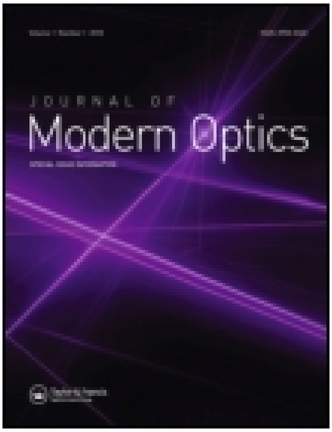

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The 2^n-element Number Neural Network Model: Recognition of the Multistate Patterns

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The 2^n-element number neural network model: recognition of the multistate patterns

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Abstract. In this paper, a 2^n-element number discrete neural network is suggested. The storage capacity ratios of the model for the various value n are the same, and the storage capacities of the 2^n-element number network decrease with the increase of n. The present model can be applied to recognize the 4, 16 or 256 level grey or colour patterns.

1. Introduction

The original Hopfield neural network model [1] consists of neurons that take on two states corresponding to 'firing' and 'quiescenting', with appropriate synaptic couplings. The network is able to remember patterns. If one treats the state as a two-dimensional picture, the two-valued neuronal activities correspond to black and white pixels. In this sense, the Hopfield network can be used to recognize black/white pictures. But for the eyes of man, the pictures of its recognition mostly are colour-level or grey-level ones. Now, in a very natural way, the question arises as whether a network model can store pictures composed of pixels of various colours or grey shades. In recent years, several techniques for grey-level neural networks have been developed [2–9]. There are two kinds of methods to process this problem. One way is to use the Hopfield model basically and so a multistate pixel is expressed by a group of two-state neurons. Some suggestions are discussed by Taketa, Zhang, and colleagues [2–4]. Another way is to propose the multistate neural network models in which a pixel is expressed by a multistate neuron. Rieger suggested a Q-state neural network model [5], in which the input signal is divided into Q intervals. The storage capacity ratio of the Q-state model decreases as Q^{-2} with the number of grey tones [6]. The Q-state Potts-glass model of a neural network is discussed too [7–9]. By defining the Potts Hamiltonian in different ways, different representations of the Potts spins are used. One way is to define the neuron Q unit vectors, pointing in Q-directions, which span a hypertetrahedron in R^{Q-1}. Another way takes the neutron the Q points on the unit circle in the complex number plane.

In this paper, a 2^n-element number neural network model is suggested by introducing the 2^n-element number [10] into the Hopfield neural network. So for the various n, we can set up the Q-state neural network with $Q = 4$, 16 or 256. The grey or colour patterns with $Q = 16$ or 256 state are widely used in the computer. The paper is organized as follows. In the Appendix a simple introduction about the 2^n-element number is presented. Section 2 is devoted to set up the 2^n-element discrete neural network. In section 3 the signal-to-noise theory and the computer numerical simulations is made to analyse the storage capacity of the model. Section 2 discusses the application for recognizing the multistate patterns.

2. The 2^n-element number neural network model

The 2^n-element number is introduced into the neural network so as to form a discrete 2^n-element number neural network. In the model, each neuron is assumed to be a multistate one, such as two-state (± 1) for the real number, i.e. the Hopfield model, four-state ($\pm 1 \pm \mathbf{i}$) for the complex number, 16-state ($\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k}$) for the Hamilton number, the 256-state ($\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k} \pm \mathbf{e} \pm \mathbf{ie} \pm \mathbf{je} \pm \mathbf{ke}$) for the Cayley number.

Suppose there are N neurons and M patterns $\mathbf{S}^{\mu}$ ($\mu = 1, 2, \ldots, M$) stored in the network. The connection matrix is given by the extended Hebbian learning rule:

$$\left.\begin{aligned} J_{mn} &= \sum_{\mu=1}^{M} S_m^{\mu}(S_n^{\mu})^*, \quad (m \neq n), \\ J_{mm} &= 0. \end{aligned}\right\} \tag{1}$$

Here, it should be emphasized that, by multiplication the vector $\mathbf{S}^{\mu}$ with its conjugate vector ($\mathbf{S}^{\mu *}$), the connection matrix of the model is constructed. Clearly, we can draw that the synaptic matrix is symmetric for $n = 0$, i.e. the Hopfield model; $J_{mn} = J^*_{mn}$ for $n > 0$.

The dynamics of the system is defined by the generalized sigmoid function SGN:

$$S_m(t+1) = \mathrm{SGN}\left[\sum_{n=1}^{N} J_{mn} S_n(t)\right]. \tag{2}$$

The operating rule of the function SGN(α) is as follows: whenever a real or imaginary component of α is non-negative, a positive unit is drawn out for the corresponding component of SGN(α); otherwise, a negative unit drawn out. For example, for the Hamilton neural network model,

$$\mathrm{SGN}(-3 + 6\mathbf{i} - 2\mathbf{j}) = -1 + \mathbf{i} - \mathbf{j} + \mathbf{k}.$$

Here, one can see that it is actually the Hopfield model with $n = 0$. And with $n = 1$, the four-state model can be treated as a speciality of the discrete-state complex phasor neural networks suggested by Noest [8], although there are some differences by appearance.

3. The storage capacity

Here, an analysis about the stability and the storage capacity of the 2^n-element number dynamic system basing on the signal-to-noise theory is presented. Equation (1) and equation (2) can be rewritten in terms of each real and imaginary parts respectively. If the pattern $\mathbf{S}(0) = \mathbf{S}^{\nu}$ is put into the dynamic system, the dynamic equation $\mathbf{S}^{\nu'} = \mathrm{SGN}(J\mathbf{S}^{\nu})$ can be expanded as follows:

$$a_{im}^{\nu'} = \mathrm{SGN}\left[2^n(N-1)a_{im}^{\nu} + \Delta_{im}^{\nu}\right], \tag{3}$$

where a_{im}^{ν} is the ith part of the mth component of the pattern $\mathbf{S}^{\nu}$, here $0 \leqslant i < 2^n$ and $0 < m \leqslant N$. Clearly, in equation (3) the first term is the signal while the second term Δ the noise. The expression of the noise term Δ can be written out respectively for the every 2^n-element number. For example, for $n=2$, i.e. the Hamilton number model, the noise term of the real part of the S_m^ν is expressed as follows:

$$\begin{aligned}\Delta_H = \sum_{\mu\neq\nu}\Bigg[& a_{0m}^\mu \sum_{n\neq m}(a_{0n}^\mu a_{0n}^\nu + a_{1n}^\mu a_{1n}^\nu + a_{2n}^\mu a_{2n}^\nu + a_{3n}^\mu a_{3n}^\nu)\\
& + a_{1m}^\mu \sum_{n\neq m}(a_{1n}^\mu a_{0n}^\nu - a_{0n}^\mu a_{1n}^\nu + a_{2n}^\mu a_{3n}^\nu - a_{3n}^\mu a_{2n}^\nu)\\
& + a_{2m}^\mu \sum_{n\neq m}(a_{2n}^\mu a_{0n}^\nu - a_{0n}^\mu a_{2n}^\nu + a_{3n}^\mu a_{1n}^\nu - a_{1n}^\mu a_{3n}^\nu)\\
& + a_{3m}^\mu \sum_{n\neq m}(a_{3n}^\mu a_{0n}^\nu - a_{0n}^\mu a_{3n}^\nu + a_{1n}^\mu a_{2n}^\nu - a_{2n}^\mu a_{1n}^\nu)\Bigg]. \end{aligned} \tag{4}$$

For $n=3$, i.e. the Cayley number, the noise term of the real part is expressed as follows:

$$\begin{aligned}\Delta_C = \sum_{\mu\neq\nu}\Bigg[& a_{0m}^\mu \sum_{n\neq m}(a_{0n}^\mu a_{0n}^\nu + a_{1n}^\mu a_{1n}^\nu + a_{2n}^\mu a_{2n}^\nu + a_{3n}^\mu a_{3n}^\nu\\
& + a_{4n}^\mu a_{4n}^\nu + a_{5n}^\mu a_{5n}^\nu + a_{6n}^\mu a_{6n}^\nu + a_{7n}^\mu a_{7n}^\nu)\\
& + a_{1m}^\mu \sum_{n\neq m}(a_{1n}^\mu a_{0n}^\nu - a_{0n}^\mu a_{1n}^\nu + a_{2n}^\mu a_{3n}^\nu - a_{3n}^\mu a_{2n}^\nu\\
& + a_{4n}^\mu a_{5n}^\nu - a_{5n}^\mu a_{4n}^\nu + a_{7n}^\mu a_{6n}^\nu - a_{6n}^\mu a_{7n}^\nu)\\
& + a_{2m}^\mu \sum_{n\neq m}(a_{2n}^\mu a_{0n}^\nu - a_{0n}^\mu a_{2n}^\nu + a_{3n}^\mu a_{1n}^\nu - a_{1n}^\mu a_{3n}^\nu\\
& + a_{4n}^\mu a_{6n}^\nu - a_{6n}^\mu a_{4n}^\nu + a_{5n}^\mu a_{7n}^\nu - a_{7n}^\mu a_{5n}^\nu)\\
& + a_{3m}^\mu \sum_{n\neq m}(a_{3n}^\mu a_{0n}^\nu - a_{0n}^\mu a_{3n}^\nu + a_{1n}^\mu a_{2n}^\nu - a_{2n}^\mu a_{1n}^\nu\\
& + a_{4n}^\mu a_{7n}^\nu - a_{7n}^\mu a_{4n}^\nu + a_{6n}^\mu a_{5n}^\nu + a_{5n}^\mu a_{6n}^\nu)\\
& + a_{4m}^\mu \sum_{n\neq m}(a_{4n}^\mu a_{0n}^\nu - a_{0n}^\mu a_{4n}^\nu + a_{5n}^\mu a_{1n}^\nu - a_{1n}^\mu a_{5n}^\nu\\
& + a_{6n}^\mu a_{2n}^\nu - a_{2n}^\mu a_{6n}^\nu + a_{7n}^\mu a_{3n}^\nu - a_{3n}^\mu a_{7n}^\nu)\\
& + a_{5m}^\mu \sum_{n\neq m}(a_{5n}^\mu a_{0n}^\nu - a_{0n}^\mu a_{5n}^\nu + a_{3n}^\mu a_{6n}^\nu - a_{6n}^\mu a_{3n}^\nu\\
& + a_{1n}^\mu a_{4n}^\nu - a_{4n}^\mu a_{1n}^\nu + a_{7n}^\mu a_{2n}^\nu - a_{2n}^\mu a_{7n}^\nu)\\
& + a_{6m}^\mu \sum_{n\neq m}(a_{6n}^\mu a_{0n}^\nu - a_{0n}^\mu a_{6n}^\nu + a_{1n}^\mu a_{7n}^\nu - a_{7n}^\mu a_{1n}^\nu\\
& + a_{2n}^\mu a_{4n}^\nu - a_{4n}^\mu a_{2n}^\nu + a_{5n}^\mu a_{3n}^\nu - a_{3n}^\mu a_{5n}^\nu)\\
& + a_{7m}^\mu \sum_{n\neq m}(a_{7n}^\mu a_{0n}^\nu - a_{0n}^\mu a_{7n}^\nu + a_{6n}^\mu a_{1n}^\nu - a_{1n}^\mu a_{6n}^\nu\\
& + a_{3n}^\mu a_{4n}^\nu - a_{4n}^\mu a_{3n}^\nu + a_{2n}^\mu a_{5n}^\nu - a_{5n}^\mu a_{2n}^\nu)\Bigg]. \end{aligned} \tag{5}$$

The factor Δ, such as expressed by equation (4) or equation (5), is a sum of $2^n(M-1)$

components of the stored patterns, each of which has a weight that equals to the inside summation of $2^n(N-1)$ terms. Owing to the random character of the stored patterns and their independence to each other, it is reasonable to suppose that the noise term Δ is governed by a Gaussian distribution with expectation value zero and standard deviation value $[2n(N-1)2^n(M-1)]^{1/2}$. Then the signal-noise-ratio SNR of the real or imaginary parts of each component of $\mathbf{S}^v$ can be obtained

$$SNR = \left(\frac{N-1}{M-1}\right)^{1/2}. \tag{6}$$

When the number of neurons is much larger than that of the stored patterns, i.e. $N \gg M$, then $SNR \gg 1$. Hence, the neural network converges to the stored pattern $\mathbf{S}^v$. If a pattern $\mathbf{S}$, very close to the stored pattern $\mathbf{S}^v$, is put into the network, the previous conclusion basically holds true. Therefore, the pattern $\mathbf{S}$ automatically converges to the pattern $\mathbf{S}^v$ after one or more retrieval processes. In conclusion, the stored pattern $\mathbf{S}^v$ is a stable attractor of the 2^n-element number neural network.

The storage capacity of the network are mainly assessed by the SNR. Because SNRs of the 2^n-element number neural network model are independent of the value n, the storage capacities of the 2^n-element number neural networks are in the same level, i.e. the storage capacities of the models are in the same level as that of the Hopfield model.

Now we consider the storage capacity ratio $SCR = M/N \approx 1/SNR^2$ of the presented model with the thermodynamic limit in which $N, M \rightarrow \infty$ and SCR finite. Without loss of generality, assume that the component $a_{im}^v = 1$, then the probability that the $a_{im}^{v'} = 1$ can be given basing on the Gaussian distribution

$$p = \frac{1}{(2\pi)^{1/2}} \int_{-(1/SCR)^{1/2}}^{\infty} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x. \tag{7}$$

Thus compared with the pattern $\mathbf{S}^v$, the expected number of error real and imaginary components in the pattern $\mathbf{S}^{v'}$ is approximately

$$\begin{aligned} E(SCR) &= 2^n N[1 - p(SCR)] \\ &= \frac{2^n N}{(2\pi)^{1/2}} \int_{(1/SCR)^{1/2}}^{\infty} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x. \end{aligned} \tag{8}$$

If the number of error components in $\mathbf{S}^{v'}$ is approximately Poisson distribution, it follows that the probability of correct components, i.e. the probability that $\mathbf{S}^v$ is indeed a stable attractor, is given approximately by the expression

$$P = \exp[-E(SCR)]. \tag{9}$$

Now suppose that the probability P is a fixed number very near 1. Then inverting the preceding expression for $0 < SCR \ll 1$, the following result can be obtained

$$\begin{aligned} SCR &\propto \frac{1}{2 \ln 2^n N} \\ &\approx \frac{1}{2 \ln N}. \end{aligned} \tag{10}$$

The result is independent of the value n and equals that of the Hopfield neural network model analysed by Bruce *et al.* [11] and Mceliece *et al.* [12]. So the storage

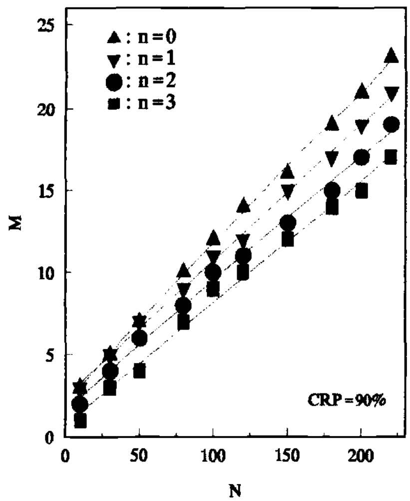

Figure 1. The statistical curves of the stored pattern number $M_{\max}$ via the neuron number N under the correct retrieval per cent $CRP = 90\%$. ▲, the Hopfield model; ▼, complex number model; ●, Hamilton number model; and ■, Cayley number model.

capacity of the 2^n-element number neural network models and the Hopfield model are almost in the same level.

Now we provide the outcomes of the numerical simulation about the storage capacity of the 2^n-element number neural network. The numerical simulations are performed on finite-size systems $N \leqslant 220$. The statistical curves of the maximum stored patterns number $M_{\max}$ via the neuron number N under the correct retrieval per cent $CRP = 90\%$ are obtained (see figure 1) for $n = 0, 1, 2$ and 3. The statistical curve $M_{\max} - N$ with $n = 0$ is exactly the Hopfield neural network model. From figure 1, one can see that the four curves for the various n are very similar, in other words, *SCR*s of the 2^n-element number neural network models are the same, and the storage capacities of the 2^n-element network decrease with the increase of n.

4. To recognize the multistate patterns

For the various value n, we can get 2^{2^n}-state neural network model. So the present models can be used to recognize the 2, 4, 16 or 256 levels grey or colour patterns. Because each component of the 2^n-element number neuron is bistable and the neuron code is similar as the binary code, the neuron encoding schemes with the 2, 4, 16 or 256 levels grey or colour pattern can be set up in a similar way to the binary encoding schemes that are widely used in the computer. Actually, for the 2^{2^n}-state patterns, no matter the grey level patterns or the colour patterns, the same binary encoding schemes are adopted and a multistate pixel is expressed by a 2^n-element number neuron. So the model makes no different to recognizing the grey-level patterns and the colour patterns, although the bits in the colour patterns can be viewed as independent.

For example, if we want to recognize the 16-colour patterns, the Hamilton number model can be used. The three imaginary parts of the Hamilton number can be treated as the three basic colours (i.e. red, blue, green) and the real part indicates the colour saturation degrees. Then the corresponding relationship between the computer code and the Hamilton neuron code of the sixteen-level colours can be set up naturally. The Cayley number model can be used to process the high precision patterns with 256-colour and the Cayley neuron code of the 256-level colours can be set up in a similar way to the 256-level computer code.

To describe the difference between two patterns A and B, a distance function D is defined as follows

$$D = \tfrac{1}{2}|B - A| = \frac{1}{2} \sum_{m=1}^{N} \sum_{i=0}^{2^n - 1} |b_{im} - a_{im}|. \tag{11}$$

Here, a_{im} or b_{im} is the ith part of the mth component of the pattern A or B. The distance D is the generalization of the Hamming distance. So if a pattern $\mathbf{S}$ has a distance D compared with the stored pattern $\mathbf{S}^\mu$, one can say that the pattern $\mathbf{S}$ has a noise per cent $D/2^nN$ compared with the memory $\mathbf{S}^\mu$.

Using the Hamilton number model, some numerical simulations to recognize the 16-state grey or colour patterns are processed. Four 16-state patterns, composed by a 12×12 dot matrix and shown in figure 2 (a), are stored in the present network. Numerical simulation results show that these four patterns are all the stable patterns stored in the Hamilton neural network. Now if a pattern slightly different to one of the stored patterns is put into the network, the system can recognize it and recall to the proper pattern. The simulations show that, for the input patterns with about 10% or 20% random noise (i.e. 60 or 110 error bits), the correct recognition ratio is about 95% or 75%. Figure 2 (b) or 2 (c) shows the set of patterns as input test objects but contaminated with about 10% or 20% noise, respectively. The Hamilton neural network model can recognize them correctly.

5. Conclusions

In this paper, a 2^n-element number discrete neural network is suggested. The stability and the storage capacity are analysed by using the signal-to-noise theory or the numerical simulation. The storage capacity ratios of the model for the various value n are the same, and the storage capacities of the 2^n-element number network decrease as n increases. The 2^n-element number discrete neural network can be applied to recognize the 2^{2^n}-level patterns. The 16 and 256 states grey or colour patterns are widely used in the computer. The neuron code of the 4, 16 or 256 level grey or colour pattern can be set up in similarly way to the 4, 16 or 256 level computer code.

There are some optical architectures have been reported for the complex-valued vector-matrix multiplication [13, 14], e.g. optical 128-elements eight-bit complex-valued vector-matrix multiplication architecture [13] has been implemented by Mosca *et al.* In that system, each complex element of the vector is represented as a two-element real vector, and each complex matrix element is represented as a 2×2 real matrix. With the present method, complex matrix-vector multiplication can be accomplished by real matrix-vector multiplication where the real matrix and vector have dimensions twice the size of those of their complex counterparts. Following this idea, the 2^n-element number vector-matrix multiplication can be implemented into optical domain. Thus the 2^n-element number neural network model can be implemented by the optical-electrical system.

Appendix: The 2^n-element number

We are familiar with the natural, integral, real and complex numbers. However, a type of multi-dimension numbers is defined in mathematics, i.e. 2^n-element

pattern 1

14	14	14	14	14	14	14	14	14	14	14	14
14	5	5	5	5	5	5	5	5	5	5	14
14	5	5	5	5	5	5	5	5	5	5	14
14	5	5	11	11	11	11	11	11	5	5	14
14	5	5	11	11	11	11	11	11	5	5	14
14	5	5	11	11	11	11	11	11	5	5	14
14	5	5	11	11	11	11	11	11	5	5	14
14	5	5	11	11	11	11	11	11	5	5	14
14	5	5	11	11	11	11	11	11	5	5	14
14	5	5	5	5	5	5	5	5	5	5	14
14	5	5	5	5	5	5	5	5	5	5	14
14	14	14	14	14	14	14	14	14	14	14	14

pattern 2

8	8	8	8	8	8	13	13	13	13	13	13
5	8	8	8	8	8	13	13	13	13	13	2
5	5	8	8	8	8	13	13	13	13	2	2
5	5	5	8	8	8	13	13	13	2	2	2
5	5	5	5	8	8	13	13	2	2	2	2
5	5	5	5	5	8	13	2	2	2	2	2
5	5	5	5	5	8	13	2	2	2	2	2
5	5	5	5	8	8	13	13	2	2	2	2
5	5	5	8	8	8	13	13	13	2	2	2
5	5	8	8	8	8	13	13	13	13	2	2
5	8	8	8	8	8	13	13	13	13	13	2
8	8	8	8	8	8	13	13	13	13	13	13

pattern 3

1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1
4	4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4	4
4	4	4	4	4	4	4	4	4	4	4	4
9	9	9	9	9	9	9	9	9	9	9	9
9	9	9	9	9	9	9	9	9	9	9	9
9	9	9	9	9	9	9	9	9	9	9	9
11	11	11	11	11	11	11	11	11	11	11	11
11	11	11	11	11	11	11	11	11	11	11	11
11	11	11	11	11	11	11	11	11	11	11	11

pattern 4

14	14	14	2	2	2	5	5	5	10	10	10
14	14	14	2	2	2	5	5	5	10	10	10
14	14	14	2	2	2	5	5	5	10	10	10
14	14	14	2	2	2	5	5	5	10	10	10
14	14	14	2	2	2	5	5	5	10	10	10
14	14	14	2	2	2	5	5	5	10	10	10
14	14	14	2	2	2	5	5	5	10	10	10
14	14	14	2	2	2	5	5	5	10	10	10
14	14	14	2	2	2	5	5	5	10	10	10
14	14	14	2	2	2	5	5	5	10	10	10
14	14	14	2	2	2	5	5	5	10	10	10
14	14	14	2	2	2	5	5	5	10	10	10

(*a*)

14	10	6	14	6	14	14	14	14	14	14	6
14	1	13	13	5	5	5	5	5	5	5	14
14	5	5	13	5	5	13	5	5	5	5	6
14	5	1	11	11	3	11	11	3	5	13	14
6	1	5	11	10	11	11	11	3	5	13	6
14	9	5	11	15	11	11	3	3	5	5	6
14	1	5	11	11	2	11	11	15	5	5	14
10	5	5	11	11	11	3	7	3	5	5	14
14	5	5	3	11	3	11	11	3	5	7	14
6	13	13	5	13	5	13	13	13	5	5	2
6	5	0	7	13	5	5	5	13	13	13	0
14	14	14	14	14	14	14	14	14	14	6	14

To 1th pattern.

4	0	4	0	8	0	9	9	5	13	1	13
5	8	8	8	8	8	13	8	13	13	5	2
5	5	12	8	8	8	13	13	15	11	2	2
5	5	7	8	0	0	13	13	13	2	6	10
13	5	1	5	8	0	13	13	2	0	2	10
5	5	1	13	5	8	13	2	2	2	2	2
5	5	5	5	5	8	13	6	6	14	2	6
1	5	13	5	8	8	4	13	2	2	2	2
1	5	13	8	8	8	13	13	13	2	6	2
5	13	8	8	8	8	13	5	13	13	6	6
13	8	8	8	8	8	13	15	13	13	11	6
1	0	8	8	9	8	5	13	13	9	1	13

To 2th pattern.

1	9	1	9	1	0	1	1	1	9	1	9
3	1	1	5	5	5	1	1	1	1	1	1
1	9	5	1	11	1	9	1	1	1	1	1
4	4	0	8	12	4	12	12	4	4	12	5
6	12	4	4	4	4	14	4	12	4	12	4
4	12	4	4	4	4	4	4	4	0	12	12
9	9	1	9	1	1	15	1	12	9	9	1
0	11	9	9	9	9	9	1	9	9	9	9
9	1	9	9	9	9	1	9	9	9	9	9
11	11	11	11	11	11	15	11	11	15	11	9
9	11	9	11	11	3	11	11	15	3	15	11
11	11	11	11	3	11	11	3	11	11	11	7

To 3th pattern.

14	14	14	10	10	3	13	4	5	10	10	10
6	14	6	2	3	3	5	13	13	10	10	10
14	6	14	2	10	3	5	13	5	10	10	10
14	15	6	2	2	2	5	5	5	2	10	2
6	14	14	2	2	2	5	5	5	10	10	10
15	14	15	2	3	2	12	13	4	10	10	3
14	13	6	10	2	2	5	5	5	10	10	10
14	6	14	2	10	3	5	13	5	11	10	10
15	15	14	10	2	2	5	13	4	10	11	10
14	6	14	3	10	3	5	5	5	2	10	10
14	14	14	2	11	2	5	13	13	10	10	10
14	14	15	2	2	10	5	12	5	11	10	3

To 4th pattern.

(*b*)

7	14	10	6	6	14	6	14	14	6	2	6
6	9	5	5	5	5	13	13	13	9	13	10
14	5	9	13	13	13	13	13	13	13	15	14
2	13	9	11	11	3	11	11	3	13	5	6
14	1	5	3	11	3	11	7	15	13	5	10
6	13	5	11	15	11	7	3	11	5	5	6
14	5	5	11	3	11	3	15	3	5	5	6
6	5	9	7	3	3	15	11	3	13	13	14
6	5	13	7	11	11	7	7	7	5	9	6
14	1	5	13	5	5	13	13	5	13	5	2
6	5	9	1	5	13	9	13	13	5	13	2
6	6	6	14	6	6	14	14	14	14	6	6

To 1th pattern.

12	12	8	8	8	8	8	13	9	14	9	9
9	8	14	8	8	12	9	5	9	13	9	2
7	1	14	12	10	8	13	8	14	14	6	6
5	4	0	12	8	8	13	13	9	2	2	2
0	7	7	13	9	8	10	13	6	2	2	6
7	1	5	1	5	12	13	2	2	2	2	2
1	1	7	1	1	8	13	0	6	6	6	7
1	5	1	7	13	12	11	13	6	2	2	3
5	5	1	8	12	8	12	13	13	6	0	3
5	6	4	10	9	12	9	12	15	13	2	2
5	13	5	9	9	12	14	13	9	13	10	2
8	12	9	8	14	8	8	13	13	13	13	9

To 2th pattern.

9	9	5	1	9	9	9	9	13	9	1	9
1	1	9	9	1	1	9	1	13	13	1	1
9	1	9	9	1	5	9	9	8	3	1	9
12	6	12	0	5	12	8	4	4	6	14	4
0	4	4	12	4	12	8	12	4	4	4	6
4	12	0	12	14	4	12	4	4	2	12	4
1	5	1	9	1	9	13	1	13	9	5	9
1	13	9	9	9	11	1	9	9	1	1	1
1	1	9	3	1	1	13	1	1	1	9	15
3	3	11	13	3	7	3	3	11	3	3	3
11	15	3	3	3	11	11	11	11	3	11	3
3	7	11	7	11	3	11	15	11	11	3	11

To 3th pattern.

7	6	14	6	2	10	5	13	5	8	2	2
6	2	14	11	10	2	5	13	13	2	10	2
7	6	6	10	14	2	13	13	5	2	10	10
6	15	6	10	10	10	5	13	5	2	2	2
14	6	6	10	10	10	5	13	13	10	2	2
14	6	14	2	10	0	13	15	13	2	2	2
6	14	14	2	10	10	5	13	13	2	2	10
2	6	6	2	10	2	13	1	5	2	10	14
14	6	14	14	10	2	5	5	13	2	2	10
14	14	2	10	10	10	13	13	5	10	2	10
6	6	7	10	10	3	5	13	5	2	2	2
14	10	6	2	10	10	1	13	13	2	2	2

To 4th pattern.

(c)

Figure 2. Computer simulation results for recognizing the four 16-level patterns: (a) the stored patterns that are composed by 12×12 dots matrix; (b) the input patterns with about 10% noise; (c) the input patterns with about 20% noise. All of these input patterns with noise (b, c) can be recognized correctly by the Hamilton neural network model.

numbers [10]. For $n = 0$ and 1, it stands for the real (R) and complex (C) numbers; for $n = 2$, 3 and 4, it stands for the four-element (the Hamilton, i.e. H), eight-element (the Cayley, i.e. A) and 16-element (the Clifford, i.e. L) numbers, respectively.

Assume the Hamilton number

$$H(R) = \{\alpha : \alpha = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}; \qquad a_0, a_1, a_2, a_3 \in R\},$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the three basic unit vectors of the Hamilton number. The addition between the Hamilton numbers is defined as usual, and also the multiplication between the real number and the Hamiltonian number. The multiplication between the basis unit vectors has been defined as follows

$$\begin{aligned} \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad & \mathbf{ij} = -\mathbf{ji} = \mathbf{k} \\ \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad & \mathbf{ki} = -\mathbf{ik} = \mathbf{j} \end{aligned} \tag{A 1}$$

So the multiplication between the Hamiltonian numbers is defined to expand the following equation by using the distribution law:

$$\begin{aligned} (a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})(b_0 + b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ = (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) + (a_0b_1 + a_1b_0 + a_2b_3 + a_3b_2)\mathbf{i} \\ + (a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1)\mathbf{j} + (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)\mathbf{k}. \end{aligned} \tag{A 2}$$

From equation (A 1), we know that the Hamilton number does not obey the exchange law of the multiplication, but obey the combination law, i.e. $(\alpha\beta)\gamma = \alpha(\beta\gamma)$. For a Hamilton number $\alpha = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, its conjugate number is defined as $\alpha^* = a_0 = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$. So its modulate is $\alpha\alpha^* = a_0^2 + a_1^2 + a_2^2 + a_3^2 = |\alpha|^2$.

The Cayley number A can be obtained by expanding the defined range of the complex number from the real number R to the Hamilton number H, i.e.

$$\begin{aligned} A(R) &= \{\Pi : \Pi = \alpha + \beta\mathbf{e};\quad \alpha, \beta \in H\} \\ &= \{\Pi : \Pi = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} + a_4\mathbf{e} + a_5\mathbf{ie} + a_6\mathbf{je} + a_7\mathbf{ke}; \\ &\quad a_i \in R,\ i = 0, \ldots, 7\} \end{aligned} \tag{A 3}$$

where, the basic unit vector $\mathbf{e}$ is the super-complex number basic unit vector. The Cayley number can also be obtained by expanding the defined range of the Hamilton number from the real number to the complex number. So the Cayley number can be taken as the super-complex-Hamilton number.

The multiplication between the basis unit vectors of the Cayley numbers can be defined as the table.

The multiplication rule of the basic unit vectors of the Cayley number.

	1	i	j	k	e	ie	je	ke
1	1	i	j	k	e	ie	je	ke
i	i	−1	k	−j	ie	−e	−ke	je
j	j	−k	−1	i	je	ke	−e	−ie
k	k	j	−i	−1	ke	−je	ie	−e
e	e	−ie	−je	−ke	−1	i	j	k
ie	ie	e	−ke	je	−i	−1	−k	j
je	je	ke	e	−ie	−j	k	−1	−i
ke	ke	−je	ie	e	−k	−j	i	−1

From the table, we known that the Cayley number obeys neither the exchange law nor the combination law of the multiplication. For example,

$$\mathbf{ie}\cdot\mathbf{je} = -\mathbf{k}, \quad \mathbf{je}\cdot\mathbf{ie} = \mathbf{k},$$

$$(\mathbf{j}\cdot\mathbf{j})\mathbf{e} = \mathbf{ke} \qquad \mathbf{i}(\mathbf{j}\cdot\mathbf{e}) = -\mathbf{ke}$$

From the table, we know also that there is a relationship between the super-complex number basic unit vector $\mathbf{e}$ and the Hamilton number α

$$\alpha\mathbf{e} = \mathbf{e}\alpha^*. \tag{A4}$$

For a Cayley number $\Pi = \alpha + \beta\mathbf{e}$, its conjugate number is defined as $\Pi^* = \alpha^* - \beta\mathbf{e}$. So its modulate is $\Pi\Pi^* = \alpha\alpha^* + \beta\beta^* = |\Pi|^2$.

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