11/16/2021

- Polynomial approximation
- Taylor Series
- Taylor Theorem

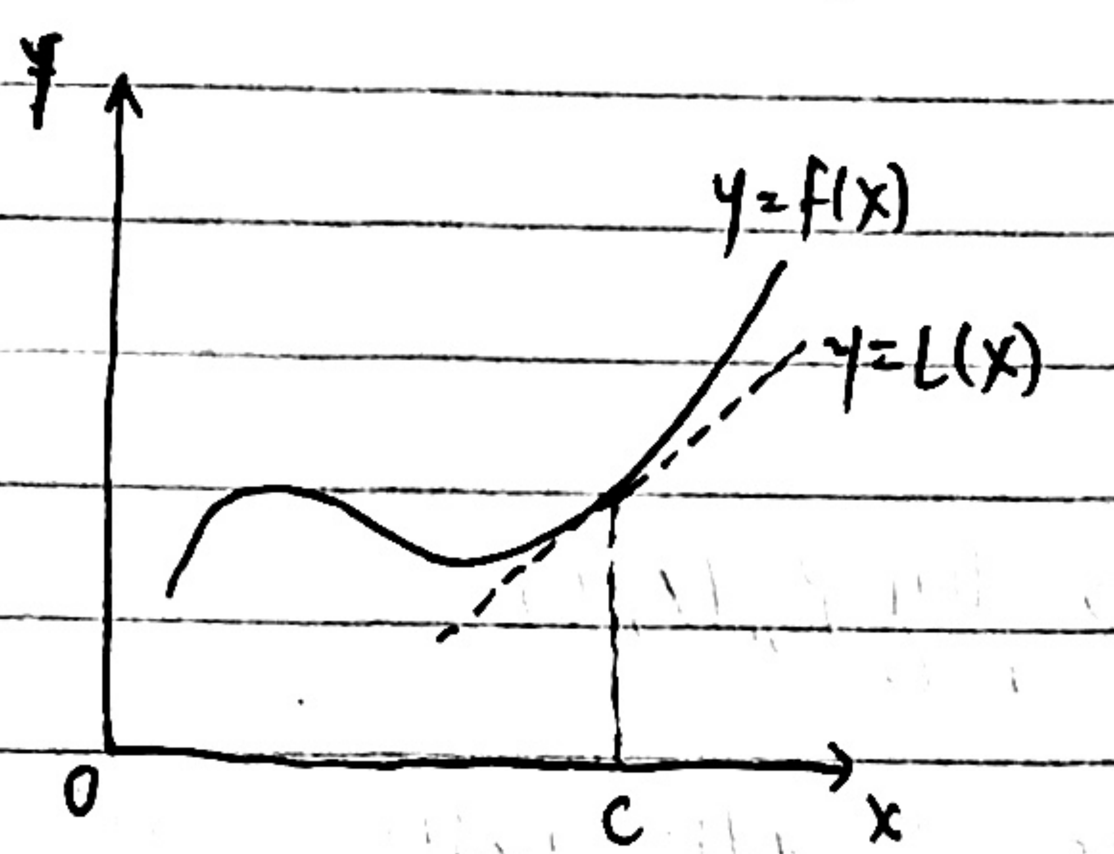
MATH 104 LECTURE NOTES 21 TAYLOR SERIES

Setup

Consider an open interval I and $c \in I$. Let $f: I \to \mathbb{R}$ be infinitely differentiable (i.e. f is differentiable, f' is differentiable ...). Denote the n^{th} derivative as $f^{(n)}$.
Goal: Approximate f by a power series centered at c. i.e. of the form $\sum_{k=0}^{\infty} a_k (x-c)^k$ near c.

Polynomial Approximation

What is the linear polynomial that best approximates f near c?



$L(x) = f'(c)(x-c) + f(c)$

Reason:
$L(c) = f(c)$
$L'(c) = f'(c)$

Both function & derivative takes the same value at c.

What is the degree n polynomial that best approximate $f(x)$ near c?
Want $f^{(i)}(c) = p^{(i)}(c)$ for $0 \leq i \leq n$.

To obtain p, better to write in terms of $(x-c)$
Write $p(x) = \cancel{a_n(x-c)^n +} \; a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$

$p(c) = a_0 = f(c)$
$p'(c) = a_1 = f'(c)$ $\qquad p^{(k)}(c) = k!\,a_k = f^{(k)}(c)$
$p''(c) = 2a_2 = f''(c)$
$p'''(c) = 6a_3 = f'''(c)$ $\qquad$ Hence $\boxed{a_k = \frac{f^{(k)}(c)}{k!}}$

$$\therefore \boxed{p(x) = \frac{f(c)}{0!} + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(k)}(c)}{k!}(x-c)^k + \cdots}$$

The reason that it is the best estimate is because all values up to the n^{th} derivative are the same.

Taylor Series

Definition: Taylor series of $f(x)$ around c is the power series $\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$
When does the Taylor Series agree with the original function? (For which x?)

Definition: For each n and $x \in I$, the n^{th} remainder of f around c is

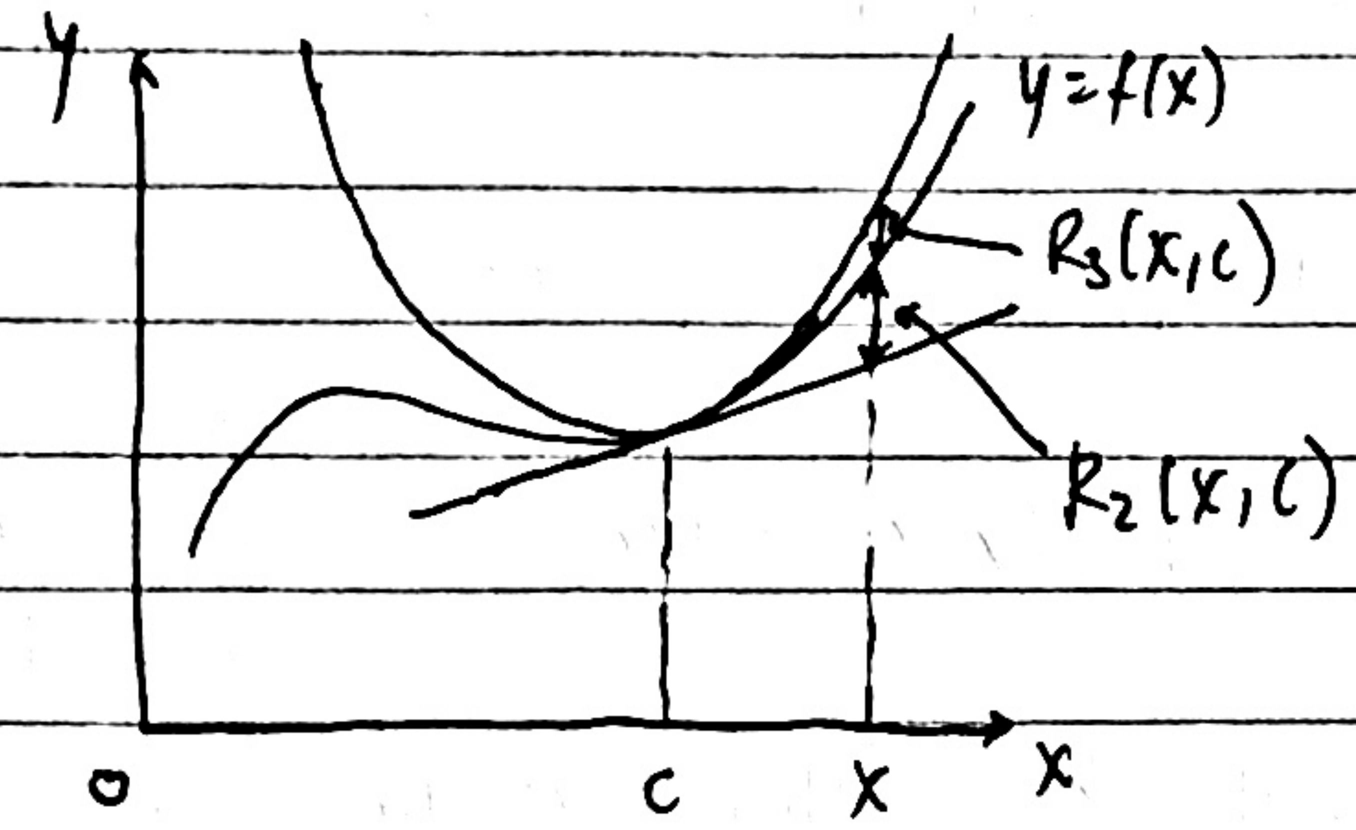
$$R_n(x,c) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

(This is basically the difference between target limit and partial sum sequence)

Example: $R_1(x,c) = f(x) - f(c)$

$R_2(x,c) = f(x) - f(c) - \frac{f'(c)}{1!}(x-c)$

$R_3(x,c) = f(x) - f(c) - \frac{f'(c)}{1!}(x-c) - \frac{f''(c)}{2!}(x-c)^2$



Property: For $x \in I$, $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \iff \lim_{n\to\infty} R_n(x,c) = 0$.

Taylor's Theorem: Let I be an open interval. Let $f: I \to \mathbb{R}$ be n^{th} differentiable ($f^{(n)}$ exists). Let $c \in I$. For each fixed $x \in I$ s.t. $x \neq c$, there exists y_n between x and c s.t. $R_n(x,c) = \frac{f^{(n)}(y_n)}{n!}(x-c)^n$

If $n=1$, $R_1(x,c) = f(x) - f(c)$.
Taylor's theorem says $R_1(x,c) = f'(y_1)(x-c)$ for some y between x and c.
This is precisely mean value theorem. In other words, Taylor's theorem is a n^{th} generalization of mean value theorem.

Proof: Set $M = \dfrac{R_n(x,c)}{\left(\frac{(x-c)^n}{n!}\right)}$ Makes sense because $x \neq c$ so denominator $\neq 0$

Note: x is fixed. t is the variable

Hence $R_n(x,c) = \frac{M}{n!}(x-c)^n$

Suffices to find y_n between x and c s.t. $M = f^{(n)}(y_n)$

Consider $g: I \to \mathbb{R}$ defined by $g(t) = -R_n(t,c) + \frac{M}{n!}(t-c)^n$ ~~f(t)~~

i.e. $g(t) = \underbrace{\left[f(c) + f'(c)(t-c) + \dots + \frac{f^{(n-1)}(c)}{(n-1)!}(t-c)^{n-1} - f(t)\right]}_{-R_n(t,c)} + \frac{M}{n!}(t-c)^n$

observe that $g(x)=0$ (since $g(x) = -R_n(x,c) + \frac{M}{n!}(x-c)^n = 0$)
$g(c)=0$ (since $g(c) = f(c) + (c-c)(\cdots) - f(c) = 0$)
$g'(c)=0$ (since $g'(c) = \left(f'(c) + h.o.t(x-c) - f'(c)\right) = 0$)
$\vdots$
$g^{(n-1)}(c)=0$

Since $g(x)=0=g(c)$, Rolle's theorem implies $\exists\, y_1$ between x and c s.t. $g'(y_1)=0$.
Since $g'(y_1)=0=g'(c)$, Rolle's theorem implies $\exists\, y_2$ between y_1 and c s.t. $g''(y_2)=0$.
$\vdots$
Since $g^{(n-1)}(y_{n-1})=0=g^{(n-1)}(c)$, Rolle's theorem implies $\exists\, y_n$ between y_{n-1} and c s.t. $g^{(n)}(y_n)=0$.

Hence, $\exists\, y_n$ between x and c s.t. $g^{(n)}(y_n)=0$.

On the other hand, we can compute g directly.
Because $g(t) = \left[f(c) + f'(c)(t-c) + \dots + \frac{f^{(n-1)}(c)}{(n-1)!}(t-c)^{n-1} + \frac{M}{n!}(t-c)^n - f(t)\right]$

$g^{(n)}(t) = M - f^{(n)}(t) \Rightarrow \exists\, y_n$ between x and c s.t. $\boxed{M = f^{(n)}(y_n)}$

upshot $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$ iff $\lim_{n\to\infty} R_n(x,c) = 0$

Taylor's theorem says $\exists\, y_n$ between x and c s.t. $R_n(x,c) = \frac{f^{(n)}(y_n)}{n!}(x-c)^n$

Example: $f(x)=e^x$, $c=0$.
Taylor series: $1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$

To show these two are the same, show $\lim_{n\to\infty} R_n(x,c)=0$.

If $x=0$, they are the same (by design).
Assume $x\neq 0$. Want $\lim_{n\to\infty} R_n(x,0)=0$.
For each n, Taylor's Theorem $\Rightarrow \exists\, y_n$ between x and 0 s.t. $R_n(x,0) = \frac{f^{(n)}(y_n)}{n!}x^n$
$= \frac{e^{y_n}}{n!}x^n$ (in this case)

Note, $y_1, y_2, \dots y_n \in [-|x|, |x|]$
and e^x is increasing. Hence $|e^{y_n}| \le e^{|x|}$
$|R_n(x,0)| = \frac{|e^{y_n}|}{n!}x^n < \frac{e^{|x|}}{n!}|x|^n$ and $\lim \frac{e^{|x|}}{n!}|x|^n = 0$ since x is constant.
Hence $\boxed{\lim_{n\to\infty} R_n(x,0)=0}$

Remarks: This argument can be generalized

Theorem: If $\exists\, c > 0$ s.t. $|f^{(n)}(t)| \leq c$ $\forall\, n$, t between x and c

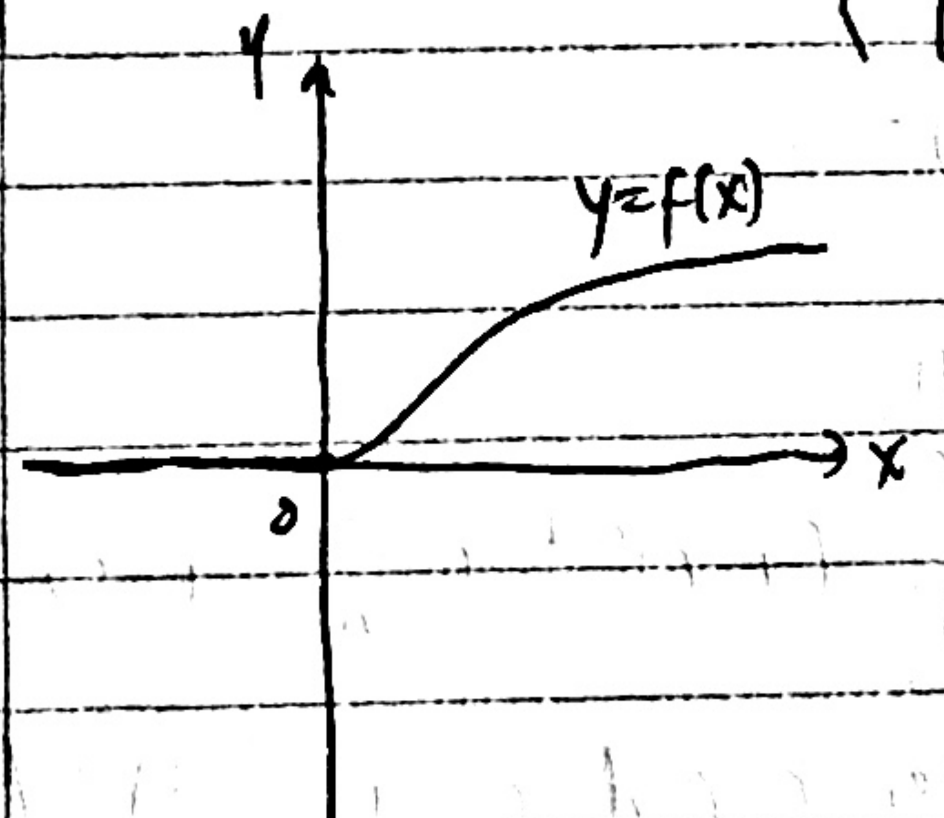
then $\lim_{n\to\infty} R_n(x, c) = 0$.

$$|R_n(x,c)| = \frac{|f^{(n)}(y_n)|}{n!}|x-c|^n \leq c\,\frac{|x-c|^n}{n!}$$

The last term tends to 0. as $n \to \infty$.

Remark: Taylor series does not always agree with original function.

<u>Example:</u> $f(x) = \begin{cases} e^{\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$



Can check $f^{(n)}(0) = 0$

$\Rightarrow$ Taylor series $= 0 \neq f(x)$

So always find the Taylor series, then check if your function agrees with Taylor series.