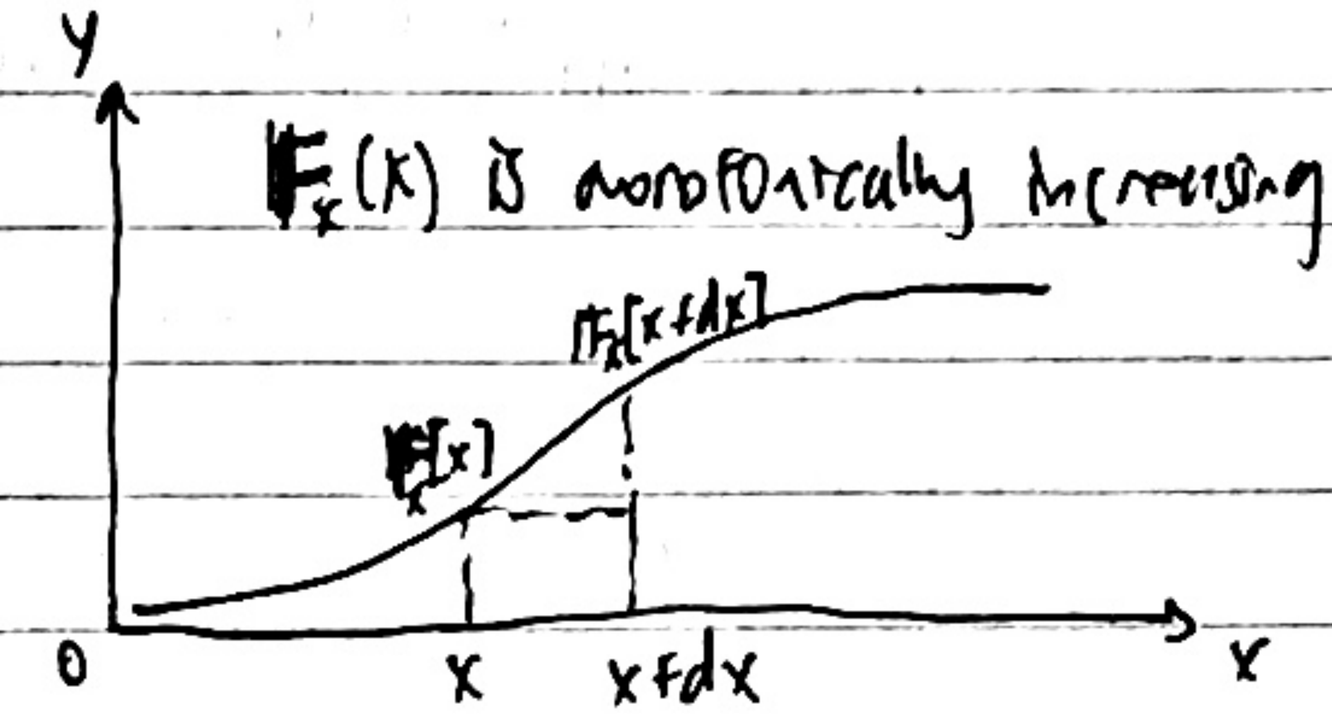
11/23/2021

CS70 LECTURE 24 : CONTINUOUS PROBABILITY

Probability Density Function

let $a = x$, $b = x + dx$

$\mathbb{P}[x < X < x + dx] = F_X[x+dx] - F_X[x]$

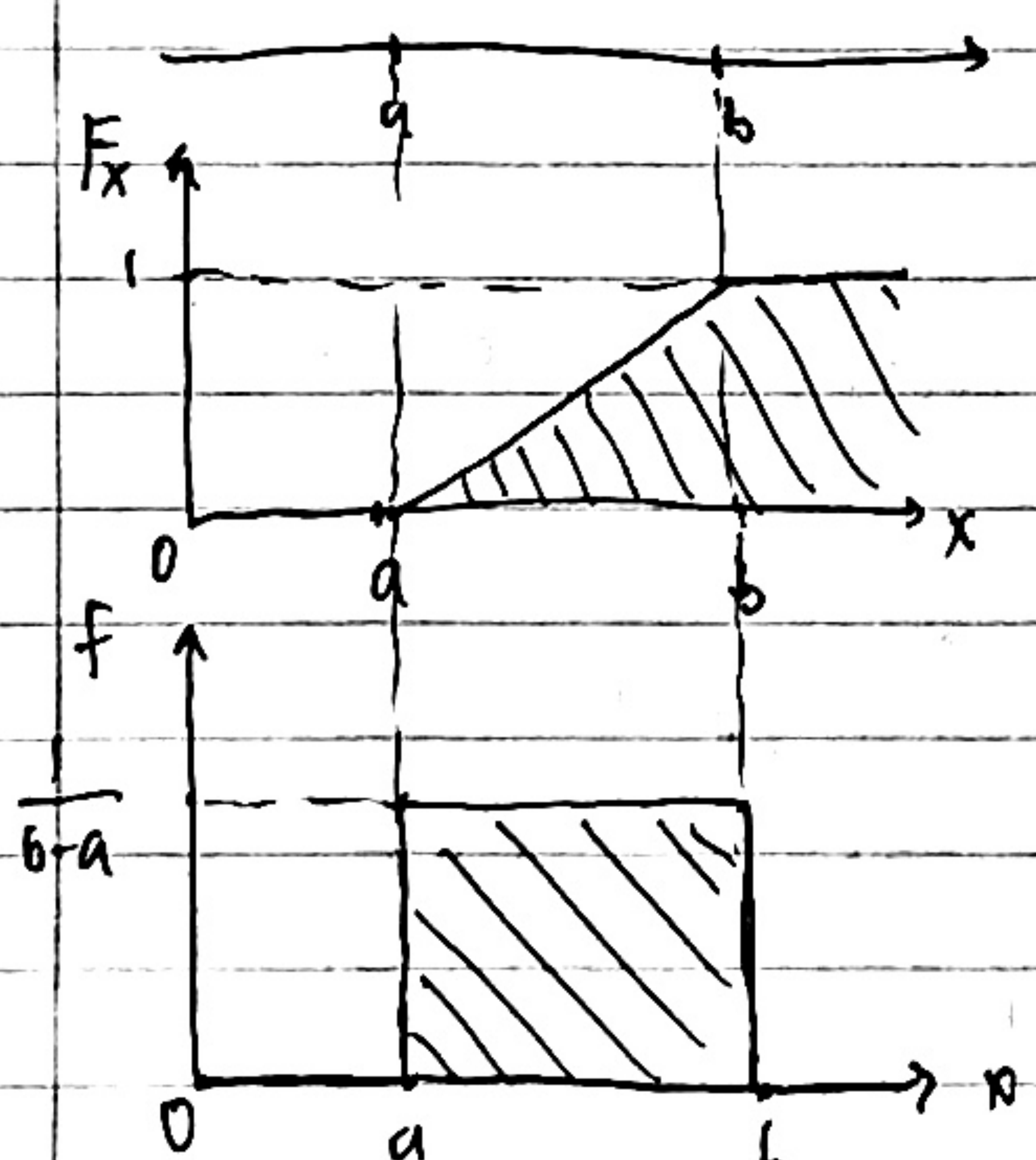


$$\underbrace{\frac{\mathbb{P}[x < X \leq x + dx]}{dx}}_{\substack{\text{Probability density} \\ f_X[x]}} = \frac{F_X[x+dx] - F_X[x]}{dx} = \frac{dF_X[x]}{dx}$$

$$f_X[x] = \frac{\mathbb{P}[x < X < x + dx]}{dx} = \frac{dF_X(x)}{dx}$$

$$F_X[x] = \int_{-\infty}^{x} f_X[\tau]\, d\tau \qquad \mathbb{P}[X \leq x] = F_X[x]$$

__Example:__ Uniform random variable, distributed between a and b.



Consider two segments $(\xi, \xi + \Delta x)$ and $(\mu, \mu + \Delta x)$.

The probability of landing in each interval is the same.

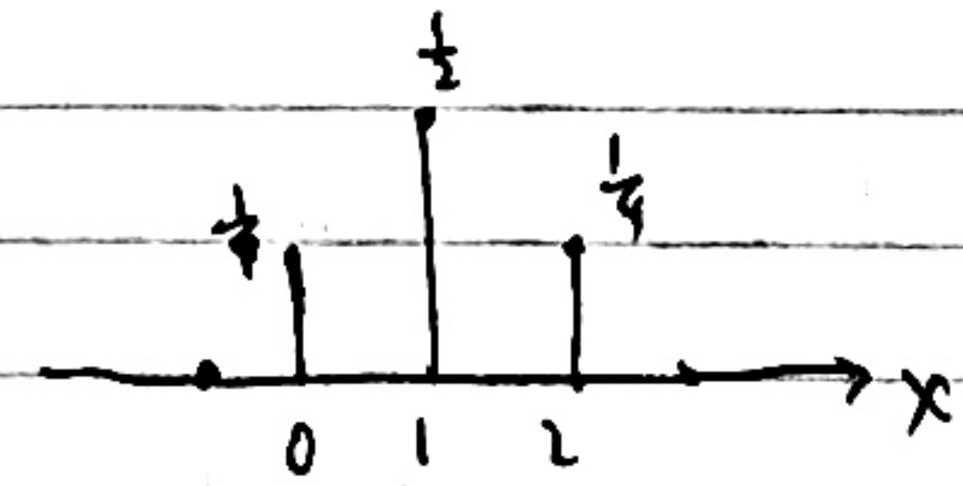
① $\int_{-\infty}^{\infty} f_X(x)\, dx = F_X(\infty) - F_X(-\infty) = 1$

② $f_X(x) \geq 0$

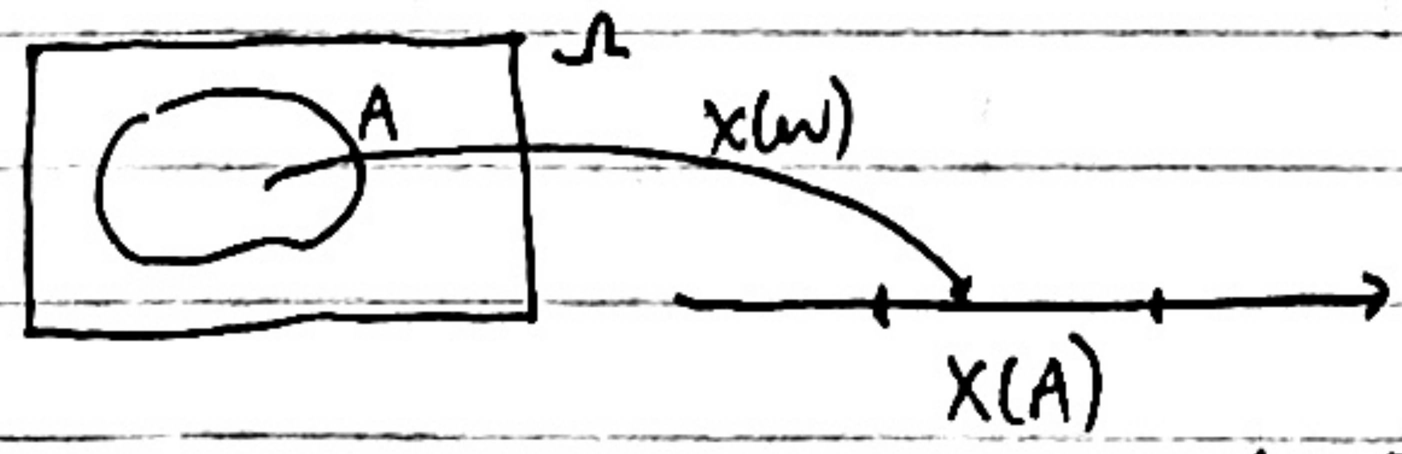
] characteristic of PDF.

__Exercise:__

Discrete random variable X



PDF conditioned on events



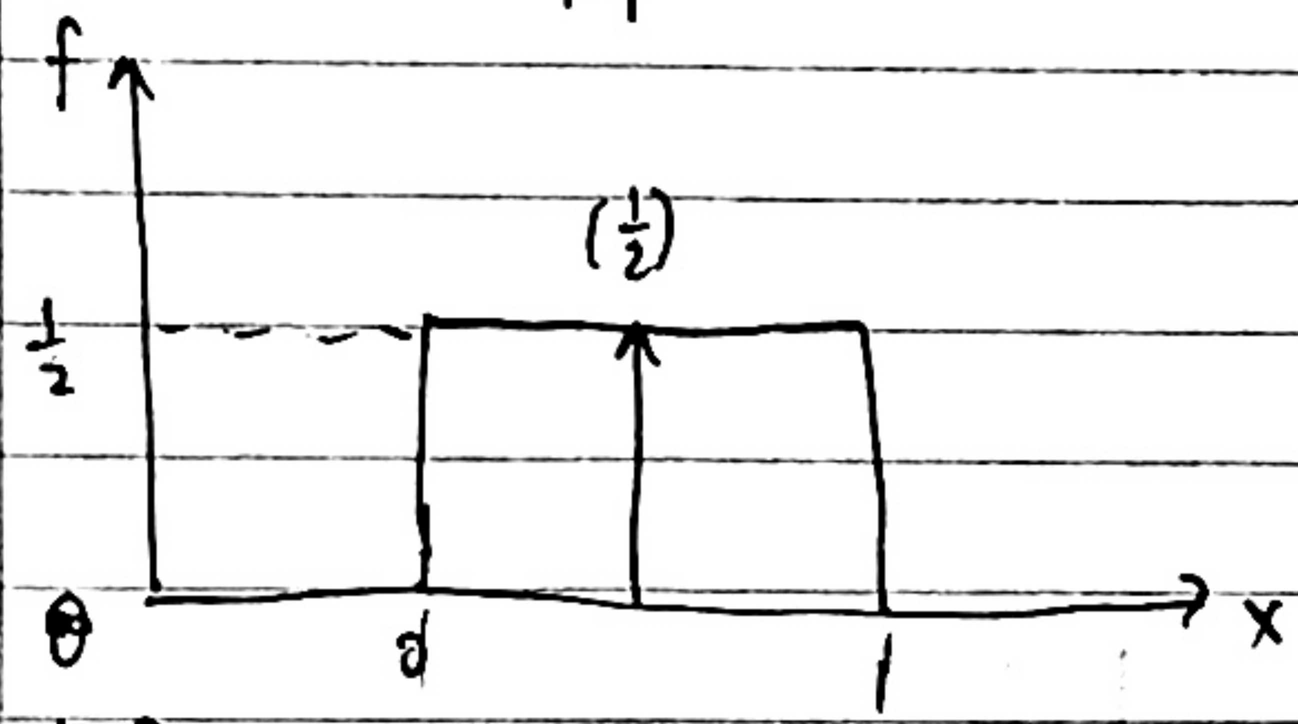
$$f_{X|A}(x)\, dx = \mathbb{P}[x < X \leq x + dx \mid A] = \frac{\mathbb{P}[\{x < X \leq x + dx\} \cap A]}{\mathbb{P}[A]}$$

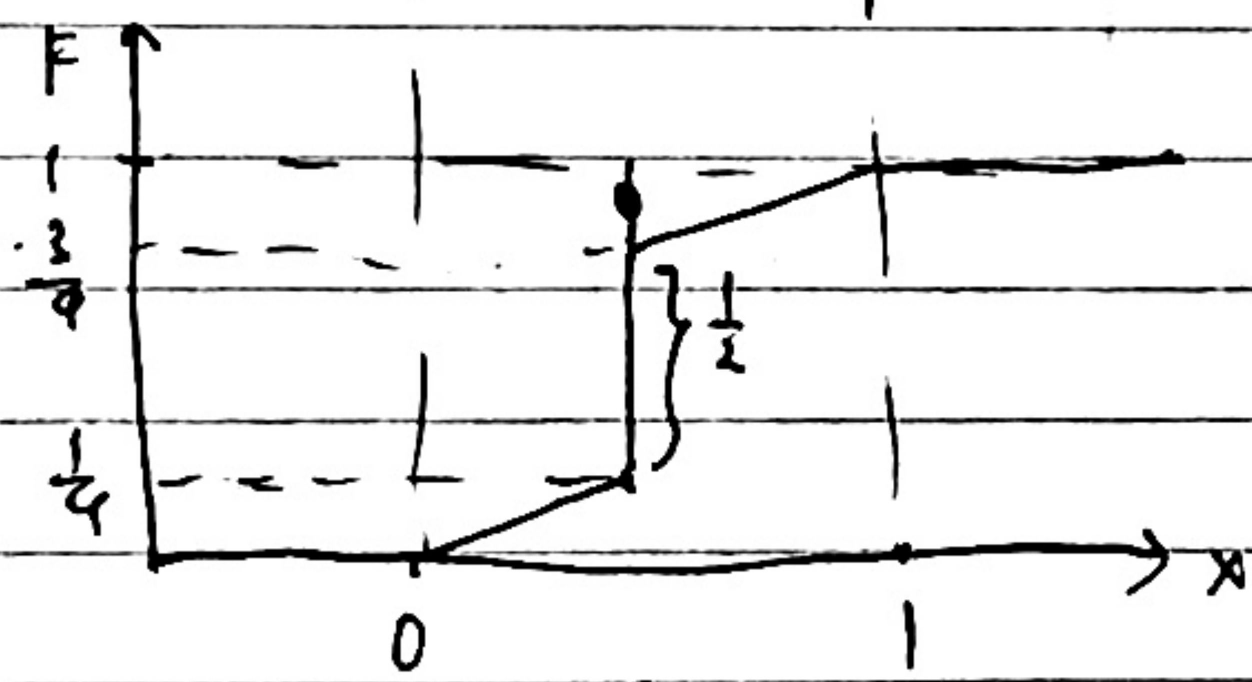
$$\therefore f_{X|A}[x]\, dx = \begin{cases} \frac{f_X[x]\, dx}{\mathbb{P}[A]} & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \Rightarrow f_{X|A}[x] = \begin{cases} \frac{f_X[x]}{\mathbb{P}[A]} & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

$f_X(x) = f_{X|H}[x]\,\mathbb{P}[H] + f_{X|T}[x]\,\mathbb{P}[T]$ (law of total probability for PDFs)

If $A_1, A_2, \ldots$ are events s.t. $A_i \cap A_j = \phi$ for $i \neq j$ and $\bigcup_i A_i = \Omega$,
Then

$$f_X[x] = \sum_{i=1}^{\infty} f_{X|A_i}[x]\,\mathbb{P}(A_i)$$



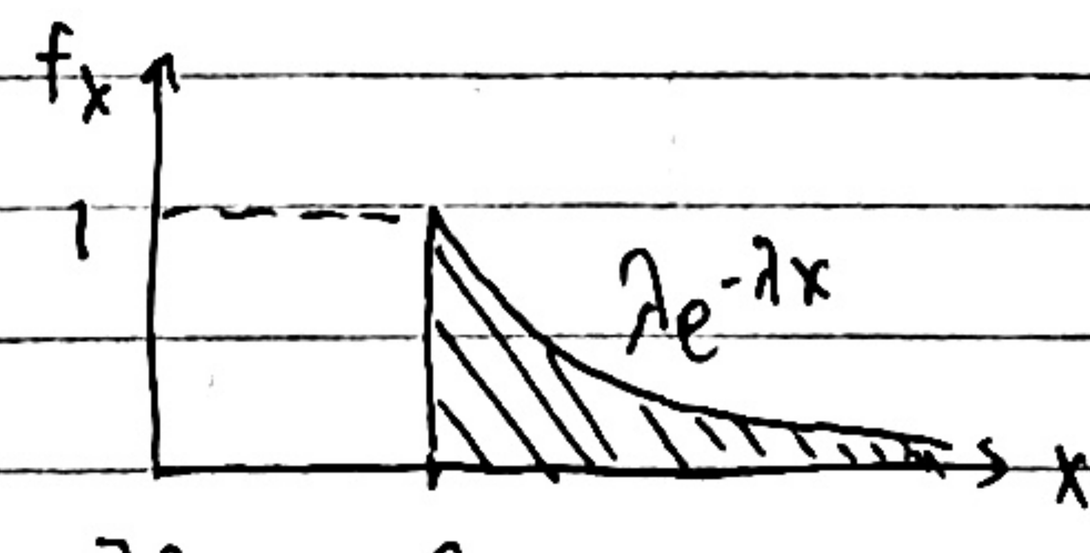


Exponential Random Variable

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

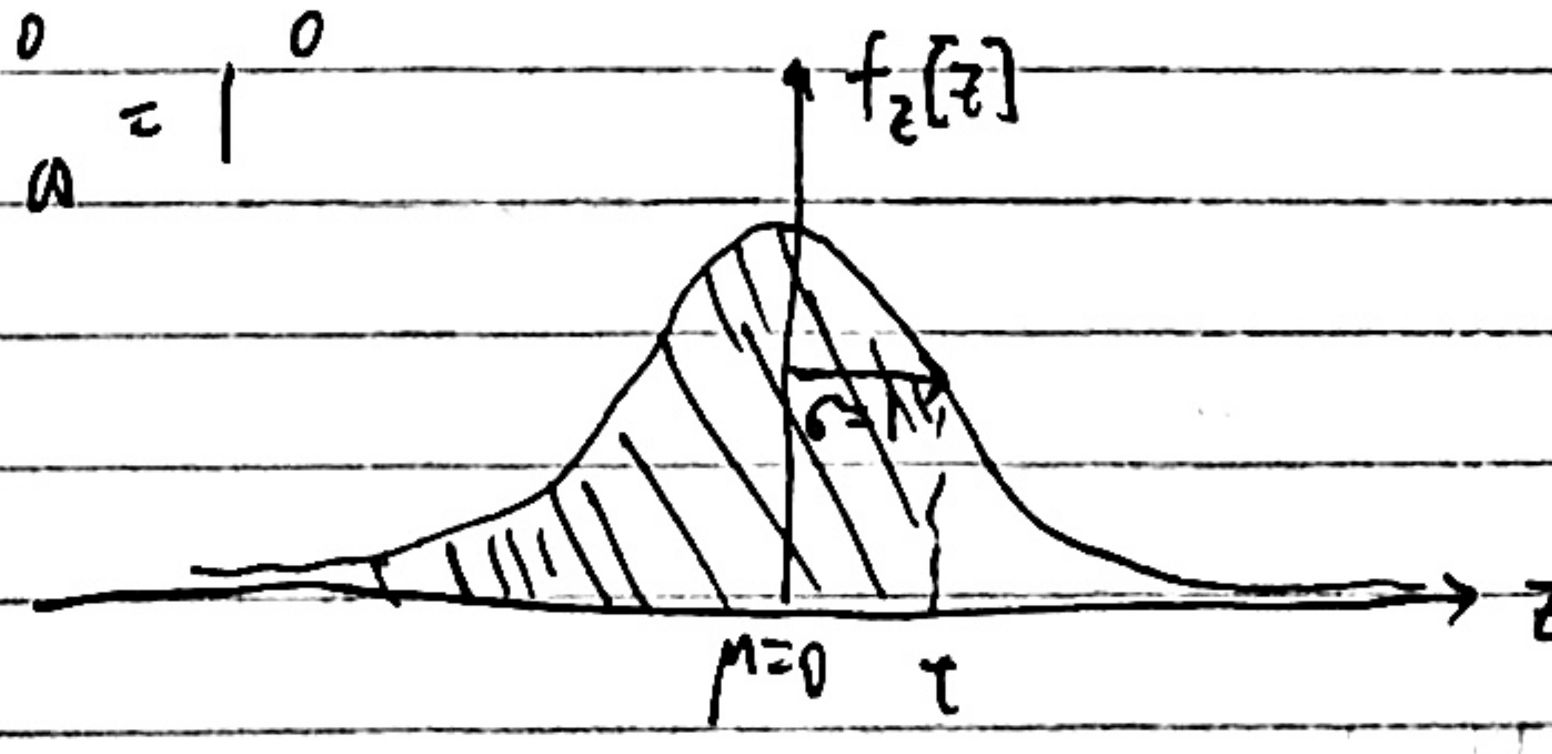
Note: $f_X[x] \geq 0 \;\; \forall x \in \mathbb{R}$.

where $\lambda > 0$.



$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{0}^{\infty} \lambda e^{-\lambda x}\,dx$$

$$= \lambda\left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^{\infty} = \left[e^{-\lambda x}\right]_{\infty}^{0} = 1$$



Gaussian Random Variable

Equivalently, normal random variable

$$f_Z[z] = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$ (mean: 0, standard dev: 1)

Even function (symmetric about 0)

$$F_Z(t) = \mathbb{P}[Z \leq t] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{z^2}{2}}\,dz$$ has no closed form expression.

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz = 1$$

More generally, $f_X[x]$ is gaussian if $f_X[x] = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ where μ: mean, σ: standard deviation

Mean of continuous RV

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \, f_X[x] \, dx$$

Law of Expectation for a function: $\mathbb{E}[g(x)] = \int_{-\infty}^{\infty} g(x) f_X[x] \, dx$.

i.e. if $g = x^2$, then $\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X[x] \, dx$.

$$\mathrm{Var}[X] = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \int_{-\infty}^{\infty} (X - \mathbb{E}(X))^2 f_X[x] \, dx$$

Intuition: λ is average arrival rate. So $\frac{1}{\lambda}$ is distance between arrival

<u>Example</u>: Mean of the Exponential X

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \, f_X[x] \, dx = \int_0^{\infty} x \lambda e^{-\lambda x} \, dx$$

$$= \left[x\left(\cancel{\tfrac{\lambda}{\lambda}} - e^{-\lambda x}\right)\right]_0^{\infty} - \int_0^{\infty} \left(-e^{-\lambda x}\right) dx = \int_0^{\infty} e^{-\lambda x} = \frac{1}{\lambda}.$$

Note from earlier $\int_0^{\infty} \lambda e^{-\lambda x} \, dx = 1$, $\int_0^{\infty} e^{-\lambda x} \, dx = \frac{1}{\lambda}$

<u>Homework</u>: Find Variance for the exponential distribution.