Classics

Poisson Process $(N_t)_{t\geq 0} \sim PP(\lambda)$

Set-up:

- $N(0) = 0, \; N([t, t+\Delta t]) \sim \text{Poisson}(\lambda \Delta t)$
- Disjoint intervals are independent
- $X_i \sim \text{Expo}(\lambda)$ i.i.d. (interarrival time)

Waiting Time W_n Analysis:

- $W_n = \sum_{i=1}^n X_i$; $W_0 = 0$ (waiting time)
- $W_n \sim \text{Erlang}(n, \lambda)$ i.e. $f_{W_n}(w) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}$
- $N(t) = \max\{n : 0 < W_n \leq t\} \sim \text{Poisson}(\lambda t)$
- $N(t) < n \Leftrightarrow W_n > t \Leftrightarrow \sum_{i=1}^n X_i > t$
- $N(t) \geq n \Leftrightarrow W_n \leq t \Leftrightarrow \sum_{i=1}^n X_i \leq t$
- $\mathbb{P}[W_n \leq t] = \mathbb{P}[N(t) \geq n] = \mathbb{P}[\text{Poisson}(\lambda t) \geq n]$
- $N(t)$ and $W_{N(t)+1}$ are independent

Conditioning on Interval $N(t) = n$:

- $\mathbb{P}[N(s) = k | N(t) = n] \sim \text{Binomial}\left(n, \frac{s}{t}\right)$ for $s \leq t, k \leq n$
- Let $W_i | N(T) = n$ denote the arrival of ith customer
- $f_{W_1,\ldots,W_n|N(T)=n}(t_1, \ldots, t_n) = \frac{n!}{T^n}$ if $0 \leq t_1 \leq \cdots \leq t_n \leq T$
- [Big Theorem] Conditional on $N(T) = n$, the n arrivals are i.i.d. uniform in $[0, T]$.
- $W_i \sim V_i = i$th order statistics of $U_1, \ldots, U_n$
- If g symmetric, $g(V_1, \ldots, V_n) = g(U_1, \ldots, U_n)$
- Symmetric functions $\mathbb{P}[g(W_1, \ldots, W_n) = k | N(t) = n] = \mathbb{P}[g(U_1, \ldots, U_n) = k]$

Current Life and Residual Life Analysis:

- δ_t: current life, γ_t: excess life
- γ_t independent of δ_t (memoryless)
- $\delta_t \sim \min(\text{Expo}(\lambda), t)$, $\gamma_t \sim \text{Expo}(\lambda)$
- $\mathbb{P}[\delta_t \leq x] = \begin{cases} 1 - e^{-\lambda x}, & 0 \leq x < t \\ 1, & t \leq x \end{cases}$
- $\mathbb{P}[\gamma_t \leq x] = 1 - e^{-\lambda x}$
- $\mathbb{P}[\gamma_t > x, \delta_t > y] = \mathbb{P}[\gamma_t > x]\mathbb{P}[\delta_t > y]$
- $\mathbb{E}[\delta_t + \gamma_t] = \frac{2}{\lambda} - \frac{1}{\lambda} e^{-\lambda t}$ (size-biased)
- $M(t) = \mathbb{E}[N(t)] = \lambda t$

Differential Analysis:

- $\mathbb{P}[N(t, t + \mathrm{d}t) = 1] = \lambda \, \mathrm{d}t$
- $\mathbb{P}[N(t, t + \mathrm{d}t) > 1] = 0$
- $\mathbb{P}[N(t, t + \mathrm{d}t) = 0] = 1 - \lambda \, \mathrm{d}t$

Poisson Merging and Splitting:

Renewal Process

Set-up:

- $N(0) = 0$, $N(t) = \max\{n | W_n \leq t\}$ is the number of replacements by time t

Current Life and Residual Life Analysis:

- $\delta_t = t - W_{N(t)}$: current life
- $\gamma_t = W_{N(t)+1} - t$: excess life
- $\delta_t + \gamma_t = W_{N(t)+1} - W_{N(t)}$: total life
- $\mathbb{E}[\gamma_t] = \mathbb{E}[X_i]\mathbb{E}[N(t) + 1] - t$
- $\mathbb{P}[\gamma_t > x] = \mathbb{P}[\text{no renewal in } (t, t + x)]$
- $\mathbb{P}[\delta_t > x] = \mathbb{P}[\text{no renewal in } (t - x, t)]$
- $\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{1}\{\delta_s > x, \gamma_s > y\} \, \mathrm{d}s = \frac{\int_{x+y}^\infty \mathbb{P}[X_i > z] \, \mathrm{d}z}{\mathbb{E}[X_i]}$ i.e. proportion of time up until t where $\delta_s > x, \gamma_s > y$
  - $r_i = \max\big(0, X_i - (x + y)\big)$
  - $t_i = X_i$
- $\lim_{t\to\infty} \mathbb{P}[\delta_t > x, \gamma_t > y] = \frac{\int_{x+y}^\infty \mathbb{P}[X_i > z] \, \mathrm{d}z}{\mathbb{E}[X_i]}$
- $f_\delta(x) = \frac{\mathbb{P}[X_i > x]}{\mathbb{E}[X_i]}$, $f_\gamma(x) = \frac{\mathbb{P}[X_i > x]}{\mathbb{E}[X_i]}$ (same)
- $f_{\gamma,\delta}(x, y) = \frac{\mathbb{P}[X_i > x + y]}{\mathbb{E}[X_i]^2}$
- Define $L(s) = \delta_s + \gamma_s$
- $\lim_{s\to\infty} \mathbb{E}[L(s)] = 2 \lim_{s\to\infty} \mathbb{E}[\gamma] = \frac{\mathbb{E}[X_i^2]}{\mathbb{E}[X_i]} \geq \mathbb{E}[X_i]$ (size-biased sampling)
- $\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{E}[L(s)] \, \mathrm{d}s = \frac{\mathbb{E}[X_i^2]}{\mathbb{E}[X_i]}$
- $\lim_{t\to\infty} \frac{1}{t} \int_0^t L(s) \, \mathrm{d}s = \frac{\mathbb{E}[X_i^2]}{\mathbb{E}[X_i]}$
  - $t_i = X_i$
  - $r_i = X_i^2$

Expected Number of Replacements by Time t:

- $M(t) = \mathbb{E}[N(t)]$
- $M(t) = \sum_{k=1}^\infty \mathbb{P}[N(t) \geq k] = \sum_{k=1}^\infty \mathbb{P}[W_k \leq t] = \sum_{k=1}^\infty F_{W_k}(t)$ where F_{W_k} is the k-fold convolution of X_i
- $\mathbb{E}\big[W_{N(t)+1}\big] = \mathbb{E}[X_i]\mathbb{E}[N(t) + 1]$
- $M(t) = F(t) + \int_0^t M(t - x) \, \mathrm{d}F(x)$

Renewal Theorem:

- $\lim_{t\to\infty} \frac{t}{N(t)} = \lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \mu$ a.s.
- Consider i.i.d. pairs $(r_i, t_i)_{i=1}^\infty$; r_i, t_i could be dependent
- t_i: length of ith cycle (can be period, time till success)

- $\left(N_1(t)\right)_{t\geq 0} \sim PP(\lambda)$ and $\left(N_2(t)\right)_{t\geq 0} \sim PP(\mu)$, then $\left(N_1(t)+N_2(t)\right)_{t\geq 0} \sim PP(\lambda+\mu)$
- [Splitting] $(Y_i)_{i=1}^{\infty}$ discrete, i.i.d independent of $\left(N(t)\right)_{t\geq 0}$ determines a type
- $N_j(t) = \sum_{i=1}^{N(t)} \mathbb{1}\{Y_i = j\}$: arrival process of the jth type. Then $\left(N_j(t)\right)_{j=1}^{k} \sim PP(\lambda\mathbb{P}[Y = j])$ and are independent of each other (**NOT** the parent stream $N(t)$)

Inhomogeneous Poisson Process $PP\left(\lambda(t)\right)$

- Rate changes with time
- $N_s - N_t \sim \text{Poisson}\left(\int_t^s \lambda(u)\,\mathrm{d}u\right)$
- $\lambda(u) \equiv \lambda_0$ reduces to homogeneous
- $\Lambda(t) = \int_0^t \lambda(u)\,\mathrm{d}u$ (rate accumulated)
- $(Y_s)_{s\geq 0}$ such that $Y_s = N_{\Lambda^{-1}(s)}$
- $(Y_s)_{s\geq 0} \sim PP(1)$
- New process lags and extends the time to make sure it is homogeneous

Queueing Theory

- [GI/G/1] Interarrival $t_i \sim \text{Expo}(\lambda)$, service time $s_i \sim \text{Expo}(\mu)$. If $\lambda < \mu$, then queue clears with probability 1 and long run average proportion of time spent working $= \frac{\lambda}{\mu} < 1$ a.s.
  - $\lambda < \mu$: positive recurrent MC
  - $\lambda = \mu$: null recurrent MC
  - $\lambda > \mu$: transient MC ($\equiv$ branching process with replacement > 1)
- [M/G/1] Only assumption is $t_i \sim \text{Expo}(\lambda)$
- Customers arriving during the nth service time
  - $\mathbb{P}[k \text{ arrivals} | S_n = s] = \frac{(\lambda s)^k e^{-\lambda s}}{k!}$
  - $\mathbb{P}[k \text{ arrivals}] = \mathbb{E}\left[\frac{(\lambda S_n)^k e^{-\lambda S_n}}{k!}\right]$
  - $X_{n+1} = \max(0, X_n - 1 + S_n)$
  - $(X_n)_{n=1}^{\infty}$ is a Markov chain

$X_n \backslash X_{n+1}$	0	1	2	3
0	$p_0 + p_1$	p_2	p_3	p_4
1	p_0	p_1	p_2	p_3
2	0	p_0	p_1	p_2
3	0	0	p_0	p_1

Checks

- Consider edge cases i.e. length 0 intervals, edge effects for δ_t; justify symmetry

- r_i some reward associated with ith cycle (can be cost, conditionals, counter)
- $R(t)$ is the reward collected by time t
- $\sum_{i=1}^{N(t)} r_i \leq R(t) \leq \sum_{i=1}^{N(t+1)} r_i$
- $\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[r_i]}{\mathbb{E}[t_i]}$ a.s.
- [Two-state system] For $(s_i, t_i)_{i=1}^{\infty}$, time spent in first system is $\frac{\mathbb{E}[s_i]}{\mathbb{E}[s_i]+\mathbb{E}[t_i]}$

Probabilistic Toolkit

- [Order statistics]
$$f_{V_r}(x) = \frac{n!}{(r-1)!\,(n-r)!} f_X(x)\left(F(x)\right)^{r-1}\left(1-F(x)\right)^{n-r}$$
- [Erlang(k,λ)]
$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}$; $\mathbb{E}[X] = \frac{k}{\lambda}$; $\text{Var}[X] = \frac{k}{\lambda^2}$
- [Binomial Approximation to Poisson]
$$\lim_{n\to\infty} \mathbb{P}\left[\text{Binomial}\left(n, \frac{\lambda}{n}\right) = k\right] = \frac{\lambda^k e^{-\lambda}}{k!}$$
- [Tail Sum] $\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X > z]\,\mathrm{d}z$
- [Tail Sum] $\mathbb{E}[X^n] = \int_0^\infty n z^{n-1} \mathbb{P}[X > z]\,\mathrm{d}z$
- [Tail Sum] $\mathbb{E}[X] = \sum_{z=0}^{\infty} \mathbb{P}[X > z]$
- [Le Cam] Let $\epsilon_i \sim \text{Bernoulli}(p_i)$ independent, p_i not necessarily equal. $S_n = \sum_{i=1}^n \epsilon_i$ and $\mu = p_1 + \cdots + p_n$. Then $\left|\mathbb{P}[S_n = k] - \frac{\mu^k e^{-\mu}}{k!}\right| \leq \sum_{i=1}^n p_i^2$ (Used to prove that Poisson Process can be constructed by n small intervals of length $\frac{1}{n}$)

Problem Solving

- Break into <u>disjoint</u> intervals
- Convert condition $N(t) < k$ and $W_k > t$
- Convert to indicators on each individual arrival $\mathbb{P}[M(t) = m | N(t) = n] = \mathbb{P}[\sum_{k=1}^n \mathbb{1}\{W_k + Y_k \geq m\} | N(t) = n]$.
- $\mathbb{E}\left[\sum_{i=1}^{N(t)+1} X_i\right] = \mathbb{E}[\sum_{i=1}^{\infty} X_i \mathbb{1}\{i \leq N(t)+1\}] = \mathbb{E}[\sum_{i=1}^{\infty} X_i \mathbb{1}\{W_{i-1} \leq t\}]$
- Total probability on $N(t) = k$ then condition
$$\mathbb{P}[\delta_t > x, \gamma_t > y] = \sum_{k=0}^{\infty} \mathbb{P}[N(t) = k, \delta_t > x, \gamma_t > y]$$
- Nest conditions $t_i = \mathbb{1}\{U_i < S_i\}(U_i + V_i \mathbb{1}\{U_i > 1\}) + \mathbb{1}\{U_i > S_i\} S_i$
- Use graphical method for order statistics
- Refine your reward and period. Shift the burden through conditioning and indicators.
- Bash renewal with limit theorems