Distribution Theory

Definitions

- [Test Function] Let $U \subset \mathbb{R}^d$ be open and $C^\infty(U)$ be the set of all infinitely smooth differentiable functions on U. Then $C_0^\infty(U) = \{\phi \in C^\infty(U) \mid \operatorname{supp}\phi \subset U, \text{compact}\}$. ϕ is said to be a test function.
  - $\psi \in C_0^\infty(U) \Rightarrow \partial_x \psi \in C_0^\infty(U)$
- [Convergence] Let $(\phi_k)_k$ and ϕ be test functions with $\phi_k, \phi: U \to \mathbb{R}$. Then $\phi_k \xrightarrow{T} \phi$ if $\exists\, S \subseteq U$ compact with $\operatorname{supp}\phi_k$, $\operatorname{supp}\phi \subseteq S$ and $\forall j \in \mathbb{N}$, $\left|D^j(\phi_k - \phi)\right| \to 0$ uniformly on S.
  - Equivalently, $\phi_k \xrightarrow{T} \phi$ if $\forall j \in \mathbb{N}$, $\max_{y \in S}\left|D^j(\phi_k - \phi)(y)\right| \to 0$
  - $\left|D^j f\right| = \max\{\partial^j_{x_1, x_2, \ldots, x_d} f\}$ e.g. $|D^2 f| = \max\{\left|\partial^2_{x_1} f\right|, \left|\partial_{x_1 x_2} f\right|, \left|\partial^2_{x_2} f\right|\}$
- [Distribution] A distribution F on U is a functional (i.e. $F: C_0^\infty(U) \to \mathbb{R}$) s.t.:
  - [Linearity] $\forall a, b \in \mathbb{R}$, $\forall \phi, \psi \in C_0^\infty(U)$, $F[a\phi + b\psi] = aF[\phi] + bF[\psi]$
  - [Continuity] For any sequence of test functions $(\phi_k)_k$ s.t. $\phi_k \xrightarrow{T} \phi$, $F[\phi_k] \to F[\phi]$
- [Derivative] Let F be a distribution, then the derivative with respect to x is another functional $\partial_x F$ which is defined by: $(\partial_x F)[\psi] \coloneqq F[-\partial_x \psi] \quad \forall \psi \in C_0^\infty(U)$
  - $\langle \partial_x F, \psi \rangle = -\langle F, \partial_x \psi \rangle \quad \forall \psi \in C_0^\infty(U)$
- [Delta Distribution at $x_0 \in \mathbb{R}^d$] $\delta_{x_0}[\phi] = \phi(x_0)$
  - $\langle \delta_{x_0}, \psi \rangle = \int_{-\infty}^{\infty} \delta_{x_0}(x)\psi(x)\,\mathrm{d}x = \psi(x_0)$
- [Heavy-side Function] $H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$
  - $\partial_x H = \delta_0$
  - $\langle H, \psi \rangle = \int_{-\infty}^{\infty} H(x)\psi(x)\,\mathrm{d}x = \int_0^\infty \psi(x)\,\mathrm{d}x$
- [Induced Distribution] Let $f \in C_0^\infty(\mathbb{R}^d)$, then the induced distribution is F_f with $F_f[\phi] = \int_{\mathbb{R}^d} f(y)\phi(y)\,\mathrm{d}y$.
  - $\langle \partial_x F_f, \psi \rangle = \langle F_{\partial_x f}, \psi \rangle$
- [Weak Derivative] Let $f(x)$ be piecewise continuous, then the weak derivative of f is the distributional derivative of F_f, denoted as $f'(x)$.
- [Convolution] Let $f, g \in C_0^\infty(\mathbb{R}^d)$, then $(f * g)(x) = \int_{\mathbb{R}^d} f(y) g(x - y)\,\mathrm{d}y$
- [Convolution of Distribution with Function] Let F be a distribution on $\mathbb{R}^d$ and $u \in C_0^\infty(\mathbb{R}^d)$, then the convolution is $(F * u)(x) = F[u(x - \cdot)] = \langle F, u(x - \cdot) \rangle$
  - If F is induced by a function w with $w \in C_0^\infty(\mathbb{R}^d)$, then $F * u = w * u$, since $F[u(x - \cdot)] = \int_{\mathbb{R}^d} u(x - y) w(y)\,\mathrm{d}y$
- [Fundamental Solution] Let P be linear, constant coefficient differentiable operator on $\mathbb{R}^d$ (e.g. ∇^2, $\partial_t - \nabla^2$), then distribution E is a fundamental solution for P if $P[E] = \delta_0$.
  - $P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$
  - E is the inverse of P under convolution
  - If $P[u] = f$, first solve $P[E] = \delta_{\{x=0\}}$, then $u = E * f$
- [Adjoint] Let P be a partial differential linear operator i.e. $Pv = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha(v)$. Then the adjoint of P is $P^* v = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha(a_\alpha(x) v)$

Properties

- Let $(f_k)_k, f \in C_0^\infty$ with $(f_k)_k \to f$ uniformly, then $\int_{\mathbb{R}^d} f_k(y)\,\mathrm{d}y \to \int_{\mathbb{R}^d} f(y)\,\mathrm{d}y$
- [Identity of Convolution] δ_0 is the identity with respect to $*$: $\delta_0 * u = u \ \forall u \in C_0^\infty(\mathbb{R}^d)$
- Let f be a function. Then $F_f * u = \int_{\mathbb{R}^d} f(y) u(x - y)\,\mathrm{d}y = f * u$
- Let $u \in C_0^\infty(\mathbb{R}^d)$, then $v = E * u$ solves $Pv = u$.
- Computations with respect to PDEs:
  - $\langle D^\alpha(F * G), v \rangle = (-1)^{|\alpha|} \langle F * G, D^\alpha v \rangle = (-1)^{|\alpha|} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(z) G(y) (D^\alpha v)(z + y)\,\mathrm{d}y\,\mathrm{d}z$
  - $D^\alpha(F * G) = (D^\alpha F) * G = F * (D^\alpha G)$

- Prove by: $\langle D^\alpha(F * G), v\rangle = \langle D^\alpha F * G, v\rangle = \langle F * D^\alpha G, v\rangle$

- Let $u, v \in C_0^\infty(\mathbb{R}^d)$ and L be any differential operator, then $\langle Lu, v\rangle = \langle u, L^* v\rangle$ wher:
  - $Lu = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha u$
  - $L^* v = \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha(a_\alpha(x) v)$

Problem Solving Strategies

- Use compact support! Evaluation at the boundaries is 0.
- Remember that the spirit of distribution is to find a fundamental solution, then superpose all of them together.
- Example: $f(x) = \begin{cases} x^2, & x < 0 \\ x+1, & 0 < x < 1 \\ \frac{3}{2}, & x > 1 \end{cases}$, then $f'(x) = h(x) + \delta_0(x) - \frac{1}{2}\delta_1(x)$ where $h(x) = \begin{cases} 2x, & x < 0 \\ 1, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$
- Just treat distribution as generalized functions: derivatives should intuitively make sense.

Separation of Variables

Ordinary Differential Equations

- [Integrating Factor] $y' + p(x)y = q(x)$
  - $\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\int P(x)\,\mathrm{d}x} y\right) = q(x)e^{\int P(x)\,\mathrm{d}x}$
- [Second Order Differential Equations]
  - $\ddot{x} + \lambda x = 0$
  - $[\lambda > 0]$ $x(t) = A\cos\left(\sqrt{\lambda}t\right) + B\sin\left(\sqrt{\lambda}t\right)$
  - $[\lambda = 0]$ $x(t) = C_1 t + C_2$
  - $[\lambda < 0]$ $x(t) = Ae^{\sqrt{-\lambda}t} + Be^{-\sqrt{-\lambda}t}$

Boundary Conditions

- Boundary conditions
  - (D) Dirichlet $\begin{cases} u(t,0) = 0, \ t \geq 0 \\ u(t,l) = 0, \ t \geq 0 \end{cases}$
    - Fixed endpoints
  - (N) Neumann $\begin{cases} \partial_x u(t,0) = 0, \ t \geq 0 \\ \partial_x u(t,l) = 0, \ t \geq 0 \end{cases}$
    - Slack endpoints, but the endpoints may move i.e. $\partial_t u(t,0)$ and $\partial_t u(t,l)$ may not necessarily be 0
  - (P) Periodic $\begin{cases} u(t,-l) = u(t,l), \ t \geq 0 \\ \partial_x u(t,-l) = \partial_x u(t,l), \ t \geq 0 \end{cases}$
    - u can be periodically extended smoothly
- Problems
  - (H) Heat equation
    - $(-\partial_t + \partial_x^2)u = 0$ where $u \in C\left((0,\infty)_t \times (0,l)\right)$
    - $u(0,x) = g(x)$ on $\{t = 0\} \times (0,l)$ or $(-l,l)$
  - (W) Wave equation
    - $(-\partial_t^2 + \partial_x^2)u = 0$ where $u \in C((0,\infty)_t \times U)$
    - $u(0,x) = g(x)$ on $\{t = 0\} \times (0,l)$ or $(-l,l)$
    - $\partial_t u(0,x) = h(x)$ on $\{t = 0\} \times (0,l)$ or $(-l,l)$
- Let $u(t,x) = T(t)X(x)$

	(D) $U = [0,l]$	(N) $U = [0,l]$	(P) $U = [-l,l]$
(H)	• $T(t) = Ae^{-\left(\frac{k\pi}{l}\right)^2 t}$, $k \in \mathbb{Z}$ • $X(x) = B\sin\left(\frac{k\pi}{l}x\right)$ • $u(t,x) = \sum_{k=1}^{\infty} C_k e^{-\left(\frac{k\pi}{l}\right)^2 t}\sin\left(\frac{k\pi}{l}x\right)$ • $g(x) = \sum_{k=1}^{\infty} C_k \sin\left(\frac{k\pi}{l}x\right)$ • $C_k = \frac{2}{l}\int_0^l g(x)\sin\left(\frac{k\pi}{l}x\right)\mathrm{d}x$	• $T(t) = Ae^{-\left(\frac{k\pi}{l}\right)^2 t}$, $k \in \mathbb{Z}$ • $X(x) = B\cos\left(\frac{k\pi}{l}x\right)$ • $u(t,x) = \frac{1}{2}C_0 + \sum_{k=1}^{\infty} C_k e^{-\left(\frac{k\pi}{l}\right)^2 t}\cos\left(\frac{k\pi}{l}x\right)$ • $g(x) = \frac{1}{2}C_0 + \sum_{k=1}^{\infty} C_k \cos\left(\frac{k\pi}{l}x\right)$ • $C_k = \frac{2}{l}\int_0^l g(x)\cos\left(\frac{k\pi}{l}x\right)\mathrm{d}x$	• $T(t) = Ae^{-\left(\frac{k\pi}{l}\right)^2 t}$, $k \in \mathbb{Z}$ • $X(x) = B_1\cos\left(\frac{k\pi}{l}x\right) + B_2\sin\left(\frac{k\pi}{l}x\right)$ • $u(t,x) = \frac{1}{2}C_0 + \sum_{k=1}^{\infty} C_k e^{-\left(\frac{k\pi}{l}\right)^2 t}\cos\left(\frac{k\pi}{l}x\right) + \sum_{k=1}^{\infty} C_k' e^{-\left(\frac{k\pi}{l}\right)^2 t}\sin\left(\frac{k\pi}{l}x\right)$ • $g(x) = \frac{1}{2}C_0 + \sum_{k=1}^{\infty} C_k \cos\left(\frac{k\pi}{l}x\right) + \sum_{k=1}^{\infty} C_k' \sin\left(\frac{k\pi}{l}x\right)$ • $C_k = \frac{1}{l}\int_{-l}^l g(x)\cos\left(\frac{k\pi}{l}x\right)\mathrm{d}x$ • $C_k' = \frac{1}{l}\int_{-l}^l g(x)\sin\left(\frac{k\pi}{l}x\right)\mathrm{d}x$
(W)	• $T(t) = A_1\cos\left(\frac{k\pi}{l}t\right) + A_2\sin\left(\frac{k\pi}{l}t\right)$, $k \in \mathbb{Z}$	• $T(t) = A_1\cos\left(\frac{k\pi}{l}t\right) + A_2\sin\left(\frac{k\pi}{l}t\right)$, $k \in \mathbb{Z}$	• $T(t) = A_1\cos\left(\frac{k\pi}{l}t\right) + A_2\sin\left(\frac{k\pi}{l}t\right)$, $k \in \mathbb{Z}$

	• $X(x) = B\sin\left(\frac{k\pi}{l}x\right)$ • $u(x,t) = \sum_{k=1}^{\infty}\left(A_k\cos\left(\frac{k\pi}{l}t\right) + B_k\sin\left(\frac{k\pi}{l}t\right)\right)\sin\left(\frac{k\pi}{l}x\right)$ • $g(x) = \sum_{k=1}^{\infty} A_k\sin\left(\frac{k\pi}{l}x\right)$ • $h(x) = \sum_{k=1}^{\infty} B_k\frac{k\pi}{l}\sin\left(\frac{k\pi}{l}x\right)$ • $A_k = \frac{2}{l}\int_0^l g(x)\sin\left(\frac{k\pi}{l}x\right)\mathrm{d}x$ • $B_k = \frac{2}{k\pi}\int_0^l h(x)\sin\left(\frac{k\pi}{l}x\right)\mathrm{d}x$	• $X(x) = B\cos\left(\frac{k\pi}{l}x\right)$ • $u(x,t) = \sum_{k=0}^{\infty}\left(A_k\cos\left(\frac{k\pi}{l}t\right) + B_k\sin\left(\frac{k\pi}{l}t\right)\right)\cos\left(\frac{k\pi}{l}x\right)$ • $g(x) = \sum_{k=0}^{\infty} A_k\cos\left(\frac{k\pi}{l}x\right)$ • $h(x) = \sum_{k=0}^{\infty} B_k\frac{k\pi}{l}\cos\left(\frac{k\pi}{l}x\right)$ • $A_k = \frac{2}{l}\int_0^l g(x)\cos\left(\frac{k\pi}{l}x\right)\mathrm{d}x$ • $A_0 = \frac{1}{l}\int_0^l g(x)\mathrm{d}x$ • $B_k = \frac{2}{k\pi}\int_0^l h(x)\cos\left(\frac{k\pi}{l}x\right)\mathrm{d}x$ • $B_0 = \frac{1}{k\pi}\int_0^l h(x)\,\mathrm{d}x$	• $X(x) = B_1\cos\left(\frac{k\pi}{l}x\right) + B_2\sin\left(\frac{k\pi}{l}x\right)$ • $u(x,t) = \sum_{k=0}^{\infty}\left(A_k\cos\left(\frac{k\pi}{l}t\right) + B_k\sin\left(\frac{k\pi}{l}t\right)\right)\cos\left(\frac{k\pi}{l}x\right) + \sum_{k=1}^{\infty}\left(C_k\cos\left(\frac{k\pi}{l}t\right) + D_k\sin\left(\frac{k\pi}{l}t\right)\right)\sin\left(\frac{k\pi}{l}x\right)$ • $g(x) = \sum_{k=0}^{\infty} A_k\cos\left(\frac{k\pi}{l}x\right) + \sum_{k=1}^{\infty} C_k\sin\left(\frac{k\pi}{l}x\right)$ • $h(x) = \sum_{k=0}^{\infty} B_k\frac{k\pi}{l}\cos\left(\frac{k\pi}{l}x\right) + \sum_{k=1}^{\infty} D_k\frac{k\pi}{l}\sin\left(\frac{k\pi}{l}x\right)$ • $A_k = \frac{1}{l}\int_{-l}^{l} g(x)\cos\left(\frac{k\pi}{l}x\right)\mathrm{d}x$ • $A_0 = \frac{1}{2l}\int_{-l}^{l} g(x)\,\mathrm{d}x$ • $B_k = \frac{1}{k\pi}\int_{-l}^{l} h(x)\cos\left(\frac{k\pi}{l}x\right)\mathrm{d}x$ • $B_0 = \frac{1}{2k\pi}\int_{-l}^{l} h(x)\,\mathrm{d}x$ • $C_k = \frac{1}{l}\int_{-l}^{l} g(x)\sin\left(\frac{k\pi}{l}x\right)\mathrm{d}x$ • $D_k = \frac{1}{k\pi}\int_{-l}^{l} h(x)\sin\left(\frac{k\pi}{l}x\right)\mathrm{d}x$

Inhomogeneous Problems

- [Inhomogeneous Heat Equation]
  - Look for the form of solution $u(t,x) = e^{t\Delta}v(t,x)$
    - $e^{t\Delta} = F^{-1}\circ e^{-|\xi|^2 t}\circ F$
  - $u(t,x) = e^{t\Delta}f(x) + \int_0^t e^{(t-s)\Delta}F(s,x)\,\mathrm{ds}$
    - $e^{t\Delta}f(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-\frac{\|x-y\|_2^2}{4t}}f(y)\,\mathrm{d}y$
    - $\int_0^t e^{(t-s)\Delta}F(s,x)\,\mathrm{ds} = \int_0^t\int_{\mathbb{R}^d}\frac{1}{(4\pi(t-s))^{\frac{d}{2}}}e^{-\frac{\|x-y\|_2^2}{4(t-s)}}\,\mathrm{ds}$
- [Inhomogeneous Wave Equation]
  - Look for solutions in the form of $\begin{bmatrix} u \\ u_t \end{bmatrix} = e^{tA}\begin{bmatrix} f \\ g \end{bmatrix}$
  - $e^{tA} = \begin{bmatrix} F^{-1} & 0 \\ 0 & F^{-1} \end{bmatrix} e^{tA(\xi)}\begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}$

Fourier Analysis

Definitions (Fourier Series)

- [Complex Valued Fourier Series] A <u>complex valued Fourier series</u> on $(-l,l)$ is a series of the form $\sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi}{l}x}$
- [Inner Product] Let f,g be functions defined on (a,b). Then $\langle f,g\rangle = \int_a^b f(x)\overline{g(x)}\,\mathrm{d}x$
    - If f,g are in addition real-valued functions, then $\langle f,g\rangle = \int_a^b f(x)g(x)\,\mathrm{d}x$
- [L_2 Norm] Let $f:(-l,l)\to\mathbb{R}$, then $\|f\|_{L^2} := \sqrt{\langle f,f\rangle} = \sqrt{\int_{-l}^{l}\big(f(x)\big)^2\mathrm{d}x}$
- [Pointwise Convergence] Let $(f_n)_n$ be a sequence of functions. Then $(f_n)_n$ <u>converges</u> to f <u>pointwise</u> if $\forall x$, $\lim_{n\to\infty} f_n(x) = f(x)$
- [Uniform Convergence] Let $(f_n)_n$ be a sequence of functions. Then $(f_n)_n$ <u>converges</u> to f <u>uniformly</u> if $\forall\epsilon > 0$, $\exists n\in\mathbb{N}$ s.t. $|f_n(x) - f(x)| < \epsilon\ \forall x$
    - $\lim_{n\to\infty}\sup_{x\in(-l,l)} |f_n(x) - f(x)| = 0$
- [L^2 Convergence] Let $f_n:(-l,l)\to\mathbb{R}$. Then $(f_n)_n$ <u>converges</u> to f <u>in L^2</u> if $\lim_{n\to\infty}\|f_n - f\|_{L^2} = 0$
- [$S_N f$] Let $f:(-l,l)\to\mathbb{R}$. Define $S_N f := \frac{A_0}{2} + \sum_{k=1}^{N} A_k\cos\left(\frac{k\pi}{l}x\right) + \sum_{k=1}^{N} B_k\sin\left(\frac{k\pi}{l}x\right)$

Formulae (Fourier Series)

- Let $n,m\in\mathbb{N}$, then:
    - $\langle\sin\left(\frac{n\pi}{l}x\right),\sin\left(\frac{m\pi}{l}x\right)\rangle = \langle\sin\left(\frac{n\pi}{l}x\right),\cos\left(\frac{m\pi}{l}x\right)\rangle = \langle\cos\left(\frac{n\pi}{l}x\right),\cos\left(\frac{m\pi}{l}x\right)\rangle = \begin{cases}\frac{1}{2}l, & n = m\\ 0, & n\neq m\end{cases}$ if the integral is taken from 0 to l
    - $\langle e^{i\frac{n\pi}{l}x}, e^{i\frac{m\pi}{l}x}\rangle = \int_{-l}^{l} e^{i\frac{n\pi}{l}x}e^{-i\frac{m\pi}{l}x}\,\mathrm{d}x = \begin{cases}2l, & n = m\\ 0, & n\neq m\end{cases}$
- [Sine Fourier] Let $g:(0,l)\to\mathbb{R}$ and $\sum_{k=1}^{\infty} B_k\sin\left(\frac{k\pi}{l}x\right)\to g(x)$ uniformly, then $B_k = \frac{2}{l}\int_0^l g(x)\sin\left(\frac{k\pi}{l}x\right)\mathrm{d}x$
- [Cosine Fourier] Let $g:(0,l)\to\mathbb{R}$ and $\frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k\cos\left(\frac{k\pi}{l}x\right)\to g(x)$ uniformly, then $A_k = \frac{2}{l}\int_0^l g(x)\cos\left(\frac{k\pi}{l}x\right)\mathrm{d}x$
- [Full Fourier] Let $g:(-l,l)\to\mathbb{R}$ and $\frac{1}{2}A_0 + \sum_{k=1}^{\infty} A_k\cos\left(\frac{k\pi}{l}x\right) + \sum_{k=1}^{\infty} B_k\sin\left(\frac{k\pi}{l}x\right)\to g(x)$ uniformly, then $A_k = \frac{1}{l}\int_{-l}^{l} g(x)\cos\left(\frac{k\pi}{l}x\right)\mathrm{d}x$ and $B_k = \frac{1}{l}\int_{-l}^{l} g(x)\sin\left(\frac{k\pi}{l}x\right)\mathrm{d}x$
- [Complex Fourier] If $\sum_{k=-\infty}^{\infty} c_k e^{i\frac{k\pi}{l}x}\to g(x)$ uniformly, then $c_k = \frac{1}{2l}\int_{-l}^{l} g(x)e^{-i\frac{k\pi}{l}x}\mathrm{d}x$
    - $c_k = \frac{1}{2}A_k + \frac{1}{2i}B_k = \frac{1}{2l}\int_{-l}^{l} g(x)e^{-i\frac{k\pi}{l}x}\,\mathrm{d}x$
    - $c_{-k} = \frac{1}{2}A_k - \frac{1}{2i}B_k = \frac{1}{2l}\int_{-l}^{l} g(x)e^{i\frac{k\pi}{l}x}\,\mathrm{d}x$
    - $A_k = c_k + c_{-k}$
    - $B_k = i(c_k - c_{-k})$

Theorems (Fourier Series)

- [Sufficient Condition for Pointwise Convergence] If f is continuous on $[-l,l]$ and f' is piecewise continuous on $[-l,l]$, then $S_N f(x)\to f(x)$ pointwise $\forall x\in(-l,l)$
- [Sufficient Condition for Uniform Convergence] If $f\in C^1([-l,l])$ and it is $2l$-periodic, then $S_N f\to f$ uniformly.
- [Sufficient Condition for L^2 Convergence] If f is a $2l$-periodic function s.t. $\|f\|_{L^2} < \infty$, then $S_N f\to f$ in L^2 i.e. $\lim_{N\to\infty}\|S_N f - f\|_{L^2} = 0$

- [Parseval's Identity] Suppose that $f: (-l, l) \to \mathbb{R}$ and $\|f\|_{L^2} < \infty$, then $\int_{-l}^{l} \left(f(x)\right)^2 \mathrm{d}x = \frac{1}{2} l (A_0^2 + \sum_{n=1}^{\infty} A_n^2 + \sum_{n=1}^{\infty} B_n^2)$

Definitions (Fourier Transform)

- [Schwartz Space in $\mathbb{R}$] $\mathcal{S}(\mathbb{R}) \coloneqq \{f \in C^\infty(\mathbb{R}) | x^n \partial_x^m f \text{ bounded } \forall m, n \in \mathbb{N}\}$
  - $\lim\limits_{|x| \to \infty} |\partial_x^m f(x)| = 0$
  - $\int_{-\infty}^{\infty} |\partial_x^m f(x)| \,\mathrm{d}x < \infty \ \forall m \in \mathbb{N}$
  - Example: $f = e^{-x^2}$; non-example: $f = \frac{1}{1+x^2}$
  - Schwartz space is closed under Fourier transform: if $f \in \mathcal{S}(\mathbb{R})$, then $\hat{f} \in \mathcal{S}(\mathbb{R})$
- [Schwartz Space in $\mathbb{R}^d$] $\mathcal{S}(\mathbb{R}^d) \coloneqq \left\{f \in C^\infty(\mathbb{R}^d) | x_1^{\alpha_1} \dots x_d^{\alpha_d} \partial_{x_1}^{\beta_1} \dots \partial_{x_d}^{\beta_d} f \text{ bounded } \forall \alpha_i, \beta_i \in \mathbb{N}\right\}$
- [Fourier Transform in $\mathbb{R}$] Define the Fourier transform F as a functor $F: \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ s.t. $F[f] = \hat{f}$ where $\hat{f}(\xi) \coloneqq \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \mathrm{d}x \ \forall \xi \in \mathbb{C}$
  - x is usually referred to as 'space', ξ as 'frequency'
  - ξ is the conjugate variable of x
  - $\hat{f}$ is also the frequency domain representation of the function f
  - F is also defined for any $f: (-\infty, \infty) \to \mathbb{C}$ with $\int_{-\infty}^{\infty} |f(x)| \,\mathrm{d}x < \infty$.
- [Fourier Transform in $\mathbb{R}^d$] $F: \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ s.t. $F[f] = \hat{f}$ where $\hat{f}(\xi) \coloneqq \int_{\mathbb{R}^d} f(x) e^{-i\xi \cdot x} \mathrm{d}x$
  - $\xi \in \mathbb{R}^d$
  - $\hat{f}(\xi) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(x) e^{-i\xi_1 x_1} \dots e^{-i\xi_d x_d} \,\mathrm{d}x_1 \dots \mathrm{d}x_d$
- [Inverse Fourier Transform in $\mathbb{R}$] Define $F^{-1}\left[\hat{f}\right] = f$ where $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} \mathrm{d}\xi$
  - If $f(x) \in \mathcal{S}(\mathbb{R})$, then $f = F^{-1}\left[\hat{f}\right]$
- [Inverse Fourier Transform in $\mathbb{R}^d$] Define $F^{-1}\left[\hat{f}\right] = f$ where $f(x) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \hat{f}(\xi) e^{-i\xi \cdot x} \mathrm{d}\xi$
  - If $f(x) \in \mathcal{S}(\mathbb{R}^d)$, then $f = F^{-1}\left[\hat{f}\right]$
- [Plancherel] Let $f, g \in \mathcal{S}(\mathbb{R})$, then $\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$ where $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \,\overline{g(x)} \,\mathrm{d}x$
  - In particular, $\|f\| = \left\|\hat{f}\right\|$ where $\|f\| = \sqrt{\int_{-\infty}^{\infty} |f(x)|^2 \,\mathrm{d}x}$

Properties (Fourier Transform)

- Let $f \in \mathcal{S}(\mathbb{R})$. Then Fourier transform interchanges differentiation and multiplication:
  - $g = \partial_x f \Rightarrow \hat{g} = i\xi \hat{f}$ (integration by parts)
  - $g = xf \Rightarrow \hat{g} = i\partial_\xi \hat{f}$ (interchange ∂_ξ and integration)
- Let $f \in \mathcal{S}(\mathbb{R})$, then $\left|\hat{f}(\xi)\right| \leq \int_{-\infty}^{\infty} |f(x)| \,\mathrm{d}x$
- Let $f, g \in \mathcal{S}(\mathbb{R})$, then:
  - $fg \in \mathcal{S}(\mathbb{R})$ and $\widehat{fg} = \frac{1}{2\pi}\left(\hat{f} * \hat{g}\right)$ (inverse Fourier Transform on fg)
  - $f * g \in \mathcal{S}(\mathbb{R})$ and $\widehat{f * g} = \hat{f}\hat{g}$
    - $\widehat{f * g}(\xi) = \int_{-\infty}^{\infty} (f * g)(x) e^{-i\xi x} \mathrm{d}x = \int_{-\infty}^{\infty} f(y) e^{-i\xi y} \mathrm{d}y \int_{-\infty}^{\infty} g(x - y) e^{-i\xi(x-y)} \mathrm{d}(x - y) = \hat{f}(\xi)\hat{g}(\xi)$
- [Higher Dimensions]
  - $\widehat{fg} = \frac{1}{(2\pi)^d}\left(\hat{f} * \hat{g}\right)$
  - $\widehat{f * g} = \hat{f}\hat{g}$
  - If $g = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} f$, then $\hat{g}(\xi) = (i\xi_1)^{\alpha_1} \dots (i\xi_d)^{\alpha_d} \hat{f}(\xi)$
  - If $g = x_1^{\alpha_1} \dots x_d^{\alpha_d} f$, then $\hat{g}(\xi) = i^{\alpha_1} \partial_{\xi_1}^{\alpha_1} \left(\dots \left(i^{\alpha_d} \partial_{\xi_d}^{\alpha_d} \hat{f}(\xi) \right) \right)$
- [Heat Kernel]
  - $F^{-1}\left[e^{-t\xi^2}\right] = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$

- $F\left[e^{-tx^2}\right] = \sqrt{\frac{\pi}{t}} e^{-\frac{\xi^2}{4t}}$

- [Function Transformations]
    - $g(x) = f(-x) \Rightarrow \hat{g}(\xi) = \hat{f}(-\xi)$
    - $g(x) = f(x - h) \Rightarrow \hat{g}(\xi) = e^{-i\xi h}\hat{f}(\xi)$
    - $g(x) = f(\lambda x), \lambda > 0 \Rightarrow \hat{g}(\xi) = \frac{1}{\lambda}\hat{f}\left(\frac{\xi}{\lambda}\right)$

Problem Solving

- Typical line of attack: Fourier transform the PDE on both sides → solve ODE → inverse transform
    - Fourier transform in x variable
    - Fourier transform the initial conditions
- Remember the $\frac{1}{2\pi}$ in the inverse Fourier transform formula
- Remember the sign switch in the exponential for Fourier and inverse Fourier transform