Motivic Cohomology Notes

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0 Introduction

Let $X \in \mathrm{Sm}/k$ be a smooth separated scheme over a field k. The study of motivic cohomology started with the hope that

Conjecture 0.1 (Beilinson and Lichtenbaum, 1982-1987)**.** There exists some complexes $\mathbb{Z}(n)$ for $n \in \mathbb{N}$ of sheaves in Zariski topology on Sm/k such that

1. $\mathbb{Z}(0)$ is (quasi-isomorphic to) the constant sheaf $\mathbb{Z}$, i.e., the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$

concentrated at degree 0;

2. $\mathbb{Z}(1)$ is the complex $\mathcal{O}^*[-1]$, i.e., the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{O}^* \longrightarrow 0 \longrightarrow \cdots$$

concentrated at degree 1;

3. for every field F/k, the hypercohomology over Zariski topology satisfies[1]

$$\mathbb{H}^n_{\mathrm{Zar}}(F, \mathbb{Z}(n)) = H^n(\mathbb{Z}(n)(\mathrm{Spec}(F))) = K_n^M(F),$$

where $K_n^M(F)$ is the nth Milnor K-theory of a field F, given by the quotient of the tensor algebra $T(F^*)/\{x \otimes (1-x) : x \in F^*\}$ over $\mathbb{Z}$; (lecture 5 of [MVW06], page 29)

Example 0.2.

a. $K_0^M(F) = K_0(F) = \mathbb{Z}$;
b. $K_1^M(F) = K_1(F) = F^\times$;
c. $K_2^M(F) = K_2(F)$.

4. $\mathbb{H}^{2n}_{\mathrm{Zar}}(X, \mathbb{Z}(n)) = \mathrm{CH}^n(X)$ (lecture 17 of [MVW06], page 135), where the nth classical Chow group $\mathrm{CH}^n(X)$ is the free group given by

$$\mathrm{CH}^n(X) = \mathbb{Z}\{\text{cycles of codimension } n\}/\text{rational equivalences};$$

[1] Here we use the convention that the (hyper)cohomology of F should be interpreted as of $\mathrm{Spec}(F)$, the corresponding space.

5. there is a natural Atiyah–Hirzebruch spectral sequence

$$E_2^{p,q} = \mathbb{H}^p_{\text{Zar}}(X, \mathbb{Z}(q)) \Rightarrow K_{2q-p}(X).$$

Moreover, tensoring with $\mathbb{Q}$, the spectral sequence degenerates and one has

$$\mathbb{H}^i_{\text{Zar}}(X, \mathbb{Z}(n))_{\mathbb{Q}} = \text{gr}^n_\gamma(K_{2n-i}(X)_{\mathbb{Q}})$$

where gr^n_γ's are the quotients (graded pieces) of γ-filtration. ([Lev94]; [Lev99], Theorem 11.7)

Remark 0.3. Such choice of complexes $\mathbb{Z}(q)$ exists, and is called the motivic complex. For a clear definition of these complexes, see Lecture 3 of [MVW06]. Moreover, by convention $\mathbb{Z}(q) = 0$ for $q < 0$.

Definition 0.4. The motivic cohomology of X is defined by $H^{p,q}(X, \mathbb{Z}) = \mathbb{H}^p_{\text{Zar}}(X, \mathbb{Z}(q))$, the hypercohomology of the motivic complexes with respect to the Zariski topology.

Remark 0.5. In general, a motivic cohomology with coefficient in an abelian group A is a family of contravariant functors $H^{p,q}(-, A) : \text{Sm}/k \to \text{Ab}$.

Remark 0.6. The motivic cohomology of X satisfies the cancellation property: set $\mathbb{G}_m = \mathbb{A}^1 \backslash \{0\}$, then

$$H^{p,q}(X \times \mathbb{G}_m, \mathbb{Z}) = H^{p,q}(X, \mathbb{Z}) \oplus H^{p-1,q-1}(X, \mathbb{Z}).$$

Remark 0.7. It turns out that the group remains unchanged if we replace the Zariski topology by Nisnevich topology.[2] If one uses étale topology instead, we retrieve Lichtenbaum motivic cohomology $H^{p,q}_L(X, \mathbb{Z})$. If $\text{char}(k) \nmid n$, it admits the comparison

$$H^{p,q}_L(X, \mathbb{Z}/n\mathbb{Z}) = H_{\text{étale}}(X, \mathbb{Z}/n\mathbb{Z}(q)),$$

where $\mathbb{Z}/n\mathbb{Z}(q)$ is the q-twist $\mu_n^{\otimes q}$.

We may compare Lichtenbaum motivic cohomology with motivic cohomology by the following theorem, formerly known as Beilinson-Lichtenbaum Conjecture[3]:

Theorem 0.8 ([Voe11])**.** The natural map

$$H^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) \to H^{p,q}_L(X, \mathbb{Z}/n\mathbb{Z})$$

is an isomorphism if $p \leqslant q$, is a monomorphism if $p = q + 1$, and gives a spectral sequence for any pair of p, q.

Corollary 0.9. For $p \leqslant q$, we have

$$H^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) = H^p_{\text{étale}}(X, \mathbb{Z}/n\mathbb{Z}(q)).$$

In particular, for $X = \text{Spec}(k)$ as a point, this is the theorem formerly known as Milnor conjecture:

Corollary 0.10 ([Voe97], [Voe03a], [Voe03b])**.**

- $H^{p,p}(k, \mathbb{Z}/n\mathbb{Z}) = K^M_p(k)/n = H^p_{\text{étale}}(X, \mathbb{Z}/n\mathbb{Z}(p))$ as the Galois cohomology;
- in general,

$$H^{p,q}(k, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} 0, & p > q \\ H^{p,p}(k, \mathbb{Z}/n\mathbb{Z}) \cdot \tau^{q-p}, & p \leqslant q \end{cases}$$

where $\tau \in \mu_n(k) = H^{0,1}(k, \mathbb{Z})$ is a primitive nth root of unity.

Remark 0.11. Unlike the case with finite coefficients, $H^{p,q}(k, \mathbb{Z})$ is quite hard to compute for small $p < q$; for $p \geqslant q$, this is 0.

[2] Recall that the Nisnevich topology is a Grothendieck topology on the category of schemes that is finer than the Zariski topology but coarser than the étale topology.

[3] This is also known as the norm residue isomorphism theorem, or (formerly) Bloch-Kato conjecture.

A current long-standing conjecture is

Conjecture 0.12 (Beilinson-Soulé Vanishing Conjecture, [Lev93])**.** $H^{p,q}(k, \mathbb{Z}) = 0$ if $p < 0$.

Remark 0.13. Here are a few known cases:

- for $\operatorname{char}(k) = 0$, this is known for number fields ([Bor74]), function fields of genus 0 ([Dég08]), curves over number fields, and their inductive limits (more precise references required); ([DG05])
- for $\operatorname{char}(k) > 0$, this is known for finite fields ([Qui72]) and global fields ([Har77]).

Remark 0.14. The motivic cohomology could be realized in a tensor triangulated category, namely the category of effective motives $\mathrm{DM}(k)$. For any pair of p, q, we can find an Eilenberg-Maclane space and a corresponding representable functor so that

$$H^{p,q}(X, \mathbb{Z}) = \mathrm{Hom}_{DM}(\mathbb{Z}(X), \mathbb{Z}(q)[p])$$

where $\mathbb{Z}(X)$ is the motive of X and $\mathbb{Z}(q)[p] = \mathbb{G}_m^{\wedge q}[p-q]$.[4]

Remark 0.15. Dually, we can define the motivic homology by

$$H_{p,q}(X, \mathbb{Z}) = \mathrm{Hom}_{DM}(\mathbb{Z}(q)[p], \mathbb{Z}(X)).$$

Remark 0.16 ([MVW06] Properties 14.5, page 110)**.** By taking the hom functor from the aspect of motives, we can derive theorems for all (co)homologies which can be represented in DM. The main derives are the following:

1. If $E \to X$ is an $\mathbb{A}^n$-bundle, then motives $\mathbb{Z}(E) = \mathbb{Z}(X)$ in DM.
2. If $\{U, V\}$ is a Zariski open covering of X, we have a Mayer-Vietoris sequence

$$\mathbb{Z}(U \cap V) \longrightarrow \mathbb{Z}(U) \oplus \mathbb{Z}(V) \longrightarrow \mathbb{Z}(X) \longrightarrow \mathbb{Z}(U \cap V)[1]$$

   in the form of a distinguished triangle in DM.
3. If $Y \subseteq X$ is a closed embedding of codimension c in Sm/k, then we have a Gysin triangle

$$\mathbb{Z}(X \backslash Y) \longrightarrow \mathbb{Z}(X) \longrightarrow \mathbb{Z}(Y)(c)[2c] \longrightarrow \mathbb{Z}(X \backslash Y)[1]$$

   which is a distinguished triangle where $\mathbb{Z}(Y)(c)[2c] := \mathbb{Z}(Y) \otimes \mathbb{Z}(c)[2c]$.
4. For any vector bundle of rank n on X, we have the projective bundle formula

$$\mathbb{Z}(\mathbb{P}(E)) = \bigoplus_{i=0}^{n} \mathbb{Z}(X)(i)[2i]$$

   which defines the Chern class of E.
5. Let X be a proper smooth scheme and let d_X be its dimension, then $\mathbb{Z}(X)$ has a strong dual $\mathbb{Z}(X)(-d_X)[-2d_X]$ in DM by stabilization. This gives a Poincaré duality[5]

$$H^{p,q}(X, \mathbb{Z}) \cong H_{2d_X - p, d_X - q}(X, \mathbb{Z}).$$

[4] Again, this notation goes back to the concise definition of the motivic complexes: see Lecture 3 from [MVW06] as well as the concept of presheaves with transfers.

[5] We can use cohomology with compact support for this.

1 Intersection Theory

1.1 Cycles of Scheme

Definition 1.1. Let X be a scheme of finite type over k. We define the ith cycle on the scheme X to be a free abelian group

$$Z_i(X) = \bigoplus_{\substack{\text{irreducible closed } c\subseteq X\\ \text{with } \dim(c)=i}} \mathbb{Z}\cdot c$$

and set $Z(X) = \bigoplus_i Z_i(X)$. Define a set $K_i(X)$ to be the set of coherent sheaves $\mathcal{F}$ on X with $\dim(\operatorname{supp}(F)) \leqslant i$.[6]

Remark 1.2. Let $(A, \mathfrak{m})$ be a Noetherian local ring and M be an A-module, then by the dimension theorem, we know $\dim(M) = d(M) = \dim(\operatorname{supp}(M))$, where $d(M)$ is the degree of the Hilbert-Samuel polynomial $P_{\mathfrak{m}}(M, n)$.

Definition 1.3. Let $X \in \mathrm{Sm}/k$ and let $U, V \subseteq X$ be irreducible and closed. Suppose $W \subseteq U \cap V$ is a irreducible and closed component. If $\dim(W) = \dim(U) + \dim(V) - \dim(X)$, i.e., $\operatorname{codim}(W) = \operatorname{codim}(U) + \operatorname{codim}(V)$, we say that U and V intersect properly at W.

Remark 1.4. This condition is weaker than saying they intersect transversely: we do not require information about tangent spaces.

Theorem 1.5. Let $A \supseteq k$ be a Noetherian regular ring, M, N be finitely-generated A-modules, and suppose $\ell(M \otimes_A N) < \infty$, then

1. $\ell(\operatorname{Tor}_i^A(M, N)) < \infty$ for all $i \geqslant 0$;
2. the Euler-Poincaré characteristic $\chi(M, N) := \sum_{i=0}^{\dim(A)} (-1)^i \ell(\operatorname{Tor}_i^A(M, N)) \geqslant 0$;
3. by Remark 1.2, we have $\dim(M) + \dim(N) \leqslant \dim(A)$;
4. in particular, we have $\dim(M) + \dim(N) < \dim(A)$ if and only if $\chi(M, N) = 0$.

Proof. See [Ser12], page 106. □

Remark 1.6. Part 3. from Theorem 1.5 implies that $\dim(W) \geqslant \dim(U) + \dim(V) - \dim(X)$, i.e., $\operatorname{codim}(W) \leqslant \operatorname{codim}(U) + \operatorname{codim}(V)$ in the notation of Definition 1.3.

Definition 1.7. Let X, U, V, W be as in Definition 1.3, then we define the intersection multiplicity $m_W(U, V)$ of U and V at W by

$$m_W(U, V) = \chi^{\mathcal{O}_{X,W}}(\mathcal{O}_{X,W}/P_U, \mathcal{O}_{X,W}/P_V)$$

where P_U and P_V are prime ideals defining U and V, respectively.

Remark 1.8. By Theorem 1.5, we know $m_W(U, V) \geqslant 0$, and $m_W(U, V) = 0$ if and only if U and V do not intersect properly at W.

1.2 Intersection Product and Cross Product

Definition 1.9. Let $X \in \mathrm{Sm}/k$, and let $U \in Z_a(X)$ and $V \in Z_b(X)$. If U and V intersect properly at every component, then we define the intersection product to be the cycle

$$U \cdot V = \sum_{\substack{W\subseteq U\cap V\\ \dim(W)=a+b-d_X}} m_W(U, V)\cdot W \in Z_{a+b-d_X}(X).$$

Example 1.10. Let X be a smooth projective surface, and let C and D be divisors on X. For any point $x \in C \cap D$, locally we think of $C = \{f = 0\}$ and $D = \{g = 0\}$ around x, then $m_x(C, D) = \ell_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(f, g))$.

[6] Despite the notation, this has nothing to do with a K-theory.

Definition 1.11. Suppose X is a scheme of finite type over k, and $\mathcal{F} \in K_n(X)$ is a coherent sheaf, then we define $Z_a(\mathcal{F}) = \sum_{\dim(\bar{\eta})=a} (\mathcal{O}_{X,\eta}(\mathcal{F}_\eta) \cdot \bar{\eta}) \in Z_a(X)$.

Therefore, we define the cycle of F as an element of the cycle of X.

Definition 1.12 ([Har13], Exercise III.6.9)**.** Every coherent sheaf $\mathcal{F}$ on $X \in \mathrm{Sm}/k$ has a resolution

$$0 \longrightarrow E_k \longrightarrow E_{k-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

where E_i's are locally free of finite rank. Therefore, for any coherent sheaf $\mathcal{G}$, we can define the Tor functor[7] of coherent sheaves by

$$\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = H_i(E_* \otimes_{\mathcal{O}_X} \mathcal{G}).$$

Proposition 1.13. Let $X \in \mathrm{Sm}/k$. Suppose $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(X)$ intersect properly, then

$$Z_a(\mathcal{F}) \cdot Z_b(\mathcal{G}) = \sum_{i=0}^{d_X} (-1)^i \cdot Z_{a+b-d_X}(\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})).$$

Proof. We only have to do it locally, so we can assume X to be affine, and count the coefficients of $\bar{\xi}$ where $\dim(\xi) = a + b - d_X$. It sufficess to show that the stalks at ξ satisfies

$$\chi(F_\xi, G_\xi) = \sum_{\substack{\dim(\bar{\lambda})=a \\ \dim(\beta\eta)=b \\ \xi \in \bar{\lambda} \cap \bar{\eta}}} \ell(\mathcal{F}_\lambda) \cdot \ell(G_\eta) \cdot m_{\bar{\xi}}(\bar{\lambda}, \bar{\eta}).$$

Because our ring is Noetherian, then $\mathcal{F}$ admits a filtration

$$0 = M_0 \subseteq \cdots \subseteq M_d = \mathcal{F}$$

such that $M_i/M_{i-1} \cong \mathcal{O}_X/\mathcal{I}$ is coherent for prime ideal $\mathcal{I}$. By the additivity of both sides of the isomorphism, we may assume $\mathcal{F} = \mathcal{O}_X/\mathfrak{p}$ with dimension at most a, where $\mathfrak{p} \sim \lambda \in X$. Similarly, we may assume $\mathcal{G} = \mathcal{O}_X/\mathfrak{q}$ with dimension at most b, where $\mathfrak{q} \sim \eta \in X$. Moreover, set $\xi \in \bar{\lambda} \cap \bar{\eta}$. By definition, we now have $\chi(\mathcal{F}_\xi, \mathcal{G}_\xi) = m_{\bar{\xi}}(\bar{\lambda}, \bar{\eta})$.

- If $\dim(\bar{\lambda}) = a$ and $\dim(\bar{\eta}) = b$, then the equality follows from the fact that $\ell(\mathcal{F}_\lambda) = \ell(\mathcal{G}_\eta) = 1$.
- If not, then either $\dim(\bar{\lambda}) < a$ or $\dim(\bar{\eta}) < b$, then $\bar{\lambda}$ and $\bar{\eta}$ do not intersect properly at $\bar{\xi}$, so both the left-hand side and the right-hand side become 0.

□

Proposition 1.14. The intersection product is commutative.

Proof. This is obvious since the Tor functor is commutative. □

Proposition 1.15. The intersection product is associative.

Proof. Suppose we pick $\mathcal{F} \in K_a(X)$, $\mathcal{G} \in K_b(X)$, and $\mathcal{H} \in K_c(X)$ with support dimension at most a, b, c, respectively, and they intersect properly. Let L_* and M_* be free resolutions of $\mathcal{F}$ and $\mathcal{H}$, respectively. Define a double complex $N_{ij} = L_i \otimes \mathcal{G} \otimes M_j$, then the associativity of tensor product allows us to calculate triple Tor

$$H_i(L_i \otimes H_j(\mathcal{G}) \otimes M_j)) \cong \mathrm{Tor}_i(\mathcal{F}, \mathcal{G}, \mathcal{H}) \cong H_i(H_j(L_i \otimes \mathcal{G}) \otimes M_j)$$

as the homology of two (tensor) double complexes. We obtain two spectral sequences

$${}^{I}E^2_{p,q} = \mathrm{Tor}_p(\mathcal{F}, \mathrm{Tor}_q(\mathcal{G}, \mathcal{H})) \Rightarrow \mathrm{Tor}_{p+q}(\mathcal{F}, \mathcal{G}, \mathcal{H})$$

[7] Since we are working over sheaves of $\mathcal{O}_X$-modules, using the same argument on the level of modules shows that the Tor functor is independent from the choice of resolution.

$$^{II}E^2_{p,q} = \mathrm{Tor}_p(\mathrm{Tor}_q(\mathcal{F},\mathcal{G}),\mathcal{H}) \Rightarrow \mathrm{Tor}_{p+q}(\mathcal{F},\mathcal{G},\mathcal{H}).$$

Recall Euler-Poincaré characteristic is invariant with respect to taking spectral sequence $(*)$, then

$$\begin{aligned} Z_a(\mathcal{F})\cdot((Z_b\mathcal{G})\cdot Z_c(\mathcal{H})) &= Z_a(\mathcal{F})\cdot\sum_q(-1)^q Z_{b+c-d_X}(\mathrm{Tor}_q(\mathcal{G},\mathcal{H})) \text{ by Proposition 1.13} \\ &= \sum_{p,q}(-1)^{p+q}Z_{a+b+c-2d_X}(^IE^2_{p,q}) \text{ by Proposition 1.13} \\ &= \sum_i(-1)^i Z_{a+b+c-2d_X}(\mathrm{Tor}_i(\mathcal{F},\mathcal{G},\mathcal{H})) \text{ by } (*) \\ &= \sum_{p,q}(-1)^{p+q}Z_{a+b+c-2d_X}(^{II}E^2_{p,q}) \text{ by } (*) \\ &= \sum_p Z_{a+b-d_X}(\mathrm{Tor}_p(\mathcal{F},\mathcal{G}))\cdot Z_c(\mathcal{H}) \text{ by Proposition 1.13} \\ &= (Z_a(\mathcal{F})\cdot Z_b(\mathcal{G}))\cdot Z_c(\mathcal{H}) \text{ by Proposition 1.13.} \end{aligned}$$

□

Definition 1.16. Suppose $X_1, X_2 \in \mathrm{Sm}/k$, with $\mathcal{F}_1 \in K_{a_1}(X_1)$ and $\mathcal{F}_2 \in K_{a_2}(X_2)$. We define the cross product of cycles to be

$$Z_{a_1}(\mathcal{F}_1)\times Z_{a_2}(\mathcal{F}_2) = Z_{a_1+d_{X_2}}(p_1^*\mathcal{F}_1)\cdot Z_{a_2+d_{X_1}}(p_2^*\mathcal{F}_2),$$

where $p_i : X_1\times X_2 \to X_i$ is the projection for $i = 1, 2$.

Exercise 1.17. One should check that this is well-defined.

Remark 1.18. Suppose $X_1, X_2 \in \mathrm{Sm}/k$, with $\mathcal{F}_1 \in K_{a_1}(X_1)$, $\mathcal{F}_2 \in K_{b_1}(X_1)$, $\mathcal{G}_1 \in K_{a_2}(X_2)$ and $\mathcal{G}_2 \in K_{a_2}(X_2)$. Suppose $Z_{a_1}(\mathcal{F}_1)\cdot Z_{a_2}(\mathcal{G}_1)$ and $Z_{b_1}(\mathcal{F}_2)\cdot Z_{b_2}(\mathcal{G}_2)$ are defined, then

- $Z_{a_1}(\mathcal{F}_1)\times Z_{a_2}(\mathcal{G}_1)$ and $Z_{b_1}(\mathcal{F}_2)\times Z_{b_2}(\mathcal{G}_2)$ intersect properly on $X_1\times X_2$, and
- $(Z_{a_1}(\mathcal{F}_1)\times Z_{a_2}(\mathcal{G}_1))\cdot(Z_{b_1}(\mathcal{F}_2)\times Z_{b_2}(\mathcal{G}_2)) = (Z_{a_1}(\mathcal{F}_1)\cdot Z_{b_1}(\mathcal{F}_2))\times(Z_{a_2}(\mathcal{G}_1)\cdot Z_{b_2}(\mathcal{G}_2))$.

1.3 Pushout and Pullback

Definition 1.19. Suppose X, Y are schemes of finite type over k, and let $f : X \to Y$ be a proper map. For every irreducible closed subset $c \subseteq X$ of dimension a, we define the direct image to be

$$f_*c = \begin{cases} [k(c):k(f(c))]\cdot f(c) \in Z_a(Y), & \dim(f(c)) = a \\ 0, & \dim(f(c)) < a \end{cases}$$

to be the direct image of c under f.

Lemma 1.20. Suppose X and Y are schemes of finite type over k of the same dimension n, and that $f : X \to Y$ is proper, then there exists an open subset $U \subseteq Y$ such that $\dim(Y\backslash U) < n$ and $f : f^{-1}(U) \to U$ is a finite morphism.

Proof. Suppose $\xi \in Y$ has $\dim(\bar{\xi}) = n$. We can find $U \ni \xi$ such that $f|_U$ has finite fibers by Exercise II.3.7 from [Har13]. By Exercise III.11.2 in [Har13], such f is finite. □

Proposition 1.21. Let $f : X \to Y$ be a proper morphism between schemes over k of finite type, and let $\mathcal{F} \in K_a(X)$, then

1. $f_*\mathcal{F} \in K_a(Y)$ and the right derived $R^if_*\mathcal{F} \in K_{a-1}(Y)$ for $i > 0$.
2. $f_*Z_a(\mathcal{F}) = Z_a(f_*\mathcal{F})$.

Proof. 1. By Theorem III.8.8 from [Har13], $R^i f_*\mathcal{F}$ is coherent for all $i \geqslant 0$. We have $\operatorname{supp}(R_i f_*\mathcal{F}) \subseteq \operatorname{supp}(\mathcal{F})$. If f is finite, then f_* is exact, so $R^i f_*\mathcal{F} = 0$ for $i > 0$. For general cases, we may assume $\dim(f(\operatorname{supp}(\mathcal{F}))) = a$ and set $W = \operatorname{supp}(\mathcal{F})$. We have a commutative diagram

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, "i"'] & f(W) \arrow[d, "j"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where h is also proper. By Lemma 1.20, there exists $V \subseteq f(W)$ such that $\dim(f(W)\backslash V) < a$ and $h|_V$ is finite. Let $\mathcal{J}$ be the ideal sheaf of W, then $\mathcal{J}^s\mathcal{F}/\mathcal{J}^{s+1}\mathcal{F} = i_* i^* \mathcal{J}^s\mathcal{F}/\mathcal{J}^{s+1}\mathcal{F}$. By the long exact sequence, it suffices to prove the case for $\mathcal{F} = i_*\mathcal{G}$. Then

$$(R^k f_*) i_*\mathcal{G} = R^k (fi)_*\mathcal{G} = j_* R^k h_*\mathcal{G}.$$

It suffices to consider h, but

$$(R^k h_*\mathcal{G})V = R^k h(\mathcal{G}|_{f^{-1}(V)}) = 0$$

for $k > 0$, so $\operatorname{supp}(R^k h_*\mathcal{G}) \subseteq f(W)\backslash V$ if $k > 0$.

2. If f is finite, let us write down the coefficients of ξ of dimension a on both sides, namely

$$\ell((f_*\mathcal{F})_\xi) = \sum_{\substack{\eta \in f^{-1}(\xi) \\ \dim(\overline{\eta}) = a}} \ell(F_\eta) \cdot [k(\overline{\eta}) : k(\overline{f(\eta)})].$$

By additivity, one reduces to the case when X is affine and $F = \mathcal{O}_X/\mathfrak{p}$. For the general case, use Lemma 1.20, and the case where f is finite.

□

Definition 1.22. Suppose $f : X \to Y$ where $Y \in \mathrm{Sm}/k$ and X is closed in $Z \in \mathrm{Sm}/k$. Define $j : X \to Z \times Y$ to be the graph map. For any $C \in Z_a(X)$ and $D \in Z_b(Y)$ such that C and $f^{-1}(D)$ intersect properly, define the intersection cycle to be

$$C \cdot_f D = j_*^{-1}(j(C) \cdot (Z \times D)) \in Z_{a+b-d_Y}(X)$$

In particular, $f^*(D) = X \cdot_f D$ for $C = X$.

Proposition 1.23. Using the notation above, for $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(Y)$, if $\mathcal{F}$ and $f^*\mathcal{G}$ intersect properly, we have

$$Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G}) = \sum_{i=0}^{d_Y} (-1)^i Z_{a+b-d_Y}(L_i(\mathcal{F} \otimes f^*)\mathcal{G})$$

Proof. Denote $p_2 : Z \times Y \to Y$ to be the projection onto the second coordinate. By linearity, $Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G}) = j_*^{-1}(Z_a(j_*\mathcal{F}) \cdot Z_{b+d_Z}(p_2^*\mathcal{G})$ for $j : X \to Z \times Y$. Suppose $L_* \to \mathcal{G}$ is the locally free resolution of $\mathcal{G}$. Note that for all $i \geqslant 0$, we have

$$j^*(j_*\mathcal{F} \otimes p_2^* L_i) + F \otimes f^* L_i,$$

which induces an isomorphism

$$j_*\mathcal{F} \otimes p_2^* L_i = j_*(\mathcal{F} \otimes f^* L_i).$$

Hence $\operatorname{Tor}_i^{\mathcal{O}_{Z\times Y}}(j_*\mathcal{F}, p_2^*\mathcal{G}) = j_* L_i(F \otimes f^*)\mathcal{G}$. So

$$j_*^{-1} Z_{a+b-d_Y}(\operatorname{Tor}_i^{\mathcal{O}_{Z\times Y}}(j_*\mathcal{F}, p_2^*\mathcal{G})) = Z_{a+b-\dim(Y)}(L_i(F \otimes f^*)\mathcal{G}).$$

Therefore the statement follows. □

Proposition 1.24. Let $X \in \mathrm{Sm}/k$, $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(X)$ such that $\mathcal{F}$ and $\mathcal{G}$ intersect properly. Let $\Delta : X \to X \times X$ be the diagonal map, then

$$\Delta^*(Z_a(\mathcal{F}) \times Z_b(\mathcal{G})) = Z_a(\mathcal{F}) \cdot Z_b(\mathcal{G}).$$

Proof. See page 115 of [Ser12]. □

Proposition 1.25. f^* is compatible with intersection product, and $f^*g^* = (gf)^*$.

Proof. See page 119 of [Ser12]. □

Lemma 1.26. Let $\mathcal{A}$ be an abelian category with enough projectives (respectively, injectives) and F be a right (respectively, left) exact functor from $\mathcal{A}$. Suppose C is chain complex in $\mathcal{A}$, then there exists a double complex $M_{*,*}$ in $\mathcal{A}$ such that

$$^{I}E^2_{p,q} = L_pFH_q(C) \quad \text{(respectively, } R^{-p}F(H_q(C))).$$

Proof. To do this when F is right exact, use the Cartan-Eilenberg resolution[8] $C_* \to C$ and consider the double complex FC_*. □

Proposition 1.27. Suppose $f : X \to Y$ is in Sm/k, suppose $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(Y)$, then

$$Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G}) = Z_a(\mathcal{F}) \cdot f^*Z_b(\mathcal{G})$$

if both sides are defined.

Proof. We may assume X is affine. Let $L_* \to \mathcal{G}$ be a free resolution and apply Lemma 1.26 to f^*L_* and $F \otimes -$, then we find a double complex such that

$$^{I}E^2_{p,q} = \mathrm{Tor}_p(\mathcal{F}, L_qf^*\mathcal{G})$$
$$^{II}E^2_{p,q} = L_p(F \otimes f^*)\mathcal{G}.$$

□

Proposition 1.28. Let $X \subseteq Z$ and $Y, Z \in \mathrm{Sm}/k$ and $f : X \to Y$ be proper. Suppose $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(Y)$, and suppose $\mathcal{F}$ and $f^*\mathcal{G}$ intersect properly, then

$$f_*(Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G})) = (f_*Z_a(\mathcal{F})) \cdot Z_b(\mathcal{G}).$$

Proof. Pick $L_* \to \mathcal{G}$ to be a resolution and apply Lemma 1.26 to $F \otimes f^*L_*$ and f_*, then we have a double complex $M_{*,*}$ such that

$$^{I}E^2_{p,q} = R^{-p}f_*L_q(F \otimes f^*)\mathcal{G}).$$

On the other hand, $H_q(M_{*,n}) = R^{-q}f_*(F \otimes f^*L_n) = (R^{-q}f_*\mathcal{F}) \otimes L_n.$, therefore

$$^{II}E^2_{p,q} = \mathrm{Tor}_p(R^{-q}f_*\mathcal{F}, \mathcal{G}).$$

□

Corollary 1.29. Under the same hypothesis as Proposition 1.28, we have

$$f_*(Z_a(\mathcal{F}) \cdot f^*(Z_b(\mathcal{G}))) = f_*(Z_a(\mathcal{F})) \cdot Z_b(\mathcal{G}).$$

[8] See Proposition 11 on page 210 of [GM13].

2 Sheaves with Transfers

We fix a base scheme $S \in \mathrm{Sm}/k$.

Definition 2.1. Let $X, Y \in \mathrm{Sm}/S$, then we define the group of finite correspondences

$$\mathrm{Cor}_S(X,Y) = \mathbb{Z}\{\text{irreducible closed } C \subseteq X \times_s Y \mid C \to X \text{ finite}, \dim(C) = \dim(X)\}$$

to be the free abelian group generated by elementary correspondences from X to Y.

Example 2.2. For any $f : X \to Y$, the graph $\Gamma_f = (x, f(x)) \subseteq X \times_S Y$ is a finite correspondence from $X \to Y$.

Example 2.3. If $f : X \to Y$ is finite and $\dim(X) = \dim(Y)$, then the graph Γ_f is also a finite correspondence from $Y \to X$.

Definition 2.4. Define an additive category Cor_S, whose objects are the same as Sm/S and the hom set $\mathrm{Hom}_{\mathrm{Sm}/S}(X,Y) = \mathrm{Cor}_S(X,Y)$ as in Definition 2.1. The contravariant additive functors

$$F : \mathrm{Cor}_S^{\mathrm{op}} \to \mathrm{Ab}$$

are called presheaves with transfers. The corresponding category is denoted by $\mathrm{PSh}(S) = \mathrm{PSh}(\mathrm{Cor}_S)$, which is abelian with enough injectives and projectives. We have a functor $\gamma : \mathrm{Sm}/S \to \mathrm{Cor}_S$ by Example 2.3.

Example 2.5. Every $X \in \mathrm{Sm}/S$ gives an element $\mathbb{Z}(X) \in \mathrm{PSh}(S)$ defined by $\mathbb{Z}(X)(Y) = \mathrm{Cor}_S(Y,X)$. Therefore, we say $\mathbb{Z}(X)$ is the presheaf with transfers represented by X. By Yoneda Lemma we know there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{PST}}(\mathbb{Z}(X), F) \cong F(X).$$

Moreover, representable functors give embeddings of Sm/S and Cor_S into $\mathrm{PSh}(S)$ via

$$\begin{array}{ccccc} \mathrm{Sm}/S & \xrightarrow{\gamma} & \mathrm{Cor}_S & \longrightarrow & \mathrm{PSh}(S) \\ X & \longmapsto & X & \longmapsto & \mathbb{Z}(X) \end{array}$$

In particular, $\mathbb{Z}(S) = \mathbb{Z}$.

Example 2.6. The presheaves $\mathcal{O}$ and $\mathcal{O}^*$ are in $\mathrm{PSh}(S)$. For any $C \in \mathrm{Cor}_S(X,Y)$ and $f \in \mathcal{O}(Y)$ (respectively, $\mathcal{O}^*(Y)$), we have a diagram

$$\begin{tikzcd} C \arrow[r, "i"] & X \times_S Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \\ & X & \end{tikzcd}$$

and can define $\mathcal{O}(C)(f) = \mathrm{Tr}_{C/X}((p_2 \circ i)^*(f))$ (respectively, $\mathcal{O}^*(C)(f) = \mathrm{N}_{C/X}((p_2 \circ i)^*(f))$).

Definition 2.7. Let us describe the composition in Cor_S. Suppose $f \in \mathrm{Cor}_S(X,Y)$ and $g \in \mathrm{Cor}_S(Y,Z)$, then from the diagram

$$\begin{tikzcd} X \times_S Z & \\ X \times_S Y \times_S Z \arrow[u, "p_{13}"] \arrow[r, "p_{23}"] \arrow[d, "p_{12}"] & Y \times_S Z \\ X \times_S Y & \end{tikzcd}$$

we define the composition $g \circ f = p_{13*}(p_{23}^*(g) p_{12}^*(f))$.

Exercise 2.8. One should check that all intersections are proper.

Proposition 2.9. The composition law is associative.

Proof. Suppose

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

are morphisms in Cor_S, then we have two Cartesian squares

$$\begin{tikzcd} X \times_S Y \times_S Z \times_S W \arrow[r] \arrow[d] & X \times_S Z \times_S W \arrow[d] \\ X \times_S Y \times_S Z \arrow[r] & X \times_S Z \end{tikzcd}$$

and

$$\begin{tikzcd} X \times_S Y \times_S Z \times_S W \arrow[r] \arrow[d] & X \times_S Y \times_S W \arrow[d] \\ Y \times_S Z \times_S W \arrow[r] & Y \times_S W \end{tikzcd}$$

□

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