Motivic Homotopy Theory Notes

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1 Commutative Monoids and Commutative Semirings as Functors

The materials from this section can be found in [EH23], Chapter 1.1-1.2.

1.1 Spans and Monoids

Definition 1.1. A commutative monoid $(M, \times, 1)$ has a multiplication operation

$$\begin{aligned} \times : M \times M &\to M \\ (a, b) &\mapsto a \times b =: ab \end{aligned}$$

that satisfies $ab = ba$, as well as the associativity by the pentagon axiom

$$\begin{tikzcd}
 & & (ab)(cd) \arrow[dll, no head] \arrow[drr, no head] & & \\
((ab)c)d \arrow[dr, no head] & & & & a(b(cd)) \\
 & (a(bc))d \arrow[rr, no head] & & a((bc)d) \arrow[ur, no head] &
\end{tikzcd}$$

Definition 1.2. Denote $\mathbb{F} = \mathbf{FinSet}$ to be the finite category of finite sets, then a commutative monoid M induces a contravariant functor

$$\begin{aligned} \bar{M} : \mathbb{F}^{\mathrm{op}} &\to \mathbf{Set} \\ I &\mapsto M^I \\ (I \xleftarrow{f} S) &\mapsto (M^I \xrightarrow{f^*} M^S) \\ (a_i)_{i \in I} &\mapsto (a_{f(s)})_{s \in S} \end{aligned}$$

and similarly a covariant functor

$$\begin{aligned} \bar{M}' : \mathbb{F} &\to \mathbf{Set} \\ I &\mapsto M^I \\ (s \xrightarrow{g} I) &\mapsto (M^S \xrightarrow{g_\otimes} M^I) \\ (b_s)_{s \in S} &\mapsto \left(\prod_{s \in g^{-1}(j)} b_s \right)_{j \in J} \end{aligned}$$

Now given the construction in Definition 1.2 above, suppose we have a zigzag

$$\begin{tikzcd}
 & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\
I & & J
\end{tikzcd} \tag{1.3}$$

we can use $\bar{M}$ and $\bar{M}'$ and obtain f^* and $g_\otimes$. One can map Diagram 1.3 to a morphism $g_\otimes f^* : M^I \to M^J$.

Remark 1.4. To define a functor precisely, we need to specify what category Diagram 1.3 lies in. As we will see later, we want a category with the same objects as $\mathbb{F}$, and morphisms are the zigzags of the form Diagram 1.3, which are called spans (or correspondences).

To define the composition of spans as morphisms, we should think of a diagram

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & & T \arrow[dl, "u"'] \arrow[dr, "v"] & \\ I & & J & & K \end{tikzcd} \tag{1.5}$$

The two zigzags give rise to $g_{\otimes}f^*$ and $v_{\otimes}u^*$. For compositions to be well-defined, we should map this diagram to $v_{\otimes}u^*g_{\otimes}f^*$. In order to obtain functoriality, we would hope

$$v_{\otimes}u^*g_{\otimes}f^* = v_{\otimes}g_{\otimes}u^*f^* = (vg)_{\otimes}(fu)^*$$

using some sort of base-change phenomenon. This is certainly not true. As a remedy, we complete Diagram 1.5 to

$$\begin{tikzcd} & & A \arrow[dl, "u'"'] \arrow[dr, "g'"] & & \\ & S \arrow[dl, "f"'] \arrow[dr, "g"] & & T \arrow[dl, "u"'] \arrow[dr, "v"] & \\ I & & J & & K \end{tikzcd} \tag{1.6}$$

as we obtain $u^*g_{\otimes} : M^S \to M^T$ defined by the composition

$$(b_s)_{s\in S} \mapsto \left(\prod_{s\in g^{-1}(j)} b_s\right)_{j\in J} \mapsto \left(\prod_{s\in g^{-1}(u(t))} b_s\right)_{t\in T}.$$

Remark 1.7. If Diagram 1.6 is a commutative diagram, then there is a restriction of u' given by $u' : g'^{-1}(t) \to g^{-1}(u(t))$. In particular, if Diagram 1.6 is a pullback diagram, then this restriction map is a bijection. In this setting, the map $u^*g_{\otimes}$ sends $(b_s)_{s\in S}$ to

$$\left(\prod_{s\in g^{-1}(u(t))} b_s\right)_{t\in T} = \left(\prod_{a\in g'^{-1}(t)} b_{u'(a)}\right)_{t\in T} = g'_{\otimes}u'^*(b_s)_{s\in S}.$$

Therefore,

$$v_{\otimes}u^*g_{\otimes}f^* = v_{\otimes}g'_{\otimes}u'^*f^* = (vg')_{\otimes}(fu')^*.$$

Definition 1.8. We define $\mathbf{Span}(\mathbb{F})$ to be the category of span of $\mathbb{F}$, where objects are finite sets as in $\mathbb{F}$, and morphisms of the form $I \to J$ are the zigzag of the form $I \leftarrow S \to J$. The composition of morphisms $I \to J \to K$ on the zigzag is now defined by $I \leftarrow A \to K$ using the diagram

$$\begin{tikzcd} & & A \arrow[dl] \arrow[dr] \arrow[dll, bend right, dashed] \arrow[drr, bend left, dashed] & & \\ & S \arrow[dl] \arrow[dr] & & T \arrow[dl] \arrow[dr] & \\ I \arrow[rr, dashed] & & J \arrow[rr, dashed] & & K \end{tikzcd}$$

whenever A is constructed as the pullback, otherwise known as the outer span $S \times_K T$.

Remark 1.9. One issue that persists from this construction is the fact that the pullback A is not unique, thus the composition of morphisms is not unique. (This may be unique up to unique isomorphism.) With this in mind, $\mathbf{Span}(\mathbb{F})$ admits a $(2,1)$-category structure instead of an ordinary category.

The 2-morphisms of $\mathbf{Span}(\mathbb{F})$ are defined by $S \to S'$ via

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] \arrow[dd] & \\ I \arrow[rr, dashed] & & J \\ & S' \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Moreover, these 2-morphisms are isomorphisms (of spans) and hence invertible, therefore admitting the $(2,1)$-category structure.

Remark 1.10. The functors we defined in Definition 1.2 can be extended to a functor

$$\tilde{M} : \mathbf{Span}(\mathbb{F}) \to \mathbf{Set}$$

such that $\tilde{M}\big|_{\mathbb{F}^{\mathrm{op}}} \in \mathbf{Fun}^{\times}(\mathbb{F}^{\mathrm{op}}, \mathbf{Set})$. To see this, recall that there is a natural inclusion

$$\begin{aligned} \mathbb{F}^{\mathrm{op}} &\hookrightarrow \mathbf{Span}(\mathbb{F}) \\ A &\mapsto A \\ (I \leftarrow S) &\mapsto (I \leftarrow S \xrightarrow{=} S) \end{aligned}$$

then the extension $\tilde{M}$ is the functor we want, as the product and coproduct of the 2-category $\mathbf{Span}(\mathbb{F})$ are both the coproduct on $\mathbf{FinSet}$, i.e., the disjoint union.

Remark 1.11. In fact, given any category $\mathscr{C}$ with finite products, then there is an identification of commutative monoids on $\mathscr{C}$ with product-preserving functors $\mathbf{Span}(\mathbb{F}) \to \mathscr{C}$. Moreover, this is true homotopically, c.f., [Cra09] and [Cra11].

This is the story of how we induce functors from commutative monoids, where the span exhibits a bivariant phenomenon. We will see below that there is a similar one for commutative semirings, which exhibits distributivity.

1.2 Bispans and Semirings

Definition 1.12. A commutative semiring $(R, +, \times, 0, 1)$ is a set R equipped with operations $+$ and $\times$ as well as additive identity 0 and multiplicative identity 1. However, we do not assume the existence of additive inverse and/or multiplicative inverse. Therefore, R is both an additive monoid and a multiplicative monoid.

Using the same construction in Definition 1.2, we have a functor

$$\begin{aligned} \mathbb{F} &\to \mathbf{Set} \\ I &\mapsto R^I \end{aligned}$$

which induces a functor

$$\begin{aligned} \tilde{R}_{\times} : \text{“}\mathbf{Span}(\mathbb{F})\text{”} &\to \mathbf{Set} \\ (I \xleftarrow{f} S \xrightarrow{g} J) &\mapsto g_{\otimes} f^* \end{aligned}$$

Now note that we still have an additive monoidal structure on R, so we would hope to define a functor of the form

$$\begin{aligned} \tilde{R}_{+} : \text{“}\mathbf{Span}(\mathbb{F})\text{”} &\to \mathbf{Set} \\ ? &\mapsto g_{\oplus} f^* \end{aligned}$$

for some unknown category "$\mathbf{Span}(\mathbb{F})$". These two functors altogether shall define a desired functor $\tilde{R}$: "$\mathbf{Span}(\mathbb{F})$" $\to$ $\mathbf{Set}$. In particular, admitting two different structures here already tells us that the spans are no longer suitable, and a natural adaptation would be bispans.

Definition 1.13. A bispan (or a polynomial diagram) from I to J is given by a diagram

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[dl, "p"'] & Y \arrow[dr, "q"] & \\ I & & & J \end{tikzcd}$$

The category of bispans, denoted $\mathbf{Bispan}(\mathbb{F})$, has objects (again) the same with objects of $\mathbb{F}$, and morphisms are bispans.

Given a semiring R, we would want to construct a functor

$$\begin{aligned} \mathbf{Bispan}(R) &\to \mathbf{Set} \\ I &\mapsto R^I \\ (I \xleftarrow{p} X \xrightarrow{f} Y \xrightarrow{q} J) &\mapsto q_{\oplus} f_{\otimes} p^* \end{aligned}$$

where

$$\begin{aligned} p^* : R^I &\to R^X \\ p^*(\varphi)(x) &= \varphi(px), \end{aligned}$$

$$\begin{aligned} f_{\otimes} : R^X &\to R^Y \\ f_{\otimes}(\varphi)(y) &= \prod_{x \in f^{-1}(y)} \varphi(x), \end{aligned}$$

and

$$\begin{aligned} q_{\oplus} : R^Y &\to R^J \\ q_{\oplus}(\varphi)(j) &= \sum_{y \in q^{-1}(j)} \varphi(y), \end{aligned}$$

which represent composition (as pullback), fiberwise multiplication (as pushforward), and fiberwise addition (as pushforward), respectively. Altogether, this gives

$$\begin{aligned} q_{\oplus} f_{\otimes} p^* : M^I &\to M^J \\ (a_i)_{i \in I} &\mapsto \left(\sum_{y \in q^{-1}(j)} \prod_{x \in f^{-1}(y)} a_{p(x)} \right)_{j \in J}. \end{aligned}$$

Again, to construct such a functor, we need to consider the composition of bispans:

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[dl, "p"'] & Y \arrow[dr, "q"] & & X' \arrow[r, "g"] \arrow[dl, "u"'] & Y' \arrow[dr, "v"] & \\ I & & & J & & & K \end{tikzcd}$$

As we have seen previously, we need to study the pullback structure so that we can resolve $v_{\oplus} g_{\otimes} u^* q_{\oplus} f_{\otimes} p^*$. Using similar construction, we have

$$v_{\oplus} g_{\otimes} u^* q_{\oplus} f_{\otimes} p^* = v_{\oplus} g_{\otimes} q'_{\oplus} u'^* f_{\otimes} p^*$$

$$\begin{aligned}
&= v_\oplus q''_\oplus g'_\otimes u'^* f_\otimes p^* \\
&= v_\oplus q''_\oplus g'_\otimes f'_\otimes u''^* p^* \\
&= (vq'')_\oplus (g'f')_\otimes (pu'')^*
\end{aligned}$$

assuming we can construct $g'_\otimes$ and $q''_\oplus$ such that $g_\otimes q'_\oplus = q''_\oplus g'_\otimes$. That is, we have constructed two pullback squares

$$\begin{tikzcd}
&& A \arrow[r, "f'"] \arrow[dl, "u''"'] & Y \times_K X' \arrow[dl, "u'"'] \arrow[dr, "q'"] &&& \\
& X \arrow[r, "f"] \arrow[dl, "p"'] & Y \arrow[dr, "q"] && X' \arrow[r, "g"] \arrow[dl, "u"'] & Y' \arrow[dr, "v"] & \\
I &&& J &&& K
\end{tikzcd} \tag{1.14}$$

To deal with this, recall that addition distributes over multiplication, therefore given any

$$I \xrightarrow{u} J \xrightarrow{v} K$$

we know $v_\otimes u_\oplus : R^I \to R^K$ is the mapping defined by

$$(a_i)_{i\in I} \mapsto \left(\prod_{j\in v^{-1}(k)} \sum_{i\in u^{-1}(j)} a_j \right)_{k\in K} = \left(\sum_{(i_j)\in \prod_{j\in v^{-1}(k)} u^{-1}(j)} \prod_{t\in v^{-1}(k)} a_{i_t} \right)_{k\in K}. \tag{1.15}$$

The goal is to identify the said image from Equation (1.15). Recall that the slice categories $\mathbf{FinSet/K}$ and $\mathbf{FinSet/J}$ are involved in a pullback/pushforward adjunction

$$\begin{tikzcd}
\mathbf{FinSet/K} \arrow[r, "\cong"] \arrow[d, shift right, "v^*"'] & \mathbf{Fun(K, Set)} \\
\mathbf{FinSet/J} \arrow[r, "\cong"] \arrow[u, shift right, "v_*"'] & \mathbf{Fun(J, Set)}
\end{tikzcd} \tag{1.16}$$

where

- $\mathbf{FinSet/J} \cong \mathbf{Fun(J, Set)}$ is a Grothendieck correspondence, where given $u : I \to J$, we obtain a functor
$$\begin{aligned} J &\to \mathbf{FinSet} \\ j &\mapsto u^{-1}(j) \end{aligned}$$
- $\mathbf{FinSet/K} \cong \mathbf{Fun(K, Set)}$ is a Grothendieck correspondence, where given $v : J \to K$, we obtain a functor
$$\begin{aligned} K &\to \mathbf{FinSet} \\ k &\mapsto v^{-1}(k) \end{aligned}$$
- the Grothendieck correspondences give rise to (co)Carteisan fibrations;
- $h = v_* u \in \mathbf{Set/K}$ is a functor, and by the correspondence we obtain a functor
$$\begin{aligned} h' : K &\to \mathbf{Set} \\ k &\mapsto \prod_{j\in v^{-1}(k)} u^{-1}(j) = \prod_{j\in h^{-1}(k)} u^{-1}(j) \end{aligned}$$

- v^* is the pullback along $v : J \to K$. In particular, consider the counit $\varepsilon : v^*v_*I \to I$ of the adjunction, then for $X = v_*I$, the pullback $v^*X = v^*v_*I$ gives a counit ε in the diagram

$$\begin{tikzcd}
& v^*v_*I \arrow[r, "\tilde{v}"] \arrow[dl, "\varepsilon"'] \arrow[dd] & v_*I \arrow[dd, "h"] \\
I \arrow[dr, "u"'] & & \\
& J \arrow[r, "v"'] & K
\end{tikzcd} \tag{1.17}$$

We now make an effort to show that Diagram 1.17 actually commutes.

For any $k \in K$, we pullback $\alpha \in X$ such that $h(\alpha) = k$, but by the correspondence we know α is in the image of k along h'. Similarly, for $k \in K$, we pullback $j \in J$, and using the same argument we would then conclude that the pullback element in v^*X is just a pair (α, j).

Now let $i = \varepsilon(\alpha, j)$, but one can identify i to be the image of α under the projection $h^{-1}(k) \to \prod_{j' \in v^{-1}(k)} u^{-1}(j') \to u^{-1}(j)$. Therefore, $\alpha = (i_j)_{j \in v^{-1}(k)}$. For any fixed α, one can then identify

$$\prod_{t \in v^{-1}(k)} a_{it} = \prod_{j \in v^{-1}(k)} a_{\varepsilon(\alpha,j)}.$$

Therefore, the image of Equation (1.15) is

$$\left(\sum_{\alpha \in h^{-1}(k)} \prod_{j \in v^{-1}(k)} a_{\varepsilon(v,j)} \right)_{k \in K} = h_\oplus \tilde{v}_\otimes \varepsilon^*(a_i)_{i \in I}.$$

In particular, we obtain

$$v_\otimes u_\oplus = h_\oplus \tilde{v}_\otimes \varepsilon^*,$$

i.e., Diagram 1.17 commutes, which describes the distributivity.

Let us go back to Diagram 1.14. Using Diagram 1.17, we extend the diagram to

$$\begin{tikzcd}
& & A \arrow[r, "f'"] \arrow[dl, "u''"'] & Y \times_K X' \arrow[dl, "u'"'] \arrow[dr, "q'"] & B \arrow[l, "\varepsilon"'] \arrow[d, "g^*g_*"] \arrow[r, "\tilde{g}"] & Z \arrow[d, "h=g_*q'"] & \\
& X \arrow[r, "f"] \arrow[dl, "p"'] & Y \arrow[dr, "q"] & & X' \arrow[r, "g"] \arrow[dl, "u"'] & Y' \arrow[dr, "v"] & \\
I & & & J & & & K
\end{tikzcd}$$

which can be extended by taking one last pullback

$$\begin{tikzcd}
& & & C \arrow[dl, "\varepsilon'"'] \arrow[dr, "f''"] \arrow[dlll, dashed, bend right, "(-)^*"'] \arrow[rr, dashed, bend left, "\otimes"] & & & \\
& & A \arrow[r, "f'"] \arrow[dl, "u''"'] & Y \times_K X' \arrow[dl, "u'"'] \arrow[dr, "q'"] & B \arrow[l, "\varepsilon"'] \arrow[d, "g^*g_*"] \arrow[r, "\tilde{g}"] & Z \arrow[d, "g_*q'"] \arrow[ddr, dashed, bend left, "\oplus"] & \\
& X \arrow[r, "f"] \arrow[dl, "p"'] & Y \arrow[dr, "q"] & & X' \arrow[r, "g"] \arrow[dl, "u"'] & Y' \arrow[dr, "v"] & \\
I \arrow[uuurrr, dashed, bend left] & & & J & & & K
\end{tikzcd}$$

and we define the composition to be the outer bispan in this diagram.

Remark 1.18. An explicit construction of this $(2,1)$-category $\mathbf{Bispan}(\mathbb{F})$ can be found in [Cra09], where it is proven that the category has a product structure given by coproducts of $\mathbf{FinSet}$. In this sense, commutative semirings in a category $\mathscr{S}$ correspond to functors $\mathbf{Bispan}(\mathbb{F}) \to \mathscr{S}$ that preserve finite products.

Definition 1.19. As a dual notion to span, a cospan is a zigzag of the form

$$\begin{tikzcd} & S & \\ I \arrow[ur, "f"] & & J \arrow[ul, "g"'] \end{tikzcd}$$

Remark 1.20. The duality shows an equivalence of categories $\mathbf{Span}(\mathscr{C}) \cong \mathbf{Cospan}(\mathscr{C}^{\mathrm{op}})$ as $(2,1)$-categories.

2 ∞-CATEGORIES

2.1 CONSTRUCTIONS ON ∞-CATEGORIES

Definition 2.1. Let $r \leqslant n \leqslant \infty$. An (n, r)-category has the usual objects and (1-)morphisms like an ordinary category, but also i-morphisms for $0 \leqslant i < \infty$, such that

- when $i > r$, every i-morphism is invertible, and
- when $i > n$, every i-morphism is trivial.

We mostly consider the $(\infty, 1)$-categories. Let $\mathbf{sSet} = \mathbf{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set})$ be the category of simplicial sets, then any simplicial set $X \in \mathbf{sSet}$ is a diagram of the form

$$\cdots \; \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} \; X_2 \; \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow} \; X_1 \; \substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow} \; X_0 \tag{2.2}$$

where $X_n = X([n]) \in \mathbf{Set}$ is the set of n-simplices of X. In Diagram 2.2, the blue arrows $X_n \to X_{n-1}$ are called the face maps, as they assign each n-simplex to the face not containing the ith vertex for $0 \leqslant i \leqslant n$; the red arrows $X_n \to X_{n+1}$ are called the degeneracy maps, as they assign each n-simplex to the degenerate $(n+1)$-simplex by duplicating the ith vertex. These maps satisfy the simplicial identity, c.f., [GJ09], that is,

- if $i < j$, then $\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$;
- if $i > j$, then $s_i \circ s_j = s_j \circ s_{i-1}$;
- face maps and degeneracy maps are compatible, as

$$\partial_i \circ s_j = \begin{cases} s_{j-1} \circ \partial_i, & i < j \\ \mathrm{id}_n, & i \in \{j, j+1\} \\ s_i \circ \partial_{i-1}, & i > j+1 \end{cases}$$

Remark 2.3. The degeneracy maps usually play a role when the algebraic structure involves a unital object, e.g., existence of monoidal structure. They are often times omitted when, for example, we study non-unital monoidal objects, in which case we only draw the face maps.

Definition 2.4. A morphism $p : X \to Y$ is called an inner fibration if for every $0 < i < n$, any commutative diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook, "i"'] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{2.5}$$

in $\mathbf{sSet}$ admits a solution.

Definition 2.6. We say a simplicial set $X \in \mathbf{sSet}$ is an $(\infty, 1)$-category[1] if $X \to * \cong \Delta^0$ is an inner fibration.

Remark 2.7. The following conditions are equivalent:

- $X \in \mathbf{sSet}$ is an $(\infty, 1)$-category;
- the induced map $i^* : X^{\Delta^n} \to X^{\Lambda^n_i}$ of i from Diagram 2.5 is a trivial Kan fibration for all $0 < i < n$;

[1] Alternatively, we also call it an ∞-category or a quasi-category, depending on sources.

- the induced map $i^*: X^{\Delta^2} \to X^{\Lambda^2_1}$ of i from Diagram 2.5 is a trivial (acyclic) Kan fibration;[2]

  Note that a vertex $v \in X^{\Lambda^2_1}$ is just a map $\Lambda^2_1 \to X$ from the inner horn. Let us draw the inner horn as

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd}$$

  then its image in X is the pair of composable morphisms in X. Now a vertex in X^{Δ^2} is a 2-simplex in X, therefore for each pair of composable morphisms

$$\begin{tikzcd} & \bullet \arrow[dr, "g"] & \\ \bullet \arrow[ur, "f"] & & \bullet \end{tikzcd}$$

  it can be completed to a 2-simplex σ

$$\begin{tikzcd} & 1 \arrow[dr, "g=d_0\sigma"] & \\ 0 \arrow[ur, "f=d_2\sigma"] \arrow[rr, dash, "d_1\sigma"'] \arrow[ur, phantom, shift right=4ex, "\sigma"] & & 2 \end{tikzcd}$$

  We then define the composite $g \circ f = d_1\sigma$. The trivial fibration condition then means that once we pick composable arrows in X, then the fiber over vertex $v \in X^{\Lambda^2_1}$ will be a trivial Kan complex. Therefore, the conditions above are equivalent to

- the composition problem has a unique solution (up to contractible space of choices).

The $(\infty, 1)$-categories in $\mathbf{sSet}$ forms a full subcategory, denoted $\mathbf{qCat}$.

Remark 2.8. For $K \in \mathbf{sSet}$, and $\mathscr{C} \in \mathbf{qCat}$, we have $\mathscr{C}^K = \mathbf{Fun}(\mathbf{K}, \mathscr{C}) \in \mathbf{sSet}$.

We call an $(\infty, 1)$-category a weak Kan complex, which describes the property that it is a simplicial set for which all inner horns have a filler. It is weaker than a Kan complex, which exhibits the property that every horn has a filler.

Definition 2.9. Let $\mathscr{C} \in \mathbf{qCat}$, we say a simplicial subset $\mathscr{C}' \subseteq \mathscr{C}$ is an $(\infty, 1)$-subcategory if $i : \mathscr{C}' \hookrightarrow \mathscr{C}$ is an inner fibration.

Recall that for any $(\infty, 1)$-category $\mathscr{C}$, its homotopy category $h\mathscr{C}$ is given by quotienting homotopy relations, which identifies 1-morphisms that are connected by some 2-morphism.[3] With this, $h\mathscr{C} \in \mathbf{Cat}$ is an ordinary 1-category. Conversely, given any ordinary category, one can show that its nerve has a simplicial set structure, and in particular becomes an $(\infty, 1)$-category. Now note that the two functors $h(-)$ and $N(-)$ give an adjunction

$$\begin{tikzcd} \mathbf{qCat} \arrow[d, shift right, "h"'] \\ \mathbf{Cat} \arrow[u, shift right, "N"'] \end{tikzcd} \tag{2.10}$$

of 1-categories. The unit of this adjunction

$$\begin{aligned} F : \mathbf{qCat} &\to \mathbf{qCat} \\ \mathscr{C} &\mapsto N(h\mathscr{C}) \end{aligned}$$

[2] This equivalence is given by the Joyal model structure on $\mathbf{sSet}$, c.f., [Lur09].

[3] In particular, this is the restriction of the natural functor $\mathbf{sSet} \to \mathbf{Cat}$ to $\mathbf{qCat}$.

is an inner fibration. For any subcategory $A \subseteq h\mathscr{C}$, we consider the mapping

$$A \mapsto F^{-1}(NA) = NA \times_{N(h\mathscr{C})} \mathscr{C}.$$

Here it is notable that $F^{-1}(NA)$ acts as the pullback diagram

$$\begin{tikzcd} F^{-1}(NA) \arrow[r, hook] \arrow[d] & \mathscr{C} \arrow[d, "F"] \\ NA \arrow[r, hook] & Nh\mathscr{C} \end{tikzcd} \tag{2.11}$$

Now the defined mapping above gives a bijection from subcategories of the ordinary category $h\mathscr{C}$ to the subcategories of the $(\infty, 1)$-category $\mathscr{C}$.

Definition 2.12. If, in addition, that A is a full subcategory of $h\mathscr{C}$, then we say $\mathscr{C} \times_{Nh\mathscr{C}} NA$ is *the* full $(\infty, 1)$-subcategory of $\mathscr{C}$.

Remark 2.13. Given a functor $F : \mathscr{C} \to \mathcal{E}$ of $(\infty, 1)$-categories, and suppose we have an ∞-subcategory $\mathscr{C}' \subseteq \mathscr{C}$, then the idea is that the restricted functor $F' : \mathscr{C}' \to \mathcal{E}$ can be realized if we just look at the effect of F on vertices and edges in $\mathscr{C}'$.

Definition 2.14. Given an $(\infty, 1)$-category $\mathscr{C}$, the core of $\mathscr{C}$ is $\mathsf{core}(\mathscr{C}) = \mathscr{C}^{\simeq}$, the underlying ∞-groupoid (an $(\infty, 0)$-category) obtained by discarding non-invertible morphisms. Alternatively, we can it the largest Kan complex contained in $\mathscr{C}$.

Remark 2.15. For an $(\infty, 1)$-category $\mathscr{C}$, let $A = (h\mathscr{C})^{\simeq}$ be the core of the homotopy category, then the core $\mathscr{C}^{\simeq}$ of $\mathscr{C}$ is just $F^{-1}(NA)$, i.e., fits into Diagram 2.11.

Remark 2.16. For a 1-category $\mathscr{C}$, there is a canonical isomorphism

$$N(\mathscr{C})^{\simeq} \cong N(\mathscr{C}^{\simeq}).$$

Remark 2.17. Given a functor $F : \mathscr{C} \to \mathscr{D}$ of $(\infty, 1)$-categories, F sends $\mathscr{C}^{\simeq}$ to $\mathscr{D}^{\simeq}$, and therefore there is a morphism $F^{\simeq} : \mathscr{C}^{\simeq} \to \mathscr{D}^{\simeq}$ of $\mathbf{sSet}$.

Now suppose $X \in \mathbf{sSet}$, then X has a dual object $X^{\mathrm{op}} \in \mathbf{sSet}$.

Definition 2.18. The dual object X^{op} is a simplicial set, with $X_n^{\mathrm{op}} = X_n$, where $d_i : X_n^{\mathrm{op}} \to X_{n-1}^{\mathrm{op}}$ is defined by $d_{n-i} : X_n \to X_{n-1}$, and $s_i : X_n^{\mathrm{op}} \to X_{n+1}^{\mathrm{op}}$ is defined by $s_{n-i} : X_n \to X_{n+1}$.

Definition 2.19. Suppose $C, D \in \mathbf{qCat}$, and fix a vertex $d \in \mathscr{D}$, which is viewed as a map $d : \Delta^0 \to \mathscr{D}$. We denote $\mathscr{D}_{/d}$ to be the slice category of $\mathscr{D}$ over d, and $\mathscr{D}_{d/}$ to be the coslice category of $\mathscr{D}$ under d. Now let $p : \mathscr{C} \to \mathscr{D}$ be a morphism in $\mathbf{sSet}$, then the pullback of the (co)slice category gives rise to another (co)slice category. To be precise,

- $\mathscr{C}_{/d} = p^* \mathscr{D}_{/d}$ is the slice $(\infty, 1)$-category of $\mathscr{C}$ over $d \in \mathscr{D}$, fitting into the Cartesian square

$$\begin{tikzcd} \mathscr{C}_{/d} \arrow[r] \arrow[d] & \mathscr{C} \arrow[d, "p"] \\ \mathscr{D}_{/d} \arrow[r] & \mathscr{D} \end{tikzcd} \tag{2.20}$$

  in $\mathbf{sSet}$;

- $\mathscr{C}_{d/} = p^*\mathscr{D}_{d/}$ is the coslice $(\infty,1)$-category of $\mathscr{C}$ under $d \in \mathscr{D}$, fitting into the Cartesian square

$$\begin{tikzcd} \mathscr{C}_{d/} \arrow[r] \arrow[d] & \mathscr{C} \arrow[d, "p"] \\ \mathscr{D}_{d/} \arrow[r] & \mathscr{D} \end{tikzcd} \tag{2.21}$$

in $\mathbf{sSet}$;

Here there exists natural functors $\mathscr{D}_{/d} \to \mathscr{D}$ and $\mathscr{D}_{d/} \to \mathscr{D}$ by forgetting the vertex d.

More generally, there exists a version of (co)slice categories over morphisms $f : K \to \mathscr{D}$ for $K \in \mathbf{sSet}$.

To define limits and colimits on $(\infty,1)$-categories, we require the notion of join.

2.2 (Co)limits

Definition 2.22. For 1-categories $\mathscr{A}, \mathscr{B}$, the join $\mathscr{A} \star \mathscr{B}$ is a 1-category with objects $\mathrm{Ob}(\mathscr{A}) \coprod \mathrm{Ob}(\mathscr{B})$ and morphisms $\mathrm{Mor}(\mathscr{A}) \coprod (\mathrm{Ob}(\mathscr{A}) \times \mathrm{Ob}(\mathscr{B})) \coprod \mathrm{Mor}(\mathscr{B})$. That is,

$$\mathrm{Hom}_{\mathscr{A}\star\mathscr{B}}(x,y) = \begin{cases} \mathrm{Hom}_{\mathscr{A}}(x,y), & x,y \in \mathrm{Ob}(\mathscr{A}) \\ \mathrm{Hom}_{\mathscr{B}}(x,y), & x,y \in \mathrm{Ob}(\mathscr{B}) \\ \{*\}, & x \in \mathrm{Ob}(\mathscr{A}), y \in \mathrm{Ob}(\mathscr{B}) \\ \varnothing, & x \in \mathrm{Ob}(\mathscr{B}), y \in \mathrm{Ob}(\mathscr{A}) \end{cases}$$

Definition 2.23. For a category $\mathscr{A}$, the left cone is $\mathscr{A}^{\triangleleft} = [0] \star \mathscr{A}$, and the right cone is $\mathscr{A}^{\triangleright} = \mathscr{A} \star [0]$.

Therefore, the cone adjoins an extra vertex onto the simplicial set.

Definition 2.24. Let $F : \mathscr{A} \to \mathscr{B}$ be a functor. A limit of F is a functor $\hat{F}' : \mathscr{A}^{\triangleleft} \to \mathscr{B}$ which is terminal among all functors that extend F. Similarly, a colimit of F is a functor $\hat{F} : \mathscr{A}^{\triangleright} \to \mathscr{B}$ which is initial among functors which extend F.

Here we need to explain what initial and terminal means in terms of universal properties of $(\infty,1)$-categories.

Definition 2.25. Let $\mathscr{C}$ be an $(\infty,1)$-category. We say $x \in \mathscr{C}$ is terminal if the canonical map $\mathscr{C}_{/x} \to \mathscr{C}$ is an acyclic Kan fibration of simplicial. That is, the mapping spaces $\mathrm{Map}_{\mathscr{C}}(c,x)$ are acyclic Kan complexes for all objects $c \in \mathscr{C}$. We say $x \in \mathscr{C}$ is initial if it is a terminal object in $\mathscr{C}^{\mathrm{op}}$.

Remark 2.26. Let $\mathscr{C}$ be an ordinary category. Note that the mapping space $\mathrm{Map} = \mathrm{Map}_{\mathscr{C}} : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathcal{S}$ is a functor into $\mathcal{S}$, the $(\infty,1)$-category of (small) spaces (anima/∞-groupoids). Therefore, the mapping space is a simplicial set. With this in mind, a general (co)limit in $\mathscr{C}$ should preserve limits in each variable. That is,

$$\mathrm{Map}_{\mathscr{C}}(\operatorname*{colim}_{i\in I}(a_i), b) \cong \lim_{i \in I^{\mathrm{op}}} \mathrm{Map}_{\mathscr{C}}(a_i, b)$$

and

$$\mathrm{Map}_{\mathscr{C}}(b, \operatorname*{colim}_{i\in I}(a_i)) \cong \lim_{i \in I^{\mathrm{op}}} \mathrm{Map}_{\mathscr{C}}(b, a_i)$$

for $a_i, b \in \mathscr{C}$. These are restatements of the universal property of (co)limits, as we view Map as hom sets.

Let $K \in \mathbf{sSet}$, then the slice category $\mathscr{D}_{/d}$ has the universal property: the hom set of 1-category $\mathbf{sSet}$ satisfies

$$\mathbf{sSet}(K, \mathscr{D}_{/d}) \cong \mathrm{Hom}_d(K^{\triangleright}, \mathscr{D}).$$

In particular, if $K = \Delta^n$, then we get all n-simplices in the simplicial set $\mathscr{D}_{/d}$, which gives a description of this category. The hom set $\mathrm{Hom}_d(K^{\triangleright}, \mathscr{D})$ is a subset of the simplicial sets $\mathbf{sSet}(K^{\triangleright}, \mathscr{D})$. Dually, there is an isomorphism

$$\mathbf{sSet}(K, \mathscr{D}_{d/}) \cong \mathrm{Hom}_d(K^{\triangleleft}, \mathscr{D}).$$

Now recall that $(\infty, 1)$-categories are weak Kan complexes, therefore $\mathbf{Kan}$, the $(\infty, 1)$-category of Kan complexes[4], becomes a subcategory of $\mathbf{qCat}$. Both categories are simplicially-enriched, i.e., as $\mathbf{sSet}$-categories. By applying the homotopy coherent nerve functor N^{hc} on the inclusion, we obtain another inclusion $\mathcal{S} \subseteq \mathbf{Cat}_\infty$ of $(\infty, 1)$-categories. This inclusion functor gives an adjunction triple

$$\begin{tikzcd} \mathcal{S} \arrow[d, shift left=3] \\ \mathbf{Cat}_\infty \arrow[u, shift left=6, "|\cdot|"] \arrow[u, shift right=3, "\mathrm{Core}(-)"'] \end{tikzcd} \tag{2.27}$$

where $|\cdot|$ is the ∞-groupoid completion and $\mathrm{Core}(-)$ is the core functor.

Finally, we study cofinality in simplicial sets.

Definition 2.28. A map $v : K' \to K$ in $\mathbf{sSet}$ is right cofinal if it satisfies all of the following (equivalent) conditions:

- v respects all colimits, i.e., for every $(\infty, 1)$-category $\mathscr{C}$ and any colimit cocone $K^{\triangleright} \to \mathscr{C}$, the composition $K'^{\triangleright} \xrightarrow{v} K^{\triangleright} \to \mathscr{C}$ is also a colimit cocone.
- v respects colimits in $\mathcal{S}^{\mathrm{op}}$.

  If, in addition, K is an $(\infty, 1)$-category, then they are equivalent to
- for every object $k \in K$, the simplicial set $K'_{x/}$ is weakly contractible.[5] That is, $K'_{x/} \sim \Delta^0$.

A map $v : K' \to K$ in $\mathbf{sSet}$ is left cofinal if $v^{\mathrm{op}} : K'^{\mathrm{op}} \to K^{\mathrm{op}}$ is right cofinal.

Remark 2.29.

- Left (respectively, right) cofinal maps are stable under products, i.e., if $v : K' \to K$ is left (respectively, right) cofinal, then so is $K' \times L \to K \times L$ for any $L \in \mathbf{sSet}$.
- A left (respectively, right) adjoint is left (respectively, right) cofinal.
- Left (respectively, right) cofinal maps are stable under pushforwards (respectively, pullbacks) along Cartesian (respectively, Cocartesian) fibrations.

Definition 2.30. A morphism $p : A \to B$ in $\mathbf{sSet}$ is proper if, for any pullback pairs in $\mathbf{sSet}$ of the form

$$\begin{tikzcd} A'' \arrow[r, "u"] \arrow[d] & A' \arrow[r] \arrow[d] & A \arrow[d, "p"] \\ B'' \arrow[r, "v"'] & B' \arrow[r] & B \end{tikzcd} \tag{2.31}$$

$u : A'' \to A'$ is right cofinal whenever $v : B'' \to B'$ is.

Definition 2.32. A morphism $p : A \to B$ in $\mathbf{sSet}$ is smooth if its opposite morphism $p^{\mathrm{op}} : A^{\mathrm{op}} \to B^{\mathrm{op}}$ in $\mathbf{sSet}$ is proper. That is, given Diagram 2.31, $u : A'' \to A'$ is left cofinal whenever $v : B'' \to B'$ is.

Remark 2.33.

[4] This is sometimes referred to as "the ∞-category of spces".

[5] Unless stated otherwise, we always mean this is in terms of Kan-Quillen model structure of $\mathbf{sSet}$.

- The class of proper morphisms in $\mathbf{sSet}$ and the class of smooth morphisms in $\mathbf{sSet}$ are both stable under composition and under base-change.
- Cocartesian fibrations are proper; Cartesian fibrations are smooth.

Definition 2.34. Let κ be a regular cardinal.

- A simplicial set K is κ-small if the number of its non-degenerate simplices is less than κ. In particular, if $\kappa = \omega = \aleph_0$ is the countable regular cardinal, then it is finite.
- A simplicial set L is κ-filtered if, for any κ-small simplical set K and any map $v : K \to L$ in $\mathbf{sSet}$, the simplicial set $L_{v/}$ is non-empty. That is, v can be extended to a cocone $\bar{v} : K^{\triangleright} \to L$. Moreover, we say L is filtered if it is ω-filtered, that is, every finite diagram in L extends to a cocone.

  Remark 2.35. An $(\infty, 1)$-category $\mathscr{C}$ is filtered if and only if for any integer $n \geqslant 0$, every morphism $\partial\Delta^n \to \mathscr{C}$ of simplicial sets can be extended to a morphism of the form $(\partial\Delta^n)^{\triangleright} \to \mathscr{C}$.

  Dually, a simplicial set L is (κ-)cofiltered if its dual L^{op} is (κ-)filtered.

  Remark 2.36.

  – Any (co)filtered $(\infty, 1)$-category is weakly contractible.
  – A Kan complex is (co)filtered if and only if it is weakly contractible.

- A simplicial set K is sifted if $K \neq \varnothing$ and its diagonal $\Delta : K \to K \times K$ is cofinal. Equivalently, the diagonal $\Delta : K \to K^I$ is right cofinal for any finite set I. A simplicial set K is cosifted if K^{op} is sifted.

  Remark 2.37.

  – A (co)sifted simplicial set is weakly contractible.
  – Let $\mathscr{C}$ be an $(\infty, 1)$-category. $\mathscr{C}$ is sifted if and only if, for every pair of objects $a, b \in \mathscr{C}$, the underlying simplicial set $\mathscr{C}_{a/} \times_{\mathscr{C}} \mathscr{C}_{b/}$ is weakly contractible. In particular, any $(\infty, 1)$-category with finite coproducts is sifted.

    To prove this, consider an $(\infty, 1)$-category K and any map $K' \to K$, then the sifted property says that $K'_{x/} \sim \Delta^0$ is weakly contractible for any $x \in K$. By definition, any arbitrary pair of vertices $a, b \in K$ gives a commutative square

$$\begin{tikzcd} K_{(a,b)/} \arrow[r] \arrow[d] & K \arrow[d, "\Delta"] \\ K_{a/} \times K_{b/} \arrow[r] & K \times K \end{tikzcd} \tag{2.38}$$

    One can check $K_{(a,b)/} \cong K_{a/} \times_K K_{b/}$ pointwise. Therefore, it is equivalent to saying that $K_{a/} \times_K K_{b/}$ is weakly contractible.
  – Let $\mathcal{A}$ be an 1-category. $\mathcal{A}$ is sifted if and only if $A_{a/} \times_A A_{b/}$ is connected for any $a, b \in A$. (That is, the diagonal functor respects all limits in 1-categories.)
  – If $v : K' \to K$ is a right cofinal map of simplicial sets and K is sifted, then K' is also sifted.
  – Let $F : \mathscr{C} \to \mathscr{D}$ be a functor of $(\infty, 1)$-categories where $\mathscr{C}$ is cocomplete, then F preserves sifted colimits if and only if it preserves filtered colimits and geometric realizations. Here the geometric realization means any colimit indexed by Δ^{op}.
  – Any colimit can be written as a sifted colimit of finite coproducts.

Example 2.39.

- $N\Delta^{\mathrm{op}}$ is sifted.
- Any non-empty filtered simplicial set is sifted.

Proposition 2.40. Let $v : \mathscr{C} \to \mathscr{D}$ be a right cofinal functor of $(\infty, 1)$-categories, then $\mathscr{D}$ is filtered if $\mathscr{C}$ is.

Theorem 2.41. Let $\mathscr{C}$ be an $(\infty, 1)$-category, then the following are equivalent:

- $\mathscr{C}$ is filtered;
- there exists a right cofinal functor $NA \to \mathscr{C}$ for some directed poset A;
- the diagonal map $\Delta : \mathscr{C} \to \mathscr{C}^K$ is right cofinal for every finite simplicial set K.

Definition 2.42. Let $p : \mathscr{C} \to \mathscr{D}$ be a functor of $(\infty, 1)$-categories. We say p is fully faithful if

$$p_* : \mathrm{Map}_{\mathscr{C}}(c, c') \to \mathrm{Map}_{\mathscr{D}}(pc, pc')$$

is an equivalence for any vertices $c, c' \in \mathscr{C}$.

Remark 2.43. Note that we did not give a precise definition of the mapping space. However, We should think of the hom set $\mathrm{Map}_{\mathscr{C}}(c, c')$ to be the pullback of

$$\begin{tikzcd} \mathrm{Map}_{\mathscr{C}}(c, c') \arrow[r] \arrow[d] & \mathscr{C}^{\Delta^1} \arrow[d] \\ \Delta^0 \arrow[r, "{(c,c')}"'] & \mathscr{C} \times \mathscr{C} \end{tikzcd} \tag{2.44}$$

Here

- $\mathscr{C}^{\Delta^1}$ gives the edges in $\mathscr{C}$, then the map $\mathscr{C}^{\Delta^1} \to \mathscr{C} \times \mathscr{C}$ is the map landing at the source and the target;
- $(c, c') : \Delta^0 \to \mathscr{C} \times \mathscr{C}$ is the map landing at the pair (c, c').

Theorem 2.45. Let

$$\begin{tikzcd} \mathscr{C} \arrow[d, shift right, "p"'] \\ \mathscr{D} \arrow[u, shift right, "q"'] \end{tikzcd}$$

be an adjunction of $(\infty, 1)$-categories with unit η and counit ε, then η is a natural equivalence if and only if p is fully faithful. In addition, the essential image of p consists of objects $d \in \mathscr{D}$ such that ε_d is an equivalence. That is, the essential image gives the full subcategory of $\mathscr{D}$ to which the restriction of q is conservative.

Lemma 2.46. Given an adjunction triple $F \dashv U \dashv G$ of $(\infty, 1)$-categories, there is an adjunction pair $UF \dashv UG$ of $(\infty, 1)$-categories.

Theorem 2.47. Let

$$\begin{tikzcd} \mathscr{C} \arrow[d, shift right, "p"'] \\ \mathscr{D} \arrow[u, shift right, "q"'] \end{tikzcd}$$

be an adjunction of $(\infty, 1)$-categories. For any $K \in \mathbf{sSet}$ and $\mathscr{E} \in \mathbf{qCat}$, we obtain adjunctions

$$\begin{tikzcd} \mathrm{Fun}(K, \mathscr{C}) \arrow[d, shift right, "p_*"'] \\ \mathrm{Fun}(K, \mathscr{D}) \arrow[u, shift right, "q_*"'] \end{tikzcd}$$

and

$$
\begin{tikzcd}
\mathrm{Fun}(\mathscr{C},\mathscr{E}) \arrow[d, shift right, "q^*"'] \\
\mathrm{Fun}(\mathscr{D},\mathscr{E}) \arrow[u, shift right, "p^*"']
\end{tikzcd}
$$

of $(\infty, 1)$-categories.

A Notations of Categories

- $\mathbf{Cat}$: 1-category of 1-categories.
- $\mathbb{F} = \mathbf{FinSet}$: category of finite sets.
- $\mathbf{FinSet}/K$: slice category of $\mathbf{FinSet}$ over $K \in \mathbb{F}$.
- $\mathbf{Span}(\mathbb{F})$: category of span of $\mathbb{F}$.
- $\mathbf{Bispan}(\mathbb{F})$: category of bispan of $\mathbb{F}$.
- $\mathbf{Cospan}(\mathbb{F})$: category of cospan of $\mathbb{F}$.
- $\mathbf{sSet}$: 1-category of simplicial sets.
- $\mathbf{qCat}$: 1-category of $(\infty, 1)$-categories.
- $\mathbf{CAT}$: 2-category of categories.
- $\mathscr{C}_{/c}$: slice category of $(\infty, 1)$-category $\mathscr{C}$ over $c \in \mathscr{C}$.
- $\mathscr{C}_{c/}$: coslice category of $(\infty, 1)$-category $\mathscr{C}$ under $c \in \mathscr{C}$.
- $\mathscr{C} \star \mathscr{D}$: (1-category) join of 1-categories $\mathscr{C}$ and $\mathscr{D}$.
- $\mathcal{S}$: $(\infty, 1)$-category of (small) spaces (anima/∞-groupoids).
- $\mathbf{Kan}$: $(\infty, 1)$-category of Kan complexes.
- $\mathbf{Cat}_\infty$: $(\infty, 1)$-category of $(\infty, 1)$-categories.

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