No.: .... Date: 09/16/2021

MATH H110 LECTURE 7 Notes

Isomorphism Theorem

Inclusion is a linear map.

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[d, "\pi"'] & W \\ V/\operatorname{Ker} A \arrow[r, "\cong", "\hat{A}"'] & A(V) \arrow[u, "\text{inclusion map}"'] \end{tikzcd}$$

i.e. If $\vec{v} \in V$

then $\pi(\vec{v}) = \vec{v} + \operatorname{Ker} A$ — affine subspace parallel to Ker A

$A\vec{v} \in A(V)$ → range.

$A(V)$ is a subspace of W

$A(V) \subseteq W$

If $\vec{u} \in \operatorname{Ker} A$

then $A(\vec{v}+\vec{u}) = A\vec{v} + A\vec{u}$

$= A\vec{v} + \vec{0} = A\vec{v}$.

If there is a nontrivial

$A\vec{v}' = 0$ then $\vec{v}' \in \operatorname{Ker} A$

Then $\vec{v}' + \operatorname{Ker} A$ is the zero vector in $V/\operatorname{Ker} A$.

Matrices

$$\begin{array}{cc} & j \\ m & \begin{bmatrix} a_{ij} \end{bmatrix} \end{array} = A.$$

Suppose we have a vector space V with $\dim V = n$. Then $V \cong \mathbb{K}^n$.

Consider $\mathbb{K}^n \xrightarrow{a} \mathbb{K}$. ($a$ is a linear form).

$$a(x) = a(\vec{e}_1)x_1 + a(\vec{e}_2)x_2 + \dots + a(\vec{e}_n)x_n$$

Let $a_i = a(\vec{e}_i)$

$$= a_1x_1 + a_2x_2 + \dots + a_nx_n \overset{\text{def}}{=} \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$(\lambda a + \mu b) = \begin{bmatrix} \lambda a_1 + \mu b_1 & \lambda a_2 + \mu b_2 & \dots & \lambda a_n + \mu b_n \end{bmatrix}$$

IF a, b are two linear forms, $a, b \in (\mathbb{K}^n)^*$

$$(\lambda a + \mu b)(\vec{v}) = \lambda a(\vec{v}) + \mu b(\vec{v})$$

Linear Maps

$A : V \to W$ where $\dim V = n$, $\dim W = m$

Let $\vec{e}_1, \vec{e}_2, \dots \vec{e}_n$ be basis in V. and $\vec{f}_1, \vec{f}_2, \dots \vec{f}_m$ be basis in W.

$\vec{v} \in V \Rightarrow \vec{v} = x_1\vec{e}_1 + x_2\vec{e}_2 + \dots + x_n\vec{e}_n$

$$A\vec{v} = A(x_1\vec{e}_1 + \dots + x_n\vec{e}_n) = x_1A\vec{e}_1 + x_2A\vec{e}_2 + \dots + x_nA\vec{e}_n$$

POP bazic

No.: .... Date: ....

Represent the images of $\vec{e_j}$ (i.e. $A\vec{e_j}$) by $A\vec{e_j} = \sum_{i=1}^{m} a_{ij} \vec{f_i}$

Then $A\vec{v} = \begin{bmatrix} & & \\ & a_{ij} & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$A\vec{v} = x_1 A\vec{e_1} + \dots + x_n A\vec{e_n} = \sum_{j=1}^{n} x_j A\vec{e_j} = \sum_{j=1}^{n} x_j \sum_{i=1}^{m} a_{ij} \vec{f_i}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij} x_j) \vec{f_i}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \longmapsto m \begin{bmatrix} a_{11}x_1 + \dots + a_{1n}x_n \\ \\ a_{m1}x_1 + \dots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

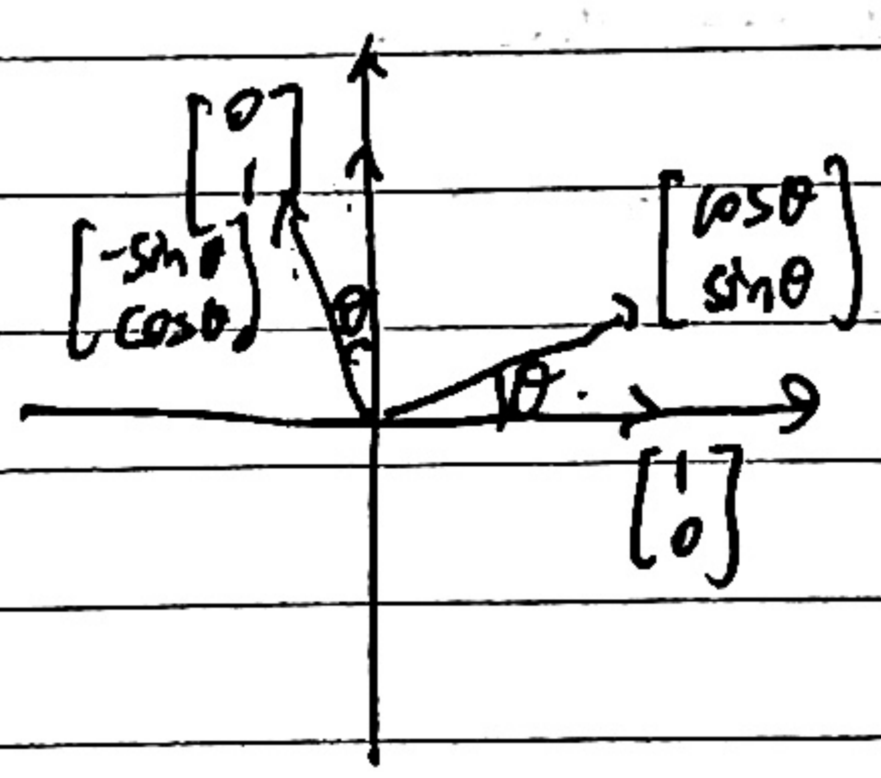
$$\begin{cases} y_1 = a_{11}x_1 + \dots + a_{1n}x_n \\ \vdots \\ y_m = a_{m1}x_1 + \dots + a_{mn}x_n \end{cases}$$

The columns of the matrix represent the unit vectors $\vec{e_1}, \vec{e_2}, \dots \vec{e_n}$ written in the form of bases $\vec{f_1}, \vec{f_2}, \dots \vec{f_m}$.

$$\begin{bmatrix} a_{ij} \end{bmatrix} = m \begin{bmatrix} A\vec{e_1} & A\vec{e_2} & \dots & A\vec{e_n} \end{bmatrix}$$

$\boxed{a_{ij} \text{ is the value of } y_i \text{ on } A\vec{e_j}}$ ← this is what the coefficients of the matrix represent geometrically.

<u>Example</u>: Rotation on a plane



∴ Matrix for transformation is

$$\begin{bmatrix} A\begin{bmatrix} 1 \\ 0 \end{bmatrix} & A\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

<u>Composition</u> (of linear maps)

$$U \xrightarrow{B} V \xrightarrow{A} W$$

C

Then $C = AB$

i.e. $C\vec{u} = A(B\vec{u})$.

C is linear, since $A(B(\lambda\vec{u} + \mu\vec{v})) = A(\lambda B\vec{u} + \mu B\vec{v})$

$= \lambda A(B\vec{u}) + \mu A(B(\vec{v}))$

$\therefore C(\lambda\vec{u} + \mu\vec{v}) = \lambda C\vec{u} + \mu C\vec{v}$.

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No.: Date:

$$m\left[\quad \overset{\ell}{C_{ij}} \quad\right] = m\left[\quad \overset{n}{A} \quad\right] n\left[\quad \overset{\ell}{B} \quad\right]$$

Let $\dim U = \ell$ Then A has dimension $m \times n$
$\dim V = n$ B has dimension $n \times \ell$
$\dim W = m$

Let bases of W be $\{y_1, y_2, \dots, y_m\}$
C_{ij} is the i^{th} coordinate of the image of the j^{th} vector $\vec{e_j}$.

$$\left[\quad C_{ij} \quad\right] = \left[\begin{array}{c} \boxed{a_{i1} \; a_{i2} \; \dots \; a_{in}} \\ \\ \end{array}\right]\left[\begin{array}{ccc} & \begin{array}{|c|} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{array} & \end{array}\right]$$

j^{th} column

$$C_{ij} = \boxed{i^{th} \text{ row of } A} \boxed{\begin{array}{c} j^{th} \\ \text{col} \\ \text{of} \\ B \end{array}} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

$i = 1 \dots m$
$j = 1 \dots \ell$

$\Rightarrow C$ is a $m \times \ell$ matrix.

<u>Exercise:</u> Prove $(AB)C = A(BC)$

$$((AB)C)_{ij} = \sum (AB)_{ik} C_{kj} = \sum\sum A_{iw} B_{wk} C_{kj} = \sum A_{iw} \sum B_{wk} C_{kj}$$
$$= \sum A_{iw} (BC)_{wj}$$
$$= (A(BC))_{ij}$$

<u>Exercise:</u> $P(\lambda Q + \mu R) = \lambda PQ + \mu PR$.

$$P(\lambda Q + \mu R)(\vec{v}) = P(\lambda Q\vec{v} + \mu R\vec{v})$$
$$= \lambda PQ\vec{v} + \mu PR\vec{v}$$

<u>Changes of Coordinate</u>

Let $\{\vec{e_1}, \vec{e_2}, \dots \vec{e_n}\}$ be a set of basis for V.
Let $\{\vec{e_1}', \vec{e_2}', \dots, \vec{e_n}'\}$ be another set of basis for V.

No.: .... Date: ....

For all $\vec{v} \in V$, $\vec{v} = x_1\vec{e_1} + x_2\vec{e_2} + \dots + x_n\vec{e_n}$
$= x_1'\vec{e_1}' + x_2'\vec{e_2}' + \dots x_n'\vec{e_n}'$

Let $\vec{e_j}' = \sum_{i=1}^{n} c_{ij}\vec{e_i}$ The matrix of coefficients is square ($n \times n$).
$j = 1, \dots, n$ expressing new bases in terms of old ones.

$\therefore \vec{v} = x_1'\vec{e_1}' + \dots + x_n'\vec{e_n}' = \sum_{j=1}^{n} x_j'\vec{e_j}' = \sum_i \left(\sum_j c_{ij} x_j'\right)\vec{e_i}$

$\therefore \boxed{x_i = \sum_j c_{ij} x_j'}$

$$\therefore \underset{\text{old coordinates}}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix} \underset{\text{new coordinates}}{\begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix}}$$

$\boxed{\vec{x} = C\vec{x}'}$

Similarly, $\exists D$ s.t. $\vec{x}' = D\vec{x}$

$\vec{x} = C\vec{x}' = CD\vec{x} \Rightarrow \boxed{CD = I_n}$

where I_n is the $n \times n$ identity matrix

Similarly, $CD = I = DC$.

Linear Transformation

:= linear map from a vector space to itself.

$V \longrightarrow V$

$\underset{\text{old}}{\vec{x}} = C\underset{\text{new}}{\vec{x}'}$

C is the transition matrix from ~~old~~ new to ~~new~~ old.

Suppose we have a linear form a,

$a(\vec{x}) = a(C\vec{x}') = (aC)\vec{x}'$ the same linear form in new coordinates

Note that C here goes in the opposite direction! Transforms linear ~~fn~~ forms from old to new, but coordinates from new to old

$$\underset{\text{old}}{\boxed{a_1\ a_2 \cdots a_n}} \begin{bmatrix} & C & \end{bmatrix} = \underset{\text{new}}{\boxed{a_1'\ a_2' \cdots a_n'}}$$

No.:

Date:

$$V \xrightarrow{A} W$$

$\{e_1, e_2, \dots\}$ $\{f_1, f_2, \dots\}$

$$\vec{x} \longmapsto \vec{y} = A\vec{x}$$

$$\vec{y} = A\vec{x}$$

$$\left.\begin{array}{l}\vec{x} = C\vec{x}' \\ \vec{y} = D\vec{y}'\end{array}\right] \Rightarrow D\vec{y}' = \vec{y} = A\vec{x} = AC\vec{x}' \Rightarrow \boxed{\vec{y}' = D^{-1}AC\vec{x}'}$$

$$\therefore A \longmapsto D^{-1}AC = A'$$

(transformation of linear form)

If A is a linear transformation, $\boxed{A \longmapsto C^{-1}AC} = A'$