MATH 502 Notes

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Throughout the notes, we assume a ring R has a multiplicative identity and is commutative.

0 Noetherian, Artinian, and Localization

Proposition 0.1. Let R be a (commutative) ring, and let M be an A-module, then the following are equivalent:

(i) Given an infinite increasing chain of submodules of M

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n \subseteq M_{n+1} \subseteq \cdots$$

then there exists some $N \in \mathbb{N}$ such that $M_N = M_{N+1} = \cdots$, i.e., for all $n \geqslant N$, $M_n = M_{n+1}$.

(ii) Every non-empty family of submodules has a maximal element.

(iii) Every submodule of M is finitely-generated.

Proof. $(i) \Rightarrow (ii)$: This is a direct result of Zorn's lemma.

$(ii) \Rightarrow (i)$: Obvious.

$(i), (ii) \Rightarrow (iii)$: Take any submodule N of M and take $x_1 \in N$. If $(x_1) \neq N$, then there exists $x_2 \in N \backslash (x_1)$, so $(x_1, x_2) \subseteq N$, now we proceed inductively, but by the given property we know this stops in finite number of steps, hence we have $N = (x_1, \dots, x_n)$ for some $n \in \mathbb{N}$, thus N is finitely-generated.

$(iii) \Rightarrow (i)$: Note that the property implies M is finitely-generated, but that means the chain of submodules must be finite. □

Definition 0.2 (Noetherian Module)**.** If any of the conditions in Proposition 0.1 holds, then M is said to be a Noetherian module. Alternatively, we say M satisfies the ascending chain condition.

Proposition 0.3. Let R be a (commutative) ring, and let M be an A-module, then the following are equivalent:

(i) Given an infinite decreasing chain of submodules of M

$$M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq M_{n+1} \supseteq \cdots$$

then there exists some $N \in \mathbb{N}$ such that $M_N = M_{N+1} = \cdots$, i.e., for all $n \geqslant N$, $M_n = M_{n+1}$.

(ii) Every non-empty family of submodules has a minimal element.

Proof. Again, Zorn's lemma. □

Definition 0.4 (Artinian Module)**.** If any of the conditions in Proposition 0.3 holds, then M is said to be a Artinian module. Alternatively, we say M satisfies the descending chain condition.

Example 0.5.
- $\mathbb{Z}$ is Noetherian.
- $\mathbb{Q}/\mathbb{Z}$ is not Noetherian.

- Let p be a prime. Let $\mathbb{Z}(p^\infty)$ be the union of chains (as direct limits)
$$\left\langle \frac{\bar{1}}{p} \right\rangle \subseteq \left\langle \frac{\bar{1}}{p^2} \right\rangle \subseteq \cdots \subseteq \left\langle \frac{\bar{1}}{p^n} \right\rangle \subseteq \cdots$$
then there is an embedding $\mathbb{Z}(p^\infty) \subseteq \mathbb{Q}/\mathbb{Z}$, where $\bar{a}$ is the image of a in $\mathbb{Q}/\mathbb{Z}$. With this construction, $\mathbb{Z}(p^\infty)$ is Artinian.

Exercise 0.6. Show that $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}(p^\infty)$ where p traverses through all the primes.

Proposition 0.7. Let N be a submodule of M. Suppose M satisfies ascending (respectively, descending) chain condition, then N and M/N also satisfy ascending (respectively, descending) chain condition. If, for some submodule N of M, we know N and M/N satisfy ascending (respectively, descending) chain condition, then M also satisfies ascending (respectively, descending) chain condition.

Proof. Suppose M satisfies ascending (respectively, descending) chain condition, and let N be a submodule of M. Let $\{N_i\}$ be an increasing (respectively, decreasing) sequence of submodules of N, then they can be regarded as submodules of M, therefore by the Noetherian (respectively, Artinian) condition, we know N satisfies ascending (respectively, descending) chain condition. Now let $\bar{M} = M/N$, and take $\{\bar{M}_i\}$ be an increasing (respectively, decreasing) sequence of submodules of $\bar{M}$. Let $\pi : M \to M/N$ be the quotient map, then the preimages give an increasing (respectively, decreasing) sequence $\{M_i\}$ of submodules of M, where $M_i = \pi^{-1}(\bar{M}_i)$, but by the Notherian (respectively, Artinian) condition, we know the sequence stops in finite steps, therefore the original sequence stops in finite steps as well, hence $\bar{M}$ satisfies the ascending (respectively, descending) chain condition.

Suppose a submodule N of M is such that N and M/N both satisfy ascending chain condition. Take a submodule T of M, then we have a short exact sequence

$$0 \longrightarrow T \cap N \hookrightarrow T \longrightarrow T/(T \cap N) \longrightarrow 0$$

Now $T \cap N$ is finitely-generated as N is finitely-generated, therefore we have an embedding $T/T \cap N \hookrightarrow M/N$, thus $T/T \cap N$ is finitely-generated, therefore T is also finitely-generated by a vector space argument.

Suppose we have a decreasing sequence $\{M_n\}$ of M, then we have a decreasing sequence $\{N \cap M_n\}$. Let $\bar{M} = M/N$, then $\bar{M}_n := (M_n + N)/N$ defines a decreasing sequence of submodules in $\bar{M}$, but N satisfies the descending chain condition, so the sequence $\{N \cap M_n\}$ stops in finite number of steps, say n_0. Moreover, the sequence of $\bar{M}_n$'s also stops in finite number of steps, so by definition the sequence of $(M_n + N)/N$ stops in finite number of steps, say m_0, but by the isomorphism theorem this shows that the sequence of $M_n/(N \cap M_n)$ stops in m_0 steps. Therefore, whenever $n \geqslant m_0, n_0$, then $N \cap M_n = N \cap M_{n+1}$, hence $M_n = M_{n+1} = \cdots$ for such n. ☐

Remark 0.8. The final argument should also work in the Noetherian case.

Definition 0.9 (Simple Module)**.** An A-module M is simple if the submodules of M are either 0 or M.

Exercise 0.10. Let A be a commutative ring, and M is an A-module, then M is simple if and only if $M \cong A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of A.

Definition 0.11 (Jordan-Hölder Chain)**.** Let A be a commutative ring and M be an A-module. We say M has a Jordan-Hölder chain if there exists a decreasing chain of submodules $\{M_i\}$ such that

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_{n-1} \supsetneq M_n = 0$$

such that M_i/M_{i+1} is simple. In such a situation, we know n is the length of the Jordan-Hölder chain, and such n is unique. We say M is a module of finite length, and the length is $\ell_A(M) = n$.

Exercise 0.12. Let A be a commutative ring, and let M be an A-module, then M is of finite length if and only if M is both Noetherian and Artinian.

Theorem 0.13. Let A be a commutative ring, then A is Artinian if and only if A is Noetherian and every prime ideal of A is maximal.

Proof. (⇐):

Lemma 0.14. Let A be Noetherian, then every ideal of A contains a product of prime ideals.

Subproof. Suppose, towards contradiction, that there exists some ideal I of A that does not contain a product of prime ideals. Let $\mathcal{J}$ be the set of such ideals of A, then $\mathcal{J} \neq \varnothing$, and we can take a maximal element of $\mathcal{J}$, namely J.[1] By definition, J is not prime, therefore there exists $a, b \in A$ such that $a \notin J$ and $b \notin J$, but $ab \in J$. Now $J \subsetneq J + Aa$ and $J \subsetneq J + Ab$, therefore $J + Aa, J + Ab \notin J$, therefore $J + Aa$ and $J + Ab$ both contain product of prime ideals. But now $(J + Aa)(J + Ab)$ should also contain products of prime ideals, but by distribution this is just $J^2 + Ja + Jb + Aab$, which is contained in J because every term is contained in J, so J contains a product of prime ideals as well, contradiction. ■

In particular, (0) contains a product of prime ideals, in particular (0) equals to this product, but every prime ideal is maximal, therefore $(0) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$ becomes the product of maximal ideals (which may not necessarily be distinct), hence we have a descending chain of ideals

$$A \supseteq \mathfrak{m}_1 \supseteq \mathfrak{m}_1\mathfrak{m}_2 \supseteq \cdots \supseteq \mathfrak{m}_1 \cdots \mathfrak{m}_n = (0),$$

and in particular $(\mathfrak{m}_1 \cdots \mathfrak{m}_{i-1})/(\mathfrak{m}_1 \cdots \mathfrak{m}_i)$ is a finite-dimensional since A is Noetherian, and it has a natural structure as a $A/\mathfrak{m}_i$-vector space. From the short exact sequence

$$0 \longrightarrow \mathfrak{m}_1 \cdots \mathfrak{m}_i \longrightarrow \mathfrak{m}_1 \cdots \mathfrak{m}_{i-1} \longrightarrow (\mathfrak{m}_1 \cdots \mathfrak{m}_{i-1})/(\mathfrak{m}_1 \cdots \mathfrak{m}_i) \longrightarrow 0$$

we know the two sides of the sequence are Artinian, hence the central term is Artinian. Proceeding inductively, we know that $\mathfrak{m}_1$ is Artinian, and $R/\mathfrak{m}_1$ would also be Artinian, hence A is Artinian.

(⇒): Now suppose A is Artinian, and we want to show that every prime ideal is maximal, and (0) is a product of maximal ideals. The result then follows from the argument above.

Lemma 0.15. Every Artinian domain is a field.

Subproof. Let $0 \neq a \in A$, then consider the chain

$$(a) \supseteq (a^2) \supseteq \cdots \supseteq (a^n) \supseteq \cdots$$

and by the Artinian property, for some large enough n the descending chain stops. Hence, we have $a^n = \lambda a^{n+1}$ for some large enough n and some $\lambda \in A$. Hence, $a^n(1 - \lambda a) = 0$, by the cancellation property of a domain, since $a \neq 0$, we must have $\lambda a = 1$, therefore a is a unit, as desired. ■

Corollary 0.16. Let A be Artinian, then every prime ideal of A is maximal.

Finally, it suffices to show that $(0) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$. Let $\mathfrak{J}$ be the set of finite products of maximal ideals, then $\mathfrak{J}$ has a minimal element, and it suffices to show that this element is (0). Suppose not, let $I \neq (0)$ be a minimal element of R. For any two ideals α, β of A, let $(\alpha : \beta) = \{a \in A \mid a\beta \subseteq \alpha\}$. Note that this has a natural structure as an ideal of A. Let $J = ((0) : I)$, and suppose $J = A$, then $I = 0$, contradiction, so $J \neq A$ is a proper ideal of A, now consider A/J which is Artinian, then let $\mathfrak{G}$ be the set of all non-zero ideals of A/J, so $\mathfrak{G}$ has a minimal element as well, call it $\bar{H}$. Let $H = \pi^{-1}(\bar{H})$ where $\pi : A \to A/J$, so we have $J \subsetneq H$, thus let $P = (J : H)$.

Claim 0.17. P is a prime ideal.

Subproof. Given $c, d \notin P$, we want to show that $cd \notin P$. Indeed, consider $J \subsetneq J + cH \subseteq H$, then since H is minimal, then $J + cH = H$, and similarly we have that $J + dH = H$. Therefore, we have that $J + cdH = J + c(dH + J) = J + cH = H$, hence we know $cd \notin P$, as desired. ■

Now $P = (J : H)$ and $J = (0 : I)$, the by definition we have $PHI = (0)$. Since P is a prime ideal, then P is maximal, and now

$$(0 : PI) \supseteq H \supsetneq J = (0 : I)$$

Therefore $PI \subsetneq I$, where I is a minimal element, contradiction, hence (0) is a product of maximal ideals. □

[1] The existence of this maximal element is the result of Zorn's lemma and ACC condition.

Definition 0.18 (Short Exact Sequence)**.** Consider the sequence

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} T \longrightarrow 0$$

This is called a short exact sequence if $\ker(f) = 0$, $\mathrm{im}(g) = T$, and $\ker(g) = \mathrm{im}(f)$. In particular, one slot of the sequence is said to be exact if the kernel of the previous map equals to the image of the subsequent map.

Definition 0.19 (Flat Module)**.** Let M be an A-module, then we say M is a flat A-module if for every short exact sequence

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

the tensored sequence

$$0 \longrightarrow M \otimes_A N_1 \longrightarrow M \otimes_A N_2 \longrightarrow M \otimes_A N_3 \longrightarrow 0$$

remains exact.

Remark 0.20. Recall that the properties of modules have the following implications: free $\Rightarrow$ projective $\Rightarrow$ flat $\Rightarrow$ torsion-free, and in the case of finitely-generated modules, torsion-free $\Rightarrow$ free.

Remark 0.21. We already know that the tensor functor is right exact, namely given the short exact sequence above, then

$$M \otimes_A N_1 \longrightarrow M \otimes_A N_2 \longrightarrow M \otimes_A N_3 \longrightarrow 0$$

is exact.

Exercise 0.22. Let M be an A-module, and if there exists a short exact sequence of A-modules

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

where N_1 and N_2 are finitely-generated as A-modules, and such that tensoring M preserves the short exact sequence, then M is flat.

Definition 0.23 (Multiplicatively Closed Subset)**.** Let A be a commutative ring and M be an A-module. Let $S \subseteq A$ be a subset. We say S is a multiplicatively closed subset of A if $1 \in S$, $0 \notin S$, and whenever $s_1, s_2 \in S$, then $s_1 s_2 \in S$.

Definition 0.24 (Localization)**.** Let $S \subseteq A$ be a multiplicatively closed subset, and let M be an A-module, then $S^{-1}M = (M \times S)/\sim$, where $\sim$ is an equivalence relation defined by the following: $(m_1, s_1) \sim (m_2, s_2)$ if and only if there exists $t \in S$ such that $t(m_1 s_2 - m_2 s_1) = 0$. $S^{-1}M$ is said to be the localization of M at S.

Given $(m, s) \in M \times S$, we write $\overline{(m, s)}$ to be the equivalence class in $S^{-1}M$ represented by (m, s).

Exercise 0.25. Similarly, one can define the localization $S^{-1}A$ of A at S. In fact, $S^{-1}A$ inherits a ring structure from A, namely

- $\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{a_1 s_2 + a_2 s_1}{s_1 s_2}$,
- $\frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{a_1 a_2}{s_1 s_2}$,
- $\frac{1}{s} \cdot \frac{s}{1} = \frac{1}{1} = 1$.

Remark 0.26. Note that a ring structure does not guarantee every element to have a multiplicative inverse. The localization of A at S ensures that every element of S now becomes invertible in the new ring $S^{-1}A$. In particular, this induces a ring homomorphism

$$\begin{aligned} f : A &\to S^{-1}A \\ a &\mapsto \frac{a}{1} \end{aligned}$$

This homomorphism is injective if A is a domain.

Remark 0.27. Let I be an ideal of A.

- Consider the ring homomorphism $f : A \to S^{-1}A$ above, then
$$S^{-1}I = IS^{-1}A = f(I)S^{-1}A.$$
In particular, $f^{-1}(IS^{-1}A) \supseteq I$.
- If $I \cap S \neq \varnothing$, then $IS^{-1}A = S^{-1}A$.
- If P is a prime ideal of A such that $P \cap S = \varnothing$, then $f^{-1}(PS^{-1}A) = P$.
- Let M be an A-module, then if $N \subseteq M$ is a submodule, then $S^{-1}N \subseteq S^{-1}M$. That is, given an exact sequence
$$0 \longrightarrow N \longrightarrow M$$
then we obtain an exact sequence
$$0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M$$
Indeed, given $0 \to N \xrightarrow{f} M$, say we have it sending $\frac{n}{1} \mapsto \frac{f(n)}{1} = 0$, then there exists $s \in S$ such that $sf(n) = 0$, so $f(sn) = 0$, therefore $sn = 0$ by injection, hence $\frac{n}{1} = 0$ in $S^{-1}N$ as well.

Exercise 0.28. The localization functor is exact.

Lemma 0.29. Let A be a commutative ring and S be a multiplicatively closed subset of A, then $S^{-1}A \otimes_A M \cong S^{-1}M$.

Proof. We define
$$\begin{aligned} \varphi : S^{-1}A \otimes_A M &\to S^{-1}M \\ \frac{a}{s} \otimes m &\mapsto \frac{am}{s}. \end{aligned}$$
For any $\frac{m}{s} \in S^{-1}M$, we have $\varphi\left(\frac{1}{s} \otimes m\right) = \frac{m}{s}$, so the map is onto. Now suppose $\varphi\left(\sum\limits_{i=1}^{n} \frac{a_i}{s_i} \otimes m_i\right) = 0$ (since this is a finite sum), then $\varphi\left(\sum\limits_{i=1}^{n} \frac{a_i}{s_i} \otimes m_i\right) = \sum\limits_{i=1}^{n} \frac{a_i m_i}{s_i} = 0$. We make $s = s_1 \cdots s_n$, so
$$\frac{a_i}{s_i} \otimes m_i = \frac{a_i s_1 \cdots s_{i-1} s_{i+1} \cdots s_n}{s} \otimes m_i =: \frac{b_i}{s} \otimes m_i,$$
then $\sum\limits_{i=1}^{n} \frac{a_i}{s_i} \otimes m_i = \sum\limits_{i=1}^{n} \frac{b_i}{s} \otimes m_i$, therefore
$$\varphi\left(\sum_{i=1}^{n} \frac{a_i}{s_i} \otimes m_i\right) = \varphi\left(\sum_{i=1}^{n} \frac{b_i}{s} \otimes m_i\right) = \frac{\sum\limits_{i=1}^{n} b_i m_i}{s} = 0,$$
so there exists $t \in S$ such that $t \sum\limits_{i=1}^{n} b_i m_i = 0$, now
$$\begin{aligned} \sum_{i=1}^{n} \frac{a_i}{s_i} \otimes m_i &= \sum_{i=1}^{n} \frac{b_i}{s} \otimes m_i \\ &= \sum_{i=1}^{n} \frac{1}{s} \otimes b_i m_i \\ &= \frac{1}{s} \otimes \sum_{i=1}^{n} b_i m_i \end{aligned}$$

$$
\begin{aligned}
&= \frac{t}{ts} \otimes \sum_{i=1}^{n} b_i m_i \\
&= \frac{1}{ts} \otimes t \sum_{i=1}^{n} b_i m_i \\
&= \frac{1}{ts} \otimes 0 \\
&= 0.
\end{aligned}
$$

□

Proposition 0.30. The map $A \to S^{-1}A$ is A-flat, i.e., $S^{-1}A$ is a flat A-module.

Proof. Consider

$$0 \longrightarrow N \longrightarrow M \longrightarrow T \longrightarrow 0$$

By Lemma 0.29 (since the isomorphism is functorial), it suffices to show the exactness of

$$0 \longrightarrow S^{-1}N \longrightarrow S^{-1}M \longrightarrow S^{-1}T \longrightarrow 0$$

and this follows from Exercise 0.28. □

Definition 0.31 (Quasi-local, Local)**.** Let A be a commutative ring. We say A is quasi-local if A has exactly one maximal ideal. In particular, if A is also Noetherian, then we say A is a local ring.

Definition 0.32 (Localization)**.** Let A be a commutative ring and $\mathfrak{p}$ be a prime ideal of A. Note that $S = A \backslash \mathfrak{p}$ is a multiplicatively closed subset, then we write $S^{-1}A = A_{\mathfrak{p}}$ (in general, we have $S^{-1}M = M_{\mathfrak{p}}$, where $M \otimes_A A_{\mathfrak{p}} \cong M_{\mathfrak{p}}$) to denote the localization of A away from the prime ideal $\mathfrak{p}$.

Exercise 0.33. $A_{\mathfrak{p}}$ is quasi-local with unique maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$.

Remark 0.34. Take $x \in M$, then the following are equivalent:

- $x = 0$;
- $\frac{x}{1} = 0$ in $M_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of A;
- $\frac{x}{1} = 0$ in $M_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p}$ of A.

Proof. We will prove the first two are equivalent. The ($\Rightarrow$) direction is obvious. Conversely, let $I = \{a \in A \mid ax = 0\}$ to be the annihilator of x in A. Suppose, towards contradiction, that $I \neq A$, then I is contained in some maximal ideal $\mathfrak{m}$ of A, then consider $M_{\mathfrak{m}}$. Since $\frac{x}{1} = 0$ in $\mathfrak{m}$, then there exists $t \in A \backslash \mathfrak{m}$ such that $tx = 0$, but $I \subseteq \mathfrak{m}$ and $t \notin \mathfrak{m}$, then we reach a contradiction, hence $I = A$, and obviously we are done. □

Exercise 0.35. 1. Given the sequence

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} T \longrightarrow 0$$

the following are equivalent:

- the sequence is exact;
- the sequence
  $$0 \longrightarrow M_{\mathfrak{m}} \xrightarrow{f_{\mathfrak{m}}} N_{\mathfrak{m}} \xrightarrow{g_{\mathfrak{m}}} T_{\mathfrak{m}} \longrightarrow 0$$
  is exact for all maximal ideals $\mathfrak{m}$ of A;
- the sequence
  $$0 \longrightarrow M_{\mathfrak{p}} \xrightarrow{f_{\mathfrak{p}}} N_{\mathfrak{p}} \xrightarrow{g_{\mathfrak{p}}} T_{\mathfrak{p}} \longrightarrow 0$$
  is exact for all prime ideals $\mathfrak{p}$ of A.

To see this, apply Remark 0.34.

2. Let A be a commutative ring and M be an A-module, then the following are equivalent:
   - M is A-flat;
   - $M_{\mathfrak{m}}$ is $A_{\mathfrak{m}}$-flat for all maximal ideals $\mathfrak{m}$ of A;
   - $M_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-flat for all prime ideals $\mathfrak{p}$ of A;

Hence, exactness is a local property.

Exercise 0.36. Let A be a commutative ring, then A is Artinian if and only if A as an A-module is of finite length, i.e., $\ell_A(A) < \infty$. Indeed, note that $(0) = \mathfrak{m}_1 \cdots \mathfrak{m}_n$, and write down the Jordan-Hölder series.

1 Primary Decomposition Theorem

Throughout Section 1, the commutative ring A is always Noetherian. In Section 1.1, M is a finitely-generated A-module; in Section 1.2, we drop this assumption.

1.1 Finitely-generated Case

Definition 1.1 (Coprimary)**.** We say M is a coprimary module if for all $a \in A$, the left multiplication $m_a : M \to M$ is either injective or nilpotent (i.e., there exists $n > 0$ such that $a^n M = 0$).

Remark 1.2. (i) If M is coprimary, then N is coprimary for all $N \subseteq M$.

(ii) If M is coprimary, let $P = \{a \in A \mid a : M \to M \text{ is nilpotent}\}$, then P is a prime ideal of A.

Proof. For $a, b \notin P$, $a, b : M \to M$ are injective maps, so $ab : M \to M$ is injective, hence $ab \notin P$. □

Hence, we usually say M is P-coprimary.

(iii) Let M be P-coprimary, then there exists an injection (as M-linear map) $A/P \hookrightarrow M$.

Proof. Take any $x \neq 0$ in M, then consider

$$\begin{aligned} a_x : A &\to M \\ 1 &\mapsto x \end{aligned}$$

Let $I = \ker(a_x)$, then we have

$$\begin{aligned} A/I &\hookrightarrow M \\ \bar{1} &\mapsto x \end{aligned}$$

Now $I \subseteq P$ since I already kills x. Since A is Noetherian, P is finitely-generated, thus consider $P = (a_1, \dots, a_r)$, then $a_i^{t_i} \cdot x = 0$ for all i and some t_i's. Let $t = t_1 + \cdots + t_r$, then $P^t \cdot x = 0$ by binomial theorem, so $P^t \subseteq I \subseteq P$, hence there exists j such that $P^j \subseteq I \subsetneq P^{j-1}$. Take $y \in P^{j-1} \backslash I$, so $\bar{y} \neq 0$ in A/P, taking the injection into M, then $\mathrm{Ann}_A(\bar{y}) = P$. We now have the composition

$$\begin{aligned} A/P \hookrightarrow A/I &\hookrightarrow M \\ \bar{1} &\mapsto \bar{y} \end{aligned}$$

to be injective. □

(iv) Suppose M is P-coprimary, and Q is a prime ideal such that $A/Q \hookrightarrow M$, then $P = Q$.

Proof. By definition of P, $Q \subseteq P$ is obvious: Q kills elements in M, therefore the mapping becomes nilpotent. The other direction is also easy. □

Definition 1.3 (Primary)**.** Let $N \subseteq M$ be a submodule. We say N is a primary submodule of M if M/N is coprimary. If M/N is P-coprimary, we say N is P-primary.

Remark 1.4. Let $\mathfrak{p}$ be a prime ideal of A. We claim that $\mathfrak{p}^t$ is P-primary. Consider

$$m_x : A/\mathfrak{p}^t \to A/\mathfrak{p}^t$$

then $x^t = 0$ on $A/\mathfrak{p}^t$.

Example 1.5. Let $A = k[X,Y,Z]/(Z^2 - XY)$, let $\mathfrak{p} = (x, z)$ where $x = \mathrm{im}(X)$ and $z = \mathrm{im}(Z)$. Now $A/\mathfrak{p} = k[Y]$. $\mathfrak{p}^2$ is not P-primary. Indeed, note that $A/\mathfrak{p}^2 = k[X,Y,Z]/(z^2 - xy, x^2, z^2) \cong k[X,Y,Z]/(X^2, XY, Z^2, XZ)$. Now the mapping given by multiplication by y on this map is not injective, so $\mathfrak{p}^2$ is not P-primary.

In particular, the represented surface is not smooth, since the origin $(0,0,0)$ is a singularity.

Theorem 1.6 (Primary Decomposition Theorem)**.** By assumption, A is Noetherian and M is finitely-generated. Let $N \subseteq M$ be a submodule, then there exists a decomposition

$$N = \bigcap_{i=1}^{r} N_i$$

where each N_i is P_i-primary, and such that

1. all P_i's are distinct, and
2. this decomposition is irredundant, i.e., minimal. In particular, this means removing any of the N_i's gives a different intersection, i.e., $\bigcap\limits_{j \neq i} N_j \nsubseteq N_i$.

This is called a primary decomposition of N. Moreover, the primary decomposition is unique up to permutation of modules, that is, if there exists another primary decomposition, i.e., $N = \bigcap\limits_{i=1}^{s} N_i'$ where N_i''s are P_i'-primary, then $r = s$ and $\{N_1, \dots, N_r\} = \{N_1', \dots, N_s'\}$.

Proof.

Definition 1.7 (Irreducible)**.** A submodule $T \subsetneq M$ is called irreducible if $T \neq T_1 \cap T_2$, where T_1, T_2 are distinct proper submodules of M.

Claim 1.8. Every submodule T of M can be expressed by $T = T_1 \cap \cdots \cap T_l$ where each T_i is irreducible.

Subproof. Suppose, towards contradiction, that there exists some T for which the claim fails, then the set of all such submodules T is a non-empty set $\mathcal{T}$. Since M is Noetherian, then $\mathcal{T}$ has a maximal element W, therefore W is not irreducible. By definition, $W = W_1 \cap W_2$ where W_1, W_2 are distinct proper submodules of M, so $W_1 \notin \mathcal{T}$ and $W_2 \notin \mathcal{T}$, therefore $W_1 = T_1 \cap \cdots \cap T_r$ for irreducible T_i's, and $W_2 = T_1' \cap \cdots \cap T_s'$ where T_i' are irreducible. Therefore, W becomes an intersection of irreducible submodules, a contradiction. ■

Claim 1.9. Suppose T is irreducible in M, then T is a primary submodule of M. That is, we need to show $\bar{M} := M/T$ is coprimary.

Subproof. It suffices to show the following: for all $a \neq 0$ in A, the multiplication map $a : \bar{M} \to \bar{M}$ is either nilpotent or injective. Note that (0) in $\bar{M}$ is irreducible. To see this, we take the ascending chain

$$\ker(a) \subseteq \ker(a^2) \subseteq \ker(a^3) \subseteq \cdots$$

and since A is Noetherian we know $\ker(a^n) = \ker(a^{n+1}) = \cdots$ for some large enough n, therefore for $g = a^n$ we know $\ker(g) = \ker(g^2)$.

Claim 1.10. $\ker(g) \cap \operatorname{im}(g) = (0)$ in $\bar{M}$.

Subproof of Subclaim. Let $x \in \ker(g) \cap \operatorname{im}(g)$, then $g(x) = 0$, and there exists $y \in \bar{M}$ such that $x = g(y)$, so $0 = g(x) = g^2(y)$, but that means $y \in \ker(g^2) = \ker(g)$, so $x = 0$. ■

Therefore, (0) is irreducible in $\bar{M}$, so either $\ker(g) = (0)$ or $\ker(g) = \bar{M}$. If $\ker(g) = (0)$, we have g to be injective, hence multiplication by a is injective; if $\ker(g) = \bar{M}$, we have $a^n \bar{M} = 0$, so a becomes nilpotent. ■

Claim 1.11. If N_1 and N_2 are both P-primary as submodules, then $N_1 \cap N_2$ is also P-primary.

Subproof. By definition, M/N_1 and M/N_2 are both P-coprimary, then it is easy to see that $M/N_1 \oplus M/N_2$ is also P-coprimary. We know there is an obvious inclusion

$$\begin{aligned} M/(N_1 \cap N_2) &\hookrightarrow M/N_1 \oplus M/N_2 \\ \bar{x} &\mapsto (\bar{x}, \bar{x}) \end{aligned}$$

so $M/(N_1 \cap N_2)$ is also coprimary by the inclusion, therefore $N_1 \cap N_2$ is P-primary. ■

Now by Claim 1.8 we have an irreducible decomposition $N = N_1 \cap \cdots \cap N_r$ and without loss of generality let it be of the smallest length, that is, the N_i's are irreducible modules that are irredundant. By Claim 1.9, we know each of the N_i's is primary with respect to some prime ideal. Now for any two P-primary modules N_i and N_j, we know the intersection is still P-primary according to Claim 1.11, therefore we obtain an irredundant intersection $N = N_1' \cap \cdots N_s'$ where each N_i' is P_i-primary (where P_i's are now distinct!), and this proves the existence.

For the uniqueness, suppose we have $N = N_1 \cap \cdots \cap N_r$ where N_i is P_i-primary, where P_i's are distinct, and suppose we have $N = N_1' \cap \cdots \cap N_s'$ where N_i' is P_i'-primary, where all P_i' are distinct as well. It is enough to show the following:

Claim 1.12. For any prime ideal p of A, $p \in \{P_1, \ldots, P_r\}$ if and only if there exists an injection $A/p \hookrightarrow M/N$.

Subproof. Let $p \in \{P_1, \ldots, P_r\}$, without loss of generality denote $p = P_1$, then we have an injection $A/p \hookrightarrow M/N_1$ by Remark 1.2. In $\bar{M} = M/N$, we have $(0) = N_1/N \cap \cdots \cap N_r/N =: \bar{N}_1 \cap \cdots \cap \bar{N}_r$, therefore $\bar{N}_2 \cap \cdots \cap \bar{N}_r \hookrightarrow \bar{M}/\bar{N}_1 = M/N_1$. But $M/N_1 = \bar{M}/\bar{N}_1$, so this gives an injection $\bar{N}_2 \cap \cdots \cap \bar{N}_r \hookrightarrow M/N_1$, but M/N_1 is P_1-coprimary, so $\bar{N}_2 \cap \cdots \cap \bar{N}_r$ is also P_1-coprimary, therefore $A/P_1 \hookrightarrow \bar{N}_2 \cap \cdots \cap \bar{N}_r \hookrightarrow \bar{M} = M/N$ by Remark 1.2.

Now suppose $A/p \hookrightarrow M/N$, to show $p \in \{P_1, \ldots, P_r\}$, it suffices to show $A/p \hookrightarrow M/N_i$ is injective for some $1 \leqslant i \leqslant r$. We have

$$\begin{tikzcd} A/p \arrow[r, hook, "\varphi"] \arrow[rr, bend left, "\varphi_i"] & M/N = \bar{M} \arrow[r, "\eta_i"] & \bar{M}/\bar{N}_i = M/N_i \end{tikzcd}$$

and we want to show there exists some injective φ_i. Suppose not, then $\ker(\varphi_i) \neq 0$ in A/p for all $1 \leqslant i \leqslant r$. But A/p is an integral domain, therefore $\bigcap_{i=1}^{r} \ker(\varphi_i) \neq 0$. Therefore, we have

$$A/p \xhookrightarrow{\varphi} M/N \xhookrightarrow{(\eta_1, \ldots, \eta_r)} \bigoplus_{i=1}^{r} M/N_i$$

Thus, the defined composition above is the injection $(\varphi_1, \ldots, \varphi_r)$. This implies $\bigcap_{i=1}^{r} \ker(\varphi_r) = \ker(\varphi_1, \ldots, \varphi_r) = 0$, a contradiction. Thus, there exists some injective φ_i, and therefore $p \in \{P_1, \ldots, P_r\}$. ■

□

Definition 1.13 (Zero-divisor)**.** Let A be Noetherian and M be a finitely-generated A-module. We say $0 \neq a \in A$ is a zero-divisor on M if there exists $0 \neq x \in M$ such that $ax = 0$. Otherwise, we say a is a non-zero-divisor on M.

Definition 1.14 (Essential prime ideal, Associated prime ideal)**.** Given a primary decomposition $N = \bigcap_{i=1}^{r} N_i$, the corresponding prime ideals $\{P_1, \ldots, P_r\}$ are called the essential prime ideals of N. In particular, if $N = (0)$, we say these are the associated prime ideals of M, denoted by $\mathrm{Ass}_A(M) = \{P_1, \ldots, P_r\}$.

Corollary 1.15. Let A be Noetherian and M be a finitely-generated A-module, and let $\mathrm{Ass}_A(M) = \{P_1, \ldots, P_r\}$, then $\bigcup_{i=1}^{r} P_i$ is the set of all zero-divisors on M.

Proof. If $p \in \mathrm{Ass}_A(M)$, then there exists an injection $A/p \hookrightarrow M$ mapping $\bar{1} \mapsto x$ by Claim 1.12. Therefore, $px = 0$, so elements of p are zero-divisors of M. Let a be a zero-divisor on M, i.e., let $0 \neq x \in M$ be such that $ax = 0$. Take the primary decomposition $(0) = N_1 \cap \cdots \cap N_r$ in M, where N_i is P_i-primary, then there exists i such that $x \notin N_i$. Since $\bar{x} \neq 0$ in M/N_i, then $a : M/N_i \to M/N_i$ is such that $a\bar{x} = 0$, so a is nilpotent on M/N_i. Therefore, M/N_i is P_i-coprimary, and by definition $a \in P_i$. □

Exercise 1.16. Let $\mathrm{Ass}_A(M) = \{P_1, \ldots, P_r\}$, then the set of all nilpotent elements of M is $\bigcap_{i=1}^{r} P_i$.

Corollary 1.17. Suppose $N \subseteq M$ is a submodule, then

$$\mathrm{Ass}_A(N) \subseteq \mathrm{Ass}_A(M) \subseteq \mathrm{Ass}_A(N) \cup \mathrm{Ass}_A(M/N).$$

Proof. The first inclusion is obvious by $A/p \hookrightarrow N \hookrightarrow M$. We now show the second inclusion. Let $p \in \operatorname{Ass}_A(M)$, and suppose $p \notin \operatorname{Ass}_A(N)$, and we have an inclusion $i : A/p \to M$.

Claim 1.18. $i(A/p) \cap N = (0)$.

Subproof. Suppose not, then let $0 \neq x \in i(A/p) \cap N$, then $x \in N$ and $x \in i(A/p)$, but A/p is an integral domain and is p-coprimary, so $i(A/p) \cap N$ is p-coprimary. Therefore, we have

$$A/p \hookrightarrow i(A/p) \cap N \hookrightarrow N$$

and so $p \in \operatorname{Ass}_A(N)$, a contradiction. ■

Therefore, we have the composition $A/p \to M \to M/N$ to be injection, thus $p \in \operatorname{Ass}_A(M/N)$. □

Corollary 1.19. Let M be finitely-generated, and let $I = \operatorname{Ann}_A(M)$, then the essential prime ideals of I is an associated prime of M.

Proof. Note that the essential prime ideals of I are just $\operatorname{Ass}_A(A/I)$, so if we write $I = I_1 \cap \cdots \cap I_r$ where I_i is a P_i-primary. Therefore, we have $A/I = \bar{I}_1 \cap \cdots \cap \bar{I}_r$, where $\bar{I}_i = I_i/I$, and $\bar{I}_i$ is P_i-primary.

Now let $M = \langle \alpha_1, \dots, \alpha_n \rangle$ be given by a set of generators, so $M = \{\sum a_i\alpha_i \mid a_i \in A\}$, now we look at the map

$$\begin{aligned}\varphi : A &\to \bigoplus_{i=1}^{n} M \\ 1 &\mapsto (\alpha_1, \dots, \alpha_n)\end{aligned}$$

then the kernel $\ker(\varphi) = I$, so $\bar{\varphi} : A/I \hookrightarrow \bigoplus_{i=1}^{n} M$ is an injection. By Corollary 1.17, $\operatorname{Ass}_A(M_1 \oplus M_2) = \operatorname{Ass}_A(M_1) \cup \operatorname{Ass}_A(M_2)$, hence we know

$$\operatorname{Ass}(A/I) \subseteq \bigcup_{i=1}^{n} \operatorname{Ass}_A(M) = \operatorname{Ass}_A(M).$$

□

Definition 1.20 (Support)**.** The support of M over A, denoted $\operatorname{Supp}_A(M)$, is the set $\{P \mid P \text{ prime ideal such that } P \supseteq I = \operatorname{Ann}_A(M)\}$.

Theorem 1.21 (Prime Filtration)**.** Let M be finitely-generated, then we have a descending chain

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{n-1} \supseteq M_n = (0)$$

of prime ideals such that $M_i/M_{i+1} \cong A/P_{i+1}$, $0 \leqslant i \leqslant n-1$, where P_i's are prime ideals of A, and $\operatorname{Ass}_A(M) \subseteq \{P_1, \dots, P_n\}$.

Proof. Note that $P \in \operatorname{Ass}_A(M)$ if and only if $i : A/P \hookrightarrow M$, therefore $i(A/P)$ satisfies the condition stated in the theorem. Therefore, take $\mathcal{A} = \{N \subseteq M \mid N \text{ satisfies the condition of the theorem}\}$. Since A is Noetherian, we take a maximal element T of $\mathcal{A}$.

Claim 1.22. $T = M$.

Subproof. Suppose, towards contradiction, that $T \neq M$, then we have a short exact sequence

$$0 \longrightarrow T \longrightarrow M \longrightarrow M/T \longrightarrow 0$$

such that $M/T \neq (0)$.

Exercise 1.23. Let L be a finitely-generated A-module, then $L = 0$ if and only if $\operatorname{Ass}_A(L) = \varnothing$.

Let $q \in \mathrm{Ass}_A(M/T)$, then we have

$$\begin{tikzcd} & & & A/q \arrow[d, hook, "j"] & \\ 0 \arrow[r] & T \arrow[r] & M \arrow[r, "\eta"] & M/T \arrow[r] & 0 \end{tikzcd}$$

and take $W = \eta^{-1}(j(A/q))$, so we have a new short exact sequence

$$0 \longrightarrow T \longrightarrow W \longrightarrow j(A/q) \cong A/q \longrightarrow 0$$

Thus, $W \supsetneq T$ satisfies the condition in the theorem. By the maximality of T, we have a contradiction. ■

□

Remark 1.24. Let A be Noetherian and $\mathfrak{m} \subseteq A$ be a maximal ideal, then for any ideal $I \subseteq A$ such that there exists n with $\mathfrak{m}^n \subseteq I \subseteq \mathfrak{m}$, then I is $\mathfrak{m}$-primary.

Proof. Consider the map

$$A/I \xrightarrow{\cdot x^n} A/I$$

for $x \in \mathfrak{m}$, then this is the zero map. Therefore, multiplication by x is nilpotent. Now suppose $x \notin \mathfrak{m}$, then we want to show that $A/I \xrightarrow{\cdot x} A/I$ is injective. Indeed, since $x \notin \mathfrak{m}$, then $\mathfrak{m} + Ax = A$, hence we have that $y + ax = 1$ for some $y \in \mathfrak{m}$ and $a \in A$, so $(y + ax)^n = 1$, $y^n + \mu x = 1$, but that means the map $A/I \to A/I$ is given by multiplication by μx, so $\bar{\mu}\bar{x} = \bar{1}$ since y vanishes. That is, $\bar{x}$ is invertible over A/I, hence multiplication by x is an isomorphism. □

Exercise 1.25. Let A be a ring and S be a multiplicatively closed subset of A, and let M be an A-module, then $S^{-1}M$ is an $S^{-1}A$-module. Let $T \subseteq S^{-1}M$ be an $S^{-1}A$-submodule, then there exists $N \subseteq M$ such that $T = S^{-1}N$.

Remark 1.26. Localization functor is fully faithful.

Remark 1.27. Let A be Noetherian and S be a multiplicatively closed subset of A.

1. Let M be P-coprimary, then

   - if $S \cap P = \varnothing$, then $S^{-1}M$ is $S^{-1}P$-coprimary;
   - if $S \cap P \neq \varnothing$, then $S^{-1}M = 0$.

   Proof. Indeed, suppose $S \cap P \neq \varnothing$, let $a : M \to M$ be the multiplication map by a, so $a \in P$ gives $a^n M = 0$ for some n, and if $a \notin P$, then this is injective. Let $\frac{a}{s} : S^{-1}M \to S^{-1}M$ be the multiplication map, but $\frac{a}{s}$ is a unit, so multiplication by s or $\frac{1}{s}$ is an isomorphism, hence we can take this to be $\frac{a}{1}$ with $s = 1$. If $s \in P$, then $s^n : M \to M$ is the zero map, therefore $s^n : S^{-1}M \to S^{-1}M$ is also the zero map, so s is a unit. This only happens if $S^{-1}M = 0$. □

2. Let N be P-primary, then

   - if $S \cap P = \varnothing$, then $S^{-1}N$ is $S^{-1}P$-primary in $S^{-1}M$;
   - if $S \cap P \neq \varnothing$, then $S^{-1}N = S^{-1}M$.

Remark 1.28. Consider the localization $S^{-1}M$. Take a submodule T of $S^{-1}M$, then by Exercise 1.25, $T = S^{-1}N$ for some $N \subseteq M$. There is now a primary decomposition on N given by $N = N_1 \cap \cdots \cap N_t$ where N_i is P_i-primary.

Exercise 1.29. Let $W_1, W_2 \subseteq M$, then $S^{-1}(W_1 \cap W_2) = S^{-1}(W_1) \cap S^{-1}(W_2)$ in $S^{-1}M$.

Remark 1.30. This is true whenever we have a flat ring extension.

Therefore, we have

$$\begin{aligned}T &= S^{-1}N\\ &= S^{-1}N_1 \cap \cdots \cap S^{-1}N_t\\ &= S^{-1}N_{i_1} \cap \cdots \cap S^{-1}N_{i_r}\end{aligned}$$

where $S^{-1}N_{i_j}$ is $S^{-1}P_{i_j}$-primary, and $P_{i_1}, \dots, P_{i_r}$ are prime ideals for which $S \cap P_j = \varnothing$, where $P_j \in \{P_1, \dots, P_t\}$.

Exercise 1.31. Let N be P-primary in M.

- if $S \cap P = \varnothing$, then $i_M : M \to S^{-1}M$ and $i_N : N \to S^{-1}N$ gives $i_M^{-1}(S^{-1}N) = N$;
- if $S \cap P \neq \varnothing$, then $i_M^{-1}(S^{-1}N) = i_M^{-1}(S^{-1}M) = M$.

Corollary 1.32. Consider a primary decomposition $N = N_1 \cap \cdots \cap N_t$ where N_i is P_i-primary. Suppose we have a different primary decomposition $N = N_1' \cap \cdots \cap N_t'$ where N_i' is also P_i-primary. Suppose P_1 is a minimal element in $\{P_1, \dots, P_t\}$, then $N_1 = N_1'$.

Proof. Let $S = A \backslash P_1$, then $S^{-1}N = S^{-1}N_1 = S^{-1}N_1'$. Now consider $i_M : M \to S^{-1}M$, this descends to $N_1 \to S^{-1}N_1 = S^{-1}N_1'$ and $N_1' \to S^{-1}N_1'$, so $i_M^{-1}(S^{-1}N_1 = S^{-1}N_1') = N_1 = N_1'$. □

Consider flat ring maps (as a ring extension) like $A \to A[x]$ and $A \to A[x_1, \dots, x_n]$ since as A-modules they are free, since we have a basis $\{x_1^{i_1}, \dots, x_n^{i_n}\}$.

Lemma 1.33. Let $A \to B$ be a flat map, and let M be an A-module. Let N_1 and N_2 be A-submodules of M, then $(N_1 \otimes_A B) \cap (N_2 \otimes_A B) = (N_1 \cap N_2) \otimes_A B$.

Proof. Consider the chain complex

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N_1 \cap N_2 \arrow[r] \arrow[d] & N_1 \arrow[r] \arrow[d] & N_1/(N_1 \cap N_2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_2 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M/N_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_2/(N_1 \cap N_2) \arrow[r] \arrow[d] & M/N_1 \arrow[r] \arrow[d] & M/(N_1 + N_2) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns. We tensor this complex by $- \otimes_A B$, then since B is flat we obtain a new chain complex

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (N_1 \cap N_2) \otimes_A B \arrow[r] \arrow[d] & N_1 \otimes_A B \arrow[r] \arrow[d] & (N/(N_1 \cap N_2)) \otimes_A B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_2 \otimes_A B \arrow[r] \arrow[d] & M \otimes_A B \arrow[r] \arrow[d] & M/N_2 \otimes_A B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_2/(N_1 \cap N_2) \otimes_A B \arrow[r] \arrow[d] & M/N_1 \otimes_A B \arrow[r] \arrow[d] & (M/(N_1 + N_2)) \otimes_A B \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Via diagram chasing, if $x \in (N_1 \otimes_A B) \cap (N_2 \otimes_A B)$, then $x \in (N_1 \cap N_2) \otimes_A B$. □

Corollary 1.34. Suppose we have a primary decomposition $N = N_1 \cap \cdots \cap N_t$ in M, let $A \to A[x]$, then $N[x] = N_1[x] \cap \cdots \cap N_t[x]$ in $M[x]$ where $N_i[x] = N_i \otimes_A A[x]$.

Proof. We want to show that if N_i is P_i-primary, then $N_i[x]$ is $P_i[x]$-primary. Take a short exact sequence

$$0 \longrightarrow P \longrightarrow A \longrightarrow A/p \longrightarrow 0$$

then we tensor it by $- \otimes_A A[x]$, then we obtain a new short exact sequence

$$0 \longrightarrow P \otimes_A A[x] \longrightarrow A[x] \longrightarrow A/p \otimes_A A[x] \longrightarrow 0$$

(Note that we are working over the commutative case, so the left tensor and the right tensor are canonically isomorphic.) We have $B \otimes_A A[x] = B[x]$, now we have $A[x] \otimes_A A/P = A[x]/PA[x] = (A/P)[x]$ which is a domain, so $PA[x]$ is a prime ideal. It now suffices to show that if M is P-coprimary, then $M[x]$ is $P[x]$-coprimary. This simplifies to showing that:

- if $f(x) \in P[x]$, then the multiplication map $M[x] \xrightarrow{f(x)} M[x]$ is nilpotent;
- if $f(x) \notin P[x]$, $M[x] \xrightarrow{f(x)} M[x]$ is an injection.

Note that $M[x] = \sum\limits_{i \geqslant 0} m_i x^i$ for some m_i's. Since $P[x]$ is a prime ideal, then $A[x]/P[x] \cong A/p[x]$. If $f(x) \in P[x]$, we have $f(X) = p_0 + p_1 x + \cdots + p_t x^t$ for p_i's in P. Consider the multiplication map via $[f(x)]^p : M[x] \to M[x]$, where $n = n_0 + n_1 + \cdots + n_t$ such that $p_i^{n_i} M = 0$ by the binomial theorem. Now suppose $f(x) \notin P[x]$, then let us write $f(x) = a_0 + a_1 x + \cdots + a_t x^t$, and we have two cases:

- if no a_i's are in P, then for all i, multiplication by a_i on M is an injection. If we multiply $f(x)$ by $m_0 + m_1 sx + \cdots$, then the constant term would be $a_0 m_0$, and for each term to be zero, we must have $f(x)$ equivalent to zero, hence that means multiplication by $f(x)$ on $M[x]$ would be injective as well.
- Now suppose there exists some a_i that is contained in P. We can write down $f(x) = u + v$ where u has coefficients in P and v does not have any coefficients in P. If possible, let $f(\alpha) = 0$ for $\alpha \in M[x]$, then we have $u\alpha = -v\alpha$, and so $u^2\alpha = v^2\alpha$ since $u^2\alpha = u(-v\alpha) = v(-u\alpha) = v^2\alpha$, and by induction we have $u^n\alpha = (-1)^n v^n \alpha$. Therefore, for large enough n such that $u^n\alpha = 0$, we know $v^n\alpha = 0$, and therefore we have a contradiction since v does not contain any coefficients in P.

□

Remark 1.35. Remark 1.24 would fail if P is not a maximal ideal: P^2 may not be P-primary in this case.

Let R be a Noetherian ring, we let $i_P : R \to R_P$ be the localization away from P, from R to the local ring with maximal ideal PR_P, then we have $(PR_P)^n = P^n R_P$ to be PR_P-primary. Therefore, this gives a mapping from P^n to $P^n R_P = (PR_P)^n$. We now denote $P^{(n)} := i_P^{-1}(P^n R_P)$ to be the nth symbolic power of P, then $P^{(n)}$ is P-primary. (Indeed, we note that P is disjoint from $R \backslash P$, so given $M \to S^{-1}M$ pulling $S^{-1}P$-primary module $S^{-1}N$ back to M gives a P-primary module.) In particular, $P^{(n)} \supseteq P^n$.[2]

Exercise 1.36.
1. - Let R be Noetherian and M be finitely-generated. Show that $\ell_R(M) < \infty$ if and only if $\mathrm{Ass}_R(M)$ consists of maximal ideals only.
   - If $\ell_A(M) < \infty$, then M is a direct sum of coprimary submodules of M.
2. Now let R be a Noetherian ring and P be a prime ideal. Prove that the following are equivalent:
   (i) P is an essential prime ideal of some submodule N of M.
   (ii) $M_P \neq 0$.
   (iii) $P \supseteq \mathrm{Ann}_R(M)$.

[2] $P^{(n)}$ is the unique P-primary component in the primary decomposition of P^n, and is the smallest P-primary ideal containing P^n. Therefore, $P^{(n)} = P^n$ if and only if P^n is primary.

(iv) P contains some $Q \in \operatorname{Ass}(M)$.

3. Let $R = k[x, y, z]$ for some field k, and let $P = (xz - y^2, x^3 - yz, z^2 - x^2y)$.
   - Prove that P is a prime ideal of R.
   - Is P^2 P-primary?

   Hint: consider
   $$\begin{aligned} \varphi : k[x, y, z] &\to k[t] \\ x &\mapsto t^3 \\ y &\mapsto t^4 \\ z &\mapsto t^5 \end{aligned}$$
   and show that $\ker(\varphi) = P$.

1.2 Infinitely-generated Case

Now let R be a Noetherian ring, and M is not finitely-generated.

Definition 1.37 (Coprimary)**.** M is called coprimary if for any $a \in R$, we have multiplication map $a : M \to M$ to be either injective, or locally nilpotent, i.e., for all $x \in M$, there exists n_x such that $a^{n_x}x = 0$.

Therefore, any submodule of M is coprimary. Now we define the associated primes to be $\operatorname{Ass}_R(M)$ to be the set of prime ideals in R such that there exists an injection $A/p \hookrightarrow M$, i.e., R/p is a cyclic submodule of M.

Theorem 1.38. Let R and M be as above. For any $P \in \operatorname{Ass}_R(M)$, there exists a P-primary submodule $N(P)$ of M such that $(0) = \bigcap_{P \in \operatorname{Ass}_R(M)} N(P)$, which may be infinite.

Example 1.39. Let A and B be Noetherian rings and M be a finitely-generated A-module, and we say have a ring homomorphism $\varphi : B \to A$. Via the pullback over φ, we make M into a B-module, but M may not be finitely-generated as a B-module. For instance, take $A = \mathbb{Z}$ and $B = \mathbb{Z}[x]$.

Exercise 1.40. Let $\varphi : B \to A$ be a homomorphism of Noetherian rings. If M is a finitely-generated A-module, then via the pullback of φ, M is a B-module. We write it as ${}_{\varphi}M$. Prove that $\operatorname{Ass}_A({}_{\varphi}M) = \varphi^{-1}(\operatorname{Ass}_A(M))$.

2 Filtered Rings and Modules, Completions

Definition 2.1 (Topological Ring)**.** Let R be a ring with addition φ and multiplication ψ. Suppose R has a topology such that φ and ψ are continuous, then we say R is a topological ring with respect to the given topology. That is, the topology respects the algebraic structure.

Similarly, we can define a topological group with respect to multiplication and inverse, and a topological module with respect to addition and scalar multiplication.

Remark 2.2. A topological ring R (respectively, topological group G, topological module M) is Hausdorff if and only if (0) is closed in R (respectively, (e) is closed in G, (0) is closed in M).

Let M be a topological module, consider

$$\begin{aligned}\varphi: M\times M &\to M\\ (x,y)&\mapsto x-y\end{aligned}$$

then the diagonal is given by $\varphi^{-1}(0)=\{(x,x)\mid x\in M\}=\Delta_M$. Now suppose (0) is closed, which gives Δ_M to be closed, hence M is Hausdorff.

Definition 2.3 (Pseudo-metric Space)**.** We say (X,d) is a pseudo-metric space if we have a function $d: X\times X\to\mathbb{R}^{\geqslant 0}$ such that

1. $d(x,y)+d(y,z)\geqslant d(x,z)$,
2. $d(x,y)=d(y,x)$,
3. $d(x,x)=0$.

This becomes a metric space if $d(x,y)=0$ if and only if $x=y$.

Remark 2.4. A pseudo-metric space is a Hausdorff if and only if it is a metric space.

Definition 2.5 (Completion)**.** Let (X,d) be a (pseudo-)metric space, then the completion $(\hat{X},\hat{d})$ of (X,d) is a complete (all Cauchy sequences converge) metric space $\hat{X}$ with a metric $\hat{d}$ with a map $\varphi: X\to\hat{X}$ such that

1. φ respects both d and $\hat{d}$,
2. $\varphi(X)$ is dense in $\hat{X}$, and
3. We have

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\psi"'] & & \hat{X} \arrow[dl, "\theta"] \\ & Y & \end{tikzcd}$$

that is, given any complete metric space Y and a continuous map $\psi: X\to Y$, there exists a unique map $\theta:\hat{X}\to Y$ such that the diagram commutes.

Remark 2.6. If $W\subseteq X$, then $\hat{W}\cong\overline{\varphi(W)}$.

Definition 2.7 (Directed Set)**.** Let $(I,\leqslant)$ be a poset, then I is called a directed set if for all pairs of $\alpha,\beta\in I$, there exists $\gamma\in I$ such that $\alpha\leqslant\gamma$ and $\beta\leqslant\gamma$.

Definition 2.8 (Inverse Limit)**.** We say $\{X_\alpha\}_{\alpha\in I}$ is an inverse family indexed by I if for all $\alpha\leqslant\beta$, there exists maps $\varphi_{\alpha,\beta}: X_\beta\to X_\alpha$ such that for all $\alpha\leqslant\beta\leqslant\gamma$, we have a commutative diagram

$$\begin{tikzcd} X_\gamma \arrow[rr, "\varphi_{\alpha\gamma}"] \arrow[dr, "\varphi_{\beta\gamma}"'] & & X_\alpha \\ & X_\beta \arrow[ur, "\varphi_{\alpha\beta}"'] & \end{tikzcd}$$

An inverse limit of $\{X_\alpha\}_{\alpha\in I}$ is an object X with maps $\varphi_\alpha : X \to X_\alpha$ for all $\alpha \in I$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\varphi_\beta"'] & & X_\alpha \\ & X_\beta \arrow[ur, "\varphi_{\alpha\beta}"'] & \end{tikzcd}$$

commutes for all $\alpha, \beta \in I$, and for all Y such that the diagram

$$\begin{tikzcd} Y \arrow[rr, "\psi_\alpha"] \arrow[dr, "\psi_\beta"'] & & X_\alpha \\ & X_\beta \arrow[ur, "\varphi_{\alpha\beta}"'] & \end{tikzcd}$$

commutes for all $\alpha, \beta \in I$, then there exists $f : Y \to X$ such that

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "\psi_\alpha"'] & & X \arrow[dl, "\varphi_\alpha"] \\ & X_\beta & \end{tikzcd}$$

commutes for all α.

Remark 2.9. To construct such inverse limits, we take $\tilde{X} = \prod\limits_{\alpha\in I} X_\alpha$, then we have an embedding $X \hookrightarrow \tilde{X}$ where

$$X = \left\{ \prod_{\alpha\in I} X_\alpha \mid \forall \alpha \leqslant \beta, \varphi(X_\beta) = X_\alpha \right\}.$$

We denote the inverse limit to be $X = \varprojlim X_\alpha$.

Exercise 2.10. Consider $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \cdots$, then the inverse limit $\varprojlim X_n = \bigcap\limits_{n\geqslant 0} X_n$.

Exercise 2.11. Let A be a commutative ring, and consider $A[x]$ or $A[x_1, \dots, x_n]$. Let $I = (x)$, or respectively the maximal ideal $(x_1, \dots, x_n)$. Then we have a map $\cdots \to A[x]/I^{n+1} \to A[x]/I^n \to A[x]/I^{n-1} \to \cdots \to A[x]/I$, so $\varprojlim A[x]/I^n \cong A\left[[x]\right]$.

Remark 2.12. By Hilbert's theorem, we know if A is Noetherian, then so is $A[x]$; similarly, if A is Noetherian, then so is $A\left[[x]\right]$.

Definition 2.13 (Graded Ring)**.** We say a commutative ring A is graded if A contains a sequence of $\{A_n\}_{n\geqslant 1}$ of subgroups such that

- $A_i \cdot A_j \subseteq A_{i+j}$,
- $A = \bigoplus\limits_{i\geqslant 0} A_i$.

By definition, this implies A_0 is a subring of A, and $A_+ = \bigoplus\limits_{i\geqslant 1} A_i$ is an ideal, usually called the irrelevant ideal.

Exercise 2.14. 1. $1 \in A_0$,

2. A is Noetherian if and only if A_0 is Noetherian and A_+ is a finitely-generated ideal of A.

2.1 Filtrations of Rings and Modules

Let A be a commutative ring, not necessarily Noetherian, and let M be an A-module.

Definition 2.15 (Filtered Ring)**.** A is called a filtered ring if it admits a filtration $\{A_n\}_{n\geqslant 0}$ where A_i's form a descending sequence of subgroups of A.

Since the descending chain satisfies $A_i \cdot A_j \subseteq A_{i+j}$, then each A_i for $i > 0$ is an ideal of A. We now write $A \sim \{A_n\}_{n\geqslant 0}$, associating A with its filtration.

Definition 2.16 (Filtered Module)**.** M is called a filtered A-module if there exists a descending chain of subgroups $M_0 \supseteq M_1 \supseteq \cdots$ of M such that $A_i \cdot M_j \subseteq M_{i+j}$.

This implies each M_j is an A-submodule.

Example 2.17. Let I be an ideal of A, and let $A_n = I^n$. Let M be an A-module, with $M_n = I^n M$. The associated filtrations are called the I-adic filtration of A and of M.

Definition 2.18 (Induced Filtration, Image Filtration)**.** Let $A \sim \{A_n\}$ and $M \sim \{M_n\}$. Let $N \subseteq M$ be a submodule. The induced filtration on N is given by $N_n = N \cap M_n$ for all n.

Let $f : M \to T$ be a surjective A-linear map of modules, then the filtration defined by $T_n = f(M_n)$ is the image filtration of T.

Definition 2.19 (Filtered Map, Strict Morphism)**.** Let $M \sim \{M_n\}$ and $N \sim \{N_n\}$ be filtrations. A map $f : M \to N$ is called a filtered map if for all n, $f(M_n) \subseteq N_n$.

If $f : M \to N$ is a filtered map, suppose $f(M)$ has an induced filtration with $f(M)_n = f(M) \cap N_n$, as well as an image filtration of $\{f(M_n)\}$. We say f is a strict morphism if for any n, $f(M_n) = f(M) \cap N_n = f(M)_n$. Note that by definition we have $f(M_n) \subseteq f(M) \cap N_n$.

2.2 Topology and metric on Filtered Rings and Modules

Definition 2.20 (Fundamental System)**.** Let $A \sim \{A_n\}$ and $M \sim \{M_n\}$. We declare $\{A_n\}$ (respectively, $\{M_n\}$) as a fundamental system of open neighborhoods of (0) in A (respectively, M). For any $x \in A$ (respectively, $x \in M$), $x + A_n$ (respectively, $x + M_n$) form a fundamental system of neighborhoods of x. This presumption defines a topology on A corresponding to $\{A_n\}$ (respectively, M corresponding to $\{M_n\}$).

Remark 2.21. A is a topological ring and M is a topological A-module with respect to this filtration.

Lemma 2.22. Let $M \sim \{M_n\}$ with $N \subseteq M$, and let $\bar{N}$ be the closure of N in M, then this is just $\bigcap\limits_{n\geqslant 0} N + M_n$.

Proof. Let $x \in \bar{N}$, then there exists n such that $(x + M_n) \cap N \neq \varnothing$. Therefore, there exists $y_n \in M_n$ and $z \in N$ such that $x + y_n = z$, therefore $x = z - y_n \in N + M_n$ for all n. Conversely, let $x \in \bigcap\limits_{n\geqslant 0} N + M_n$. When $x \in N + M_n$, then we can write $x = z + y_n$ for $z \in N$ and $y_n \in M_n$. Therefore, $x - y_n = z$, so $(x + M_n) \cap N \neq \varnothing$. ☐

Corollary 2.23. $\overline{(0)} = \bigcap\limits_{n\geqslant 0} M_n = \bigcap\limits_{n\geqslant 0} A_n$. Therefore, A (respectively, M) is Hausdorff if and only if $\bigcap\limits_{n\geqslant 0} A_n = 0$ (respectively, $\bigcap\limits_{n\geqslant 0} M_n = 0$).

Exercise 2.24. Let $f : M \to N$ be a filtered map, then f is continuous.

Let $0 < c < 1$.

If we assume A (or M) is Hausdorff, i.e., $\bigcap\limits_{n\geqslant 0} A_n = 0$ ($\bigcap\limits_{n\geqslant 0} M_n = 0$). Denote $d(x, y) = c^n$, where n is the largest integer such that $x - y \in M_n$.

If we assume A (or M) is not Hausdorff, i.e., $\bigcap\limits_{n\geqslant 0} A_n \neq 0$ ($\bigcap\limits_{n\geqslant 0} M_n \neq 0$). We can still define the notion of distance as above, but in addition we need: if $x - y \in \bigcap\limits_{n\geqslant 0} M_n$, then $d(x, y) = 0$.

Recall that a sequence $\{x_n\}$ is Cauchy if for any $\varepsilon > 0$, there exists N such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geqslant N$. Therefore, given by M_n, there exists N such that for all $s, r \geqslant N$, then $x_r - x_s \in M_n$. Note that it suffices to have $x_{N+1} - x_N \in M_n$, since by telescoping we get what we want over the additive structure of the module. Hence, $\{x_n\}$ is Cauchy if and only if $\{x_n - x_{n-1}\} \to 0$ as $n \to \infty$.

Exercise 2.25. Let M be a complete metric space with respect to $\{M_n\}$, then $\{x_n\} \in M$ has a convergent sum $\sum_{n\geqslant 0} x_n$ if and only if $x_n \to 0$.

Theorem 2.26. Let $M \sim \{M_n\}$ be filtered and Hausdorff. Suppose M is complete with respect to $\{M_n\}$. Let N be a closed submodule of M, then $\bar{M} = M/N$ with respect to the image filtration $\{\bar{M}_n\}$ is also complete (Hausdorff).

Proof. $\bar{M}$ is Hausdorff since $N = \bar{N} = \bigcap_{n\geqslant 0}(N + M_n)$. Consider $\eta : M \to \bar{M}$, then this is Hausdorff and we want to show this is complete. Let $\{\bar{x}_n\}$ be a Cauchy sequence in $\bar{M}$, then $\bar{x}_{n+1} - \bar{x}_n \in \bar{M}_{i(n)}$ for all $n \geqslant N$, for some $i(n)$ corresponding to n. In particular, $i(n) \to \infty$ as $n \to \infty$. Let x_i be the lift of $\bar{x}_i$ in M, then we have $x_{n+1} - x_n = y_n + z_n$ for some $y_n \in M_{i(n)}$ and $z_n \in N$. By telescoping, we have $x_n - x_1 = \sum_{i=1}^{n-1} y_i + \tilde{z}$ for some $\tilde{z} \in N$. But for $n \to \infty$, we have large enough $i(n) \gg 0$, therefore the sequence $\{y_n\}$ satisfies $y_n \in M_{i(n)}$, therefore $y_n \to 0$ for $n \to \infty$, thus the sequence $\sum_{n=1}^{\infty} y_n$ converges. Hence, as $n \to \infty$, we have $\lim_{n\to\infty} \bar{x}_n = \bar{x}_1 + \sum_{n=1}^{\infty} \bar{y}_n + \tilde{z} = \bar{x}_1 + \bar{y}$. □

2.3 (I-adic) Completion

Definition 2.27 (Null Sequence, Completion)**.** A Cauchy sequence $\{x_n\}$ with $x_n \to 0$ is called a null sequence.

Let $M \sim \{M_n\}$ not necessarily be Hausdorff, then we obtain the completion $\hat{M}$ of M with respect to $\{M_n\}$ (or the metric defined on $\{M_n\}$) by defining $\hat{M}$ as the set of equivalence classes of all Cauchy sequences in M, over the submodules generated by null sequences.

Remark 2.28. Recall that we define the completion $\hat{X}$ of a space X as the equivalence class of sets of all Cauchy sequences over the relation $x = (x_n) \sim y = (y_n)$ if and only if $d(x_n, y_n) \to 0$ as $n \to \infty$. In our case, we have $\{x_n - y_n\}$ forming a null sequence.

Similarly, we can define the completion $\hat{A}$ of a ring A to be the equivalence class of the sets of all Cauchy sequences over the ideal generated by the null sequences.

Remark 2.29. $\hat{M}$ is a topological $\hat{A}$-module. In particular, if $\{a_n\}$'s define a Cauchy sequence in A and $\{m_n\}$'s define a Cauchy sequence in M, then $\{a_n m_n\}$'s define a Cauchy sequence in M.

The corresponding mapping is given by

$$\begin{aligned} i : M &\to \hat{M} \\ x &\mapsto \{x\}, \end{aligned}$$

that is, the image is the constant sequence defined by $x_n = x$ for all n. Note that this is not necessarily injective. However, $i(M)$ is dense in $\hat{M}$.

Remark 2.30. The completion $\tilde{M}$ of M satisfies the following property: given any complete space T, there is $g : M \to T$ and $f : \hat{M} \to T$ such that $g = fi$ is a commutative diagram. In particular, if $\{x_n\}$ is Cauchy in M, then the image $g(x_n)$ is Cauchy in T. If we define $f(x = (x_n)) = y$, then $g(x_n) \to y$ in T.

Note that given any M_n in M, we have $\overline{i(M_n)} = \hat{M}_n$.

Definition 2.31 (Hausdorffication)**.** The quotient $M/\ker(i)$ is called the Hausdorffication of M.

Remark 2.32. By Theorem 2.26, $\hat{M}/\hat{M}_n$ is complete, then there is an induced mapping $\bar{i}_n : M/M_n \to \hat{M}/\hat{M}_n$. Now $\operatorname{im}(\bar{i}_n)$ is dense in $\hat{M}/\hat{M}_n$, then $\widehat{M/M_n} = \hat{M}/\hat{M}_n$. Recall that M_n is defined to be open in M via the fundamental system, now cosets of M_n are of the form $x + M_n \cong M_n$ with respect to a homeomorphism, hence $M \backslash M_n$ is open, so M_n is also closed in M. Therefore, M/M_n is discrete, so $\overline{(0)}$ is clopen, therefore M/M_n is complete, therefore $M/M_n \cong \hat{M}/\hat{M}_n$, i.e., isomorphic to the completion. In particular, $i^{-1}(\hat{M}_n) = M_n$ (with $M \cap \hat{M}_n = M_n$).

Remark 2.33. $\bigcap \hat{M}_n = (0)$ and $\{\hat{M}_n\}$ constitutes a fundamental system of open neighborhoods in $\hat{M}$.

Definition 2.34. Let $A \sim \{A_n\}$ and $M \sim \{M_n\}$, with $\bar{A} \sim \{\bar{A}_n\}$ and $\bar{M} \sim \{\bar{M}_n\}$. We define $E_0(A) = A/A_1 \oplus A_1/A_2 \oplus \cdots \oplus A_n/A_{n+1} \oplus \cdots$ as a graded ring, and similarly we can define $E_0(M)$. This is called the graded ring (respectively, module) associated to the filtration.

Remark 2.35. In particular, $E_0(M)$ is a graded $E_0(A)$-module. We have

$$\begin{aligned} A_i/A_{i+1}\times A_i/A_{j+1} &\to A_{i+j}/A_{i+j+1} \\ (\bar{\lambda},\bar{\mu}) &\mapsto \overline{\lambda\mu} \end{aligned}$$

and

$$\begin{aligned} A_i/A_{i+1}\times M_i/M_{j+1} &\to M_{i+j}/M_{i+j+1} \\ (\bar{\lambda},\bar{x}) &\mapsto \overline{\lambda x} \end{aligned}$$

We have $E_0(A)\cong E_0(\hat{A})$ and $E_0(M)\cong E_0(M)$ since $A_i/A_{i+1}\cong \hat{A}_i/\hat{A}_{i+1}$ and $M_i/M_{i+1}\cong \hat{M}_i/\hat{M}_{i+1}$.

Remark 2.36. Note that $k[x]$ has transcendental degree 1 over k and $k[[x]]$ has infinite transcendental degree over k, but by Remark 2.35 we know

$$\bigoplus \frac{x^n\cdot k[x]}{x^{n+1}\cdot k[x]}\cong \bigoplus \frac{x^n\cdot k[[x]]}{x^{n+1}\cdot k[[x]]}.$$

Definition 2.37 (Inverse Limit)**.** Let $A\sim\{A_n\}$ and $M\sim\{M_n\}$, then we can construct the completion of A (and similarly of M) via inverse limit. We denote $M^*=\varprojlim M/M_n=\{\prod \bar{x}_n : (\bar{x}_n)\in\prod M/M_n, \eta_{n+1}(\bar{x}_{n+1})=\bar{x}_n\ \forall n\}$ associated with the directed system

$$\cdots \longrightarrow M/M_{n+1}\xrightarrow[\bar{x}_{n+1}\mapsto\bar{x}_n]{\eta_{n+1}} M/M_n \xrightarrow{\eta_n} M/M_{n-1}\longrightarrow\cdots$$

Therefore this is true if and only if $x_{n+1}-x_n\in M_n$ for any n, so we obtain a Cauchy sequence as mentioned previously. Now M/M_n is discrete hence complete, therefore the associated topology $\prod M/M_n$ of countable products is complete in the product topology. Therefore, since each M/M_n is a metric space, then the countable product is still a metric space $\prod M/M_n$.

Exercise 2.38. Show that M^* is a closed submodule of $\prod M/M_n$. In particular, since $\prod M/M_n$ is complete, then M^* is also complete.

Remark 2.39. The associated map is

$$\begin{aligned} i: M &\to M^* \\ x &\mapsto (\bar{x},\bar{x},\bar{x},\ldots) \end{aligned}$$

and $i(M)$ is dense in M^*. For any M_n, the image $i(M_n)=(\bar{0},\ldots,\bar{0},\bar{x},\bar{x},\ldots)$ for some $x\in M_n$ with the first n coordinates as 0. In general, we have the mapping

$$M^*\xhookrightarrow{j}\prod M/M_n\xrightarrow{\pi_n} M/M_n$$

and $\overline{i(M_n)}=(\pi_n j)^{-1}(\bar{0})=j^{-1}\pi_n^{-1}(\bar{0})$. For any $Z_n\in M/M_n$, the preimage

$$\pi_n^{-1}(Z_n)=M/M_1\times M/M_{n-1}\times Z_n\times M/M_{n+1}\times\cdots,$$

so

$$j^{-1}(\pi_n^{-1}(0))=j^{-1}(M/M_1\times M/M_{n-1}\times\bar{0}\times M/M_{n+1}\times\cdots)=\overline{j(M_n)}=M_n^*.$$

It now follows that $\bigcap M_n^*=(0)$.

Remark 2.40. We now have the following universal property: for any $M\to M^*$ and mapping $f:M\to N$ for some complete Hausdorff space N, then there exists a unique $g:M^*\to N$ such that the diagram commutes.

$$\begin{tikzcd} M \arrow[rr] \arrow[dr, "f"'] & & M^* \arrow[dl, dashed, "\exists! g"] \\ & N & \end{tikzcd}$$

Indeed, M^* is the set of elements $(\bar{x}_n)$ with $\eta_{n+1}(\bar{x}_{n+1})=\bar{x}_n$, therefore this is the set of elements (x_n) with $x_{n+1}-x_n\in M_n$ for all n, therefore $\{x_n\}$ is a Cauchy sequence, so for $y=\varprojlim f(x_n)$, therefore $g((\bar{x}_n))=y$. Now if $\{x_n'\}$ is another lift of $(\bar{x}_n)\in M^*$, then we can check that $\{x_n-x_n'\}\to 0$ for $n\to\infty$, hence $\varprojlim f(x_n)=\varprojlim f(x_n')$, so $M^*=\bar{M}$, $M_n^*=\hat{M}_n$ and so on.

Lemma 2.41. Let $R = A[x_1, \dots, x_n]$, $I = (x_1, \dots, x_n)$, then the I-adic completion is equivalent to the completion with respect to I-adic filtration corresponding to the topology. i.e., the completion of $A[x_1, \dots, x_n]$ is $A\,[[x_1, \dots, x_n]]$.

Lemma 2.42. Say $A \sim \{A_n\}$, and suppose A is Hausdorff, i.e., $\bigcap A_n = (0)$, then if $E_0(A)$ is a domain, then A is also a domain.

Proof. Suppose not, then we can pick $x \neq 0$ and $y \neq 0$ such that $xy = 0$, then $x \in A_n \backslash A_{n+1}$ and $y \in A_m \backslash A_{m+1}$ for some n, m, then considering the decomposition of $E_0(A)$ we have $\bar{x} \neq 0$ in A_n/A_{n+1} and $\bar{y} \neq 0$ in A_m/A_{m+1}, so $\bar{y}\bar{x} = \overline{yx} = 0$, this is a contradiction to the fact that $E_0(A)$ is a domain, therefore A is a domain. □

Definition 2.43. Let A and M be filtered and Hausdorff, say $x \in M$ be such that $x \in M_n \backslash M_{n+1}$ with largest such n, then we say n is the filtered degree of x.

Theorem 2.44. Let $A \sim \{A_n\}$ and $M \sim \{M_n\}$ and $N \sim \{N_n\}$, and $f : M \to N$ be a filtered map. Suppose that M is complete, N is Hausdorff, and $E_0(f) : E_0(M) \to E_0(N)$ is onto, so we can write $E_0(M) = M/M_1 \oplus M_1/M_2 \oplus \cdots \oplus M_m/M_{m+1}$ and $E_0(N) = N/N_1 \oplus N_1/N_2 \oplus \cdots \oplus M_n/M_{n+1}$, then we have corresponding maps

$$\begin{aligned} E_0(f)_n : M_n/M_{n+1} &\to N_n/N_{n+1} \\ (\bar{x}) &\mapsto \overline{f(x)}, \end{aligned}$$

then f is onto, N is complete, and f is strict.

Proof. Since $E_0(f)$ is onto, take $x \in N$ and since N is Hausdorff, then $x \in N_n \backslash N_{n+1}$ for some n. Therefore, the induced mapping $E_0(f)_n : M_n/M_{n+1} \to N_n/N_{n+1}$ is onto. Therefore, for $\bar{x} \in N_n/N_{n+1}$, we can pick $y_n \in M_n$ such that $x - f(y_n) \in N_{n+1}$. Therefore, on the level of $E_0(f)_{n+1}$, we know $x - f(y_n) \in N_{n+1}/N_{n+2}$, therefore we can pick $y_{n+1} \in M_{n+1}$ such that $x - f(y_n) - f(y_{n+1}) \in N_{n+2}$. Proceeding inductively, we have a sequence of elements with $y_{n+t} \in M_{n+t}$ such that $x - \sum_{k=0}^{t} f(y_{n+k}) \in N_{n+t+1}$. Hence, we have a Cauchy sequence in M, and so this is a Cauchy sequence in M_n, so $y_{n+t} \to 0$ as $t \to \infty$, then $\sum_t y_{n+t}$ converges, thus the sum $y \in M_n$. One can check that $f(y) = \bar{x}$, so f is onto. But that means $f(M_n) = N_n$, so f is strict. We also note that $f^{-1}(0)$ is a closed submodule of M since N is Hausdorff, therefore by Theorem 2.26 we know N is complete. □

Corollary 2.45. Let A be complete with respect to the filtration, let M be Hausdorff. Suppose $E_0(M)$ is a finitely-generated graded module over $E_0(A)$, that is, there exists $x_1, \dots, x_t$, where the degree of $\bar{x}_i$ is r_i, such that $E_0(M)$ is a graded module over $E_0(A)$ generated by $\bar{x}_1, \dots, \bar{x}_t$. If this is the case, then M is generated by $x_1, \dots, x_t$ over A.

Proof. Denote $F = \bigoplus_{i=1}^{t} Ae_i$, then this induces a mapping

$$\begin{aligned} \varphi : F &\to M \\ e_i &\mapsto x_i \end{aligned}$$

defined on the generators. Since this is a finite sum over complete ring A, then F is complete. Let r_i be the degree of x_i, then this imposes a filtration on Ae_i as follows:

$$(Ae_i)_j = \begin{cases} 0, & j \leqslant r_i \\ A_{j-r_i}e_i, & j > r_i \end{cases}$$

We implement this on all i's, then the filtered degree of e_i is just r_i. Using this filtration, we induce a filtration on F, then we have a commutative diagram

$$\begin{tikzcd} E_0(F) \arrow[r, "E_0(\varphi)"] \arrow[d, equal] & E_0(M) \arrow[d, equal] \\ E_0(\bigoplus_{i=1}^{t} Ae_i) \arrow[r, "\varphi'"'] & E_0(M) \end{tikzcd}$$

with induced map φ', where φ' sends $\bar{\varphi}_i \mapsto \bar{x}_i$ for all $1 \leqslant i \leqslant t$. Therefore, φ is onto as a $E_0(A)$-module map. By Theorem 2.44 we are done. □

Corollary 2.46. Let $A \sim \{A_n\}$ be complete with respect to filtration, let M be Hausdorff with filtration $\{M_n\}$, and suppose $E_0(M)$ is Noetherian, then M is Noetherian as well.

Proof. Take submodule $N \subseteq M$, define $N_n = N \cap M_n$, then we have an induced filtration of N, therefore $E_0(N)$ is a submodule of $E_0(M)$ with $N_n/N_{n+1} \hookrightarrow M_n/M_{n+1}$ for all n. Hence, N is Hausdorff with respect to $\{N_n\}$, and $E_0(N)$ is a finitely-generated $E_0(A)$-module, since $E_0(N)$ is a submodule of $E_0(M)$. By Corollary 2.45, this implies N is finitely-generated and complete. □

Corollary 2.47. Under the same assumptions as in Corollary 2.46, every submodule N of M is a closed submodule.

Proof. By Corollary 2.46, N is complete, and every complete subspace of a Hausdorff space is closed, thus N is closed. □

Corollary 2.48. Let $(A, \mathfrak{m})$ be quasi-local, i.e., $\mathfrak{m}$ is the unique maximal ideal of a commutative ring (not necessarily Noetherian) A. In addition, suppose A is complete and Hausdorff with a $\mathfrak{m}$-adic filtration, i.e., $\bigcap \mathfrak{m}^n = (0)$. Let M be an A-module with respect to the filtration $\{\mathfrak{m}^n M\}$, and assume M is Hausdorff. If $\dim_{A/\mathfrak{m}}(M/\mathfrak{m}M)$ is finite, and suppose $\mathfrak{m}$ is a finitely-generated ideal in A, then M is a finitely-generated A-module.

Proof. We write down the decomposition

$$E_0(M) = M/\mathfrak{m}M \oplus \frac{\mathfrak{m}M}{\mathfrak{m}^2 M} \oplus \cdots \oplus \frac{\mathfrak{m}^n M}{\mathfrak{m}^{n+1} M} \oplus \cdots$$

and

$$E_0(A) = A/\mathfrak{m} \oplus \frac{\mathfrak{m}}{\mathfrak{m}^2} \oplus \cdots \oplus \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}} \oplus \cdots$$

Denote $\mathfrak{m} = (x_1, \ldots, x_n)$ to be the finitely-generated ideal, and since $A/\mathfrak{m} \cong k$ is a field, then we have a ring homomorphism

$$\begin{aligned} \eta : k[x_1, \ldots, x_n] &\to E_0(A) \\ x_i &\mapsto \bar{x}_i \in \mathfrak{m}/\mathfrak{m}^2 \end{aligned}$$

then η is onto, hence $E_0(A)$ is Noetherian. If we write $M/\mathfrak{m}M = k\{\bar{\alpha}_1, \ldots, \bar{\alpha}_r\}$, then one can check that $E_0(M)$ is generated by $\bar{\alpha}_1, \ldots, \bar{\alpha}_r$ for $\bar{\alpha}_i \in M/\mathfrak{m}M$ over $E_0(A)$. This implies $E_0(M)$ is Noetherian and thus M is finitely-generated over A by Corollary 2.46. □

Corollary 2.49. Let A be a commutative ring and I be a finitely-generated ideal over A such that A/I is Noetherian. Suppose A is I-adically complete, i.e., A is complete with respect to the filtration $\{I^n\}$, then A is Noetherian.

Proof. We write down

$$E_0(A) = A/I \oplus I/I^2 \oplus \cdots \oplus I^n/I^{n+1} \oplus \cdots$$

for $I = (x_1, \ldots, x_n)$, then using the same argument we have a ring homomorphism

$$\begin{aligned} \eta : A/I[x_1, \ldots, x_n] &\to E_0(A) \\ x_i &\mapsto \bar{x}_i \in I/I^2 \end{aligned}$$

which is also surjective. Since A/I is Noetherian, then $A/I[x_1, \ldots, x_n]$ is also Noetherian, thus $E_0(A)$ is Noetherian, and by Corollary 2.46, we conclude that A is Noetherian. □

Remark 2.50. Suppose A is Noetherian, and consider the completion $B = A[[x_1, \ldots, x_n]]$ of $A[x_1, \ldots, x_n]$ with respect to the I-adic filtration where $I = (x_1, \ldots, x_n)$. Therefore, $A[[x_1, \ldots, x_n]] = \varprojlim A[x]/I^n$. Now B/IB is A-Noetherian, so by Corollary 2.49 we conclude that $A[[x_1, \ldots, x_n]]$ is also Noetherian.

Exercise 2.51. Let A be a commutative ring, and we assume it is Noetherian. Let $I \subsetneq J$ be ideals of A, and that $\bigcap J^n = (0)$. Suppose A is complete with respect to the J-adic topology. Prove that A is complete with respect to the I-adic topology as well.

Remark 2.52. We saw in Remark 2.50 that $A[[x_1, \ldots, x_n]]$ is complete with respect to $(x_1, \ldots, x_n)$, then the completeness holds for any $I \subseteq (x_1, \ldots, x_n)$.

Proposition 2.53. Let A be commutative ring and M be a finitely-generated A-module, and suppose I is an ideal of A such that $M = IM$, then there exists $a \in I$ such that $(1-a)M = 0$.

Remark 2.54. Proposition 2.53 itself is a direct application of Cayley-Hamilton Theorem, and the proof below follows the same approach. This is also sometimes referred to as Nakayama Lemma (c.f., Corollary 2.55).

Proof. We write $M = \langle \alpha_1, \ldots, \alpha_n \rangle$ and let I be such that $IM = M$, then

$$\alpha_1 = a_{11}\alpha_1 + \cdots + a_{1n}\alpha_n$$

where $a_{1i} \in I$. In general, we have

$$\alpha_j = a_{j1}\alpha_1 + \cdots + a_{jn}\alpha_n$$

for $a_{ji} \in I$. Therefore,

$$\begin{cases} (1-a_{11})\alpha_1 - a_{12}\alpha_2 - \cdots - a_{1n}\alpha_n & = 0 \\ -a_{21}\alpha_1 + (1-a_{22})\alpha_2 - \cdots - a_{2n}\alpha_n & = 0 \\ & \vdots \\ -a_{n1}\alpha_1 - a_{n2}\alpha_2 - \cdots + (1-a_{nn})\alpha_n & = 0 \end{cases}$$

and this gives a matrix

$$C = \begin{pmatrix} 1-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & 1-a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & 1-a_{nn} \end{pmatrix}$$

such that

$$CX := C \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = 0.$$

If we do the cofactor decomposition with respect to the first column, we have $\det(C) \cdot \alpha_1 + 0 \cdot \alpha_2 + \cdots + 0 \cdot \alpha_n = 0$, hence $\det(C) \cdot \alpha_1 = 0$. If we do this for each column, we have $\det(C) \cdot \alpha_i = 0$ for all i, hence $\det(C) \cdot M = 0$. But note that $\det(C) = 1 - a$ for some $a \in I$, therefore $(1-a)M = 0$.[3] □

Corollary 2.55 (Nakayama Lemma)**.** Suppose I is an ideal of A contained in the Jacobson radical of A, and M is a finitely-generated A-module such that $M = IM$, then $M = 0$.

Proof. By Proposition 2.53, there exists $a \in I$ such that $(1-a)M = 0$. Note that the Jacobson radical is the intersection of all maximal ideals of A, so I is contained in all maximal ideals of A. Since $a \in I$, then $1-a$ is a unit in A, so $M = 0$. □

Exercise 2.56. Let A be a commutative ring and M be a finitely-generated A-module. Suppose $f : M \to M$ is a surjective A-linear map, then f is an isomorphism. *Hint*: use Proposition 2.53.

From now on, we assume A is Noetherian, M is a finitely-generated A-module. Usually, we assume A and M have I-adic filtrations for some ideal $I \subseteq A$.

Lemma 2.57 (Artin-Rees)**.** Let A be Noetherian and M is a finitely-generated A-module, and $I \subseteq A$ is an ideal. Given submodule $N \subsetneq M$, suppose there exists $k > 0$ such that for every n we have $N \cap I^{n+k}M = I^n(N \cap I^k M)$.

Remark 2.58. The proof essentially refers to the blow-up algebra, i.e., Rees algebra.

[3] The cleanest way to finish the proof would be to observe that $I \cdot \det(C) = (\mathrm{adj}(C))C$ and so $I \cdot \det(C)X = (\mathrm{adj}(C))CX = 0$. In particular, $\det(C) \cdot X = 0$ and since X generates M, then $\det(C) \cdot M = 0$. Note that this is equivalent to the given approach since the cofactor matrix induces $\mathrm{adj}(C)$.

Proof. Note that the ($\supseteq$) direction is true by definition, so we only need to show the ($\subseteq$) direction. Let us write $\tilde{A} = A \oplus I \oplus I^2 \oplus \cdots$, more formally this is $A \oplus It \oplus I^2t^2 \oplus \cdots \oplus I^nt^n \oplus \cdots \subseteq A[t]$.[4] This is a graded ring. Similarly, we write $\tilde{M} = M \oplus IM \oplus I^2M \oplus \cdots \oplus I^nM \oplus \cdots$.

Claim 2.59. $\tilde{A}$ is a graded Noetherian ring.

Subproof. Let $I = (x_1, \ldots, x_n)$, then the ring homomorphism

$$\begin{aligned} \eta : A[x_1, \ldots, x_n] &\to \tilde{A} \\ x_i &\mapsto x_i, \end{aligned}$$

is onto. Since A is Noetherian, then $A[x_1, \ldots, x_n]$ is also Noetherian. Therefore, $\tilde{A}$ is a graded Noetherian ring. ■

Suppose M is generated by $\alpha_1, \ldots, \alpha_r$, then $\tilde{M}$ is a finitely-generated graded $\tilde{A}$-module, generated by $\alpha_1, \ldots, \alpha_r \in M$ by the surjectivity of η. This implies that $\tilde{M}$ is a graded Noetherian module. Now define

$$\tilde{N} = N \oplus (N \cap IM) \oplus (N \cap I^2M) \oplus \cdots \oplus (N \cap I^kM) \oplus \cdots \oplus (N \cap I^{n+k}M) \oplus \cdots,$$

then $\tilde{N} \subseteq \tilde{M}$, so $\tilde{N}$ is a finitely-generated graded $\tilde{A}$-module. Now each generator is a finite sum given by decomposition above, so each of the generating set must be a graded element. Hence, $\tilde{N}$ is generated by finitely many elements, which are graded elements, say $\beta_1, \ldots, \beta_t$ where $\deg(\beta_i) = r_i$. Let $k = \max\limits_{1 \leqslant i \leqslant t} r_i$, and we think of ways to obtain elements in $N \cap I^{n+k}M$. Considering the multiplicity of the degree, we know $I^{n+k-r_i}\beta_i \subseteq N \cap I^{n+k}$ for each $1 \leqslant i \leqslant t$. Therefore, we have

$$N \cap I^{n+k}M = I^{n+k}N + I^{n+k-1}(N \cap IM) + \cdots + I^n(N \cap I^kM) = \sum_{j=0}^{k} I^{n+k-j}(N \cap I^jM).$$

Each $I^{n+k-j}(N \cap I^jM) = I^n \cdot I^{k-j}(N \cap I^jM) \subseteq I^n(N \cap I^kM)$, so the sum $N \cap I^{n+k}M \subseteq I^n(N \cap I^kM)$. □

Corollary 2.60. Using the same assumption as in Lemma 2.57, let I be an ideal of A contained in the Jacobson radical of Noetherian ring A, then $\bigcap I^nM = (0)$.

Proof. Let $N = \bigcap I^nM$, then by Lemma 2.57, $I^nN = N = N \cap I^{n+k}M = I^n(N \cap I^kM)$, then by Corollary 2.55, $N = 0$. □

Remark 2.61. In particular, Corollary 2.60 implies M is Hausdorff with respect to the I-adic topology, so the map $M \hookrightarrow \hat{M}$ is an injection by the mapping

$$\begin{aligned} M &\to \varprojlim M/I^nM \subseteq \prod M/M^nM \\ x &\mapsto (x, x, \ldots) \end{aligned}$$

Corollary 2.62. Using the same assumption as in Lemma 2.57, let A be a domain with ideal I, then $\bigcap I^n = (0)$.

Proof. Let $J = \bigcap I^n$, then $J \cap I^{n+k}A = I^n(J \cap I^k)$, so $J = I^nJ$, then by Proposition 2.53 there exists $a \in I^n$ such that $(1-a)J = 0$, and since A is a domain, then $J = 0$. □

Remark 2.63. Corollary 2.62 implies that under I-adic topology, the map $A \to \hat{A}$ is injective.

Definition 2.64. Let $A \sim \{I^n\}$ and $M \sim \{M_n\}$, not necessarily with respect to the I-adic filtration, then $\{M_n\}$ is called I-good if there exists $h > 0$ such that $M_{n+h} = I^nM_h$.

Remark 2.65. By Lemma 2.57, induced filtration is I-good. Topologically, given $A \sim \{I^n\}$ and $M \sim \{M_n\}$ such that $\{M_n\}$ is I-good, then $I^nM \subseteq M_h$ for some $h > 0$, so $M_{n+h} = I^nM_h \subseteq I^nM$. In this case, $\{I^nM\}$ and $\{M_n\}$ are cofinal with respect to each other and hence give the same topology on M. Moreover,

$$\varprojlim M/I^nM \cong \varprojlim M/M_n.$$

That is, the I-adic completion of M is equivalent to the completion of M with respect to $\{M_n\}$.

[4] For instance, we usually write $A[t]$ for $A \oplus At \oplus At^2 \oplus \cdots$.

Remark 2.66. Given an I-good filtration and a submodule N of M, $\{I^nN\}$ and $\{N \cap I^nM\}$ define the same topology on N, and hence the I-adic completion of N is equivalent to the completion of M with respect to $\{M_n\}$.

Proposition 2.67. Let A be Noetherian and a short exact sequence

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} T \longrightarrow 0$$

of finitely-generated A-modules, and let I be an ideal of A, then we have a short exact sequence

$$0 \longrightarrow \hat{N} \xrightarrow{\hat{f}} \hat{M} \xrightarrow{\hat{g}} \hat{T} \longrightarrow 0$$

where all completions are I-adic completions.

Proof. By Lemma 2.57, we know $\hat{N} = \varprojlim N/I^nN = \varprojlim N/(N \cap I^nM)$, then we have a short exact sequence

$$0 \longrightarrow N/(N \cap I^nM) \longrightarrow M/I^nM \longrightarrow T/I^nT \longrightarrow 0$$

for every $n > 0$. It now suffices to show that

$$0 \longrightarrow \varprojlim N/(N \cap I^nM) \longrightarrow \varprojlim M/I^nM \longrightarrow \varprojlim T/I^nT \longrightarrow 0$$

Exercise 2.68. $\ker(\bar{f}) = 0$ and $\operatorname{im}(\hat{f}) = \ker(\hat{f})$.

We now show that $\hat{g}$ is onto. Taking $\{z_n\}$ in $\varprojlim T/I^nT$, we want to show that there exists $\{y_n\}$ in $\varprojlim M/I^nM$ with image $\{z_n\}$, and we proceed inductively. Suppose we have constructed $\{y_i\}_{i \leqslant n}$ such that $\operatorname{im}(y_i) = z_i$ with system $y_n \to y_{n-1} \to \cdots \to y_1$, then there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N/(N \cap I^{n+1}M) \arrow[r, "f_{n+1}"] \arrow[d] & M/I^{n+1}M \arrow[r, "g_{n+1}"] \arrow[d] & T/I^{n+1}T \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N/(N \cap I^nM) \arrow[r] & M/I^nM \arrow[r] & T/I^nT \arrow[r] & 0
\end{tikzcd}$$

where $y_n \in M/I^nM$ and $z_n \in T/I^nT$. Here all rows are exact and the vertical mappings are surjective. We proceed by diagram chasing. To find $y_{n+1} \in M/I^{n+1}M$ such that $\operatorname{im}(y_{n+1}) = z_{n+1}$, since $g_{n+1} : M/I^{n+1}M \to T/I^{n+1}M$ is onto, then we lift it back to $x_{n+1} \in M/I^{n+1}M$ such that $g_{n+1}(x_{n+1}) = z_{n+1}$, and now there is x_n landing in M/I^nM by the vertical mapping. Note that by definition x_n now lands in z_n by the vertical mapping, so we have both $y_n \to z_n$ and $x_n \to z_n$, therefore $y_n - x_n \to 0$, now we lift it back to w_n in $N/(N \cap I^nM)$, which lifts to $w_{n+1} \in N/(N \cap I^{n+1}M)$, and let the image of w_{n+1} with respect to f_{n+1} be x'_{n+1}, then the element $x'_{n+1} + x_{n+1}$ in $M/I^{n+1}M$ is now such that we have

$$\begin{tikzcd}
x'_{n+1} + x_{n+1} \arrow[r] \arrow[d] & z_{n+1} \arrow[d] \\
y_n \arrow[r] & z_n
\end{tikzcd}$$

via diagram chasing as desired. This is the element y_{n+1} we want. ∎

Remark 2.69. Refer to the Mittag-Leffler condition, as well as the complex analysis analogue, i.e., Mittag-Leffler Theorem.

Proposition 2.70. Let A be Noetherian and M be a finitely-generated A-module, and let I be an ideal of A. Let $\hat{A}$ and $\hat{M}$ be I-adic completions of A and M, respectively, then

$$\begin{aligned}
\varphi : \hat{A} \otimes_A M &\xrightarrow{\sim} \hat{M} \\
\{a_n\} \otimes x &\mapsto \{a_n x\}
\end{aligned}$$

Remark 2.71. If we are working over direct limits, we would note

$$(\varinjlim M_\alpha) \otimes_A N = \varinjlim M_\alpha \otimes_A N.$$

This is not the case here, we do not necessarily have

$$(\varprojlim M_\alpha) \otimes_A N = \varprojlim M_\alpha \otimes_A N.$$

Proof. Since M is finitely-generated over Noetherian ring A, then we have an exact sequence

$$A^r \xrightarrow{\psi} A^s \xrightarrow[e_i \mapsto m_i]{\eta} M \longrightarrow 0$$

where M is generated by $m_1, \dots, m_s$. Tensoring by $\hat{A}$, we have an exact sequence

$$\hat{A} \otimes A^r \longrightarrow \hat{A} \otimes A^s \longrightarrow \hat{A} \otimes M \longrightarrow 0$$

Let $K = \ker(\eta)$ and take L to be the kernel of $A^r \to K$, then we have exact sequences

$$0 \longrightarrow L \longrightarrow A^r \longrightarrow K \longrightarrow 0$$

and

$$0 \longrightarrow K \longrightarrow A^s \longrightarrow M \longrightarrow 0$$

By Proposition 2.67, the I-adic filtration gives exact sequences

$$0 \longrightarrow \hat{L} \longrightarrow \hat{A}^r \longrightarrow \hat{K} \longrightarrow 0$$

and

$$0 \longrightarrow \hat{K} \longrightarrow \hat{A}^s \longrightarrow \hat{M} \longrightarrow 0$$

therefore

$$\hat{A}^r \longrightarrow \hat{A}^s \longrightarrow \hat{M} \longrightarrow 0$$

is exact and we have a diagram

$$\begin{tikzcd}
\hat{A} \otimes A^r \arrow[r] \arrow[d, "\varphi_{A^r}"'] & \hat{A} \otimes A^s \arrow[r] \arrow[d, "\varphi_{A^s}"] & \hat{A} \otimes M \arrow[r] \arrow[d, "\varphi_M"] & 0 \\
\hat{A}^r \arrow[r] & \hat{A}^s \arrow[r] & \hat{M} \arrow[r] & 0
\end{tikzcd}$$

Now

$$\begin{aligned}
\hat{A} \otimes A^s &= \hat{A} \otimes (A \oplus \cdots \oplus A) \\
&= (\hat{A} \otimes_A A) \oplus \cdots \oplus (\hat{A} \otimes_A A) \\
&= (\hat{A})^s
\end{aligned}$$

and similarly $\hat{A} \otimes A^r = (\hat{A})^r$. One can check that φ_{A^r} and φ_{A^s} are isomorphisms. Now the mapping $A^s = \bigoplus_s A \to \bigoplus_s \hat{A}$ has dense image, which implies φ_M is an isomorphism by diagram chasing. $\square$

Theorem 2.72. Let A be Noetherian and I be an ideal, then $A \to \hat{A}$, the mapping into the I-adic completion, is a flat map, that is, $\hat{A}$ is a flat A-module.

Proof. For flatness, we can assume that

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{g} T \longrightarrow 0$$

is a short exact sequence of finitely-generated modules (since we are working over Noetherian rings), and we want to show that

$$0 \longrightarrow \hat{A} \otimes_A N \xrightarrow{\hat{f}} \hat{A} \otimes_A M \xrightarrow{\hat{g}} \hat{A} \otimes_A T \longrightarrow 0$$

is a short exact sequence as well. But we know this is just

$$0 \longrightarrow \hat{N} \longrightarrow \hat{M} \longrightarrow \hat{T} \longrightarrow 0$$

by Proposition 2.70, which is exact by Proposition 2.67. □

Corollary 2.73. The map

$$A[x_1, \ldots, x_n] \to A\,[[x_1, \ldots, x_n]]$$

is flat.

2.4 Faithfully Flat Modules

Proposition 2.74. Let A be a commutative ring and M be an A-module, then the following are equivalent:

1. $$N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3$$
   is exact if and only if
   $$M \otimes N_1 \xrightarrow{f} M \otimes N_2 \xrightarrow{g} M \otimes N_3$$
   is exact;
2. $$0 \longrightarrow N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3 \longrightarrow 0$$
   is exact if and only if
   $$0 \longrightarrow M \otimes N_1 \xrightarrow{f} M \otimes N_2 \xrightarrow{g} M \otimes N_3 \longrightarrow 0$$
   is exact;
3. M is an A-flat module and for any A-module N, $M \otimes_A N = 0$ implies $N = 0$;
4. M is an A-flat module and for any ideal I of A, $M \otimes_A A/I = 0$ implies $A = I$.

Proof. The equivalence of (1) and (2) is obvious.

(1), (2) $\Rightarrow$ (3): the flatness is obvious. Suppose $M \otimes_A N = 0$, then consider

$$0 \longrightarrow N \longrightarrow 0$$

and we tensor it with M, then we have

$$0 \longrightarrow M \otimes N \longrightarrow 0$$

which is exact, so

$$0 \longrightarrow N \longrightarrow 0$$

is exact and so $N = 0$.

(3) $\Rightarrow$ (4): obvious, take $N = A/I$.

(4) $\Rightarrow$ (3): let $N = \varinjlim N_\alpha$ where each N_α is a finitely-generated submodule of N, then $N = \bigcup\limits_\alpha N_\alpha$. We know $M \otimes_A N = \varinjlim M \otimes_A N_\alpha$, and by flatness this is just $\bigcup\limits_\alpha (M \otimes_A N_\alpha)$. It is now enough to show that if N is finitely-generated, then $M \otimes N = 0$ implies $N = 0$. We proceed by induction. This is obvious when N is cyclic; suppose N is generated by a minimal set of generators $\{x_1, \dots, x_n\}$, then let N' be generated by $\{x_1, \dots, x_{n-1}\}$, so $N' \neq N$, now we have a short exact sequence

$$0 \longrightarrow N' \longrightarrow N \longrightarrow A/I \cong N/N' \longrightarrow 0$$

for some ideal I of A, and since M is A-flat, then we have a short exact sequence

$$0 \longrightarrow M \otimes N' \longrightarrow M \otimes N \longrightarrow M \otimes (A/I) \cong 0 \longrightarrow 0$$

but that means $A = I$, so $N' = N$, which is a contradiction unless $M \otimes_A N = 0$ implies $N = 0$.

Exercise 2.75. Show that (3) $\Rightarrow$ (1), (2).

$\square$

Definition 2.76 (Faithfully Flat)**.** Let A be a commutative ring, an A-module M is called faithfully flat if M satisfies one of the (equivalent) conditions in Proposition 2.74.

Definition 2.77 (Faithful)**.** Let A be a commutative ring, an A-module M is called faithful if $\operatorname{Ann}_A(M) = \{a \in A \mid aM = 0\} = (0)$.

Remark 2.78. Faithfully flat implies faithful. Indeed, let M be faithfully flat, let $I = \operatorname{Ann}_A(M)$, then consider the short exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$$

and therefore

$$\begin{tikzcd} 0 \arrow[r] & I \otimes_A M \arrow[r] \arrow[dr, "x\otimes m \mapsto xm"'] & A \otimes_A M \cong M \arrow[r] \arrow[d, "a\otimes m\mapsto am", "\cong"'] & A/I \otimes_A M \arrow[r] & 0 \\ & & M & & \end{tikzcd}$$

is a short exact sequence. In particular, $I \otimes_A M = 0$ by definition, therefore $I = 0$ since M is flat, hence M is faithful.

Example 2.79. Note that M being flat and faithful does not imply M is faithfully flat. Let $A = \mathbb{Z}$ and $M = \mathbb{Q}$, so $\mathbb{Q}$ is faithful and is $\mathbb{Z}$-flat, but $\mathbb{Q}$ is not faithfully flat over $\mathbb{Z}$ since $\mathbb{Q} \otimes \mathbb{Z}/n\mathbb{Z} = 0$ but $\mathbb{Z}/n\mathbb{Z} \neq 0$ for $n > 1$.

Theorem 2.80. Let $f : A \to B$ be a homomorphism of commutative rings. The following are equivalent:

(i) B is a faithfully flat A-module via f;

(ii) B is A-flat, and for every ideal I of A, $f^{-1}(IB) = I$;

(iii) B is A-flat, and for every A-module M, $M \to M \otimes_A B$ is injective;

(iv) f is injective and $B/f(A) \cong B/A$ is A-flat.

Proof. $(i) \Rightarrow (ii)$: B being A-flat is obvious; let $J = f^{-1}(IB)$, then there is a short exact sequence

$$0 \longrightarrow I \longrightarrow J \longrightarrow J/I \longrightarrow 0$$

and tensoring it with B gives

$$\begin{tikzcd} 0 \arrow[r] & I \otimes_A B \arrow[r] \arrow[dr] & J \otimes_A B \arrow[r] \arrow[d, "j\otimes b\mapsto jb"] & J/I \otimes_A B \arrow[r] & 0 \\ & & B & & \end{tikzcd}$$

where $J \otimes_A B \cong B \cong A \otimes_A B$, and so $\operatorname{im}(J \otimes_A B) = JB$, and $\operatorname{im}(I \otimes_A B) = IB$, therefore having $J = f^{-1}(IB)$ implies $JB = IB$. We have $I \otimes_A B = J \otimes_A B$, so $J/I \otimes_A B = 0$. Since B is faithfully flat, then $J/I = 0$, so $I = J$.

$(ii) \Rightarrow (iii)$: we want to show that $i_M : M \to M \otimes_A B$ is injective. Suppose, towards contradiction, that there exists some element $0 \neq x \in M$ such that $i_M(x) = x \otimes 1 = 0$, then define $I = \{a \in A \mid ax = 0\}$. We have a commutative diagram

$$\begin{tikzcd} A/I \arrow[r, "\bar{f}"] \arrow[d, hook] & A/I \otimes_A B \arrow[d, hook] \\ M \arrow[r] & M \otimes_A B \end{tikzcd}$$

Note that $A/I \otimes_A B \hookrightarrow M \otimes_A B$ is injective since B is A-flat. This gives a diagram chasing

$$\begin{tikzcd} \bar{1} \arrow[r, "\bar{f}"] \arrow[d, hook] & \bar{1} \otimes 1 \arrow[d, hook] \\ x \arrow[r] & x \otimes 1 = 0 \end{tikzcd}$$

By the commutative diagram, $\bar{f}(A/I) = 0$, so $\bar{f}$ is the zero map, and since $A/I \otimes_A B = B/IB$, then $f^{-1}(IB) = A \supsetneq I$, contradiction.

$(iii) \Rightarrow (iv)$: let B be A-flat and suppose every A-module M, every map $M \to M \otimes_A B$ is an injection, then $A \to A \otimes_A R = R$ is injective. Consider

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$$

to show that B/A is A-flat, take the following short exact sequence

$$0 \longrightarrow N \longrightarrow T \longrightarrow M \longrightarrow 0$$

and by tensoring via the first short exact sequence we obtain

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & N \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & N \otimes_A B \arrow[r] \arrow[d] & T \otimes_A B \arrow[r] \arrow[d] & M \otimes_A B \arrow[r] \arrow[d] & 0 \\ & N \otimes_A B/A \arrow[r] \arrow[d] & T \otimes_A B/A \arrow[r] \arrow[d] & M \otimes_A B/A \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

and it suffices to show exactness at $N \otimes_A B/A$. Let $x \in N \otimes B/A$ map to 0 in $T \otimes_A B/A$, then lift it to $y \in N \otimes_A B$, send it to z in $T \otimes_A B$, by exactness it sends to 0 in $M \otimes_A B$. Now z has a preimage of w in T, sending it to m in M, but injectivity of $M \to M \otimes_A B$ implies $m = 0$, therefore w lifts to some $n \in N$, here $n \in N$ is mapped to y' in $N \otimes_A B$, but that means n is mapped to 0 in $T \otimes_A B$ as well, by injectivity of $N \otimes_A B \to T \otimes_A B$, we have $y' = y$. Hence, n maps to $y' = y$ maps to x in the column, and by exactness this forces $x = 0$.[5]

$(iv) \Rightarrow (iii)$: it suffices to show the following lemma.

Lemma 2.81. Let

$$0 \longrightarrow N \longrightarrow M \longrightarrow T \longrightarrow 0$$

be a short exact sequence of A-modules, and suppose T is A-flat, then for all A-module L, we have the short exact sequence

$$0 \longrightarrow L \otimes_A N \longrightarrow L \otimes_A M \longrightarrow L \otimes_A T \longrightarrow 0$$

to be exact.

[5] Instead of diagram chasing, one can apply the snake lemma instead.

Subproof. Suppose we have a short exact sequence

$$0 \longrightarrow V \longrightarrow F \longrightarrow L \longrightarrow 0$$

where F is free. Then consider

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & V\otimes N \arrow[r] \arrow[d] & F\otimes N \arrow[r] \arrow[d] & L\otimes N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V\otimes M \arrow[r] \arrow[d] & F\otimes M \arrow[r] \arrow[d] & L\otimes M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V\otimes T \arrow[r] \arrow[d] & F\otimes T \arrow[r] \arrow[d] & L\otimes T \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

We want to show $L\otimes N$ is exact in the column, i.e., $L\otimes N \to L\otimes M$ is injective. Note that the last row is exact since T is A-flat. We can use a similar argument. Take x in $L\otimes N$ mapping to 0 in $L\otimes M$, lift it to y in $F\otimes N$, map it to z in $F\otimes M$ with image 0 in $L\otimes M$, lift it to w in $V\otimes M$, send it to $t\in V\otimes T$ which maps into 0 in $F\otimes T$ by exactness of middle row, by injectivity we know $t=0$, then lift it to n in $V\otimes N$, send it to y' in $F\otimes N$ which maps to z in $F\otimes M$. The middle row is exact since F is free, so $y'=y$ by injectivity, so by exactness of the row we know $x=0$. ■

Therefore, consider

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$$

where B/A is A-flat.

Exercise 2.82. If A and B/A are both A-flat, then B is also A-flat.

By Lemma 2.81, we know the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & M\otimes_A A \arrow[r] \arrow[d, equal] & M\otimes_A B \arrow[r] & M\otimes_A B/A \arrow[r] & 0 \\
& M \arrow[ur] & & &
\end{tikzcd}$$

is exact, therefore $M\to M\otimes_A B$ is injective.

$(iii),(iv) \Rightarrow (i)$: let B be A-flat and $M\to M\otimes_A B$ be injective. We want to show that for any N such that $N\otimes_A B = 0$, we have $N=0$. Consider

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$$

to be a short exact sequence, and we know B/A is A-flat, so we now know that

$$\begin{tikzcd}
0 \arrow[r] & N\otimes_A A \arrow[r] \arrow[d, equal] & N\otimes_A B \arrow[r] & N\otimes_A B/A \arrow[r] & 0 \\
& N \arrow[ur, hook] & & &
\end{tikzcd}$$

is exact, therefore $N\otimes_A B = 0$ implies $N=0$ by injectivity. □

Theorem 2.83. Let A be a Noetherian ring and I be an ideal of A. Then $A\to \hat{A}$ is faithfully flat if and only if I is contained in the Jacobson radical of A.

Proof. Suppose I is contained in the Jacobson radical of A, then I is contained in the intersection of all maximal ideals of A. For any finitely-generated A-module M, we know $\bigcap_{n \geqslant 1} I^n M = (0)$. Therefore, $M \hookrightarrow \tilde{M} \cong M \otimes_A \hat{A}$ is an injection by Theorem 2.80. Suppose M is not necessarily finitely-generated, then M is the union (hence direct limit) of finitely-generated A-modules M_α's. We want to show that $M \to M \otimes_A \hat{A}$ is an injection. Suppose $x \in M$ is mapped to 0, so let $N = Ax = A/J$ where $J = \operatorname{Ann}_A(x)$, then we have a diagram

$$\begin{tikzcd}
1 \in N \arrow[r, hook] \arrow[d] & y \in N \otimes_A \hat{A} \arrow[d, hook] \\
x \in M \arrow[r] & 0 \in M \otimes_A \hat{A}
\end{tikzcd}$$

Since $N \hookrightarrow M$ and since $\hat{A}$ is A-flat, so $N \otimes_A \hat{A} \hookrightarrow M \otimes_A \hat{A}$ is injective as well. By chasing the diagram, we know $y = 0$, therefore by the injection we know $N = 0$, hence $x = 0$.

Suppose I is not contained in the Jacobson radical of A, then there exists some maximal ideal $\mathfrak{m}$ of A such that $I \nsubseteq \mathfrak{m}$. Consider $A/\mathfrak{m}$ with I-adic topology of filtration, then $\mathfrak{m} + IA = A$, therefore $\mathfrak{m} + I^n A = A$, hence $A/(\mathfrak{m} + I^n) = 0$. Therefore, $\widehat{(A/\mathfrak{m})} = \varprojlim (A/(\mathfrak{m} + I^n)) = 0$. But note that $\widehat{(A/\mathfrak{m})} = A/\mathfrak{m} \otimes_A \hat{A} = 0$, with $A/\mathfrak{m} \neq 0$, therefore $\hat{A}$ is not faithfully flat. ☐

Example 2.84. The map $k[x_1, \ldots, x_n] \to k[[x_1, \ldots, x_n]]$ is flat but not faithfully flat. Indeed, the ideal $(x_1, \ldots, x_n)$, the ideal is not contained in $(x_1 - a_1, \ldots, x_n - a_n)$ whenever a_i's are non-zero.

However, if we factor it via the localization

$$\begin{tikzcd}
k[x_1, \ldots, x_n] \arrow[r] \arrow[d] & k[[x_1, \ldots, x_n]] \\
k[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)} \arrow[ur, hook] &
\end{tikzcd}$$

then $k[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)} \to k[[x_1, \ldots, x_n]]$ is faithfully flat.

3 Dimension Theory

3.1 Graded Rings and Hilbert-Samuel Polynomial

Definition 3.1. Let $\mathcal{F}$ be the set of functions $f : \mathbb{Z} \to \mathbb{Z}$, let $\mathcal{P}$ be the set of functions $f : \mathbb{Z} \to \mathbb{Z}$ such that there exists a polynomial $g \in \mathbb{Q}[x]$ such that $f(n) = g(n)$ for $n \gg 0$.

Remark 3.2. Obviously such g is unique, since any such choices would agree for all sufficiently large values.

Definition 3.3. $f \in \mathcal{P}$ is called an essentially polynomial, or an essentially polynomial function.

Definition 3.4 (Degree)**.** We define the degree of f to be the degree of function g.

Remark 3.5. If $f = 0$ for $n \gg 0$, then $\deg(f) = -1$; if $f = a$ is a non-zero constant function, then $\deg(f) = 0$.

Example 3.6. Say $f(n) = \binom{n}{i}$ where we fix i. For $n \geqslant i$, $f(n)$ is an integer; for $n < i$, $f(n) = 0$. Therefore, the function $f(x) = \binom{x}{i}$ is a function with rational coefficients.

Definition 3.7. For $f \in \mathcal{F}$, we define $\Delta f : \mathbb{Z} \to \mathbb{Z}$ to be a function such that $\Delta f(n) = f(n+1) - f(n)$.

Remark 3.8. If $f \in \mathcal{P}$, then $\Delta f \in \mathcal{P}$. For $n \gg 0$, $f(n) = a_0 n^r + a_1 n^{r-1} + \cdots + a_r$ for $a_i \in \mathbb{Q}$, then $\Delta f(n) = r a_0 n^{r-1} + \cdots$. Hence, $\Delta^r(f) = r! a_0$. But we know $\Delta^r : \mathbb{Z} \to \mathbb{Z}$ if we proceed inductively, so $r! a_0$ is an integer. Note that $\Delta^{r+1}(f) = 0$.

Definition 3.9 (Multiplicity)**.** We say $\Delta^r(f) \equiv \mu(f)$ is the multiplicity of f, that is, $\mu(f) = r! a_0$.

Lemma 3.10. Let $f : \mathbb{Z} \to \mathbb{Z}$, then the following are equivalent:

(i) $f \in \mathcal{P}$;

(ii) $\Delta(f) \in \mathcal{P}$;

(iii) there exists $r > 0$ such that either $\Delta^{r+1} f = 0$ for $n \gg 0$, or $\Delta^r(f)$ is constant.

Proof. It is enough to show that $\Delta f \in \mathcal{P}$ implies $f \in \mathcal{P}$, and we will induct on degree of Δf. If the degree of Δf is -1, then $\Delta f(n) = 0$ for $n \gg 0$, so if $f(n+1) - f(n) = 0$ for $n \gg 0$, then $f(n+1) = f(n)$ for $n \gg 0$, thus f is constant for $n \gg 0$, by definition $f \in \mathcal{P}$. Now suppose this holds for polynomial f with degree of Δf at most $r-1$. Suppose Δf is of the form $a_0 n^r + a_1 n^{r-1} + \cdots + a_r$, then $r! a_0 = \Delta^{r+1} f = \Delta^r(\Delta f) = r! a_1$ which are integers. We write $g(x) = r! a_0 \binom{x}{n+1}$ then $\Delta g(n)$ is dominated by the term $r! a_0 \frac{r+1}{(r+1)!} n^r$, which is just $a_0 n^r$. We know $\Delta(f-g) = \Delta(f) - \Delta(g)$ which is a polynomial of degree at most $r-1$, so by induction $f - g \in \mathcal{P}$, hence $f = g + h$ for some $h \in \mathcal{P}$, hence $f \in \mathcal{P}$. ☐

Exercise 3.11. Show that $\mathcal{P}$ is a free abelian group with basis $\binom{x}{i}$ where $i \geqslant 0$.

Recall that A is Artinian if and only if A is Noetherian and A has finitely many prime ideals such that each of which is maximal. Note that $(0) = \mathfrak{m}_1^{i_1} \cdots \mathfrak{m}_r^{i_r}$ is a decomposition of maximal ideals, if and only if $\ell_A(A) < \infty$. Moreover, if M is a finitely-generated A-module, then $\ell_A(M) < \infty$.

Definition 3.12. Suppose A has a decomposition $A = A_0 \oplus A_1 \oplus \cdots \oplus A_n \oplus \cdots$ and M is a graded module $M = M_0 \oplus M_1 \oplus \cdots \oplus M_n \oplus \cdots$ where $A_i M_j \subseteq M_{i+j}$. Suppose $N \subseteq M$ is a submodule. Let $x \in N$ be written as $x = x_{i_1} + \cdots + x_{i_t}$, then we say N is a graded submodule if every $x_{i_j} \in N$. In particular, this is equivalent to $N = \bigoplus_i M \cap N_i$.

Remark 3.13. Under this definition, M/N is also a graded module over A. Moreover, let $B = A[X_1, \ldots, X_n]$, and suppose I is a graded ideal of B, then B/I is graded. Moreover, we view B as an A-module generated by the x_i's, i.e., $B = A[x_1, \ldots, x_n]$ where each x_i has degree 1.

Theorem 3.14 (Hilbert-Serre)**.** Let A_0 be an Artinian ring and $A = A_0[x_1, \ldots, x_r]$ be a finitely-generated graded ring over A_0 with $\deg(x_i) = 1$ for all i.[6] Let M be a finitely-generated A-module, and denote $M = M_0 \oplus M_1 \oplus \cdots$, then we have the following:

[6] Alternatively, we have $A = A_0 \oplus (x_1, \ldots, x_r) \oplus (x_1, \ldots, x_r)^2 \oplus \cdots$

(i) each M_n is a module of finite length over A_0;

(ii) let $\chi(M,n)=\ell_{A_0}(M_n)$ be the Hilbert function, then $\chi(M,n)$ is essentially polynomial of degree at most $r-1$;

(iii) suppose M_0 generates M over A, then $\Delta^{r-1}\chi(M,n)\leqslant\ell_{A_0}(M_0)$. Moreover, the equality holds if and only if

$$\begin{aligned} M_0[X_1,\dots,X_r]&\to M\\ mX_1^{i_1}\cdots X_r^{i_r}&\mapsto mx_1^{i_1}\cdots x_r^{i_r}, \end{aligned}$$

where $m\in M_0$, is an isomorphism. It is obvious that φ is an onto graded map.

Proof. (i) Let $m_1,\dots,m_t$ be the graded homogeneous generators of M over A. For each M_n, we can write $x=\sum\limits_{i,j}c_{i_1,\dots,i_r}x_1^{i_1}x_2i_2\cdots x_r^{i_r}m_j$ where $c_{i_1,\dots,i_r}\in A_0$, such that each x_i has degree 1. Suppose $\deg(m_j)=h_j$, then $n=\sum\limits_{j,k}i_k+h_j$. The solution of this equation consists of finite number of $(i_1,\dots,i_r)$ and h_j's. Therefore, M_n is finitely-generated over A_0, hence $\ell_{A_0}(M_n)<\infty$.

(ii) We proceed by induction on r. Suppose $r=0$, then $A=A_0$, and $M=M_0\oplus M_1\oplus\cdots M_t\oplus 0\oplus 0\oplus\cdots$. This means $\chi(M,n)=0$ for $n\gg 0$, so the degree of $\chi(M,n)=-1$. Suppose this is true degree at most $r-1$, then let $N=\ker(x_r)$ and $\bar{M}=M/x_rM$, then

$$0\longrightarrow N\longrightarrow M\xrightarrow{x_r}M\longrightarrow\bar{M}\longrightarrow 0$$

Now $\bar{M}$ and N are finitely-generated modules over $A_0[x_1,\dots,x_r]/x_rA_0[x_1,\dots,x_r]=A_0[\bar{x}_1,\dots,\bar{x}_{r-1}]$. For any n, we have

$$0\longrightarrow N_n\longrightarrow M_n\longrightarrow M_n\longrightarrow\bar{M}_n\longrightarrow 0$$

therefore

$$\begin{aligned} \ell(\bar{M}_n)-\ell(N_n)&=\ell_{A_0}(M_{n+r})-\ell_{A_0}(M_n)\\ &=\Delta\chi(M,n)\\ &=\chi(\bar{M}_n)-\chi(N,n). \end{aligned}$$

By induction, $\chi(\bar{M},n)$ and $\chi(N,n)$ are essentially polynomials of degree at most $r-1$, so $\Delta\chi(M,n)$ is essentially polynomial of degree at most $r-2$, therefore $\chi(M,n)$ is essentially polynomial of degree at most $r-1$.

(iii) Suppose M_0 generates M over A, then it is obvious that

$$\begin{aligned} M_0[X_1,\dots,X_r]&\to M\\ mX_1^{i_1}\cdots X_r^{i_r}&\mapsto mx_1^{i_1}\cdots x_r^{i_r} \end{aligned}$$

is an onto graded map where $m\in M_0$. This implies $\varphi_n:(M_0[X_1,\dots,X_r])_n\twoheadrightarrow M_n$ is onto as well. Hence, $\ell_{A_0}(M_n)\leqslant\ell_{A_0}(M_0[X_1,\dots,X_r])_n$. (Note that $k_{[x,y]}$ has a basis given by $x^n,x^{n-1}y,\dots,xy^{n-1},y^n$.) We observe that $(M_0[X_1,\dots,X_r])_n$ is just $M_0\otimes_{A_0}[A_0[X_1,\dots,X_r]]_n$ (where $[-]_n$ is the completion on the nth grading), so $\ell_{A_0}(M_0[X_1,\dots,X_r])_n$ is just $\ell_{A_0}(M_0)$ multiplied by the number of monomials of (total) degree n in $X_1,\dots,X_r$, and by stars-and-bars that is just $\ell_{A_0}(M_0)\binom{n+r-1}{r-1}$. By part (ii), we know that the degree of $\chi(M,n)$ is at most $r-1$. Also, we have $\chi(M_0[X_1,\dots,X_r],n)=\ell_{A_0}(M_0)\binom{n+r-1}{r-1}$, which is a polynomial of degree $r-1$. We then conclude that $\Delta^{r-1}\chi(M_0[X_1,\dots,X_r],n)=\ell_{A_0}(M_0)$. Hence, $\Delta^{r-1}\chi(M,n)\leqslant\ell_{A_0}(M_0)$.

Now suppose φ is an isomorphism, then $\chi(M,n)=\chi(M_0[X_1,\dots,X_r],n)=\ell_{A_0}(M_0)\binom{n+r-1}{r-1}$, therefore $\Delta^{r-1}\chi(M,n)=\ell_{A_0}(M_0)$. Conversely, if $\Delta^{r-1}\chi(M,n)=\ell_{A_0}(M_0)$, then we want to show φ is an isomorphism. Since φ is onto, the kernel L gives a short exact sequence

$$0\longrightarrow L\longrightarrow M_0[X_1,\dots,X_r]\longrightarrow M\longrightarrow 0$$

where all terms are all graded components, so have positive lengths. Now we know $\chi(M_0[X_1,\dots,X_r],n)=\chi(M,n)+\chi(L,n)$, so $\Delta^{r-1}\chi(M_0[X_1,\dots,X_r],n)=\Delta^{r-1}\chi(M,n)+\Delta^{r-1}\chi(L,n)$, therefore $\Delta^{r-1}\chi(L,n)=$

0 since $\Delta^{r-1}\chi(M,n)=\ell_{A_0}(M_0)$. We claim that this is not true if $L\neq 0$. Induct on $\ell_{A_0}(M_0)$. If $\ell_{A_0}(M_0)=1$, then $M_0=k$ a field, so

$$0 \longrightarrow L \longrightarrow B=k[X_1,\dots,X_n] \longrightarrow M \longrightarrow 0$$

If $L\neq 0$, then L is a graded ideal of B, then for some $d>0$ we have $L_d\neq 0$. Let $0\neq f\in L_d$ be homogeneous of degree d, then $B_{n-d}f\in L_n$. This implies $\chi(L_n)=\dim_k(L_n)\geqslant \dim_k(B_{n-d})=\binom{n-d+r-1}{r-1}$. This gives $\Delta^{r-1}\chi(L,n)\geqslant 1$, contradiction. Now suppose $\ell_{A_0}(M_0)>1$, then take a Jordan-Hölder series

$$M_0\supset M_0^{(1)}\supset M_0^{(2)}\supset\cdots\supset M_0^{(n)}=0,$$

such that $M_0^{(i)}/M_0^{(i+1)}\cong A/\mathfrak{m}_i\cong k_i$, where $\mathfrak{m}_i$ is maximal and k_i is a field (but is only isomorphic as modules). Therefore,

$$M_0[X_1,\dots,X_r]\supset M_0^{(1)}[X_1,\dots,X_r]\supset M_0^{(2)}[X_1,\dots,X_r]\supset\cdots$$

is a series such that $M_0^{(i)}[X_1,\dots,X_r]/M_0^{(i+1)}[X_1,\dots,X_r]=k_i[X_1,\dots,X_r]$.[7] If we now denote $L^{(i)}=L\cap M_0^{(i)}[X_1,\dots,X_r]$, then there is a filtration $L\supset L^{(1)}\supset L^{(2)}\supset\cdots$, so

$$L^{(i)}/L^{(i+1)}\hookrightarrow M_0^{(i)}[X_1,\dots,X_r]/M^{(i+1)}[X_1,\dots,X_r]\cong k_i[X_1,\dots,X_r].$$

Hence, $\chi(L,n)=\sum\limits_i\chi(L^{(i)}/L^{(i+1)},n)$, therefore $\Delta^{r-1}\chi(L,n)=\sum\limits_i\Delta^{r-1}\chi(L^{(i)}/L^{(i+1)},n)$. But $L\neq 0$, so there exists some i such that $L^{(i)}/L^{(i+1)}\neq 0$. By the base case (of the induction on $\ell_{A_0}(M_0)$), we know $\Delta^{r-1}\chi(L^{(i)}/L^{(i+1)},n)>0$, therefore $\Delta^{r-1}\chi(L,n)>0$, contradiction.

□

Definition 3.15 (Hilbert Multiplicity)**.** Suppose $\deg(\chi(M,n))=d$, then $\chi(M,n)=a_0n^d+$ linear terms with higher degrees, where $n\gg 0$. Then $A^d=\chi(M,n)=d!a_0$. We say $e_d(M)=d!a_0$ is the Hilbert multiplicity of M over A, i.e., $a_0=\frac{e_d(M)}{d!}$.

Remark 3.16.

1. Let A be Noetherian and M and N be (non-zero) finitely-generated A-modules, then the support of M is $\operatorname{supp}(M)=V(M)$, the set of prime ideals P of A such that $M_P\neq 0$, which is equivalent to the set of prime ideals P of A where $P\supseteq \operatorname{Ann}_A(M)$.

   In particular, if $I=\operatorname{Ann}_A(M)$, then $\operatorname{supp}(M)=\operatorname{supp}(A/I)=V(A/I)\approx V(I)$.

2. Under the above assumption, $\operatorname{supp}(M\otimes_A N)=\operatorname{supp}(M)\cap\operatorname{supp}(N)$. Indeed, let P be in the support of $M\otimes_A N$, then $(M\otimes_A N_P\neq 0$, so $(M\otimes_A N)_P=M_P\otimes_{A_P}N_P\neq 0$, so $M_P\neq 0$ and $N_P\neq 0$, therefore $P\in\operatorname{supp}(M)\cap\operatorname{supp}(N)$. Now suppose $P\in\operatorname{supp}(M)\cap\operatorname{supp}(N)$, then $M_P\neq 0$ and $N_P\neq 0$.

   Lemma 3.17. Let A be a local ring and M,N be (non-zero) finitely-generated A-modules, then $M\otimes_A N\neq 0$.

   Remark 3.18. We know $\mathbb{Q}\otimes\mathbb{Z}/n\mathbb{Z}=0$, but $\mathbb{Q}$ is not finitely-generated as a $\mathbb{Z}$-module.

   Proof. Let $\mathfrak{m}$ be the maximal ideal of A. If $M\otimes_A N=0$, then $A/\mathfrak{m}\otimes_A(M\otimes_A N)=0$, therefore $M/\mathfrak{m}M\otimes_{A/\mathfrak{m}}M/\mathfrak{m}N=0$. We run a dimension argument on the vector space, then either $M/\mathfrak{m}M=0$ or $N/\mathfrak{m}N=0$. By Corollary 2.55, either $M=0$ or $N=0$. □

   This implies $\operatorname{supp}(M)\cap\operatorname{supp}(N)=\operatorname{supp}(M\otimes N)$.

3. (a) Let $\mathfrak{q}$ be an ideal of A, and M be a finitely-generated A-module. Suppose $\ell(M/\mathfrak{q}M)<\infty$, then $\ell(M/q^nM)<\infty$ for all n.

   (b) Consider the short exact sequence

   $$0\longrightarrow N\longrightarrow M\longrightarrow T\longrightarrow 0$$

   and $\mathfrak{q}$ is an ideal of A such that $\ell(M/\mathfrak{q}M)<\infty$, then $\ell(N/\mathfrak{q}N)<\infty$ and $\ell(T/\mathfrak{q}T)<\infty$.

[7] Consider the quotient of modules as a short exact sequence, and then tensor it by the polynomial ring structure, then we retrieve a short exact sequence represented by this quotient.

Proof. (a) Note that $\ell(M/\mathfrak{q}M) < \infty$ if and only if $\operatorname{supp}(M/\mathfrak{q}M)$ consists of finitely many maximal ideals only, therefore $\operatorname{supp}(M/\mathfrak{q}M) = \operatorname{supp}(A/\mathfrak{q} \otimes_A M) = \operatorname{supp}(A/\mathfrak{q}) \cap \operatorname{supp}(M)$. Therefore,

$$\begin{aligned}\operatorname{supp}(M/\mathfrak{q}^n M) &= \operatorname{supp}(A/\mathfrak{q}^n) \cap \operatorname{supp}(M) \\ &= \operatorname{supp}(A/\mathfrak{q}) \cap \operatorname{supp}(M),\end{aligned}$$

so it consists of maximal ideals only as well, therefore $\ell(M/\mathfrak{q}^n M) < \infty$ for all $n > 0$.

(b) Note that $\operatorname{supp}(N/\mathfrak{q}N) = \operatorname{supp}(A/\mathfrak{q}) \cap \operatorname{supp}(N) \subseteq \operatorname{supp}(A/\mathfrak{q}) \cap \operatorname{supp}(M)$, which consists of maximal ideals only, therefore $\operatorname{supp}(N/\mathfrak{q}N)$ consists of maximal ideals only as well. That is, $\ell(N/\mathfrak{q}N) < \infty$.

□

Theorem 3.19. Let A be a Noetherian ring, $\mathfrak{q}$ be an ideal of A, and let M be a finitely-generated A-module. Suppose $A \sim \{\mathfrak{q}^n\}$ and $M \sim \{M_n\}$ where the filtration is given by $\mathfrak{q}^i M_j \subseteq M_{i+j}$. We further assume that $\ell(M/\mathfrak{q}M) < \infty$, and that $\{M_n\}$ is $\mathfrak{q}$-good. Define $P_{\mathfrak{q}}((M_n), n) := \ell_A(M/M_n)$, then $\mathfrak{q}^n M \subseteq M_n$, therefore there is a surjection $M/\mathfrak{q}^n M \twoheadrightarrow M/M_n$. Then

- $P_{\mathfrak{q}}((M_n), n)$ is essentially polynomial that depends on $E_0(M)$, and
- if $\ell_A(M/\mathfrak{q}^n M) < \infty$, then $\ell_A(M/M_n)$ is finite.

Proof. We have

$$\begin{aligned}\Delta P_n((M_n), n) &= \ell_A(M/M_{n+1}) - \ell_A(M/M_n) \\ &= \ell_A(M_n/M_{n+1}),\end{aligned}$$

and take the decomposition $E_0(M) = M/M_1 \oplus M_1/M_2 \oplus \cdots$, and $E_0(A) = A/\mathfrak{q} \oplus \mathfrak{q}/\mathfrak{q}^2 \oplus \cdots$, then $E_0(M)$ is an $E_0(A)$-module. Since A is Noetherian, then $\mathfrak{q}$ is finitely-generated and so we write $\mathfrak{q} = (x_1, \ldots, x_n)$, and so

$$\begin{aligned}\varphi : A/\mathfrak{q}[x_1, \ldots, x_n] &\to E_0(A) \\ x_i &\mapsto \bar{x}_i \in \mathfrak{q}/\mathfrak{q}^2\end{aligned}$$

is an onto map. Note that $A/\mathfrak{q}[x_1, \ldots, x_n]$ is Noetherian, so $E_0(A)$ is Noetherian as well. Since $\{M_n\}$ is $\mathfrak{q}$-good, then there exists some h such that $M_{n+h} = \mathfrak{q}^n M_h$ for all $n > 0$. Therefore, $M/M_1 \oplus M_1/M_2 \oplus \cdots \oplus M_h/M_{h+1}$ generates $E_0(M)$ over $E_0(A)$. For $x \in M_n$, we have $0 \neq \bar{x} \in M_n/M_{n+1}$, and $M_n = \mathfrak{q}^{n-h} M_h$, so $x = \sum y_i w_i$ where $y_i \in \mathfrak{q}^{n-j}$ and $w_i \in M_h$. Therefore, $\bar{x} = \sum \bar{y}_i \bar{w}_i$ in $E_0(M)$ for $\bar{y}_i \in \mathfrak{q}^{n-h}/\mathfrak{q}^{n-h+1}$ and $\bar{w}_i \in M_h/M_{h+1}$. This shows that $E_0(M)$ is a finitely-generated $E_0(A)$-module with generators from $M/M_1, \ldots, M_h/M_{h+1}$, where each of them is a finitely-generated $A/\mathfrak{q}$-module.

Remark 3.20. Note that $A/\mathfrak{q}$ is not necessarily Artinian, so we cannot apply Theorem 3.14 right now.

Recall $\ell(M/\mathfrak{q}M) < \infty$, if we denote $I = \operatorname{Ann}_A(M)$, then

$$\begin{aligned}\operatorname{supp}(M/\mathfrak{q}M) &= \operatorname{supp}(A/\mathfrak{q}) \cap \operatorname{supp}(M) \\ &= \operatorname{supp}(A/\mathfrak{q}) \cap \operatorname{supp}(A/I) \\ &= \operatorname{supp}(A/\mathfrak{q} \otimes_A A/I) \\ &= \operatorname{supp}(A/(\mathfrak{q} + I)).\end{aligned}$$

If we denote $\bar{A} = A/I$, then $\bar{A}/\bar{\mathfrak{q}} = A/(q + I)$, therefore $\ell_{\bar{A}}(\bar{A}/\bar{\mathfrak{q}}) < \infty$. We write down $E_0(\bar{A}) = \bar{A}/\bar{\mathfrak{q}} \oplus \bar{\mathfrak{q}}/\bar{\mathfrak{q}}^2 \oplus \cdots$.

Claim 3.21. $E_0(M)$ is a finitely-generated $E_0(\bar{A})$-module.

Subproof. Since $IM = 0$, then for any i, $(\mathfrak{q} + I)^n M_i = \mathfrak{q}^n M$. ■

Since $\ell_{\bar{A}}(\bar{A}/\bar{\mathfrak{q}}) < \infty$, then $\bar{A}/\bar{\mathfrak{q}}$ is Artinian, and now by Theorem 3.14 we know $\Delta P_{\mathfrak{q}}((M_n), n)$ is essentially polynomial. Therefore, $P_{\mathfrak{q}}((M_n), n)$ is essentially polynomial.

Let $M_n = \{\mathfrak{q}^n M\}$, then $E_0(M) = M/\mathfrak{q}M \oplus \mathfrak{q}M/\mathfrak{q}^2 M \oplus \cdots$, and $E_0(\bar{A}) = \bar{A}/\bar{\mathfrak{q}} \oplus \bar{\mathfrak{q}}/\bar{\mathfrak{q}}^2 \oplus \cdots$, then $E_0(M)$ is generated by $M/\mathfrak{q}M$ over $E_0(\bar{A})$. Write $P_{\mathfrak{q}}(M, n) = \ell(M/\mathfrak{q}^n M)$, then $\Delta P_{\mathfrak{q}}(M, n) = \ell(\mathfrak{q}^n M/\mathfrak{q}^{n+1} M)$. Suppose

$(\mathfrak{q}+I)/I$, that is, $\bar{q}$ in $\bar{A}$, is minimally generated by r elements $\bar{x}_1, \ldots, \bar{x}_r$, so $E_0(\bar{A}) = \bar{A}[\bar{x}_1, \ldots, \bar{x}_r]$, then $\Delta P_{\mathfrak{q}}(M,n)$ is of degree at most $r-1$, and $\Delta^{r-1}(\Delta P_q(M,n)) \leqslant \ell(M/\mathfrak{q}M)$, and note that the equality holds if and only if

$$\varphi : M/\mathfrak{q}M \otimes_{\bar{A}/\bar{\mathfrak{q}}} \bar{A}/\bar{\mathfrak{q}}[x_1, \ldots, x_n] \to E_0(M) = M/\mathfrak{q}M \oplus \mathfrak{q}M/\mathfrak{q}^2M \oplus \cdots$$

is an isomorphism. In particular, $\Delta^r(P_{\mathfrak{q}}(M,n)) \leqslant \ell(M/\mathfrak{q}M)$ therefore $\ell_A(M/M_n)$ is finite. ☐

Corollary 3.22. Under the same assumption, $\ell(M/\mathfrak{q}^nM) \geqslant \ell(M/M_n)$. Moreover, if we write down the polynomials of $P_{\mathfrak{q}}(M,n)$ and $P_{\mathfrak{q}}((M_n),n)$, then

- the degree of $P_{\mathfrak{q}}(M,n)$ is the degree of $P_{\mathfrak{q}}((M_n),n)$, the leading coefficient of $P_{\mathfrak{q}}(M,n)$ is the leading coefficient of $P_{\mathfrak{q}}((M_n),n)$, hence $\Delta^r(P_{\mathfrak{q}}(M,n)) = \Delta^r(P_q((M_n),n))$ where r is the degree of $P_{\mathfrak{q}}(M,n)$;
- $P_{\mathfrak{q}}(M,n) = P_{\mathfrak{q}}((M_n),n) + R(n)$ where $R(n)$ is essentially polynomial whose degree is less than the degree of $P_{\mathfrak{q}}(M,n)$, and the leading coefficient is non-negative.

Proof.
- Let $P_{\mathfrak{q}}(M,n)$ has degree d and leading coefficient a_0, and let $P_{\mathfrak{q}}((M_n),n)$ has degree d' and leading coefficient b_0. Since $\ell(M/\mathfrak{q}^nM) \geqslant \ell(M/M_n)$ for all n, then $d \geqslant d'$. Now $M_{n+h} = \mathfrak{q}^nM_h \subseteq \mathfrak{q}^nM$ since this is a good filtration, therefore $\ell(M/M_{n+h}) \geqslant \ell(M/\mathfrak{q}^nM)$, therefore $d' \geqslant d$, hence $d = d'$. Similarly, the argument above implies $a_0 \geqslant b_0$ and $b_0 \geqslant a_0$, so $a_0 = b_0$.

  This implies $\Delta^d(P_{\mathfrak{q}}(M,n)) = \Delta^d(P_{\mathfrak{q}}((M_n),n)) = a_0 \cdot d!$.
- Consider
$$0 \longrightarrow M_n/\mathfrak{q}^nM \longrightarrow M/\mathfrak{q}^nM \longrightarrow M/M_n \longrightarrow 0$$
  therefore $\ell(M/\mathfrak{q}^nM) = \ell(M/M_n) + \ell(M_n/\mathfrak{q}^nM)$. Let $R(n) = \ell(M_n/\mathfrak{q}^nM)$, then $P_{\mathfrak{q}}(M,n) = P_{\mathfrak{q}}(M_n,n) + R(n)$, therefore the degree of $R(n)$ is less than d, the degree of $P_{\mathfrak{q}}(M,n)$, and by definition of $R(n)$, the coefficient of the leading term of $R(n)$ is non-negative.

☐

Definition 3.23 (Hilbert-Samuel Polynomial)**.** Let A be a Noetherian ring, $\mathfrak{q}$ be an ideal of A, M be a finitely-generated A-module, with $\ell(M/\mathfrak{q}M) < \infty$, then $P_{\mathfrak{q}}(M,n)$ is called the Hilbert-Samuel polynomial of M with respect to $\mathfrak{q}$. We define the degree of $P_{\mathfrak{q}}(M,n) = a_0n^d + a_1n^{d-1} + \cdots$ to be d, then $\Delta^d(P_{\mathfrak{q}}(M,n)) = d!a_0$ is called the Hilbert-Samuel multiplicity of M with respect to $\mathfrak{q}$.

Proposition 3.24. Let A be a Noetherian ring, $\mathfrak{q}$ be an ideal of A, M be a finitely-generated A-module, with $\ell(M/\mathfrak{q}M) < \infty$. Let $\mathfrak{q}'$ be another ideal of A such that $\ell(M/\mathfrak{q}'M) < \infty$. Suppose $\operatorname{supp}(M/\mathfrak{q}M) = \operatorname{supp}(M/\mathfrak{q}'M)$, then the degree of $P_{\mathfrak{q}}(M,n)$ equals to the degree of $P_{\mathfrak{q}'}(M,n)$.

Proof. Let $I = \operatorname{Ann}_A(M)$. Recall that

$$\begin{aligned}\operatorname{supp}(M/\mathfrak{q}M) &= \mathrm{A}/\mathfrak{q} \otimes_{\mathrm{A}} \mathrm{M} \\ &= \operatorname{supp}(A/\mathfrak{q}) \cap \operatorname{supp}(M) \\ &= \operatorname{supp}(A/\mathfrak{q}) \cap \operatorname{supp}(A/I) \\ &= \operatorname{supp}(A/\mathfrak{q} \otimes A/I) \\ &= \operatorname{supp}(A/\mathfrak{q} + I),\end{aligned}$$

then similarly $\operatorname{supp}(M/\mathfrak{q}'M) = \operatorname{supp}(A/(q'+I))$. Since $I = \operatorname{Ann}_A(M)$, then $IM = 0$, so we can assume M to be an A/I-module, that is, M is an A-module such that $\operatorname{Ann}_A(M) = 0$. In that case, then $\operatorname{supp}(M/\mathfrak{q}M) = \operatorname{supp}(A/\mathfrak{q})$ and $\operatorname{supp}(M/\mathfrak{q}'M) = \operatorname{supp}(A/\mathfrak{q}')$. Recall that $\ell(M/\mathfrak{q}M) < \infty$, so $\operatorname{supp}(A/\mathfrak{q})$ consists of maximal ideals only. (Since it is Artinian, there are finitely many of them.) Similarly, $\ell(M/\mathfrak{q}'M) < \infty$, so $\operatorname{supp}(A/\mathfrak{q}')$ consists of maximal ideals only as well. In particular, $\operatorname{supp}(A/\mathfrak{q})$ is the set of prime ideals containing $\mathfrak{q}$, and $\operatorname{supp}(A/\mathfrak{q}')$ is the set of prime ideals containing $\mathfrak{q}'$, but they are the same, so the radicals agree, i.e., $\sqrt{\mathfrak{q}} = \sqrt{\mathfrak{q}'}$. Since A is Noetherian, then $\mathfrak{q}^r \subseteq \mathfrak{q}'$ for some $r > 0$ and ${\mathfrak{q}'}^{r'} \subseteq \mathfrak{q}$ for some $r' > 0$ as well.

Claim 3.25. The degree of $P_{\mathfrak{q}}(M,n)$ equals to the degree of $P_{\mathfrak{q}^r}(M,n)$.

Subproof. If we write $P_{\mathfrak{q}}(M,n) = a_0 n^d + \cdots$, with lower degree terms, and $P_{\mathfrak{q}^r}(M,n) = \ell(M/\mathfrak{q}^{rn}M) = P_{\mathfrak{q}}(M,rn) = a_0(rn)^d + \cdots = a_0 r^d \cdot n^d + \cdots$, with lower degree terms. Therefore, the degree of $P_{\mathfrak{q}}(M,n)$ is the degree of $P_{\mathfrak{q}^r}(M,n)$, and the degree of $P_{\mathfrak{q}'}(M,n)$ is the degree of $P_{\mathfrak{q}'^{r'}}(M,n)$. ■

Recall that $\mathfrak{q}^r \subseteq \mathfrak{q}'$ for some $r > 0$ and $\mathfrak{q}'^{r'} \subseteq \mathfrak{q}$ for some $r' > 0$, therefore the degree of $P_{\mathfrak{q}}(M,n)$ is at least the degree of $P_{\mathfrak{q}'}(M,n)$, and the degree of $P_{\mathfrak{q}'}(M,n)$ is at least the degree of $P_{\mathfrak{q}}(M,n)$, therefore the degree of $P_{\mathfrak{q}}(M,n)$ is the degree of $P_{\mathfrak{q}'}(M,n)$. □

Remark 3.26. If $\ell(M/\mathfrak{q}M) < \infty$, then we can assume that $\operatorname{Ann}_A(M) = \mathfrak{q}$. Therefore, $\operatorname{supp}(M/\mathfrak{q}M) = \operatorname{supp}(A/\mathfrak{q})$, consists of maximal ideals only.

If we write $\mathfrak{q} = I_1 \cap I_2 \cap \cdots \cap I_r$ where each I_i is $\mathfrak{m}_i$-primary for maximal ideal $\mathfrak{m}_i$. By the Chinese Remainder Theorem, we have $\mathfrak{q} = I_1 I_2 \cdots I_r$. Thus, $\mathfrak{q}6n = I_1^n I_2^n \cdots I_r^n$, and $A/\mathfrak{q} \cong A/I_1 \oplus \cdots \cdots \oplus A/I_r$, and so $A/\mathfrak{q}^n = A/I_1^n \oplus \cdots \oplus A/I_r^n$. Therefore, $I_i = \mathfrak{q}A_{\mathfrak{m}_i}$, and $M/\mathfrak{q}^n M \cong \bigoplus_i M/I_i^n M$ by tensoring M. Therefore, $P_{\mathfrak{q}}(M,n) = \sum_i P_{\mathfrak{q}A_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i}, n)$. Therefore, it suffices to understand the Hilbert-Samuel polynomials in the local case (assuming $M/\mathfrak{q}M$ has finite length).

Proposition 3.27. Let A be Noetherian, $\mathfrak{q}$ be an ideal. Consider the short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow T \longrightarrow 0$$

of finitely-generated A-modules. Suppose $\ell(M/\mathfrak{q}M) < \infty$, (so $\ell(T/\mathfrak{q}T)$ and $\ell(N/\mathfrak{q}N)$ are also finite,) then $P_{\mathfrak{q}}(M,n) = P_{\mathfrak{q}}(T,n) + P_{\mathfrak{q}}(N,n) - R(n)$, where $R(n)$ is an essentially polynomial of degree less than degree of $P_{\mathfrak{q}}(N,n)$, and the leading term of $R(n)$ has non-negative coefficient.

Proof. Consider

$$0 \longrightarrow N/(N \cap \mathfrak{q}^n M) \longrightarrow M/\mathfrak{q}^n M \longrightarrow T/\mathfrak{q}^n T \longrightarrow 0$$

The corresponding filtrations $\{N_n = N \cap \mathfrak{q}^n M\}$ and $\{\mathfrak{q}^n N\}$ are $\mathfrak{q}$-good. By Corollary 3.22, $P_{\mathfrak{q}}(N,n) = P_{\mathfrak{q}}(N_n,n) + R(n)$. From the short exact sequence above, $P_{\mathfrak{q}}(M,n) = P_{\mathfrak{q}}(T,n) + P_{\mathfrak{q}}(N_n,n)$, thus $\ell(M/\mathfrak{q}^n M) = \ell(T/\mathfrak{q}^n T) + \ell(N/N_n)$, so one can write $P_{\mathfrak{q}}(M,n) = P_{\mathfrak{q}}(T,n) + P_{\mathfrak{q}}(N,n) - R(n)$ with $R(n)$ as specified above. □

3.2 Dimension over Zariski Topology

Definition 3.28 (Zariski Topology)**.** Let A be a commutative ring, then the Zariski spectrum is the set $\operatorname{Spec}(A) = \{P \mid P \text{ is a prime ideal in } A\}$. This becomes a topological space $X = \operatorname{Spec}(A)$ with the following (Zariski) topology: we declare the closed sets of X to be $V(I) = \{P \in \operatorname{Spec}(A) \mid P \supseteq I\}$, i.e., the vanishing set of an ideal I.

Exercise 3.29.
- $\bigcap_{i \in I} V(I_i) = V(\sum_{i \in I} I_i)$,
- $V(I) \cup V(J) = V(I \cap J) = V(IJ)$.

If $I = (f_i) i \in I$, then $V(I) = V(\sum_{i \in I} Af_i) = \bigcap_{i \in I} V(f_i)$, so $X \backslash V(I) = X \backslash \bigcap_{i \in I} V(f_i) = \bigcup_{i \in I} (X \backslash V(f_i)) = \bigcup_{i \in I} D(f_i)$, where we define $D(f_i) = X \backslash V(f_i) = \{p \in \operatorname{Spec}(A) \mid f_i \notin p\}$. Therefore, $\{D(f_i)\}$ forms a family of basic open subsets of X. Therefore, $D(f_i)$ corresponds to $\operatorname{Spec}(A_{f_i})$.

Exercise 3.30. Let $Y \subseteq X$ be a subset, then $\bar{Y} = V(I)$ where $I = \bigcap_{p \in Y} p$. Therefore, $V(I) = V(\sqrt{I})$. In particular, $V(I) \subsetneq V(J)$ if and only if $\sqrt{J} \subsetneq \sqrt{I}$. One can check that there exists a one-to-one inclusion-reversing correspondence between closed subsets of X and radical ideals of A.

Exercise 3.31. $[p] \in X$ is a closed point if and only if p is a maximal ideal of A. In particular, the spectrum as a topological space is non-Hausdorff.

Definition 3.32 (Irreducible Subset)**.** Let X be a topological space and $Y \subseteq X$ be a subset. Then Y is called irreducible if Y cannot be expressed as a union of two proper closed subsets of Y.

Exercise 3.33.
- Y is irreducible if and only if any two non-empty open subsets of Y has a non-empty intersection.
- Y being irreducible implies $\bar{Y}$ irreducible.

Example 3.34. Let $X = \operatorname{Spec}(A)$ be a topological space and Y be a closed subset of X, with $Y = V(I)$. Then Y is irreducible if and only if $\sqrt{I}$ is a prime ideal of A.

Therefore, we have an increasing sequence of closed subsets $Y_0 \subsetneq Y_1 \subsetneq Y_2 \subsetneq \cdots \subseteq Y_r$ in $X = \operatorname{Spec}(A)$ if and only if $P_r \subsetneq P_{r-1} \subsetneq \cdots \subsetneq P_0$ for $V(P_i) = Y_i$ for all $0 \leqslant i \leqslant r$.

Remark 3.35.

- Let X be a topological space and let $\mathcal{F}$ be the family of irreducible closed subsets Y of X, then $\mathcal{F}$ has a maximal element. Let $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \cdots$ be an increasing chain of irreducible closed subsets, then one can check that $Y = \bigcup_{i \geqslant 0} Y_i$ is irreducible and closed. By Zorn's lemma, there exists a maximal element of $\mathcal{F}$.
- For any $x \in X$, $\{x\}$ irreducible does not imply $\overline{\{x\}}$ irreducible. (In contrast, in Hausdorff spaces, every singleton set is closed.)

Definition 3.36 (Component)**.** A maximal irreducible closed subset of a space X is called a component of X. Therefore, a space X is the union of its components.

Definition 3.37 (Noetherian)**.** Let X be a topological space, then X is Noetherian if

(i) every non-empty of open subsets of X has a maximal element, or equivalently,

(ii) every non-empty of closed subsets of X has a minimal element.

Remark 3.38.

(i) If X is Noetherian, then any subset Y of X is Noetherian as well.

(ii) Conversely, if $X = \bigcup_{i=1}^{n} X_i$ where each X_i is Noetherian, then X is Noetherian.

(iii) If X is Noetherian, then every subset of X is quasi-compact.

Example 3.39. If A be a Noetherian ring, then $\operatorname{Spec}(A)$ is Noetherian. The converse is not necessarily true.

Remark 3.40. Suppose A is Noetherian, then $(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ where $\mathfrak{q}_i$ is P_i-primary. Let $\{P_1, \ldots, P_t\} = \min\{P_1, \ldots, P_r\}$ be the minimal primes, then $\operatorname{Spec}(A) = V(0) = V(\mathfrak{q}_1) \cup \cdots \cup V(\mathfrak{q}_r)$, but since $\mathfrak{q}_i$ is P_i-primary for all i, then $V(\mathfrak{q}_i) = V(P_i)$, so $P_i = \operatorname{Ass}(A/\mathfrak{q}_i) = V(P_1) \cup \cdots V(P_r)$. But if $P_i \subsetneq P_j$, then $V(P_j) \subsetneq V(P_i)$, so the union is just $V(P_1) \cup \cdots V(P_t)$, where each $V(P_i)$ is a component of $\operatorname{Spec}(A)$ for $1 \leqslant i \leqslant t$.

Proposition 3.41. A Noetherian space X has finite components, i.e., $X = X_1 \cup \cdots \cup X_n$ is a finite union.

Proof. Let $\mathcal{F}$ be the collection of closed subsets Z of X for which the proposition is not true, that is, each Z is a finite union of its components. Suppose, towards contradiction, that $\mathcal{F}$ is non-empty. Since X is Noetherian, then there exists a minimal element Z_0 of $\mathcal{F}$, therefore Z_0 is not irreducible, otherwise $Z_0 \notin \mathcal{F}$, so $Z_0 = W_0 \cup V_0$ is the union of two proper closed subsets. By minimality $W_0, V_0 \notin \mathcal{F}$, therefore W_0 and V_0 should be the finite union of their (finitely many) irreducible components, but that means $\mathcal{F}$ is also a finite union of irreducible components, contradiction. ☐

Definition 3.42 (Dimension)**.** Let X be a topological space, then the dimension of X, denoted $\dim(X)$, is defined as

$$\dim(X) = \sup\{r \mid \text{there exists a decreasing chain of irreducible closed subsets } X_r \supsetneq X_{r-1} \supsetneq \cdots \supsetneq X_1 \supsetneq X_0\}.$$

Exercise 3.43. Let A be a commutative ring, $X = \operatorname{Spec}(A)$. Show that X is quasi-compact, i.e., every open cover has a finite subcover.

Definition 3.44 (Dimension)**.** Let A be a commutative ring and $X = \operatorname{Spec}(A)$, then

$$\dim(X) = \sup\{r \mid \text{there exists an increasing chain of prime ideals } P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_r\}.$$

This follows from the definition above.

Definition 3.45 (Krull Dimension)**.** The Krull dimension of a commutative ring A, denoted $\dim(A)$, is $\dim(\operatorname{Spec}(A))$.

Remark 3.46. For any space X, $\dim(X) = \sup_i(\dim(X_i))$ where each X_i is a component of X.

Remark 3.47. Let A be a commutative ring, $X = \operatorname{Spec}(A)$, then

$$\dim(X) = \sup\{\dim(A/P_i) \mid P_1, \ldots, P_t \text{ are minimal prime ideals of } A\}.$$

Remark 3.48 (Nagata)**.** There exists Noetherian rings A such that $\dim(A) = \infty$.

Definition 3.49 (Krull Dimension)**.** Let A be a Noetherian ring (this would probably be the implicit assumption from now on) and let M be an A-module, then the Krull dimension of M is $\dim(M) = \dim(A/I)$ where $I = \operatorname{Ann}_A(M)$.

Exercise 3.50. $\dim(M) = \sup_{\mathfrak{m}}(\dim(M_\mathfrak{m}))$ where $\mathfrak{m}$ is a maximal ideal. Note that now the dimension of M can be studied locally. This is similar to the case of studying the degree of $P_\mathfrak{q}(M, n)$, where $\operatorname{supp}(\mathfrak{q} + I) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ we just need to study $P_{\mathfrak{q}A_\mathfrak{m}}(M_\mathfrak{m}, n)$ for maximal ideals $\mathfrak{m}$ in the support.

Definition 3.51 (Length)**.** Let $(A, \mathfrak{m})$ be a local ring, i.e., A is Noetherian with a unique maximal ideal $\mathfrak{m}$, and let M be a finitely-generated A-module. We denote the length $s(M) = \inf\{n \mid \exists x_1, \ldots, x_n \in \mathfrak{m} \text{ such that } \ell(M/(x_1, \ldots, x_n)M) < \infty\}$. Note that since M is finitely-generated, then $\dim_{A/\mathfrak{m}}(M/\mathfrak{m}M) < \infty$, so $s(M)$ is always finite.

Definition 3.52 (System of Parameters)**.** We say $x_1, \ldots, x_r \in \mathfrak{m}$ is a system of parameters of M if $r = s(M)$ and $\ell(M/(x_1, \ldots, x_r)M) < \infty$.

Let $(A, \mathfrak{m})$ be a local ring, M be a finitely-generated A-module, then we denote $d(M) = \deg(P_\mathfrak{m}(M, n))$

Remark 3.53. For Noetherian ring A (but not necessarily quasi-local), we have $\dim(A) = \sup(\dim(A_\mathfrak{m}))$ and $d(M) = \sup(d(M_\mathfrak{m}))$.

Theorem 3.54 (Dimension Theorem)**.** Let $(A, \mathfrak{m})$ be a local ring, M be a finitely-generated A-module, then $\dim(M) = d(M) = s(M)$.

Proof. We will show that $\dim(M) \leqslant d(M) \leqslant s(M) \leqslant \dim(M)$.

- To show $\dim(M) \leqslant d(M)$, we will induct on $d(M)$. If $d(M) = 0$, then $P_\mathfrak{m}(M, n) = \ell(M/\mathfrak{m}^n M)$, and since $d(M) = 0$ is the degree of $P_\mathfrak{m}(M, n)$, then $\ell(M/\mathfrak{m}^n M) = \ell(M/\mathfrak{m}^{n+1}M) = \cdots$, therefore $\ell(\mathfrak{m}^n M/\mathfrak{m}^{n+1}M) = 0$, hence we have a short exact sequence

$$0 \longrightarrow \mathfrak{m}^n M/\mathfrak{m}^{n+1}M \longrightarrow M/\mathfrak{m}^{n+1}M \longrightarrow M/\mathfrak{m}^n M \longrightarrow 0$$

  therefore $\mathfrak{m}^n M/\mathfrak{m}^{n+1}M = 0$, so $\mathfrak{m}^n M = \mathfrak{m}^{n+1}M = \mathfrak{m}(\mathfrak{m}^n M)$, then by Nakayama Lemma (Corollary 2.55), we have $\mathfrak{m}^n M = 0$, so $\operatorname{supp}(M) = \{\mathfrak{m}\}$. Therefore, $\dim(M) = 0$.

  Now suppose $d(M) > 0$, and we have shown the case for dimension $0, \ldots, d(M) - 1$. Since $(A, \mathfrak{m})$ is local, then it has finitely many components. Let $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ be a chain of prime ideals in $\operatorname{supp}(M)$ such that P_0 is a minimal prime ideal in $\operatorname{supp}(M)$. We need to show that $n \leqslant d(M)$. Denote $N = A/P_0$ and take $x \in P_1 \backslash P_0$, then x is a non-zero-divisor of N, therefore

$$0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0$$

  is a short exact sequence. By Proposition 3.27, $d(N/xN) \leqslant d(N) - 1$. By the inductive hypothesis, $\dim(N/xN) \leqslant d(N/xN) \leqslant d(N-1)$, then note that $N/xN = A/(P_0 + x_1 A)$, so $P_0 + x_1 A \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq P_n$, therefore $n - 1 \leqslant \dim(N/xN) \leqslant d(N/xN) \leqslant d(N) - 1$, therefore $n \leqslant d(N) \leqslant d(M)$.

- To show $d(M) \leqslant s(M)$, let $x_1, \ldots, x_n$ be a system of parameters of M, i.e., $n = s(M)$ and $\ell(M/(x_1, \ldots, x_n)M) < \infty$. This implies $\deg(P_{(x_1,\ldots,x_n)}(M, n)) \leqslant n$, but $V(M/(x_1, \ldots, x_n)M) = V(M/\mathfrak{m}M)$, therefore we have $\operatorname{supp}(M/(x_1, \ldots, x_n)M) = \{\mathfrak{m}\} = \operatorname{supp}(M/\mathfrak{m}M)$, thus by Proposition 3.24 we conclude $\deg(P_\mathfrak{m}(M, n)) = \deg(P_{(x_1,\ldots,x_n)}(M, n))$, so $d(M) \leqslant s(M) = n$.

- To show $s(M) \leqslant \dim(M)$, we proceed by induction on $\dim(M)$. If $\dim(M) = 0$, then $\operatorname{supp}(M) = \{\mathfrak{m}\}$, so $\ell_A(M) < \infty$, therefore $s(M) = 0$. Let $\{P_1, \ldots, P_r\}$ be the minimal primes of $\operatorname{supp}(M)$. Take $x \in \mathfrak{m} \backslash \bigcup_{i=1}^{r} P_i$, then $s(M) - 1 \leqslant s(M/xM) \leqslant \dim(M/xM) \leqslant \dim(M - 1)$[8], hence $s(M) \leqslant \dim(M)$.

[8] The first inequality follows from definition, and the second inclusion follows from the inductive hypothesis.

□

Remark 3.55. If A is a PID, then every prime has height 1, therefore $\dim(A) = 1$. For instance, $\dim(\mathbb{Z}) = \dim(k[x]) = 1$. For $A = k[x_1, \ldots, x_n]$, we have $(x_1, \ldots, x_n) \supseteq (x_1, \ldots, x_{n-1}) \supseteq \cdots \supseteq (x_1) \supseteq (0)$, so $\dim(A) \geqslant n$.

Corollary 3.56. Let $(A, \mathfrak{m})$ be a local ring with M a finitely-generated A-module, then $\dim_A(M) = \dim_{\hat{A}}(\hat{M})$.

Proof. Note $\dim_A(M) = d(M) = \deg(P_{\mathfrak{m}}(M, n))$, $P_{\mathfrak{m}}(M, n) = \ell(M/\mathfrak{m}^n M)$; similarly $\dim_{\hat{A}}(\hat{M}) = d(\hat{M}) = \deg(P_{\mathfrak{m}}(\hat{M}, n)) = \ell(\hat{M}/\hat{\mathfrak{m}}^n \hat{M})$, therefore $M/\hat{\mathfrak{m}}^n M \cong \hat{M}/\hat{\mathfrak{m}}^n M$. □

Corollary 3.57. Let $(A, \mathfrak{m})$ be a local ring, then $\dim(A)$ is the minimal number of elements required to generate an $\mathfrak{m}$-primary ideal.

Proof. Note $\dim(A) = s(A)$ is the minimal number n such that $x_1, \ldots, x_n \in \mathfrak{m}$ gives $\ell(A/(x_1, \ldots, x_n)) < \infty$. Since $s(A) = d$, then there exists $x_1, \ldots, x_d$ such that $\ell(A/(x_1, \ldots, x_d)) < \infty$, so $\{\mathfrak{m}\} = \mathrm{Ass}_A(A/(x_1, \ldots, x_d))$, i.e., $(x_1, \ldots, x_d)$ is $\mathfrak{m}$-primary. □

Corollary 3.58. Let A be Noetherian, any descending chain of prime ideals must stop after a finite number of steps.

Proof. Take a descending chain $P = P_0 \supseteq P_1 \supseteq P_2 \supseteq \cdots$, then taking the localization at P, we have $PA_P \supseteq P_1 A_P \supseteq P_2 A_P \supseteq \cdots$ in A_P. But A_P is a local ring with maximal ideal PA_P, therefore $\dim(A_P) < \infty$, so there exists some $r > 0$ such that $P_r A_P = P_{r+1} A_P = \cdots$. This implies $P_r = P_{r+1} = \cdots$, by pulling back via $i_P : A \to A_P$. (One needs to check that $i_P^{-1}(P_r A_P) = P_r$.) □

Definition 3.59 (Height)**.** Let A be Noetherian, $P \subseteq A$ be a prime ideal. The height of P, denoted $\mathrm{ht}(P)$, is $\dim(A_P)$. Alternatively, it is $\sup\{r \mid \exists \text{ a chain of prime ideals } P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_r \subsetneq P_r = P\}$.

Let I be an ideal of A, then $\mathrm{ht}(I) = \inf\limits_{P \supseteq I} \mathrm{ht}(P) = \inf\limits_{\text{minimal } P \supseteq I} \mathrm{ht}(P)$. By the primary decomposition, if we write down $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ with minimal primes $P_1, \ldots, P_r$, then this is just $\inf\limits_{\text{minimal primes} P_i} \mathrm{ht}(P_i)$ in a primary decomposition of I.

Corollary 3.60 (Generalized Krull's Principal Ideal Theorem)**.** Let A be a Noetherian ring and P be a prime ideal, then $\mathrm{ht}(P) \leqslant n$ if and only if there exists $a_1, \ldots, a_n \in P$ such that P contains $(a_1, \ldots, a_n)$ minimally.

Proof. ($\Rightarrow$): note that $\mathrm{ht}(P) \leqslant n$ if and only if $\dim(A_P) \leqslant n$, which implies $s(A_P) \leqslant n$. Let $\frac{a_1}{1}, \ldots, \frac{a_d}{1}$ be a system of parameters for A_P where $d \leqslant n$. Therefore, $\mathrm{Ass}_{A_P}(A_P/(a_1, \ldots, a_d)A_P) = PA_P$, that is, PA_P contains $(a_1, \ldots, a_d)_{A_P}$ minimally. This implies $P \supseteq (a_1, \ldots, a_d)$ minimally.

($\Leftarrow$): suppose $P \supseteq (a_1, \ldots, a_n)$ minimally, then $PA_P \supseteq (a_1, \ldots, a_n)A_P$ minimally, therefore we have $PA_P = \mathrm{Ass}_{A_P}(A_P/(a_1, \ldots, a_n)A_P)$, therefore $\ell(A_P/(a_1, \ldots, a_n)A_P) < \infty$, thus $\dim(A_P) \leqslant n$. □

Exercise 3.61. Let $(A, \mathfrak{m})$ be a local ring. Suppose there exists a principal prime ideal P, then A is a domain.

Exercise 3.62. Let A be a Noetherian ring with $\dim(A) \geqslant 2$. Show that A has infinitely many prime ideals of height 1.

Exercise 3.63. Let $(A, \mathfrak{m})$ be a local ring and M be a finitely-generated A-module. Let $x_1, \ldots, x_i \in \mathfrak{m}$, then show that $\dim(M/(x_1, \ldots, x_i)) \geqslant \dim(M) - i$. The equality holds if and only if $x_1, \ldots, x_i$ form a part of a system of parameters of M.

Theorem 3.64. Let A be a Noetherian ring, then $\dim(A[x]) = \dim(A) + 1$.

Proof. First, we need two lemmas.

Lemma 3.65. Let $\mathfrak{p} \supsetneq \mathfrak{q}$ be two prime ideals in $A[x]$ such that $\mathfrak{q}_0 = \mathfrak{q} \cap A = P \cap A$, then $\mathfrak{q} = \mathfrak{q}_0[x]$.

Remark 3.66. In particular, this implies there is no prime ideal between $\mathfrak{p}$ and $\mathfrak{q}$. Otherwise, say $\mathfrak{p} \supseteq \mathfrak{q}' \supseteq \mathfrak{q}$, then $\mathfrak{q}' = \mathfrak{q}_0[x]$, so $\mathfrak{q} = \mathfrak{q}'$.

Subproof. Suppose, towards contradiction, that $\mathfrak{q}_0[x] \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}$, then $\bar{A} := A/\mathfrak{q}_0 \to A/\mathfrak{q}_0[x] = A[x]/\mathfrak{q}_0[x] = \bar{A}[x]$. Now $\bar{A}[x]$ has a strict chain:

$$\bar{0} \subseteq \bar{\mathfrak{q}} \subseteq \bar{\mathfrak{q}}$$

where $\bar{\mathfrak{q}}$ is the image of $\mathfrak{q}$ in $\bar{A}[x]$ and $\bar{\mathfrak{p}}$ is the image of $\mathfrak{p}$ in $\bar{A}[x]$. Also note that $(\bar{0}) = (\bar{0}) \cap \bar{A} = \bar{\mathfrak{q}} \cap \bar{A} = \bar{\mathfrak{p}} \cap \bar{A}$. Let $k = S^{-1}\bar{A}$ for $S = \bar{A}\backslash\{0\}$, then by tensoring with $\bar{A}$ on $k \to k[x]$ (as $\bar{A} \hookrightarrow \bar{A}[x]$ where $S^{-1}\bar{A}$ is $\bar{A}$-flat), we have a strict chain

$$\bar{0} \subsetneq S^{-1}\bar{\mathfrak{q}} \subsetneq S^{-1}\bar{\mathfrak{p}}$$

of length 2. Therefore $\dim(k[x]) \geqslant 2$, but $\dim(k[x]) = 1$, contradiction. Therefore $\mathfrak{q} = \mathfrak{q}_0[x]$. ■

Lemma 3.67. Let A be a Noetherian ring and I be an ideal, then $\mathrm{ht}(I) = \mathrm{ht}(I[x])$.

Subproof. We have $\mathrm{I} = \inf_{P \supseteq I} \mathrm{ht}(P) = \inf_{\text{minimal } P \supseteq I} \mathrm{ht}(P)$ and $\mathrm{I[x]} = \inf_{A[x] \supseteq \mathfrak{q} \supseteq I[x]} \mathrm{ht}(\mathfrak{q}) = \inf_{\text{minimal } P[x] \supseteq I[x]} \mathrm{ht}(P)$, therefore it is enough to show that $\mathrm{ht}(P) = \mathrm{ht}(P[x])$.

Given any chain $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_r = P$, then $P_0[x] \subsetneq P_1[x] \subsetneq \cdots \subsetneq P_r[x] = P[x]$. Thsi says $\mathrm{ht}(P[x]) \geqslant \mathrm{ht}(P)$. Also, suppose $\mathrm{ht}(P) = t$, then there exists $a_1, \ldots, a_t \in P$ such that $P \supseteq (a_1, \ldots, a_t)$ minimally. By the primary decomposition, we know $P[x] \supseteq (a_1, \ldots, a_t)[x]$ minimally, then $\mathrm{ht}(P[x]) \leqslant t = \mathrm{ht}(P)$, thus $\mathrm{ht}(P) = \mathrm{ht}(P[x])$. ■

Suppose $\dim(A) = \infty$, then take a strict chain of prime ideals in A, i.e., $P_0 \subsetneq \cdots \subsetneq P_r$, so $P_0[x] \subsetneq \cdots \subsetneq P_r[x]$ is also a strict chain in $A[x]$, so $\dim(A[x]) = \infty$.

Now suppose $\dim(A) < \infty$. Take any chain $P_0 \subsetneq \cdots \subsetneq P_r$, then we have another chain $P_0[x] \subsetneq P_1[x] \subsetneq \cdots \subsetneq P_r[x] \subsetneq (P_r[x], x)$, so $\dim(A[x]) \geqslant \dim(A) + 1$. We now proceed by induction on $\dim(A)$. Suppose $\dim(A) = 0$, then it is equivalent to $\ell_A(A) < \infty$, i.e., all the associated primes of A are maximal. By Lemma 3.65, $\dim(A) = 1$.[9]

We now want to show that $\dim(A[x]) \leqslant \dim(A) + 1$. Take a strict chain of ideals in $A[x]$ of any length (say r), that is $P_r \supsetneq \cdots \supsetneq P_1 \supsetneq P_0$, then by intersecting with A we have another chain $\mathfrak{p}_r \supseteq \cdots \supseteq \mathfrak{p}_1 \supseteq \mathfrak{p}_0$, where $\mathfrak{p}_i = P_i \cap A$. We now want to show that $r \leqslant \dim(A) + 1$. We have two cases:

- suppose $\mathfrak{p}_r \neq \mathfrak{p}_{r-1}$, so $\mathrm{ht}(P_{r-1}) < \dim(A)$. By induction, $\dim(A_{\mathfrak{p}_{r-1}}[x]) = \dim(A_{\mathfrak{p}_{r-1}})+1$, so $\dim(A_{\mathfrak{p}_{r-1}}[x]) \leqslant \dim(A)$, and by localization we have a chain $A_{\mathfrak{p}_{r-1}}[x] \supseteq P_{r-1}A_{\mathfrak{p}_{r-1}}[x] \supsetneq \cdots \supsetneq P_0 A_{\mathfrak{p}_{r-1}}[x]$, therefore $r - 1 \leqslant \dim(A_{\mathfrak{p}_{r-1}}[x]) \leqslant \dim(A)$, so $r \leqslant \dim(A) + 1$.
- suppose $\mathfrak{p}_r = \mathfrak{p}_{r-1}$, so $P_{r-1} = \mathfrak{p}_{r-1}[x]$ by Lemma 3.65, with $\mathrm{ht}(P_{r-1}) = \mathrm{ht}(\mathfrak{p}_{r-1})$. Therefore, $r - 1 \leqslant \mathrm{ht}(P_{r-1}) = \mathrm{ht}(P_{r-1}) \leqslant \dim(A)$, so $r \leqslant \dim(A) + 1$.

□

Corollary 3.68.

- Let A be a Noetherian ring, then $\dim(A[x_1, \ldots, x_n]) = \dim(A) + n$.
- Let k be a field, then $\dim(k[x_1, \ldots, x_n]) = n$.
- $\dim(\mathbb{Z}[x_1, \ldots, x_n]) = n + 1$.

Exercise 3.69. Let A be a Noetherian ring, then $\dim(A\,[[x]]) = \dim(A) + 1$.
Hint: is X contained in the Jacobson radical of $A\,[[x]]$?

Corollary 3.70.

- For a Noetherian ring A, $\dim(A\,[[x]]) = \dim(A) + n$.
- For a field k, $\dim(k\,[[x]]) = n$.
- $\dim(\mathbb{Z}\,[[x_1, \ldots, x_n]]) = n + 1$.

Remark 3.71. For rings like $k[x_1, \ldots, x_n]$, the dimension and the transcendental degree are both n. For rings like $k\,[[x]]$, the degree is still n, but the transcendental degree is ∞.

[9] Indeed, take the primary decomposition $0 = I_1 \cap \cdots \cap I_r$ where I_i is $\mathfrak{m}_i$-primary, then pushing it out to the polynomial ring, we have $0 = I_1[x] \cap \cdots I_r[x]$, where $I_r[x]$ is $\mathfrak{m}_i[x]$-primary. Take the chain given by $P = (\mathfrak{m}_1[x], x) \supsetneq \mathfrak{m}_1[x]$, but they both collapse onto $\mathfrak{m}_1$, so by Lemma 3.65 this is the maximal chain, thus has length 1.

4 Integral Extensions

5 Noether's Normalization Lemma

6 Homological Algebra

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