11/23/2021

ε-P
uniform continuity

- integrability
- basic properties

MATH 104 LECTURE 23 PROPERTIES OF THE DARBOUX INTEGRAL

Recall Worked on bounded functions on a finite closed interval $f:[a,b] \to \mathbb{R}$.
Wanted to define $\int_a^b f$ if possible

For a partition $P = \{a = t_0 < \cdots < t_n = b\}$

$$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1})$$

M: sup value, m: inf value } max, min might not exist

$$L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k])(t_k - t_{k-1})$$

$U(f) = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}$
$L(f) = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}$
} consider all partitions

Say f is integrable if $U(f) = L(f)$.

Properties:
- $L(f,P) \leq L(f) \leq U(f) \leq U(f,P)$
- f is integrable if ε-P property holds: i.e. $\forall \varepsilon > 0$, $\exists$ partition P s.t. $U(f,P) - L(f,P) < \varepsilon$.

Qn
(1) Which f is integrable?
(2) If f is integrable, how to compute? $\int_a^b f$

Theorem Consider a function on a finite closed interval $f:[a,b] \to \mathbb{R}$. Assume either (1) f is bounded and monotone OR (2) f is continuous. Then f is integrable.

Two ways to prove integrability: (1) Definition
(2) ε-P

ε-P usually used for theoretical proofs, in which you do not need the value of integration (since ε-P only computes the difference).

Note For $P = \{a = t_0 < \cdots < t_n = b\}$

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} \left(M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])\right)(t_k - t_{k-1})$$

Proof (1):

For simplicity, assume f is increasing (decreasing is similar)
Since f is bounded, exist M, m s.t. $m < f(x) < M$ for every $x \in [a,b]$.

Take $\varepsilon > 0$. Take $n \in \mathbb{N}$ s.t. $n = \frac{\varepsilon}{M-m}$.
Take a partition $P = \{a = t_0 < \cdots < t_n = b\}$ s.t. $t_k - t_{k-1} < \frac{\varepsilon}{M-m}$ For example can consider equipartition

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} \left(M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])\right)(t_k - t_{k-1})$$

$$< \sum_{k=1}^{n} \left[f(t_k) - f(t_{k-1})\right] \frac{\varepsilon}{M-m} = \frac{\varepsilon}{M-m}\left[f(t_n) - f(t_0)\right]$$

$$\cancel{= \frac{\varepsilon}{M-m}(M-m) = \varepsilon}. < \frac{\varepsilon}{M-m}(M-m) = \varepsilon.$$

where the last inequality comes from $m < f(t_0) \leq f(t_n) < M$

Proof (2): Assume f is continuous on $[a,b]$, then f is uniformly continuous.
Take $\varepsilon > 0$. $\exists \delta > 0$ s.t. $\forall x,y \in [a,b]$, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a}$

Take a partition $P = \{a = t_0 < \cdots < t_n = b\}$ with the property that $t_k - t_{k-1} < \delta$.
Then

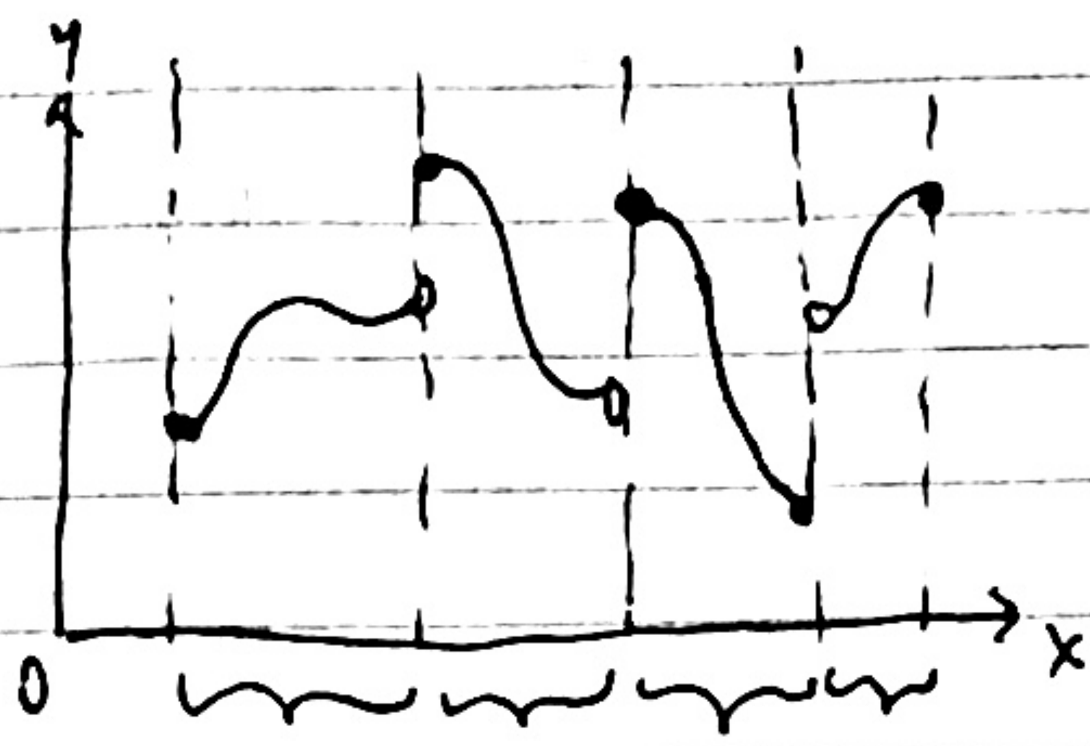
$$U(f,P) - L(f,P) = \sum_{k=1}^{n} \left(M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])\right)(t_k - t_{k-1})$$

f is continuous on $[t_{k-1}, t_k] \Rightarrow f$ achieves min and max (say at P, Q).
Then $M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k]) = f(Q) - f(P)$.
Since $|P-Q| \leq t_k - t_{k-1} < \delta$, $|f(P) - f(Q)| < \frac{\varepsilon}{b-a}$.

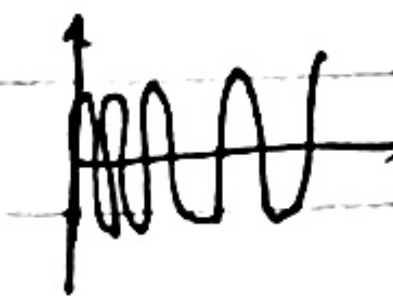
$$\therefore U(f,P) - L(f,P) < \sum_{k=1}^{n} \frac{\varepsilon}{b-a}(t_k - t_{k-1}) = \frac{\varepsilon}{b-a}\sum_{k=1}^{n}(t_k - t_{k-1}) = \frac{\varepsilon}{b-a}(t_n - t_0)$$

$$= \frac{\varepsilon}{b-a}(b-a) = \varepsilon. \quad \text{(as desired)}$$

Definition: A bounded function f $f:[a,b] \to \mathbb{R}$ is called piecewise continuous if there is a partition $P = \{a = t_0 < \cdots < t_n = b\}$ s.t. for each k, f on (t_{k-1}, t_k) admits a continuous extension on $[t_{k-1}, t_k]$.



Consider $f(x) = \begin{cases} \sin\frac{1}{x} & 0 < x \le 1 \\ 0 & x = 0 \end{cases}$



Equivalent to: If you assign the value at 0, can you make the function continuous?

Theorem Every piecewise continuous function on $[a,b]$ is integrable.

Theorem Let $f: [a,b] \to \mathbb{R}$ be a bounded function. Take $a < c < b$.

[READING ASSIGNMENT] f is integrable on $[a,b] \Leftrightarrow f$ is integrable both on $[a,c]$ & $[c,b]$.

In this case, $\int_a^b f = \int_a^c f + \int_c^b f$

Basic Properties

Theorem Suppose we have functions $f, g: [a,b] \to \mathbb{R}$ which are bounded. Assume f, g are integrable. Then

(1) cf, $f+g$ are integrable (forms a vector space)

$$\int_a^b cf = c\int_a^b f \quad \text{and} \quad \int_a^b f+g = \int_a^b f + \int_a^b g$$

(2) If $f(x) \geq g(x)$ $\forall x \in [a,b]$, then $\int_a^b f(x) \geq \int_a^b g(x)$

(3) $|f|$ is integrable and $\left|\int_a^b f\right| \leq \int_a^b |f|$

<u>Proof of (1) for $f+g$</u> (Cannot just use ε-P property since ε-P property does not give value)

Suffices to show $U(f+g) = \int_a^b f + \int_a^b g = L(f+g)$

Note: $U(f+g) = \inf\{U(f+g, P) : P \text{ is a partition of } [a,b]\}$

Since $\int_a^b f + \int_a^b g = U(f) + U(g)$ and $U(f) + U(g) = \inf\{U(f,P_1)\} + \inf\{U(g,P_2)\}$
$= \inf\{U(f,P_1) + U(g,P_2) : P_1, P_2 \text{ are partitions}\}$

Take any partition P_1, P_2, set $P = P_1 \cup P_2$.
P is another partition: a refinement of P_1 & P_2.

Write $P = \{a = t_0 < \cdots < t_n = b\}$.

$U(f+g, P) = \sum_{k=1}^{n} M(f+g, [t_{k-1}, t_k])(t_k - t_{k-1})$ $\qquad M(f, [t_{k-1}, t_k]) + M(g, [t_{k-1}, t_k])$

$\sup\{f(x) + g(x) : t_{k-1} \le x \le t_k\} \le \sup\{f(x) : t_{k-1} \le x \le t_k\} + \sup\{g(y) : t_{k-1} \le y \le t_k\}$

$$\therefore U(f+g,P) \leq \sum_{k=1}^{n} \Big[M(f,[t_{k-1},t_k]) + M(g,[t_{k-1},t_k])\Big](t_k - t_{k-1})$$

$$= U(f,P) + U(g,P) \leq U(f,P_1) + U(g,P_2)$$

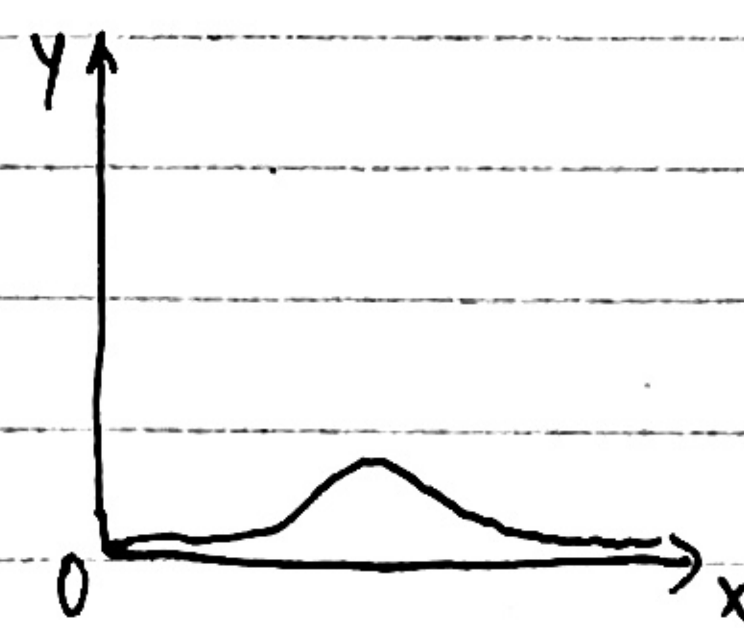
Hence, $U(f+g) \leq U(f) + U(g)$.
Similarly, $L(f+g) \geq L(f) + L(g)$.
Since f, g are integrable, $U(f) = L(f)$, $U(g) = L(g)$, so

$$U(f) + U(g) \geq U(f+g) \geq L(f+g) \geq L(f) + L(g) = U(g) + U(f)$$

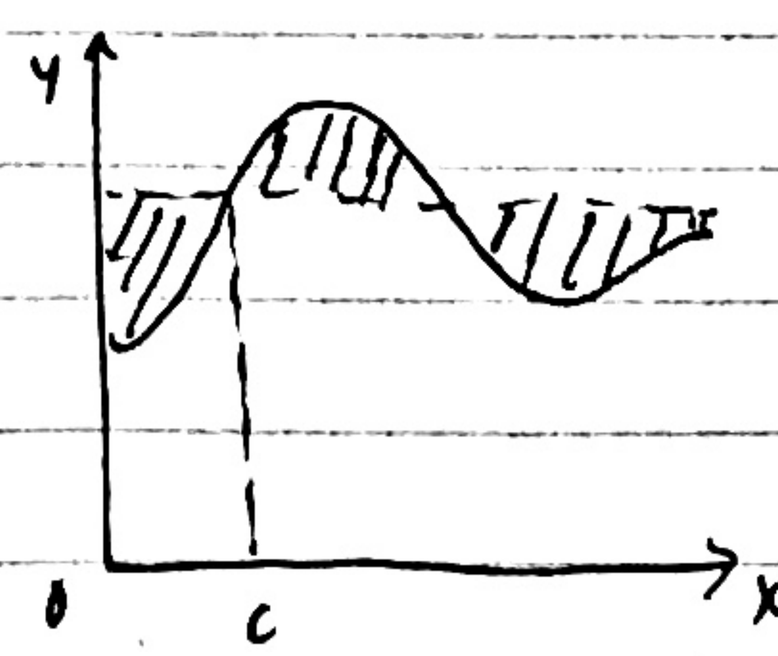
$$\therefore U(f+g) = L(f+g) = U(f) + U(g).$$

$$\therefore \boxed{\int_a^b f+g = \int_a^b f + \int_a^b g}$$

Theorem Let $f : [a,b] \to \mathbb{R}$ be continuous. (hence integrable)
(1) If $f(x) \geq 0 \ \forall x \in [a,b]$ and $\int_a^b f = 0$. Then $f(x) = 0 \ \forall x \in [a,b]$
(2) [Intermediate Value Theorem for Integral].
$\exists c \in [a,b]$ s.t. $f(c) = \frac{1}{b-a}\int_a^b f$.



Proof (2)
Let M be the max value of f on $[a,b]$
and m be the min value of f on $[a,b]$
(Since f is continuous, max, min exist)



If $M = m$, f is constant $\Rightarrow$ any c works.
Else, $M > m$. Then $m \leq f(x) \leq M \ \forall x \in [a,b]$

$$\Rightarrow \int_a^b m \leq \int_a^b f(x) \leq \int_a^b M$$

$$\Rightarrow \frac{\int_a^b m}{b-a} \leq \frac{\int_a^b f(x)}{b-a} \leq \frac{\int_a^b M}{b-a} \Rightarrow m \leq \frac{1}{b-a}\int_a^b f(x) \leq M$$

Since $\exists p$ s.t. $f(p) = m$ and $\exists q$ s.t. $f(q) = M$. By intermediate value theorem,
$\exists c \in [p,q]$ s.t. $f(c) = \frac{1}{b-a}\int_a^b f(x)$. ■