Probability

Definitions

- [π-Class] A collection P is a <u>π-class</u> if:
  - P is nonempty
  - $A, B \in P \Rightarrow A \cap B \in P$
- [λ-Class] A collection L is a <u>λ-class</u> if:
  - $\phi \in L$
  - $A, B \in L$ with $A \subset B \Rightarrow B \backslash A \in L$
  - $A_1 \subset A_2 \subset \cdots \in L \Rightarrow \bigcup_{i=1}^{\infty} A_i \in L$
- [Measurable Map] Let $X: \Omega \to S$. Then X is a <u>measurable map</u> from $(\Omega, \mathcal{F})$ to $(S, \mathcal{B})$ if $X^{-1}(B) = \{\omega: X(\omega) \in B\} \in \mathcal{F}\ \forall B \in \mathcal{B}$
- [Random Vector] X is a <u>random vector</u> if X is a measurable map with $(S, \mathcal{B}) = \left(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\right)$
- [Random Variable] X is a <u>random variable</u> if X is a measurable map with $(S, \mathcal{B}) = \left(\mathbb{R}, \mathcal{B}(\mathbb{R})\right)$
- [Generates] Let $\mathcal{S}$ be a σ-field and $\mathcal{A}$ a collection of subsets of $\mathcal{S}$. Then $\mathcal{A}$ <u>generates</u> $\mathcal{S}$ if $\mathcal{S}$ is the smallest σ-field containing $\mathcal{A}$
  - $\mathcal{S} = \bigcap_{\mathcal{S}': \mathcal{A} \subset \mathcal{S}'} \mathcal{S}'$
- [σ-Field Generated by X] Let $X: (\Omega, \mathcal{F}) \to (S, \mathcal{B})$. Then, $\sigma(X)$ is the <u>σ-field generated by X</u> $\sigma(X) = \{X^{-1}(B): B \in \mathcal{B}\} = \left\{\{\omega: X(\omega) \in B\}: B \in \mathcal{B}\right\}$
- [Independence] Let $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{F}$ be sub-σ-fields. $\mathcal{B}_1, \mathcal{B}_2$ are <u>independent</u> if $\mathbb{P}[B_1 \cap B_2] = \mathbb{P}[B_1]\mathbb{P}[B_2]\ \forall B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2$
- [Independence] The σ-fields $\mathcal{B}_1, \mathcal{B}_2, \dots \mathcal{B}_n$ are <u>independent</u> when $\mathbb{P}[\bigcap_{i=1}^n B_i] = \prod_{i=1}^n \mathbb{P}[B_i]$ $\forall B_i \in \mathcal{B}_i$
- [Independence] The events $A_1, \dots, A_n$ are <u>independent</u> if $\mathbb{P}[\bigcap_{i=1}^n A_i] = \prod_{i=1}^n \mathbb{P}[A_i]$
- [Independence] Two random variables X, Y are <u>independent</u> if $\sigma(X), \sigma(Y)$ are independent
  - $\mathbb{P}[B_1 \cap B_2] = \mathbb{P}[B_1]\mathbb{P}[B_2]$ for $\forall B_1 \in \sigma(X), B_2 \in \sigma(Y)$
- [Infinite Product Measure] Let $\left((\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha)\right)_{\alpha \in \mathcal{A}}$ be probability spaces. $\exists!$ measure on $\mathcal{F}$. $\Omega = \prod_{\alpha \in \mathcal{A}} \Omega_\alpha$ is the Cartesian product. $\mathcal{F} = \sigma(\bigcup_{\alpha \in \mathcal{A}} \mathcal{F}_\alpha)$. $\mathbb{P}[E] = \prod_{\alpha \in \mathcal{A}'} \mathbb{P}[E_\alpha]$ where $\mathcal{A}' \subset \mathcal{A}$ is a finite subset.

Theorems

- [1.3.1] Let $X: \Omega \to S$ and $\mathcal{A}$ be a collection of subsets of S that generates $\mathcal{S}$. If $\{\omega: X(\omega) \in A\} \subset \mathcal{F}\ \forall A \in \mathcal{A}$, then X is measurable.
- If X, Y independent, then $\sigma(X), \sigma(Y)$ are independent.
- [Kolmogorov's Extension Theorem] Suppose given probability measure μ_n on $\left(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)\right)$ that are consistent i.e. $\mu_{n+1}\left((a_1, b_1] \times \dots \times (a_n, b_n] \times \mathbb{R}\right) = \mu_n\left((a_1, b_1] \times \dots \times (a_n, b_n]\right)$, then there is a unique probability measure $\mathbb{P}$ on $\left(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}})\right)$ with $\mathbb{P}[\{\omega: \omega \in (a_i, b_i], 1 \leq i \leq n\}] = \mu_n\left((a_1, b_1] \times \dots \times (a_n, b_n]\right)$
- [2.1.22] Let S be a Borel subset of a complete separable metric space M and $\mathcal{S}$ be the collection of Borel subsets of S, then $(S, \mathcal{S})$ is a Borel space.
- [Dynkin's π-λ Theorem] Let P be a π-class and L be a λ-class with $P \subset L$. Then $\sigma(P) \subset L$.
- [6.1] Let X_1, X_2 be random variables with range $(S_i, \mathcal{S}_i)$. The following are equivalent:
  - X_1, X_2 are independent
  - $\mathbb{P}[X_1 \in B_1, X_2 \in B_2] = \mathbb{P}[X_1 \in B_1]\mathbb{P}[X_2 \in B_2]\ \forall B_i \in \mathcal{S}_i$
  - $\mathbb{P}[X_1 \in B_1, X_2 \in B_2] = \mathbb{P}[X_1 \in B_1]\mathbb{P}[X_2 \in B_2]\ \forall B_i \in \mathcal{A}_i$ where $\mathcal{A}_i$ is a λ-class and $\sigma(\mathcal{A}_i) = \mathcal{S}_i$
  - $\mathbb{E}[h_1(X_1)h_2(X_2)] = \mathbb{E}[h_1(X_1)]\mathbb{E}[h_2(X_2)]$ for all bounded measurable $h_i: S_i \to \mathbb{R}$

Convergence

Definitions

- [Uncorrelated] Let $\mathbb{E}[X_i^2] < \infty$. Then X_1, X_2 are uncorrelated if $\mathbb{E}[X_1X_2] = \mathbb{E}[X_1]\mathbb{E}[X_2]$.
- [Convergence in Probability] $X_n \xrightarrow{\mathbb{P}} X \Rightarrow \lim_{n\to\infty} \mathbb{P}[|X_n - X| > \epsilon] = 0\ \forall \epsilon > 0$
- [Convergence in L^p] $X_n \xrightarrow{L^p} X \Rightarrow \lim_{n\to\infty} \mathbb{E}[|X_n - X|^p] = 0$
- [Infinitely Often] Let $A_1, A_2, \dots$ be events. Then A_n i.o $= \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} A_j$
- [Ultimately] Let $A_1, A_2, \dots$ be events. Then A_n ult $= \bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} A_j$
  - In other words, only finite number of failures
- [lim sup] $\limsup a_n = \lim_{n\to\infty} \sup_{m\geq n} a_m$
- [lim sup] $\limsup A_n = \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} A_j = A_n \text{ i.o.}$
- [lim inf] $\liminf a_n = \lim_{n\to\infty} \inf_{m\geq n} a_m$
- [lim inf] $\liminf A_n = \bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} A_j = A_n \text{ ult.}$
- [Atom] A point $x \in \mathbb{R}$ is an <u>atom</u> of distribution function F if $F(x) - F(x^-) = \mathbb{P}[X = x] > 0$
- [Bernstein Polynomial] The <u>Bernstein polynomial</u> of degree n associated with f is: $f_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$
- [Truncation] Let X be a random variable. Then define the <u>truncated random variable at level M</u> to be $\bar{X} = X\mathbb{1}_{\{|X|\leq M\}} = \begin{cases} X, & |X| \leq M \\ 0, & |X| > M \end{cases}$

Theorems

- [6.6] Let $p > 0$, then $X_n \xrightarrow{L^p} X \Rightarrow X_n \xrightarrow{\mathbb{P}} X$
- [L^2 WLLN 6.8] Let $(X_n)_n$ be a pairwise uncorrelated sequence of random variables for which $\sup \mathbb{E}[X_n^2] \leq c$ for some constant c. Let $\mu_i = \mathbb{E}[X_i]$ and $S_n = \sum_{i=1}^{n} X_i$ and $\bar{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} \mu_i$. Then $\frac{S_n}{n} - \bar{\mu}_n \xrightarrow{L^2} 0$ and hence $\frac{S_n}{n} - \bar{\mu}_n \xrightarrow{\mathbb{P}} 0$
- [Weierstrass Approximation Theorem] Any continuous function $f: [0,1] \to \mathbb{R}$ may be approximated in the supremum norm by polynomials.
- [Bernstein's Theorem] Let $f: [0,1] \to \mathbb{R}$ be continuous. Define $f_n(x) = \sum_{m=1}^{n} \binom{n}{m} x^m (1-x)^{n-m} f\left(\frac{m}{n}\right)$. Then $\lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| = 0$.
- [7.2]
  - $\mathbb{P}[A_n \text{ i.o.}] \geq \limsup \mathbb{P}[A_n]$ i.e. $\mathbb{P}[\limsup A_n] \geq \limsup \mathbb{P}[A_n]$
  - $\mathbb{P}[A_n \text{ ult}] \leq \liminf \mathbb{P}[A_n]$ i.e. $\mathbb{P}[\liminf A_n] \leq \liminf \mathbb{P}[A_n]$
  - If $\mathbb{P}[A_n] = 1\ \forall n$, then $\mathbb{P}[\bigcap_{i=1}^{\infty} A_i] = 1$
- [Borel Cantelli I] Let $(A_i)_{i=1}^{\infty}$ be events s.t. $\sum_{i=1}^{\infty} \mathbb{P}[A_i] < \infty$. Then $\mathbb{P}[A_i \text{ i.o.}] = 0$.
- [Borel Cantelli II] Let $(A_i)_{i=1}^{\infty}$ be independent, then $\sum_{i=1}^{\infty} \mathbb{P}[A_i] = \infty$ implies $\mathbb{P}[A_i \text{ i.o.}] = 1$
- [7.5] Let $(Y_n)_n$ be a sequence of $\mathbb{R}$-valued random variables and $y \in (-\infty, \infty)$. If $\sum_{n=1}^{\infty} \mathbb{P}[Y_n \geq y + \epsilon] < \infty\ \forall \epsilon > 0$, then $\limsup Y_n \leq y$ a.s.
  - $\mathbb{P}[\{\omega: \limsup Y_n(\omega) \leq y\}] = 1$
  - $\sum_{n=1}^{\infty} \mathbb{P}[|Y_n| \geq \epsilon] < \infty\ \forall \epsilon > 0$, then $Y_n \to 0$ a.s.
- [SLLN 7.7] Let $(X_n)_n$ be i.i.d. Let X has the common law of X_i. Suppose $\mathbb{E}[X] = 0$ and $\mathbb{E}[X^4] < \infty$. Write $S_n = \sum_{i=1}^{n} X_i$. Then:
  - $\mathbb{E}[S_n^4] \leq 3n^2 \mathbb{E}[X^4]$
  - $\frac{S_n}{n} \to 0$ a.s.
  - If $\mathbb{E}[X] = \mu$, then $\frac{S_n}{n} \to \mu$ a.s.
- [8.1] Let F_n and F be distribution functions. Then, $\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |F_n(x) - F(x)| = 0$ if:

 - $\lim_{n\to\infty} F_n(x) = F(x)$ for $x \in \mathbb{Q}$
 - $\lim_{n\to\infty} F_n(x) = F(x)$ and $\lim_{n\to\infty} F_n(x^-) = F(x^-)$ for
- [Glivenko-Cantelli Theorem] Let $(X_n)_n$ be i.i.d with arbitrary distribution function G. Let $G_n(\omega, x)$ be the empirical distribution of $\left(X_1(\omega), \dots, X_n(\omega)\right)$ with $G_n(\omega, x) = \frac{1}{n}\sum_{i=1}^n \mathbb{1}\{X_i(\omega) \le x\}$. Then $\lim_{n\to\infty} G_n(\omega, x) = G(x)$
 - $\mathbb{P}\left[\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |G_n(\omega, x) - G(x)| = 0\right] = 1$
- [8.4] Let $x_n \ge 0$ and $0 < b_n$ with $\lim_{n\to\infty} b_n = \infty$. Then $\limsup \frac{\max\{x_1,\dots,x_n\}}{b_n} = \limsup \frac{x_n}{b_n}$
- [8.6] Let $(a_n)_n$ be a sequence. Then, if $\exists$ subsequence $\left(a_{n_k}\right)_k$ s.t. $\lim_{k\to\infty} \frac{a_{n_k}}{n_k} = 0$ and $\lim_{k\to\infty} \frac{\max_{n_k \le n' < n_{k+1}} \left|a_{n'} - a_{n_k}\right|}{n_k} = 0$, then $\lim_{n\to\infty} \frac{a_n}{n} = 0$
- [SLLN 8.7] Let $(X_n)_n$ be a sequence of random variables with $\mathbb{E}[X_i] = 0$ and $\sup \mathbb{E}[X_i^2]$ finite. Suppose $\mathbb{E}\left[X_i X_j\right] = 0$ for $i \ne j$. Let $S_n = \sum_{i=1}^n X_i$. Then $\frac{S_n}{n} \to 0$ a.s.
- [Durrett 2.3.2] $X_n \xrightarrow{\mathbb{P}} X$ if and only if for every subsequence $\left(X_{n_m}\right)_m$, there is a further subsequence $\left(X_{n_{m_k}}\right)_k$ that converges almost surely to X.
- [Durrett 2.3.4] If f is continuous and $X_n \xrightarrow{\mathbb{P}} X$ in probability, then $f(X_n) \xrightarrow{\mathbb{P}} f(X)$. If f is bounded, then $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]$
- [2.3.5] Let $X_1, X_2, \dots$ i.i.d. with $\mathbb{E}[X_i] = \mu$ and $\mathbb{E}[X_i^4] < \infty$. Let $S_n = X_1 + \cdots + X_n$, then $\frac{S_n}{n} \to \mu$ a.s.
- [2.3.8] Let $X_1, X_2, \dots$ i.i.d. with $\mathbb{E}[|X_i|] = \infty$, then $\mathbb{P}[|X_n| \ge n \text{ i.o.}] = 1$. Let $S_n = X_1 + \cdots + X_n$, then $\mathbb{P}\left[\lim_{n\to\infty} \frac{S_n}{n} \in (-\infty, \infty)\right] = 0$
- [2.3.9] Let $A_1, A_2, \dots$ be pairwise independent and $\sum_{n=1}^\infty \mathbb{P}[A_n] = \infty$, then $\frac{\sum_{i=1}^n \mathbb{1}_{A_m}}{\sum_{i=1}^n \mathbb{P}[A_m]} \to 1$ a.s.
- [HW 5 P3] The following are equivalent:
 - $X_n \xrightarrow{\mathbb{P}} X$
 - $\exists (\epsilon_n)_n$ with $\lim_{n\to\infty} \epsilon_n = 0$ s.t. $\mathbb{P}[|X_n - X| > \epsilon_n] \le \epsilon_n$
 - $\lim_{n\to\infty} \mathbb{E}[\min(|X_n - X|, 1)] = 0$

Triangular Arrays

- [2.2.6] Let $(X_n)_n$ be any sequence of random variables and $S_n = X_1 + \cdots + X_n$. Denote $\mu_n = \mathbb{E}[S_n]$ and $\sigma_n^2 = \mathrm{Var}[S_n]$. If $\lim_{n\to\infty} \frac{\sigma_n^2}{b_n^2} = 0$, then $\frac{S_n - \mu_n}{b_n} \xrightarrow{\mathbb{P}} 0$
- [2.2.11] For each n, let $\left(X_{n,k}\right)_{1\le k\le n}$ be independent. Let $(b_n)_n$ be s.t. $b_n > 0$ and $\lim_{n\to\infty} b_n = \infty$ and $\bar{X}_{n,k} = X_{n,k}\mathbb{1}_{\{|X_{n,k}| \le b_n\}}$. Define $S_n = \sum_{i=1}^n X_{n,i}$ and $a_n = \sum_{k=1}^n \mathbb{E}\left[\bar{X}_{n,k}\right]$. Suppose that as $n \to \infty$, the following two conditions hold:
 - $\lim_{n\to\infty} \sum_{k=1}^n \mathbb{P}\left[\left|X_{n,k}\right| > b_n\right] = 0$
 - $\lim_{n\to\infty} \frac{1}{b_n^2} \sum_{k=1}^n \mathbb{E}\left[\bar{X}_{n,k}^2\right] = 0$

 Then $\frac{S_n - a_n}{b_n} \xrightarrow{\mathbb{P}} 0$
- [2.2.13] Let $Y \ge 0$ and $p > 0$. Then $\mathbb{E}[Y^p] = \int_0^\infty p y^{p-1} \mathbb{P}[Y > y]\, \mathrm{d}y$
- [2.2.12] Let $X_1, X_2, \dots$ be i.i.d. with $\lim_{x\to\infty} x\mathbb{P}[|X_i| > x] = 0$ as $x \to \infty$. Let $S_n = X_1 + \cdots + X_n$ and $\mu_n = \mathbb{E}\left[X \mathbb{1}_{\{|X| \le n\}}\right]$. Then $\frac{S_n}{n} - \mu_n \xrightarrow{\mathbb{P}} 0$

- [2.2.14 WLLN] Let $X_1, X_2, \ldots$ i.i.d. with $\mathbb{E}[|X_i|] < \infty$. Let $S_n = X_1 + \cdots + X_n$ and $\mu = \mathbb{E}[X_i]$. Then $\frac{S_n}{n} \xrightarrow{\mathbb{P}} \mu$

Convergence Theorems

- Let $X_1, X_2, \ldots$ be uncorrelated RVs with $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}[X_i] \le C < \infty$. If $S_n = X_1 + \cdots + X_n$, then $\frac{S_n}{n} \xrightarrow{L^2} \mu$ and $\frac{S_n}{n} \xrightarrow{\mathbb{P}} \mu$
- [2.2.6] Let $\mu_n = \mathbb{E}[S_n]$ and $\sigma_n^2 = \mathrm{Var}[S_n]$. If $\lim_{n\to\infty} \frac{\sigma_n^2}{b_n^2} = 0$, then $\frac{S_n - \mu_n}{b_n} \xrightarrow{\mathbb{P}} 0$
- [2.2.11] Let $X_{n,k}$ be independent. Let $b_n > 0$ with $\lim_{n\to\infty} b_n = \infty$ and $\bar{X}_{n,k} = X_{n,k}\mathbb{1}\{|X_{n,k}| \le b_n\}$. Define $S_n = \sum_{i=1}^n X_{n,i}$ and $a_n = \sum_{i=1}^n \mathbb{E}[\bar{X}_{n,i}]$. Suppose that:
  - $\lim_{n\to\infty} \sum_{k=1}^n \mathbb{P}[|X_{n,k}| > b_n] = 0$
  - $\lim_{n\to\infty} \frac{1}{b_n^2} \sum_{k=1}^n \mathbb{E}[\bar{X}_{n,k}^2] = 0$

  Then $\frac{S_n - a_n}{b_n} \xrightarrow{\mathbb{P}} 0$

Convergence of Random Series

- [Tail σ-Field] Let $\mathcal{F}_n = \sigma(X_n, X_{n+1}, \ldots)$ and $\mathcal{T} = \bigcap_{t=1}^\infty \mathcal{F}_t$. Then $\mathcal{T}$ is the <u>tail σ-field</u>.
  - $A \in \mathcal{T}$ if and only if changing a finite number of random variables doesn't affect the knowledge of occurrence of A
- [Finite Permutation] A <u>finite permutation</u> of $\mathbb{N}$ is a map $\pi: \mathbb{N} \to \mathbb{N}$ s.t. $\pi(i) \ne i$ for only finitely many i
- [Permutable] An event A is <u>permutable</u> if $\pi^{-1}(A) \equiv \{\omega: \pi\omega \in A\} = A$ for any finite permutation π i.e. occurrence of A is not affected by rearranging finitely many of the random variables
- [Exchangeable σ-Field] ξ is the collection of permutable events.
  - ξ is a σ-field
- [Properties]
  - Let $A \in \sigma(X_1, \ldots, X_k)$ and $B \in \sigma(X_{k+1}, \ldots)$, then A, B are independent
- [Kolmogorov 0-1 Law] Let $X_1, X_2, \ldots$ be independent and $A \in \mathcal{T}$, then $\mathbb{P}[A] = 0$ or 1
- [Hewitt-Savage 0-1 Law] Let $X_1, X_2, \ldots$ be i.i.d. and $A \in \xi$, then $\mathbb{P}[A] = 0$ or 1
- [9.1 Kolmogorov's Maximal Inequality] Suppose $X_1, \ldots, X_n$ are independent with $\mathbb{E}[X_i] = 0$ and $\mathrm{Var}[X_i] < \infty$. If $S_n = X_1 + \cdots + X_n$, then $\mathbb{P}\left[\max_{1\le k\le n} |S_k| \ge x\right] \le \frac{1}{x^2}\mathrm{Var}[S_n]$
- [9.2 / 2.5.6] Suppose $X_1, X_2, \ldots$ are independent and $\mathbb{E}[X_n] = 0$. If $\sum_{n=1}^\infty \mathrm{Var}[X_n] < \infty$, then $\sum_{i=1}^\infty X_i$ converges a.s.
  - i.e. $\mathbb{P}[\{\omega: \sum_{i=1}^\infty X_i(\omega) \text{ converges}\}] = 1$
- [Kolmogorov's Three-Series Theorem] Let $X_1, X_2, \ldots$ be independent. Let $A > 0$ and $Y_i = X_i \mathbb{1}\{|X_i| \le A\}$. $\sum_{i=1}^\infty X_i$ converges a.s. if and only if:
  - $\sum_{i=1}^\infty \mathbb{P}[|X_i| > A] < \infty$
  - $\sum_{i=1}^\infty \mathbb{E}[Y_i]$ converges
  - $\sum_{n=1}^\infty \mathrm{Var}[Y_i] < \infty$,
- [9.3 Kronecker's Lemma] Let $\lim_{n\to\infty} a_n \uparrow \infty$ and $\sum_{i=1}^\infty \frac{x_i}{a_i}$ converges, then $\lim_{n\to\infty} \frac{1}{a_n}\sum_{i=1}^n x_i = 0$
  - [9.4] Let $(X_n)_n$ independent with $\mathbb{E}[X_n] = 0$ and $\mathrm{Var}[X_n] < \infty$ and $S_n = \sum_{i=1}^n X_i$. Suppose $(a_n)_n$ is a sequence s.t. $a_n > 0$ and $\lim_{n\to\infty} a_n \uparrow \infty$ s.t. $\sum_{i=1}^n \frac{\mathbb{E}[X_n^2]}{a_n^2} < \infty$, then $\lim_{n\to\infty} \frac{S_n}{a_n} = 0$ a.s.
- [9.5 / 2.5.10 SLLN] Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbb{E}[|X_i|] < \infty$. Let $\mathbb{E}[X_i] = \mu$ and $S_n = \sum_{i=1}^n X_i$. Then $\frac{S_n}{n} \to \mu$ a.s.

Renewal Theory

- [Set-up] Let $X_1, X_2, \ldots$ be i.i.d. with $0 < X_i < \infty$; consider X_i as the ith waiting time. Let $T_n = X_1 + \cdots + X_n$ be the nth occurrence of event. Call $(T_n)_n$ renewals. $N_t = \inf\{k : T_k > t\}$ is the number of renewals in $[0, t]$ counting the one at 0.
  - N_t is a stopping time i.e. $\{N_t = k\}$ is measurable w.r.t. $\mathcal{F}_k$
  - $U(t) = \mathbb{E}[N_t]$ is expected number of renewals at time t
- [2.4.7] Let $\mathbb{E}[X_1] = \mu \leq \infty$, then $\frac{N_t}{t} \to \frac{1}{\mu}$ a.s. as $t \to \infty$.
- [2.6.2] Let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{E}[|X_i|] < \infty$. If N is a stopping time with $\mathbb{E}[N] < \infty$, then $\mathbb{E}[S_N] = \mathbb{E}[X_i]\mathbb{E}[N]$
- [Renewal Measure] $U(A) = \sum_{i=0}^{\infty} \mathbb{P}[T_i \in A]$
  - $U(t) = U([0, t])$

Wald's Identities

- [4.8] Assume $(X_n)_n$ i.i.d. with finite first moments and T stopping time s.t. $\mathbb{E}[T] < \infty$. Then $\mathbb{E}[S_T] = \mathbb{E}[X]\mathbb{E}[T]$
- [4.10] Assume $(X_n)_n$ i.i.d. with $\mathbb{E}[X] = 0$ and $\mathbb{E}[X^2] < \infty$. If T is a stopping time s.t. $\mathbb{E}[T] < \infty$, then $\mathbb{E}[S_T^2] = \sigma^2 \mathbb{E}[T]$

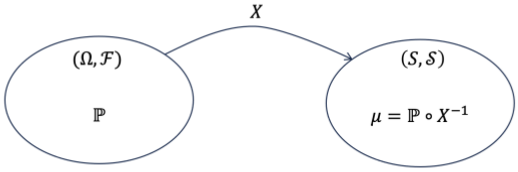
Theory of Large Deviations

- [Set-up] Let $X_1, X_2, \ldots$ be i.i.d. and $S_n = X_1 + \cdots + X_n$. Define $\pi_n = \mathbb{P}[S_n \geq na]$, $\gamma(a) = \lim\limits_{n \to \infty} \frac{\pi_n}{n}$.
- [Observation] $\pi_{m+n} \geq \pi_m \pi_n$
- [2.7.1] Let $\gamma_{m+n} \geq \gamma_m + \gamma_n$, then $\lim\limits_{n \to \infty} \frac{\gamma}{n} = \sup\limits_m \frac{\gamma_m}{m}$
- [12.1] As $n \to \infty$, $\frac{1}{n} \log \mathbb{P}\left[\frac{S_n}{n} \geq a\right] \to \inf\limits_{\theta \in \mathbb{R}} \{\log \phi(\theta) - a\theta\}$
- [12.2] $\phi'(0_+) = \mu$, $\phi'(\theta) = \mathbb{E}\left[Xe^{\theta X}\right]$, $\phi''(\theta) = \mathbb{E}\left[X^2 e^{\theta X}\right]$
- [Tilted Random Variable] Fix $\theta \in (0, \infty)$. Define $\hat{X}$ by $\mathbb{P}\left[\hat{X} = x\right] = \frac{e^{\theta x}\mathbb{P}[X = x]}{\phi(\theta)}$
- [12.3] $\mathbb{E}\left[\hat{X}\right] = \frac{\mathrm{d}}{\mathrm{d}\theta} \log \phi(\theta)$, $\mathrm{Var}\left[\hat{X}\right] = \frac{\mathrm{d}^2}{\mathrm{d}\theta^2} \log \phi(\theta)$

Conditional

Definitions

- [Distribution] Let $X: (\Omega, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{S})$ be a random variable. X has distribution or law μ if $\mu(B) = \mathbb{P}[\{\omega \in \Omega: X(\omega) \in B\}]\ \forall B \in \mathcal{S}$
  - $\mu = \mathbb{P} \circ X^{-1}$ is the pushforward measure
    - Pushforward because it brings the measure $\mathbb{P}$ in $(\Omega, \mathcal{F})$ to measure in $(S, \mathcal{S})$



- [Borel Space] Let $(S, \mathcal{S})$ be a measurable space. Then $(S, \mathcal{S})$ is a Borel space if exists a Borel-measurable $A \subset \mathbb{R}$ and a bijection $\phi: A \to S$ s.t. ϕ and ϕ^{-1} are measurable.
  - $(S, \mathcal{S})$ can be identified as a subspace of $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$
- [Product Measurable Space] Let $(S_1, \mathcal{S}_1)$ and $(S_2, \mathcal{S}_2)$ be measurable spaces. Then the product measurable space is $(S_1 \times S_2, \mathcal{S}_1 \otimes \mathcal{S}_2)$:
  - $\mathcal{S}_1 \otimes \mathcal{S}_2 = \sigma(A \times B: A \in \mathcal{S}_1, B \in \mathcal{S}_2)$
- [Kernel] A kernel Q from S_1 to S_2 is a map $Q: S_1 \times \mathcal{S}_2 \to [0,1]$ s.t.
  - For fixed $s \in S_1$, the map $Q(s, \cdot)$ is a probability measure on $\mathcal{S}_2$
  - For fixed $B \in \mathcal{S}_2$, the map $Q(\cdot, B)$ is measurable function from S_1 to $\mathbb{R}$
  - Think of $Q(s_1, B)$ as $\mathbb{P}[Y \in B | X = s_1]$
- [Product Measure] Let μ_1 and μ_2 be probability measures on $(S_1, \mathcal{S}_1)$ and $(S_2, \mathcal{S}_2)$. Then, exists product measure $\mu = \mu_1 \otimes \mu_2$ on $S_1 \times S_2$ s.t.:
  - $\mu(A \times B) = \mu_1(A) \times \mu_2(B)$ for $A \in \mathcal{S}_1, B \in \mathcal{S}_2$
  - If $D \in \mathcal{S}_1 \otimes \mathcal{S}_2$, then $\mu(D) = \int \mu_2(D_{s_1}) \mathrm{d}\mu_1(s_1)$
  - Let $h: S_1 \times S_2 \to \mathbb{R}$ be measurable, $h \geq 0$, $|h|$ is μ-integrable, then $\int_{S_1} \int_{S_1} h(s_1, s_2)\, \mathrm{d}\mu_2(s_2)\, \mathrm{d}\mu_1(s_1)$

Borel Space Properties

- [14.6] Every complete separable metric space is a Borel space.
- [14.7] Given probability measure ν on Borel space $(S, \mathcal{S})$, there exists a measurable $h: [0,1] \to S$ s.t. $h(U)$ has distribution ν.

Propositions

- [Basic Relations] Given a probability measure μ on $S_1 \times S_2$, a probability measure μ_1 on S_1, and a kernel Q from S_1 to S_2. The following are equivalent:
  - $\mu(A \times B) = \int_A Q(s_1, B)\, \mathrm{d}\mu_1(s_1)\ \forall A \in \mathcal{S}_1, B \in \mathcal{S}_2$
  - $\mu(D) = \int_{S_1} Q(s_1, D_{s_1})\, \mathrm{d}\mu_1(s_1)\ \forall D \in \mathcal{S}_1 \otimes \mathcal{S}_2$
    - $D_{s_1} = \{s_2: (s_1, s_2) \in D\}$ is the slice of D across s_1
  - $\int_{S_1 \times S_2} h(s_1, s_2)\, \mathrm{d}\mu(s) = \int_{S_1} \int_{S_2} h(s_1, s_2) Q(s_1, \mathrm{d}s_2)\ \mathrm{d}\mu_1(s_1)$
    - Note: $Q(s_1, \mathrm{d}s_2)$ is a probability measure over $\mathcal{S}_2$
- [13.4] Let $D \in \mathcal{S}_1 \otimes \mathcal{S}_2$, then:
  - $D_{s_1} \in \mathcal{S}_2\ \forall s_1 \in S_1$ i.e. each slice is measurable
  - The map $s_1 \mapsto Q(s_1, D_{s_1})$ is measurable
- [13.5] Let μ_1 be a probability measure on S_1 and Q a kernel from S_1 to S_2. Then, μ is a probability measure on $S_1 \times S_2$ via $\mu(D) = \int_{S_1} Q(s_1, D_{s_1}) \mathrm{d}\mu_1(s_1)\ \forall D \in \mathcal{S}_1 \otimes \mathcal{S}_2$
  - Can construct joint measure from the marginal measure and kernel
- [13.6] Let μ be a probability measure on $S_1 \times S_2$, then define the marginal measure $\mu_1(A) = \mu(A \times S_2)$. If S_2 is a Borel space, then $\exists$ kernel Q from S_1 to S_2 satisfying the basic relations.
  - Can get kernel from joint probability measure, given that S_2 is a Borel space

- [15.1] Let μ be a probability measure on $S \times \mathbb{R}$ and $X: \Omega \to S$ be a random variable. Let $U: \Omega \to [0,1]$ denote a random variable uniformly distributed on $[0,1]$, with U, X independent. Then, exists $f: S \times [0,1] \to \mathbb{R}$ s.t. by defining $Y = f(X, U)$, (X, Y) has distribution μ

Definitions (Conditional Expectation)

- [Conditional Expectation] Let $X: (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ with $\mathbb{E}[|X|] < \infty$ and $\mathcal{G} \subset \mathcal{F}$. Then $\mathbb{E}[X|\mathcal{G}]$ is a random variable with $\mathbb{E}[X|\mathcal{G}]: (\Omega, \mathcal{G}, \mathbb{P}) \to \mathbb{R}$ satisfying:
  - $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$-measurable
    - $\forall B \in \mathcal{B}(\mathbb{R})$, $\{\omega \in \Omega: \mathbb{E}[X|\mathcal{G}](\omega) \in B\} \in \mathcal{G}$
  - $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]\mathbb{1}_G] = \mathbb{E}[X\mathbb{1}_G]\ \forall G \in \mathcal{G}$
- [Conditional Variance] Let $\mathrm{Var}[X|\mathcal{G}] = \mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])^2|\mathcal{G}]$
- [Conditionally Independent] Let X, Y be random variables. Then they are conditionally independent given $\mathcal{G}$ if $\forall$ bounded h_1, h_2, $\mathbb{E}[h_1(X)h_2(Y)|\mathcal{G}] = \mathbb{E}[h_1(X)|\mathcal{G}]\mathbb{E}[h_2(Y)|\mathcal{G}]$

Propositions (Conditional Expectation)

- [15.6] Let V be a bounded $\mathcal{G}$-measurable function. Then $\mathbb{E}\big[V\mathbb{E}[X|\mathcal{G}]\big] = \mathbb{E}[VX]$
- [15.6] Let Z be $\mathcal{G}$-measurable and $\mathcal{A} \subset \mathcal{G}$ be a π-class s.t. $\mathcal{G} = \sigma(\mathcal{A})$. Then $Z = \mathbb{E}[X|\mathcal{G}]$ if $\mathbb{E}[Z\mathbb{1}_A] = \mathbb{E}[X\mathbb{1}_A]\ \forall A \in \mathcal{A}$

Martingale

Definitions

• [Filtration] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A <u>filtration</u> is a sequence of sub-σ-fields s.t. $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}$. Also, define $\mathcal{F}_\infty = \bigcup_{i=1}^{\infty} \mathcal{F}_i$.
  ○ $\mathcal{F}_n$ is the information known at time n
  ○ [Natural Filtration] Let $(X_n)_n$ be a martingale. Then $(\mathcal{F}_n)_n$ where $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$ is the natural filtration for $(X_n)_n$
  ○ [Final σ-Field] $\mathcal{F}_\infty$
• [Adapted] A sequence of random variables $(X_n)_n$ is <u>adapted</u> to $(\mathcal{F}_n)_n$ i.e. $\sigma(X_n) \in \mathcal{F}_n \ \forall n$
• [Martingale] A $\mathbb{R}$-valued process $(X_n)_n$ is a <u>martingale</u> if:
  ○ $\mathbb{E}[\lvert X_n\rvert] < \infty \ \forall n$
  ○ $(X_n)_n$ is adapted to $(\mathcal{F}_n)_n$
  ○ $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$ for $0 \leq n < \infty$
• [Submartingale] $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \geq X_n$
  ○ $\mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n] \geq 0$
• [Supermartingale] $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \leq X_n$
  ○ $\mathbb{E}[X_{n+1} - X_n \mid \mathcal{F}_n] \leq 0$
• [Martingale Difference Sequence] Let $\Delta_n^X := X_n - X_{n-1}$. Then $(\Delta_n^X)_n$ is a martingale difference sequence.
  ○ $(X_n)_n$ is a martingale if and only if:
    ▪ $\Delta_n^X \in \mathcal{F}_n$
    ▪ $\mathbb{E}[\lvert\Delta_n^X\rvert] < \infty$
    ▪ $\mathbb{E}[\Delta_{n+1}^X \mid \mathcal{F}_n] = 0$

Theorems

• [18.3] Let $(X_n)_n$ be a random process adapted to $(\mathcal{F}_n)_n$, Let $\phi: \mathbb{R} \to \mathbb{R}$ be convex with $\mathbb{E}[\lvert\phi(X_n)\rvert] < \infty$
  ○ If $(X_n)_n$ is a martingale, then $\big(\phi(X_n)\big)_n$ is a submartingale.
  ○ If $(X_n)_n$ is a submartingale and ϕ increasing, then $\big(\phi(X_n)\big)_n$ is also a submartingale.
• [20.1 / 4.2.10 Upcrossing Inequality] Let $(X_n)_n$ be a martingale and $a < b$ be two thresholds. Let U_n be the number of up-crossings from a to b completed by time n. Then $(b-a)\mathbb{E}[U_n] \leq \mathbb{E}[(X_n - a)^+] - \mathbb{E}[(X_0 - a)^+] \leq \mathbb{E}[X_n^+] + \lvert a\rvert$.

Martingale Convergence Theorems
• [Martingale Convergence Theorem] Let $(X_n)_n$ be a submartingale with $\sup_n \mathbb{E}[X_n^+] < \infty$, then $X_n \to X_\infty$ a.s. for some X_∞ with $\mathbb{E}[\lvert X_\infty\rvert] < \infty$.   ○ If $(X_n)_n$ is a supermartingale with $X_n \geq 0$ a.s., then $X_n \to X_\infty$ a.s. for some X_∞ with $0 \leq \mathbb{E}[X_\infty] \leq \mathbb{E}[X_0]$ • [L^p Convergence Theorem] Let $(X_n)_n$ be a martingale with $\sup \mathbb{E}[\lvert X_n\rvert^p] < \infty$ where $p > 1$. Then $(X_n)_n \to X$ a.s. and in L^p.
Other Results

• [Doob Decomposition] Let $(X_n)_n$ be a sequence of random variables that is adapted to $(\mathcal{F}_n)_n$ and $\mathbb{E}[\lvert X_n\rvert] < \infty$. Then, we construct $(Y_n)_n$ and $(Z_n)_n$ s.t.
  ○ $Y_0 = X_0$
  ○ $\Delta_n^Y = \Delta_n^X - \mathbb{E}[\Delta_n^X \mid \mathcal{F}_{n-1}]$
  ○ $Z_0 = 0$
  ○ $\Delta_n^Z = \mathbb{E}[\Delta_n^X \mid \mathcal{F}_{n-1}]$
In other words, Z_n increments by the predicted change of X_n from time $t = n-1$ to $t = n$. Y_n increments by the difference between the actual change and the predicted change.
  ○ $X_n = Y_n + Z_n$
  ○ $(Y_n)_n$ martingale

  - $Z_n \in \mathcal{F}_{n-1}$ i.e. $(Z_n)_n$ predictable and $\mathbb{E}[|Z_n|] < \infty$
    - $(Z_n)_n$ handles the average drift in X_n
- [21.1] Let $(X_n)_n$ be a martingale s.t. $\exists K > 0$ s.t. $|X_n - X_{n-1}| \leq K$ $\forall n$. Denote $C = \left\{\omega \in \Omega: \lim_{n\to\infty} X_n(\omega) < \infty\right\}$ and $D = \left\{\omega \in \Omega: \limsup_n X_n(\omega) = +\infty, \liminf_n X_n(\omega) = -\infty\right\}$. Then $\mathbb{P}[C \cup D] = 1$.
  - i.e. for martingales with bounded differences, they either converge or oscillate infinitely
  - i.e. cannot get a behaviour that is forever finitely sinusoidal without converging
- [21.2 Conditional Borel Cantelli] Let $(A_n)_n$ be a sequence of events adapted to filtration $(\mathcal{F}_n)_n$. Define $B = \{A_i \text{ i.o.}\} = \bigcap_n \bigcup_{m\geq n} A_m$. Then:
  - $\{A_i \text{ i.o.}\} = \{\sum_{i=1}^{\infty} \mathbb{P}[A_i|\mathcal{F}_{n-1}] = \infty\}$ a.s.
  - $\lim_{n\to\infty} \mathbb{P}[\bigcup_{m\geq n} A_m \,|\mathcal{F}_{n-1}] = \mathbb{1}_{\{A_i \text{ i.o.}\}}$ a.s.
    - Borel Cantelli extends to processes via conditional probability

Problem Solving

- Given $(Y_n)_n$, apply some transformation h s.t. $\left(h(Y_n)\right)_n$ is a martingale. Take $\mathcal{F}_n = \sigma(Y_0, \dots, Y_n)$. Then $\left(h(Y_n)\right)_n$ is adapted to $(\mathcal{F}_n)_n$

Martingale (Inequalities)

Maximal Inequalities

- [19.3] Let $(X_n)_n$ be a supermartingale, with $X_n \geq 0$ a.s. Let $X_N^* = \max_{0 \leq i \leq N} X_i$ and $X^* = \sup_{n \in \mathbb{N}} X_n$. Then:
  - $\mathbb{P}[X_N^* \geq \lambda] \leq \frac{\mathbb{E}[X_0]}{\lambda} \forall \lambda > 0$
  - $\mathbb{P}[X^* \geq \lambda] \leq \frac{\mathbb{E}[X_0]}{\lambda} \forall \lambda > 0$.
  - Gets a handle on probability of reaching above a certain level for martingale
- [Doob L^1 Maximal Inequality] Let $(X_n)_n$ be a submartingale. Let $N \in \mathbb{N}$ and $\lambda > 0$, then $\lambda \mathbb{P}[X_N^* \geq \lambda] \leq \mathbb{E}\left[X_N \mathbb{1}_{X_N^* \geq \lambda}\right] \leq \mathbb{E}[X_N^+] = \mathbb{E}[\max(X_N, 0)]$
- [19.5] Let $(X_n)_n$ be a martingale. Then $\mathbb{P}\left[\max_{0 \leq n \leq N} |X_n| \geq \lambda\right] \leq \frac{\mathbb{E}[|X_N|]}{\lambda}$
- [19.5] Let $(X_n)_n$ be a martingale. Then $\mathbb{P}\left[\max_{0 \leq n \leq N} |X_n| \geq \lambda\right] \leq \frac{\mathbb{E}[X_N^2]}{\lambda^2}$
- [Doob L^2 Maximal Inequality] Let $(X_n)_n$ be a martingale. Let $N \in \mathbb{N}$, $\mathbb{E}[\max(0, X_N^*)^2] \leq 4\mathbb{E}[(X_N^+)^2]$
- [L^p Maximal Inequality 4.4.4] Let $(X_n)_n$ be a submartingale and let $1 < p < \infty$. Denote $X_n^* = \max_{0 \leq i \leq n} X_i^+$. Then $\mathbb{E}[(X_n^*)^p] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[(X_n^+)^p]$.
  - Let $(Y_n)_n$ be a martingale, then $\mathbb{E}[|Y_n^*|^p] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|Y_n|^p]$
- [Azuma's Inequality] Let $S_n = \sum_{i=1}^n X_i$ be a martingale with $|X_i| \leq 1$ a.s. Then $\mathbb{P}\left[S_n \geq \lambda\sqrt{n}\right] \leq e^{-\frac{\lambda^2}{2}} \forall \lambda > 0$.
  - $\mathbb{P}\left[|S_n| \geq \lambda\sqrt{n}\right] \leq 2e^{-\frac{\lambda^2}{2}}$ for $\lambda > 0$

Martingale (Stopping Time)

Definitions

• [Predictable] Let $(\mathcal{F}_n)_n$ be a filtration. Then, $(H_n)_n$ is <u>predictable</u> if $H_n \in \mathcal{F}_{n-1}$.
  ○ i.e. value of H_n can be predicted with certainty from information at time $n-1$
• [Martingale Transform] $(H \cdot X)_n = \sum_{i=1}^{n} H_i(X_i - X_{i-1})$
  ○ $(H \cdot X)_{n+1} = (H \cdot X)_n + H_{n+1}(X_{n+1} - X_n)$
• [Stopping Time] Let $T: \Omega \to \mathbb{N} \cup \{\infty\}$ be a random variable. Then T is a <u>stopping time</u> if $\{T = n\} \in \mathcal{F}_n \; \forall n \in \mathbb{N} \cup \{\infty\}$.
  ○ Equivalently, $\{T \leq n\} \in \mathcal{F}_n$
• [Pre-T σ-Field] Let T be a stopping time. Then the <u>pre-T σ-field</u> is the σ-field $\mathcal{F}_T = \{A \in \mathcal{F}: A \cap \{T = n\} \in \mathcal{F}_n\}$
  ○ Equivalently, $\mathcal{F}_T = \{A \in \mathcal{F}: A \cap \{T \leq n\} \in \mathcal{F}_n\}$
  ○ Intuitively, $\mathcal{F}_T$ are the events that the observer knows about the random process until its stopping time (inclusive)
• [Stopped Process] Let $(X_n)_n$ be a random process adapted to $(\mathcal{F}_n)_n$ and T be a stopping time. Then $\left(X_{\min(n,T)}\right)_n$ is the <u>stopped process</u> and is also adapted to $(\mathcal{F}_n)_n$.

Properties

• [Pre-T σ-Field Properties]
  ○ Let $(X_n)_n$ be adapted to $(\mathcal{F}_n)_n$ and T be a stopping time with $T < \infty$. Then X_T is $\mathcal{F}_T$-measurable.
  ○ Let $T_1 \leq T_2$ be stopping times. Then $\mathcal{F}_{T_1} \subset \mathcal{F}_{T_2}$.
  ○ Let T_1, T_2 be stopping times. Then $\{T_1 = T_2\} \in \mathcal{F}_{T_1} \cap \mathcal{F}_{T_2}$
    ▪ Intuitively, when one of the processes stop, we would know if $\{T_1 = T_2\}$ occurred.
    ▪ If $A \subset \{T_1 = T_2\}$, then $A \in \mathcal{F}_{T_1} \Leftrightarrow A \in \mathcal{F}_{T_2}$
  ○ [19.1] Let $(X_n)_n$ be a submartingale and $0 \leq T_1 \leq T_2$ be bounded stopping times. Then $\mathbb{E}\left[X_{T_2} \mid \mathcal{F}_{T_1}\right] \geq X_{T_1}$

Theorems

• [18.7 / 4.2.8] Let $(X_n)_n$ be a random process adapted to $(\mathcal{F}_n)_n$ and $(H_n)_n$ predictable and $H_i \leq c_i \; \forall i$. Consider $Y = H \cdot X$ i.e. $Y_n = X_0 + \sum_{i=1}^{n} H_i(X_i - X_{i-1})$ i.e. $\Delta_n^Y = H_n \Delta_n^X$.
  ○ If $(X_n)_n$ is a martingale, then $(Y_n)_n$ is a martingale.
  ○ If $(X_n)_n$ is a submartingale and $H_n \geq 0$, then $(Y_n)_n$ is also a submartingale.
  ○ If $(X_n)_n$ is a supermartingale and $H_n \geq 0$, then $(Y_n)_n$ is also a supermartingale.
• [18.8 / 4.2.9] Let $(X_n)_n$ be a (sub/super) martingale and T be a stopping time. Then $\left(X_{\min(T,n)}\right)_n$ is a (sub/super) martingale.
  ○ i.e. "a stopped martingale is a martingale"
  ○ $H_n = \mathbb{1}\{0 \leq n \leq T\}$
• [Optional Sampling Theorem] Let $(X_n)_n$ be a (sub)martingale. Let $0 \leq T_0 \leq T_1 \leq \cdots$ be stopping times, with $T_i \leq t_i$ for t_i constant. Then $\left(X_{T_n}\right)_n$ is a (sub)martingale with respect to $\left(\mathcal{F}_{T_n}\right)_n$
  ○ Example: $T_i = \min(i, T)$ with $T \leq c$
  ○ Basically, this theorem details conditions in which the expected value of a martingale at stopping time is equal to its initial expected value; this one being bounded stopping time.
• [4.4.1] Let $(X_n)_n$ be a submartingale and T be a stopping time s.t. $\mathbb{P}[T \leq k] = 1$. Then $\mathbb{E}[X_0] \leq \mathbb{E}[X_T] \leq \mathbb{E}[X_k]$

Martingale (L^1 Theory)

Definitions

- [Uniformly Integrable] A family $(X_\alpha)_{\alpha\in\mathcal{A}}$ is <u>uniformly integrable</u> if $\lim_{M\to\infty} \sup_{\alpha\in\mathcal{A}} \mathbb{E}[|X_\alpha|\mathbb{1}_{|X_\alpha|\geq M}] = 0$
  - i.e. $\forall\epsilon > 0$, $\exists M \in [0,\infty)$ s.t. $\mathbb{E}\big[|X_\alpha|\mathbb{1}_{|X_\alpha|\geq M}\big] < \epsilon\ \forall\alpha \in \mathcal{A}$
  - Equivalently:
    - $\exists M < \infty$ s.t. $\mathbb{E}[|X_\alpha|] < M$
    - $\forall\epsilon > 0, \exists\delta > 0$ s.t. $\forall$ measurable A and $\forall\alpha \in \mathcal{A}$, $\mathbb{P}[A] < \delta \Rightarrow \mathbb{E}[|X_\alpha|\mathbb{1}_A] < \epsilon$

Properties

- [Uniformly Integrable Properties]
  - Let $(Y_\alpha)_\alpha$ be s.t. $\sup_\alpha \mathbb{E}[|Y_\alpha|^q] < \infty$ for some $q > 1$, then $(Y_\alpha)_\alpha$ uniform integrable, and therefore $\sup_\alpha \mathbb{E}[|Y_\alpha|] < \infty$
  - Let $(Y_n)_n$ be a random process s.t. $Y_n \to Y_\infty$ a.s. and $(Y_n)_n$ uniform integrable. Then $\mathbb{E}[|Y_\infty|] < \infty$ and $\lim_{n\to\infty} \mathbb{E}[|Y_n - Y_\infty|] = 0$ (i.e. $Y_n \xrightarrow{L_1} Y_\infty$)
  - Let $(Y_n)_n$ be a random process s.t. $Y_n \xrightarrow{L_1} Y_\infty$, then $(Y_n)_n$ is uniform integrable.
- Let Y be a random variable s.t. $\mathbb{E}[|Y|] < \infty$, then $(\mathbb{E}[Y|\mathcal{G}])_\mathcal{G}$ where $\mathcal{G} \subset \mathcal{F}$ is uniform integrable.
- [4.6.5] Let $(X_n)_n, X$ be integrable random variables s.t. $(X_n)_n \to X$ in L^1. Then $\lim_{n\to\infty} \mathbb{E}[X_n\mathbb{1}_A] = \mathbb{E}[X\mathbb{1}_A]$.
- [4.6.6] Let $(X_n)_n$ be a martingale. If $(X_n)_n \to X$ in L^1, then $X_n = \mathbb{E}[X|\mathcal{F}_n]$.
  - Used in the proof of 4.6.7

Theorems

- [4.6.1] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X \in L^1$. Then $\{\mathbb{E}[X|\mathcal{F}']\}_{\mathcal{F}'}$ is uniform integrable, where $\mathcal{F}' \subset \mathcal{F}$ is a sub-σ-field
- [4.6.2] Let $\phi \geq 0$ be a function s.t. $\lim_{x\to\infty} \frac{\phi(x)}{x} = \infty$. If $\mathbb{E}[\phi(|X_\alpha|)] \leq C\ \forall\alpha \in \mathcal{A}$, then $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ is uniformly integrable.
- [4.6.4] Let $(X_n)_n$ be a submartingale. Then the following are equivalent:
  - $(X_n)_n$ is uniform integrable
  - $(X_n)_n$converges a.s. and in L^1
  - $(X_n)_n$ converges in L^1
- [L^1 Martingale Convergence Theorem 20.5 / 4.6.7] Let $(X_n)_n$ be a martingale. Then the following are equivalent:
  - $(X_n)_n$ is uniform integrable
  - $(X_n)_n$converges a.s. and in L^1
  - $(X_n)_n$ converges in L^1
  - $\exists X_\infty$ s.t. $\mathbb{E}[|X_\infty|] < \infty$ (i.e. integrable) s.t. $X_k = \mathbb{E}[X_\infty|\mathcal{F}_k]\ \forall k$

  If any of the above conditions hold, $\exists X_\infty$ s.t. $X_n \to X_\infty$ both a.s. and in L^1
- [4.6.8] Let $\mathcal{F}_n \uparrow \mathcal{F}_\infty$. Then $\mathbb{E}[X|\mathcal{F}_n] \to \mathbb{E}[X|\mathcal{F}_\infty]$ a.s. and in L^1
  - $(\mathbb{E}[X|\mathcal{F}_n])_n$ is a martingale
- [Lévy's 0-1 Law] Let $(Y_n)_n$ be a random process. Let Z be a random variable s.t. $\mathbb{E}[|Z|] < \infty$ and $Z \in \sigma(Y_1, Y_2, \ldots)$. Let $X_n = \mathbb{E}[Z|Y_1, \ldots, Y_n]$. Then $(X_n)_n$ is a uniformly integrable martingale with $X_n \to Z$ a.s. and in L^1.
  - Let $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ and $A \in \mathcal{F}_\infty$. Then $\mathbb{E}[\mathbb{1}_A|\mathcal{F}_n] \to \mathbb{1}_A$ a.s.
- [4.6.10] Let $(Y_n) \to Y$ a.s. and $|Y_n| \leq Z\ \forall n$ and $\mathbb{E}[Z] < \infty$. If $\mathcal{F}_n \uparrow \mathcal{F}_\infty$, then $\mathbb{E}[Y_n|\mathcal{F}_n] \to \mathbb{E}[Y|\mathcal{F}_\infty]$ a.s.

- [Kakutani Theorem] Let $(X_n)_n$ be independent with $X_i > 0$ and $\mathbb{E}[X_i] = 1$. Then $(M_n)_n$ is a martingale with $M_n = \prod_{i=1}^{n} X_i$. Hence $(M_n)_n \to M_\infty$ with $\mathbb{E}[M_\infty] \leq 1$. The following are equivalent:
  - $\mathbb{E}[M_\infty] = 1$
  - $(M_n)_n \overset{L^1}{\to} M_\infty$
  - $(M_n)_n$ uniform integrable
  - $\prod_{i=1}^{\infty} \mathbb{E}\left[X_i^{\frac{1}{2}}\right] > 0$
  - $\sum_{i=1}^{\infty} \left(1 - \mathbb{E}\left[X_i^{\frac{1}{2}}\right]\right) < \infty$

Martingale (Miscellaneous)

Reversed Martingale

- [Reversed Martingale] Let $\mathcal{G}_0 \supseteq \mathcal{G}_1 \supseteq \cdots$ and $\mathcal{G}_\infty = \bigcap_n \mathcal{G}_n$. Then $(X_n)_n$ is a reversed martingale if $\mathbb{E}[|X_n|] < \infty$, $\mathbb{E}[X_m|\mathcal{G}_n] = X_n$ for $m \le n$ and $(X_n)_n$ is adapted to $(\mathcal{G}_n)_n$.
  - $X_n = \mathbb{E}[X_0|\mathcal{G}_n]$ a.s.
- [24.5] Let $(X_n)_n$ be a reversed martingale. Then $(X_n)_n \to \mathbb{E}[X_0|\mathcal{G}_\infty]$ a.s. and in L^1.
- [4.7.3] Let $\mathcal{G}_0 \supseteq \mathcal{G}_1 \supseteq \cdots$ and $\mathcal{G}_\infty = \bigcap_n \mathcal{G}_n$. Then $(\mathbb{E}[Y|\mathcal{G}_n])_n \to \mathbb{E}[Y|\mathcal{G}_\infty]$ a.s. and in L^1

Exchangeable Sequences Results

- [Permutable] Let $\omega \in S^{\mathbb{N}}$ and π be a finite permutation. Then, $\pi\omega \in S^{\mathbb{N}}$ with $(\pi\omega)_i \coloneqq \omega_{\pi(i)}$. An event A is permutable if $\pi^{-1}A = A\ \forall \pi$ finite permutation i.e. $\{\omega \in S^{\mathbb{N}} : \pi\omega \in A\} = A$
  - The occurrence of the event A is not affected by rearranging finitely many of the random variables
  - Same as permutating the random variables
- [Exchangeable σ-Field] Let $\mathcal{E}_n$ denote the σ-field generated by events invariant under permutations leaving $n+1, n+2, \ldots$ fixed. Then, $\mathcal{E} = \bigcap_{n=1}^{\infty} \mathcal{E}_n$ is the exchangeable σ-field.
  - It is exactly the collection of permutable events.
  - $\mathcal{E}_{n+1} \subset \mathcal{E}_n$
- [Exchangeable] A sequence of random variables $(X_n)_n$ is exchangeable if $(X_1, \ldots, X_n) \sim (X_{\pi(1)}, \ldots, X_{\pi(n)})$ in distribution $\forall n$ and permutation π.
- [Facts]
  - If (Z_1, W) and (Z_2, W) are equal in distributions and $\mathbb{E}[|Z_1|] < \infty$, then $\mathbb{E}[Z_1|W] = \mathbb{E}[Z_2|W]$ a.s.
    - $\mathbb{E}[\phi(Z_1)|W] = \mathbb{E}[\phi(Z_2)|W]$ a.s.
  - Let X be a random variable s.t. $\mathbb{E}[|X|] < \infty$ and $\mathcal{G}$ be a σ-field. If $X = \mathbb{E}[X|\mathcal{G}]$ in distribution, then $X = \mathbb{E}[X|\mathcal{G}]$ a.s.
    - If $\mathcal{G} \subset \mathcal{H}$ and $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X|\mathcal{H}]$ in distribution, then $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X|\mathcal{H}]$ a.s.
- Let $X_1, X_2, \ldots$ i.i.d. and $A_n(\phi) = \frac{1}{{}^nP_k} \sum_i \phi(X_{i_1}, \ldots, X_{i_k})$. If ϕ is bounded, then $\lim_{n\to\infty} A_n(\phi) = \mathbb{E}[\phi(X_1, \ldots, X_k)]$ a.s.
- [24.6] Let $(X_n)_n$ be an exchangeable sequence of random variables, with $\mathbb{E}[|X_i|] < \infty$. Let $S_n = \sum_{i=1}^n X_i$. Then $\lim_{n\to\infty} \frac{S_n}{n} = \mathbb{E}[X_1|\tau]$ a.s. and in L^1.
  - Recall: $\tau = \bigcap_{i=1}^{\infty} \sigma(X_i, X_{i+1}, \ldots)$ is the tail σ-field
- [24.7] Let $(X_n)_n$ be i.i.d. with $\mathbb{E}[|X_i|] < \infty$. Then τ is trivial, and hence $\mathbb{E}[X_1|\tau] = \mathbb{E}[X_1]$. Hence, $\lim_{n\to\infty} \frac{S_n}{n} = \mathbb{E}[X_1]$ a.s. and in L^1.
- [24.8] Let $\mathcal{G}_n = \sigma(S_n, X_{n+1}, X_{n+2}, \ldots)$. Then $\mathbb{E}[X_i|\mathcal{G}_n] = \mathbb{E}[X_1|\mathcal{G}_n]$ a.s. for $i \in \{1, \ldots, n\}$
- [25.1] Let $(X_1, \ldots, X_n)$ be an exchangeable sequence. Let $T \in \{0, \ldots, n-1\}$ be a stopping time. Then $X_{T+1} = X_1$ in distribution.
- [25.2 de Finetti] Let $(X_n)_n$ be an exchangeable sequence of random variables. Let τ be the tail σ-field. Then, conditional on τ, $(X_i)_i$ are i.i.d.
  - $X_1, \ldots$ are conditionally independent given τ
  - For each i, $\exists$ kernel $Q(\omega,\cdot)$ s.t. $Q(\omega,\cdot)$ is the regular conditional distribution of X_i given τ for each i i.e. $\mathbb{P}[X_i \in A|\tau](\omega) = Q(\omega, A)$
- [Hewitt-Savage 0-1 Law] Let $X_1, X_2, \ldots$ be i.i.d. and $A \in \mathcal{E}$, then $\mathbb{P}[A] \in \{0,1\}$

Martingale (L^2 Theory)

Definitions
• [L^2 Bounded] Let $(M_n)_n$ be a martingale. Then $(M_n)_n$ is <u>L^2 bounded</u> if $\sup_n \mathbb{E}[M_n^2] < \infty$ ○ Equivalently, $\sum_{i=1}^{\infty} \mathbb{E}[(M_n - M_{n-1})^2] < \infty$
L^2 Theory
• Let $(M_n)_n$ be a L^2 bounded martingale. Then $\exists M_\infty$ s.t. $M_n \to M_\infty$ a.s. and in L^1 and in L^2.

Martingale (L^p Theory)

Definitions

- [L^p Norm] Let (X, μ) be a measure space and $p \in [1, \infty)$. The L^p-norm of a measurable function f is $\|f\|_p = \left(\int_X |f(x)|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}}$
- [L^p Functions] $f \in L^p(X)$ if $\|f\|_p < \infty$
  - Equivalently, $\int_X |f(x)|^p \, \mathrm{d}\mu < \infty$
- [L^p Norm] Let $(a_n)_n$ be a sequence with $a_n \in \mathbb{R}$ and $p \in [1, \infty)$. Then the L^p norm of $(a_n)_n$ is $\|(a_n)_n\|_p := \left(\sum_{i=1}^{\infty} |a_i|^p\right)^{\frac{1}{p}}$
- [L^p Sequence] $(a_n)_n$ is a L^p sequence if $\|(a_n)_n\|_p < \infty$
  - Equivalently, $\sum_{i=1}^{\infty} |a_i|^p < \infty$
- [L^p Cauchy] Let (X, μ) be a measure space and $(f_n)_n$ be a sequence of measurable functions on X with $p \in [1, \infty)$. Then $(f_n)_n$ is L^p Cauchy sequence if $\forall \epsilon > 0, \exists N > 0$ s.t. $n, m > N \Rightarrow \|f_n - f_m\|_p < \epsilon$
- [Bounded L^p Variation] Let (X, μ) be a measure space and $(f_n)_n$ be a sequence of measurable functions on X with $p \in [1, \infty)$. Then $(f_n)_n$ has bounded L^p variation if $\sum_{i=1}^{\infty} \|f_{i+1} - f_i\|_p < \infty$
- [Convergence in L^p] Let (X, μ) be a measure space and $(f_n)_n$ be a sequence of measurable functions on X with $p \in [1, \infty)$. Then $(f_n)_n$ converges in L^p to f if $\lim\limits_{n \to \infty} \|f_n - f\|_p = 0$
  - In probability terms, $\lim\limits_{n \to \infty} \mathbb{E}[\|X_n - X\|^p]^{\frac{1}{p}} = 0$
  - Equivalently, $\lim\limits_{n \to \infty} \mathbb{E}[\|X_n - X\|^p] = 0$
- [L^∞ Norm] Let (X, μ) be a measure space and f be a measurable function on X. The L^∞ norm of f is $\|f\|_\infty = \min\{M \in [0, \infty] : \mu(\{x : |f(x)| > M\}) = 0\}$
- [L^∞ Function] $f \in L^\infty$ if $\|f\|_\infty < \infty$
- [Convergence in L^∞] Let (X, μ) be a measure space and $(f_n)_n$ be a sequence of measurable functions on X. Then $(f_n)_n \to f$ in L^∞ if $\lim\limits_{n \to \infty} \|f_n - f\|_\infty = 0$
  - Equivalent to uniform convergence i.e. $(f_n)_n \to f$ in L^∞ if and only if $(f_n)_n \to f$ uniformly except on a measure 0 set i.e. $\lim\limits_{n \to \infty} \sup\limits_{x \in X}\{f_n(x) - f(x)\} = 0$

Proposition

- Let (X, μ) be a measure space and $1 \le p < q < \infty$. If $\mu(X) \in (0, \infty)$, then $\|f\|_p \le \left(\mu(X)\right)^r \|f\|_q$ $\forall$ measurable f where $r = \frac{1}{p} - \frac{1}{q}$
  - If $\mu(X) = 1$, then $\|f\|_p \le \|f\|_q$ $\forall$ measurable f
  - $f \in L^q \Rightarrow f \in L^p$
- Let $1 \le p < q < \infty$, then every L^p sequence is also L^q
- [L^p Maximal Inequality 4.4.4] Let $(X_n)_n$ be a submartingale and let $1 < p < \infty$. Then $\mathbb{E}\left[\left(\max\limits_{1 \le i \le n} X_i^+\right)^p\right] = \mathbb{E}\left[\max\limits_{1 \le i \le n} (X_i^+)^p\right] \le \left(\frac{p}{p-1}\right)^p \mathbb{E}[(X_n^+)^p]$.
  - Let $(X_n)_n$ be a martingale, then $\mathbb{E}\left[\max\limits_{1 \le i \le n} |X_i|^p\right] \le \left(\frac{p}{p-1}\right)^p \mathbb{E}[|X_n|^p]$

Theorem

- Let (X, μ) be a measure space and $(f_n)_n$ be a sequence of measurable functions on X with bounded L^p variation. Then $(f_n)_n$ converges pointwise almost everywhere to a measurable function f and $(f_n)_n \to f$ in L^p
- Let (X, μ) be a measure space and let $(f_n)_n$ be a L^p Cauchy sequence on X. Then $(f_n)_n$ converges in L^p to some measurable function f on X.
- Let $(f_n)_n$ be an L^∞ Cauchy sequence of measurable functions. Then $(f_n)_n$ converges in L^∞ to some measurable function f

- [Parimal 2.6] Let $(X_n)_n$ be a sequence of random variables with $X_n \in L^p$ for $p \geq 1$. Then the following are equivalent:
    - $(X_n)_n \xrightarrow{L^p} X$ i.e. $\lim_{n\to\infty} (\mathbb{E}[|X_n - X|^p])^{\frac{1}{p}} = 0$
    - $(X_n)_n$ Cauchy in L^p i.e. $\lim_{m,n\to\infty} (\mathbb{E}[|X_m - X_n|^p])^{\frac{1}{p}} = 0$
    - $(X_n)_n \xrightarrow{\mathbb{P}} X$ and $(|X_n|^p)_n$ uniformly integrable

Tools

- [Minkowski] Given $f, g \in L^p$, $\|f\|_p + \|g\|_p \geq \|f + g\|_p$
    - Essentially the triangle inequality in L^p
    - $\|f\|_\infty + \|g\|_\infty \geq \|f + g\|_\infty$
- [Hölder] Let $(a_n)_n, (b_n)_n$ be sequences and $p, q \in [1, \infty)$ so that $\frac{1}{p} + \frac{1}{q} = 1$. If $\sum_{i=1}^\infty |a_i|^p$ and $\sum_{i=1}^\infty |b_i|^p$ both converge, then $\sum_{i=1}^\infty a_i b_i$ converges absolutely with $|\sum_{i=1}^\infty a_i b_i| \leq (\sum_{i=1}^\infty |a_i|^p)^{\frac{1}{p}} (\sum_{i=1}^\infty |b_i|^q)^{\frac{1}{q}}$
    - $|\langle f, g\rangle| \leq \|f\|_1 \|g\|_\infty$

Remarks

- Convergence in L^p does not imply convergence a.e. (typewriter sequence)
- Convergence a.e. does not imply convergence in L^p
- Convergence in probability does not imply convergence in L^p
- Convergence in L^p implies convergence in probability

Brownian Motion

Definitions

- [Brownian Motion] Let $B: [0,\infty) \to \mathbb{R}$ be a standard Brownian motion
  - $B(0) = 0$
  - Let $0 = t_0 \le t_1 \le \cdots \le t_n$, then $\{B(t_{i+1}) - B(t_i): 0 \le i \le n-1\} \sim N(0, t_{i+1} - t_i)$ are independent
  - The mapping $t \longmapsto B(t)$ is almost surely continuous
- [Continuous Time Martingale] Let $\{\mathcal{F}_t: t \in [0,\infty)\}$ be a filtration s.t. $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \le t$ and $\mathcal{F}_t$ is a σ-field $\forall t \in [0,\infty)$. Then, a process $(M_t)_t$ is a <u>continuous time martingale</u> if:
  - $\mathbb{E}[|M_t|] < \infty \ \forall t \ge 0$
  - $(M_t)_t$ is adapted to $(\mathcal{F}_t)_t$ i.e. $\sigma(M_t) \subset \mathcal{F}_t \ \forall t$
  - $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$ a.s. for $0 \le s \le t < \infty$
- [Stopping Time] Let $T: \Omega \to [0,\infty)$. T is a <u>stopping time</u> if $\{T \le t\} \in \mathcal{F}_t \ \forall t \in [0,\infty)$

Propositions

- [27.2] The following are continuous time martingales:
  - B_t
  - $B_t^2 - t$
  - $e^{\theta B_t - \frac{\theta^2 t}{2}}$ for $\theta \in \mathbb{R}$
  - $B_t^3 - 3tB_t$
  - $B_t^4 - 6tB_t^2 + 3t^2$
- [27.4] Let $c > 0$, $d \in \mathbb{R}$ and $T = \inf\{t: B_t = c + dt\} \le \infty$. Then, for $\lambda \in [0,\infty)$, the Laplace transform of T is $\mathbb{E}\left[e^{-\lambda t}\right] = e^{-c\left(d + \sqrt{d^2 + 2\lambda}\right)}$

Theorems

- [27.1 Optional Stopping Theorem] Let $(M_t)_t$ be a bounded martingale and T be a bounded stopping time. Then $\mathbb{E}[M_T] = \mathbb{E}[M_0]$
- [27.1* Optional Stopping Theorem] Let $(M_t)_t$ be a bounded martingale and T be a stopping time that is a.s. finite i.e. $\mathbb{P}[T = \infty] = 0$ and $\lim_{t\to\infty} \mathbb{E}[|M_t|\mathbb{1}_{T>t}] = 0$. Then $\mathbb{E}[M_T] = \mathbb{E}[M_0]$
- [28.1 Reflection Principle] Let $a, b > 0$, $t > 0$. Then $\mathbb{P}[T_a \le t, B_t \ge a + b] = \mathbb{P}[T_a \le t, B_t \le a - b]$.

Variant Processes

- [Brownian Bridge] The process $(B_t^0)_{t\in[0,1]}$ where $B_t^0 \sim (B_t|B_1 = 0)$
  - [Maximum] Let $M^0 = \sup_{t\in[0,1]} B_t^0$. Then $\mathbb{P}[M^0 \ge a] = e^{-2a^2}$
- [Brownian Meander] The process $(B_t^M)_{t\in[0,1]}$ where $B_t^M \sim (B_t|B_{t'} \ge 0 \ \forall t' \in [0,1])$