

Prep: pen, watch, laptop with full battery, MATH 105 sheet, lecture notes, homework

Strategy: Intuition → Identify problem type → Apply treatments → Patience and resolve → Win

Lebesgue Measure

Axioms and Properties
<ul style="list-style-type: none"> • Axioms of Measurable Sets <ul style="list-style-type: none"> ◦ [Borel Property] Every open set and closed set in \mathbb{R}^n measurable. ◦ [Complementarity] If Ω measurable, then $\mathbb{R}^n \setminus \Omega$ measurable. ◦ [Boolean Algebra Property] If $(\Omega_j)_{j \in J}$ finite collection of measurable sets (i.e. J finite), then $\bigcup_{j \in J} \Omega_j$ and $\bigcap_{j \in J} \Omega_j$ measurable. ◦ [σ-Algebra Property] If $(\Omega_j)_{j \in J}$ countable collection of measurable sets (i.e. J countable), then $\bigcup_{j \in J} \Omega_j$ and $\bigcap_{j \in J} \Omega_j$ measurable. ◦ [Empty Set] $m(\phi) = 0$ ◦ [Positivity] $0 \leq m(\Omega) \leq +\infty$ for every measurable set Ω ◦ [Monotonicity] If $A \subset B$ and A, B both measurable, $m(A) \leq m(B)$ ◦ [Finite Sub-additivity] If $(A_j)_{j \in J}$ finite collection of measurable sets, then $m(\bigcup_{j \in J} A_j) \leq \sum_{j \in J} m(A_j)$ ◦ [Finite Additivity] If $(A_j)_{j \in J}$ finite collection of disjoint measurable sets i.e. $A_j \cap A_k = \phi$ for $j \neq k$, then $m(\bigcup_{j \in J} A_j) = \sum_{j \in J} m(A_j)$ ◦ [Countable Sub-additivity] If $(A_j)_{j \in J}$ countable collection of measurable sets, then $m(\bigcup_{j \in J} A_j) \leq \sum_{j \in J} m(A_j)$ ◦ [Countable Additivity] If $(A_j)_{j \in J}$ countable collection of disjoint measurable sets i.e. $A_j \cap A_k = \phi$ for $j \neq k$, then $m(\bigcup_{j \in J} A_j) = \sum_{j \in J} m(A_j)$ ◦ [Normalization] $m([0,1]^n) = 1$ (i.e. the unit cube has measure 1) ◦ [Translational Invariance] If Ω measurable and $x \in \mathbb{R}^n$, then $x + \Omega$ also measurable with $m(x + \Omega) = m(\Omega)$ • Properties of Outer Measure <ul style="list-style-type: none"> ◦ [Empty Set] $m^*(\phi) = 0$ ◦ [Positivity] $0 \leq m^*(\Omega) \leq +\infty$ for every measurable set Ω ◦ [Monotonicity] If $A \subset B$ and A, B both measurable, $m^*(A) \leq m^*(B)$ ◦ [Finite Sub-additivity] If $(A_j)_{j \in J}$ finite collection of measurable sets, then $m^*(\bigcup_{j \in J} A_j) \leq \sum_{j \in J} m^*(A_j)$ ◦ [Countable Sub-additivity] If $(A_j)_{j \in J}$ countable collection of measurable sets, then $m^*(\bigcup_{j \in J} A_j) \leq \sum_{j \in J} m^*(A_j)$ ◦ [Translational Invariance] If Ω measurable and $x \in \mathbb{R}^n$, then $x + \Omega$ also measurable with $m^*(x + \Omega) = m^*(\Omega)$ • Properties of Measurable Sets <ul style="list-style-type: none"> ◦ Every open box and every closed box is measurable. ◦ Any set E with $m^*(E) = 0$ is measurable with $m(E) = 0$
Definitions
<ul style="list-style-type: none"> • [Open Box] An open box in \mathbb{R}^n is a set of the form $B = \prod_{i=1}^n (a_i, b_i)$, $b_i \geq a_i$. Define the volume of the box $\text{vol}(B) = \prod_{i=1}^n (b_i - a_i)$ • [Covering by Boxes] Let $\Omega \subset \mathbb{R}^n$. Say collection of boxes $(B_j)_{j \in J}$ covers Ω if and only if $\Omega \subset \bigcup_{j \in J} B_j$.

- [Outer Measure] Let $\Omega \subset \mathbb{R}^n$. Define outer measure $m^*(\Omega) := \inf \left\{ \sum_{j \in J} \text{vol}(B_j) : (B_j)_{j \in J} \text{ covers } \Omega, J \text{ at most countable} \right\}$
- [Lebesgue Measurability] Let $E \subset \mathbb{R}^n$. Say E is Lebesgue measurable if and only if the following identity holds: $m^*(A) = m^*(A \cap E) + m^*(A \setminus E) \forall A \subset \mathbb{R}^n$ i.e. for every test set A , it separates E cleanly based on the outer measure.
 - If E measurable, define the Lebesgue measure of E as $m(E) = m^*(E)$
- [Measurable Functions] Let $\Omega \subset \mathbb{R}^n$ measurable and $f: \Omega \rightarrow \mathbb{R}^m$. Say f is measurable if and only if $f^{-1}(V)$ is measurable for every open $V \subset \mathbb{R}^m$.
- $[\mathbb{R}^*] \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$
- [Measurable Functions in \mathbb{R}^*] Let $\Omega \subset \mathbb{R}^n$. $f: \Omega \rightarrow \mathbb{R}^*$ measurable if and only if $f^{-1}((a, +\infty])$ measurable $\forall a \in \mathbb{R}$.

Alternate Definitions

- [Outer Measure] $m^*(A) = \inf \{m^*(U) | U \supset A, U \text{ open}\}$
- [Measurable Set] Say $E \subset \mathbb{R}^n$ measurable if $\forall \epsilon > 0, \exists$ open set $U \supset E$ s.t. $m^*(U \setminus E) < \epsilon$

General Lemmas and Theorems

- [Outer Measure of Closed Box] For any closed box $B = \prod_{i=1}^n [a_i, b_i]$, $m^*(B) = \prod_{i=1}^n (b_i - a_i)$
- [Outer Measure of Open Box] For any open box $B = \prod_{i=1}^n (a_i, b_i)$, $m^*(B) = \prod_{i=1}^n (b_i - a_i)$
- [Failure of Outer Measure] Outer measure does not satisfy countable additivity i.e. $\exists (A_j)_{j \in J}$ countable collection of disjoint subsets of \mathbb{R} s.t. $m^*(\bigcup_{j \in J} A_j) \neq \sum_{j \in J} m^*(A_j)$
- [Half-spaces are Measurable] $\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$ is measurable.
- [Finite Additivity] If $(E_j)_{j \in J}$ finite collection of disjoint measurable sets and any set A (not necessarily measurable), then $m^*(A \cap \bigcup_{j \in J} E_j) = \sum_{j \in J} m^*(A \cap E_j)$. Furthermore, $m^*(\bigcup_{j \in J} E_j) = \sum_{j \in J} m^*(E_j)$.
- $A \subset B$ are two measurable sets, then $B \setminus A$ also measurable and $m(B \setminus A) = m(B) - m(A)$
- Every open set can be written as a countable or finite union of open boxes.
- [Continuous Functions are Measurable] Let $\Omega \subset \mathbb{R}^n$ measurable and $f: \Omega \rightarrow \mathbb{R}^m$ continuous. Then f is automatically measurable.
- Let $\Omega \subset \mathbb{R}^n$ measurable and $f: \Omega \rightarrow \mathbb{R}^m$. Then f is measurable if and only if $f^{-1}(B)$ measurable for every open box B .
- Let $\Omega \subset \mathbb{R}^n$ measurable and $f: \Omega \rightarrow \mathbb{R}^m$. Suppose that $f = (f_1, \dots, f_m)$ where $f_i: \Omega \rightarrow \mathbb{R}$ is the i th coordinate of f . Then f measurable if and only if all of f_i are individually measurable.
- Let $\Omega \subset \mathbb{R}^n$ measurable and $W \subset \mathbb{R}^m$ open. If $f: \Omega \rightarrow W$ measurable and $g: W \rightarrow \mathbb{R}^p$ continuous, then $g \circ f: \Omega \rightarrow \mathbb{R}^p$ measurable.
- Let $\Omega \subset \mathbb{R}^n$ measurable. If $f: \Omega \rightarrow \mathbb{R}$ measurable, then so is $|f|, \max(f, 0), \min(f, 0)$
- Let $\Omega \subset \mathbb{R}^n$ measurable. If $f, g: \Omega \rightarrow \mathbb{R}$ measurable, then so is $f + g, f - g, fg, \max(f, g), \min(f, g)$. If $g(x) \neq 0 \forall x \in \Omega$, f/g measurable.
- Let $\Omega \subset \mathbb{R}^n$ measurable and $f: \Omega \rightarrow \mathbb{R}$. Then f is measurable if and only if $f^{-1}((a, \infty))$ measurable $\forall a \in \mathbb{R}$.
- Let $\Omega \subset \mathbb{R}^n$ measurable and $\forall n, f_n: \Omega \rightarrow \mathbb{R}^*$ measurable. Then $\sup_{n \geq 1} f_n, \inf_{n \geq 1} f_n, \limsup_{n \rightarrow \infty} f_n, \liminf_{n \rightarrow \infty} f_n$ all measurable. In particular, if $(f_n)_n \rightarrow f$ pointwise, then f measurable.

Major Theorems

- [Continuity of Measure] Let $A_1 \subseteq A_2 \subseteq \dots$ be an increasing sequence of measurable sets (i.e. $A_j \subseteq A_{j+1}$), then $m(\bigcup_{j=1}^{\infty} A_j) = \lim_{j \rightarrow \infty} m(A_j)$
- [Continuity of Measure] Let $A_1 \supseteq A_2 \supseteq \dots$ be a decreasing sequence of measurable sets (i.e. $A_j \supseteq A_{j+1}$) and $m(A_1) < +\infty$, then $m(\bigcap_{j=1}^{\infty} A_j) = \lim_{j \rightarrow \infty} m(A_j)$

Regularity and Slice

- $[G_\delta \text{ Set}]$ A G_δ set is a countable intersection of open sets.

- [F_σ Set] A F_σ set is a countable union of closed sets.
- [Regularity] Every measurable set E can be sandwiched between a F_σ set F and G_δ set G with $F \subset E \subset G$ s.t. $m^*(G \setminus F) = 0$ (i.e. $G \setminus F$ is a zero set). Conversely, if $\exists G \in G_\delta, F \in F_\sigma$ s.t. $F \subset E \subset G$ and $m^*(G \setminus F) = 0$, then E measurable.
 - All measurable sets are F_σ sets and G_δ sets modulo a zero set.

Vitali Covering

- [Vitali Covering] A covering \mathcal{V} of A is a Vitali covering if $\forall p \in A$ and $r > 0$, $\exists V \in \mathcal{V}$ s.t. $p \in V \subset B_r(p)$ and V is not merely $\{p\}$.
- [Vitali Covering Lemma] A Vitali covering of a bounded set $A \subset \mathbb{R}^n$ by closed balls reduces to an efficient disjoint sub-covering of almost all of A satisfying the following. Given $\epsilon > 0$, \mathcal{V} can be reduced to countable sub-covering $\{V_k\}$:
 - V_k disjoint i.e. $V_i \cap V_j = \emptyset$ for $i \neq j$
 - $m(\cup_k V_k) \leq m^*(A) + \epsilon$
 - $m(\cup_k V_k) = 0$
- [Vitali Covering Lemma for Cubes] A Vitali covering of $A \subset \mathbb{R}^n$ by closed cubes also reduces to an efficient disjoint sub-covering of almost all of A .

Lebesgue Density Theorem

- [Density] Let $E \subset \mathbb{R}^n$ measurable. Define the density of E at $p \in \mathbb{R}^n$ as $\delta(p, E) = \lim_{Q \downarrow p} \frac{m(E \cap Q)}{m(Q)}$ if the limit exists.
- [Lebesgue Density Theorem] If E measurable, then almost every $p \in E$ is a density point of E .
 - If E measurable, then for almost every p , we have $\chi_E(p) = \lim_{Q \downarrow p} \frac{m(E \cap Q)}{m(Q)}$.
 - $\text{dp}(\text{dp}(E)) = \text{dp}(E)$
- [Average] Define the average of a locally integrable function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ over measurable set $A \subset \mathbb{R}^n$ with finite positive measure as: $\oint_A f = \frac{1}{m(A)} \int_A f$
- [Average Value Theorem] If $f: \mathbb{R}^n \rightarrow \mathbb{R}$ locally integrable, then for almost every $p \in \mathbb{R}^n$, the density of f at p exists and $f(p) = \delta(p, f) = \lim_{Q \downarrow p} \oint_Q f$
- If $g: \mathbb{R}^n \rightarrow [0, \infty)$ is integrable, then for every small $\alpha > 0$, the set $X(\alpha, g) = \{p: \bar{\delta}(p, g) > \alpha\}$ has outer measure $m^*(X(\alpha, g)) \leq \frac{1}{\alpha} \int g$

Lebesgue Integration

Definition

- [Simple Function] Let $\Omega \subset \mathbb{R}^n$ measurable and $f: \Omega \rightarrow \mathbb{R}$ measurable. Say f is simple if $f(\Omega)$ is finite i.e. exists $\{c_1, \dots, c_N\}$ s.t. $f(x) = c_j$ for some $1 \leq j \leq N$.
- [Indicator Function] $\chi_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$
- [Lebesgue Integral for Simple Functions] $\int_{\Omega} f := \sum_{\lambda \in f(\Omega); \lambda > 0} \lambda \cdot m(\{x \in \Omega: f(x) = \lambda\})$.
- [Majorize] Say f majorizes g if $f(x) \geq g(x)$ for all $x \in \Omega$
- [Minorize] Say f minorizes g if $f(x) \leq g(x)$ for all $x \in \Omega$
- [Lebesgue Integral for Nonnegative Functions]

$$\int_{\Omega} f := \sup \left\{ \int_{\Omega} s : s \text{ simple, nonnegative and minorizes } f \right\}$$

- [Absolutely Integrable Function] Let $\Omega \subset \mathbb{R}^n$ measurable and $f: \Omega \rightarrow \mathbb{R}^*$. Say f is absolutely integrable if $\int_{\Omega} |f|$ finite.
- [Lebesgue Integral] Define $f^+ := \max(f, 0)$ and $f^- := -\min(f, 0)$. Let $f: \Omega \rightarrow \mathbb{R}^*$ be an absolutely integrable function, then $\int_{\Omega} f := \int_{\Omega} f^+ - \int_{\Omega} f^-$
- [Upper and Lower Lebesgue Integral] Let $\Omega \subset \mathbb{R}^n$ measurable and $f: \Omega \rightarrow \mathbb{R}$ not necessarily measurable. Define the upper and lower Lebesgue integrals

$$\begin{aligned} \overline{\int}_{\Omega} f &:= \inf \left\{ \int_{\Omega} g : g: \Omega \rightarrow \mathbb{R} \text{ absolutely integrable function, } g \geq f \right\} \\ \underline{\int}_{\Omega} f &:= \sup \left\{ \int_{\Omega} g : g: \Omega \rightarrow \mathbb{R} \text{ absolutely integrable function, } g \leq f \right\} \end{aligned}$$

Properties

- [Lebesgue Integral for Simple Functions] Let $f, g: \Omega \rightarrow \mathbb{R}$ be nonnegative simple functions.
 - $0 \leq \int_{\Omega} f \leq \infty$ with $\int_{\Omega} f = 0$ if and only if $m(\{x \in \Omega: f(x) \neq 0\}) = 0$ i.e. $f = 0$ a.e.
 - $\int_{\Omega} f + g = \int_{\Omega} f + \int_{\Omega} g$
 - Let $c > 0$, then $\int_{\Omega} cf = c \int_{\Omega} f$
 - If $f(x) \leq g(x)$ for all $x \in \Omega$, then $\int_{\Omega} f \leq \int_{\Omega} g$
- [Lebesgue Integral for Nonnegative Functions] Let $f, g: \Omega \rightarrow [0, \infty]$ be nonnegative measurable functions.
 - $0 \leq \int_{\Omega} f \leq \infty$ with $\int_{\Omega} f = 0$ if and only if $f(x) = 0$ a.e.
 - Let $c > 0$, then $\int_{\Omega} cf = c \int_{\Omega} f$
 - $f(x) \leq g(x)$ for all $x \in \Omega$, then $\int_{\Omega} f \leq \int_{\Omega} g$
 - $f(x) = g(x)$ a.e. then $\int_{\Omega} f = \int_{\Omega} g$
 - Let $\Omega' \subset \Omega$ measurable, then $\int_{\Omega'} f = \int_{\Omega} f \chi_{\Omega'} \leq \int_{\Omega} f$
- [Lebesgue Integral] Let Ω measurable, $f, g: \Omega \rightarrow \mathbb{R}$ absolutely integrable.
 - [Triangle Inequality] $|\int_{\Omega} f| \leq \int_{\Omega} |f|$
 - $\forall c \in \mathbb{R}$, cf absolutely integrable and $\int_{\Omega} cf = c \int_{\Omega} f$
 - $f + g$ absolutely integrable with $\int_{\Omega} f + g = \int_{\Omega} f + \int_{\Omega} g$
 - If $f(x) \leq g(x) \forall x \in \Omega$, then $\int_{\Omega} f \leq \int_{\Omega} g$
 - If $f(x) = g(x)$ a.e., then $\int_{\Omega} f = \int_{\Omega} g$

Theorems and Lemmas

- [Vector Field] $f, g: \Omega \rightarrow \mathbb{R}$ simple. Then $f + g, cf$ are also simple where $c \in \mathbb{R}$
- [Decomposition into Simple Functions] Let $f: \Omega \rightarrow \mathbb{R}$ simple. Then $\exists \{c_1, \dots, c_N\}$ and a finite number of disjoint measurable sets $E_1, \dots, E_N \subset \Omega$ s.t. $f = \sum_{i=1}^N c_i \chi_{E_i}$

- [Pointwise Approximation by Simple Functions] Let $f: \Omega \rightarrow \mathbb{R}$ measurable, nonnegative. Then $\exists (f_n)_n$ of simple functions s.t. f_n nonnegative and increasing $0 \leq f_1(x) \leq f_2(x) \leq \dots$ $\forall x \in \Omega$ and converges pointwise to f i.e. $\lim_{n \rightarrow \infty} f_n(x) = f(x)$.
- [Lebesgue Integral of Simple Functions II] Let E_1, \dots, E_N be a finite number of disjoint subsets in Ω and c_1, \dots, c_N be nonnegative numbers. Then $\int_{\Omega} f = \int_{\Omega} \sum_{j=1}^N c_j \chi_{E_j} = \sum_{j=1}^N c_j m(E_j)$
- [Interchange of Addition and Integration] Let $f, g: \Omega \rightarrow [0, \infty]$ be measurable, then $\int_{\Omega} f + g = \int_{\Omega} f + \int_{\Omega} g$
- [Interchange of Integration and Series] Let g_1, g_2, \dots be a sequence of nonnegative measurable functions $\int_{\Omega} \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int_{\Omega} g_n$
- [Finite Integral] Let $\Omega \subset \mathbb{R}^n$ measurable and $f: \Omega \rightarrow [0, \infty]$ be nonnegative measurable s.t. $\int_{\Omega} f$ finite. Then f finite a.e. (i.e. $m\{x \in \Omega: f(x) = +\infty\} = 0$)
- $\int_{\underline{\Omega}} f \leq \int_{\Omega} f$
- If f absolutely integrable, then $\int_{\overline{\Omega}} f = \int_{\underline{\Omega}} f$.
- Let $\Omega \subset \mathbb{R}^n$ measurable and $f: \Omega \rightarrow \mathbb{R}$ not necessarily measurable. Suppose $\int_{\underline{\Omega}} f = \int_{\overline{\Omega}} f \in \mathbb{R}$, then f is absolutely integrable and $\int_{\Omega} f = \int_{\overline{\Omega}} f = \int_{\underline{\Omega}} f$
- Let $I \subset \mathbb{R}$ be an interval and $f: I \rightarrow \mathbb{R}$ be Riemann integrable. Then f also absolutely integrable with $\int_I f = R. \int_I f$

Major Theorems

- [Lebesgue Monotone Convergence Theorem] Let $\Omega \subset \mathbb{R}^n$ measurable and let $(f_n)_{n=1}^{\infty}$ be nonnegative measurable and increasing $0 \leq f_1(x) \leq f_2(x) \leq \dots$ for all $x \in \Omega$. Then $0 \leq \int_{\Omega} f_1 \leq \int_{\Omega} f_2 \leq \dots$ and $\int_{\Omega} \sup_n f_n = \sup_n \int_{\Omega} f_n$.
- [Fatou's Lemma] Let $\Omega \subset \mathbb{R}^n$ measurable and f_1, f_2, \dots be a sequence of nonnegative functions from Ω to $[0, \infty]$. Then $\int_{\Omega} \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f_n$.
 - Intuitively, integrating over the worst of every function at each pointwise position will definitely be no better than taking the worst of the integral of a function
- [Borel-Cantelli Lemma] Let $\Omega_1, \Omega_2, \dots$ be measurable subsets of \mathbb{R}^n s.t. $\sum_{n=1}^{\infty} m(\Omega_n) < \infty$. Then $m(\{x \in \mathbb{R}^n: x \in \Omega_n \text{ i.o.}\}) = 0$, i.e. almost every point $x \in \mathbb{R}^n$ belongs to finitely many Ω_n .
- [Lebesgue Dominated Convergence Theorem] Let $\Omega \subset \mathbb{R}^n$ measurable and let $(f_n)_{n=1}^{\infty}$ measurable with $f_i: \Omega \rightarrow \mathbb{R}^*$ and $\lim_{n \rightarrow \infty} f_n = f$ pointwise. Suppose \exists an absolutely integrable function $F: \Omega \rightarrow [0, \infty]$ s.t. $|f_n(x)| \leq F(x) \forall x \in \Omega$, then $\int_{\Omega} \lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \int_{\Omega} f_n$
- [Fubini's Theorem] Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ absolutely integrable. Then \exists absolutely integrable functions $F, G: \mathbb{R} \rightarrow \mathbb{R}$ s.t. for almost every x , $f(x, y)$ absolutely integrable in y with $F(x) = \int_{\mathbb{R}} f(x, y) dy$ and for almost every y , $g(x, y)$ absolutely integrable in x with $G(y) = \int_{\mathbb{R}} f(x, y) dx$. Finally, $\int_{\mathbb{R}} F(x) dx = \int_{\mathbb{R}^2} f = \int_{\mathbb{R}} G(y) dy$

Extra Concepts

Slice Theorem

Lebesgue Density Theorem

Counter Examples

- [Interchange of Limit and Integrals] $\int_{\Omega} \lim_{n \rightarrow \infty} f_n \neq \lim_{n \rightarrow \infty} \int_{\Omega} f_n$ Consider the moving bump example.

Multivariable Calculus

Definitions

- [Norm] A norm on vector space V , $|\cdot|: V \rightarrow \mathbb{R}$, satisfies three properties
 - [Positive Definite] $\forall v \in V, |v| \geq 0$ with $|v| = 0$ if and only if $v = 0$
 - [Positive Homogeneity of Degree 1] $|\lambda v| = |\lambda||v|$
 - [Triangle Inequality] $|v + w| \leq |v| + |w|$
- [Normed Space] A vector space with a norm. It is a metric space with $d(v_1, v_2) = |v_1 - v_2|$
- [Operator Norm] Let V, W be normed spaces and $T: V \rightarrow W$ linear transformation. Define the operator norm of T as $\|T\| = \sup \left\{ \frac{|Tv|_W}{|v|_V} : v \neq 0 \right\}$ (i.e. maximum stretch of T)
- [Derivative] Let $f: U \rightarrow \mathbb{R}^m$ where $U \subset \mathbb{R}^n$ open. Say f is differentiable at $p \in U$ with derivative $(Df)_p = T$ if $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear transformation and $f(p + v) = f(p) + T(v) + R(v)$ where $\lim_{|v| \rightarrow 0} \frac{|R(v)|}{|v|} = 0$. Note: $(Df)_p$ if it exists is a linear transformation from $U \rightarrow \mathbb{R}^m$.
- [Partial Derivative] $\frac{\partial f_i(p)}{\partial x_j} = \lim_{t \rightarrow 0} \frac{f_i(p + te_j) - f_i(p)}{t}$ if it exists.
- [Lipschitz Condition] Say $f(t, y)$ satisfies a Lipschitz condition in y on set $D \subset \mathbb{R}^2$ if $\exists L > 0$ with $|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$ whenever $(t, y_1), (t, y_2) \in D$. Define L as the Lipschitz constant.

Operator Analysis

- $\|T \circ S\| \leq \|T\| \|S\|$
- Let V, W normed spaces and $T: V \rightarrow W$ linear transformation. The following are equivalent:
 - $\|T\| < \infty$
 - T is uniformly continuous
 - T is continuous
 - T is continuous at the origin
- Every linear transformation $T: \mathbb{R}^n \rightarrow W$ is continuous.
- Let $T: \mathbb{R}^n \rightarrow W$ linear transformation. If T is an isomorphism, then T^{-1} is continuous. (i.e. T is a homeomorphism; bi-continuous bijection)
- In finite dimensional normed spaces, all linear transformations are continuous and all isomorphisms are homeomorphism.

Theorems and Lemmas

- If f differentiable at p , then $(Df)_p(u) = \lim_{t \rightarrow 0} \frac{f(p+tu) - f(p)}{t}$.
- If f is differentiable at p , then it is automatically continuous at p .
- If Df exists, then partial derivatives exist and in matrix form $[(Df)_p]_{ij} = \frac{\partial f_i(p)}{\partial x_j}$.
- If partial derivatives of $f: U \rightarrow \mathbb{R}^m$ exist and continuous, then f differentiable.
- Let f, g differentiable. Then:
 - $D(f + cg) = Df + cDg$
 - $D(\text{constant}) = 0$
 - [Chain Rule] $D(g \circ f) = Dg \circ Df$
 - [Leibniz Rule] $D(fg) = (Df)g + f(Dg)$
- $f: U \rightarrow \mathbb{R}^m$ is differentiable at $p \in U$ if and only if each of its components f_i is differentiable at p . The derivative of its i th component is the i th component of the derivative.
- [Mean Value Theorem] If $f: U \rightarrow \mathbb{R}^m$ differentiable on U and $[p, q] \subset U$, then $|f(q) - f(p)| \leq M|q - p|$ where $M = \sup_{x \in U} \|(Df)_x\|$
- [C^1 Mean Value Theorem] If $f: U \rightarrow \mathbb{R}^m$ is of class C^1 (i.e. derivative exists and continuous) and $[p, q] \subset U$, then $f(q) - f(p) = T(q - p)$ where $T = \int_0^1 (Df)_{p+t(q-p)} dt$ is the average derivative of f on $[p, q]$

- If U connected and $f: U \rightarrow \mathbb{R}^m$ differentiable and for each point $x \in U$, $(Df)_x = 0$, then f is constant.
- [Differentiation Past Integral] If $f: [a, b] \times (c, d) \rightarrow \mathbb{R}$ continuous and $\frac{\partial f(x, y)}{\partial y}$ exists and continuous, then $F(y) = \int_a^b f(x, y) dx$ is of class C^1 and $\frac{dF}{dy} = \int_a^b \frac{\partial f(x, y)}{\partial y} dx$
- [Schwarz Theorem] If $f \in C^2$ (i.e. all second partial derivatives exist and continuous), then second order partial derivatives commute $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$.

Inequalities

- Hölder's inequality: For $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $(\sum_{i=1}^n |x_i y_i|) \leq (\sum_i |x_i|^p)^{\frac{1}{p}} (\sum_i |y_i|^q)^{\frac{1}{q}}$
 - $\|fg\|_1 \leq \|f\|_p \|g\|_q$
 - Let $\Omega \subset \mathbb{R}^n$ Lebesgue measurable and f, g measurable, then $\int_{\Omega} |f(x)g(x)| dx \leq (\int_{\Omega} |f(x)|^p dx)^{\frac{1}{p}} (\int_{\Omega} |g(x)|^q dx)^{\frac{1}{q}}$
- Minkowski's Inequality: For $p \geq 1$, $(\sum_{i=1}^n |x_i + y_i|^p)^{\frac{1}{p}} \leq (\sum_i |x_i|^p)^{\frac{1}{p}} + (\sum_i |y_i|^p)^{\frac{1}{p}}$

Contraction Mapping Theorem

- [Contraction] Let (X, d) metric space and $f: X \rightarrow X$. Say f is a contraction if $d(f(x), f(y)) \leq d(x, y) \forall x, y \in X$.
- [Strict Contraction] Say f is a strict contraction if $\exists c$ with $0 < c < 1$ s.t. $d(f(x), f(y)) \leq c \cdot d(x, y) \forall x, y \in X$.
- [Contraction Constant] Say c is the contraction constant of f .
- [Fixed Points] Let $f: X \rightarrow X$ and $x \in X$. Say x is a fixed point of f if $f(x) = x$.
- [Contraction Mapping Theorem] Let (X, d) metric space and $f: X \rightarrow X$ strict contraction. Then f has at most 1 fixed point. If X nonempty and complete, then f has exactly one fixed point $x = \lim_{n \rightarrow \infty} \underbrace{f(f(\dots f(x)))}_{n \text{ times}}$.
- [Hairy Ball Theorem] Any continuous map $f: S^2 \rightarrow S^2$ (i.e. from the sphere $\{(x, y, z) \in \mathbb{R}^3: x^2 + y^2 + z^2 = 1\}$ to itself) must contain either a fixed point or an anti-fixed point (i.e. a point x in S^2 s.t. $f(x) = -x$)
- Let $B_r(0) \subset \mathbb{R}^n$ and $g: B_r(0) \rightarrow \mathbb{R}^n$ with $g(0) = 0$ and $\|g(x) - g(y)\| \leq \frac{1}{2} \|x - y\| \forall x, y \in B_r(0)$, then $f: B_r(0) \rightarrow \mathbb{R}^n$ defined by $f(x) = g(x) + x$ is injective and the image $f(B_r(0))$ contains $B_{\frac{r}{2}}(0)$

Inverse Function Theorem

- [1D Inverse Function Theorem] If $f: \mathbb{R} \rightarrow \mathbb{R}$ invertible, differentiable and $f'(x_0)$ nonzero, then f^{-1} differentiable at $f(x_0)$ with $(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}$
- Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ invertible linear transformation. Then the inverse transformation $T^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is also linear.
- [Inverse Function Theorem] Let $E \subset \mathbb{R}^n$ open and $f: E \rightarrow \mathbb{R}^n$ continuously differentiable on E . Suppose $x_0 \in E$ is such that the linear transformation $f'(x_0): \mathbb{R}^n \rightarrow \mathbb{R}^n$ invertible. Then \exists open $U \subset E$ containing x_0 and open $V \subset \mathbb{R}^n$ containing $f(x_0)$ s.t. f bijection from U to V . In particular, there is an inverse map $f^{-1}: V \rightarrow U$ that is differentiable at $f(x_0)$ and $(f^{-1})'(f(x_0)) = (f'(x_0))^{-1}$
- If $f'(x_0)$ not invertible, then impossible that inverse f^{-1} can exist and be differentiable at x_0
- If $f'(x_0)$ exists but non-invertible, then inverse function theorem does not apply.
 - Not possible for f^{-1} to exist and be differentiable at x_0
 - Still possible for f to be invertible however (consider $f(x) = x^3$ despite $f'(0)$ not invertible)

- Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ continuous differentiable function s.t. $f'(x)$ invertible linear transformation $\forall x \in \mathbb{R}^n$. Then whenever $V \subset \mathbb{R}^n$ open, $f(V)$ open.

Implicit Function Theorem

- [Implicit Function Theorem] Let $E \subset \mathbb{R}^n$ open and $f: E \rightarrow \mathbb{R}$ continuously differentiable. Let $y = (y_1, \dots, y_n) \in E$ s.t. $f(y) = 0$ and $\frac{\partial f}{\partial x_n}(y) \neq 0$. Then $\exists U \subset \mathbb{R}^{n-1}$ open containing (y_1, \dots, y_{n-1}) , $V \subset E$ containing y and $g: U \rightarrow \mathbb{R}$ s.t. $g(y_1, \dots, y_{n-1}) = y_n$ and $\{(x_1, \dots, x_n) \in V: f(x_1, \dots, x_n) = 0\} = \{(x_1, \dots, x_{n-1}, g(x_1, \dots, x_{n-1})) : (x_1, \dots, x_{n-1}) \in U\}$ i.e. the set $\{x \in V: f(x) = 0\}$ is a graph of a function over U . Moreover, g differentiable at (y_1, \dots, y_{n-1}) with
$$\frac{\partial g}{\partial x_j}(y_1, \dots, y_{n-1}) = -\frac{\frac{\partial f}{\partial x_j}(y)}{\frac{\partial f}{\partial x_n}(y)} \text{ for } 1 \leq j \leq n-1.$$
- Given $f(x_1, \dots, x_n) = 0$, $\frac{\partial f}{\partial x_j} + \frac{\partial f}{\partial x_n} \cdot \frac{\partial f}{\partial x_j} = 0$

Differential Forms

- [k -cell in \mathbb{R}^n] Smooth map $\phi: [0,1]^k \rightarrow \mathbb{R}^n$ (set of k -cells in \mathbb{R}^n is $C_k(\mathbb{R}^n)$)
- [k -form] $\omega = \sum_I f dy_I$ where $dy_I = dy_{i_1} \wedge \dots \wedge dy_{i_k}$ (just see the number of differentials)
 - ω is a k -form, then $d\omega$ is a $(k+1)$ -form
- [Jacobian] $\frac{\partial \phi_I}{\partial u} = \frac{\partial(\phi_{i_1}, \dots, \phi_{i_k})}{\partial(u_1, \dots, u_k)} = \begin{bmatrix} \frac{\partial \phi_{i_1}}{\partial u_1} & \dots & \frac{\partial \phi_{i_1}}{\partial u_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_{i_k}}{\partial u_1} & \dots & \frac{\partial \phi_{i_k}}{\partial u_k} \end{bmatrix}$
- [Integration] $\int_\phi f dy_I = \int_{[0,1]^k} f(\phi(u)) \frac{\partial \phi_I}{\partial u} du = \int_{[0,1]^k} f(\phi(u)) \frac{\partial(\phi_{i_1}, \dots, \phi_{i_k})}{\partial(u_1, \dots, u_k)} du_1 \dots du_k$

Exterior Derivative d	Pullbacks T^*
<ul style="list-style-type: none"> $d(\alpha + c\beta) = d\alpha + cd\beta$ $d(\sum_I f_I dy_I) = \sum_I df_I \wedge dy_I$ $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$ $d^2 = 0$ (i.e. $d(d\omega) = 0 \forall \omega \in \Omega^k$) 	<ul style="list-style-type: none"> $(T^*\alpha)(\phi) = \alpha(T_*\phi)$ $T^*(\alpha \wedge \beta) = T^*\alpha \wedge T^*\beta$ $dT^* = T^*d$ $\int_{T \circ \phi} \alpha = \int_\phi T^*\alpha$

Stokes' Theorem

- $$\int_\Omega d\omega = \int_{\partial\Omega} \omega$$
- The boundary of a $(k+1)$ -cell ϕ is the k -chain $\partial\phi = \sum_{j=1}^{k+1} (-1)^{j+1} (\phi \circ i^{j,1} - \phi \circ i^{j,0})$
 - $i^{j,1}, i^{j,0}: [0,1]^k \rightarrow \mathbb{R}^{k+1}$ are k -cells; $i^{j,l}(u_1, \dots, u_k) = (u_1, \dots, u_{j-1}, l, u_j, \dots, u_k)$
 - [Stokes' Formula for Cubes] If ω is a $(n-1)$ -form in \mathbb{R}^n and $i: [0,1]^n \rightarrow \mathbb{R}^n$ is the identity-inclusion n -cell in \mathbb{R}^n , then $\int_i d\omega = \int_{\partial i} \omega$
 - [Stokes' Formula for General Cell] If ω is a $(n-1)$ -form in \mathbb{R}^m and ϕ is an n -cell in \mathbb{R}^m , then $\int_\phi d\omega = \int_{\partial\phi} \omega$
 - [Stokes' Formula for General Manifold] If $M \subset \mathbb{R}^n$ divides into m -cells diffeomorphic to $[0,1]^m$ and its boundary divides into $(m-1)$ -cells diffeomorphic to $[0,1]^{m-1}$ and ω is a $(m-1)$ -form, then $\int_M d\omega = \int_{\partial M} \omega$
 - [Homotopy Operator] Integration along one component $H_z(\omega) = (\int_0^z g(x, y, z) dz) dx + (\int_0^z h(x, y, z) dz) dy$

Application of Stokes' Theorem

- [Fundamental Theorem of Calculus] $M = [a, b] \subset \mathbb{R}$, $\omega = f$, then $f(b) - f(a) = \int_{\partial M} \omega = \int_M d\omega = \int_{[a,b]} df$
- [Path Independence] Let $p, q \in \mathbb{R}^2$ and M be 1-cell representing path from p to q . Then $f(q) - f(p) = \int_{\partial M} \omega = \int_M d\omega = \int_{[p,q]} f_x dy + f_y dx$

- [Green's Theorem] $M = D$ region, $\omega = f dx + g dy$, then $\partial M = C$ the curve bounding D .
 $\int_C f dx + g dy = \int_{\partial M} \omega = \int_M d\omega = \int_D (g_x - f_y) dx dy$
- [Divergence Theorem] $M = D$ Gaussian surface, $\omega = f dy \wedge dz + g dz \wedge dx + h dx \wedge dy$, then $\partial M = S$ the boundary of Gaussian surface. $\int_S f dy \wedge dz + g dz \wedge dx + h dx \wedge dy = \int_{\partial M} \omega = \int_M d\omega = \int_D (f_x + g_y + h_z) dx \wedge dy \wedge dz = \int_D \nabla \cdot F$
- [Curl Theorem] $M = S$ Gaussian surface, $\omega = f dx + g dy + h dz$, then $\partial M = C$ the curve bounding S . $\int_C f dx + g dy + h dz = \int_{\partial M} \omega = \int_M d\omega = \int_S (h_y - g_z) dy \wedge dz + (f_z - h_x) dz \wedge dx + (g_x - f_y) dx \wedge dy$

Poincaré's Lemma

Definitions:

- Say a form ω is closed if $d\omega = 0$. Intuitively, ω is closed as in a loop
- Say a form ω is exact if $d\omega = \lambda$ for some other form λ . Intuitively, ω is the boundary

Facts and Theorems:

- Every exact form is closed, since $d\omega = d(d\lambda) = d^2\lambda = 0$
- [Poincaré's Lemma] If ω is a closed k -form on \mathbb{R}^n , then it is exact.
- If U diffeomorphic to \mathbb{R}^n , then all closed forms on U are exact.
- Say a set $U \subset \mathbb{R}^n$ is star-like if $\exists p \in U$ s.t. the line segment from p to each $q \in U$ lies in U
- Every star-like open set in \mathbb{R}^n is diffeomorphic to \mathbb{R}^n
- If $U \subset \mathbb{R}^n$ open and convex, then closed forms on U exact.

Fourier Analysis

Definitions

- [L-periodic] Let $L > 0 \in \mathbb{R}$. Say $f: \mathbb{R} \rightarrow \mathbb{C}$ is L-periodic if $f(x + L) = f(x) \forall x \in \mathbb{R}$
- [\mathbb{Z} -periodic] Say $f: \mathbb{R} \rightarrow \mathbb{C}$ is \mathbb{Z} -periodic if $f(x + 1) = f(x) \forall x \in \mathbb{R}$
- [$C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$] The space of \mathbb{C} -valued continuous \mathbb{Z} -periodic functions
- [Sup-norm] $d_\infty(f, g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)| = \sup_{x \in [0, 1]} |f(x) - g(x)|$
- [Inner Product] For $f, g \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$, define $\langle f, g \rangle = \int_{[0, 1]} f(x) \overline{g(x)} dx$
- [L^2 Norm] $\|f\|_2 := \sqrt{\langle f, f \rangle} = \left(\int_{[0, 1]} |f(x)|^2 dx \right)^{\frac{1}{2}}$
- [L^2 Metric] $d_{L^2}(f, g) := \|f - g\|_2 = \left(\int_{[0, 1]} |f(x) - g(x)|^2 dx \right)^{\frac{1}{2}}$
- [Convergence in L^2 Metric] Let $f_n, f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$. Say $(f_n)_{n=1}^\infty \rightarrow f$ in L_2 metric if $\lim_{n \rightarrow \infty} d_{L^2}(f_n, f) = 0$. Equivalently, if $\int_{[0, 1]} |f_n(x) - f(x)|^2 dx = 0$.
- [Characters] Define $e_n \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$ as the character of frequency n where $e_n(x) := e^{2\pi i n x}$
- [Trigonometric Polynomial] Say $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$ is a trigonometric polynomial if it can be expressed as a finite linear combination of characters: $f = \sum_{n=-N}^N c_n e_n$ for some $N \geq 0$.
- [Fourier Transform] Let $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$, define $\hat{f}: \mathbb{Z} \rightarrow \mathbb{C}$ as the Fourier transform of f and define the n th Fourier coefficient of f as:

$$\hat{f}(n) := \langle f, e_n \rangle = \int_{[0, 1]} f(x) e^{-2\pi i n x} dx$$

- [Periodic Convolution] Let $f, g \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$. Define $f * g: \mathbb{R} \rightarrow \mathbb{C}$ by

$$(f * g)(x) := \int_{[0, 1]} f(y) g(x - y) dy$$
- [Periodic Approximation to the Identity] Let $\epsilon > 0$ and $0 < \delta < \frac{1}{2}$. Say $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$ is a periodic (ϵ, δ) approximation to the identity if:
 - $f(x) \geq 0 \forall x \in \mathbb{R}$ and $\int_{[0, 1]} f = 1$
 - $f(x) < \epsilon$ for all $\delta \leq |x| < 1 - \delta$
- [Fejér Kernel] $F_N = \sum_{n=-N}^N \left(1 - \frac{|n|}{N}\right) e_n = \frac{1}{N} |\sum_{n=0}^{N-1} e_n|^2$

Properties

- [Basic Properties of $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$]
 - [Boundedness] $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C}) \Rightarrow f$ is bounded
 - [Vector Space, Algebra] $f + g, f - g, fg, cf \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$
 - [Closure under uniform limit] $(f_n)_{n=1}^\infty \rightarrow f$ uniformly with $f_n \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C}) \forall n$ implies $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$
 - [Metric Space] $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$ with d_∞ is a complete metric space
 - $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$ is not complete in L_2 metric, but is complete in L_∞ metric.
- [Properties of Inner Product]
 - [Hermitian Symmetry] $\langle f, g \rangle = \overline{\langle g, f \rangle}$
 - [Positive Definite] $\langle f, f \rangle \geq 0$ with equality if and only if $f = 0$
 - [Linearity in First Argument] $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle, \langle cf, g \rangle = c \langle f, g \rangle$
 - [Anti-linearity in Second Argument] $\langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle, \langle f, cg \rangle = \bar{c} \langle f, g \rangle$
- [Properties of L_2 Norm]
 - [Positive Definite] $\|f\|_2 \geq 0$ with equality if and only if $f = 0$
 - [Cauchy-Schwarz] $|\langle f, g \rangle| \leq \|f\|_2 \|g\|_2$
 - [Triangle Inequality] $\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$
 - [Pythagoras] If $\langle f, g \rangle = 0$, then $\|f + g\|_2^2 = \|f\|_2^2 + \|g\|_2^2$
 - [Positive Homogeneity of Degree 1] $\|cf\|_2 = |c| \|f\|_2$

- [Less than L_∞ Norm] $0 \leq \|f\|_2 \leq \|f\|_\infty$
- [Properties of Periodic Convolution]
 - [Closure] $f, g \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C}) \Rightarrow f * g \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$
 - [Commutativity] $f * g = g * f$
 - [Bilinearity] $f * (g + h) = f * g + f * h$, $(f + g) * h = f * h + g * h$, $c(f * g) = (cf) * g = f * (cg)$

MATH 54 Materials

- $f(x) = \frac{a_0}{2} + \sum_{m=1}^k \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right) \right)$
- $\left\langle f(x), \cos\left(\frac{n\pi x}{L}\right) \right\rangle = \begin{cases} \frac{a_0}{2} \cdot 2L, & n = 0 \\ a_n \cdot L, & 0 < n \leq k \\ 0, & n > k \end{cases}$
- $\left\langle f(x), \sin\left(\frac{n\pi x}{L}\right) \right\rangle = \begin{cases} b_n \cdot L, & 0 < n \leq k \\ 0, & n > k \end{cases}$
- $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$, $n = 0, 1, \dots$
- $b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$, $n = 1, 2, \dots$
- If f continuous on $[-L, L]$ with $f(-L) = f(L)$ i.e. it wraps around, then Fourier series converges to $f \forall x \in [-L, L]$
- If f continuous $2L$ -periodic on $(-\infty, \infty)$, then Fourier series converges to $f \forall x \in (-\infty, \infty)$

Theorems and Lemmas

- [Orthonormal Basis] $\langle e_n, e_m \rangle = \delta_{nm}$
- [Projection] Let $f = \sum_{n=-N}^N c_n e_n$ be a trigonometric polynomial. Then:
 - $c_n = \langle f, e_n \rangle$ for $-N \leq n \leq N$
 - $\langle f, e_n \rangle = 0$ for $|n| > N$
 - $\|f\|_2^2 = \sum_{n=-N}^N |c_n|^2$
- [Fourier Transform of Trigonometric Polynomials]
 - $f = \sum_{n=-N}^N \langle f, e_n \rangle e_n = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}$
 - $\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$
- [Identity] Let $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$, then $f * e_n = \hat{f}(n) e_n$
 - $f * \sum_{n=-N}^N c_n e_n = \sum_{n=-N}^N c_n (f * e_n) = \sum_{n=-N}^N \hat{f}(n) c_n e_n$
- [Arbitrarily Close Approximation to Identity] For every $\epsilon > 0$ and $0 < \delta < \frac{1}{2}$, \exists trigonometric polynomial P that is an (ϵ, δ) approximation to the identity. (Take the Fejér kernel F_N and take $N \rightarrow \infty$)

Main Theorems

- [Weierstrass Approximation Theorem] Let $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$ and $\epsilon > 0$. Then \exists trigonometric polynomial P s.t. $\|f - P\|_\infty \leq \epsilon$.
 - Any continuous periodic function can be uniformly approximated by trigonometric polynomials.
 - The closure of space of all trigonometric polynomial in L^∞ metric is $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$
 - Consider $P = f * P'$ where P' is a (ϵ, δ) approximation to the identity.
- [Fourier Theorem] Let $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$. The series $\sum_{n=-\infty}^{\infty} \hat{f}(n) e_n$ converges in the L^2 metric to f i.e. $\lim_{N \rightarrow \infty} \|f - \sum_{n=-N}^N \langle f, e_n \rangle e_n\|_2 = 0$
 - Find trigonometric polynomial approximation $\|f - P\|_2 \leq \|f - P\|_\infty < \epsilon$
 - $\langle f - S_N, S_N - P \rangle = 0 \Rightarrow \|f - P\|_2^2 = \|f - S_N\|_2^2 + \|S_N - P\|_2^2 \Rightarrow \|f - S_N\|_2 < \epsilon$
 - If f continuously differentiable, then it is also pointwise convergent.
 - If f continuously twice differentiable, then it is also uniformly convergent.
- Let $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$. Suppose $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|$ absolutely convergent, then $\sum_{n=-\infty}^{\infty} \hat{f}(n) e_n$ converges uniformly to f , i.e. $\lim_{N \rightarrow \infty} \|f - \sum_{n=-N}^N \hat{f}(n) e_n\|_\infty = 0$
 - Closure under uniform limit and apply uniqueness of limit point.

- [Plancherel Theorem] Let $f \in \mathcal{C}(\mathbb{R}/\mathbb{Z}; \mathbb{C})$. The series $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$ is absolutely convergent and $\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$
 - Apply Fourier's theorem and manipulate the convergence of L^2 norm.

Last Resorts**Last Resorts**

- Intuition/Instinct -> identify which topic -> think of all theorems: which ones apply here? -> try to use those theorems
- Have time, calm down
- Leave sufficient time (1 hour) to whack for partials and list down ideas.