MATH 104 - LECTURE 1 NOTES

Date: No.

Notation

$\mathbb{N}$: set of natural numbers $\{1, 2, 3, \dots\}$

$\mathbb{Z}$: set of integers $\{0, \pm 1, \pm 2, \dots\}$ — $(+, -, \times)$ ring

$\mathbb{Q}$: set of rational numbers $\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$ — $(+, -, \times, \div)$ field.

$\mathbb{R}$: set of real numbers — $(+, -, \times, \div, \underset{a \leq b}{\text{inequality}})$

$\mathbb{N}$ can be defined rigorously by peano's axioms. used in mathematical induction. to prove statement P_n for all $n \in \mathbb{N}$.

Exercise. Prove $\forall n \in \mathbb{N}$, $1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

< will prove by induction (specify method)

Let P_n be the statement $1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

P_1 is clearly true since $LHS = 1^2 = 1 = \frac{6}{6} = \frac{1 \cdot 2 \cdot 3}{6} = \frac{1(1+1)(2 \cdot 1+1)}{6} = RHS$.

Suppose P_k is true for some $k \in \mathbb{N}$. i.e. $1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$.

Then for P_{k+1},

$$LHS = [1^2 + 2^2 + \dots + k^2] + (k+1)^2 \overset{\text{by inductive hypothesis}}{=} \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)}{6}[2k^2 + k + 6k + 6]$$

$$= \frac{k+1}{6}(2k^2 + 7k + 6) = \frac{k+1}{6}(k+2)(2k+3) = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} = RHS$$

Hence $P_k \Rightarrow P_{k+1}$, and P_n is true $\forall n \in \mathbb{N}$.

Exercise Prove $\sqrt{2}$ is not rational.

< Assume on the contrary.

Prove by contradiction. Suppose $\sqrt{2}$ is rational. let $\sqrt{2} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$. and p, q are in lowest terms. Then $2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2 \Rightarrow 2 \mid p$. Let $p = 2k$. $4k^2 = 2q^2 \Rightarrow 2k^2 = q^2 \Rightarrow 2 \mid q$. But p, q are in lowest form $\Rightarrow$ contradiction.

Hence $\sqrt{2}$ is irrational

Without loss of generality (by dividing by a power of 2), we may assume at least m or n is odd.

Properties of $\mathbb{R}$ $(a, b, c \in \mathbb{R})$

- $a \leq a$
- $a \leq b$ and $b \leq a \Rightarrow a = b$.
- $a \leq b$ and $b \leq c \Rightarrow a \leq c$ (transitive)
- $a \leq b \Rightarrow a + c \leq b + c$ (with addition)
- $a \leq b$ and $0 \leq c \Rightarrow ac \leq bc$ (with multiplication)

Total Order
any two elements can be compared
$\forall a, b$ either $a \leq b$ or $b \leq a$.

Absolute Value

For $a \in \mathbb{R}$, $$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0. \end{cases}$$

$-|a| \leq a \leq |a|$

$|a+b| \leq |a| + |b|$ (triangle inequality) $a, b \in \mathbb{R}$

we know $$\left.\begin{array}{l} -|a| \leq a \leq |a| \\ -|b| \leq b \leq |b| \end{array}\right\} \Rightarrow -|a|-|b| \leq a+b \leq |a|+|b|$$

$$\Rightarrow \left.\begin{array}{l} (a+b) \leq |a|+|b|. \\ -a-b \leq |a|+|b| \end{array}\right] \Rightarrow \boxed{|a+b| \leq |a|+|b|}$$