02/07/2022

MATH 185 LECTURE 10 NOTES

Definition Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$. Then f is complex differentiable if $f'(z) = \lim\limits_{h \to 0} \frac{f(z+h) - f(z)}{h}$ exists.

Definition Let $\Omega \subset \mathbb{C}$ be a region and $f : \Omega \to \mathbb{C}$. If f is complex differentiable in a neighbourhood of $z_0 \in \Omega$, then f is holomorphic / analytic at z_0.
↳ automatically have a power series equal to at that point

If f is complex differentiable on Ω, say f is holomorphic on Ω.

Definition If f is complex differentiable everywhere on $\mathbb{C}$, then f is an entire function.

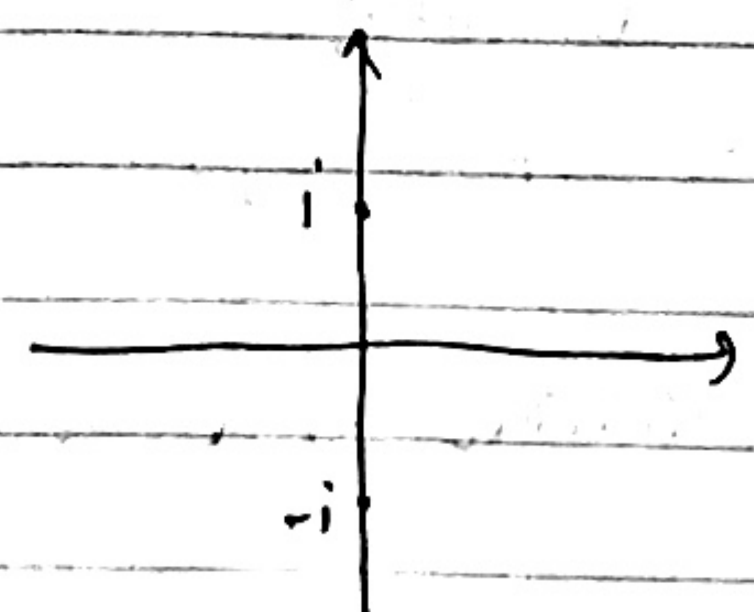
Example $f(z) = \log z$ is not entire since f is not continuous everywhere in $\mathbb{C}$.

$f'(z) = \frac{1}{z} \in H(\mathbb{C} \setminus \{0\})$ larger place on which it is continuous.

Example If I have a power series, automatically holomorphic on its radius of convergence
$f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(B_R(0))$ where $R = \frac{1}{\alpha}$ where $\alpha = \limsup |a_n|^{\frac{1}{n}}$.

Example. If p is a polynomial, then it is entire.

Example $f(z) = \frac{1}{z^2+1} \in H(\mathbb{C} \setminus \{\pm i\})$



in $\mathbb{R}$, it would be infinitely differenti

Example. $f(z) = \begin{cases} \frac{(\bar{z})^2}{z} & z \neq 0 \\ 0 & z = 0 \end{cases}$ Satisfies Cauchy Riemann but is not complex differentiable at $(0,0)$

$$f(z) = f(x+iy) = \frac{(x-iy)^2}{x+iy} = \frac{1}{x^2+y^2}(x-iy)^3 = \overbrace{\frac{x^3-3xy^2}{x^2+y^2}}^{u} + i\overbrace{\left(\frac{y^3-3x^2y}{x^2+y^2}\right)}^{v}$$

$$\cancel{\frac{\partial f}{\partial x}} = \quad u_x = \frac{\partial u}{\partial x} = \frac{(x^2+y^2)(3x^2-3y^2) - (x^3-3xy^2)(2x)}{(x^2+y^2)^2}$$

$$u_x(0,0) = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h} = \frac{\frac{h^3}{h^2} - 0}{h} = 1$$

$$u_y(0,0) = \lim_{h\to 0} \frac{u(0,h)-u(0,0)}{h} = 0$$

$$v_y(0,0) = \lim_{h\to 0} \frac{v(0,h)-v(0,0)}{h} = \frac{\frac{h^3}{h^2} - 0}{h} = 1$$

$$f'(0) = \lim_{h\to 0} \frac{f(h+ih)-f(0)}{h+ih} = \frac{\frac{\overline{(h+ih)}^2}{h+ih} - 0}{h+ih} = \frac{(h-ih)^2}{(h+ih)^2} = \frac{(1-i)^2}{(1+i)^2} = \neq 1$$

$$= \frac{-2i}{2i} = -1$$

Theorem ☆ Let $\Omega \subset \mathbb{C}$ be open, $z_0 \in \Omega$ and $f: \Omega \to \mathbb{C}$.
Suppose f_x, f_y exist and are continuous in a neighborhood z_0 and $if_x = f_y$. Then f is differentiable at z_0.

Example The function $f(z) = e^{-z}$ is entire.

$$f(z) = f(x+iy) = e^{-(x+iy)} = e^{-x} e^{-iy} = u + iv.$$

where $u = e^{-x}\cos(-y) = e^{-x}\cos y$

$v = e^{-x}\sin(-y) = -e^{-x}\sin y.$

$u_x = -e^{-x}\cos y \quad v_x = e^{-x}\sin y$

$u_y = -e^{-x}\sin y \quad v_y = -e^{-x}\cos y.$

$\Big]\ u_x = v_y$, $u_y = -v_x$ $\therefore$ Cauchy Riemann Holds everywhere

Since u_y, u_x, v_y, v_x are continuous on $\mathbb{C}$ and Cauchy Riemann holds everywhere $f(z) = e^{-z}$ is entire.

What is the value of $\frac{d}{dz}e^{-z}$?

$$\frac{d}{dz}e^{-z} = \boxed{-e^{-z}.}$$

Trick!

Ok this is not entirely 0. If f is differentiable, then $f'(z) = \frac{\partial}{\partial x} f$

In this case

$$f'(z) = \frac{\partial}{\partial x} f = u_x + iv_x = -e^{-x}(\cos y - i\sin y) = -e^{-x-iy} = -e^{-z}$$

Example

$f(z) = f(x+iy) = x^3 + i(1-y)^3$ is continuous everywhere but is $\mathbb{C}$ differentiable only at a point. In particular, this function is not holomorphic in any neighborhood of that point.

$$\left.\begin{array}{l} u = x^3 \\ v = (1-y)^3 \end{array}\right] \Rightarrow \begin{array}{l} u_x = 3x^2 \\ u_y = 0 \\ v_x = 0 \\ v_y = 3(1-y)^2(-1) \end{array} \qquad \begin{array}{l} u_x = v_y \\ u_y = -v_x. \end{array} \begin{array}{l} \Rightarrow 3x^2 = -3(1-y)^2 \\ \Rightarrow 3x^2 + 3(1-y)^2 = 0 \\ \Rightarrow \boxed{\begin{array}{l} x = 0 \\ y = 1 \end{array}} \end{array}$$

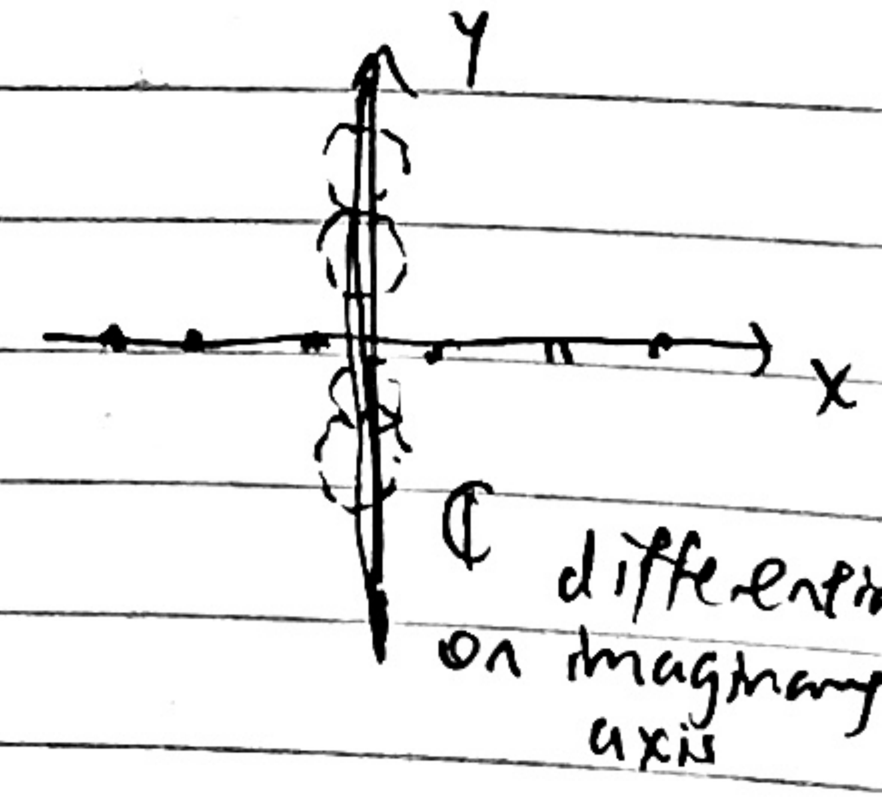
for differentiable

and Cauchy Riemann is satisfied

Since u_x, v_y, u_y, v_x are continuous, the function is complex differentiable at $(0,1)$.
(i.e. u, v smooth)
Partial derivatives exist and are continuous

Example

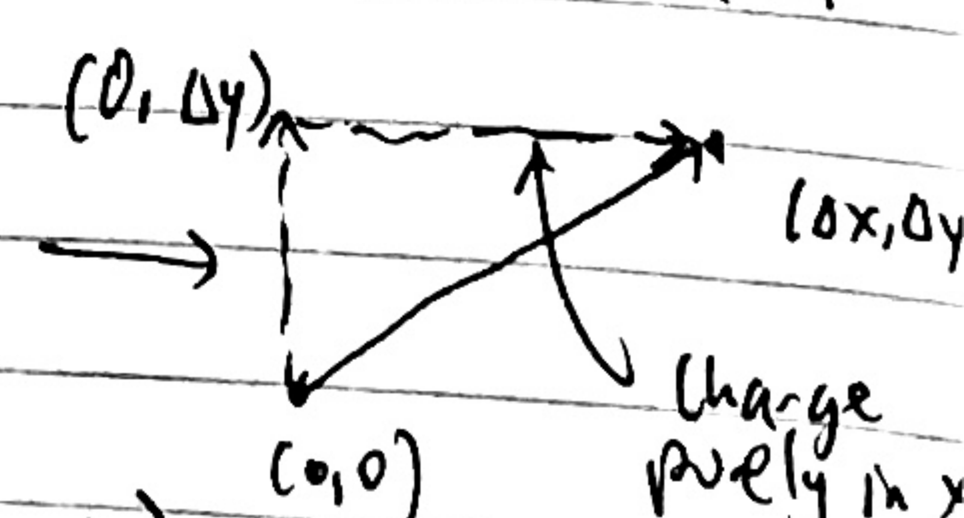
The function $f(z) = f(x+iy) = x^2+y^2-2xyi$ is $\mathbb{C}$ differentiable at infinitely many points, but nowhere holomorphic.



$f(z) = f(x+iy) = x^2+y^2-2xyi$

$$\left.\begin{array}{l} if_x = i(2x-2yi) = 2ix+2y \\ f_y = 2y-2xi \end{array}\right] \text{ hold when } x=0$$

Since u_x, u_y, v_x, v_y are continuous and CR holds, differentiable everywhere on the imaginary axis. But nowhere holomorphic.



Proof of ★ key idea: Mean Value Theorem.

Let $z_0 = 0$. and $h = \Delta x + i\Delta y$.

$$\frac{f(h)-f(0)}{h} = \frac{u(\Delta x, \Delta y) - u(0,0)}{\Delta x + i\Delta y} + i\left(\frac{v(\Delta x,\Delta y) - v(0,0)}{\Delta x + i\Delta y}\right)$$

note: $u(\Delta x, \Delta y) - u(0,0) = \underbrace{u(\Delta x, \Delta y) - u(0,\Delta y)} + \underbrace{u(0,\Delta y) - u(0,0)}$

Since partial derivatives are continuous, we can apply MVT (to be continued on wednesday)