MATH 212B Notes

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1 Examples of Tensor-triangulated Categories

[1]We aim to discuss examples of tensor-triangulated categories as an entryway into the theory of tensor-triangulated geometry. These examples often involve two categories, a small (compact) category $\mathcal{K}$ and a large (triangulated) category $\mathcal{T}$.

1.1 Examples in Commutative Algebra and Algebraic Geometry

Definition 1.1. An element $x \in \mathcal{T}$ is *compact* if $\mathbf{Hom}_{\mathcal{T}}(x, -)$ commutes with coproducts.[2]

Example 1.2. Let A be a commutative ring. The large category is $\mathcal{T} = D(A\textbf{-Mod})$, the derived category of A-modules. Note that this is the derived category of an Abelian (and Grothendieck) category, made up of complexes of A-modules, with quasi-isomorphisms inverted. $\mathcal{K}$ is the subcategory consisting the compact elements of $\mathcal{T}$[3], i.e., $\mathcal{T}^c$, which happens to be $D_{perf}(A)$, the derived category of perfect complexes of A, which is just $K_b(A\textbf{-proj})$, the bounded complexes of finitely-generated projective A-modules. Therefore, on each degree of the complex we have finitely generated projective modules, and far enough on the left (and the right) there are zero terms. The maps in this complex are up to homotopy simply because quasi-isomorphisms between such complexes have to be homotopy-equivalent. $\mathcal{K}$ is now a triangulated category.

[1]This lecture coincides to Professor Paul Balmer's Talk. It is also based on his notes. A shortened version of the notes can be found here.

[2]We usually assume that $\mathcal{T}$ contains all coproducts.

[3]A theorem due to Amnon Neeman shows that this construction coincides with the collection of compact elements.

Remark 1.3. Note that the construction above does not require commutativity. What requires this property is the construction of the symmetric monoidal tensor product.

The category has a tensor product $\otimes$ induced from the tensor product of A, i.e., $- \otimes_A^L -$, given by the left derived functor of the derived category.

We can now generalize this example in algebraic geometry.

Example 1.4. Let X be a quasi-compact and quasi-separated scheme, i.e., the underlying space $|X|$ has a quasi-compact open basis. For example, let $X = \mathbf{Spec}(A)$ be the spectrum of a commutative ring. Denote $\mathcal{T} = D(X)$, (actually) the derived category of complexes of $\mathcal{O}_X$-modules with quasi-coherent homology. $\mathcal{K}$ is still the compact subcategory of $\mathcal{T}$, equivalent to $D_{perf}(X)$, those that are in $D_{perf}(A)$ for every affine $\mathbf{Spec}(A)$.

The triangular structures on $\mathcal{T}$ are really the traces that survived from the exact sequences of modules, and the tensor product is exact in each variable, therefore tensoring a fixed object preserves exact triangles.

These considerations of larger categories go hand-in-hand with the modern development of algebraic geometry like K-theory or homological algebra. One of the early motivations (other than the ones in homological algebra) was the pushforward. When we look at a vector bundle, we have things working nicely on the closed subschemes or on given schemes. We try pushing it to another scheme, like in the following example:

Example 1.5. Consider $i : \mathbf{Spec}(\mathbb{Z}/p\mathbb{Z}) \hookrightarrow \mathbf{Spec}(\mathbb{Z})$. Let $A = \mathbb{Z}$ and $k = \mathbb{Z}/p\mathbb{Z}$, then this is associated with the quotient $A \twoheadrightarrow k$. If V is a finite-dimensional k-vector space, we can view it as an A-module i_*V. A acts on V by projecting onto k and acts correspondingly. The k-dual of V has k-dimension $\dim_k(V^*) = \dim_k(V)$. But if we look the homomorphisms over A instead, we have $\mathbf{Hom}_A(i_*V, A) = 0$ because the module i_*V is killed by p since it is torsion, so every element lands in elements killed by p in A, but there is no such element. Therefore, the information about the dual gets lost.

We can look at an even easier example.

Example 1.6. If we take $V = \mathbb{Z}/p\mathbb{Z}$ itself, then i_*V is $\mathbb{Z}/p\mathbb{Z}$ as an A-module, but in the derived category of A, this is equivalent (quasi-isomorphic) to the complex

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to 0$$

This is a perfect complex, i.e., contained in $D_{perf}(A)$. If we try to dualize this perfect complex, we have $(i_*V)^*$ to dualize on every degree, but because it is contravariant, we have the complex

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to 0$$

Note that the two complexes has different degree 0's: if the first complex has degree 0 to be $\mathbb{Z}$ on the right, then the second complex has degree 0 to be $\mathbb{Z}$ on the left. In other words, it shifted by one. We can denote the dual to be $i_*V[-1]$.

Remark 1.7. Example 1.6 works for all finitely generated $V \in D_{perf}(k)$.

Example 1.8. Take $A = k[X_1, \dots, X_n]$ and $k = A/\langle X_1, \dots, X_n\rangle$. We then get $(i_*V)^* = (i_*V)[-n]$, i.e., shifted by $-n$.

Remark 1.9. In fact, a more precise way of writing the isomorphisms in the examples above is $(i_*V)^* \cong i_*(V^*)[-1]$ and $(i_*V)^* \cong (i_*(V^*))[-n]$, because the functor is contravariant.

This is interesting because the value n is the difference between dimensions of the two schemes we are looking at. For example, the first one is the difference between the codimensions of $\mathbf{Spec}(\mathbb{Z}/p\mathbb{Z})$ (which is 0) and $\mathbf{Spec}(\mathbb{Z})$ (which is 1).

We can make the following observations.

Remark 1.10.
- There are phenomena that make sense on derived (triangulated) categories but not on the level of modules (c.f. Example 1.5, where we lost information about the dual as a module).
- Some geometric information appears in the derived category $D(X)$, e.g., the relative dimension as seen in Remark 1.9.

Another classical example comes from K-theory. K-theory was born from Grothendieck's theory on Grothendieck–Riemann–Roch theorem (formalized by Borel–Serre in 1958), where he also looked at f_* for vector bundles.

Example 1.11.
- For example, let us look at a vector bundle over X and a (smooth enough) map $f : X \to Y$. We push the vector bundle down and get a perfect complex (which may not be a vector bundle anymore) over Y, then we look at the alternate sum of elements of this complex (resolution).
- Another example comes from the Thomason-Trobaugh paper in 1990s, where they developed the higher algebraic K-theory of schemes in algebraic geometry. This goes hand-in-hand with the development of perfect complexes with more theoretical information, i.e., under localizations.

Neeman concluded in the early 1990s that we could not expect certain K-theories to factor via homotopy categories because there are certain functors in these categories with sections, but no sections in those K-theories.

Very recently, Muro and Raptis give a big reconciliation on the K-theory of derivators.

Following the observations above, one can now ask: how much geometry of X survives in $D(X)$ or $D_{perf}(X)$? Note that the work *duality between* $D(X)$ *and* $D(\hat{X})$ by Mukai in 1981 shows that there are non-isomorphic schemes X and X' (in particular, Abelian varieties and their duals) such that $D(X)$ and $D(X')$ are equivalent as triangulated categories. However, this construction was not $\otimes$-compatible. Thomason (1997) highlighted the importance of $\otimes$ when classifying the triangulated subcategories of the derived category of perfect complexes, as he classified the tensor ideals of $D^{perf}(X)$. This is a very important precursor of tensor triangular geometry. An important corollary is the following:

Theorem 1.12. If $D(X) \cong D(X')$ as tensor triangulated categories (i.e., preserving the tensor), then the schemes are isomorphic, i.e., $X \cong X'$. Alternatively, the same result holds if $D_{perf}(X) \cong D_{perf}(X')$.

We now go over a few non-geometric examples.

1.2 Examples in Modular Representation Theory

Let G be a finite group and k be a field of positive characteristic ($p > 0$). In particular, we look at the case where $p \mid |G|$. Recall

Theorem 1.13 (Maschke)**.** If $p = 0$ or $p \nmid |G|$, then kG is semisimple. In particular, all modules are projective and injective, and the finitely generated ones decompose uniquely as a sum of irreducible (or simple) ones (according to Krull-Schmidt).

Therefore, the theory studies the case when kG is not semisimple. That is to say, there are non-projective modules. We look at the category of kG-modules and mod out the projective ones. Therefore, objects are still kG-modules, but if a map differs from another map by factoring via a projective, then it is zero, i.e., $f \sim 0 : M \to M'$ if there exists a projective kG-module P and maps such that the diagram

$$\begin{tikzcd} M \arrow[rr] \arrow[dr, dashed] & & M' \\ & P \arrow[ur, dashed] & \end{tikzcd}$$

commutes. In particular, the identity of all projective modules will factor by itself, and therefore become zero. Hence, all projective modules disappear and give the idea of an additive quotient. That is to say, the quotient category kG-**Mod**/kG-**Proj** is an additive category, i.e., receiving kG-modules and all projective modules become zero. Amusingly, the quotient category is a triangulated category $\mathcal{T}$. Again, the compact portion $\mathcal{K}$ of this quotient category is actually the finitely-generated ones, i.e., kG-**mod**/kG-**Proj**, where kG-**mod** is the category of finitely-generated kG-modules. The tensor product $\otimes$ is given over the field, i.e., as $\otimes_k$, with diagonal G-action. This means that $g \cdot (m_1 \otimes m_2) = (gm_1) \otimes (gm_2)$ in $M_1 \otimes_k M_2$. This tensor product is nice because it allows us to pass into the quotient. Therefore, these quotients have a tensor structure and are tensor triangulated categories, with the tensor compatible with the triangulation. Denoting $\mathrm{stab}(kG) = \mathcal{K}$ and $\mathrm{Stab}(kG) = \mathcal{T}$, this stable module category is the measure of modularity, i.e, how non-semisimple kG is. Note that we can have $\mathrm{Stab}(kG) = 0$ if $p \nmid |G|$. In fact, the restriction $\mathrm{Res}^G_H : \mathrm{Stab}(kG) \to \mathrm{Stab}(kH)$ can be an equivalence if $H \cap H^g$, i.e., intersecting with the conjugate, has order relatively prime to p for all $g \in G \backslash H$. In some sense, the modular representation theory of G and H are the same. For example, this happens when $p = 2$ and $G = S_3$ with $H = C_2$.

By Krull-Schmidt, every finitely generated kG-module can be decomposed in an essentially unique way as a sum of indecomposables (even in modular case). Therefore, we can apply the same idea to $\mathrm{stab}(kG)$. In some sense, knowing the decomposition of modules in there is the same as studying non-projectives in the indecomposables. If we look at the quotient $\otimes$-functor kG-**mod** $\twoheadrightarrow \mathrm{stab}(kG)$, (even if it is from an Abelian category to a triangulated category), if M is such that $M \otimes -$ is an equivalence on $\mathrm{stab}(kG)$, then if N is indecomposable in the stable category $\mathrm{stab}(kG)$, then so are $M^{\otimes n} \otimes N$ for all $n \in \mathbb{Z}$. Therefore, the invertible (as an equivalence) elements in kG-**mod** are mapped to the invertible elements in $\mathrm{stab}(kG)$. We see that $\otimes$-invertible in kG-**mod** is exactly saying that $\dim_k(M) = 1$. But there are more invertible elements in $\mathrm{stab}(kG)$, which are called endotrivial and crucial in modular representation theory.

1.3 Stable Homotopy Theory

Consider $\mathcal{T} = SH$, the stable homotopy category, also known as the homotopy category of **Top**-spectra. We can start with topological spaces and ask whether we can study them up to homotopy. This is possible for pointed spaces, as we can just suspend them. Therefore, in general, we consider "spaces" up to homotopy with the suspension $\sum = S^1 \wedge -$ (essentially the smash product) inverted. The compact portion $\mathcal{K} = SH^{fin}$ is classified by looking at finite CW-complexes and attaching finitely many disks to finitely many points[4], then we can look at the homotopy and stabilizes.

The motivation is that studying spaces (even up to homotopy) is too hard. Working stably, we can look at the spheres and their suspensions, where the homomorphisms (of the stable homotopy group of spheres) $\pi_i^{st} = \mathbf{Hom}_{SH}(S^i, S^0)$ are hard but interesting to study.

[4]This is known as the Spanier-Whitehead stable homotopy category of finite pointed CW-complexes.

One can also look at Chromatic theory, which motivates all of this, and provides overall organization to the finite spectrum SH^{fin}. This helps us to study the homotopy groups, and even the tensor triangular categories and stable homotopy theory, and therefore we see it has the same role as $\mathbb{Z}$ in commutative algebra. To make this idea precise, we would need more structure, but we can just look at the tensor triangular categories with certain enrichment.

The category SH has its significance because of Brown Representability Theorem. This theorem had been generalized by Neeman on such categories (see theorem 4.1 from his work in 1996).

Remark 1.14. There are relations in equivariant versions of the same categories. Let $\mathcal{T} = SH(G)$ and $\mathcal{K} = SH(G)^c$. Although it should be similar to what we have seen before, there is a bit of subtlety in what we mean by stabilization: one is not stabilizing with respect to smashing with spheres, but with the ones that have a G-action in general. This construction helps us look at actions like restriction and induction.

1.4 Motivic Theory

Let S be a base scheme, e.g., the spectrum of the ground field $\mathbf{Spec}(k)$. Note that we sometimes want it to be a perfect field. We want to do similar things, but to study smooth schemes over S, and their homological properties. In particular, we want to make $\mathbb{A}^1 \times X \cong X$, so we can look at an algebraic form of homotopy, i.e., $\mathbf{Spec}(\mathbb{Z})(t)$ for some variable t, instead of the traditional $[0, 1]$. To do this, we have an algebraic theory called derived category of motives, with $\mathcal{K} = DM^{gm}(S) \subseteq DM(S) = \mathcal{T}$, and there is a topological theory where $\mathcal{K} = SH(S)^c \subseteq SH(S) = \mathcal{T}$. The first one is called the derived category of motives by Voevodsky, and the second one is the motivic stable homotopy category.

In both cases, each of those categories

- contains an object $[X]$ for every smooth scheme X over S, in a way that the motives satisfy $[\mathbb{A}^1 \times X] \cong [X]$.
- algebraic "coefficients" in complexes, and topological "coefficients" in spectra (spaces).

Remark 1.15. In some sense, our example in motivic theory has the same role as our example of stable homotopy theory in the algebraic geometry examples.

1.5 More Examples

- KK-theory of C^*-algebras.
- Homological mirror symmetry.

2 Pre-triangulated Categories

Definition 2.1 (Suspended Category)**.** A suspended (or stable) category is a pair (K, Σ) where K is an additive category and $\Sigma : K \xrightarrow{\cong} K$ is an equivalence.

Example 2.2. Let $\mathcal{A}$ be an additive category. We can consider $\mathbf{Ch}(\mathcal{A})$ whose objects are complexes:

$$\cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots$$

such that $d^2 = 0$, i.e., $d_n \circ d_{n+1} = 0$, and with morphisms from $A. \to B.$ which are collection of $f_n : A_n \to B_n$ for all $n \in \mathbb{Z}$, and such that $df = fd$, i.e., a commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A_n & \xrightarrow{d} & A_{n-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \cdots & \longrightarrow & B_n & \xrightarrow{d} & B_{n-1} & \longrightarrow & \cdots \end{array}$$

The suspension is (almost) shifting the degree (with a sign difference). Indeed, it is suspended with $(\Sigma A)_n = A_{n-1}$ and $d_n^{\Sigma A} = -d_{n-1}^A$.

Remark 2.3. $f \sim g : A. \to B.$ are homotopic if there exists $\varepsilon_n : A_n \to B_{n+1}$ for all $n \in \mathbb{Z}$, such that $f - g = d\varepsilon + \varepsilon d$.

This notion appears when we discuss the uniqueness of resolutions, maps lifted up to homotopy, which are themselves unique up to homotopy.

Alternatively, we can define $f \sim g$ to be $f - g \in \mathcal{I} = \{h \sim 0\}$ in an additive construction, to be an ideal.

We define the category $\mathbf{K}(\mathcal{A}) = \mathbf{Ch}(\mathcal{A})/\sim$ with the same objects (chain complexes), and morphisms are up to homotopy, i.e., we get $\mathbf{Hom}_{\mathbf{K}(\mathcal{A})}(A, B) = \mathbf{Hom}_{\mathbf{Ch}(\mathcal{A})}(A, B)/\mathcal{I}(A, B)$.

Definition 2.4 (Triangle)**.** A triangle Δ is a diagram in K of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

or alternatively,

$$\begin{tikzcd} & C \arrow[dl, "h"'] & \\ A \arrow[rr, "f"'] & & B \arrow[ul, "g"'] \end{tikzcd}$$

A morphism of triangles is of the form $(u, v, w) : \Delta \to \Delta'$, denoted

$$\begin{array}{lccccccc} \Delta : & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\ \downarrow{\scriptstyle (u,v,w)} & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \Sigma u} \\ \Delta' : & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C & \xrightarrow{h'} & \Sigma A \end{array}$$

An isomorphism of triangles (u, v, w) is just a morphism where u, v, w are all triangles.

Remark 2.5. A pre-triangulated category is a structure on the categories (in the given models). If you take the category of chain complexes from an Abelian category, adjoin the inversions of quasi-isomorphisms, we do not get an Abelian category anymore, but a pre-triangulated category.

Definition 2.6 (Pre-triangulated Category)**.** A pre-triangulated category is a suspended category (K, Σ) together with a chosen (distinguished) class of triangles, called exact triangles, satisfying some axioms.

1. Book-keeping Axiom:
    - Exact triangles should be replete: if $\Delta \cong \Delta'$ and Δ is exact, then Δ' is exact.
    - For every object A, the triangle

      $$A \xrightarrow{\mathbf{id}} A \longrightarrow 0 \longrightarrow \Sigma A$$

      is exact.
    - Rotation Axiom: Note that the triangle is essentially a long exact sequence

      $$\cdots \longrightarrow \Sigma^{-1}C \xrightarrow{\Sigma^{-1}h} A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A \longrightarrow \Sigma B \longrightarrow \cdots$$

      and so the base (the first morphism in the triangle) does not really matter. The axiom says that

      $$\begin{tikzcd} & C \arrow[dl, "h"'] & \\ A \arrow[rr, "f"'] & & B \arrow[ul, "g"'] \end{tikzcd}$$

      is exact if and only if

      $$\begin{tikzcd} & \Sigma A \arrow[dl, "-\Sigma f"'] & \\ B \arrow[rr, "g"'] & & C \arrow[ul, "h"'] \end{tikzcd}$$

      is exact.

      Observe that by the replete axiom, the triangle above is essentially made by changing two signs, e.g., changing f and g to $-f$ and $-g$, which is equivalent to the triangle above. The replete axiom says that we can only change an even number of signs.

2. Existence Axiom: Every morphism $f : A \to B$ can be completed in an exact triangle:

$$\begin{tikzcd} & C \arrow[dl, "h"'] & \\ A \arrow[rr, "f"'] & & B \arrow[ul, "g"'] \end{tikzcd}$$

   However, the axiom does not guarantee it to be unique.

3. Morphism Axiom: Given exact rows

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[r, "g"] \arrow[d, "v"] & C \arrow[r, "h"] & \Sigma A \arrow[d, "\Sigma u"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C \arrow[r, "h'"] & \Sigma A \end{tikzcd}$$

   and u, v such that $f'u = vf$, then there exists a morphism $w : C \to C'$ such that (u, v, w) is a morphism of triangles, i.e., w completes the commutative diagram. (Again, no uniqueness.)

Remark 2.7. For any $f : A \to B$ we have a morphism of triangles

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[r, "\mathbf{id}"] \arrow[d, "f"] & A \arrow[r] \arrow[d] & 0 \arrow[d] \\ B \arrow[r] & B \arrow[r] & 0 \arrow[r] & \Sigma B \end{tikzcd}$$

Remark 2.8. We "conjugate" by rotation, that is, if we pre-compose a morphism with a rotation, and post-compose the inverse of the rotation, then we get a conjugated version of the original morphism. For instance, if we have

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[r, "g"] & C \arrow[r, "h"] \arrow[d, "w"] & \Sigma A \arrow[d, "\Sigma u"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C \arrow[r, "h'"] & \Sigma A \end{tikzcd}$$

then we can rotate the figure one step to the right, then by the morphism axiom, we get the desired morphism v, then we can rotate it back to the original form, saying there exists such morphism $v : B \to B'$ that makes this a morphism of triangles.

Example 2.9. $\mathbf{K}(\mathcal{A})$ for $\mathcal{A}$ additive is pre-triangulated. Let Σ be the shifting, then the exact triangles are those isomorphic (i.e., hotomopy equivalent) to the following: Suppose we have a morphism of complexes $f : A. \to B.$, that is,

$$
\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r, "d"] \arrow[d, "f"] & A_{n-1} \arrow[r] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r] & B_n \arrow[r, "d"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}
$$

and we want to build the composition, i.e., with $g : B. \to C.$, $h : C. \to D.$, and so on, we can construct C as the cone of f, denoted $\mathrm{cone}(f)$, and then $D = \Sigma A$. To construct them, we obviously have D as the suspension by 1 with sign of differential changed, i.e.,

$$
\begin{tikzcd}
A. : \arrow[d, "f"] & \cdots \arrow[r] & A_n \arrow[r, "d"] \arrow[d, "f"] & A_{n-1} \arrow[r] \arrow[d, "f"] & \cdots \\
B. : \arrow[d, "g"] & \cdots \arrow[r] & B_n \arrow[r, "d"] & B_{n-1} \arrow[r] & \cdots \\
\mathrm{cone}(f) : \arrow[d, "h"] & & & & \\
\Sigma A : & \cdots \arrow[r, "-d"] & A_{n-1} \arrow[r, "-d"] & A_{n-2} \arrow[r] & \cdots
\end{tikzcd}
$$

To make it additive, we do not have much choice on the cone. The easiest way to get the composition as 0 is

$$
\begin{tikzcd}
A. : \arrow[d, "f"] & \cdots \arrow[r] & A_n \arrow[r, "d"] \arrow[d, "f"] & A_{n-1} \arrow[r] \arrow[d, "f"] & \cdots \\
B. : \arrow[d, "g"] & \cdots \arrow[r] & B_n \arrow[r, "d"] \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & B_{n-1} \arrow[r] \arrow[d, "\begin{pmatrix} 0 \\ 1 \end{pmatrix}"] & \cdots \\
\mathrm{cone}(f) : \arrow[d, "h"] & \cdots \arrow[r] & A_{n-1} \oplus B_n \arrow[r, "\begin{pmatrix} -d & 0 \\ f & d \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & A_{n-2} \oplus B_{n-1} \arrow[r] \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"] & \cdots \\
\Sigma A : & \cdots \arrow[r, "-d"] & A_{n-1} \arrow[r, "-d"] & A_{n-2} \arrow[r] & \cdots
\end{tikzcd}
$$

Considering the composite from A to $\mathrm{cone}(f)$, the morphism is given by $\begin{pmatrix} 0 \\ f \end{pmatrix}$. Then the natural homotopy from A_{n-1} to $A_{n-1} \oplus B_n$ is given by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Therefore, the composition is just homotopic to 0, thus showing the sequence is exact vertically. Horizontally, we have

$$
\begin{pmatrix} -d & 0 \\ f & d \end{pmatrix} \begin{pmatrix} -d & 0 \\ f & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
$$

so the entire diagram is a complex indeed.

Definition 2.10 (Weak Kernel/Cokernel)**.** Recall that a kernel is the universal map into the source such that the composition is zero, and a cokernel is the universal map out of the target such that the composition is zero.

A weak kernel (respectively, cokernel) is just a kernel (respectively, cokernel) without the universal property, i.e., existence of factorization without the uniqueness. For instance, for the weak kernel triangle of A, B, C, for every map $t : B \to T$, there exists $\bar{t}$ so that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "0"] \arrow[ddr, bend right, "0"'] & B \arrow[d, "g"] \arrow[ddr, bend left, "\forall t"] & \\
& C \arrow[dr, dashed, "\exists \bar{t}"] & \\
& & T
\end{tikzcd}$$

Proposition 2.11. Let K be a pre-triangulated category and consider an exact triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

(a) $g \circ f = 0$ and $h \circ g = 0$, and $(\Sigma f) \circ h = 0$, and so on (in the long exact triangle sequence).

(b) g is a weak cokernel of f, and $\Sigma^{-1} h$ is a weak kernel of f, and so on.

Proof. Left as an exercise. Hit the morphism axiom on the given triangle, and all triangles of the form

$$X \xrightarrow{\mathbf{id}} X \longrightarrow 0 \longrightarrow \Sigma X$$

for $X \in \{A, B, C\}$ (in part (a), and $X = T$ in part (b)) and rotations. For instance, we look at the two exact rows below:

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d] & C \arrow[r, "h"] \arrow[d] & \Sigma \\
0 \arrow[r] & C \arrow[r, "1"] & C \arrow[r] & 0
\end{tikzcd}$$

We can then take the identity map from C to C, and the map $g : B \to C$, and we can take rest of the morphisms to be the zero morphisms. $\square$

Corollary 2.12. For K pre-triangulated and $X \in K$, the functors

$$\mathbf{Hom}_K(X, -) : K \to \mathbf{Ab}$$

and

$$\mathbf{Hom}_K(-, X) : K^{op} \to \mathbf{Ab}$$

map triangles (written as long exact sequences) to long exact sequences. That is, if

$$\begin{tikzcd} & C \arrow[dl, "h"'] & \\ A \arrow[rr, "f"'] & & B \arrow[ul, "g"'] \end{tikzcd}$$

is exact, then

$$\cdots \xrightarrow{(\Sigma^{-1}h)_*} \mathbf{Hom}(X,A) \xrightarrow{f_*} \mathbf{Hom}(X,B) \xrightarrow{g_*} \mathbf{Hom}(X,C) \xrightarrow{h_*} \mathbf{Hom}(X,\Sigma A) \longrightarrow \cdots$$

is exact.

Remark 2.13. This is the source to almost all long exact sequences we found in homological algebra. It is powerful to see that pre-triangulated categories can give us long exact sequences, and therefore give us spectral sequences in the right situation, and so on.

Remark 2.14. Using the above, Yoneda Lemma, and the Five Lemma in **Ab**, it is easy to see that for a morphism $(u, v, w) : \Delta \to \Delta'$ of exact triangles, if u and v are isomorphisms, so is w.

Lemma 2.15. Let Δ, Δ', and Δ'' be exact triangles in a pre-triangulated category, and two morphisms $(u, v, 0) : \Delta \to \Delta'$ and $(0, v', w') : \Delta' \to \Delta''$, then $v' \circ v = 0$.

Proof. Consider

$$\begin{tikzcd} \Delta: & A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[r, "g"] \arrow[d, "v"] \arrow[dl, dashed, "\exists\tilde{v}"] & C \arrow[r, "h"] \arrow[d, "0"] & \Sigma A \arrow[d, "\Sigma u"] \\ \Delta': & A' \arrow[r, "f'"] \arrow[d, "0"] & B' \arrow[r, "g'"] \arrow[d, "v'"] & C' \arrow[r, "h'"] \arrow[d, "w'"] \arrow[dl, dashed, "\exists\bar{v}'"] & \Sigma A \arrow[d, "0"] \\ \Delta'': & A'' \arrow[r, "f''"] & B'' \arrow[r, "g''"] & C'' \arrow[r, "h''"] & \Sigma A'' \end{tikzcd}$$

Note that $g' \circ v = 0$, and therefore v lands in the kernel of g', but we only have a weak kernel, so there exists $\tilde{v}'' : B \to A'$ such that $f' \circ \tilde{v} = v$. Similarly, we have $v' \circ f' = 0$, and with the same reasoning (on the weak cokernel) shows that there exists $\bar{v}'$ such that $\bar{v}' \circ g' = v'$. We compute $v' \circ v = (\bar{v}' \circ g') \circ (f' \circ \tilde{v}) = 0$ because $g' \circ f' = 0$. □

Corollary 2.16. If $(0, 0, w) : \Delta \to \Delta$ is an endomorphism of an exact triangle, then (by rotating to $(0, w, 0)$ and $(0, w, 0)$) $w^2 = 0$.

Corollary 2.17. If $(\mathbf{id}, \mathbf{id}, w) : \Delta \to \Delta$ is an endomoprhism of an exact triangle, then $w = \mathbf{id} + x$ such that $x^2 = 0$. (This can be done by taking the difference of this morphism and $(\mathbf{id}, \mathbf{id}, \mathbf{id})$.) In particular, w is an automorphism.

Corollary 2.18. If $(u, v, w) : \Delta \to \Delta'$ is a morphism of exact triangles, and two of them are isomorphisms, then so is the third.

Proof. Without loss of generality, say u, v are isomorphisms. Then we can extend the morphism back to Δ (by taking some w'), using the morphism axiom:

$$\Delta \xrightarrow{(u,v,w)} \Delta' \xrightarrow{(\mu^{-1},v^{-1},w')} \Delta$$

Then the composition is $(\mathbf{id}, \mathbf{id}, w' \circ w)$. By the corollary, $w' \circ w$ is an isomorphism, and similarly, $w \circ w'$ is an isomorphism. Hence, w is an isomorphism. $\square$

Remark 2.19. Looking at the existence axiom again: given $f : A \to B$, suppose

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, equal] & C \arrow[r, "h"] \arrow[d, dashed, "\exists w"] & \Sigma A \arrow[d, equal] \\ A \arrow[r, "f"] & B \arrow[r, "g'"] & C' \arrow[r, "h'"] & \Sigma A \end{tikzcd}$$

are two exact triangles extending f, then there exists w by the morphism axiom, and w is an isomorphism by the above.

Therefore, the triple $(C, g, h) \cong (C', g', h')$. This is usually referred to as the cone of f. In notation, we write $C = \text{cone}(f)$, which is only up to isomorphism. The map $g : B \to C$ is called the homotopy cofiber of f, and $\Sigma^{-1}h : \Sigma^{-1}C \to A$ is called the homotopy fiber of f.

Proposition 2.20. *A morphism $f : A \to B$ is an isomorphism if and only if $\text{cone}(f) \cong 0$.*

Proof. By replete axiom, if f is an isomorphism, we have two exact triangles

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] & 0 \arrow[r, "h"] & \Sigma A \arrow[d, equal] \\ A \arrow[r, "\mathbf{id}"] & A \arrow[r, "g'"] \arrow[u, "f"] & 0 \arrow[r, "h'"] \arrow[u, "\cong"] & \Sigma A \end{tikzcd}$$

and forces the cone to be 0. Conversely, we use the 2-out-of-3 and compare the exact sequences

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] & 0 \arrow[r, "h"] \arrow[d, equal] & \Sigma A \arrow[d, equal] \\ A \arrow[r, "\mathbf{id}"] & A \arrow[r, "g'"] \arrow[u, "f"] & 0 \arrow[r, "h'"] & \Sigma A \end{tikzcd}$$

then f is an isomorphism. $\square$

We now make a few remarks comparing exact triangles and exact sequences.

Exercise 2.21. Given two triangles Δ and Δ', then $\Delta \oplus \Delta'$ is exact if and only if Δ and Δ' are exact.

Exercise 2.22. For every A, B, the following is exact:

$$A \xrightarrow{0} B \longrightarrow ? \longrightarrow \Sigma A$$

if and only if $? = B \oplus \Sigma A$.

Proposition 2.23. Let $\Delta : A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ be exact. Then the following are equivalent:

1. $f = 0$,
2. g is a split monomorphism,
3. h is a split epimorphism,
4. g is a monomorphism,
5. h is an epimorphism.

Proof. Obviously (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5). Note that (4) $\Rightarrow$ (1) since $g \circ f = 0 = g \circ 0$ and so $f = 0$. Similarly, (5) $\Rightarrow$ (1). Finally, (1) implies (2) and (3) because for any object (on the third slot) that makes the bottom row an exact triangle, it must be of the form $C \cong B \oplus \Sigma A$

$$\begin{tikzcd}
A \arrow[r, "0"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, equal] & C \arrow[r, "h"] \arrow[d, "\cong"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "0"] & B \arrow[r] & B \oplus \Sigma A \arrow[r] & \Sigma A
\end{tikzcd}$$

by the uniqueness of the triangle of the zero morphism. However, that means the bottom row splits. $\square$

Remark 2.24. There are no interesting monomorphisms or epimorphisms in pre-triangulated categories.

Example 2.25. In $\mathbf{K}(\mathbb{Z}\text{-}\mathbf{Mod})$, the map

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "\cdot 2"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

is not a monomorphism.

Remark 2.26. A pre-triangulated category can only be Abelian (exact) if it is Abelian semisimple, in which case every exact triangle is just a direct sum of the trivial ones (and their rotations):

$$A \xrightarrow{\mathbf{id}} A \longrightarrow 0 \longrightarrow \Sigma A$$

$$0 \longrightarrow B \xrightarrow{\mathbf{id}} B \longrightarrow 0$$

$$\Sigma^{-1}C \longrightarrow 0 \longrightarrow C \xrightarrow{\mathbf{id}} C$$

Remark 2.27. Suppose K is a pre-triangulated category (or just additive) such that its arrow category $\mathbf{Arr}(K)$ is pre-triangulated. Then $K = 0$!

Proof. Pick $A \in K$, look at the identity morphism on A as an object in the arrow category, look at the morphism from this object to the zero morphism from A to 0, then that morphism is an epimorphism in the arrow category for obvious reasons.

$$\begin{tikzcd}
A \arrow[r, "\mathbf{id}"] \arrow[d, "\mathbf{id}"] & A \arrow[d] \\
A \arrow[r] \arrow[u, dashed, bend left, "u"] & 0
\end{tikzcd}$$

By the proposition, it must be split, so there exists some splitting (morphism of arrow category) backwards, but the backwards map u should be 0 because $\mathbf{id} \circ u = \mathbf{id}$ and $\mathbf{id} \circ u = 0$ as a backwards commutative square. In particular, this says that the object $A = 0$. $\square$

3 Verdier Octahedron Axiom and Triangulated Categories

So far, in a pre-triangulated category, for every morphism $f : A \to B$, we have an object $\text{cone}(f)$ (unique up to isomorphism) that measures f "homologically". For example, it provides a weak (co)kernel, vanishes if and only if $f = 0$, etc. A natural question to ask is: what about compositions? That is, if we have two composable morphisms

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$$

would there be a relation between the cones of f_1, f_2, and $f_2 \circ f_1$. This is what the Verdier Octahedron Axiom tells us about. In some sense, this should be called a composition axiom.

Definition 3.1. A Verdier triangulated category is a pre-triangulated category (K, Σ, Δ) such that for any composable morphisms

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3$$

there exists a Verdier octahedron:

$$
\begin{tikzcd}
A_1 \arrow[rr, "f_1"] \arrow[rrrr, bend left, "f_3"] & & A_2 \arrow[rr, "f_2"] \arrow[dl, "g_1"] & & A_3 \arrow[dl, "g_2"] \arrow[ddll, bend left, "g_3"] \\
& C_1 \arrow[ul, "h_1"] \arrow[dr, "f_4"] & & C_2 \arrow[ul, "h_2"] \arrow[ll, "(\Sigma g_1)\circ h_2"] & \\
& & C_3 \arrow[ur, "g_4"] \arrow[uull, bend left, "h_3"] & &
\end{tikzcd}
$$

where $A_1A_2A_3, A_1C_1C_3, A_3C_2C_3$, and $A_2C_1C_2$ commutes, and $A_1A_2C_1$, $A_2A_3C_2$, $C_1C_2C_3$, and $A_1A_3C_3$ are exact. Moreover, $f_4 \circ g_1 = g_3 \circ f_2 : A_2 \to C_3$, and $\Sigma f_1 \circ h_3 = h_2 \circ g_4 : C_3 \to \Sigma A_2$.

More precisely, the axiom says that given the composable morphisms and the three cones, there exists morphisms f_4 and g_4 such that the diagram of cones is exact, and everything commutes well.

Remark 3.2. In other words, the first part of the commutativity says that there exists an exact triangle

$$\mathrm{cone}(f_1) \xrightarrow{f_4} \mathrm{cone}(f_2 \circ f_1) \xrightarrow{g_4} \mathrm{cone}(f_2) \xrightarrow{\Sigma g_1 \circ h_2} \Sigma\mathrm{cone}(f_1)$$

also known as the bottom triangle $C_1C_2C_3$, that is compatible with the rest of the structures. The second part of the commutativity says that we have the following diagram:

$$
\begin{tikzcd}
& & A_3 \arrow[dl, "g_2"] \arrow[dr, "g_3"] & \\
A_2 \arrow[urr, "f_2"] \arrow[ddr, "g_1"] & C_2 \arrow[l, "h_2"] \arrow[dd, "h_4"] & & C_3 \arrow[ll, dashed, "g_4"] \arrow[dl, "h_3"] \\
& & A_1 \arrow[uu, "f_3"] \arrow[ull, "f_1"] & \\
& C_1 \arrow[ur, "h_1"] \arrow[uurr, dashed, bend right, "f_4"] & &
\end{tikzcd}
$$

Remark 3.3. Alternatively, we can think of the diagram as in the following form:

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "f_1"] \arrow[d] & A_2 \arrow[r, "f_2"] \arrow[d, "g_1"] & A_3 \arrow[r] \arrow[d] & 0 \arrow[d] \\
& 0 \arrow[r] & C_1 = A_{12} \arrow[r, "f_4"] \arrow[d] & C_3 = A_{13} \arrow[r, "h_3"] \arrow[d, "g_4"] & \Sigma A_1 \arrow[d, "\Sigma f_1"] \\
& & 0 \arrow[r] & C_2 = A_{23} \arrow[r, "h_2"] \arrow[d] & \Sigma A_2 \arrow[d, "\Sigma f_2"] \\
& & & 0 \arrow[r] & \Sigma A_3 \arrow[d] \\
& & & & 0
\end{tikzcd}$$

Problem 3.4 (Take-home Problem 1)**.** Suppose K is (Verdier) triangulated. Let

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
A' \arrow[r] & B'
\end{tikzcd}$$

be a commutative square. Then there exists an extended diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d] & C \arrow[r, "h"] \arrow[d] & \Sigma A \arrow[d] \\
A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & \Sigma A' \arrow[d] \\
A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] & \Sigma A'' \arrow[d] \\
\Sigma A \arrow[r] & \Sigma B \arrow[r] & \Sigma C \arrow[r] & \Sigma^2 A
\end{tikzcd}$$

with exact rows and columns (that is, first three of each, with the last one being the suspension of the first), and all squares commute, except the bottom-right that anti-commutes.

Hint: Use 3 Octahedra.

4 Derivators

In mathematics we often have to examine the relationship between axioms and models. For example, when studying vector spaces, yes it is more favorable to look at the theoretical properties of the groups and fields underneath, but it is still important to look at specific examples of vector spaces which allows us to verify these properties, and to wave our hands and say the same idea goes for other fields.

We may now answer the question: why triangulated category? If we also look at the spectrum between models and axioms, we would get something like this:

stable quiver model   stable ∞-categories   stable homotopy theory   stable derivations   triangulated categories

It is hard to justify this, but stable homotopy theory lands in the middle as a perfect model separating the models and the axioms. In order to look at stable homotopy theory, we would need to push from right the left, and try to formulate the constructions. We would first need to explain what derivators are.[5]

Example 4.1 (Derivators)**.** Suppose given a "model" (category) $\mathcal{M}$ of your favorite homotopy theory. The homotopy category of $\mathcal{M}$ is the category $\mathbf{Ho}(\mathcal{M})$, that is, adjoining $\mathcal{M}$ with inverted weak equivalences (analogous to the derived category).

This category itself may not be fun: we have seen some ideas last time. Instead, we look at the functor category $\mathcal{M}^I$ for some small category I. (On the constant category I, we just have $\mathcal{M}$.)

It now seems that $\mathbf{Ho}(\mathcal{M}^I)$ is a better category. Let us look at a functor between two small categories $u : I \to J$. The restriction along u, i.e., $u^* : \mathcal{M}^J \to \mathcal{M}^I$ preserves limits. This now induces $u^* : \mathbf{Ho}(\mathcal{M}^J) \to \mathbf{Ho}(\mathcal{M}^I)$.

What about adjoints? (On the constant diagram, we just have limit and colimit functors as adjoints.) Sometimes we do have adjoints:

$$\begin{tikzcd}
\mathbf{Ho}(\mathcal{M}^I) \arrow[d, dashed, bend right, "u_!"'] \arrow[d, dashed, bend left, "u^*"] \\
\mathbf{Ho}(\mathcal{M}^J) \arrow[u, "u^*"']
\end{tikzcd}$$

Definition 4.2 (Prederivator)**.** A predeviator is a 2-functor $\mathbb{D} : \mathbf{Cat}^{op} \to \mathbf{CAT}$, sending $I \mapsto \mathbb{D}(I)$, and sending 1-morphisms $u : I \to J$ to $u^* : \mathbb{D}(J) \to \mathbb{D}(I)$, and sending 2-morphisms $\alpha : u \Longrightarrow v$ to $\alpha^* : u^* \Longrightarrow v^*$.

Definition 4.3 (Derivator)**.** A derivator is a prederivator satisfying some axioms:

1. $\mathbb{D}(I \sqcup J) \cong \mathbb{D}(I) \times \mathbb{D}(J)$.

[5]See Professor Paul Balmer's paper for some ideas explained in this lecture.

2. For all $I \in \mathbf{Cat}$, the functor $* \to I$ sending the point to a specified object $i \in I$ induces a functor $\mathbb{D}(I) \to \prod_{x \in \mathbf{Ob}(I)} \mathbb{D}(*)$. In particular, this functor detects isomorphisms.

3. For every $u : I \to J$, the functor u^* admits adjoints on both sides:

$$\begin{tikzcd} \mathbf{Ho}(\mathcal{M}^I) \arrow[d, dashed, bend right, "u_!"'] \arrow[d, dashed, bend left, "u^*"] \\ \mathbf{Ho}(\mathcal{M}^J) \arrow[u, "u^*"] \end{tikzcd}$$

4. For every diagram

$$\begin{tikzcd} I \arrow[dr, "u"'] & & K \arrow[dl, "v"] \\ & J & \end{tikzcd}$$

there is a formulation $v^* \circ u_!$ and $v^* \circ u_*$, given by

$$\begin{tikzcd} & (u/v) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ I \arrow[dr, "u"'] \arrow[rr, phantom, "\gamma \Longrightarrow"] & & K \arrow[dl, "v"] \\ & J & \end{tikzcd}$$

where (u/v) is a (comma) category with objects $(i, k, g : u(i) \to v(k))$, given by an object in I, an object in K, and a morphism in J, and with morphisms (i', k', g'), defined such that for $f : i \to i'$ in I and $l : k \to k'$ in K, we have a commutative diagram in J:

$$\begin{tikzcd} u(i) \arrow[r, "g"] \arrow[d, "u(f)"'] & v(k) \arrow[d, "v(l)"] \\ u(i') \arrow[r, "g'"'] & v(k') \end{tikzcd}$$

The axiom then says that the diagram

$$
\begin{tikzcd}
 & & \mathbb{D}(u/v) \arrow[rr, bend left, "q_!"] \arrow[rr, phantom, "\Longrightarrow \varepsilon"] & & \mathbb{D}(K) \\
 & \mathbb{D}(I) \arrow[ur, "p^*"] \arrow[rr, phantom, "\Longrightarrow \mathbb{D}(\gamma)"] \arrow[d, phantom, "\Longrightarrow \eta : u_! \to u^*"] & & \mathbb{D}(K) \arrow[ul, "q^*"'] \arrow[ur, equal] & \\
 & & \mathbb{D}(J) \arrow[ul, "u^*"'] \arrow[ur, "v^*"'] & & \\
 \mathbb{D}(I) \arrow[urr, "u_!"'] \arrow[uur, equal] & & & &
\end{tikzcd}
$$

induces an isomorphism

$$q_! \circ p^* \Longrightarrow v^* \circ u_!.$$

Example 4.4. In our previous example, we note that $\mathbf{Ho}(\mathcal{M}^I) = \mathbb{D}(I)$, and $\mathbf{Ho}(\mathcal{M}) = \mathbf{Ho}(\mathcal{M}^*) = \mathbb{D}(*)$.

5 Origins of Triangles

The questions we may want to ask starts with the following: why “cone”?

Consider the mapping of topological space $f : X \to Y$ from some manifold to another manifold. We now want a further mapping so that the composite is identity (more precisely, homotopic to the identity map), then the obvious thing to do is to contract whatever many genus we have to points. Therefore, we image there is a cylinder inside that void region, and contract the upper surface to a point, then we get a cone-shape structure upon the manifold indeed. Essentially, this is what we called the cone of f. We now invert this cone, then we get something homotopic to X, namely the suspension of X[6].

What does the nth suspension look like if we do it repeatedly? Consider the tuple (X, n), giving the information of $\Sigma^n X$ for some $n \in \mathbb{Z}$. Consider the mapping between this tuple and another, i.e., $(X, n) \to (X', n')$, then we obtain the mapping $\Sigma^{n+k} X \to \Sigma^{n'+k} X'$.

Imagine the mapping from the small diagram I to its pushout square diagram J, then there corresponds to the mapping $\mathbf{Ho}(\mathbf{Top}^J) \to \mathbf{Ho}(\mathbf{Top}^I)$, and one question would be to build the left adjoint of this functor, called hocolim.

[6]More precisely, the projection onto the original space maintains the original shape of X, and therefore gives the similar structure in this sense.

We now think of this in terms of the derivator setting. Recall the derivator is a 2-functor $\mathbb{D} : \mathbf{Cat}^{op} \to \mathbf{CAT}$.

Remark 5.1. What is the suspension? Consider the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ 0 & \end{tikzcd}$$

and we apply the hocolim functor to get the pushout diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & \mathrm{cone}(h) \end{tikzcd}$$

Now the suspension is essentially the diagram after we apply the functor (to push it out) again:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \mathrm{cone}(h) \arrow[r] & \Sigma X \end{tikzcd}$$

One may also think of strong derivators. Here "strong" indicates the functor $\mathbb{D}(\cdot \to \cdot) \to \mathbb{D}(e)^{\cdot \to \cdot}$, which is essentially surjective and full. Here $\cdot \to \cdot$ is called the coherent diagram of given shape. In general, for the functor $\mathbf{Ho}(\mathcal{M}^I) \to \mathbf{Ho}(\mathcal{M})^I$, the first is the coherent object, and the second is a naive object.

Remark 5.2 (How to Build Triangles?)**.** Consider

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \\ 0 & \end{tikzcd}$$

We now do a homotopy pushout: for $i : \begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \\ \cdot & \end{tikzcd} \to \begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$, we have an adjoint pair $i_! \dashv i^* : \mathbb{D}\left(\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \\ \cdot & \end{tikzcd}\right) \Longleftrightarrow \mathbb{D}\left(\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}\right)$ and so we see by applying $i_!$ to the original diagram, we have a square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & \mathrm{cone}(f) \end{tikzcd}$$

Again, applying the the hocolim for the second time, we get the square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, "g"] & 0 \arrow[d] \\
0 \arrow[r] & Z \arrow[r, "h"'] & \Sigma X
\end{tikzcd}$$

For details on derivators, one should read papers by Movitz Groth.

Remark 5.3 (Higher Triangles)**.**

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d] & X_2 \arrow[r, "f_2"] \arrow[d, "g_1"] & X_3 \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{cone}(f_1) \arrow[r] \arrow[d] & \mathrm{cone}(f_2 \circ f_1) \arrow[r] \arrow[d] & \Sigma X_1 \arrow[r] \arrow[d] & 0 \arrow[d] \\
& 0 \arrow[r] & \mathrm{cone}(f_2) \arrow[r] \arrow[d] & \Sigma X_2 \arrow[r] \arrow[d] & \Sigma \mathrm{cone}(f_1) \\
& & 0 \arrow[r] & \Sigma X_3 &
\end{tikzcd}$$

and therefore induces the sequence

$$\mathrm{cone}(f_1) \to \mathrm{cone}(f_2 \circ f_1) \to \mathrm{cone}(f_2) \to \Sigma X_2 \to \Sigma \mathrm{cone}(f_1)$$

We have two book-keeping axioms on higher triangles:

1. Existence Axiom: given the sequence $X_1 \to X_2 \to X_3$, we can obtain the n-triangle.

2. Morphism Axiom: there is a way to complete the n-triangle morphism mappings.

Definition 5.4 (Exact Category)**.** An exact category is an additive category $\mathcal{E}$ together with a chosen class of diagrams of the following shape (called admissible exact sequences):

$$A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$$

where f is called the admissible monomorphism and g is called the admissible epimorphism, such that

1. The Replete Axiom: $A \rightarrowtail A \to 0$ and $0 \to B \twoheadrightarrow B$ are admissible.

2. They are short exact sequences, i.e., $g \circ f = 0$ and $g = \mathrm{coker}(f)$ and $f = \ker(g)$.

3. Composition of admissible monomorphisms (respectively, epimorphisms) are admissible.

4. Pushouts of admissible monomorphisms exist and remain admissible: for any diagram of the shape

$$\begin{tikzcd} A \arrow[r, tail, "f"] \arrow[d, "u"] & B \\ A' & \end{tikzcd}$$

there exists a pushout

$$\begin{tikzcd} A \arrow[r, tail, "f"] \arrow[d, two heads, "u"] & B \arrow[d, "u'"] \\ A' \arrow[r, tail, "f'"'] & B' \end{tikzcd}$$

such that f' is admissible.

5. Dually, the pullbacks of admissible epimorphisms exist.

Moreover, a split exact sequence should be admissible.[7]

Example 5.5. If $\mathcal{A}$ is an additive category, then $\mathcal{E} = \mathcal{A}^{\oplus}$ with the split sequences as exact.

Example 5.6. If $\mathcal{A}$ is an Abelian category, then $\mathcal{A}$ is exact with the intrinsical short exact sequences.

Example 5.7. For instance, we can let X is a scheme and $\mathcal{E} = VB(X)$ is the category of locally free filtrated $\mathcal{O}_X$-module, i.e., on vector bundles.

Problem 5.8 (Take-home Problem 2)**.** Let $\mathcal{A}$ be an additive category and $\mathcal{E}$ be the category of chain complexes in $\mathcal{A}$. Show that it admits an exact structure where the admissible short exact sequences are the short sequences that are split-exact in every degree (without requiring the splitting to be compatible with the differentials, hence not the split-exact structure on the additive category $\mathcal{E}$). Moreover, prove that for a complex $E \in \mathcal{E}$ the following are equivalent:

(i) E is contractible (its identity is homotopic to zero).

(ii) E is projective in $\mathcal{E}$ ($\mathbf{Hom}_{\mathcal{E}}(E, -)$ sends admissible short exact sequences to exact sequences, i.e., E has the lifting property for admissible epimorphisms).

(iii) E is injective in $\mathcal{E}$ (projective in $\mathcal{E}^{op}$).

Conclude that $\mathcal{E}$ is a Frobenius category.

[7] This axiom seems to be abandoned because it becomes a direct consequence.

6 Frobenius Exact Category

Remark 6.1. We can adopt the (homological algebra) notions of "projective", "injective", etc., over Abelian categories to exact ones.

Example 6.2. We say $F : \mathcal{E}_1 \to \mathcal{E}_2$ is exact if it maps admissible exact sequences to admissible exact sequences.

Example 6.3. $P \in \mathcal{E}$ is called projective if $\mathbf{Hom}_{\mathcal{E}}(P, -) : \mathcal{E} \to \mathbf{Ab}$ is exact.

Dually, we get to define an injective object.

Definition 6.4. A Frobenius exact category is an exact category with enough injectives (for every $X \in \mathcal{E}$, there exists $X \rightarrowtail I$ with I injective) and enough projectives, and injectives and projectives coincide.

Example 6.5. Let G be a group and k be a field. Looking at the group algebra, kG-**Mod** is the Abelian category of kG-modules, and there is a restriction map $\mathbf{Res}_1^G : kG\text{-}\mathbf{Mod} \to k\text{-}\mathbf{Mod} = k\text{-}\mathbf{Vector\ Space}$. This map is exact and faithful. There is a left adjoint $\mathbf{Ind}_1^G = kG \otimes_k -$, given by the tensor-hom adjunction. Similarly, there is a right adjoint structure $\mathbf{CoInd}_1^G(-) = \mathbf{Hom}_k(kG, -)$.

We now know that the structure kG-**Mod** has enough projectives and enough injectives. The left adjoint preserves projectives, and the right adjoint preserves injectives.

Remark 6.6. Looking at the map $\varepsilon : kG \otimes_k M \to M$, we are now given an epimorphism in the category of kG-modules. And since $\mathbf{Res}_1^G(M)$ is projective, then the category of kG-modules has enough projectives, all direct summand of induced $\mathbf{Ind}_1^G(V)$ for some k-vector space V.

Similarly, the category of kG-modules has enough injectives, the direct summands of $\mathbf{CoInd}_1^G$ for some k-vector space V.

Note that if G is a finite group, the induction is isomorphic (canonically) to coinduction, and so the left adjoint and the right adjoint coincides. Namely, we obtain the Frobenius reciprocity for finite groups. In particular, we have $\mathbf{Hom}_k(kG-,) \cong \mathbf{Hom}_k(kG, k) \otimes_k - \cong kG \otimes_k -$, where the first isomorphism is given as kG is finite-dimensional over k.

Note that all of the constructed adjunctions above preserves finitely-generated modules.

Proposition 6.7. Let G be a finite group and k be a field. Then kG-**Mod** and kg-**mod** (the category of finitely-generated kG-modules) are Frobenius categories.

Proof. We identified the projectives and the injectives as the direct summands of the frees. $\square$

Proposition 6.8. Let $\mathcal{A}$ be an additive category and let $\mathbf{Ch}(\mathcal{A})$ be the additive category of complexes in $\mathcal{A}$. It is an exact category with degreewise-split exact sequences

$$A \rightarrowtail B \twoheadrightarrow C$$

such that $A_n \rightarrowtail B_n \twoheadrightarrow C_n$ is split exact in $\mathcal{A}$ for all $n \in \mathbb{Z}$.

An object $E \in \mathbf{Ch}(\mathcal{A})$ is projective if and only if it is injective, if and only if it is contractible (i.e., $\mathbf{id}_E \sim 0$). In fact, $\mathbf{Ch}(\mathcal{A})$ is Frobenius.

Proof. Look at the mapping cone on identity, we should get the contractible property.

Enough injectives: for any $A \in \mathbf{Ch}(\mathcal{A})$, the map $A \to \mathrm{cone}(\mathbf{id}_\Delta)$ has the contractible property, i.e., injective, and projective. $\square$

Definition 6.9 (The Stable Category of a Frobenius Exact Category)**.** Let $\mathcal{E}$ be a Frobenius exact category. We construct a new category $\underline{\mathcal{E}} = \mathcal{E}/\mathbf{Proj}(\mathcal{E}) = \mathcal{E}/\mathbf{Inj}(\mathcal{E})$ as additive categories, with the same objects as $\mathcal{E}$ and new morphisms:

$$\mathbf{Hom}_{\underline{\mathcal{E}}}(X, Y) = \mathbf{Hom}_{\mathcal{E}}(X, Y)/\{f : X \to Y \mid \exists P \text{ projective that factors } f\}$$

in $\mathbf{Ab}$. In particular, the quotient structure is given by the existence of a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \\ & P \arrow[ur] & \end{tikzcd}$$

with P projective. More over, the composition is given in the obvious way.

Choose for every $A \in \mathcal{E}$ a monomorphism into an injective, i.e., $i_A : A \rightarrowtail I_A$. Define $\Sigma A = I_A/A = \mathrm{coker}(i_A)$. To make it a functor, for every $f : A \to B$,

$$\begin{tikzcd} A \arrow[r, tail, "i_A"] \arrow[d, "f"'] & I_A \arrow[r, two heads] \arrow[d, dashed, "\exists \tilde{f}"] & \Sigma A \arrow[d, dashed, "\exists \bar{f}"] \\ B \arrow[r, tail, "i_B"'] & I_B \arrow[r, two heads] & \Sigma B \end{tikzcd}$$

this induces 1) $\tilde{f} : I_A \to I_B$, not necessarily unique, and 2) $\bar{f} : \Sigma A \to \Sigma B$ by applying the cokernel. This construction $\bar{f}$ is unique up to a morphism factoring via i_B. Therefore, $\Sigma : \underline{\mathcal{E}} \to \underline{\mathcal{E}}$ is well-defined.

As for the exact triangles, let $[f] : A \to B$ in $\underline{\mathcal{E}}$ be a class of morphisms. Pick a representative $f : A \to B$ in $\mathcal{E}$, do the homotopy pushout:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail] & B \arrow[r, tail] \arrow[d, "g"] & I_B \arrow[d] \\ I_A \arrow[r] & C \arrow[r] & \Sigma' A \cong \Sigma A \end{tikzcd}$$

where C is the pushout from A, and $\Sigma' A \cong \Sigma A$ is the pushout from B. This induces the triangle $A \to B \to C \to \Sigma A$ as desired.

For the octahedron, consider

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, tail] & A_2 \arrow[r, "f_2"] \arrow[d] & A_3 \arrow[r, tail] \arrow[d] & J \arrow[d] \\
I_1 \arrow[r] & C_{12} \arrow[r] \arrow[d, tail] & C_{13} \arrow[r] \arrow[d] & \Sigma' A \arrow[d] \\
& I_2 \arrow[r] & C_{23} \arrow[r] & \Sigma' A_2
\end{tikzcd}$$

where every square is a pushout square.

Problem 6.10 (Take-home Problem 3)**.** Check the Morphism Axiom (look at the triangle that lifts a commutative square) and the rotation axiom (why is there no sign change?) to show that this is a triangulated category.

7 Tensor-triangulated Categories

Last time we did the construction as follows.

Remark 7.1. Let ε be a Frobenius exact category. Given $f : A \to B$ in $\mathcal{E}$ we construct by choosing any monomorphism $A \rightarrowtail I$ (e.g., $A \rightarrowtail I_A$), then

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, tail] & B \arrow[d, dashed] \\
I \arrow[r, dashed] \arrow[d, two heads] & C \arrow[d, dashed] \\
\Sigma A \arrow[r, equal, dashed] & \Sigma A
\end{tikzcd}$$

and therefore constructs the cone sequence. Now the exact triangles in $\underline{\mathcal{E}} = \mathcal{E}/\mathbf{Proj}(\mathcal{E})$ are all triangles isomorphic in $\underline{\mathcal{E}}$ to one of these.

Theorem 7.2. $\underline{\mathcal{E}}$ is triangulated (even with "higher triangles").

Proposition 7.3. If

$$A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$$

is an exact sequence in $\mathcal{E}$, then there exists $h : C \to \Sigma A$ such that

$$A \xrightarrow{[f]} B \xrightarrow{[g]} C \xrightarrow{[h]} \Sigma A$$

is an exact triangle in $\underline{\mathcal{E}}$. (Note that monomorphisms (and epimorphisms) are split in triangulated, so we lose the information in a trivial sense.)

Proof. Look at

$$\begin{tikzcd}
A \arrow[r, tail, "f"] \arrow[d, tail] & B \arrow[r, two heads] \arrow[d, "g'"] & C \arrow[d, equal] \\
I \arrow[r, "C'"] \arrow[d, two heads] & C' \arrow[r, two heads] \arrow[d, two heads, "h'"] & C \\
\Sigma A \arrow[r, equal] & \Sigma A &
\end{tikzcd}$$

and note that the middle sequence splits: $C' \cong I \oplus C \cong C$ in $\underline{\mathcal{E}}$. $\square$

Example 7.4. Let $\mathcal{E}$ be kG-**Mod** for G finite and k field. Note that kG is self-injective as a module.

Therefore, the stable category kG-**Mod** $=$ **StMod**$(kG) = kG$-**Mod**/**Proj** is a triangulated category.

Example 7.5. For an additive category $\mathcal{A}$, $\mathcal{E} = \mathbf{Ch}(\mathcal{A})$ is exact with admissible sequences the degreewise split short exact sequences. Then $\underline{\mathcal{E}} = \mathbf{Ch}(\mathcal{A})/\{\text{contractibles}\} = K(\mathcal{A})$. The objects are the complexes in $\mathcal{A}$, and the morphisms are the maps up to homotopy.

Remark 7.6 (What if you remember that $\mathcal{A}$ was Abelian?)**.** In that case, $\mathbf{Ch}(\mathcal{A})$ is also Abelian, but with a different class of exact sequences and a richer exact category structure, but not Frobenius anymore.

If $A \rightarrowtail B \twoheadrightarrow C$ is an intrinsic short exact sequence, in $K(\mathcal{A})$

$$\begin{tikzcd}
A \arrow[r, "{[f]}"] \arrow[d, equal] & B \arrow[r, "{[g]}"] \arrow[d, equal] & C & \Sigma A \arrow[d, equal] \\
A \arrow[r, "{[f]}"'] & B \arrow[r] & \mathrm{cone}(f) \arrow[r] \arrow[u, dashed, "q"] & \Sigma A
\end{tikzcd}$$

and note that the mapping induced by q has to be a quasi-isomorphism by the five lemma. By inverting the quasi-isomorphisms from $K(\mathcal{A})$, we arrive at a derived category and obtain the triangles. In particular, it has calculus of fractions with it, which gives the following description: the derived category has the same objects as $\mathbf{Ch}(\mathcal{A})$, and the morphisms are fractions from X to Y, meaning there exists quasi-isomorphism such that $X \xleftarrow{q} Z \to Y$.

Definition 7.7 (Exact Functor)**.** For an exact category, an exact functor preserves the admissible short exact sequences. Formally, an exact functor between triangulated categories is one which commutes with suspension and preserves exact triangles.

Definition 7.8 (Tensor-triangulated Category)**.** A tensor-triangulated category $\mathcal{T}$ is a triangulated category together with a symmetric monoidal structure $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ given by

$(a, b) \mapsto a \otimes b$, such that $a \otimes - : \mathcal{T} \to \mathcal{T}$ and $- \otimes b : \mathcal{T} \to \mathcal{T}$ are exact functors, and (because of commutativity) such that

$$\begin{tikzcd}
(\Sigma a) \otimes (\Sigma b) \arrow[r, "\cong"] \arrow[d, "\cong"] & \Sigma((\Sigma a) \otimes b) \arrow[d, "\cong"] \\
\Sigma(a \otimes (\Sigma b)) \arrow[r, "\cong"] & \Sigma^2(a \otimes b)
\end{tikzcd}$$

In particular, $\mathbb{1}$ acts as the tensor unit. There are some trivial relations $\lambda : \mathbb{1} \otimes a \cong a$ and $\rho : a \otimes 1 \cong a$, therefore

$$\begin{tikzcd}
 & \mathbb{1} \otimes \mathbb{1} \arrow[dl, "\lambda"'] \arrow[dr, "\rho"] & \\
\mathbb{1} \arrow[rr, equal] & & \mathbb{1}
\end{tikzcd}$$

and we have $\alpha_{a,b,c} : (a \otimes b) \otimes c \cong a \otimes (b \otimes c)$ such that

$$\begin{tikzcd}
a \otimes (b \otimes (c \otimes d)) \arrow[rr, "\cong"] \arrow[d] & & a \otimes ((b \otimes c) \otimes d) \arrow[d, "\cong"] \\
(a \otimes b) \otimes (c \otimes d) \arrow[dr, "\cong"] & & (a \otimes (b \otimes c)) \otimes d \arrow[dl, "\cong"] \\
 & ((a \otimes b) \otimes c) \otimes d &
\end{tikzcd}$$

Finally, there is a symmetric structure $\sigma : a \otimes b \cong b \otimes a$ such that $\sigma^2 = \mathbf{id}$. Therefore, this induces the commutative diagram

Lemma 7.9. $\mathbf{End}_K(\mathbb{1})$ is a commutative ring.

Proof. Let $f, g : \mathbb{1} \to \mathbb{1}$ be morphisms, then we have

$$\begin{tikzcd}
 & \mathbb{1} \arrow[r, "f"] & \mathbb{1} \arrow[dr, equal] & \\
\mathbb{1} \arrow[ur, equal] \arrow[d, "g"] & \mathbb{1} \otimes \mathbb{1} \arrow[l, "\cong"'] \arrow[u, "\rho"', "\cong"] \arrow[r, "f \otimes \mathbb{1}"] \arrow[d, "\mathbb{1} \otimes g"] \arrow[dr, "f \otimes g"] & \mathbb{1} \otimes \mathbb{1} \arrow[u, "\rho"', "\cong"] \arrow[r, "\cong", "\lambda"'] \arrow[d, "\mathbb{1} \otimes g"] & \mathbb{1} \arrow[d, "g"] \\
\mathbb{1} \arrow[dr, equal] & \mathbb{1} \otimes \mathbb{1} \arrow[l, "\cong"'] \arrow[r, "f \otimes \mathbb{1}"] \arrow[d] & \mathbb{1} \otimes \mathbb{1} \arrow[r, "\cong", "\lambda"'] \arrow[d] & \mathbb{1} \\
 & \mathbb{1} \arrow[r, "f"] & \mathbb{1} \arrow[ur, equal] &
\end{tikzcd}$$

Therefore $fg = gf$. $\square$

8 Spectrum of a Commutative Ring

Let R be a commutative ring. A question would be: want to draw pictures associated to elements of R. What do we want from the picture?

- We want a topological space X and closed "$a = 0$" for every $a \in R$, i.e., $Z(a) \subseteq X$ for every $a \in R$.
- $Z(0) = X$.
- $Z(1) = \varnothing$.
- $Z(a \cdot b) = Z(a) \cup Z(b)$ for all $a, b \in R$.
- $Z(a + b) \supseteq Z(a) \cap Z(b)$ for all $a, b \in R$.

Remark 8.1. Given a topological space X and $(X, Z : R \to \mathbf{Closed}(X))$ and a continuous function $f : Y \to X$ we can build on Y a Z' by $Z'(a) = f^{-1}(Z(a))$ for all $a \in R$.

This makes a category of such "zero data" (X, Z) for R, and a trivial construction would be an initial object. The most interesting object would be the terminal object.

Theorem 8.2. There exists a terminal object in this category, namely the Zariski spectrum, i.e., $(\mathbf{Spec}(R), V)$, i.e., for any zero data (X, Z), there exists a unique continuous map $f : X \to \mathbf{Spec}(R)$ such that $Z(a) = f^{-1}(V(a))$ for all $a \in R$.

Constructive Proof. Let us build this pair $(\mathbf{Spec}(R), V)$. The set $\mathbf{Spec}(R)$ is the set of all prime ideals φ in R. The topology of $\mathbf{Spec}(R)$ is given as follows: for every subset $E \subseteq R$, we let $V(E) = \{\varphi \mid \varphi \supseteq E\}$, e.g., E could be taken to be an ideal. In particular, if $E = \{a\}$, then $V(a) = \{\varphi \mid \varphi \ni a\}$. It is a zero data. It is universal: pick any such zero data (X, Z), and we want a unique $f : X \to \mathbf{Spec}(R)$ such that $Z(a) = f^{-1}(V(a))$.

This is indeed unique: pick $x \in X$, $f(x) \subseteq R$ is a prime and for every $a \in R$, we have $a \in f(x)$ if and only if $f(x) \in V(a)$, given by definition of V, and this is true if and only if $x \in f^{-1}(V(a))$, and we want it to be $Z(a)$. Therefore, this forces a unique construction: we must have $f(x) = \{a \in R : x \in Z(a)\}$, hence uniqueness and a candidate for existence.

One checks that $f^{-1}(V(a)) = Z(a)$ for all $a \in R$, hence $f^{-1}(V(E)) = f^{-1}\left(\bigcap_{a \in E} V(a)\right) = \bigcap_{a \in E} f^{-1}(V(a))$ closed for all E. $\square$

We now look at the tensor triangulated category. We have a way of adding stuff, and the cone gives us a way of subtracting stuff, can we make it a ring? To formalize the question, is there a universal space where objects of our tensor triangulated category K have "supports".

Definition 8.3. Fix K. A support data is a pair (X, σ) where X is a topological space and σ is an assignment $\sigma : \mathbf{Ob}(K) \to \mathbf{Closed}(X)$, i.e., $\sigma(a) \subseteq X$ for all $a \in K$, such that

- $\sigma(0) = \varnothing$ and $\sigma(\mathbb{1}) = X$,
- $\sigma(\Sigma a) = \sigma(a)$ for all $a \in K$,
- $\sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$ for all $a, b \in K$,
- $\sigma(c) \subseteq \sigma(a) \cup \sigma(b)$ for all exact triangles $a \to b \to c \to \Sigma a$.

The morphism of support data is given by $(X, \sigma) \to (Y, \tau)$ by a continuous map $f : X \to Y$ such that $f^{-1}(\tau(a)) = \sigma(a)$ for all $a \in K$.

Theorem 8.4. Let K be an essentially small tensor-triangulated category. There exists a terminal support data, called the Balmer spectrum of K, $(\mathbf{Spc}(K), \mathrm{supp})$.

Constructive Proof. We call a prime $P \subseteq K$ a subcategory which is triangulated ($0 \in P$, closed under 2-out-of-3 in exact triangles, and thick ($a \oplus b \in P$ implies $a \in P$), and $\otimes$-ideal ($P \otimes K \subseteq P$), and prime: $\mathbb{1} \notin P$, and if $a \otimes b \in P$, then $a \in P$ or $b \in P$.

The topology is given by $\mathrm{supp}(a) = \{P \mid P \not\ni a\}$ and general closed are $Z(E) = \{P \mid P \cap E = \varnothing\}$ for any $E \subseteq K$.

Exercise 8.5. Check that this is a support data (as topology).

The universality is as follows: given a support data (X, σ), we build $f : X \to \mathbf{Spc}(K)$ via $x \mapsto \{a \in K \mid x \notin \sigma(a)\}$. □

9 Topological Properties of the Spectrum

Recall: let K be an essentially small tensor triangulated category. We constructed the Balmer spectrum

$$\mathbf{Spc}(K) = \{P \subsetneq K \mid P \text{ thick } \otimes\text{-ideals triangulated} : a \otimes b \in P \Rightarrow, a \in P \text{ or } b \in P\}.$$

We now denote these thick triangulated $\otimes$-ideals P as tt-ideals P. The topology on this structure is given by $\mathrm{supp}(a) = \{P \in \mathbf{Spc}(K) \mid a \notin P\}$, and we use these as a basis of closed subsets.

We denote $U(a) = \mathrm{open}(a) = \mathbf{Spc}(K) \backslash \{\mathrm{supp}(a)\} = \{P \mid a \in P\}$. Now a general open set is given by $\bigcup_{a \in S} U(a) = \{P \mid P \cap S \neq \varnothing\}$ for every $S \subseteq \mathbf{Ob}(K)$.

Lemma 9.1 (Existence Lemma)**.** Let $J \subsetneq K$ be a thick $\otimes$-ideal and $S \subseteq K$ is a $\otimes$-multiplicative set of objects (i.e., $1 \in S \supseteq S \otimes S$) such that $J \cap S = \varnothing$. Then there exists $P \in \mathbf{Spc}(K)$ such that $P \supseteq J$ and $P \cap S = \varnothing$.

Proof. Let $\mathcal{F} = \{\mathcal{A} \subseteq K \mid \mathcal{A} \supseteq J \text{ and is tt-ideal, and } \mathcal{A} \cap S = \varnothing\}$. By Zorn's Lemma, we find $P \in \mathcal{F}$ maximal in terms of inclusion.

Claim 9.2. P is prime.

Subproof. Let $a, b \in K$ be such that $a \otimes b \in P$. Ab absurdo, suppose $a \notin P$ and $b \notin P$. Consider the tt-ideals $\langle P, a \rangle$ and $\langle P, b \rangle$. By assumption, these ideals are strictly greater than $P \supseteq J$, then these two ideals are not in the family, and therefore $\langle P, a \rangle$ and $\langle P, b \rangle$ are not in $\mathcal{F}$. Therefore, there exists $s, t \in S$ such that $s \in \langle P, a \rangle$ and $t \in \langle P, b \rangle$, then $s \otimes t \in \langle P, a \otimes b \rangle = P$. But $s \otimes t \in S$, we reach a contradiction. ■

□

Corollary 9.3. If $K \neq 0$, then $\mathbf{Spc}(K) \neq \varnothing$.

Proof. Apply the lemma to the case where $J = 0$ and $S = \{\mathbb{1}\}$. □

Proposition 9.4. For every $a \in K$, the open set $U(a)$ is quasi-compact. Conversely, any quasi-compact open is of the form $U(a)$ for some $a \in K$.

Proof. Suppose $U(a) \subseteq \bigcup_{s \in S} U(s) = \{P \mid P \cap S \neq \varnothing\}$. Let $S' = \{s_1 \otimes \ldots \otimes s_n \mid n \geq 0, s_i \in S\} \supseteq S$ be $\otimes$-multiplicative.

Claim 9.5. $\langle a \rangle \cap S' \neq \varnothing$.

Subproof. Suppose not, then by the existence lemma, there exists $P \in \mathbf{Spc}(K)$ such that $a \in P$ (so $u(a) \ni P$), but $P \cap S' = \varnothing$, therefore $P \cap S = \varnothing$, hence $P \notin \bigcup_{s \in S} u(s)$, contradiction. ■

Therefore, there exists $n \geq 0$ and $s_1, \ldots, s_n \in S$ such that $s_1 \otimes \ldots \otimes s_n \in \langle a \rangle$, so

$$U(a) \subseteq U(s_1 \otimes \ldots \otimes s_n) = U(s_1) \cup \ldots \cup U(s_n).$$

Conversely, if U is quasi-compact, then $U = \bigcup_{s \in K : U(s) \subseteq U} U(s)$. Therefore, there exists $s_1, \ldots, s_n$ such that $U = U(s_1) \cup \ldots \cup U(s_n) = U(s_1 \otimes \ldots \otimes s_n)$. □

Proposition 9.6. For every $E \subseteq \mathbf{Spc}(K)$, we have $\bar{E} = \bigcap_{\mathrm{supp}(a) \supseteq E} \mathrm{supp}(a)$. In particular, $\overline{\{P\}} = \{Q \in \mathbf{Spc}(K) \mid Q \subseteq P\}$. Thus, $\overline{\{P\}} = \overline{\{Q\}}$ implies $P = Q$.

Proof. The first statement is formal. The second statement is true because (since $\mathrm{supp}(a) \ni P$ if and only if $a \notin P$) $\overline{\{P\}} = \bigcap_{\mathrm{supp}(a) \ni p} \mathrm{supp}(a) = \{Q \mid \forall a \notin P, a \notin Q\}$, therefore this is the set of Q such that $Q \subseteq P$. The last statement is obvious. □

Proposition 9.7. In $\mathbf{Spc}(K)$, every irreducible closed subset Z ($Z \neq \varnothing$, and $Z \subseteq Z_1 \cup Z_2$ closed implies $Z \subseteq Z_1$ or $Z \subseteq Z_2$) admits a unique generic point $Z = \overline{\{P\}}$. In case, we have $P = \{a \in K \mid Z \cap U(a) \neq \varnothing\}$.

Proof. Let Z be irreducible, so $Z \cap U(a) \cap U(b) = \varnothing$, and taking contrapositive of a previous statement gives: if $Z \cap U(a) \neq \varnothing$ and $Z \cap U(b) \neq \varnothing$, then $Z \cap U(a) \cap U(b) \neq \varnothing$, i.e., $Z \cap U(a \otimes b) \neq \varnothing$. In set-theoretic studies, we note $U(a \otimes b) = U(a) \cup U(b)$, so we note P is prime and $\otimes$-ideal for free. Therefore, P is triangulated because the contrapositive statement we just observed: $a \to b \to c \to \Sigma a$ then $U(c) \supseteq U(a) \cap U(b)$.

Therefore, $Z = \bigcap_{\mathrm{supp}(a) \supseteq Z} \mathrm{supp}(a) = \{Q \mid \forall a \notin P, a \notin Q\} = \{Q \mid Q \subseteq P\} = \overline{\{P\}}$ (note that $\mathrm{supp}(a) \supseteq Z$ implies $Z \cap U(a) = \varnothing$, and $a \notin P$). □

Remark 9.8 (Summary)**.** $\mathbf{Spc}(K)$ has a basis of quasi-compact open, and is quasi-compact itself (quasi-compacts are closed under finite intersection), and every irreducible has a unique generic point.

These are the spectral topological spaces in the sense of Hochster.

Theorem 9.9 (Hochster)**.** These are exactly the topological spaces homeomorphic to Zariski spectra of commutative rings.

10 Classification of tt-ideals

Remark 10.1 (Why Balmer Spectrum?)**.**

1. They behave like Zariski's spectrum over a ring R.
2. It is the universal support data.
3. "Mankind's Best Hope".

We will explain what the last point actually means.

Remark 10.2. Most tensor-triangulated categories K we encounter are hard to handle, e.g., no easy way to decide when two objects are isomorphic, no classification of objects up to isomorphisms.

Instead, we study "tt-classification", i.e., the classification of tt-ideals: triangulated, thick, $\otimes$-ideals as a subcategory of K.[8]

[8] This lecture follows Section 2 of the notes here.

Example 10.3. If $a \in K$, $\langle a \rangle$ is a tt-ideal generated by a. We can ask $\langle a \rangle = \langle b \rangle$ instead of $a \cong b$. That is, we can build a out of b by taking cones, direct sums, summands, tensors, with other other objects and representations.

Remark 10.4. For every $Y \subseteq \mathbf{Spc}(K)$, we denote $K_Y = \{a \in K \mid \operatorname{supp}(a) \subseteq Y\}$. This is a tt-ideal.

Conversely, if $J \subseteq K$ is a tt-ideal, then $\operatorname{supp}(J) = \bigcup_{a \in J} \operatorname{supp}(a)$. This is a subset of $\mathbf{Spc}(K)$.

Note that for $Y \subseteq \mathbf{Spc}(K)$, the tt-ideal $J = K_Y$ is radical: $a^{\otimes n} \in J$ indicates $a \in J$ for $n \geq 1$. Indeed, $\operatorname{supp}(a^{\otimes n}) = \operatorname{supp}(a)$.

Definition 10.5 (Rigid)**.** We say that K is rigid if there exists an exact functor $(-)^t : K^{\mathrm{op}} \to K$ mapping $a \mapsto a^t$, i.e., objects to duals, such that $a \otimes - \dashv a^v \otimes -$ for all $a \in K$. This gives

$$\mathbf{Hom}_K(a \otimes b, c) \cong \mathbf{Hom}_K(b, a^v \otimes c).$$

One can also show that $(a \otimes b)^v \cong a^v \otimes b^v$, and $(a^v)^v \cong a$.

The unit-counit adjunction gives

$$\begin{tikzcd} a \arrow[r, "\eta"] \arrow[rr, bend left, "\mathbf{id}"] & a \otimes a^v \otimes a \arrow[r, "\varepsilon"] & a \end{tikzcd}$$

at $\mathbb{1}$. In particular, $a \in \langle a^{\otimes 2} \rangle$. Hence, $a \in \langle a^{\otimes n} \rangle$ for all $n \geq 1$.

Remark 10.6. If K is rigid, then every tt-ideal is radical.

Remark 10.7. For every tt-ideal J, $\operatorname{supp}(J)$ is a union of closed subsets, $\operatorname{supp}_{a \in J}(a)$, each having a quasi-compact complement.

Definition 10.8 (Thomason Subset)**.** A subset Y in a topological space (e.g., $\mathbf{Spc}(K)$, or a spectral space) is called a Thomason subset (or a Hochester-dual open subset) if $Y = \bigcup_{\alpha \in A} Z_\alpha$ where Z_α is closed with quasi-compact open complement, i.e., Y is the union of all Z's where $Z \subseteq Y$ is closed and Z^c is quasi-compact.

Theorem 10.9 (Classificaiton of tt-ideals)**.** Let K be an essentially small tt-category. The above yields an inclusion-preserving bijection between Thomason subsets $Y \subseteq \mathbf{Spc}(K)$ and the radical tt-ideals $J \subseteq K$. In particular, Y is sent to $K_Y = \{a \mid \operatorname{supp}(a) \subseteq Y\}$, and J is sent to $\bigcup_{a \in J} \operatorname{supp}(a)$.

Proof. Let $Y \subseteq \mathbf{Spc}(K)$ be a Thomason subset. Since Y is Thomason, it is equivalent to $\bigcup_{Z \subseteq Y} Z$ for $Z \subseteq Y$ closed and have $\mathbf{Spc}(K) \backslash Z$ quasi-compact. By the result last time, this is equivalent to $\bigcup_{a \in K, \mathrm{supp}(a) \subseteq Y} \mathrm{supp}(a)$, which is just $\bigcup_{a \in K_Y} \mathrm{supp}(a) = \mathrm{supp}(K_Y)$.

Let $J \subseteq K$ be a radical tt-ideal, and consider $K_{\mathrm{supp}(J)}$. Then $J \subseteq K_{\mathrm{supp}(J)}$. Conversely, let $a \in K$ be such that $a \in K_{\mathrm{supp}(J)}$, then this means $\mathrm{supp}(a) \subseteq \mathrm{supp}(J)$. Let $S = \{\mathbb{1}, a, a^{\otimes 2}, \ldots, a^{\otimes n}, \ldots\}$ for $n \in \mathbb{N}$. If $S \cap J \neq \varnothing$, then there exists n such that $a^{\otimes n} \in J$ thus $a \in J$ because J is radical. Let us show that $S \cap J = \varnothing$ is impossible. Indeed, suppose $S \cap J = \varnothing$, then by the existence lemma, there exists $P \in \mathbf{Spc}(K)$ such that $J \subseteq P$ and $P \cap S = \varnothing$, so $a \notin P$, thus $P \in \mathrm{supp}(a)$. But since $\mathrm{supp}(a) \subseteq \mathrm{supp}(J) = \bigcup_{b \in J} \mathrm{supp}(b)$, then there exists $b \in J$ with $P \in \mathrm{supp}(b)$, so $b \notin P$. But $J \subseteq P$, then $b \in P$, contradiction. $\square$

Corollary 10.10. Let $a, b \in K$, then $\langle a \rangle = \langle b \rangle$ (a and b generate the same radical tt-ideal) if and only if $\mathrm{supp}(a) = \mathrm{supp}(b)$. More precisely, $b \in \langle a \rangle$ if and only if $\mathrm{supp}(b) \subseteq \mathrm{supp}(a)$.

This theorem says that the spectrum gives the classification. There is a corresponding inverse result, which says the classification gives the spectrum.

Problem 10.11 (Take-home Problem 4)**.** Suppose that the poset of radical tt-ideals of K is totally ordered (i.e., for every two I, J we either have $I \subseteq J$ or $J \subseteq I$), then every proper tt-ideal is prime.

11 Classical Examples

So far: let K be an essentially small tt-category, then $\mathbf{Spc}(K)$ is a spectral topological space (quasi-compact and quasi-separated and every irreducible closed subset has unique generic point) carrying a support data for K (closed $\mathrm{supp}(a) \subseteq \mathbf{Spc}(K)$ for all $a \in K$ satisfying certain rules), the universal one. We get a classification of (radical) tt-ideals, where the Thomason subsets of $\mathbf{Spc}(K)$ with radical tt-ideals.

Note that the correspondence above asks the space to be spectral. Moreover, if we have such a classification, the space would be spectral.[9]

Remark 11.1. If X is a spectral space, the Thomason subsets are the open subsets of another topology called the Hochster-dual topology. We usually denote X^* to be the space X with the dual topology. Hochster proved:

1. X^* is again spectral,

[9]This lecture follows Section 2 and 3 of the notes here.

2. $X^{**} = X$.

Remark 11.2. The lattice of thick (non-tensor) triangulated subcategories of a triangulated category is not *distributive* (i.e., $I_1 \wedge (I_2 \vee I_3) = (I_1 \wedge I_2) \vee (I_1 \wedge I_3)$) in general, and in particular cannot be in bijection with the lattice of open subsets of any space.

Example 11.3. Let $K = D_{\text{perf}}(P^1_k)$ for k a field. We think of the sheaves $\mathcal{O}$ as $\mathbb{1}$, then note that we have $\mathcal{O}(n)$ for $n \in \mathbb{Z}$. Looking at the Koszul sequence

$$\mathcal{O}(n) \rightarrowtail \mathcal{O}(n+1) \oplus \mathcal{O}(n+1) \twoheadrightarrow \mathcal{O}(n+2),$$

then any thick triangulated subcategory containing $\mathcal{O}(n)$ and $\mathcal{O}(n+1)$for some n contains K.

In fact, we look at the thick subcategory of $\mathcal{O}(n)$ and thick subcategory of $\mathcal{O}(m)$, for $n \neq m$, their intersection is 0. Therefore, if we let I_j be the thick subcategory of $\mathcal{O}(j)$, then $I_1 = I_1 \wedge (I_2 \vee I_3) = I_1 \wedge K \neq (I_1 \wedge I_2) \vee (I_1 \wedge I_3) = 0$.

Theorem 11.4 (Converse of Classification of tt-ideals)**.** Let K be an essentially small tt-category and (X, σ) be a support data on K such that

1. X is a spectral space,
2. the map $Y \mapsto \{a \mid \sigma(a) \subseteq Y\}$ and $J \mapsto \bigcup\limits_{a \in J} \sigma(a)$ yield a bijection between the Thomason subsets of X and the radical tt-ideals of K,

then the canonically continuous map $f : X \to \mathbf{Spc}(K)$ (such that $f^{-1}(\text{supp}(a)) = \sigma(a)$ for all $a \in K$) is a homeomorphism.

Proof Outline. Note that there is a correspondence between the radical tt-ideals and the Thomason subsets of $\mathbf{Spc}(K)$ (via the classification theorem) and a correspondence between the radical tt-ideals and the Thomason subsets of X (given by the hypothesis), then we can look for the map from the Thomason subsets of $\mathbf{Spc}(K)$ and the Thomason subsets of X via $Y \mapsto f^{-1}(Y)$. In particular, the diagram now commutes. Now $f : X \to \mathbf{Spc}(K)$ gives a bijection on the dual open subsets of those two spectral spaces.

Exercise 11.5. $g : Y \to Z$ is a continuous map of spectral spaces that induces a bijection $U \mapsto g^{-1}(U)$ on open subsets, then g is a bijection, thus a homeomorphism.

Taking the dual again, we get the map. □

Example 11.6. Let SH^c be the stable homotopy category of finite pointed CW-complexes. Let p be a prime. Consider $K = (\mathrm{SH}^c_{(p)})$, the p-local version: invert all primes different from p.

The unit $\mathbb{1} = S^0$, the sphere spectrum, generates the category $\mathrm{SH}^c_{(p)}$ as a thick triangulated subcategory. Consequently, every thick subcategory is automatically a tt-ideal.

Theorem 11.7 (Hopkins-Smith)**.** The complete list of thick subcategories of K is as follows:

$$K = C_0 \supsetneq C_1 \supsetneq \ldots \supsetneq C_n \supsetneq C_{n+1} \supsetneq \ldots \supsetneq 0 = C_\infty,$$

i.e., $\bigcap\limits_{n\geq 1} C_n = 0$. Hence, $\mathbf{Spc}(\mathrm{SH}^c_{(p)})$ is just those sequences with $\overline{\{C_n\}} = \{C_m \mid m \geq n\}$. This space is not Notherian: $\{C_\infty\}$ is closed with non-quasi-compact open complement. Globally, $\mathbf{Spc}(\mathrm{SH}^c)$ looks like

$$\begin{tikzcd}
C_{2,\infty} \arrow[d, dash] & C_{3,\infty} \arrow[d, dash] & \cdots & C_{n,\infty} \arrow[d, dash] & \cdots \\
\vdots \arrow[d, dash] & \vdots \arrow[d, dash] & \vdots & \vdots \arrow[d, dash] & \vdots \\
C_{2,m} \arrow[d, dash] & C_{3,m} \arrow[d, dash] & \cdots & C_{n,m} \arrow[d, dash] & \cdots \\
\vdots \arrow[d, dash] & \vdots \arrow[d, dash] & \vdots & \vdots \arrow[d, dash] & \vdots \\
C_{2,2} \arrow[drr, dash] & C_{3,2} \arrow[dr, dash] & \cdots & C_{n,2} \arrow[dl, dash] & \cdots \\
 & & C_1 & &
\end{tikzcd}$$

12 Localization of Triangulated Category

We now try to invert maps in a category to get an initial category in which those maps are isomorphisms. That is, for $\mathcal{C}$ a triangulated category and S a collection of morphisms, we want $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ such that $Q(S)$ is a collection of isomorphisms, and for all $F : \mathcal{C} \to \mathcal{D}$ such that $F(S)$ is a collection of isomorphisms, there exists a unique (up to isomorphism) $\bar{F} : \mathcal{C}[S^{-1}] \to \mathcal{D}$ such that $\bar{F} \circ Q \cong F$:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "Q"'] & & \mathcal{D} \\
 & \mathcal{C}[S^{-1}] \arrow[ur, dashed, "\exists ! \bar{F}"'] &
\end{tikzcd}$$

To be more useful: we say a functor $Q : \mathcal{C} \to \bar{\mathcal{C}}$ is a localization if $\bar{\mathcal{C}} = \mathcal{C}[S^{-1}]$ where $S = Q^{-1}\{(\text{ isomorphisms })\}$.

A rough solution is to use long (finite) zig-zags to create the relations. This would be almost useless. An alternative solution is to do ore calculus of fractions: some S allow for every morphism in $\mathcal{C}[S^{-1}]$ to be of the form $\cdot \xleftarrow{s} \cdot \xrightarrow{f} \cdot$ (with only one denominator) which allows composition of fractions and relations such as amplification (see 212A notes). However, this would not make the diagram commute usually because $F(S)$ is usually not a subset of the isomorphisms.

Can we move away from F (either via left or right derived functors) to a functor $L \Rightarrow F \Rightarrow R$) that does factor via $\mathcal{C}[S^{-1}]$? Again, this is to say we want the derived functors $L(S)$ and $R(S)$ to be sets of isomorphisms, i.e., $L \cong \tau \circ Q / R = \bar{R} \circ Q$, i.e., we have $\tau \circ Q \Rightarrow F \Rightarrow \bar{R} \circ Q$. That is, we want $\tau \circ Q$ to be as terminal as possible (and $R = \bar{R} \circ Q$ as initial as possible).

If there exists $LF : \mathcal{C}[S^{-1}] \to \mathcal{D}$ and $\lambda : LF \circ Q \Rightarrow F$ such that for all $G : \mathcal{C}[S^{-1}] \to \mathcal{D}$ and $\gamma : G \circ Q \Rightarrow F$, there exists unique $\bar{\gamma} : G \Rightarrow LF$ such that

$$\begin{tikzcd} G \circ Q \arrow[rr, Rightarrow, "\bar{\gamma} \circ Q"] \arrow[dr, Rightarrow, "\gamma"'] & & LF \circ Q \arrow[dl, Rightarrow, "\lambda"] \\ & F & \end{tikzcd}$$

we call LF, i.e., (LF, λ) the left derived functor of F with respect to S.

Proposition 12.1. Given $F : \mathcal{C} \to \mathcal{D}$ with an adjoint $G : \mathcal{D} \to \mathcal{C}$ (either left or right), then F is a localization if and only if G is fully faithful.

These are materials discussed in 212A already. We now discuss the triangulated setting.

Suppose $F : \mathcal{T} \to \mathcal{S}$ be an exact functor of triangulated categories. For $s : A \to B$ in $\mathcal{T}$, we have $F(s)$ isomorphism if and only if $\operatorname{cone}(F(s)) = 0$ (so $F(\operatorname{cone}(s)) = 0$) if and only if $\operatorname{cone}(s) \in F^{-1}(0) = \{C \mid F(C) \cong 0\}$.

Let $J = F^{-1}(0)$, then J is a thick triangulated subcategory of $\mathcal{T}$.

Remark 12.2 (Verdier Localization)**.** Let $J \subseteq \mathcal{T}$ be a thick subcategory and $\mathcal{T}/J = T[S^{-1}]$ where $S = S(J)$. Define $S(J) = \{s \mid \operatorname{cone}(s) \in J\}$, then $S = S(J)$ satisfies Ore Calculus:

- $S \ni \mathrm{id}$, $S \circ S \subseteq S$, and S satisfies the 2-out-of-3 property.
- Turn-right-into-left: given $X \xrightarrow{f} Y \xleftarrow{s} Z$, construct the homotopy pullback W with $t : W \to X$ and $-g : W \to Z$, then we obtain the diagram

$$\begin{tikzcd} & Z \arrow[r, "s"] \arrow[d, "(1\ 0)^T"] & Y \arrow[d, equal] & \\ W \arrow[r, "(t\ g)^T"] & X \oplus Z \arrow[r, "(f\ s)"'] & Y \arrow[r, dashed, "h"] & \Sigma W \end{tikzcd}$$

with $\operatorname{cone}(t) \cong \operatorname{cone}(s)$, so $t \in S(J)$ too.

- Composition with triangles: $\Sigma(S) = S$.
- If $s, t \in S$ are maps of exact triangles

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "s"] & Y \arrow[r] \arrow[d, "t"] & Z \arrow[r] & \Sigma X \arrow[d, "\Sigma s"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X'
\end{tikzcd}$$

then there exists $u \in S$ such that (s, t, u) is a morphism.

We then get $Q : \mathcal{T} \to \mathcal{T}[S^{-1}]$ to be identity on objects: $f : X \to y$ maps to $x \xleftarrow{=} x \xrightarrow{f} y$.

Define a triangle in $\mathcal{T}[S^{-1}]$ to be exact if it is isomorphic to Q adjoined by the exact triangles in $\mathcal{T}$. This makes $\mathcal{T}[S^{-1}]$ a triangular localization.

Remark 12.3. $s \in Q^{-1}(\cong)$ if and only if $s \in S(J)$ because J is thick, and for similar reasons $Q(C) = 0$ if and only if $C \in J$.

Remark 12.4. If J is a tt-ideal, then the quotient $\mathcal{T}/J = \mathcal{T}[S(J)^{-1}]$ inherits a tensor product because every $A \in \mathcal{T}$ satisfies $A \otimes S(J) \subseteq S(J)$.

Remark 12.5. Take $P \in \mathbf{Spc}(K)$ for K an essentially small tt-category, we can consider $K \twoheadrightarrow K/P$, which gives the local category K at P.

Problem 12.6 (Take-home Problem 5)**.** Let R be a commutative ring and $K = K_b(R\text{-}\mathbf{Proj})$ and $\varphi \in \operatorname{Spec}(R)$. Consider the localization at φ, i.e., $(-)_\varphi : K(R) \to K(R_\varphi)$. Check that $P(\varphi) = \ker(K(R) \to K(R_\varphi))$ is prime and $K(R)/P(\varphi) \cong K(R_\varphi)$.

Note that there is an isomorphism $\operatorname{Spec}(R) \cong \mathbf{Spc}(K(R))$ mapping from φ to $P(\varphi)$.

Remark 12.7. For $x \in K$ and $P \in \mathbf{Spc}(K)$, we have $K \to K/P$ given by $x \mapsto 0$ (i.e., $x = 0$ locally at P), which is just saying $x \in P$. Thus, x is non-zero localization at P if and only if $x \notin P$, i.e., $P \in \operatorname{supp}(x)$.

13 Localization, Continued

For reference, see Krause's "localization theory for triangulated categories".

Recall: sometimes the localization functor $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ admits a right adjoint, which is fully faithful. The idea is to note that $\mathcal{C}[S^{-1}]$ is realized as a subcategory of $\mathcal{C}$.

Proposition 13.1. Let $\mathcal{T}$ be a triangulated category and $J \subseteq \mathcal{T}$ a thick triangulated subcategory. Consider $Q : \mathcal{T} \to \mathcal{T}/J = \mathcal{T}[S^{-1}]$ where $S = S(J) = \{s \mid \mathrm{cone}(s) \in J\}$. The following are equivalent:

1. Q has a right adjoint.

2. $\mathrm{inc} : J \hookrightarrow \mathcal{T}$ has a right adjoint.

3. every object $t \in \mathcal{T}$ fits in an exact triangle $x \to t \to y \to \Sigma x$ where $x \in J$, $y \in J^{\perp} = \{y \in \mathcal{T} : \mathbf{Hom}(j, y) = 0\ \forall j \in J\}$.

When this happens, the triangle in (3) is functorial in t, in fact, for all $f : t \to t'$ and for exact triangles where $x, x' \in J$ and $y, y' \in J^{\perp}$

$$\begin{tikzcd} x \arrow[r] \arrow[d, dashed, "\exists! g"'] & t \arrow[r] \arrow[d, "f"] & y \arrow[r] \arrow[d, dashed, "\exists! h"] & \Sigma x \arrow[d, dashed, "\Sigma g"] \\ x' \arrow[r] & t' \arrow[r] & y' \arrow[r] & \Sigma x' \end{tikzcd}$$

this induces unique g and h. In particular, the triangle is unique up to unique isomorphism of the form $(*, \mathrm{id}, *)$. In fact, $x : \mathcal{T} \to J$ and $y : \mathcal{T} \to J^{\perp}$ yield adjoints

$$\begin{tikzcd} J \arrow[d, shift right, hook, "\mathrm{inc}"'] \\ \mathcal{T} \arrow[u, shift right, two heads, "x"'] \arrow[d, shift right, two heads, "y"'] \\ J^{\perp} \arrow[u, shift right, hook, "\mathrm{inc}"'] \end{tikzcd}$$

and $\mathrm{inc} \circ x \xrightarrow{\varepsilon} \mathrm{id} \xrightarrow{\eta} \mathrm{inc} \circ y \xrightarrow{\omega} \Sigma \mathrm{inc} \circ x$ is a functorial exact triangle. Finally, $x : \mathcal{T} \to J$ realizes $\mathcal{T}/J^{\perp}$ and $y : \mathcal{T} \to J^{\perp}$ realizes $\mathcal{T}/J$, i.e., the adjoints $Q \dashv R : \mathcal{T} \Leftrightarrow \mathcal{T}/J$ and $x \dashv \mathrm{inc} : \mathcal{T} \Leftrightarrow J^{\perp}$ induces an equivalence $\mathcal{T}/J \cong J^{\perp}$ given by the mappings R and $Q \circ \mathrm{inc}$.

Remark 13.2 (Key Observation)**.** Just using (iii) under the assumption $\Sigma(J) \subseteq J$, we can finish the proof. We now show the existence of the morphisms. Given two exact triangles where $x, x' \in J$ and $y, y' \in J^{\perp}$,

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d, dashed, "\exists! g"'] & t \arrow[r, "\beta"] \arrow[d, "f"] & y \arrow[r, "\gamma"] \arrow[d, dashed, "\exists! h"] & \Sigma x \arrow[d, dashed, "\Sigma g"] \\ x' \arrow[r, "\alpha"] & t' \arrow[r, "\beta"] & y' \arrow[r, "\gamma"] & \Sigma x' \end{tikzcd}$$

Note that g is induced by $\beta' f \alpha = 0$ and the fact that $\mathbf{Hom}(x, y') = 0$, and h is induced by the morphism axiom. To show the uniqueness, look at

$$\begin{tikzcd} x \arrow[r, "\alpha"] \arrow[d, dashed, "g"'] & t \arrow[r, "\beta"] \arrow[d, "0"] & y \arrow[r, "\gamma"] \arrow[d, dashed, "h"] & \Sigma x \arrow[d, dashed, "\Sigma g"] \\ x' \arrow[r, "\alpha"] & t' \arrow[r, "\beta"] & y' \arrow[r, "\gamma"] & \Sigma x' \end{tikzcd}$$

Obviously $g = 0$. To see $h = 0$, we note that the diagram induces $h\beta = 0$, and $\bar{h} : \Sigma x \to y'$ via factoring such that $h = \bar{h}\gamma$, so $\mathbf{Hom}(\Sigma x, y') = 0$, so $h = 0$.

Example 13.3. For Abelian category $\mathcal{A}$ and $J = K_{ac}(\mathcal{A}) \subseteq K(\mathcal{A}) = \mathcal{T}$, where K_{ac} is the category of acyclic complexes (i.e., with zero homology), then $S(J)$ is the set of quasi-isomorphisms. Now $K(\mathcal{A})/K_{ac}(\mathcal{A}) = K(\mathcal{A})[\text{ quasi-isomorphisms}^{-1}] = D(\mathcal{A})$.

If $\mathcal{A}$ is Grothendick, for example $\mathcal{A} = R$-**Mod**, then (3) holds (non-trivial fact).

In cash, for every complex $t \in K(\mathcal{A})$, i.e., in $\mathbf{Ch}(\mathcal{A})$, we need $\beta : t \to y$ to be a quasi-isomorphism with $y \in K_{ac}(A)^{\perp} = \{y \mid \mathbf{Hom}(x, y) = 0 \ \forall x \in K_{ac}(\mathcal{A})\} \supseteq K^{+}(\text{inj}(\mathcal{A})) = \{\cdots \to 0 \to I^n \to I^{n+1} \to \cdots\}$, where $K_{ac}(\mathcal{A})^{\perp}$ is the category of K-injectives.

Then the category of K-injectives, i.e., $\mathbf{KInj}(\mathcal{A}) = (K_{ac}(\mathcal{A}))^{\perp} \cong D(\mathcal{A})$. This motivates an isomorphism $K^{+}(\mathbf{Inj}(\mathcal{A})) \cong D^{+}(\mathcal{A})$.

Remark 13.4. Similar stories work on the left of t. If every $t \in \mathcal{T}$ fits in an exact triangle $w \to t \to z \to \Sigma w$ with $z \in J$ and $w \in {}^{\perp}J = \{w \in \mathcal{T} \mid \mathbf{Hom}(w, j) = 0 \ \forall j \in J\}$. This gives a notion of K-projectives:

$$\begin{tikzcd} J \arrow[d, shift left, "\text{inc}"] \\ \mathcal{T} \arrow[u, shift left, two heads, "z"] \arrow[d, shift left, two heads, "w"] \\ {}^{\perp}J \cong \mathcal{T}/J \arrow[u, shift left, "\text{inc}"] \end{tikzcd}$$

Let us go back to the setting described in Proposition 13.1, suppose $Q : \mathcal{T} \to \mathcal{T}[S^{-1}]$ admits a right adjoint R. What about derived functors? Let $J = \ker(Q)$ and we can ask $S = S(J) = Q^{-1}(\cong)$. Then consider an exact functor $F : \mathcal{T} \to \mathcal{S}$ with

$$\begin{tikzcd} J \arrow[d, shift right, "\text{inc}"'] & \\ \mathcal{T} \arrow[u, shift right, two heads, "x"'] \arrow[r, "F"] \arrow[d, shift right, two heads, "y"'] & \mathcal{S} \\ \mathcal{T}/J = J^{\perp} \arrow[u, shift right, "\text{inc}"'] \arrow[ur, "RF=F\circ\text{inc}=F\circ R"'] & \end{tikzcd}$$

so F localizes (i.e., factors via Q) if and only if $F(J) = 0$, i.e., $F(S) \subseteq \{\cong\}$. This induces the unit $\eta : \mathrm{id} \Longrightarrow \mathrm{inc} \circ y = R \circ Q$, and so $F\eta : F \Longrightarrow F \circ R \circ Q = RF \circ Q$.

14 Brown-Neeman Representability

In this section, we assume $\mathcal{T}$ is a triangulated category with arbitrary coproducts.

Definition 14.1 (Compact)**.** An object $c \in \mathcal{T}$ is called compact if $\mathbf{Hom}(c, -) : \mathcal{T} \to \mathbf{Ab}$ commutes with coproducts. Equivalently, for any morphism $c \to \coprod_{i \in I} t_i$ and the inclusion of finite subcover $\coprod_{j=1}^{n} t_{i_j} \hookrightarrow \coprod_{i \in I} t_i$, there exists a morphism such that the diagram commutes:

$$\begin{tikzcd} c \arrow[rr] \arrow[dr, dashed] & & \coprod_{i\in I} t_i \\ & \coprod_{j=1}^{n} t_{i_j} \arrow[ur, hook] & \end{tikzcd}$$

We denote $\mathcal{T}^c$ to be the collection of compact objects in $\mathcal{T}$. Let $\mathcal{G}$ be a set of compact objects (closed under $\oplus$, Σ). (Later on, we will note $\mathcal{G} = \mathcal{T}^c$.) Consider the restricted Yoneda embedding $\mathcal{T} \to \mathbf{Add}(\mathcal{G}^{op}, \mathbf{Ab})$, then there is a diagram

$$\begin{tikzcd} \mathcal{T} \arrow[r] \arrow[dr, "\text{restricted Yoneda}"'] & \mathbf{Add}(\mathcal{T}^{op}, \mathbf{Ab}) \arrow[d, "\mathrm{res}_{\mathcal{G}}"] \\ & \mathbf{Add}(\mathcal{G}^{op}, \mathbf{Ab}) \end{tikzcd}$$

mapping $X \in \mathcal{T}$ to $\mathbf{Hom}_{\mathcal{T}}(-, X) =: y(X) \in \mathbf{Add}(\mathcal{T}^{op}, \mathbf{Ab})$ and into $\mathbf{Hom}(-, X) \mid_{\mathcal{G}} = \hat{X} \in \mathbf{Add}(\mathcal{G}^{op}, \mathbf{Ab})$, giving objectwise limits and colimits. The restricted Yoneda map preserves coproducts. Since it is homological (maps exact triangles to long exact sequences), we note that the key condition is having the functor as conservative, which is equivalent to the fact that if $\mathbf{Hom}(c, X) = 0$ for all $c \in \mathcal{G}$ then $X = 0$, which is equivalent to $\mathcal{G}^{\perp} = 0$.

Definition 14.2 (Compactly Generated)**.** $\mathcal{T}$ is compactly generated if there exists a set $\mathcal{G}$ of compacts such that $\mathcal{G}^{\perp} = 0$ and $\mathcal{T}$ has small coproducts.

Definition 14.3 (Localizing)**.** A subcategory $\mathcal{L} \subseteq \mathcal{T}$ is called localizing if it is triangulated (and thick, automatically) and closed under coproducts. We denote $\mathrm{Loc}(\mathcal{E})$ as the intersection of $\bigcap_{\substack{\mathcal{L} \text{ localizing} \\ \text{such that } \mathcal{L} \supseteq \mathcal{E}}} \mathcal{L}$ the localizing subcategory generated by $\mathcal{E}$.

Remark 14.4. $\mathrm{Loc}(\mathcal{G}) = \mathcal{T}$ implies $\mathcal{G}^{\perp} = 0$.

Proof. Indeed, for $X \in \mathcal{G}^{\perp}$, we have $\mathcal{G} \subseteq {}^{\perp}X$ which is always localizing, so $X \in \mathcal{T} = \mathrm{Loc}(\mathcal{G}) \subseteq {}^{\perp}X$, therefore $X = 0$. $\square$

Remark 14.5. The inverse of Remark 14.4 is true if $\mathcal{G} \subseteq \mathcal{T}^c$.

Now given $L : \mathcal{T} \to \mathcal{S}$ with a right adjoint $L \dashv R$, we note $\mathbf{Hom}_{\mathcal{S}}(L(-), s) \cong \mathbf{Hom}_{\mathcal{T}}(-, R(s))$. We may ask: which functors $F : \mathcal{T}^{op} \to \mathbf{Ab}$ are representable, i.e, there exists $X \in \mathcal{T}$ such that $F \cong y(X)$?

Theorem 14.6 (Brown Representability Theorem, due to Neeman)**.** Let $\mathcal{T}$ be a compactly generated triangulated category. Let $F : \mathcal{T}^{op} \to \mathbf{Ab}$ be a cohomological functor, i.e., sending exact triangles to long exact sequences, and turns coproducts into products, i.e., $F(\coprod_i t_i) = \prod_i F(t_i)$, then F is representable, i.e., $F \cong \mathbf{Hom}_{\mathcal{T}}(-, X)$ for some object $X \in \mathcal{T}$ (essentially in $\mathrm{Loc}(\mathcal{G})$.[10]

Proof. First observe that there exists $X_0 \in \mathcal{T}$, a coproduct of objects of $\mathcal{G}$, and a natural transformation $f_0 : \hat{X}_0 = \mathbf{Hom}_{\mathcal{T}}(-, X_0) \Rightarrow F : \mathcal{T}^{op} \to \mathbf{Ab}$ which is a $\mathcal{G}$-epimorphism, i.e., $f_0 \mid_{\mathcal{G}}$ is an epimorphism. To see this, just take $\mathcal{G} = \Sigma\mathcal{G}$.

Let $X_0 = \coprod_{\substack{c \in \mathcal{G} \\ \alpha \in F(c)}} c$. Note that this induces a canonical element $\tilde{\alpha}$ in $F(X_0) \cong \prod_{(c,\alpha)} F(c) \ni (\alpha)_{(c,\alpha)} =: \tilde{\alpha}$. By the Yoneda Lemma, $\tilde{\alpha}$ yields a map $f_0 : \hat{X}_0 \Rightarrow F$, since f_0 is a $\mathcal{G}$-epimorphism.

We obtain an exact sequence in the presheaves: $\ker(f_0) \to \hat{X}_0 \xrightarrow{f_0} F$. By applying our observation to $\ker(f_0)$, this induces a diagram

$$\begin{tikzcd}
\hat{Y}_1 \arrow[rr, "\hat{g}_1"] \arrow[dr, two heads] & & \hat{X}_0 \arrow[dr, "f_0"] & \\
& \ker(f_0) \arrow[ur, tail] & & F
\end{tikzcd}$$

Looking at the restrictions at $\mathcal{G}$, we have

$$\begin{tikzcd}
\hat{Y}_1 \mid_{\mathcal{G}} \arrow[r, "\hat{g}_1"] & \hat{X}_0 \mid_{\mathcal{G}} \arrow[dr, "f_0"] & \\
& & F \mid_{\mathcal{G}}
\end{tikzcd}$$

[10]See the detailed paper here.

Now using the Yoneda Lemma, we have Y_1 as a coproduct of objects of $\mathcal{G}$ such that there is a map $g_1 : Y_1 \to X_0$, now taking the cone of this map, we obtain

$$Y_1 \xrightarrow{g_1} X_0 \xrightarrow{h_1} X_1 = \mathrm{cone}(g_1)$$

and this construction makes X_0 vanish. Note that there is $F(X_1) \to F(X_0) \to F(Y_1)$ given by $\tilde{f}_1 \mapsto \tilde{f}_0 \mapsto 0$, which induces a commutative extension on the restriction diagram, and in turn induces the commutative diagram

$$\begin{tikzcd}
\hat{Y}_1 \arrow[rr, "\hat{g_1}"] \arrow[dr, two heads] & & \hat{X}_0 \arrow[rr, "\hat{h_1}"] \arrow[dr, "f_0"] & & \hat{X}_1 \arrow[dl, dashed, "f_1"] \\
& \ker(f_0) \arrow[ur, tail] & & F &
\end{tikzcd}$$

By induction, we get a diagram in $\mathcal{T}$:

$$\begin{tikzcd}
Y_1 \arrow[d, "g_1"] & Y_2 \arrow[d, "g_2"] & \cdots & \cdots & Y_{n+1} \arrow[d, "g_{n+1}"] & \cdots & \cdots \\
X_0 \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & \cdots \arrow[r] & X_n \arrow[r] & X_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

where Y_i's are in $\mathcal{G}^{\sqcup}$, constructed from the kernel just like the ones above, and $X_i = \mathrm{cone}(g_i)$ for all $i \geq 1$. Note that $X_n \in \mathrm{Loc}(\mathcal{G})$ for all n. Let X be the homotopy colimit of X_n.[11] Using the fact that F is cohomological and preserving coproducts, we get $f : \hat{X} \Rightarrow F$ making all the diagrams commute, i.e., such that

$$\begin{tikzcd}
\hat{X}_n \arrow[rr] \arrow[dr, "f_n"'] & & \hat{X} \arrow[dl, "f"] \\
& F &
\end{tikzcd}$$

commutes for all n. (Note that F is the direct limit of $\tilde{X}_n$'s, and note that each f_i is an epimorphism.) In particular, f is a $\mathcal{G}$-epimorphism.

Now observe that $\hat{X} \mid_{\mathcal{G}}$ is isomorphic to the colimit of $\hat{X}_n$. One can then show that $f \mid_{\mathcal{G}}$ is a monomorphism (by construction).[12]

Remark 14.7 (Sidenote)**.** If $F = \bar{Z}$ for some $Z \in \mathcal{T}$, we found $X \in \mathrm{Loc}(\mathcal{G})$ such that $\hat{f} : \hat{X} \to \hat{Z}$ is a $\mathcal{G}$-isomorphism, therefore $\mathrm{cone}(f) \in \mathcal{G}^{\perp} = 0$, so f is an isomorphism. Therefore, $\mathrm{Loc}(\mathcal{G}) = \mathcal{T}$.

Then a $\mathcal{G}$-isomorphism between two cohomological coproduct-preserving functors $F_1 \to F_2$ is an isomorphism: $\{t \in \mathcal{T} \mid F_1(t) \xrightarrow{\cong} F_2(t)\}$ is triangulated and coproduct-preserving, i.e., localizing. But this set contains $\mathcal{G}$, so it contains $\mathrm{Loc}(\mathcal{G}) = \mathcal{T}$, proving the theorem. $\square$

[11] See Remark 15.1.

[12] This is also known as the "small object argument".

15 Sequential Homotopy Colimits

Remark 15.1 (Construction of Homotopy Colimit)**.** Let $\mathcal{T}$ be a triangulated category with coproducts. Let

$$X_0 \xrightarrow{h_0} X_1 \xrightarrow{h_1} X_2 \xrightarrow{h_2} \cdots \longrightarrow X_n \xrightarrow{h_n} \cdots$$

be a sequence of morphisms in $\mathcal{T}$.

If $\mathcal{C}$ is cocomplete and $I \to \mathcal{C}$ maps $i \in I$ to $X_i \to \mathcal{C}$, then there is

$$\coprod_{\alpha: i \to j \in I} X_i \overset{\mathbf{id}}{\underset{\alpha}{\rightrightarrows}} \coprod_{i \in I} X_i \longrightarrow \operatorname{colim}(X_i) \longrightarrow 0$$

In particular, if $I = \mathbb{N}$ and $\mathcal{C}$ is additive, then we have a cokernel diagram

$$\coprod_{i \in \mathbb{N}} X_i \xrightarrow{\mathbf{id} - h_i} \coprod_{i \in I} X_i \longrightarrow \operatorname{colim}(X_i) \longrightarrow 0$$

where $\mathbf{id} - h_i$ is a split monomorphism on every finite sum $\bigoplus_{i=1}^{n} X_i$. Therefore, the colimit is given by the cone.

We now consider an exact triangle on the morphism $\mathbf{id} - h_i$.

$$\begin{tikzcd}
X_i \arrow[r, "(1\ -h_i)^T"] \arrow[d, hook] & X_i \oplus X_{i+1} \arrow[d, hook] & & \\
\coprod_{i \in \mathbb{N}} X_i \arrow[r, "\mathbf{id} - h_i"] & \coprod_{i \in I} X_i \arrow[r] & \operatorname{colim}(X_i) \arrow[r] & 0
\end{tikzcd}$$

and the colimit of X_i in this case is the homotopy colimit $\operatorname{hocolim}(X_i)$ we want. In particular, the homotopy colimit is the direct limit of the sequence $(X_i)_{i \in I}$.

Remark 15.2 (Behavior of Yoneda)**.** For every $c \in \mathcal{T}^c$, applying $\mathbf{Hom}_{\mathcal{T}}(c, -)$ to the triangle above gives a short exact sequence

$$0 \longrightarrow \coprod_i \mathbf{Hom}(c, X_i) \longrightarrow \coprod_i \mathbf{Hom}(c, X_i) \longrightarrow \mathbf{Hom}(c, \operatorname{hocolim}(X_i)) \longrightarrow 0$$

Therefore,

$$\operatorname{colim}(\mathbf{Hom}(c, X_i)) \cong \mathbf{Hom}(c, \operatorname{hocolim}(X_i)).$$

Corollary 15.3. In

$$\begin{tikzcd}
\mathcal{T} \arrow[r] \arrow[dr, "\text{restricted Yoneda}"'] & \mathbf{Add}(\mathcal{T}^{op}, \mathbf{Ab}) \arrow[d, "\operatorname{res}_{\mathcal{G}}"] \\
& \mathbf{Add}(\mathcal{G}^{op}, \mathbf{Ab})
\end{tikzcd}$$

we obtain $\widehat{\operatorname{colim}(X_i)} \mid_{\mathcal{G}} \cong \operatorname{colim}(\hat{X}_i \mid \mathcal{G})$.

Remark 15.4. Now if $F : \mathcal{T}^{op} \to \mathbf{Ab}$ is cohomological, and F takes coproducts to products, then applying functor F obtains

$$\prod_{i\in\mathbb{N}} F(X_i) \xleftarrow{\mathbf{id}-h} \prod_{i\in I} F(X_i) \longleftarrow F(\mathrm{hocolim}(X_i)) \qquad \cdots$$

Therefore, if we have in $\mathbf{Add}(\mathcal{T}^{op}, \mathbf{Ab})$ a collection of $f_i : \hat{X}_i \to F$, we note that the homotopy colimit is compatible with the systems h_i (and $\hat{h_i}$), then we induce a (not necessarily unique) morphism $f_\infty : \widehat{\mathrm{colim}(X_i)} \to F$ such that the directed system commutes.

Therefore, the key fact in Brown Representability Theorem is that in $\mathbf{Add}(\mathcal{G}^{op}, \mathbf{Ab})$, the same diagram (of direct system with homotopy colimit) commutes with restrictions on $\mathcal{G}$.

Remark 15.5 (Final Remark on Brown Representability Theorem)**.** Note that the triangle above induces mapping $X \mapsto \hat{X} = \mathbf{Hom}(-, X)$, and the mapping $F \mapsto F \mid_{\mathcal{G}}$, and note that there is a mapping $f_\infty : \hat{X} \mapsto F$, and by our construction it is an isomorphism, and therefore the diagram flows with representability.

Remark 15.6. We can adapt the proof (with $\mathcal{T}$ compactly generated) above to so-called "perfectly generated" category $\mathcal{T}$ (because compactly generated implies perfectly generated), and note that the opposite category $\mathcal{T}^{op}$ would also be perfectly generated.

Moreover, the proof implies that $\mathrm{Loc}(\mathcal{E})$ is perfectly generated, as long as $\mathcal{E}$ is a set of objects. In particular, if $\mathcal{T}$ is compactly generated, then the inclusion map $\mathrm{Loc}(\mathcal{E}) \to \mathcal{T}$ admits a right adjoint.

Corollary 15.7. Let $\mathcal{T}$ be compactly generated and $F : \mathcal{T} \to \mathcal{S}$ is a functor such that F is exact and preserves coproducts (assuming $\mathcal{S}$ has coproducts), then F has a right adjoint.

Proof. $\mathbf{Hom}_{\mathcal{S}}(F(-), s) = \mathbf{Hom}_{\mathcal{T}}(-, X(s))$. □

Remark 15.8. Also, any $F : \mathcal{T} \to \mathcal{S}$ that preserves products (assuming $\mathcal{S}$ has coproducts) and is exact has a left adjoint.

16 Local tt-Categories

Remark 16.1 (What should it mean for an essentially small tt-category to be local?)**.** In commutative algebra, a commutative ring R is local if $\mathfrak{m} = R \backslash R^\times$ is the unique maximal ideal.

Note that geometrically, R is local if and only if $\mathrm{Spec}(R)$ has a unique closed point, if and only if $X = \mathrm{Spoec}(R)$ is a local space, i.e., for all $X = \bigcup_{i=1}^{n} U_i$ with all U_i's open, there exists i such that $X = U_i$.

Therefore, for a tt-category to be local, we would want $\mathbf{Spc}(K)$ to be a local space, i.e., it admits a unique closed point (as a minimal prime). In particular, in $\mathbf{Spc}(K)$, every non-empty closed Z contains a closed point.

Example 16.2. If P is the unique minimal prime, then P is the nilradical, i.e., $\{a \mid a^{\otimes n} = 0, n \geq 1\}$ or $\{a \in K \mid \mathrm{supp}(a) = \varnothing\}$.

Definition 16.3 (Local tt-category)**.** K is local if $\mathrm{Spc}(K)$ to be local, i.e., K admits the nilradical as a unique minimal prime, or alternatively, for all $a, b \in K$, if $a \otimes b = 0$, then either a or b is $\otimes$-nilpotent. (Note that if K is rigid, then $\otimes$-nilpotent is just 0.)

Example 16.4. If K is rigid, then K is lcoal if and only if $a \otimes b = 0$ implies $a = 0$ or $b = 0$.

Remark 16.5. If we can show that $\mathbf{Spc}(D_{perf}(X)) \cong |X|$, the underlying space of X, then $D_{perf}(X)$ is local if and only if $X = \mathrm{Spec}(R)$ for R local.

Proposition 16.6. $K(R) = K_b(R\text{-}\mathbf{Proj})$ is local if and only if R is local.

Proof. ($\Rightarrow$): Suppose $r, s \in R \backslash R^{\times}$. We want to show that $r + s$ is not invertible. Let a be the cone of $r : \mathbb{1} \to \mathbb{1}$ for $\mathbb{1} = R[0]$, then the cone is equivalent to $(0 \to R \xrightarrow{r} R \to 0)$, with the second R on position 0. Now the multiplication by r on a gives a homotopy on the complex:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "r"] \arrow[d, "r"] & R \arrow[r] \arrow[d, "r"] \arrow[dl, dashed, "1"] & 0 \\
0 \arrow[r] & R \arrow[r, "r"] & R \arrow[r] & 0
\end{tikzcd}$$

That is, $r \cdot \mathrm{id}_a = 0$, and similarly $s \cdot \mathrm{id}_b = 0$ for $b = \mathrm{cone}(s)$. Therefore, for $\mathrm{id}_{a \otimes b} = \mathrm{id}_a \otimes \mathrm{id}_b$, we have $r \cdot \mathrm{id}_{a\otimes b} = 0$ and $s \cdot \mathrm{id}_{a \otimes b} = 0$, thus $(r + s) \cdot \mathrm{id}_{a \otimes b} = 0$. Note that for $a, b \neq 0$, as $\mathrm{cone}(r) = 0$ implies r is invertible in $K(R)$, thus $r \in R^{\times}$, then since $K(R)$ is local we note that $a \otimes b \neq 0$, therefore $(r + s) \cdot \mathrm{id}_{a \otimes b}$ is not an isomorphism, hence $r + s$ is not contained in $R^{\times}$.

($\Leftarrow$): Structure fact about perfect complexes over local rings: every complex $a \in \mathbf{Ch}_b(R\text{-}\mathbf{Proj})$ is homotopy equivalent (isomorphic in $K(R)$) to a so-called minimal perfect complex:

$$0 \longrightarrow R^{n_p} \xrightarrow{\ d\ } \cdots \longrightarrow R^{n_1} \xrightarrow{\ d\ } R^{n_0} \xrightarrow{\ d\ } \cdots \longrightarrow R^{n_{-q}} \longrightarrow 0$$

where every $d_i : R^{n_i} \to R^{n_{i-1}}$ belongs to $M_{n_{i-1} \times n_i}(\mathfrak{m}) \subseteq M_{n_{i-1} \times n_i}(R)$, i.e., have all entries in $\mathfrak{m} = R \backslash R^{\times}$, unique up to isomorphism (since homotopy equivalence between minimal complexes are isomorphisms).

To see why a minimal perfect complex exists, note that there is an elementwise addition operation on the complexes, which reduces any complex here to one of the form above (because the other summand would be homotopic to zero).

For $k = R/\mathfrak{m}$, we obtain a tensor ($\otimes$) functor $F = k \otimes_R - : K(R) \to K(k)$. There is a shifting happening by mapping $\cdots \to R^{n_i} \to \cdots$ to $\cdots \xrightarrow{0} k^{n_i} \xrightarrow{0} \cdots$, which is just $\bigoplus_i k^{n_i}[i]$ with $[i]$ the shifting. It follows that F is conservative: $F(a) = 0$ implies $a = 0$. Finally, if $a \otimes b = 0$, then $F(a) \otimes F(b) = 0$ in $K(k)$, i.e., isomorphic to the k-graded modules, and so $F(a) = 0$ or $F(b) = 0$, hence $a = 0$ or $b = 0$. (This is given by the operation $k^n[p] \otimes k^m[q] = k^{n \cdot m}[p+q]$.) $\square$

17 Examples of Local tt-category

Recall: an essentially small tt-category K is local if $\mathbf{Spc}(K)$ is a local space, equivalently, admits a unique closed point $\sqrt{0}$. If K is rigid, then this means 0 is prime: $a \otimes b = 0$ implies $a = 0$ or $b = 0$.

Example 17.1. Let X be a quasi-compact and quasi-separable scheme (e.g., $\mathrm{Spec}(R)$ for commutative ring R), then X is local if and only if $X = \mathrm{Spec}(R)$ for R commutative local $\mathcal{O}_X(X)$.

Example 17.2. For every prime P, the Verdier quotient K/P is local.

Example 17.3. If $K = K_b(P\text{-}\mathbf{Proj}) = K(R)$, and prime $P = P(\varphi)$, then $K/P \cong K(R_P)$.

Example 17.4. For a prime p, the tt-category $\mathrm{SH}^c_{(p)} = \mathrm{SH}^c / \langle \mathrm{cone}(s) \mid p \nmid s, s \in \mathbb{Z} \rangle$ gives $\mathbf{Spc}(\mathrm{SH}^c_{(p)})$, presented as

$$\begin{tikzcd}[column sep=small]
 & (0) \arrow[d, dash] \\
 & \vdots \arrow[d, dash] \\
 & P_n \arrow[d, dash] \\
 & \vdots \arrow[d, dash] \\
 & P_1 \arrow[dl, dash] \\
P_0 &
\end{tikzcd}$$

as local. But SH^c is not: $\mathrm{cone}(p) \otimes \mathrm{cone}(q) = 0$ with $p \neq q$.

Example 17.5. Let G be a finite group and k a field, then $D_b(kG\text{-}\mathbf{mod}) = D_b(kG)$ is local. Indeed, $\mathrm{Res}_1^G : D_b(kG) \to D_b(k)$ is conservative: $\mathrm{Res}_1^G(a) = 0$ implies $a = 0$. In particular, $D_b(k)$ is a tt-field and is certainly local.

Remark 17.6. For $K = D_b(kG)$, $Y = * = (0)$ as the unique closed point, then K_Y is just the category of perfect complexes, i.e., $K_b(kG\text{-}\mathbf{Proj}) \subseteq K$.

Recall that $K_b(kG\text{-}\mathbf{mod})$ is Frobenius (as in Proposition 6.7), then similarly $K_b(kG\text{-}\mathbf{Proj})$ is also Frobenius: the tensor $kG \otimes_k - \cong \mathrm{Ind}_1^G \mathrm{Res}_1^G$.

Theorem 17.7 (Rickard)**.** $D_b(kG\text{-}\mathbf{mod})/K_b(kG\text{-}\mathbf{Proj}) \cong kG\text{-}\mathbf{mod}/kG\text{-}\mathbf{Proj} = kG\text{-}\mathbf{stab}$.

Remark 17.8 (When is kG-**stab** local?)**.** kC_p-**stab** is local, for example. It is enough to see that $M_i \otimes b = 0$ implies $b = 0$.

Remark 17.9. In general, $kC_p = k[x]/x^p - 1 = k[t]/t^p$ for $t = x - 1$. All the indecomposable objects in the category can be represented as $M_i = k[t]/t^i$ for $1 \leq i \leq p - 1$.

Problem 17.10 (Take-home Problem 6)**.** Take $K = \mathrm{stab}(kC_5)$ for $\mathrm{char}(k) = 5$. The object M_3 would have dimension (as in the trace of the identity) equals to $\frac{1 \pm \sqrt{5}}{2}$ (in any F-vector space).[13] In fact, $M_3 \otimes M_3 \cong \mathbb{1} \oplus M_3$ when formally speaking, i.e, $d^2 = d + 1$.

Remark 17.11. If $M \dashv M^*$ as adjoints, then the characteristic map is constructed via $\chi(M) : \mathbb{1} \to \mathbb{1}$ is constructed via $\mathbb{1} \xrightarrow{\eta} M^* \otimes M \cong M \otimes M^* \xrightarrow{\varepsilon} \mathbb{1}$. This map is characterized as $i \in K^\times \pmod{q}$ in K.

In fact, $\mathbb{1} \xrightarrow{\eta} M_i^* \otimes M_i \cong M_i \otimes M_i^* \xrightarrow{\varepsilon} \mathbb{1}$. Therefore, $M_i \otimes - : kC_p\text{-}\mathbf{stab} \to kC_p\text{-}\mathbf{stab}$ is a faithful functor, and $\mathbf{Spc}(kC_p\text{-}\mathbf{mod}) = \{(0)\}$.

Example 17.12 (Mackey Formula)**.** In kG-**stab**, we have for every subgroup $H \subseteq G$ the object $k(G/H)$ where G acts on the G-set G/H by left multiplication $g \cdot [x]_H = [gx]_H$.

$$k(G/H) \otimes_k k(G/K) \cong \bigsqcup_{[g] \in H \backslash G/K} k(G/H^g \cap K)$$

Example 17.13. For $G = C_p \times C_p$, let $H = C_p \times 1$, $K = 1 \times C_p$, then $H^g \cap K = 1$. Hence, $k(G/H) \otimes k(G/K)$ is projective in kG-**Mod**. Therefore, $a = k(G/H)$ and $b = k(G/K)$ in kG-**stab** satisfying $a \otimes b = 0$ but $a, b \neq 0$.

[13] In the sense of $\mathrm{End}(\mathbb{1})$, it has dimension 3.