Mechanics

Definitions

- [Operator] An operator is a mapping from a set of functions to a set of functions
- [Linear operator] $\mathcal{F}$ is a linear operator if $\mathcal{F}[au+bv] = a\mathcal{F}[u] + b\mathcal{F}[v]$ for all $a, b \in \mathbb{R}$ and u, v are functions.
- [Translation operator] Let $\mathcal{L}_t$ be operator for translation by t, so $L_t: (\mathbb{R}^d \to \mathbb{R}) \to (\mathbb{R}^d \to \mathbb{R})$
  - $(\mathcal{L}_t u)(x) = u(x-t)$ where $u(x): \mathbb{R}^d \to \mathbb{R}$, $u(x-t): \mathbb{R}^d \to \mathbb{R}$, $x \in \mathbb{R}^d$ is the argument
- [Translational invariance] $\mathcal{F}$ is translational invariant if $\mathcal{F} \circ \mathcal{L}_{x_0} = \mathcal{L}_{x_0} \circ \mathcal{F}\ \forall x_0$
  - $\mathcal{F}[u(x-x_0)] = (\mathcal{F}[u])(x-x_0)\ \forall u \in (\mathbb{R}^d \to \mathbb{R}), x_0 \in \mathbb{R}^d$
  - $\mathcal{F} \circ \mathcal{L}_{x_0}(u) = \mathcal{F}[u(x-x_0)] \in (\mathbb{R}^d \to \mathbb{R})$
  - $\mathcal{L}_{x_0} \circ \mathcal{F}\ (u) = (\mathcal{F}[u])(x-x_0) \in (\mathbb{R}^d \to \mathbb{R})$
- [Constant coefficients] If functor $\mathcal{F}$ is both translational invariant and linear, then $\mathcal{F}$ has constant coefficients.
- [Linear PDE] A PDE $\mathcal{F}[u] = 0$ is linear if $\mathcal{F}$ is linear.
- [$\mathcal{C}^0$] $\mathcal{C}^0(U)$ is the space of scalar functions which are continuous on U
- [$\mathcal{C}^k$] $f \in \mathcal{C}^k(U)$ if f is continuous and all its partial derivatives of order k and lower are continuous
- [$\mathcal{C}_0^k$] $f \in \mathcal{C}_0^k(U)$ if $f \in \mathcal{C}^k(U)$ and there is a compact set $K \subset U$ such that $f(x) = 0$ for all $x \in U \backslash K$.
- [Support] $\operatorname{supp} u \coloneqq \{(t,x) \in [0,\infty) \times \mathbb{R} \mid u(t,x) \neq 0\}$
- [Well-Posed] A boundary value problem OR initial value problem is well-posed if:
  - There exists a solution
  - The solution is unique
  - The solution depends continuously on initial conditions OR boundary conditions

Topology

- [Ball] $B_r(x_0) = \{x \in \mathbb{R}^d \,|\, \|x - x_0\| < r\}$
- [Interior Point] Let $U \subset \mathbb{R}^d$ be a set. Then $x \in U$ is an interior point of U if $\exists r > 0$ such that $B_r(x) \subset U$.
- [Interior] The interior of U is $\operatorname{int} U = \{x \in U | x \text{ is an interior point of } U\}$
- [Open] U is open if $U = \operatorname{int} U$
- [Closed] U is closed if $\mathbb{R}^d \backslash U$ is open.
  - U closed $\Leftrightarrow (x_n)_n \subset U$ converges to $x \Rightarrow x \in U$
- [Limit Point] $x \in \mathbb{R}^d$ is a limit point of U if $\forall r > 0$, $B_r(x) \cap U \neq \phi$
  - If $x \in \mathbb{R}^d$ is a limit point of U, then $\exists$ sequence $(x_n)_n \subset U$ such that $\lim_{n\to\infty} x_n = x$
- [Closure] The closure of U is $\overline{U} = \{x \in \mathbb{R}^d | x \text{ is a limit point of } U\}$
  - $\overline{U} = \mathbb{R}^d \backslash \operatorname{int}(\mathbb{R}^d \backslash U)$
- [Boundary] The boundary of U is $\partial U = \overline{U} \backslash \operatorname{int} U$
- [Compact] $U \subset \mathbb{R}^d$ is compact if it is closed and bounded
  - Any sequence $(x_n)_n \subset U$ admits a convergent subsequence.
- [Continuity] Let $U \subset \mathbb{R}^d$ and $f: U \to \mathbb{R}$, then f is continuous at $x \in U$ if for all sequences $(x_n)_n$ with $x_n \in U$ and $(x_n)_n \to x \Rightarrow \big(f(x_n)\big)_n \to f(x)$. Write $f \in C(U)$.
- [Pointwise Convergence] Let $f_n: U \to \mathbb{R}$ be a sequence of functions and $f: U \to \mathbb{R}$, then $(f_n)_n \to f$ pointwise if $\lim_{n\to\infty} f_n(x) = f(x)\ \forall x \in U$.
- [Uniform Convergence] Let $f_n: U \to \mathbb{R}$ be a sequence of functions and $f: U \to \mathbb{R}$, then $(f_n)_n \to f$ uniformly if $\lim_{n\to\infty} \sup_{x \in U} |f_n(x) - f(x)| = 0$
  - If f_n is continuous and $f_n \to f$ uniformly, then f is also continuous.
  - If f is a continuous map on a compact set, then f is also uniformly continuous.

Multivariable Calculus

- [Partial Derivative] Let $x \in \operatorname{int} U$, $f: U \to \mathbb{R}$, then $\partial_{x_j} f(x) \coloneqq \lim_{h \to 0} \frac{f(x+he_j)-f(x)}{h}$
- [Directional Derivative] Let $v \in \mathbb{R}^d$, then $D_v f \coloneqq \lim_{h \to 0} \frac{f(x+hv)-f(x)}{h}$
- [Gradient] $\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_d} \end{bmatrix}$
- [Surface] A k-dimensional surface $\Gamma \subset \mathbb{R}^d$ is the image of a continuous map $\gamma: \widetilde{U} \to \mathbb{R}^d$ where $\widetilde{U} \subset \mathbb{R}^k$ and γ is a parametrization of Γ.
- [Normal] Let $U \subset \mathbb{R}^d$ open and $\partial U \in C^1$. Then ∂U is a $(d-1)$-dimensional surface with parametrization $\gamma: \widetilde{U} \to \partial U$ such that $\gamma \in C^1(\widetilde{U})$ and $D_y \gamma = \begin{bmatrix} \frac{\partial \gamma_1}{\partial y_1} & \cdots & \frac{\partial \gamma_1}{\partial y_{d-1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \gamma_d}{\partial y_1} & \cdots & \frac{\partial \gamma_d}{\partial y_{d-1}} \end{bmatrix}$ has full rank $\forall y \in \widetilde{U}$. $v \in \mathbb{R}^d$ is normal to ∂U at $x \in \partial U$ if $v \cdot \frac{\partial \gamma}{\partial y_i}(y) = 0$ for $i \in \{1, \ldots, d-1\}$ where $\gamma(y) = x$.
    - $\left\{\frac{\partial \gamma}{\partial y_i}(y)\right\}_{i \in [d-1]}$ spans the tangent space of ∂U at x
- [Outward Pointing] $v \in \mathbb{R}^d$ is outward-pointing at $x \in \partial U$ if $\exists \delta > 0$ such that $0 < h < \delta \Rightarrow x + hv \notin U$
- [Outward Pointing Unit Normal] $v \in \mathbb{R}^d$ is outward-pointing unit normal to U at $x \in \partial U$ if $\|v\| = 1$ and it is normal and outward pointing at $x \in \partial U$.
    - If ∂U is a graph, then $\partial U = \operatorname{im}(\gamma)$ where $\gamma(x_1, \ldots, x_{d-1}) = \big(x_1, \ldots, x_{d-1}, h(x_1, \ldots, x_{d-1})\big)$. Let $\bar{x} \in \partial U$ and $\bar{x}' = (x_1, \ldots, x_{d-1})$, then:
    $$v(\bar{x}) = \frac{1}{\sqrt{1 + |Dh|^2(\bar{x})}} \begin{bmatrix} -\partial_{x_1} h(\bar{x}') \\ \vdots \\ -\partial_{x_{d-1}} h(\bar{x}') \\ 1 \end{bmatrix}$$
- [Integral] If $\int_{\mathbb{R}^d} |f| \mathrm{d}x < \infty$, then f is absolutely integrable and the order of integration with respect to the coordinates does not matter.
- [Change of Variables] Let $\Gamma \subset \mathbb{R}^d$, $\widetilde{U} \subset \mathbb{R}^k$, $\gamma: \widetilde{U} \to \Gamma$ be a proper parametrization, $\gamma \in C^1(\widetilde{U})$ and $\frac{\partial \gamma}{\partial y}(y)$ full rank $\forall y \in \widetilde{U}$, then $\int_\Gamma f(x) \mathrm{d}S(x) = \int_{\widetilde{U}} f(\gamma(y)) \left|\frac{\partial \gamma}{\partial y}(y)\right| \mathrm{d}y$
    - $\left|\frac{\partial \gamma}{\partial y}(y)\right|$ is the k-dimensional volume of the parallelopiped formed by the k column vectors $\frac{\partial \gamma}{\partial y_i}(y)$
    - $|X| \coloneqq \sqrt{\det X^T X}$, $X = \begin{bmatrix} \frac{\partial \gamma}{\partial y_1} & \cdots & \frac{\partial \gamma}{\partial y_k} \end{bmatrix}$, $X \in \mathbb{R}^{d \times k}$, $d > k$
    - $[d = 1]$ $\int_\Gamma f(x) \mathrm{d}S(x) = \int_I f(\gamma(y)) |\gamma'(y)| \mathrm{d}y$
- [Divergence Theorem] Let U open and bounded set $\subset \mathbb{R}^d$, ∂U a C^1 surface, $X: \overline{U} \to \mathbb{R}^d$, $x \in C^1(\overline{U})$. Let $\nabla \cdot X = \sum_{i=1}^d \partial_{x_j} X_j$, then $\int_U (\nabla \cdot X)(x) \mathrm{d}x = \int_{\partial U} X(x) \cdot \nu(x) dS(x)$ where $\nu(x)$ is the outward unit normal to ∂U at $x \in \partial U$.
    - Let $X: U \to \mathbb{R}^d$, with $U \subset \mathbb{R}^d$ and $X \in C^1(\overline{U})$. Then $\nabla \cdot X = 0$ on U if and only if for all $r > 0$ and x_0 such that $B_r(x_0) \subset U$, $\int_{\partial B_r(x_0)} x \cdot \nu(x) \mathrm{d}S(x) = 0$
- [Gauss-Green] Let U bounded, open set $\subset \mathbb{R}^d$ and ∂U is C^1 surface. Let $u: \overline{U} \to \mathbb{R}$, $u \in C^1(\overline{U})$, then $\int_U \partial_{x_j} u(x) \mathrm{d}x = \int_{\partial U} u(x) \nu_i(x) dS(x)$
- [Green] $\oint_C (L\, \mathrm{d}x + M\, \mathrm{d}y) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) \mathrm{d}x\, \mathrm{d}y$

Wave Equation

Set-up

- $u: \mathbb{R}^{1+d} \to \mathbb{R}$
  - $u: (t,x) \to \mathbb{R}$
- [Homogeneous Wave Equation] $\left(-\partial_t^2 + \sum_{i=1}^d \partial_{x_i}^2\right)u = 0$
  - $\square u = 0$
  - $\square \coloneqq -\partial_t^2 + \sum_{i=1}^d \partial_{x_i}^2$
- [Inhomogeneous Wave Equation] $\left(-\partial_t^2 + \sum_{i=1}^d \partial_{x_i}^2\right)u = f$
- [Representation Formula] Assume that a solution to problem exists and is nice, find an explicit expression for u in terms of f, g, h (i.e f, g, h arbitrary functions)
- [Solution Formula] Given f, g, define u.
- [Global Solution] A solution that satisfies equation for all positive time and does not go to ∞ in finite time.
- [Singularity Formation] Solution starting from "regular" initial conditions becomes infinity in finite time.

Wave Equation

Initial Value Problem (IVP)	Initial Boundary Value Problem (IBVP)
• $\left(-\partial_t^2 + \sum_{i=1}^d \partial_{x_i}^2\right)u = 0$ in $(0,\infty)_t \times \mathbb{R}_x^d$ • $u = g$ on $\{t=0\} \times \mathbb{R}_x^d$ • $\partial_t u = h$ on $\{t=0\} \times \mathbb{R}_x^d$	• $\left(-\partial_t^2 + \sum_{i=1}^d \partial_{x_i}^2\right)u = 0$ in $(0,\infty)_t \times \mathbb{R}_+$ • $u = g$ on $\{t=0\} \times \mathbb{R}_+$ • $\partial_t u = h$ on $\{t=0\} \times \mathbb{R}_+$ • $u(t,0) = 0$ for $t \in (0,\infty)$

- [d'Alembert] Assume $u \in C^2([0,\infty)_t \times \mathbb{R}_x)$ solve (IVP), then the representation formula is:
$$u(t,x) = \frac{1}{2}\big(g(x+t) + g(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y$$
- [Solution Formula] Given $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$, then $u(t,x)$ defined by $u(t,x) = \frac{1}{2}\big(g(x+t) + g(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y$ satisfies (IVP)
  - $g(x-t)$ is a wave travelling to the right
  - $g(x+t)$ is a wave travelling to the left
- [Singularity Formation] Let $T > 0$, then there exists $g, h \in C_0^2(\mathbb{R})$, $u \in C^2([0,T) \times \mathbb{R}_x)$ solving the following equations such that $\lim_{t\to T} |u(t,0)| = \infty$

Singularity Example
• $(-\partial_t^2 + \partial_x^2)u = (\partial_t u)^2 - (\partial_x u)^2$ in $(0,T)_t \times \mathbb{R}_x$ • $u = g$ on $\{t=0\} \times \mathbb{R}_x$ • $\partial_t u = h$ on $\{t=0\} \times \mathbb{R}_x$

- [IBVP] Assume that $u \in C^2\left(\overline{(0,\infty)_t \times \mathbb{R}_+}\right)$ solve (IBVP), then the representation formula is:
$$u(t,x) = \frac{1}{2}\left(g(x+t) + \frac{x-t}{|x-t|}g(|x-t|)\right) + \frac{1}{2}\int_{|x-t|}^{x+t} h(y)\,\mathrm{d}y$$
- [IBVP] If g, h are "regular enough", then the above formula is also a solution formula.

Radial Wave Equation in $\mathbb{R}^3$	Wave Equation in $\mathbb{R}^3$
• $\left(-\partial_t^2 + \sum_{i=1}^d \partial_{x_i}^2\right)u = 0$ in $(0,\infty)_t \times \mathbb{R}^3$ • $u = g$ on $\{t=0\} \times \mathbb{R}^3$ • $\partial_t u = h$ on $\{t=0\} \times \mathbb{R}^3$ • u radial $\Rightarrow \Delta u = \partial_r^2 u + \frac{2}{r}\partial_r u$	• $\left(-\partial_t^2 + \sum_{i=1}^3 \partial_{x_i}^2\right)u = 0$ in $(0,\infty)_t \times \mathbb{R}^3$ • $u = g$ on $\{t=0\} \times \mathbb{R}^3$ • $\partial_t u = h$ on $\{t=0\} \times \mathbb{R}^3$

- [Radial Wave Equation] If $u, g, h \in C^2(\mathbb{R}^3)$, then $u(t,r) = \frac{1}{2r}\big((r+t)g(r+t) + (r-t)g(|r-t|)\big) + \frac{1}{2r}\int_{|r-t|}^{r+t} y\, h(y)\, \mathrm{d}y$
  - $\bar{u} = ru$, $\bar{g} = rg$, $\bar{h} = rh$
- [Kirchhoff] Assume that $u \in C^2((0,\infty)_t \times \mathbb{R}^3)$, then representation formula is:
$$u(t,x) = \frac{1}{4\pi t}\int_{\partial B_t(x)} h(y)\mathrm{d}S(y) + \frac{1}{4\pi t^2}\int_{\partial B_t(x)} \big(g(y) + (y-x)\cdot\nabla g(y)\big)\mathrm{d}S(y)$$
  - $\bar{u}(t,r) \coloneqq \frac{1}{4\pi r^2}\int_{\partial B_r(0)} u(t,y)\, \mathrm{d}S(y) = \frac{1}{4\pi}\int_{\partial B_1(0)} u(t, r\omega)\, \mathrm{d}S(\omega)$
    - $\left(-\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r\right)\bar{u} = 0$
    - $\bar{u}$ obeys RWE
  - $\bar{g}(r) \coloneqq \frac{1}{4\pi r^2}\int_{\partial B_r(0)} g(y)\, \mathrm{d}S(y)$
  - $\bar{h}(r) \coloneqq \frac{1}{4\pi r^2}\int_{\partial B_r(0)} h(y)\, \mathrm{d}S(y)$
  - $\lim_{r\to 0}\bar{u}(t,r) = u(t,0)$

Wave Equation in $\mathbb{R}^2$
• $\big(-\partial_t^2 + \sum_{i=1}^2 \partial_{x_i}^2\big)u = 0$ in $(0,\infty)\times\mathbb{R}^2$ • $u = g$ on $\{t=0\}\times\mathbb{R}^2$ • $\partial_t u = h$ on $\{t=0\}\times\mathbb{R}^2$

- [Poisson] Let $u \in C^2\big(\overline{(0,\infty)_t \times \mathbb{R}^2}\big)$ solve (IVP-2D), then the representation formula is:
$$u(t,x) = \frac{1}{2\pi t}\int_{B_t(x)} \left(g(y) + \frac{(y-x)\cdot\nabla g(y)}{\sqrt{t^2 - \|y-x\|^2}}\right)\mathrm{d}y + \frac{1}{2\pi}\int_{B_t(x)} \frac{h(y)}{\sqrt{t^2 - \|y-x\|^2}}\mathrm{d}y$$

Properties

- [Finite Speed of Propagation] World line has gradient $\frac{1}{c}$
- [Strong Huygen's Principle] For odd dimensions d, the solution for the wave equation $\Box u = 0$ at the point (t,x) depends only on the value of the initial condition on $\partial B_t(x)$
- [Weak Huygen's Principle] For even dimensions d, the solution for the wave equation $\Box u = 0$ at the point (t,x) depends only on the initial data in the ball $B_t(x)$

Energy Method

- [1D Energy] $E(t) = \frac{1}{2}\int_{\mathbb{R}}\left[\big((\partial_t u)(t,x)\big)^2 + \big((\partial_x u)(t,x)\big)^2\right]\mathrm{d}x$
- [1D Energy] $E(t) = \frac{1}{2}\int_{\mathbb{R}^d}\left[\big((\partial_t u)(t,x)\big)^2 + \|\nabla u(t,x)\|^2\right]\mathrm{d}x$
- [Conservation of Energy] For $u \in C^2\big(\overline{(0,\infty)\times\mathbb{R}^d}\big)$, integrable such that energy is well-defined, $g \in C_0^2(\mathbb{R}^d)$, $h \in C_0^1(\mathbb{R}^d)$, then $\frac{\mathrm{d}}{\mathrm{d}t}E(t) = 0$

Heat Equation

Set-up $u: \mathbb{R}^{1+d} \to \mathbb{R}$; $u: (t,x) \to \mathbb{R}$; $f: (t, x_1, \dots, x_d) \to \mathbb{R}$

- [Homogeneous Heat Equation] $\left(-\partial_t + \sum_{i=1}^{d} \partial_{x_i}^2\right)u = 0$
  - $(-\partial_t + \Delta)u = 0$
- [Inhomogeneous Heat Equation] $\left(-\partial_t + \sum_{i=1}^{d} \partial_{x_i}^2\right)u = f$
- u: temperature
- Δu: gives difference between average value of a function in the neighbourhood of a point and its value at that point.
- [Initial Boundary Value Problem]
  - $\partial_t u - \Delta u = 0$ on $(0,\infty)_t \times U$
  - $u = g$ on $\{t = 0\} \times U$
  - $u = h$ on $(0,\infty)_t \times \partial U$

Properties

- [Scaling Symmetry] If $u(t,x)$ is a solution, then $u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$ is a solution $\forall \lambda > 0$
- [Rotational Symmetry] If $u(t,x)$ is a solution, then $u(t, Ux)$ is a solution for unitary U
- [Conversation of Mass] If $u(t,x)$ "tends to 0" as $|x| \to \infty$, then $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u(t,x)\mathrm{d}x = 0$
- [Linear] Superposition of solutions is a solution
- [Infinitely Smoothing] $\partial_x^m u(t,x) \in C\big((0,\infty) \times \mathbb{R}\big)$ $\forall m \in \mathbb{N}$
- Infinite speed of propagation
- Not time-reversible

Theorems

- [Heat Kernel] $\phi(t,x) = \begin{cases} \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{\|x\|^2}{4t}}, \ t > 0 \\ 0, \ t \le 0 \end{cases}$
  - $\phi \in C^k(\mathbb{R}^d)$ $\forall k \in \mathbb{N}$ on $\mathbb{R}^{1+d} \backslash \{0,0\}$
  - $(\partial_t - \Delta)\phi = 0$ in $\mathbb{R}^{1+d} \backslash \{0,0\}$
  - $\int_{\mathbb{R}^d} \phi(t,x)\mathrm{d}x = 1$ $\forall t > 0$
  - ϕ is a solution to (1) $\partial_t u - \Delta u = 0$ on $(0,\infty)_t \times \mathbb{R}^d$ (2) $u = \delta_{\{x=0\}}$ on $\{t = 0\} \times \mathbb{R}^d$
- [General Solution Formula] $u(t,x) = \int_{\mathbb{R}^d} \phi(t, x-y) g(y)\mathrm{d}y$
  - If $g(y) \in C(\mathbb{R}^d)$ bounded, then $u(t,x)$ is $C^0\left(\overline{(0,\infty) \times \mathbb{R}^d}\right) \cap C^2\left((0,\infty) \times \mathbb{R}^d\right)$ and solves (1) $\partial_t u - \Delta u = 0$ on $(0,\infty)_t \times \mathbb{R}^d$ (2) $u = g$ on $\{t = 0\} \times \mathbb{R}^d$
  - $\lim_{t \to 0^+} u(t,x) = g(x)$ $\forall x \in \mathbb{R}^d$
- [Extremal Principle] Let $T > 0$ and U bounded open set $\subset \mathbb{R}^d$. Let $u \in C_t^1 C_x^2([0,T) \times U) \cap C\big(\overline{(0,T) \times U}\big)$ solve $(\partial_t - \Delta)u = 0$, then $\max_{[0,T] \times \overline{U}} u(t,x) = \max_{([0,T] \times \partial U) \cup (\{t=0\} \times \overline{U})} u(t,x)$
  - Similarly, $\min_{[0,T] \times \overline{U}} u(t,x) = \min_{([0,T] \times \partial U) \cup (\{t=0\} \times \overline{U})} u(t,x)$
- If u is a solution to IBVP on $(0,T)$, then its maximum and minimum are located on $(\partial U \times [0,T]) \cup (\{t = 0\} \times \overline{U})$
- [Uniqueness] If $u_1, u_2 \in C_t^1 C_x^2(\mathbb{R} \times \mathbb{R}^d)$ s.t. u_1, u_2 bounded and solve $(0,\infty) \times \mathbb{R}^d$, then $u_1(t,x) = u_2(t,x)$ $\forall t, x$
  - If $g \in C(\mathbb{R}^d)$ bounded, then the unique solution is $u(t,x) = \int_{\mathbb{R}^d} \phi(t, x-y) g(y)\mathrm{d}y$
- [Energy Inequality] Let $U \subset \mathbb{R}^d$ be bounded and open, $T > 0$ and $u \in C_t^1 C_x^2\big((0,T) \times U\big) \cap C([0,T] \times \overline{U})$ and g continuous. Given the IBVP: $\phi(t,x) = \begin{cases} \partial_t u - \Delta u = 0, \ (0,T) \times U \\ u = g, \ \{t = 0\} \times U \\ u = 0, \ [0,T] \times \partial U \end{cases}$, then for $t \in [0,T)$, $\frac{1}{2}\int_U u(t,x)^2 \mathrm{d}x = -\int_0^t \int_U |\nabla u(s,y)|^2 \mathrm{d}y\,\mathrm{d}s + \frac{1}{2}\int_U g(x)^2 \mathrm{d}x$

Laplace and Poisson Equations

Definitions

- $u: \mathbb{R}^d \to \mathbb{R}$
- [Laplace Equation] $-\Delta u = 0$ (also, say u is harmonic)
- [Poisson Equation] $-\Delta u = f$
  - u: electric potential that arises from a point charge of unit strength
  - f: charge density
- [Dirichlet Problem] $\begin{cases} -\Delta u = f, \text{ in } U \\ \quad u = g, \text{ on } \partial U \end{cases}$
- [Path Connected] A set U is path connected if for all $x_1, x_2 \in U$, $\exists \gamma: [0,1] \to U$ s.t. γ continuous with $\gamma(0) = x_1$ and $\gamma(1) = x_2$.
- [Radial Fundamental Solution]
  - $\phi(x) = \begin{cases} \frac{1}{\mathrm{Vol}(\partial B_1(0))(d-2)} \|x\|^{-(d-2)}, \ d > 2 \\ -\frac{1}{2\pi} \log \|x\|, \ d = 2 \end{cases}$
  - ϕ is not defined at $x = 0$ and is also not continuous there.
  - ϕ is smooth in $\mathbb{R}^d \backslash \{0\}$
  - $\nabla\phi(x) = \frac{1}{\mathrm{Vol}(\partial B_1(0))} \|x\|^{-(d-1)} \frac{x}{\|x\|} \quad (d > 2)$
  - $-\Delta\phi = 0$ in $\mathbb{R}^d \backslash \{0\}$
  - $-\Delta\phi(x) = \delta_{\{x=0\}}$
  - $-\int_{\partial B_r(0)} \nabla\phi(y) \cdot \nu(y) \, \mathrm{d}S(y) = 1 \ \forall r > 0$
  - $\int_{B_\epsilon(0)} \phi(y) \, \mathrm{d}y = c \int_0^\epsilon r^{-(d-2)} r^{d-1} \mathrm{d}r \le c' \epsilon^2$
  - $\int_{\partial B_\epsilon(0)} \phi(y) \, \mathrm{d}y = c \int_0^\epsilon r^{-(d-2)} r^{d-2} \mathrm{d}r \le c' \epsilon$
- [Corrector Function] Define $h_x(y): \mathbb{R}^d \to \mathbb{R}$ as the corrector function where:
  - $-\Delta_y h_x(y) = 0 \ \forall x, y \in U$
    - $\int_U (-\Delta u)(y) h_x(y) \, \mathrm{d}y + \int_{\partial U} \nu(y) \cdot \nabla u(y) \, h_x(y) \, \mathrm{d}S(y) - \int_{\partial U} u(y) \nu(y) \cdot \nabla h_x(y) \, \mathrm{d}S(y) = 0$ (just integrate by parts twice)
  - $h_x(y) = -\phi(y - x) \ \forall y \in \partial U$ i.e. $h_x(y)$ annihilates ϕ on ∂U
- [Green's Function] Let $x, y \in U$ with $x \ne y$. Define $G(x, y) := \phi(y - x) + h_x(y)$ as the Green's function on region U.
  - $G(x, y) = 0 \ \forall x \in U, y \in \partial U$
  - $u(x) = \int_U (-\Delta u)(y) G(x, y) \, \mathrm{d}y - \int_{\partial U} u(y) \nu(y) \cdot \nabla_y G(x, y) \, \mathrm{d}S(y)$
  - An interpretation of Green's function is: first fix $x \in U$ and regard $G = G(x, \cdot)$, then:
    - $-\Delta G = \delta_x$ in U
    - $G = 0$ on ∂U
  - Properties:
    - [Symmetric] $\forall x, y \in U$, $x \ne y$, we have $G(x, y) = G(y, x)$

Properties

- [Laplace operator Δ]
  - [Linearity] For $a, b \in \mathbb{R}$, $u, v \in C^2(\mathbb{R}^d)$, $\Delta(au + bv) = a\Delta u + b\Delta v$
  - [Translation symmetry] Fix $\bar{x} \in \mathbb{R}^d$, then $\Delta\big(u(x - \bar{x})\big) = (\Delta u)(x - \bar{x})$
  - [Rotational symmetry] For U unitary matrix, $\Delta\big(u(Ux)\big) = (\Delta u)(Ux)$
  - [Scaling symmetry] Let $\lambda \in \mathbb{R}$, then $\Delta\big(u(\lambda x)\big) = \lambda^2 (\Delta u)(\lambda x)$

- [Radial Fundamental Solution] Let $f \in C_0^2(\mathbb{R}^d)$ and $u(x) = \int_{\mathbb{R}^d} f(y)\phi(x - y) \mathrm{d}y$, then:
  - $u \in C^2(\mathbb{R}^d)$
  - $-\Delta u = f$ in $\mathbb{R}^d$; $\Delta u = \int_{\mathbb{R}^d} \Delta f(x - y) \phi(y) \, \mathrm{d}y$
  - $\partial_{x_i} u(x) = \int_{\mathbb{R}^d} \partial_{x_i} f(x - y) \phi(y) \, \mathrm{d}y$; $\partial_{x_j} \partial_{x_i} u(x) = \int_{\mathbb{R}^d} \partial_{x_j} \partial_{x_i} f(x - y) \phi(y) \, \mathrm{d}y$

- [Harmonic functions on U]
  - [Mean value property] Let $u \in C^2(U)$ s.t. $-\Delta u = 0$ in U, then $\forall B_r(x) \subseteq U$, $u(x) = \frac{1}{\mathrm{Vol}(B_r(x))}\int_{B_r(x)} u(y)\,\mathrm{d}y = \frac{1}{\mathrm{Vol}(\partial B_r(x))}\int_{\partial B_r(x)} u(y)\,dS(y)$
  - [Strong maximum principle] Suppose that $U \subseteq \mathbb{R}^d$ is open and path connected and $u \in C^2(U) \cap C(U)$ and $-\Delta u = 0$ in U. If $\exists x' \in U$ s.t. $u(x') = \max_{x\in\overline{U}} u(x)$, then $u(x) = u(x')\ \forall x \in \overline{U}$.
    - If maximum is achieved at the interior, then it must be everywhere constant.
    - Similarly for minimum.
  - [Weak maximum principle] Let $U \subset \mathbb{R}^d$ be bounded, open and $u \in C^2(U) \cap C(\overline{U})$ with $-\Delta u = 0$ in U, then $\max_{\overline{U}} u(x) = \max_{\partial U} u(x)$.
    - Maximum is attained at the boundary (it may also be attained at the interior simultaneously).
    - Similarly for minimum.
  - [Uniqueness] Let $U \subset \mathbb{R}^d$ be bounded, open. If $u_1, u_2 \in C^2(U) \cap C(\overline{U})$ are solutions to $\begin{cases} -\Delta u = f, & \text{in } U \\ u = g, & \text{on } \partial U \end{cases}$, then $u_1 = u_2$.
  - [Smoothness] Let $U \subseteq \mathbb{R}^d$ open and $u \in C^2(U)$. If $-\Delta u = 0$ on U, then u is infinitely differentiable on U.

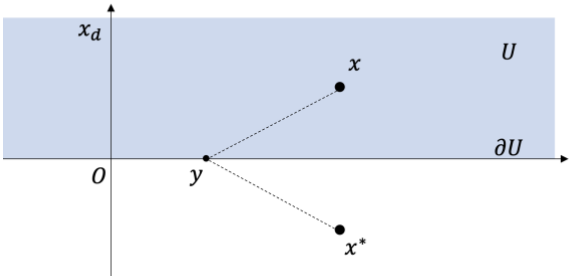
Theorems

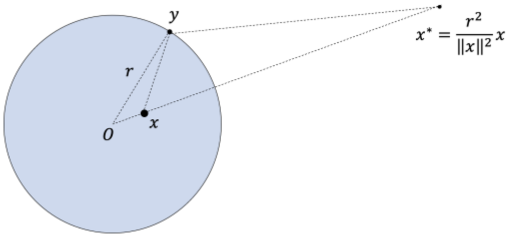
- Let $(f_n)_n : U \to \mathbb{R}$ be a sequence of continuous functions, absolutely integrable on U. Suppose $(f_n)_n \to f$ uniformly with f absolutely integrable on U, then $\int_U f_n(x)\,\mathrm{d}x = \int_U f(x)\,\mathrm{d}x$
- Let $I \subset \mathbb{R}$ be a compact interval and $(f_n)_n$ be a sequence of functions with $f_n : I \to \mathbb{R}$. If $\lim_{n\to\infty} f_n = f$ uniformly, then $\lim_{n\to\infty} \int_I f_n(x)\,\mathrm{d}x = \int_I f(x)\,\mathrm{d}x$.
- [Liouville's Theorem] Let $u \in C^2(\mathbb{R}^d)$ bounded and $-\Delta u = 0$, then $u = c$ for constant c.
  - If $u_1, u_2 \in C^2(\mathbb{R}^d)$ bounded, $f \in C_0^2(\mathbb{R}^d)$ and $-\Delta u_1 = f$, $-\Delta u_2 = f$, then $u_1 = u_2 + c$
- [Converse of Mean Value Property] If $u \in C^2(U)$ satisfies $u(x) = \int_{\partial B_r(x)} u(y)\,\mathrm{d}S(y)$ for every $B_r(x) \subset U$, then u is harmonic i.e. $\Delta u = 0$.
- [Solution Formula for Poisson, $\mathbb{R}^d$] Assuming $f \in C_0^2(\mathbb{R}^d)$, then a solution to $-\Delta u = f$ on $\mathbb{R}^d$ is $u(x) = \int_{\mathbb{R}^d} f(y)\phi(x-y)\,\mathrm{d}y$
  - If no boundedness assumptions are made on u, then the solution is non-unique.
  - If bounded, then u unique up to an additive constant.
- [Representation Formula for Poisson, $\mathbb{R}^d$] Suppose $d > 2$ and $f \in C_0^2(\mathbb{R}^d)$, then all bounded solutions to $-\Delta u = f$ are given by $u(x) = \int_{\mathbb{R}^d} \phi(x-y) f(y)\mathrm{d}y + c$ where c is an arbitrary constant.
- [Representation Formula, U] Let $U \subseteq \mathbb{R}^d$ be an open and bounded domain with $d > 2$, $\partial U \in C^1$ and $u \in C^2(\overline{U})$. Then, the following representation formula holds $\forall x \in U$:
  - $u(x) = \int_U (-\Delta u)(y)\phi(y-x)\,\mathrm{d}y + \int_{\partial U} \nu(y)\cdot\nabla u(y)\,\phi(y-x)\,\mathrm{d}S(y) - \int_{\partial U} u(y)\nu(y)\cdot(\nabla\phi)(y-x)\,\mathrm{d}S(y)$
  - (Special case) If $\phi(y-x) = 0\ \forall y \in \partial U$, then $u(x) = \int_U (-\Delta u)(y)\phi(y-x)\,\mathrm{d}y$
  - Note: $\nu(y)\cdot(\nabla\phi)(y-x) = \frac{\partial\phi}{\partial\nu}(y-x)$
- [Poisson Integral Formula] Let $u \in C^2(\overline{U}), \partial U \in C^1, f \in C(\overline{U}), g \in C(\partial U)$ and suppose:
  - $-\Delta u = f$ in U
  - $u = g$ on ∂U

  Then $\forall x \in U$, the <u>representation formula</u> for the Dirichlet problem is: $u(x) = \int_U f(y)G(x,y)\,\mathrm{d}y - \int_{\partial U} g(y)\nu(y)\cdot\nabla_y G(x,y)\,\mathrm{d}y$ where $G(x,y) = \phi(y-x) + h_x(y)$

Method of Images

- Key idea: build $h_x(y)$ as a superposition of $\phi\big(\alpha_k(y-x_k)\big)$ with $x_k \notin U$
  - Intuitively, add a point charge of value $\frac{1}{\alpha_k}$ at position $x_k \notin U$ to counter the potential due to charge at $x \in U$, in doing so, annihilate the value of ϕ at the boundary ∂U
  - $h_x(y)$ will be defined $\forall y \in U$ since $x_k \notin U$ so $y - x_k \neq 0$.
- [Reflection] $U = \mathbb{R}^d_+ = \{x \in \mathbb{R}^d | x_d > 0\}$
  - $h_x(y) = -\phi(y - x^*)$ where $x^* = (x_1, x_2, \dots, x_{d-1}, -x_d)$





- [Pole-Polar] $U = B_1(0) \subseteq \mathbb{R}^d$
  - $h_x(y) = -\phi\big(\|x\|(y - x^*)\big)$ where $x^* = \frac{1}{\|x\|^2}x$
  - In general, if $U = B_r(0)$, then $h_x(y) = -\phi\left(\frac{\|x\|}{r}(y - x^*)\right)$ with $x^* = \frac{r^2}{\|x\|^2}x$