No.: Date: 09/16/2021

MATH 104 LECTURE 7 NOTES (Subsequences)

- Subsequences
- Bolzano Weierstrass
- Subsequential limits

Let $(S_n)_{n\in\mathbb{N}}$ be a sequence. We always consider sequence of infinite length.
Notation: $(S_{n_k})_{k\in\mathbb{N}}$

(S_n): $S_1 \quad S_2 \quad S_3 \quad S_4 \quad S_5 \quad S_6 \quad S_7 \quad S_8 \quad S_9 \quad S_{10} \quad S_{11} \ldots$
Subsequence: $S_2, \; S_3, \; S_5, \; S_7, \; S_{11}, \ldots$ $\quad n_k = k^{th}$ smallest prime.

Theorem. Any subsequence of a convergent sequence converges to the same limit.
Proof:
Take any convergent (S_n) and let $\lim S_n = S$.
Take any subsequence $(S_{n_k})_{k\in\mathbb{N}}$ and let $\varepsilon > 0$ be any positive real.

Since $\lim S_n = S$, $\exists N$ s.t. $n > N \Rightarrow |S_n - S| < \varepsilon$
Then $k > N$, $n_k \geq k > N \Rightarrow |S_{n_k} - S| < \varepsilon$

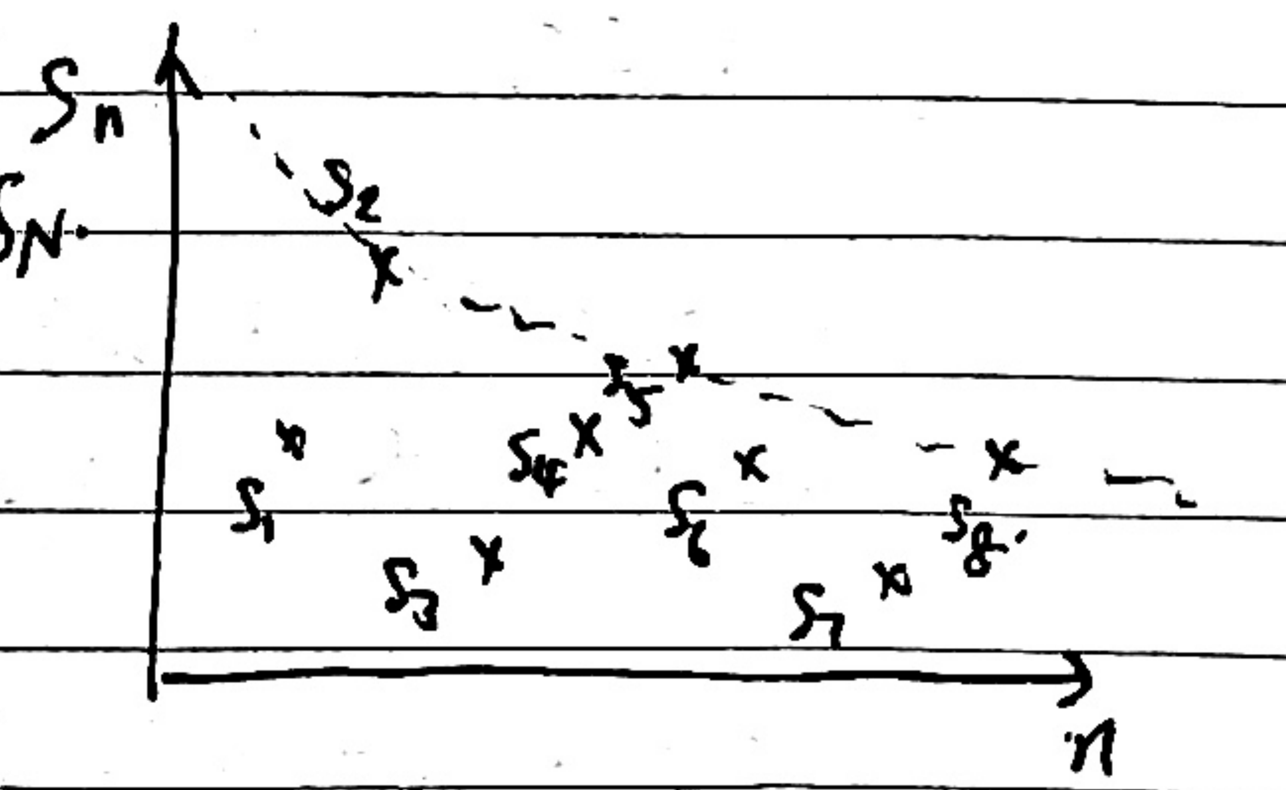
The sequence indexed by k satisfies the ε–N definition.

$\therefore \boxed{\lim S_{n_k} = S}$

Bolzano-Weierstrass

We cannot always find convergent subsequence from any sequence. Consider $S_n = n$. Then any subsequence diverges to $+\infty$. The next best option is to find monotone subsequence.

Theorem. Any sequence (S_n) admits a monotone subsequence.
Proof:
Say a term S_N is dominant if $m > N \Rightarrow S_m < S_N$.
i.e. all terms following S_N is less than it.



Case 1: There are infinitely many dominant terms.
Then, the set of all dominant terms form a subsequence $S_{n_1}, S_{n_2}, \ldots$. Since $n_{k+1} > n_k$, $S_{n_{k+1}} < S_{n_k}$ because S_{n_k} dominant
$\therefore (S_{n_k})_k$ is decreasing

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Case 2: There are finitely many dominant terms

Take $N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow S_n$ is not dominant. We can always do so since there are only finitely many dominant terms.

Set $n_1 = N$. Since $S_{n_1} = S_N$ is non-dominant, $\exists\, n_2 > n_1$ s.t. $S_{n_2} \geq S_{n_1}$

Since S_{n_2} is non-dominant, $\exists\, n_3 > n_2$ s.t. $S_{n_3} \geq S_{n_2}$.

Repeating the above procedures yields an increasing subsequence $(S_{n_k})_k$.

$\therefore$ Any sequence admits a monotone subsequence.

Bolzano Weierstrass

Any bounded sequence admits a convergent subsequence.

Proof:

Any sequence admits a monotone subsequence. Since the original sequence is also bounded, the subsequence is bounded and hence converges.

We can use Bolzano-Weierstrass to show that every continuous function on $[a, b]$ admits maximum and minimum.

Bolzano Weierstrass $\Leftarrow$ Any bounded monotone sequence converges $\Leftarrow$ Completeness Axiom

Subsequential Limits

Definition: A subsequential limit of a sequence (S_n) is a real number or $\pm\infty$ that is obtained as the limit of a subsequence $(S_{n_k})_k$

i.e. $\lim_{k \to \infty} S_{n_k}$

Theorem

$\limsup S_n$ is the largest subsequential limit.

$\liminf S_n$ is the smallest subsequential limit.

In particular, if (S_n) has only one subsequential limit, then $\lim S_n$ is defined and equal to that subsequential limit.

Proof: Let $L = \limsup S_n$. We focus on $L \in \mathbb{R}$ ($L = \pm\infty$ is reading assignment)

Set $b_n = \sup\{S_n, S_{n+1}, \dots\}$. By definition, $L = \limsup S_n = \lim b_n$

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Need to prove 2 implications: The limit of any subsequence that converges must $\leq L$ and also there exists a subsequence that converges to L.

① [largest] Take any subsequence $(S_{n_k})_k$ for which $\lim_{k\to\infty} S_{n_k}$ is defined.

$\forall k,\ S_{n_k} \leq b_{n_k} \Rightarrow \lim S_{n_k} \leq \lim b_{n_k} = L$

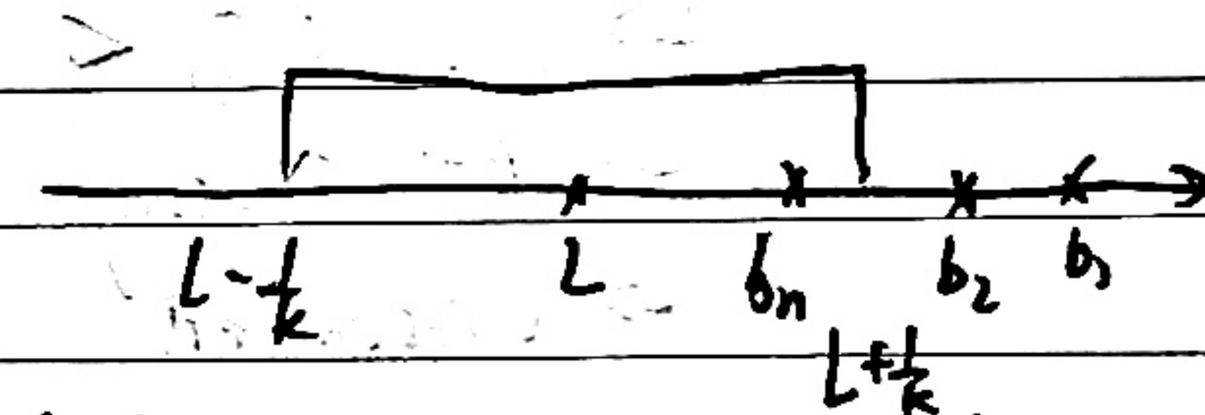
Since $b_{n_k} = \sup\{S_{n_k}, \dots\} \geq S_{n_k}$

(b_n) converges to L.
(b_{n_k}) a subsequence also converges to L.

② [L is a subsequential limit] We must construct explicitly a sequence that converges to L

Proof:

We proceed by constructing inductively a subsequence $(S_{n_k})_k$ s.t. $L - \frac{1}{k} \leq S_{n_k} \leq L + \frac{1}{k}\ \forall k$.



Base case: $k=1$. Since $\lim b_n = L$,

$\exists N_1$ s.t. $n \geq N_1 \Rightarrow L \leq b_n \leq L+1$

$\therefore\ b_{N_1} \leq L+1$

Since $b_{N_1} = \sup\{S_N, S_{N+1}, \dots\}$, in particular $b_N - 1$ is not an upperbound $\Rightarrow \exists n_1 \geq N_1$ s.t. $b_{N_1} - 1 < S_{n_1} \leq b_{N_1} \leq L+1$

$\therefore\ L-1 \leq S_{n_1} \leq L+1$

Inductive Step: Assume we picked $S_{n_1}, S_{n_2}, \dots S_{n_k}$.

Since $\lim b_n = L$, $\exists N_{k+1} \in \mathbb{N} > N_k$ s.t. $L \leq b_{N_{k+1}} \leq L + \frac{1}{k+1}$

Since $b_{N_{k+1}} - \frac{1}{k+1}$ is not an upperbound of $\{S_{N_{k+1}}, S_{N_{k+1}+1}, \dots\}$

$\exists n_{k+1} \geq N_{k+1}$ s.t. $b_{N_{k+1}} - \frac{1}{k+1} \leq S_{n_{k+1}} \leq b_{N_{k+1}} \leq L + \frac{1}{k+1}$

$\therefore \boxed{L - \frac{1}{k+1} \leq S_{n_{k+1}} \leq L + \frac{1}{k+1}}$

By squeeze lemma, $\lim S_{n_k} = L$ (Since $\lim(L - \frac{1}{n}) = \lim(L + \frac{1}{n}) = L$)

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Midterm Review

- $\mathbb{R}$, Completeness Axiom → Archimedean Property; → Denseness of $\mathbb{Q}$
- Convergence of sequences

↳ Convergence

↳ ε-N definition

↳ Bounded Monotone

↳ Cauchy

↳ $\liminf s_n = \limsup s_n \in \mathbb{R}$

↳ $\liminf \left|\frac{s_{n+1}}{s_n}\right| \le \liminf |s_n|^{\frac{1}{n}} \le \limsup |s_n|^{\frac{1}{n}} \le \limsup \left|\frac{s_{n+1}}{s_n}\right|$

↳ Limit theorems, basic examples, squeeze lemma

$\lim \frac{1}{n^p} = 0, \ p > 0$

$\lim a^n = 0, \ |a| < 1$

$\lim \sqrt[n]{n} = 1$

$\lim \sqrt[n]{a} = 1 \quad a > 0$

↳ Non convergence

↳ ε-N definition

↳ $\liminf s_n < \limsup s_n$

↳ Not Cauchy

↳ Unbounded