

# MATH 526 Notes

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Let  $X$  be a topological space with basepoint  $x_0 \in X$ . We already know two invariants,

- the fundamental group  $\pi_1(X, x_0)$ , and
- the homology groups  $H_n(X)$  for  $n \geq 0$ , which are abelian groups.

We will look at two more invariants,

- the cohomology groups  $H^n(X)$  for  $n \geq 0$ , and
- the higher homotopy groups  $\pi_n(X, x_0)$  for  $n \geq 0$ .

In particular,  $\pi_*(X, x_0)$  is a very good invariant in the following sense:

**Theorem 1.1** (Whitehead). If  $f : (X, x_0) \rightarrow (Y, y_0)$  is a map of CW-complexes, then  $f$  is a homotopy equivalence if and only if  $\pi_*(f) : \pi_*(X, x_0) \rightarrow \pi_*(Y, y_0)$  is an isomorphism.

However,  $\pi_*$  is very hard to compute. On the other hand,  $H^*(X)$  is relatively easy to compute, but this is not a complete invariant. For instance,  $\mathbb{C}P^2$  and  $S^2 \vee S^4$  have isomorphic cohomology groups, but they are not equivalent.  $H^*(X)$  is closely related to  $H_*(X)$ , but  $H^*(X)$  is a graded ring structure with cup product. It is contravariant in  $X$ , where  $H_*(X)$  is covariant. The cup product is defined by the composition of induced diagonal map with an external product:

$$H^i(X) \times H^j(X) \xrightarrow{\times} H^{i+j}(X \times X) \xrightarrow{\Delta^*} H^{i+j}(X)$$

Other things we will talk about include:

- Natural transformations  $H^i(-) \rightarrow H^j(-)$  encoded by Steenrod operations.
- $H^n(-)$  becomes a representable functor, i.e.,  $H^n(X) = [X, K(\mathbb{Z}, n)]$ , where  $K(\mathbb{Z}, n)$  is the Eilenberg-MacLane space, and the bracket indicates the homotopy classes of maps.
- Poincaré duality in  $H^*(M)$  for compact manifold  $M$ , namely the cup product gives

$$H^i(M) \otimes H^{\dim(M)-i}(M) \xrightarrow{\sim} H^{\dim(M)}(M).$$

- Characteristic classes in  $H^*(X)$  associated to vector bundles over  $X$ .

Recall for a topological space  $X$ , we obtain a collection of (singular) homology groups  $H_n(X)$ , with  $H_*(X) = \bigoplus_{n \geq 0} H_n(X)$ . The functoriality of morphisms says that  $X \xrightarrow{f} Y \xrightarrow{g} Z$  induces  $f_*g_* = (fg)_* : H_*(X) \xrightarrow{f_*} H_*(Y) \xrightarrow{g_*} H_*(Z)$ . So

$$H_*(-) : \mathbf{Top} \rightarrow \mathbf{Ab}$$

is a well-defined functor. This factors into

$$\begin{array}{ccc} \mathbf{Top} & \xrightarrow{H_*(-)} & \mathbf{Ab} \\ & \searrow C_*(-) & \nearrow H_*(-) \\ & \mathbf{Ch} & \end{array}$$

Here  $C_*(-)$  is usually the singular chain, given by  $\partial : C_n(X) \rightarrow C_{n-1}(X)$ , where  $C_n(X)$  is the free abelian group generated by  $\text{Hom}_{\mathbf{Top}}(\Delta^n, X) \cong \bigoplus \mathbb{Z}\sigma$ .  $\Delta^n \subseteq \mathbb{R}^{n+1}$  is the set of tuples  $(t_0, \dots, t_n)$  such that the coordinates sum to 1. The boundary is  $\partial\sigma = \sum_{0 \leq i \leq n} (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$ .

We say  $C_*(-)$  is homotopy invariant, i.e., if  $f : X \rightarrow Y$  is a homotopy equivalence, then the induced map  $C_*(X) \rightarrow C_*(Y)$  on chain complexes is a chain equivalence.

**Remark 1.2.**  $C_*^\Delta(X)$  and  $C_*^{\text{CW}}(X)$  are both chain equivalent to  $C_*(X)$ .

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Here is a list of properties of  $C_*(-) : \mathbf{Top} \rightarrow \mathbf{Ch}$ :

- Functoriality: given a continuous map  $f : X \rightarrow Y$ , there is an induced map

$$\begin{aligned} f_* : C_*(X) &\rightarrow C_*(Y) \\ (\sigma : \Delta^n \rightarrow X) &\mapsto (f\sigma : \Delta^n \rightarrow Y) \end{aligned}$$

- Homotopy invariance: given  $f, g : X \rightarrow Y$  such that  $f \simeq g$ , i.e., there is  $H : X \times [0, 1] \rightarrow Y$  such that  $H|_0 = f$  and  $H|_1 = g$ , then  $f_* \simeq g_*$  as a chain homotopy equivalence, i.e., there exists maps  $h_n : C_n(X) \rightarrow C_{n+1}(Y)$  making a diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{n+1}(X) & \longrightarrow & C_n(X) & \longrightarrow & C_{n-1}(X) \longrightarrow \cdots \\ & & \swarrow h & \downarrow g \parallel f & \swarrow h & \downarrow g \parallel f & \swarrow h \downarrow g \parallel f \\ \cdots & \longrightarrow & C_{n+1}(Y) & \longrightarrow & C_n(Y) & \longrightarrow & C_{n-1}(Y) \longrightarrow \cdots \end{array}$$

such that  $f - g = \partial h + h\partial$ . Therefore  $f_* = g_* : H_*(X) \rightarrow H_*(Y)$ .

**Remark 2.1.**  $f : A_* \rightarrow B_*$  is a chain equivalence if there exists  $g : B_* \rightarrow A_*$  and  $fg \simeq \text{id}_B$  and  $gf \simeq \text{id}_A$ , then  $f_* : H_*(A_*) \rightarrow H_*(B_*)$  is an isomorphism, i.e.,  $f$  is a quasi-isomorphism.

**Example 2.2.** The complexes  $A : 0 \rightarrow \mathbb{Z} \xrightarrow{-2} \mathbb{Z} \rightarrow 0$  and  $B : 0 \rightarrow 0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$  gives a quasi-isomorphism  $f : A \rightarrow B$  in the canonical way, but this is not a chain equivalence, since the backwards map has to be zero.

- Additivity:  $C_*(\coprod_\alpha X_\alpha) \cong \bigoplus_\alpha C_*(X_\alpha)$ .
- Excision: given a pair  $(X, A)$  with  $Z \subseteq A$  such that  $\bar{Z} \subseteq \text{int}(A)$ , then we have  $C_*(X \setminus Z, A \setminus Z) \cong C_*(X, A)$ .
- Mayer-Vietoris: given  $A, B \subseteq X$ , with  $X = \text{int}(A) \cup \text{int}(B)$ , then we have a short exact sequence

$$0 \longrightarrow C_*(A \cap B) \longrightarrow C_*(A) \oplus C_*(B) \xrightarrow{*} C_*(X) \longrightarrow 0$$

The cochain complex is obtained via inverting the indices and maps  $\delta$  from a chain complex. This induces a cohomology  $H^*(C^*) = \ker(\delta)/\text{im}(\delta)$  as the quotient of cocycles over coboundaries. Now  $f : A^* \rightarrow B^*$  is a quasi-isomorphism if  $f^* : H^*(A^*) \rightarrow H^*(B^*)$  is an isomorphism. Similarly, one can define the cochain homotopy equivalence.

**Example 2.3.** If  $C_* \in \mathbf{Ch}$ , and  $k \in \mathbf{Ab}$ , then we can form cochain complex  $C_k^* := \text{Hom}(C_*, k)$ , where  $C_k^n = \text{Hom}_{\mathbf{Ab}}(C_n, k) \xrightarrow{\delta} C_k^{n+1}$  by sending  $f : C_n \rightarrow k$  to  $f\partial : C_{n+1} \rightarrow C_n \rightarrow k$ .

- $\text{Hom}(-, k) : \mathbf{Ch} \rightarrow \mathbf{coCh}$  is a functor.
- The functor preserves quasi-isomorphisms between chain complexes of free abelian groups.

**Definition 2.4.** For  $k \in \mathbf{Ab}$ , the singular cochains with coefficients in  $k$  is

$$\begin{array}{ccc} C^*(-, k) : \mathbf{Top} & \xrightarrow{\quad} & \mathbf{coCh} \\ & \searrow C_*(-) & \nearrow \text{Hom}(-, k) \\ & \mathbf{Ch} & \end{array}$$

The cohomology of  $X$  with coefficients in  $k$  is defined by  $H^*(X; k) = H^*(C^*X, k)$ . We have the convention  $C^*(X) = C^*(X, \mathbb{Z})$ .

Alternatively, we take the opposite categories  $\mathbf{Top}^*$  and  $\mathbf{Ch}^*$  so that the functors are viewed as covariant.

The corresponding map  $\delta : C^n(X; k) \rightarrow C^{n+1}(X; k)$  is given by  $\delta f$  that maps  $\sigma \in C_{n+1}(X)$  to  $(-1)^{n+1}f(\partial\sigma)$ . Although the cochains are in general the dual of chains, the cohomology is not going to be the dual of the homology.

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Recall:

$$\begin{array}{ccccc} & & H^*(-, k) & & \\ & \nearrow & \text{arc} & \searrow & \\ \mathbf{Top}^{\text{op}} & \xrightarrow{C_*} & \mathbf{Ch}^{\text{op}} & \xrightarrow{\text{Hom}(-, k)} & \mathbf{coCh} & \xrightarrow{H^*} & \mathbf{GrAb} \end{array}$$

Properties of  $H^*(-, k) : \mathbf{Top} \rightarrow \mathbf{GrAb}$ :

- Dimension:

**Claim 3.1.**  $H^i(\{*\}, k) = \begin{cases} 0, & i \neq 0 \\ k, & i = 0 \end{cases}$

*Proof.* Note that each degree of cohomology is given the free abelian group generated by  $\text{Hom}(\Delta^n, \{*\})$ , but the singleton set is the terminal object in the category of topological spaces, so there is always a unique generator, thus the chain complex is given by  $\mathbb{Z}$ 's on each degree  $n \geq 0$ .

Now the generating map at degree  $n$  is  $\sigma_n : \Delta^n \rightarrow \{*\}$ , and see Homework 1 where we proved the homology. Now looking at  $C^*(\{*\}, k)$ , we have

$$k \xrightarrow{0} k \xrightarrow{\cong} k \xrightarrow{0} k \longrightarrow \dots$$

and this gives the cohomology. □

- Homotopy: if  $f \simeq g : X \rightarrow Y$ , then  $f^* = g^* : H^*(Y, k) \rightarrow H^*(X, k)$ .

*Proof.* We have  $f_* = g_* : C_*X \rightarrow C_*Y$ , and then  $\text{Hom}(f_*, k) \cong \text{Hom}(g_*, k)$ , so  $H^*(-)$  is invariant under cochain homotopies. □

- Additivity:  $H^*(\coprod_{\alpha} X_{\alpha}, k) \cong \prod_{\alpha} H^*(X_{\alpha}, k)$ .

*Proof.* We know that for chains there is  $C_*(\coprod_{\alpha} X_{\alpha}) = \bigoplus_{\alpha} C_*(X_{\alpha})$ , so the cochain version says that  $C^*(\coprod_{\alpha} X_{\alpha}, k) \cong \text{Hom}(\bigoplus_{\alpha} C_*(X_{\alpha}), k) \cong \prod_{\alpha} \text{Hom}(C_*(X_{\alpha}), k) \cong \prod_{\alpha} C^*(X_{\alpha})$  and  $H^* : \mathbf{coCh} \rightarrow \mathbf{GrAb}$  commutes with the product. □

- Exactness: for a pair  $(X, A)$ , there is a natural long exact sequence

$$\cdots \longrightarrow H^n(X, A; k) \longrightarrow H^n(X; k) \longrightarrow H^n(A; k) \longrightarrow \cdots$$

*Proof.* We have a short exact sequence

$$0 \longrightarrow C_*A \longrightarrow C_*X \longrightarrow C_*(X, A) \longrightarrow 0$$

where  $C_*A \rightarrow C_*X$  is an inclusion of summands. Therefore, the quotient  $C_*(X, A)$  is also a chain complex of free abelian groups. Therefore, taking the cochains also gives a short exact sequence. We then obtain a short exact sequence of cochain complexes

$$0 \longrightarrow C^*(X, A; k) \longrightarrow C^*(X; k) \longrightarrow C^*(A; k) \longrightarrow 0$$

and can then apply cohomology functor. □

- Excision: given a pair  $(X, A)$  and  $Z$  such that  $\bar{Z} \subseteq \text{int}(A)$ , we have  $H^*(X, A; k) \cong H^*(X \setminus Z, A \setminus Z; k)$ .
- Mayer-Vietoris: given  $A, B \subseteq X$  such that  $\text{int}(A) \cup \text{int}(B) = X$ , then we have a natural long exact sequence

$$\cdots \longrightarrow H^n(X; k) \longrightarrow H^n(A; k) \oplus H^n(B; k) \longrightarrow H^n(A \cap B; k) \longrightarrow \cdots$$

**Definition 3.2.** A functor  $E^* : \mathbf{Top}^{\text{op}} \rightarrow \mathbf{GrAb}$  is called a generalized cohomology theory if it satisfies the four middle property (except the dimension property and Mayer-Vietoris).

**Remark 3.3.** If  $E^*$  also satisfies the dimension property, then  $E^*$  is naturally isomorphic to the cohomology  $H^*(-; k)$ . There are also other generalized cohomology theories like  $K$ -theory, cobordism, etc.

The Mayer-Vietoris becomes a consequence of the first five properties.

We will now try to use homological algebra to relate  $H_*(X) = H_*(CX)$  and  $H^*(X; k) = H^*(\text{Hom}(C_*X, k))$ .

**Definition 3.4.** We say  $C_*(X; k) \cong C_*(X) \otimes_{\mathbb{Z}} k$  and  $H_*(X; k) \cong H_*(C_*X \otimes k)$  gives the singular homology of  $X$  with coefficients in  $k$ .

**Lemma 3.5.**  $-\otimes k : \mathbf{Ab} \rightarrow \mathbf{Ab}$  is a right exact functor.  $\text{Hom}(-, k) : \mathbf{Ab}^{\text{op}} \rightarrow \mathbf{Ab}$  is left exact.

*Proof.* Exercise. □

**Remark 3.6.** The covariant hom functor is also left exact.

**Remark 3.7.** The left adjoint is right exact, the right adjoint is left exact. In particular, we have the hom-tensor adjunction

$$\text{Hom}(A, \text{Hom}(B, C)) \cong \text{Hom}(A \otimes B, C).$$

Note that

$$\text{Hom}(A, \text{Hom}(B, C)) \cong \text{Hom}(A \otimes B, C) \cong \text{Hom}(B \otimes A, C) \cong \text{Hom}(B, \text{Hom}(A, C))$$

**Example 3.8.** Consider

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

Tensoring with  $\mathbb{Z}/n\mathbb{Z}$ , we do not have exactness.

**Example 3.9.**

$$0 \longrightarrow A \longrightarrow A \oplus C \longrightarrow C \longrightarrow 0$$

is always exact after tensoring  $-\otimes k$  or applying the hom functor  $\text{Hom}(-, k)$ .

**Definition 3.10.** A short exact sequence  $0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0$  is split if any of the following equivalence conditions hold:

- (i)  $p$  has a section  $s : C \rightarrow B$  such that  $ps = 1$ ;
- (ii)  $i$  has a retraction  $r : B \rightarrow A$  such that  $ri = 1$ ;
- (iii)  $B \cong A \oplus C$ , i.e.,

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C \longrightarrow 0 \\ & & \parallel & & \downarrow & & \parallel \\ 0 & \longrightarrow & A & \xrightarrow{i} & A \oplus C & \xrightarrow{p} & C \longrightarrow 0 \end{array}$$

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We will prove that (ii) implies (iii).

Suppose  $b \in B$ , then  $b = (b - irb) + irb$ , which is a decomposition of elements in  $\ker(r)$  and in  $\text{im}(i)$ , respectively. Also,  $\ker(r) \cap \text{im}(i) = 0$ , therefore  $B = \ker(r) \oplus \text{im}(i)$ . Since  $i$  is an inclusion, then  $\text{im}(i) \cong A$ . Now  $p : B \rightarrow C$  factors through the projection onto  $\ker(r)$  since  $ri = 0$ . By restricting  $p$  onto  $\ker(r)$ , we see  $p$  is also injective, thereby an isomorphism.

**Lemma 4.1.** If we have a split exact sequence

$$0 \longrightarrow A \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{r} \end{array} B \begin{array}{c} \xrightarrow{p} \\ \xleftarrow{s} \end{array} C \longrightarrow 0$$

then  $- \otimes k$  and  $\text{Hom}(-, k)$  preserves the split exactness, i.e.,

$$0 \longrightarrow A \otimes k \longrightarrow B \otimes k \longrightarrow C \otimes k \longrightarrow 0$$

and

$$0 \longrightarrow \text{Hom}(C, k) \longrightarrow \text{Hom}(B, k) \longrightarrow \text{Hom}(A, k) \longrightarrow 0$$

The point is tensors and homs preserve retracts.

*Proof.* •  $(r \otimes \text{id}_k)(i \otimes \text{id}_k) = ri \otimes \text{id}_k = \text{id}_{A \otimes k}$ , so  $i \otimes \text{id}_k$  is split injective.

- Similarly,  $\text{Hom}(i, \text{id})$  is split surjective.

□

**Example 4.2.** Given a surjection  $B \rightarrow C \rightarrow 0$  such that  $C$  is free abelian, then there is always a section  $s : C \rightarrow B$  making the exact sequence split. (That is,  $C$  is projective.) That is, if  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is an exact sequence where  $C$  is free, then the sequence is split exact.

**Definition 4.3.** Let  $C \in \mathbf{Ab}$ . A free resolution of  $C$  is a chain complex of free objects

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

and an augmentation  $F_0 \rightarrow C$ , so that

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0$$

is acyclic, i.e., exact everywhere.

**Example 4.4.**

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow 0$$

is a free resolution of  $\mathbb{Z}/n\mathbb{Z}$ . So is

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

as well as

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\text{id} \oplus (\times n)} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0$$

**Lemma 4.5.** Any  $C \in \mathbf{Ab}$  admits a free resolution, and moreover, it admits a resolution of length 1; given by

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \longrightarrow 0$$

*Proof.* Choose a surjection  $p : F_0 \rightarrow C$  from a free abelian group  $F_0$  to  $C$ . Let  $F_1 = \ker(p)$ , then  $F_1$  is free, so we are done.  $\square$

**Lemma 4.6.** Free resolutions are essentially unique, i.e., if  $F \rightarrow C$  and  $F' \rightarrow C$  are free resolutions, then there is a quasi-isomorphism  $F \xrightarrow{\sim} F'$  which commutes with the augmentations to  $C$ .

**Definition 4.7.** Let  $C \in \mathbf{Ab}$  and let  $F \rightarrow C$  be a free resolution, then we define the torsion groups to be  $\mathrm{Tor}_n^{\mathbb{Z}}(C, k) = H_n(F \otimes k)$ , and the ext groups to be  $\mathrm{Ext}_{\mathbb{Z}}^n(C, k) = H^n(\mathrm{Hom}_{\mathbb{Z}}(F, k))$ .

**Remark 4.8.** • Tor and Ext are independent of the choice of resolutions.

- $\mathrm{Tor}_n^{\mathbb{Z}}$  and  $\mathrm{Ext}_{\mathbb{Z}}^n$  are zero for  $n > 1$ .
- $\mathrm{Tor}_n^{\mathbb{Z}}(C, k) \cong \mathrm{Tor}_n^{\mathbb{Z}}(k, C)$ .
- $\mathrm{Tor}_0^{\mathbb{Z}}(C, k) \cong C \otimes k$ .
- $\mathrm{Ext}_{\mathbb{Z}}^0(C, k) \cong \mathrm{Hom}(C, k)$ .

**Example 4.9.** • If  $C$  is free, then  $\mathrm{Tor}_1(C, k) = \mathrm{Ext}^1(C, k) = 0$ .

- $\mathrm{Tor}_1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$ .
- $\mathrm{Tor}_1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = 0$ .
- $\mathrm{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$ .
- $\mathrm{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$ .
- $\mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = 0$ .

*Proof.* Look at

$$0 \longrightarrow F_1 = \mathbb{Z} \longrightarrow F_0 = \mathbb{Z} \longrightarrow C = \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

then  $\mathrm{Tor}_*(\mathbb{Z}/p\mathbb{Z}, k) = H_*(F_1 \otimes k = k \xrightarrow{\times p} F_0 \otimes k = k) = \begin{cases} k[p], * = 0 \\ k/pk, * = 1 \end{cases}$ . Here  $k[p]$  denotes  $p$ -torsion subgroup

of  $k$ . Moreover,  $\mathrm{Ext}^*(\mathbb{Z}/p\mathbb{Z}, k) = H^*(\mathrm{Hom}(F_1, k) = k \xleftarrow{\times p} \mathrm{Hom}(F_0, k) = k) = \begin{cases} k[p], * = 0 \\ k/pk, * = 1 \end{cases}$ .  $\square$

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Recall that cohomology are basically the dual of homology, where the difference originates from the failure of exactness of the hom functor.

**Theorem 5.1** (Universal Coefficient Theorem). Let  $C_*$  be a chain of free abelian groups and  $k \in \mathbf{Ab}$ , then there exists a natural short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(H_{n-1}(C_*), k) \longrightarrow H^n(\mathrm{Hom}(C_*, k)) \xrightarrow{h} \mathrm{Hom}(H_n(C_*), k) \longrightarrow 0$$

that splits in an unnatural sense.

Here we define  $h \in \mathrm{Hom}(H^n(\mathrm{Hom}(C_*, k)), \mathrm{Hom}(H_n(C_*), k))$ . Note that this hom set is isomorphic to the hom set  $\mathrm{Hom}(H^n(\mathrm{Hom}(C_*, k)) \otimes H_n(C_*), k)$  via the tensor-hom adjunction. That is,  $h$  is given by a bilinear pairing  $H^n(\mathrm{Hom}(C_*, k)) \times H_n(C_*) \rightarrow k$ . We use the Kronecker pairing  $([f], [x]) \mapsto f(x)$ . To see this is well-defined, let  $f \in \mathrm{Hom}(C_n, k)$  with  $\delta f = 0$ , for  $x \in C_n$ , we have  $\partial x = 0$ . Now replace  $x$  by  $x + \partial y$ , then  $f(x + \partial y) = f(x) = f(\partial y) = f(x) \pm (\delta f)(y) = f(x)$ . Also, replace  $f$  by  $f + \delta(g)$  gives  $(f + \delta(g))(x) = f(x) + (\delta g)(x) = f(x) + g(\delta x) = f(x)$ .

**Lemma 5.2.**  $h$  is a split surjection.

*Proof.* Write  $C_k^* = \text{Hom}(C_*, k)$ . Now  $h : \ker(\delta, C_k^n \rightarrow C_k^{n+1}) \rightarrow \text{Hom}(H_n(C_*), k)$  via  $h : f \mapsto (x \mapsto f(x))$ , then we will construct a section of  $h$  via  $\varphi \mapsto \tilde{\varphi}$ . Let  $Z_n = \ker(\partial)$  and  $B_n = \text{im}(\partial)$ , then  $H_n(C_*) = Z_n/B_n$ , and the short exact sequence of free abelian groups

$$0 \longrightarrow Z_n \xrightarrow{i} C_n \longrightarrow B_{n-1} \longrightarrow 0$$

and this splits so  $C_n \cong Z_n \oplus B_{n-1}$ . Given  $\varphi : H_n(C_*) \rightarrow k$ , we have

$$C_n \xrightarrow{r} Z_n \longrightarrow Z_n/B_n \xrightarrow{\varphi} k$$

where  $r$  is the retraction to  $i$ , and we define the composition to be  $\tilde{\varphi}$ . Now the composition

$$C_{n+1} \xrightarrow{\partial} C_n \longrightarrow Z_n \longrightarrow Z_n/B_n \longrightarrow k$$

is still zero since  $C_{n+1} \rightarrow Z_n$  is zero, but that means  $\delta\tilde{\varphi}$  is also zero.  $\square$

We will now prove the universal coefficient theorem.

*Proof.* Since  $h$  is a split surjection, then we know this extends to a short exact sequence, hence we just need to identify the kernel of  $h$ , i.e., to show that  $\ker(h) \cong \text{Ext}^1(H_{n-1}(C_*), k)$ . Given the split short exact sequence

$$0 \longrightarrow Z_n \xrightarrow{i} C_n \xrightarrow{\partial} B_{n-1} \longrightarrow 0$$

we have a diagram

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow 0 & & \downarrow \delta & & \downarrow 0 \\ 0 & \longrightarrow & \text{Hom}(B_{n-1}, k) & \longrightarrow & \text{Hom}(C_n, k) & \longrightarrow & \text{Hom}(Z_n, k) \longrightarrow 0 \\ & & \downarrow 0 & & \downarrow \delta & & \downarrow 0 \\ 0 & \longrightarrow & \text{Hom}(B_n, k) & \longrightarrow & \text{Hom}(C_{n+1}, k) & \longrightarrow & \text{Hom}(Z_{n+1}, k) \longrightarrow 0 \\ & & \downarrow 0 & & \downarrow \delta & & \downarrow 0 \\ & & \vdots & & \vdots & & \vdots \end{array}$$

which is a short exact sequence of complexes. By the snake lemma, we have the long exact sequence of cohomology  $\cdots \rightarrow H^n(B_k^{*-1}) \rightarrow H^n(C_k^*) \rightarrow H^n(Z_k^*) \rightarrow H^{n+1}(B_k^{*-1}) \rightarrow \cdots$ . We claim that the connecting homomorphism  $H^n(Z_k^*) \rightarrow H^{n+1}(B_k^{*-1})$  is  $\text{Hom}(B_n \subseteq Z_n, k)$ . But  $0 \rightarrow B^n \rightarrow Z^n \rightarrow H_n(C_*) \rightarrow 0$  is a free resolution of  $H_n(C_*)$  of length 1. Then  $H^*(\beta : \text{Hom}(Z_n, k) \rightarrow \text{Hom}(B_n, k)) = \text{Ext}^*(H_n(C_*), k)$  where  $\beta$  has kernel  $\text{Hom}(H_n(C_*), k)$  and cokernel  $\text{Ext}^1(H_n(C_*), k)$ . Therefore, the long exact sequence of cohomology is the splicing (as epi-mono factorization) of

$$0 \longrightarrow \text{coker}(\beta_{n-1}) \longrightarrow H_n(C_k^*) \longrightarrow \ker(\beta_n) \longrightarrow 0$$

and by identification we are done.  $\square$

**Corollary 5.3.** If  $C_* \rightarrow C'_*$  is a quasi-isomorphism, then  $\text{Hom}(C'_*, k) \rightarrow \text{Hom}(C_*, k)$  is a quasi-isomorphism.

**Corollary 5.4.** Let  $X \in \mathbf{Top}$  and  $A \subseteq X$ , then there exists a short exact sequence

$$0 \longrightarrow \text{Ext}^1(H_{n-1}(X, A), k) \longrightarrow H^n(X, A; k) \longrightarrow \text{Hom}(H_n(X, A); k) \longrightarrow 0$$

which is natural in  $(X, A)$ . This also splits in  $(X, A)$  in an unnatural way.

**Theorem 5.5.** If  $C_*$  is a chain complex of free abelian groups, then there is a short exact sequence

$$0 \longrightarrow H_n(C_*) \otimes k \longrightarrow H_n(C_* \otimes k) \longrightarrow \mathbf{Tor}_1(H_{n-1}(C_*, k) \longrightarrow 0$$

which is natural. It splits unnaturally.

**Corollary 5.6.** For any pair  $(X, A)$ , there is a natural short exact sequence

$$0 \longrightarrow H_n(X, A) \otimes k \longrightarrow H_n(X, A; k) \longrightarrow \mathbf{Tor}_1(H_{n-1}(X, A), k) \longrightarrow 0$$

which splits in an unnatural way.

## 6 SEPT 1, 2023

**Example 6.1.** Take  $X = \mathbb{C}P^2$ , then the Tor and Ext terms go away, so the cohomology is equivalent to the homology.

**Example 6.2.** Take  $X = \mathbb{R}P^2$ , the Tor term gives  $\mathbf{Tor}_1(\mathbb{Z}/2\mathbb{Z}, k) = k/2 \cong k[2]$ , as the 2-torsion of  $k$ , i.e., the set of  $a \in k$  such that  $2a = 0$ . Also,  $\mathbf{Ext}^1(\mathbb{Z}/2\mathbb{Z}, k) = k/2k$ .

Indeed, the Tor is given by the homology on multiplication by 2 map over  $k$  via tensor, and the Ext is given by the cohomology on multiplication by 2 map over  $k$  via hom.

Tor stands for torsion and Ext stands for extension.

Went on to talk about the limits and colimits.

**Remark 6.3.** In many abelian categories (and in particular, the category of abelian groups), we find a short exact sequence

$$0 \longleftarrow \operatorname{colim}_I \longleftarrow \bigoplus_{i \geq 0} X_i \longleftarrow \bigoplus_{i \geq 0} X_i \longleftarrow 0$$

and note that taking the dual version in the opposite category, we should obtain a sequence in the covariant sense. However, there is an asymmetry given by

$$0 \longrightarrow \lim_{I^{\text{op}}} X \longrightarrow \prod_{i \geq 0} X_i \longrightarrow \prod_{i \geq 0} X_i \longrightarrow \lim_{I^{\text{op}}}^1 X \longrightarrow 0$$

which is not short anymore. This is called a Milnor sequence.

## 7 SEPT 6, 2023

The colimit of the empty diagram is the initial object; dually, the limit of the empty diagram is the terminal object.

**Definition 7.1.** We say  $X : I \rightarrow \mathcal{C}$  is a filtered diagram if

- $\text{Ob}(\mathcal{C}) \neq \emptyset$ ,
- for all  $i, j \in I$ , there exists  $k \in I$  and morphisms  $i \rightarrow k$  and  $j \rightarrow k$ , and
- for parallel morphisms  $a, b : i \rightarrow j$  in  $I$ , then there exists coequalizers.

**Example 7.2.** A poset (as a category)  $P$  is a directed set if for any  $i, j \in P$ , there exists  $k \in P$  such that  $i \leq k$  and  $j \leq k$ .

For a filtered diagram  $X : I \rightarrow \mathbf{Set}$ , the colimit  $\operatorname{colim}_I X$  exists and is isomorphic to  $\coprod_{i \in I} X_i / \sim$ , where  $x_i \in X_i$  and  $x_j \in X_j$  are equivalent if for some  $k \in I$ , we have  $a : i \rightarrow k$  and  $b : j \rightarrow k$  and that  $a(x_i) = b(x_j)$

For concrete categories, we forget the additional structure to the category of sets, and find the colimits there, and give it the additional structure we want.



**Lemma 7.3.** If  $I$  is a directed set, then

$$0 \longrightarrow \bigoplus_{i \in I} A_i \longrightarrow \bigoplus_{i \in I} A_i \longrightarrow \operatorname{colim}_{i \in I} A_i \longrightarrow 0$$

$$(a_i)_{i \in I} \longrightarrow (a_j - f_{ij}(a_i))$$

where  $f_{ij} : i \rightarrow j$ .

**Example 7.4.** The colimit of a sequence given by  $A \xrightarrow{\times n} A$  is  $A \left[ \frac{1}{n} \right]$ .

**Lemma 7.5.** Colimit functor is exact in category of abelian groups.

8 SEPT 8, 2023

- For a sequential diagram

$$\cdots \longrightarrow A_2 \longrightarrow A_1 \longrightarrow A_0$$

the limit of  $A_i$ 's is the terminal cone, and in fact is the kernel of

$$\begin{aligned} \prod_{i \geq 0} A_i &\rightarrow \prod_{i \geq 0} A_i \\ (a_i) &\mapsto (a_i - f_{i+1}(a_{i+1}))_i \end{aligned}$$

However, this sequence is not exact, as we discussed before.

**Lemma 8.1.** Let

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_i & \longrightarrow & B_i & \longrightarrow & C_i \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A_{i-1} & \longrightarrow & B_{i-1} & \longrightarrow & C_{i-1} \longrightarrow 0 \end{array}$$

then we have a long exact sequence

$$0 \longrightarrow \lim A_i \longrightarrow \lim B_i \longrightarrow \lim C_i \longrightarrow \lim^1 A_i \longrightarrow \lim^1 B_i \longrightarrow \lim^1(C_1) \longrightarrow 0$$

*Proof.* Take the products to get

$$\begin{array}{ccccccc} 0 & \longrightarrow & \prod_i A_i & \longrightarrow & \prod_i B_i & \longrightarrow & \prod_i C_i \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \prod_i A_{i-1} & \longrightarrow & \prod_i B_{i-1} & \longrightarrow & \prod_i C_{i-1} \longrightarrow 0 \end{array}$$

and now use the snake lemma. □

**Example 8.2.** The  $p$ -adic integers  $\mathbb{Z}_p = \lim(\cdots \rightarrow \mathbb{Z}/p^k \rightarrow \mathbb{Z}/p^{k+1} \rightarrow \cdots)$  is a limit.

**Theorem 8.3** (Mittag-Leffler Condition). If  $\{A_{i+1} \rightarrow A_i\}$  satisfies for each  $k$ , there is  $i \geq k$  such that  $\operatorname{im}(A_i \rightarrow A_k) \rightarrow \operatorname{im}(A_j \rightarrow A_k)$  for all  $j \geq i \leq k$ , then  $\lim^1(A_i) = 0$ .

**Example 8.4.** 1. This is true if all maps are surjections.

2. This is also true if all  $A_i$ 's are finite.

**Definition 8.5.** Recall that a mapping cylinder is  $M_f = (X \times s[0, 1] \amalg Y) / ((x, 1) \sim f(x))$ , so there is an inclusion  $X \hookrightarrow M_f \cong Y$ . Now given a sequence with  $f_i : X_i \rightarrow X_{i+1}$ , then the mapping telescope is

$$T = \text{Tel}(X_*) = \left( \coprod_{n \geq 0} X_n \times [0, 1] \right) / ((n, x, 1) \sim (n+1, f_n(x), 0)),$$

with

$$\begin{aligned} i_n : X_n &\rightarrow T \\ x &\mapsto (n, x_n, 0) \end{aligned}$$

and homotopies  $(i_n \circ f_{n-1}) \cong i_{n-1} : X_{n-1} \rightarrow T$ . Therefore, the diagrams

$$\begin{array}{ccc} H_*(X_{n-1}) & & \\ \downarrow (f_{n-1})_* & \searrow (i_{n-1})_* & \\ & H_*(T) & \\ \uparrow (i_n)_* & \nearrow & \\ H_*(X_n) & & \end{array}$$

commute. This induces a map  $\text{colim}_n(H_*(X_n)) \rightarrow H_*(T)$ . We claim that this is an isomorphism.

*Proof.* Indeed, consider the refinement

$$\begin{aligned} \lambda : T = \coprod_n X_n \times [0, 1] / \sim &\rightarrow \mathbb{R}_{\geq 0} \\ (n, x, t) &\mapsto n + t \end{aligned}$$

Let  $T_{\leq a} = \lambda^{-1}([0, a])$  or  $T_{< a} = \lambda^{-1}([0, a])$ . We observe that  $T_{\leq n}$  has a homotopy equivalence via  $X_n \hookrightarrow T_{\leq n}$  with a deformation retraction. But  $T_{\leq n}$  is also homotopy equivalent to  $T_{< n+1}$ . The upshot is that it suffices to show that  $\text{colim}(H_*(T_{< n})) \rightarrow H_*(T)$  is an isomorphism.  $\square$

**Proposition 8.6.** Let  $Y$  be a space and let  $\mathcal{A}$  be a collection of subspaces forming a direct system under inclusion. Assume that  $Y = \bigcup_{A \in \mathcal{A}} A$ , and for any compact  $K \subseteq Y$ ,  $K \subseteq A$  for some  $A \in \mathcal{A}$ . Then the map  $\text{colim}_{A \in \mathcal{A}} C_*(A \rightarrow C_*(Y))$  is an isomorphism, hence induces an isomorphism on the level of homology:  $\text{colim}(H_*(A)) \cong H_*(Y)$ .

9 SEPT 11, 2023

Recall that  $H_*(\text{Tel}(X_n)) \cong \text{colim}_n H_*(X_n)$ , with the proof replying on  $C_*(\text{Tel}(X_n)) \cong \text{colim}_n C_*(X_n)$ .

**Example 9.1.**  $\text{Tel}(S^1 \xrightarrow{p} S^1 \xrightarrow{p} \dots) = T = S^1 \left[ \frac{1}{p} \right]$ . Correspondingly, we have  $\text{colim}(H_0(S^1) \cong \mathbb{Z} \xrightarrow{p*} H_0(S^1) \cong \mathbb{Z} \xrightarrow{p*} \dots) = \mathbb{Z}$ , where the induced maps are just identities. Also,  $\text{colim}(H_1(S^1) \cong \mathbb{Z} \xrightarrow{p*} H_1(S^1) \cong \mathbb{Z} \xrightarrow{p*} \dots) = \mathbb{Z} \left[ \frac{1}{p} \right] \cong H_1(T)$ , where the induced maps are multiplications by  $p$ .

By the Universal Coefficient theorem, we can calculate the cohomology of  $T$  as follows:

$$0 \longrightarrow \text{Ext}^1(H_n^1(S^1 \left[ \frac{1}{p} \right]), \mathbb{Z}) \longrightarrow H^n(S^1 \left[ \frac{1}{p} \right]) \text{Hom}(H_n(S^1 \left[ \frac{1}{p} \right]), \mathbb{Z}) \longrightarrow 0$$

Here

- $H^0 * (S^1 \left[ \frac{1}{p} \right]) = \text{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z};$
- $H^1(S^1 \left[ \frac{1}{p} \right]) \cong \text{Hom}(\mathbb{Z} \left[ \frac{1}{p} \right], \mathbb{Z}) = 0$ , since the Ext term is 0;

- Higher homologies are zero, so  $H^2(S^1 \left[ \frac{1}{p} \right]) \cong \text{Ext}(\mathbb{Z} \left[ \frac{1}{p} \right], \mathbb{Z}) \cong \mathbb{Z}_p/\mathbb{Z}$ , the  $p$ -adic integers over  $\mathbb{Z}$ .

We are interested in calculating  $H^*(\text{Tel})$  in terms of  $H^*(X_i)$ 's. Note that the chain complex  $C_*(\text{Tel}(X_i)) \cong \text{colim}_i(C_*X_i)$ , so

$$\begin{aligned} C^*(\text{Tel}(X_i)) &= \text{Hom}(\text{colim}_i(C_*X_i), \mathbb{Z}) \\ &= \lim_i(C^*(X_i)). \end{aligned}$$

Therefore, the question becomes, what is  $H^*(\lim_i(C_i^*))$ ?

**Theorem 9.2** (Milnor Exact Sequence). Suppose  $\{C_i^*\}$  is an inverse system of cochain complexes, such that for each  $n$ ,  $\{C_i^n\}$  is an inverse system that satisfies Mittag-Leffler condition, i.e., we need  $\lim^1_i = 0$ , then we have a short exact sequence

$$0 \longrightarrow \lim_i^1(H^{n-1}(C_i^*)) \longrightarrow H^n(\lim_i C_i^*) \longrightarrow \lim_i(H^n(C_i^*)) \longrightarrow 0$$

*Proof.* We set  $B_i^n = \text{im}(\delta : C_i^{n-1} \rightarrow C_i^n)$ , and  $Z_i^n = \ker(\delta : C_i^n \rightarrow C_i^{n+1})$ . With this notation, we have a system of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z_i^n & \longrightarrow & C_i^n & \xrightarrow{\delta} & B_i^{n+1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Z_{i-1}^n & \longrightarrow & C_{i-1}^n & \xrightarrow{\delta} & B_{i-1}^{n+1} \longrightarrow 0 \end{array}$$

Therefore we have a long exact sequence

$$0 \longrightarrow \lim_i Z_i^n \longrightarrow \lim_i C_i^n \longrightarrow \lim_i B_i^{n+1} \longrightarrow \lim_i^1 Z_i^n \longrightarrow \lim_i^1 C_i^n \longrightarrow \lim_i^1 B_i^{n+1} \longrightarrow 0$$

By assumption,  $\lim_i^1 C_i^n = 0$ , so  $\lim_i^1 B_i^{n+1} = 0$ , and we have the sequence

$$0 \longrightarrow \lim_i Z_i^n \longrightarrow \lim_i C_i^n \longrightarrow \lim_i B_i^{n+1} \longrightarrow \lim_i^1 Z_i^n \longrightarrow 0$$

Denote  $C^* = \lim_i C_i^*$ , and  $Z^n = \ker(C^n \xrightarrow{\delta} C^{n+1})$ , and  $B^n = \text{im}(C^{n-1} \rightarrow C^n)$ . This gives

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z^n & \longrightarrow & C^n & & \lim_i B_i^{n+1} \longrightarrow \lim_i^1 Z_i^n \longrightarrow 0 \\ & & & & \searrow \delta & \nearrow & \\ & & & & B_{n+1} & & \end{array}$$

We know have  $0 \subseteq B^{n+1} \subseteq \lim_i B_i^{n+1} \subseteq \lim_i Z_i^{n+1} = Z^{n+1}$ , therefore this gives an exact sequence

$$0 \longrightarrow \lim_i B_i^{n+1}/B^{n+1} \longrightarrow Z^{n+1}/B^{n+1} \longrightarrow Z^{n+1}/\lim_i B_i^{n+1} \longrightarrow 0$$

so this is

$$0 \longrightarrow \lim_i^1 Z_i^n \longrightarrow H^{n+1}(C^*) \longrightarrow Z^{n+1}/\lim_i B_i^{n+1} \longrightarrow 0$$

From the canonical exact sequence

$$0 \longrightarrow B_i^n \longrightarrow Z_i^n \longrightarrow H^n(C_i^*) \longrightarrow 0$$

we induce

$$0 \longrightarrow \lim_i B_i^n \longrightarrow Z^n \longrightarrow \lim_i H^n(C_i^n) \longrightarrow \lim_i^1 B_i^n \longrightarrow \lim_I^1 Z_i^n \longrightarrow \lim_i^1 H^n(C_i^*) \longrightarrow 0$$

but we have  $\lim_i^1 B_i^n = 0$ , so  $\lim_i^1 Z_i^n \cong \lim_i^1 H^n(C_i^*)$ , therefore we identify  $Z^n / \lim_i B_i^{n+1} \cong \lim_i H^{n+1}(C_i^*)$ .  $\square$

**Corollary 9.3.** Let  $X \in \mathbf{Top}$  and  $X = \bigcup_i X_i$  such that if there is compact  $K \subseteq X$ , then there exists some  $i$  such that  $K \subseteq X_i$ . If this is the case, then we have a short exact sequence in cohomology given by

$$0 \longrightarrow \lim_i^1 H^{n-1}(X_i) \longrightarrow H^n(X) \longrightarrow \lim_i H^n(X_i) \longrightarrow 0$$

*Proof.* We have  $C_*(X) \cong \text{colim}(C_*(X_i))$ , and  $C^*(X) \cong \lim C^*(X_i)$ .

**Claim 9.4.**  $\lim_i^1 (C^n(X_i)) = 0$  for all  $n$ .

*Subproof.* We want the open cover of  $X$  to be a direct system, i.e., nested in some sense, so that we have a telescope and by the Mittag-Leffler condition we win. For instance, if we have telescopes, then  $T = \text{Tel}(X_0 \rightarrow X_1 \rightarrow \cdots)$ , then  $\bigcup_n T_{\leq n}$  gives  $T_{\leq 0} \subseteq T_{\leq 1} \subseteq \cdots \subseteq T = \bigcup T_{\leq i}$ . The point being, now we have  $T_{\leq i} \cong X_i$  by deformation retraction, so we have a Milnor exact sequence on the level of cohomology of  $T$ , and we are done.  $\blacksquare$

$\square$

**Example 9.5.**

$$0 \longrightarrow \lim^1 H^1(S^1) \xrightarrow{\cong} H^2(S^1 \left[ \frac{1}{p} \right]) \longrightarrow H^2(S^1) \longrightarrow 0$$

where  $\lim^1 H^1(S^1)$  is  $\lim^1(\cdots \rightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \rightarrow \cdots) \cong \mathbb{Z}_p / \mathbb{Z}$ .

10 SEPT 13, 2023

We now want to define a map on cohomology groups. Let  $R$  be a commutative ring, and let  $\varphi_i \in C^{n_i}(X, R)$  be with  $i = 1, 2$ , then we can define the cup product on  $\smile$  with

$$\begin{aligned} C^{n_1}(X, R) \times C^{n_2}(X, R) &\rightarrow C^{n_1+n_2}(X, R) \\ (\varphi_1 \smile \varphi_2)(\sigma) &= \varphi_1(\sigma|_{[v_0, \dots, v_{n_1}]}) \varphi_2(\sigma|_{[v_{n_1}, \dots, v_{n_1+n_2}]}) \end{aligned}$$

and we extend it linearly. Note that if  $n_1 = 0$ , then the map sends  $\sigma$  to  $\varphi_1(\sigma|_{v_0})\varphi_2(\sigma)$ . Moreover, if  $\varphi_1 = e$  is the constant mapping with image 1, then  $e \smile \varphi = \varphi = \varphi \smile e$ . By associativity, we know  $C^*(X, R)$  is a graded ring.

**Lemma 10.1.**  $\smile$  is functorial in  $X$ , that is, if  $f : X \rightarrow Y$ , then  $f^* : C^*(Y, R) \rightarrow C^*(X, R)$  is a ring homomorphism.

**Lemma 10.2.**  $\partial(\varphi_1 \smile \varphi_2) = \partial\varphi_1 \smile \varphi_2 + (-1)^{|\varphi_1|} \varphi_1 \smile \partial\varphi_2$ .

**Corollary 10.3.** • If  $\varphi_1, \varphi_2 \in Z^*$  are cocycles, then the cup product  $\varphi_1 \smile \varphi_2 \in Z^*$ .

• If  $\varphi_i \in Z^*$ , and one is in  $B^*$ , then  $\varphi_1 \smile \varphi_2 \in B^*$ .

Using these two facts, we know that  $\smile : H^{n_1}(X, \mathbb{R}) \times H^{n_2}(X, \mathbb{R}) \rightarrow H^{n_1+n_2}(X, \mathbb{R})$  is an induced map. In particular, if  $X$  is connected, then  $H^0(X, R) \cong R$ , and the cup product becomes the product on  $R$ . This has a graded ring structure.

**Theorem 10.4.** The cohomology cup product satisfies:

1. naturality in  $X$ ,

2.  $1 \smile \alpha = \alpha = \alpha \smile 1$  for  $\alpha \in H^*(X, R)$ . This is given by  $1 : C_0 X \rightarrow R$  with  $\sigma : \Delta^0 \rightarrow X$  sent to 1. Therefore,  $1 = [1]$ .
3.  $\alpha \smile (\beta \smile \gamma) = (\alpha \smile \beta) \smile \gamma$ .
4.  $\alpha \smile \beta = (-1)^{|\alpha||\beta|} \beta \smile \alpha$ .
5. For any pair  $(X, A)$  with  $i : A \hookrightarrow X$  with  $\delta : H^*(A; R) \rightarrow H^{*+1}(X, A; R)$ , then for  $\alpha \in H^*(A; R)$  and  $\beta \in H^*(X; R)$ , then  $\delta(\alpha \smile i^* \beta) = \delta(\alpha) \smile \beta$ , and  $\delta(i^* \beta \smile \alpha) = (-1)^{|\beta|} \beta \smile \delta(\alpha)$ .

**Remark 10.5.** The cup product  $\smile$  comes from  $C^*(X) \otimes C^*(X) \rightarrow C^*(X)$ , also regarded as  $\text{Hom}(C_* X, R) \otimes \text{Hom}(C_* X, R) \rightarrow \text{Hom}(C_* X, R)$ , which is given by the factoring via  $\text{Hom}(C_* X \otimes C_* X, R)$ . This gives a pairing on  $C^* X$  if we have a commutative diagram

$$\begin{array}{ccc} C_* X & \longrightarrow & C_* X \otimes C_* X \\ \downarrow \sigma_n \mapsto 0 & & \downarrow \\ \mathbb{Z} & \xlongequal{\quad} & \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z} \end{array}$$

The map  $C_* X \rightarrow C_* X \otimes C_* X$  is called the diagonal approximation. More generally, if we think of  $X$  and  $Y$ , then we have

$$\begin{array}{ccc} C_*(X \times Y) & \longrightarrow & C_* X \otimes C_* Y \\ \downarrow & & \downarrow \\ \mathbb{Z} & \xlongequal{\quad} & \mathbb{Z} \end{array}$$

In particular, if  $X = Y$ , then we have a diagonal mapping  $X \rightarrow X \times X$ , therefore induces  $C_* X \rightarrow C_*(X \times X)$ .

**Definition 10.6.** The Alexander-Whitney map is given by

$$AW_{X,Y} : C_*(X \times Y) \rightarrow C_* X \otimes C_* Y$$

where  $C_* X \otimes C_* Y$  is given by total complex of degree  $n$ , i.e.,  $\bigoplus_{i+j=n} C_i X \otimes C_j Y$ , and differential  $\partial(a \otimes b) = \partial a \otimes b + (-1)^{|a|} a \otimes \partial b$ .

$$\begin{array}{ccccc} & & X & & \\ & \nearrow \sigma & \uparrow \pi_X & & \\ \Delta^n & \xrightarrow{(\sigma, \tau)} & X \times Y & & \\ & \searrow \tau & \downarrow \pi_Y & & \\ & & Y & & \end{array}$$

The Alexander-Whitney map defines  $AW(\sigma, \tau) = \sum_{i+j=n} \sigma|_{[v_0, \dots, v_i]} \otimes \tau|_{[v_i, \dots, v_n]}$ . On the level of cochains, the cup product is  $\text{Hom}(-, R)$  of composition of Alexander-Whitney map and the induced diagonal mapping.

Similarly, we can define the cochain version, with a pair  $(X, A)$ , then

$$\begin{array}{ccc} C_*(X \times Y, A \times Y) & \dashrightarrow & C_*(X, A) \otimes C_* Y \\ \uparrow & & \uparrow \\ C_*(X \times Y) & \xrightarrow{AW_{X \times Y}} & C_* X \otimes C_* Y \\ \uparrow & & \uparrow \\ C_*(A \times Y) & \xrightarrow{AW_{A,Y}} & C_* A \otimes C_* Y \end{array}$$

We now want  $(X, A) \times (Y, B) = (X \times Y, A \times Y \cup X \times B)$  to have the suitable mapping. Naturally, we get the Alexander-Whitney map

$$\begin{array}{ccc} C_*(X \times Y)/(C_*(X \times B) + C_*(A \times Y)) & \longrightarrow & C_*(X, A) \otimes C_*(Y, B) \\ \downarrow & \dashrightarrow & \\ C_*(X \times Y)/C_*(A \times Y \cup X \times B) & & \end{array}$$

The summation is not the direct sum but not summation in complex.

11 SEPT 15, 2023

Recall that the Alexander-Whitney map is the natural transformation of functors  $\mathbf{Top} \times \mathbf{Top} \rightarrow \mathbf{Ch}$  via  $C_*(X \times Y) \rightarrow C_*(X) \rightarrow C_*(Y)$ , where

$$AW(\sigma, \tau) = \sum_{i+j=n} \sigma|_{[v_0, \dots, v_i]} \otimes \tau|_{[v_i, \dots, v_n]}$$

for  $\sigma, \tau : \Delta^n \rightarrow X \times Y$ . We also note that the cross product is defined as the composition

$$\begin{array}{ccc} H^*(\text{Hom}(C_*X, R)) \otimes H^*(\text{Hom}(C_*Y, R)) & \longrightarrow & H^*(\text{Hom}(C_*X \otimes C_*Y, R)) \\ & \searrow \times & \downarrow AW^* \\ & & H^*(\text{Hom}(C_*(X, Y), R)) \end{array}$$

where the horizontal map is induced by homological algebra. The cup product is composed by the diagonal inclusion and the cross product:

$$\begin{array}{ccccc} H^*(X) \otimes H^*(X) & \xrightarrow{\times} & H^*(X \times X) & \xrightarrow{\Delta^*} & H^*(X) \\ & \searrow & & \nearrow & \\ & & & & \end{array}$$

given by

$$(f \smile g)(\sigma) = f(\sigma|_{[v_0, \dots, v_i]}) g(\sigma|_{[v_i, \dots, v_{i+j}]})$$

for  $f \in H^i(X)$ ,  $g \in H^j(X)$ ,  $\sigma : \Delta^{i+j} \rightarrow X$ .

**Remark 11.1.** • If  $X$  is connected, then  $H^0(X, R) = R$ .

• The cup product gives the  $R$ -module structure on  $H^n(X)$ .

**Example 11.2.** Let  $X = S^n$ , then

$$H^*(X, R) = \begin{cases} R, & * = 0, n \\ 0, & \text{otherwise} \end{cases}$$

This says that the induced multiplication map  $R \otimes R \rightarrow R$  on cohomology has the same behavior, i.e.,  $H^n(S^n; R) \otimes H^n(S^n; R) \rightarrow H^{2n}(S^n, R) = 0$ . That is, we have  $H^*(S^n; R) \cong R[e_n]/e_n^2$ .

For the unit interval  $I = [0, 1]$ , then

$$\tilde{H}^*(S^1) \cong H^*(I, \partial I) = \begin{cases} \mathbb{Z}, & * = 1 \\ 0, & \text{otherwise} \end{cases}$$

**Claim 11.3.**

$$H^1(I, \partial I) \otimes H^n(Y) \xrightarrow{\times} H^{n+1}(I \times Y, \partial I \times Y)$$

is an isomorphism for any  $Y$ .

**Corollary 11.4.**

$$H^*(S^1) \otimes H^*(Y) \xrightarrow{\times} H^*(S^1 \times Y)$$

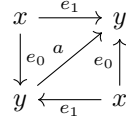
is an isomorphism for any space  $Y$ .

**Example 11.5.** Consider the Moore spaces. For any  $m \in \mathbb{Z}$ , we have  $X_m = S^1 \cup_m e^2$ , so we have

$$\begin{array}{ccc} S^1 & \longrightarrow & D^2 \\ m \downarrow & & \downarrow \\ S^1 & \longrightarrow & X_m \end{array}$$

We can give this a cell structure, so for instance  $m = 2$ , we have  $X_m = \mathbb{R}P^2$ . For general  $m$ , we have the cell structure with vertices  $x$  and  $y$ ,  $m+1$  edges  $a, e_0, \dots, e_m$ , and  $m$  faces  $C_0, \dots, C_{m-1}$ , then the boundary map is given by  $\partial(a) = 0$ ,  $\partial(e_i) = y - x$ , and  $\partial(C_i) = a - e_{i+1} + e_i$ .

In the case  $m = 2$ , we have



where the upper triangle is the face  $C_0$  and the bottom triangle is the face  $C_1$ . We look at the chain equivalences

$$\begin{array}{ccccccc} C_0 X_2 & \longleftarrow & C_1 X_2 & \longleftarrow & C_0 X_2 & \longleftarrow & \dots \\ \uparrow & & \uparrow & & \uparrow & & \\ \mathbb{Z}\{x, y\} & \longleftarrow & \mathbb{Z}\{a, e_0, e_1\} & \longleftarrow & \mathbb{Z}\{C_0, C_1\} & \longleftarrow & \dots \end{array}$$

The integral cohomology is just the cohomology of the above chain with respect to the dual basis, then checking the kernel and image, we know  $\delta(x^\vee) = -e_0^\vee - e_1^\vee$ ,  $\delta(y^\vee) = e_0^\vee + e_1^\vee$ ,  $\delta(a^\vee) = C_0^\vee + C_1^\vee$ , and  $\delta(e_0^\vee) = C_0^\vee - C_1^\vee$ ,  $\delta(e_1^\vee) = C_1^\vee - C_0^\vee$ , therefore  $x^\vee + y^\vee$  generates  $H^0$ . Similarly, we can show that  $H^1 = 0$  and  $H^2 = \mathbb{Z}/2$ .

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We need to prove that  $\alpha \smile \beta = (-1)^{|\alpha||\beta|} \beta \smile \alpha$  in  $H^*(X; k)$ . Define  $\rho : \Delta^n \rightarrow \Delta^n$  by sending  $[v_0, \dots, v_n]$  to  $[v_n, \dots, v_0]$ . Using this, we can define a map

$$\begin{aligned} \rho : C_* X &\rightarrow C_* X \\ \sigma &\mapsto (-1)^{\varepsilon_n} \sigma|_{[v_n, \dots, v_0]} \end{aligned}$$

where  $\varepsilon_n$  is the number of permutations required to permute  $(0, \dots, n)$  into  $(n, \dots, 0)$ . This should just be  $\binom{n+1}{2}$ .

**Exercise 12.1.**  $\rho$  is a chain map.

This induces  $\rho : C^* X \rightarrow C^* X$  with  $\rho(\alpha)(\sigma) = (-1)^{\varepsilon_i} \alpha(\sigma)|_{[v_n, \dots, v_0]}$ . Therefore,

$$\begin{aligned} \rho(\alpha \smile \beta)(\sigma) &= (-1)^{\varepsilon_n} (\alpha \smile \beta)(\sigma|_{[v_n, \dots, v_0]}) \\ &= (-1)^{\varepsilon_n} \alpha(\sigma|_{[v_n, \dots, v_j]}) \beta(\sigma|_{[v_j, \dots, v_0]}) \\ &= (-1)^{\varepsilon_n} (-1)^{\varepsilon_i} \alpha(\sigma|_{[v_j, \dots, v_n]}) \cdot (-1)^{\varepsilon_j} \rho(\beta)(\sigma|_{[v_0, \dots, v_j]}) \\ &= (-1)^{\varepsilon_n + \varepsilon_i + \varepsilon_j} \rho(\beta) \smile \rho(\alpha)(\sigma). \end{aligned}$$

**Claim 12.2.**  $\varepsilon_i + \varepsilon_j - \varepsilon_{i+n} \equiv ij \pmod{2}$ .

In particular, this proves the claim. Moreover,  $\rho$  is a chain equivalence.

**Proposition 12.3.** If  $f, g : C_* X \rightarrow C_* X$  are natural transformations of functors  $\mathbf{Top} \rightarrow \mathbf{Ch}$ , such that  $f_0$  and  $g_0$  are naturally isomorphic (as components of the natural transformations), then  $f$  and  $g$  are naturally equivalent. Here  $f_0, g_0 : \mathbf{Top} \rightarrow \mathbf{Ab}$ .

**Theorem 12.4.** Given a functor  $F : \mathcal{C} \rightarrow \mathbf{Ch}$ , there is an equivalence of categories  $\mathbf{Func}(\mathcal{C}, \mathbf{Ch}) \cong \mathbf{Ch}(\mathbf{Func}(\mathcal{C}, \mathbf{Ab}))$ .

To prove the theorem, we introduce acyclic models. Suppose we have a functor  $F : \mathcal{C} \rightarrow \mathbf{Ch}$ . We regard  $\mathcal{C}$  as  $\mathbf{Top}$ , or  $\mathbf{Top} \times \mathbf{Top}$ .

**Definition 12.5.** A functor  $F : \mathcal{C} \rightarrow \mathbf{Ab}$  is called free on models  $M$  if

- there exists a set  $M \subseteq \mathbf{Ob}(\mathcal{C})$  such that  $F$  is naturally isomorphic to the functor defined by the mapping  $X \mapsto \bigoplus_{A \in M} \mathbb{Z}\{\mathrm{Hom}_{\mathcal{C}}(A, X)\}$ .

**Remark 12.6.** Note that if  $G : \mathcal{C} \rightarrow \mathbf{Set}$  is representable with respect to  $A \in \mathcal{C}$ , then the composition of the free functor  $\mathbf{Set} \rightarrow \mathbf{Ab}$  and  $G : \mathcal{C} \rightarrow \mathbf{Set}$  is free on model  $A$ .

- A functor  $F_* : \mathcal{C} \rightarrow \mathbf{Ch}$  is free on models  $\{M_n\}_{n \in \mathbb{Z}}$  if each  $F_n : \mathcal{C} \rightarrow \mathbf{Ab}$  is free on  $M_n$ .
- Given  $M \subseteq \text{Ob}(\mathcal{C})$ , a functor  $F : \mathcal{C} \rightarrow \mathbf{Ch}$  is  $M$ -acyclic in positive degrees if for all  $A \in M$ ,  $H_q(F(A)) = 0$  for all  $q > 0$ .

**Example 12.7.**  $C_* : \mathbf{Top} \rightarrow \mathbf{Ch}$  is acyclic in positive degrees on  $\{\Delta^n\}_{n \in \mathbb{Z}}$ .

**Example 12.8.** Consider  $\mathbf{Top}^2 \rightarrow \mathbf{Ch}$ .

1. If we have  $(X, Y) \mapsto C_*(X \times Y)$ , then  $C_n(- \times -)$  is free on the model  $\Delta^n \times \Delta^n$ , and  $C_*(- \times -)$  is acyclic on  $\{\Delta^p \times \Delta^q\}_{p, q \geq 0}$ .
2. If we have  $(X, Y) \mapsto C_*(X) \otimes C_*(Y)$ , then  $(C_*(-) \otimes C_*(-))_n$  is free on the models  $\{\Delta^p \times \Delta^{n-p}\}_p$ , which is acyclic in positive degrees on  $\{\Delta^p \times \Delta^q\}$ .

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**Theorem 13.1** (Acyclic Models). Suppose  $F_*, G_* : \mathcal{C} \rightarrow \mathbf{Ch}$  are functors, and assume  $F_n = 0 = G_n$  for  $n < 0$ . Assume

- (a) each  $F_n : \mathcal{C} \rightarrow \mathbf{Ab}$  is free on models  $M_n \subseteq \text{ob}(\mathcal{C})$ , and
- (b)  $G_*$  is acyclic in positive degrees on  $\bigcup_{n \geq 0} M_n$ ,

then

1. any natural transformation  $H_0 F_* \rightarrow H_0 G_*$  of functors  $\mathcal{C} \rightarrow \mathbf{Ab}$  is induced by a natural transformation  $F_* \rightarrow G_*$ , and
2. if  $f, g : F_* \rightarrow G_*$  are natural transformations such that  $H_0 f = H_0 g$ , then there exists a natural chain homotopy  $f \simeq g$ , and
3. assume, in addition, that  $G_*$  is free on some model  $N$ , then if  $f : F_* \rightarrow G_*$  is a natural transformation such that  $H_0 f : H_0 F_* \rightarrow H_0 G_*$  is a natural isomorphism, then  $f$  is a natural chain equivalence.

**Claim 13.2.** Any natural transformation  $C_* X \rightarrow C_* X$  that induces an isomorphism  $H_0 X \rightarrow H_0 X$  is a chain equivalence.

**Example 13.3.** Take  $\rho : C_* X \rightarrow C_* X$  that inverts orientation, then  $\rho$  induces identity on cohomology, so

$$\alpha \smile \beta = \rho(\alpha \smile \beta) = (-1)^{|\beta| \times |\alpha|} \rho(\beta) \smile \rho(\alpha) = (-1)^{|\beta| \times |\alpha|} \beta \smile \alpha$$

**Claim 13.4.**

$$AW : C_*(X \times Y) \rightarrow C_*(X) \otimes C_*(Y)$$

is a natural chain equivalence.

*Proof.* Apply acyclic models. □

**Lemma 13.5** (Yoneda). If  $G : \mathcal{C} \rightarrow \mathbf{Set}$  is a functor, and let  $C \in \text{Ob}(\mathcal{C})$  be a representation of the functor, that is,  $F_c(d) = \text{Hom}_{\mathcal{C}}(c, d)$ , then there is a natural bijection of sets  $\text{Nat}(F_c, G) \cong G(c)$  by  $f : F_c \rightarrow G \mapsto f(\text{id}_c)$ .

**Corollary 13.6.** If  $F : \mathcal{C} \rightarrow \mathbf{Ab}$  is free on models  $M$ , that is,  $F(X) = \mathbb{Z} \left\{ \coprod_{A \in M} \text{Hom}_{\mathcal{C}}(A, X) \right\} \cong \bigoplus_{A \in M} \mathbb{Z} \{F_A(X)\}$ , which induces

$$F : \mathcal{C} \xrightarrow{\coprod_{A \in M} F_A} \mathbf{Set} \xrightarrow{\text{Free}} \mathbf{Ab}$$

then for any  $G : \mathcal{C} \rightarrow \mathbf{Ab}$ , then we have a natural isomorphism  $\text{Nat}(F, G) \cong \prod_{A \in M} G(A)$  given by  $(f : F \rightarrow G) \mapsto (f(\text{id}_A))_{A \in M}$ .



We will now prove the acyclic models theorem.

*Proof.* 1. Take  $F_* \rightarrow G_*$ , then we are given a natural transformation  $\bar{\varphi}_{-1} : H_0 F_* \rightarrow H_0 G_*$  with

$$\begin{array}{ccccccc} 0 & \longleftarrow & H_0 F_* & \longleftarrow & F_0 & \longleftarrow & F_1 & \longleftarrow & \cdots \\ & & \downarrow \bar{\varphi}_{-1} & & & & & & \\ 0 & \longleftarrow & H_0 G_* & \longleftarrow & G_0 & \longleftarrow & G_1 & \longleftarrow & \cdots \end{array}$$

We want to lift  $\varphi_0 \in \text{Nat}(F_0, G_0)$ , we take a look into the commutative diagram

$$\begin{array}{ccc} \text{Nat}(F_0, G_0) & \xrightarrow{\cong} & \prod_{A \in M_0} G_0(A) \\ \partial_G \downarrow & & \downarrow \\ \text{Nat}(F_0, H_0 G_0) & \xrightarrow{\cong} & \prod_{A \in M_0} H_0 G_0(A) \end{array}$$

so we take  $\varphi_{-1} \circ \partial_F \in \prod_{A \in M_0} H_0 G_0(A)$ , then lift it to  $\varphi_0 \in \prod_{A \in M_0} G_0(A)$ , then we obtain  $\varphi_0 : F_0 \rightarrow G_0$  as desired.

By construction,  $\partial_G \varphi_0 = \varphi_{-1} \partial_F$ . Proceeding inductively, we complete the diagram.

2. Now given  $f, g : F_* \rightarrow G_*$ , with  $H_0 f = H_0 g$ , we want  $f \simeq g$ . We want  $h_i : F_i \rightarrow G_i$  to be such that  $f_i - g_i = h_{i-1} \partial_F + \partial_F h_i$ . □

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A complex  $C_*$  that is chain equivalent to 0 implies it is acyclic, i.e.,  $H_q(C_*) = 0$  for all  $q$ .

**Proposition 14.1.** If  $C_*$  is a complex of free abelian groups with  $C_n = 0$  for  $n \ll 0$ , then  $C_*$  is acyclic if and only if it is chain equivalent to 0.

*Proof.* We can assume  $C_n = 0$  for  $n < 0$ . Now consider  $F : \mathcal{C} = \{*\} \rightarrow \mathbf{Ch}$  where  $F(*) = C_*$ . Now  $F$  is free and acyclic on models  $\{*\}$ , then the identity and zero map gives the same map on  $H_0$ , and by the acyclic model theorem we are done. □

**Example 14.2.** If  $X \in \mathbf{Top}$ , then  $X$  is acyclic if  $H_* X = \begin{cases} \mathbb{Z}, * = 0 \\ 0, \text{otherwise} \end{cases}$ , and so we extend the kernel and get a short exact sequence

$$0 \longrightarrow \tilde{C}_* X \longrightarrow C_* X \twoheadrightarrow C_* \{*\} \longrightarrow 0$$

Note that the last map admits a section with respect to a choice of a point  $x_0 \in X$ . Therefore,  $X$  is acyclic if and only if  $\tilde{C}_* X$  is acyclic. Also,  $\tilde{C}_* X$  is a complex of free abelian groups, so  $\tilde{C}_* X$  being acyclic implies  $\tilde{C}_* X$  is chain equivalent to 0. Therefore,  $C_* X$  is chain homotopic to zero, as a complex concentrated at degree 0.

For instance, let  $X = \Delta^p$  or  $\Delta^p \times \Delta^q$ .

**Corollary 14.3** (Eilenberg-Zilber). For any  $X, Y \in \mathbf{Top}$ ,  $C_*(X \times Y) \cong C_* X \otimes C_* Y$ .

**Claim 14.4.** There is an anti-commutative diagram

$$\begin{array}{ccc} H^p(X) \times H^q(Y) & \xrightarrow{\times} & H^{p+q}(X \times Y) \\ s \downarrow \cong & & \downarrow s^* \\ H^q(Y) \times H^p(X) & \xrightarrow[\times]{} & H^{q+p}(Y \times X) \end{array}$$

with  $\alpha \times \beta = (-1)^{|\alpha||\beta|} s^*(\beta \times \alpha)$ .

This follows from

**Lemma 14.5.**

$$\begin{array}{ccc} C_*(X \times Y) & \xrightarrow{AW} & C_*(X) \otimes C_*(Y) \\ s_* \downarrow & & \uparrow T \\ C_*(Y \times X) & \xrightarrow{AW} & C_*(Y) \otimes C_*(X) \end{array}$$

where  $T$  is a twist map via  $T(y \otimes x) = (-1)^{|x||y|}x \otimes y$ .

**Theorem 14.6** (Kunneth). Let  $C_*, D_* \in \mathbf{Ch}$ , say  $C_*$  is built out of free abelian groups, then

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(C_*) \otimes H_j(D_*) \longrightarrow H_n(C_* \otimes D_*) \longrightarrow \bigoplus_{i+j=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_i C_*, H_j D_*) \longrightarrow 0$$

splits unnaturally.

**Remark 14.7.**  $\mathrm{Tor}(M, A) \cong \mathrm{Tor}(A, M)$ .

**Example 14.8.**  $\mathrm{Tor}(A, \mathbb{Z}) = 0 = \mathrm{Tor}(\mathbb{Z}, A)$ , and  $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$ .

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**Remark 15.1.** Similar results for chain complexes of  $R$ -modules for PID  $R$  holds. If  $R$  is not a PID, then there may be extra terms.

**Example 15.2.** If  $C_*, D_*$  are chain complexes of  $k$ -modules for a field  $k$ , then  $H_*(C_* \otimes_k D_*) \cong H_*(C_*) \otimes_k H_*(D_*)$ .

**Theorem 15.3** (Kunneth). If  $X, Y \in \mathbf{Top}$ , there is a short exact sequence

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \longrightarrow H_n(X \times Y) \longrightarrow \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(X), H_j(Y)) \longrightarrow 0$$

which splits unnaturally.

*Proof.* Identify  $C_*(X \times Y) \cong C_*(X) \otimes C_*(Y)$  and use the previous Kunneth theorem.  $\square$

**Remark 15.4.** Let  $(X, A) \times (Y, B) = (X \times Y, A \times Y \cup X \times B)$ , then  $H_*(X, A) \cong \tilde{H}_*(X) = \ker(H_*(X) \rightarrow H_*(A)) \cong \mathrm{coker}(H_*(A) \rightarrow H_*(X))$ .

**Definition 15.5** (Smash Product). We denote  $(X, *) \times (Y, *) \cong (X \times Y, C := X \times * \cup * \times Y)$ , then  $X \wedge Y = X \times Y / C$  is the smash product.

**Theorem 15.6** (Kunneth). We have an unnatural short exact sequence

$$0 \longrightarrow \bigoplus_{i+j=n} \tilde{H}_i(X) \otimes \tilde{H}_j(Y) \longrightarrow \tilde{H}_n(X \wedge Y) \longrightarrow \bigoplus_{i+j=n-1} \mathrm{Tor}(\tilde{H}_i(X), \tilde{H}_j(Y)) \longrightarrow 0$$

**Example 15.7.** Take  $X = S^k$ , then  $S^k \wedge Y = \Sigma^k Y$  with  $Y = S^m$ , i.e.,  $S^k \wedge S^m \cong S^{k+m}$ . Therefore,  $\tilde{H}_n(\Sigma^k Y) \cong \tilde{H}_{n-k}(Y)$  as the suspension isomorphism.

**Example 15.8.** Let  $A$  be an abelian group, then there is a space  $M(A, n)$  with the property that  $\tilde{H}_*(M(A, n)) = A$  if  $*$  is  $n$  and is 0 otherwise.

- If  $A = \mathbb{Z}$ , then  $M(A, n) = S^n$ ;
- $A = \mathbb{Z} \left[ \frac{1}{p} \right]$ , then  $M(A, n)$  is the mapping telescope of  $S^n \xrightarrow{p} S^n \rightarrow \cdots$ ;
- if  $A = \mathbb{Z}/k\mathbb{Z}$ , then  $M(A, n) \cong S^n \cup e^{n+1}$ .

Therefore,

$$0 \longrightarrow \tilde{H}_r(Y) \otimes A \longrightarrow \tilde{H}_{n+r}(M(A, n) \wedge Y) \longrightarrow \text{Tor}(A, \tilde{H}_{r-1}(Y)) \longrightarrow 0$$

One can compare this to the universal coefficient theorem for homology, i.e.,  $\tilde{H}_*(Y \wedge M(A, n)) = \tilde{H}_*(Y, A)$ .

**Definition 15.9.** If  $A^*$  and  $B^*$  are cochain complexes, with a multiplication structure, then  $A^* \otimes B^*$  also has a multiplicative structure by  $(a \otimes b)(a' \otimes b') = (-1)^{|a||b|}(aa' \otimes bb')$ .

For instance,  $H^*X \otimes H^*Y$  is a graded commutative ring.

**Proposition 15.10.** The cross product  $\times : H^*X \otimes H^*Y \rightarrow H^*(X \times Y)$  is a map of graded rings via  $(a \times b) \smile (a' \times b') = (-1)^{|a'||b|}(a \smile a') \times (b \smile b')$ .

*Proof.* Consider the diagonal maps  $\text{diag}_{X \times Y} : X \times Y \rightarrow X \times Y \times X \times Y$  and  $\text{diag}_X \times \text{diag}_Y : X \times Y \rightarrow X \times X \times Y \times Y$ , then the left-hand side is just  $\text{diag}_{X \times Y}^*(a \times b \times a' \times b')$ , and

$$\begin{aligned} (a \smile a') \times (b \smile b') &= \text{diag}_X^*(a \times a') \times \text{diag}_Y^*(b \times b') \\ &= (\text{diag}_X \times \text{diag}_Y)^*(a \times a' \times b \times b') \\ &= (\text{diag}_{X \times Y})^*(1 \times \tau \times 1)^*(a \times a' \times b \times b') \\ &= (\text{diag}_{X \times Y})^*(-1)^{|a'||b|}a \times b \times a' \times b' \end{aligned}$$

where  $\tau$  swaps  $X \times Y$  to  $Y \times X$ , therefore  $1 \times \tau \times 1$  factors  $\text{diag}_X \times \text{diag}_Y$  via  $\text{diag}_{X \times Y}$ .  $\square$

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**Theorem 16.1** (Kunneth). Let  $X$  and  $Y$  be topological spaces such that  $H_n(Y)$  is finitely-generated as abelian groups, then we have a short exact sequence

$$0 \longrightarrow \bigoplus_{i+j=n} H^i(X) \otimes H^j(Y) \xrightarrow{\times} H^n(X \times Y) \longrightarrow \bigoplus_{i+j=n+1} \text{Tor}(H^i(X), H^j(Y)) \longrightarrow 0$$

**Remark 16.2.** 1. One can think of this as

$$0 \longrightarrow H^*(X) \otimes H^*(Y) \xrightarrow{\times} H^*(X \times Y) \longrightarrow \text{Tor}(H^*(X), H^{*+1}(Y)) \longrightarrow 0$$

2. same for coefficients in a PID;

3. If  $k$  is a field, then we have a Kunneth isomorphism  $H^*(X, k) \otimes_k H^*(Y, k) \cong H^*(X \times Y)$ .

*Proof.* Consider

$$\begin{array}{ccc} C^*(X) \otimes C^*(Y) & \xrightarrow{\quad \quad \quad} & C^*(X \times Y) \\ & \searrow & \nearrow \cong \\ & \text{Hom}(C_*(X) \otimes C_*(Y), \mathbb{Z}) & \text{AW}^* \end{array}$$

note that the first map in the splitting is not an equivalence in general. If  $H_n(Y)$  is finitely-generated in each degree, then there is a complex  $D_*$  such that each  $D_n$  is finitely-generated, and  $D_*D \cong C_*Y$ , so

$$\begin{array}{ccc} \text{Hom}(C_*X \otimes D_*, \mathbb{Z}) & \xleftarrow{\sim} & \text{Hom}(C_*X \otimes C_*Y, \mathbb{Z}) \\ \uparrow \cong & & \uparrow \text{dashed} \\ C^*X \otimes \text{Hom}(D_*, \mathbb{Z}) & \xleftarrow{\sim} & C^*X \otimes C_*Y \end{array}$$

Note that in this case, the dashed map is an equivalence. By the algebraic Kunneth theorem, we are done.  $\square$

- Example 16.3.** 1. If  $X = S^m$ , then  $\times : H^*(S^m) \otimes H^*(Y) \xrightarrow{\sim} H^*(S^m \times Y)$ , so  $H^*(S^m \times Y) \cong H^*(Y)[e_m]/e_m^2$ .
2.  $H^*(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{F}_2) = \mathbb{F}_2[x, y]/(x^3, y^3)$ , with  $|x| = |y| = 1$ .
3.  $H^*(\mathbb{C}P^2) = \mathbb{Z}[x]/x^3$ , with  $|x| = 2$ . Note that  $\mathbb{C}P^2$  and  $S^2 \vee S^4$  have the same cohomology, but different cohomology rings. The first one is obtained by attaching a 3-cell on  $S^2$  by the Hopf map, and the second one is obtained by attaching a 4-cell on  $S^2$  by the trivial map.

**Definition 16.4** (Bockstein Operation). Consider a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of abelian groups, we then have two connecting homomorphisms  $H_n(X, C) \rightarrow H_{n-1}(X, A)$  and  $H^n(X, C) \rightarrow H^{n+1}(X, A)$ . For instance, consider

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

and therefore we have Bockstein maps  $\beta : H_n(X, \mathbb{Z}/p\mathbb{Z}) \rightarrow H_{n-1}(X, \mathbb{Z}/p\mathbb{Z})$  as well as  $\beta : H^n(X, \mathbb{Z}/p\mathbb{Z}) \rightarrow H^{n+1}(X, \mathbb{Z}/p\mathbb{Z})$ . Consider another sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

where  $\tilde{\beta} : H^n(X; \mathbb{Z}/p\mathbb{Z}) \rightarrow H^{n+1}(X, \mathbb{Z})$ . This is also called a Bockstein map. In particular, they agree in the sense that

$$\begin{array}{ccc} H^n(X; \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\tilde{\beta}} & H^{n+1}(X, \mathbb{Z}) \\ & \searrow \beta & \downarrow \text{mod } p \\ & & H^{n+1}(X, \mathbb{Z}/p\mathbb{Z}) \end{array}$$

Considering

$$\cdots \longrightarrow H^n(X, \mathbb{Z}) \longrightarrow H^n(X, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\tilde{\beta}} H^{n+1}(X, \mathbb{Z}) \longrightarrow \cdots$$

then  $\tilde{\beta}x = 0$  if and only if  $x$  lifts to an integral cohomology class.

**Proposition 16.5.**  $\beta$  is a derivation with respect to  $\smile$  and  $\times$ , that is,  $\beta(x \smile y) = (\beta x) \smile y + (-1)^{|x|} x \smile (\beta y)$ , and similarly for  $\times$ .

*Proof.* We will prove this for  $\smile$ . Let  $x = [f]$  and  $y = [g]$  for  $f \in C^n(X, \mathbb{Z}/p\mathbb{Z})$  and  $g \in C^m(X, \mathbb{Z}/p\mathbb{Z})$  as cocycles. Given

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

this induces maps on the cochain level with respect to the connecting homomorphisms. Use the fact that  $\delta(\tilde{f} \smile \tilde{g}) = \delta\tilde{f} \smile \tilde{g} + (-1)^{|\tilde{f}|} \tilde{f} \smile \delta\tilde{g}$ .  $\square$

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**Lemma 17.1.**  $\beta^2 = 0$ .

*Proof.* Identify the cycles as the cochains over the boundaries, then  $\beta(x) = \frac{1}{n} \delta(\tilde{x})$  where  $\delta$  is the connecting map between them, then  $\beta^2(x)$  is identified to be  $\frac{1}{n^2} \times \delta^2(\tilde{x}) = 0$ .  $\square$

**Example 17.2.** We know  $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z})$  is just  $\mathbb{Z}/2\mathbb{Z}$  for all  $* \geq 0$ , so this is  $\mathbb{Z}/2\mathbb{Z}[x]$ . For every  $n$ , we know  $H^*(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[x]/(x^{n+1})$  for  $|x| = 1$ , with  $\beta(x^k) = x^{k+1}$  if  $k$  is odd and is zero otherwise.

Motivated by this, we will work on Steenrod operations with coefficients in  $\mathbb{Z}/2\mathbb{Z}$ , in particular the Steenrod squares. (The Steenrod powers are over  $\mathbb{Z}/p\mathbb{Z}$  in general.)

A cohomology operation  $\theta$  is a natural transformation  $H^i(-, A) \rightarrow H^j(-, B)$ . We usually want the functors **Top**  $\rightarrow$  **Set** to be additive. For instance, the Bockstein map would be.

**Definition 17.3.** The Steenrod squares are additive cohomology operations  $\text{Sq}^i : H^n(X, A; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^{n+i}(X, A; \mathbb{Z}/2\mathbb{Z})$  for  $i \geq 0$ , satisfying

1.  $\text{Sq}^0 = \text{id}$ ;
2. if  $|x| = i$ , then  $\text{Sq}^i x = x^2$ ;
3. if  $|x| < i$ , then  $\text{Sq}^i x = 0$ ;
4.  $\text{Sq}^k(x \smile y) = \sum_{k=i+j} \text{Sq}^i x \smile \text{Sq}^j y$ . Alternatively,  $\text{Sq}^k(x \times y) = \sum_{k=i+j} \text{Sq}^i x \times \text{Sq}^j y$ .

**Corollary 17.4.** 1.  $\beta = \text{Sq}^1$ ;

2. Adem relation: if  $0 < a < 2b$ , then  $\text{Sq}^a \text{Sq}^b = \binom{b-j-1}{a-2j} \sum_{j=0}^{\lfloor \frac{a}{2} \rfloor} \text{Sq}^{a+b-j} \text{Sq}^j$ . For example, if  $a = b = 1$  and  $j = 0$ , then  $\binom{0}{1} = 0$ , so  $\text{Sq}^1 \text{Sq}^1 = 0$ .

**Proposition 17.5.** For any  $(X, A)$ , the diagram

$$\begin{array}{ccc} H^s(A; \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\delta} & H^{s+1}(X, A; \mathbb{Z}/2\mathbb{Z}) \\ \downarrow \text{Sq}^i & & \downarrow \text{Sq}^i \\ H^{s+i}(A; \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\delta} & H^{s+i+1}(X, A; \mathbb{Z}/2\mathbb{Z}) \end{array}$$

commutes.

*Proof.* For  $A \hookrightarrow X$ , we have a mapping cylinder  $M = X \smile_{A \times \{1\}} (A \times I)$ . Then

$$(X, A) \xleftarrow{\sim} (M, A \times I) \xleftarrow{\sim} (M, A \times \{0\}) \longrightarrow (M, A \times \{0\}) \cup X \cup (A \times [0, \tfrac{1}{2}]) \xleftarrow{\sim} (A \times I, A \times \partial I)$$

where the last map is the excision. Then

$$\begin{array}{ccccccccc} H^s(A) & \xrightarrow{\sim} & H^s(A \times I) & \longrightarrow & H^s(A \times \{0\}) & \xleftarrow{\sim} & H^s(A \times \{0\} \sqcup Z) & \xrightarrow{\cong} & H^s(A \times \partial I) \\ \delta \downarrow & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\ H^{s+1}(X, A) & \xrightarrow{\sim} & H^{s+1}(M, A \times I) & = & H^{s+1}(M, A \times \{0\}) & \xleftarrow{\sim} & H^{s+1}(M, A \times \{0\} \cup Z) & \xrightarrow{\cong} & H^{s+1}(A \times I, A \times \partial I) \end{array}$$

□