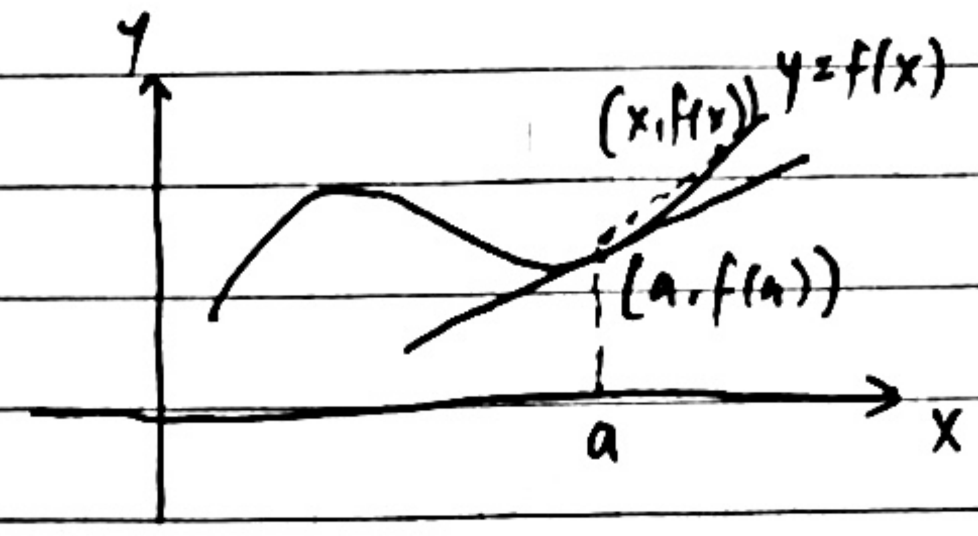
MATH 104 LECTURE 18 DERIVATIVES

Basic properties & examples of derivatives



Let I be an open interval.
$f: I \to \mathbb{R}$ $(a \in I)$

Qn: what is derivative $f'(a)$?

Definition: Say $f(x)$ is differentiable at $x \in I$ if $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ is a real number, in which case denote $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

Say f is differentiable if it is differentiable at every $a \in I$.

Note: $f'(a) = \lim \frac{f(x)-f(a)}{x-a}$ means:

Sequential Definition: $\forall (x_n)_n$ in I with $\lim x_n = a$ and $x_n \neq a$ $\forall n$
$$\lim_{n \to \infty} \frac{f(x_n)-f(a)}{x_n - a} = f'(a)$$

ε-δ Definition: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in I$, $0 < |x-a| < \delta \Rightarrow \left| \left\{\frac{f(x)-f(a)}{x-a}\right\} - f'(a) \right| < \varepsilon$

Example (1) $f(x) = x$ on $I = \mathbb{R}$. Then $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a} \frac{x-a}{x-a} = \lim_{x \to a} 1 = 1$

$$\Rightarrow \boxed{f'(x) = 1}$$

(2) $f(x) = \sqrt{x}$ on $I = (0, \infty)$

Then, $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} = \lim_{x \to a} \frac{(\sqrt{x}-\sqrt{a})}{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})} = \lim_{x \to a} \frac{1}{\sqrt{x}+\sqrt{a}}$

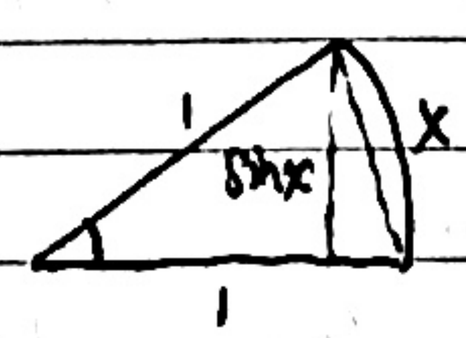
$= \frac{1}{2\sqrt{a}}$. $\therefore \boxed{f'(x) = \frac{1}{2\sqrt{x}}}$

function is continuous at a.

Note: $\lim_{x \to a} g(x) = g(a)$ if and only if g is continuous at a.

(3) $f(x) = \sin x$, $a = 0$.

$f'(0) = \lim_{x \to 0} \frac{\sin x - \sin 0}{x - 0} = \lim_{x \to 0} \frac{\sin x}{x} = 1$



Basic Properties

Theorem: Let $f, g: I \to \mathbb{R}$ and $a \in I$. Assume f and g are both differentiable at a. Then kf $(k \in \mathbb{R})$, $f+g$, fg are all differentiable at a.

$(kf)'(a) = kf'(a)$; $(f+g)'(a) = f'(a) + g'(a)$; $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.

Moreover, if $g(a) \neq 0$, $\frac{f}{g}$ is also differentiable at a. $\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{g^2(a)}$

Theorem: If f is differentiable at a, then f is continuous at a.

Proof:

By assumption, $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

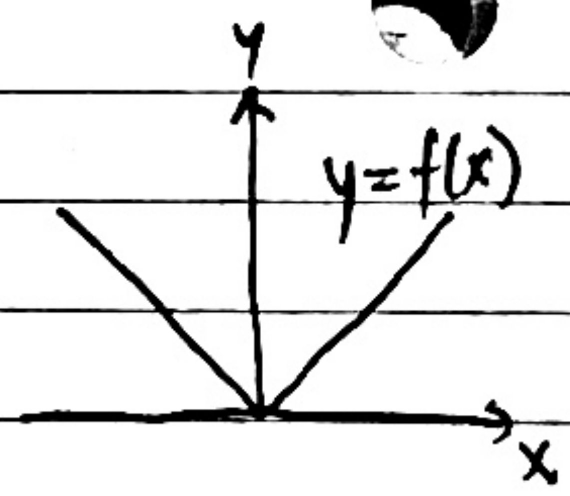
Write, for $x \neq a$, $f(x) = \frac{f(x)-f(a)}{x-a}(x-a) + f(a)$

The function $x-a$ is continuous, so $\lim_{x\to a}(x-a) = 0$

$$\Rightarrow \lim_{x\to a} f(x) = \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}(x-a) + f(a)\right] = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\lim_{x\to a}(x-a) + \lim_{x\to a} f(a)$$

$$= f'(a)\cdot 0 + f(a) = f(a).$$

Since $\lim_{x\to a} f(x) = f(a)$, $\boxed{f \text{ is continuous at } a.}$



Note: continuity $\not\Rightarrow$ Differentiability. (consider $f(x) = |x|$ at $a=0$)

$f(x) = |x| \Rightarrow f(x) = \begin{cases} x & (x \geq 0) \\ -x & (x < 0) \end{cases}$ $\Rightarrow \cancel{\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x =}$

$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+}\frac{x}{x} = 1$ but $\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-}\frac{-x}{x} = -1$

Since $\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} \neq \lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}$, $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$ DNE $\Rightarrow$ f is not differentiable at $x=0$.

Proof (of Product Rule)

Need to show $\lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a} = f'(a)g(a) + f(a)g'(a)$

Note: $f(x)g(x) - f(a)g(a) = f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)$

$= (f(x)-f(a))\,g(x) + f(a)\,(g(x)-g(a))$

Hence, for $x \neq a$,

$$\frac{f(x)g(x)-f(a)g(a)}{x-a} = \frac{f(x)-f(a)}{x-a}g(x) + \frac{g(x)-g(a)}{x-a}f(a)$$

Hence $\lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a} = \cancel{ff} \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}g(x) + \frac{g(x)-g(a)}{x-a}f(a)\right]$

$= f'(a)g(a) + g'(a)f(a)$

because $g(x)$ is differentiable and thus continuous

$\therefore$ $\boxed{(fg) \text{ is differentiable at } a.}$

Consequence: know $(1)' = 0$ and $(x)' = 1 \Rightarrow (x^n)' = nx^{n-1}$ (product rule and induction on n)
$\therefore$ All polynomial functions are differentiable.

Fact: $\sin x$, $\cos x$, $\log_e x$, e^x are all differentiable.

<u>Chain Rule</u>
Let I, J be open intervals and f, g be functions s.t. $f: I \to J$, $g: J \to \mathbb{R}$.
Assume f is differentiable at a
and g is differentiable at $f(a)$

Then $g \circ f : I \to \mathbb{R}$ is differentiable at a with $(g \circ f)'(a) = g'(f(a))\, f'(a)$.

Example: $f(x) = x^2 \cos\left(\frac{1}{x}\right)$.

$$f'(x) = 2x\cos\left(\tfrac{1}{x}\right) + x^2\left(-\sin\left(\tfrac{1}{x}\right)\right)\left(-\tfrac{1}{x^2}\right) = 2x\cos\left(\tfrac{1}{x}\right) + \sin\left(\tfrac{1}{x}\right).$$

<u>Proof</u>: There are two cases:
(1) $f(x) \neq f(a)$ for x "close to" a
★ (DONE) (2) Otherwise (Reading Assignment)

(1) Want to show: $\displaystyle\lim_{x\to a} \frac{g(f(x)) - g(f(a))}{x-a} = g'(f(a))\, f'(a)$

Shrinking I if necessary, may assume $f(x) \neq f(a)$ $\forall x \in I$, $x \neq a$.
Then, $x \in I$, $x \neq a$,

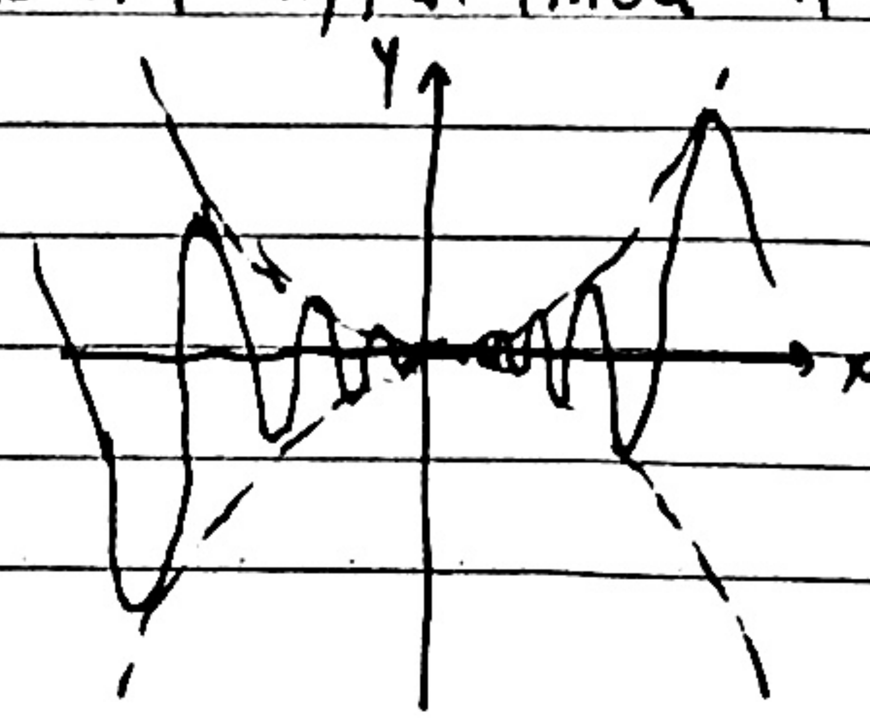
$$\frac{g(f(x)) - g(f(a))}{x-a} = \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x-a}$$

Take any sequence (x_n) in I s.t. $\lim x_n = a$ and $x_n \neq a$ $\forall n$.
Since f is continuous at a, $\lim_{n\to\infty} f(x_n) = f(a)$ and $f(x_n) \neq f(a)$ $\forall n$

$$\Rightarrow \lim_{n\to\infty} \frac{g(f(x_n)) - g(f(a))}{f(x_n) - f(a)} = \lim_{y \to f(a)} \frac{g(y) - g(f(a))}{y - f(a)} \underset{\text{by definition}}{=} g'(f(a))$$

$$\text{So } \lim_{n\to\infty} \frac{g(f(x_n)) - g(f(a))}{x_n - a} = g'(f(a)) \lim_{n\to\infty} \frac{f(x_n) - f(a)}{x_n - a} = g'(f(a))\, f'(a).$$

Qn: Is there a function f that is differentiable but $f'(x)$ is not differentiable at $a = 0$?
Consider $f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$



$f'(0) = 0$
$f'(x) = 2x\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right)$ $(x \neq 0)$
But $f'(x)$ is not continuous at $x = 0$
In particular, $f'(x)$ is not differentiable at $x = 0$.

NO: DATE:

Reading Assignment:

Suppose $f(x) = f(a)$ for x arbitrarily close to a.

Then $\exists$ a sequence (z_n) in $J \setminus \{a\}$ s.t. $\lim_{n\to\infty} z_n = a$ and $\lim_{n\to\infty} f(z_n) = f(a)$

Then, $f'(a) = \lim_{n\to\infty} \frac{f(z_n)-f(a)}{z_n - a} = 0$ (This is because f' is differentiable at a, so any sequence converging to a must work)

Since g is differentiable at $f(a)$, $g'(f(a))$ exists so $\frac{g'(y)-g(f(a))}{y-f(a)}$ is bounded near $f(a)$. Hence, replacing the open interval I with a smaller one if necessary, there is a constant $C>0$ so that $\left|\frac{g(y)-g(f(a))}{y-f(a)}\right| \leq C$ for $y \in I \setminus f(a)$

Hence: $\left|\frac{(g\circ f)(x_n) - (g\circ f)(a)}{x_n - a}\right| \leq C \left|\frac{f(x_n)-f(a)}{x-a}\right|$

The above inequality holds if $f(x_n) = f(a)$ since both sides are 0 otherwise by the product.

Since $f'(a) = 0$, $\lim_{n\to\infty}\left|\frac{f(x_n)-f(a)}{x_n-a}\right| = 0 \Rightarrow \lim_{n\to\infty}\left|\frac{(g\circ f)(x_n) - (g\circ f)(a)}{x_n-a}\right| = 0$

Since (x_n) is any sequence in $J \setminus \{a\}$ converging to a, $\boxed{(g\circ f)'(a) = 0}$ as desired.