Fourier Transform

Definitions

- [Fourier Transform] $\mathcal{F}: \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$, $\mathcal{F}: \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$
  - Let $f \in C_0^\infty(\mathbb{R}^d)$, the Fourier transform of f is $\mathcal{F}[f](\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-i\xi\cdot x}\mathrm{d}x$
  - Let $f \in \mathcal{S}(\mathbb{R}^d)$, the Fourier transform of f is $\mathcal{F}[f](\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-i\xi\cdot x}\mathrm{d}x$
  - Let $u \in \mathcal{S}'(\mathbb{R}^d)$, the Fourier transform of f is $\langle \mathcal{F}[u], \phi\rangle_\xi = \langle u, \mathcal{F}^*\phi\rangle$ for $\phi \in \mathcal{S}(\mathbb{R}^d)$
    - Typically, approximate u via $(u_n)_n \to u$ for $u_n \in L^1(\mathbb{R}^d)$. Then compute $\mathcal{F}[u_n]$ via formula, then compute $\mathcal{F}[u] = \lim_{n\to\infty} \mathcal{F}[u_n]$
- [Adjoint] $\mathcal{F}^*[f](x) = \int_{\mathbb{R}^d} f(\xi)e^{i\xi\cdot x}\frac{\mathrm{d}\xi}{(2\pi)^d}$
  - $\mathcal{F}: (\mathbb{R}_x^d, \mathbb{C}) \to \left(\mathbb{R}_\xi^d, \mathbb{C}\right)$
  - $\mathcal{F}^*: \left(\mathbb{R}_\xi^d, \mathbb{C}\right) \to (\mathbb{R}_x^d, \mathbb{C})$
  - $\langle \mathcal{F}f, g\rangle_\xi = \langle f, \mathcal{F}^*g\rangle$
  - $\langle f, g\rangle_\xi = \int_{\mathbb{R}^d} f\bar{g}\frac{\mathrm{d}\xi}{(2\pi)^d}$
  - $\langle f, g\rangle = \int_{\mathbb{R}^d} f\bar{g}\,\mathrm{d}x$
  - $\mathcal{F}^*[f](x) = \int_{\mathbb{R}^d} f(\xi)e^{i\xi\cdot x}\frac{\mathrm{d}\xi}{(2\pi)^d}$
  - Let $u \in \mathcal{S}'(\mathbb{R}^d)$, $\langle \mathcal{F}^*u, \phi\rangle := \langle u, \mathcal{F}\phi\rangle_\xi$ for $\phi \in \mathcal{S}(\mathbb{R}^d)$
  - $\mathcal{F}^*[f](x) = \frac{1}{(2\pi)^d}\mathcal{F}[f](-x)$ (but rarely think of it this way)
- [Inverse Fourier Transform]
  - Let $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$, the inverse Fourier transform $F^{-1}\left[\hat{f}\right](x) = f(x) := \int_{\mathbb{R}^d} \hat{f}(\xi)e^{i\xi\cdot x}\frac{\mathrm{d}\xi}{(2\pi)^d}$
- [Time Space]
  - $\langle a, b\rangle_X = \int_{\mathbb{R}^d} a\bar{b}\,\mathrm{d}x$
- [Frequency Space] $\mathbb{R}_\xi^d$ with measure $\frac{\mathrm{d}\xi}{(2\pi)^d}$
  - $\langle a, b\rangle_\xi = \int_{\mathbb{R}^d} a\bar{b}\frac{\mathrm{d}\xi}{(2\pi)^d}$
- [Schwarz Class] $\mathcal{S}(\mathbb{R}^d; \mathbb{C}) = \left\{\phi \in C^\infty(\mathbb{R}^d; \mathbb{C}): \sup_{x\in\mathbb{R}^d}\left|x^\alpha\partial^\beta\phi\right| < \infty\ \forall\alpha,\beta\right\}$
  - "rapidly decreasing functions"
  - [Convergence] A sequence $(\phi_n)_n \to \phi$ if $\left|x^\alpha\partial^\beta(\phi_n - \phi)\right| \to 0\ \forall\alpha,\beta$ multi-indices
  - Closed under Fourier transform i.e. $\mathcal{F}: \mathcal{S}(\mathbb{R}^d; \mathbb{C}) \to \mathcal{S}(\mathbb{R}^d; \mathbb{C})$
    - $\mathcal{F}^*: \mathcal{S}(\mathbb{R}^d; \mathbb{C}) \to \mathcal{S}(\mathbb{R}^d; \mathbb{C})$
  - $C_0^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d; \mathbb{C})$ and is dense
  - $\mathcal{S}(\mathbb{R}^d; \mathbb{C}) \subset \mathcal{S}'(\mathbb{R}^d; \mathbb{C})$
- [Tempered Distribution] $\mathcal{S}'(\mathbb{R}^d; \mathbb{C})$ is the dual space of $\mathcal{S}(\mathbb{R}^d; \mathbb{C})$. It is the set of continuous conjugate-linear functional on $\mathcal{S}(\mathbb{R}^d; \mathbb{C})$
  - i.e. given $\left(\phi_j\right)_j \to \phi$, $\lim_{j\to\infty}\langle u, \phi_j\rangle = \langle u, \phi\rangle$ for $\left(\phi_j\right)_j, \phi \in \mathcal{S}(\mathbb{R}^d; \mathbb{C})$
  - $\langle u, \phi\rangle_\xi := \frac{1}{(2\pi)^d}u(\phi)$ for $u \in \mathcal{S}'(\mathbb{R}^d; \mathbb{C})$, $\phi \in \mathcal{S}(\mathbb{R}^d; \mathbb{C})$
  - "slowly growing": each derivative of T grows at most as fast as some polynomial
  - $T \in \mathcal{S}'(\mathbb{R}^d; \mathbb{C}) \Leftrightarrow T: \mathcal{S}(\mathbb{R}^d; \mathbb{C}) \to \mathbb{C}$
  - $\exists k, C_k$ s.t. $|T(\phi)| \le C_k\|\phi\|_k\ \forall\phi \in \mathcal{S}(\mathbb{R}^d; \mathbb{C})$
  - $\|\phi\|_k = \max_{|\alpha|+|\beta|\le k}\sup_{x\in\mathbb{R}^d} x^\alpha\partial^\beta\phi$
  - If $(\phi_n)_n, \phi \in \mathcal{S}(\mathbb{R}^d; \mathbb{C})$ with $(\phi_n)_n \to \phi$ in $\mathcal{S}(\mathbb{R}^d; \mathbb{C})$, then $\lim_{j\to\infty}\langle u, \phi_j\rangle = \langle u, \phi\rangle$
  - Given $u \in \mathcal{S}'(\mathbb{R}^d; \mathbb{C})$, $\phi \in \mathcal{S}(\mathbb{R}^d; \mathbb{C})$, $\langle u, \phi\rangle_\xi = \frac{1}{(2\pi)^d}u(\phi) = \int u\bar{\phi}\frac{\mathrm{d}\xi}{(2\pi)^d}$
  - Given $u \in \mathcal{S}'(\mathbb{R}^d; \mathbb{C})$, $\phi \in \mathcal{S}(\mathbb{R}^d; \mathbb{C})$, $\langle \mathcal{F}u, \phi\rangle_\xi := \langle u, \mathcal{F}^*\phi\rangle$

- Given $u \in \mathcal{S}'(\mathbb{R}^d; \mathbb{C})$, $\phi \in \mathcal{S}(\mathbb{R}^d; \mathbb{C})$, $\langle \mathcal{F}^* u, \phi \rangle := \langle u, \mathcal{F}\phi \rangle_\xi$
  - $\mathcal{S}'(\mathbb{R}^d) \subset \mathcal{D}(\mathbb{R}^d)$
- [Convolution] Let $f \in \mathcal{S}'(\mathbb{R}^d)$, $g \in \mathcal{S}(\mathbb{R}^d; \mathbb{C})$. Then $(f * g)(x) = \langle f, \bar{g}(x - \cdot) \rangle$
  - $\mathcal{F}[f * g] = \mathcal{F}[f]\mathcal{F}[g]$
- [Fourier Multiplier] Let $T: \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ be a linear operator. Then T is a Fourier multiplier operator if $\exists\ m \in \mathcal{S}'(\mathbb{R}^d)$ s.t. $\mathcal{F}[Tf] = m\mathcal{F}[f]$
  - [Symbol] Say m is the symbol of T

Theorems

- [8.3] Let $f \in L^1(\mathbb{R}^d)$.
  - $\mathcal{F}[f]$ is well-defined by $\mathcal{F}[f] = \int_{\mathbb{R}^d} f(y) e^{-i\xi \cdot y} \mathrm{d}y$.
  - $\sup_{\xi \in \mathbb{R}^d} |\mathcal{F}[f](\xi)| = \|\mathcal{F}[f]\|_{L^\infty} \le \|f\|_{L^1} = \int_{\mathbb{R}^d} |f(y)| \mathrm{d}y$
  - If $f, \partial_{x^j} f \in L^1(\mathbb{R}^d)$, then $\mathcal{F}[\partial_{x^j} f] = i\xi_j \mathcal{F}[f]$
  - If $f, \partial_{x^j} f \in L^1(\mathbb{R}^d)$, then $\mathcal{F}[f]$ continuously differentiable in ξ_j and $\mathcal{F}[x^j f] = i\partial_{\xi_j}\mathcal{F}[f]$
- Any tempered distribution is of finite order.
- [Fourier Inversion] Let $f \in \mathcal{S}(\mathbb{R}^d; \mathbb{C})$, then $f = \mathcal{F}^*\mathcal{F}[f] = \mathcal{F}\mathcal{F}^*[f]$
- [Fourier Inversion] Let $f \in \mathcal{S}'(\mathbb{R}^d; \mathbb{C})$, then $f = \mathcal{F}^*\mathcal{F}[f] = \mathcal{F}\mathcal{F}^*[f]$
- [Plancherel] Let $f \in \mathcal{S}(\mathbb{R}^d)$, then $\langle f, g \rangle = \int_{\mathbb{R}^d} f(x)\overline{g(x)} \mathrm{d}x = \int_{\mathbb{R}^d} \mathcal{F}[f]\overline{\mathcal{F}[g]} \frac{\mathrm{d}\xi}{(2\pi)^d} = \langle \mathcal{F}[f], \mathcal{F}[g] \rangle_\xi$
  - Let $f \in L^2(\mathbb{R}^d)$, then $\langle f, g \rangle = \langle \mathcal{F}[f], \mathcal{F}[g] \rangle_\xi$
  - Let $f \in L^2(\mathbb{R}^d)$, then $\langle f, g \rangle_\xi = \langle \mathcal{F}^*[f], \mathcal{F}^*[g] \rangle$
- [Schwarz Representation Theorem] For any $u \in \mathcal{S}'(\mathbb{R}^d; \mathbb{C})$, there is a finite collection $u_{\alpha,\beta}: \mathbb{R}^d \to \mathbb{C}$ of bounded continuous functions, $|\alpha| + |\beta| \le k$ s.t. $u = \sum_{|\alpha|+|\beta|\le k} x^\beta \partial^\alpha u_{\alpha,\beta}$
- [1.3] Suppose $u \in \mathcal{S}'(\mathbb{R}^d; \mathbb{C})$ and $x_j u = 0\ \forall j$, then $u = c\delta(x)$ for some constant c
- [1.3] Fourier transform extends by continuity from dense subspace $\mathcal{S}'(\mathbb{R}^d; \mathbb{C}) \subset L^2(\mathbb{R}^d)$ to an isomorphism $\mathcal{F}: L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$
- [Parseval]
- A homogeneous distribution on $\mathbb{R}^d$ is a tempered distribution
- [8.12] A bounded linear operator $T: L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is translation-invariant if and only if it is a Fourier multiplier operator with a symbol $m \in L^\infty(\mathbb{R}^d)$
- [8.17] Let u be a harmonic function on $\mathbb{R}^d$ that is also a tempered distribution. Then u is a polynomial.

Examples

- $e^{-\|x\|^2} \in \mathcal{S}(\mathbb{R}^d; \mathbb{C})$
- $\delta_0 = \frac{1}{(2\pi)^d}\mathcal{F}^*[1] = \mathcal{F}^{-1}[1]$
- $\mathcal{F}^{-1} = \frac{1}{(2\pi)^d}\mathcal{F}^*$
- $\mathcal{F}[1] = 2\pi\delta_0$

Energy Methods

Definitions

- [Schrödinger]
  - $i\partial_t u - \Delta u = f$ in $\mathbb{R}^{1+d}_+$
  - $u = g$ on $\{t = 0\} \times \mathbb{R}^d_+$
- [Translation Operator] Let $y \in \mathbb{R}^d$ and $u \in L^1_{\text{loc}}(\mathbb{R}^d)$. Then $\tau_y u(x) \coloneqq u(x-y)$
  - Let $u \in \mathcal{D}'(U)$, then $\tau_y u$ is implicitly defined via: $\langle \tau_y u, \phi \rangle \coloneqq \langle u, \tau_{-y}\phi \rangle$
- [Open Covering] Let $U \subset \mathbb{R}^d$. A collection $\{V_j\}_{j\in\mathcal{J}}$ of open sets $V_j \subset U$ (w.r.t. the subspace topology) is an <u>open covering</u> if $U = \bigcup_{j\in\mathcal{J}} V_j$
- [Smooth Partition of Unity] A collection of functions $\{\chi_j\}_{j\in\mathcal{J}}$ is a <u>smooth partition of unity</u> subordinate to $\{V_j\}_{j\in\mathcal{J}}$ if:
  - χ_j is smooth $\forall j \in \mathcal{J}$
  - $\operatorname{supp} \chi_j \subset V_j$
  - $\chi_j(x) \in [0,1]\ \forall x \in U$
  - $\sum_{j\in\mathcal{J}} \chi_j(x) = 1$ and at most finitely many summands are non-zero

A Priori Estimates

- [Heat] $\frac{1}{2}\int_U |u(t_1)|^2 \,\mathrm{d}x + \int_{t_0}^{t_1}\int_U \|\nabla u\|^2 \,\mathrm{d}x\,\mathrm{d}t = \frac{1}{2}\int_U |u(t_0)|^2 \,\mathrm{d}x + \int_{t_0}^{t_1}\int_{\partial U} (\nu\cdot\nabla u)u \,\mathrm{d}S\,\mathrm{d}t + \int_{t_0}^{t_1}\int_U fu \,\mathrm{d}x\,\mathrm{d}t$
- [Heat] Let $f \in L^1_t\big((0,T); L^2(\mathbb{R}^d)\big)$ and $g \in L^2(\mathbb{R}^d)$. Then, the solution $u \in C_t\big([0,T], L^2(\mathbb{R}^d)\big)$ and $Du \in L^2\big((0,T)\times\mathbb{R}^d\big)$ is unique. Moreover, exists $C > 0$ s.t. $\sup_{t\in[0,T]} \|u(t)\|_{L^2(\mathbb{R}^d)} + \|Du\|_{L^2((0,T)\times\mathbb{R}^d)} \le C\left(\|g\|_{L^2(\mathbb{R}^d)} + \|f\|_{L^1((0,T);L^2(\mathbb{R}^d))}\right)$
- [Heat] Let $D^\alpha f \in L^1_t\big((0,T); L^2(\mathbb{R}^d)\big)$ and $D^\alpha g \in L^2(\mathbb{R}^d)\ \forall|\alpha| \le k$. Then, the unique solution to the heat equation $u \in C_t\big([0,T], L^2(\mathbb{R}^d)\big)$ and $Du \in L^2\big((0,T)\times\mathbb{R}^d\big)$ also obeys $D^\alpha u \in C_t\big([0,T]; L^2(\mathbb{R}^d)\big)$ and $DD^\alpha u \in L^2\big((0,T)\times\mathbb{R}^d\big)$. Moreover, exists $C_k > 0$ s.t. $\sum_{\alpha:|\alpha|\le k}\left(\sup_{t\in[0,T]} \|D^\alpha u(t)\|_{L^2(\mathbb{R}^d)} + \|DD^\alpha u\|_{L^2((0,T)\times\mathbb{R}^d)}\right) \le C_k \sum_{\alpha:|\alpha|\le k}\left(\|D^\alpha g\|_{L^2(\mathbb{R}^d)} + \|D^\alpha f\|_{L^1((0,T);L^2(\mathbb{R}^d))}\right)$
  - *Prove via applying energy method to $D^\alpha u$ since $D^\alpha(\partial_t - \Delta) = (\partial_t - \Delta)D^\alpha$*
- [Wave]
  - [Local Energy Identity] $\partial_t\left(\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}\sum_{j=1}^d (\partial_j u)^2\right) = \sum_{j=1}^d \partial_j(\partial_j u \partial_t u) - f\partial_t u = \nabla\cdot(\partial_t u \nabla u) - f\partial_t u$
  - $\frac{1}{2}\int_{\mathbb{R}^d} ((\partial_t u)^2(t_1) + \|\nabla u(t_1)\|^2) \,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^d} ((\partial_t u)^2(t_0) + \|\nabla u(t_0)\|^2) \,\mathrm{d}x - \int_{t_0}^{t_1}\int_{\mathbb{R}^d} f\partial_t u \,\mathrm{d}x\,\mathrm{d}t$
- [Schrödinger] Let $f \in L^1_t\big((0,T); L^2(\mathbb{R}^d)\big)$ and $g \in L^2(\mathbb{R}^d)$. The solution $u \in C_t\big([0,T]; L^2(\mathbb{R}^d)\big)$ to the Schrödinger equation is unique. Moreover, $\exists C > 0$ s.t. $\sup_{t\in[0,T]} \|u(t)\|_{L^2(\mathbb{R}^d)} \le C\left(\|g\|_{L^2(\mathbb{R}^d)} + \|f\|_{L^1((0,T);L^2(\mathbb{R}^d))}\right)$
- [Schrödinger] Let $D^\alpha f \in L^1_t\big((0,T); L^2(\mathbb{R}^d)\big)$ and $D^\alpha g \in L^2(\mathbb{R}^d)\ \forall|\alpha| \le k$. Then, the unique solution to the Schrödinger equation $u \in C_t\big([0,T]; L^2(\mathbb{R}^d)\big)$ also obeys $D^\alpha u \in$

$C_t\left([0,T];L^2(\mathbb{R}^d)\right)$. Moreover, $\exists C_k>0$ s.t. $\sum_{\alpha:\lvert\alpha\rvert\le k}\sup_{t\in[0,T]}\lVert D^\alpha u(t)\rVert_{L^2(\mathbb{R}^d)}\le$ $C_k\sum_{\alpha:\lvert\alpha\rvert\le k}\left(\lVert D^\alpha g\rVert_{L^2(\mathbb{R}^d)}+\lVert D^\alpha f\rVert_{L^1\left((0,T);L^2(\mathbb{R}^d)\right)}\right)$
Theorems (L^p Spaces)
• Let $1\le p<\infty$. $C_0(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$.
Tools

• $\int_U u\Delta v=\int_{\partial U}(\nu\cdot\nabla v)u-\int_U\nabla u\cdot\nabla v$
• $\int_U u\Delta u=\int_{\partial U}(\nu\cdot\nabla u)u-\int_U\lVert\nabla u\rVert^2$
• [Hölder] $\lVert fg\rVert_1\le\lVert f\rVert_p\lVert g\rVert_q$
• [Young] Let $a,b\ge 0$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Then $ab\le\frac{a^p}{p}+\frac{b^q}{q}$ with equality if and only if $a^p=b^q$
• [Minkowski] Let $1\le p<\infty$. Then $\lVert f+g\rVert_{L^p}\le\lVert f\rVert_{L^p}+\lVert g\rVert_{L^p}$
• Let $u\in\mathcal{D}'(\mathbb{R}^d)$. Then $\{\phi_\epsilon*u\}_{\epsilon\to 0}$ provides an approximation of u by smooth functions.
    ○ $\phi_\epsilon*u\in C^\infty(\mathbb{R}^d)$
    ○ $\phi_\epsilon*u\to u$ in $\mathcal{D}'(\mathbb{R}^d)$ as $\epsilon\to 0$
    ○ $D^\alpha(\phi_\epsilon*u)=\phi_\epsilon*D^\alpha u$
• [Mollifier] Let $\phi\in C_0^\infty(\mathbb{R}^d)$
    ○ $\int_{\mathbb{R}^d}\phi=1$
    ○ $\phi_\epsilon(x)=\frac{1}{\epsilon^d}\phi\left(\frac{x}{\epsilon}\right)$

Sobolev Spaces

Definitions

- [Sobolev Space $W^{k,p}(U)$] Let $k \in \mathbb{N} \cup \{0\}$ and $p \in [1,\infty]$. The Sobolev space with regularity index k and integrability index p is denoted $W^{k,p}(U) = \{u \in \mathcal{D}'(U): D^\alpha u \in L^p\ \forall \alpha \text{ s.t.} |\alpha| \le k\}$
  - [Norm] $\|u\|_{W^{k,p}(U)} \coloneqq \begin{cases} \left(\sum_{\alpha:|\alpha|\le k} \|D^\alpha u\|_{L^p}^p\right)^{\frac{1}{p}}, & p < \infty \\ \sum_{\alpha:|\alpha|\le k} \|D^\alpha u\|_{L^\infty}, & p = \infty \end{cases}$
  - [Convergence] $(u_n) \to u$ in $W^{k,p}(U)$ if $\|u_n - u\|_{W^{k,p}(U)} \to 0$
  - Space of functions possessing sufficiently many derivatives and equipped with a norm that measures both size and regularity of the function
  - Remark: $L^p \equiv W^{0,p}$
- [$W_0^{k,p}(U)$] $W_0^{k,p}(U) = \left\{u \in W^{k,p}(U): \exists u_j \in C_0^\infty(U) \text{ s.t. } \left(u_j\right)_j \to u \text{ in } W^{k,p}(U)\right\}$
  - $W_0^{k,p}(U)$ is the closure of $C_0^\infty(U)$ in $W^{k,p}(U)$
  - Intuitively, $W_0^{k,p}(U)$ is a closed subspace of $W^{k,p}(U)$ containing functions whose values at the boundary ∂U vanish up to all relevant orders
- [$H^k(U)$] Define $H^k(U) \coloneqq W^{k,2}(U)$ i.e. $p = 2$
  - $H^k(U)$ is a Hilbert space w.r.t $\langle\cdot,\cdot\rangle_{H^k(U)} \coloneqq \langle\cdot,\cdot\rangle_{W^{k,2}(U)}$
  - $\langle u, v\rangle_{H^k(U)} \coloneqq \sum_{\alpha:|\alpha|\le k} \int_U D^\alpha u \cdot D^\alpha v \, \mathrm{d}x$
  - $H_0^k(U) \coloneqq W_0^{k,2}(U)$
- [Hölder Space $C^{0,\alpha}(K)$] Let $K \subset \mathbb{R}^d$ be closed. Let $\alpha \in (0,1)$. Let $f \in C(K)$.
  - [$[\cdot]_{C^{0,\alpha}(K)}$] Define the Hölder semi-norm of regularity α for $f \in C(K)$ as: $[f]_{C^{0,\alpha}(K)} \coloneqq \sup\left\{\frac{|f(x)-f(y)|}{|x-y|^\alpha}: x, y \in K, x \neq y\right\}$
  - [$\|\cdot\|_{C^{0,\alpha}(K)}$] Define the Hölder norm $\|\cdot\|_{C^{0,\alpha}(K)}$ as: $\|f\|_{C^{0,\alpha}(K)} \coloneqq \|f\|_{L^\infty} + [f]_{C^{0,\alpha}(K)}$

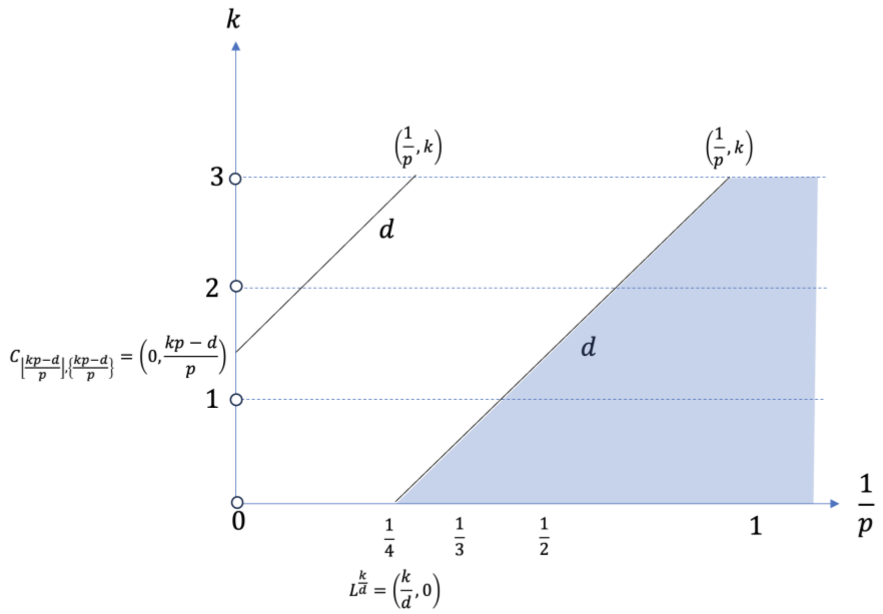
  Then, the Hölder space is $C^{0,\alpha}(K) = \left\{f \in C(K): \|f\|_{C^{0,\alpha}(K)} < \infty\right\}$, equipped with norm $\|\cdot\|_{C^{0,\alpha}(K)}$
  - $\|\cdot\|_{L^\infty}$ controls the amplitude, $[f]_{C^{0,\alpha}(K)}$ controls the frequency
  - $f \in C^{0,\alpha}(K)$ if f bounded, continuous and obeys Hölder continuity bound i.e. $|f(x) - f(y)| \le C|x-y|^\alpha$ for some $C > 0$ and $\forall x, y \in K$
- [Hölder Space $C^{k,\alpha}(K)$] The Hölder space is $C^{k,\alpha}(K) = \left\{f \in C^k(K): \sum_{\beta:|\beta|\le k} \left\|\partial_\beta f\right\|_{C^{0,\alpha}(K)} < \infty\right\}$, equipped with norm $\|\cdot\|_{C^{k,\alpha}(K)}$
  - $\|f\|_{C^{k,\alpha}(K)} = \sum_{\beta:|\beta|\le k} \left\|\partial_\beta f\right\|_{C^{0,\alpha}(K)}$
- [Morrey] Let $d < p \le \infty$. Then $\exists$ constant $c_{p,d}$ s.t. $\|u\|_{C^{0,\frac{p-d}{p}}(\mathbb{R}^d)} \le c_{d,p} \|u\|_{W^{1,p}(\mathbb{R}^d)}\ \forall u \in C^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$
  - Take $f \in C^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $d < q < \infty$. Then $[f]_{\dot{C}^\alpha} \le c\|Df\|_{L^p}$, where $\alpha = 1 - \frac{d}{q}$
  - i.e. $W^{1,p}(\mathbb{R}^d) \subset C^{0,\frac{p-d}{p}}(\mathbb{R}^d)$

Theorems

- [Properties of Sobolev Space] Let $k \in \mathbb{N} \cup \{0\}$ and $p \in [1,\infty]$.
  - $\left(W^{k,p}(U), \|\cdot\|_{W^{k,p}(U)}\right)$ is complete i.e. it is a Banach space
  - $\left(H^k(U), \langle\cdot,\cdot\rangle_{H^k(U)}\right)$ is complete i.e. it is a Hilbert space
  - $u \in H^k(\mathbb{R}^d)$ if and only if $\left\|(1 + \|\xi\|^2)^{\frac{k}{2}} \hat{u}(\xi)\right\|_{L^2} \in L^2(\mathbb{R}^d)$
    - $\exists C_{d,k}$ s.t. $C_{d,k}^{-1} \|u\|_{H^k(\mathbb{R}^d)} \le \left\|(1 + \|\xi\|^2)^{\frac{k}{2}} \hat{u}(\xi)\right\|_{L^2(\mathbb{R}^d)} \le C_{d,k} \|u\|_{H^k(\mathbb{R}^d)}$

- [11.3] Let $1 \le p < \infty$. The mapping $y \mapsto \tau_y$ is continuous as a linear map on $L_p(\mathbb{R}^d)$
  - Equivalently, $\forall u \in L^p(\mathbb{R}^d)$, $\lim_{y\to 0}\|\tau_y u - u\|_{L^p(\mathbb{R}^d)} = 0$
  - *Prove by $\frac{\epsilon}{3}$ argument*
- [11.4] Let $u \in L^p(\mathbb{R}^d)$, then $\phi_\epsilon * u \to u$ in $L^p(\mathbb{R}^d)$
- [11.5] Let $k \in \mathbb{N} \cup \{0\}$ and $p \in [1,\infty)$. If $u \in W^{k,p}(\mathbb{R}^d)$, then $(\phi_\epsilon * u)_{\epsilon\to 0} \to u$ in $W^{k,p}(\mathbb{R}^d)$.
  - $C^\infty(\mathbb{R}^d)$ is dense in $W^{k,p}(\mathbb{R}^d)$
- [11.7] Let U be a nonempty subspace in $\mathbb{R}^d$ and $\{V_j\}_{j\in\mathcal{J}}$ be an open covering of U. Then $\exists$ smooth partition of unity subordinate to $\{V_j\}_{j\in\mathcal{J}}$.
- [11.9] Let $k \in \mathbb{N} \cup \{0\}$ and $p \in [1,\infty)$. Let $U \subset \mathbb{R}^d$ be a domain. If $u \in W^{k,p}(U)$, then $\exists$ sequence $(u_j)_j \in C^\infty(U)$ s.t. $(u_j)_j \to u$ in $W^{k,p}(\mathbb{R}^d)$
  - i.e. $C^\infty(U)$ is dense in $W^{k,p}(U)$
  - $u \in W^{k,p}(U)$ can be approximated by smooth functions
- [11.10] Let $k \in \mathbb{N} \cup \{0\}$ and $p \in [1,\infty)$. Let $U \subset \mathbb{R}^d$ be a C^1 domain. If $u \in W^{k,p}(U)$, then $\exists$ sequence $(u_j)_j \in C^\infty(\overline{U})$ s.t. $(u_j)_j \to u$ in $W^{k,p}(\mathbb{R}^d)$.
  - i.e. $C^\infty(\overline{U})$ is dense in $W^{k,p}(U)$
  - $u \in W^{k,p}(U)$ can be approximated by functions smooth up to and including boundary of U
- [11.11] Let $k \in \mathbb{N} \cup \{0\}$ and $p \in [1,\infty)$. Let $\chi \in C_0^\infty(\mathbb{R}^d)$ with $\chi(0) = 1$. If $u \in W^{k,p}(\mathbb{R}^d)$, then $\chi\left(\frac{x}{R}\right) u \to u$ in $W^{k,p}(\mathbb{R}^d)$ as $R \to \infty$.
  - $u \in W^{k,p}(\mathbb{R}^d)$ can be approximated by compactly supported functions
- [11.12] Let $k \in \mathbb{N} \cup \{0\}$ and $p \in [1,\infty)$. Then $C_0^\infty(\mathbb{R}^d)$ is dense in $W^{k,p}(\mathbb{R}^d)$ i.e. $W_0^{k,p}(\mathbb{R}^d) = W^{k,p}(\mathbb{R}^d)$
  - Warning: this fails for any other C^1 domain U
- [Extension Mapping 11.13] Let $k \in \mathbb{N} \cup \{0\}$ and $p \in [1,\infty)$. Let U be a C^k domain in $\mathbb{R}^d$ and V be a domain in $\mathbb{R}^d$ s.t. $\overline{U} \subset V$. Then $\exists$ linear mapping $\mathcal{E}: W^{k,p}(U) \to W^{k,p}(\mathbb{R}^d)$ with the following properties:
  - $\mathcal{E}$ is bounded i.e. $\exists C_{d,k,p,U,V} > 0$ s.t. $\forall u \in W^{k,p}(U)$, $\|\mathcal{E}[u]\|_{W^{k,p}(\mathbb{R}^d)} \le C_{d,k,p,U,V}\|u\|_{W^{k,p}(U)}$
  - $\mathcal{E}[u]|_U = u$
  - $\operatorname{supp} \mathcal{E}[u] \subset V$
    - i.e. we can extend an element $u \in W^{k,p}(U)$ to a larger space $W^{k,p}(\mathbb{R}^d)$
    - $\mathcal{E}$ is the extension map
- [11.20 Gagliardo-Nirenberg-Sobolev for $C_0^\infty(\mathbb{R}^d)$] Let $d \ge 2$ and $u \in C_0^\infty(\mathbb{R}^d)$, then $\|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \le \|Du\|_{L^1(\mathbb{R}^d)}$
  - *Upshot: can bound some L^{p^*} norm of u with the L^1 norm of Du*
  - $W^{1,1}(U) \subset L^{\frac{d}{d-1}}(U)$
- [Gagliardo-Nirenberg-Sobolev] Let $1 \le p < d$. Then $\exists$ constant $c_{p,d}$ s.t. $\|u\|_{L^{\frac{pd}{d-p}}(\mathbb{R}^d)} \le c_{p,d}\|Du\|_{L^p(\mathbb{R}^d)}$ $\forall u \in C_0^1(\mathbb{R}^d)$.
  - *Upshot: can bound some L^{p^*} norm of u with the L^p norm of Du, provided $p < d$*
  - $W^{1,p}(U) \subset L^{\frac{pd}{d-p}}(U)$
- [11.22 Loomis-Whitney] Let $f_1, \dots, f_d: \mathbb{R}^{d-1} \to \mathbb{R}$ where $f_j \coloneqq f_j(x^1, \dots, \hat{x}^j, \dots, x^d)$ measurable. Then $\int_{\mathbb{R}^d} \prod_{i=1}^d |f_i| \,\mathrm{d}x_1 \dots \mathrm{d}x_d \le \prod_{i=1}^d \|f_i\|_{L^{d-1}(\mathbb{R}^{d-1})}$
  - $\left(\int_{\mathbb{R}^d} |g_1|^{\frac{1}{d-1}} \dots |g_d|^{\frac{1}{d-1}} \mathrm{d}x_1 \dots \mathrm{d}x_d\right)^{d-1} \le \prod_{i=1}^d \|g_i\|_{L^1(\mathbb{R}^{d-1})}$

- *Prove via integrating one variable at a time, then repeat Hölder*
- [11.26 Sobolev Inequalities for $W^{1,p}(U)$, $1 \leq p < d$] Let $U \subset \mathbb{R}^d$ be a domain and $1 \leq p < d$. $p^* = \frac{pd}{d-p}$. Then:
  - $W_0^{1,p}(U) \subset L^{\frac{pd}{d-p}}(U)$
  - $\forall u \in W_0^{1,p}(U)$, $\exists$ constant $c_{d,p}$ s.t. $\|u\|_{L^{\frac{pd}{d-p}}(U)} \leq c_{d,p}\|Du\|_{L^p(U)}$
  - If U is in addition a bounded C^1 domain, then:
    - $W^{1,p}(U) \subset L^{\frac{pd}{d-p}}(U) = W^{0,\frac{pd}{d-p}}(U)$
    - $\forall u \in W^{1,p}(U)$, $\exists$ constant $c_{d,p,U}$ s.t. $\|u\|_{L^{\frac{pd}{d-p}}(U)} \leq c_{d,p,U}\|Du\|_{W^{1,p}(U)}$
- [$p^* = \infty$]
  - If $d = 1$, $\|f\|_{L^\infty} \leq c\|\nabla f\|_{L^1}$
  - If $d = 2$, Sobolev embedding fails i.e. $\|f\|_{L^\infty}$ is not a constant factor of $\|\nabla f\|_{L^d}$
- [11.30 Properties of Hölder Space] Let $K \subset \mathbb{R}^d$ be closed. Let $k \in \mathbb{N} \cup \{0\}$ and $\alpha \in (0,1)$.
  - $\left(C^{k,\alpha}(K), \|\cdot\|_{C^{k,\alpha}(K)}\right)$ is a Banach space (complete normed space)
  - $\|u\|_{C^k(K)} \leq \|u\|_{C^{k,\alpha}(K)} \leq C\|u\|_{C^{k+1}(K)}$
  - For $0 < \alpha' < \alpha$, $\|u\|_{C^{k,\alpha'}(K)} \leq c\|u\|_{C^{k,\alpha}(K)}$
    - i.e. $0 < \alpha' < \alpha \Rightarrow C^{k,\alpha}(K) \subset C^{k,\alpha'}(K)$
  - For $L \subset K$, $\|u\|_{C^{k,\alpha}(L)} \leq \|u\|_{C^{k,\alpha}(K)}$
    - i.e. $L \subset K \Rightarrow C^{k,\alpha}(K) \subset C^{k,\alpha}(L)$
- [11.27] Let $u \in C^1\left(\overline{B_r(x)}\right)$. Then $\frac{1}{|B_r(x)|}\int_{B_r(x)} |u(y) - u(x)|\, \mathrm{d}y \leq \frac{1}{d\alpha(d)}\int_{B_r(x)} \frac{\|Du(y)\|}{\|x-y\|^{d-1}}\mathrm{d}y$
  - $|u(x)| \leq c\int_{\mathbb{R}^d} \frac{\|Du(y)\|}{\|x-y\|^{d-1}}\mathrm{d}y$
- [11.31] Let $u \in C^\infty(\mathbb{R}^d) \cap W^{1,p}(\mathbb{R}^d)$ and $p > d$. Then $\exists$ constant $c_{d,p} > 0$ s.t. $\|u\|_{C^{0,\frac{p-d}{p}}(\mathbb{R}^d)} \leq c_{d,p}\|u\|_{W^{1,p}(\mathbb{R}^d)}$
  - i.e. $W^{1,p}(\mathbb{R}^d) \subset C^{0,\frac{p-d}{p}}(\mathbb{R}^d)$
- [11.32 Sobolev Inequalities for $W^{1,p}(U)$, $p > d$] Let $U \subset \mathbb{R}^d$ be a domain and let $p > d$. Let $\alpha = 1 - \frac{d}{p}$. Then:
  - For any $u \in W_0^{1,p}(U)$, $\exists$ function $u^* \in C^{0,\alpha}(\overline{U})$ agreeing with u almost everywhere in U. Moreover, $\exists$ constant $c_{d,p} > 0$ s.t. $\|u\|_{C^{0,\alpha}(\overline{U})} \leq c_{d,p}\|u\|_{W^{1,p}(U)}$
  - Assume in addition that $U$is bounded C^1 domain. Then for any $u \in W^{1,p}(U)$, $\exists$ function $u^* \in C^{0,\alpha}(\overline{U})$ that agrees with u almost everywhere in U. Moreover, $\exists$ constant $c_{d,p,U}$ s.t. $\|u\|_{C^{0,\alpha}(\overline{U})} \leq c_{d,p,U}\|u\|_{W^{1,p}(U)}$
- [Sobolev Inequality for $W^{k,p}$ 11.39] Let $k \in \mathbb{N} \cup \{0\}$ and $p \in [1, \infty)$. Assume U is either (1) a domain in $\mathbb{R}^d$ and $u \in W_0^{k,p}(U)$ or (2) bounded C^k domain in $\mathbb{R}^d$ and $u \in W^{k,p}(U)$. Then the following holds:
  - Let $l \in \mathbb{N} \cup \{0\}$ s.t. $l \leq k$ and $q \in [1, \infty)$. If $\frac{d}{q} - l \geq \frac{d}{p} - k$, then $u \in W^{l,q}(U)$. Moreover, $\exists$ constant $c_{d,k,l,p,q,U}$ s.t. $\|u\|_{W^{l,q}(U)} \leq c_{d,k,l,p,q,U}\|u\|_{W^{k,p}(U)}$
    - i.e. if $l \leq k$ and $\frac{d}{q} - l \geq \frac{d}{p} - k$, $W^{k,p}(U) \subset W^{l,q}(U)$
  - Let $l \in \mathbb{N} \cup \{0\}$ s.t. $l \leq k$ and $\alpha \in (0,1)$. If $-l - \alpha \geq \frac{d}{p} - k$, then $\exists$ function $u^* \in C^{k,\alpha}(U)$ s.t. $u^* = u$ almost everywhere in U. Moreover, $\exists$ constant $c_{d,k,l,p,\alpha,U}$ s.t. $\|u^*\|_{C^{l,\alpha}(U)} \leq c_{d,k,l,p,\alpha,U}\|u\|_{W^{k,p}(U)}$
    - i.e. if $l \leq k$ and $-l - \alpha \geq \frac{d}{p} - k$, $W^{k,p}(U) \subset C^{l,\alpha}(U)$



Toolbox

- [Young] Let $a, b \geq 0$ and $p, q > 1$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$. Then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, with equality if and only if $a^p = b^q$
- [Hölder] Let $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let $u \in L^p(U)$, $v \in L^p(U)$. Then $\|uv\|_{L^1} = \int_U |uv| \mathrm{d}x \leq \|u\|_{L^p} \|v\|_{L^q}$
- Let U be bounded, $f \in L^r(U)$ for $1 \leq r \leq p$. Then $\exists$ constant c_U s.t. $\|f\|_{L^r(U)} \leq c_U \|f\|_{L^p(U)}$
- [Generalised Hölder] Let $1 \leq p_1, \dots, p_m \leq \infty$ with $\sum_{i=1}^{\infty} \frac{1}{p_i} = 1$. Let $u_k \in L^{p_k}(U)$ for $k = 1, \dots, m$. Then $\|u_1 \dots u_k\|_{L^1} = \int_U |u_1 \dots u_k| \mathrm{d}x \leq \prod_{i=1}^{k} \|u_i\|_{L^{p_i}(U)}$