

Storage capacity of the Hopfield neural network

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Abstract

In this paper, a new approximate formula to probability integral is deduced using theoretical analysis combining with computer numerical simulation. The absolute storage capacity of the Hopfield neural network is analyzed with this approximate formula and a more strict result is obtained.

Keywords: Neural network; Storage capacity; Normal distribution

1. Introduction

Storage capacity is an important parameter of neural networks. The statistical storage capacity of the Hopfield neural network (HNN) [1] analyzed by means of equilibrium statistical mechanics is about 0.14 [2,3]. It is found that, when the storage capacity of HNN is about 0.14, there are many metastable states, which are separated by high-energy barriers and exist around the memorized patterns with very small Hamming distance. The retrieval patterns often fall into these metastable states. However, as pointed in Ref. [4], strictly speaking, the replica method's prediction for the statistical storage capacity of the HNN with zero-temperature dynamics is approximately 0.05 against the numerical result 0.14. In some cases, the network is usually needed to recollect the storage exactly. So there are also many discussions on the absolute storage capacity which determines how many patterns can be really stored in the networks [5–11]. Theoretically, the signal-to-noise theory is often used to resolve the absolute storage capacity of network. In this process, the approximate analytic equation of standard normal distribution is needed. But different approximate analytic equation often gives different results. The absolute storage capacity of discrete HNN given in Refs. [5,6] is $\frac{1}{2} \ln N$, where N is the number of neurons and that in Ref. [7] is $1/(2 \ln N - \ln \ln N)$.

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In this paper, we further analyze the approximate analytic equation and find a better approximate equation of the probability integral. The storage capacity of HNN based on this equation is much stricter.

2. Approximate analytic equation of standard normal distribution

The positive distribution is an essential and important probability. Many random variables obey this principle exactly or approximately. Moreover, it is the maximum distribution of many probabilities. The probability integral of standard normal distribution can be obtained from integral tables generally, and can be calculated by stage value closing. But in many cases, simple approximate analytic equation is favorable for finding the answer. Here, we find a better equation according to the theoretical calculation combined with numerical simulation.

The standard normal function is

$$\Phi(r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^r \exp\left(-\frac{t^2}{2}\right) dt, \quad -\infty < r < +\infty. \quad (1)$$

Let

$$I(r) = \frac{1}{\sqrt{2\pi}} \int_0^r \exp\left(-\frac{t^2}{2}\right) dt. \quad (2)$$

Then $\Phi(z) = 0.5 + I(z)$. Eq. (2) can be translated into integral equation with binary variables:

$$\begin{aligned} I^2 &= \frac{1}{2\pi} \int_0^r \exp\left(-\frac{x^2}{2}\right) dx \int_0^r \exp\left(-\frac{y^2}{2}\right) dy \\ &= \frac{1}{2\pi} \int_0^r \int_0^r \exp\left(-\frac{x^2 + y^2}{2}\right) dx dy. \end{aligned} \quad (3)$$

The integral area is the square OBCA as shown in Fig. 1. Expressed in plane polar coordinate system, Eq. (3) can be rewritten as follows:

$$I^2 = \frac{1}{2\pi} \int_0^{\pi/2} \int_0^{r'(\theta, r)} \exp\left(-\frac{r^2}{2}\right) r dr d\theta. \quad (4)$$

When the integral area is the sector OAB and the upper limit of integral is $r' \equiv r$, we have

$$I_{\min}^2 = \frac{1}{2\pi} \int_0^{\pi/2} \int_0^r \exp\left(-\frac{r^2}{2}\right) r dr d\theta = \frac{1}{4} \left[1 - \exp\left(-\frac{r^2}{2}\right) \right]. \quad (5)$$

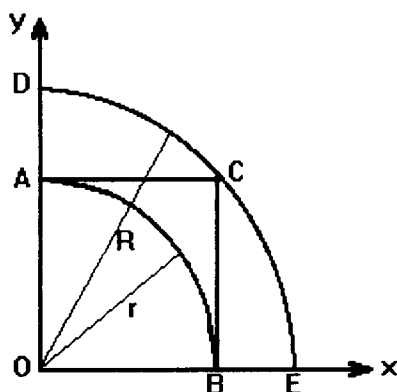


Fig. 1. Sketch map of Eqs. (4)–(6).

While if the integral area is the sector ODE and the upper limit of integral changes to $r' \equiv R = \sqrt{2}r$, then

$$I_{\max}^2 = \frac{1}{2\pi} \int_0^{\pi/2} \int_0^{\sqrt{2}r} \exp\left(-\frac{r'^2}{2}\right) r' dr' d\theta = \frac{1}{4} [1 - \exp(-r^2)]. \quad (6)$$

As in Eq. (4), x and y vary in square OACB, r' is limited in the range $r \leq r' \leq R$, and the area is between the two sectors, i.e., $I_{\min} < I < I_{\max}$, we can set

$$I = \frac{1}{2} \left[1 - \exp\left(-\frac{\lambda(r)r^2}{2}\right) \right]^{1/2}. \quad (7)$$

Thus,

$$\Phi(r) = \frac{1}{2} + \frac{1}{2} \left[1 - \exp\left(-\frac{\lambda(r)r^2}{2}\right) \right]^{1/2}. \quad (8)$$

Here $1 \leq \lambda(r) \leq 2$. To obtain the expression of $\lambda(r)$, From (8) $\lambda(r)$ can be written as follows:

$$\lambda(r) = \frac{2 \ln[1 - (2\Phi_{\text{num}}(r) - 1)^2]}{r^2}, \quad (9)$$

where $\Phi_{\text{num}}(r)$ is the probability integral value of standard normal distribution obtained from integral tables [6]. Using the numerical value of the probability integral, the value of $\lambda(r)$ can be obtained. Fig. 2 shows $\lambda(r)$ numerical values as a function of r .

From Fig. 2 one can see that the curves of $\lambda(r)$ is similar to that of exponential function. It can be expressed as $\lambda(r) = a + b \exp(-cr^2)$ approximately, where a , b and c are constant coefficients. The coefficients of $\lambda(r)$ can be obtained with computer

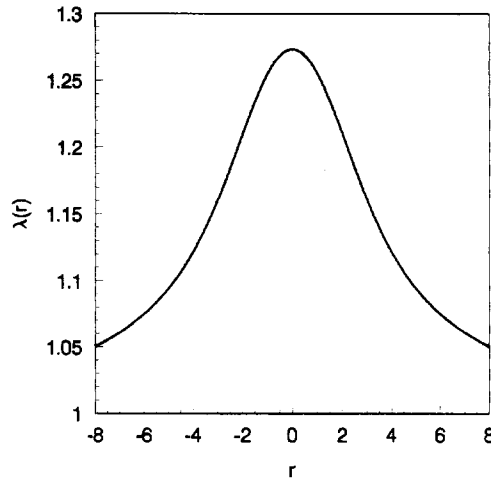


Fig. 2. Variation of $\lambda(r)$ as a function of r .

numerical fitting method, $a=1.045$, $b=0.2282$, $c=0.0813$. $\lambda(r)$ can be written as

$$\lambda(r) = 1.045 + 0.2282 \exp(-0.0813r^2), \quad (10)$$

and the probability integral can be obtained as

$$\Phi_z(r) = [1 + Z(r)]/2, \quad (11)$$

where

$$Z(r) = \left[1 - \exp\left(-\frac{\lambda(r)r^2}{2}\right) \right]^{1/2}. \quad (12)$$

Ref. [12] presents the approximate formula of normal distribution as

$$\Phi_w(r) = [1 + W_1(r)]/2, \quad (13)$$

where $W_1(r) = [1 - \exp(-2/\pi r^2)]^{1/2}$. The first step approximation of $\lambda(r)$ can be obtained as 1.2732, $Z(r) = [1 - \exp(-0.6366r^2)]^{1/2}$ is similar to $W_1(r)$. The maximum absolute error is about $3 \times 10^{-3} \Phi_w(r)$, while is only 10^{-5} with our approximate formula $\Phi_z(r)$.

3. Absolute storage capacity of Hopfield model

In this section, we analyze the absolute storage capacity of discrete HNN based on the Eqs. (10)–(12). Suppose there are N neurons and M random pattern S^μ stored in HNN. Its connection matrix is $J_{ij} = \sum_\mu S_i^\mu S_j^\mu$, in which $i, j = 1, 2, \dots, N$. For any input state, the dynamic equation of NN is $S_i(t+1) = \Theta(\sum J_{ij} S_j(t))$, where Θ is a symbol

function. When N and M are large, the probability of the network that every neuron can iterate correctly can be expressed as [5]

$$P = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{1/\alpha}}^{\infty} \exp\left(-\frac{t^2}{2}\right) dt. \quad (14)$$

According to Eq. (11), the above equation can be approximated by

$$P = [1 + Z(u)]/2, \quad (15)$$

where $u = \sqrt{1/\alpha}$. So the error probability is

$$\rho = 1 - P = [1 - Z(u)]/2. \quad (16)$$

The condition of the single neuron iterating correctly is $\rho \rightarrow 0$. If the number of error components follows approximately the Poisson distribution [5], then the condition for a stable attractor of M stored patterns is $\exp(-N\rho) = \beta \rightarrow 1$. Let $C = -\ln\beta$, we get

$$[1 - Z(u)]/2 = \frac{C}{N}. \quad (17)$$

Here C, β are constant. The above equation can be rewritten as

$$\frac{(1.045 + 0.2282 \exp(-0.0813/\alpha))}{2\alpha} = -\ln\left(\frac{4C}{N} - \frac{4C^2}{N^2}\right). \quad (18)$$

The approximate solution of the equation can be obtained as

$$\alpha = \frac{1.033}{2 \ln N - 2 \ln[4C(1 - C/N)]} + \frac{1.327}{\{2 \ln N - 2 \ln[4C(1 - C/N)]\}^2} + \text{smaller order terms}. \quad (19)$$

For a fixed correct probability, $C < 1$. As N is large, C and $4C^2/N^2$ can be neglected. Thus, the approximate value of storage capacity can be written as

$$\alpha \approx \frac{1.033}{2 \ln N} + \frac{1.327}{(2 \ln N)^2}. \quad (20)$$

Here we can compare our result with that in Refs. [5,7]. Computer simulations show that, the largest result is that of Ref. [7] when $N < 10^{16}$. But when $N > 10^{17}$, our result is the largest. Considering the number of the neurons of the brain is about 10^{11-12} , our result is larger than Ref. [5], but smaller than Ref. [7]. A plot of $\ln(N)$ vs. the log of the absolute storage capacities obtained in Refs. [5,7], and here is given in Fig. 3. As a fact, in order to obtain the absolute storage capacity, $N \rightarrow \infty$ is required. With this infinity, the result in Ref. [7] is the same as that of Ref. [5,6]; While our result shows a little larger than them with 0.33%.

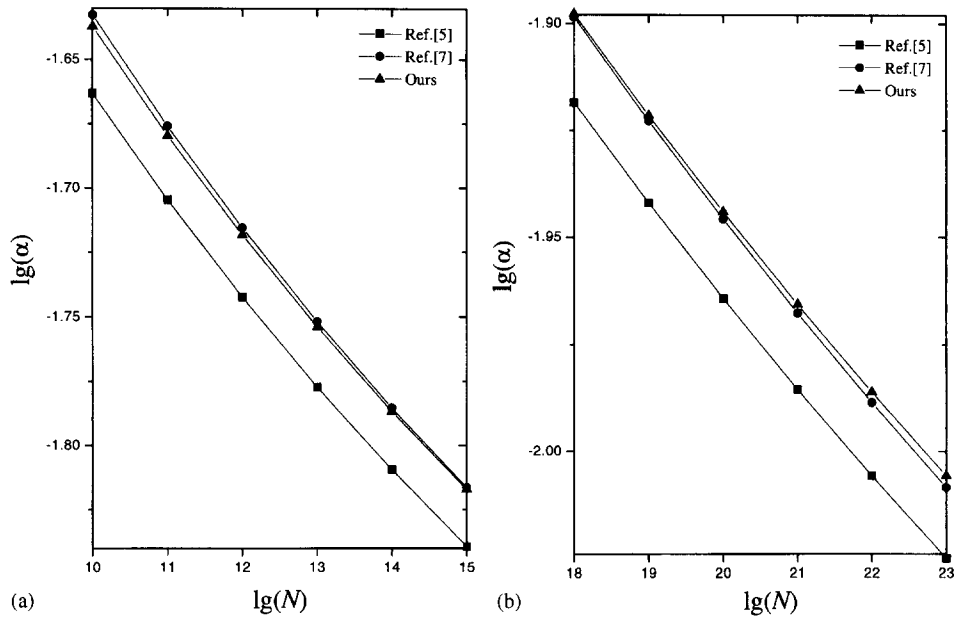


Fig. 3. The comparison of three results of absolute storage capacity. Here (a) N is from 10^{10} – 10^{15} ; (b) N is from 10^{18} – 10^{25} .

4. Conclusions

In this paper, an approximate expression of standard normal distribution is developed using theoretical analysis and computer numerical simulation. It is much more precise than those adopted by other researchers. Using this formula, the absolute storage capacity of Hopfield neural network is analyzed. The result is a little larger than those previously reported when $N > 10^{17}$.

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