NO. ....

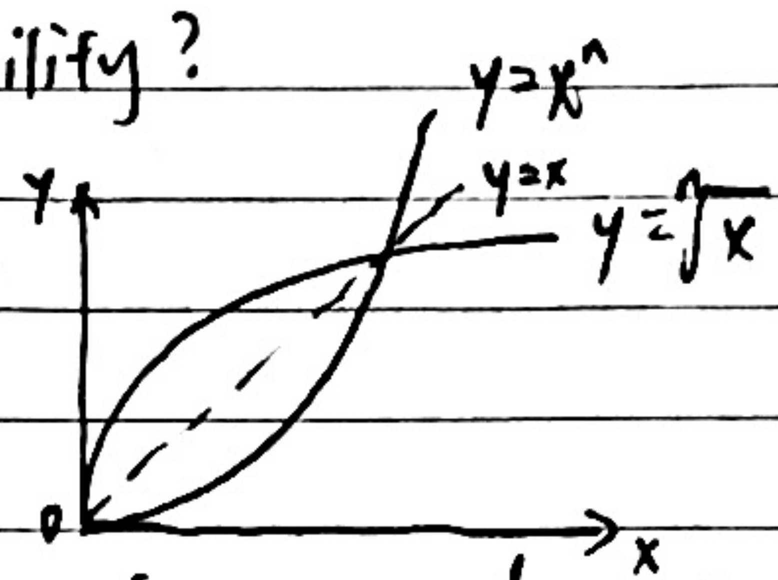
DATE: 11/04/2021

MATH 104 LECTURE NOTES 19 — MEAN VALUE THEOREM — APPLICATIONS

Qn. How to find $(\sqrt[n]{x})'$? And how to show differentiability?

From calculus, $(\sqrt[n]{x})' = \frac{1}{n}(x)^{-\frac{n-1}{n}} = \frac{1}{n}x^{\frac{1}{n}-1}$

$\sqrt[n]{x}$ is the inverse function of x^n.



Theorem [Reading Assignment] 18.6, 18.3, 18.4

Let I be an open interval and $f: I \to \mathbb{R}$ be one-to-one and continuous. Set $J = f(I)$. Then J is another open interval and $f^{-1}(J)$ is defined $f^{-1}: J \to \mathbb{R}$ and continuous.

Take $a \in I$. Assume f is differentiable at a. Then f^{-1} is differentiable at $f(a)$ with the derivative being $(f^{-1})'(f(a)) = \frac{1}{f'(a)}$

Note: $f: I \to \mathbb{R}$ is one-to-one and continuous $\Rightarrow$ f is strictly increasing or strictly decreasing.

Otherwise, can find $x_1 < x_2 < x_3$ s.t. $f(x_1) < f(x_2) > f(x_3)$

or $f(x_1) > f(x_2) < f(x_3)$

Then, can use intermediate value theorem.

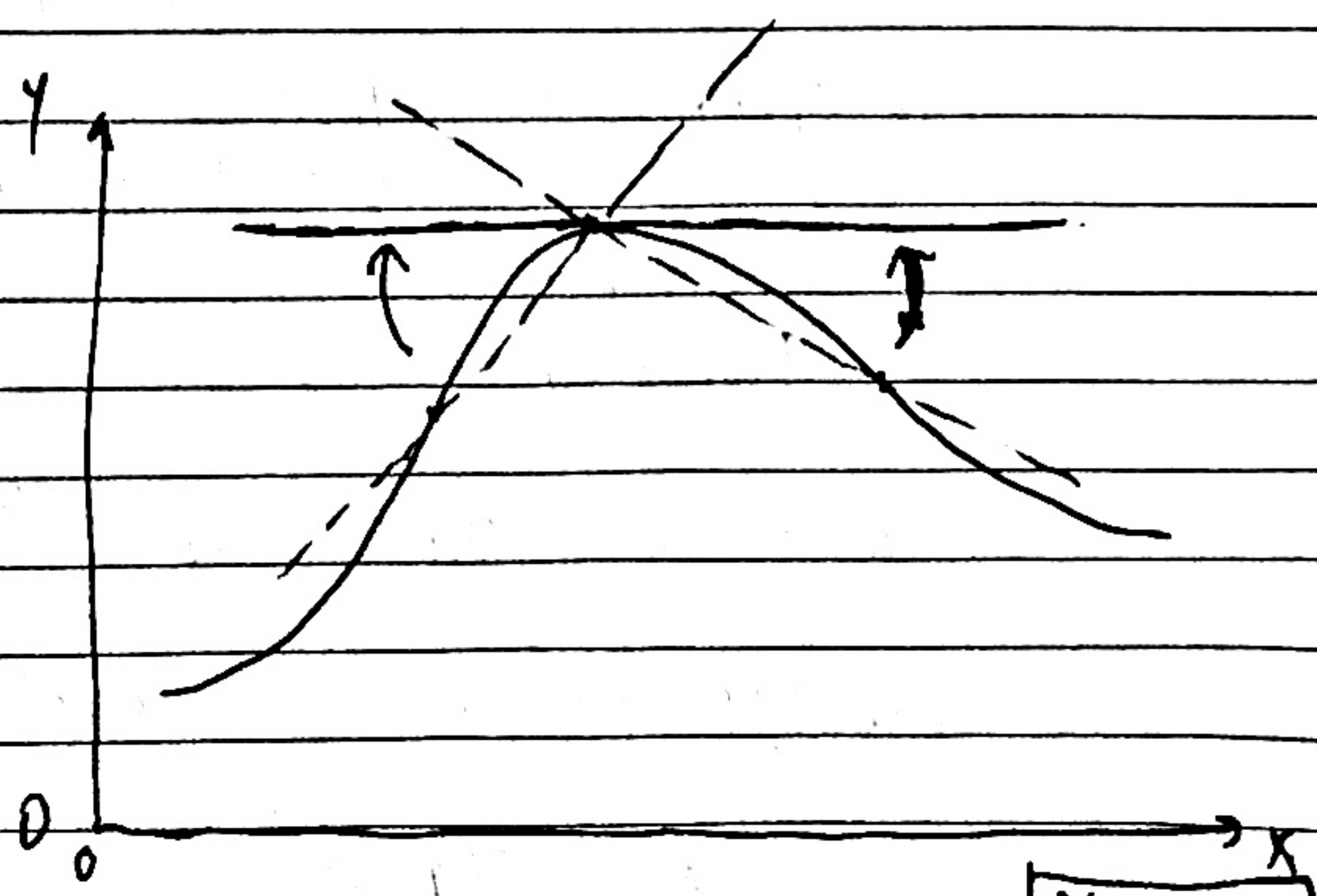
Example: $f(x) = x^n \Rightarrow f'(x) = nx^{n-1} \Rightarrow (f^{-1})'(f(x)) = \frac{1}{f'(x)} = \frac{1}{nx^{n-1}}$

$(f^{-1})'(x) = \frac{1}{n \cdot x^{\frac{n-1}{n}}} = \frac{1}{n}x^{-\frac{n-1}{n}} = \boxed{\frac{1}{n}x^{\frac{1}{n}-1}}$ as required

Theorem: Let $f: (a,b) \to \mathbb{R}$ be a differentiable function. Assume $c \in (a,b)$ attains max (or min). Then c is a critical point, i.e. $f'(c) = 0$.

Example: Let $f: [a,b] \to \mathbb{R}$ be a continuous and differentiable function on (a,b). Then candidates of max/min are a, b or critical points c, where $c \in (a,b)$, $f'(c) = 0$.

Since any continuous function on closed interval always admits max/min, they always exist.



Proof: For $x \neq c$.

Assume f attains max at c.

$f(x) - f(c) \leq 0$

$$\Rightarrow \frac{f(x)-f(c)}{x-c} = \begin{cases} \leq 0 & x > c \\ \geq 0 & x < c \end{cases}$$

$$0 \leq \lim_{x \to 0^-} \frac{f(x)-f(c)}{x-c} = f'(c) = \lim_{x \to 0^+} \frac{f(x)-f(c)}{x-c} \leq 0$$

$\therefore \boxed{f'(c) = 0}$

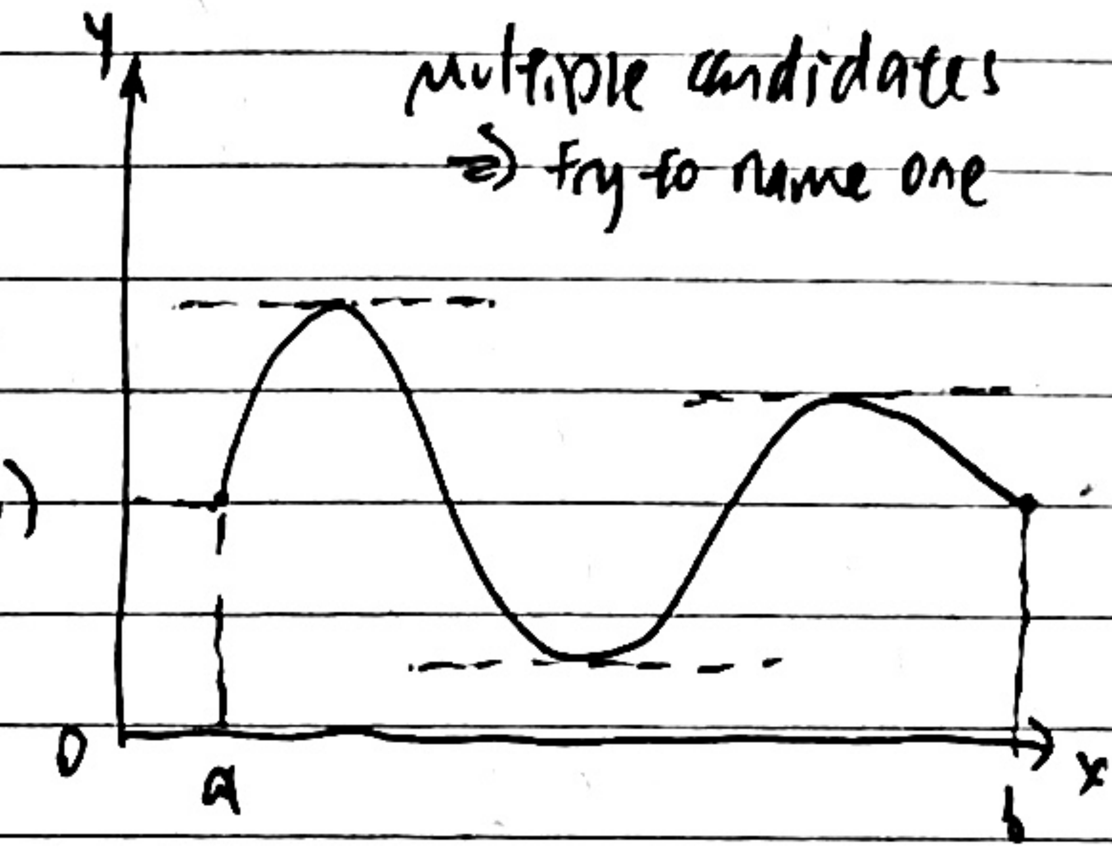
This limit exists by the assumption of differentiability of f — both limits can be justified by the sequential definition in $(-\delta, 0)$ and $(0, \delta)$

NO: .... DATE: ....

Mean Value Theorem

Rolle's Theorem

Let $f:[a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ and differentiable on (a,b)
If $f(a) = f(b)$, then $\exists c \in (a,b)$ s.t. $f'(c) = 0$.



Proof: Since f is continuous on $[a,b]$, it admits both maximum & minimum on $[a,b]$.

Case (1): a is both max and min
$\Rightarrow f(a) \leq f(x) \leq f(a) \quad \forall x \Rightarrow f(x) = f(a) \quad \forall x$

$\therefore f(x)$ is a constant function.
$f'(x) = 0 \quad \forall x$ in (a,b) ⇒ simply choose any point in the interior.

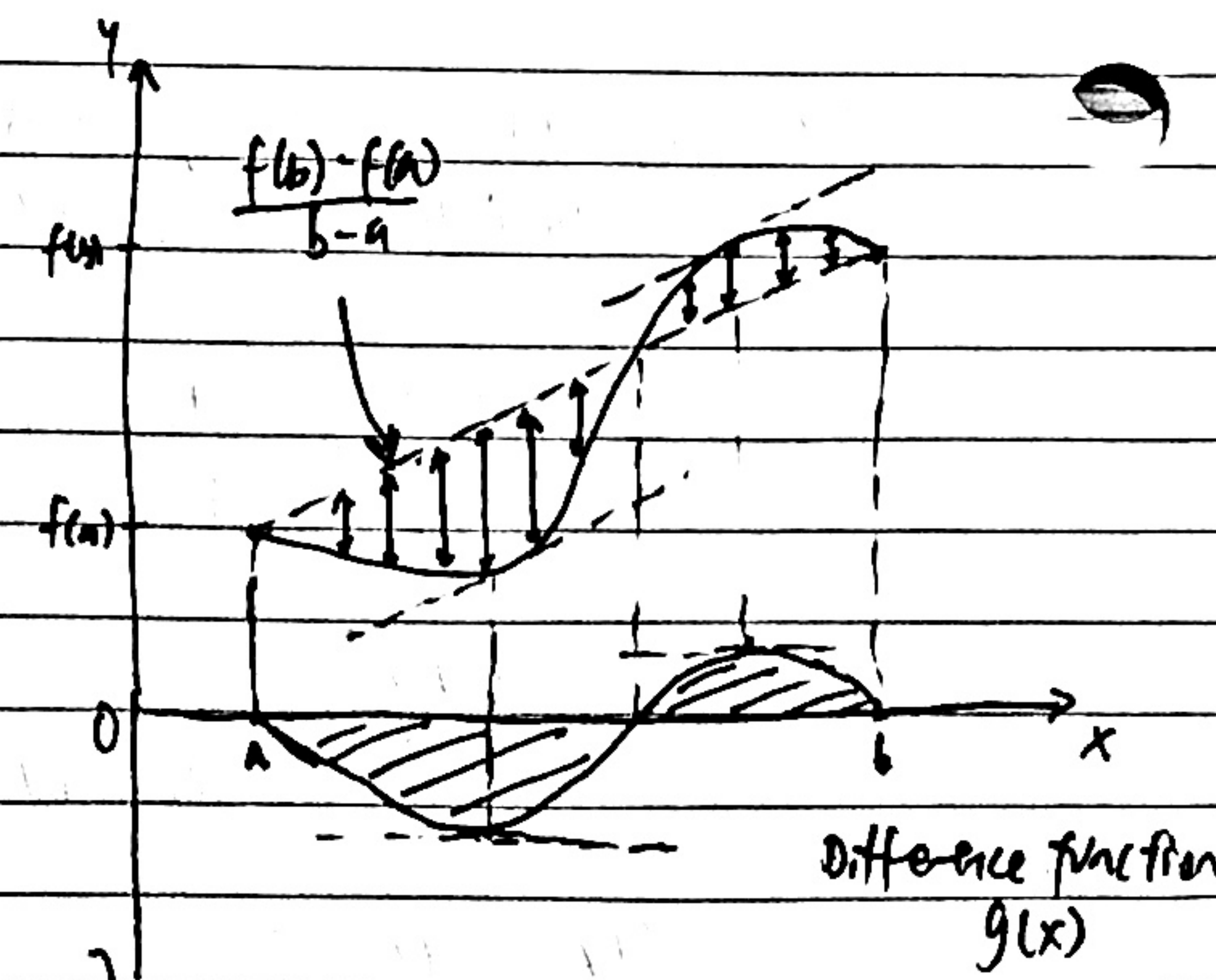
Case (2): Otherwise, $\exists c \in (a,b)$ s.t. c is either maximum or minimum point
(since $f(a) = f(b)$ so a, b can't be both maximum or minimum by assumption)
$\Rightarrow f'(c) = 0$ by earlier theorem

$\therefore \boxed{\exists c \in (a,b) \text{ s.t. } f'(c) = 0.}$

Mean Value Theorem

Let $f:[a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b). Then $\exists c \in (a,b)$ s.t.
$$f'(c) = \frac{f(b)-f(a)}{b-a}$$
If $f(a) = f(b)$, then $\exists c \in (a,b)$ s.t. $f'(c) = 0$ [reduction to Rolle's Theorem]

Proof: Consider the difference function $g(x)$ defined by:
$$g(x) = f(x) - \underbrace{\left[\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right]}_{\text{(equation for the line)}}$$



Then $g(x)$ is a function on the same interval that is also continuous and differentiable on the interior (a,b)
$g:[a,b] \to \mathbb{R}$

$g(a) = 0. \quad g(b) = f(b) - \left[\frac{f(b)-f(a)}{b-a}(b-a) + f(a)\right] = 0$

Since $g(a) = g(b) = 0$, by Rolle's theorem, $\exists c \in (a,b)$ s.t. $g'(c) = 0$.

Since $g(x) = f(x) - \left[\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right]$, $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$
$\Rightarrow f'(c) = g'(c) + \frac{f(b)-f(a)}{b-a} = \frac{f(b)-f(a)}{b-a}$.
$\therefore \boxed{\exists c \in (a,b) \text{ s.t. } f'(c) = \frac{f(b)-f(a)}{b-a}}$

NO. DATE:

Applications of MVT

Let I be an open interval and f be a differentiable function $f: I \to \mathbb{R}$.

(1) If $f'(x) = 0 \ \forall x \in I$, then f is constant.
($\Rightarrow$ anti-derivatives differ by a constant).

Proof: Take any two points $a<b \in I$. Suffices to show $f(a) = f(b)$.
Applying MVT to the restriction of f on $[a,b]$,
$\exists c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a} = 0 \Rightarrow \boxed{f(b) = f(a)}$

(2) If $f'(x) > 0 \quad \forall x \in I$, then $f(x)$ is strictly increasing.
If $f'(x) \geq 0 \quad \forall x \in I$, then $f(x)$ is increasing.
If $f'(x) < 0, \ \forall x \in I$, then $f(x)$ is strictly decreasing
If $f'(x) \leq 0 \quad \forall x \in I$, then $f(x)$ is decreasing

derivatives determine the local behaviour of functions.

Proof: Assume $f(x) > 0 \ \forall x \in I$
Suffices to show $f(a) < f(b)$ for $\forall a < b \in I$.

By MVT, $\exists c \in (a,b)$ s.t. $\frac{f(b)-f(a)}{b-a} = f'(c) > 0$ by ~~the choice~~ assumption

Since $b-a > 0$, $f(b) - f(a) > 0 \Rightarrow \boxed{f(b) > f(a)}$

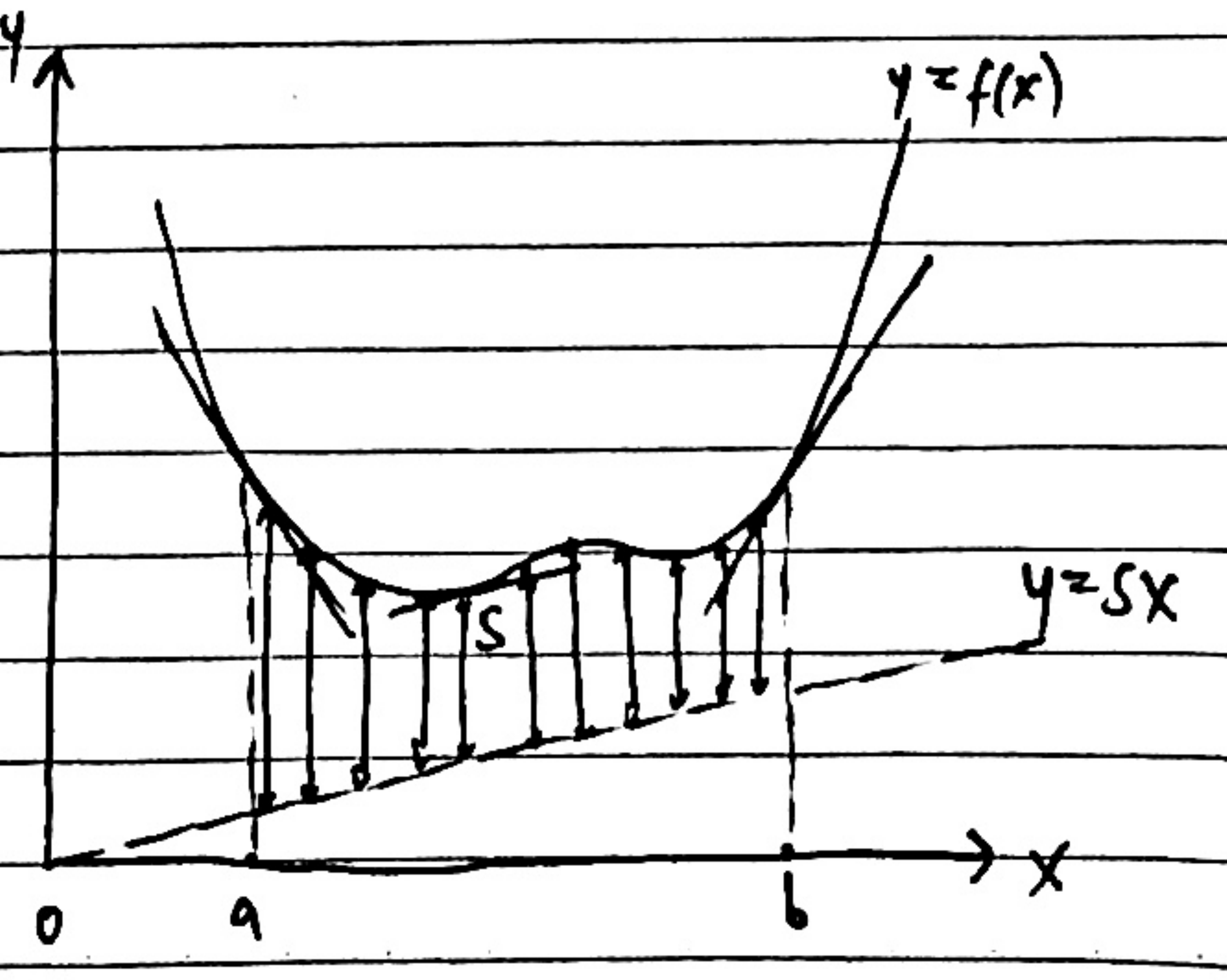
Mean value theorems help to determine the relationship between 2 points. Consider restriction of MVT on a smaller interval.

Intermediate Value Theorem for Derivatives

Let I be an open interval and f be a differentiable function on I.
$f: I \to \mathbb{R}$. Take $a < b$ in I.

If $s \in \mathbb{R}$ satisfies $f'(a) < s < f'(b)$, then $\exists c \in (a,b)$ s.t. $f'(c) = s$.
(same holds for $f'(b) < s < f'(a)$)

Proof: Consider $g(x) = f(x) - sx$ on I. This is differentiable with $g'(x) = f'(x) - s$.
Suffices to find $c \in (a,b)$ s.t. $g'(c) = 0$.



Since g is continuous on $[a,b]$, g attains a minimum say at $c \in [a,b]$. Suffices to show $c \neq a, b$

$g'(a) = f'(a) - s < 0$ by assumption
$g'(b) = f'(b) - s > 0$ by assumption

NO:.............................. DATE:..............................

$g'(a) = f'(a) - s < 0$ by assumption.

Since $g'(a) = \lim_{x \to a^+} \frac{g(x) - g(a)}{x - a} < 0$

In particular, $\exists\, d \in (a,b)$ s.t. $\frac{g(d) - g(a)}{d - a} < 0 \Rightarrow g(d) - g(a) < 0$
$\Rightarrow g(d) < g(a)$

So a cannot be a minimum point.

Similarly, can prove b is not a minimum point of g.

$g'(b) = \cancel{g'(b)}\, f'(b) - s > 0$ by assumption.
Since $g'(b) = \lim_{x \to b^-} \frac{g(x) - g(b)}{x - b} > 0$, ~~in~~ particular, $\exists\, d \in (a,b)$ s.t. $\frac{g(d) - g(b)}{d - b} > 0$
$\Rightarrow g(d) - g(b) < 0$ (since $d < b$) $\Rightarrow g(d) < g(b) \Rightarrow b$ cannot be a minimum point of g.

Hence $\boxed{\exists\, c \in (a,b) \text{ s.t. } g'(c) = 0.}$