Lebesgue Measure

Axioms and Properties

- Axioms of Measurable Sets
  - [Borel Property] Every open set and closed set in $\mathbb{R}^n$ measurable.
  - [Complementarity] If Ω measurable, then $\mathbb{R}^n \backslash \Omega$ measurable.
  - [Boolean Algebra Property] If $(\Omega_j)_{j\in J}$ finite collection of measurable sets (i.e. J finite), then $\bigcup_{j\in J} \Omega_j$ and $\bigcap_{j\in J} \Omega_j$ measurable.
  - [σ-Algebra Property] If $(\Omega_j)_{j\in J}$ countable collection of measurable sets (i.e. J countable), then $\bigcup_{j\in J} \Omega_j$ and $\bigcap_{j\in J} \Omega_j$ measurable.
  - [Empty Set] $m(\phi) = 0$
  - [Positivity] $0 \le m(\Omega) \le +\infty$ for every measurable set Ω
  - [Monotonicity] If $A \subset B$ and A, B both measurable, $m(A) \le m(B)$
  - [Finite Sub-additivity] If $(A_j)_{j\in J}$ finite collection of measurable sets, then $m\left(\bigcup_{j\in J} A_j\right) \le \sum_{j\in J} m(A_j)$
  - [Finite Additivity] If $(A_j)_{j\in J}$ finite collection of disjoint measurable sets i.e. $A_j \cap A_k = \phi$ for $j \ne k$, then $m\left(\bigcup_{j\in J} A_j\right) = \sum_{j\in J} m(A_j)$
  - [Countable Sub-additivity] If $(A_j)_{j\in J}$ countable collection of measurable sets, then $m\left(\bigcup_{j\in J} A_j\right) \le \sum_{j\in J} m(A_j)$
  - [Countable Additivity] If $(A_j)_{j\in J}$ countable collection of disjoint measurable sets i.e. $A_j \cap A_k = \phi$ for $j \ne k$, then $m\left(\bigcup_{j\in J} A_j\right) = \sum_{j\in J} m(A_j)$
  - [Normalization] $m([0,1]^n) = 1$ (i.e. the unit cube has measure 1)
  - [Translational Invariance] If Ω measurable and $x \in \mathbb{R}^n$, then $x + \Omega$ also measurable with $m(x+\Omega) = m(\Omega)$
- Properties of Outer Measure
  - [Empty Set] $m^*(\phi) = 0$
  - [Positivity] $0 \le m^*(\Omega) \le +\infty$ for every measurable set Ω
  - [Monotonicity] If $A \subset B$ and A, B both measurable, $m^*(A) \le m^*(B)$
  - [Finite Sub-additivity] If $(A_j)_{j\in J}$ finite collection of measurable sets, then $m^*\left(\bigcup_{j\in J} A_j\right) \le \sum_{j\in J} m^*(A_j)$
  - [Countable Sub-additivity] If $(A_j)_{j\in J}$ countable collection of measurable sets, then $m^*\left(\bigcup_{j\in J} A_j\right) \le \sum_{j\in J} m^*(A_j)$
  - [Translational Invariance] If Ω measurable and $x \in \mathbb{R}^n$, then $x + \Omega$ also measurable with $m^*(x+\Omega) = m^*(\Omega)$
- Properties of Measurable Sets
  - Every open box and every closed box is measurable.
  - Any set E with $m^*(E) = 0$ is measurable with $m(E) = 0$

Definitions

- [Open Box] An open box in $\mathbb{R}^n$ is a set of the form $B = \prod_{i=1}^n (a_i, b_i)$, $b_i \ge a_i$. Define the volume of the box $\mathrm{vol}(B) = \prod_{i=1}^n (b_i - a_i)$
- [Covering by Boxes] Let $\Omega \subset \mathbb{R}^n$. Say collection of boxes $(B_j)_{j\in J}$ covers Ω if and only if $\Omega \subset \bigcup_{j\in J} B_j$.
- [Outer Measure] Let $\Omega \subset \mathbb{R}^n$. Define outer measure $m^*(\Omega) \coloneqq \inf\left\{\sum_{j\in J} \mathrm{vol}(B) : (B_j)_{j\in J} \text{ covers } \Omega,\ J \text{ at most countable}\right\}$
- [Lebesgue Measurability] Let $E \subset \mathbb{R}^n$. Say E is Lebesgue measurable if and only if the following identity holds: $m^*(A) = m^*(A \cap E) + m^*(A \backslash E)\ \forall A \subset \mathbb{R}^n$ i.e. for every test set A, it separates E cleanly based on the outer measure.

- If E measurable, define the Lebesgue measure of E as $m(E) = m^*(E)$
- [Measurable Functions] Let $\Omega \subset \mathbb{R}^n$ measurable and $f: \Omega \to \mathbb{R}^m$. Say f is measurable if and only if $f^{-1}(V)$ is measurable for every open $V \subset \mathbb{R}^m$.
- [$\mathbb{R}^*$] $\mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$
- [Measurable Functions in $\mathbb{R}^*$] Let $\Omega \subset \mathbb{R}^n$. $f: \Omega \to \mathbb{R}^*$ measurable if and only if $f^{-1}\big((a, +\infty]\big)$ measurable $\forall a \in \mathbb{R}$.

Alternate Definitions

- [Outer Measure] $m^*(A) = \inf\{m^*(U) | U \supset A, U \text{ open}\}$
- [Measurable Set] Say $E \subset \mathbb{R}^n$ measurable if $\forall \epsilon > 0$, $\exists$ open set $U \supset E$ s.t. $m^*(U \backslash E) < \epsilon$

General Lemmas and Theorems

- [Outer Measure of Closed Box] For any closed box $B = \prod_{i=1}^n [a_i, b_i]$, $m^*(B) = \prod_{i=1}^n (b_i - a_i)$
- [Outer Measure of Open Box] For any open box $B = \prod_{i=1}^n (a_i, b_i)$, $m^*(B) = \prod_{i=1}^n (b_i - a_i)$
- [Failure of Outer Measure] Outer measure does not satisfy countable additivity i.e. $\exists \big(A_j\big)_{j\in J}$ countable collection of disjoint subsets of $\mathbb{R}$ s.t. $m^*\big(\bigcup_{j\in J} A_j\big) \neq \sum_{j\in J} m^*\big(A_j\big)$
- [Half-spaces are Measurable] $\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$ is measurable.
- [Finite Additivity] If $\big(E_j\big)_{j\in J}$ finite collection of disjoint measurable sets and any set A (not necessarily measurable), then $m^*\big(A \cap \bigcup_{j\in J} E_j\big) = \sum_{j\in J} m^*\big(A \cap E_j\big)$. Furthermore, $m^*\big(\bigcup_{j\in J} E_j\big) = \sum_{j\in J} m^*\big(E_j\big)$.
- $A \subset B$ are two measurable sets, then $B \backslash A$ also measurable and $m(B \backslash A) = m(B) - m(A)$
- Every open set can be written as a countable or finite union of open boxes.
- [Continuous Functions are Measurable] Let $\Omega \subset \mathbb{R}^n$ measurable and $f: \Omega \to \mathbb{R}^m$ continuous. Then f is automatically measurable.
- Let $\Omega \subset \mathbb{R}^n$ measurable and $f: \Omega \to \mathbb{R}^m$. Then f is measurable if and only if $f^{-1}(B)$ measurable for every open box B.
- Let $\Omega \subset \mathbb{R}^n$ measurable and $f: \Omega \to \mathbb{R}^m$. Suppose that $f = (f_1, \dots, f_m)$ where $f_i: \Omega \to \mathbb{R}$ is the ith coordinate of f. Then f measurable if and only if all of f_i are individually measurable.
- Let $\Omega \subset \mathbb{R}^n$ measurable and $W \subset \mathbb{R}^m$ open. If $f: \Omega \to W$ measurable and $g: W \to \mathbb{R}^p$ continuous, then $g \circ f: \Omega \to \mathbb{R}^p$ measurable.
- Let $\Omega \subset \mathbb{R}^n$ measurable. If $f: \Omega \to \mathbb{R}$ measurable, then so is $|f|, \max(f, 0), \min(f, 0)$
- Let $\Omega \subset \mathbb{R}^n$ measurable. If $f, g: \Omega \to \mathbb{R}$ measurable, then so is $f + g, f - g, fg, \max(f, g), \min(f, g)$. If $g(x) \neq 0\ \forall x \in \Omega$, f/g measurable.
- Let $\Omega \subset \mathbb{R}^n$ measurable and $f: \Omega \to \mathbb{R}$. Then f is measurable if and only if $f^{-1}\big((a, \infty)\big)$ measurable $\forall a \in \mathbb{R}$.
- Let $\Omega \subset \mathbb{R}^n$ measurable and $\forall n$, $f_n: \Omega \to \mathbb{R}^*$ measurable. Then $\sup_{n\geq 1} f_n$, $\inf_{n\geq 1} f_n$, $\limsup_{n\to\infty} f_n$, $\liminf_{n\to\infty} f_n$ all measurable. In particular, if $(f_n)_n \to f$ pointwise, then f measurable.

Major Theorems

- [Continuity of Measure] Let $A_1 \subseteq A_2 \subseteq \cdots$ be an increasing sequence of measurable sets (i.e. $A_j \subseteq A_{j+1}$), then $m\big(\bigcup_{j=1}^\infty A_j\big) = \lim_{j\to\infty} m\big(A_j\big)$
- [Continuity of Measure] Let $A_1 \supseteq A_2 \supseteq \cdots$ be a decreasing sequence of measurable sets (i.e. $A_j \supseteq A_{j+1}$) and $m(A_1) < +\infty$, then $m\big(\bigcap_{j=1}^\infty A_j\big) = \lim_{j\to\infty} m\big(A_j\big)$

Regularity and Slice

- [G_δ Set] A G_δ set is a countable intersection of open sets.
- [F_σ Set] A F_σ set is a countable union of closed sets.
- [Regularity] Every measurable set E can be sandwiched between a F_σ set F and G_δ set G with $F \subset E \subset G$ s.t. $m^*(G \backslash F) = 0$ (i.e. $G \backslash F$ is a zero set). Conversely, if $\exists G \in G_\delta$, $F \in F_\sigma$ s.t. $F \subset E \subset G$ and $m^*(G \backslash F) = 0$, then E measurable.
  - All measurable sets are F_σ sets and G_δ sets modulo a zero set.

Vitali Covering

- [Vitali Covering] A covering $\mathcal{V}$ of A is a Vitali covering if $\forall p \in A$ and $r > 0$, $\exists V \in \mathcal{V}$ s.t. $p \in V \subset B_r(p)$ and V is not merely $\{p\}$.
- [Vitali Covering Lemma] A Vitali covering of a bounded set $A \subset \mathbb{R}^n$ by closed balls reduces to an efficient disjoint sub-covering of almost all of A satisfying the following. Given $\epsilon > 0$, $\mathcal{V}$ can be reduced to countable sub-covering $\{V_k\}$:
  - V_k disjoint i.e. $V_i \cap V_j = \phi$ for $i \neq j$
  - $m(\bigcup_k V_k) \leq m^*(A) + \epsilon$
  - $m(\bigcup_k V_k) = 0$
- [Vitali Covering Lemma for Cubes] A Vitali covering of $A \subset \mathbb{R}^n$ by closed cubes also reduces to an efficient disjoint sub-covering of almost all of A.

Lebesgue Density Theorem

- [Density] Let $E \subset \mathbb{R}^n$ measurable. Define the density of E at $p \in \mathbb{R}^n$ as $\delta(p, E) = \lim_{Q \downarrow p} \frac{m(E \cap Q)}{m(Q)}$ if the limit exists.
- [Lebesgue Density Theorem] If E measurable, then almost every $p \in E$ is a density point of E.
  - If E measurable, then for almost every p, we have $\chi_E(p) = \lim_{Q \downarrow p} \frac{m(E \cap Q)}{m(Q)}$.
  - $\mathrm{dp}(\mathrm{dp}(E)) = \mathrm{dp}(E)$
- [Average] Define the average of a locally integrable function $f: \mathbb{R}^n \to \mathbb{R}$ over measurable set $A \subset \mathbb{R}^n$ with finite positive measure as: $\oint_A f = \frac{1}{m(A)} \int_A f$
- [Average Value Theorem] If $f: \mathbb{R}^n \to \mathbb{R}$ locally integrable, then for almost every $p \in \mathbb{R}^n$, the density of f at p exists and $f(p) = \delta(p, f) = \lim_{Q \downarrow p} \oint_Q f$
- If $g: \mathbb{R}^n \to [0, \infty)$ is integrable, then for every small $\alpha > 0$, the set $X(\alpha, g) = \{p: \bar{\delta}(p, g) > \alpha\}$ has outer measure $m^*(X(\alpha, g)) \leq \frac{1}{\alpha} \int g$

Lebesgue Integration

Definition

- [Simple Function] Let $\Omega \subset \mathbb{R}^n$ measurable and $f: \Omega \to \mathbb{R}$ measurable. Say f is simple if $f(\Omega)$ is finite i.e. exists $\{c_1, \dots, c_N\}$ s.t. $f(x) = c_j$ for some $1 \le j \le N$.
- [Indicator Function] $\chi_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$
- [Lebesgue Integral for Simple Functions] $\int_\Omega f := \sum_{\lambda \in f(\Omega); \lambda > 0} \lambda \cdot m(\{x \in \Omega: f(x) = \lambda\})$.
- [Majorize] Say f majorizes g if $f(x) \ge g(x)$ for all $x \in \Omega$
- [Minorize] Say f minorizes g if $f(x) \le g(x)$ for all $x \in \Omega$
- [Lebesgue Integral for Nonnegative Functions]
$$\int_\Omega f := \sup\left\{\int_\Omega s : s \text{ simple, nonnegative and minorizes } f\right\}$$
- [Absolutely Integrable Function] Let $\Omega \subset \mathbb{R}^n$ measurable and $f: \Omega \to \mathbb{R}^*$. Say f is absolutely integrable if $\int_\Omega |f|$ finite.
- [Lebesgue Integral] Define $f^+ := \max(f, 0)$ and $f^- := -\min(f, 0)$. Let $f: \Omega \to \mathbb{R}^*$ be an absolutely integrable function, then $\int_\Omega f := \int_\Omega f^+ - \int_\Omega f^-$
- [Upper and Lower Lebesgue Integral] Let $\Omega \subset \mathbb{R}^n$ measurable and $f: \Omega \to \mathbb{R}$ not necessarily measurable. Define the upper and lower Lebesgue integrals
$$\overline{\int}_\Omega f := \inf\left\{\int_\Omega g : g: \Omega \to \mathbb{R} \text{ absolutely integrable function}, g \ge f\right\}$$
$$\underline{\int}_\Omega f := \sup\left\{\int_\Omega g : g: \Omega \to \mathbb{R} \text{ absolutely integrable function}, g \le f\right\}$$

Properties

- [Lebesgue Integral for Simple Functions] Let $f, g: \Omega \to \mathbb{R}$ be nonnegative simple functions.
  - $0 \le \int_\Omega f \le \infty$ with $\int_\Omega f = 0$ if and only if $m(\{x \in \Omega: f(x) \ne 0\}) = 0$ i.e. $f = 0$ a.e.
  - $\int_\Omega f + g = \int_\Omega f + \int_\Omega g$
  - Let $c > 0$, then $\int_\Omega cf = c\int_\Omega f$
  - If $f(x) \le g(x)$ for all $x \in \Omega$, then $\int_\Omega f \le \int_\Omega g$
- [Lebesgue Integral for Nonnegative Functions] Let $f, g: \Omega \to [0, \infty]$ be nonnegative measurable functions.
  - $0 \le \int_\Omega f \le \infty$ with $\int_\Omega f = 0$ if and only if $f(x) = 0$ a.e.
  - Let $c > 0$, then $\int_\Omega cf = c\int_\Omega f$
  - $f(x) \le g(x)$ for all $x \in \Omega$, then $\int_\Omega f \le \int_\Omega g$
  - $f(x) = g(x)$ a.e. then $\int_\Omega f = \int_\Omega g$
  - Let $\Omega' \subset \Omega$ measurable, then $\int_{\Omega'} f = \int_\Omega f\chi_{\Omega'} \le \int_\Omega f$
- [Lebesgue Integral] Let Ω measurable, $f, g: \Omega \to \mathbb{R}$ absolutely integrable.
  - [Triangle Inequality] $\left|\int_\Omega f\right| \le \int_\Omega |f|$
  - $\forall c \in \mathbb{R}$, cf absolutely integrable and $\int_\Omega cf = c\int_\Omega f$
  - $f + g$ absolutely integrable with $\int_\Omega f + g = \int_\Omega f + \int_\Omega g$
  - If $f(x) \le g(x)\ \forall x \in \Omega$, then $\int_\Omega f \le \int_\Omega g$
  - If $f(x) = g(x)$ a.e., then $\int_\Omega f = \int_\Omega g$

Theorems and Lemmas

- [Vector Field] $f, g: \Omega \to \mathbb{R}$ simple. Then $f + g$, cf are also simple where $c \in \mathbb{R}$
- [Decomposition into Simple Functions] Let $f: \Omega \to \mathbb{R}$ simple. Then $\exists\ \{c_1, \dots, c_N\}$ and a finite number of disjoint measurable sets $E_1, \dots, E_N \subset \Omega$ s.t. $f = \sum_{i=1}^N c_i \chi_{E_i}$

- [Pointwise Approximation by Simple Functions] Let $f: \Omega \to \mathbb{R}$ measurable, nonnegative. Then $\exists (f_n)_n$ of simple functions s.t. f_n nonnegative and increasing $0 \leq f_1(x) \leq f_2(x) \leq \cdots$ $\forall x \in \Omega$ and converges pointwise to f i.e. $\lim_{n\to\infty} f_i(x) = f(x)$.
- [Lebesgue Integral of Simple Functions II] Let $E_1, \ldots E_N$ be a finite number of disjoint subsets in Ω and $c_1, \ldots, c_N$ be nonnegative numbers. Then $\int_\Omega f = \int_\Omega \sum_{j=1}^N c_j \chi_{E_j} = \sum_{j=1}^N c_j m(E_j)$
- [Interchange of Addition and Integration] Let $f, g: \Omega \to [0, \infty]$ be measurable, then $\int_\Omega f + g = \int_\Omega f + \int_\Omega g$
- [Interchange of Integration and Series] Let $g_1, g_2, \ldots$ be a sequence of nonnegative measurable functions $\int_\Omega \sum_{n=1}^\infty g_n = \sum_{n=1}^\infty \int_\Omega g_n$
- [Finite Integral] Let $\Omega \subset \mathbb{R}^n$ measurable and $f: \Omega \to [0, \infty]$ be nonnegative measurable s.t. $\int_\Omega f$ finite. Then f finite a.e. (i.e. $m\{x \in \Omega: f(x) = +\infty\} = 0$)
- $\underline{\int}_\Omega f \leq \overline{\int}_\Omega f$
- If f absolutely integrable, then $\overline{\int}_\Omega f = \underline{\int}_\Omega f$.
- Let $\Omega \subset \mathbb{R}^n$ measurable and $f: \Omega \to \mathbb{R}$ not necessarily measurable. Suppose $\underline{\int}_\Omega f = \overline{\int}_\Omega f \in \mathbb{R}$, then f is absolutely integrable and $\int_\Omega f = \overline{\int}_\Omega f = \underline{\int}_\Omega f$
- Let $I \subset \mathbb{R}$ be an interval and $f: I \to \mathbb{R}$ be Riemann integrable. Then f also absolutely integrable with $\int_I f = R.\int_I f$

Major Theorems

- [Lebesgue Monotone Convergence Theorem] Let $\Omega \subset \mathbb{R}^n$ measurable and let $(f_n)_{n=1}^\infty$ be nonnegative measurable and increasing $0 \leq f_1(x) \leq f_2(x) \leq \cdots$ for all $x \in \Omega$. Then $0 \leq \int_\Omega f_1 \leq \int_\Omega f_2 \leq \cdots$ and $\int_\Omega \sup_n f_n = \sup_n \int_\Omega f_n$.
- [Fatou's Lemma] Let $\Omega \subset \mathbb{R}^n$ measurable and $f_1, f_2, \ldots$ be a sequence of nonnegative functions from Ω to $[0, \infty]$. Then $\int_\Omega \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_\Omega f_n$.
    - Intuitively, integrating over the worst of every function at each pointwise position will definitely be no better than taking the worst of the integral of a function
- [Borel-Cantelli Lemma] Let $\Omega_1, \Omega_2, \ldots$ be measurable subsets of $\mathbb{R}^n$ s.t. $\sum_{n=1}^\infty m(\Omega_n) < \infty$. Then $m(\{x \in \mathbb{R}^n: x \in \Omega_n \ i.o.\}) = 0$, i.e. almost every point $x \in \mathbb{R}^n$ belongs to finitely many Ω_n.
- [Lebesgue Dominated Convergence Theorem] Let $\Omega \subset \mathbb{R}^n$ measurable and let $(f_n)_{n=1}^\infty$ measurable with $f_i: \Omega \to \mathbb{R}^*$ and $\lim_{n\to\infty} f_n = f$ pointwise. Suppose $\exists$ an absolutely integrable function $F: \Omega \to [0, \infty]$ s.t. $|f_n(x)| \leq F(x)$ $\forall x \in \Omega$, then $\int_\Omega \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_\Omega f_n$
- [Fubini's Theorem] Let $f: \mathbb{R}^2 \to \mathbb{R}$ absolutely integrable. Then $\exists$ absolutely integrable functions $F, G: \mathbb{R} \to \mathbb{R}$ s.t. for almost every x, $f(x, y)$ absolutely integrable in y with $F(x) = \int_\mathbb{R} f(x, y)\, \mathrm{d}y$ and for almost every y, $g(x, y)$ absolutely integrable in x with $G(y) = \int_\mathbb{R} f(x, y)\, \mathrm{d}x$. Finally, $\int_\mathbb{R} F(x)\, \mathrm{d}x = \int_{\mathbb{R}^2} f = \int_\mathbb{R} G(y)\, \mathrm{d}y$

Counter Examples

- [Interchange of Limit and Integrals] $\int_\Omega \lim_{n\to\infty} f_n \neq \lim_{n\to\infty} \int_\Omega f_n$ Consider the moving bump example.

Multivariable Calculus

Definitions

- [Norm] A norm on vector space V, $|\cdot|: V \to \mathbb{R}$, satisfies three properties
  - [Positive Definite] $\forall v \in V$, $|v| \geq 0$ with $|v| = 0$ if and only if $v = 0$
  - [Positive Homogeneity of Degree 1] $|\lambda v| = |\lambda||v|$
  - [Triangle Inequality] $|v + w| \leq |v| + |w|$
- [Normed Space] A vector space with a norm. It is a metric space with $d(v_1, v_2) = |v_1 - v_2|$
- [Operator Norm] Let V, W be normed spaces and $T: V \to W$ linear transformation. Define the operator norm of T as $\|T\| = \sup\left\{\frac{|Tv|_W}{|v|_V} : v \neq 0\right\}$ (i.e. maximum stretch of T)
- [Derivative] Let $f: U \to \mathbb{R}^m$ where $U \subset \mathbb{R}^n$ open. Say f is differentiable at $p \in U$ with derivative $(Df)_p = T$ if $T: \mathbb{R}^n \to \mathbb{R}^m$ linear transformation and $f(p + v) = f(p) + T(v) + R(v)$ where $\lim_{|v| \to 0} \frac{R(v)}{|v|} = 0$. Note: $(Df)_p$ if it exists is a linear transformation from $U \to \mathbb{R}^m$.
- [Partial Derivative] $\frac{\partial f_i(p)}{\partial x_j} = \lim_{t \to 0} \frac{f_i(p + te_j) - f_i(p)}{t}$ if it exists.
- [Lipschitz Condition] Say $f(t, y)$ satisfies a Lipschitz condition in y on set $D \subset \mathbb{R}^2$ if $\exists L > 0$ with $|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$ whenever $(t, y_1), (t, y_2) \in D$. Define L as the Lipschitz constant.

Operator Analysis

- $\|T \circ S\| \leq \|T\|\|S\|$
- Let V, W normed spaces and $T: V \to W$ linear transformation. The following are equivalent:
  - $\|T\| < \infty$
  - T is uniformly continuous
  - T is continuous
  - T is continuous at the origin
- Every linear transformation $T: \mathbb{R}^n \to W$ is continuous.
- Let $T: \mathbb{R}^n \to W$ linear transformation. If T is an isomorphism, then T^{-1} is continuous. (i.e. T is a homeomorphism; bi-continuous bijection)
- In finite dimensional normed spaces, all linear transformations are continuous and all isomorphisms are homeomorphism.

Theorems and Lemmas

- If f differentiable at p, then $(Df)_p(u) = \lim_{t \to 0} \frac{f(p + tu) - f(p)}{t}$.
- If f is differentiable at p, then it is automatically continuous at p.
- If Df exists, then partial derivatives exist and in matrix form $\left[(Df)_p\right]_{ij} = \frac{\partial f_i(p)}{\partial x_j}$.
- If partial derivatives of $f: U \to \mathbb{R}^m$ exist and continuous, then f differentiable.
- Let f, g differentiable. Then:
  - $D(f + cg) = Df + cDg$
  - $D(\text{constant}) = 0$
  - [Chain Rule] $D(g \circ f) = Dg \circ Df$
  - [Leibniz Rule] $D(fg) = (Df)g + f(Dg)$
- $f: U \to \mathbb{R}^m$ is differentiable at $p \in U$ if and only if each of its components f_i is differentiable at p. The derivative of its ith component is the ith component of the derivative.
- [Mean Value Theorem] If $f: U \to \mathbb{R}^m$ differentiable on U and $[p, q] \subset U$, then $|f(q) - f(p)| \leq M|q - p|$ where $M = \sup_{x \in U} \|(Df)_x\|$
- [C^1 Mean Value Theorem] If $f: U \to \mathbb{R}^m$ is of class C^1 (i.e. derivative exists and continuous) and $[p, q] \subset U$, then $f(q) - f(p) = T(q - p)$ where $T = \int_0^1 (Df)_{p + t(q - p)} \mathrm{d}t$ is the average derivative of f on $[p, q]$
- If U connected and $f: U \to \mathbb{R}^m$ differentiable and for each point $x \in U$, $(Df)_x = 0$, then f is constant.

- [Differentiation Past Integral] If $f:[a,b]\times(c,d)\to\mathbb{R}$ continuous and $\frac{\partial f(x,y)}{\partial y}$ exists and continuous, then $F(y)=\int_a^b f(x,y)\,\mathrm{d}x$ is of class C^1 and $\frac{\mathrm{d}F}{\mathrm{d}y}=\int_a^b \frac{\partial f(x,y)}{\partial y}\,\mathrm{d}x$
- [Schwarz Theorem] If $f\in C^2$ (i.e. all second partial derivatives exist and continuous), then second order partial derivatives commute $\frac{\partial^2 f}{\partial x_i \partial x_j}=\frac{\partial^2 f}{\partial x_j \partial x_i}$.

Inequalities

- Hölder's inequality: For $p,q\ge 1$ with $\frac{1}{p}+\frac{1}{q}=1$, $\left(\sum_{i=1}^n |x_i y_i|\right)\le\left(\sum_i |x_i|^p\right)^{\frac{1}{p}}\left(\sum_i |y_i|^q\right)^{\frac{1}{q}}$
  - $\|fg\|_1\le\|f\|_p\|g\|_q$
  - Let $\Omega\subset\mathbb{R}^n$ Lebesgue measurable and f,g measurable, then $\int_\Omega |f(x)g(x)|\,\mathrm{d}x\le\left(\int_\Omega |f(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_\Omega |g(x)|^q\,\mathrm{d}x\right)^{\frac{1}{q}}$
- Minkowski's Inequality: For $p\ge 1$, $\left(\sum_{i=1}^n |x_i+y_i|^p\right)^{\frac{1}{p}}\le\left(\sum_i |x_i|^p\right)^{\frac{1}{p}}+\left(\sum_i |y_i|^p\right)^{\frac{1}{p}}$

Contraction Mapping Theorem

- [Contraction] Let (X,d) metric space and $f:X\to X$. Say f is a contraction if $d\big(f(x),f(y)\big)\le d(x,y)\ \forall x,y\in X$.
- [Strict Contraction] Say f is a strict contraction if $\exists c$ with $0<c<1$ s.t. $d\big(f(x),f(y)\big)\le c\cdot d(x,y)\ \forall x,y\in X$.
- [Contraction Constant] Say c is the contraction constant of f.
- [Fixed Points] Let $f:X\to X$ and $x\in X$. Say x is a fixed point of f if $f(x)=x$.

- [Contraction Mapping Theorem] Let (X,d) metric space and $f:X\to X$ strict contraction. Then f has at most 1 fixed point. If X nonempty and complete, then f has exactly one fixed point $x=\lim\limits_{n\to\infty}\underbrace{f\Big(f\big(\dots f(x)\big)\Big)}_{n\text{ times}}$.
- [Hairy Ball Theorem] Any continuous map $f:S^2\to S^2$ (i.e. from the sphere $\{(x,y,z)\in\mathbb{R}^3: x^2+y^2+z^2=1\}$ to itself) must contain either a fixed point or an anti-fixed point (i.e. a point x in S^2 s.t. $f(x)=-x$)
- Let $B_r(0)\subset\mathbb{R}^n$ and $g:B_r(0)\to\mathbb{R}^n$ with $g(0)=0$ and $\|g(x)-g(y)\|\le\frac{1}{2}\|x-y\|\ \forall x,y\in B_r(0)$, then $f:B_r(0)\to\mathbb{R}^n$ defined by $f(x)=g(x)+x$ is injective and the image $f\big(B_r(0)\big)$ contains $B_{\frac{r}{2}}(0)$

Inverse Function Theorem

- [1D Inverse Function Theorem] If $f:\mathbb{R}\to\mathbb{R}$ invertible, differentiable and $f'(x_0)$ nonzero, then f^{-1} differentiable at $f(x_0)$ with $(f^{-1})'\big(f(x_0)\big)=\frac{1}{f'(x_0)}$
- Let $T:\mathbb{R}^n\to\mathbb{R}^n$ invertible linear transformation. Then the inverse transformation $T^{-1}:\mathbb{R}^n\to\mathbb{R}^n$ is also linear.
- [Inverse Function Theorem] Let $E\subset\mathbb{R}^n$ open and $f:E\to\mathbb{R}^n$ continuously differentiable on E. Suppose $x_0\in E$ is such that the linear transformation $f'(x_0):\mathbb{R}^n\to\mathbb{R}^n$ invertible. Then $\exists$ open $U\subset E$ containing x_0 and open $V\subset\mathbb{R}^n$ containing $f(x_0)$ s.t. f bijection from U to V. In particular, there is an inverse map $f^{-1}:V\to U$ that is differentiable at $f(x_0)$ and $(f^{-1})'\big(f(x_0)\big)=\big(f'(x_0)\big)^{-1}$
- If $f(x_0)$ not invertible, then impossible that inverse f^{-1} can exist and be differentiable at x_0
- If $f'(x_0)$ exists but non-invertible, then inverse function theorem does not apply.
  - Not possible for f^{-1} to exist and be differentiable at x_0
  - Still possible for f to be invertible however (consider $f(x)=x^3$ despite $f'(0)$ not invertible)
- Let $f:\mathbb{R}^n\to\mathbb{R}^n$ continuous differentiable function s.t. $f'(x)$ invertible linear transformation $\forall x\in\mathbb{R}^n$. Then whenever $V\subset\mathbb{R}^n$ open, $f(V)$ open.

Implicit Function Theorem

- [Implicit Function Theorem] Let $E \subset \mathbb{R}^n$ open and $f: E \to \mathbb{R}$ continuously differentiable. Let $y = (y_1, \dots, y_n) \in E$ s.t. $f(y) = 0$ and $\frac{\partial f}{\partial x_n}(y) \neq 0$. Then $\exists U \subset \mathbb{R}^{n-1}$ open containing $(y_1, \dots, y_{n-1})$, $V \subset E$ containing y and $g: U \to \mathbb{R}$ s.t. $g(y_1, \dots, y_{n-1}) = y_n$ and $\{(x_1, \dots, x_n) \in V: f(x_1, \dots, x_n) = 0\} = \{(x_1, \dots, x_{n-1}, g(x_1, \dots, x_{n-1})): (x_1, \dots, x_{n-1}) \in U\}$ i.e. the set $\{x \in V: f(x) = 0\}$ is a graph of a function over U. Moreover, g differentiable at $(y_1, \dots, y_{n-1})$ with $\frac{\partial g}{\partial x_j}(y_1, \dots, y_{n-1}) = -\frac{\frac{\partial f}{\partial x_j}(y)}{\frac{\partial f}{\partial x_n}(y)}$ for $1 \leq j \leq n-1$.
- Given $f(x_1, \dots, x_n) = 0$, $\frac{\partial f}{\partial x_j} + \frac{\partial f}{\partial x_n} \cdot \frac{\partial f}{\partial x_j} = 0$

Differential Forms

- [k-cell in $\mathbb{R}^n$] Smooth map $\phi: [0,1]^k \to \mathbb{R}^n$ (set of k-cells in $\mathbb{R}^n$ is $C_k(\mathbb{R}^n)$)
- [k-form] $\omega = \sum_I f \mathrm{d}y_I$ where $\mathrm{d}y_I = \mathrm{d}y_{i_1} \wedge \dots \wedge \mathrm{d}y_{i_k}$ (just see the number of differentials)
  - ω is a k-form, then $\mathrm{d}\omega$ is a $(k+1)$-form
- [Jacobian] $\frac{\partial \phi_I}{\partial u} = \frac{\partial(\phi_{i_1}, \dots, \phi_{i_k})}{\partial(u_1, \dots, u_k)} = \begin{vmatrix} \frac{\partial \phi_{i_1}}{\partial u_1} & \cdots & \frac{\partial \phi_{i_1}}{\partial u_k} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_{i_k}}{\partial u_1} & \cdots & \frac{\partial \phi_{i_k}}{\partial u_k} \end{vmatrix}$
- [Integration] $\int_\phi f \mathrm{d}y_I = \int_{[0,1]^k} f(\phi(u)) \frac{\partial \phi_I}{\partial u} \mathrm{d}u = \int_{[0,1]^k} f(\phi(u)) \frac{\partial(\phi_{i_1}, \dots, \phi_{i_k})}{\partial(u_1, \dots, u_k)} \mathrm{d}u_1 \dots \mathrm{d}u_k$

Exterior Derivative d	Pullbacks T^*
• $\mathrm{d}(\alpha + c\beta) = \mathrm{d}\alpha + c\mathrm{d}\beta$	• $(T^*\alpha)(\phi) = \alpha(T_*\phi)$
• $\mathrm{d}(\sum_I f_I \mathrm{d}y_I) = \sum_I \mathrm{d}f_I \wedge \mathrm{d}y_I$	• $T^*(\alpha \wedge \beta) = T^*\alpha \wedge T^*\beta$
• $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta$	• $\mathrm{d}T^* = T^*\mathrm{d}$
• $\mathrm{d}^2 = 0$ (i.e. $\mathrm{d}(\mathrm{d}\omega) = 0\ \forall \omega \in \Omega^k$)	• $\int_{T \circ \phi} \alpha = \int_\phi T^*\alpha$

Stokes' Theorem

$$\int_\Omega \mathrm{d}\omega = \int_{\partial\Omega} \omega$$

- The boundary of a $(k+1)$-cell ϕ is the k-chain $\partial\phi = \sum_{j=1}^{k+1} (-1)^{j+1} (\phi \circ i^{j,1} - \phi \circ i^{j,0})$
- $i^{j,1}, i^{j,0}: [0,1]^k \to \mathbb{R}^{k+1}$ are k-cells; $i^{j,l}(u_1, \dots, u_k) = (u_1, \dots, u_{j-1}, l, u_j, \dots, u_k)$
- [Stokes' Formula for Cubes] If ω is a $(n-1)$-form in $\mathbb{R}^n$ and $i: [0,1]^n \to \mathbb{R}^n$ is the identity-inclusion n-cell in $\mathbb{R}^n$, then $\int_i \mathrm{d}\omega = \int_{\partial i} \omega$
- [Stokes' Formula for General Cell] If ω is a $(n-1)$-form in $\mathbb{R}^m$ and ϕ is an n-cell in $\mathbb{R}^m$, then $\int_\phi \mathrm{d}\omega = \int_{\partial\phi} \omega$
- [Stokes' Formula for General Manifold] If $M \subset \mathbb{R}^n$ divides into m-cells diffeomorphic to $[0,1]^m$ and its boundary divides into $(m-1)$-cells diffeomorphic to $[0,1]^{m-1}$ and ω is a $(m-1)$-form, then $\int_M \mathrm{d}\omega = \int_{\partial M} \omega$
- [Homotopy Operator] Integration along one component $H_z(\omega) = \left(\int_0^z g(x,y,z)\, \mathrm{d}z\right)\mathrm{d}x + \left(\int_0^z h(x,y,z)\, \mathrm{d}z\right)\mathrm{d}y$

Application of Stokes' Theorem

- [Fundamental Theorem of Calculus] $M = [a,b] \subset \mathbb{R}$, $\omega = f$, then $f(b) - f(a) = \int_{\partial M} \omega = \int_M \mathrm{d}\omega = \int_{[a,b]} \mathrm{d}f$
- [Path Independence] Let $p, q \in \mathbb{R}^2$ and M be 1-cell representing path from p to q. Then $f(q) - f(p) = \int_{\partial M} \omega = \int_M \mathrm{d}\omega = \int_{[p,q]} f_x \mathrm{d}y + f_y \mathrm{d}x$
- [Green's Theorem] $M = D$ region, $\omega = f\mathrm{d}x + g\mathrm{d}y$, then $\partial M = C$ the curve bounding D. $\int_C f\mathrm{d}x + g\mathrm{d}y = \int_{\partial M} \omega = \int_M \mathrm{d}\omega = \int_D (g_x - f_y)\mathrm{d}x\,\mathrm{d}y$

- [Divergence Theorem] $M = D$ Gaussian surface, $\omega = f\ \mathrm{d}y \wedge \mathrm{d}z + g\ \mathrm{d}z \wedge \mathrm{d}x + h\ \mathrm{d}x \wedge \mathrm{d}y$, then $\partial M = S$ the boundary of Gaussian surface. $\int_S f\ \mathrm{d}y \wedge \mathrm{d}z + g\ \mathrm{d}z \wedge \mathrm{d}x + h\ \mathrm{d}x \wedge \mathrm{d}y = \int_{\partial M} \omega = \int_M \mathrm{d}\omega = \int_D (f_x + g_y + h_z)\ \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \int_D \nabla \cdot F$
- [Curl Theorem] $M = S$ Gaussian surface, $\omega = f\mathrm{d}x + g\mathrm{d}y + h\mathrm{d}z$, then $\partial M = C$ the curve bounding S. $\int_C f\mathrm{d}x + g\mathrm{d}y + h\mathrm{d}z = \int_{\partial M} \omega = \int_M \mathrm{d}\omega = \int_S (h_y - g_z)\mathrm{d}y \wedge \mathrm{d}z + (f_z - h_x)\mathrm{d}z \wedge \mathrm{d}x + (g_x - f_y)\mathrm{d}x \wedge \mathrm{d}y$

Poincaré's Lemma

Definitions:

- Say a form ω is closed if $\mathrm{d}\omega = 0$. Intuitively, ω is closed as in a loop
- Say a form ω is exact if $\mathrm{d}\omega = \lambda$ for some other form λ. Intuitively, ω is the boundary

Facts and Theorems:

- Every exact form is closed, since $\mathrm{d}\omega = \mathrm{d}(\mathrm{d}\lambda) = \mathrm{d}^2\lambda = 0$
- [Poincaré's Lemma] If ω is a closed k-form on $\mathbb{R}^n$, then it is exact.
- If U diffeomorphic to $\mathbb{R}^n$, then all closed forms on U are exact.
- Say a set $U \subset \mathbb{R}^n$ is star-like if $\exists p \in U$ s.t. the line segment from p to each $q \in U$ lies in U
- Every star-like open set in $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$
- If $U \subset \mathbb{R}^n$ open and convex, then closed forms on U exact.

Fourier Analysis

Definitions

- [L-periodic] Let $L > 0 \in \mathbb{R}$. Say $f: \mathbb{R} \to \mathbb{C}$ is L-periodic if $f(x+L) = f(x)\ \forall x \in \mathbb{R}$
- [$\mathbb{Z}$-periodic] Say $f: \mathbb{R} \to \mathbb{C}$ is $\mathbb{Z}$-periodic if $f(x+1) = f(x)\ \forall x \in \mathbb{R}$
- [$C(\mathbb{R}/\mathbb{Z};\mathbb{C})$] The space of $\mathbb{C}$-valued continuous $\mathbb{Z}$-periodic functions
- [Sup-norm] $d_\infty(f,g) = \sup_{x\in\mathbb{R}}|f(x)-g(x)| = \sup_{x\in[0,1)}|f(x)-g(x)|$
- [Inner Product] For $f, g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$, define $\langle f,g\rangle = \int_{[0,1]} f(x)\,\overline{g(x)}\,\mathrm{d}x$
- [L^2 Norm] $\|f\|_2 \coloneqq \sqrt{\langle f,f\rangle} = \left(\int_{[0,1]}|f(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}$
- [L^2 Metric] $d_{L^2}(f,g) \coloneqq \|f-g\|_2 = \left(\int_{[0,1]}|f(x)-g(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}$
- [Convergence in L^2 Metric] Let $f_n, f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$. Say $(f_n)_{n=1}^\infty \to f$ in L_2 metric if $\lim_{n\to\infty} d_{L^2}(f_n,f) = 0$. Equivalently, if $\int_{[0,1]}|f_n(x)-f(x)|^2\,\mathrm{d}x = 0$.
- [Characters] Define $e_n \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ as the character of frequency n where $e_n(x) \coloneqq e^{2\pi inx}$
- [Trigonometric Polynomial] Say $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ is a trigonometric polynomial if it can be expressed as a finite linear combination of characters: $f = \sum_{n=-N}^{N} c_n e_n$ for some $N \geq 0$.
- [Fourier Transform] Let $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$, define $\hat{f}: \mathbb{Z} \to \mathbb{C}$ as the Fourier transform of f and define the nth Fourier coefficient of f as:
$$\hat{f}(n) \coloneqq \langle f, e_n\rangle = \int_{[0,1]} f(x)e^{-2\pi inx}\,\mathrm{d}x$$
- [Periodic Convolution] Let $f, g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$. Define $f * g: \mathbb{R} \to \mathbb{C}$ by
$$(f*g)(x) \coloneqq \int_{[0,1]} f(y)g(x-y)\,\mathrm{d}y$$
- [Periodic Approximation to the Identity] Let $\epsilon > 0$ and $0 < \delta < \frac{1}{2}$. Say $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ is a periodic (ϵ,δ) approximation to the identity if:
    - $f(x) \geq 0\ \forall x \in \mathbb{R}$ and $\int_{[0,1]} f = 1$
    - $f(x) < \epsilon$ for all $\delta \leq |x| < 1-\delta$
- [Fejér Kernel] $F_N = \sum_{n=-N}^{N}\left(1-\frac{|n|}{N}\right)e_n = \frac{1}{N}|\sum_{n=0}^{N-1} e_n|^2$

Properties

- [Basic Properties of $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$]
    - [Boundedness] $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C}) \Rightarrow f$ is bounded
    - [Vector Space, Algebra] $f+g, f-g, fg, cf \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$
    - [Closure under uniform limit] $(f_n)_{n=1}^\infty \to f$ uniformly with $f_n \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})\ \forall n$ implies $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$
    - [Metric Space] $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ with d_∞ is a complete metric space
    - $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ is not complete in L_2 metric, but is complete in L_∞ metric.
- [Properties of Inner Product]
    - [Hermitian Symmetry] $\langle f,g\rangle = \overline{\langle g,f\rangle}$
    - [Positive Definite] $\langle f,f\rangle \geq 0$ with equality if and only if $f = 0$
    - [Linearity in First Argument] $\langle f+g,h\rangle = \langle f,h\rangle + \langle g,h\rangle$, $\langle cf,g\rangle = c\langle f,g\rangle$
    - [Anti-linearity in Second Argument] $\langle f,g+h\rangle = \langle f,g\rangle + \langle f,h\rangle$, $\langle f,cg\rangle = \bar{c}\langle f,g\rangle$
- [Properties of L_2 Norm]
    - [Positive Definite] $\|f\|_2 \geq 0$ with equality if and only if $f = 0$
    - [Cauchy-Schwarz] $|\langle f,g\rangle| \leq \|f\|_2\|g\|_2$
    - [Triangle Inequality] $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$
    - [Pythagoras] If $\langle f,g\rangle = 0$, then $\|f+g\|_2^2 = \|f\|_2^2 + \|g\|_2^2$
    - [Positive Homogeneity of Degree 1] $\|cf\|_2 = |c|\|f\|_2$
    - [Less than L_∞ Norm] $0 \leq \|f\|_2 \leq \|f\|_\infty$

- [Properties of Periodic Convolution]
  - [Closure] $f, g \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C}) \Rightarrow f * g \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$
  - [Commutativity] $f * g = g * f$
  - [Bilinearity] $f * (g + h) = f * g + f * h$, $(f + g) * h = f * h + g * h$, $c(f * g) = (cf) * g = f * (cg)$

MATH 54 Materials

- $f(x) = \frac{a_0}{2} + \sum_{m=1}^{k} \left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_n \sin\left(\frac{m\pi x}{L}\right)\right)$
- $\left\langle f(x), \cos\left(\frac{n\pi x}{L}\right)\right\rangle = \begin{cases} \frac{a_0}{2} \cdot 2L, & n = 0 \\ a_n \cdot L, & 0 < n \le k \\ 0, & n > k \end{cases}$
- $\left\langle f(x), \sin\left(\frac{n\pi x}{L}\right)\right\rangle = \begin{cases} b_n \cdot L, & 0 < n \le k \\ 0, & n > k \end{cases}$
- $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \quad n = 0, 1, \ldots$
- $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \quad n = 1, 2, \ldots$
- If f continuous on $[-L, L]$ with $f(-L) = f(L)$ i.e. it wraps around, then Fourier series converges to f $\forall x \in [-L, L]$
- If f continuous $2L$-periodic on $(-\infty, \infty)$, then Fourier series converges to f $\forall x \in (-\infty, \infty)$

Theorems and Lemmas

- [Orthonormal Basis] $\langle e_n, e_m \rangle = \delta_{nm}$
- [Projection] Let $f = \sum_{n=-N}^{N} c_n e_n$ be a trigonometric polynomial. Then:
  - $c_n = \langle f, e_n \rangle$ for $-N \le n \le N$
  - $\langle f, e_n \rangle = 0$ for $|n| > N$
  - $\|f\|_2^2 = \sum_{n=-N}^{N} |c_n|^2$
- [Fourier Transform of Trigonometric Polynomials]
  - $f = \sum_{n=-N}^{N} \langle f, e_n \rangle\, e_n = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle\, e_n = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}$
  - $\|f\|_2^2 = \sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right|^2$
- [Identity] Let $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$, then $f * e_n = \hat{f}(n) e_n$
  - $f * \sum_{n=-N}^{N} c_n\, e_n = \sum_{n=-N}^{N} c_n (f * e_n) = \sum_{n=-N}^{N} \hat{f}(n) c_n\, e_n$
- [Arbitrarily Close Approximation to Identity] For every $\epsilon > 0$ and $0 < \delta < \frac{1}{2}$, $\exists$ trigonometric polynomial P that is an (ϵ, δ) approximation to the identity. (Take the Fejér kernel F_N and take $N \to \infty$)

Main Theorems

- [Weierstrass Approximation Theorem] Let $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$ and $\epsilon > 0$. Then $\exists$ trigonometric polynomial P s.t. $\|f - P\|_\infty \le \epsilon$.
  - Any continuous periodic function can be <u>uniformly approximated</u> by trigonometric polynomials.
  - The closure of space of all trigonometric polynomial in L^∞ metric is $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$
  - Consider $P = f * P'$ where P' is a (ϵ, δ) approximation to the identity.
- [Fourier Theorem] Let $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$. The series $\sum_{n=-\infty}^{\infty} \hat{f}(n)\, e_n$ converges in the L^2 metric to f i.e $\lim_{N\to\infty} \|f - \sum_{n=-N}^{N} \langle f, e_n \rangle\, e_n\|_2 = 0$
  - Find trigonometric polynomial approximation $\|f - P\|_2 \le \|f - P\|_\infty < \epsilon$
  - $\langle f - S_N, S_N - P \rangle = 0 \Rightarrow \|f - P\|_2^2 = \|f - S_N\|_2^2 + \|S_N - P\|_2^2 \Rightarrow \|f - S_N\|_2 < \epsilon$
  - If f continuously differentiable, then it is also pointwise convergent.
  - If f continuously twice differentiable, then it is also uniformly convergent.
- Let $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$. Suppose $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|$ absolutely convergent, then $\sum_{n=-\infty}^{\infty} \hat{f}(n)\, e_n$ converges uniformly to f, i.e. $\lim_{N\to\infty} \left\|f - \sum_{n=-N}^{N} \hat{f}(n) e_n\right\|_\infty = 0$
  - Closure under uniform limit and apply uniqueness of limit point.

- [Plancherel Theorem] Let $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$. The series $\sum_{n=-\infty}^{\infty}\left|\hat{f}(n)\right|^2$ is absolutely convergent and $\|f\|_2^2 = \sum_{n=-\infty}^{\infty}\left|\hat{f}(n)\right|^2$
    - Apply Fourier's theorem and manipulate the convergence of L^2 norm.