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Du Jianxing **Feng Jianyu**

ABSTRACT

This is a note about [1]. The conctent is in Chapter 3 Section 1.

1 Definition of Lattices

First, we start with the difinition of lattices.

Definition 1. *we say that $\Gamma \subset \mathbb{R}^d$ is a lattice if there exists d vectors $b_1, \dots, b_d$ such that*

$$\Gamma = \mathbb{Z}b_1 \oplus \cdots \oplus \mathbb{Z}b_d$$

and $\langle \Gamma \rangle_{\mathbb{R}} = \mathbb{R}^d$.

Let Γ be a lattice in $\mathbb{R}^d$, then the group $\mathbb{R}^d/\Gamma$ is a compact Abelian group, namely a d-dimension torus. We could choose representatives of $\mathbb{R}^d/\Gamma$ in $\mathbb{R}^d$ may be like

$$\mathrm{FD}_\Gamma := \left\{ \sum_{i=1}^{d} \alpha_i b_i | 0 \leq \alpha_i < 1 \text{ for all } i \right\}$$

which is called the fundamental domain of Γ.Therefore, a lattice in$\mathbb{R}^d$ is a co-compact discrete subgroup of it.

Remark. *I don't know whether that a co-compact discrete subgroup of $\mathbb{R}^d$ is always a lattice, but in the proof of Proposition 1 one use this claim.*

Lemma 1. *Any lattice is a Meyer set.*

Since the translate of a lattice is not always a lattice, we define the generalisation of lattices.

Definition 2. *A non-empty point set $\Lambda \subset \mathbb{R}^d$ is called a crystallographic point packing in $\mathbb{R}^d$ if there is a lattice Γ in $\mathbb{R}^d$ and a finite point set F such that $\Lambda = \Gamma + F$.*

2 Periodicity

The objects we considering here are point sets $\Lambda \in \mathbb{R}^d$. We say that $t \in \mathbb{R}^d$ is a period of Λ when $t + \Lambda = \Lambda$. The set

$$\mathrm{per}(\Lambda) := \{t \in \mathbb{R}^d | t + \Lambda = \Lambda\}.$$

is called the set of period Λ. $\mathrm{per}(\Lambda)$ has a group structure in $\mathbb{R}^d$. Its $\mathbb{R}$-span is a subspace of $\mathbb{R}^d$, which dimension is called the rank of $\mathrm{per}(\Lambda)$.

A point set Λ is called periodic if $\mathrm{per}(\Lambda)$ is non-trivial. A point set Λ is called crystallographic if $\mathrm{per}(\Lambda)$ is a lattice. So, even a non-crystallographic set can still have non-trival periods. For example, we can choose a point set in $\mathbb{R}^2$ whose set of periodic only lies in one dimension. The term aperiodic which we will introduce later is a stronger property than non-periodic.

The next proposition explains the relation between crystallographic point sets and crystallographic point packings.

Proposition 1. *A locally finite point set $\Lambda \subset \mathbb{R}^d$ is crystallographic if and only if there is a lattice Γ in $\mathbb{R}^d$ and a finite point set $F \subset \mathbb{R}^d$ such that $\Lambda = \Gamma \oplus F$.*

Proof. Let $\Gamma \subset \mathbb{R}^d$ be a lattice and $F \subset \mathbb{R}^d$ be a finite set. Let $\Lambda = \Gamma + F$. Considering the decompose of Λ, we get that Λ is locally finite. Thus $\mathrm{per}(\Lambda)$ is a discrete subgroup of $\mathbb{R}^d$. Since Λ is a lattice, $\Lambda - \Lambda = \Lambda$, so $\Gamma \subset \mathrm{per}(\Lambda)$. Therefore $\mathrm{per}(\Lambda)/\Gamma$ is finite, and consequently $\mathbb{R}^d/\mathrm{per}(\Lambda)$ is compact. So that $\mathrm{per}(\Lambda)$ is a lattice, whence Λ is crystallographic.

Conversely, let Λ be a crystallographic with $\Gamma := \mathrm{per}(\Lambda)$ as a lattice. Let C be a fundamental domain of Γ. Then, one has $\mathbb{R}^d = \sqcup_{t\in\Gamma}(t + C))$, and $F := C \cap \Lambda$ is a finite set that satisfies $\Lambda = F \oplus \Gamma$.

□

This proposition shows that the locally finite, crystallographic point sets are precisely the crystallographic point packings.

3 More Lattices

A subset Γ' of a lattice Γ that is itself a lattice is called a sublattice of Γ. The gruop index $[\Gamma : \Gamma']$ is referred to as the index of Γ' in Γ.

Lemma 2. *If Γ' is a sublattice of the lattice $\Gamma \subset \mathbb{R}^d$ of index n, the lattice $n\Gamma$ is a sublattice of Γ'.*

The index is related to the volumes of the fundamental domains via

$$[\Gamma : \Gamma'] = \frac{\mathrm{vol}(\mathrm{FD}_{\Gamma'})}{\mathrm{vol}(\mathrm{FD}_{\Gamma})}.$$

If Γ is a lattice in $\mathbb{R}^d$, an important related lattice is the so-called dual lattice $\Gamma*$, which is defined as

$$\Gamma^* = \left\{ y \in \mathbb{R}^d | \langle x|y\rangle \in \mathbb{Z} \text{ for all } x \in \Gamma \right\} .$$

Clearly, the unique basis of Γ^* is vectors $\{b_1^*, \dots, b_d^*\}$ satisfying $\langle b_i^*|b_j\rangle = \delta_{i,j}$,which is called the dual basis and implies that Γ^* is indeed a lattice.

In general, if B is the basis matrix of a lattice Γ that contains the basis vectors columnwise, the dual basis matrix is given by $(B^{-1})^T$.

A useful quantity is the corresponding Gram matrix

$$G := B^T B,$$

which satisfies $G_{ij} = \langle b_i|b_j\rangle$. One has

$$\det(G) = (\det(B))^2 = (\mathrm{vol}\,(\mathrm{FD}_\Gamma))^2 > 0,$$

which is the discriminant of Γ.

4 Through Measure Eyes

Given a locally finite point set $\Lambda \subset \mathbb{R}^d$, we can define a measure

$$\delta_\Lambda = \sum_{x\in\Lambda} \delta_x,$$

where δ_x is the normalised point measure at x. Let $\mathcal{M}(\mathbb{R}^d)$ be the set of all measures on $\mathbb{R}^d$. Notice that

$$\delta_t * \delta_\Lambda = \delta_{t+\Lambda},$$

where $*$ denotes the convolution of measures. We can reconstruct the contents above through measure eyes.

If $\mu \in \mathcal{M}(\mathbb{R}^d)$, the set

$$\mathrm{per}(\mu) := \left\{ t \in \mathbb{R}^d | \delta_t * \mu = \mu \right\}$$

is called the set of periods of μ. It is a subgroup of $\mathbb{R}^d$.

Definition 3. *A measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ is called periodic when $\mathrm{per}(\mu)$ is not trivial. Moreover, the measure μ is called crystallographic when $\mathrm{per}(\mu)$ contains a lattice in $\mathbb{R}^d$.*

Remark. *Why does the definition of crystallographic in Definition 3 require* $\mathrm{per}(\mu)$ *"contains" a lattice but not "be" a lattice?*

It is beacause that there is no obvious ananlogue of Proposition 1. If $\Gamma \subset \mathbb{R}^d$ is a lattice and ϱ a finite measure, the convolution $\mu = \varrho * \delta_\Gamma$ is well-defined and a crystallographic ($\mathrm{per}(\mu) \subset \Gamma$). However, $\mathrm{per}(\mu)$ need not be a lattice, but can be a much larger group. An example is

$$\mu = \varrho * \delta_{\mathbb{Z}^2} \quad \text{with} \quad \varrho = 1_{[0,1)\times[0,\frac{1}{2})}\lambda,$$

where λ is Lebesgue measure and $1_{\{\cdot\}}$ denotes the characteristic function. $\mathrm{per}(\mu) = \mathbb{R} \times \mathbb{Z}$. The geometric meaning of $\mu(A)$ is the size of area belongs to both A and $\cup_{\mathbb{Z}}(\mathbb{R} \times [n, n + 1/2])$. It is reasonable to distinguish measures according to their groups of periods.

References

[1] Michael Baake and Uwe Grimm. *Aperiodic order*, volume 1. Cambridge University Press, 2013.