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MATH H110 LECTURE 4 NOTES

Quadratic forms

Classification of quadratic forms $ax^2 + 2bxy + cy^2$ — can rotate extra 90°.

- up to rotations of the coordinate system. $(AX^2 + CY^2,\ A \geq C)$

$$AX^2 + CY^2 = \begin{cases} X^2 + Y^2 \\ X^2 - Y^2 \\ -X^2 - Y^2 \\ X^2,\ -Y^2,\ 0 \end{cases}$$

Can absorb the constant but not the sign.

- up to linear changes of coordinate.

classification

set of objects to classify
notion of equivalence (up to some transformations)
$\Rightarrow$ Exhibit one representative in each equivalence class, called their "normal forms" and prove the classification theorem

I. Given m linear functions in n variables

$$y_1 = a_{11}x_1 + \dots + a_{1n}x_n \qquad x_1 = c_{11}X_1 + \dots + c_{1n}X_n \qquad y_1 = b_{11}Y_1 + \dots + b_{1m}Y_m$$
$$y_2 = a_{21}x_1 + \dots + a_{2n}x_n \qquad \vdots \qquad \vdots$$
$$\vdots$$
$$y_m = a_{m1}x_1 + \dots + a_{mn}x_n \qquad x_n = c_{n1}X_1 + \dots + c_{nn}X_n \qquad y_m = b_{m1}Y_1 + \dots + b_{mm}Y_m$$

What is the simplest form that this system of linear equations can be reduced to given change of coordinates to both independent & dependent variables.

We should be able to go back from X_i to x_i (invertible)

Same number of variables to ensure that it is invertible. It's just a new coordinate system

Example: $y = ax$
Then $x = cX$
$y = bY$ note $b \neq 0$, $c \neq 0$, otherwise cannot go back
$bY = acX \Rightarrow Y = b^{-1}acX$

If $a = 0$, it's the 0 function $\quad Y = 0$
Otherwise choose b, c s.t. $b^{-1}ac = 1 \Rightarrow Y = X$.

$\therefore Y = \begin{cases} 0 & \text{if it is the zero function in the beginning} \\ X & \text{if otherwise.} \end{cases}$

Example: $y = 3x_1 + 5x_2$
Let $X_1 = 3x_1 + 5x_2$
$X_2 = x_2$
Then $\boxed{y = X_1}$

The Rank Theorem

Every system of m linear functions in n variables can be transformed to exactly one of the normal form by linear changes of dependent and independent variables:

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= x_2 \\ &\vdots \\ y_r &= x_r \\ y_{r+1} &= 0 \\ &\vdots \\ y_m &= 0 \end{aligned}$$

$$\underbrace{x_1, x_2, \dots x_r,}_{\text{only these are participating in the final normal form}} \underbrace{x_{r+1}, \dots x_n}_{\text{these are still coordinates but are not participating.}}$$

Here $0 \leq r \leq \min(m,n)$ is the rank.

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

I Given a quadratic form $Q = \sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij} x_i x_j$ $\quad q_{ij} = q_{ji}$ assume symmetric coefficients.

What is the simplest form that Q can be transformed by linear changes of coordinates?

$$\left.\begin{aligned} x_1 &= c_{11} X_1 + \dots + c_{1n} X_n \\ &\vdots \\ x_n &= c_{n1} X_1 + \dots + c_{nn} X_n \end{aligned}\right\} \text{This transformation is invertible.}$$

The Inertia Theorem

Each quadratic form in n variables can be transformed by linear changes of coordinates to exactly 1 of the normal forms

$$X_1^2 + X_2^2 + \dots + X_p^2 - X_{p+1}^2 - \dots - X_{p+q}^2 \qquad 0 \leq p+q \leq n$$

$$(X_1, X_2, \dots, X_p, X_{p+1}, \dots, X_{p+q}, \underbrace{\dots X_n}_{\text{does not depend on these unlisted variables}})$$

$$\begin{matrix} p \\ q \end{matrix} \begin{bmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& -1 &&&& \\ &&&& -1 &&& \\ &&&&& -1 && \\ &&&&&& 0 & \\ &&&&&&& 0 \end{bmatrix} = \begin{bmatrix} I_p & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

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n	#
1	3
2	6

Examples $n=1$. $y=qx^2$ $x=cX$ $c\neq 0$.

$\Rightarrow y = qc^2X^2$

c^2 can change the magnitude but not the sign of it.

$\Rightarrow$ choose $c=\frac{1}{\sqrt{|q|}}$ $\Rightarrow \boxed{Y=X^2 \text{ or } Y=-X^2 \text{ or } Y=0.}$

$n=2$. X^2+Y^2, X^2-Y^2, $-X^2-Y^2$, X^2, $-Y^2$, 0

As what we had done for conic sections

III Given two quadratic forms $Q=\sum\sum q_{ij}x_ix_j > 0$

$S=\sum\sum s_{ij}x_ix_j$

and s.t. Q is positive everywhere other than origin.

Since $Q>0$ everywhere other than origin, ~~the~~ we can transform Q to $X_1^2+X_2^2+\dots+X_n^2$

For $n=2$. $Q=x^2+y^2 \rightsquigarrow X^2+Y^2$

$S=ax^2+2bxy+cy^2 \rightsquigarrow AX^2+CY^2$

Orthogonal Diagonalization Theorem

Q and S can be transformed by a linear change of coordinates at the same time to achieve

$Q=X_1^2+X_2^2+\dots+X_n^2$

$S=\underbrace{\lambda_1X_1^2+\lambda_2X_2^2+\dots+\lambda_nX_n^2}_{\text{spectrum}}$, $\lambda_1\geq\lambda_2\geq\dots\geq\lambda_n$.

continuous equivalence class, since $\lambda_i\in\mathbb{R}$.

the coefficients are unique up to ordering.

I can take a symmetric matrix S, apply a orthogonal matrix U to it

$U^{-1}SU=\begin{bmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{bmatrix}$ to achieve a diagonal matrix.

Compared to II. I. this is continuous rather than discrete.

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IV $\mathbb{R}^n \to \mathbb{R}^n$

$a_{11}x_1 + \dots + a_{1n}x_n = \dot{x}_1$

$\vdots$

$a_{n1}x_1 + \dots + a_{nn}x_n = \dot{x}_n$

$\dot{\vec{x}} = A\vec{x}$

If we change coordinates s.t. $\vec{x} = C\vec{X}$

$\dot{\vec{x}} = C\dot{\vec{X}}$

$\therefore C\dot{\vec{X}} = AC\vec{X} \Rightarrow \dot{\vec{X}} = C^{-1}AC\vec{X}$

Does not make sense to transform if (velocity and position)

What is the simplest form to which the system can be transformed by a linear change of coordinates

$x_1 = c_{11}X_1 + \dots + c_{1n}X_n$

$x_2 = c_{21}X_1 + \dots + c_{2n}X_n$

$\vdots$

$x_n = c_{n1}X_1 + \dots + c_{nn}X_n$

We need to allow complex numbers.

Jordan Canonical Form Theorem

Every ODE system with possibly complex coefficients can be transformed by linear (complex) changes of coordinates ~~can b~~ to exactly 1 of the Jordan Canonical normal forms

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \boxed{m_1} & & & 0 \\ & \boxed{m_2} & & \\ & & \ddots & \\ 0 & & & \boxed{m_r} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

within 2 Jordan cells, λ could be the same but not necessarily.

$n = m_1 + m_2 + \dots + m_r$

within a ~~[illegible]~~ Jordan cell, λ are the same.

Each block is made up of

$$\boxed{\begin{matrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda \end{matrix}}$$ "Jordan cell"

$\left(\frac{d}{dt} - \lambda_i\right)^{m_i} y_i(t) = 0$

$i = 1, 2, \dots, r$

$m_1 + m_2 + \dots + m_r = n$.

Unique up to the permutations of the blocks

The system falls apart into separate systems.

$$\boxed{\begin{matrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & \lambda_n \end{matrix}}$$ $\dot{x}_1 = \lambda_1 x_1$