MATH 580 Notes

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To start with, we have several basic principles:

1. sum principle: suppose $B_1 \dot\cup \cdots \dot\cup B_k$, then $|A| = \sum |B_i|$. Here $\dot\cup$ denote the disjoint union of sets.
2. product rule: suppose $A = B_1 \times \cdots \times B_k$, then $|A| = \prod_i |B_i|$, i.e., the product of the number of choices of each coordinate of A gives the number of elements of A. This is true even if we choose element one by one, such that later choices depend on the earlier ones.
3. method of double counting (Fubini's theorem): given a set, suppose we have two ways of counting it, then the two formulas are the same.

   Example 1.1. Consider $\sum_{i=1}^{n-1} i$. Note that the number is just the size of the set $\{(a,b) : a < b\}$, so this is just $\binom{n}{2}$. Alternatively, we can partition the set into subsets $B_{i+1} := \{(a,b) : a < b, b = i+1\}$, then this is just $\sum_{i=0}^{n-1} |B_{i+1}|$.
4. bijection principle: let A and B be (finite) sets, and we hope to check that they have the same size, then it suffices to check that there exists a bijection $f : A \to B$.

   Example 1.2. Let A be the collection of k-set, i.e., subsets of size k, of $[n] = \{1, \dots, n\}$, then it has $\binom{n}{k}$ elements. Let B be the $(n-k)$-set of $[n]$. To find a bijection, we take

   $$\begin{aligned} f : A &\to B \\ X &\mapsto [n] \backslash X. \end{aligned}$$

   In particular, this proves that $\binom{n}{k} = \binom{n}{n-k}$.

   Sometimes it may be hard to find a bijection, so we have the following trick: suppose $f : A \to B$ and $g : B \to A$ are both injections, then we know both are bijections, hence the two sets have the same size.
5. pigeonhole principle: given a set of numbers, then the maximal number should be no less than the average number.
6. polynomial principle: let $p(x)$ and $q(x)$ be polynomials of degree at most d, and that they agree at $d+1$ places, then the two polynomials coincide.

Definition 1.3.

- A k-word is an ordered tuple of k elements from the given alphabets.
- A simple word is a word where the letters are not repeated.
- A k-set is a k-element subset. Given a parentset $[n]$ with $n \geqslant k$, a k-set is denoted $\binom{[n]}{k}$, with size $\binom{n}{k}$.

Remark 1.4. Given a k-set, it is called a simple k-word or a permutation if it is ordered with no repetition; it is called a k-word if it is ordered with repetition allowed; it is a subset if it is unordered with no repetition allowed; it is a multiset if it is an unordered set with repetitions allowed.

Notation. We denote $n \cdot (n-1) \cdot \cdots \cdot (n-k+1)$ to be both $n_{(k)}$ and $(n-k+1)^{(k)}$.

Example 1.5. For instance, $n_{(n)} = n!$, and $n_{(0)} = n^{(0)} = 1$. Also, we have $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n_{(k)}}{k!}$.

Remark 1.6. To conclude,

k-sets of $[n]$	no repetition	with repetition
ordered	simple k-words/permutations, size $n_{(k)}$	k-words, size n^k
unordered	k-subsets, size $\binom{n}{k}$	multisets, size $\binom{k+n-1}{k}$, c.f., Theorem 1.7

7. binomial theorem: let $\mathbb{N}$ be the set of natural numbers containing 0, then for all $n \in \mathbb{N}$, for all x, y, we have $(x+y)^n = \sum\limits_k \binom{n}{k} x^k y^{n-k}$. Note that the summation can be understood as over $k \in \mathbb{Z}$.

Theorem 1.7. The number of k-element multisets from $[n]$ is the number of non-negative integer solutions of $\sum\limits_{i=1}^{n} x_i = k$, which is just $\binom{k+n-1}{n-1} = \binom{k+n-1}{k}$.

Proof. Consider x_i as the number of times the element i is chosen, this proves the first part. To prove the second part, we need to count the number of solutions, which is done by considering a tuple $(x_1, \ldots, x_n)$ as a classic star-and-bar problem. $\square$

Definition 1.8. A composition of $k \in \mathbb{Z}^+ = \mathbb{N} \backslash \{0\}$ is an ordered list of positive integers summing to k.

Example 1.9. $1+2=3=2+1$ gives two different compositions $(1,2)$ and $(2,1)$ as ordered lists.

Theorem 1.10. The number of compositions of k with n parts is $\binom{k-1}{n-1}$.

Proof. Consider the number of solutions of $y_1 + \cdots + y_n = k$ for $y_i \geqslant 1$ for all i. $\square$