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MATH H110 LECTURE 6 NOTES

Quiz Confirmed

The axiom $1\vec{u} = \vec{u}$ cannot be derived from the remaining ones.
Consider scalar · vector = 0 ∀ scalar, vector. The other axioms are satisfied.

Basis & Dimensions

Basis

Given a $\mathbb{K}$ vector space V, a subset of vectors $S \subset V$ is a basis of V if every vector from V can be written uniquely as a linear combination of vectors from S.

can be → existence
uniquely → uniqueness

Example

For $\mathbb{K}[x]$, (i.e. polynomials in x $a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$)
Take $S = \{1, x, x^2, \dots x^n \dots\}$
For $\mathbb{K}[x,y]$, $S = \{x^i y^j, i \in \mathbb{Z}, j \in \mathbb{Z}\}$.

For $\mathbb{K}^n$ $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = x_1 \vec{e_1} + x_2 \vec{e_2} + \dots + x_n \vec{e_n}$
$\{\vec{e_1}, \vec{e_2}, \dots, \vec{e_n}\}$ forms a basis.

Definition

V is finite dimensional if it ~~has a~~ can be spanned by finitely many vectors. i.e. $\exists V \in V, |V| < \infty, V = \text{span } V$

Existence

Consider B, a subset of V. B can be any subset.
(subset → no repeated elements)
Denote $\text{Span } B = \{\text{all linear combinations of elements of } V\}$
It is clear $\text{Span } B$ is a subspace.

If B is a basis, $\text{span } B = V$.

Uniqueness Linear Independence

B is linearly independent if no vector can be represented as a linear combination of elements from ~~V~~ B in more than one way.

- no two distinct linear combinations of elements from B are equal to each other
- $\alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \dots + \alpha_k \vec{v_k} = \beta_1 \vec{v_1} + \beta_2 \vec{v_2} + \dots + \beta_k \vec{v_k}$
  $\Rightarrow \alpha_i = \beta_i \ \forall i$
- $\gamma_1 \vec{v_1} + \gamma_2 \vec{v_2} + \dots + \gamma_k \vec{v_k} = 0 \Rightarrow \gamma_i = 0 \ \forall i$
  (called trivial linear combination)
- $\vec{0}$ can be represented as a linear combination of elements from V in only the trivial way.

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- Every nontrivial linear combination $\gamma_1\vec{v_1} + \gamma_2\vec{v_2} + \cdots + \gamma_k\vec{v_k} \neq 0$ where at least one of $\gamma_i \neq 0$.

B is linearly dependent if B is not linearly independent

- Some nontrivial linear combination $\gamma_1\vec{v_1} + \gamma_2\vec{v_2} + \cdots + \gamma_k\vec{v_k} = 0$ $\quad v_i \in B$
  (at least one of $\gamma_i \neq 0$)

  In particular, $\{\vec{0}\}$ is always linearly dependent.
  $\{\vec{u}, \vec{v}\}$ is linearly dependent if $\vec{v} = k\vec{u} \Rightarrow k\cdot\vec{u} - 1\cdot\vec{v} = 0$.

- WLOG, $\gamma_k \neq 0$.
  Then $\vec{v_k} = \left(-\frac{\gamma_1}{\gamma_k}\right)\vec{v_1} + \left(-\frac{\gamma_2}{\gamma_k}\right)\vec{v_2} + \cdots + \left(-\frac{\gamma_{k-1}}{\gamma_k}\right)\vec{v_{k-1}}$

  i.e. if one of the vectors in B can be written as a linear combination of the others

(1) If V' contains a linearly dependent subset, then V' is linearly dependent
(2) If V is linearly independent, then any subset of V is linearly independent.

V is finite dimensional if it can be spanned by finitely many vectors.
i.e. $\mathcal{V} \subset V$, $|\mathcal{V}| < \infty$, $V = \text{Span}\,\mathcal{V}$.

In $\mathcal{V}$, we can find a linearly independent subset which still spans V
so it is a finite basis.

Suppose $\{\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}\}$ is such a basis, ordered in such manner.
$\forall \vec{v} \in V$,

$$\vec{v} = x_1\vec{v_1} + x_2\vec{v_2} + \cdots + x_n\vec{v_n} \qquad (x_1, x_2, \ldots x_n) \text{ unique.}$$

$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is the coordinate of $\vec{v}$ w.r.t. the basis $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix} \qquad \lambda\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{pmatrix}$$

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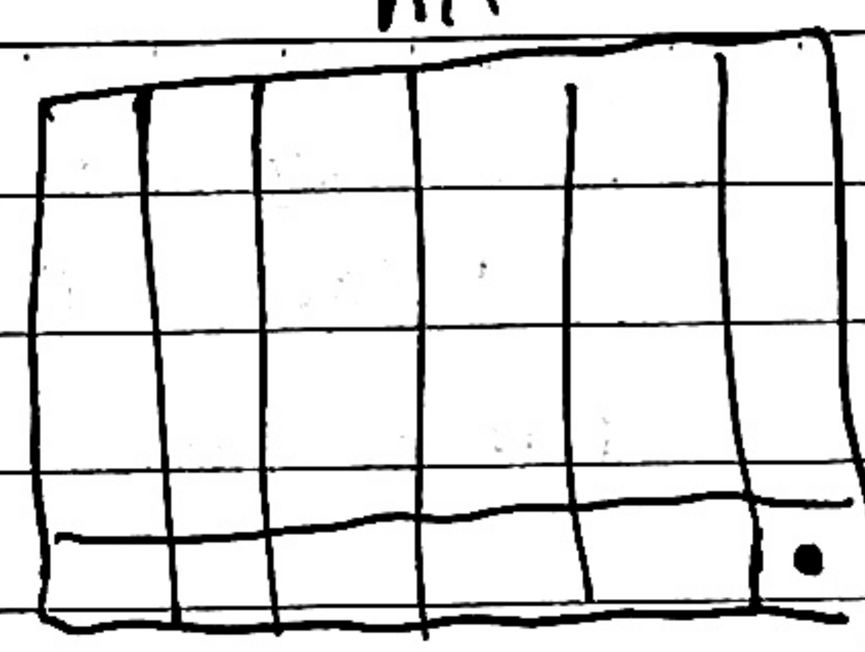
n+1



Consider $\mathbb{K}^n \longmapsto V$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longmapsto x_1\vec{v_1} + x_2\vec{v_2} + \cdots + x_n\vec{v_n}$$

i.e. $\vec{e_1} \mapsto \vec{v_1}$ $\quad \vec{e_i} \mapsto v_i$

One-to-one correspondence that respects the piecewise operations

$\Rightarrow$ Isomorphism. $\quad \mathbb{K}^n \xrightarrow{\cong} V$

Idea: Induction

Lemma A set of n+1 vectors in $\mathbb{K}^n$ is linearly dependent.

Proof When $n=1$, $\mathbb{K}^1 = \mathbb{K}$. Any two scalars are ~~linearly~~ proportional dependent. ✓

Assume that any n vectors in $\mathbb{K}^{n-1}$ are linearly dependent, prove any n+1 vectors in $\mathbb{K}^n$ are linearly dependent.

If all elements in the last row is 0, the vectors are equivalent to $\mathbb{K}^{n-1}$. Then we are done by induction hypothesis.

Otherwise, suppose at least one of the entry in the last row is nonzero. WLOG, let it be the last column. We can take suitable coefficients to make the remaining elements in the last row. ($\mathbb{K}$ is a field, so it is possible to do so since all nonzero elements are invertible)

$$\vec{u_1}, \vec{u_2}, \ldots, \vec{u_n} \longrightarrow \vec{v_1} - \alpha_1\vec{v_{n+1}}, \vec{v_2} - \alpha_2\vec{v_{n+1}}, \ldots, \vec{v_n} - \alpha_n\vec{v_{n+1}}$$

and $\vec{u_1}, \vec{u_2}, \ldots \vec{u_n} \in \mathbb{K}^{n-1}$

$\therefore \exists \beta_1, \beta_2, \ldots \beta_n$ not all 0 s.t. $\beta_1\vec{u_1} + \beta_2\vec{u_2} + \cdots + \beta_n\vec{u_n} = 0$ by induction hypothesis

$$\Rightarrow \beta_1\vec{v_1} + \beta_2\vec{v_2} + \cdots + \beta_n\vec{v_n} + (-\alpha_1\beta_1 - \alpha_2\beta_2 - \cdots - \alpha_n\beta_n)\vec{v_{n+1}} = 0.$$

Some of β_i are nonzero $\Rightarrow$ non-trivial combination $= 0 \Rightarrow$ linearly dependent.

Corollary (1) Any set of $m > n$ vectors in $\mathbb{K}^n$ is linearly dependent.

(2) $\mathbb{K}^n$ and $\mathbb{K}^m$ are not isomorphic unless $m = n$

(3) Every finite dimensional vector space is isomorphic to exactly one of $\mathbb{K}^n$ (equivalence defined to isomorphism)

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(4) In a finite dimensional vector space, every basis has the same number of elements and equal dim V

~~Otherwise, isomophic to distinct~~ ~~$\mathbb{K}^m$, $\mathbb{K}^n$ $(m \neq n)$.~~

(5) Two finite dimensional vector spaces are isomorphic if and only if they have the same dimension.