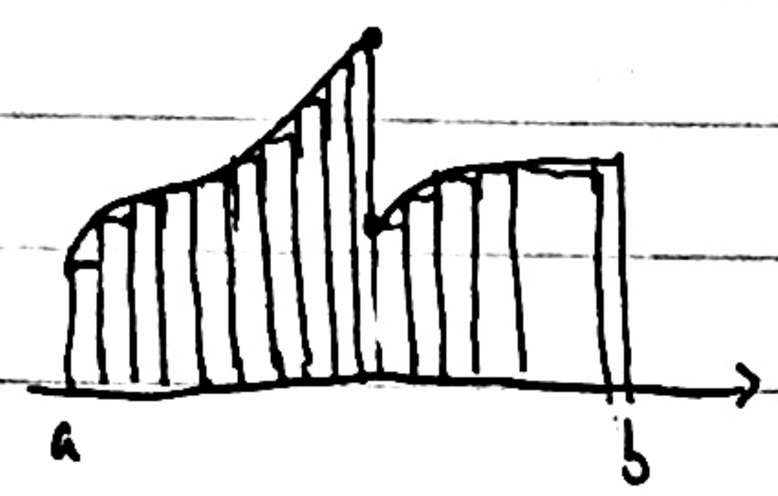
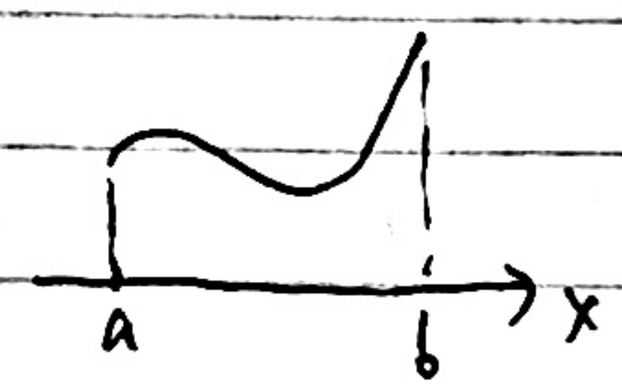
11/18/2021

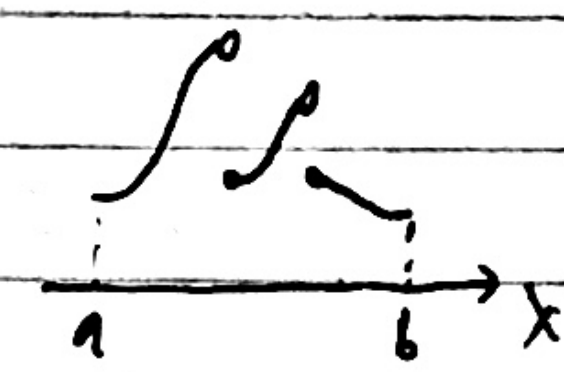
MATH 104 LECTURE 22 : DARBOUX INTEGRAL

- Darboux Integral
- Basic Properties

Let f be a bounded function on a finite closed interval $f:[a,b] \to \mathbb{R}$ (no need to assume that it is continuous)







How do we make the definition of limit rigorous? $\to$ $\Delta x \to 0$ (Riemann Integral) ; $\to$ Darboux Integral.

Darboux Integral Terminology & Notation

A <u>partition</u> of $[a,b]$ is a finite subset of $[a,b]$ of the form

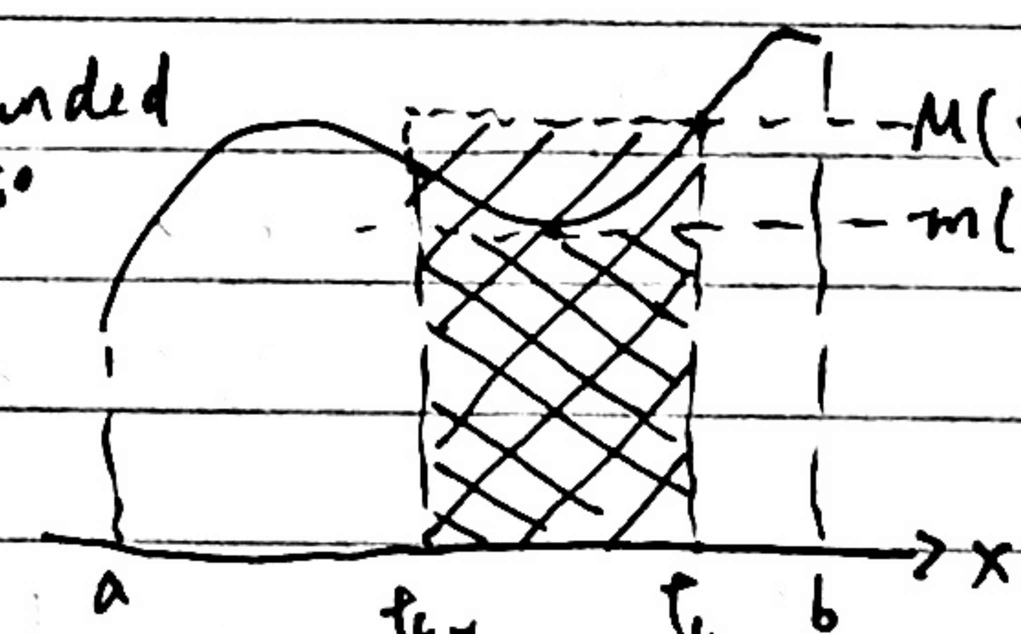
$$P = \{a = t_0 < t_1 < t_2 < \dots < t_n = b\}$$

Does not matter if the points are equidistant.

For $S \subseteq [a,b]$ where S is nonempty, denote

- $M(f,S) = \sup\{f(x) : x \in S\}$
- $m(f,S) = \inf\{f(x) : x \in S\}$

Exists since S is bounded so these sets are also bounded.



$-M(f,[t_{k-1},t_k])$

$-m(f,[t_{k-1},t_k])$

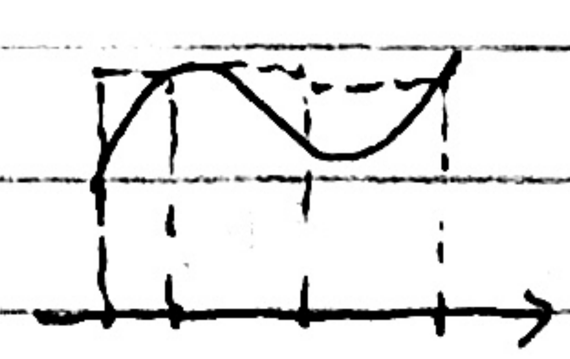
<u>Example</u>: $S = [t_{k-1}, t_k]$

Area under the curve is sandwiched between the Darboux upper and lower sum

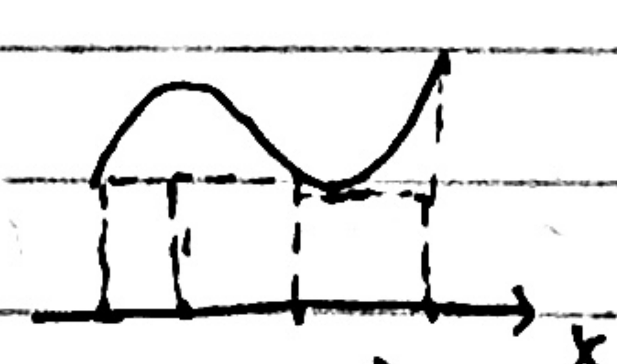
Definition

For a partition $P = \{a = t_0 < t_1 < t_2 < \dots < t_n = b\}$, denote

- upper Darboux sum w.r.t. P as $U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1})$
- lower Darboux sum w.r.t. P as $L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k])(t_k - t_{k-1})$



$U(f,P)$



$L(f,P)$

Observation

$$\underbrace{m(f,[a,b])(b-a)}_{\text{the most crude lower rectangle approximation}} \le L(f,P) \le U(f,P) \le \underbrace{M(f,[a,b])(b-a)}_{\text{most crude upper rectangle approximation}}$$

$\Rightarrow L, U$ are bounded.

Definition

The <u>upper Darboux integral</u> is $U(f) = \inf\{U(f,P),\ P \text{ is a partition of } [a,b]\}$.
The <u>lower Darboux integral</u> is $L(f) = \sup\{L(f,P),\ P \text{ is a partition of } [a,b]\}$.

Say f is integrable if $U(f) = L(f)$. In this case, write
$$\int_a^b f = U(f) = L(f)$$

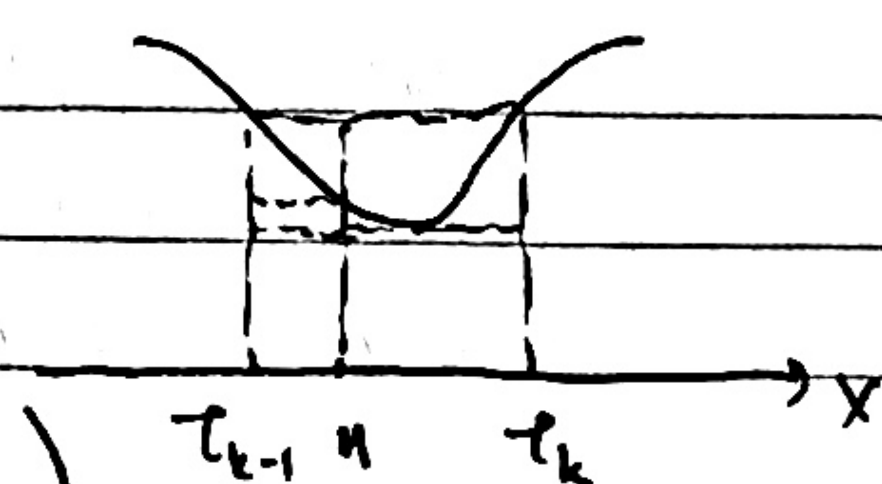
<u>Remark:</u> This definition only stems from the concepts of supremum and infimum. There's no "taking limits" or "making partitions finer & finer"

Basic Properties

Let P, Q be two partitions of $[a,b]$. If Q contains P $(Q \supseteq P)$, then
$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P).$$
"More rigorous way to say 'finer partition gives better approximation'"

<u>Proof:</u>

Let $P = \{a = t_0 < t_1 < \cdots < t_{k-1} < t_k < \cdots < t_n = b\}$
$Q = \{a = t_0 < t_1 < \cdots < t_{k-1} < u < t_k < \cdots < t_n = b\}$ (i.e. $Q - P = \{u\}$)



Since $M(f,[t_{k-1},t_k]) \ge M(f,[t_{k-1},u]), M(f,[u,t_k])$
we see that $U(f,P) \ge U(f,Q)$
$$\left(\text{Since } M(f,[t_{k-1},t_k])(t_k - t_{k-1}) \ge \begin{matrix} M(f,[t_{k-1},u])(u - t_{k-1}) \\ + M(f,[u,t_k])(t_k - u) \end{matrix}\right)$$

Similarly, we can get $L(f,P) \le L(f,Q)$
The general case follows by induction on $|Q-P|$. (number of points Q has more than P)

Theorem

$L(f) \le U(f)$
In particular, $L(f,P) \le \cancel{U(f,P)}\ L(f) \le U(f) \le U(f,P) \quad \forall P$.

<u>Proof:</u>

$U(f) = \cancel{\sup}\inf\{U(f,P)\}$ Take any partition P, P' of $[a,b]$
$L(f) = \sup\{L(f,P)\}$ Set $Q = P \cup P'$. Q is also a partition since it is a finite set and contains the endpoints.

In particular, $Q \supseteq P$ and $Q \supseteq P'$

Hence $L(f,P') \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$
Hence $U(f,P)$ is bigger than any $L(f,P')$
$\Rightarrow U(f,P)$ is an upperbound for any P
$\Rightarrow \boxed{U(f) \geq L(f).}$

<u>Example</u>: Show $f(x) = x$ is integrable on $[0,1]$ and $\int_0^1 x = \frac{1}{2}$.

<u>Proof</u>: For each n, consider $P_n = \{0 = t_0 < t_1 = \frac{1}{n} < \cdots < t_k = \frac{k}{n} < \cdots < t_n = 1\}$.

Then $U(f,P_n) = \sum_{k=1}^{n} M(x,[t_{k-1},t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n} \frac{k}{n}\cdot\frac{1}{n} = \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{n+1}{2n}$

$L(f,P_n) = \sum_{k=1}^{n} m(x,[t_{k-1},t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n} \frac{k-1}{n}\cdot\frac{1}{n} = \frac{1}{n^2}\sum_{k=1}^{n}(k-1) = \frac{n-1}{2n}$

By Theorem, $\forall n$, $\frac{1}{2} - \frac{1}{2n} \leq L(f) \leq U(f) \leq \frac{1}{2} + \frac{1}{2n}$
By Squeeze lemma $(n \to \infty)$, $\frac{1}{2} \leq L(f) \leq U(f) \leq \frac{1}{2}$
In particular, the inequalities must be strict $\Rightarrow \boxed{L(f) = U(f) = \frac{1}{2}.}$

Qn Which function f is integrable?
- continuous functions
- monotone functions (Any good criterion for integrability?)

Theorem Let f be a bounded function ($f : [a,b] \to \mathbb{R}$).
f is integrable $\Leftrightarrow$ ε-P property holds for f.

(Intuitively, can find a rectangular approximation ~~to~~ to obtain any accuracy)

$\forall \varepsilon > 0$, $\exists P$ partition of $[a,b]$ s.t. $U(f,P) - L(f,P) < \varepsilon$

<u>Proof</u>:
($\Rightarrow$) Assume f is integrable, i.e. $U(f) = L(f)$
$\Rightarrow \sup\{L(f,P)\} = L(f) = U(f) = \inf\{U(f,P)\}$
Take any $\varepsilon > 0$. Since $L(f) - \frac{1}{2}\varepsilon < L(f)$, $\exists P_1$ s.t. $L(f) - \frac{\varepsilon}{2} < L(f,P_1) \leq L(f)$.
$U(f) + \frac{1}{2}\varepsilon > U(f)$, $\exists P_2$ s.t. $U(f) \leq U(f,P_2) < U(f) + \frac{1}{2}\varepsilon$.

Consider $P = P_1 \cup P_2$. This is another partition of $[a,b]$.
we have: Since $P \supseteq P_1$, $P \supseteq P_2$,

$$L(f,P_1) \leq L(f,P) \leq U(f,P) \leq U(f,P_2)$$

$$\Rightarrow U(f,P) - L(f,P) \leq U(f,P_2) - L(f,P_1)$$

$$< \left(U(f) + \frac{\varepsilon}{2}\right) - \left(L(f) - \frac{\varepsilon}{2}\right) = \varepsilon$$

Since $U(f) = L(f)$.

($\Leftarrow$) Take any $\varepsilon > 0$.
By ε-P property, $\exists P$ s.t. $U(f,P) - L(f,P) < \varepsilon$.
Since $L(f,P) \leq L(f) \leq U(f) \leq U(f,P)$

$$\Rightarrow 0 \leq U(f) - L(f) \leq U(f,P) - L(f,P) < \varepsilon \quad \forall \varepsilon > 0.$$

Since the above holds true for any $\varepsilon > 0$, $\boxed{U(f) - L(f) = 0} \Rightarrow U(f) = L(f)$.