Classics

Conditional Expectations and Random Sum

- $\mathbb{P}[X = x] = \sum_i \mathbb{P}[X = x|Y = i]\mathbb{P}[Y = i]$
- $f_X(x) = \sum_i f_{X|N}(x|n)\mathbb{P}[N = n]$
- $\mathbb{E}[X] = \sum_i \mathbb{E}[X|Y]\,\mathbb{P}[Y = i]$
  - $\mathbb{E}[X] = \mathbb{E}\big[\mathbb{E}[X|Y]\big]$
- $\mathrm{Var}[X] = \mathrm{Var}\big[\mathbb{E}[X|Y]\big] + \mathbb{E}\big[\mathrm{Var}[X|Y]\big]$
- $\mathbb{P}[\cup_i A_i] \le \sum_i \mathbb{P}[A_i]$
- Let $S = \sum_{i=1}^{N} X_i$ where N, X_i independent, then $\mathbb{E}[S] = \mathbb{E}[N]\mathbb{E}[X_i]$
- If X, Y independent and $Z = X + Y$, then $f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\mathrm{d}x$

Random Walk

- [Stirling] $n! \sim n^n e^{-n}\sqrt{2\pi n}$
- [$\mathbb{Z}^1$] $P_{0,0}^{(2n)} = \binom{2n}{n} p^n q^n \sim \frac{(4pq)^n}{\sqrt{\pi n}}$
  Only recurrent when $p = q = \frac{1}{2}$
- [$\mathbb{Z}^2$] $P_{0,0}^{(2n)} = \binom{2n}{n} p^n q^n \sim \frac{(4pq)^n}{\sqrt{\pi n}}$
  Only recurrent when symmetric
- [$\mathbb{Z}^3$] All transient

Gambler's Ruin

- [$p = q = \frac{1}{2}$] $\mathbb{P}[X_T = 0|X_0 = k] = \frac{N-k}{N}$
- $\mathbb{E}[T|X_0 = k] = k(N - k)$
- [$p \ne \frac{1}{2}$] $\mathbb{P}[X_T = 0|X_0 = i] = \left(\frac{q}{p}\right)^k \frac{1-\left(\frac{q}{p}\right)^{N-k}}{1-\left(\frac{q}{p}\right)^{N}}$

Final Checks and Last Resorts

- Check you modelled the problem correctly
- Return what the problem asked for
- First-step and previous-step analysis
- Understand dynamics of MC first
- Method of differences
- GFs: go back to first principle
- If have time, rigorously prove MC with limits

Branching Process and Generating Functions

- $X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n)}$ with $X_0 = 1$
- $\mathbb{E}[X_{n+1}] = \mu\mathbb{E}[X_n]$ where $\mu = \mathbb{E}[\xi]$
- $\mathbb{E}[X_n] = \mu^n$
- $\mathrm{Var}[X_{n+1}] = \sigma^2\mathbb{E}[X_n] + \mu^2\mathrm{Var}[X_n]$
- $\mathrm{Var}[X_n] = \begin{cases} \sigma^2 n, & \mu = 1 \\ \sigma^2\mu^{n-1}\frac{1-\mu^n}{1-\mu}, & \mu \ne 1 \end{cases}$
- [Time of Extinction] $N = \min_n\{n|X_n = 0\}$
- [Extinction Probability] $u_n = \mathbb{P}[N \le n] = \mathbb{P}[X_n = 0]$ i.e. $u_0 = 0$, $u_1 = p_0$
- $0 \le u_n \le u_{n+1} \le 1\ \forall n$ (monotonicity)
- $u_n = \sum_{k=0}^{\infty} p_k (u_{n-1})^k = \phi_\xi(u_{n-1})$
- u_∞ is the smallest solution to $u = \phi_\xi(u)$
- $\phi_{X_{n+1}}(s) = \phi_{X_n}\left(\phi_\xi(s)\right)$
- If $\phi_{X_0}(s) = s$, $\phi_{X_n}(s) = \phi_\xi^n(s)$
- [Pure Death Process] $\phi_\xi(s) = (1 - p) + ps$; $\phi_{X_n}(s) = 1 - p^n + p^n s$
- [Time of extinction] $\mathbb{P}[T = n|X_0 = k] = \mathbb{P}[X_n = 0|X_0 = k] - \mathbb{P}[X_{n-1} = 0|X_0 = k] = \phi_{X_n}(0) - \phi_{X_{n-1}}(0)$

Case	Result	Remarks
$\mu < 1$	$u_\infty = 1$	1 is sole fixed point
$\mu = 1, \sigma^2 = 0$	$u_\infty = 0$	Self-regeneration
$\mu = 1, \sigma^2 > 0$	$u_\infty = 1$	$\phi_\xi''(1) > 0$
$\mu > 1$	$u_\infty < 1$	$\phi_\xi'(1) = \mu > 1$

- $\phi_X(s) = \mathbb{E}[s^X] = \sum_{x=0}^{\infty} s^x\mathbb{P}[X = x]$; $s \in [0,1]$
- $\phi_X(0) = \mathbb{P}[X = 0]$, $\phi_X(1) = 1$
- $\mathbb{P}[X = k] = \frac{1}{k!}\phi_X^{(k)}(0)$
- If $X = \xi_1 + \dots + \xi_n$ and ξ_i are independent, then $\phi_X(s) = \prod_{i=1}^{n}\phi_{\xi_i}(s)$
- $\phi_X'(1) = \mathbb{E}[X]$
- $\phi_X''(1) = \mathbb{E}[X^2] - \mathbb{E}[X]$
- $\mathrm{Var}[X] = \phi_X''(1) + \phi_X'(1) - \left(\phi_X'(1)\right)^2$
- If $X = \sum_{i=1}^{N}\xi_i$, then $\phi_X(s) = \phi_N\left(\phi_\xi(s)\right)$
- One-to-one correspondence with PMF

Markov Chain (Definitions)

- [Regularity] P regular if $\exists n$ s.t. $P_{i,j}^n > 0\ \forall i, j$
- [Stationary Distribution] π s.t. $\pi P = \pi$
- [Limiting Distribution] $\lim_{n\to\infty} \tau P^n = \pi\ \forall\tau$
- [Accessible] $i \to j \Rightarrow \exists n > 0$ s.t. $P_{i,j}^{(n)} > 0$
- [Communicate] $i \leftrightarrow j \Leftrightarrow i \to j, j \to i$

Markov Chain (Theorems)

- Period and recurrence/transience are class properties i.e. $i \leftrightarrow j \Rightarrow d(i) = d(j)$
- $\exists N$ s.t. $\forall n > N$, $P_{i,i}^{(nd(i))} > 0$
- [Finite] Regular finite MC have unique limiting distribution.

- [Irreducible] $\forall i, j \in S$, $\exists n$ s.t. $P_{i,j}^{(n)} > 0$
- [Period] $d(i) = \gcd\left\{n | P_{ii}^{(n)} > 0\right\}$
  - If $P_{ii}^{(n)} = 0\ \forall n$, define $d(i) = 0$.
  - [Aperiodic] $d(i) = 1$
- [Probability of return in n steps $f_{i,i}^{(n)}$]
  $f_{i,i}^{(n)} \coloneqq \mathbb{P}[X_n = i, X_{n-1} \neq i, \ldots, X_1 \neq i | X_0 = i]$
- [Probability of return $f_{i,i}$]
$$f_{i,i} = \sum_{k=0}^{\infty} f_{i,i}^{(k)}$$
- [Recurrent] State i recurrent $\Leftrightarrow f_{i,i} = 1$
  - $\Leftrightarrow \mathbb{P}[R_i < \infty | X_0 = i] = 1$
  - $\Leftrightarrow \sum_{m=0}^{\infty} P_{i,i}^{(m)} = \infty$
  - $\Leftrightarrow \lim_{m \to \infty} \prod_{j=0}^{m} \left(1 - P_{i,i}^{(j)}\right) = 0$
- [Transient] State i transient $\Leftrightarrow f_{i,i} < 1$
  - $\Leftrightarrow \mathbb{P}[R_i < \infty | X_0 = i] < 1$
  - $\Leftrightarrow \sum_{m=0}^{\infty} P_{i,i}^{(m)} < \infty$
  - $\Leftrightarrow \lim_{m \to \infty} \prod_{j=0}^{m} \left(1 - P_{i,i}^{(j)}\right) > 0$
- $R_i = \min\{n > 0 | X_n = i\}$
- $m_i = \mathbb{E}[R_i | X_0 = i]$
- [Positive Recurrent] $m_i < \infty$
- [Null Recurrent] $m_i = \infty$
- M_i is number of visits to state i
- $M_i = \sum_{k=1}^{\infty} \mathbb{1}\{X_k = i\}$
- $\mathbb{E}[M_i | X_0 = i] = \sum_{k=1}^{\infty} \left(f_{i,i}\right)^k$

- [Infinite/Mean Return Time] Let $(X_n)_{n \geq 0}$ be an irreducible aperiodic recurrent MC, then $\lim_{n \to \infty} P_{ii}^{(n)} = \frac{1}{m_i}$ and $\lim_{n \to \infty} P_{ji}^{(n)} = \frac{1}{m_i}$
- [Infinite/Limiting] In an irreducible, aperiodic, positive recurrent MC, $\exists$ unique limiting distribution π (satisfying $\pi P = \pi$)
- [Infinite/Positive] In a positive recurrent aperiodic class, $\lim_{n \to \infty} P_{jj}^{(n)} = \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}$ and π is uniquely determined by the equations $\pi_i \geq 0$, $\sum_{i=0}^{\infty} \pi_i = 1$ and $\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}$
- [Infinite/Transient] In an irreducible, aperiodic, transient MC, $\lim_{n \to \infty} \mathbb{P}[X_n = j | X_0 = i] = 0\ \forall i, j$
- [Infinite/Null] In an irreducible, aperiodic, null recurrent MC, $\lim_{n \to \infty} \mathbb{P}[X_n = j | X_0 = i] = 0\ \forall i, j$
- [Infinite/Periodic] In an irreducible, periodic MC with period d, $P_{i,i}^{m} = 0$ if $d \nmid m$. Else, $\lim_{n \to \infty} P_{i,i}^{nd} = \frac{d}{m_i}$