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MATH 104 LECTURE 2 NOTES

intervals
max, min, sup, inf
completeness axiom.

Completeness Axiom

$\mathbb{R}$ has the properties $+, -, \times, \div, \leq$. $\mathbb{Q}$ also has them
$\Rightarrow$ what's the difference between $\mathbb{Q}$ and $\mathbb{R}$?

Intervals. For $a<b$,
$(a,b) = \{x \in \mathbb{R} : a < x < b\}$. (open interval)
$[a,b] = \{x \in \mathbb{R} ; a \leq x \leq b\}$. (closed interval)
$[a,b) = \{x \in \mathbb{R}, a \leq x < b\}$ (semi-open interval)
$(a,\infty) = \{x \in \mathbb{R}, a < x\}$.

Max, min.

Definition Let $S \subseteq \mathbb{R}$. be a nonempty set.
$x \in \mathbb{R}$ is called maximum of S if ① $x \in S$ and ② $\forall y \in S$, $y \leq x$.
$z \in \mathbb{R}$ is called minimum of S if ① $z \in S$ and ② $\forall y \in S$, $z \leq y$.

Ex: Prove $S=(0,1)$ has no max.
Suppose on the contrary, x is the max. $0<x<1$
Then consider the number $\frac{x+1}{2}$. $\frac{x+1}{2} > x \Leftrightarrow 1>x$ which is true.
$0 < \frac{1}{2} < \frac{x+1}{2} < \frac{1+1}{2} = 1$ $\Rightarrow \frac{x+1}{2} \in S$.
Hence we found a number within $(0,1)$ that is bigger than x
$\Rightarrow$ contradiction.

Assume the contrary, $\exists \Delta_0 = \max S$.
$\Rightarrow 0 < \Delta_0 < 1$.
But, consider $x = \frac{\Delta_0+1}{2} \in S$. $\Delta_0 < x$ $(\rightarrow\leftarrow)$. $\therefore S$ has no max.

Definition Say $M \in \mathbb{R}$ is an upperbound of S if ② $\forall x \in S$, $x \leq M$
If such M exists, say S is bounded above.

Say $m \in \mathbb{R}$ is a lower bound of S if ② $\forall x \in S$, $m \leq x$.
If m exists, S is bounded below.

extracting the 2nd condition.

If S is bounded both above and below, then S is bounded.

If set S admits a maximum/minimum, then it is bounded above/below respectively.

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Definition. $M \in \mathbb{R}$ is the supremum of S (that is bounded above) if M is the smallest upperbound. ($\sup S$)

$m \in \mathbb{R}$ is the infimum of a bounded below set S if m is the largest lower bound ($\inf S$)

Exercise: $1 = \sup S$ for $S = (0,1)$

$\forall x \in (0,1)$, $1 > x$ (since $0 < x < 1$) $\Rightarrow$ 1 is an upperbound.

To show 1 is the smallest upperbound, suffices to show any $x < 1$ cannot be an upperbound.

In fact, if $x < 0$, $\frac{1}{2} \in S$ and $\frac{1}{2} > x$.

If $0 \leq x < 1$, consider $\frac{1+x}{2}$, $x < \frac{1+x}{2} \in S$.

$\Rightarrow$ if $x < 1$, x cannot be an upperbound.

$$M = \sup S \iff \begin{cases} M \text{ is an upperbound of } S \\ M \text{ is the smallest among such} \end{cases}$$

$$\iff \begin{cases} M \text{ is an upperbound} \\ \text{Any } x < M \text{ is \underline{not} an upperbound.} \end{cases}$$

Completeness Axiom of $\mathbb{R}$ (very important)

Fundamental property of real numbers that you can use in the course.

Any nonempty subset of $\mathbb{R}$ that is bounded above (admits at least 1 upperbound) admits supremum. (as a real).

Completeness axiom fails for rational numbers. $S = \{x \in \mathbb{Q} : x^2 < 2\}$. S does not admit supremum.

rational numbers have a gap at $\sqrt{2}$.

★ Dedekind cuts is a rigorous way of defining $\mathbb{R}$ from $\mathbb{Q}$.

Consequences ① Any nonempty subset $S \subseteq \mathbb{R}$ bounded below admits infimum. (as a real)

② Archimedean property: For any $a, b > 0$, $\exists n \in \mathbb{N}$ s.t. $na > b$.

③ Denseness of $\mathbb{Q}$: For every $a < b$, $\exists r \in \mathbb{Q}$ s.t. $a < r < b$.

$\mathbb{Q}$ has gaps but not too many.

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<u>Proof of (2)</u>

Assume on the contrary, $\forall n \in \mathbb{N}$, assume $na < b$.

let $S = \{na : n \in \mathbb{N}\}$.

S is not empty, since $a = 1 \cdot a \in S$. Also, b is an upper bound.

By completeness axiom, $\sup S$ exists. let $M = \sup S$.

note, $M - a \ (< M)$ is not an upper bound. So $\exists n_0 a \in S$ s.t. $M - a < n_0 a$.

$\Rightarrow M < (n_0 + 1) a \in S$. (Contradicting the fact that $M = \sup S$)

$\therefore$ Using the completeness axiom

<u>Proof of (3)</u>

Suffices to show $\exists m, n$ s.t. $an < m < bn$
$(n > 0)$

Since $b - a > 0$, we can pick $n(b-a) > 1$ by Archimedean principle
$\Rightarrow nb - na > 1$.

By Archimedean principle, exists $k > \max\{|an|, |bn|\}$ so that $-k < an < bn <$
$(k \cdot 1)$

Consider the sets $K_1 = \{j \in \mathbb{Z} ; -k \le j \le k\}$

$K_2 = \{j \in K ; an < j\}$.

Both sets are non empty since $k \in K_1, K_2$.

let $m = \min\{j \in K : an < j\}$ Then $-k < an < m$.

Since $m > -k$, $m - 1 \in K$. Since m is the minimum element of K_2,

$m - 1 \notin K_2 \Rightarrow$ $m - 1 \le an$

$\Rightarrow m \le a_n + 1 < bn$

$\therefore$ $\boxed{an < m < bn}$

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