Laplace Equation

Definitions

- [Laplace Equation] $-\Delta u = 0$
- [Fundamental Solution] Let $\mathcal{P}$ be an operator and $y \in \mathbb{R}^d$. The fundamental solution E_y for $\mathcal{P}$ at y is a distribution $E_y \in \mathcal{D}'(U)$ satisfying $\mathcal{P}E_y = \delta_y$.
- [Subharmonic] u is subharmonic in U if $-\Delta u \le 0$ in U
  - Maximum principle holds
- [Superharmonic] u is superharmonic in U if $-\Delta u \ge 0$ in U
  - Minimum principle holds
- [Green's Function] Let U be a domain. Then $G(x,y)$ is a Green's function on U if $G(\cdot,y): \in \mathcal{D}'(U) \cap C^1(\overline{U} \setminus \{y\})$ and $-\Delta G(\cdot,y) = \delta_y$ in U and $G(\cdot,y) = 0$ on ∂U
  - Green's function is unique
  - Smooth for $x \in U \setminus \{y\}$ for each $y \in U$
  - $G(\cdot,y)$ is smooth in $U \setminus \{y\}$
  - [4.18] $G(x,y) = G(y,x)$

Properties

- [Translation Invariance] $-\Delta\big(u(x-x_0)\big) = (-\Delta u)(x-x_0)$
- [Rotational Invariance] Given $O^T O = \mathbb{I}_d$, then $-\Delta\big(u(Ox)\big) = (-\Delta u)(Ox)$
- [Homogeneity] For $\lambda > 0$, $-\Delta\big(u(\lambda x)\big) = \lambda^2(-\Delta u)(\lambda x)$
- $f(x) = \lim_{\epsilon\to 0} \int f(y)\phi_\epsilon(x-y)\,\mathrm{d}y = \lim_{\epsilon\to 0}(f * \phi_\epsilon)(x)$

Results

- [Fundamental Solution] $E_0(r) = \begin{cases} -\frac{1}{2\pi}\log r, & d = 2 \\ \frac{1}{d(d-2)\alpha(d)r^{d-2}}, & d \ge 3 \end{cases}$
  - $E_0(r)$ is locally integrable near 0 i.e. is a distribution
  - $-\Delta E_0 = \delta_0$ in $\mathbb{R}^d$
- [Solution] A solution for $-\Delta u = f$ on $\mathbb{R}^d$ is $u = f * E$
- [Uniqueness] If $u \in C_0^\infty(\mathbb{R}^d)$, then $u = (-\Delta u) * E$ i.e. u can be recovered from the Laplacian by convolving with the fundamental solution
- [Regularity] If $-\Delta u = 0$ and $u \in \mathcal{D}'(U)$, then u is smooth
- [Derivative Estimates] Let u be a harmonic function on U. Then, for any ball $B_r(x)$ s.t. $\overline{B_r(x)} \subset U$, we have: $|D^\alpha u(x)| \le \frac{C_\alpha}{r^{d+|\alpha|}}\|u\|_{L^1(B_r(x_0))}$ for some constant C_α
- [Liouville] Let u be harmonic on $\mathbb{R}^d$ and bounded. Then $u \equiv c$.
- [4.8] Let $f \in C(\mathbb{R}^d)$ be compactly supported.
  - If $d \ge 3$, then any bounded solution of $-\Delta u = f$ has form $u = E_0 * f + c$ for some constant c
  - If $d = 2$, then any locally integrable solution of $-\Delta u = f$ satisfying $\sup_{x\in\mathbb{R}^d}|Du(x)| < \infty$ has form $u = E_0 * f + b^T x + c$
- [Mean Value Property] Let u be such that $-\Delta u = 0$ in U, $\overline{B_r(x)} \subset U$, then $u(x) = \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)\,\mathrm{d}y = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)\,\mathrm{d}S(y)$
- [Maximum Principle] Let U be a connected, open, bounded domain and $u \in C(\overline{U})$ with $\Delta u = 0$ in U. Then u admits maximum and minimum.
  - [Weak] $\max_{\overline{U}} u = \max_{\partial U} u$
  - [Strong] If $u(x_0) = \max_{\overline{U}} u$ for $x_0 \in U$, then $u \equiv c$ for some constant c

Poisson Equation

Definitions

- [Poisson]
  - $-\Delta u = f$ in U
  - $u = g$ in ∂U
- [Green's Function] Let U be a domain. The Green's function is $G(x, y): U \times U \to \mathbb{R}$ s.t. for all fixed $y \in U$
  - $-\Delta G(\cdot, y) = \delta_y(\cdot) = \delta_0(\cdot - y)$
  - $G(\cdot, y) = 0$ on ∂U
  - If $f \in C_0^\infty(U)$, then $u(x) = \int_U G(x, y) f(y)\, \mathrm{d}y$ solves $-\Delta u = f$ in U and $u = 0$ on ∂U
  - (Physical Idea)
    - $G(\cdot, y)$ is the potential from a point charge at position y
    - $f(y)$ is the charge at position y
    - $u(x)$ is the sum of potential contributions from all point charges
  - Remark: in general, G may be solved in a larger domain. In that case, cut-off $\bar{G} = G\mathbb{1}_U$

- [Properties of Green's Function]
  - $G(x, y) - E(x, y) = h(x, y)$ with h satisfying $-\Delta_x h(x, y) = 0$
  - [Symmetric] $G(x, y) = G(y, x)$
  - [Uniqueness] $\exists$ at most one Green's function $G(x, y)$
  - If exists $G(x, y)$ on U, then $u(x) = \int_U G(x, y) f(y)\, \mathrm{d}y$ gives a solution to:
    - $-\Delta u = f$ in U
    - $u = 0$ in ∂U
- [Poisson Integral Formula] Let U be C^1 domain and suppose exists $G(x, y)$ on U. Then, $\forall u \in C^2(U) \cap C(\bar{U})$, $u(x) = \int_U (-\Delta u)(z) G(z, x)\, \mathrm{d}z - \int_{\partial U} u(z) \nu(z) \cdot \nabla_z G(z, x)\, \mathrm{d}S$
  - If $-\Delta u = 0$, then $u(x) = -\int_{\partial U} u(z) \nu(z) \cdot \nabla_z G(z, x)\, \mathrm{d}S$
  - [Poisson Kernel] $K: \partial U \to \mathbb{R}$ with $K(x, y) \coloneqq \nu(y) \cdot \nabla_y G(x, y)$

Method of Images

- Typically, the form of $G(x, y)$ is: $G(x, y) = E(y - x) + \sum q_i E(y - \bar{x}_i)$ for $x_i \notin U$
- [Half-Plane] $G(x, y) = E(y - x) - E(y - \bar{x})$
- [Unit Ball] $G(x, y) = E(y - x) - E\big(|x|(y - \bar{x})\big)$

Complex Analysis

Definitions

• [Holomorphic] f is holomorphic in U if $(\partial_x + i\partial_y)f = 0$ in U
  ○ f is a smooth solution to the Cauchy-Riemann equations
• [Cauchy-Riemann Operator] $\partial_x + i\partial_y$
• [Cauchy-Riemann Equations]
  ○ $f_y = if_x$
  ○ $(\partial_x + i\partial_y)f = 0$

Theorems

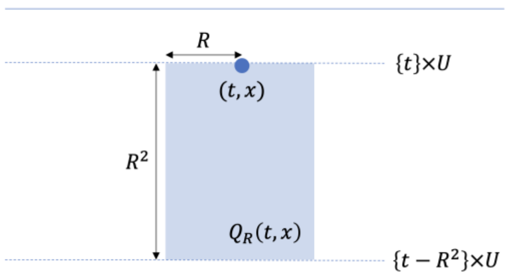
• [4.25] If $f \in \mathcal{D}'(U)$ is a solution to $(\partial_x + i\partial_y)f = 0$ in U, then f is smooth in U
• [Morera] Let f be a continuous function on U s.t. $\forall\Omega$ bounded domain, $\Omega \subset U$, $\partial\Omega$ triangle, $\int_{\partial\Omega} f \, \mathrm{d}z = 0$. Then f is holomorphic in U
• Let f be a continuous, complex-valued function on open set U satisfying $\oint_\gamma f(z) \, \mathrm{d}z = 0 \ \forall\gamma$ closed, piecewise C^1. Then f must be holomorphic.
• [4.27] Let Ω. Then $\int h(z)(\partial_x + i\partial_y)1_\Omega \mathrm{d}z = i\int_{\partial\Omega} h(z) \, \mathrm{d}z$
• [Cauchy Integral Formula] Let f be holomorphic in U. Then, for every bounded piecewise C^1 domain Ω and $z_0 \in \Omega$, $f(z_0) = \frac{1}{2\pi i}\int_\Omega \frac{f(z)}{z-z_0} \mathrm{d}z$
• [4.30] Let f be a continuous function on domain U. The following are equivalent:
  ○ f is complex-differentiable
  ○ f is a solution to the Cauchy-Riemann equation i.e. f is holomorphic
  ○ f is complex-analytic i.e. at every point $z_0 \in U$, $\exists r > 0$ and coefficients c_j s.t. $f(z) = \sum_{i=0}^{\infty} c_i(z-z_0)^i$ for $\lvert z - z_0\rvert < r$
• [Schwartz Reflection Principle]

Results
• [Fundamental Solution] $E_0 = \frac{1}{2\pi}\frac{1}{z}$ is a fundamental solution for $\partial_x + i\partial_y$. • If $f = u + iv$ solves the Cauchy-Riemann equations, then u, v harmonic.

Heat Equation

Definitions

- [Heat Equation] $(-\partial_t + \Delta)u = f$
- [Initial Value Problem]
  - $(-\partial_t + \Delta)u = f$ in $(0,\infty)_t \times \mathbb{R}^d_x$
  - $u = g$ on $\{t = 0\} \times \mathbb{R}^d_x$
- [Initial Boundary Value Problem]
  - $(-\partial_t + \Delta)u = f$ in $(0,\infty)_t \times U$
  - $u = g$ on $\{t = 0\} \times U$
  - $u = h$ on $(0,\infty)_t \times \partial U$
- [Forward (Support) Condition] $E = 0$ in $(-\infty, 0)_t \times \mathbb{R}^d$
- [Forward Fundamental Solution] E_+ is the forward fundamental solution if it satisfies:
  - $(-\partial_t + \Delta)E_+ = \delta^d_0(t - a)$
  - $E_+ = 0$ on $\{t < a\} \times \mathbb{R}^d_x$
  - $E_+(t,x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{\|x\|^2}{4t}} \mathbb{1}_{(a,\infty)}(t)$
- [Duhamel's Principle]
- [Heat Ball] Denote $Q_R(t,x) \coloneqq (t - R^2, t) \times B_R(x)$ with $\overline{Q_R(t,x)} \subset U$ the heat ball of radius R centered at (t,x).



- [Ancient Solution] u is an ancient solution to heat equation if $(-\partial_t + \Delta)u = 0$ in $(-\infty, a) \times \mathbb{R}^d$
- [Gaussian Growth Condition]
  - Let $g \in \mathcal{D}'(\mathbb{R}^d)$. Then g satisfies the Gaussian Growth Condition if $\forall A > 0$, $\langle g, \left(\chi_{>1}\left(\frac{x}{R'}\right) - \chi_{>1}\left(\frac{x}{R}\right)\right) e^{-A|x|^2} \rangle \to 0$ as $R, R' \to \infty$
  - Let $f \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d)$, then f satisfies the Gaussian growth condition in space if $\forall \eta \in C^\infty_0(\mathbb{R}_t)$ and $A > 0$, $\langle f, \eta(t) \left(\chi_{>1}\left(\frac{x}{R'}\right) - \chi_{>1}\left(\frac{x}{R}\right)\right) e^{-A|x|^2} \rangle \to 0$ as $R, R' \to \infty$

Properties

- [Properties of $-\partial_t + \Delta$]
  - [Rotation] $(\partial_t - \Delta)\big(u(t, Ox)\big) = \big((\partial_t - \Delta)u\big)(t, Ox)$
  - [Homogeneity] $(\partial_t - \Delta)\left(u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)\right) = \frac{1}{\lambda^2}\big((\partial_t - \Delta)(u)\big)\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$
    - $[t] = [x]^2$
  - [Linearity] $(\mu u)(t,x) = \mu\big(u(t,x)\big)$
  - [Conservation of Mass] $(\partial_t - \Delta)u = 0 \Rightarrow \partial_t \int u(t,x)\,\mathrm{d}x$
- [Properties of Heat Kernel]
  - [Infinite Speed of Propagation] $\operatorname{supp} E_+(t,\cdot) = \mathbb{R}^d$ for $t > 0$
  - [Smooth] E_+ is smooth except at $(t,x) = (0,0)$
-

Results

- [Solution Formula] Let $f \in C_0^\infty(\mathbb{R}^{1+d})$, then for the heat equation $(\partial_t - \Delta)u = f$, a forward solution is $u(t,x) \coloneqq (E_+ * f)(t,x) = \iint_{-\infty}^{t-a} \frac{1}{(4\pi(t-s))^{\frac{d}{2}}} e^{-\frac{\|x-y\|^2}{4(t-s)}} \mathbb{1}_{(a,\infty)}(t-s) f(s,y)\, \mathrm{d}s\, \mathrm{d}y$
- [Solution Formula IVP] Let $f \in \mathcal{D}_0'(\mathbb{R}^{1+d}), g \in \mathcal{D}_0'(\mathbb{R}^d)$

$$u(t,x) = \int_a^t \int_{\mathbb{R}^d} \frac{1}{(4\pi(t-s))^{\frac{d}{2}}} e^{-\frac{\|x-y\|^2}{4(t-s)}} f(s,y)\, \mathrm{d}y\ \mathrm{d}s + \int_{\mathbb{R}^d} \frac{1}{(4\pi(t-a))^{\frac{d}{2}}} e^{-\frac{\|x-y\|^2}{4(t-a)}} g(y)\, \mathrm{d}y$$

- [Representation Formula] Let $u \in \mathcal{D}_0'(\mathbb{R}^{1+d})$, can recover via $u = E_+ * \big((\partial_t - \Delta)u\big)$
- [Smoothness] Let u be a solution to $(-\partial_t + \Delta)u = 0$ in $U \subset \mathbb{R}^{1+d}$ open, then u is smooth.
- [Derivative Bounds] Let $(-\partial_t + \Delta)u = 0$ in U. Then, for $(t,x) \in U$ with $\overline{Q_R(t,x)} \subset U$, $\left|\partial_x^\alpha \partial_t^k u(t,x)\right| \le C_{\alpha,k} R^{-|\alpha|-2k} \sup_{(s,y)\in Q_R(t,x)} |u(s,y)|$
- [Mean Value Property] Let $(-\partial_t + \Delta)u = 0$ in U. Let $(t,x) \in \mathbb{R}^{1+d}$. Define $\xi_r(t,x) = \left\{(s,y) \in \mathbb{R}^{1+d} : s \le t, E_+(t-s, x-y) \ge \frac{1}{r^d}\right\}$. Then $u(t,x) = \frac{1}{4r^d} \iint_{\xi_r(t,x)} u(s,y) \frac{\|x-y\|^2}{(t-s)^2}\, \mathrm{d}s\, \mathrm{d}y$
- [5.9]

Theorems

- [Liouville] If u s a bounded ancient solution, then u must be constant.
- [Coarea] Let $f: \mathbb{R}^d \to \mathbb{R}$ be Lipschitz continuous and assume for $\rho \in \mathbb{R}$, the level set $\{x \in \mathbb{R}^d : f(x) = \rho\}$ is smooth, $(d-1)$-dimensional hypersurface in d. Assume $u: \mathbb{R}^d \to \mathbb{R}$ continuous, integrable. Then $\int_{\mathbb{R}^d} u|Df|\mathrm{d}x = \int_{-\infty}^{\infty} \int_{\{f=\rho\}} u\, \mathrm{d}\sigma\, \mathrm{d}\rho$

Wave Equation

Definitions

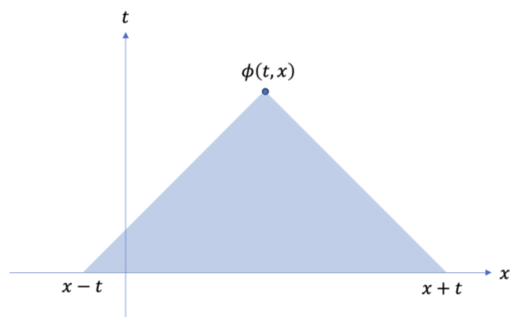
- [Wave Equation] $\phi: \mathbb{R}_t \times \mathbb{R}_x^d \to \mathbb{R}$
    - $\Box\phi = (-\partial_t^2 + \Delta)\phi = f$ in $(0,\infty)_t \times \mathbb{R}_x^d$
    - $\phi = g$ on $\{t=0\} \times \mathbb{R}_x^d$
    - $\partial_t \phi = h$ on $\{t=0\} \times \mathbb{R}_x^d$
- [Null Coordinates]
    - $u = t - r$
    - $v = t + r$
    - $\Box = -4\partial_u \partial_v$
    - $\partial_t = \partial_u + \partial_v$
    - $\partial_x = -\partial_u + \partial_v$
- [Forward Fundamental Solution] E_+ is a forward fundamental solution if E_+ satisfies:
    - $\Box E_+ = \delta_0$
    - $\operatorname{supp} E_+ \subset \{(t,x) \in \mathbb{R}^{1+d} : t \geq 0\}$
- [Finite Speed of Propagation] $\operatorname{supp} E_+ \cap \{t \in I\}$ is compact for bounded interval I

Results

- [Fundamental Solution]
    - $E_0(t,x) = -\frac{1}{2}(H(t-x) + c_1)(H(t+x) + c_2)$
- [Forward Fundamental Solution in $\mathbb{R}^{1+1}$] $\operatorname{supp} E_+ \subset \{t \geq 0\}$
    - $E_+(t,x) = \frac{1}{2}\mathbb{1}_{(0,\infty)}(t-x)\mathbb{1}_{(0,\infty)}(t+x) = \frac{1}{2}H(t-x)H(t+x)$ (i.e. $c_1 = c_2 = 0$)
- [Forward Fundamental Solution in $\mathbb{R}^{1+d}$]
    - $E_+(t,x) = -\frac{1}{2\pi^{\frac{d-1}{2}}}\mathbb{1}_{(0,\infty)}\chi_+^{-\frac{d-1}{2}}(t^2 - \|x\|^2)$ in $\mathbb{R}^{1+d} \backslash \{(0,0)\}$

Properties

- [Properties of Forward Fundamental Solution]
    - Finite speed of propagation: $\operatorname{supp}_x E_+(t,\cdot) \subset [-t,t]$ for $t \geq 0$
        - Disturbance at $x=0$ at $t=0$ can reach at most $|x| \leq t'$ at time t'
    - Propagation of singularity
- [Symmetries of $\Box$]
    - [Lorentz Transformation]
        - A Lorentz transformation $L: \mathbb{R}^{1+d} \to \mathbb{R}^{1+d}$ is a linear transformation that keeps $s^2(t,x) := t^2 - \|x\|^2$ invariant i.e. $s^2(t,x) = s^2\big(L(t,x)\big)$
        - $\Box(\phi \circ L) = (\Box\phi) \circ L$
        - Generated by rotations, reflections and $\mathbb{R}^{2\times 2}$ Lorentz boosts
    - [Scaling] $\Box\left(\phi\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right)\right) = \frac{1}{\lambda^2}(\Box\phi)\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right)$ for $\lambda > 0$
- [7.8 Finite Speed of Propagation]
- [Strong Huygens] Let $d \geq 3$ be odd. Let $\phi \in C^\infty(\mathbb{R}^{1+d})$ be a solution to $\Box\phi = f$ with initial data (g,h). If $f(s,y) = 0$ for $\{(s,y): 0 < s < t, \|y - x\| = t - s\}$ And $g(y) = h(y) = 0$ for $y \in \partial B_t(x)$, then $\phi(t,x) = 0$



Homogeneity

- [Homogeneous of Degree a] A smooth function h on $\mathbb{R}^d\backslash\{0\}$ is homogeneous of degree a if $h(\lambda x) = \lambda^a h(x)$ for $x \neq 0$, $\lambda > 0$
- [Homogeneous of Degree a] Let $h \in \mathcal{D}'(\mathbb{R}^{1+d})$. Then h is homogeneous of degree $a \in \mathbb{C}$ if for every test function $\phi \in C_0^\infty(\mathbb{R}^{1+d})$ and $\lambda > 0$, $\lambda^{-d}\langle h, \phi(\lambda^{-1}\,\cdot)\rangle = \lambda^a\langle h, \phi\rangle$
  - Denote h_λ by the distribution that satisfies $\langle h_\lambda, \phi\rangle = \lambda^{-d}\langle h, \phi(\lambda^{-1}\,\cdot)\rangle$.
- The delta distribution δ_0 on $\mathbb{R}^d$ is homogeneous of degree $-d$
- Let h be homogeneous of degree a, then $D^\alpha h$ is homogeneous of degree $a - |\alpha|$
- [Euler's Identity] Let h be homogeneous of degree a, then $\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}\langle h_\lambda, \phi\rangle = a\langle h_\lambda, \phi\rangle$
- [6.7 Classification] Let $h \in \mathcal{D}'(\mathbb{R})$ be a homogeneous distribution of degree a, then:
  - h agrees with a smooth homogeneous function of degree a on $\mathbb{R}\backslash\{0\}$.
  - If $a \geq 0$, h is uniquely determined by $h(1)$ and $h(-1)$
  - If $a = -k$ integer, then any two homogeneous distributions $h, \bar{h}$ of degree $-k$ with $h(1) = \bar{h}(1)$, $h(-1) = \bar{h}(-1)$ differ by a multiple of $\delta_0^{(k-1)}$
- [6.11 Classification] Let $h \in \mathcal{D}'(\mathbb{R})$ be a homogeneous distribution of degree a, then
  - If $a \notin \{-1, -2, \dots\}$, $h = c_+(x_+)^a + c_-(x_-)^a$ for some $c_+, c_- \in \mathbb{R}$

Theorems

- [d'Alembert] Let $\phi \in C^\infty(\mathbb{R}_+^{1+1})$ and $(t, x) \in \mathbb{R}_+^{1+1}$:
  - $\phi(t,x) = -\frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} (\Box\phi)(s,y)\mathrm{d}y\ \mathrm{d}s + \frac{1}{2}\big(\phi(0, x-t) + \phi(0, x+t)\big) + \frac{1}{2}\int_{x-t}^{x+t} \partial_t\phi(0,y)\,\mathrm{d}y$
  - $\phi(t,x) = -\frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(s,y)\mathrm{d}y\ \mathrm{d}s + \frac{1}{2}\big(g(x-t) + g(x+t)\big) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y$
- [Solution Formula] Given E_+, $g, h \in C^\infty(\mathbb{R}^d)$, $f \in C^\infty(\mathbb{R}^{1+d})$, exists unique ϕ to the initial value problem: $\phi = -E_+ * \big(h\delta_0(t)\big) - \partial_t\Big(E_+ * \big(g\delta_0(t)\big)\Big) + E_+ * \Big(f\mathbb{1}_{(0,\infty)}(t)\Big)$
- [Representation Formula]
  - $\phi = -\Big(E_+ * \partial_t\phi|_{\{t=0\}}\delta_0(t)\Big) - \partial_t\Big(E_+ * \phi|_{\{t=0\}}\delta_0(t)\Big) + \Big(E_+ * \Big(\Box\phi\mathbb{1}_{(0,\infty)}(t)\Big)\Big)$
- [Uniqueness] The forward fundamental solution E_+ is unique.
- [Poisson $\mathbb{R}^{1+2}$] Let ϕ be a solution to equation $\Box\phi = f$ with $\phi, f \in C^\infty(\mathbb{R}^{1+2})$. Then:
  - $\phi(t,x) = \partial_t\left(\frac{1}{2\pi}\int_{B_t(x)} \frac{g(y)}{\sqrt{t^2 - \|x-y\|^2}}\mathrm{d}y\right) + \frac{1}{2\pi}\int_{B_t(x)} \frac{h(y)}{\sqrt{t^2 - \|x-y\|^2}}\mathrm{d}y -$
    $\frac{1}{2\pi}\int_0^t \int_{B_s(x)} \frac{f(s,y)}{\sqrt{(t-s)^2 - \|x-y\|^2}}\ \mathrm{d}y\ \mathrm{d}s$
- [Kirchhoff $\mathbb{R}^{1+3}$] Let ϕ be a solution to equation $\Box\phi = f$ with $\phi, f \in C^\infty(\mathbb{R}^{1+3})$. Then:
  - $\phi(t,x) = \partial_t\left(\frac{1}{2\pi t}\int_{\partial B_t(x)} g(y)\,\mathrm{d}y\right) + \frac{1}{2\pi t}\int_{\partial B_t(x)} h(y)\,\mathrm{d}y - \frac{1}{2\pi t}\int_0^t \int_{\partial B_{t-s}(x)} f(s,y)\,\mathrm{d}y\,\mathrm{d}s$
- [$\mathbb{R}^{1+(2k+1)}$]
- [Method of Descent $\mathbb{R}^{1+2k}$]

Distribution Theory

Definitions
• [6.1] • [6.2] • [Homogeneous Function of Degree α] A function $h: \mathbb{R}^d \backslash \{0\} \to \mathbb{R}$ is homogeneous of degree α if $h(\lambda x) = \lambda^\alpha h(x)\ \forall x \neq 0, \lambda > 0$ • [Homogeneous Distribution of Degree α]
Lemmas
• Let $\chi \in C_0^\infty(\mathbb{R}^2)$ with $\int \chi = 1$. Then, $\delta_0(t, x) = \lim\limits_{\epsilon \to 0^+} \frac{1}{\epsilon^2} \chi\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right)$. • [7.3] Let f be a distribution with $\operatorname{supp} f \subset \{t \in [L, \infty)\}$ for some $L \in \mathbb{R}$. Then $E_+ * f$ is well-defined.   ○ $(E_+ * f)(t, x) = \iint \mathbb{1}_{(0,t-L)}(t - s) E_+(t - s, x - y) f(s, y)\, \mathrm{d}s\, \mathrm{d}y$

Exam

Checks
• [Jacobian] $J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$ • [Change of Variables] $\int f(u,v)\,\mathrm{d}u\,\mathrm{d}v = \int f(t,x)\lvert\det J\rvert\,\mathrm{d}t\,\mathrm{d}x$ • Check that you transformed the PDEs correctly – don't miss some terms. Transform the constraints / initial conditions too.