote $Z = \mathrm{Spc}(K) \backslash U$. We define the tt-category $K(U) := (K/K_Z)^\natural$, where $K_Z = \{a \in K \mid \mathrm{supp}(a) \subseteq Z\}$ is a thick $\otimes$-ideal.

Lemma 3.4. (a) (see [BF07]) We have a natural functor $\mathrm{res}_U : K \to K(U)$, and actually $\mathrm{Spc}(K(U)) \cong U$ as a homeomorphism.

(b) (see [Bal10a]) $\mathrm{Spc}(K)$ is a local (every open cover contains the whole space) topological space if and only if $a \otimes b = 0$ implies $a = 0$ or $b = 0$. Hence, $\{0\}$ is the unique closed point of $\mathrm{Spc}(K)$.

We will now construct a sheaf on $\mathrm{Spc}(K)$ using $K(U)$.

Remark 3.5 ([Bal10a])**.** Take quasi-compact open subset U of $\mathrm{Spc}(K)$, we know via Lemma 2.5 that $\mathrm{End}_{K(U)}(\mathbb{1})$ is a commutative ring. We know $\mathbb{1}$ of $K(U)$ is just the restriction of $\mathbb{1}$ in K, and we know that $(K(U))(V) \cong K(V)$ for all $V \subseteq U$ $\mathrm{Spc}(K(U))$, therefore we obtain a presheaf of commutative rings $p\mathcal{O}_K$ on the open basis consisting of quasi-compact open subsets, with $p\mathcal{O}_K(U) = \mathrm{End}_U(\mathbb{1})$. By sheafification, we obtain a sheaf $\mathcal{O}_K$ of commutative rings on $\mathrm{Spc}(K)$. Therefore, we have a locally ringed space $\mathrm{Spec}(K) = (\mathrm{Spc}(K), \mathcal{O}_K)$.

Theorem 3.6. Suppose $a \in K$ has disconnected support, i.e., $\mathrm{supp}(a) = Y_1 \sqcup Y_2$ with Y_1, Y_2 closed and disjoint. Then $a \cong a_1 \oplus a_2$ with $\mathrm{supp}(a_1) = Y_1$ and $\mathrm{supp}(a_2)$.

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