Prep: bring ID, water, pen, jacket and this sheet

You got this!

Conditional Expectations and Random Sum

- $\mathbb{P}[X=x]=\sum_i \mathbb{P}[X=x|Y=i]\mathbb{P}[Y=i]$
- $f_X(x)=\sum_i f_{X|N}(x|n)\mathbb{P}[N=n]$
  - $\mathbb{E}[X]=\sum_i \mathbb{E}[X|Y]\,\mathbb{P}[Y=i]$
  - $\mathbb{E}[X]=\mathbb{E}\big[\mathbb{E}[X|Y]\big]$
  - $\mathrm{Var}[X]=\mathrm{Var}\big[\mathbb{E}[X|Y]\big]+\mathbb{E}\big[\mathrm{Var}[X|Y]\big]$
- $\mathbb{P}[\cup_i A_i]\le\sum_i \mathbb{P}[A_i]$
- Let $S=\sum_{i=1}^N X_i$ where N, X_i independent, then $\mathbb{E}[S]=\mathbb{E}[N]\mathbb{E}[X_i]$
  - If X, Y independent and $Z=X+Y$, then $f_Z(z)=\int_{-\infty}^{\infty} f_X(x)f_Y(z-x)\mathrm{d}x$

Random Walk

- [Stirling] $n!\sim n^n e^{-n}\sqrt{2\pi n}$
- [$\mathbb{Z}^1$] $P_{0,0}^{(2n)}=\binom{2n}{n}p^nq^n\sim\frac{(4pq)^n}{\sqrt{\pi n}}$
  Only recurrent when $p=q=\frac{1}{2}$
- [$\mathbb{Z}^2$] $P_{0,0}^{(2n)}=\binom{2n}{n}p^nq^n\sim\frac{(4pq)^n}{\sqrt{\pi n}}$
  Only recurrent when symmetric
- [$\mathbb{Z}^3$] All transient

Gambler's Ruin

- [$p=q=\frac{1}{2}$] $\mathbb{P}[X_T=0|X_0=k]=\frac{N-k}{N}$
- $\mathbb{E}[T|X_0=k]=k(N-k)$
- [$p\ne\frac{1}{2}$] $\mathbb{P}[X_T=0|X_0=i]=\left(\frac{q}{p}\right)^k\frac{1-\left(\frac{q}{p}\right)^{N-k}}{1-\left(\frac{q}{p}\right)^N}$

Final Checks and Last Resorts

- Check you modelled the problem correctly
- Return what the problem asked for
- First-step and previous-step analysis
- Understand dynamics of MC first
- Method of differences
- GFs: go back to first principle
- If have time, rigorously prove MC with limits

Branching Process and Generating Functions

- $X_{n+1}=\sum_{i=1}^{X_n}\xi_i^{(n)}$ with $X_0=1$
- $\mathbb{E}[X_{n+1}]=\mu\mathbb{E}[X_n]$ where $\mu=\mathbb{E}[\xi]$
- $\mathbb{E}[X_n]=\mu^n$
- $\mathrm{Var}[X_{n+1}]=\sigma^2\mathbb{E}[X_n]+\mu^2\mathrm{Var}[X_n]$
- $\mathrm{Var}[X_n]=\begin{cases}\sigma^2 n, & \mu=1\\ \sigma^2\mu^{n-1}\frac{1-\mu^n}{1-\mu}, & \mu\ne1\end{cases}$
- [Time of Extinction] $N=\min_n\{n|X_n=0\}$
- [Extinction Probability] $u_n=\mathbb{P}[N\le n]=\mathbb{P}[X_n=0]$ i.e. $u_0=0$, $u_1=p_0$
- $0\le u_n\le u_{n+1}\le1\ \forall n$ (monotonicity)
- $u_n=\sum_{k=0}^{\infty}p_k(u_{n-1})^k=\phi_\xi(u_{n-1})$
- u_∞ is the smallest solution to $u=\phi_\xi(u)$
- $\phi_{X_{n+1}}(s)=\phi_{X_n}\left(\phi_\xi(s)\right)$
- If $\phi_{X_0}(s)=s$, $\phi_{X_n}(s)=\phi_\xi^n(s)$
- [Pure Death Process] $\phi_\xi(s)=(1-p)+ps$; $\phi_{X_n}(s)=1-p^n+p^ns$
- [Time of extinction] $\mathbb{P}[T=n|X_0=k]=\mathbb{P}[X_n=0|X_0=k]-\mathbb{P}[X_{n-1}=0|X_0=k]=\phi_{X_n}(0)-\phi_{X_{n-1}}(0)$

Case	Result	Remarks
$\mu<1$	$u_\infty=1$	1 is sole fixed point
$\mu=1,\sigma^2=0$	$u_\infty=0$	Self-regeneration
$\mu=1,\sigma^2>0$	$u_\infty=1$	$\phi_\xi''(1)>0$
$\mu>1$	$u_\infty<1$	$\phi_\xi'(1)=\mu>1$

- $\phi_X(s)=\mathbb{E}[s^X]=\sum_{x=0}^{\infty}s^x\mathbb{P}[X=x]$; $s\in[0,1]$
- $\phi_X(0)=\mathbb{P}[X=0]$, $\phi_X(1)=1$
- $\mathbb{P}[X=k]=\frac{1}{k!}\phi_X^{(k)}(0)$
- If $X=\xi_1+\cdots+\xi_n$ and ξ_i are independent, then $\phi_X(s)=\prod_{i=1}^n\phi_{\xi_i}(s)$
- $\phi_X'(1)=\mathbb{E}[X]$
- $\phi_X''(1)=\mathbb{E}[X^2]-\mathbb{E}[X]$
- $\mathrm{Var}[X]=\phi_X''(1)+\phi_X'(1)-\left(\phi_X'(1)\right)^2$
- If $X=\sum_{i=1}^N\xi_i$, then $\phi_X(s)=\phi_N\left(\phi_\xi(s)\right)$
- One-to-one correspondence with PMF

Markov Chain (Definitions)

- [Regularity] P regular if $\exists n$ s.t. $P_{i,j}^n>0\ \forall i,j$
- [Stationary Distribution] π s.t. $\pi P=\pi$
- [Limiting Distribution] $\lim_{n\to\infty}\tau P^n=\pi\ \forall\tau$
- [Accessible] $i\to j\Rightarrow\exists n>0$ s.t. $P_{i,j}^{(n)}>0$

Markov Chain (Theorems)

- Period and recurrence/transience are class properties i.e. $i\leftrightarrow j\Rightarrow d(i)=d(j)$
- $\exists N$ s.t. $\forall n>N$, $P_{i,i}^{(nd(i))}>0$

- [Communicate] $i \leftrightarrow j \Leftrightarrow i \to j, j \to i$
- [Irreducible] $\forall i, j \in S$, $\exists n$ s.t. $P_{i,j}^{(n)} > 0$
- [Period] $d(i) = \gcd\left\{n | P_{ii}^{(n)} > 0\right\}$
  - If $P_{ii}^{(n)} = 0\ \forall n$, define $d(i) = 0$.
  - [Aperiodic] $d(i) = 1$
- [Probability of return in n steps $f_{i,i}^{(n)}$]
  $f_{i,i}^{(n)} \coloneqq \mathbb{P}[X_n = i, X_{n-1} \neq i, \dots, X_1 \neq i | X_0 = i]$
- [Probability of return $f_{i,i}$]
  $$f_{i,i} = \sum_{k=0}^{\infty} f_{i,i}^{(k)}$$
- [Recurrent] State i recurrent $\Leftrightarrow f_{i,i} = 1$
  - $\Leftrightarrow \mathbb{P}[R_i < \infty | X_0 = i] = 1$
  - $\Leftrightarrow \sum_{m=0}^{\infty} P_{i,i}^{(m)} = \infty$
  - $\Leftrightarrow \lim_{m\to\infty} \prod_{j=0}^{m} \left(1 - P_{i,i}^{(j)}\right) = 0$
- [Transient] State i transient $\Leftrightarrow f_{i,i} < 1$
  - $\Leftrightarrow \mathbb{P}[R_i < \infty | X_0 = i] < 1$
  - $\Leftrightarrow \sum_{m=0}^{\infty} P_{i,i}^{(m)} < \infty$
  - $\Leftrightarrow \lim_{m\to\infty} \prod_{j=0}^{m} \left(1 - P_{i,i}^{(j)}\right) > 0$
- $R_i = \min\{n > 0 | X_n = i\}$
- $m_i = \mathbb{E}[R_i | X_0 = i]$
- [Positive Recurrent] $m_i < \infty$
- [Null Recurrent] $m_i = \infty$
- M_i is number of visits to state i
- $M_i = \sum_{k=1}^{\infty} \mathbb{1}\{X_k = i\}$
- $\mathbb{E}[M_i | X_0 = i] = \sum_{k=1}^{\infty} \left(f_{i,i}\right)^k$

- [Finite] Regular finite MC have unique limiting distribution.
- [Infinite/Mean Return Time] Let $(X_n)_{n\geq 0}$ be an irreducible aperiodic recurrent MC, then $\lim_{n\to\infty} P_{ii}^{(n)} = \frac{1}{m_i}$ and $\lim_{n\to\infty} P_{ji}^{(n)} = \frac{1}{m_i}$
- [Infinite/Limiting] In an irreducible, aperiodic, positive recurrent MC, $\exists$ unique limiting distribution π (satisfying $\pi P = \pi$)
- [Infinite/Positive] In a positive recurrent aperiodic class, $\lim_{n\to\infty} P_{jj}^{(n)} = \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}$ and π is uniquely determined by the equations $\pi_i \geq 0$, $\sum_{i=0}^{\infty} \pi_i = 1$ and $\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}$
- [Infinite/Transient] In an irreducible, aperiodic, transient MC, $\lim_{n\to\infty} \mathbb{P}[X_n = j | X_0 = i] = 0\ \forall i, j$
- [Infinite/Null] In an irreducible, aperiodic, null recurrent MC, $\lim_{n\to\infty} \mathbb{P}[X_n = j | X_0 = i] = 0\ \forall i, j$
- [Infinite/Periodic] In an irreducible, periodic MC with period d, $P_{i,i}^{m} = 0$ if $d \nmid m$. Else, $\lim_{n\to\infty} P_{i,i}^{nd} = \frac{d}{m_i}$