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DETERMINATION OF PIECEWISE HOMOGENEOUS SOURCES FOR ELASTIC AND ELECTROMAGNETIC WAVES

Jian Zhai

School of Mathematical Science
Fudan University, Shanghai 200433, China

Yue Zhao

School of Mathematics and Statistics
Central China Normal University, Wuhan 430079, China

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Abstract. This paper is concerned with inverse source problems for the time-harmonic elastic wave equations and Maxwell's equations with a single boundary measurement at a fixed frequency. We show the uniqueness and a Lipschitz-type stability estimate under the assumption that the source function is piecewise constant on a domain which is made of a union of disjoint convex polyhedral subdomains.

1. **Introduction.** We are interested in the inverse source problems for the time-harmonic elastic wave equations and electromagnetic wave equations using boundary measurement at a single frequency . It is well known that a general source function can not be uniquely determined at a fixed frequency since non-radiating sources may exist [16]. However, it is still possible to obtain the uniqueness for certain source functions with special forms. For instance, point sources may be uniquely determined from a single measurement [26]. Moreover, the support of a source function can also be determined if it satisfies certain geometric conditions [21, 29]. Recently a Lipschitz-type stability estimate was proved in [25] for a piecewise constant source function on a domain which is a union of disjoint convex polyhedral subdomains. For more recent results on related topics on inverse electromagnetic and elastic scattering problems we refer the reader to [4, 14, 18, 27]. In this paper we intend to extend the results in [25] for Helmholtz equation to elastic wave equations and Maxwell's equations. Since elastic waves and electromagnetic waves are both vector waves, the analysis will become more sophisticated.

We briefly review the existing literature on inverse source problems of elastic wave equations and electromagnetic wave equations. For the elastic wave equations, a characterization of the radiating and non-radiating sources was given in [23]. The increasing stability estimates of the inverse source problems for the elastic wave equations were obtained in [6, 19] using multi-frequency boundary measurements. The authors in [12] investigated the vanishing property at corners and edges of the

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* Corresponding author: Yue Zhao.

support of the source function. For the electromagnetic wave equations, the authors in [1] characterized the radiating and non-radiating sources. A uniqueness result and relevant numerical methods for reconstructing the dipole sources were obtained in [5]. For the increasing stability estimates of the inverse source problems of the electromagnetic wave equations we refer the readers to [22]. For other interesting topics on inverse source problems, we refer the reader to [31] on the study of inverse source problems using a machine-learning approach and [30] on an inverse moving source problem. The inverse source problems have found applications in a variety of scientific and industrial branches such as antenna design and synthesis, seismic imaging, biomedical imaging, and photo-acoustic tomography [20].

The main tool for this work is the construction of singular solutions, which is widely used for inverse coefficient problems [2,3,7–11]. We also want to point out here that "corner scattering" [15] is utilized in our study of inverse source problems as already mentioned in [25]. To be more precise, the argument relies heavily on the lower bound on the integral of the singular solutions in a conic domain (cf. [25, (8)]).

The rest of paper is organized as follows. In Section 2, we introduce the geometric assumptions on the domain partitioning needed for the piecewise homogeneous source functions. In Sections 3 and 4, we study the inverse source problems for elastic waves and electromagnetic waves respectively.

2. **Geometric setup.** We will use the same geometric assumptions as in [25], and hereby briefly summarize them. Assume $\Omega \subset \mathbb{R}^3$ is a domain such that

$$\overline{\Omega} = \cup_{j=1}^{N} \overline{D}_j,$$

where D_j are mutually disjoint bounded open subsets. Denote $B_R := \{x \in \mathbb{R}^3 : |x| < R\}$ where $R > 0$ is a constant. Assume that $\operatorname{dist}(\Omega, \mathbb{R}^3 \setminus B_R) \geq r_0$ for some constant $r_0 > 0$. We use the geometric setup of the domains D_j that has been used in [13].

Assumption 1. *We assume that*

1. *the subdomains $D_j \subset \mathbb{R}^3, 1 \leq j \leq N$ are convex polyhedrons;*
2. *for each $k = 0, \cdots, N-1$, $\cup_{j=k+1}^{N} \overline{D_j}$ is simply connected, and there exists a constant r_0 such that $\{x \in \mathbb{R}^3 | \operatorname{dist}(x, \cup_{j=k+1}^{N} \overline{D_j}) > 2r_0\}$ is connected;*
3. *each D_j has a vertex, denoted by $P^{(j)}$, such that $B_{3r_0}(P^{(j)}) \cap D_k = \emptyset$ for any $k > j$.*

For a sample domain in $\mathbb{R}^2$ satisfying the above assumptions, we refer to [25, Figure 1].

Let (x_1, x_2, x_3) be the Cartesian coordinates in $\mathbb{R}^3$, the corresponding spherical coordinates are

$$x_1 = \rho \sin\theta \cos\varphi, \quad x_2 = \rho \sin\theta \sin\varphi, \quad x_3 = \rho\cos\theta.$$

Assume $\alpha = \alpha(\varphi)$ is a continuous function on $[0, 2\pi]$, such that $\alpha(\varphi) \in (0, \frac{\pi}{2})$ for any $\varphi \in [0, 2\pi]$. We let

$$\mathcal{C}(r_0, \alpha) := \{(\rho, \theta, \varphi) : 0 \leq \rho \leq r_0,\, 0 \leq \theta \leq \alpha(\varphi), 0 \leq \varphi \leq 2\pi\}$$

denote the cone with radius r_0 and vertical angle α. Furthermore, we assume:

Assumption 2. *Let α_1, α_2 be two constants satisfying $0 < \alpha_1 < \alpha_2 < \frac{\pi}{2}$. For each D_j, $j = 1, 2 \cdots, N$, let $P_\ell^{(j)}$ be a vertex. Assume that, after a rigid transform,*

$P_\ell^{(j)} = (0,0,0)$, and $B_{r_0} \cap D_j = \mathcal{C}(r_0, \alpha_\ell^{(j)})$ with $\alpha_1 < \alpha_\ell^{(j)}(\varphi) < \alpha_2$ for any $\varphi \in [0, 2\pi]$.

We refer to [25, Figure 2 and 3] for an illustration for above assumptions.

3. **Elastic waves.** In this section, we consider the inverse source problem for the elastic wave equations in three dimensions

$$\mu \Delta \mathbf{u} + (\lambda + \mu)\nabla\nabla\cdot\mathbf{u} + \omega^2 \rho \mathbf{u} = \mathbf{f}(x), \quad x \in \mathbb{R}^3, \tag{1}$$

where $\omega > 0$ is the frequency, λ and μ are the Lamé constants satisfying $\mu > 0$ and $2\lambda + 3\mu > 0$, ρ is the density; $\mathbf{u}$ denotes the elastic wavefield, and the source function $\mathbf{f} \in L^\infty(\mathbb{R}^3)$ is assumed to have the compact support $\overline{\Omega}$.

It is well known that in $\mathbb{R}^3 \setminus \overline{\Omega}$, the solution $\mathbf{u}$ to the equation (1) admits a decomposition into P-wavefield $\mathbf{u}_p$ and S-wavefield $\mathbf{u}_s$ (cf. [6] for example),

$$\mathbf{u} = \mathbf{u}_p + \mathbf{u}_s \tag{2}$$

where $\nabla \times \mathbf{u}_p = 0$, $\nabla \cdot \mathbf{u}_s = 0$ and

$$\Delta \mathbf{u}_p + \kappa_p^2 \mathbf{u}_p = 0, \quad \Delta \mathbf{u}_s + \kappa_s^2 \mathbf{u}_s = 0, \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}.$$

Here $\kappa_p = \omega\sqrt{\frac{\rho}{\lambda+2\mu}}$, $\kappa_s = \omega\sqrt{\frac{\rho}{\mu}}$ are the wave numbers for P- and S- waves. Actually, one has

$$\mathbf{u}_p = -\frac{1}{\kappa_p^2}\nabla\nabla\cdot\mathbf{u}, \quad \mathbf{u}_s = -\frac{1}{\kappa_s^2}\nabla\times\nabla\times\mathbf{u} \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}. \tag{3}$$

To ensure the well-posedness of the direct scattering problem [6], the wavefields $\mathbf{u}_p$ and $\mathbf{u}_s$ are required to satisfy the Kupradze-Sommerfeld radiation condition

$$\lim_{r\to\infty} r(\partial_r \mathbf{u}_p - \mathrm{i}\kappa_p \mathbf{u}_p) = 0, \quad \lim_{r\to\infty} r(\partial_r \mathbf{u}_s - \mathrm{i}\kappa_s \mathbf{u}_s) = 0, \quad r = |x| \tag{4}$$

uniformly in all directions $\hat{x} = x/|x|$.

Given the source $\mathbf{f}$, the direct scattering problem is to determine the wave field $\mathbf{u}$ which satisfies (1)–(4). It is known that the direct scattering problem has a unique solution $\mathbf{u} \in H^2(B_R)$ for an arbitrary frequency $\omega > 0$ and the solution $\mathbf{u}$ satisfies the following estimate by standard elliptic regularity theory (see e.g. [28, (7.13)])

$$\|\mathbf{u}\|_{H^2(B_R)} \le C\|\mathbf{f}\|_{L^2(\Omega)}, \tag{5}$$

where C is a positive constant. This paper is concerned with the inverse source scattering problem, which is to determine $\mathbf{f}$ from the boundary measurement of $\mathbf{u}$ on $\partial B_R = \{x \in \mathbb{R}^3 : |x| = R\}$ at a fixed frequency ω.

In this work, we consider the case where the source $\mathbf{f}$ is a piecewise constant function. Using the notations in Section 2, we assume

$$\mathbf{f}(x) = \sum_{j=1}^N \mathbf{c}_j \chi_{D_j}(x) = \sum_{j=1}^N \begin{pmatrix} c_{j,1} \\ c_{j,2} \\ c_{j,3} \end{pmatrix} \chi_{D_j}(x), \tag{6}$$

where $\mathbf{c}_j = (c_{j,1}, c_{j,2}, c_{j,3})$, $j = 1, \cdots, N$, are constants. The goal is to establish the Lipschitz stability of determining the constants $c_{j,k}$, $j = 1, \cdots, N$, $k = 1, 2, 3$ from the measurement of $\mathbf{u}$ on ∂B_R at a fixed frequency ω.

In addition, we also make the following assumption on the source function.

Assumption 3. *The source function f has the compact support $\overline{\Omega}$ with $|\Omega| \le A$ and satisfies $\|f\|_{L^\infty(\Omega)} \le E$, where A and E are positive constants.*

3.1. **Statement of the main result.** Denote

$$\epsilon := \|\mathbf{u}\|_{H^1(\partial B_R)}.$$

The following theorem is the main result for the inverse source problem for elastic waves.

Theorem 3.1. *Let* $\mathbf{f}$ *satisfy Assumptions 1–3 and the subdomains* $D_j, j = 1, \ldots, N$ *are given. If* $\epsilon = 0$ *then* $\mathbf{f} = 0$*. Moreover, the following estimate holds:*

$$\|\mathbf{f}\|_{L^\infty(\Omega)} \lesssim \epsilon. \tag{7}$$

Hereafter, the notation $a \lesssim b$ stands for $a \leq Cb$, where $C > 0$ is a positive constant which depends on the following parameters: $\kappa, A, E, N, r_0, R, \alpha_1, \alpha_2$.

3.2. **Construction of singular solutions.** As in [25], we need to construct singular solutions for the equation

$$\mu\Delta\mathbf{u} + (\lambda + \mu)\nabla\nabla \cdot \mathbf{u} + \omega^2\rho\mathbf{u} = 0.$$

The solutions to be constructed need to solve the above equation for $x \neq 0$, and blow up when $x \to 0$.

We start with the functions

$$\Gamma_p(x) = \frac{e^{\mathrm{i}\kappa_p|x|}}{|x|}, \quad \Gamma_s(x) = \frac{e^{\mathrm{i}\kappa_s|x|}}{|x|}.$$

Note that these are fundamental solutions for the Helmholtz equation with wavenumbers κ_p, κ_s respectively,

$$(\Delta + \kappa_p^2)\Gamma_p(x) = \delta(x), \quad (\Delta + \kappa_s^2)\Gamma_s(x) = \delta(x),$$

where $\delta(x)$ is the delta function.

Now let

$$\begin{aligned}
\mathbf{\Phi}^{(1)}(x) &= \partial_{x_3}^2 \nabla\Gamma_p(x), \\
\mathbf{\Phi}^{(2)}(x) &= \partial_{x_3}^2 \nabla \times \begin{pmatrix} \Gamma_s(x) \\ 0 \\ 0 \end{pmatrix}, \\
\mathbf{\Phi}^{(3)}(x) &= -\partial_{x_3}^2 \nabla \times \begin{pmatrix} 0 \\ \Gamma_s(x) \\ 0 \end{pmatrix}.
\end{aligned}$$

where $\partial_{x_3}^2$ denotes the second order partial derivative with respect to the variable x_3. Notice that, for $x \neq 0$, $\mathbf{\Phi}^{(1)}$ is curl-free, and thus a P-wave solution; $\mathbf{\Phi}^{(2)}$, $\mathbf{\Phi}^{(3)}$ are divergence-free, and thus S-wave solutions. By calculation, we have the following asymptotic behaviors as $x \to 0$,

$$\begin{aligned}
\partial_{x_3}^2 \partial_{x_1}\Gamma_{p/s}(x) &= \frac{3x_1}{|x|^5} - \frac{15x_1x_3^2}{|x|^7} + \mathcal{O}(|x|^{-3}), \\
\partial_{x_3}^2 \partial_{x_2}\Gamma_{p/s}(x) &= \frac{3x_2}{|x|^5} - \frac{15x_2x_3^2}{|x|^7} + \mathcal{O}(|x|^{-3}), \\
\partial_{x_3}^3 \Gamma_{p/s}(x) &= \frac{9x_3}{|x|^5} - \frac{15x_3^3}{|x|^7} + \mathcal{O}(|x|^{-3}).
\end{aligned}$$

Consequently, we have the following asymptotics

$$\mathbf{\Phi}^{(1)}(x) = \begin{pmatrix} \frac{3x_1}{|x|^5} - \frac{15x_1x_3^2}{|x|^7} \\ \frac{3x_2}{|x|^5} - \frac{15x_2x_3^2}{|x|^7} \\ \frac{9x_3}{|x|^5} - \frac{15x_3^3}{|x|^7} \end{pmatrix} + \mathcal{O}(|x|^{-3}),$$

$$\mathbf{\Phi}^{(2)}(x) = \begin{pmatrix} 0 \\ \frac{9x_3}{|x|^5} - \frac{15x_3^3}{|x|^7} \\ -\frac{3x_2}{|x|^5} + \frac{15x_2x_3^2}{|x|^7} \end{pmatrix} + \mathcal{O}(|x|^{-3}),$$

$$\mathbf{\Phi}^{(3)}(x) = \begin{pmatrix} \frac{9x_3}{|x|^5} - \frac{15x_3^3}{|x|^7} \\ 0 \\ -\frac{3x_1}{|x|^5} + \frac{15x_1 2x_3^2}{|x|^7} \end{pmatrix} + \mathcal{O}(|x|^{-3}).$$

Lemma 3.2. *The functions $\mathbf{\Phi}^{(j)}$, $j = 1, 2, 3$ are solutions to the equation*

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} + \omega^2\rho\mathbf{u} = 0,$$

for $x \neq 0$.

Proof. First, let us consider $\mathbf{\Phi}^{(1)}$. We have, for $x \neq 0$,

$$\begin{aligned}(\mu\Delta + (\lambda+\mu)\nabla\nabla\cdot)\mathbf{\Phi}^{(1)}(x) =& \mu\partial_{x_3}^2\Delta\nabla\Gamma_p(x) + (\lambda+\mu)\partial_{x_3}^2\nabla\nabla\cdot\nabla\Gamma_p(x) \\ =& \mu\partial_{x_3}^2\nabla\Delta\Gamma_p(x) + (\lambda+\mu)\partial_{x_3}^2\nabla\Delta\Gamma_p(x) \\ =& \partial_{x_3}^2\nabla(\lambda+2\mu)\Delta\Gamma_p(x) \\ =& -\partial_{x_3}^2\nabla(\lambda+2\mu)\kappa_p^2\Gamma_p(x) \\ =& -\rho\omega^2\partial_{x_3}^2\nabla\Gamma_p(x) \\ =& -\rho\omega^2\mathbf{\Phi}^{(1)}(x)\end{aligned}$$

where we have used the fact $(\lambda+2\mu)\kappa_p^2 = \omega^2\rho$.

Next, we consider $\mathbf{\Phi}^{(2)}$. For $x \neq 0$,

$$\begin{aligned}(\mu\Delta + (\lambda+\mu)\nabla\nabla\cdot)\mathbf{\Phi}^{(2)} =& \partial_{x_3}^2(\mu\Delta + (\lambda+\mu)\nabla\nabla\cdot)\nabla\times\begin{pmatrix}\Gamma_s(x)\\0\\0\end{pmatrix} \\ =& \partial_{x_3}^2\mu\Delta\begin{pmatrix}\Gamma_s(x)\\0\\0\end{pmatrix} \\ =& \partial_{x_3}^2\mu\begin{pmatrix}-\kappa_s^2\Gamma_s(x)\\0\\0\end{pmatrix} \\ =& -\omega^2\rho\partial_{x_3}^2\begin{pmatrix}\Gamma_s(x)\\0\\0\end{pmatrix} \\ =& -\omega^2\rho\mathbf{\Phi}^{(2)},\end{aligned}$$

where we have used the fact $\mu\kappa_s^2 = \omega^2\rho$. The proof for $\mathbf{\Phi}^{(3)}$ is similar, and thus omitted. □

3.3. **Proof of Theorem 3.1.** As in [25], we define a sequence of domains which will be used in the proof.

Let

$$U_0=\Omega,\quad W_0=\emptyset,\quad U_k=\Omega\setminus\cup_{j=1}^k D_j,\quad W_k=\Omega\setminus U_k,\quad k=1,...,N.$$

For each $k\in\{0,1,2,...,N-1\}$, consider the vertex $P^{(k+1)}$ of the cell D_{k+1}. By choosing appropriate Cartesian coordinates $(x_1^{(k+1)},x_2^{(k+1)},x_3^{(k+1)})$, we assume $D_{k+1}\cap B_{r_0}(P^{(k+1)})=P^{(k+1)}+\mathcal{C}(r_0,\alpha^{(k+1)})$, with $\alpha^{(k+1)}=\alpha^{(k+1)}(\varphi)$, $\varphi\in[0,2\pi]$, i.e., a cone with its vertex at $P^{(k+1)}$. By Assumption 2, we have

$$\alpha_1<\alpha^{(k+1)}(\varphi)<\alpha_2$$

for $\varphi\in[0,2\pi]$.

Denote $P^{(k+1)}=(p_1^{(k+1)},p_2^{(k+1)},p_3^{(k+1)})$,

$$\begin{aligned}Q_{k+1}^-=\{x=(x_1^{(k+1)},x_2^{(k+1)},x_3^{(k+1)}):|x_1^{(k+1)}-p_1^{(k+1)}|^2+|x_2^{(k+1)}-p_2^{(k+1)}|^2<r_0^2,\\ -2r_0<x_3^{(k+1)}-p_3^{(k+1)}<0\},\end{aligned}$$

and

$$\mathcal{K}_k=\{x\in B_{R+r_0}:\mathrm{dist}(x,U_k)>r_0\}\cup Q_{k+1}^-.$$

We refer to [25, Figure 4] for an illustrative example of the domains U_k and $\mathcal{K}_k$. We note the connectedness of $\mathcal{K}_k$ due to Assumption 1.

We use repetitively the following quantitative estimate of unique continuation for the solution of the *Helmholtz* equation (cf. [25, Proposition 1]).

Proposition 1. *Let $\mathcal{K}_k$ be defined as before and let $v\in H^1(\mathcal{K}_k)$ be a weak solution to the Helmholtz equation*

$$\Delta v+\kappa^2 v=0\quad in\quad \mathcal{K}_k.$$

Assume that, for given positive constants ε_0 and E_1, v satisfies

$$\|v\|_{L^\infty(B_{R+r_0}\setminus B_{R+\frac{r_0}{2}}))}\le\varepsilon_0$$

and

$$|v(x)|\le E_1|x-P^{(k+1)}|^{-1},\quad x\in\mathcal{K}_k.$$

Then the following inequality holds for small enough $r>0$:

$$|v(x_r)|\lesssim\varepsilon^{\tau_r}E_1^{1-\tau_r}r^{-(1-\tau_r)},$$

where $x_r=P^{(k+1)}+(0,0,-r)$ and $\tau_r=\theta r^\delta$ with $0<\theta<1$ and $\delta>0$ depending on r_0,κ,N,A.

For a fixed $k\in\{0,1,\cdots,N-1\}$, we just denote the Cartesian coordinates $(x_1,x_2,x_3)=(x_1^{(k+1)},x_2^{(k+1)},x_3^{(k+1)})$ for brevity. After the coordinate is fixed, in the following, let

$$\mathbf{\Phi}_k^{(j)}(x,y):=\mathbf{\Phi}^{(j)}(x-y),\quad j=1,2,3.$$

Define

$$S_k^{(j)}(y)=\int_{U_k}\mathbf{f}(x)\cdot\mathbf{\Phi}_k^{(j)}(x,y)\mathrm{d}x.$$

Note that $S_k^{(j)}$ is scalar function, and solves the scalar Helmholtz equation. More precisely, we have the following lemma.

Lemma 3.3. *For $y \in \mathcal{K}_k$, it holds that*

$$
\begin{aligned}
(\Delta+\kappa_p^2)S_k^{(1)}(y) &= 0,\\
(\Delta+\kappa_s^2)S_k^{(2)}(y) &= 0,\\
(\Delta+\kappa_s^2)S_k^{(3)}(y) &= 0.
\end{aligned}
$$

Proof. We just take $j=1$ for example, and the proof for $j=2,3$ is similar. Noticing that for any $x \in U_k$, $y \in \mathcal{K}_k$, we have

$$
\begin{aligned}
&\mathbf{f}(x)\cdot(\Delta_y+\kappa_p^2)\mathbf{\Phi}_k^{(1)}(x,y)\\
=&\mathbf{f}(x)\cdot(\partial_{x_3}^2(\Delta_y+\kappa_p^2)\nabla_x\Gamma_p(x-y))\\
=&\mathbf{f}(x)\cdot(\partial_{x_3}^2\nabla_x(\Delta_y+\kappa_p^2)\Gamma_p(x-y))\\
=&0,
\end{aligned}
$$

since U_k and $\mathcal{K}_k$ are disconnected. The proof is completed if we change the order of integration and differentiation. □

We will use the following lemma (cf. [25, Lemma 2]).

Lemma 3.4. *If for some $\varepsilon_0>0$ and $k \in \{1,...,N-1\}$, it holds*

$$
|S_k^{(j)}(y)| \le \varepsilon_0, \quad \forall\, y \in B_{R+r_0}\setminus B_{R+\frac{r_0}{2}},
$$

then

$$
|S_k^{(j)}(y_r)| \lesssim E^{1-\tau_r}\varepsilon_0^{\tau_r} r^{-(1-\tau_r)},
$$

where $y_r = P^{(k+1)}+(0,0,-r)$ with r being small enough and $\tau_r=\theta r^\delta$ with the positive constants $\theta \in (0,1)$ and δ depending on r_0,κ,N,A.

We will use the following lemma (cf. [25, Lemma 2]).

Lemma 3.5. *If for some $\varepsilon_0>0$ and $k \in \{1,...,N-1\}$, it holds*

$$
|S_k^{(j)}(y)| \le \varepsilon_0, \quad \forall\, y \in B_{R+r_0}\setminus B_{R+\frac{r_0}{2}},
$$

then

$$
|S_k^{(j)}(y_r)| \lesssim E^{1-\tau_r}\varepsilon_0^{\tau_r} r^{-(1-\tau_r)},
$$

where $y_r = P^{(k+1)}+(0,0,-r)$ with r being small enough and $\tau_r=\theta r^\delta$ with the positive constants $\theta \in (0,1)$ and δ depending on r_0,κ,N,A.

Taking dot product of both sides of (1) with $\mathbf{\Phi}_k^{(j)}(x,y)$ for $y \in B_{R+r_0} \setminus B_{R+\frac{r_0}{2}}$ and using integration by parts, we have

$$\begin{aligned}
&\int_{\Omega} \mathbf{f}(x)\cdot \mathbf{\Phi}_k^{(j)}(x,y)\mathrm{d}x \\
=&\int_{B_R} \mathbf{f}(x)\cdot \mathbf{\Phi}_k^{(j)}(x,y)\mathrm{d}x \\
=&\int_{B_R} \left[(\mu\Delta + (\lambda+\mu)\nabla\nabla\cdot + \omega^2\rho)\mathbf{u}(x)\right]\cdot \mathbf{\Phi}_k^{(j)}(x,y)\mathrm{d}x \\
=&\int_{B_R} \mathbf{u}(x)\cdot (\mu\Delta + (\lambda+\mu)\nabla\nabla\cdot + \omega^2\rho)\mathbf{\Phi}_k^{(j)}(x,y)\mathrm{d}x \\
&+\int_{\partial B_R} \Big[(\mu\partial_{\nu(x)}\mathbf{u} + (\lambda+\mu)(\nabla\cdot\mathbf{u})\nu(x))\cdot \mathbf{\Phi}_k^{(j)}(x,y) \\
&\qquad - \left(\mu\partial_{\nu(x)}\mathbf{\Phi}_k^{(j)}(x,y) + (\lambda+\mu)(\nabla_x\cdot \mathbf{\Phi}_k^{(j)}(x,y))\nu\right)\cdot \mathbf{u}(x)\Big]\mathrm{d}s(x) \\
=&\int_{\partial B_R} \Big[(\mu\partial_{\nu(x)}\mathbf{u} + (\lambda+\mu)(\nabla\cdot\mathbf{u})\nu(x))\cdot \mathbf{\Phi}_k^{(j)}(x,y) \\
&\qquad - \left(\mu\partial_{\nu(x)}\mathbf{\Phi}_k^{(j)}(x,y) + (\lambda+\mu)(\nabla_x\cdot \mathbf{\Phi}_k^{(j)}(x,y))\nu\right)\cdot \mathbf{u}(x)\Big]\mathrm{d}s(x),
\end{aligned} \tag{8}$$

where ν is the unit outer normal vector on ∂B_R.

First, note that for $k=0$,

$$S_0(y) = \int_{\Omega} \mathbf{f}(x)\cdot \mathbf{\Phi}_0^{(j)}(x,y)\mathrm{d}x.$$

Also notice that

$$\int_{\partial B_R} |\mathbf{\Phi}_0^{(j)}(\cdot,y)|^2 + |\partial_\nu \mathbf{\Phi}_0^{(j)}(\cdot,y)|^2 \mathrm{d}s \le C$$

for $y \in B_{R+r_0} \setminus B_{R+\frac{r_0}{2}}$, where C depends on R, κ, r_0. Notice that $\mathbf{u}|_{\mathbb{R}^3\setminus B_R}$ is the solution to the exterior problem for the time-harmonic elastic wave equation

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} + \omega^2\rho\mathbf{u} = 0 \quad \text{in } \mathbb{R}^3\setminus B_R$$

along with the radiation condition (4). For the above exterior problem, the solution $\mathbf{u}|_{\mathbb{R}^3\setminus B_R}$ is uniquely determined. Then we define the operator $\mathcal{N}: H^1(\partial B_R)\to L^2(\partial B_R)$ such that

$$\mathcal{N}\mathbf{u} := \mu\partial_\nu\mathbf{u} + (\lambda+\mu)(\nabla\cdot\mathbf{u})\nu \quad \text{on } \partial B_R.$$

It is known that the operator $\mathcal{N}$ is bounded. For an explicit representation and a detailed deduction of the Dirichlet-to-Neumann map, we refer to [24, (18)]. Hence, the Neumann data $\mu\partial_\nu\mathbf{u}+(\lambda+\mu)(\nabla\cdot\mathbf{u})\nu$ on ∂B_R can be obtained once the Dirichlet data $\mathbf{u}$ is available on ∂B_R. Therefore, we obtain the following estimate

$$\int_{\partial B_R} (|\mu\partial_\nu\mathbf{u}+(\lambda+\mu)(\nabla\cdot\mathbf{u})\nu|^2 + |\mathbf{u}|^2)\mathrm{d}s = \int_{\partial B_R} (|\mathcal{N}\mathbf{u}|^2+|\mathbf{u}|^2)\mathrm{d}s \le C\|\mathbf{u}\|^2_{H^1(\partial B_R)} \le C\epsilon^2,$$

where C depends on κ and R. Therefore by (8), we obtain

$$\left|S_0^{(j)}(y)\right| \lesssim \epsilon, \quad y \in B_{R+r_0}\setminus B_{R+\frac{r_0}{2}}. \tag{9}$$

First we prove a logarithmic-type stability. Denote $\delta_0 = \epsilon$ and $\delta_j = \|\mathbf{f}\|_{L^\infty(W_j)}$, $j = 1,\cdots,N$. We will inductively prove that the following estimates hold:

$$\delta_j \le \omega_j(\epsilon), \tag{10}$$

where $\omega_0(\epsilon) \leq \omega_1(\epsilon) \leq \cdots \leq \omega_N(\epsilon)$ for any small $\epsilon > 0$ and

$$\lim_{\epsilon \to 0} \omega_j(\epsilon) = 0$$

for each j. The estimate (10) is clearly true for $j = 0$, for which $\omega_0(\epsilon) = \epsilon$, by invoking (9). We now assume that the estimate (10) is valid for $j = k$, and deduce the estimate for $j = k + 1$.

Recall that

$$\begin{aligned} S_k^{(j)}(y) &= \int_{U_k} \mathbf{f}(x) \cdot \mathbf{\Phi}_k^{(j)}(x, y) \mathrm{d}x \\ &= \int_{\Omega} \mathbf{f}(x) \cdot \mathbf{\Phi}_k^{(j)}(x, y) \mathrm{d}x - \int_{W_k} \mathbf{f}(x) \cdot \mathbf{\Phi}_k^{(j)}(x, y) \mathrm{d}x. \end{aligned}$$

Thus we have the estimate

$$|S_k^{(j)}(y)| \leq \left| \int_{\Omega} \mathbf{f}(x) \cdot \mathbf{\Phi}_k^{(j)}(x, y) \mathrm{d}x \right| + \left| \int_{W_k} \mathbf{f}(x) \cdot \mathbf{\Phi}_k^{(j)}(x, y) \mathrm{d}x \right|. \tag{11}$$

Similar to (9), we have

$$\left| \int_{\Omega} \mathbf{f}(x) \cdot \mathbf{\Phi}_k^{(j)}(x, y) \mathrm{d}x \right| \leq C\epsilon \tag{12}$$

for $y \in B_{R+r_0} \setminus B_{R+\frac{r_0}{2}}$. For the estimate of the second term in the right hand side of (11), first notice that $|x - y| > Cr_0$ for $x \in W_k$ and $y \in B_{R+r_0} \setminus B_{R+\frac{r_0}{2}}$, and therefore

$$|\mathbf{\Phi}_k^{(j)}(x, y)| \leq \frac{C}{|x - y|^4} \leq \frac{C}{r_0^4}.$$

Also we have $|\mathbf{f}(x)| \leq C\omega_k(\epsilon)$ for $x \in W_k$ by the hypothesis for induction. Therefore

$$\left| \int_{W_k} \mathbf{f}(x) \cdot \mathbf{\Phi}_k^{(j)}(x, y) \mathrm{d}x \right| \leq C\omega_k(\epsilon) \tag{13}$$

for $y \in B_{R+r_0} \setminus B_{R+\frac{r_0}{2}}$. Combining the estimates (11)–(13), we obtain

$$|S_k^{(j)}(y)| \lesssim (\epsilon + \omega_k(\epsilon)), \quad y \in B_{R+r_0} \setminus B_{R+\frac{r_0}{2}}.$$

Note that the above estimate is also valid for $k = 0$, for which $W_0 = \emptyset$. Now let $y_r = P^{(k+1)} + (0, 0, -r)$. By Lemma 3.5, we have

$$|S_k^{(j)}(y_r)| \lesssim r^{-1} \omega_k(\epsilon)^{\tau_r}, \tag{14}$$

if $0 < r < \frac{1}{C_2}$ for some constant $C_2 > 0$.

Next, we write

$$S_k^{(j)}(y_r) = I_1^{(j)} + I_2^{(j)},$$

where

$$\begin{aligned} I_1^{(j)} &= \int_{B_{r_0}(y_r) \cap D_{k+1}} \mathbf{f}(x) \cdot \mathbf{\Phi}_k^{(j)}(x, y_r) \mathrm{d}x, \\ I_2^{(j)} &= \int_{U_k \setminus (B_{r_0}(y_r) \cap D_{k+1})} \mathbf{f}(x) \cdot \mathbf{\Phi}_k^{(j)}(x, y_r) \mathrm{d}x. \end{aligned}$$

For a depiction of the region $B_{r_0}(y_r) \cap D_{k+1}$ we refer the reader to [25, Figure 5]. First it is easy to verify that

$$|I_2^{(j)}| \lesssim 1. \tag{15}$$

Combining (14) and (15) yields

$$|I_1^{(j)}| \lesssim r^{-1}\omega_k(\epsilon)^{\tau_r} + 1. \tag{16}$$

Since $\mathbf{f}(x) = (c_{k+1,1}, c_{k+1,2}, c_{k+1,3})$ on D_{k+1}, we have

$$\begin{pmatrix} I_1^{(1)} \\ I_1^{(2)} \\ I_1^{(3)} \end{pmatrix} = \begin{pmatrix} c_{k+1,1}\alpha_3(r) + c_{k+1,3}\alpha_1(r) \\ c_{k+1,2}\alpha_3(r) + c_{k+1,3}\alpha_1(r) \\ c_{k+1,1}\alpha_1(r) + c_{k+1,2}\alpha_2(r) - c_{k+1,3}\alpha_3(r) \end{pmatrix} = \mathbf{M}(r) \begin{pmatrix} c_{k+1,2} \\ c_{k+1,2} \\ c_{k+1,3} \end{pmatrix}$$

where

$$\mathbf{M}(r) = \begin{pmatrix} \alpha_3(r) & 0 & \alpha_1(r) \\ 0 & \alpha_3(r) & \alpha_2(r) \\ \alpha_1(r) & \alpha_2(r) & -\alpha_3(r) \end{pmatrix},$$

and

$$\alpha_1(r) = \int_{B_{r_0}(y_r)\cap D_{k+1}} -\frac{3x_1}{|x|^5} + \frac{15x_1x_3^2}{|x|^7}\mathrm{d}x + \mathcal{O}(|\log r|),$$

$$\alpha_2(r) = \int_{B_{r_0}(y_r)\cap D_{k+1}} -\frac{3x_2}{|x|^5} + \frac{15x_2x_3^2}{|x|^7}\mathrm{d}x + \mathcal{O}(|\log r|),$$

$$\alpha_3(r) = \int_{B_{r_0}(y_r)\cap D_{k+1}} \frac{9x_3}{|x|^5} - \frac{15x_3^3}{|x|^7}\mathrm{d}x + \mathcal{O}(|\log r|).$$

Note that we have proved in [25] that $|\alpha_3(r)| > C_0 r^{-1} - C_1 r^{-1/2}$ for some $C_0 > 0$. Therefore,

$$|\det \mathbf{M}(r)| = |\alpha_3(r)(\alpha_3^2(r) + \alpha_1^2(r) + \alpha_2^2(r))| \geq C_0 r^{-3} - C_1 r^{-5/2} \gtrsim r^{-5/2}$$

for r small enough. In particular the matrix $\mathbf{M}(r)$ is non-singular. Therefore

$$\begin{pmatrix} c_{k+1,1} \\ c_{k+1,2} \\ c_{k+1,3} \end{pmatrix} = \mathbf{M}(r)^{-1} \begin{pmatrix} I_1^{(1)} \\ I_1^{(2)} \\ I_1^{(3)} \end{pmatrix} = (\det \mathbf{M}(r))^{-1}\mathrm{adj}(\mathbf{M}(r)) \begin{pmatrix} I_1^{(1)} \\ I_1^{(2)} \\ I_1^{(3)} \end{pmatrix},$$

and consequently,

$$|\mathbf{c}_{k+1}| \lesssim r^{1/2}|(I_1^{(1)}, I_1^{(2)}, I_1^{(3)})|.$$

Here we have also used the fact that $|\alpha_j(r)| \lesssim r^{-1}$ for $j = 1, 2, 3$.

Together with (16), we obtain

$$|\mathbf{c}_{k+1}| \lesssim r^{-1/2}\omega_k(\epsilon)^{\tau_r} + r^{1/2},$$

where $r \in (0, \frac{1}{C_2})$. Define

$$\sigma(t) = \begin{cases} |\log t|^{-\frac{1}{4\delta}} \text{ for } 0 < t < e^{-1}, \\ t - e^{-1} + 1 \text{ for } t > e^{-1}. \end{cases}$$

If $\omega_k(\epsilon) < e^{-1}$, by taking

$$r = \frac{|\log \omega_k(\epsilon)|^{-\frac{1}{2\delta}}}{C_2} < \frac{1}{C_2},$$

we obtain

$$|\mathbf{c}_{k+1}| \lesssim |\log \omega_k(\epsilon)|^{-\frac{1}{4\delta}} = \sigma(\omega_k(\epsilon)).$$

Remember that $\delta > 0$ depends on r_0, κ, N, A. If $\omega_k(\epsilon) > e^{-1}$, we have

$$|\mathbf{c}_{k+1}| \lesssim \sigma(\omega_k(\epsilon))$$

since $|\mathbf{c}_{k+1}|$ is bounded. Hence

$$\delta_{k+1} \lesssim \omega_{k+1}(\epsilon) := \sigma(\omega_k(\epsilon)).$$

Then it is easy to verify that $\lim_{\epsilon\to 0} \omega_{k+1}(\epsilon) = 0$, which completes the induction. Now we conclude that there exists some positive constant C^* such that

$$\|\mathbf{f}\|_{L^\infty(\Omega)} \le C^* \omega_N(\epsilon), \tag{17}$$

where $\lim_{\epsilon\to 0} \omega_N(\epsilon) = 0$.

The Lipschitz stability estimate can be derived immediately following the same consideration as in [25]. For the self-containedness of the paper, we give a brief proof here. Consider the linear operator

$$\begin{aligned} &T : \mathbb{C}^{3N} \to H^1(\partial B_R), \\ &T(\mathbf{c}_1, \cdots, \mathbf{c}_N) \mapsto u|_{\partial B_R}, \end{aligned}$$

where $\mathbf{u}$ solves (1) with $\mathbf{f}$ given by (6). Then, by (17) and using the fact that $\omega_N(\cdot)$ is strictly monotonically increasing, we have

$$\inf_{\|(\mathbf{c}_1,\cdots,\mathbf{c}_N)\|_\infty = 1} \|T(\mathbf{c}_1, \cdots, \mathbf{c}_N)\|_{H^1(\partial B_R)} \ge \omega_N^{-1}\left(\frac{1}{C^*}\right) =: C'' > 0.$$

Therefore we have

$$\|(\mathbf{c}_1, \cdots, \mathbf{c}_N)\|_\infty \le C'' \|T(\mathbf{c}_1, \cdots, \mathbf{c}_N)\|_{H^1(\partial B_R)}.$$

This completes the proof of Theorem 3.1.

4. **Electromagnetic waves.** In this section, we turn to the inverse source problem for electromagnetic waves. Consider the time-harmonic Maxwell's equations in a homogeneous medium

$$\nabla \times \mathbf{E} - \mathrm{i}\kappa \mathbf{H} = 0, \quad \nabla \times \mathbf{H} + \mathrm{i}\kappa \mathbf{E} = \mathbf{J} \quad \text{in } \mathbb{R}^3. \tag{18}$$

The Silver-Müller radiation reads

$$\lim_{r\to\infty} ((\nabla \times \mathbf{E}) \times x - \mathrm{i}\kappa r \mathbf{E}) = 0, \quad r = |x|. \tag{19}$$

Eliminating the magnetic field $\mathbf{H}$, we obtain

$$\nabla \times (\nabla \times \mathbf{E}) - \kappa^2 \mathbf{E} = -\mathrm{i}\kappa \mathbf{J}. \tag{20}$$

Assume now that the source $\mathbf{J}$ is piecewise constant. More precisely, we assume

$$\mathbf{J}(x) = \sum_{j=1}^N \begin{pmatrix} c_{j,1} \\ c_{j,2} \\ c_{j,3} \end{pmatrix} \chi_{D_j}(x). \tag{21}$$

For the well-posedness of the direct scattering problem we refer the readers to [6].

4.1. **Construction of singular solutions.** Denote $\Gamma(x) = \frac{e^{\mathrm{i}\kappa|x|}}{|x|}$. Notice that

$$\Delta\Gamma(x) + \kappa^2\Gamma(x) = 0$$

for $x \neq 0$. Now we take three singular solutions

$$\mathbf{\Phi}^{(1)}(x) = \partial_{x_3}\begin{pmatrix} \partial_{x_1}\partial_{x_3}\Gamma(x) \\ \partial_{x_2}\partial_{x_3}\Gamma(x) \\ \partial^2_{x_3}\Gamma(x) + \kappa^2\Gamma(x) \end{pmatrix} = \begin{pmatrix} \frac{3x_1}{|x|^5} - \frac{15x_1x_3^2}{|x|^7} \\ \frac{3x_2}{|x|^5} - \frac{15x_2x_3^2}{|x|^7} \\ \frac{9x_3}{|x|^5} - \frac{15x_3^3}{|x|^7} \end{pmatrix} + \mathcal{O}(|x|^{-3}),$$

$$\mathbf{\Phi}^{(2)}(x) = \partial^2_{x_3}\nabla\times\begin{pmatrix} \Gamma(x) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{9x_3}{|x|^5} - \frac{15x_3^3}{|x|^7} \\ -\frac{3x_2}{|x|^5} + \frac{15x_2x_3^2}{|x|^7} \end{pmatrix} + \mathcal{O}(|x|^{-3}),$$

$$\mathbf{\Phi}^{(3)}(x) = -\partial^2_{x_3}\nabla\times\begin{pmatrix} 0 \\ \Gamma(x) \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{9x_3}{|x|^5} - \frac{15x_3^3}{|x|^7} \\ 0 \\ -\frac{3x_1}{|x|^5} + \frac{15x_12x_3^2}{|x|^7} \end{pmatrix} + \mathcal{O}(|x|^{-3}).$$

We emphasize here that these solutions have the same leading order asymptotic behaviors as those used for elastic waves.

Lemma 4.1. *The functions $\mathbf{\Phi}^{(j)}$, $j = 1, 2, 3$ are solutions to the equation*

$$\nabla\times(\nabla\times\mathbf{u}) - \kappa^2\mathbf{u} = 0,$$

for $x \neq 0$.

Proof. First, let us consider $\mathbf{\Phi}^{(1)}$. It can be rewritten as

$$\mathbf{\Phi}^{(1)} = \partial_{x_3}\nabla\partial_{x_3}\Gamma(x) + \partial_{x_3}\begin{pmatrix} 0 \\ 0 \\ \kappa^2\Gamma(x) \end{pmatrix}.$$

Then

$$\begin{aligned} \nabla\times\nabla\times\mathbf{\Phi}^{(1)}(x) =& \kappa^2\partial_{x_3}\nabla\times\nabla\times\begin{pmatrix} 0 \\ 0 \\ \Gamma(x) \end{pmatrix} \\ =& \kappa^2\partial_{x_3}(-\Delta + \nabla\nabla\cdot)\begin{pmatrix} 0 \\ 0 \\ \Gamma(x) \end{pmatrix} \\ =& \kappa^2\partial_{x_3}\begin{pmatrix} 0 \\ 0 \\ \kappa^2\Gamma(x) \end{pmatrix} + \kappa^2\partial_{x_3}\nabla\partial_{x_3}\Gamma(x) \\ =& \kappa^2\mathbf{\Phi}^{(1)}(x). \end{aligned}$$

Next, we consider $\mathbf{\Phi}^{(2)}$. We have

$$\begin{aligned}\nabla\times\nabla\times\mathbf{\Phi}^{(2)} =&\nabla\nabla\cdot\mathbf{\Phi}^{(2)}-\Delta\mathbf{\Phi}^{(2)}\\ =&\partial_{x_3}^2\nabla\nabla\cdot\nabla\times\begin{pmatrix}\Gamma(x)\\0\\0\end{pmatrix}-\partial_{x_3}^2\Delta\nabla\times\begin{pmatrix}\Gamma(x)\\0\\0\end{pmatrix}\\ =&-\partial_{x_3}^2\nabla\times\begin{pmatrix}\Delta\Gamma(x)\\0\\0\end{pmatrix}\\ =&\partial_{x_3}^2\nabla\times\begin{pmatrix}\kappa^2\Gamma(x)\\0\\0\end{pmatrix}\\ =&\kappa^2\mathbf{\Phi}^{(2)}.\end{aligned}$$

The proof for $\mathbf{\Phi}^{(3)}$ is similar, and thus omitted. □

4.2. **Statement of the main result.** Denote

$$\epsilon:=\|\mathbf{E}\times\nu\|_{H^1(\partial B_R)}.$$

The following theorem is the main result for the inverse source problem for the Maxwell's equations.

Theorem 4.2. *Let $\mathbf{J}$ satisfy Assumptions 1–3 and the subdomains $D_j, j=1,\dots,N$ are given. If $\epsilon=0$ then $\mathbf{J}=0$. Moreover, the following estimate holds:*

$$\|\mathbf{J}\|_{L^\infty(\Omega)}\lesssim\epsilon. \tag{22}$$

4.3. **Proof of Theorem 4.2.** For a fixed $k\in\{0,1,\cdots,N-1\}$, we just denote the Cartesian coordinates $(x_1,x_2,x_3)=(x_1^{(k+1)},x_2^{(k+1)},x_3^{(k+1)})$ for brevity. After the coordinate is fixed, in the following, let

$$\mathbf{\Phi}_k^{(j)}(x,y):=\mathbf{\Phi}^{(j)}(x-y),\quad j=1,2,3.$$

Define

$$S_k^{(j)}(y)=\int_{U_k}\mathrm{i}\kappa\mathbf{J}(x)\cdot\mathbf{\Phi}_k^{(j)}(x,y)\mathrm{d}x.$$

Lemma 4.3. *For $y\in\mathcal{K}_k$, it holds that for $j=1,2,3$,*

$$(\Delta+\kappa^2)S_k^{(j)}(y)=0.$$

Taking dot product of both sides of (20) with $\mathbf{\Phi}_k^{(j)}(x,y)$ for $y \in B_{R+r_0} \setminus B_{R+\frac{r_0}{2}}$ and using integration by parts, we have

$$\begin{aligned}
&\int_\Omega \mathrm{i}\kappa \mathbf{J}(x) \cdot \mathbf{\Phi}_k^{(j)}(x,y)\mathrm{d}x \\
=& \int_{B_R} \mathrm{i}\kappa \mathbf{J}(x) \cdot \mathbf{\Phi}_k^{(j)}(x,y)\mathrm{d}x \\
=& \int_{B_R} (\nabla \times \nabla \times \mathbf{E}(x) - \kappa^2 \mathbf{E}(x)) \cdot \mathbf{\Phi}_k^{(j)}(x,y)\mathrm{d}x \\
=& \int_{B_R} \mathbf{E}(x) \cdot (\nabla_x \times \nabla_x \times \mathbf{\Phi}_k^{(j)}(x,y) - \kappa^2 \mathbf{\Phi}_k^{(j)}(x,y))\mathrm{d}x \\
&+ \int_{\partial B_R} \Big[-((\nabla \times \mathbf{E}(x)) \times \nu(x)) \cdot \mathbf{\Phi}_k^{(j)}(x,y) - (\nabla_x \times \mathbf{\Phi}_k^{(j)}(x,y)) \cdot (\mathbf{E}(x) \times \nu(x)) \Big] \mathrm{d}s(x) \\
=& \int_{\partial B_R} \Big[-((\nabla \times \mathbf{E}(x)) \times \nu(x)) \cdot \mathbf{\Phi}_k^{(j)}(x,y) - (\nabla_x \times \mathbf{\Phi}_k^{(j)}(x,y)) \cdot (\mathbf{E}(x) \times \nu(x)) \Big] \mathrm{d}s(x).
\end{aligned} \tag{23}$$

First, note that for $k = 0$,

$$S_0(y) = \int_\Omega \mathrm{i}\kappa \mathbf{J}(x) \cdot \mathbf{\Phi}_0^{(j)}(x,y)\mathrm{d}x.$$

Also notice that

$$\int_{\partial B_R} |\mathbf{\Phi}_0^{(j)}(\cdot\,,y)|^2 + |\nabla \times \mathbf{\Phi}_0^{(j)}(\cdot\,,y)|^2 \mathrm{d}s \le C$$

for $y \in B_{R+r_0} \setminus B_{R+\frac{r_0}{2}}$, where C depends on R, κ, r_0. Notice that $\mathbf{E}|_{\mathbb{R}^3 \setminus B_R}$ is the solution to the exterior problem for the Maxwell's equation

$$\nabla \times \nabla \times \mathbf{E} - \kappa^2 \mathbf{E} = 0 \quad \text{in } \mathbb{R}^3 \setminus B_R$$

along with the radiation condition (19). For the above exterior problem, it is shown in [17] that there exists a bounded operator $\mathcal{N} : H^1(\partial B_R) \to L^2(\partial B_R)$ such that

$$(\nabla \times \mathbf{E}) \times \nu = \mathcal{N}(\mathbf{E} \times \nu) \quad \text{on } \partial B_R.$$

Hence, the tangential trace of the magnetic field $\mathbf{H} \times \nu = \frac{1}{\mathrm{i}\kappa}(\nabla \times \mathbf{E}) \times \nu$ on ∂B_R can be obtained once the tangential trace of the electric field $\mathbf{E} \times \nu$ is available on ∂B_R. Therefore, we obtain the following estimate

$$\begin{aligned}
\int_{\partial B_R} (|(\nabla \times \mathbf{E}) \times \nu|^2 + |\mathbf{E} \times \nu|^2)\mathrm{d}s &= \int_{\partial B_R} (|\mathcal{N}(\mathbf{E} \times \nu)|^2 + |(\mathbf{E} \times \nu)|^2)\mathrm{d}s \\
&\le C\|\mathbf{E} \times \nu\|^2_{H^1(\partial B_R)} \le C\epsilon^2,
\end{aligned}$$

The rest of the proof is similar to the proof for the elastic wave equations in previous section.

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E-mail address: jianzhai@fudan.edu.cn
E-mail address: zhaoyueccnu@163.com