No.: Date: 09/14/2021

MATH 104 LECTURE 6 NOTES (lim inf, lim sup, Cauchy)

Admin

1^{st} midterm Thursday Sept 23! 75min in class

practice exam 1 & 2 @ Files

Topics up to the next lecture. HW ≤ 3

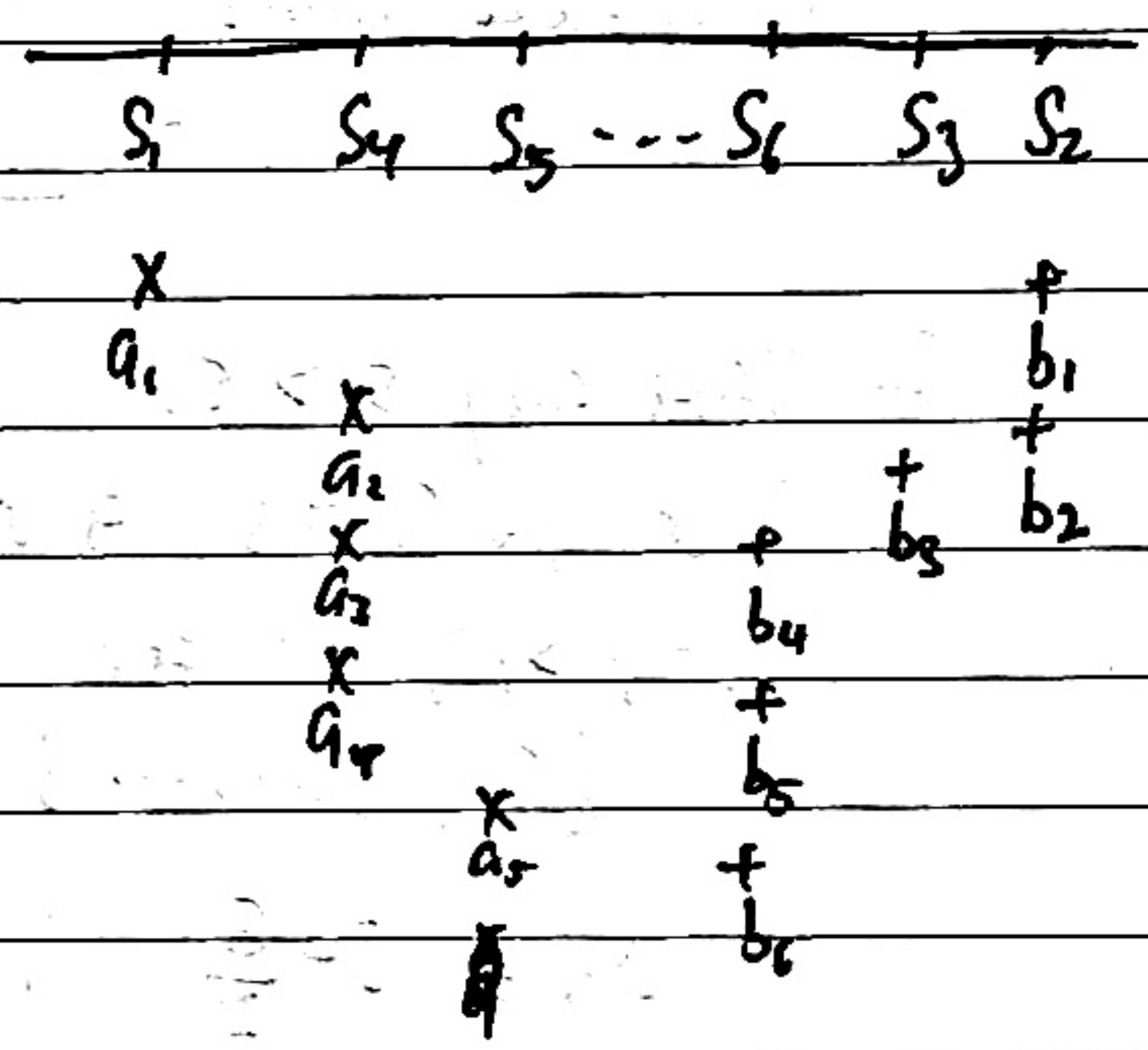
Recall

$\lim s_n$ is defined (as a real number or $\pm\infty$) for monotone sequence (s_n). Converges to a real number only if (s_n) is bounded, and $\pm\infty$ if unbounded.

Motivation

This theorem is very good, but can we apply it for sequences that are not monotone? Can we create monotone sequences out of (s_n)?

lim sup & lim inf



Let (s_n) be any sequence.

$a_n = \inf\{s_n, s_{n+1}, s_{n+2}, \dots\}$

i.e. ignore $s_1, s_2, \dots s_{n-1}$

$b_n = \sup\{s_n, s_{n+1}, s_{n+2}, \dots\}$

(a_n) is increasing, since

$a_n = \inf\{s_n, s_{n+1}, \dots\}$

$a_{n+1} = \inf\{s_{n+1}, \dots\} \geq \inf\{s_n, \dots\} = a_n$

Properties

(1) (a_n) is increasing. (b_n) is decreasing.

(2) $a_n \leq s_n \leq b_n \quad \forall n.$

Since (a_n) and (b_n) are monotone, $\lim a_n$ and $\lim b_n$ are defined.

Definition

$$\left.\begin{aligned} \liminf s_n &= \lim_{n\to\infty} a_n \\ \limsup s_n &= \lim_{n\to\infty} b_n \end{aligned}\right\} \text{real number or } \pm\infty.$$

Example

(1) $s_n = (-1)^n$. $\left.\begin{aligned} a_n &= -1 \\ b_n &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} \liminf s_n &= -1 \\ \limsup s_n &= 1 \end{aligned}$

(2) $s_n = n$ $\quad a_n = \inf\{n, n+1, \dots\} = n \quad \liminf s_n = \infty$

$b_n = \sup\{n, n+1, \dots\} \underset{\uparrow\ \text{symbolically}}{=} \infty \quad \limsup s_n = \infty.$

Remark: $\sup s_n$ does not make sense. sup is defined on a set.

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Theorem Let $s \in \mathbb{R} \cup \{+\infty, -\infty\}$

(1) If $\liminf s_n = s = \limsup s_n$, then $\lim s_n = s$ (i.e. (s_n) converges to s)

(2) If $\lim s_n = s$, then $\liminf s_n = \limsup s_n = s$.

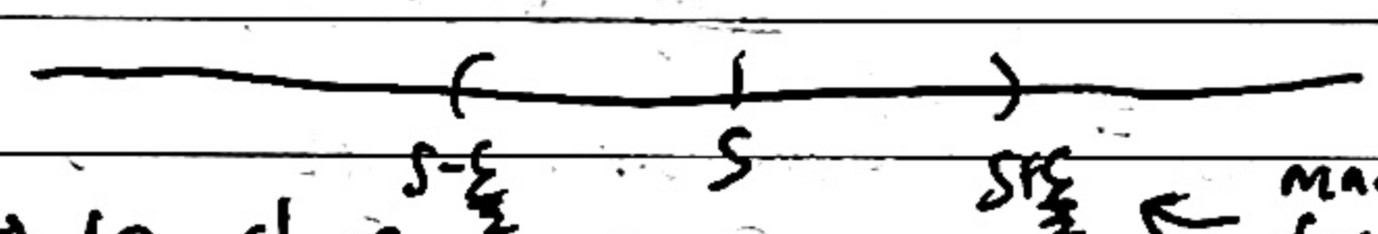
To show convergence, another way is to show $\liminf s$ and $\limsup s$ gives the same answer.

Proof (1) Know $a_n \leq s_n \leq b_n$

By squeeze lemma, $\liminf s_n = \lim a_n \leq \lim s_n \leq \lim b_n = \limsup s_n$

$\Rightarrow \boxed{\liminf s_n = \lim s_n = \limsup s_n}$

(2) Take any $\varepsilon > 0$.



← mainly introduced for making inequality strict.

Since $\lim s_n = s$, $\exists N$ s.t. $n > N \Rightarrow |s_n - s| < \frac{\varepsilon}{2}$.

For $n > N$, the set

$\{s_n, s_{n+1}, \dots\}$ has a lower bound of $s - \frac{\varepsilon}{2}$ and upperbound $s + \frac{\varepsilon}{2}$.

$\left.\begin{array}{l} \Rightarrow a_n \geq s - \frac{\varepsilon}{2} \\ b_n \leq s + \frac{\varepsilon}{2} \end{array}\right]$ $s - \frac{\varepsilon}{2} \leq a_n \leq b_n \leq s + \frac{\varepsilon}{2}$

$\Rightarrow s - \varepsilon < s - \frac{\varepsilon}{2} \leq a_n \leq b_n \leq s + \frac{\varepsilon}{2} < s + \varepsilon$

$\therefore |a_n - s| < \varepsilon$ and $|b_n - s| < \varepsilon$

__Cauchy Sequence__

Motivation Can we find a condition equivalent to convergence without knowing the limit beforehand?

Definition A sequence (s_n) is Cauchy if $\forall \varepsilon > 0$, $\exists N$ s.t. for $m, n \in \mathbb{N}$, $m, n > N$ $\Rightarrow |s_m - s_n| < \varepsilon$.

Example $s_1 = 1.4$ where s_n = decimal expansion of $\sqrt{2}$ up to 10^{-n}.

$s_2 = 1.41$ $\Rightarrow m, n > N$, s_m and s_n agrees up to 10^{-N}

$s_3 = 1.414$ $\Rightarrow |s_m - s_n| < 10^{-N} < \varepsilon$ holds if $N = \log_{10} \frac{1}{\varepsilon}$.

Theorem (s_n) is Cauchy iff (s_n) is convergent

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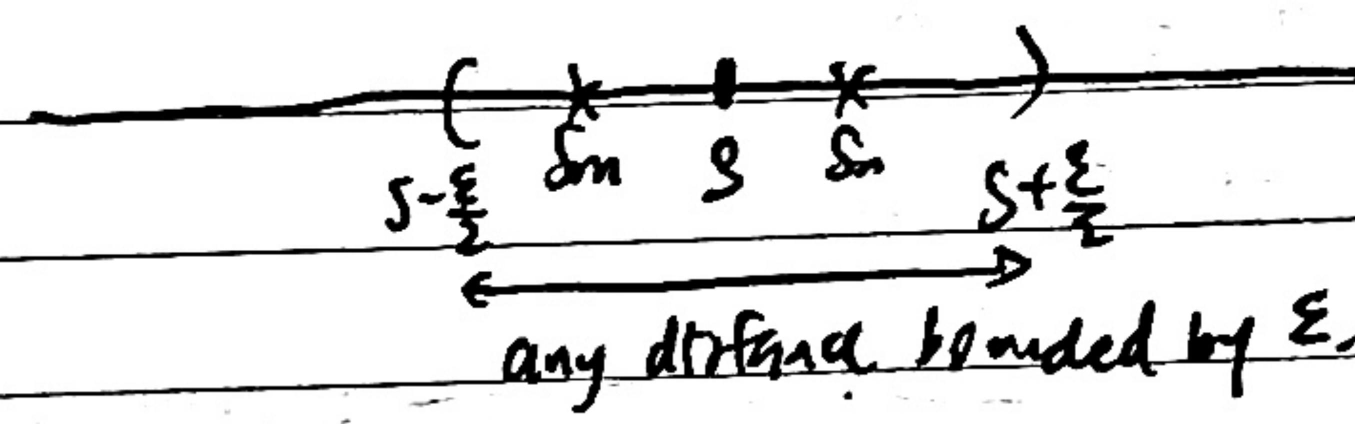
Proof ($\Leftarrow$) Assume (s_n) converges to a real number s. ($s \in \mathbb{R}$)

Take any $\varepsilon > 0$.



Since $\lim s_n = s$, $\exists N$ s.t.

$n > N \Rightarrow |s_n - s| < \frac{\varepsilon}{2}$.

$$\therefore \underset{m,n>N}{|s_m - s_n|} \underset{\Delta\text{-inequality}}{\leq} |s_m - s| + |s - s_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \therefore (s_n) \text{ Cauchy.}$$

($\Rightarrow$) Note we cannot use ε-N definition since we are unclear about the limit.

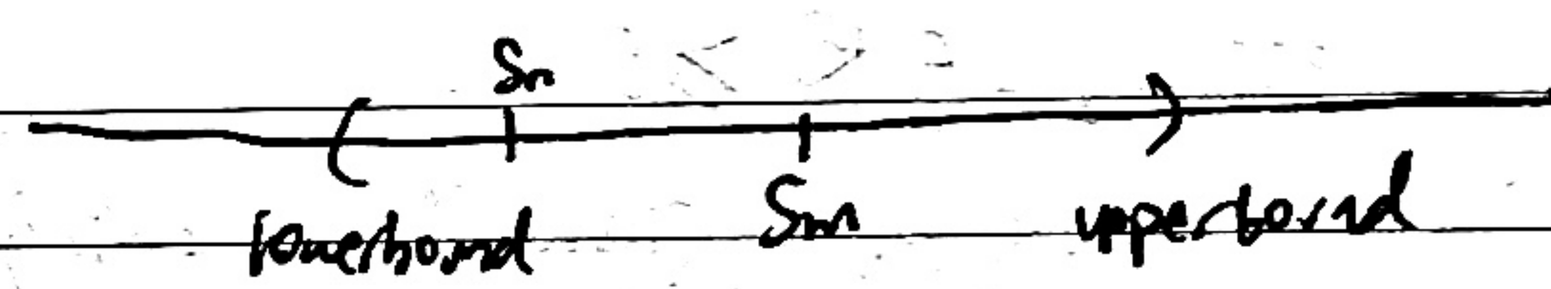
Suffices to show $\lim\inf s_n = \lim\sup s_n \in \mathbb{R}$.

Set $a_n = \inf\{s_n, s_{n+1}, \dots\}$

$b_n = \sup\{s_n, s_{n+1}, \dots\}$

To show a_n is equal b_n, show difference $\to 0$.

Take any $\varepsilon > 0$.



Since (s_n) Cauchy, $\exists N$ s.t. $m, n > N$, $|s_m - s_n| < \frac{\varepsilon}{3}$

Fix m (i.e. set $m = N+1$)

Then $n > N \Rightarrow |s_n - s_{N+1}| < \frac{\varepsilon}{3}$

$\therefore \{s_n, s_{n+1}, \dots\} \subseteq \left[s_{N+1} - \frac{\varepsilon}{3}, s_{N+1} + \frac{\varepsilon}{3}\right]$

$$\Rightarrow \boxed{s_{N+1} - \frac{\varepsilon}{3} \leq a_n \leq b_n \leq s_{N+1} + \frac{\varepsilon}{3}.}$$

$\Rightarrow (a_n)$ and (b_n) are bounded $\Rightarrow$ both $\lim\inf s_n$ and $\lim\sup s_n$ are real numbers.

$$|b_n - a_n| \leq \left(s_{N+1} + \frac{\varepsilon}{3}\right) - \left(s_{N+1} - \frac{\varepsilon}{3}\right) = \frac{2\varepsilon}{3} < \varepsilon$$

$$\therefore \lim (b_n - a_n) = 0 \Leftrightarrow \lim a_n = \lim b_n \Leftrightarrow \lim\inf s_n = \lim\sup s_n$$

Theorem Assume $s_n \neq 0 \ \forall n$. Then we have ~~lim inf~~

$$\lim\inf \left|\frac{s_{n+1}}{s_n}\right| \leq \lim\inf |s_n|^{\frac{1}{n}} \leq \lim\sup |s_n|^{\frac{1}{n}} \leq \lim\sup \left|\frac{s_{n+1}}{s_n}\right|$$

Good for analysing $(\ )^{\frac{1}{n}}$ sequences. Important theorem for series.

In particular, if $\lim \left|\frac{s_{n+1}}{s_n}\right|$ exists, then so does $\lim |s_n|^{\frac{1}{n}}$ and they are equal.

Example $t_n = (n!)^{\frac{1}{n}}$

Let $s_n = n!$

Here $\left|\frac{s_{n+1}}{s_n}\right| = \frac{(n+1)!}{n!} = n+1$. $\lim \left|\frac{s_{n+1}}{s_n}\right| = \infty$

$\Rightarrow \boxed{\lim t_n = \infty.}$

Proof: Know $\liminf |s_n|^{\frac{1}{n}} \le \limsup |s_n|^{\frac{1}{n}}$.

Suffices to show $\limsup |s_n|^{\frac{1}{n}} \le \limsup \left|\frac{s_{n+1}}{s_n}\right|$

Let $L = \limsup \left|\frac{s_{n+1}}{s_n}\right|$ (nonnegative real or $+\infty$)

If $L = \infty$, we are trivially done.

Assume $L \in \mathbb{R}_{\ge 0}$

Enough to show the following: $\forall \varepsilon > 0$, $\limsup \sqrt[n]{|s_n|} \le L+\varepsilon$

eventually all b_n here.

$L-\varepsilon \quad L \quad L+\varepsilon$

Take any $\varepsilon > 0$,

Set $b_n = \cancel{\sup \left|\frac{s_{n+1}}{s_n}\right|} = \sup\left\{\left|\frac{s_{n+1}}{s_n}\right|, \left|\frac{s_{n+2}}{s_{n+1}}\right|, \dots\right\}$, so $L = \lim b_n$.

$\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |b_n - L| < \varepsilon$

$\Rightarrow b_n < L+\varepsilon$

For $n > N$, $|s_n| = \left|\frac{s_n}{s_{n-1}}\right|\left|\frac{s_{n-1}}{s_{n-2}}\right| \cdots \left|\frac{s_{N+2}}{s_{N+1}}\right| |s_{N+1}|$

$< \cancel{L^{n-N-1}} (L+\varepsilon)^{n-N-1} |s_{N+1}| = (L+\varepsilon)^n C$ where $C = \frac{|s_{N+1}|}{(L+\varepsilon)^{N+1}}$ is constant.

$\Rightarrow |s_n|^{\frac{1}{n}} < (L+\varepsilon)\, C^{\frac{1}{n}}$

From basic examples, $\lim C^{\frac{1}{n}} = 1$ for constant c, $|c| > 0$.

$\Rightarrow \cancel{\lim |s_n|^{\frac{1}{n}} < L+\varepsilon}$ $\boxed{\limsup |s_n|^{\frac{1}{n}} < L+\varepsilon}$

$\therefore \boxed{\limsup |s_n|^{\frac{1}{n}} \le \limsup \left|\frac{s_{n+1}}{s_n}\right|}$