gradPrep: Laptop (charged to full battery), pen, watch
Strategy: Intuition → Identify problem type → Apply relevant treatments → Win

Definitions

Adjoint map	$A^T: W^* \to V^*$ $(A^T a)(\vec{x}) = a(A\vec{x})$
Quadratic form	$Q(\vec{x}) = \vec{x}^T Q\vec{x}$
Hermitian quadratic form	$Q(\vec{x}) = \bar{\vec{x}}^T Q\vec{x}$
Hermitian matrix	$a_{ij} = \overline{a_{ji}}$, $A = A^\dagger$
Positive definite (quadratic form)	$x^T Q x \geq 0$ for all x with equality only if $x = 0$ $z^* Q z \geq 0$ for all z with equality only if $z = 0$
Inertia indices (p, q)	equal to the maximal dimensions of the subspaces in $\mathbb{R}^n$ where the quadratic form Q is positive/negative
Orthogonal	$\langle U\vec{x}, U\vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle$ for all $\vec{x}, \vec{y} \in V$
Orthogonal complement	$U^\perp = \{\vec{v} \in V: \langle \vec{u}, \vec{v} \rangle = 0\ \forall u \in U\}$
Adjoint of $T \in L(V, W)$	$T^*: W \to V$ such that $\langle T\vec{v}, \vec{w} \rangle = \langle \vec{v}, T^*\vec{w} \rangle\ \forall \vec{v}, \vec{w}$
Self-adjoint	$T = T^*$ i.e. $\langle T\vec{v}, \vec{w} \rangle = \langle \vec{v}, T\vec{w} \rangle$
Normal	$TT^* = T^*T$ (includes self-adjoint)
Positive definite	T is self-adjoint **and** $\langle T\vec{v}, \vec{v} \rangle \geq 0\ \forall \vec{v} \in V$
Square root	$R^2 = T$
Isometry	$\|S\vec{v}\| = \|\vec{v}\|\ \forall \vec{v} \in V$
Singular values	Eigenvalues of $\sqrt{T^*T}$. Equivalent to nonnegative square roots of eigenvalues of T^*T
Minimal polynomial	The unique monic polynomial of smallest degree such that $p(T) = 0$.

Rank Theorem

1. A linear map $A: V \to W$ of rank r is given by the matrix E_r in suitable bases of space V and W.
2. Linear maps between finite dimensional spaces are equivalent if and only if they have the same rank.
3. Adjoint linear maps have the same rank
4. Maximal number of linearly independent columns = maximal number of independent rows.
5. Rank of matrix equal maximal size k for which there exists a $k \times k$ submatrix with non-zero determinant.
6. [Dimension Counting] Let $V, W \subset U$ with $\dim U = n$, $\dim V = k$, $\dim W = l$ and V and W spans U, then $\dim V \cap W = k + l - n$.
7. [LPU Decomposition] Every invertible matrix M can be factored into the product $M = LPU$ of a lower triangular matrix L, a permutation matrix P and an upper triangular matrix U.

Flags

1. Every complete flag in $\mathbb{F}^n$ can be obtained from any other by an invertible linear transformation.
2. Every complete flag in $\mathbb{F}^n$ can be transformed to exactly one of $n!$ coordinate flags by invertible linear transformations preserving one of them. If preserving the opposite flag, then every complete flag can be transformed to its equivalent class by a lower triangular matrix L.

Inertia Theorem (Sylvester's Law of Inertia)

(solves two matrices describe the same bilinear form up to a change of basis)

1. Every quadratic form in a finite dimensional vector space has an orthogonal basis.
2. For every symmetric $n \times n$ matrix Q with entries from $\mathbb{F}$ there exists an invertible matrix C such that $C^T QC$ is diagonal. The diagonal entries are $Q(\vec{f_i})$
3. Every quadratic form in $\mathbb{R}^n$ can be transformed to $X_1^2 + \cdots + X_p^2 - X_{p+1}^2 - \cdots - X_{p+q}^2$ where $0 \leq p + q \leq n$.
4. Every real symmetric matrix Q can be transformed to exactly one of the following form by transformations of the form $Q \to C^T QC$ where C is an invertible real matrix.
$$\begin{bmatrix} I_p & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
5. Every quadratic form in $\mathbb{C}^n$ can be transformed by linear changes of coordinates to exactly one of the normal forms: $z_1^2 + \cdots + z_r^2$ where $0 \leq r \leq n$.
6. Every complex symmetric matrix Q can be transformed to exactly one of $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by transformations of the form $Q \to C^T QC$ defined by invertible complex matrix C.
7. Every Hermitian quadratic form H in $\mathbb{C}^n$ can be transformed by a linear change of coordinates to exactly one of $|z_1|^2 + \cdots + |z_p|^2 - |z_{p+1}^2| - \cdots - |z_{p+q}|^2$, $0 \leq p + q \leq n$.
8. Every anti-Hermitian quadratic form Q in $\mathbb{C}^n$ can be transformed by a linear change of coordinates to $i|z_1|^2 + \cdots + i|z_p|^2 - i|z_{p+1}^2| - \cdots - i|z_{p+q}|^2$, $0 \leq p + q \leq n$
9. Any Hermitian matrix can be transformed to exactly one of the following forms by transformations of the from $T \to C^\dagger TC$ defined by invertible complex matrix C.
$$\begin{bmatrix} I_p & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
10. Any anti-Hermitian matrix can be transformed to exactly one of the following forms by transformations of the from $T \to C^\dagger TC$ defined by invertible complex matrix C.
$$\begin{bmatrix} iI_p & 0 & 0 \\ 0 & -iI_q & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
11. [Sylvester's Rule] Suppose that an Hermitian $n \times n$ matrix H has non-zero leading minors. Then the negative inertia index of the corresponding Hermitian form is equal to the number of sign changes in the sequence $\Delta_0 = 1, \Delta_1, \ldots, \Delta_n$
12. A Hermitian form non-degenerate on each space of the standard coordinate flag can be transformed to exactly one of the 2^n normal forms $\pm|z_1|^2 \pm \cdots \pm |z_n|^2$ by a linear change of coordinates preserving the flag.
13. Any positive definite Hermitian form in $\mathbb{C}^n$ can be transformed into $|z_1|^2 + \cdots + |z_n|^2$ by a linear change of coordinates preserving a complete flag.
14. A Hermitian form in $\mathbb{C}^n$ is positive definite if and only if all of its leading minors are positive.
15. Every positive definite Hermitian form in $\mathbb{C}^n$ has an orthonormal basis $\{\vec{f_1}, \ldots, \vec{f_n}\}$ such that $\vec{f_k} \in Span(\vec{e_1}, \ldots, \vec{e_k})$.

Inner Products

- $\langle \vec{v}, \alpha\vec{w} \rangle = \alpha \langle \vec{v}, \vec{w} \rangle$
- $\langle \vec{v}, \vec{w} \rangle = \overline{\langle \vec{w}, \vec{v} \rangle}$
- [Triangle Inequality] $|\vec{z} - \vec{w}| \leq |\vec{z}| + |\vec{w}|$
- [Cauchy-Schwarz Inequality] $|\langle \vec{u}, \vec{v} \rangle|^2 \leq \|\vec{u}\|^2 \|\vec{v}\|^2$
- [Pythagoras] If $\langle \vec{u}, \vec{v} \rangle = 0$, $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$
- [Parallelogram Law] $\|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2(\|\vec{u}\|^2 + \|\vec{v}\|^2)$

1. If $\{\overrightarrow{e_1}, \dots, \overrightarrow{e_n}\}$ is an orthonormal basis of V, then
$$\vec{v} = \langle \overrightarrow{e_1}, v \rangle \overrightarrow{e_1} + \cdots + \langle \overrightarrow{e_n}, v \rangle \overrightarrow{e_n}$$
$$\|\vec{v}\|^2 = |\langle \overrightarrow{e_1}, v \rangle|^2 + \cdots + |\langle \overrightarrow{e_n}, v \rangle|^2$$
2. Suppose U is a finite-dimensional subspace of V, then $V = U \oplus U^\perp$
3. If V is finite dimensional, then $\dim U^\perp = \dim V - \dim U$
4. If a subspace U is A-invariant, then $U^\perp$ is $A^\dagger$-invariant.
5. λ is an eigenvalue of T if and only if $\bar{\lambda}$ is an eigenvalue of T^*.
6. A subspace U is invariant under T if and only if $U^\perp$ is invariant under T^*.
7. Let $T \in L(V, W)$ and $\{\overrightarrow{e_i}\}$ is a set of orthonormal basis in V and $\{\overrightarrow{f_i}\}$ is a set of orthonormal basis in W, then the matrix of T is the conjugate transpose of T^* on these two bases.
8. For complex vector space, if $\langle T\vec{v}, \vec{v} \rangle = 0\ \forall \vec{v}$ then $T = 0$.

$\dim \operatorname{null} T^* = \dim \operatorname{null} T + \dim W - \dim V$ $\dim \operatorname{range} T^* = \dim \operatorname{range} T$	$\operatorname{null} T^* = (\operatorname{range} T)^\perp$ $\operatorname{range} T^* = (\operatorname{null} T)^\perp$ $\operatorname{null} T = (\operatorname{range} T^*)^\perp$ $\operatorname{range} T^* = (\operatorname{null} T^*)^\perp$

Spectral Theorem

Normal	Self-Adjoint	Symmetric / Anti- Matrices

• Normal if and only if $\|T\vec{v}\| = \|T^*\vec{v}\|$ for all $\vec{v} \in V$.
• $T - \lambda I$ is also normal
• T and T^* have the same eigenvectors with conjugate eigenvalues.
• $\overrightarrow{v_i} \perp \overrightarrow{v_j}$ for $\overrightarrow{v_i} \in W_i, \overrightarrow{v_j} \in W_j$, $i \neq j$ (eigenvectors with distinct λ are orthogonal)
• If $U \subset V$ invariant under T:
  ○ $U^\perp$ is invariant under T
  ○ U is invariant under T^*
  ○ $(T\|_U)^* = (T^*)\|_U$
  ○ $T\|_U \in L(U)$, $T\|_{U^\perp} \in L(U^\perp)$ are normal operators
• T is normal if and only if V has an orthonormal basis of eigenvectors of T.
• T has a diagonal matrix with respect to some orthonormal basis of V.
• Let W_i be the eigenspace for λ_i which has multiplicity m_i. Then $\dim W_i = m_i$ and $\Sigma \dim W_i = \dim V$
• A matrix A commuting with $A^\dagger$ can be transformed to a diagonal form by $A \to U^\dagger A U$ defined by unitary U
	• All eigenvalues are real
	• If U is a subspace invariant under T, then:
	  ○ $U^\perp$ is invariant under T
	  ○ $T\|_U$ is self-adjoint
	  ○ $T\|_{U^\perp}$ is also self-adjoint.
	• [Spectral Theorem] V has an orthonormal basis of eigenvectors of T.
	• [Spectral Theorem] T has a diagonal matrix with respect to some orthonormal basis of V.
		$A^T = A$
		• Eigenvalues are real, can always get an orthonormal basis of n eigenvectors. $A = Q\Lambda Q^T$ $(Q^{-1} = Q^T)$
		
		A is positive definite if any of the criteria holds:
		• All pivots are positive
		• All eigenvalues positive
		• All upper left determinants are positive
		• $x^T A x > 0$ unless $x = 0$
		
		$A^T = -A$
		• Eigenvalues purely imaginary, can always get an orthonormal basis of n eigenvectors.

- An operator is Hermitian if and only if in some orthonormal basis its matrix is diagonal with all real diagonal entries.
- An operator is anti-Hermitian if and only if in some orthonormal basis its matrix is diagonal with all imaginary diagonal entries.
- A Hermitian form in a n-dimensional Hermitian space can be transformed by unitary changes of coordinates to exactly one of the normal forms
$$\lambda|z_1|^2+\cdots+\lambda_n|z_n|^2,\ \lambda_1\geq\cdots\geq\lambda_n$$
- A square matrix A is Hermitian if and only if it is unitarily diagonalizable with real eigenvalues.
- An anti-Hermitian form can be transformed to
$$i\omega_1|z_1|^2+\cdots+i\omega_n|z_n|^2,\ \omega_1\geq\cdots\geq\omega_n$$
- [Orthogonal Diagonalization Theorem] In a complex vector space of dimension n, a pair of Hermitian forms, the first of which is positive definite, can be transformed by a choice of coordinate system to exactly one of:
$$|z_1|^2+\cdots+|z_n|^2,\ \lambda_1|z_1|^2+\cdots+\lambda_n|z_n|^2,\ \ \lambda_1\geq\cdots\geq\lambda_n$$

- [Spectral Theorem for Real Operators] V decomposes into direct orthogonal sum of invariant lines and planes, each of which the transformation acts as multiplication by a real or complex scalar respectively.
- Transformation is <u>orthogonal</u> if and only if V can be decomposed into the direct orthogonal sum of invariant lines and planes on each of which the transformation acts as multiplication by ± 1 or rotation.
- In Euclidean space, every symmetric operator has an orthonormal basis of eigenvectors.
- Every quadratic form in a Euclidean space of dimension n can be transformed by an orthogonal change of coordinates to exactly one of the normal forms $\lambda_1x_1^2+\cdots+\lambda_nx_n^2,\ \lambda_1\geq\cdots\geq\lambda_n$.
- In an Euclidean space of dimension n, every anti-symmetric bilinear form can be transformed by an orthogonal change of coordinates to exactly one of the normal forms.
$$\langle\vec{x},\vec{y}\rangle=\sum_{i=1}^{r}\omega_i(x_{2i-1}y_{2i}-x_{2i}y_{2i-1}),\quad \omega_1\geq\cdots\geq\omega_r>0,\quad 2r\leq n$$
- Every real normal matrix A can be written in the form $A=U^TMU$ where U is an orthogonal matrix and M is block diagonal with each block being either size 1 or size 2 of the form
$$\begin{bmatrix}\alpha & -\beta\\ \beta & \alpha\end{bmatrix},\ \beta>0$$
    - If A symmetric (corresponding to self-adjoint operators), only blocks of size 1 are present.
    - If A is a real symmetric matrix, then the eigenvalues are real.
    - If A is anti-symmetric, then blocks of size 1 is 0, and size 2 are of the form $\begin{bmatrix}0 & -\omega\\ \omega & 0\end{bmatrix}$ where $\omega>0$.
    - If A is orthogonal (isometry), then all blocks of size 1 are ±1 and blocks of size 2 are $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ where $\theta\in(0,\pi)$

Bottom line
If T is normal in $\mathbb{C}$ or self-adjoint in $\mathbb{R}$, then there exists an orthonormal basis of eigenvectors in V for T. Then the application of T can be just thought of as **<u>projections onto the eigenvectors</u>** then multiplying by eigenvalues. $$T=\sum_{i=1}^{n}\lambda_iP_{\vec{v_i}}$$

Unitary / Isometry / Orthogonal

Characterization of Isometry
• S is an isometry • $\langle S\vec{u}, S\vec{v}\rangle = \langle \vec{u}, \vec{v}\rangle\ \forall \vec{u}, \vec{v} \in V$ • $\{S\vec{e_i}\}$ is orthonormal for every orthonormal $\{\vec{e_i}\}$ • Exists orthonormal $\{\vec{e_i}\}$ such that $\{S\vec{e_i}\}$ is orthonormal • $S^*S = I$ • $SS^* = I$ • S^* is an isometry • S is invertible and $S^{-1} = S^*$ • A transformation is unitary if and only if in some orthonormal basis its matrix is diagonal, and the diagonal entries are complex with absolute value 1. • Unitary transformations in a Hermitian space of dimension n are exactly unitary rotations (through possibly different angles) in n mutually perpendicular directions

Characteristic and Minimal Polynomials

$$p(\lambda) = \det(\lambda I - A) = \lambda^n + tr(A)\lambda^{n-1} + \cdots + \det A$$
$$V = W_{\lambda_0} \oplus U_{\lambda_0}$$
$$p(\lambda) = (\lambda - \lambda_1)^{m_1} \dots (\lambda - \lambda_r)^{m_r} \Rightarrow V = \oplus_{i=1}^{r} W_{\lambda_i}$$

- Coefficients do not change under similarity transformation.
- For any $\lambda \neq \lambda_0$, $W_\lambda \subset U_{\lambda_0}$
- $\dim W_{\lambda_i} = m_i$
- If $m_i = 1\ \forall i$, then matrix is diagonalizable.
- If minimal polynomial has all distinct roots, then matrix is diagonalizable.

Nilpotent Operators ($N^k = 0$)

- V can be decomposed to the direct sum of N-invariant subspaces, on each of which N is regular.
- Matrix of N in suitable basis is block diagonal with regular nilpotent blocks $n_1 \geq \cdots \geq n_r > 0$.
- Number of equivalence classes of nilpotent operators on V is equal to the number of partitions of $\dim V$.

Jordan Canonical Form

- Every complex matrix is similar to a block diagonal normal form (unique up to permutations of the block) with each diagonal block of the form:

$$\begin{bmatrix} \lambda_0 & 1 & 0 & \cdots & 0 \\ 0 & \lambda_0 & 1 & \cdots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \cdots & \lambda_0 & 1 \\ 0 & 0 & \cdots & 0 & \lambda_0 \end{bmatrix}$$

- Every operator $A: \mathbb{C}^n \to \mathbb{C}^n$ in a suitable basis is the sum of $D + N$ where D is diagonal, N is nilpotent and they commute.

- A linear operator $A: \mathbb{R}^n \to \mathbb{R}^n$ is represented in suitable basis by block-diagonal matrix with diagonal blocks either Jordan cells with real eigen values or have the form with $\beta \neq 0$ where $B = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ and $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} B & I_2 & 0 & \cdots & 0 \\ 0 & B & I_2 & \cdots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \cdots & B & I_2 \\ 0 & 0 & \cdots & 0 & B \end{bmatrix}$$

Projections P_U

- $P_U \in L(V)$ is the orthogonal projection of V onto U (a finite-dimensional subspace of V)
- $\|\vec{v} - P_U\vec{v}\| \leq \|\vec{v} - \vec{u}\| \; \forall \vec{v} \in V, \vec{u} \in U$ with equality if and only if $\vec{u} = P_U\vec{v}$.
- A projection matrix is square and symmetric.

Basic properties		$\vec{v} - P_U\,\vec{v} \in U^\perp$
$P_U\vec{u} = \vec{u}\;\forall\vec{u} \in U$	range $P_U = U$	$P_U^2 = P_U$
$P_U\vec{w} = 0\forall\vec{w} \in U^\perp$	null $P_U = U^\perp$	$\|P_U\vec{v}\| \leq \|\vec{v}\|$

Positive Operators $\langle T\vec{v}, \vec{v}\rangle \geq 0$

- T is positive
- T is self-adjoint and all eigenvalues are nonnegative (i.e. can take square roots)
- T has a positive square root (which is unique)
- T has a self-adjoint square root
- There exists R such that $T \in R^*R$

Polar Decomposition $T = S\sqrt{T^*T}$

- T^*T is positive semi-definite.
- Every linear transformation T of a Hermitian space has a polar decomposition $T = RS$ where R is positive and S is unitary.
- When T is invertible, polar decomposition is unique and $R = \sqrt{TT^*}$
- Similarly, if $T \in L(V)$, then exists an isometry $S \in L(V)$ s.t. $T = S\sqrt{T^*T}$

Singular Value Decomposition

- Let $\{\sigma_i^2\}$ be the eigenvalues of T^*T, then $\{\sigma_i\}$ are the singular values of T.
- If $T \in L(V)$ has singular values $\sigma_1, \dots, \sigma_n$, then exists orthonormal bases $\{\vec{e_i}\}$ and $\{\vec{f_i}\}$ of V such that $T\vec{v} = \sigma_1\langle\vec{e_1}, \vec{v}\rangle\vec{f_1} + \cdots + \sigma_n\langle\vec{e_n}, \vec{v}\rangle\vec{f_n}$ for every $\vec{v} \in V$.
- [Matrix Formulation] Every $m \times n$ matrix A can be decomposed into the following, where U, V are matrices with orthonormal columns and Σ comprises the singular values.

$$A = U\Sigma V^T$$
$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$
$$D = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r \end{bmatrix}$$
$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$
$$V = [v_1\; v_2 \dots v_n]$$
$$A\vec{v_i} = \sigma_i\vec{u_i} \Rightarrow \vec{u_i} = \frac{A\vec{v_i}}{\sigma_i}$$

- $\{\vec{v_i}\}$ are orthonormal bases of A^TA with eigenvalues $\sigma_i^2 \geq 0$ (i.e. V diagonalizes A^TA). Similarly U diagonalizes AA^T (i.e. the columns of U forms an orthonormal basis of AA^T)
$$A^TA = (U\Sigma V^T)^T(U\Sigma V^T) = V\Sigma^TU^TU\Sigma V^T = V(\Sigma^T\Sigma)V^T$$
- [Geometric Interpretation] Every linear transformation decomposes into rotation (and possibly reflection) composed with stretching (to ellipse) composed with a (possibly different) rotation.
- $U = V$ when matrix is symmetric and positive definite.
- For positive definite symmetric matrices, $A = Q\Sigma Q^T$.
- [Reduction to Polar Decomposition] $A = U\Sigma V^T = U\Sigma U^TUV^T = (U\Sigma U^T)(UV^T) = SQ$

Determinants

General	Cramer's Rule
$\det A = \sum_\sigma \varepsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}$ $\varepsilon(\sigma) = (-1)^{l(\sigma)}$ $\varepsilon(\sigma\sigma') = \varepsilon(\sigma)\varepsilon(\sigma')$ $\det(AB) = \det A \det B$	$A\vec{x} = \vec{b}$ $x_i = \dfrac{\det[\overrightarrow{a_1}, \dots, \overrightarrow{a_{i-1}}, \vec{b}, \overrightarrow{a_{i+1}}, \dots, \overrightarrow{a_n}]}{\det A}$ **Cofactor Theorem** $A \operatorname{adj}(A) = (\det A) I = \operatorname{adj}(A) A$

For block matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where D^{-1} exists, $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A - BD^{-1}C) \det D$

Schur Decomposition

Suppose V is a finite dimensional complex vector space and $T \in L(V)$, then T has an upper-triangular matrix with respect to some orthonormal basis of V (may not be unique)

- An $n \times n$ square matrix A with complex entries can be expressed as QUQ^{-1} where Q is unitary and U an upper triangular matrix.
- U is similar to A (same spectrum), thus eigenvalues are the diagonal entries of U.
- There is a nested sequence of A-invariant subspaces $\{0\} = V_0 \subset V_1 \subset \dots \subset V_n = \mathbb{C}^n$ and there exists an orthonormal basis (for the standard Hermitian form of $\mathbb{C}^n$).
- If A is normal matrix, then U must be a diagonal matrix and the column vectors of Q are the eigenvectors of A.
- If A is positive definite, the Schur decomposition, the spectral decomposition and singular value decomposition coincide.
- A commuting family $\{A_i\}$ of matrices can be simultaneously triangularized i.e. exists a unitary matrix Q such that for every A_i, QA_iQ^* is upper triangular.

Matrix Incarnations

- Row space and null space are orthogonal complements
- Column space and null space of transpose are orthogonal complements
- $A^T A$ is symmetric and square and positive definite
- $A^T A$ is invertible if and only if A has independent columns.

$$\operatorname{null}(A^T A) = \operatorname{null}(A)$$
$$\operatorname{rank}(A^T A) = \operatorname{rank}(A)$$

Applications

1. <u>Conics</u>

Every conic in $\mathbb{R}^n$ is equivalent to either a cylinder over a conic in $\mathbb{R}^{n-1}$, or to one of the conics (hyperboloids, cones, paraboloids):

$$x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_n^2 = 1, \quad 0 \le p \le n$$
$$x_1^2 + \dots + x_p^2 = x_{p+1}^2 + \dots + x_n^2, \quad 0 \le p \le \frac{n}{2}$$
$$x_n = x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{n-1}^2, \quad 0 \le p \le \frac{n-1}{2}$$

Every conic in $\mathbb{C}^n$ is equivalent to either the cylinder over a conic in $\mathbb{C}^{n-1}$ or to one of the three conics:

$$z_1 + \dots + z_n^2 = 1, \quad z_1^2 + \dots + z_n^2 = 0, \quad z_n = z_1^2 + \dots + z_{n-1}^2$$

2. <u>Linear Recurrence / Discrete Time Systems</u>

$$\lambda_i^n n^{k-1}, k \le m_i$$
$$\vec{v}(n) = \sum C_i \lambda_i^n n^{k-1} \vec{v}_i$$

3. <u>Coupled Oscillations</u>

The solution to $\dot{\vec{x}} = A\vec{x}$ is $\vec{x}(t) = e^{tA}\vec{x}(0)$

Formulae (just Taylor)

$$e^{tA} = I + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \cdots + \frac{t^kA^k}{k!} + \cdots$$

$$e^{t(\lambda I + N)} = e^{\lambda t}e^{tN}$$

$$e^{t\Sigma} = \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

$$e^{tN} = \begin{bmatrix} 1 & \frac{t}{1!} & \frac{t^2}{2!} & \cdots & \frac{t^{m-1}}{(m-1)!} \\ 0 & 1 & \frac{t}{1!} & \frac{t^2}{2!} & \vdots \\ 0 & 0 & 1 & \frac{t}{1!} & \frac{t^2}{2!} \\ \vdots & \vdots & 0 & 1 & \frac{t}{1!} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$J^n = (D+N)^n = \begin{bmatrix} \lambda^n & \binom{n}{1}\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} & \binom{n}{3}\lambda^{n-3} & \cdots \\ 0 & \lambda^n & \binom{n}{1}\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} & \binom{n}{3}\lambda^{n-3} \\ 0 & 0 & \lambda^n & \binom{n}{1}\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} \\ \vdots & \vdots & 0 & \lambda^n & \binom{n}{1}\lambda^{n-1} \\ 0 & 0 & \cdots & 0 & \lambda^n \end{bmatrix}$$

4. Higher Order Oscillations

All components to the ODE are expressible as linear combinations of $t^k e^{\lambda t}, t^k e^{at}\cos bt, t^k e^{at}\sin bt$ where λ are real eigenvalues, $a \pm ib$ are complex eigenvalues and k is smaller than the multiplicity of the corresponding eigenvalue.

5. Kinetic and Potential Energy

$K(\dot{x}) = \frac{1}{2}\langle \dot{x}, M\dot{x}\rangle$	$P(x) = \frac{1}{2}\langle x, Qx\rangle$	$M\ddot{x} = -Qx$

K is positive definite, so (K, Q) can be simultaneously transformed to the following, where the ODE splits into unlinked second order ODEs:

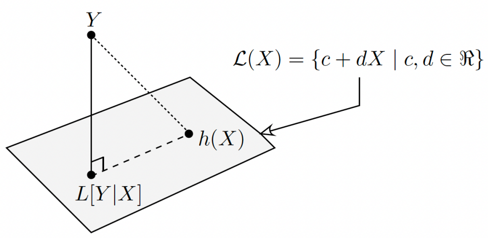
$$K = \frac{1}{2}(\dot{X}_1^2 + \cdots + \dot{X}_n^2) \qquad Q = \frac{1}{2}(\lambda_1 X_1^2 + \cdots + \lambda_n X_n^2)$$

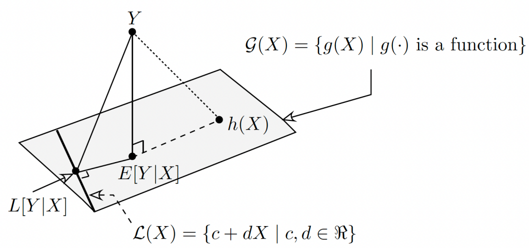
$$\ddot{X}_i = -\lambda_i X_i$$

The consequence is that small oscillations in any conservative mechanical system near a local minimum of potential energy are described as *superpositions of independent harmonic oscillations*.

6. Minimization Problems

Consider an inner-product space where $\langle X, Y\rangle = \mathbb{E}[XY]$. $L[Y|X]$ can be interpreted as the projection of Y onto $L(X)$, the set of linear functions of X.





- $L[Y|X] = a + bX$ if and only if $\langle Y - (a + bX), c + dX\rangle = 0 \ \forall\, c + dX \in L(X)$
- $\mathbb{E}[Y] = a + b\mathbb{E}[X]$ and $\mathbb{E}\big[\big(Y - (a + bX)\big)X\big] = 0$
- $\|Y - L[Y|X]\| \le \|Y - h(X)\| \ \forall h(X) \in L(X) \quad \Rightarrow \quad \mathbb{E}[|Y - L[Y|X]|^2] \le \mathbb{E}(|Y - h(X)|^2)$

The MMSE of Y given X is given by $g(X) = \mathbb{E}[Y|X]$, which satisfies $\langle Y - g(X), \phi(X)\rangle = 0 \ \forall\, \phi(\cdot)$

- $\langle Y - g'^{(X)}, \phi(X)\rangle \quad \Leftrightarrow \quad g'(X) = g(X)$

- $\|Y-\phi(X)\| \geq \|Y-\mathbb{E}[Y|X]\|$
- $L[Y|X] = L[\mathbb{E}[Y|X]|X]$
- $L[Y|X]$ is the projection of $\mathbb{E}[Y|X]$ onto $L(X)$
- If $\mathbb{E}[Y|X]$ happens to be linear, then Y and X are jointly Gaussian. Each linear combination of X, Y is Gaussian $\sim \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Let $\vec{Y} \in \mathbb{R}^{m\times 1}, X \in \mathbb{R}^{n\times 1}$, want to find $L[\vec{Y}|\vec{X}] = \vec{a} + B\vec{X}$ where $B \in \mathbb{R}^{m\times n}$ and $\vec{a}$ is a vector in $\mathbb{R}^{m\times 1}$ that minimizes $\mathbb{E}\left[\|\vec{Y}-\vec{a}-B\vec{X}\|^2\right]$

- $\mathbb{E}[\vec{Y}] = [\mathbb{E}[Y_1]\ \mathbb{E}[Y_2] \ldots \mathbb{E}[Y_m]]^T$
- $\left(\mathbb{E}[\vec{Y}\vec{X}^T]\right)_{ij} = \mathbb{E}\left[\left(\vec{Y}\vec{X}^T\right)_{ij}\right] = \mathbb{E}[\vec{Y}_i\vec{X}_j]$
- Covariance matrix of $\vec{X}$: $\Sigma_{\vec{X}} \coloneqq \mathbb{E}[\vec{X}\vec{X}^T] - \mathbb{E}[\vec{X}]\mathbb{E}[\vec{X}^T]$
- Covariance matrix of $\vec{Y}$ and $\vec{X}$: $\Sigma_{\vec{Y},\vec{X}} \coloneqq \mathbb{E}[\vec{Y}\vec{X}^T] - \mathbb{E}[\vec{Y}]\mathbb{E}[\vec{X}^T]$

If $\vec{Y}, \vec{X}$ are random vectors such that $\Sigma_{\vec{X}}$ is invertible, then

$$L[\vec{Y}|\vec{X}] = \mathbb{E}[\vec{Y}] + \Sigma_{\vec{Y},\vec{X}}\Sigma_{\vec{X}}^{-1}\left(\vec{X} - \mathbb{E}[\vec{X}]\right)$$

$$\mathbb{E}\left[\|\vec{Y} - L[\vec{Y}|\vec{X}]\|^2\right] = tr\left(\Sigma_{\vec{Y}} - cov[\vec{Y},\vec{X}]\Sigma_{\vec{X}}^{-1}cov[\vec{X},\vec{Y}]\right)$$

For a zero-mean random vector $\vec{V}$

$$\mathbb{E}\left[\|\vec{V}\|^2\right] = \mathbb{E}\left[tr\left(\vec{V}\vec{V}^T\right)\right] = tr\left(\mathbb{E}[\vec{V}\vec{V}^T]\right) = tr\left(\Sigma_{\vec{V}}\right)$$

Miscellaneous Toolkits

1. Gram-Schmidt Orthogonalization Algorithm
   - Constructs an orthonormal basis form a given basis.
   - Constructs an orthonormal set of vectors from a linearly independent set of vectors.

$$\vec{e_j} = \frac{\vec{v_j} - \langle\vec{e_1}, \vec{v_j}\rangle\vec{e_1} - \cdots - \langle\overrightarrow{e_{j-1}}, v_j\rangle\overrightarrow{e_{j-1}}}{\left\|v_j - \langle\vec{e_1}, \vec{v_j}\rangle\vec{e_1} - \cdots - \langle\overrightarrow{e_{j-1}}, v_j\rangle\overrightarrow{e_{j-1}}\right\|}$$

2. Cayley-Hamilton Theorem

Suppose V is a complex vector space and $T \in L(V)$. Let q denote the characteristic polynomial of T. Then $q(T) = 0$. Hence, $p|q$, where p is the minimal polynomial.

3. Polarization
4. Companion Matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \vdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}$$

$$\det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

5. Courant-Fischer's Minimax Principle

The kth greatest spectral number is given by

$$\lambda_k = \max_{W:\dim W = k} \min_{x \in W - \{0\}} \frac{S(x)}{Q(x)}$$

Final Checks / Last Resorts

<table>
<tr><th>Final Checks</th><th>Last Resorts</th></tr>
<tr><td>
Did you forget any conjugation? (Hermitian quadratic form, inner product on $\mathbb{C}$)
</td><td>

Intuition
Start with a basis or an orthonormal basis.
Think of all theorems: which ones apply here?
Consider a geometric perspective to the problem

Real $\approx$ symmetric
Complex $\approx$ anti-symmetric
Rotation $\approx$ orthogonal/unitary
Stretching $\approx$ diagonal
Commuting $=$ Invariant eigenspaces

Whack + Partial

</td></tr>
</table>