Classics

Continuous Time Markov Chain (CTMC)

<u>Set-up</u>:

- $(X_t)_{t\geq 0}$, $\left(p_t(i,j)\right)_{t\geq 0, i, j}$, $p_t(i,j) = \mathbb{P}[X_{s+t} = j | X_s = i]$

<u>Construction of CTMC</u>:

From DTMC:

- DTMC: $(X_n)_{n=0}^{\infty}$, $(Y_t)_{t\geq 0}$, $PP(\lambda)$
- $(Z_t)_{t\geq 0} = \left(X_{Y(t)}\right)_{t\geq 0}$ is CTMC with $q(i,j) = \lambda p(i,j)$

From $Expo(1)$ $t_1, \dots, t_n$ and $\left(q(i,j)\right)_{i\neq j}$:

- Define DTMC $(Y_n)_{n=0}^{\infty}$ with $p(i,j) = \frac{q(i,j)}{\sum_{j\in S} q(i,j)} = \frac{q(i,j)}{\lambda_i}$
- Stay for $\lambda_i = \sum_{j\in S} q(i,j)$
- Start at x_0, stay for $\frac{t_i}{\lambda_{x_0}} \sim \mathrm{Expo}(\lambda x_0)$ time

Obtaining jump chain from CTMC $(X_t)_{t\geq 0}$ and Q. $P(i,j) = \frac{q(i,j)}{\lambda_i}$ for $i \neq j$.

<u>Properties</u>:

- $p_{t+h}(i,j) = \sum_{k\in S} p_t(i,k) p_h(k,j)$
- $q(i,j) = \lim_{h\to 0} \frac{p_h(i,j)}{h}$ is the rate of flow from state i to state j
- $\lambda_i = \sum_{j\in S} q(i,j)$ is the rate of exit from state i
- $Q(i,j) = \begin{cases} q(i,j), & i \neq j \\ -\lambda_i, & i = j \end{cases}$
- Rows of Q sums to 0
- $p_t' = Q p_t$
- $p_t = e^{Qt} = \mathbb{I} + \frac{1}{1!} Qt + \frac{1}{2!} Q^2 t^2 + \cdots$
- CTMCs are aperiodic for any $t \geq 0$ since $p_t(i,i) > 0$.
- $(X_t)_{t\geq 0}$ is irreducible if for any $i, j \in S$, $\exists i_1, \dots, i_m, q_{i,i_1}, q_{i_1,i_2}, \dots, q_{i_m,j} > 0$ i.e. exists a path with positive flow.
- CTMC is irreducible if the jump chain is irreducible.
- A stationary distribution is a π such that $\pi P_t = \pi \ \ \forall t \geq 0 \Leftrightarrow \pi Q = 0$
- If $(X_t)_{t\geq 0}$ is irreducible, then $p_t(i,j) > 0$ for any i, j and $t > 0$ i.e. $(X_t)_{t\geq 0}$ is regular.
- If an irreducible CTMC has a stationary distribution π, then $\lim_{t\to 0} p_t(i,j) = \pi_j$ for any i.
- Let $(X_t)_{t\geq 0}$ be a irreducible CTMC.

Martingale

<u>Conditional Expectation (revisited)</u>:

- $\mathbb{E}[X; A] = \mathbb{E}[X \mathbb{1}\{A\}]$, $\quad \mathbb{E}[X|A] = \frac{\mathbb{E}[X;A]}{\mathbb{P}[A]}$
- [Linearity] $\mathbb{E}[\sum_{i=1}^m a_i X_i\, ; A] = \sum_{i=1}^m a_i \mathbb{E}[X_i; A]$
- [Linearity] $\mathbb{E}[\sum_{i=1}^m a_i X_i\, | A] = \sum_{i=1}^m a_i \mathbb{E}[X_i | A]$
- [Jensen] ϕ convex, $\mathbb{E}[\phi(X)|A] \geq \phi(\mathbb{E}[X|A])$
- If $B = \cup_{i=1}^n A_i$ and $A_i \cap A_j = \phi$ for $i \neq j$, then $\mathbb{E}[X; B] = \sum_{i=1}^n \mathbb{E}[X; A_i]$.
- $\mathbb{1}_B = \mathbb{1}_{\cup_{i=1}^n A_i} = \mathbb{1}_{A_1} + \cdots + \mathbb{1}_{A_n}$
- $\mathbb{E}[X|B] = \sum_{i=1}^n \mathbb{E}[X|A_i] \frac{\mathbb{P}[A_i]}{\mathbb{P}[B]}$
  - $\mathbb{E}[X] = \sum_{i=1}^n \mathbb{E}[X|A_i] \mathbb{P}[A_i]$ if $B = \Omega$
- Let $\mathcal{A} = \{A_1, \dots, A_n\}$ be collectively exhaustive, pairwise disjoint partition.
- $\mathbb{E}[X|\mathcal{A}] = \sum_{i=1}^n \mathbb{E}[X|A_i] \mathbb{1}_{A_i}$

<u>Definition</u>:

- Say $(M_i)_{i=0}^{\infty}$ is a <u>martingale</u> w.r.t. $(X_i)_{i=0}^{\infty}$ if
  - $\forall n \geq 0$, $\mathbb{E}[|M_i|] < \infty$
  - M_n depends on $(X_i)_{i=0}^{n}$ and M_0 only
  - $\mathbb{E}[M_{n+1} | M_0, X_1, \dots, X_n] = M_n$ OR
  - $\mathbb{E}[M_{n+1} - M_n | M_0, X_1 \dots, X_n] = 0$ OR
  - $\mathbb{E}[M_{n+1} - M_n | M_0 = m_0, X_1 = x_1 \dots, X_n = x_n] = 0 \ \forall m_0, x_1, \dots, x_n$
- [Super] $\mathbb{E}[M_{n+1} - M_n | M_0, X_1, \dots, X_n] \leq 0$
- [Sub] $\mathbb{E}[M_{n+1} - M_n | M_0, X_1, \dots, X_n] \geq 0$
- [Admissible] $(H_n)_{n=1}^{\infty}$ is <u>admissible</u> if H_n can be determined from $M_0, X_1, \dots, X_{n-1}$
- [Wealth] Let $(M_i)_{i=0}^{\infty}$ be a sequence and H_n be an admissible strategy. Then <u>wealth</u> is: $W_n = W_0 + \sum_{m=1}^n H_m (M_m - M_{m-1})$
  - M_i: price of stock at time i
  - H_i: amount of stock held at time i
- [Stopping Time] T is a <u>stopping time</u> w.r.t. $(X_i)_{i=1}^{\infty}$ if the event $\{T = m\}$ can be determined from $M_0, X_1, \dots, X_m$ $\forall m$.
  - $T = \min\{m | X_m = 1\}$
  - $T = \min\{m | X_m = X_{m-1} = X_{m-2} = 1\}$

<u>Properties and Theorems</u>:

- $(M_n)_{n=1}^{\infty}$ is a super $\Leftrightarrow$ $(-M_n)_{n=1}^{\infty}$ is a sub
- $(M_n)_{n=1}^{\infty}$ is a martingale $\Leftrightarrow$ $(M_n)_{n=1}^{\infty}$ is a supermartingale and a submartingale.
- If M_m is a supermartingale and $m \leq n$, then $\mathbb{E}[M_m] \geq \mathbb{E}[M_n]$
- If M_m is a submartingale and $0 \leq m < n$, then $\mathbb{E}[M_m] \leq \mathbb{E}[M_n]$
- If M_m is a martingale and $0 \leq m < n$, then $\mathbb{E}[M_m] = \mathbb{E}[M_n]$

- $S_0 = \sup\{t | X_t = X_0\}$ (first time which you leave the initial state)
- $R_i = \min\{t > S_0 | X_t = i, X_0 = i\}$ (time it takes for you to return to i)
- $m_i = \mathbb{E}[R_i]$ i.e. expected return time starting at i

- If $m_j > 0$, CTMC is positive recurrent and there will be a limiting distribution π
  - $\pi_j = \frac{1}{\lambda_j m_j}$
  - $\lim_{t\to\infty} p_t(i,j) = \frac{1}{\lambda_j m_j} = \pi_j$
- If exists π such that $\pi Q = 0$, then CTMC must be positive recurrent and π must be the limiting distribution. $\pi_j = \lim_{t\to\infty} \mathbb{E}\left[\frac{1}{T}\int_0^T \mathbb{1}_{X_t=j} \mathrm{d}t \,|X_0 = i\right]$ i.e. proportion of time spent in j starting from i.

Detailed balance:

- If $\pi_k q(k,j) = \pi_j q(j,k)$ for $j \neq k$, then $\pi Q = 0$.

Hitting Time:

- $S = \{1,2,\dots,n\} \cup \{n+1,\dots,N\}$ partition into transient and absorbing states
- $T = \min\{t | X_t \geq n+1\}$ is time of absorption
- $\mathbb{P}[X_T = k | X_0 = i] = u_{i,k}, \quad i \in [n], k \in \{n+1,\dots,N\}$ so $u_{i,k}$ is the probability of getting absorbed at state k starting from i.
- $\mathbb{E}[T | X_0 = i] = w_{i,k}$
- Absorbing $\Leftrightarrow \lambda_k = \sum_{j\neq k} q(k,j) = 0$
- $Q = \begin{bmatrix} R & S \\ 0 & 0 \end{bmatrix}$
- $U = (-R)^{-1} S$
- $u_{i,k} = \frac{q_{i,k}}{\lambda_i} + \sum_{j\in[n]\setminus i} \frac{q(i,j)}{\lambda_i} u_{j,k}$
- $w_i = \frac{1}{\lambda_i} + \sum_{j\in[n]} \frac{q(i,j)}{\lambda_i} w_j$
- $w_i = \mathbb{E}[g(Y_i,i)] + \sum_{j\in[n]} \frac{q_{i,j}}{\lambda_i} w_j$ where $Y_i \sim \mathrm{Expo}(\lambda_i)$ where $g(Y_i,i)$ is the cost of staying Y_i time at state i.
- $w = (-R)^{-1} \begin{bmatrix} \lambda_1 \mathbb{E}[g(Y_1,1)] \\ \vdots \\ \lambda_n \mathbb{E}[g(Y_n,n)] \end{bmatrix}$
- $(-R)_{ij}$ is the expected amount of time spent in state j starting from state i.

- Let $(M_n)_{n=1}^\infty$ be a martingale w.r.t. $(X_n)_{n=1}^\infty$ and ϕ convex. Then $\left(\phi(M_n)\right)_{n=1}^\infty$ is a submartingale w.r.t. $(X_n)_{n=1}^\infty$
- Let $(M_n)_{n=1}^\infty$ be a supermartingale w.r.t. $(X_n)_{n=1}^\infty$ and $(H_n)_{n=1}^\infty$ admissible with $0 \leq H_n \leq c_n$ (i.e. H_n is bounded $\forall n$), then $(W_n)_{n=1}^\infty$ is a supermartingale.
  - If $(M_n)_{n=1}^\infty$ is a submartingale, then $(W_n)_{n=1}^\infty$ also submartingale
  - If $(M_n)_{n=0}^\infty$ is a martingale and $|H_n| \leq c_n$, then $(W_n)_{n=1}^\infty$ is a martingale
- Let $(M_n)_{n=1}^\infty$ be a supermartingale w.r.t $(X_n)_{n=1}^\infty$ and T is a stopping time, then the stopped process $(M_n)_{n=1}^{\min(T,n)}$ is a supermartingale w.r.t. $(X_n)_{n=1}^\infty$.
  - $\mathbb{E}[W_n] = \mathbb{E}\left[M_{\min(T,n)}\right] \leq \mathbb{E}[M_0] = \mathbb{E}[W_0]$
  - If $(M_n)_{n=1}^\infty$ martingale, then $\left(M_{\min(n,T)}\right)_{n=1}^\infty$ is also a martingale and $\mathbb{E}\left[M_{\min(T,n)}\right] = \mathbb{E}[M_0] \; \forall n$
  - In general, $\mathbb{E}[M_T] \neq \mathbb{E}[M_0]$
- Let $(M_n)_{n=1}^\infty$ be a martingale and T be a stopping time with $\mathbb{P}[T < \infty] = 1$ and $\left|M_{\min(T,n)}\right| \leq K$ for some constant K, then $\mathbb{E}[M_T] = \mathbb{E}[M_0]$
- [Wald] If T is a stopping time with $\mathbb{E}[T] < \infty$, then $\mathbb{E}[S_T - S_0] = \mathbb{E}[X_i]\mathbb{E}[T]$

Examples:

- Let $(X_n)_{n=1}^\infty$ be i.i.d. with mean μ. Then $(M_n)_{n=1}^\infty$ is a martingale where $M_0 = S_0$, $M_n = S_0 + X_1 + \cdots + X_n - n\mu$.
- Let $(X_n)_{n=1}^\infty$ be i.i.d. with mean 0 and variance σ^2. Let $S_n = S_0 + X_1 + \cdots + X_n$. Then $(M_n)_{n=1}^\infty$ where $M_n = S_n^2 - n\sigma^2$ is a martingale with respect to $(X_n)_{n=1}^\infty$.
- Let $(X_n)_{n=1}^\infty$ be i.i.d. with mean 1 nonnegative, then $(M_n)_{n=1}^\infty$ be such that $M_n = M_0 X_1 \dots X_n$ is a martingale.
- Let $(X_n)_{n=1}^\infty$ be i.i.d. and $\theta \in \mathbb{R}$ such that $\phi(\theta) = \mathbb{E}\left[e^{\theta X_i}\right] < \infty$. Then $M_n = \frac{e^{\theta(X_1+\cdots+X_n)}}{\phi(\theta)^n}$ is a martingale w.r.t. $(X_n)_{n=1}^\infty$
- $H_m = \mathbb{1}\{T \geq m\}$ admissible, $W_n = M_{\min(n,T)}$
- $M_n = M_0 + X_1 + \cdots + X_n$ and $\mathbb{P}[X_i = \pm 1] = \frac{1}{2}$. $M_0 = x$. $T = \min\{n | M_n \notin (a,b)\}$.
  - $\left|M_{\min(n,T)}\right| \leq \max(|a|,|b|) \Rightarrow \mathbb{E}[M_T] = \mathbb{E}[M_0]$
- $\mathbb{P}[X_i = 1] = p \neq \frac{1}{2}$. Then $\left(\left(\frac{q}{p}\right)^{M_n}\right)_{n=1}^\infty$ is a martingale. $T = \min\{n | M_n \notin (a,b)\}$.