Integration

Definition

- [Degree of Precision] The <u>degree of precision</u> of a quadrature formula is the largest positive integer n s.t. the formula is exact for x^k for $k \in \{0,1,\dots,n\}$

Method of Quadrature

- [Quadrature] Select set of distinct nodes $x_0 < \cdots < x_n$ from interval $[a,b]$ and integrate the Lagrange interpolating polynomial $P_n(x) = \sum_{i=0}^{n} f(x_i)L_i(x)$
  - $\int_a^b f(x)\,\mathrm{d}x = \sum_{i=0}^{n} a_i f(x_i) + \frac{1}{(n+1)!}\int_a^b \prod_{i=0}^{n}(x-x_i)\, f^{(n+1)}\big(\xi(x)\big)\,\mathrm{d}x$ where $a_i = \int_a^b L_i(x)\,\mathrm{d}x$
- [Trapezoidal Rule] $x_0 = a, x_1 = b, h = b - a$
  - $\int_a^b f(x)\,\mathrm{d}x = \frac{h}{2}\big(f(x_0) + f(x_1)\big) - \frac{h^3}{12} f''(\xi)$
  - Degree of precision: 1
- [Simpson Rule] $x_0 = a$, $x_1 = \frac{a+b}{2}$, $x_2 = b$, $h = \frac{b-a}{2}$
  - $\int_a^b f(x)\,\mathrm{d}x = \frac{h}{3}\big(f(x_0) + 4f(x_1) + f(x_2)\big) - \frac{h^5}{90} f^{(4)}(\xi)$
  - Degree of precision: 3
- [Closed Newton-Cotes Formula] $x_0 = a$, $x_n = b$, $h = \frac{b-a}{n}$
  - n even: $\int_a^b f(x)\,\mathrm{d}x = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+3} f^{(n+2)}(\xi)}{(n+2)!}\int_0^n t^2(t-1)\dots(t-n)\,\mathrm{d}t$
  - n odd: $\int_a^b f(x)\,\mathrm{d}x = \sum_{i=0}^{n} a_i f(x_i) + \frac{h^{n+2} f^{(n+1)}(\xi)}{(n+1)!}\int_0^n t(t-1)\dots(t-n)\,\mathrm{d}t$
- [Open Newton-Cotes Formula]

Composite Methods

- [Composite Simpson's Rule] Let $f \in C^4[a,b]$, n even, $h = \frac{b-a}{n}$ and $x_j = a + jh$. Then $\exists \xi \in (a,b)$ s.t.
  - $\int_a^b f(x)\,\mathrm{d}x = \frac{h}{3}\left(f(a) + 2\sum_{i=1}^{\frac{n}{2}-1} f(x_{2i}) + 4\sum_{i=1}^{\frac{n}{2}} f(x_{2i-1}) + f(b)\right) - \frac{b-a}{180} h^4 f^{(4)}(\xi)$
- [Composite Trapezoidal Rule]
  - $\int_a^b f(x)\,\mathrm{d}x = \frac{h}{2}\Big(f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\Big) - \frac{b-a}{12} h^2 f''(\xi)$
- [Composite Midpoint Rule]
  - $\int_a^b f(x)\,\mathrm{d}x = 2h\sum_{i=0}^{\frac{n}{2}} f(x_{2i}) + \frac{b-a}{6} h^2 f''(\xi)$
- [Round Off Error Stability] Round off error does not depend on number of calculations performed i.e. independent of composite integration techniques and n

Romberg Integration

- $R_{n,k}$: $n+1$ number of points subdividing the interval, k
- $R_{k,j} = R_{k,j-1} + \frac{1}{4^{j-1}-1}\big(R_{k,j-1} - R_{k-1,j-1}\big)$ for $k = j, j+1, \dots$
- [Algorithm]
  - $R_{1,1} = \frac{h}{2}\big(f(a) + f(b)\big)$
  - $R_{n,1} = \frac{1}{2}\big(R_{n-1,1} + 2h_n\sum_{j=1}^{2^{n-2}} f(a + (k-0.5)h)\big)$ where $h_n = \frac{b-a}{2^{n-1}}$
    - $R_{n,1}$ is just dividing the interval into 2^{n-1} pieces and use composite method
  - $R_{n,i} = R_{n,i-1} + \frac{R_{n,i-1} - R_{1,i-1}}{4^{i-1}-1}$
    - Other Romberg terms come for free
    - For Simpson's, $R_{n,i} = I + O\big(h_i^{2i+2}\big)$

$O(h_k^4)$	$O(h_k^6)$	$O(h_k^8)$	$O(h_k^{10})$
$R_{1,1} \searrow \rightarrow$			
$R_{2,1} \searrow \rightarrow$	$R_{2,2} \searrow \rightarrow$		
$R_{3,1} \searrow$	$R_{3,2} \searrow$	$R_{3,3} \searrow$	
$R_{4,1} \rightarrow$	$R_{4,2} \rightarrow$	$R_{4,3} \rightarrow$	$R_{4,4}$

Adaptive Methods

- Adding more points only when necessary

Gaussian Quadrature

- Choose $(x_i)_{i=1}^n, (c_i)_{i=1}^n$ s.t. $\int_a^b f(x)\,\mathrm{d}x \approx \sum_{i=1}^n c_i f(x_i)$
- [Legendre Polynomials]
  - $P_n(x)$: monic polynomial of degree n s.t. $\int_{-1}^1 P(x)P_n(x)\mathrm{d}x = 0\ \forall P$ with $\deg P < n$
  - Roots of $P_n(x)$ are distinct and lie in $(-1,1)$, symmetric about origin
  - $(x_i)_{i=1}^n$ are chosen to be roots of P_n, exact for polynomials P with $\deg P < 2n$
- [4.7] Let $x_1, \dots, x_n$ be roots of P_n. Then $\forall P$ with $\deg P < 2n$, $\int_{-1}^1 P(x)\mathrm{d}x = \sum_{i=1}^n c_i P(x_i)$ with $c_i = \int_{-1}^1 \prod_{j=1, j\neq i}^n \frac{x - x_j}{x_i - x_j}\,\mathrm{d}x$

Multiple Integrals

- [Simpson Double Integral]
- [Triple Integral Approximation]

Ordinary Differential Equations

Definitions

- [Lipschitz Continuous] A function $f(t,y)$ is Lipschitz in y on set $D \subset \mathbb{R}^2$ if $\exists$ constant L s.t. $|f(t,y_1) - f(t,y_2)| \le L|y_1 - y_2|$
- [Convex] A set $D \subset \mathbb{R}^2$ is convex if $(t_1,y_1),(t_2,y_2) \in D$ implies $(\lambda t_1 + (1-\lambda)t_2, \lambda y_1 + (1-\lambda)y_2) \in D\ \forall \lambda \in (0,1)$
- [Initial Value Problem]
  - $y'(t) = f(t,y)$
  - $a \le t \le b$
  - $y(a) = \alpha$
- [Well-Posed] The initial value problem is well-posed if a unique solution $y(t)$ exists and $\exists \epsilon_0 > 0, k > 0$ s.t. for any $\epsilon \in (0,\epsilon_0)$ and continuous function $\delta(t)$ with $|\delta(t)| < \epsilon\ \forall t \in [a,b]$ and $|\delta_0| < \epsilon$, the perturbed initial value problem
  - $z'(t) = f(t,z) + \delta(t)$
  - $a \le t \le b$
  - $z(a) = \alpha + \delta_0$

  has unique solution $z(t)$ satisfying $|z(t) - y(t)| < k\epsilon\ \forall t \in [a,b]$
- [Local Truncation Error] Let $(w_n)_n$ be a method. The local truncation error is $\tau_{i+1}(h) = \frac{y_{i+1} - w_{i+1}}{h}$ assuming $w_i = y_i$ i.e. the method was exact in the previous step.
  - [Order] A method has order α if $\tau(h) \in O(h^\alpha)$
  - Let $w_{i+1} = w_i + h\phi(t_i,w_i)$ be a difference method. Then the local truncation error is: $\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\phi(t_i,y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i,y_i)$
    - [Euler] $\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - f(t_i,y_i) = \frac{h}{2} y''(\xi_i)$ for $\xi \in (t_i, t_{i+1})$
- [Consistent] A method is consistent if $\lim_{h\to 0} \max_{1 \le i \le N} |\tau_i(h)| = 0$ where $\tau_i(h)$ is the local truncation error at the ith step.
  - i.e. local truncation error uniformly converges to 0 if we take smaller step sizes
  - Note that this is still a local definition, since τ_i assumes exact value at t_{i-1}.
- [Convergent] A method is convergent if $\lim_{h\to 0} \max_{1 \le i \le N} |y(t_i) - w_i| = 0$
  - This is a global definition
  - As step size gets smaller, expect w_i to converge to $y(t_i)$ at all timesteps
- [Characteristic Polynomial] Given $w_{i+1} = a_{m-1}w_i + \cdots + a_0 w_{i+1-m}$, the characteristic polynomial is $P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - \cdots - a_0$, with solutions $w_n = \sum_{i=1}^m c_i \lambda_i^n$ (assuming roots are distinct)
- [Root Condition] A multistep difference method of the form $w_{i+1} = a_{m-1}w_i + \cdots + a_0 w_{i+1-m} + hF(t_i, h, w_{i+1}, w_i, \dots, w_{i+1-m})$ satisfies the root condition if the roots of the characteristic equation $(\lambda^m - a_{m-1}\lambda^{m-1} - \cdots - a_0)$ are s.t. $|\lambda_i| \le 1\ \forall i$ and $|\lambda_i| = 1$ only if λ_i is a simple root.
- [Strongly Stable] Methods that satisfy the root condition and have $\lambda = 1$ as the only root are strongly stable.
- [Weakly Stable] Methods that satisfy the root condition with more than one distinct root with magnitude 1 are weakly stable.
- [Unstable] Methods that do not satisfy the root condition are unstable.
- [mth-Order System] $\frac{\mathrm{d}u_i}{\mathrm{d}t} = f_i(t, u_1, \dots, u_m)$ for $i \in \{1, \dots, m\}$ with initial conditions $u_i(a) = \alpha_i$
- [Lipschitz Continuous] The function $f(t, y_1, \dots, y_m)$ on $D = \{(t, u_1, \dots, u_m) | a \le t \le b, -\infty < u_i < \infty\}$ satisfies Lipschitz condition on D in $y_1, \dots, y_m$ if $\exists L > 0$ s.t. $|f(t,\vec{y}) - f(t,\vec{u})| \le L\|\vec{y} - \vec{u}\|_{L^1}$

Methods

- [Euler] Take $h = \frac{b-a}{N}$ and $t_i = a + ih$. $y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i)$

  - $w_0 = \alpha$
  - $w_{i+1} = w_i + hf(t_i, w_i)$
- [Euler with Round Off] δ_i denotes the round-off error associated with u_i
  - $u_0 = \alpha + \delta_0$
  - $u_{i+1} = u_i + hf(t_i, u_i) + \delta_{i+1}$
- [Taylor]
  - $y(t_{i+1}) = y(t_i) + hf\big(t_i, y(t_i)\big) + \frac{h^2}{2} f'\big(t_i, y(t_i)\big) + \cdots$
  - $f'\big(t, y(t)\big) = \frac{\mathrm{d}}{\mathrm{d}t} f\big(t, y(t)\big) = \frac{\partial}{\partial t} f\big(t, y(t)\big) + \frac{\partial}{\partial y} f\big(t, y(t)\big) y'(t)$
  - $w_0 = \alpha$
  - $w_{i+1} = w_i + hT^{(n)}(t_i, w_i)$
  - $T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i)$
- [Midpoint] $O(h^2)$
  - $w_0 = \alpha$
  - $w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right)$
  - $O(h^2)$
- [Modified Euler] $O(h^2)$
  - $w_0 = \alpha$
  - $w_{i+1} = w_i + \frac{h}{2}\Big(f(t_i, w_i) + f\big(t_{i+1}, w_i + hf(t_i, w_i)\big)\Big)$
- [Heun] $O(h^3)$
  - $w_0 = \alpha$
  - $w_{i+1} = w_i + \frac{h}{4}\left(f(t_i, w_i) + 3\left(f\left(t_i + \frac{2h}{3}, w_i + \frac{2h}{3} f\left(t_i + \frac{h}{3}, w_i + \frac{h}{3} f(t_i, w_i)\right)\right)\right)\right)$
- [Runge-Kutta] $O(h^4)$
  - $w_0 = \alpha$
  - $k_1 = hf(t_i, w_i)$
  - $k_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2} k_1\right)$
  - $k_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2} k_2\right)$
  - $k_4 = hf(t_{i+1}, w_i + k_3)$
  - $w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
- [Adaptive Methods] Adjust step size h_j as necessary to minimise estimate of local truncation error
  - $q = \left|\frac{\epsilon h}{2(\tilde{w}_{j+1} - w_{j+1})}\right|^{\frac{1}{n}}$
  - $h \leftarrow qh$ where $q = \begin{cases} 0.1, & q \le 0.1 \\ 4, & q \ge 4 \\ q, & 0.1 < q < 4 \end{cases}$
  - $h = \min(h, h_{\max})$
  - If $h < h_{\min}$, declare failure
- [Explicit m-Step] Given $\Big(f\big(t_i, y(t_i)\big)\Big)_{i=j-m+1}^{j}$, let $P(t)$ interpolate $f\big(t, y(t)\big)$. Then
  - $y(t_{j+1}) - y(t_j) = \int_{t_j}^{t_{j+1}} P(t)\,\mathrm{d}t = h\sum_{i=j-m+1}^{j} b_i f\big(t_i, y(t_i)\big)$
  - $w_{j+1} = w_j + h\sum_{i=j-m+1}^{j} b_i f(t_i, w_i)$
  - [Adams-Bashforth Explicit 4-Step]
- [Implicit $(m-1)$-Step] Given $\Big(f\big(t_i, y(t_i)\big)\Big)_{i=j-m+2}^{j+1}$, let $P(t)$ interpolate $f\big(t, y(t)\big)$. Then

- $y(t_{j+1}) - y(t_j) = \int_{t_j}^{t_{j+1}} P(t)\,\mathrm{d}t = h\sum_{i=j-m+2}^{j+1} b_i f(t_i, y(t_i))$
- $w_{j+1} = w_j + h\sum_{i=j-m+2}^{j+1} b_i f(t_i, w_i)$
- [Adams-Moulton Implicit 3-Step]

- [Predictor-Corrector] First run Runge-Kutta, then use Adams-Bashforth predictor, then use Adams-Moulton corrector.
- [Implicit Trapezoidal Method]
  - $w_0 = \alpha$
  - $w_{j+1} = w_j + \frac{h}{2}\left(f(t_{j+1}, w_{j+1}) + f(t_j, w_j)\right)$

Theorems

- [5.3] Suppose f is defined on convex set $D \subset \mathbb{R}^2$. If $\exists L > 0$ s.t. $\left|\frac{\partial f}{\partial y}(t, y)\right| \le L\ \forall (t, y) \in D$, then f is Lipschitz in y with Lipschitz constant L
- [5.4] Let $D = \{(t, y): t \in [a, b], y \in (-\infty, \infty)\}$ and $f(t, y)$ continuous on D. If f satisfies Lipschitz condition in y, then the IVP $y'(t) = f(t, y)$, $a \le t \le b$, $y(a) = \alpha$, has a unique solution $y(t)$ for $a \le t \le b$.
- [5.6] Let $D = \{(t, y): t \in [a, b], y \in (-\infty, \infty)\}$. If f continuous and satisfies Lipschitz condition in y, then the initial value problem is well-posed.
- [5.9] Suppose f continuous and satisfies Lipschitz condition with constant L on $D = \{(t, y): t \in [a, b], y \in (-\infty, \infty)\}$ and $\exists M$ with $|y''(t)| \le M\ \forall t \in [a, b]$, where $y(t)$ is the unique solution to IVP. Let $w_0, \dots, w_N$ be approximations of Euler's method. Then $|y(t_i) - w_i| \le \frac{hM}{2L}\left|e^{L(t_i - a)} - 1\right|$
- [5.10 Euler Method Bound] Let $u_0, \dots, u_N$ be approximations. If $|\delta_i| < \delta\ \forall i \in \{0, \dots, N\}$, then $|y(t_i) - u_i| \le \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left(e^{L(t_i - a)} - 1\right) + |\delta_0| e^{L(t_i - a)}$
- [5.12] If Taylor's method of order n is used to approximate the solution to IVP with step size h and $y \in C^{n+1}[a, b]$, then local truncation error $= O(h^n)$
- [Multivariable Taylor] Let $f(t, y)$ and all its partial derivatives of order $\le n + 1$ be continuous on $D = \{(t, y): a \le t \le b, c \le y \le d\}$ and $(t_0, y_0) \in D$. Then, $\forall (t, y) \in D$, $\exists \xi, \mu$ s.t. $f(t, y) = P_n(t, y) + R_n(t, y)$, where:
  - $P_n(t, y) = f(t_0, y_0) + \left(\frac{\partial f}{\partial t}(t_0, y_0)\cdot(t - t_0) + \frac{\partial f}{\partial y}(t_0, y_0)\cdot(y - y_0)\right) + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial t^2}(t_0, y_0)\cdot(t - t_0)^2 + 2\frac{\partial^2 f}{\partial t \partial y}(t_0, y_0)\cdot(t - t_0)(y - y_0) + \frac{\partial^2 f}{\partial y^2}(t_0, y_0)\cdot(y - y_0)^2\right) + \cdots + \frac{1}{n!}\sum_{\alpha:|\alpha| = n}\frac{\partial^{|\alpha|} f}{\partial^{\alpha_t} t\, \partial^{\alpha_y} y}(t_0, y_0)\cdot(t - t_0)^{\alpha_t}(y - y_0)^{\alpha_y}$
  - $R_n(t, y) = \frac{1}{(n+1)!}\sum_{\alpha:|\alpha| = n+1}\frac{\partial^{|\alpha|} f}{\partial^{\alpha_t} t\, \partial^{\alpha_y} y}(t_0, y_0)\cdot(t - t_0)^{\alpha_t}(y - y_0)^{\alpha_y}$
- If f and $\frac{\partial f}{\partial u_i}$ continuous on D and the partial derivatives satisfy $\left|\frac{\partial f}{\partial u_i}(t, u_1, \dots, u_m)\right| \le L$, then f is Lipschitz on D with Lipschitz constant L
- [5.17] Let $D = \{(t, u_1, \dots, u_m) | a \le t \le b, -\infty < u_i < \infty\}$ and f_i are continuous and satisfy Lipschitz condition on D. Then the system of ODEs has a unique solution $\left(u_i(t)\right)_{i=1}^m$
- [5.20 One-Step Method Stability] Let $w_{j+1} = w_j + h\phi(t_j, w_j, h)$ be a one-step method, with $\phi(t, w, h)$ continuous. Suppose $\exists h_0 > 0$ s.t. ϕ satisfies Lipschitz condition in w with Lipschitz constant L on $\mathcal{D} = \{(t, w, h) | a \le t \le b, -\infty < w < \infty, 0 \le h \le h_0\}$. Then:
  - ϕ is stable
  - ϕ is convergent if and only if it is consistent i.e. $\phi(t, y, 0) = f(t, y)\ \forall a \le t \le b$
    - $\phi(t, y, 0) = f(t, y)\ \forall a \le t \le b$ is an easy condition to check
  - If $\exists \tau$ s.t. local truncation error $\tau_i(h)$ satisfies $|\tau_i(h)| \le \tau(h)$ whenever $0 \le h \le h_0$, then $|y(t_i) - w_i| \le \frac{\tau(h)}{L} e^{L(t_i - a)}$ (e.g. $\tau(h) \coloneqq \max_{0 \le j \le N}|\tau_j(h)|$

- [5.24 Multistep Consistency] A multi-step method of the form $w_{i+1} = a_{m-1}w_i + \cdots + a_0 w_{i+1-m} + hF(t_i, h, w_{i+1}, \dots, w_{i+1-m})$ is stable if and only if it satisfies the root condition.
  - Moreover, if the method is consistent, then it is stable if and only if it is convergent.
- [Local Truncation Error Estimate] Let $\phi(t, w, h)$ be a method of order n and $\tilde{\phi}(t, w, h)$ be a method of order $n+1$. Then $\tau_{j+1}(h) \approx \frac{\tilde{w}_{j+1} - w_{j+1}}{h}$.
  - Factor to change step size $q = \left|\frac{\epsilon}{\tilde{\phi}(t_j, w_j, h) - \phi(t_j, w_j, h)}\right|^{\frac{1}{n}}$

Stiff Equations

- [Stiff Equations] Magnitude of derivative of solution increases, but the solution does not.
  - Forces step size h to be very small ⇒ not good!
- Given multistep method $w_{j+1} = a_{m-1}w_j + \cdots + a_0 w_{j+1-m} + h\left(b_m f(t_{j+1}, w_{j+1}) + \cdots + b_0 f(t_{j+1-m}, w_{j+1-m})\right)$
  - $(1 - \lambda h b_m)w_{j+1} - (a_{m-1} + \lambda h b_{m-1})w_j - \cdots - (a_0 + \lambda h b_0)w_{j+1-m} = 0$
  - $Q(z, \lambda h) \coloneqq (1 - \lambda h b_m)z^m - (a_{m-1} + \lambda h b_{m-1})z^{m-1} - \cdots - (a_0 + \lambda h b_0) = 0$
  - Let $\beta_1, \dots, \beta_m$ be roots of $Q(z, \lambda h)$. Then, $|\beta_i| < 1$ for convergence and numerical stability
- [Region of Absolute Stability]
  - [One Step] $R = \{h\lambda \in \mathbb{C} : |Q(z, \lambda h)| < 1\}$
  - [Multistep] $R = \{h\lambda \in \mathbb{C} : |\beta_k| < 1 \ \forall \beta_k \text{ s.t. } Q(\beta_k, h\lambda) = 0\}$
  - [A-Stable] A numerical method is A-stable if its region of absolute stability R contains the entire left halfplane i.e. any choice of $h > 0$ is valid.

Example

- $\dot{y} = y - t^2 + 1$, $y(0) = 0.5$
  - $y(t) = (1+t)^2 - \frac{1}{2}e^t$
- [Test Equation for Stiff ODEs] $\dot{y} = \lambda y$, $y(0) = \alpha$, $\lambda < 0$
  - $y(t) = \alpha e^{\lambda t}$

Linear Equations

Definitions
• [Strictly Diagonally Dominant] A matrix A is strictly diagonally dominant if $\lvert a_{ii}\rvert > \sum_{j=1,j\neq i}^{n}\lvert a_{ij}\rvert$ i.e. a_{ii} has a larger magnitude than the sum of the other elements in its row. • [Pivoting] Choose largest entry in absolute value. Exchange E_1 and E_{piv}. • [Partial Pivoting] • [Full Pivoting] • [Tri-diagonal] $A \in \mathbb{R}^{n\times n}$ is <u>tri-diagonal</u> if $A_{ij} = 0$ for $\lvert i-j\rvert > 1$ ○ LU factorisation takes $O(3n)$ for no pivoting or $O(4n)$ for partial pivoting
Theory
• [PA = LU] • Let $A \in \mathbb{R}^{n\times n}$ be a non-singular matrix. Then, GEPP computes $PA = LU$. $O\left(\frac{2}{3}n^3\right)$ ○ L is all 1 on diagonal • For SDD matrix, Gaussian elimination succeeds without pivoting. • For symmetric positive definite matrix, Gaussian elimination succeeds without pivoting. • [Cholesky] Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. Then, $A = LDL^T$ $O\left(\frac{1}{3}n^3\right)$