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MATH H110 (lecture 2)

Quadratic curves

satisfies $ax^2 + 2bxy + cy^2 + dx + ey + f = 0.$

What is the simplest form to which you can rotate / translate.

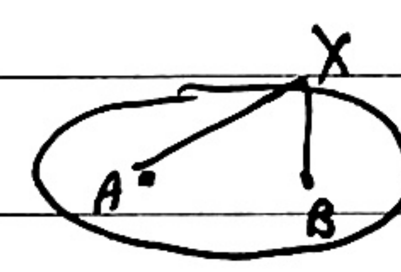
Rotations are linear (consider parallelograms)

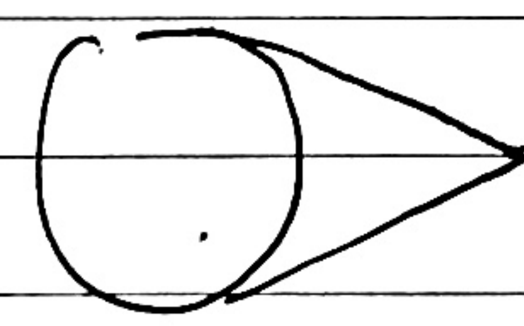
$$x = \alpha X + \beta Y + \mu$$
$$y = \gamma X + \delta Y + \nu$$

$$\underbrace{\alpha X + \beta Y}_{\text{rotation}} + \underbrace{\mu}_{\text{translations}}$$

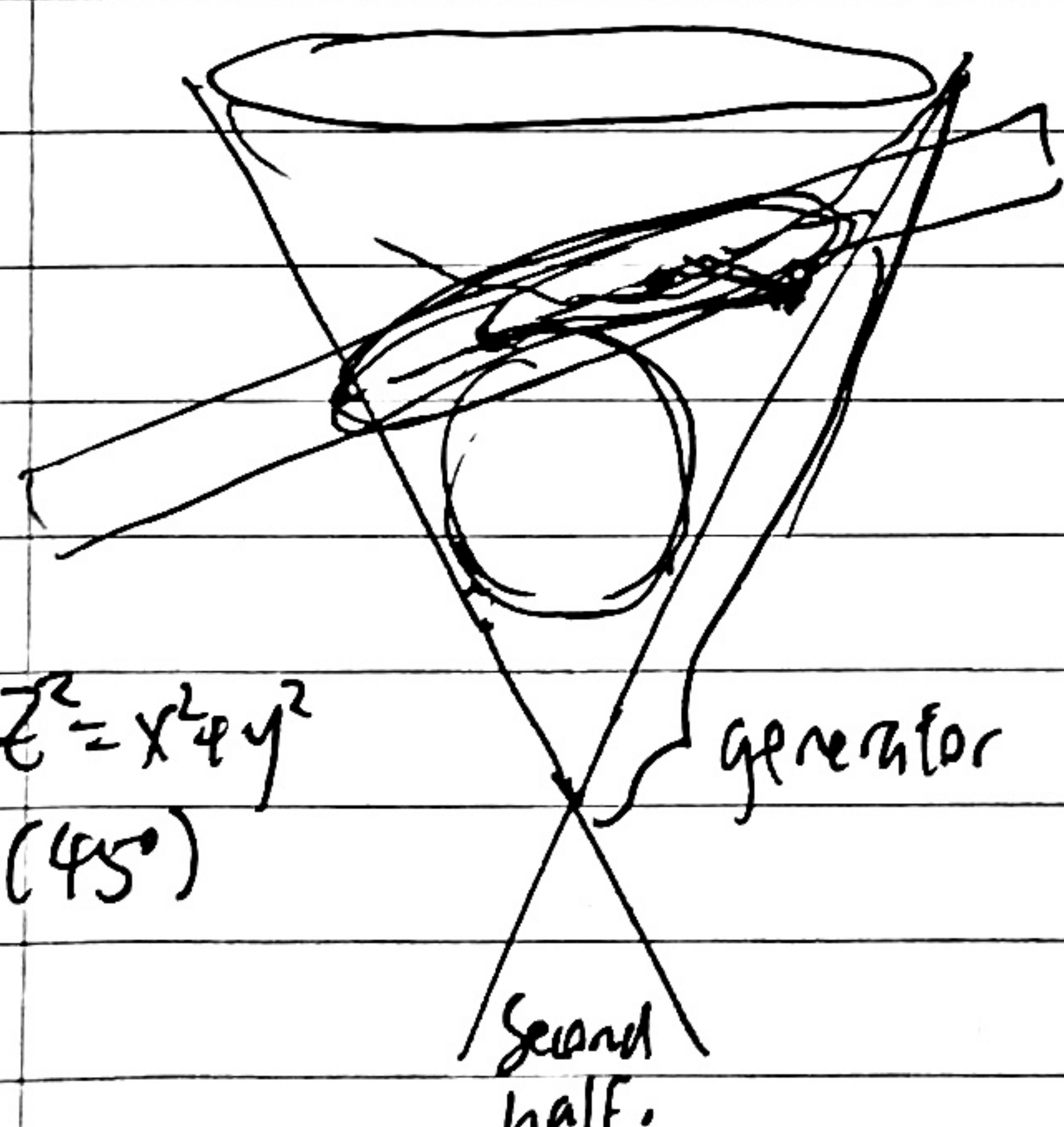
then after substitution, the quadratic curves are still quadratic.

Conic Sections

Ellipse  $|AX| + |BX| = \text{constant}.$

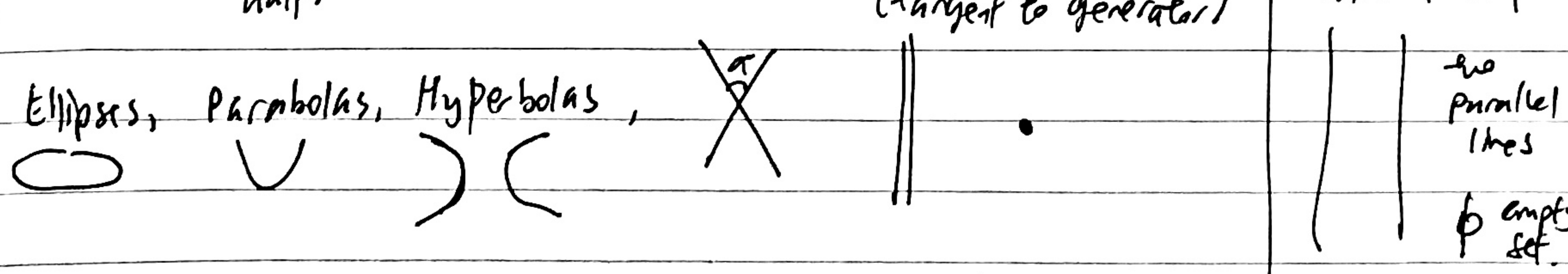


all tangents are of same length.



$z^2 = x^2 + y^2$
$(45°)$

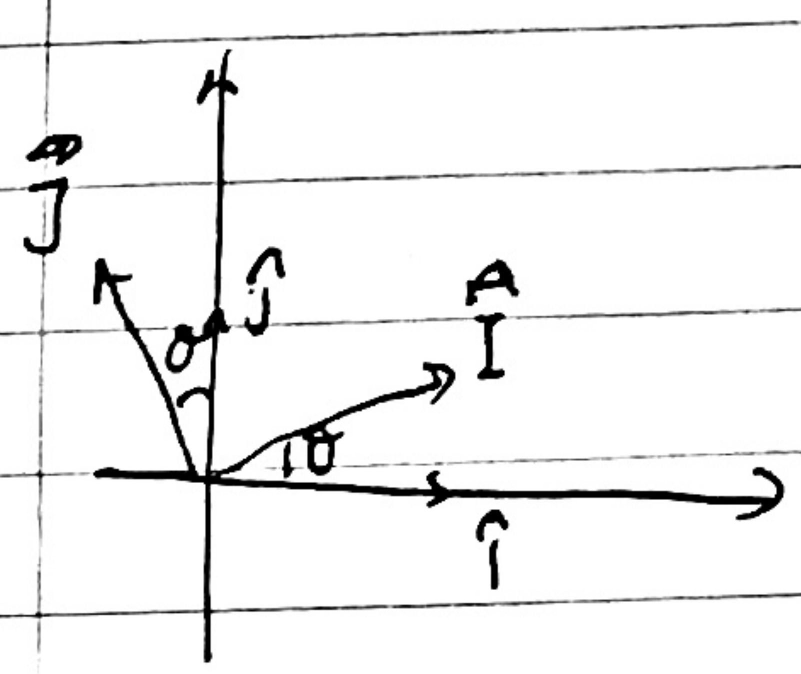
Consider any point X on the ellipse. The tangents to the spheres sum up to the generatrix from that point on X.
= distance along the generatrix between the balls



usually refer to homogenous objects

$ax^2 + 2bxy + cy^2$: quadratic forms

$$x\hat{i} + y\hat{j} = X\hat{I} + Y\hat{J}$$



$$\hat{I} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$
$$\hat{J} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}.$$

$$\therefore X\hat{I} + Y\hat{J} = X(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) + Y(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$
$$= (X\cos\theta - Y\sin\theta)\hat{i} + (X\sin\theta + Y\cos\theta)\hat{j}.$$

$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}}$$

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$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

Find θ such that it benefits us.
Motivation: want to get rid of cross terms XY.

$$a(X\cos\theta - Y\sin\theta)^2 + 2b(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) + c(X\sin\theta + Y\cos\theta)^2$$

$$= X^2 [\quad \cdots \quad]$$

$$Y^2 [\quad \cdots \quad]$$

$$+ XY [\cancel{2aXY} - 2a\cos\theta\sin\theta + 2b(\cos^2\theta - \sin^2\theta) + 2c\cos\theta\sin\theta].$$

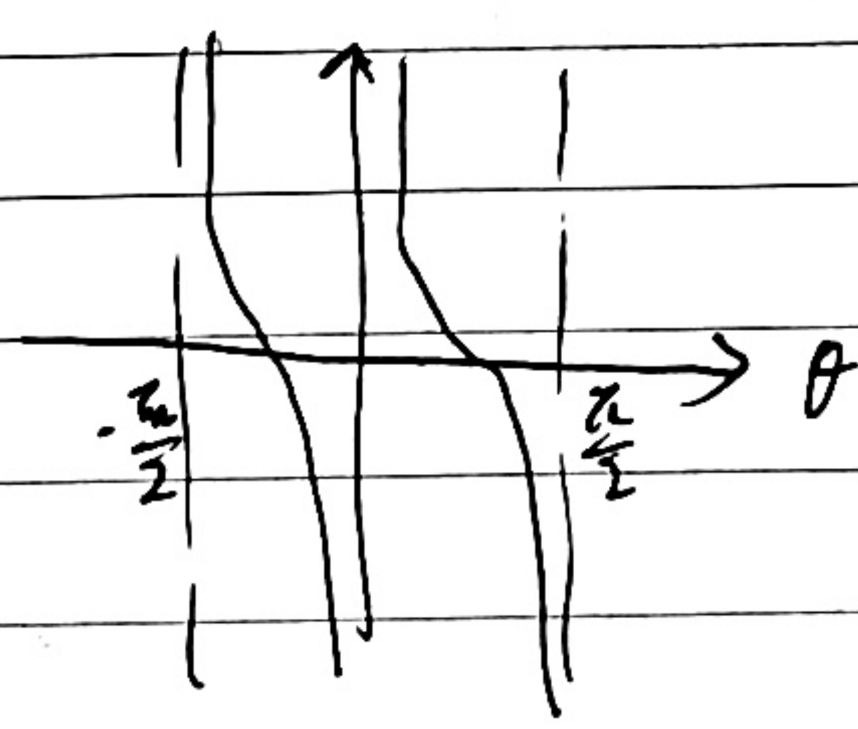
$$= AX^2 + 2BXY + CY^2$$

Can we make $2B = 0$?

$$2B = -a\sin 2\theta + 2b\cos 2\theta + c\sin 2\theta = 2b\cos 2\theta + (c-a)\sin 2\theta.$$

If $b = 0$, take $\theta = 0$.
Else, $2b\cos 2\theta + (c-a)\sin 2\theta = 0$
$\Leftrightarrow 2b\cos 2\theta = (a-c)\sin 2\theta \Rightarrow \boxed{\tan 2\theta = \frac{2b}{a-c}}$

$$\cot 2\theta = \frac{a-c}{2b}.$$



Can always find θ. (actually, exists 2 values between $-\pi$ and π.

Theorem. For a quadratic form $ax^2 + 2bxy + cy^2$. It is possible to take a coordinate system such that the quadratic form assumes $AX^2 + CY^2$.

Corollary. Any quadratic form has two perpendicular symmetry axes (since $X \leftrightarrow -X$, $Y \leftrightarrow -$

Corollary. Any quadratic form can be transformed by a rotation of the coordinate system to one (and only one) of $AX^2 + CY^2$ where $A \geq C$.

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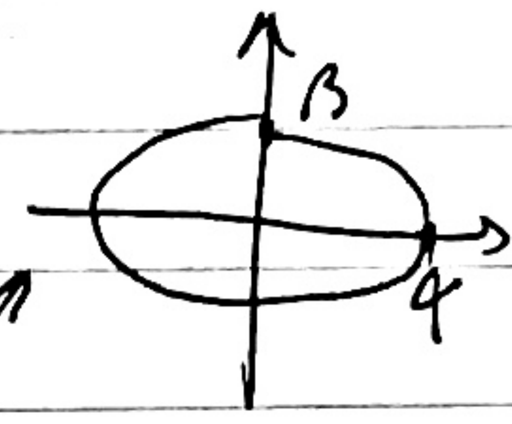
$ax^2 + 2bxy + cy^2 + dx + ey + f = 0.$

WLOG, $ax^2 + cy^2 + dx + ey + f = 0$ (can rotate coordinate system s.t. $b=0$).

Case #1: $a \neq 0 \neq c$.



$$a\left(x+\frac{d}{2a}\right)^2 + c\left(y+\frac{e}{2c}\right)^2 + \left(f - \frac{d^2}{4a} - \frac{e^2}{4c}\right) = 0.$$

$\Leftrightarrow aX^2 + cY^2 = F.$

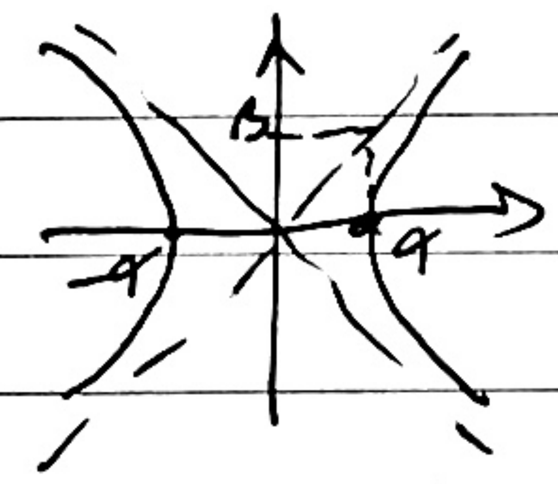
Subcase 1: $F \neq 0$.

$\Rightarrow \frac{X^2}{\alpha^2} + \frac{Y^2}{\beta^2} = 1$ or $\frac{X^2}{\alpha^2} - \frac{Y^2}{\beta^2} = 1$

(standard ellipse with axes α, β)

or $-\frac{X^2}{\alpha^2} - \frac{Y^2}{\beta^2} = 1$.

(solution is an empty set in the real)



$Y = \pm\frac{\beta}{\alpha}X$

Subcase 2: $F = 0$.

$ac < 0 \Rightarrow Y = \pm\sqrt{\frac{a}{c}}\,X$ (two intersecting lines)

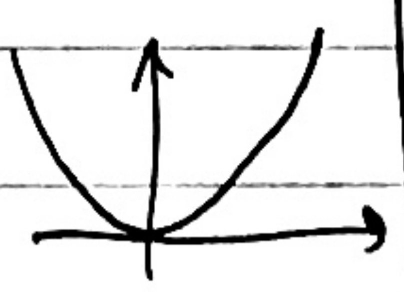
$ac \geq 0 \Rightarrow (X, Y) = (0,0)$.

solution lies in the complex domain.

Case #2: $a \neq 0$, $c = 0$.

$aX^2 + eY = F.$

Subcase 1: $e \neq 0$.



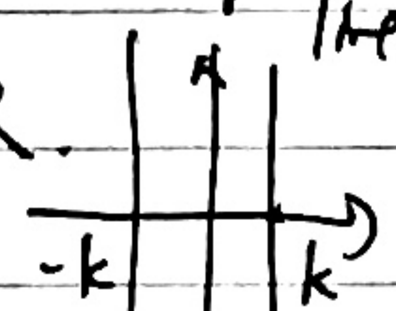
$$a x^2 + e\underbrace{\left(y + \frac{F}{e}\right)}_{Y} = 0$$ parabola.

$\Rightarrow \frac{a}{e}X^2 + Y = 0 \Rightarrow Y = kX^2$ (WLOG $k > 0$)

Subcase 2: $e = 0$.

$aX^2 = F \Rightarrow X^2 = k^2 \Rightarrow$ ~~either~~ 2 parallel lines

If $k \in \mathbb{R}$.



Else, empty set ϕ.

When $k = 0 \Rightarrow$ double line.

Theorem. Every quadratic curve in a suitably rotated and/or translated coordinate system is either a standard ellipse, hyperbola, parabola or X or | | or | or • or ϕ.

Every quadratic equation can be transformed to $\frac{X^2}{\alpha^2} \pm \frac{Y^2}{\beta^2} = \begin{cases} +1 \\ -1 \\ 0 \end{cases}$; $Y = kX^2$; $X^2 = k$

Trying to understand the geometric properties that does not depend on the coordinate system. Classification by choosing the best coordinate system.

For parabola, the second symmetry is lost because of linearity in Y rather than Y^2.

Ex. $x^2+3xy+y^2=1$

Symmetric w.r.t. x and y.

$\left(x+\frac{3y}{2}\right)^2 - \frac{5}{4}y^2 = 1 \Rightarrow$ hyperbola.