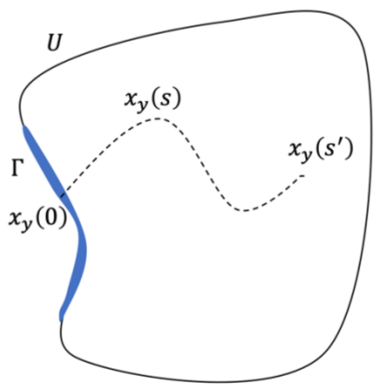
Method of Characteristics

General Form

- [Set Up] $U \subseteq \mathbb{R}^d$ open, $\partial U \in C^1$, $\Gamma \subseteq \partial U$, $F: U \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $u: U \to \mathbb{R}$
  - $x \in U \subseteq \mathbb{R}^d$, $u(x) \in \mathbb{R}$, $\nabla u = \begin{bmatrix} \partial_{x_1} u \\ \vdots \\ \partial_{x_d} u \end{bmatrix} \in \mathbb{R}^d$
  - $F(x, u(x), \nabla u(x)) = 0$ in U
  - $u = g$ on Γ
  - Idea: want to find $(\dot{x}(s), \dot{z}(s), \dot{p}(s)) = Q(x(s), z(s), p(s))$, so as to apply ODE
- [Procedure]
  - Write ODE for $x(s)$ and $z(s)$
  - $\forall y \in \Gamma$, find trajectory $x_y(s)$ and $z_y(s)$ such that:

Linear First-Order Scalar Equations

- $U \subseteq \mathbb{R}^d$, $a: U \to \mathbb{R}$, $b: U \to \mathbb{R}^d$, $\Gamma \subseteq \partial U$
  - $a(x)u + b(x) \cdot \nabla u - f(x) = 0$ in U
  - $u = g$ on Γ
- $x(s)$: path parametrized by s s.t. $x(0) \in \Gamma \subseteq \partial U$
- $z(s) = u(x(s))$: value function along the path $x(s)$
- $p(s) = (\nabla u)(x(s))$: gradient of value function evaluated at point $x(s)$
  - $p_j(s) = (\partial_{x_j} u)(x(s))$
  - $\dot{z} = (\nabla u)(x) \cdot \dot{x} = p \cdot \dot{x}$



$$z_y(s') = z_y(0) + \int_0^{s'} \left(\frac{\mathrm{d}}{\mathrm{d}s} z(s)\right) \mathrm{d}s$$

- Choose $x(s)$ such that $\dot{x} = b(x)$
- Solve system of ODEs via x in terms of s first, then z in terms of s
  - Pick $y \in \Gamma$
  - $x_y(0) = y$, $z_y(0) = u\left(x_y(0)\right) = g(y)$
  - Invert $(s, y) \mapsto x_y(s) \in U$ to solve for $u(x)$ $\forall x \in U$

Expanded	Compact
• $\frac{\mathrm{d}}{\mathrm{d}s} x(s) = b(x(s))$ • $\frac{\mathrm{d}}{\mathrm{d}s} z(s) = -a(x(s))z(s) + f(x(s))$	• $\dot{x} = b(x)$ • $\dot{z} = -a(x)z + f(x)$

- Examples:
  - [Transport / Advection] $\partial_t u + b(x) \cdot \nabla_x u = 0$

Quasilinear Equations

- Equations that are linear with respect to higher order derivatives.
  - $a(x, u)u + b(x, u) \cdot \nabla u = 0$ in U
  - $u = g$ on $\Gamma = \partial U$
- Choose $x(s)$ such that $\dot{x} = b(x, z)$

Expanded	Compact
• $\frac{\mathrm{d}}{\mathrm{d}s} x(s) = b(x(s), z(s))$ • $\frac{\mathrm{d}}{\mathrm{d}s} z(s) = -a(x(s), z(s))$	• $\dot{x} = b(x, z)$ • $\dot{z} = -a(x, z)$

- Examples:
  - [Burger] $\partial_t u + u\partial_x u = 0$

Fully Nonlinear Scalar Equations

- $F\big(x, u(x), \nabla u(x)\big) = 0$ in U
- $u = g$ on Γ

- $\dot{p}_j(s) = \frac{\mathrm{d}}{\mathrm{ds}}\left(\partial_{x_j}\right)\left(x(s)\right) = \sum_{k=1}^{d}\left(\partial_{x_j}\partial_{x_k}u\right)\left(x(s)\right)\dot{x}_k(s)$
- $0 = \partial_{x_j}\big(F(x,z,p)\big) = \left(\partial_{x_j}F\right)(x,z,p) + (\partial_z F)(x,z,p)p_j(s) + \sum_{k=1}^{d}\left(\partial_{p_k}F\right)(x,z,p)\partial_{x_j}\partial_{x_k}u\big(x(s)\big)$
- Pick $x(s)$ s.t. $\dot{x}_k(s) = \left(\partial_{p_k}F\right)\big(x(s), z(s), p(s)\big)$

Expanded	Compact
• $\dot{x}_k(s) = \left(\partial_{p_k}F\right)\big(x(s), z(s), p(s)\big)$ • $\dot{p}_j(s) = -\left(\partial_{x_j}F\right)\big(x(s), z(s), p(s)\big) - p_j(s)(\partial_z F)\big(x(s), z(s), p(s)\big)$ • $\dot{z}(s) = \sum_{j=1}^{d} p_j\left(\partial_{p_j}F\right)\big(x(s), z(s), p(s)\big)$	• $\dot{x} = \left(\nabla_p F\right)(x,z,p)$ • $\dot{z} = p \cdot \left(\nabla_p F\right)(x,z,p)$ • $\dot{p} = -(\nabla_x F)(x,z,p) - p(\partial_z F)(x,z,p)$

- Pick $y \in \partial U$, then set $x_y(0) = y$, $z_y(0) = g(y)$
- $p_y(0)$ is the solution to:
  - $F(x_y(0), z_y(0), p_y(0)) = 0$
  - $\forall \nu$ tangent to Γ at y, $\nu \cdot p_y(0) = \nu \cdot \nabla g$

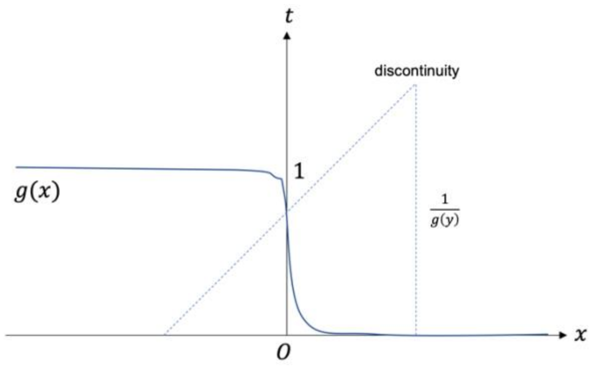
Scalar Conservation Law

- Let $u\colon t, x \to \mathbb{R}$, $f\colon t, x \to \mathbb{R}$
  - u: density, f: flux
  - [1D Conservation Law] $\partial_t u + \partial_x f = 0$
- [1D Scalar Conservation Law] $f\colon \mathbb{R} \to \mathbb{R}$, quasilinear PDE
  - $\partial_t u + f'(u)\partial_x u = 0$ in U
  - $u = g$ on Γ
- All characteristics are straight lines.
  - $x_y(s) = \begin{bmatrix} 1 \\ f'\big(g(y)\big) \end{bmatrix} s + \begin{bmatrix} 0 \\ y \end{bmatrix}$
  - $z_y(s) = g(y)$
  - $u(t,x) = g(y)$ for some y such that $x = tf'\big(g(y)\big) + y$

Singularity Formation

- [Bounded Integral Solution] Given a scalar conservation law with f, g, a bounded and locally integrable function is a function $u\colon (0,\infty)_t \times \mathbb{R} \to \mathbb{R}$ is an <u>integral solution</u> if $\int_0^\infty \int_{\mathbb{R}_x} (u\,\partial_t\phi + f(u)\,\partial_x\phi)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}_x} g(x)\phi(0,x)\,\mathrm{d}x = 0\ \forall \phi \in C_0^\infty(\mathbb{R}_t \times \mathbb{R}_x)$
  - If $u \in C^1([0,\infty) \times \mathbb{R})$ is bounded, then u is a classical solution and hence is a bounded integral solution.
- [Lemma] Let $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$. Assume that $\exists x_0 \in \mathbb{R}$ s.t. $f''\big(g(x_0)\big)g'(x_0) < 0$, then $\sup_{x\in\mathbb{R}}|\partial_x u(t,\cdot)| \to +\infty$ as $t \to T^-$ where $T = -\frac{1}{f''(g(x_0))g'(x_0)}$
- [Shock Curve] The <u>shock curve</u> is a curve $\{(t,x) | x = \sigma(t)\}$ where the solution u is not continuous i.e. there is a jump discontinuity.
- [Rankine-Hugonoit] For a shock solution $u(t,x)$, the speed of the shockwave $\sigma'(t)$ is given by $\sigma'(t) = \frac{f(u_+(t)) - f(u_-(t))}{u_+(t) - u_-(t)}$
  - $u_+(t) = \lim_{x\to\sigma(t)^+} u(t,x)$

- $u_-(t) = \lim_{x \to \sigma(t)^-} u(t,x)$
- Characteristic lines crash into the shock curve from left and right



- Example: Burger's equation
    - $\partial_t u + u\,\partial_x u = 0$ in $U = (0,\infty)_t \times \mathbb{R}_x$
        - $f(u) = \frac{1}{2}u^2$, $f'(u) = u$
    - $u = g$ on $\Gamma = \{t = 0\} \times \mathbb{R}_x$
    - $x_y(s) = \begin{bmatrix} 1 \\ g(y) \end{bmatrix} s + \begin{bmatrix} 0 \\ y \end{bmatrix}$
    - $z_y(s) = g(y)$

Change of Coordinates

- See method of characteristics as a change of coordinates
- Note: x here refers to the actual x and not the x in the characteristic equations
- $(s,y) \leftrightarrow (t,x)$
    - Write out the equations for change of coordinates, usually $t = f_1(s,y)$, $x = f_2(s,y)$
    - Write out the equations for $u(s,y)$ from $z = u\left(s, x_y(s)\right)$
    - When querying values, remember which coordinates you are querying in
- $\partial_y u = \frac{\partial t}{\partial y}\partial_t u + \frac{\partial x}{\partial y}\partial_x u$
- $\partial_s u = \frac{\partial t}{\partial s}\partial_t u + \frac{\partial x}{\partial s}\partial_x u$
- Can solve for $\partial_x u$ and $\partial_t u$

Dispersion

General Form

• Given a constant coefficient linear equation, plug in $u(t,x) = Ae^{i(\xi x - \omega t)}$
  ○ ξ: wavenumber
  ○ ω: frequency
  ○ $\frac{\partial \omega}{\partial \xi}$: group velocity (velocity of the envelope)
• [Dispersive] $\frac{\partial^2 \omega}{\partial \xi^2} \neq 0$
• [Ehrenfest] Let u solve the following PDEs. Then $\frac{\mathrm{d}}{\mathrm{d}t}\langle xu, u\rangle = \langle v\left(\frac{1}{\mathrm{i}}\partial_x\right)u, u\rangle$ where $v(\xi) = \frac{\partial \omega}{\partial \xi}$ and $\langle f, g\rangle = \int_{\mathbb{R}} f\bar{g}\,\mathrm{d}x$
  ○ $-i\partial_t u + \omega\left(\frac{1}{i}\partial_x\right)u = 0$ on $(0,\infty)_t \times \mathbb{R}_x$
  ○ $u(0,x) = g$ on $\mathbb{R}_x$