Prep: bring ID, water, pen and the handwritten version of this sheet; don't be afraid of wasting paper! **You got this!**

Known Distributions

Bernoulli Distribution: $X \sim \text{Bernoulli}(p)$

$\mathbb{P}[X=0]=(1-p)$	$\mathbb{E}[X]=p$
$\mathbb{P}[X=1]=p$	$\text{Var}[X]=p(1-p)$
$F(x)=\begin{cases}0, & x<0\\ 1-p, & 0\le x<1\\ 1, & x\ge 1\end{cases}$	

Binomial Distribution: $X \sim \text{Binomial}(n,p)$

$\mathbb{P}[X=i]=\binom{n}{i}p^i(1-p)^{n-i},\ \ i=0,1,\dots,n$	
$\mathbb{E}[X]=np$	$\text{Var}[X]=np(1-p)$

Geometric Distribution: $X \sim \text{Geometric}(p)$

$\mathbb{P}[X=i]=(1-p)^{i-1}p, i=1,2,\dots$	
$\mathbb{E}[X]=\frac{1}{p}$	$\text{Var}[X]=\frac{1-p}{p^2}$
$pgf(x)=\frac{px}{1-(1-p)x}$	$F(x)=1-(1-p)^x$
$\mathbb{P}[X>n+m \mid X>n]=\mathbb{P}[X>m]\quad n,m>0$	

Poisson Distribution: $X \sim \text{Poisson}(\lambda)$

$\mathbb{P}[X=i]=\frac{\lambda^i}{i!}e^{-\lambda}, i=0,1,2,\dots$	
$\mathbb{E}[X]=\lambda$	$\text{Var}[X]=\lambda$
$X+Y \sim \text{Poisson}(\lambda+\mu)$	$\text{Binomial}\left(n,\frac{\lambda}{n}\right)\xrightarrow{n\to\infty}\text{Poisson}(\lambda)$

Exponential Distribution: $X \sim \text{Expo}(\lambda)$

$f(x)=\begin{cases}\lambda e^{-\lambda x}, & x\ge 0\\ 0, & \text{otherwise}\end{cases}$		
$\mathbb{E}[X]=\frac{1}{\lambda}$	$\text{Var}[X]=\frac{1}{\lambda^2}$	$\mathbb{E}[X^2]=\frac{2}{\lambda^2}$
$F(x)=\mathbb{P}[X\le x]=1-e^{-\lambda x}$		
$X\sim\text{Expo}(\lambda), Y\sim\text{Expo}(\mu)\Rightarrow \min(X,Y)\sim\text{Expo}(\mu+\lambda)$		
$X\sim\text{Expo}(\lambda), Y\sim\ \text{Expo}(\mu)\Rightarrow \mathbb{P}[X\le Y]=\frac{\lambda}{\lambda+\mu}$		

Discrete Uniform: $X \sim \text{Uniform}(\{a,\dots,b\})$

$\mathbb{E}[X]=\frac{a+b}{2}$	$\text{Var}[X]=\frac{(b-a)(b-a+2)}{12}$
$f(x)=\frac{1}{b-a}$	$F(x)=\frac{x-a}{b-a}$
$p_X(k)=\begin{cases}\frac{1}{b-a+1}, & k=a,\dots,b\\ 0, & \text{otherwise}\end{cases}$	

Continuous Uniform: $X \sim \text{Uniform}([a,b])$

$\mathbb{E}[X]=\frac{a+b}{2}$	$\text{Var}[X]=\frac{(b-a)^2}{12}$
$f(x)=\frac{1}{b-a}$	$F(x)=\frac{x-a}{b-a}$

Normal Distribution: $X \sim N(\mu,\sigma^2)$

$X\sim N(\mu,\sigma^2)\to Z=\frac{X-\mu}{\sigma}\sim N(0,1)$	
$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$f_Z(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$

Probabilistic Bounding

- [Markov] Nonnegative RV X, finite mean
$$\mathbb{P}[X\ge c]\le\frac{\mathbb{E}[X]}{c},\quad c>0$$
- [Generalized Markov] Y not necessarily nonnegative, finite mean; $c,r>0$
$$\mathbb{P}[|Y|\ge c]\le\frac{\mathbb{E}[|Y|^r]}{c^r}$$
- [Extended Markov] X not necessarily nonnegative; $\Phi(X)$ nonnegative function, monotonically increasing for $x>0$; $\alpha>0$
$$\mathbb{P}[X\ge\alpha]\le\frac{\mathbb{E}[\Phi(X)]}{\Phi(\alpha)}$$
- [Chebyshev] $c>0$
$$\mathbb{P}[|X-\mu|\ge c]\le\frac{\text{Var}[X]}{c^2}$$
$$\mathbb{P}[|X-\mu|\ge k\sigma]\le\frac{1}{k^2}$$
- [Cantelli] $\alpha>0$
$$\mathbb{P}[X-\mathbb{E}[X]\ge\alpha]\le\frac{\sigma^2}{\alpha^2+\sigma^2}$$
- [Law of Large Numbers] $X_1,\dots,X_n$ i.i.d. RV with $\mathbb{E}[X_i]=\mu<\infty$. Define $S_n=X_1+\dots+X_n$
$$\forall\varepsilon\ \lim_{n\to\infty}\mathbb{P}\left[\left|\frac{1}{n}S_n-\mu\right|<\varepsilon\right]=1$$
- [Central Limit Theorem] Distribution of sample average $\frac{S_n}{n}$ approaches a **normal distribution** with mean μ and variance $\frac{\sigma^2}{n}$.
$$\frac{\frac{S_n}{n}-\mu}{\sqrt{\sigma^2/n}}=\frac{S_n-n\mu}{\sigma\sqrt{n}}\sim N(0,1)$$
$$\mathbb{P}\left[\frac{S_n-n\mu}{\sigma\sqrt{n}}\le c\right]\to\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{c}e^{-\frac{x^2}{2}}\mathrm{d}x$$
Let $(X_n)_n$ be a sequence of i.i.d. RV with common mean μ and variance σ^2. Define $Z_n=\frac{X_1+\dots+X_n-n\mu}{\sigma\sqrt{n}}$. Then $(Z_n)_n$ converges to $N(0,1)$ in distribution.
- [Chernoff]
$$\mathbb{P}[X\ge a]=\mathbb{P}[e^{sX}\ge e^{sa}]\le\frac{M_X(s)}{e^{sa}}\ \forall s\ge 0$$
- [Jensen]
$$f(\mathbb{E}[X])\le\mathbb{E}[f(X)]\quad\forall f\text{ convex}$$

$X \sim N(0,\sigma^2) \Rightarrow \mathbb{E}[X^{2n}] = (2n-1)!!\,\sigma^{2n}$

Erlang: $X \sim \text{Erlang}(k,\lambda)$ sum of k i.i.d $\text{Expo}(\lambda)$

$\mathbb{E}[X] = \frac{k}{\lambda}$	$\text{Var}[X] = \frac{k}{\lambda^2}$
$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}$	$F(x) = 1 - \sum_{n=0}^{k-1} \frac{1}{n!} e^{-\lambda x}(\lambda x)^n$
$X \sim \text{Erlang}(k,\lambda) \Rightarrow aX \sim \text{Erlang}\left(k, \frac{\lambda}{a}\right)$	
$X \sim Erlang(k_1,\lambda), Y \sim Erlang(k_2,\lambda)$ independent $\Rightarrow X+Y \sim Erlang(k_1+k_2,\lambda)$	

Pascal: $X \sim \text{Pascal}(k,p)$ sum of k i.i.d $\text{Geometric}(p)$

$\mathbb{E}[X] = \frac{k}{p}$	$\text{Var}[X] = \frac{k(1-p)}{p^2}$
$\mathbb{E}[X^2] = \frac{k^2 + k(1-p)}{p^2}$	$M_X(s) = \left[\frac{pe^s}{1-(1-p)e^s}\right]^k$
$p_X(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$ $k = 1,2,\ldots;\ x = k, k+1, \ldots$	
$F_X[x] = \mathbb{P}[X \le x] = \sum_{n=k}^{x} \binom{n-1}{k-1} p^k (1-p)^{n-k}$	

- [WLLN] Let $(X_n)_n$ be a sequence of i.i.d. random variables with mean μ. $\forall \epsilon > 0$
$$\mathbb{P}\left[\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| \ge \epsilon\right] \to 0 \quad \text{as } n \to \infty$$
- [SLLN] Let $(X_n)_n$ be a sequence of i.i.d. random variables with mean μ. Then $M_n = \frac{X_1 + \cdots + X_n}{n}$ converges to μ with probability 1.
$$\mathbb{P}\left[\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \mu\right] = 1$$
i.e. $\forall \epsilon > 0$, $\mathbb{P}[|M_n - \mu| > \epsilon\ i.o] = 0$
- [De Movire-Laplace approximation]
If $S_n \sim \text{Binomial}(n,p)$ and $n \gg 1$ and k, l nonnegative integers, then:
$$\mathbb{P}[k \le S_n \le l] \approx \Phi\left(\frac{l + \frac{1}{2} - np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{k - \frac{1}{2} - np}{\sqrt{np(1-p)}}\right)$$

Joint and Conditional Probability

- $p_{X|Y}(x|y) = \frac{\mathbb{P}[X=x,Y=y]}{\mathbb{P}[Y=y]} = \frac{p_{X,Y}(x,y)}{p_Y(y)}$
- $p_{X,Y}(x,y) = p_Y(y) p_{X|Y}(x|y)$
- $f_{X,Y}(x,y) = \frac{\partial F_{X,Y}}{\partial x \partial y}(x,y)$
- $f_{X,Y}(x,y) = f_Y(y) f_{X|Y}(x|y)$
- $f_X(x) = \int_{-\infty}^{\infty} f_Y(y) f_{X|Y}(x|y)\,\mathrm{d}y$
- $f_X(x) = \sum_{i=1}^{n} f_{X|A_i}(x)\,\mathbb{P}[A_i]$

Bayes and Continuous Bayes

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[B|A]\mathbb{P}[A]}{\mathbb{P}[B]}$$

- $f_Y(y) f_{X|Y}(x|y) = f_X(x) f_{Y|X}(y|x)$
- $f_{X|Y}(x|y) = \frac{f_X(x) f_{Y|X}(y|x)}{\int_{-\infty}^{\infty} f_X(t) f_{Y|X}(y|t)\mathrm{d}t}$
- $f_Y(y)\mathbb{P}[N=n|Y=y] = \mathbb{P}[N=n] f_{Y|N}(y|n)$
- $\mathbb{P}[N=n|Y=y] = \frac{\mathbb{P}[N=n] f_{Y|N}(y|n)}{\sum_i \mathbb{P}[N=i] f_{Y|N}(y|i)}$
- $f_{Y|N}(y|n) = \frac{f_Y(y)\mathbb{P}[N=n|Y=y]}{\int_{-\infty}^{\infty} f_Y(t)\mathbb{P}[N=n|Y=t]\mathrm{d}t}$

Differential Probability and Convolution

- $f_X(x)\,\mathrm{d}x = \mathbb{P}[x \le X \le x + \mathrm{d}x]$
- $f_{X|Y}(x|y)\,\mathrm{d}x = \frac{\mathbb{P}[x \le X \le x+\mathrm{d}x, y \le Y \le y+\mathrm{d}y]}{\mathbb{P}[y \le Y \le y+\mathrm{d}y]}$
- $\frac{\mathrm{d}}{\mathrm{d}z}\mathbb{P}[Z \le z|X=x] = f_{Z|X}(z|x)$
- $f_Y(y) = f_X(f^{-1}(y))\left|\frac{\mathrm{d}f^{-1}}{\mathrm{d}y}(y)\right|$

Conditional Expectation and Variance

Note	$\mathbb{E}[Y|X] = f(X)$ i.e. is a function of X
Note	$\mathbb{E}[Y|X=x]$ is a real number

To find $\mathbb{E}[X|Y]$, generalize pattern from $\mathbb{E}[X|Y=y]$

1. (Linearity)
$\mathbb{E}[a_1 Y_1 + a_2 Y_2|X] = a_1\mathbb{E}[Y_1|X] + a_2\mathbb{E}[Y_2|X]$
2. (Factoring known values)
$\mathbb{E}[h(X)Y|X] = h(X)\mathbb{E}[Y|X]$
3. (Independence) If X, Y independent:
$\mathbb{E}[Y|X] = \mathbb{E}[Y]$

$\mathbb{E}\big[\mathbb{E}[X|Y]\big] = \mathbb{E}[X]$
$\text{Var}[X] = \mathbb{E}\big[\text{Var}[X|Y]\big] + \text{Var}\big[\mathbb{E}[X|Y]\big]$
$\mathbb{E}[XY] = \mathbb{E}\big[\mathbb{E}[XY|Y]\big] = \mathbb{E}\big[Y\mathbb{E}[X|Y]\big]$

Conditioning on event

- $\mathbb{E}[X|A] = \int_{-\infty}^{\infty} x f_{X|A}(x)\mathrm{d}x$
- $\mathbb{E}[g(X)|A] = \int_{-\infty}^{\infty} g(x) f_{X|A}(x)\mathrm{d}x$
- $\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{P}[A_i]\mathbb{E}[X|A_i]$

Conditioning on RV

- $\mathbb{E}[Y] = \mathbb{E}\big[\mathbb{E}[Y|X]\big] = \int_{-\infty}^{\infty} \mathbb{E}[Y|X=x] f_X(x)\mathrm{d}x$
- $\mathbb{E}[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\mathrm{d}x$
- $\mathbb{E}[g(X)|Y=y] = \int_{-\infty}^{\infty} g(x) f_{X|Y}(x|y)\mathrm{d}x$
- $\mathbb{E}[g(X,Y)|Y=y] = \int_{-\infty}^{\infty} g(x,y) f_{X|Y}(x|y)\,\mathrm{d}x$

Convolution $Z = X + Y$

$$\mathbb{P}[Z = z] = \mathbb{P}[X + Y = z] = \sum_x \mathbb{P}[X = x]\mathbb{P}[Y = z - x]$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\mathrm{d}x$$

Moment Generating Function (MGF)

Properties

$$e^{sX} = 1 + sX + \frac{s^2X^2}{2!} + \frac{s^3X^3}{3!} + \cdots$$

$$\mathbb{E}[e^{sX}] = 1 + s\mathbb{E}[X] + \frac{s^2}{2!}\mathbb{E}[X^2] + \frac{s^3}{3!}\mathbb{E}[X^3] + \cdots$$

$$\left(\frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathbb{E}[e^{sX}]\right)(0) = \mathbb{E}[X^n] \qquad M_X(0) = 1$$

$$Y = aX + b \Rightarrow M_Y(s) = e^{sb} M_X(as)$$

$Z = \sum_i X_i$, independent $\Rightarrow M_Z(s) = \prod_i M_{X_i}(s)$

Distribution	MGF
$\mathrm{Bernoulli}(p)$	$M(s) = (1-p) + pe^s$
$\mathrm{Binomial}(n, p)$	$M(s) = (1 - p + pe^s)^n$
$\mathrm{Geometric}(p)$	$M(s) = \frac{pe^s}{1-(1-p)e^s}$
$\mathrm{Poisson}(\lambda)$	$M(s) = e^{\lambda(e^s - 1)}$
$\mathrm{Expo}(\lambda)$	$M(s) = \frac{\lambda}{\lambda - s}, \lambda > s$
$\mathrm{Uniform}([a, b])$	$M(s) = \frac{e^{sb} - e^{sa}}{s(b-a)}$
$N(\mu, \sigma^2)$	$M(s) = e^{\mu s + \frac{\sigma^2 s^2}{2}}$
$\mathrm{Erlang}(k, \lambda)$	$M(s) = \left(1 - \frac{s}{\lambda}\right)^{-k}$
$\mathrm{Pascal}(k)$	$M_X(s) = \left[\frac{pe^s}{1-(1-p)e^s}\right]^k$

Convergence Definitions

Convergence in Distribution

A sequence $(X_n)_n$ <u>converges in distribution</u> to X, denoted as $X_n \xrightarrow{d} X$, if $\forall x \lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ (i.e. CDF of X_n converges to CDF of X)

<u>Theorem</u>: For integer valued X, $(X_n)_n$ suffices to show: $\forall x \lim_{n\to\infty} \mathbb{P}[X_n = x] = \mathbb{P}[X = x]$

- $\mathbb{E}[g(X, Y)] = \mathbb{E}[\mathbb{E}[g(X, Y)|Y]] = \int_{-\infty}^{\infty} \mathbb{E}[g(X, Y)|Y = y] f_Y(y)\,\mathrm{d}y$
- $\mathbb{E}[X|N > k] = \mathbb{E}[\mathbb{E}[X|N]|N > k] = \sum_{n=1}^{\infty} \mathbb{E}[X|N = n, N > k]\mathbb{P}[N = n|N > k]$

Series of RV

- $Y = X_1 + \cdots + X_N$
- $\mathbb{E}[Y|N] = N \cdot \mathbb{E}[X]$
- $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|N]] = \mathbb{E}[N]\mathbb{E}[X]$
- $\mathrm{Var}[Y] = \mathbb{E}[N]\mathrm{Var}[X] + (\mathbb{E}[X])^2\mathrm{Var}[N]$
- $M_Y(s) = \sum_{n=0}^{\infty} (M_X(s))^n \mathbb{P}[N = n]$

Covariance and Correlation

<u>Covariance</u> (bilinear)

$$\mathrm{Cov}[X, Y] = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

$$\mathrm{Cov}[X, X] = \mathrm{Var}[X]$$

$$\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X, Y]$$

$$\begin{aligned}\mathrm{Cov}[aX_1 + bX_2, cY_1 + dY_2] &= ac\,\mathrm{Cov}[X_1, Y_1] + ad\,\mathrm{Cov}[X_2, Y_1] \\ &+ bc\,\mathrm{Cov}[X_2, Y_1] + bd\,\mathrm{Cov}[X_2, Y_2]\end{aligned}$$

X, Y independent $\Rightarrow \mathrm{Cov}[X, Y] = 0$

<u>Correlation</u>

$$\rho[X, Y] = \frac{\mathrm{Cov}[X, Y]}{\sigma_X \sigma_Y}$$

$$X' = \frac{X - \mu_X}{\sigma_X},\ Y' = \frac{Y - \mu_Y}{\sigma_Y}$$

$$-1 \le \rho[X, Y] = \mathrm{Cov}[X', Y'] \le 1$$

$$\rho[X, Y] = 1 \Rightarrow Y = AX + B, A > 0\ (Y' = X')$$

$$\rho[X, Y] = -1 \Rightarrow Y = AX + B, A < 0\ (Y' = -X')$$

Borel Cantelli Lemmas and Continuity

First Lemma: Let $(A_n)_n$ be a sequence of events. If $\sum_{n=1}^{\infty} \mathbb{P}[A_n] < \infty$, then $\mathbb{P}[A_n\ i.o] = 0$ (i.e. the probability that infinitely many of A_n occurring is 0).

Second Lemma: Let $(A_n)_n$ be a sequence of events. If A_n are independent and $\sum_{n=1}^{\infty} \mathbb{P}[A_n] = \infty$, then $\mathbb{P}[A_n\ i.o] = 1$ (i.e. probability that infinitely many of them occurring is 1)

Let $h: \mathbb{R} \to \mathbb{R}$ be a continuous:

If $X_n \xrightarrow{d} X$ then $h(X_n) \xrightarrow{d} h(X)$

If $X_n \xrightarrow{p} X$ then $h(X_n) \xrightarrow{p} h(X)$

If $X_n \xrightarrow{a.s.} X$ then $h(X_n) \xrightarrow{a.s.} h(X)$

Classics

- [Ballot] Let A, B be players such that A scored n points, B scored $m < n$ points. Then, probability that A is strictly ahead of B at all times is $\frac{n-m}{n+m}$.
- [Gambler's Ruin] Let A be a player who starts at state i and at each step increments

Convergence in Probability

A sequence $(X_n)_n$ <u>converges in probability</u> to X, denoted as $X_n \xrightarrow{p} X$, if $\forall \epsilon > 0$, $\lim_{n\to\infty} \mathbb{P}[|X_n - X| \geq \epsilon] = 0$

<u>Theorem</u>: If $X_n \xrightarrow{d} c$ constant, then $X_n \xrightarrow{p} c$.

Convergence with Probability 1

A sequence $(X_n)_n$ <u>converges almost surely</u> to X, denoted as $X_n \xrightarrow{a.s.} X$, if under sample space Ω, $\mathbb{P}\left[\left\{\omega \in \Omega: \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right] = 1$

<u>Theorem</u>: Consider $(X_n)_n$. If $\forall \epsilon > 0$, $\sum_{n=1}^{\infty} \mathbb{P}[|X_n - X| > \epsilon] < \infty$, then $X_n \xrightarrow{a.s.} X$

<u>Theorem</u>: Consider $(X_n)_n$. For $\epsilon > 0$, define $A_m = \{|X_n - X| < \epsilon \;\; \forall n \geq m\}$, then $X_n \xrightarrow{a.s.} X$ if and only if $\lim_{m\to\infty} \mathbb{P}[A_m] = 1$.

by $\{-1, +1\}$ with probability $\frac{1}{2}$. Game ends when he reaches 0 or n. Then, probability he reaches n is $\frac{i}{n}$.

- [Gambler's Ruin (unfair)] If probability is p, then probability of him reaching n is
$$\frac{1 - \left(\frac{p}{1-p}\right)^{n-i}}{1 - \left(\frac{p}{1-p}\right)^{n}}$$
- [Secretary] Optimal cutoff is ne^{-1}.
- [Coupon collector] $nH_n = n\sum_{i=1}^{n} \frac{1}{i}$

Miscellaneous

Tail Sum (X, Z nonnegative)

Discrete	$\mathbb{E}[X] = \sum_{i=1}^{\infty} \mathbb{P}[X \geq i]$
Continuous	$\mathbb{E}[Z] = \int_0^{\infty} \mathbb{P}[Z \geq z]\,\mathrm{d}z$
$\mathbb{E}[X] = \int_0^{\infty} \mathbb{P}[X > x]\mathrm{d}x - \int_0^{\infty} \mathbb{P}[X < -x]\mathrm{d}x$	

If X is a random variable, then $\mathbb{P}[X = 0] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2}$

Union bound
$$\mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right] \leq \sum_{i=1}^{n} \mathbb{P}[A_i]$$

kth Order Statistics

$$f_{X^{(i)}}(x) = n\binom{n-1}{i-1} f(x)F(x)^{i-1}\left(1 - F(x)\right)^{n-i}$$

$\mathbb{E}\left[X^{(i)}\right] = \frac{i}{n+1}$ (uniform distribution)

$X \sim N(0,1)$	$f_{X^2}(x) = \frac{1}{\sqrt{2\pi x}} e^{-\frac{x}{2}}$	$f_{X^2+Y^2}(z) = \frac{1}{2} e^{-\frac{z}{2}}$

$$\int_0^1 y^{\alpha}(1-y)^{\beta}\,\mathrm{d}y = \frac{\alpha!\,\beta!}{(\alpha+\beta+1)!}$$

$$I_k = \int_0^{\infty} x^k e^{-\theta x}\mathrm{d}x$$
$$I_k = \frac{k}{\theta} I_{k-1} \Rightarrow I_k = \frac{k!}{\theta^{k+1}}$$

Last Resort and Final Checks

- Think indicators, even bizarre ones (e.g. I_i indicate whether ith bad ball appears before first good ball)
- $\mathrm{Var}[X] = \mathrm{Cov}[X, X]$, then whack using indicators.
- Symmetry; Bijection arguments; Coupling (bijection to [0,1])
- Draw pictures: represent each RV by $\mathbb{R}$, then reduce to geometry problem
- **<u>Remember</u>** to define all the range, especially for f_X, F_X! ($= 0$ elsewhere)
- Reduce to classic problem
- Isolate objects, especially in sequences or lines
- Guard against one-off errors, especially in indicators. Check small cases.
- Consider conditioning on some events.
- Just use intuition and take care of momentum!