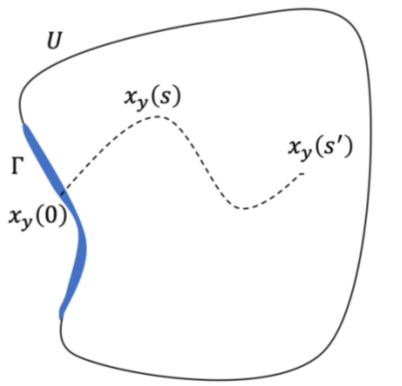
Method of Characteristics

General Form

- [Set Up] $U \subseteq \mathbb{R}^d$ open, $\partial U \in C^1$, $\Gamma \subseteq \partial U$, $F: U \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $u: U \to \mathbb{R}$
  - $x \in U \subseteq \mathbb{R}^d$, $u(x) \in \mathbb{R}$, $\nabla u = \begin{bmatrix} \partial_{x_1} u \\ \vdots \\ \partial_{x_d} u \end{bmatrix} \in \mathbb{R}^d$
  - $F\big(x, u(x), \nabla u(x)\big) = 0$ in U
  - $u = g$ on Γ
  - Idea: want to find $\big(\dot{x}(s), \dot{z}(s), \dot{p}(s)\big) = Q\big(x(s), z(s), p(s)\big)$, so as to apply ODE
- [Procedure]
  - Write ODE for $x(s)$ and $z(s)$
  - $\forall y \in \Gamma$, find trajectory $x_y(s)$ and $z_y(s)$ such that:

Linear First-Order Scalar Equations

- $U \subseteq \mathbb{R}^d$, $a: U \to \mathbb{R}$, $b: U \to \mathbb{R}^d$, $\Gamma \subseteq \partial U$
  - $a(x)u + b(x) \cdot \nabla u - f(x) = 0$ in U
  - $u = g$ on Γ
- $x(s)$: path parametrized by s s.t. $x(0) \in \Gamma \subseteq \partial U$
- $z(s) = u(x(s))$: value function along the path $x(s)$
- $p(s) = (\nabla u)\big(x(s)\big)$: gradient of value function evaluated at point $x(s)$
  - $p_j(s) = \big(\partial_{x_j} u\big)\big(x(s)\big)$
  - $\dot{z} = (\nabla u)(x) \cdot \dot{x} = p \cdot \dot{x}$



$$z_y(s') = z_y(0) + \int_0^{s'} \left(\frac{\mathrm{d}}{\mathrm{d}s} z(s)\right) \mathrm{d}s$$

- Choose $x(s)$ such that $\dot{x} = b(x)$
- Solve system of ODEs via x in terms of s first, then z in terms of s
  - Pick $y \in \Gamma$
  - $x_y(0) = y$, $z_y(0) = u\big(x_y(0)\big) = g(y)$
  - Invert $(s, y) \mapsto x_y(s) \in U$ to solve for $u(x)$ $\forall x \in U$

Expanded	Compact
• $\frac{\mathrm{d}}{\mathrm{d}s} x(s) = b\big(x(s)\big)$ • $\frac{\mathrm{d}}{\mathrm{d}s} z(s) = -a\big(x(s)\big)z(s) + f\big(x(s)\big)$	• $\dot{x} = b(x)$ • $\dot{z} = -a(x)z + f(x)$

- Examples:
  - [Transport / Advection] $\partial_t u + b(x) \cdot \nabla_x u = 0$

Quasilinear Equations

- Equations that are linear with respect to higher order derivatives.
  - $a(x, u)u + b(x, u) \cdot \nabla u = 0$ in U
  - $u = g$ on $\Gamma = \partial U$
- Choose $x(s)$ such that $\dot{x} = b(x, z)$

Expanded	Compact
• $\frac{\mathrm{d}}{\mathrm{d}s} x(s) = b\big(x(s), z(s)\big)$ • $\frac{\mathrm{d}}{\mathrm{d}s} z(s) = -a\big(x(s), z(s)\big)$	• $\dot{x} = b(x, z)$ • $\dot{z} = -a(x, z)$

- Examples:
  - [Burger] $\partial_t u + u\partial_x u = 0$

Fully Nonlinear Scalar Equations

- $F\big(x, u(x), \nabla u(x)\big) = 0$ in U
- $u = g$ on Γ

- $\dot{p}_j(s) = \frac{\mathrm{d}}{\mathrm{ds}}\left(\partial_{x_j}\right)\big(x(s)\big) = \sum_{k=1}^{d}\left(\partial_{x_j}\partial_{x_k}u\right)\big(x(s)\big)\,\dot{x}_k(s)$
- $0 = \partial_{x_j}\big(F(x,z,p)\big) = \left(\partial_{x_j}F\right)(x,z,p) + (\partial_z F)(x,z,p)p_j(s) + \sum_{k=1}^{d}\left(\partial_{p_k}F\right)(x,z,p)\partial_{x_j}\partial_{x_k}u\big(x(s)\big)$
- Pick $x(s)$ s.t. $\dot{x}_k(s) = \left(\partial_{p_k}F\right)\big(x(s), z(s), p(s)\big)$

Expanded	Compact
• $\dot{x}_k(s) = \left(\partial_{p_k}F\right)\big(x(s), z(s), p(s)\big)$ • $\dot{p}_j(s) = -\left(\partial_{x_j}F\right)\big(x(s), z(s), p(s)\big) - p_j(s)(\partial_z F)\big(x(s), z(s), p(s)\big)$ • $\dot{z}(s) = \sum_{j=1}^{d} p_j\left(\partial_{p_j}F\right)\big(x(s), z(s), p(s)\big)$	• $\dot{x} = \left(\nabla_p F\right)(x,z,p)$ • $\dot{z} = p \cdot \left(\nabla_p F\right)(x,z,p)$ • $\dot{p} = -(\nabla_x F)(x,z,p) - p(\partial_z F)(x,z,p)$

- Pick $y \in \partial U$, then set $x_y(0) = y$, $z_y(0) = g(y)$
- $p_y(0)$ is the solution to:
  - $F(x_y(0), z_y(0), p_y(0)) = 0$
  - $\forall \nu$ tangent to Γ at y, $\nu \cdot p_y(0) = \nu \cdot \nabla g$

Scalar Conservation Law

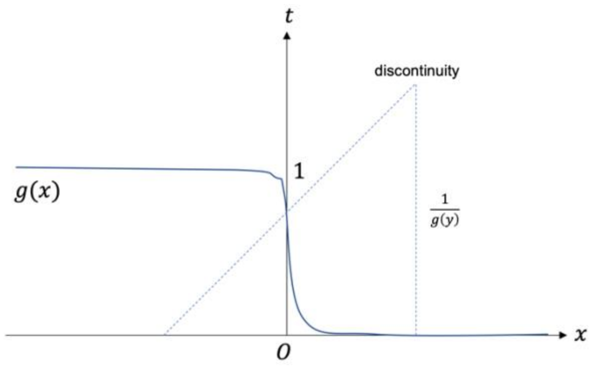
- Let $u: t, x \to \mathbb{R}$, $f: t, x \to \mathbb{R}$
  - u: density, f: flux
  - [1D Conservation Law] $\partial_t u + \partial_x f = 0$
- [1D Scalar Conservation Law] $f: \mathbb{R} \to \mathbb{R}$, quasilinear PDE
  - $\partial_t u + f'(u)\partial_x u = 0$ in U
  - $u = g$ on Γ
- All characteristics are straight lines.
  - $x_y(s) = \begin{bmatrix} 1 \\ f'\big(g(y)\big) \end{bmatrix} s + \begin{bmatrix} 0 \\ y \end{bmatrix}$
  - $z_y(s) = g(y)$
  - $u(t,x) = g(y)$ for some y such that $x = tf'\big(g(y)\big) + y$

Singularity Formation

- [Bounded Integral Solution] A bounded and locally integrable function is a function $u: (0,\infty)_t \times \mathbb{R} \to \mathbb{R}$ is an <u>integral solution</u> if $\int_0^\infty \int_{\mathbb{R}_x} (u\,\partial_t\phi + f(u)\,\partial_x\phi)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}_x} g(x)\phi(0,x)\,\mathrm{d}x = 0\ \forall \phi \in C_0^\infty(\mathbb{R}_t \times \mathbb{R}_x)$
  - If $u \in C^1([0,\infty) \times \mathbb{R})$ is bounded, then u is a classical solution and hence is a bounded integral solution.
- [Lemma] Let $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$. Assume that $\exists x_0 \in \mathbb{R}$ s.t. $f''\big(g(x_0)\big)g'(x_0) < 0$, then $\sup_{x\in\mathbb{R}}|\partial_x u(t,\cdot)| \to +\infty$ as $t \to T^-$ where $T = -\frac{1}{f''\big(g(x_0)\big)g'(x_0)}$
- [Shock Curve] The <u>shock curve</u> is a curve $\{(t,x) | x = \sigma(t)\}$ where the solution u is not continuous i.e. there is a jump discontinuity.
- [Rankine-Hugonoit] For a shock solution $u(t,x)$, the speed of the shockwave $\sigma'(t)$ is given by $\sigma'(t) = \frac{f\big(u_+(t)\big) - f\big(u_-(t)\big)}{u_+(t) - u_-(t)}$
  - $u_+(t) = \lim_{x \to \sigma(t)^+} u(t,x)$

- $u_-(t) = \lim_{x \to \sigma(t)^-} u(t, x)$
- Characteristic lines crash into the shock curve from left and right

- Example: Burger's equation
  - $\partial_t u + u\, \partial_x u = 0$ in $U = (0, \infty)_t \times \mathbb{R}_x$
    - $f(u) = \frac{1}{2}u^2$, $f'(u) = u$
  - $u = g$ on $\Gamma = \{t = 0\} \times \mathbb{R}_x$
  - $x_y(s) = \begin{bmatrix} 1 \\ g(y) \end{bmatrix} s + \begin{bmatrix} 0 \\ y \end{bmatrix}$
  - $z_y(s) = g(y)$



Line of Attack

General Form
- Method of characteristics   - Any PDE of the form $F(x, u, \nabla u) = 0$ - Separation of Variables   - Try a solution of the form $u(t, x) = T(t)X(x)$ - Beware: this might not comprise all solutions   - Fourier Transform   - Keep t, transform x - See if variant of Heat / Wave / Laplacian   - Inverse of a distribution