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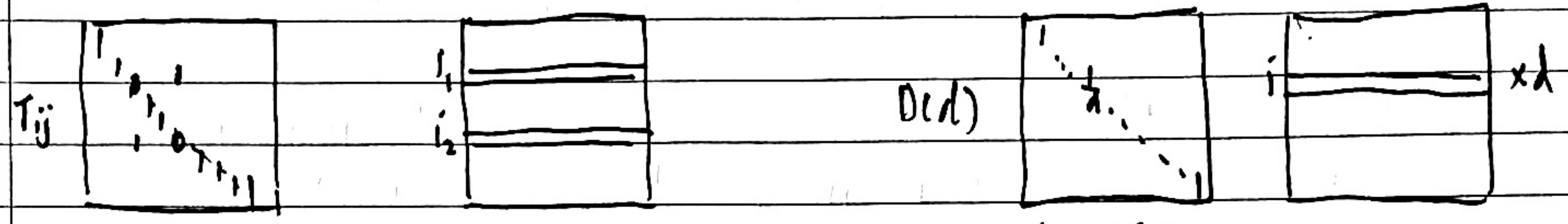
MATH 1410 LECTURE 17 NOTES

Consider M, a $n \times n$ invertible matrix.

LPU Decomposition

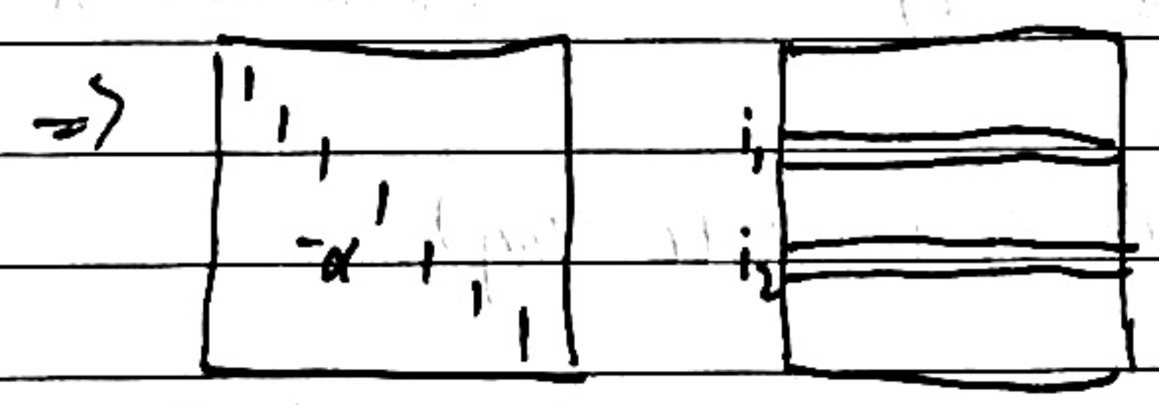
Every invertible $n \times n$ matrix can be represented in the form $M = LPU$ where L is lower-triangular, P is a permutation matrix, and U is upper-triangular.



Transposition is equivalent to multiplying the matrix by T_{ij}

Multiplying a row by d is equivalent to factoring out a $\frac{1}{d}$ (denoted by $D(d)$)

Consider two rows $i_1 < i_2$. To eliminate an element α in row i_2, perform row i_2 − α row i_1 (assuming element of i_1 is 1)

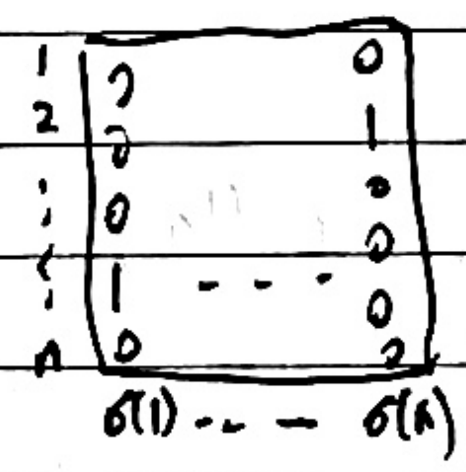


Result is $\boxed{A_N \cdots A_1} M = U$ where U is an upper-triangular matrix with all 1 on the diagonal.

$\Rightarrow \boxed{M = A_1^{-1} A_2^{-1} \cdots A_N^{-1} U}$

$T_{ij}^{-1} = T_{ij}$ (inverse of transposition is itself)

$P_\sigma^{-1} = P_\sigma^T$ (inverse of permutation matrix is its transpose)



$D_i(d)^{-1} = D_i(\frac{1}{d})$

$L_{i_1 i_2}(\alpha)^{-1} = L_{i_1 i_2}(-\alpha)$

Algorithm: ① For all column $j = 1$ to n

Find the first nonzero entry in the column

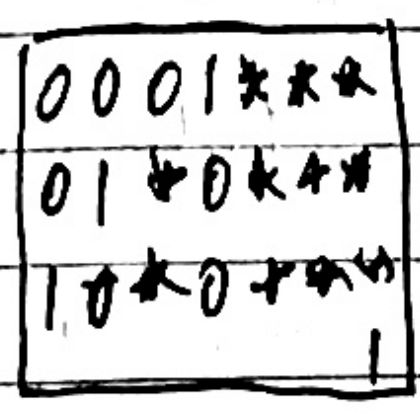
(definitely can find one since otherwise, contradict the fact for matrix is invertible)

Make the entry 1.

Subtract from the remaining rows to make the entries zero.

$\Rightarrow$ product of L and D.

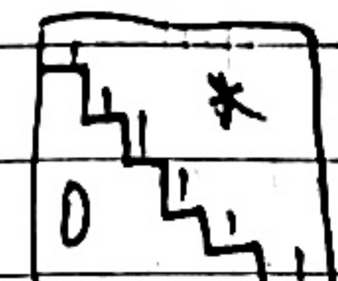
② We obtain matrix



If we only see the 1s, it is a permutation matrix.

Apply the inverse permutation matrix so all the 1s are on the diagonal.

③ Obtain



Unipotent (subtract I becomes nilpotent)

$\Rightarrow P_{\sigma^{-1}} L M = U \Rightarrow M = L^{-1} P_{\sigma^{-1}}^{-1} U \Rightarrow \boxed{M = L' P_\sigma U}$

To get other forms, consider $(M^{-1})^* = LPU$.

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$$\begin{tikzcd} V \arrow[r, "M"] \arrow[d, "I"'] & W \\ V' \arrow[r, "M'"'] & W' \arrow[u] \end{tikzcd}$$

M: invertible

$$V \xrightarrow{M} W$$

$0 < V^1 \subset V^2 \subset \dots \subset V^{n-1} \subset V^n$ is a complete flag. — standard flag associated to a basis

Let $\{\vec{f_1}, \vec{f_2}, \dots \vec{f_n}\}$ be a flag in V^n.

Can associate a flag with the basis (coordinate complete flag)

$V^i = \mathrm{Span}(\{\vec{f_1}, \vec{f_2}, \dots \vec{f_i}\})$

Invertible upper triangular matrices preserve all the subspaces preserve coordinate complete flag

Opposite flag

$V^1 = \mathrm{Span}\{\vec{f_n}\}$

$V^2 = \mathrm{Span}\{\vec{f_n}, \vec{f_{n-1}}\}$

Invertible lower triangular matrices. preserve the opposite coordinate flags.

Notation: Let $\{\vec{e_1}, \vec{e_2}, \dots \vec{e_n}\}$ be in $\mathbb{K}^n$.

$\mathbb{F}_{id}: V^k = \mathrm{Span}(\vec{e_1}, \vec{e_2}, \dots \vec{e_k})$

$\mathbb{F}_\sigma: V^k = \mathrm{Span}(\vec{e_{\sigma(1)}}, \vec{e_{\sigma(2)}}, \dots, \vec{e_{\sigma(k)}})$ $\quad \sigma = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ n & n-1 & \dots & 2 & 1 \end{pmatrix}$

$\mathbb{F}_{opp}: V^k = \mathrm{Span}(\vec{e_n}, \vec{e_{n-1}}, \dots \vec{e_{n-k+1}})$

Invertible upper triangular matrices preserve $\mathbb{F}_{id}$.

Invertible lower triangular matrices preserve $\mathbb{F}_{opp}$.

We call two flags equivalent if the subspaces of a flag can be transformed from one to another via an invertible transformation.

All complete flags of a space are equivalent (show they are all equivalent to standard flag)

$0 \subset V_1 \subset V_2 \subset \dots \subset V_{n-1} \subset V_n = V \cong \mathbb{K}^n$.

Find basis $\vec{f_1}$ in V_1, complete it to basis $\{\vec{f_1}, \vec{f_2}\}$ in V_2, ...

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we can obtain $\{\vec{f_1}, \vec{f_2}, \dots, \vec{f_n}\}$ in V. Then can transform this basis to $\{\vec{e_1}, \vec{e_2}, \dots \vec{e_n}\}$ which is the basis of the standard coordinate flag. Hence all flags are equivalent to the standard coordinate flag (in a non-unique way, since we have choice of $\vec{f_1}, \vec{f_2}, \dots \vec{f_n}$ up to scalar multiplications)

$\therefore$ $\boxed{\text{All flags are equivalent.}}$

Consider

$$0 \subset V^1 \subset \cdots \subset V^{n-1} \subset V \xrightarrow{M} W \supset W^{n-1} \supset \cdots \supset W \supset 0$$

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can only choose basis that preserves the flags transformations

$D^{-1} M C = M'$

Can always choose basis s.t. M is standard coordinate

$\Rightarrow$ C must be upper triangular $= U$

D^{-1} must be a lower triangular $= L$

$\Rightarrow LMC \Rightarrow M \cong P_\sigma \Rightarrow$ $\boxed{n! \text{ equivalence classes.}}$

Qn. Find the equivalence of pairs of flags $(\mathbb{F}_1, \mathbb{F}_2)$ $\quad V_1^1 \subset V_1^2 \subset \cdots \subset V_1^{n-1} \subset V_1^n = V$

i.e. invertible transformation M s.t. $\mathbb{F}_1' = M\mathbb{F}_1$ $\quad \dim(V_1^k \cap V_2^l) = \text{constant}.$

$\mathbb{F}_2' = M\mathbb{F}_2.$

Since single flags are equivalent. WLOG, $\mathbb{F}_1 = \mathbb{F}_1'$

$\therefore$ want to transform $\mathbb{F}_2$ to $\mathbb{F}_2'$ by a transformation preserving $\mathbb{F}_1$.

$\mathbb{F}_1 \rightsquigarrow F_{opp} \leftsquigarrow \mathbb{F}_1'$

$\Rightarrow$ Allowed transformations $\mathbb{F} \underset{L}{\rightsquigarrow} \mathbb{F}'$ Since L preserves F_{opp}

Since any flags are equivalent, $\exists M$ st. $\mathbb{F} = M\mathbb{F}_{id}$

$= LPU\mathbb{F}_{id}$ (since M has LPU decomposition)

$= LP\mathbb{F}_{id}$ (U preserves $\mathbb{F}_{id}$)

$\therefore \mathbb{F} = L\mathbb{F}_\sigma$

$\Rightarrow$ $\boxed{L^{-1}\mathbb{F} = \mathbb{F}_\sigma}$ Hence, any $\mathbb{F}$ is equivalent to $\mathbb{F}_\sigma$ $\Rightarrow$ $\boxed{n! \text{ possible ways}}$

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Consider (F_{opp}, F_{σ}) $\dim(W^k \cap V^k)$ is invariant

W^k V^k

$W^{n-k} = \text{Span}(\vec{e_n}, \vec{e_{n-1}}, \ldots, \vec{e_{k+1}})$

$W^{n-k+1} = \text{Span}(\vec{e_n}, \vec{e_{n-1}}, \ldots, \vec{e_k})$

Consider $V^i = \text{Span}(\vec{e_{\sigma(1)}}, \vec{e_{\sigma(2)}}, \ldots \vec{e_{\sigma(i)}})$

$V^1 \subset V^2 \subset V^3 \subset \ldots \subset V^i \subset \ldots$

Find i s.t. $\vec{e_k}$ appears for the first time in V^i (have not appeared anywhere else before, i.e. $\vec{e_k} \notin V^j$, $j<i$)

$\Rightarrow \sigma(i) = k$

$$\dim(V^i \cap W_{n-k+1}) - \dim(V^i \cap W_{n-k}) = \begin{cases} 0 & i < \sigma^{-1}(k) \\ 1 & i \geq \sigma^{-1}(k) \end{cases}$$

Only when $\vec{e_k}$ suddenly appears in the subspace.

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