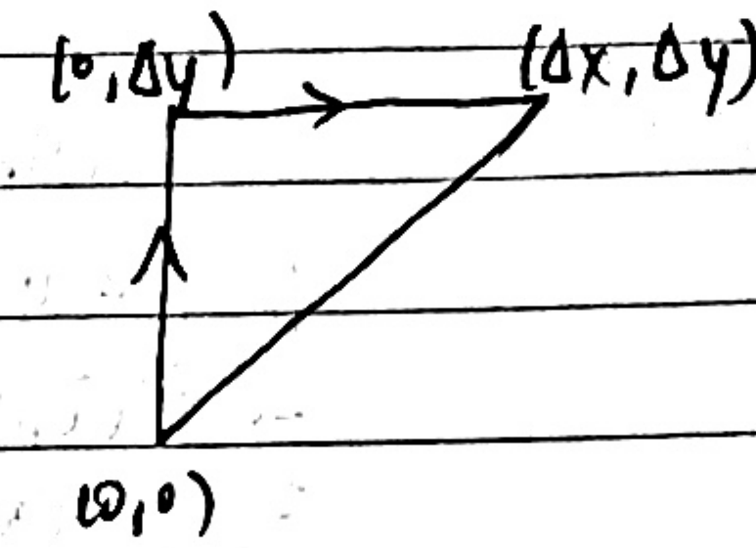
MATH 185 LECTURE 11 NOTES Chapter 3

Theorem Let $\Omega \subset \mathbb{C}$ be open, $z_0 \in \Omega$ and $f : \Omega \to \mathbb{C}$. Suppose f_x, f_y exist and are continuous in a neighborhood of z_0 and $if_x = f_y$. Then f is differentiable at z_0.



Proof:

$$\frac{f(h)-f(0)}{h} = \frac{u(\Delta x,\Delta y) - u(0,0) + i\big(v(\Delta x,\Delta y) - v(0,0)\big)}{\Delta x + i\Delta y}$$

$$= \frac{\big(u(\Delta x,\Delta y) - u(0,\Delta y)\big) + \big(u(0,\Delta y) - u(0,0)\big) + i\Big(\big(v(\Delta x,\Delta y) - v(0,\Delta y)\big) + \big(v(0,\Delta y) - v(0,0)\big)\Big)}{\Delta x + i\Delta y}$$

By Mean Value Theorem,

$$u(\Delta x,\Delta y) - u(0,\Delta y) = (\Delta x)\, u'(\bar{x},\Delta y) = \Delta x \underbrace{u_x(\bar{x},\Delta y)}_{\text{Goal: estimate this by continuity}}$$

$$\Rightarrow u_x(\bar{x},\Delta y) = u_x(0,0) + \varepsilon_1$$

Similarly,

$$u(0,\Delta y) - u(0,0) = \Delta y\,(u_y(0,0) + \varepsilon_2)$$
$$v(\Delta x,\Delta y) - v(0,\Delta y) = \Delta x\,(v_x(0,0) + \varepsilon_3)$$
$$v(0,\Delta y) - v(0,0) = \Delta y\,(v_y(0,0) + \varepsilon_4)$$

$$\Rightarrow \frac{f(h)-f(0)}{h} = \frac{\Delta x\,(u_x + iv_x + \varepsilon_1 + \varepsilon_2) + \Delta y\,(u_y + iv_y + \varepsilon_3 + \varepsilon_4)}{\Delta x + i\Delta y}$$

(Used Cauchy Riemann $if_x = f_y$)
$$= \frac{\Delta x\,(u_x + iv_x + \varepsilon_1 + \varepsilon_2) + \Delta y\,(i(u_x + iv_x) + \varepsilon_3 + \varepsilon_4)}{\Delta x + i\Delta y}$$

$$= \frac{(\Delta x + i\Delta y)(u_x + iv_x)}{\Delta x + i\Delta y} + \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4}{\Delta x + i\Delta y}$$

$$= u_x + iv_x + \frac{\lambda}{\Delta x + i\Delta y} = f_x + \frac{\lambda}{\Delta x + i\Delta y}$$

$$\left|\frac{\lambda}{\Delta x + i\Delta y}\right| \leq |(\varepsilon_1 + i\varepsilon_3)| \frac{\Delta x}{|\Delta x + i\Delta y|} + |\varepsilon_2 + i\varepsilon_4| \frac{\Delta y}{|\Delta x + i\Delta y|} \leq |\varepsilon_1 + i\varepsilon_3| + |\varepsilon_2 + i\varepsilon_4|$$

since these two terms ≤ 1.

$\Delta x + i\Delta y$
$\parallel$
As $h \to 0$, the $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \to 0$ by the continuity of partial derivatives.
$\Rightarrow f' = u_x + iv_x = f_x$ i.e. derivatives exist for any direction $h \in \mathbb{C}$ and equal f_x.

Remark The following concepts are not the same:
1. $f = u + iv$ induces a differentiable map from $\mathbb{R}^2 \to \mathbb{R}^2$ — consider conjugation.
2. $f'(z_0)$ exists $\longrightarrow$ can be nowhere holomorphic
3. f' exists in a neighbourhood of z_0 everywhere in a region or on f's domain. (related to power series expansions of f).

Consider $f(x,y) = (u(x,y), v(x,y))$ is differentiable. ($\mathbb{R}$)

$$J_f = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$ Hence, if reflection in $\mathbb{R}^2$, then naturally also differentiable.

For complex differentiable, must satisfy CR $\Rightarrow J_f = \begin{bmatrix} u_x & u_y \\ -u_y & u_x \end{bmatrix}$ $\longrightarrow$ equivalent to the isomorphism of complex numbers to 2x2 matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

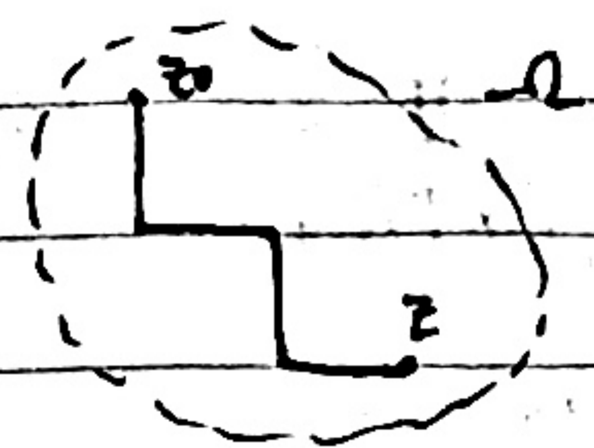
which is a rotation, dilation matrix.

$$f(z) \approx f(z_0) + \underbrace{(z - z_0)}_{\text{rotation and dilation}} \frac{f'(z_0)}{1!}$$

Remark Mean Value Theorem does not hold for $f : \mathbb{R} \to \mathbb{C}$. i.e. not necessary $\exists c$ s.t.
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$ for c between a, b.

Theorem Let $\Omega \subset \mathbb{C}$ be a region, and $f \in H(\Omega)$. If $f' = 0$ everywhere on Ω, then f is constant on Ω.

<u>Proof:</u> $f'(z) = 0 = f_x \Rightarrow u_x = v_x = 0 \Rightarrow u_y = v_y = 0$ (by Cauchy Riemann)

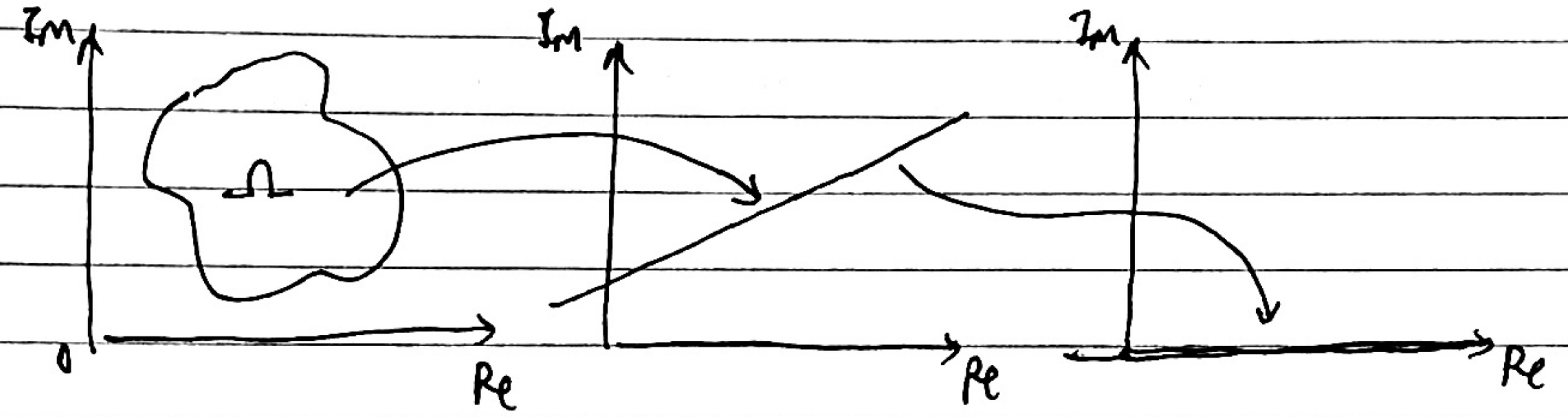


Since Ω is a region, $\exists$ a polygonal path between z and z_0. Along each line, $u_x, v_x, u_y, v_y = 0$ hence apply MVT $\Rightarrow u, v$ constant along each of the line.

$\therefore f(z) = f(z_0) \Rightarrow f$ is constant on Ω.

Example

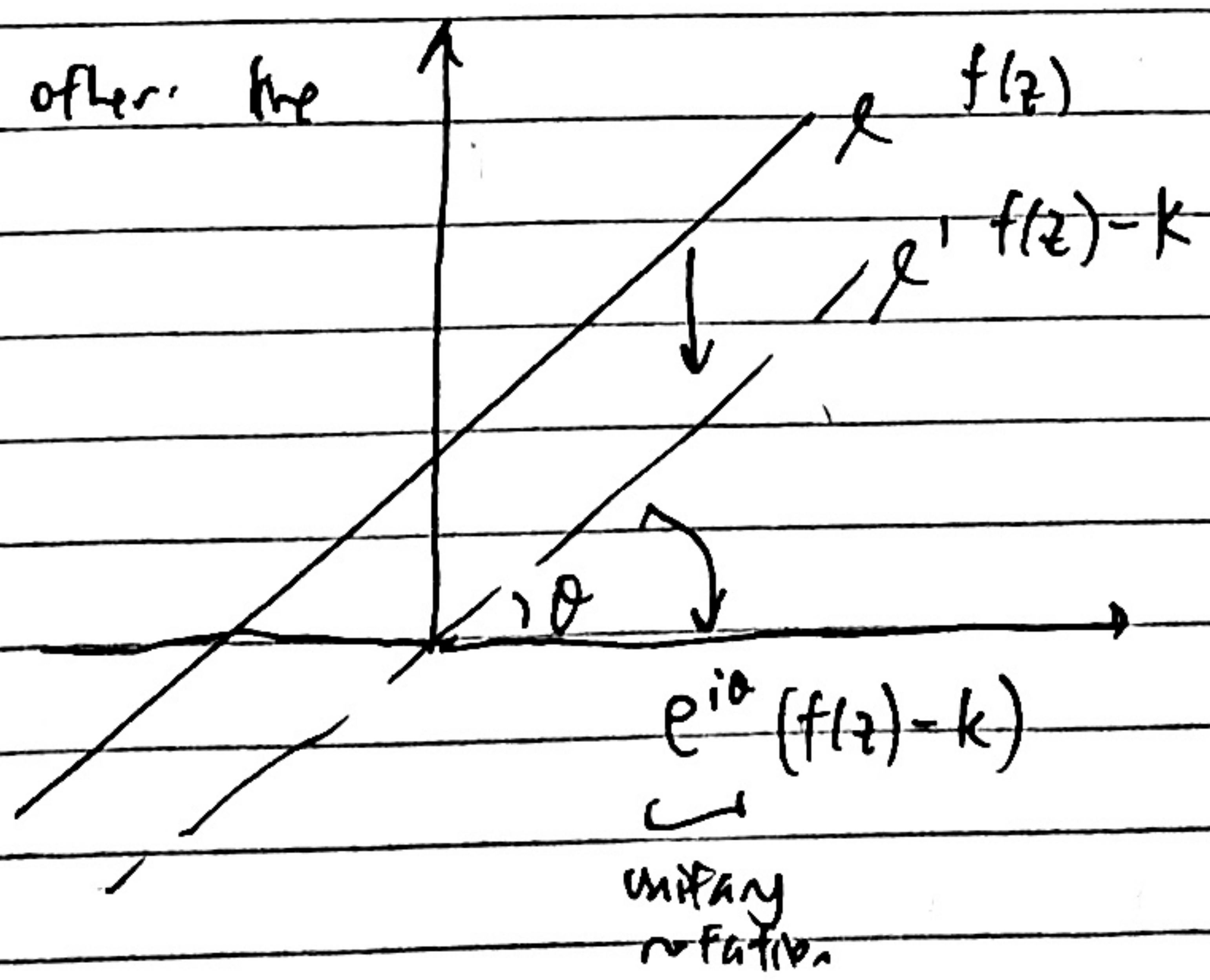
Let Ω be a region and $f \in H(\Omega)$.
Suppose $f(\Omega)$ is mapped onto a line in $\mathbb{C}$, find f.



Suppose f maps Ω onto $\mathbb{R}$.
Then $f(x+iy) = u(x,y) + \underbrace{iv(x,y)}_{0}$. In particular, $v_x, v_y = 0$.
Applying Cauchy Riemann, $v_x = u_y = 0$
$\Rightarrow f$ must be a constant function.

Key problem: Mapping an open Ω to a line which is closed. (Open Mapping Thm)

For any other line



$\therefore$ Transformation that maps ℓ to ℓ' to $\mathbb{R}$

If $f \in H(\Omega)$ and $f'(z) \neq 0$ on Ω then $f(E)$ is open for all $E \subset \Omega$
[Open Mapping Theorem]