No.: Date: 09/07/2024

MATH 104: LIMIT THEOREMS (LECTURE 4)

Recall: $\lim s_n = s \Rightarrow \forall \varepsilon > 0, \exists N$ s.t. $\forall n > N \Rightarrow |s_n - s| < \varepsilon$.

Visually, consider the ε-neighbourhood $\overset{(\ |\ )}{s}$ ← eventually, all numbers in sequence lands within this ε-neighbourhood.

Exercise

(1) $s_n = \dfrac{3n^2-1}{n^2-2n+3}$.

(2) Assume $\lim s_n = s$, show $(4s_n)$ converges to $4s$.

(3) $s_n = (-1)^n$ does not converge.

My workings.

(1) $s_n = \dfrac{3n^2-1}{n^2-2n+3} = 3 + \dfrac{6n-10}{n^2-2n+3}$ $|s_n - 3| = \left|\dfrac{6n-10}{n^2-2n+3}\right| < \dfrac{6n}{\left(\frac{n^2}{2}\right)} = \dfrac{12}{n}$

Let $\varepsilon > 0$. Set $N = \max\left(5, \dfrac{12}{\varepsilon}\right)$ $\forall n > N$

Then $|s_n - 3| = \left|\dfrac{3n^2-1}{n^2-2n+3} - 3\right| = \left|\dfrac{6n-10}{n^2-2n+3}\right| = \dfrac{6n-10}{n^2-2n+3}$ Since $6n-10>0$ $\forall n>5$, $n^2-2n+3 = (n-1)^2+2>0$

$< \dfrac{6n}{\frac{n^2}{2}}$ (Since $n^2-2n+3 > n^2-2n = \dfrac{n^2}{2}+\dfrac{n^2}{2}-2n > \dfrac{n^2}{2}$ for $n \geq 5$ $= \dfrac{n^2}{2} + \dfrac{(n-4)n}{2}$

$= \dfrac{12}{n} < \dfrac{12}{N} \leq \dfrac{12}{\left(\frac{12}{\varepsilon}\right)} = \varepsilon$.

Hence $\boxed{\lim s_n = 3}$

(2) $\lim s_n = s \Rightarrow \forall \varepsilon > 0, \exists N$ s.t. $\forall n > N$, $|s_n - s| < \varepsilon$.

Let $\varepsilon = \frac{\varepsilon}{4}$. $\exists N$ s.t. $\forall n > N$ $|s_n - s| < \frac{\varepsilon}{4}$

$\Leftrightarrow |4s_n - 4s| < \varepsilon$

$\Rightarrow \boxed{\lim 4s_n = 4s}$

(3) Assume on the contrary that s_n converges to s.
Let $\varepsilon = 1$. For any N, pick $2k$ and $2k+1 > N$

$s_{2k} = 1$ $|s_{2k} - s| < 1$ $2 = |s_{2k} - s_{2k+1}|$

$s_{2k+1} = -1$ $|s_{2k+1} - s| < 1$ $\leq |s_{2k} - s| + |s - s_{2k+1}| < 1+1 = 2$

contradiction!

$\therefore$ $\boxed{s_n = (-1)^n \text{ does not converge.}}$

No.: .... Date: ....

<u>Formal Proof:</u>

(1) Take any $\varepsilon > 0$.

Set $N = \max(3, \frac{12}{\varepsilon})$

Then $n > N \Rightarrow \left|\frac{3n^2-1}{n^2-2n+3} - 3\right| = \left|\frac{6n-10}{n^2-2n+3}\right| \overset{n\geq 1}{=} \frac{6n-10}{n^2-2n+3} \overset{n\geq 4 \Rightarrow \frac{n^2}{2}\geq 2n}{\leq} \frac{6n}{\frac{n^2}{2}} = \frac{12}{n} \overset{n > \frac{12}{\varepsilon}}{<} \varepsilon$

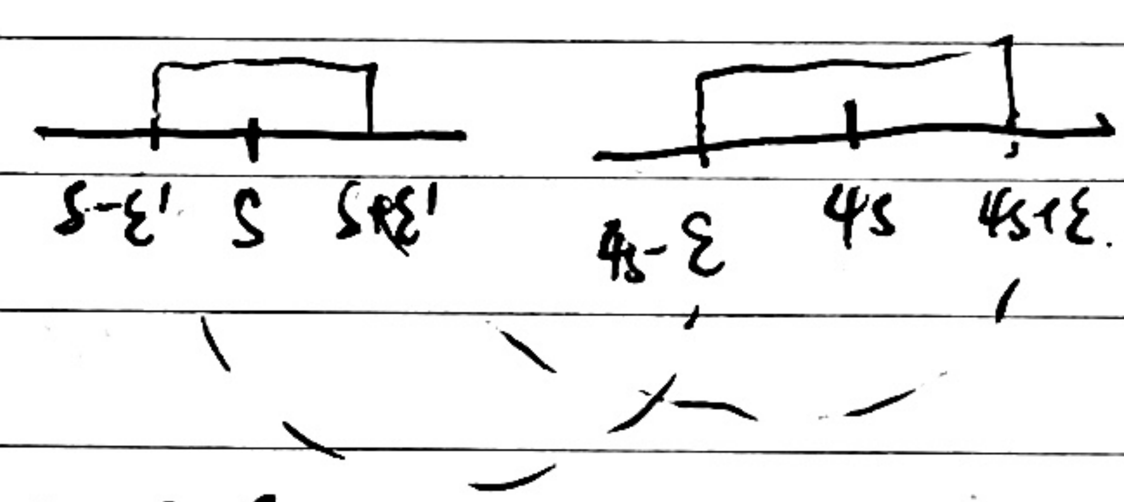
$\therefore \boxed{\lim s_n = 3}$

(2) Take any $\varepsilon > 0$.

Since $\lim s_n = s$, $\exists N \in \mathbb{R}$ s.t. $\forall n > N$, $|s_n - s| < \underbrace{\frac{\varepsilon}{4}}_{\varepsilon'}$.

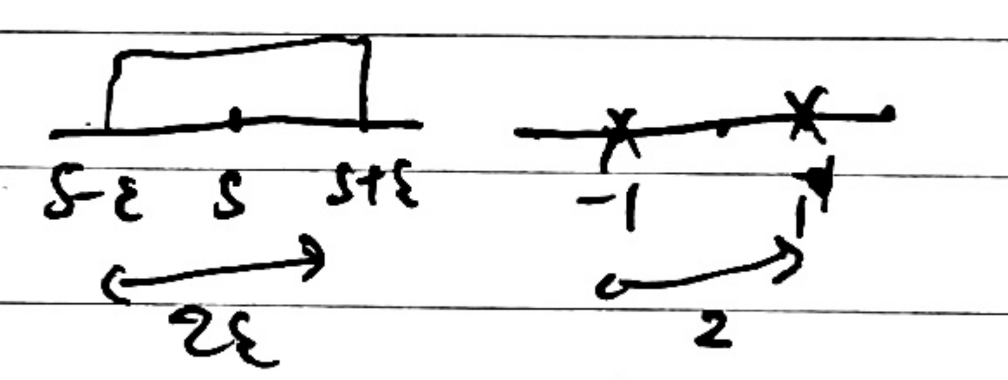
$\Rightarrow |4s_n - 4s| < \frac{\varepsilon}{4} \cdot 4 = \varepsilon$

$\therefore \boxed{\lim 4s_n = 4s.}$



(3) Assume on the contrary, let $\lim s_n = s \in \mathbb{R}$.

Applying $\varepsilon = 1$ to ε-N definition, $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |s_n - s| < 1$.



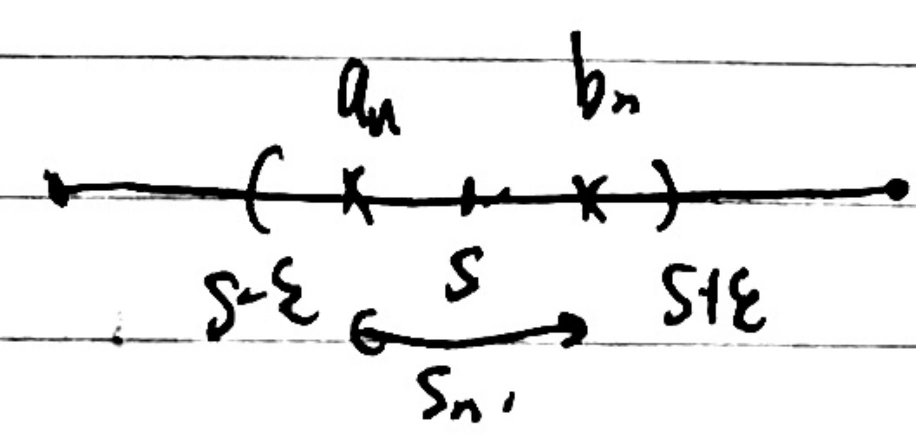
$2 < 2\varepsilon$ if converge $\forall \varepsilon > 0$.

In particular, $|s_{N+1} - s| < 1$ and $|s_{N+2} - s| < 1$.

$$2 = |s_{N+1} - s_{N+2}| = |(s_{N+1} - s) + (s - s_{N+2})|$$
$$\leq |s_{N+1} - s| + |s - s_{N+2}| < 1 + 1 = 2. \text{ (Contradiction!)}$$

<u>Squeeze Lemma</u>

Let (a_n), (b_n) and (s_n) be sequences s.t. $a_n \leq s_n \leq b_n$ $\forall n \in \mathbb{N}$.
If (a_n) and (b_n) are convergent and to the same real number s, then (s_n) converges to s.



<u>Formal Proof</u>

Take $\varepsilon > 0$.

Since $\lim a_n = s$, $\exists N_1 \in \mathbb{R}$ s.t. $n > N_1 \Rightarrow |a_n - s| < \varepsilon \Rightarrow a_n > s - \varepsilon$

Since $\lim b_n = s$, $\exists N_2 \in \mathbb{R}$ s.t. $n > N_2 \Rightarrow |b_n - s| < \varepsilon \Rightarrow b_n < s + \varepsilon$

Set $N = \max(N_1, N_2)$. Then $n > N \Rightarrow s - \varepsilon < a_n$ and $b_n < s + \varepsilon$.

Since $a_n \leq s_n \leq b_n$, $s - \varepsilon < a_n \leq s_n \leq b_n < s + \varepsilon \Rightarrow s - \varepsilon < s_n < s + \varepsilon \Rightarrow |s_n - s| < \varepsilon$

$\therefore \boxed{\lim s_n = s}$

Limit Theorems

Assume (s_n) and (t_n) are sequences s.t. $\lim s_n = s$ and $\lim t_n = t$.
Then

(1) $\forall k \in \mathbb{R}$. $\lim k s_n = k \lim s_n = ks$

(2) $\lim (s_n + t_n) = \lim s_n + \lim t_n = s+t$

(3) $\lim (s_n t_n) = (\lim s_n)(\lim t_n) = st$

(4) Assume $s_n \neq 0 \ \forall n$ and $s \neq 0$, then $\lim \frac{1}{s_n} = \frac{1}{s}$ and $\lim \frac{t_n}{s_n} = \frac{t}{s}$.

Example: $s_n = \frac{3n^2-1}{n^2-2n+3}$

$$s_n = \frac{3n^2-1}{n^2-2n+3} = \frac{3-\frac{1}{n^2}}{1-\frac{2}{n}+\frac{3}{n^2}}$$

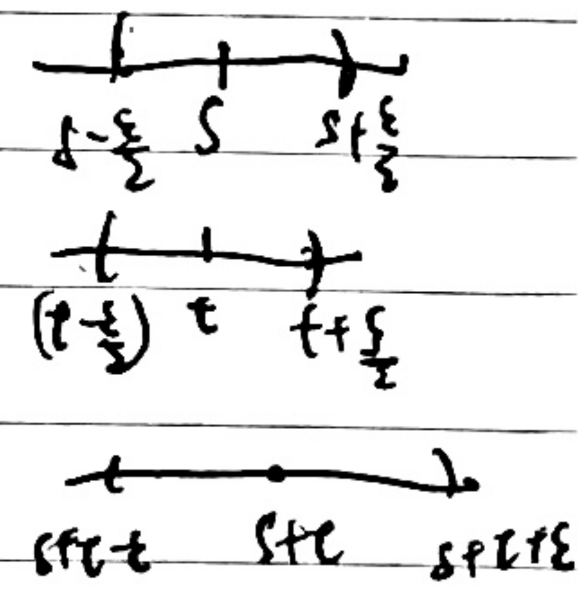
know $\lim \frac{1}{n} = 0 \xRightarrow{\text{limit theorem}} \lim(3-\frac{1}{n^2}) = 3$

$\lim(1-\frac{2}{n}+\frac{3}{n^2}) = 1$

$$\Rightarrow \lim s_n = \frac{\lim(3-\frac{1}{n^2})}{\lim(1-\frac{2}{n}+\frac{3}{n^2})} = \frac{3}{1} = 3.$$

For (1), note that when proving, need to isolate the case $k=0$ (even though it's trivial)

Formal Proof



(2) Take $\varepsilon > 0$.

Since $\lim s_n = s$, $\exists N_1 \in \mathbb{R}$ s.t. $n > N_1 \Rightarrow |s_n - s| < \frac{\varepsilon}{2}$.

$\lim t_n = t$, $\exists N_2 \in \mathbb{R}$ s.t. $n > N_2 \Rightarrow |t_n - t| < \frac{\varepsilon}{2}$.

Let $N = \max(N_1, N_2)$. Then $n > N$

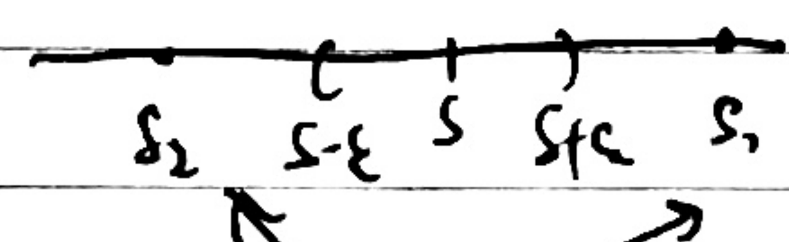
$$\Rightarrow |(s_n+t_n)-(s+t)| = |(s_n-s)+(t_n-t)| \overset{\Delta\text{ inequality}}{\leq} |s_n - s| + |t_n - t| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

$\therefore \boxed{\lim(s_n+t_n) = s+t}$

does not matter if $\leq$ or $<$

Theorem: Any convergent sequence (s_n) is bounded i.e. $\exists M > 0$ s.t. $|s_n| \leq M \ \forall n$.

Formal Proof:



wild terms are finitely many

Let $s = \lim s_n$. Then $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |s_n - s| < 1$.

$\Rightarrow s-1 < s_n < s+1 \Rightarrow |s_n| < |s|+1$ for $n > N$.

Set $M = \max(|s_1|, |s_2|, \dots, |s_N|, |s|+1)$

For $n = 1, 2, \dots, N$, $|s_n| \leq M$. For $n > N$, $|s_n| \leq M \Rightarrow |s_n| \leq M \ \forall n \in \mathbb{N}$

No.: Date:

$\lim s_n t_n = st$

Formal Proof

Take any $\varepsilon > 0$.

Since (s_n) converges, $\exists M > 0$ s.t. $|s_n| < M \ \forall n$.

Since $\lim t_n = t$, $\exists N_1 \in \mathbb{R}$ s.t. $n > N_1$

$\Rightarrow |t_n - t| < \frac{\varepsilon}{2M}$

Since $\lim s_n = s$, $\exists N_2 \in \mathbb{R}$, s.t. $n > N_2$

$\Rightarrow |s_n - s| < \frac{\varepsilon}{2|t|+1}$

because $|t|$ might be 0, so the +1 ensures positive

Idea:

$$s_n t_n - st = s_n t_n - s_n t + s_n t - st = s_n(t_n - t) + (s_n - s)t$$

$$|s_n t_n - st| \leq |s_n(t_n - t)| + |(s_n - s)t| = |s_n||t_n - t| + |s_n - s||t|$$

Both $t_n - t$ and $s_n - s$ can be arbitrarily small, and we know $|s_n|$ and $|t|$ are bounded ($|s_n|$ is bounded, $|t|$ is constant)

Set $N = \max(N_1, N_2)$. Then $n > N \Rightarrow |s_n t_n - st| = |s_n(t_n - t) + (s_n - s)t|$

$$\leq |s_n||t_n - t| + |s_n - s||t|$$

$$< M \cdot \frac{\varepsilon}{2M} + \frac{\varepsilon}{2(|t|+1)}|t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\frac{|t|}{|t|+1}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore \boxed{\lim s_n t_n = st.}$