CS748: Research Project Proposal Sequential Tests and Confidence Sequences for Prediction of Election Results

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***Abstract*—Predicting the winner of an election from limited samples of data in the form of polls is a fundamental problem of interest. Our work models polling as a community sampling-with-replacement model, and we investigate the sample complexity of queries needed to ensure a fixed mistake probability δ. The core of this work relies on sequential tests to ensure fixed mistake probability bounds. This work is split into two parts, with the second building on the first - (a) estimating the winner of a single constituency, which is equivalent to estimating the mode of a discrete distribution, and (b) providing a sampling rule to sample constituencies to estimate the overall winner while minimizing the expected number of queries made. We plan to investigate the problem in a twofold manner: theoretical guarantees of the proposed algorithm in terms of upper and lower bounds for sample complexity, along with simulations with practical data to back the performance of such algorithms.**

***Index Terms*—sequential tests, confidence sequences, martingales, hypothesis testing, best arm identification, bandits**

I. Introduction

National elections are a topic of great significance and interest in democracies, which are generally accompanied by an air of anticipation due to the immediate and long-term effects for all the stakeholders. This particular interest in elections can be demonstrated by constant updates by all the media houses, spending of billions of dollars by political parties for rallies, fierce debates and discussion among the subjects of that particular nation. Due to the weight associated with these proceedings, polls are conducted to ask a finite sample of the population about their preferences. The results of these polls are used to get an overall estimate of the probability of a particular political party winning the elections. Hence, the prediction of election results prior to the official announcements is of significant interest to a large part of the population. We plan to investigate this problem from a statistical framework, giving guarantees for estimating the winner with high probability from data obtained from polls.

Consider the general elections in a country like India. The entire nation is partitioned into a number of small geographical locations named constituencies. Each of these constituencies participates in the elections where each adult citizen can choose to vote for 1 of the k total parties contesting in the elections. Based on the cumulative votes that each party receives in each constituency, a party is declared to be a winner of that particular constituency. The overall winner party of the elections is the one that wins in the maximum number of constituencies, while at least winning in some threshold number of constituencies. The problem which we try to tackle is that of estimating the winner of the elections given that we can ask sample the population to know about their preferences. We break this problem into two sequential sub-problems -

- Estimate the winner of each constituency, which is equivalent to mode estimation of a discrete probability mass function, as formally defined in II. The model we look at is the following: consider an urn having a fixed number of balls N of k different colors. At each turn, you can sample a ball from the urn with replacement and note down the color. The question we want to ask is: which color is present in the maximum number in the urn, and how many times do you need to sample balls in order to be wrong with probability at most δ.
- Estimating the party which wins the majority number of seats across all m constituencies. In this problem, we also need to define a sampling rule which tells us which constituency to sample at each step. Intuitively, the key idea about why this problem is harder than the first is that we can leave constituencies where two parties are very

close to each other in terms of probability of winning (as long as we can predict the overall winner from other constituencies) or where we are sure that some *minority* party shall win which does not have much chance to make it as the final winner.

Specifically, we aim to make the following contributions with this work -

- Modify prior works about best arm identification in multi-armed bandits and estimation of parameters associated with certain distributions in the sequential setting to estimate the winning party in a single constituency. An example of this kind of work is [3].
- Derive sample complexity bounds for estimating the winning party in a single constituency with the method of confidence sequences. Some of the papers listed in III have already derived bounds for sample complexity. Waubdy et al. [1] have presented confidence sequence based techniques for estimating parameters of a distribution under sampling without replacement setting. We aim to modify the algorithm in [1] to solve the problem of identifying the winning party with high performance under sampling with replacement setting and derive sample complexity bounds for the same.
- Propose an algorithm for determining the winning party with the highest number of seats across all constituencies. This will enable us to predict the winning party for many real-world elections. We additionally aim to derive sample complexity bounds for the proposed algorithm.

II. Problem Statement

Given the framework laid out in I, we formally define both the sub-problems in this section. Let us consider the simpler problem of estimating the winner in a single constituency. As mentioned earlier, we consider the case where we have a fixed population size, say N, and have k political parties contesting in the elections for each of the m constituencies. We index each constituency as C_i and each party as P_j where $1 \leq i \leq m$ and $1 \leq j \leq k$ by definition of the problem. Let constituency C_i have n_i population such that $\sum_{i=1}^{i=m} n_i = N$.

We now consider the first sub-problem of estimating the winner in single constituencies. This can be viewed as the problem of mode estimation. Let $p_X(x)$ be a discrete probability distribution and $X_1, X_2, .., X_t$ be i.i.d random variables with $X_i \sim p_X(x)$. The support of this distribution is the set $\{1, 2, 3, ..., k\}$ where X_i denotes the index of the party a sample person is going to vote for. Estimating the mode of the distribution $p_X(x)$ is equivalent to the single constituency winner problem.

The overall problem can be defined as follows. Let us say that for each party P_j we have a indicator random variable $Y_{i,j}$ where $Y_{i,j} = 1$ if party P_j won in constituency C_i and 0 otherwise. Hence for each party P_j, we define $Z_j = \sum_i Y_{i,j}$ as the total *score* for party P_j and the winner of the election is $P_{\widehat{j}}$ where $\widehat{j} \in \arg\max Z_j$ which we wish to estimate. Note that we assume that there is a unique winner of the elections and hence $\widehat{j} = \arg\max Z_j$.

A query $\mathcal{Q}$ is defined as choosing a constituency, sampling a person from that constituency, and asking for the party they want to vote for. Our main goal is to design a good adaptive policy to control the number of queries $\mathcal{Q}$ asked to predict the winner based on confidence results based on single constituency estimation (for example, if the decision of a particular constituency is decided with high probability, performing $\mathcal{Q}$ operations on that constituency would be wasteful). We do this under the fixed mistake probability setting, as opposed to working with a fixed budget and trying to minimize the number of queries asked. To the best of our knowledge, no work deals with this exact situation although there has been considerable work done in predicting the winner for a single constituency. The subsequent section shall outline prior research which shall act as baselines for this project.

III. Relevant Literature

There has been work in the past attempting to deal with the problem of sequential mode estimation of a distribution, which is the first sub-problem that we had defined earlier. The reader may relate this problem with the well studied problem of best-arm identification in a multi-armed bandit setting by considering each party in a constituency as an arm - the difference being that, here there is no choice of which arm to choose while sampling.

Shah et al. [2] proposed an algorithm for adaptively learning a distribution's mode by extracting sequential samples from it. For a discrete distribution, the algorithm involves maintaining a confidence interval for each of the classes. The value of probability for observing a particular class is contained in its respective confidence interval with high probability. The authors use refined Bernstein bounds for deriving the confidence intervals. The stopping criteria for the algorithm is the instant when the confidence intervals for one of the classes becomes disjoint with other confidence intervals. The authors have also proposed both lower and upper bounds for the sample complexity, which are tight upto logarithmic factors.

Waudby-Smith et al. [1] have presented a suite of tools for estimating parameters of a distribution by providing a sequence of confidence sets that contain the parameters of the underlying distribution with high probability uniformly across time, which shrink under the setting of sampling without replacement. The work proposes a prior-posterior ratio (PPR) martingale which is constructed using the prior belief on the parameters and the posterior distribution over parameters after sampling sequentially from the distribution without replacement. Let $\theta \in \Theta$ be the parameter to be estimated (true parameter being θ^*). The samples X_t are sampled according to $X_1 \sim f_{\theta^*}(x)$ and $X_{t+1} \sim f_{\theta^*}(x|X_1^t) \, \forall \, t \in \tau$, τ being the horizon where X_1^t is a shorthand notion of the history $\{X_1, X_2, ..., X_t\}$. Let us denote the prior distribution of the belief with $\pi_0(\theta)$. The expression of posterior distribution for θ is given by:

$$\pi_t(\theta) = \frac{\pi_0(\theta) f_\theta(X_1^t)}{\int_{\eta \in \Theta} \pi_0(\eta) f_\eta(X_1^t) d\eta} \tag{1}$$

The PPR evaluated at a given $\theta \in \Theta$ is defined as:

$$R_t(\theta) := \frac{\pi_0(\theta)}{\pi_t(\theta)} \quad (2)$$

The paper asserts that for any prior π_0 on Θ that assigns non-zero mass everywhere, the sequence of prior-posterior ratios evaluated at the true θ^*, $((R_t(\theta^*))_{t=0}^{N})$ is a non-negative martingale with respect to the canonical filtration $(\mathcal{F}_t)_{t=0}^{N}$ where $\mathcal{F}_t := \sigma(X_1, X_2, ...X_t)$. Further, the sequence of sets

$$C_t := \{\theta \in \Theta : R_t(\theta) < 1/\delta\}$$

forms a (1 - δ) confidence sequence for θ^*, meaning that $Pr(\exists t \in \tau : \theta^* \notin C_t) \leq \delta$.

Kaufmann et al. [3] present deviation inequalities that are valid uniformly in time under adaptive sampling in a multi-armed bandit model. They are obtained by constructing a mixture martingale for each arm based on a hierarchial prior. Let the true mean of an arm a be μ_a and it's estimate at time t be $\hat{\mu}_a(t)$. $N_a(t)$ is the number of times arm a has been sampled upto the tth round. The paper present bounds for estimating the self-normalised sum which is defined as follows -

$$\sum_{a=1}^{a=K} N_a(t) d(\mu_a(t), \mu_a) = \log \frac{p(X_1, X_2, ..., X_t; \hat{\mu}(t))}{p(X_1, X_2, ..., X_t; \mu(t))} \quad (3)$$

where $d(.)$ is the Kullback-Leibler divergence between two distributions. The paper claims that under certain sampling rules, this self-normalized sum is bounded as follows:

$$Pr_\mu\{\exists t \in \mathcal{N} : \sum_{a=1}^{a=K} [N_a(t) d(\mu_a(t), \mu_a) - \mathcal{O}(\ln \ln N_a(t)] \geq K\mathcal{T}\left(\frac{\ln \frac{1}{\delta}}{K}\right)\} \leq \delta$$

where $\mathcal{T}$ is a threshold function that obeys $\mathcal{T}(x) = x + o(x)$, and the exact form of $\mathcal{T}(x)$ is chosen based on the problem. In specific, this paper presents the GLR stopping rule, which is formulated as a hypothesis testing problem. For our specific application, we use the "Best arm identification" section of the paper, which presents better threshold functions for the GLR stopping rule in this specific case, and claims that they are tight in terms of sample complexity under some associated sampling rule. To apply this algorithm for two communities, we model our problem as trying to estimate the parameters of two bernoulli arms. Here, the algorithm does not use the fact that the two bernoulli parameters will add up to 1.

IV. Rough Plan

The nature of the specific problem that we attempt to solve involves sampling with replacement. We observed that the stochastic process (PPR) continues to be a martingale when evaluated at true parameter while sampling with replacement. Thus, we make suitable modifications in the method proposed by the authors before applying it for our problem statement.

We have implemented the Sequential Mode Estimation paper [2], PPR [1] and GLR stopping [3] for the 2-community case to get an estimate of the practical performance of the algorithms. Currently, GLR stopping seems to perform slightly better than PPR and significantly better than the sequential mode estimation paper. However, the threshold functions need to be investigated more thoroughly in our code of GLR stopping - we have not substituted the full expressions for $o(x)$. GLR stopping rule also takes significantly more time to run, due to the multidimensional grid search for lambda parameters in the rule. We plan to focus on and show sample complexity bounds for PPR [1] for the two parties' case to explain its practical behavior, and investigate into why GLR stopping seems to be slightly better (and if the proofs in the paper can be extended to our case with suitable threshold functions) before the mid-stage evaluations. If time permits, we also try to extend these bound for the multiple parties' case.

In specific, to prove sample complexity bounds for PPR, we are starting with showing bounds on $E[R_t(\theta)]$, whose rate of increase is closely related to the sample complexity.

After this stage, we think we will have sufficient tools to deal with the Indian election system. Our major work is to find good strategies to extend PPR [1] from a single constituency to multiple constituencies and prove meaningful bounds for the sample complexity of the final proposed algorithm.

References

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