Theory for Taut String

Direction Definitions

The +y axis is defined to be the upward direction.

The +x axis is defined to be the right direction.

The +z axis is defined to be the out of the page direction.

Anticlockwise is taken to be the positive angular direction.

Variable Definitions

Suppose the impulse by the string is $\int T dt > 0$.

Let $\dot{x}_1'$ and $\dot{x}_2'$ be the initial x component of the velocity of the first plank and second plank respectively. Let $\dot{x}_1$ and $\dot{x}_2$ be the final x component of the velocity of the first plank and second plank respectively.

Let $\dot{y}_1'$ and $\dot{y}_2'$ be the initial y component of the velocity of the first plank and second plank respectively. Let $\dot{y}_1$ and $\dot{y}_2$ be the final y component of the velocity of the first plank and second plank respectively.

Let $\dot{\omega}_1'$ and $\dot{\omega}_2'$ be the initial angular velocity of the first plank and second plank respectively. Let $\dot{\omega}_1$ and $\dot{\omega}_2$ be the final angular of the first plank and second plank respectively.

Let u_1 and u_2 be the initial velocity of the first and second plank and v_1 and v_2 be the final velocity of the first and second plank.

w in this case is half of the width of the rod between the strings.

l is half the length of the rod.

θ_1 is the angle between the string vector and the horizontal unit vector.

$$\theta_1 = \cos^{-1}\left(\frac{\vec{s} \cdot (1\ 0\ 0)}{|\vec{s}|}\right)$$

θ_2 is the angle between the **negated** string vector and the horizontal unit vector. ($0 < \theta_2 < \pi$ too)

$$\theta_2 = \cos^{-1}\left(\frac{\overline{-\vec{s}} \cdot (1\ 0\ 0)}{|\vec{s}|}\right) = \pi - \theta_1$$

Actual Theory

Considering the first (lower) rod:

Actually need to consider quite a lot of cases:

If the string vector is pointing towards the left (if $\theta_1 > \frac{\pi}{2}$), then $\dot{x}_1 < \dot{x}_1'$, we have:

$$\cos(\pi - \theta_1) \int T dt = m(\dot{x}_1' - \dot{x}_1)$$

If the string vector is pointing towards the right (if $\theta_1 < \frac{\pi}{2}$), then $\dot{x}_1 > \dot{x}_1'$ and we have:

$$\cos(\theta_1) \int T dt = m(\dot{x}_1 - \dot{x}_1')$$

The above 2 conditions can be combined into 1 single equation:

$$\cos(\theta_1) \int T dt = m(\dot{x}_1 - \dot{x}_1')$$

Both impulses on left and right string serve to pull rod upwards so:

$$\sin(\theta_1) \int T dt = m(\dot{y}_1 - \dot{y}_1')$$

If the impulse is on the left string, then angular velocity will decrease (go clockwise so will become more negative). So for left string, we have:

$$w \sin(\theta_1 - \phi_1) \int T dt = I(\omega_1' - \omega_1)$$

For right string, then angular velocity will increase, so we have:

$$w \sin(\theta_1 - \phi_1) \int T dt = I(\omega_1 - \omega_1')$$

Considering the second (upper) rod:

If the string vector (not the negated one) is pointing towards the left, then the second rod is being pulled to the right, then $\dot{x}_2 > \dot{x}_2'$, we have:

$$\cos(\pi - \theta_1) \int T dt = m(\dot{x}_2 - \dot{x}_2') \quad \Longrightarrow \quad \cos(\theta_1) \int T dt = m(\dot{x}_2' - \dot{x}_2)$$

If the string vector is pointing towards the right, then rod is going to the left, so $\dot{x}_2 < \dot{x}_2'$, and we have:

$$\cos(\theta_1) \int T dt = m(\dot{x}_2' - \dot{x}_2)$$

We can combine these into 1 equation:

$$\cos(\theta_1)\int T dt = m(\dot{x}_2 - \dot{x}_2')$$

Both impulses on the left and right string serve to pull rod downwards so:

$$\sin(\theta_2)\int T dt = \sin(\theta_1)\int T dt = m(\dot{y}_2' - \dot{y}_2)$$

Checking with conservation of momentum, we have:

$$\left(\int T dt\right)^2 = m^2[(\Delta\dot{x}_2)^2 + (\Delta\dot{y}_2)^2] = m^2(\Delta v_2)^2$$

Similarly we have:

$$\left(\int T dt\right)^2 = m^2[(\Delta\dot{x}_1)^2 + (\Delta\dot{y}_1)^2] = m^2(\Delta v_1)^2$$

Since v_1 and v_2 are in opposite directions, $m(\Delta v_1) = -m(\Delta v_2)$ as by conservation of momentum. Hence our equations agree with law of conservation of momentum.

If the impulse is on the left string, then angular velocity will increase (go anticlockwise so will become more positive). So for left string, we have:

$$w \sin\,(\theta_2 + \phi_2)\int T dt = I(\omega_2 - \omega_2')$$

$$w \sin\,(\pi - \theta_1 + \phi_2)\int T dt = I(\omega_2 - \omega_2')$$

$$w \sin(\theta_1 - \phi_2)\int T dt = I(\omega_2 - \omega_2')$$

For right string, then angular velocity will decrease, so we have:

$$w \sin\,(\theta_1 - \phi_2)\int T dt = I(\omega_2' - \omega_2)$$

We cannot trivially check conservation of angular momentum, because angular momentum is conserved about a particular axis, but we have 2 rods here.

<u>Conservation of energy:</u>

$$\frac{1}{2}m(u_1^2 + u_2^2) + \frac{1}{2}I\left({\omega_1'}^2 + {\omega_2'}^2\right) = \frac{1}{2}m(v_1^2 + v_2^2) + \frac{1}{2}I(\omega_1^2 + \omega_2^2)$$

Final Conservation of Energy (u, v decomposed)

$$\frac{1}{2}m\left({\dot{x}_1'}^2 + {\dot{y}_1'}^2 + {\dot{x}_2'}^2 + {\dot{y}_2'}^2\right) + \frac{1}{2}I\left({\omega_1'}^2 + {\omega_2'}^2\right)$$
$$= \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}I(\omega_1^2 + \omega_2^2)$$

<u>Solving the equations:</u>

We have 7 equations:

$$\cos(\theta_1)\int Tdt = m(\dot{x}_1 - \dot{x}_1')$$

$$\sin(\theta_1)\int Tdt = m(\dot{y}_1 - \dot{y}_1')$$

$$w\sin(\theta_1 - \phi_1)\int Tdt = I(\omega_1' - \omega_1) \quad \text{or} \quad w\sin(\theta_1 - \phi_1)\int Tdt = I(\omega_1 - \omega_1')$$

$$\cos(\theta_2)\int Tdt = m(\dot{x}_2 - \dot{x}_2')$$

$$\sin(\theta_2)\int Tdt = m(\dot{y}_2' - \dot{y}_2)$$

$$w\sin(\theta_2 + \phi_2)\int Tdt = I(\omega_2 - \omega_2') \quad \text{or} \quad w\sin(\theta_2 + \phi_2)\int Tdt = I(\omega_2' - \omega_2)$$

$$\frac{1}{2}m\left({\dot{x}_1'}^2 + {\dot{y}_1'}^2 + {\dot{x}_2'}^2 + {\dot{y}_2'}^2\right) + \frac{1}{2}I\left({\omega_1'}^2 + {\omega_2'}^2\right)$$
$$= \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}I(\omega_1^2 + \omega_2^2)$$

and 7 unknown variables:

$\int Tdt, \dot{x}_1, \dot{x}_2, \dot{y}_1, \dot{y}_2, \omega_1, \omega_2$

Clearly, can solve.