ARR week Review:
Tuesday 9:40 – 11:00
Thursday 8:10 – 9:30

12/02/2021

MATH 104 LECTURE NOTES 25 INTEGRATION AND DIFFERENTIATION OF POWER SERIES

Recall What we have discussed in the course:

- $\mathbb{R}$, completeness axiom
- sequences (ε-N)
- series
- continuity / uniform continuity (ε-δ)
- limits of functions
- uniform convergence
- power series
- differentiation
- Integration

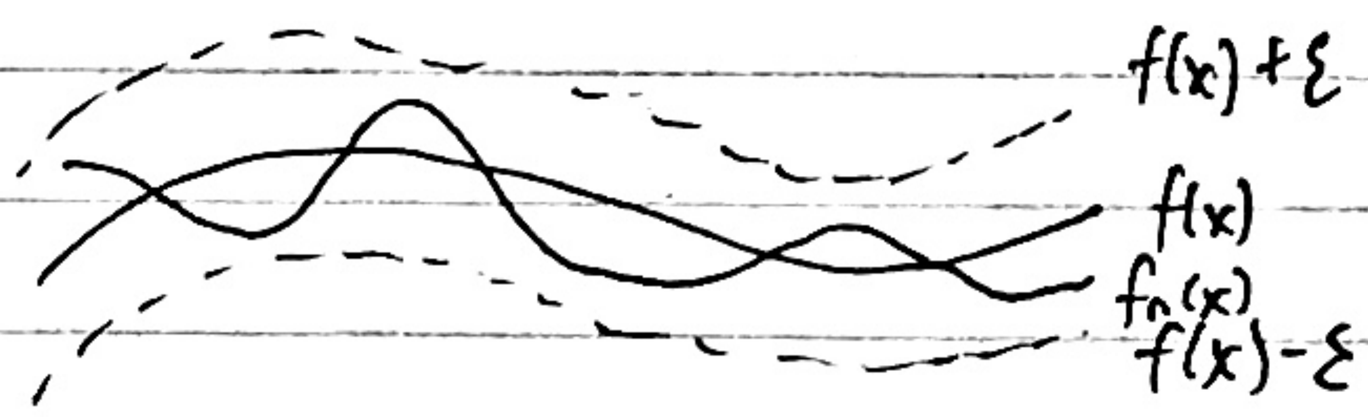
Limits of Integrals Ex. For $n \in \mathbb{N}$, let $f_n(x) = \dfrac{n + \sin nx}{2n - \cos(x)}$ $\qquad$ f_n is continuous $\Rightarrow$ integrable

Find (1) $\int_0^\pi f_n$ $\qquad$ (2) $\lim_{n\to\infty} \int_0^\pi f_n$

<u>Warning</u>: $\lim_{n\to\infty} \int_a^b \neq \int_a^b \lim_{n\to\infty}$ in general

There are two notions of convergence: pointwise and uniform convergence.
(stronger notion of convergence)

Let $f, f_n : S \to \mathbb{R}$.
$f_n \to f$ uniformly means
$\forall \varepsilon > 0$, $\exists N$ s.t. $n > N$
$\Rightarrow |f_n(x) - f(x)| < \varepsilon \quad \forall x \in S$



<u>Proof</u> (That (2) is uniformly convergent)
Take $\varepsilon > 0$. Set $N = \frac{3}{2\varepsilon}$.
Then for $n > N$ and $x \in \mathbb{R}$

$$|f_n(x) - f(x)| = \left| \frac{2\sin(nx) + \cos x}{2(2n - \cos x)} \right| \leq \frac{3}{2(2n-1)} < \frac{3}{2n} < \varepsilon$$

Hence $f_n \to f$ uniformly on $\mathbb{R}$.

Theorem Let $f_n : [a,b] \to \mathbb{R}$ be continuous. If $f_n \to f$ uniformly on $[a,b]$ for some f, then f is continuous (hence integrable) and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$$

<u>Note</u>: Showed $f_n \to f$ on S + f_n continuous $\forall n$
$\Rightarrow$ f is continuous on S by $\frac{\varepsilon}{3}$ argument.

<u>Example</u>: $\displaystyle \lim_{n\to\infty} \int_0^\pi \frac{n + \sin nx}{2n - \cos x}\, dx = \int_0^\pi \frac{1}{2}\, dx = \frac{\pi}{2}$

Note: discussed
g is integrable $\Rightarrow$ $|g|$ is integrable
$\left|\int_a^b g\right| \leq \int_a^b |g|$

Also, if g, h integrable and $g \leq h$.
$\int_a^b g \leq \int_a^b h$.
If $g < h$ and both continuous,
$\int_a^b g < \int_a^b h$
becomes strict.

Proof:
Know f is continuous.
Take $\varepsilon > 0$. Since $f_n \to f$ uniformly on $[a,b]$,

$\exists N$ s.t. $\forall x \in S$, $n > N \Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{b-a}$

Then $n > N \Rightarrow \left|\int_a^b f_n - \int_a^b f\right| = \left|\int_a^b (f_n - f)\,dx\right| \leq \left|\int_a^b |f_n - f|\right|$

$= \int_a^b |f_n(x) - f(x)|\,dx < \int_a^b \frac{\varepsilon}{b-a}\,dx = \varepsilon$

$\therefore \boxed{\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.}$

Power Series
$\sum_{k=0}^{\infty} a_k x^k$

Recall: Radius of convergence $R = \frac{1}{\beta}$ where $\beta = \limsup |a_n|^{\frac{1}{n}}$
(lim sup makes sense for any sequence $\Rightarrow$ the notion always make sense)

Qn: Is $\sum^{\infty} a_k x^k$ continuous? differentiable? integrable?
Know that polynomials are continuous, differentiable and integrable

Theorem
For each $0 < r < R$, the power series $\sum^{\infty} a_k x^k$ converges uniformly on $[-r, r]$
In particular, it is continuous on $(-R, R)$

Note: $\sum^{\infty} a_k x^k$ converges uniformly means if we define $f_n = \sum^{n} a_k x^k$,
$f_n \to \sum^{\infty} a_k x^k$ uniformly on $[-r, r]$

★ Reading Assignment
Fact: [Abel's Theorem]
If $\sum a_k x^k$ converges at $x = R$ or $x = -R$, then it is continuous at $x = R$ or $x = -R$

Proof:
Note that $\sum^{\infty} |a_k| x^k$ has the same radius of convergence R as the original series
$\Rightarrow$ the series $\sum^{\infty} |a_k| r^k$ converges. (i.e. the partial sum converge)

$$\lim_{m\to\infty} \sum_{k=0}^{m} |a_k| r^k = \sum_{k=0}^{\infty} |a_k| r^k$$

Take any $\varepsilon > 0$. Then, $\exists N$ s.t.

$$m > N \Rightarrow \left| \sum_{k=0}^{m} |a_k| r^k - \sum_{k=0}^{\infty} |a_k| r^k \right| < \varepsilon$$

$$\Rightarrow \sum_{k=m+1}^{\infty} |a_k| r^k < \varepsilon$$

For every $n > N$ and $x \in [-r, r]$,

$$|f_n(x) - f(x)| = \left| \sum_{k=0}^{n} a_k x^k - \sum_{k=0}^{\infty} a_k x^k \right| = \left| -\sum_{k=n+1}^{\infty} a_k x^k \right|$$

$$= \lim_{m \to \infty} \left| -\sum_{k=n+1}^{m} a_k x^k \right|$$

Cannot skip directly because it is an infinite sum

$$\underset{\triangle\text{-inequality}}{\le} \cancel{\sum_{k=n}} \lim_{m \to \infty} \sum_{k=n+1}^{m} |a_k| r^k$$

$$= \sum_{k=n+1}^{\infty} |a_k| r^k < \varepsilon$$

Observation $\sum_{k=1}^{\infty} k a_k x^{k-1}$ and $\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$ have the same radius of convergence as $\sum_{k=0}^{\infty} a_k x^k$.

<u>Proof:</u>
Note $\sum_{k=1}^{\infty} k a_k x^{k-1}$ and $\sum_{k=1}^{\infty} k a_k x^k$ have the same radius of convergence
(partial sum of the second series is x times partial sum of the first)

Suffices to consider the radius of $\sum_{k=1}^{\infty} k a_k x^k$ (i.e. $\limsup |a_k|^{\frac{1}{k}}$)

$$\limsup \sqrt[k]{k|a_k|} = \limsup \sqrt[k]{k} \sqrt[k]{|a_k|} = \lim_{k \to \infty} \sqrt[k]{k} \limsup \sqrt[k]{|a_k|} = \limsup \sqrt[k]{|a_k|}$$

$\Rightarrow$ same β $\Rightarrow$ same R.

Theorem (1) For each $x \in (-R, R)$ $\quad \int_0^x \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$

(2) The power series $\sum_{k=0}^{\infty} a_k x^k$ is differentiable on $(-R, R)$ with derivative $\sum_{k=0}^{\infty} k a_k x^{k-1}$

Proof of (1):

For simplicity, assume $x > 0$.

Know $\sum_{k=0}^{n} a_k t^k$ converges uniformly to $\sum_{k=0}^{\infty} a_k t^k$ on $[0, x]$

Let $f_n(t) = \sum_{k=0}^{n} a_k t^k$ and $f(t) = \lim_{n\to\infty} f_n(t)$

Hence, $\int_0^x \sum_{k=0}^{\infty} a_k t^k \, dt = \int_0^x f(t) \, dt$

uniform convergence on continuous fns $\longleftrightarrow$ $= \lim_{n\to\infty} \int_0^x f_n(t) \, dt = \lim_{n\to\infty} \int_0^x \sum_{k=0}^{n} a_k t^k \, dt$

$\longleftarrow$ $= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{k+1} x^{k+1} = \sum_{k=1}^{\infty} \frac{a_k}{k+1} x^{k+1}$

from Fundamental Theorem of Calculus I

Proof of (2)

Set $g(t) = \sum_{k=1}^{\infty} k a_k x^{k-1}$

By (1), $\int_0^x g(t) \, dt = \sum_{k=1}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^k - a_0.$

Since $g(t)$ is continuous,

By Fundamental Theorem of Calculus I,

$\frac{d}{dx} \int_0^x g(t) \, dt = g(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$ $\therefore$ $\boxed{\frac{d}{dx} \sum_{k=0}^{\infty} a_k x^k = \sum_{k=1}^{\infty} k a_k x^{k-1}}$