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MATH 191 LECTURE 4 NOTES

Last time: $a_1 = 3$

$a_{n+1} = a_n(a_n+2)$

We guessed $a_n = 2^{(2^n)} - 1$, $n \geq 1$

Base case: $n=1$, $a_1 = 2^{(2^1)} - 1 = 2^2 - 1 = 3$ ✓

Suppose $n=k$, $a_k = 2^{(2^k)} - 1$

Then for $n = k+1$

$$LHS = a_{k+1} = a_k(a_k+2) = \left(2^{(2^k)} - 1\right)\left(2^{(2^k)} - 1 + 2\right)$$

$$= \left(2^{(2^k)} - 1\right)\left(2^{(2^k)} + 1\right) = 2^{2^{k+1}} - 1 = RHS$$

$\therefore \boxed{a_n = 2^{(2^n)} - 1}$ $\forall n \geq 1$.

Bernoulli's inequality: $(1+x)^n \geq 1 + nx$. $x \geq -1$, $\forall n \in \mathbb{N}$

Base case: $n = 1$. $LHS = (1+x)^1 = 1 + x \geq 1 + 1 \cdot x = RHS$ ✓.

Suppose $n = k$, is true, $(1+x)^k \geq 1 + kx$

For $n = k+1$,

$$LHS = (1+x)^{k+1} = (1+x)^k(1+x)$$ because $(1+x) \geq 0$

$$\geq (1+kx)(1+x)$$

$$= 1 + kx + x + kx^2 = 1 + (k+1)x + kx^2$$ because $kx^2 \geq 0$ $\forall x$.

$$\geq 1 + (k+1)x = RHS$$

Hence $\boxed{(1+x)^n \geq 1 + nx \quad \forall n \in \mathbb{N}.\ x \geq -1}$

well ordering principle.

Induction $\Leftrightarrow$ well ordering principle

Induction principle: Let $T \subseteq \mathbb{N}$ satisfy

1) $1 \in T$
2) If $n \in T$, then $n+1 \in T$

Then $T \in \mathbb{N}$

Well ordering principle: Every nonempty subset of $\mathbb{N}$ has a smallest element.

Well-ordering $\Rightarrow$ Induction.

Assume on the contrary that for induction, $T \neq \mathbb{N}$.

Let $S = \mathbb{N} \setminus T$. Then S is nonempty by our assumption.

By wellordering principle, S has a minimum element, say m. (Note $m \neq 1$ since $1 \in T$)

Note $m-1 \notin T$ (for if $m-1 \in T$, then $m \in T$ contradiction)

Hence $m-1 \in S \Rightarrow$ contradiction! Since m is the smallest element in S.

$\therefore T = \mathbb{N}$. (wellordering $\Rightarrow$ induction)