Method of Characteristics

Definitions

• [Cauchy-Lipschitz Theorem]
• [Characteristic Equations]
  ○ $\dot{x}^j(s) = \partial_{p_j} F\big(x(s), z(s), p(s)\big)$
  ○ $\dot{z}(s) = \sum_j \partial_{p_j} F\big(x(s), z(s), p(s)\big) p_j(s)$
  ○ $\dot{p}_j(s) = -\partial_{x^j} F\big(x(s), z(s), p(s)\big) - \partial_z F\big(x(s), z(s), p(s)\big) p_j(s)$
• [Noncharacteristic] A triple (x_0, z_0, p_0) where $x_0 \in \Gamma$ and $z_0 = u(x_0)$ is noncharacteristic if $\nabla_p F(x_0, z_0, p_0) \cdot \nu(x_0) \neq 0$

Special Cases

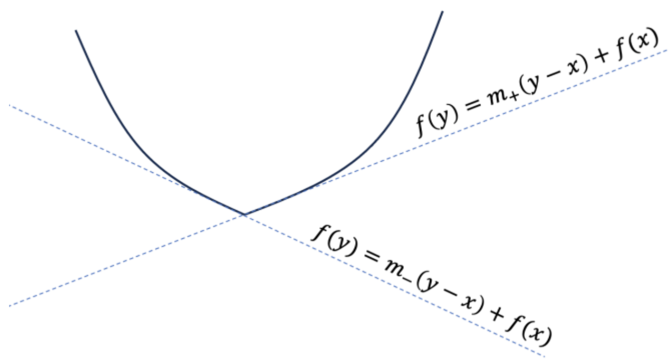
• [Linear] $F(x, u, Du) = b(x) \cdot Du + c(x)u - f$
  ○ $\dot{x} = b(x)$
  ○ $\dot{z} = -c(x)z + f$
• [Quasilinear] $F(x, u, Du) = b(x, u) \cdot Du + c(x, u)$
  ○ $\dot{x} = b(x, z)$
  ○ $\dot{z} = -c(x, z)$

Theorems
• Let $u \in C^2(U)$ be a solution to $F(x, u, Du) = 0$ in U, where $F \in C^1$. If $x(s)$, which lies in U for $s \in I$, solves ODE $\dot{x}^j(s) = \partial_{p_j} F\big(x(s), z(s), p(s)\big)$, then $z(s) = u\big(x(s)\big)$ and $p_j\big(x(s)\big)$ obey the other two characteristic equations respectively. • [Implicit Function Theorem] Let $F: \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$ be a C_1 function, and $x_0 \in \mathbb{R}^d, y_0 \in \mathbb{R}^n$ satisfy $F(x_0, y_0) = 0$, $\det \partial_{y^j} F^k(x_0, y_0) \neq 0$. Then exists neighborhoods $U \ni x$ and $V \ni y$ and C^1 function $y: U \to V$ s.t. $F\big(x, y(x)\big) = 0$ and if $(x, y) \in U \times V$ satisfies $F(x, y) = 0$, then $y = y(x)$ • [Inverse Function Theorem] • [Local Existence Theorem]

Hamilton-Jacobi Equations

Definitions

- [Initial Value Problem] Given Hamiltonian $H: \mathbb{R}^n \to \mathbb{R}$, initial value function $g: \mathbb{R}^n \to \mathbb{R}$
  - $u_t + H(\nabla u) = 0$ in $\mathbb{R}^n \times (0, \infty)$
  - $u = g$ on $\mathbb{R}^n \times \{t = 0\}$
- [Lagrangian] The Lagrangian is a smooth function $L: \mathbb{R}^n_x \times \mathbb{R}^n_v \to \mathbb{R}$
- [Generalised Momentum] $p(s) \coloneqq D_v L\big(x(s), v(s)\big)$
- [Hamiltonian] The Hamiltonian H associated with Lagrangian L is:
  - $H(x, p) \coloneqq p \cdot v(x, p) - L\big(x, v(x, p)\big)$
  - where $v(x, p)$ is the v such that $p = \nabla_v L(v, x)$
- [Hamilton ODEs]
  - $\dot{x}(s) = \nabla_p H\big(p(s), x(s)\big)$
  - $\dot{p}(s) = -\nabla_x H\big(p(s), x(s)\big)$
- [Legendre Transform] Let $L: \mathbb{R}^n_v \to \mathbb{R}$. The Legendre transform of L is:
$$L^*(p) = \sup_{v \in \mathbb{R}^n} \{p \cdot v - L(v)\}$$
- [Lipschitz Continuous] A function $f: X \to Y$ is Lipschitz continuous if $\exists$ constant $K \geq 0$ s.t. $\forall x_1, x_2 \in X$, $d_Y\big(f(x_1), f(x_2)\big) \leq K \cdot d_X(x_1, x_2)$
- [Uniformly Convex] A C^2 convex function $H: \mathbb{R}^n \to \mathbb{R}$ is uniformly convex (with constant $\theta > 0$) if $\xi^T\big(\nabla^2 H(p)\big)\xi \geq \theta \|\xi\|_2^2 \ \forall p, \xi \in \mathbb{R}^n$
- [Semiconcave] A function g is semiconcave if $\exists$ constant C s.t. $g(x+z) - 2g(x) + g(x-z) \leq C\|z\|_2^2$ for all $x, z \in \mathbb{R}^n$.
  - g is semiconcave if and only if the mapping $x \to g(x) - \frac{C}{2}\|x\|^2$ is concave for some constant C
- [Subdifferential] $m \in \partial f(x)$ if $f(y) \geq m(y - x) + f(x) \ \forall y \in \mathbb{R}$
  - $\partial f(x)$ collects all the gradients of hyperplanes that supports f at x. In the picture before, $\partial f(x) = [m_-, m_+]$



  - f is differentiable at v if and only if $\partial f(v)$ is a singleton
  - Let $f: \mathbb{R} \to \mathbb{R}$ be convex with $\lim_{|v| \to \infty} \frac{f(v)}{|v|} = +\infty$. Then, f is differentiable on $\mathbb{R}$ if and only if f^* is strictly convex

Results

- [Characteristic Equations]
  - $\dot{x} = \nabla_p H(p)$
  - $\dot{z} = p \cdot \nabla_p H(p) - H(p)$
- [Duality of Hamiltonian and Lagrangian]
  - $H = L^*$
  - $L = H^*$

- $u(x,t) = \inf\left\{\int_0^t L\big(\dot{w}(s)\big)\mathrm{d}s + g\big(w(0)\big) : w(t) = x\right\}$
- [Hopf-Lax Formula] Let $x \in \mathbb{R}^n$ and $t > 0$. Then the solution $u = u(x,t)$ of minimization problem is $u(x,t) = \min_{y\in\mathbb{R}^n}\left\{tL\left(\frac{x-y}{t}\right) + g(y)\right\}$
  - This formula provides a reasonable weak solution of initial value problem for Hamilton-Jacobi equation
- [Rademacher] A Lipschitz function is differentiable almost everywhere
- [Finite Propagation Speed]
- [Bump Function] $\phi(x) = \begin{cases} Ce^{-\frac{1}{1-|x|^2}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$ is infinitely differentiable, compactly supported with vanishing derivatives at $|x| = 1$
- [Properties of Mollifiers]
  - [Smooth] Let T be a distribution, then the sequence $(T_\epsilon)_\epsilon$ by $T_\epsilon = T * \phi_\epsilon$ is a family of smooth functions.
  - [Approximation to Identity] $\lim_{\epsilon\to 0} T_\epsilon = \lim_{\epsilon\to 0} T * \phi_\epsilon = T \in \mathcal{D}'(\mathbb{R}^n)$ in the sense of distributions
  - [Support] $\operatorname{supp} T_\epsilon = \operatorname{supp}(T * \phi_\epsilon) \subset \operatorname{supp} T + \operatorname{supp} \phi_\epsilon$
- [Cut-off Functions] χ_B

Distributions

Definitions

- [Support] Let $f \in C(U)$. The <u>support</u> of f is the closure of subset of U where f is non-zero i.e. $\operatorname{supp} f = \overline{\{x \in U : f(x) \neq 0\}}$
  - [Compactly Supported] If $\operatorname{supp} f$ compact, then f is <u>compactly supported</u>.
- [C_0^∞] The space of all infinitely differentiable functions with compact support.
- [Convolution] Let f be a continuous function in U and $\phi \in C_0^\infty(U)$. Define:
  - $(f * \phi)(x) = \int f(y)\phi(x-y)\mathrm{d}y$
- [Convergence of Test Functions] A sequence $\{\phi_n\}_n$ with $\phi_n \in C_0^\infty(U)$ <u>converges</u> to $\phi \in C_0^\infty(U)$ if:
  - [Support] $\exists$ compact $K \subset U$ s.t. $\operatorname{supp} \phi_n$, $\operatorname{supp} \phi \subset K$
  - [Regularity] $\lim\limits_{n\to\infty} \sup\limits_{x\in K} \|\nabla^\alpha \phi_n(x) - \nabla^\alpha \phi(x)\| = 0 \ \forall$ multi-index α
    - i.e. I can find a compact set K containing the support of all functions s.t. there is uniform convergence for any multi-index gradient
    - $\forall \alpha, \forall \epsilon > 0, \exists N$ s.t. $n > N \Rightarrow \|\nabla^\alpha \phi_n(x) - \nabla^\alpha \phi(x)\| < \epsilon \ \forall x \in U$
- [Distribution] A <u>distribution</u> $u: C_0^\infty(U) \to \mathbb{R}$ on U is a linear functional that is continuous in the following sense:
  - For any sequence $\{\phi_j\}_j, \phi \in C_0^\infty(U)$ s.t. $\phi_j \to \phi$, we have $\lim\limits_{n\to\infty} u(\phi_n) = u(\phi)$
  - [3.8] Equivalent definition: A linear functional $u: C_0^\infty(U) \to \mathbb{R}$ is a distribution if and only if it is bounded in the following sense: for any compact $K \subset U$, $\exists N, C_{N,K}$ s.t. $\forall \phi \in C_0^\infty(U)$ with $\operatorname{supp}\phi \subset K$, $|\langle u, \phi\rangle| \leq C_{N,K} \sum_{|\alpha|\leq N} \sup\limits_{x\in K} |\nabla^\alpha \phi(x)|$
    - i.e. for every compact set, the range of the functional $\langle u, \cdot\rangle$ is bounded by sup of N multi derivatives is bounded
- [$\mathcal{D}'(U)$] $\mathcal{D}'(U)$ is the space of distributions on U
  - $C_0^\infty(U)$ is dense in $\mathcal{D}'(U)$
  - Given $u \in \mathcal{D}'(U)$, construct $u_n = (u\chi_n) * \phi_{\epsilon_n} \in C_0^\infty(U)$
- [Order] Let $u \in \mathcal{D}'(U)$. If $\exists N$ s.t. $\forall K \subset U$ compact, $\exists\ C_{N,K}$ constant s.t. $|\langle u, \phi\rangle| \leq C_{N,K} \sum_{|\alpha|\leq N} \sup\limits_{x\in K} |D^\alpha \phi(x)|$, then u has order $\leq N$. <u>Order of distribution</u> u is smallest such N.
- [Vanishes] A distribution $u \in \mathcal{D}'(U)$ <u>vanishes</u> in an open subset $V \subset U$ if $\langle u, \phi\rangle = 0$ for every test function ϕ s.t. $\operatorname{supp}\phi \subset V$.
- [Support of Distribution] The <u>support of a distribution</u> $u \in \mathcal{D}'(U)$ is $\operatorname{supp} u = U \backslash V_{\max} = U(\bigcup\{V : V \subset U \text{ open}, u \text{ vanishes in } V\})$
- [$L^1_{\text{loc}}(U)$] A function $u: U \to \mathbb{R}$ is <u>locally integrable</u> if it is measurable and absolutely integrable on every compact subset K of U with respect to the Lebesgue measure.
  - Absolutely integrable means $\int_K |f| < \infty$
  - Any locally integrable function u defines a distribution by $\langle u, \phi\rangle \coloneqq \int u\phi \ \mathrm{d}x$
  - Any such distribution has order 0
- [Delta Distribution] $\langle \delta_y, \phi\rangle = \phi(y)$
- [Adjoint Method] Let $\mathcal{A}$ be an operator. We can compute the adjoint operator $\mathcal{A}'$ where
  - $\int_U (\mathcal{A}u)\phi \ \mathrm{d}x = \int_U u(\mathcal{A}'\phi)\ \mathrm{d}x \ \forall \phi \in C_0^\infty(U), \forall u \in C^\infty(U)$
  - $\langle \mathcal{A}u, \phi\rangle = \langle u, \mathcal{A}'\phi\rangle$
  - If $\mathcal{P}u = \sum_{\alpha:|\alpha|\leq k} a_\alpha(x)\nabla^\alpha u$, then $\mathcal{P}'v = \sum_{\alpha:|\alpha|\leq k} (-1)^{|\alpha|}\nabla^\alpha(a_\alpha v)$
- [Adjoint Convolution] $(f *' \phi)(x) = \int_{\mathbb{R}^d} f(y-x)\phi(y)\ \mathrm{d}y = \int_{\mathbb{R}^d} f(y)\phi(x+y)\ \mathrm{d}y$
- [Convergence of Distributions] A sequence $(u_n)_n$ converges to u with $u_n, u \in \mathcal{D}'(U)$ if $\lim\limits_{n\to\infty} \langle u_n, \phi\rangle = \langle u, \phi\rangle \ \forall \phi \in C_0^\infty(U)$. Write $u_n \rightharpoonup u$.
  - $u_n \rightharpoonup u \Rightarrow \partial^\alpha u_n \rightharpoonup \partial^\alpha u$

- [Approximation Method] Let $\mathcal{A}$ be an operator on $C^\infty(U)$ and $u \in \mathcal{D}'(U)$ and $u_n \rightharpoonup u$ with $u_n \in C^\infty(U)$. Then, $\mathcal{A}$ can be extended to act on u via $\mathcal{A}u = \lim_{n\to\infty} \mathcal{A}u_n$
    - i.e. if not sure what is $\langle \mathcal{A}u, \phi\rangle$, find sequence $(u_n)_n \rightharpoonup u$ and compute $\lim_{n\to\infty}\langle \mathcal{A}u_n, \phi\rangle$
    - Can take $u_n = (u\chi_n) * \phi_{\frac{1}{n}}$ for example
- [Principal Value Distribution]
- [Fundamental Solution] Let $\mathcal{P}$ be an operator and $y \in \mathbb{R}^d$. The <u>fundamental solution</u> E_y for $\mathcal{P}$ at y is a distribution $E_y \in \mathcal{D}'(U)$ satisfying $\mathcal{P}E_y = \delta_y$.
    - With this, can define $u = \int f(y)E_y(x)\mathrm{d}y$ as the integral of point charges
- [Standard Mollifier] The standard mollifier is a smooth function $\phi: \mathbb{R}^n \to \mathbb{R}$ satisfying:
    - [Compact Support] $\phi \in C_0^\infty(\mathbb{R}^n)$
    - $\int_{\mathbb{R}^n} \phi(x)\mathrm{d}x = 1$
    - $\lim_{\epsilon\to 0}\phi_\epsilon(x) = \lim_{\epsilon\to 0}\frac{1}{\epsilon^n}\phi\left(\frac{x}{\epsilon}\right) = \delta(x)$
- [Positive Mollifier] A mollifier is positive if $\phi \geq 0$ in $\mathbb{R}^n$
- [Symmetric Mollifier] A mollifier is symmetric if $\phi(x) = \psi(|x|)$ with $\psi \in C^\infty(\mathbb{R}_+)$
- Mollifier is a smooth function used to create a sequence of smooth functions approximating nonsmooth functions via convolution/
    - Intuitively, sharp features of nonsmooth functions are mollified

Properties

- [Properties of Convolution]
    - [Commutativity] $f * \phi = \phi * f$
    - [Associativity] $(f * g) * h = f * (g * h)$
    - [Support] $\operatorname{supp} f * \phi \subset \operatorname{supp} f + \operatorname{supp} \phi$
    - [Mollification] If one of $f, g \in C_0^\infty(\mathbb{R}^d)$, then $f * g \in C_0^\infty(\mathbb{R}^d)$
    - [Differentiation] $\partial_{x_j}(f * g) = \partial_{x_j} f * g = f * \partial_{x_j} g$ assuming $f, g \in C_0^1(\mathbb{R}^d)$
- [Adjoint Properties]
    - $\langle fu, \phi\rangle = \langle u, f\phi\rangle$ for $f \in C^\infty(U)$
    - $\langle \partial_j u, \phi\rangle = -\langle u, \partial_j\phi\rangle\ \forall \phi \in C_0^\infty(U)$
        - Every distribution is differentiable
    - $\langle f * u, \phi\rangle = \langle u, f *' \phi\rangle\ \forall\phi \in C_0^\infty(U)$ where $f \in C_0^\infty(\mathbb{R}^d)$
- [Bump Function] $\psi(r) = e^{-\frac{1}{1-r^2}} \cdot 1\{|r| < 1\}$
- $\phi(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$
- [Properties of Mollifiers]
    - $\int \phi_\epsilon \,\mathrm{d}x = 1$
    - Given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\psi \in C_0(\mathbb{R}^n)$, then $f * \psi$ is continuous
    - Given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\psi \in C_0^\infty(\mathbb{R}^n)$, $\frac{\partial(f*\psi)}{\partial x_i}(x) = \left(f * \frac{\partial\psi}{\partial x_i}\right)(x)$
    - Given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\psi \in C_0^\infty(\mathbb{R}^n)$, $f * \psi \in C^\infty(\mathbb{R}^d)$
    - For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, $\operatorname{supp}(f * \phi_\epsilon) \subset \operatorname{supp}(f) + \overline{B_\epsilon(0)} = \left\{x + y : x \in \operatorname{supp}(f), y \in \overline{B_\epsilon(0)}\right\}$ where ϕ_ϵ is the standard mollifier
    - For any function $f \in C(\mathbb{R}^n)$, $f_\epsilon = f * \phi_\epsilon \to f$ converges uniformly on all compact subsets of $\mathbb{R}^n$ where ϕ_ϵ is the standard mollifier
    - For any $1 \leq p < \infty$, $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$
    - For any function $f \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$, we have $\|f * \phi_\epsilon - f\|_{L^p(\mathbb{R}^n)} \to 0$ as $\epsilon \to 0$ where ϕ_ϵ is the standard mollifier
    - If $U \subset \mathbb{R}^n$ open and $K \subset U$ compact, then $\exists \psi \in C_0^\infty(\mathbb{R}^n)$ with $0 \leq \psi \leq 1$ s.t. $\psi = 1$ on a neighbourhood of K

- Given $U \subset \mathbb{R}^n$ open and $f \in L^1_{\text{loc}}(U)$, if $\int_U f\phi \, \mathrm{d}x = 0 \; \forall \phi \in C_0^\infty(U)$, then $f = 0$ almost everywhere in U

Propositions

- Any distribution is infinitely differentiable.
- Let $U \subset \mathbb{R}^d$ be open with C^1 boundary. Then $\partial_j 1_U = -(\nu_{\partial U})_j \mathrm{d}S_{\partial U}$ where $\mathrm{d}S_{\partial U}$ is the Euclidean surface element on ∂U
- If $u, v \in C(U)$ with $\langle u, \phi \rangle = \langle v, \phi \rangle \; \forall \phi \in C_0^\infty(U)$, then $u = v$
- $C_0^\infty(U)$ is dense in $\mathcal{D}'(U)$
  - Given any $u \in \mathcal{D}'(U)$, $\exists (u_n)_n \in C_0^\infty(U)$ s.t. $u_n \rightharpoonup u$ i.e. you can approximate every distribution with "nice" functions
  - For construction, $u_n = (u\chi_n) * \phi_{\epsilon_n} \in C_0^\infty(U)$
    - $\chi_n = \mathbb{1}_{K_n}$ where $(K_n)_n$ compact, $K_n \subset K_{n+1}$ and $\bigcup_n K_n = U$
    - $\langle u_n, \psi \rangle = \langle (u\chi_n) * \phi_{\epsilon_n}, \psi \rangle = \langle u\chi_n, \phi_{\epsilon_n} *' \psi \rangle = \langle u, \chi_n(\phi_{\epsilon_n} *' \psi) \rangle$
- [3.20] Let $\phi \in C_0^\infty(\mathbb{R}^d)$ satisfy $\int_{\mathbb{R}^d} \phi = 1$. For every $\delta > 0$, define $\phi_\delta(x) = \frac{1}{\delta^d}\phi\left(\frac{x}{\delta}\right)$. Then $\phi_\delta * u \rightharpoonup u$ as $\delta \to 0$.
- [3.21] Let U be an open set, $u \in \mathcal{D}'(U)$ and $(K_n)_n$ be a sequence of compact subsets of U s.t. $K_n \subset \operatorname{int} K_{n+1}$ and $\bigcup_{n=1}^\infty K_n = U$. Define $\chi_n \in C_0^\infty(U)$ s.t. $\chi_n = 1$ on K_n and $\operatorname{supp} \chi_n \subset \operatorname{int} K_{n+1}$. Let $u_n = \phi_{\delta_n} * (\chi_n u)$, where $\phi_{\delta_n}(x) = \frac{1}{\delta_n^d}\phi\left(\frac{x}{\delta_n}\right)$. Then $u_n \rightharpoonup u$.
  - $C_0^\infty(U)$ is dense in $\mathcal{D}'(U)$
  - We approximated $u \in \mathcal{D}'(U)$ by smooth and compactly supported functions in $C_0^\infty(U)$
- [3.28] Let $u, v \in \mathcal{D}'(\mathbb{R}^d)$ s.t. at least one of them has compact support, then $u * v$ is well-defined with:
  - $u * v = v * u$
  - $\operatorname{supp}(u * v) \subset \operatorname{supp}(u) + \operatorname{supp}(v)$
  - $\delta_{\{x=0\}} * u = u * \delta_{\{x=0\}} = u$
- [3.30] Let $\mathcal{P}$ be a constant coefficient partial differential operator on $\mathbb{R}^d$. Let E_0 be a fundamental solution for $\mathcal{P}$ at 0 and f be a compactly supported distribution. Then $u = E_0 * f$ solves $\mathcal{P}u = 0$.
  - If u is a compactly supported distribution, then $u = E_0 * \mathcal{P}u$

Lemmas

- [3.4] Let $f \in C^k(\mathbb{R}^d)$, $k < \infty$. Let ϕ be smooth with $\operatorname{supp} \phi \subset \overline{B(0,1)}$ and $\int_{\mathbb{R}^d} \phi = 1$. Define $\phi_\delta(x) = \frac{1}{\delta^d}\phi\left(\frac{x}{\delta}\right)$ and $f_\delta = \phi_\delta * f$
  - $\operatorname{supp} \phi_\delta \subset \overline{B(0,\delta)}$
  - $f_\delta(x) = \int_{\mathbb{R}^d} \frac{1}{\delta^d}\phi\left(\frac{x-y}{\delta}\right) f(y) \mathrm{d}y = \int_{\overline{B(0,1)}} \phi(z) f(x - \delta z) \mathrm{d}z$
  - $f_\delta \in C^\infty$, $\operatorname{supp} f_\delta \subset \operatorname{supp} f + B_\delta(0)$
  - Let $|\alpha| \le k$, then $\partial^\alpha f_\delta \to \partial^\alpha f$ uniformly on each compact set K as $\delta \to 0$
    - $\forall \epsilon > 0$, $\exists D$ s.t. $\delta < D \Rightarrow |\partial^\alpha f_\delta(x) - \partial^\alpha f(x)| < \epsilon \; \forall x \in K$
    - $\lim_{\delta \to 0} \sup_{x \in K} |\partial^\alpha f_\delta(x) - \partial^\alpha f(x)| = 0$
- [3.4'] Let $f \in C_0(\mathbb{R}^d)$. Take $\phi_1 \in C_0^\infty(\mathbb{R}^d)$, $\operatorname{supp} \phi_1 \subset \overline{B_1(0)}$ with $\int \phi \, \mathrm{d}x = 1$. Define $\phi_\epsilon = \frac{1}{\epsilon^d}\phi\left(\frac{x}{\epsilon}\right)$. Then $f * \phi_\epsilon \to f$ uniformly.
  - $\int \phi_\epsilon \, \mathrm{d}x = 1$
- [3.11] Let $f \in C^\infty(\mathbb{R}^d)$ and $u \in \mathcal{D}'(U)$, then $f * u \in C^\infty(\mathbb{R}^d)$
  - $D^\alpha(f * u)(x) = \big((D^\alpha f) * u\big)(x)$ for any multi-index α
  - $(f * u)(x) = \int f(x-y)u(y)\mathrm{d}y = \langle u, f(x - \cdot) \rangle$
- [3.12] Let $f \in C^\infty(\mathbb{R}^d)$ and $u \in \mathcal{D}'(\mathbb{R}^d)$, then $\operatorname{supp}(f * u) \subset \operatorname{supp}(f) + \operatorname{supp}(u)$
- [3.17] If $u_n \rightharpoonup u$, then $D^\alpha u_n \rightharpoonup D^\alpha u$

- [3.18] Let $u_n(x) \in L^1_{\text{loc}}(\mathbb{R})$ satisfies $\lim\limits_{n\to\infty} u_n(x) = u(x)$. If $\exists v \in L^1_{\text{loc}}(\mathbb{R})$ s.t. $|u_n(x)| \le v(x)$ for almost every $x \in \mathbb{R}$, then $u_n \rightharpoonup u$.
- [3.19] Let $\phi \in C_0^\infty(\mathbb{R}^d)$ and $\phi_\delta = \frac{1}{\delta^d}\phi\left(\frac{x}{\delta}\right)$. Then $\phi_\delta \rightharpoonup (\int \phi(x)\mathrm{d}x)\delta_0$ as $\delta \to 0$.
- [3.22] If $u \in \mathcal{D}'(U)$ s.t. $\partial_j u = 0$, then u is a constant
- If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\psi \in C_0(\mathbb{R}^n)$, then $f * \psi$ is continuous.
- Given an open set U and $f \in L^1_{\text{loc}}(U)$ and $\int_U f\phi \,\mathrm{d}x = 0$ $\forall \phi \in C_0^\infty(U)$, then $f \equiv 0$ everywhere in U

Theorems

- [3.16, Sequential Compactness] Let $(u_n)_n \in \mathcal{D}'(U)$ be a sequence with the following property: for each $\phi \in C_0^\infty(U)$, the sequence $\langle u_n, \phi\rangle$ converges as $n \to \infty$. Then $\exists u \in \mathcal{D}'(U)$ characterised by $\langle u, \phi\rangle = \lim\limits_{n\to\infty}\langle u_n, \phi\rangle$ $\forall \phi \in C_0^\infty(U)$
  - A corollary is that u is automatically continuous
  - For every $K \subset U$ compact, $\exists N, C$ (independent of n) s.t. $|\langle u_n, \phi\rangle|, |\langle u, \phi\rangle| \le C\sum_{|\alpha|\le N} \sup\limits_{x\in K}|D^\alpha \phi(x)|$
  - If $\phi_n \to \phi$ in $C_0^\infty(U)$, then $\lim\limits_{n\to\infty}\langle u_n, \phi\rangle = \langle u, \phi\rangle$
  - To check that u is continuous, can instead try to check for existence of $\langle u_n, \phi\rangle$ for each $\phi \in C_0^\infty(U)$.
  - $\lim\limits_{n\to\infty}\langle u_n, \phi_n\rangle = \langle \lim\limits_{n\to\infty} u_n, \lim\limits_{n\to\infty}\phi_n\rangle$
- [Malgrange-Ehrenpreis] Every constant coefficient scalar linear partial differential operator has a fundamental solution.

Tricks

- Given a good function $h(x)$, consider $h_\delta(x) = h\left(\frac{x}{\delta}\right)$

Laplace Equation

Definitions
• [Laplace Equation] $-\Delta u = 0$ • [Fundamental Solution] Let $\mathcal{P}$ be an operator and $y \in \mathbb{R}^d$. The fundamental solution E_y for $\mathcal{P}$ at y is a distribution $E_y \in \mathcal{D}'(U)$ satisfying $\mathcal{P}E_y = \delta_y$.
Properties
• [Translation Invariance] $-\Delta\big(u(x - x_0)\big) = (-\Delta u)(x - x_0)$ • [Rotational Invariance] Given $O^T O = \mathbb{I}_d$, then $-\Delta\big(u(Ox)\big) = (-\Delta u)(Ox)$ • [Homogeneity] For $\lambda > 0$, $-\Delta\big(u(\lambda x)\big) = \lambda^2(-\Delta u)(\lambda x)$ • $f(x) = \lim_{\epsilon \to 0} \int f(y)\phi_\epsilon(x - y)\,\mathrm{d}y = \lim_{\epsilon \to 0}(f * \phi_\epsilon)(x)$
Results

• [Fundamental Solution] $E_0(r) = \begin{cases} -\frac{1}{2\pi}\log r, & d = 2 \\ \frac{1}{d(d-2)\alpha(d)r^{d-2}}, & d \geq 3 \end{cases}$
    ○ $E_0(r)$ is locally integrable near 0 i.e. is a distribution
    ○ $-\Delta E_0 = \delta_0$ in $\mathbb{R}^d$
• [Solution] A solution for $-\Delta u = f$ on $\mathbb{R}^d$ is $u = f * E$
• [Uniqueness] If $u \in C_0^\infty(\mathbb{R}^d)$, then $u = (-\Delta u) * E$ i.e. u can be recovered from the Laplacian by convolving with the fundamental solution
• [Regularity] If $-\Delta u = 0$ and $u \in \mathcal{D}'(U)$, then u is smooth

Exam

Checks

- When using method of characteristics, make sure to specify the region, usually apparent from where you can invert $(y,s) \to (x,t)$
- Differentiate termwise i.e. $\frac{\partial f}{\partial p_j}$ for example instead of $\nabla_p f$
- [Jacobian] $J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$
- For test functions, consider Carathéodory functions.
- [Cutoff] For $\phi \in C_0^\infty(\mathbb{R})$, the cutoff function is $\chi(x) = \begin{cases} 1, & x \in \operatorname{supp} \phi \\ 0, & x \notin \operatorname{supp} \phi \end{cases}$
- [Fubini] If $\int_{\mathbb{R}^2} |f(x,y)| \mathrm{d}x \, \mathrm{d}y < \infty$, then $\int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y) \, \mathrm{d}x \;\; \mathrm{d}y = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y) \, \mathrm{d}y \;\; \mathrm{d}x$
- Check that the term defined is actually a distribution.
- Derivative of distribution is just the intuitive derivative with delta boosts
- Can you guess the solution?
- $\phi_n = \frac{1}{n^{2N}} \cos(2\pi n^2 x) \, \chi\left((n+1)^2 \left(x - \frac{1}{n}\right)\right)$
  - $\chi(x) = 1_{\left(-\frac{1}{2},\frac{3}{2}\right)}$ and 0 outside $[-1,1]$ smooth cutoff
  - $\operatorname{supp} \chi\left((n+1)^2 \left(x - \frac{1}{n}\right)\right) \subset \left(\frac{1}{n+1}, \frac{1}{n-1}\right)$
  - $\sum_{\alpha:|\alpha| \le N} \sup_{x \in K} |\partial^\alpha \phi(x)| < C_N$