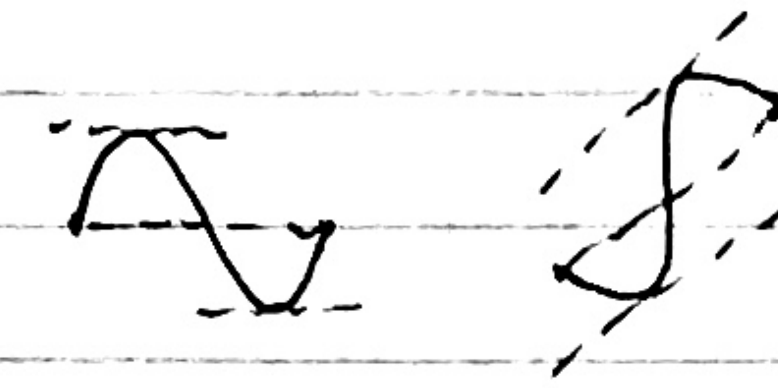
11/09/2021

- generalized Mean Value Theorem
- L'Hospital's Rule
- Polynomial Approximation

MATH 104 LECTURE NOTES 20: L'Hospital's Rule

Recall

Mean Value Theorem:



→ If c is max/min of f on (a,b), then $f'(c) = 0$

For a continuous function on closed interval, always admits max and min.

→ Rolle's theorem (special case of Mean Value Theorem) → used to prove Mean Value Theorem by using it on the difference function.

→ Consequences of Mean Value Theorem

$f' = 0 \Rightarrow f$ is constant

$f' > 0 \Rightarrow f$ is strictly increasing

→ Intermediate value theorem for derivatives

Generalized Mean Value Theorem

Let f, g be continuous and differentiable functions on $[a,b]$ (→ on (a,b))

Then $\exists c \in (a,b)$ s.t.

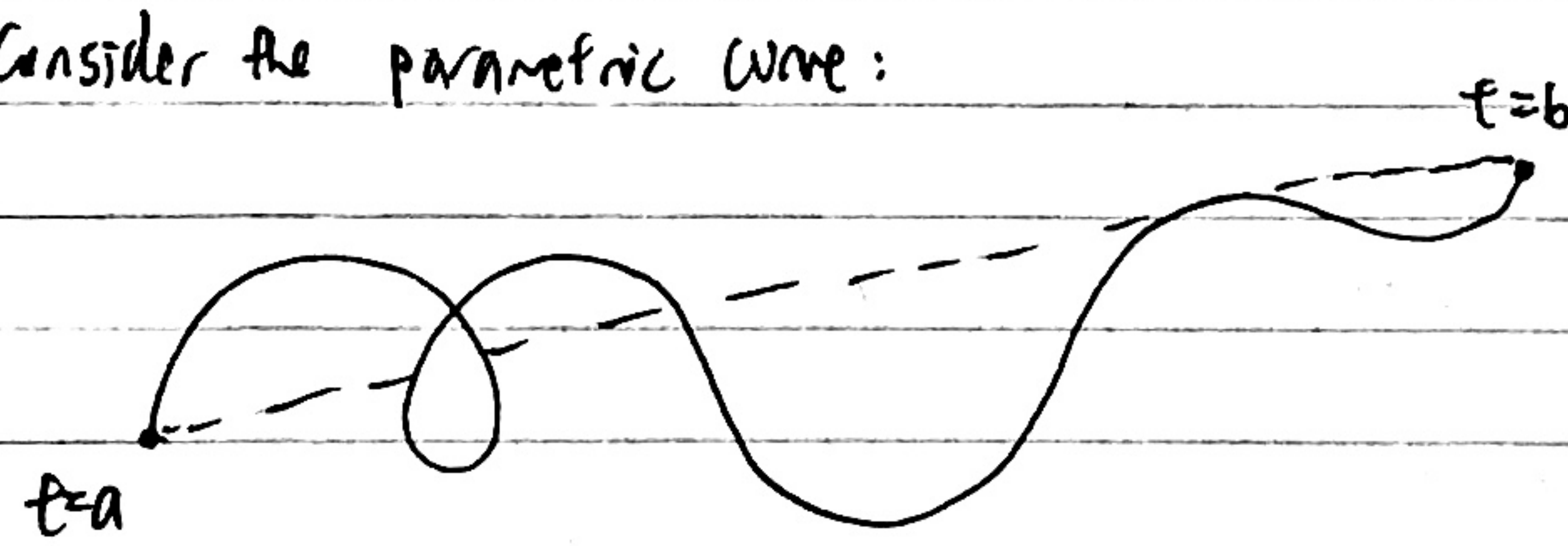
$$g'(c)\big(f(b)-f(a)\big) = f'(c)\big(g(b)-g(a)\big)$$

When $g(x) = x$, then $g'(x) = 1$

$$\Rightarrow f(b)-f(a) = f'(c)(b-a) \Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a} \text{ for } a \neq b.$$

Visualization: Consider two vectors $\begin{bmatrix} f(b)-f(a) \\ g(b)-g(a) \end{bmatrix} /\!/ \begin{bmatrix} f'(c) \\ g'(c) \end{bmatrix}$ (parallel comes from the equation).

Consider the parametric curve:



$\begin{bmatrix} f(b)-f(a) \\ g(b)-g(a) \end{bmatrix}$ is simply the displacement vector

$\begin{bmatrix} f'(c) \\ g'(c) \end{bmatrix}$ is the velocity vector at a point $c \in (a,b)$

i.e. $\exists$ a time such that the velocity vector is parallel to the displacement vector.

(*) We want to consider a difference function which is sort of an antiderivative of the difference function → Apply Rolle / MVT

Proof:

Set $h(x) = g(x)\big(f(b)-f(a)\big) - f(x)\big(g(b)-g(a)\big)$

Then h is continuous on $[a,b]$ and differentiable on (a,b).

$$h(a) = g(a)\big(f(b)-f(a)\big) - f(a)\big(g(b)-g(a)\big) = g(a)f(b) - f(a)g(b)$$

$$h(b) = g(b)\big(f(b)-f(a)\big) - f(b)\big(g(b)-g(a)\big) = g(a)f(b) - f(a)g(b)$$

Since $h(a)=h(b)$, by Rolle's theorem, $\exists c \in (a,b)$ s.t. $h'(c)=0$.
Note that since $h(x) = (f(b)-f(a))\,g(x) - (g(b)-g(a))\,f(x)$
$h'(x) = (f(b)-f(a))\,g'(x) - (g(b)-g(a))\,f'(x)$.

Hence $h'(c) = (f(b)-f(a))\,g'(c) - (g(b)-g(a))\,f'(c) = 0$
$\Rightarrow \exists c \in (a,b)$ s.t. $\boxed{g'(c)(f(b)-f(a)) = f'(c)(g(b)-g(a))}$ as desired.

L'Hospital's Rule (L'Hôpital)

Let s denote one of the following: $a \in \mathbb{R}$, a^+ or a^- or ∞ or $-\infty$.
Let $L \in \mathbb{R} \cup \{\pm\infty\}$
Take functions f and g for which $\lim_{x\to s} \frac{f'(x)}{g'(x)} = L$

Assume either ① $\lim_{x\to s} f(x) = 0$ and $\lim_{x\to s} g(x) = 0$
② $\lim_{x\to s} |g(x)| = \infty$

Then $\lim_{x\to s} \frac{f(x)}{g(x)} = L = \lim_{x\to s} \frac{f'(x)}{g'(x)}$

<u>Example:</u> $\lim_{x\to 0^+} x\ln x$ $\quad x\ln x = \frac{\ln x}{\frac{1}{x}}$ and $\frac{(\ln x)'}{(\frac{1}{x})'} = \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -x$

Since $\lim_{x\to 0^+} -x = 0$, by L'Hospital's Rule, $\boxed{\lim_{x\to 0^+} x\ln x = 0.}$

Consider a special case $s = a^+$, $f,g : (a,b) \to \mathbb{R}$, $L \in \mathbb{R}$
(#) Assume $\lim_{x\to a^+} f(x) = 0 = \lim_{x\to a^+} g(x)$

<u>Proof:</u> Need to show $\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$ where $L = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$
Take any $\varepsilon > 0$.
Need to check $\exists \delta > 0$ s.t. $a < \delta < b$ and $\forall x \in (a, a+\delta)$, $\left|\frac{f(x)}{g(x)} - L\right| < \varepsilon$

Since $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, by decreasing b if necessary, we may assume the following
$g'(x) \neq 0 \ \forall x \in (a, b')$ where b' is the restriction after decreasing b.

By intermediate value theorem for derivatives, $g'(x)$ is always positive or always negative
(otherwise, $\exists c$ s.t. $g'(c) = 0$)

we need this reduction to define our δ.

This means that g is strictly increasing or strictly decreasing.
$\Rightarrow$ $\boxed{\text{at most}}$ one x s.t. $g(x) = 0$.

By restricting b' again, we may assume $g(x) \neq 0 \quad \forall x \in (a, b'')$ where b'' is b' after the above restriction.

Since $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$, $\exists \delta > 0$ s.t. $a + \delta < b$ and $x \in (a, a+\delta) \Rightarrow \frac{f'(x)}{g'(x)} \in (L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2})$

Take any x, y s.t. $a < x < y < a + \delta$

By Generalized Mean Value Theorem, $\exists c \in (x, y)$ s.t.

$$g'(c)(f(y) - f(x)) = f'(c)(g(y) - g(x))$$

By our reduction, $g'(c) \neq 0$ (since $g'(x) \neq 0 \;\forall x \in (a, b)$)

$g(y) - g(x) \neq 0$ (since $g(x)$ is strictly increasing)

Hence, $\frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f'(c)}{g'(c)} \in (L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2})$

This is because all terms $\frac{f(y) - f(x)}{g(y) - g(x)} \in (L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2})$

By assumption (#), if we let $x \to a^+$, $\lim_{x \to a^+} \frac{f(y) - \overset{\to 0}{f(x)}}{g(y) - \underset{\to 0}{g(x)}} = \lim_{x \to a^+} \frac{f(y)}{g(y)} = \frac{f(y)}{g(y)}$

$\in [L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2}]$.

In other words, $\forall y \in (a, a+\delta)$, $\left| \frac{f(y)}{g(y)} - L \right| \leq \frac{\varepsilon}{2} < \varepsilon$

Hence $\boxed{\lim_{y \to 0^+} \frac{f(y)}{g(y)} = L}$

limit is confined in the interval, including the boundaries

Exercises

① $\lim_{x \to 0} \frac{\sin(2x)}{e^x - \cos x}$ ② $\lim_{x \to 0} \frac{e^{x^2} - 1}{x^2}$ ③ $\lim_{x \to 0} \frac{2 - \cos x}{x^2}$

④ $\lim_{x \to 0^+} x^x$ ⑤ $\lim_{x \to \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$

① Note $\frac{(\sin(2x))'}{(e^x - \cos x)'} = \frac{2\cos(2x)}{e^x + \sin x}$. This is a continuous function, so $\lim_{x \to 0} \frac{2\cos(2x)}{e^x + \sin x} = 2$

By L'Hospital's rule, $\boxed{\lim_{x \to 0} \frac{\sin(2x)}{e^x - \cos x} = 2}$

② $\lim_{x \to 0} \frac{e^{x^2} - 1}{x^2} = \lim_{y \to 0^+} \frac{e^y - 1}{y} = \lim_{y \to 0^+} \frac{e^y - 1}{y - 0} = \frac{d}{dy} e^y \Big|_{y=0} = e^y \Big|_{y=0} = \boxed{1}$

OR By L'Hospital's rule: $\lim_{x\to 0} \frac{e^{x^2}-1}{x^2} = \lim_{x\to 0} \frac{\cancel{2x} e^{x^2}}{\cancel{2x}} = \lim_{x\to 0} e^{x^2} = 1.$

③ Does not satisfy L'Hospital's conditions.

Take any sequence (x_n) in $(0,\infty)$ with $\lim x_n = 0$

Note: $\frac{2-\cos x_n}{x_n^2} \geq \frac{1}{x_n^2} \to \infty$ (as $n\to\infty$)

So $\lim_{n\to\infty} \frac{2-\cos x_n}{x_n^2} = \infty$. Hence $\boxed{\lim_{x\to 0^+} \frac{2-\cos x}{x^2} = \infty}$

④ $x^x = e^{x\ln x}$.

We know $\lim_{x\to 0^+} x\ln x = 0$. Since e^x is continuous at $x=0$,

$\lim_{x\to 0^+} e^{x\ln x} = e^{\lim_{x\to 0^+} x\ln x} = e^0 = 1$. $\therefore \boxed{\lim_{x\to 0^+} x^x = 1}$

⑤ $\frac{e^x-e^{-x}}{e^x+e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1} = 1 - \frac{2}{e^{2x}+1}$ $\cancel{\frac{e^x - e^{-x}}{e^x+e^{-x}}}$

$\to \boxed{1}$ (as $x\to\infty$)