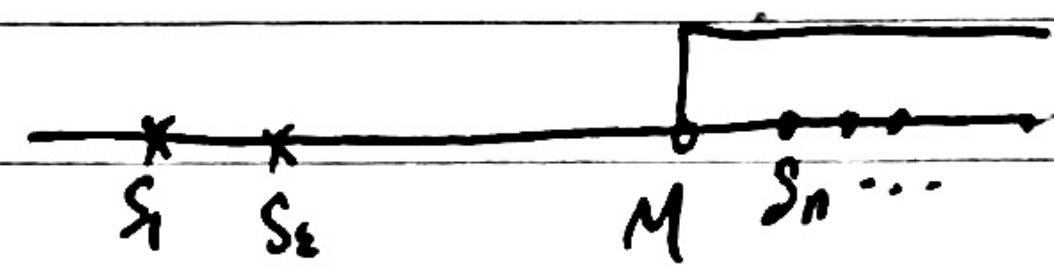
No.: Date: 09/09/2021

MATH 104 LECTURE NOTES 5 (MONOTONE SEQUENCES)

$\lim S_n = \infty$



__Definition__: Say (s_n) diverges to ∞ ($\lim s_n = \infty$) if for every $M>0$, there exists some $N \in \mathbb{R}$ s.t. $n > N \Rightarrow s_n > M$.

__Exercise__ $s_n = \dfrac{n^2-2n+3}{4n+5}$

My workings: Let $M>0$ and set $N = \max(5, 10M)$. $\forall n > N$

Then $s_n = \dfrac{n^2-2n+3}{4n+5} > \dfrac{\frac{n^2}{2}}{5n} = \dfrac{n}{10} > \dfrac{N}{10} \geq \dfrac{10M}{10} = M$

Hence $\boxed{\lim s_n = \infty}$

__Formal Proof__:

Take any $M>0$. Set $N = \max(5, 10M)$

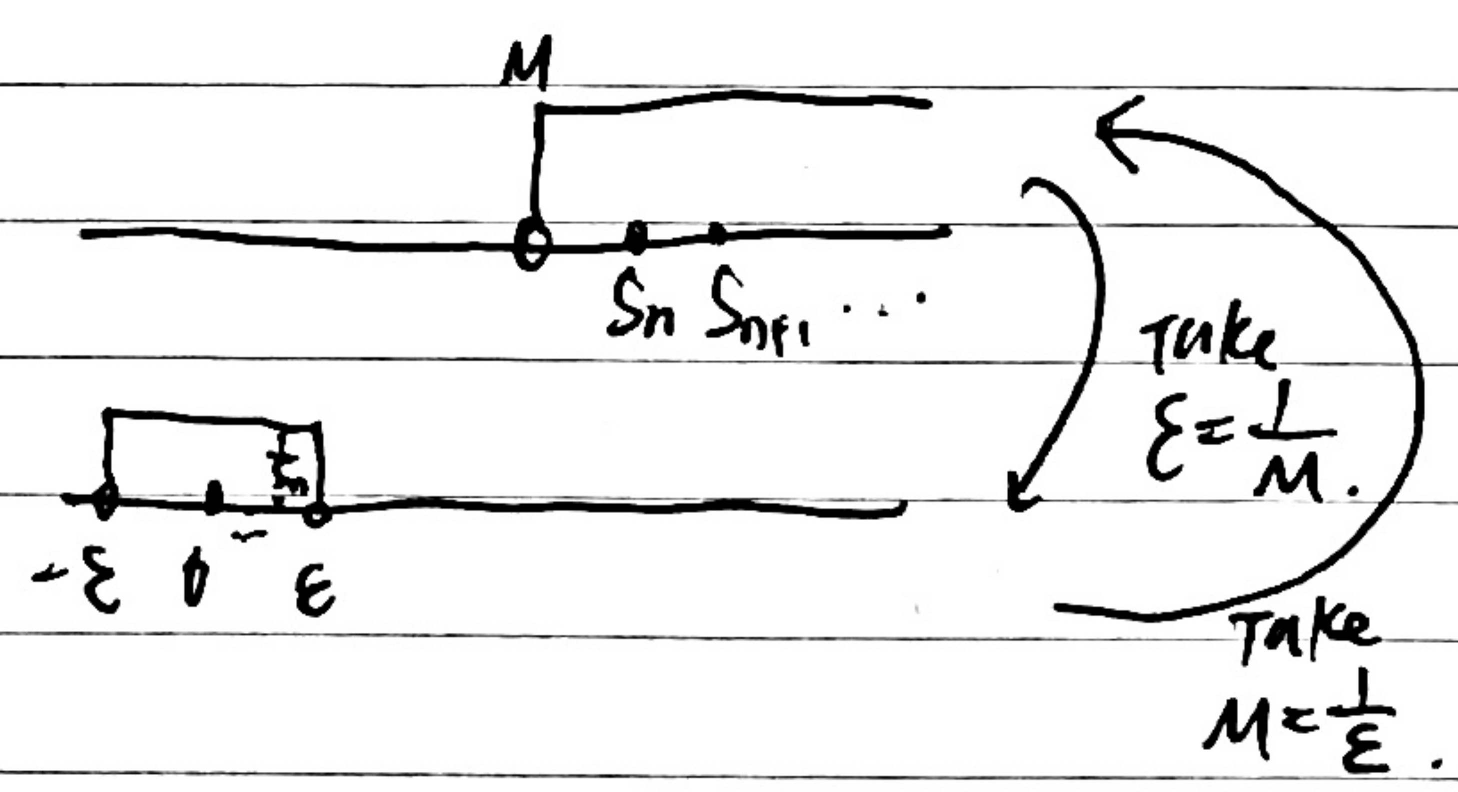
Then $n > M \Rightarrow s_n = \dfrac{n^2-2n+3}{4n+5} \underset{\uparrow}{\geq} \dfrac{\frac{n^2}{2}}{4n+5} \underset{\uparrow}{\geq} \dfrac{\frac{n^2}{2}}{5n} = \dfrac{n}{10} > \dfrac{N}{10} \geq \dfrac{10M}{10} = M$.

$\frac{n^2}{2} - 2n + 3 > 0$ if $n > 4$ $\qquad$ $4n+5 \leq 5n$ when $n \geq 5$

$\therefore \boxed{\lim s_n = \infty}$

Reading Assignment ☆

Basic properties of $\lim s_n = \pm\infty$



__Proposition__: For (s_n) with $s_n > 0$ $\forall n$,

$\lim s_n = \infty$ iff $\lim \frac{1}{s_n} = 0$.

Basic Examples

(1) For $p > 0$, $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

(2) For $-1 < a < 1$ ($|a| < 1$), $\lim_{n\to\infty} a^n = 0$.

(3) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

(4) For any positive real a, $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$

Review of Binomial Theorem

For $x>0$, $(1+x)^n > \frac{n(n-1)}{2}x^2$.

Bernoulli's theorem: $|x|<1$ $(1+x)^n \geq 1+nx$.

(3) $\lim n^{\frac{1}{n}} = 1$.

Set $S_n = n^{\frac{1}{n}} - 1$, we know $n^{\frac{1}{n}} > 1$ since $n>1 \Rightarrow n^{\frac{1}{n}} > 1$. So $S_n > 0$

It suffices to show $\lim S_n = 0$ by Limit Theorem

Since $n^{\frac{1}{n}} = 1 + S_n$

$$\Rightarrow n = (1+S_n)^n \underset{(n\geq 2)}{\geq} \frac{n(n-1)}{2} S_n^2 \Rightarrow \frac{2}{n-1} \geq S_n^2 \Rightarrow \sqrt{\frac{2}{n-1}} \geq S_n$$

Since $\sqrt{\frac{2}{n-1}} \geq S_n > 0$. By squeeze theorem $\boxed{\lim S_n = 0}$

Since $\pm\sqrt{\frac{2}{n-1}} \to 0$ $(n_{\geq 2} \to \infty)$, we get $\lim S_n = 0$ by squeeze lemma.

(4) For $a>0$, $\lim\limits_{n\to\infty} a^{\frac{1}{n}} = 1$

Case I: $(a \geq 1)$

By the Archimedean property, $\exists n_0 \in \mathbb{N}$ s.t. $1 \leq a \leq n_0$.

Then for $n > n_0$, $1 \leq a^{\frac{1}{n}} \leq n_0^{\frac{1}{n}} < n^{\frac{1}{n}}$

By squeeze lemma, since $\lim 1 = 1$, $\lim n^{\frac{1}{n}} = 1$, $\boxed{\lim a^{\frac{1}{n}} = 1}$ for $a \geq 1$.

Case II $(a<1)$

Note $\frac{1}{a} > 1$. By case I, $\lim \left(\frac{1}{a}\right)^{\frac{1}{n}} = 1. = \lim \frac{1}{a^{\frac{1}{n}}}$

Since $\lim \frac{1}{a^{\frac{1}{n}}} = 1$, by Limit Theorem, $\boxed{\lim a^{\frac{1}{n}} = \frac{1}{1} = 1.}$

First midterm in 2weeks!!

Monotone Sequences

Definition Say (S_n) is increasing if $S_{n+1} \geq S_n$ $\forall n$.

Say (S_n) is monotone if it is either increasing or decreasing.

Theorem: Given any increasing sequence (S_n)

(1) If (S_n) is bounded above, then (S_n) converges.

(2) If (S_n) is not bounded above, then (S_n) diverges to infinity.

Similar statement holds for decreasing sequences.

<u>Consequence:</u> If (s_n) is monotone (i.e. either increasing or decreasing), then $\lim s_n$ is defined (either as a real number or $\pm\infty$)

(s_n) converges iff (s_n) is bounded.

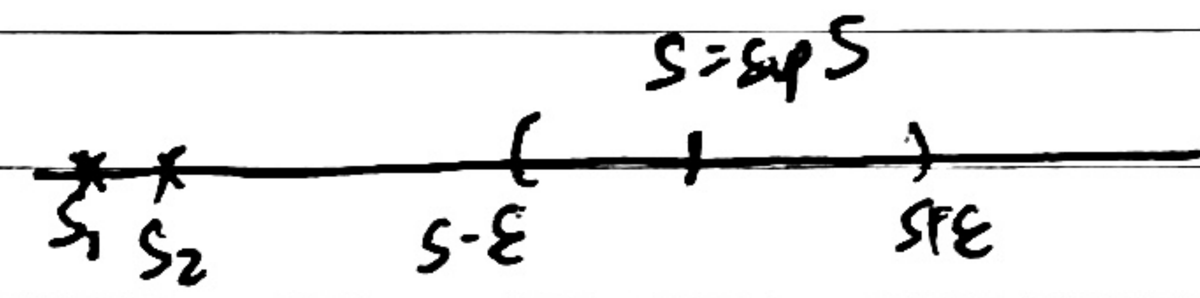
<u>Proof of Theorem</u>

Set $S = \{s_1, s_2, \dots\} \neq \phi$.

By assumption, S is nonempty and is bounded above.

By the completeness axiom, $\sup S$ exists and $\in \mathbb{R}$

Let $s = \sup S$.



Take any $\varepsilon > 0$.

Since $s - \varepsilon < s = \sup S$, $\exists N \in \mathbb{N}$ s.t. $s_N > s - \varepsilon$

Since (s_n) is increasing, $\forall n > N$, $s_n > s_N > s - \varepsilon$. At the same time, S is the supremum $\Rightarrow s_n < s$

$\therefore |s_n - s| < \varepsilon \Rightarrow \boxed{\lim s_n = s = \sup S}$

<u>Example</u>

Define (s_n) by $s_1 = 4$, $s_{n+1} = \dfrac{s_n^2 + 4}{2 s_n}$

$$s_{n+1} = \frac{s_n^2+4}{2s_n} < s_n \Leftrightarrow s_n^2 + 4 < 2 s_n^2 \Leftrightarrow 4 < s_n^2 \Leftrightarrow 2 < s_n$$ which can be proven by induction

$s_1 > 4 \Rightarrow s_1 > 2 \Rightarrow s_2 > 2 \Rightarrow \dots$

<u>Formal</u>

$\cancel{s_{n+1} = \frac{s_n^2+4}{2s_n} \Rightarrow \lim s_{n+1} = \lim \frac{s_n^2+4}{2s_n}}$ $\Rightarrow \lim s_{n+1} = \lim \frac{s_n}{2} + \lim \frac{2}{s_n}$

$s = \frac{s}{2} + \frac{2}{s}$ (by limit theorem)

$\Rightarrow \frac{s}{2} = \frac{2}{s} \Rightarrow s = 2$