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MATH H110 LECTURE 5 NOTES

$\vec{u}, \vec{v} \in V \quad \lambda, \mu$

vector space — A vector space is a set V equipped with operations of addition '+' and multiplication by scalars satisfying axioms.

(1) $\vec{u} + \vec{v} \in V$ (sum of vectors also $\in V$)
$\lambda \vec{u} \in V$ (multiplication by scalars also $\in V$)
(2) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ for $\vec{u}, \vec{v}, \vec{w} \in V$
(3) There exists the vector $\vec{0}$ s.t. $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$
(4) For every $\vec{u} \in V$, there exists $-\vec{u}$ s.t. $(-\vec{u}) + \vec{u} = \vec{0}$ (for each vector there exists an opposite vector)
(5) $(\lambda + \mu)(\vec{u} + \vec{v}) = \lambda\vec{u} + \lambda\vec{v} + \mu\vec{u} + \mu\vec{v} \quad \forall \lambda, \mu \quad \vec{u}, \vec{v} \in V$
(6) $(\lambda\mu)\vec{u} = \lambda(\mu\vec{u})$
(7) $0 \cdot \vec{u} = \vec{0} \quad 1\vec{u} = \vec{u} \quad \forall \vec{u} \in V$

$\mathbb{Q}, \mathbb{R}(x)$

Scalars — Scalars form a field, which we will denote as $\mathbb{K}$. $\mathbb{K}$ can be $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{Z}_p$ ← integers mod prime
In a field, every nonzero element must have a multiplicative inverse. || $\mathbb{Z}_2 = \{0, 1\}$.

Hence, $\mathbb{Z}, \mathbb{Z}_m, \mathbb{R}[x]$ are not fields
For example $\mathbb{Z}_6 = \{0,1,2,3,4,5\}$ but $2 \times 3 = 0$!

Example of the Axioms

Show: ~~$\vec{u} + \vec{u} = 0$~~ $(-1)\vec{u} = -\vec{u}$.

$-\vec{u} = -\vec{u} + \vec{0} = -\vec{u} + 0 \cdot \vec{u} = -\vec{u} + (1 + (-1))\vec{u}$
$= -\vec{u} + 1 \cdot \vec{u} + (-1)\vec{u} = (-\vec{u} + \vec{u}) + (-1)\vec{u} = (-1)\vec{u}$.

The definitions of vector space is abstract in 2 ways: what is the set?, and what are the operations $+, \times$? (doubly abstract).

$+, \times$ with functions

Let S be any set and $\mathbb{K}^S$ denote the set of all functions $S \to \mathbb{K}$
$f, g \in \mathbb{K}^S$.
$(f+g)(s) := f(s) + g(s)$
$(\lambda f)(s) := \lambda f(s)$

can be any set of objects

$\mathbb{R}^{\mathbb{R}}$: all functions from $\mathbb{R} \to \mathbb{R}$
V^S: all functions from $S \to V$
$V^{\mathbb{R}}$: all functions from $\mathbb{R} \to V$

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If we consider $S = \{1, 2, \dots n\} \xrightarrow{x} \mathbb{K}$. Consider $(x(1), x(2), \dots, x(n))$

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right\} = \mathbb{K}^n \text{ (standard n space)}$$

$S = \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} = \{(i, j)\}$ $\begin{bmatrix} & \\ & \end{bmatrix}$ upper triangular
lower triangular

m×n matrices

Subspace Let V be a $\mathbb{K}$ vector space.
$W \subset V$ is a subspace if $\vec{u}, \vec{v} \in W$, $\lambda, \mu \in \mathbb{K} \Rightarrow \lambda \vec{u} + \mu \vec{v} \in W$.

W satisfies all axioms of vector space as well.
$\vec{u} \in W \Rightarrow 0 \cdot \vec{u} = \vec{0} \in W$
$\vec{u} \in W$, $(-1)\vec{u} \in W$ $(-1)\vec{u} = -\vec{u} \in W$.

Morphism Morphisms are linear maps
$A : V \to W$ is linear if it sends linear combination of vectors to linear combination of their images

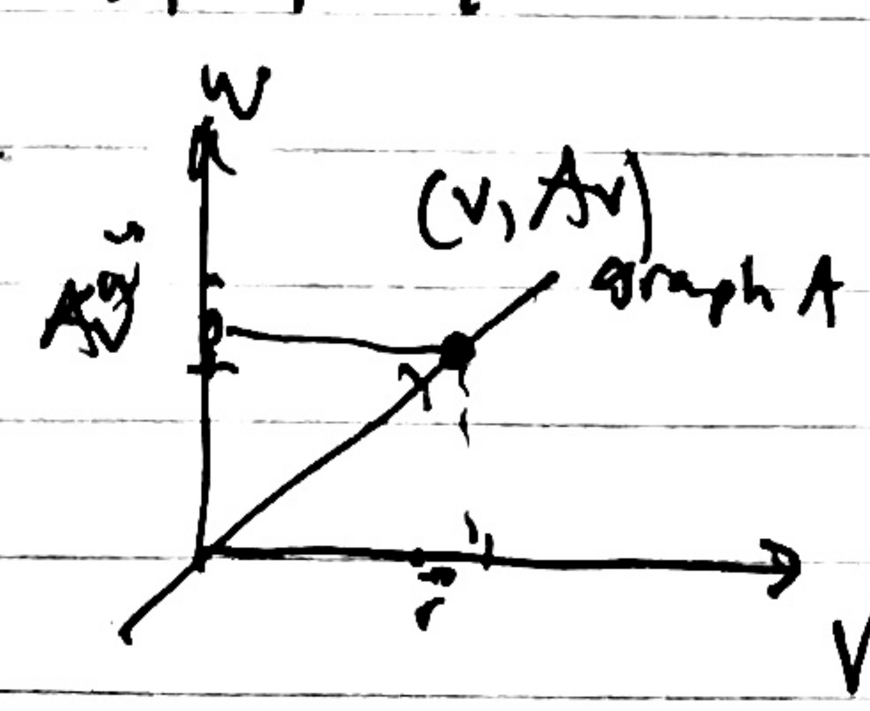
$A(\lambda \vec{u} + \mu \vec{v}) = \lambda A(\vec{u}) + \mu A(\vec{v})$ $\forall \vec{u}, \vec{v} \in V$ $\forall \vec{u}, \vec{v} \in V$
$\lambda, \mu \in \mathbb{K}$

Kernel $\operatorname{Ker} A := \{\vec{v} \in V \mid A\vec{v} = 0\}$. is a subspace
Any linear combination of objects in Ker A maps to 0.

Range $A(V) \subset W$ is also a subspace.

Direct Sum $V \oplus W = \{(\vec{v}, \vec{w}) \mid \vec{v} \in V, \vec{w} \in W\}$ $\mathbb{K}^n = \mathbb{K} \oplus \cdots \oplus \mathbb{K}$.
Operations defined s.t. $(\vec{v}_1, \vec{w}_1) + (\vec{v}_2, \vec{w}_2) = (\vec{v}_1 + \vec{v}_2, \vec{w}_1 + \vec{w}_2)$

Graph Graph is a subspace of direct sum.
Graph $A = \{(\vec{v}, A\vec{v})\}$



Surjective $A(V) = W$

Injective 1-to-1. iff $\operatorname{Ker} A = \{\vec{0}\}$. If $\exists$ 2 objects in Ker $A \Rightarrow$ not injective
If $A\vec{x} = A\vec{y}$, $\vec{x} \neq \vec{y}$, $A(\vec{x} - \vec{y}) = 0$

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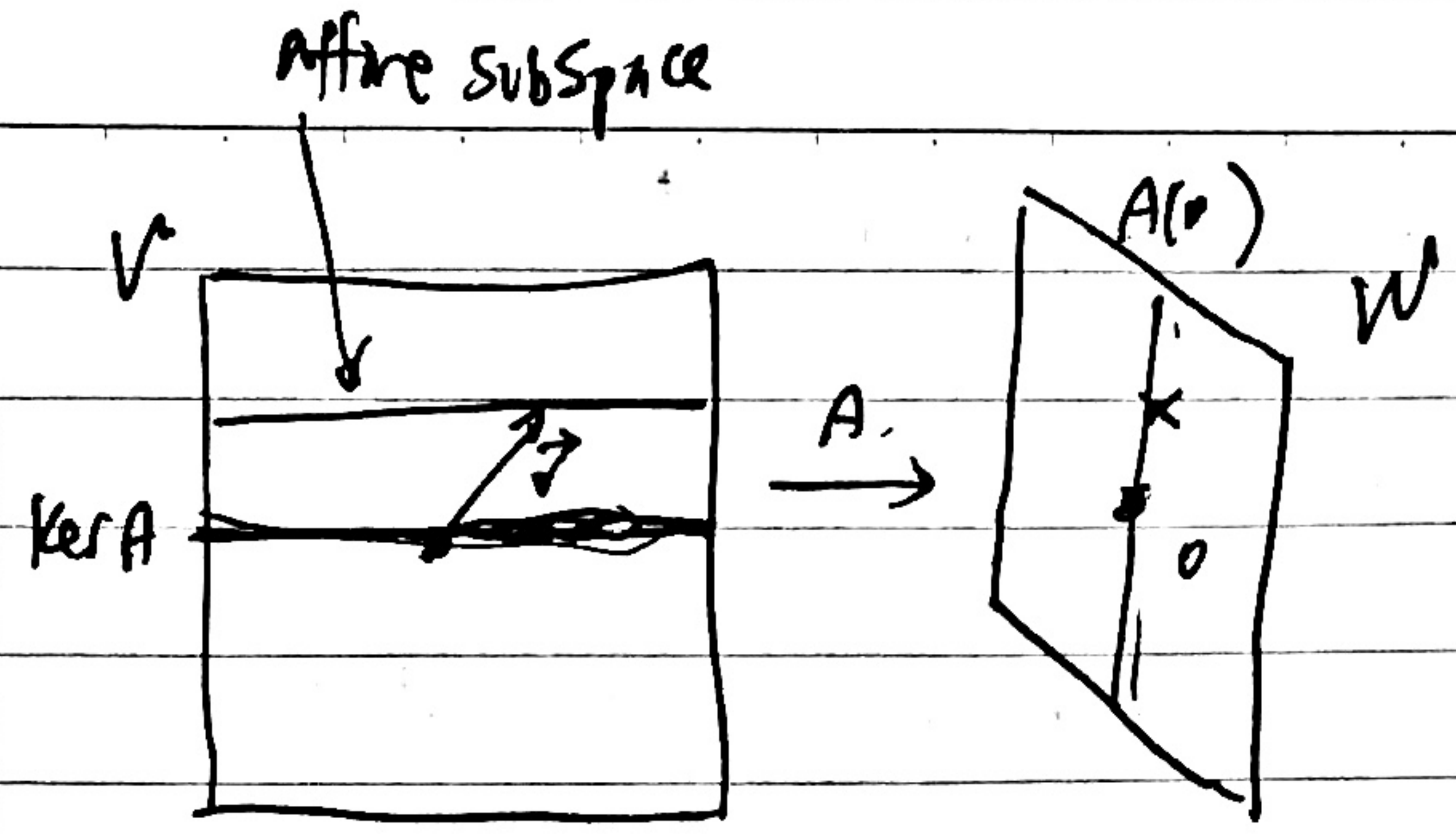

consider the vector space $W^V = \{ V \xrightarrow{f} W \}$. (f not necessarily linear)

$Hom(V, W) = \{ V \xrightarrow{A} W,\ A$ is linear function$\}$. (Hom is homomorphism, linear in linear alegebra)

$Hom(V, W)$ is a subspace of W^V $(Hom(V,W) \subset W^V)$

Suppose $A, B \in Hom(V, W)$

$$kA(\lambda \vec{u} + \mu \vec{v}) = k(\lambda A(\vec{u}) + \mu A(\vec{v}))$$
$$= k\lambda A(\vec{u}) + k\mu A(\vec{v}) = \lambda(kA)(\vec{u}) + \lambda(kA)(\vec{v})$$

Sum of two linear functions perhaps is linear.

consider the linear ~~maps~~ functions $V \to \mathbb{K}^1 = \mathbb{K}$

$A \in \{ V \to \mathbb{K} \} = Hom(V, \mathbb{K}) = V^*$ (dual space)

Example: The space $\mathbb{K}^n$ consists of $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

$$A\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = x_1 A\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) + x_2 A\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) + \dots + x_n A\left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\right).$$

we can choose $A\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) \dots A\left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}\right)$, say $a_1, a_2, \dots a_n$.

$$A\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = a_1 x_1 + a_2 x_2 + \dots + a_n x_n.$$

In particular, we can choose $a_1 = 1,\ a_2 = a_3 = \dots = 0.$ $\Rightarrow A\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = x_1$

These set of n functions form a basis for the dual space.

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Consider a vector space V and its dual space V^*.

$\vec{v} \in V$

$f \in V^*$. Since $f \in V^*$, $f(\vec{v}) \in \mathbb{K}$

Instead of thinking of acting f on $\vec{v}$, we fix $\vec{v}$ and treat it as a pointwise evaluation of $\vec{v}$.

$E_{\vec{v}}$ is the evaluation of all linear function on a given vector

$$\vec{v} \longmapsto E_{\vec{v}}(f) = f(\vec{v})$$

Claim: $E_{\vec{v}}$ is a linear function of V^*

$$E_{\vec{v}}(\lambda f + \mu g) = \lambda f(\vec{v}) + \mu g(\vec{v}) = \lambda(E_{\vec{v}}(f)) + \mu(E_{\vec{v}}(g)) \Rightarrow E_{\vec{v}} \text{ is linear}$$

$$\therefore E_{\vec{v}} \in (V^*)^*$$

For each vector $\vec{v}$, there exists an element V^{**} ($E_{\vec{v}}$). We can associate each vector $\vec{v}$ to an element in the dual dual space

$$\begin{tikzcd} V \arrow[r] \arrow[rr, bend left, "E"] & V^* \arrow[r, "E_{\vec{v}}"] & V^{**} \end{tikzcd}$$

$\vec{v}$ $\qquad$ f

$$E_{(\lambda\vec{v}+\mu\vec{w})}(f) = f(\lambda\vec{v} + \mu\vec{w}) = \lambda f(\vec{v}) + \mu f(\vec{w})$$
$$= \lambda E_{\vec{v}}(f) + \mu E_{\vec{w}}(f)$$

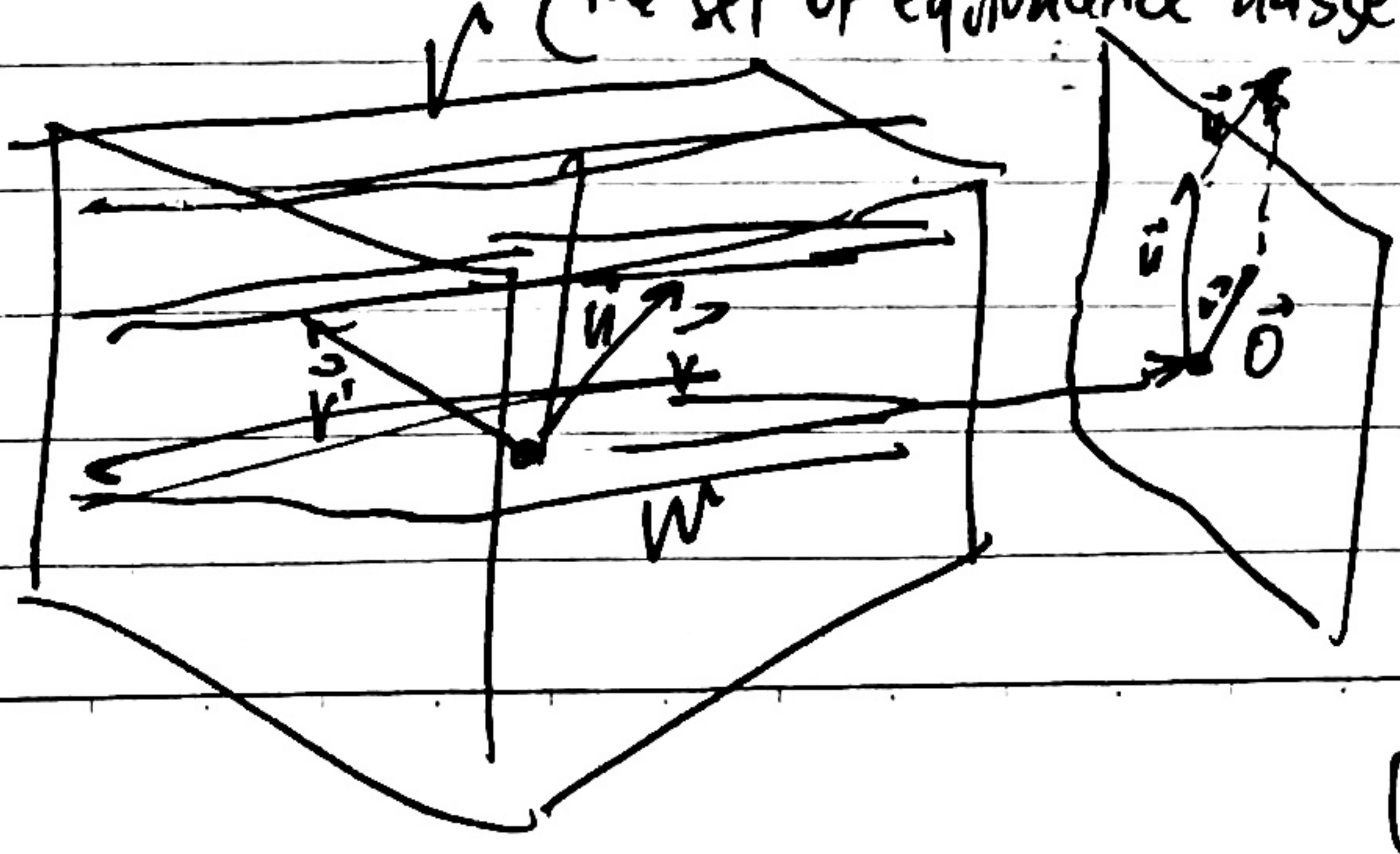
$$\therefore V \xrightarrow{E} (V^*)^*$$

Easy to check that it is injective (consider the kernel)

Quotient Space

Let W be a subspace of V. $W \subset V$

Then V/W is the quotient space. (the set of equivalence classes)



equivalent

$\vec{v} \sim \vec{v}'$ if $\vec{v} - \vec{v}' \in W$

Partitions into equivalence classes

Quotient is the set of equivalent classes

$\vec{v} + W$.

If $\vec{v} \in W$, then $\vec{v} + W = W$.

Otherwise disjoint.

$(\vec{v} + W) + (\vec{u} + W) = (\vec{u} + \vec{v}) + W$

Quotient space comes with a linear mapping

$$V \xrightarrow{\pi} V/W$$

Note W is the null space of the quotient space.

Consider $\mathbb{R}[x]$ / polynomials divisible by x^2+1.

claim : the above $= \mathbb{C}$

$P \mapsto P(i) = 0 \quad P(-i) = 0.$

$(x^2+1)Q \mapsto 0 \cdot Q(i)$

Any linear map from $V \to W$

$$A : \begin{tikzcd} V \arrow[r, "\tilde{A}"] \arrow[d, "\pi"'] & W \\ V/\ker A \arrow[r, "\simeq"] & A(V) \arrow[u, hook] \end{tikzcd}$$

can be decomposed into

equivalence