

Motivic Cohomology Notes

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0 INTRODUCTION

Let $X \in \mathbf{Sm}/k$ be a smooth separated scheme over a field k . The study of motivic cohomology started with the hope that

Conjecture 0.1 (Beilinson and Lichtenbaum, 1982-1987). There exists some complexes $\mathbb{Z}(n)$ for $n \in \mathbb{N}$ of sheaves in Zariski topology on \mathbf{Sm}/k such that

1. $\mathbb{Z}(0)$ is (quasi-isomorphic to) the constant sheaf \mathbb{Z} , i.e., the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$

concentrated at degree 0;

2. $\mathbb{Z}(1)$ is the complex $\mathcal{O}^*[-1]$, i.e., the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{O}^* \longrightarrow 0 \longrightarrow \cdots$$

concentrated at degree 1;

3. for every field F/k , the hypercohomology over Zariski topology satisfies¹

$$\mathbb{H}_{\mathrm{Zar}}^n(F, \mathbb{Z}(n)) = H^n(\mathbb{Z}(n)(\mathrm{Spec}(F))) = K_n^M(F),$$

where $K_n^M(F)$ is the n th Milnor K-theory of a field F , given by the quotient of the tensor algebra $T(F^*)/\{x \otimes (1-x) : x \in F^*\}$ over \mathbb{Z} ; (lecture 5 of [\[MVW06\]](#), page 29)

Example 0.2.

- a. $K_0^M(F) = K_0(F) = \mathbb{Z}$;
 - b. $K_1^M(F) = K_1(F) = F^\times$;
 - c. $K_2^M(F) = K_2(F)$.
4. $\mathbb{H}_{\mathrm{Zar}}^{2n}(X, \mathbb{Z}(n)) = \mathrm{CH}^n(X)$ (lecture 17 of [\[MVW06\]](#), page 135), where the n th classical Chow group $\mathrm{CH}^n(X)$ is the free group given by

$$\mathrm{CH}^n(X) = \mathbb{Z}\{\text{cycles of codimension } n\}/\text{rational equivalences};$$

¹Here we use the convention that the (hyper)cohomology of F should be interpreted as of $\mathrm{Spec}(F)$, the corresponding space.

5. there is a natural Atiyah–Hirzebruch spectral sequence

$$E_2^{p,q} = \mathbb{H}_{\text{Zar}}^p(X, \mathbb{Z}(q)) \Rightarrow K_{2q-p}(X).$$

Moreover, tensoring with \mathbb{Q} , the spectral sequence degenerates and one has

$$\mathbb{H}_{\text{Zar}}^i(X, \mathbb{Z}(n))_{\mathbb{Q}} = \text{gr}_{\gamma}^n(K_{2n-i}(X)_{\mathbb{Q}})$$

where gr_{γ}^n 's are the quotients (graded pieces) of γ -filtration. ([Lev94]; [Lev99], Theorem 11.7)

Remark 0.3. Such choice of complexes $\mathbb{Z}(q)$ exists, and is called the motivic complex. For a clear definition of these complexes, see Lecture 3 of [MVW06]. Moreover, by convention $\mathbb{Z}(q) = 0$ for $q < 0$.

Definition 0.4. The motivic cohomology of X is defined by $H^{p,q}(X, \mathbb{Z}) = \mathbb{H}_{\text{Zar}}^p(X, \mathbb{Z}(q))$, the hypercohomology of the motivic complexes with respect to the Zariski topology.

Remark 0.5. In general, a motivic cohomology with coefficient in an abelian group A is a family of contravariant functors $H^{p,q}(-, A) : \text{Sm}/k \rightarrow \text{Ab}$.

Remark 0.6. The motivic cohomology of X satisfies the cancellation property: set $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$, then

$$H^{p,q}(X \times \mathbb{G}_m, \mathbb{Z}) = H^{p,q}(X, \mathbb{Z}) \oplus H^{p-1, q-1}(X, \mathbb{Z}).$$

Remark 0.7. It turns out that the group remains unchanged if we replace the Zariski topology by Nisnevich topology.² If one uses étale topology instead, we retrieve Lichtenbaum motivic cohomology $H_L^{p,q}(X, \mathbb{Z})$. If $\text{char}(k) \nmid n$, it admits the comparison

$$H_L^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) = H_{\text{étale}}^p(X, \mathbb{Z}/n\mathbb{Z}(q)),$$

where $\mathbb{Z}/n\mathbb{Z}(q)$ is the q -twist $\mu_n^{\otimes q}$.

We may compare Lichtenbaum motivic cohomology with motivic cohomology by the following theorem, formerly known as Beilinson-Lichtenbaum Conjecture³:

Theorem 0.8 ([Voe11]). The natural map

$$H^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) \rightarrow H_L^{p,q}(X, \mathbb{Z}/n\mathbb{Z})$$

is an isomorphism if $p \leq q$, is a monomorphism if $p = q + 1$, and gives a spectral sequence for any pair of p, q .

Corollary 0.9. For $p \leq q$, we have

$$H^{p,q}(X, \mathbb{Z}/n\mathbb{Z}) = H_{\text{étale}}^p(X, \mathbb{Z}/n\mathbb{Z}(q)).$$

In particular, for $X = \text{Spec}(k)$ as a point, this is the theorem formerly known as Milnor conjecture:

Corollary 0.10 ([Voe97], [Voe03a], [Voe03b]).

- $H^{p,p}(k, \mathbb{Z}/n\mathbb{Z}) = K_p^M(k)/n = H_{\text{étale}}^p(X, \mathbb{Z}/n\mathbb{Z}(p))$ as the Galois cohomology;
- in general,

$$H^{p,q}(k, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} 0, & p > q \\ H^{p,p}(k, \mathbb{Z}/n\mathbb{Z}) \cdot \tau^{q-p}, & p \leq q \end{cases}$$

where $\tau \in \mu_n(k) = H^{0,1}(k, \mathbb{Z})$ is a primitive n th root of unity.

Remark 0.11. Unlike the case with finite coefficients, $H^{p,q}(k, \mathbb{Z})$ is quite hard to compute for small $p < q$; for $p \geq q$, this is 0.

²Recall that the Nisnevich topology is a Grothendieck topology on the category of schemes that is finer than the Zariski topology but coarser than the étale topology.

³This is also known as the norm residue isomorphism theorem, or (formerly) Bloch-Kato conjecture.

A current long-standing conjecture is

Conjecture 0.12 (Beilinson-Soulé Vanishing Conjecture, [Lev93]). $H^{p,q}(k, \mathbb{Z}) = 0$ if $p < 0$.

Remark 0.13. Here are a few known cases:

- for $\text{char}(k) = 0$, this is known for number fields ([Bor74]), function fields of genus 0 ([Dég08]), curves over number fields, and their inductive limits (more precise references required); ([DG05])
- for $\text{char}(k) > 0$, this is known for finite fields ([Qui72]) and global fields ([Har77]).

Remark 0.14. The motivic cohomology could be realized in a tensor triangulated category, namely the category of effective motives $\text{DM}(k)$. For any pair of p, q , we can find an Eilenberg-MacLane space and a corresponding representable functor so that

$$H^{p,q}(X, \mathbb{Z}) = \text{Hom}_{DM}(\mathbb{Z}(X), \mathbb{Z}(q)[p])$$

where $\mathbb{Z}(X)$ is the motive of X and $\mathbb{Z}(q)[p] = \mathbb{G}_m^{\wedge q}[p - q]$.⁴

Remark 0.15. Dually, we can define the motivic homology by

$$H_{p,q}(X, \mathbb{Z}) = \text{Hom}_{DM}(\mathbb{Z}(q)[p], \mathbb{Z}(X)).$$

Remark 0.16 ([MVW06] Properties 14.5, page 110). By taking the hom functor from the aspect of motives, we can derive theorems for all (co)homologies which can be represented in DM . The main derives are the following:

1. If $E \rightarrow X$ is an \mathbb{A}^n -bundle, then motives $\mathbb{Z}(E) = \mathbb{Z}(X)$ in DM .
2. If $\{U, V\}$ is a Zariski open covering of X , we have a Mayer-Vietoris sequence

$$\mathbb{Z}(U \cap V) \longrightarrow \mathbb{Z}(U) \oplus \mathbb{Z}(V) \longrightarrow \mathbb{Z}(X) \longrightarrow \mathbb{Z}(U \cap V)[1]$$

in the form of a distinguished triangle in DM .

3. If $Y \subseteq X$ is a closed embedding of codimension c in Sm/k , then we have a Gysin triangle

$$\mathbb{Z}(X \setminus Y) \longrightarrow \mathbb{Z}(X) \longrightarrow \mathbb{Z}(Y)(c)[2c] \longrightarrow \mathbb{Z}(X \setminus Y)[1]$$

which is a distinguished triangle where $\mathbb{Z}(Y)(c)[2c] := \mathbb{Z}(Y) \otimes \mathbb{Z}(c)[2c]$.

4. For any vector bundle of rank n on X , we have the projective bundle formula

$$\mathbb{Z}(\mathbb{P}(E)) = \bigoplus_{i=0}^n \mathbb{Z}(X)(i)[2i]$$

which defines the Chern class of E .

5. Let X be a proper smooth scheme and let d_X be its dimension, then $\mathbb{Z}(X)$ has a strong dual $\mathbb{Z}(X)(-d_X)[-2d_X]$ in DM by stabilization. This gives a Poincaré duality⁵

$$H^{p,q}(X, \mathbb{Z}) \cong H_{2d_X - p, d_X - q}(X, \mathbb{Z}).$$

⁴Again, this notation goes back to the concise definition of the motivic complexes: see Lecture 3 from [MVW06] as well as the concept of presheaves with transfers.

⁵We can use cohomology with compact support for this.

1 INTERSECTION THEORY

1.1 CYCLES OF SCHEME

Definition 1.1. Let X be a scheme of finite type over k . We define the i th cycle on the scheme X to be a free abelian group

$$Z_i(X) = \bigoplus_{\substack{\text{irreducible closed } c \subseteq X \\ \text{with } \dim(c)=i}} \mathbb{Z} \cdot c$$

and set $Z(X) = \bigoplus_i Z_i(X)$. Define a set $K_i(X)$ to be the set of coherent sheaves \mathcal{F} on X with $\dim(\text{supp}(\mathcal{F})) \leq i$.⁶

Remark 1.2. Let (A, \mathfrak{m}) be a Noetherian local ring and M be an A -module, then by the dimension theorem, we know $\dim(M) = d(M) = \dim(\text{supp}(M))$, where $d(M)$ is the degree of the Hilbert-Samuel polynomial $P_{\mathfrak{m}}(M, n)$.

Definition 1.3. Let $X \in \text{Sm}/k$ and let $U, V \subseteq X$ be irreducible and closed. Suppose $W \subseteq U \cap V$ is a irreducible and closed component. If $\dim(W) = \dim(U) + \dim(V) - \dim(X)$, i.e., $\text{codim}(W) = \text{codim}(U) + \text{codim}(V)$, we say that U and V intersect properly at W .

Remark 1.4. This condition is weaker than saying they intersect transversely: we do not require information about tangent spaces.

Theorem 1.5. Let $A \supseteq k$ be a Noetherian regular ring, M, N be finitely-generated A -modules, and suppose $\ell(M \otimes_A N) < \infty$, then

1. $\ell(\text{Tor}_i^A(M, N)) < \infty$ for all $i \geq 0$;
2. the Euler-Poincaré characteristic $\chi(M, N) := \sum_{i=0}^{\dim(A)} (-1)^i \ell(\text{Tor}_i^A(M, N)) \geq 0$;
3. by Remark 1.2, we have $\dim(M) + \dim(N) \leq \dim(A)$;
4. in particular, we have $\dim(M) + \dim(N) < \dim(A)$ if and only if $\chi(M, N) = 0$.

Proof. See [Ser12], page 106. □

Remark 1.6. Part 3. from Theorem 1.5 implies that $\dim(W) \geq \dim(U) + \dim(V) - \dim(X)$, i.e., $\text{codim}(W) \leq \text{codim}(U) + \text{codim}(V)$ in the notation of Definition 1.3.

Definition 1.7. Let X, U, V, W be as in Definition 1.3, then we define the intersection multiplicity $m_W(U, V)$ of U and V at W by

$$m_W(U, V) = \chi^{\mathcal{O}_{X,W}}(\mathcal{O}_{X,W}/P_U, \mathcal{O}_{X,W}/P_V)$$

where P_U and P_V are prime ideals defining U and V , respectively.

Remark 1.8. By Theorem 1.5, we know $m_W(U, V) \geq 0$, and $m_W(U, V) = 0$ if and only if U and V do not intersect properly at W .

1.2 INTERSECTION PRODUCT AND CROSS PRODUCT

Definition 1.9. Let $X \in \text{Sm}/k$, and let $U \in Z_a(X)$ and $V \in Z_b(X)$. If U and V intersect properly at every component, then we define the intersection product to be the cycle

$$U \cdot V = \sum_{\substack{W \subseteq U \cap V \\ \dim(W)=a+b-d_X}} m_W(U, V) \cdot W \in Z_{a+b-d_X}(X).$$

Example 1.10. Let X be a smooth projective surface, and let C and D be divisors on X . For any point $x \in C \cap D$, locally we think of $C = \{f = 0\}$ and $D = \{g = 0\}$ around x , then $m_x(C, D) = \ell_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(f, g))$.

⁶Despite the notation, this has nothing to do with a K-theory.

Definition 1.11. Suppose X is a scheme of finite type over k , and $\mathcal{F} \in K_n(X)$ is a coherent sheaf, then we define $Z_a(\mathcal{F}) = \sum_{\dim(\bar{\eta})=a} (\mathcal{O}_{X,\eta}(\mathcal{F}_\eta) \cdot \bar{\eta}) \in Z_a(X)$.

Therefore, we define the cycle of \mathcal{F} as an element of the cycle of X .

Definition 1.12 ([Har13], Exercise III.6.9). Every coherent sheaf \mathcal{F} on $X \in \text{Sm}/k$ has a resolution

$$0 \longrightarrow E_k \longrightarrow E_{k-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

where E_i 's are locally free of finite rank. Therefore, for any coherent sheaf \mathcal{G} , we can define the Tor functor⁷ of coherent sheaves by

$$\text{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = H_i(E_* \otimes_{\mathcal{O}_X} \mathcal{G}).$$

The rest of Section 1 follows from Chapter V.C in [Ser12].

Proposition 1.13. Let $X \in \text{Sm}/k$. Suppose $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(X)$ intersect properly, then

$$Z_a(\mathcal{F}) \cdot Z_b(\mathcal{G}) = \sum_{i=0}^{d_X} (-1)^i \cdot Z_{a+b-d_X}(\text{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})).$$

Proof. We only have to do it locally, so we can assume X to be affine, and count the coefficients of $\bar{\xi}$ where $\dim(\xi) = a + b - d_X$. It suffices to show that the stalks at ξ satisfies

$$\chi(F_\xi, G_\xi) = \sum_{\substack{\dim(\bar{\lambda})=a \\ \dim(\bar{\eta})=b \\ \xi \in \bar{\lambda} \cap \bar{\eta}}} \ell(\mathcal{F}_\lambda) \cdot \ell(\mathcal{G}_\eta) \cdot m_{\bar{\xi}}(\bar{\lambda}, \bar{\eta}).$$

Because our ring is Noetherian, then \mathcal{F} admits a filtration

$$0 = M_0 \subseteq \cdots \subseteq M_d = \mathcal{F}$$

such that $M_i/M_{i-1} \cong \mathcal{O}_X/\mathcal{I}$ is coherent for prime ideal \mathcal{I} . By the additivity of both sides of the isomorphism, we may assume $\mathcal{F} = \mathcal{O}_X/\mathfrak{p}$ with dimension at most a , where $\mathfrak{p} \sim \lambda \in X$. Similarly, we may assume $\mathcal{G} = \mathcal{O}_X/\mathfrak{q}$ with dimension at most b , where $\mathfrak{q} \sim \eta \in X$. Moreover, set $\xi \in \bar{\lambda} \cap \bar{\eta}$. By definition, we now have $\chi(\mathcal{F}_\xi, \mathcal{G}_\xi) = m_{\bar{\xi}}(\bar{\lambda}, \bar{\eta})$.

- If $\dim(\bar{\lambda}) = a$ and $\dim(\bar{\eta}) = b$, then the equality follows from the fact that $\ell(\mathcal{F}_\lambda) = \ell(\mathcal{G}_\eta) = 1$.
- If not, then either $\dim(\bar{\lambda}) < a$ or $\dim(\bar{\eta}) < b$, then $\bar{\lambda}$ and $\bar{\eta}$ do not intersect properly at $\bar{\xi}$, so both the left-hand side and the right-hand side become 0.

□

Proposition 1.14. The intersection product is commutative.

Proof. This is obvious since the Tor functor is commutative.

□

Proposition 1.15. The intersection product is associative.

Proof. Suppose we pick $\mathcal{F} \in K_a(X)$, $\mathcal{G} \in K_b(X)$, and $\mathcal{H} \in K_c(X)$ with support dimension at most a, b, c , respectively, and they intersect properly. Let L_* and M_* be free resolutions of \mathcal{F} and \mathcal{H} , respectively. Define a double complex $N_{ij} = L_i \otimes \mathcal{G} \otimes M_j$, then the associativity of tensor product allows us to calculate triple Tor

$$H_i(L_i \otimes H_j(\mathcal{G}) \otimes M_j) \cong \text{Tor}_i(\mathcal{F}, \mathcal{G}, \mathcal{H}) \cong H_i(H_j(L_i \otimes \mathcal{G}) \otimes M_j)$$

as the homology of two (tensor) double complexes. We obtain two spectral sequences

$${}^I E_{p,q}^2 = \text{Tor}_p(\mathcal{F}, \text{Tor}_q(\mathcal{G}, \mathcal{H})) \Rightarrow \text{Tor}_{p+q}(\mathcal{F}, \mathcal{G}, \mathcal{H})$$

⁷Since we are working over sheaves of \mathcal{O}_X -modules, using the same argument on the level of modules shows that the Tor functor is independent from the choice of resolution.

$${}^{II}E_{p,q}^2 = \mathrm{Tor}_p(\mathrm{Tor}_q(\mathcal{F}, \mathcal{G}), \mathcal{H}) \Rightarrow \mathrm{Tor}_{p+q}(\mathcal{F}, \mathcal{G}, \mathcal{H}).$$

Recall Euler-Poincaré characteristic is invariant with respect to taking spectral sequence $(*)$, then

$$\begin{aligned} Z_a(\mathcal{F}) \cdot ((Z_b \mathcal{G}) \cdot Z_c(\mathcal{H})) &= Z_a(\mathcal{F}) \cdot \sum_q (-1)^q Z_{b+c-d_X}(\mathrm{Tor}_q(\mathcal{G}, \mathcal{H})) \text{ by Proposition 1.13} \\ &= \sum_{p,q} (-1)^{p+q} Z_{a+b+c-2d_X}({}^I E_{p,q}^2) \text{ by Proposition 1.13} \\ &= \sum_i (-1)^i Z_{a+b+c-2d_X}(\mathrm{Tor}_i(\mathcal{F}, \mathcal{G}, \mathcal{H})) \text{ by } (*) \\ &= \sum_{p,q} (-1)^{p+q} Z_{a+b+c-2d_X}({}^{II} E_{p,q}^2) \text{ by } (*) \\ &= \sum_p Z_{a+b-d_X}(\mathrm{Tor}_p(\mathcal{F}, \mathcal{G})) \cdot Z_c(\mathcal{H}) \text{ by Proposition 1.13} \\ &= (Z_a(\mathcal{F}) \cdot Z_b(\mathcal{G})) \cdot Z_c(\mathcal{H}) \text{ by Proposition 1.13.} \end{aligned}$$

□

Definition 1.16. Suppose $X_1, X_2 \in \mathrm{Sm}/k$, with $\mathcal{F}_1 \in K_{a_1}(X_1)$ and $\mathcal{F}_2 \in K_{a_2}(X_2)$. We define the cross product of cycles to be

$$Z_{a_1}(\mathcal{F}_1) \times Z_{a_2}(\mathcal{F}_2) = Z_{a_1+d_{X_2}}(p_1^* \mathcal{F}_1) \cdot Z_{a_2+d_{X_1}}(p_2^* \mathcal{F}_2),$$

where $p_i : X_1 \times X_2 \rightarrow X_i$ is the projection for $i = 1, 2$.

Exercise 1.17. One should check that this is well-defined.

Remark 1.18. Suppose $X_1, X_2 \in \mathrm{Sm}/k$, with $\mathcal{F}_1 \in K_{a_1}(X_1)$, $\mathcal{F}_2 \in K_{b_1}(X_1)$, $\mathcal{G}_1 \in K_{a_2}(X_2)$ and $\mathcal{G}_2 \in K_{a_2}(X_2)$. Suppose $Z_{a_1}(\mathcal{F}_1) \cdot Z_{a_2}(\mathcal{G}_1)$ and $Z_{b_1}(\mathcal{F}_2) \cdot Z_{b_2}(\mathcal{G}_2)$ are defined, then

- $Z_{a_1}(\mathcal{F}_1) \times Z_{a_2}(\mathcal{G}_1)$ and $Z_{b_1}(\mathcal{F}_2) \times Z_{b_2}(\mathcal{G}_2)$ intersect properly on $X_1 \times X_2$, and
- $(Z_{a_1}(\mathcal{F}_1) \times Z_{a_2}(\mathcal{G}_1)) \cdot (Z_{b_1}(\mathcal{F}_2) \times Z_{b_2}(\mathcal{G}_2)) = (Z_{a_1}(\mathcal{F}_1) \cdot Z_{b_1}(\mathcal{F}_2)) \times (Z_{a_2}(\mathcal{G}_1) \cdot Z_{b_2}(\mathcal{G}_2)).$

1.3 PUSHOUT AND PULLBACK

Definition 1.19. Suppose X, Y are schemes of finite type over k , and let $f : X \rightarrow Y$ be a proper map. For every irreducible closed subset $c \subseteq X$ of dimension a , we define the direct image to be

$$f_* c = \begin{cases} [k(c) : k(f(c))] \cdot f(c) \in Z_a(Y), & \dim(f(c)) = a \\ 0, & \dim(f(c)) < a \end{cases}$$

to be the direct image of c under f .

Lemma 1.20. Suppose X and Y are schemes of finite type over k of the same dimension n , and that $f : X \rightarrow Y$ is proper, then there exists an open subset $U \subseteq Y$ such that $\dim(Y \setminus U) < n$ and $f : f^{-1}(U) \rightarrow U$ is a finite morphism.

Proof. Suppose $\xi \in Y$ has $\dim(\bar{\xi}) = n$. We can find $U \ni \xi$ such that $f|_U$ has finite fibers by Exercise II.3.7 from [Har13]. By Exercise III.11.2 in [Har13], such f is finite. □

Proposition 1.21. Let $f : X \rightarrow Y$ be a proper morphism between schemes over k of finite type, and let $\mathcal{F} \in K_a(X)$, then

1. $f_* \mathcal{F} \in K_a(Y)$ and the right derived $R^i f_* \mathcal{F} \in K_{a-1}(Y)$ for $i > 0$.
2. $f_* Z_a(\mathcal{F}) = Z_a(f_* \mathcal{F})$.

Proof. 1. By Theorem III.8.8 from [Har13], $R^i f_* \mathcal{F}$ is coherent for all $i \geq 0$. We have $\text{supp}(R^i f_* \mathcal{F}) \subseteq \text{supp}(\mathcal{F})$. If f is finite, then f_* is exact, so $R^i f_* \mathcal{F} = 0$ for $i > 0$. For general cases, we may assume $\dim(f(\text{supp}(\mathcal{F}))) = a$ and set $W = \text{supp}(\mathcal{F})$. We have a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{h} & f(W) \\ i \downarrow & & \downarrow j \\ X & \xrightarrow{f} & Y \end{array}$$

where h is also proper. By Lemma 1.20, there exists $V \subseteq f(W)$ such that $\dim(f(W) \setminus V) < a$ and $h|_V$ is finite. Let \mathcal{I} be the ideal sheaf of W , then $\mathcal{I}^s \mathcal{F} / \mathcal{I}^{s+1} \mathcal{F} = i_* i^* \mathcal{I}^s \mathcal{F} / \mathcal{I}^{s+1} \mathcal{F}$. By the long exact sequence, it suffices to prove the case for $\mathcal{F} = i_* \mathcal{G}$. Then

$$(R^k f_*) i_* \mathcal{G} = R^k (fi)_* \mathcal{G} = j_* R^k h_* \mathcal{G}.$$

It suffices to consider h , but

$$(R^k h_* \mathcal{G})_V = R^k h_*(\mathcal{G}|_{f^{-1}(V)}) = 0$$

for $k > 0$, so $\text{supp}(R^k h_* \mathcal{G}) \subseteq f(W) \setminus V$ if $k > 0$.

2. If f is finite, let us write down the coefficients of ξ of dimension a on both sides, namely

$$\ell((f_* \mathcal{F})_\xi) = \sum_{\substack{\eta \in f^{-1}(\xi) \\ \dim(\eta) = a}} \ell(F_\eta) \cdot [k(\bar{\eta}) : k(\overline{f(\eta)})].$$

By additivity, one reduces to the case when X is affine and $F = \mathcal{O}_X / \mathfrak{p}$. For the general case, use Lemma 1.20, and the case where f is finite. \square

Definition 1.22. Suppose $f : X \rightarrow Y$ where $Y \in \text{Sm}/k$ and X is closed in $Z \in \text{Sm}/k$. Define $j : X \rightarrow Z \times Y$ to be the graph map. For any $C \in Z_a(X)$ and $D \in Z_b(Y)$ such that C and $f^{-1}(D)$ intersect properly, define the intersection cycle to be

$$C \cdot_f D = j_*^{-1}(j(C) \cdot (Z \times D)) \in Z_{a+b-d_Y}(X)$$

In particular, $f^*(D) = X \cdot_f D$ for $C = X$.

Proposition 1.23. Using the notation above, for $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(Y)$, if \mathcal{F} and $f^* \mathcal{G}$ intersect properly, we have

$$Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G}) = \sum_{i=0}^{d_Y} (-1)^i Z_{a+b-d_Y}(L_i(\mathcal{F} \otimes f^* \mathcal{G}))$$

Proof. Denote $p_2 : Z \times Y \rightarrow Y$ to be the projection onto the second coordinate. By linearity, $Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G}) = j_*^{-1}(Z_a(j_* \mathcal{F}) \cdot Z_{b+d_Z}(p_2^* \mathcal{G}))$ for $j : X \rightarrow Z \times Y$. Suppose $L_* \rightarrow \mathcal{G}$ is the locally free resolution of \mathcal{G} . Note that for all $i \geq 0$, we have

$$j^*(j_* \mathcal{F} \otimes p_2^* L_i) = F \otimes f^* L_i,$$

which induces an isomorphism

$$j_* \mathcal{F} \otimes p_2^* L_i = j_*(\mathcal{F} \otimes f^* L_i).$$

Hence $\text{Tor}_i^{\mathcal{O}_{Z \times Y}}(j_* \mathcal{F}, p_2^* \mathcal{G}) = j_* L_i(F \otimes f^* \mathcal{G})$. So

$$j_*^{-1} Z_{a+b-d_Y}(\text{Tor}_i^{\mathcal{O}_{Z \times Y}}(j_* \mathcal{F}, p_2^* \mathcal{G})) = Z_{a+b-\dim(Y)}(L_i(F \otimes f^* \mathcal{G})).$$

Therefore the statement follows. \square

Proposition 1.24. Let $X \in \text{Sm}/k$, $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(X)$ such that \mathcal{F} and \mathcal{G} intersect properly. Let $\Delta : X \rightarrow X \times X$ be the diagonal map, then

$$\Delta^*(Z_a(\mathcal{F}) \times Z_b(\mathcal{G})) = Z_a(\mathcal{F}) \cdot Z_b(\mathcal{G}).$$

Proof. See page 115 of [Ser12]. □

Proposition 1.25. f^* is compatible with intersection product, and $f^*g^* = (gf)^*$.

Proof. See page 119 of [Ser12]. □

Lemma 1.26. Let \mathcal{A} be an abelian category with enough projectives (respectively, injectives) and F be a right (respectively, left) exact functor from \mathcal{A} . Suppose C is chain complex in \mathcal{A} , then there exists a double complex $M_{*,*}$ in \mathcal{A} such that

$${}^I E_{p,q}^2 = L_p F H_q(C) \quad (\text{respectively, } R^{-p} F(H_q(C))).$$

Proof. To do this when F is right exact, use the Cartan-Eilenberg resolution⁸ $C_* \rightarrow C$ and consider the double complex FC_* . □

Proposition 1.27. Suppose $f : X \rightarrow Y$ is in Sm/k , suppose $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(Y)$, then

$$Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G}) = Z_a(\mathcal{F}) \cdot f^* Z_b(\mathcal{G})$$

if both sides are defined.

Proof. We may assume X is affine. Let $L_* \rightarrow \mathcal{G}$ be a free resolution and apply Lemma 1.26 to f^*L_* and $F \otimes -$, then we find a double complex such that

$$\begin{aligned} {}^I E_{p,q}^2 &= \text{Tor}_p(\mathcal{F}, L_q f^* \mathcal{G}) \\ {}^{II} E_{p,q}^2 &= L_p(F \otimes f^* \mathcal{G}). \end{aligned}$$

□

Proposition 1.28. Let $X \subseteq Z$ and $Y, Z \in \text{Sm}/k$ and $f : X \rightarrow Y$ be proper. Suppose $\mathcal{F} \in K_a(X)$ and $\mathcal{G} \in K_b(Y)$, and suppose \mathcal{F} and $f^* \mathcal{G}$ intersect properly, then

$$f_*(Z_a(\mathcal{F}) \cdot_f Z_b(\mathcal{G})) = (f_* Z_a(\mathcal{F})) \cdot Z_b(\mathcal{G}).$$

Proof. Pick $L_* \rightarrow \mathcal{G}$ to be a resolution and apply Lemma 1.26 to $F \otimes f^* L_*$ and f_* , then we have a double complex $M_{*,*}$ such that

$${}^I E_{p,q}^2 = R^{-p} f_* L_q(F \otimes f^* \mathcal{G}).$$

On the other hand, $H_q(M_{*,n}) = R^{-q} f_*(F \otimes f^* L_n) = (R^{-q} f_* \mathcal{F}) \otimes L_n$, therefore

$${}^{II} E_{p,q}^2 = \text{Tor}_p(R^{-q} f_* \mathcal{F}, \mathcal{G}).$$

□

Corollary 1.29. Under the same hypothesis as Proposition 1.28, we have

$$f_*(Z_a(\mathcal{F}) \cdot f^*(Z_b(\mathcal{G}))) = f_*(Z_a(\mathcal{F})) \cdot Z_b(\mathcal{G}).$$

2 SHEAVES WITH TRANSFERS

We fix a base scheme $S \in \text{Sm}/k$.

Definition 2.1. Let $X, Y \in \text{Sm}/S$, then we define the group of finite correspondences

$$\text{Cor}_S(X, Y) = \mathbb{Z}\{\text{irreducible closed } C \subseteq X \times_S Y \mid C \rightarrow X \text{ finite, } \dim(C) = \dim(X)\}$$

to be the free abelian group generated by elementary correspondences from X to Y .

Example 2.2. For any $f : X \rightarrow Y$, the graph $\Gamma_f = (x, f(x)) \subseteq X \times_S Y$ is a finite correspondence from $X \rightarrow Y$.

⁸See Proposition 11 on page 210 of [GM13].

Example 2.3. If $f : X \rightarrow Y$ is finite and $\dim(X) = \dim(Y)$, then the graph Γ_f is also a finite correspondence from $Y \rightarrow X$.

Definition 2.4. Define an additive category Cor_S , whose objects are the same as Sm/S and the hom set $\text{Hom}_{\text{Sm}/S}(X, Y) = \text{Cor}_S(X, Y)$ as in [Definition 2.1](#). The contravariant additive functors

$$F : \text{Cor}_S^{\text{op}} \rightarrow \text{Ab}$$

are called presheaves with transfers. The corresponding category is denoted by $\text{PSh}(S) = \text{PSh}(\text{Cor}_S)$, which is abelian with enough injectives and projectives. We have a functor $\gamma : \text{Sm}/S \rightarrow \text{Cor}_S$ by [Example 2.3](#).

Example 2.5. Every $X \in \text{Sm}/S$ gives an element $\mathbb{Z}(X) \in \text{PSh}(S)$ defined by $\mathbb{Z}(X)(Y) = \text{Cor}_S(Y, X)$. Therefore, we say $\mathbb{Z}(X)$ is the presheaf with transfers represented by X . By Yoneda Lemma we know there is a natural isomorphism

$$\text{Hom}_{\text{PST}}(\mathbb{Z}(X), F) \cong F(X).$$

Moreover, representable functors give embeddings of Sm/S and Cor_S into $\text{PSh}(S)$ via

$$\begin{aligned} \text{Sm}/S &\xrightarrow{\gamma} \text{Cor}_S \longrightarrow \text{PSh}(S) \\ X &\longmapsto X \longmapsto \mathbb{Z}(X) \end{aligned}$$

In particular, $\mathbb{Z}(S) = \mathbb{Z}$.

Example 2.6. The presheaves \mathcal{O} and \mathcal{O}^* are in $\text{PSh}(S)$. For any $C \in \text{Cor}_S(X, Y)$ and $f \in \mathcal{O}(Y)$ (respectively, $\mathcal{O}^*(Y)$), we have a diagram

$$\begin{array}{ccc} C & \xrightarrow{i} & X \times_S Y \xrightarrow{p_2} Y \\ & & \downarrow p_1 \\ & & X \end{array}$$

and can define $\mathcal{O}(C)(f) = \text{Tr}_{C/X}((p_2 \circ i)^*(f))$ (respectively, $\mathcal{O}^*(C)(f) = \text{N}_{C/X}((p_2 \circ i)^*(f))$).

Definition 2.7. Let us describe the composition in Cor_S . Suppose $f \in \text{Cor}_S(X, Y)$ and $g \in \text{Cor}_S(Y, Z)$, then from the diagram

$$\begin{array}{ccc} & X \times_S Z & \\ & \uparrow p_{13} & \\ X \times_S Y \times_S Z & \xrightarrow{p_{23}} & Y \times_S Z \\ & \downarrow p_{12} & \\ & X \times_S Y & \end{array}$$

we define the composition $g \circ f = p_{13*}(p_{23}^*(g)p_{12}^*(f))$. (Check all intersections are proper.)

Proposition 2.8. The composition law is associative.

Proof. Suppose

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

are morphisms in Cor_S , then we have two Cartesian squares

$$\begin{array}{ccc} X \times_S Y \times_S Z \times_S W & \longrightarrow & X \times_S Z \times_S W \\ \downarrow & & \downarrow \\ X \times_S Y \times_S Z & \longrightarrow & X \times_S Z \end{array}$$

and

$$\begin{array}{ccc} X \times_S Y \times_S Z \times_S W & \longrightarrow & X \times_S Y \times_S W \\ \downarrow & & \downarrow \\ Y \times_S Z \times_S W & \longrightarrow & Y \times_S W \end{array}$$

□

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