No.: Date:

MATH 104 LECTURE 3 NOTES

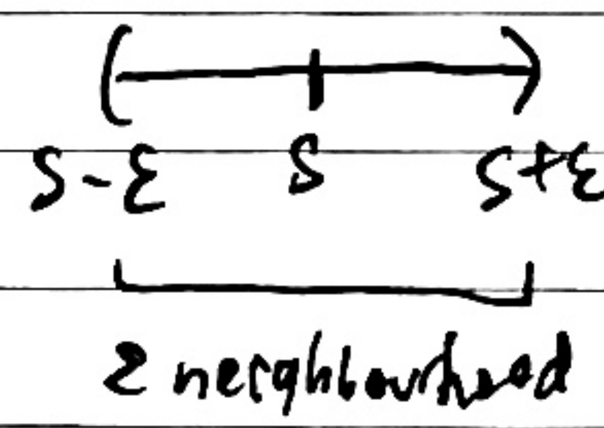
Definition Say a sequence (s_n) converges to some real number s if: for every $\varepsilon > 0$, $\exists N \in \mathbb{R}$ s.t. $\forall n > N \Rightarrow |s_n - s| < \varepsilon$. Then we say

Merely set a threshold, no need for natural no.

$$\lim_{n\to\infty} s_n = s \qquad (\text{Equivalently, } \forall \varepsilon > 0\ \exists N \text{ s.t. } \forall n > N \Rightarrow |s_n - s| < \varepsilon)$$

Otherwise, we say (s_n) diverges. (If sequence does not converge to any real number, then the sequence diverges).

$|s_n - s|$ = distance between s_n and s.
Consider ε as an error neighborhood around s
All s_n, $n > N$ falls within this neighbourhood.



N = threshold

Example $\lim_{n\to\infty} s_n = 0$ for $s_n = \frac{1}{n^2}$.

Suffices to show $\left|s_n - 0\right| = \left|\frac{1}{n^2} - 0\right| < \varepsilon \quad \forall \varepsilon > 0$

$$\Leftrightarrow \frac{1}{n^2} < \varepsilon \Leftrightarrow n > \frac{1}{\sqrt{\varepsilon}}.$$

Formal proof:
Take any $\varepsilon > 0$.
Set $N = \frac{1}{\sqrt{\varepsilon}}$

Then $n > N \Rightarrow n > \frac{1}{\sqrt{\varepsilon}} \Leftrightarrow n^2 > \frac{1}{\varepsilon} \Leftrightarrow \varepsilon > \frac{1}{n^2} = s_n$

$\therefore \varepsilon > |s_n - 0|$

Hence $\lim s_n = 0$.

Example $s_n = \frac{3n}{5n+2}$. Suffices to show $\left|\frac{3n}{5n+2} - \frac{3}{5}\right| < \varepsilon \ \forall \varepsilon > 0$.

$$\left|\frac{3n}{5n+2} - \frac{3}{5}\right| = \left|-\frac{6}{5(5n+2)}\right| = \frac{6}{5(5n+2)} < \varepsilon \Leftrightarrow \frac{6}{5\varepsilon} < (5n+2) \Leftrightarrow \frac{1}{5}\left(\frac{6}{5\varepsilon} - 2\right) < n$$

Formal proof:
Take any $\varepsilon > 0$. Set $N = \frac{1}{5}\left(\frac{6}{5\varepsilon} - 2\right)$

Then $n > N \Rightarrow n > \frac{1}{5}\left(\frac{6}{5\varepsilon} - 2\right)$

$\Rightarrow \frac{6}{5\varepsilon} < (5n+2) \Rightarrow \frac{6}{5(5n+2)} < \varepsilon$

$\Rightarrow \left|-\frac{6}{5(5n+2)}\right| < \varepsilon \Rightarrow \left|s_n - \frac{3}{5}\right| < \varepsilon.$

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No.: Date:

Take any $\varepsilon > 0$. Set $N = \frac{1}{5}\left(\frac{6}{5\varepsilon} - 2\right)$. Since $n > N$, $n > \frac{1}{5}\left(\frac{6}{5\varepsilon} - 2\right) \Rightarrow 5n+2 > \frac{6}{5\varepsilon}$

Then $n > N$ implies

$$\left|S_n - \frac{3}{5}\right| = \left|\frac{3n}{5n+2} - \frac{3}{5}\right| = \left| - \frac{6}{5(5n+2)}\right| = \frac{6}{5(5n+2)} < \frac{6}{5\left(\frac{6}{5\varepsilon}\right)} = \varepsilon.$$

Hence $\boxed{\lim S_n = \frac{3}{5}.}$

Key Remark

No need to solve "$|S_n - S| < \varepsilon$" for n, but rather you can just give an estimate for n

$$\left|S_n - \frac{3}{5}\right| = \frac{6}{5(5n+2)} < \frac{6}{5 \cdot 5n} = \frac{6}{25n} < \varepsilon \iff \frac{6}{25\varepsilon} < n$$

Formal Proof #2

Take any $\varepsilon > 0$. Set $N = \frac{6}{25\varepsilon}$.

Then, for $n > N$, $n > \frac{6}{25\varepsilon}$

$$\Rightarrow |S_n - S| = \frac{6}{5(5n+2)} < \frac{6}{5(5n)} < \frac{6}{25\left(\frac{6}{25\varepsilon}\right)} = \varepsilon$$

$\therefore \boxed{\lim S_n = \frac{3}{5}}$

There is no need to find the smallest possible threshold. Just need to find one particular threshold.

Example

(1) $a_n = \frac{5n-6}{4n-3}$

(2) $b_n = \frac{n^2-n+1}{2n^2-1}$.

$4n-3 > n \quad 4n > 1$

(1) Limit for $a_n = \frac{5}{4}$.

$$\left|a_n - \frac{5}{4}\right| = \left|\frac{5n-6}{4n-3} - \frac{5}{4}\right| = \left|\frac{-9}{4(4n-3)}\right| = \frac{9}{4(4n-3)} < \frac{9}{4n} < \varepsilon$$

$\Rightarrow N > \frac{9}{4\varepsilon}$

Take ~~Set~~ $\varepsilon > 0$. Set $N = \frac{9}{4\varepsilon}$. For $n > N$

Then $\left|a_n - \frac{5}{4}\right| = \frac{9}{4(4n-3)} < \frac{9}{4n} < \frac{9}{4\left(\frac{9}{4\varepsilon}\right)} = \varepsilon$ Hence $\boxed{\lim a_n = \frac{5}{4}}$

No.: Date:

(2) $b_n = \frac{n^2-n+1}{2n^2-1}$ limit is $\frac{1}{2}$.

$$\left|b_n - \frac{1}{2}\right| = \left|\frac{n^2-n+1}{2n^2-1} - \frac{1}{2}\right| = \left|\frac{-n+\frac{3}{2}}{2n^2-1}\right| = \left|\frac{-2n+3}{2(2n^2-1)}\right|$$

For $n > 2$, $\left|\frac{-2n+3}{2(2n^2-1)}\right| = \frac{2n-3}{2(2n^2-1)} < \frac{2n}{2n^2}$ $\left(\begin{array}{l} 2n-3 < 2n \\ 2n^2-1 > n^2 \end{array}\right)$

$= \frac{1}{n} < \varepsilon$

$\iff \frac{1}{\varepsilon} < n.$

Take $\varepsilon > 0$. Set $N = \max(2, \frac{1}{\varepsilon})$. For $n > N$: $(n > N \geq 2)$

$$\left|b_n - \frac{1}{2}\right| = \left|\frac{n^2-n+1}{2n^2-1} - \frac{1}{2}\right| = \left|\frac{-2n+3}{2(2n^2-1)}\right| = \frac{2n-3}{2(2n^2-1)}$$

$$< \frac{2n}{2n^2} = \frac{1}{n} < \frac{1}{N} \leq \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

$\therefore \boxed{\lim b_n = \frac{1}{2}.}$

<u>Formal Proof for (1)</u>

Take any $\varepsilon > 0$. Set $N = \frac{9}{4\varepsilon}$.

Then $n > N \Rightarrow \frac{9}{4n} < \varepsilon$. So $\left|a_n - \frac{5}{4}\right| = \left|\frac{-24+15}{4(4n-3)}\right| = \frac{9}{4(4n-3)}$

$(n \geq 1) \leq \frac{9}{4n} < \varepsilon$

Hence $\boxed{\lim a_n = \frac{5}{4}.}$

<u>Formal Proof for (2)</u>

Take $\varepsilon > 0$. Set $N = \max(\frac{1}{\varepsilon}, 1)$.

$n > N \Rightarrow \frac{1}{n} < \varepsilon$, and $n > 1$

So $\left|b_n - \frac{1}{2}\right| = \left|\frac{3-2n}{2(2n^2-1)}\right| = \frac{2n-3}{2(2n^2-1)} < \frac{2n}{2n^2} = \frac{1}{n} < \varepsilon.$

So. $\boxed{\lim b_n = \frac{1}{2}}$