11/30/2021

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MATH 104 LECTURE 24: FUNDAMENTAL THEOREM OF CALCULUS

Recap

Let f, g be two functions (bounded) $[a,b] \longrightarrow \mathbb{R}$.
If f is piecewise continuous or piecewise monotone, then f is integrable. (verified using ε-P)

If f, g are integrable, then, cf, $f+g$, f^2, $f \cdot g$ are all integrable.

Once we know f^2 is integrable (HW 11), $(f+g)^2 = f^2 + 2fg + g^2$, $(f-g)^2 = f^2 - 2fg + g^2$] polarisation
are also integrable

$\Rightarrow fg = \dfrac{(f+g)^2 - (f-g)^2}{4}$ is integrable.

Intermediate Value theorem for integrals

$a < c < b$

f is integrable on $[a,b] \Leftrightarrow f$ is integrable on $[a,c]$, and $[c,b]$

Notation

Assume f is integrable on $[a,b]$. Take $x, y \in [a,b]$ with $y \leq x$.
Then set

$$\int_x^y f = \begin{cases} -\int_y^x f & (y < x) \\ 0 & (x = y) \end{cases}$$

This is defined because if f is integrable on $[a,b]$, f is also integrable on any smaller closed interval.
$\Rightarrow$ Can check $\int_x^y f + \int_y^z f = \int_x^z f$ for any $x, y, z \in [a,b]$

Fundamental Theorem of Calculus

Both versions of fundamental theorem of calculus kind of says differentiation and integration are "inverse operations".

FTC I

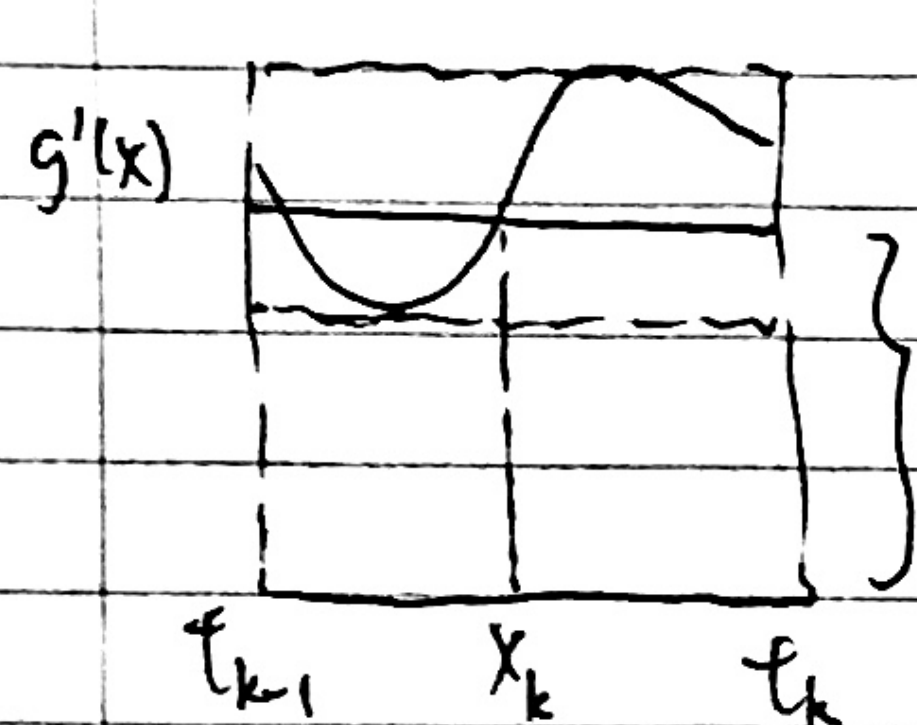
Let $g: [a,b] \to \mathbb{R}$ be a continuous function on the interval $[a,b]$ and differentiable on the interior (i.e. (a,b)). If g' is integrable, then $\int_a^b g' = g(b) - g(a)$

i.e. we can compute the integral of the derivative, which is just the difference between the end values.

Suppose we have function $f = g'$ and we can find antiderivative F of f, then
$\int_a^b f = F(b) - F(a)$

Remark "g' is integrable": g' may only be defined on (a,b), so "g' is integrable" means
* Some extension of g' on $[a,b]$ is integrable (Exercise in Ross!)

Idea Need to show $\forall P$, $L(g', P) \leq g(b) - g(a) \leq U(g', P)$



Area $= g'(x_k)(t_k - t_{k-1})$

If we can use magic s.t.
$g'(x_k)(t_k - t_{k-1}) = g(t_k) - g(t_{k-1})$
Can employ telescoping.

<u>Proof:</u>
Take any partition $P = \{a = t_0 < \cdots < t_n = b\}$.
By Mean Value Theorem applied to g on $[t_{k-1}, t_k]$, $\exists\, x_k \in (t_{k-1}, t_k)$ s.t.

$$g'(x_k) = \frac{g(t_k) - g(t_{k-1})}{t_k - t_{k-1}}$$

Then
$$L(g', p) = \cancel{\sum \text{inf}} = \sum_{k=1}^{n} m(g', [t_{k-1}, t_k])(t_k - t_{k-1}) \leq \sum_{k=1}^{n} g'(x_k)(t_k - t_{k-1})$$
$$= \sum_{k=1}^{n} \frac{g(t_k) - g(t_{k-1})}{t_k - t_{k-1}}(t_k - t_{k-1}) = \sum_{k=1}^{n} g(t_k) - g(t_{k-1}) = g(t_n) - g(t_0)$$
$$= g(b) - g(a)$$

Similarly,
$$U(g', p) = \sum_{k=1}^{n} M(g', [t_{k-1}, t_k])(t_k - t_{k-1})$$
$$\geq \sum_{k=1}^{n} \frac{g(t_k) - g(t_{k-1})}{t_k - t_{k-1}}(t_k - t_{k-1}) = \sum_{k=1}^{n} g(t_k) - g(t_{k-1}) = g(t_n) - g(t_0)$$
$$= g(b) - g(a).$$

Hence $L(f, P) \leq g(b) - g(a) \leq U(f, P)$ $\forall P$
$\Rightarrow L(f) \leq g(b) - g(a) \leq U(f)$.
Since g' is integrable, $L(f) = U(f) = \int_a^b g'$ $\Rightarrow$ $\boxed{\int_a^b g' = g(b) - g(a)}$

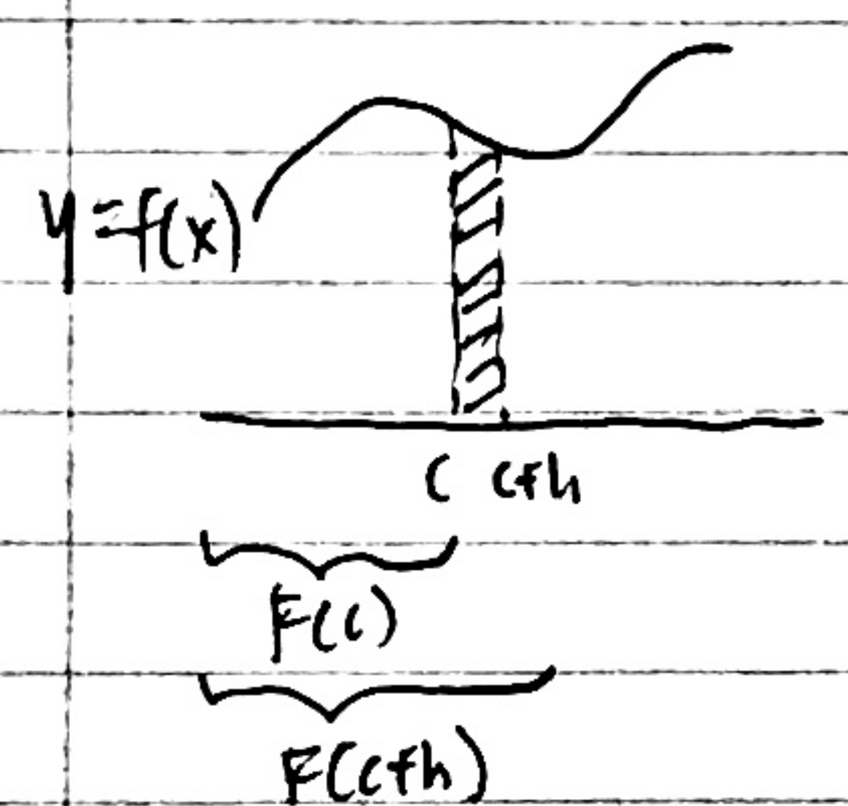
FTC II Let $f:[a,b]\to\mathbb{R}$ be a bounded and integrable. For $x\in[a,b]$, set $F(x)=\int_a^x f$ (defined because since f is integrable over $[a,b]$, it is integrable over smaller intervals)

(1) F is uniformly continuous on $[a,b]$
(2) If f is continuous at $c\in(a,b)$, then F is differentiable at c with $F'(c)=f(c)$.

Idea $F'(c)=\lim\limits_{h\to 0}\dfrac{F(c+h)-F(c)}{h}$

This quantity visually represents the height of a rectangle with the same area as the shaded area.

The continuous assumption is necessary



__Proof:__ (1)
Since f is bounded by assumption, $\exists B>0$ s.t. $|f(x)|\le B$ $\forall x\in[a,b]$.
Take any $\varepsilon>0$. Set $\delta=\frac{\varepsilon}{B}>0$. Then for $x,y\in[a,b]$, $|x-y|<\delta=\frac{\varepsilon}{B}$

$$\Rightarrow |F(x)-F(y)|=\left|\int_a^x f-\int_a^y f\right|=\left|\int_y^x f\right|\le\left|\int_y^x |f|\right|\le\left|\int_y^x B\right|=B|(x-y)|$$
$$<B\cdot\frac{\varepsilon}{B}=\varepsilon$$

$\Rightarrow F$ is uniformly continuous on $[a,b]$.

__Proof:__ (2)
Note $\dfrac{F(c+h)-F(c)}{h}=\dfrac{1}{h}\displaystyle\int_c^{c+h} f$ for $h\neq 0$

Hence $\dfrac{F(c+h)-F(c)}{h}-f(c)=\dfrac{F(c+h)-F(c)}{h}-\dfrac{h\cdot f(c)}{h}=\dfrac{1}{h}\displaystyle\int_c^{c+h}(f(t)-f(c))\,dt$

Want to show this quantity $\to 0$ as $h\to 0$.
Take $\varepsilon>0$. Since f is continuous at c, $\exists\delta>0$ s.t. $\forall x\in[a,b]$, $|x-c|<\delta\Rightarrow|f(x)-f(c)|$
$<\varepsilon$

$$\left|\frac{F(c+h)-F(c)}{h}-f(c)\right|=\frac{1}{h}\left|\int_c^{c+h}(f(t)-f(c))\,dt\right|$$

Then $\forall h \neq 0$, $0 < |h| < \delta$

$$\Rightarrow \left| \frac{F(c+h) - F(c)}{h} - f(c) \right| = \frac{1}{|h|} \left| \int_c^{c+h} (f(t) - f(c))\, dt \right|$$

$$\leq \frac{1}{|h|} \left| \int_c^{c+h} |f(t) - f(c)|\, dt \right|$$

bounded by δ since $|t-c| < \delta$.

$$\leq \frac{1}{|h|} \left| \int_c^{c+h} \varepsilon \, dt \right| = \frac{1}{|h|} |h| \varepsilon = \varepsilon$$

This means $\boxed{\lim_{h \to 0} \frac{F(c+h) - F(c)}{h} = f(c)}$

Applications <u>Integration by Parts & Change of Variables</u>

differentiable $\Rightarrow$ continuity $\Rightarrow$ bounded

$\Rightarrow$ integrability

Integration by Parts

$u, v : [a,b] \to \mathbb{R}$ be bounded, and differentiable.
If u' and v' are integrable, then

$$\int_a^b u'(t) v(t)\, dt + \int_a^b u(t) v'(t)\, dt = u(b)v(b) - u(a)v(a)$$

Example $\int_a^b x \sin x \, dx$

Apply theorem to $u(x) = x$ and $v(x) = -\cos x$. Then $\int_a^b uv' + \int_a^b u'v = u(b)v(b) - u(a)v(a)$

$$\therefore \int_a^b x \sin x \, dx = -b\cos b + a\cos a - \int_a^b -\cos x \, dx$$

$$= \boxed{-b\cos b + a\cos a + \sin b - \sin a}$$

<u>Proof:</u>

Apply Fundamental theorem of calculus to $g(x) = u(x)v(x)$
Chain rule implies $g'(x) = u(x)v'(x) + u'(x)v(x)$

Change of Variables

Let I, J be open interval. $J \xrightarrow{u} I \xrightarrow{f} \mathbb{R}$
If f is ~~differentiable~~ continuous, u is differentiable, u' is continuous
Then $\forall a, b \in J$

$$\int_a^b f(u(t))\, u'(t)\, dt = \int_{u(a)}^{u(b)} f(u)\, du$$

Symbolically $\frac{du}{dt} \cdot dt = du$.

Example $\int t e^{t^2} dt$

Let $u(t) = t^2$, $u'(t) = 2t$

$\Rightarrow \int t e^{t^2} dt = \int e^{t^2}(2t)\frac{1}{2} dt = \int e^u \frac{1}{2} du = \frac{1}{2}e^u + c = \frac{1}{2}e^{t^2} + c.$

Idea: want to express $g(t) = f(u(t))\, u'(t)$.

Proof:

Define $g(x) = \int_c^x f(u)\, du$ for some $c \in I$

Fundamental Theorem of Calculus II $\Rightarrow g'(x) = f(x)$.

Using Chain Rule, $\frac{d}{dt}(g(u(t))) = g'(u(t))\, u'(t) = f(u(t))\, u'(t)$ (!)

$\Rightarrow \int_a^b f(u(t))\, u'(t)\, dt = \int_a^b \frac{d}{dt}(g(u(t)))\, dt = g(u(b)) - g(u(a))$

$= \int_c^{u(b)} f\, du - \int_c^{u(a)} f\, du$

$= \int_{u(a)}^{u(b)} f\, du.$

$\therefore \boxed{\int_a^b f(u(t))\, u'(t)\, dt = \int_{u(a)}^{u(b)} f\, du}$