UCLA Algebra Seminar

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Preliminary Information

This is a summary document for the UCLA Participating Algebra Seminar (i.e., MATH 290C, *Participating Seminar: Current Literature in Algebra*) in Winter 2023. The seminar is organized by Logan Hyslop, focusing on semisimple and reductive groups. Most of the information in this section are taken from the syllabus document.

Resources:

- Chapter 6 of *The Book of Involutions* ([boi])
- Milne's Notes on *Reductive Groups* ([rd])
- Milne's Book *Algebraic Groups: the theory of group schemes of finite type over a field* [(ag)], with a preliminary version available.

Seminar Lineup:

1. **Introduction.** Group schemes, subgroups, connected component of the identity, examples in group theory, c.f. section 20 of [boi].
2. **Specific Kinds of Groups and Lie Algebras.** Diagonalizable group schemes, groups of multiplicative type, Lie Algebra and smoothness, c.f. section 20 and 21 of [boi].
3. **Factor Groups.** Factor groups, representations, representations of diagonizable group schemes, c.f. section 22 of [boi].
4. **Root Systems, Split Semisimple and Reductive Groups.** Root systems, semisimple, reductive, split groups, c.f. section 25 of [boi], section 14-15 of [rd].

5. **Root Systems, Split Semisimple and Reductive Groups.** Split semisimple groups, root systems for split semisimple and split reductive groups, c.f. section 25 of [boi], section 18-19 of [rd].

6. **Semisimple and Reductive Groups over Arbitrary Fields.** Classification of semisimple groups over an arbitrary field, classification of reductive groups, c.f. section 24 and 26 of [rd].

1 Affine Group Scheme: Feb. 8, 2023

Definition 1.1 (Group Scheme)**.** An S-scheme G together with the unit map $u : S \times G \to G$, the inverse map $i : G \times G \to G$, and the multiplication map $m : G \otimes_S G \to G$, is a (affine) *group scheme* over S if the following diagrams commutes:

$$\begin{tikzcd}
G \times G \times G \arrow[r, "\mathrm{id}\times m"] \arrow[d, "m\times\mathrm{id}"'] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"'] & G
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r, "\cong"] & G \times S \arrow[r, "\mathrm{id}\times u"] \arrow[d, "\cong"'] \arrow[ddr, "\mathrm{id}"] & G \times G \arrow[dd] \\
 & S \times G \arrow[d, "u\times\mathrm{id}"'] & \\
 & G \times G \arrow[r] & G
\end{tikzcd}$$

$$\begin{tikzcd}
 & G \times G \arrow[rr, "i\times\mathrm{id}"] & & G \times G \arrow[dr, "m"] & \\
G \arrow[ur, "\Delta"] \arrow[rr] \arrow[dr] & & S \arrow[rr, "u"] & & G \\
 & G \times G \arrow[rr, "\mathrm{id}\times u"'] & & G \times G \arrow[ur, "m"'] &
\end{tikzcd}$$

Proposition 1.2. If T is a S-scheme, then

$$G(T) := \mathbf{Hom}(T, G)$$

is a group.

Remark 1.3. A *group scheme homomorphism* $\rho : G \to H$ over S is a morphism of schemes respecting the multiplication map. The group schemes and group scheme homomorphisms form a category, the category of group schemes over S.

Definition 1.4 (Hopf Algebra)**.** Let F be a field and A be a commutative F-algebra. Take $G = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(F)$, then there are dual maps co-unit $u : A \to F$, co-inverse $i : A \to A$, and co-multiplication $c : A \to A \otimes_F A$. Suppose the dual diagrams of Definition 1.1 commutes, then A is a *Hopf algebra* over F. In particular, $\mathbf{Hom}_{F\text{-}\mathbf{Alg}}(A, -)$ are groups.

Example 1.5. Let $\mathbf{1} = \mathrm{Spec}(F)$. Note that F is trivially an F-Hopf algebra. In particular, $\mathbf{1}(T) = \{e\}$. We have $c : F[x] \to F[y, z]$ via $x \mapsto y + z$, $u : x \mapsto 0$, and $i : x \mapsto -x$ over $G_a = \mathrm{Spec}(F[x])$, where $G_a(A)$ is the additive group $(A, +)$.

Let $G_m = \operatorname{Spec}(F[x, x^{-1}])$ with $c : x \mapsto x \otimes x$, we have $G_m(A)$ as the multiplicative group $(A^\times, \cdot)$. Now G_a and G_m correspond to $\mathbb{A}^1$ and $\mathbb{A}^1 \backslash \{0\}$, respectively.

Example 1.6. In general, let V be a F-vector space, then the symmetric algebra $S(V^*)$ of the dual space V^* satisfies

$$\mathbf{Hom}_{\mathbf{Alg}_F}(S(V^*), R) = \mathbf{Hom}_F(V^*, R) = V \otimes_F R,$$

and $S(V^*)$ represents $R \mapsto (R \otimes_F V, +)$, and so $\operatorname{Spec}(S(V^*))$ is a group scheme.

Example 1.7. Given an F-algebra A, $\mathrm{GL}_1(A)$ is described $\mathrm{GL}_1(A)(R) = (A \otimes R)^\times$, represented by $S(A^*)\left[\frac{1}{N}\right]$. In general, $\mathrm{GL}_n(F)$ is given by $F\left[X_{ij} \mid 1 \leq i, j \leq n\right]\left[\frac{1}{\det(X)}\right]$.

Remark 1.8. An (affine) group scheme G over F is a functor $G : \mathbf{Alg}_F \to \mathbf{Grp}$ isomorphic to $\mathbf{Hom}_{\mathbf{Alg}_F}(A, -)$ for some Hopf algebra A over F. By the Yoneda Lemma, the Hopf algebra A is uniquely determined by G (up to isomorphism), and is therefore denoted $F[G]$. In this sense, a group scheme homomorphism is a natural transformation of functors.

Remark 1.9 (Correspondence)**.**

$$\begin{aligned} \text{Group Schemes over } F &\Longleftrightarrow \text{Commutative Hopf Algebras over } F \\ G &\mapsto F[G] \\ G^A &\leftarrow\!\shortmid A \end{aligned}$$

defines an equivalence of categories.

Definition 1.10 (Algebraic Group Scheme)**.** An (affine) group scheme A is algebraic if its coordinate ring is a finitely-generated F-algebra.

Remark 1.11. If L/K is a field extension, then any L-algebra is a K-algebra, and so we say $A \otimes L$ is the restriction of A to L.

Definition 1.12 (Closed Subgroup, Normal)**.** A *closed subgroup* H of a group scheme G is just a closed subscheme such that u, i, m restrict down to H. In particular, if G is affine, i.e., $G = \operatorname{Spec}(A)$, then $H = \operatorname{Spec}(A/J)$ where J is a Hopf ideal, i.e., $c(J) \subseteq J \otimes A + A \otimes J$.

We now denote $H \subseteq G$. We say $H \subseteq G$ is *normal* if $H(T) \subseteq G(T)$ is normal for all spectrums of F-algebras T.

Remark 1.13 (Trivial Subgroup)**.** Every Hopf algebra has an augmented ideal $I = \ker(u : A \to F)$, which is a Hopf ideal. Therefore, $\mathbf{1} \to \operatorname{Spec}(A)$ always gives a trivial normal subgroup.

Remark 1.14 (Kernel)**.** Let $\varphi : G \to H$ be a morphism of group schemes (over discrete groups). The kernel is constructed as the pullback:

$$\begin{array}{ccc} \ker(\varphi) & \longrightarrow & G \\ \downarrow & & \downarrow \\ \mathbf{1} & \longrightarrow & H \end{array}$$

Example 1.15. Consider $F[x, x^{-1}] \to F[x, x^{-1}]$ given by $x \mapsto x^k$, then by applying the spectrum functor, we obtain a morphism of group schemes $\exp(k) : G_m \to G_m$. The kernel of this map is the unit group μ_k.

Now $\mathcal{O}(\mu_k) = F[x, x^{-1}] \otimes_{F[x^k, x^{-k}]} F = F[x]/(x^k - 1)$.

Recall if G is a Lie group, then $\pi_0(G)$ is a group and there is a surjection $G \twoheadrightarrow \pi_0(G)$ whose kernel is the connected component of identity (in the Zariski sense). There is a similar story for group schemes.

Definition 1.16 (étale)**.** A finitely-generated F-algebra A is *étale* if $A \cong L_1 \times \cdots \times L_n$ where L_i/F is separable.

Example 1.17. $A \otimes_F F^{\text{sep}} \cong (F^{\text{sep}})^n$.

Proposition 1.18. If A is a finitely-generated F-algebra, then A has a unique maximal étale subgroup $\pi_0(A)$.

If $B \subseteq A$ is étale, then $\dim_F(B)$ is bounded by the number of essential idempotents, and the compositum of two étale subalgebras is again étale. Therefore, there is a maximal étale subalgebra.

Further, if A is a Hopf algebra, then $\pi_0(A)$ is also a Hopf algebra, then the identity of an affine group scheme is the kernel of the map

$$G \to \operatorname{Spec}(\pi_0 \mathcal{O}(G)) =: \pi_0(G).$$

Proposition 1.19. If G is a finite group and $A := \mathbf{Hom}(G, F) \cong F[e_g \mid g \in G]$, then A has a Hopf algebra structure given by $A \ni e_g \mapsto \sum\limits_{hk=g} e_h \otimes e_k$. Furthermore, $\operatorname{Spec}(A(T)) \cong G$ for Zariski connected T.

2 Groups and Lie Algebras

In this section, we construct all concepts upon a based field F.

2.1 Diagonalizable Groups

Definition 2.1 (Diagonalizable Group Schemes)**.** Let H be an Abelian group, then the functor $R \to \mathbf{Hom}(H, R^\times)$ is representable by H_{diag}, for $R \in \mathbf{Alg}_F$. That is, we have $H_{\text{diag}}(R) = \mathbf{Hom}(H, R^\times)$. In particular, H_{diag} is the group scheme representing the group algebra $F\langle H\rangle$ over F to be the the group algebra of H, given by $c(h) = h \otimes h$, $i(h) = h^{-1}$, and $u(h) = 1$. Therefore, this gives a Hopf algebra structure on the group algebra. Group schemes of the form H_{diag} are called diagonalizable.

Remark 2.2. Note that the elements in $F\langle H\rangle$ are given of the form $h \otimes h$ for $h \in H$. Therefore, we have a natural isomorphism from $(H_{\text{diag}})^*$ and H.

Example 2.3. $\mathbb{Z}_{\text{diag}} = G_m$ and $(\mathbb{Z}/n\mathbb{Z})_{\text{diag}} = \mu_n$.

Definition 2.4 (Multiplicative Type)**.** We say a group scheme G is of multiplicative type if $G_{\text{sep}} := G_{F_{\text{sep}}}$ is diagonalizable.

Remark 2.5. In particular, diagonalizable group schemes are of multiplicative type.

Let $\Gamma = \text{Gal}(F^{\text{sep}}/F)$ and H be an Abelian group equipped with a continuous Γ-action, then $R \to \mathbf{Hom}_\Gamma(H, (R \otimes_F F^{\text{sep}})^\times)$ is represented by H_{mult}. In cash, $H_{\text{mult}}(R) = \mathbf{Hom}_\Gamma(H, (R \otimes_F F_{\text{sep}})^\times)$.

Proposition 2.6. There is an equivalence of categories between

- group schemes of multiplicative type over F, and
- Abelian groups with a continuous Γ-action,

defined by two contravariant functors, with $G \mapsto (G_{\text{sep}})^* = \mathbf{Hom}_{\mathbf{GrpSch}}(G_{\text{sep}}, G_m)$ for group scheme G, and with $H \mapsto H_{\text{mult}}$ for Abelian group H.

Remark 2.7. In particular, diagonalizable group schemes correspond to Abelian groups with trivial Γ-action.

2.2 Lie Algebra

Let G be an algebraic group scheme.

Definition 2.8 (Lie Algebra)**.** The Lie algebra of G, denoted $\mathrm{Lie}(G)$ is the tangent space of G at the identity.

Proposition 2.9. $\mathrm{Lie}(G) = (I/I^2)^*$ where I is the augmentation ideal of A, the Hopf Algebra of G, i.e., $G = \mathrm{Spec}(A)$.

Proof. A_I is the local ring at the identity, so $\mathrm{Lie}(A) = (IA_I/I^2A_I)^* = (I/I^2)^*$. □

Definition 2.10 (Derivator)**.** If A is an F-algebra and M is an A-module, then a derivator D from A to M is a F-linear map $A \to M$ satisfying $D(fg) = fD(g) + gD(f)$.

Proposition 2.11. There are natural isomorphisms between

1. Lie group G,
2. $\mathrm{der}(A, F)$, where F is an A-algebra using $u : A \to F$ with $D(fg) = u(f)D(g) + u(g)D(f)$,
3. The left-invariant derivations, given by $\{D \in \mathrm{Der}(A, A) \mid c \circ D = (\mathrm{id} \otimes D) \circ c\}$,
4. $\ker(G(F[\varepsilon]) \to G(F))$ where $F[\varepsilon]$ is the dual numbers over F given by $\varepsilon^2 = 0$. This is a kernel of groups induced by $F[\varepsilon] \to F$. In particular, the kernel carries a natural F-vector space structure: the addition operation is the multiplication on $G(F[\varepsilon])$, and the scalar multiplication operation is defined by the following action: for $a \in F$ and g in the kernel, $a \cdot g = G(l_a)g$, where $l_a : F[\varepsilon] \to F[\varepsilon]$ is the multiplication map defined by $\varepsilon \mapsto a\varepsilon$.

Proposition 2.12. $\mathrm{Lie}(G)$ has finite dimension over F.

Proof. Note that A is Noetherian, so I is finite-dimensional, so I/I^2 is finite-dimensional over $A/I = F$, so $(I/I^2)^*$ is finite-dimensional over F. □

Remark 2.13. The Lie group has the following properties.

1. $\mathrm{Lie}(G_1 \times G_2) = \mathrm{Lie}(G_1) \times \mathrm{Lie}(G_2)$.
2. $\mathrm{Lie}(G^0) = \mathrm{Lie}(G)$.
3. Given a field extension L/F, $\mathrm{Lie}(G_L) = \mathrm{Lie}(G) \otimes_F L$.

Remark 2.14. In particular, the Lie bracket is induced via the structure on vector space and the bracket on deriviators given by $[D_1, D_2] = D_1D_2 - D_2D_1$.

Example 2.15.
1. For the general linear group $\mathrm{GL}_n(R)$, the associated Lie algebra is $\mathfrak{gl}_n = M_n(F)$.
2. For the standard linear group $\mathrm{SL}_n(R)$, the associated Lie algebra is $\mathfrak{sl}_n = \{M \in M_n(F) \mid \mathrm{tr}(M) = 0\}$.
3. For the orthogonal group $\mathrm{O}_n(R)$, the associated Lie algebra is $\mathfrak{O}_n = \{M \in M_n(F) \mid M + M^T = 0\}$.

   In particular, for even $n = 2m$, we have the special linear group $\mathfrak{sp}_n(R) = \{A \in M_n(R) \mid A^T\Omega A = \Omega\}$ for $\Omega = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, then the associated Lie algebra is given by $\mathfrak{sp}_n = \{M \in M_n(F) \mid \Omega M + M^T\Omega = 0\}$.

2.3 Dimension and Smoothness

Definition 2.16 (Dimension)**.** If G is a connected group, then the reduced structure $F[G]_{\text{red}}$ is a domain. In particular, the dimension $\dim(G)$ of G is the transcendence degree of F over the field of fractions of $F[G]_{\text{red}}$. If G is not connected, we define $\dim(G) = \dim(G^0)$.

Example 2.17.
- $\dim(V) = \dim_F(V)$.
- $\dim(\mathrm{GL}_n(R)) = n^2$.
- $\dim(\mathrm{SL}_n(R)) = n^2 - 1$.
- $\dim(G_m) = \dim(G_a) = 1$.
- $\dim(\mu_n) = 0$.

Remark 2.18. The dimensions satisfy many important properties.

1. $\dim(G) = \dim(F[G])$, given by the Krull dimension.
2. G is finite if and only if $\dim(G) = 0$.
3. Given a field extension L/F, $\dim(G_L) = \dim(G)$.
4. $\dim(G_1 \times G_2) = \dim(G_1) + \dim(G_2)$.

Recall that we say a commutative local ring R we have $\dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \geq \dim(R)$. In particular, we say R is regular if the equality holds.

Definition 2.19 (Smooth)**.** We say a group G is smooth if $\dim(\text{Lie}(G)) = \dim(G)$. Equivalently, $F[G]_L$ is reduced for any field extension L/F.

Proposition 2.20. If F is a perfect field, then G is smooth if and only if $F[G]$ is reduced.

Remark 2.21. Suppose F has characteristic 0, every algebraic group is smooth.

Example 2.22. Suppose F has characteristic $p > 0$, then $\mu_p = \text{Spec}(k[x]/x^p - 1) = k[x]/(x-1)^p$, which is not smooth.

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