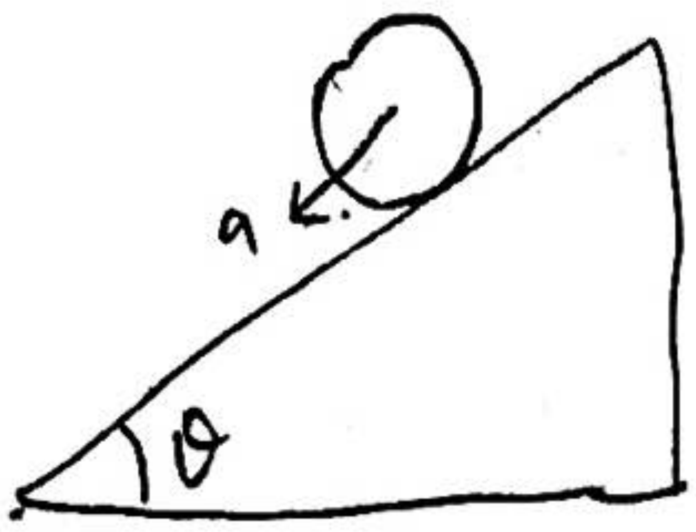
Mechanics II

(1)



τ) $fr = \frac{1}{2}mr^2\alpha$

x) $mg\sin\theta - f = ma$

$a = r\alpha$ (from $v = r\omega$)

$\Rightarrow f = \frac{1}{2}ma \Rightarrow \boxed{a = \frac{2}{3}g\sin\theta}$

Energy method

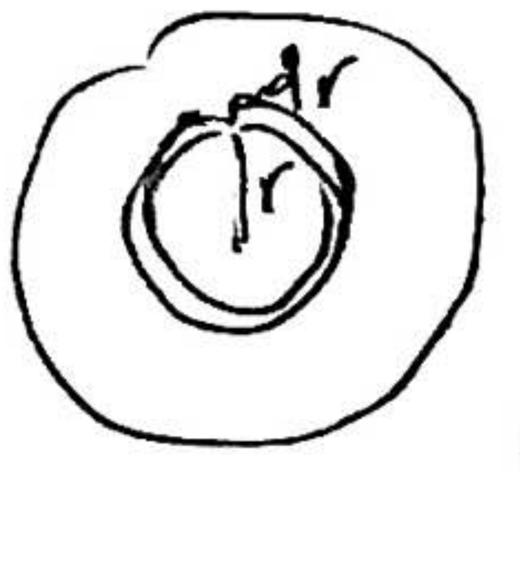
cylinder moves d down the plane

loss in PE $= mgd\sin\theta$

$= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 =$ Gain in KE

$\Rightarrow v = \sqrt{\frac{4}{3}gd\sin\theta} \Rightarrow \boxed{a = \frac{2}{3}g\sin\theta}$

(2a)



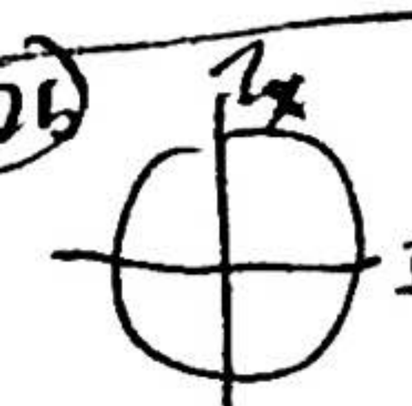
$dm = \sigma(2\pi r)\,dr$

$I = \int_0^R r^2\,dm = \int_0^R \sigma 2\pi r^3\,dr$

$= \sigma 2\pi\left[\frac{r^4}{4}\right]_0^R = \sigma\frac{\pi R^4}{2}$

$= \frac{M}{\pi R^2}\,\frac{\pi R^4}{2} = \boxed{\frac{1}{2}MR^2}$

(2b)



using perpendicular axis thm,

$I_x + I_y = I_z = \frac{1}{2}MR^2$

$I_x = I_y \Rightarrow \boxed{I_x = \frac{1}{4}MR^2}$

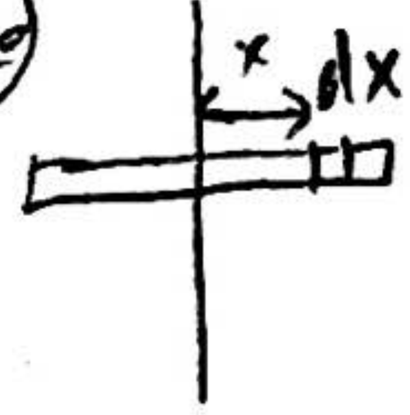
(2c)

$I = \int r^2\,dm = \boxed{MR^2}$

(2d)

Similarly, by perpendicular axis theorem! $I_x = I_y = \boxed{\frac{1}{2}MR^2}$

(2e)



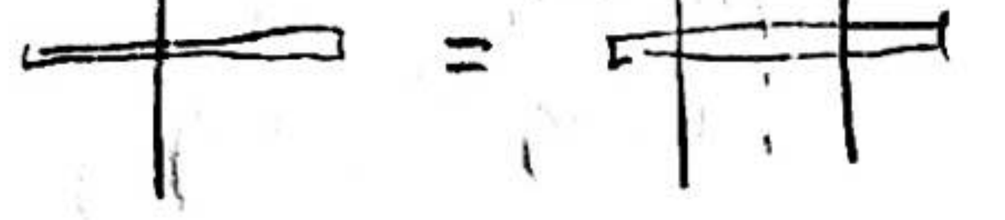
$dm = \lambda\,dx$

$I = \int x^2\,dm = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2\lambda\,dx$

$= \left[\frac{x^3}{3}\right]_{-\frac{L}{2}}^{\frac{L}{2}}\lambda = \lambda\frac{L^3}{12}$

$= \frac{M}{L}\,\frac{L^3}{12} = \boxed{\frac{1}{12}ML^2}$

Alternatively, by scaling method,



$=$ + parallel axi

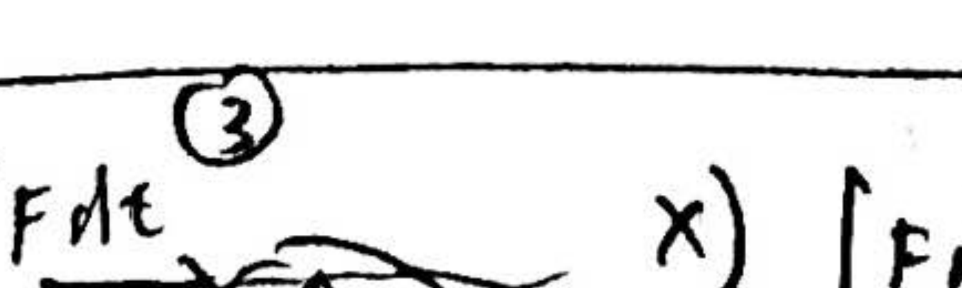
Suppose moment of inertia is βML^2.

Then $\beta ML^2 = \beta\left(\frac{M}{2}\right)\left(\frac{L}{2}\right)^2 \times 2$] inertia of smaller rods

$+ \frac{M}{2}\left(\frac{L}{4}\right)^2 \times 2$] parallel axis thm

$\Rightarrow \beta = \frac{1}{12} \Rightarrow \boxed{I = \frac{1}{12}ML^2}$

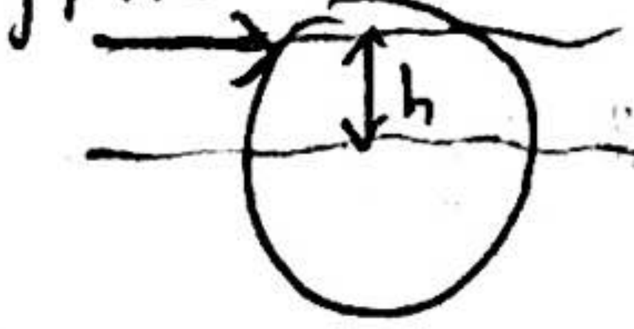
(2f)



$=$ + parallel axis theorem

$\Rightarrow I = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \boxed{\frac{1}{3}ML^2}$

(3)



x) $\int F\,dt = mV_{cm}$

τ) $h\int F\,dt = I\omega_{cm}$

$\Rightarrow h = \frac{I\omega_{cm}}{mV_{cm}} = \frac{\frac{2}{5}mr^2\omega_{cm}}{mr\omega_{cm}} = \boxed{\frac{2}{5}r}$

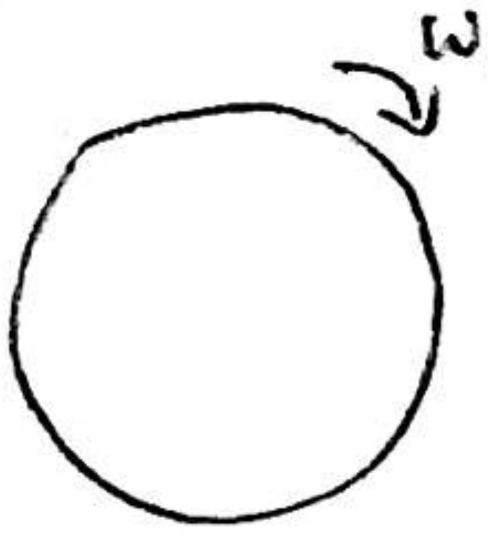
Alternatively

Conservation of angular momentum

$p(r+h) = mv(r+h) = I\omega$

$= \left(\frac{2}{5}mr^2 + mr^2\right)\left(\frac{v}{r}\right)$

④



$\int f\,dt = f\Delta t$
$= \mu N \Delta t$
$= \mu m g \Delta t$

x) $\int f\,dt = m V_{cm}$

τ) $R\int f\,dt = I\left(w_i - \frac{V_{cm}}{R}\right)$
$= \frac{1}{2} m R^2 \left(w_i - \frac{V_{cm}}{R}\right)$

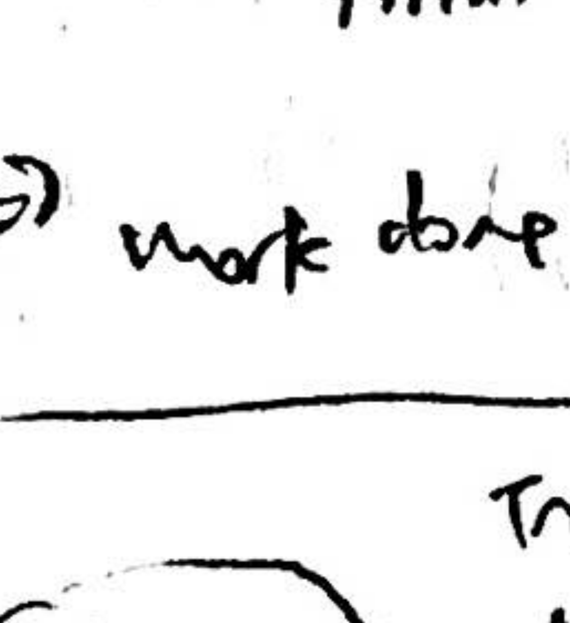
$\Rightarrow V_{cm} = \frac{R w_i}{3}$

$\therefore \Delta t = \frac{\int f\,dt}{\mu m g} = \frac{m R w_i}{3 \mu m g} = \boxed{\frac{R w_i}{3 \mu g}}$

$\dot{v} = \mu g$

$d = \frac{1}{2} \mu g (\Delta t)^2 = \boxed{\frac{1}{18} \frac{R^2 w_i^2}{\mu g}}$

⑤

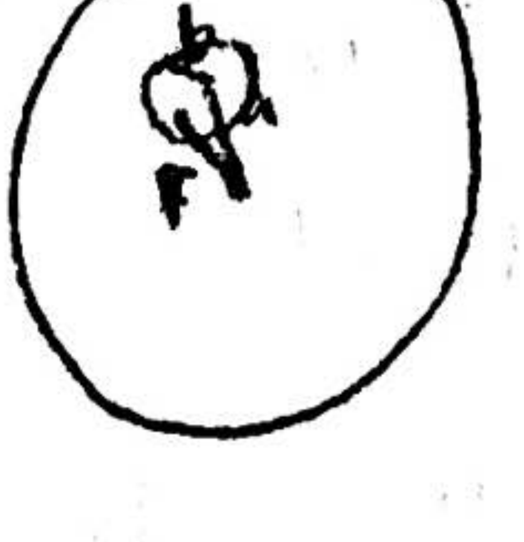


COAM)
$m R V_i = m \frac{R}{2} V_f$
$\Rightarrow V_i = \frac{1}{2} V_f$.

Initial energy $= E_0 = \frac{1}{2} m V_i^2$
Final energy $= \frac{1}{2} m V_f^2 = 2 m V_i^2$

$\Rightarrow$ work done $= \frac{3}{2} m V_i^2 = \boxed{3 E_0}$

(B9)



Treat cavity as negative mass

$g(\vec{r}) = -\frac{G\left(\rho \frac{4}{3}\pi r^3\right)}{r^2}\hat{r}$
$- \frac{G\left(-\rho \frac{4}{3}\pi |\vec{r}-\vec{a}|^3\right)}{|\vec{r}-\vec{a}|^2}\widehat{(\vec{r}-\vec{a})}$

$= \left(\rho \frac{4}{3}\pi G\right)\left(-\vec{r} + (\vec{r}-\vec{a})\right)$

$= \boxed{-\rho \frac{4}{3}\pi G a \hat{a}}$

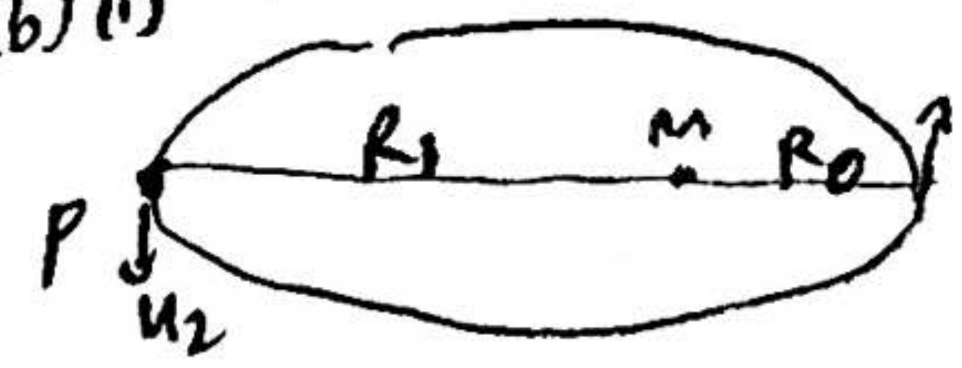
⑥

(a)

$\frac{GMm}{R_0^2} = \frac{m u_0^2}{R_0}$

$\Rightarrow \boxed{u_0 = \sqrt{\frac{GM}{R_0}}}$

(b) (i)



let u_2 be velocity at P.

COE) $-\frac{GMm}{R_0} + \frac{1}{2} m u_1^2 = -\frac{GMm}{R_1} + \frac{1}{2} m u_2^2$

COAM) $m R_0 u_1 = m R_1 u_2$

$\Rightarrow u_2 = \frac{R_0}{R_1} u_1$

$\Rightarrow u_1^2 = \frac{2GM}{R_0} \frac{1}{\left(\frac{R_0}{R_1}+1\right)} = \frac{2 u_0^2}{\left(\frac{R_0}{R_1}+1\right)}$

$\therefore \boxed{u_1 = \sqrt{\frac{2}{\frac{R_0}{R_1}+1}}\, u_0}$

(ii) To escape, $R_1 \to \infty$

$\Rightarrow u_1 = \sqrt{2} u_0 = \boxed{\sqrt{\frac{2GM}{R_0}}}$

(iii) $u_2 = \frac{R_0}{R_1} u_1 = \boxed{\frac{R_0}{R_1} u_0 \sqrt{\frac{2}{\frac{R_0}{R_1}+1}}}$

(c) $u_3 = \sqrt{\frac{GM}{R_1}}$

$u_2 = u_3 \sqrt{\frac{2}{\frac{R_1}{R_0}+1}} = u_3 \sqrt{\frac{2 R_0}{R_1 + R_0}}$

$\Rightarrow \boxed{u_3 = u_2 \sqrt{\frac{R_0 + R_1}{2 R_0}}}$

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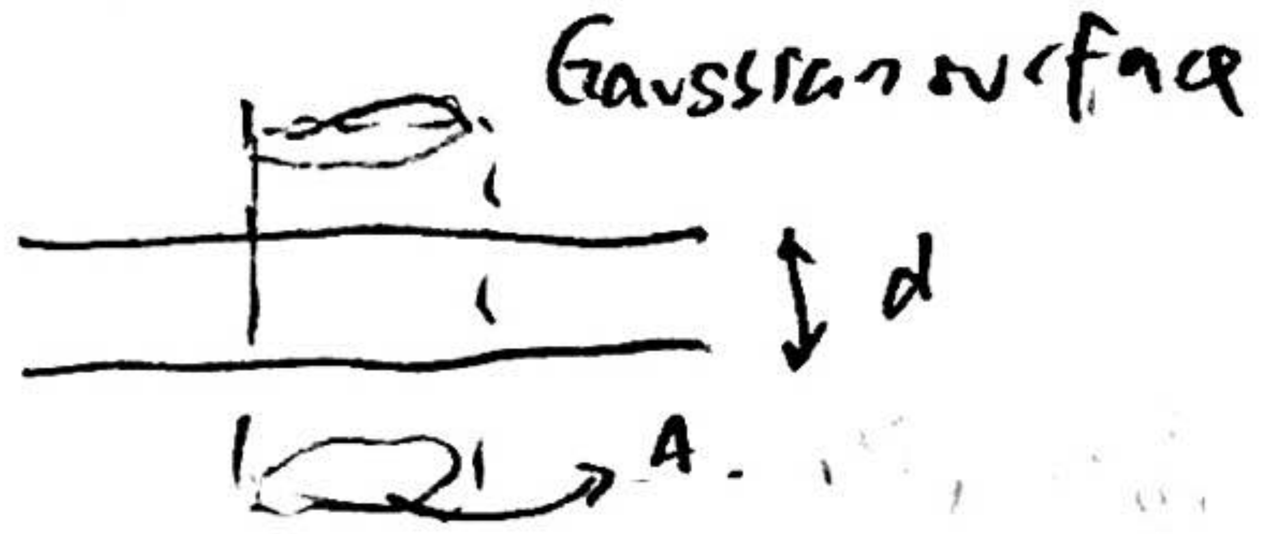


$$\left|\oint \vec{g}\cdot d\vec{A}\right| = g(2A) = 4G M_{enc}\pi$$
$$= 4\pi G \rho d A$$

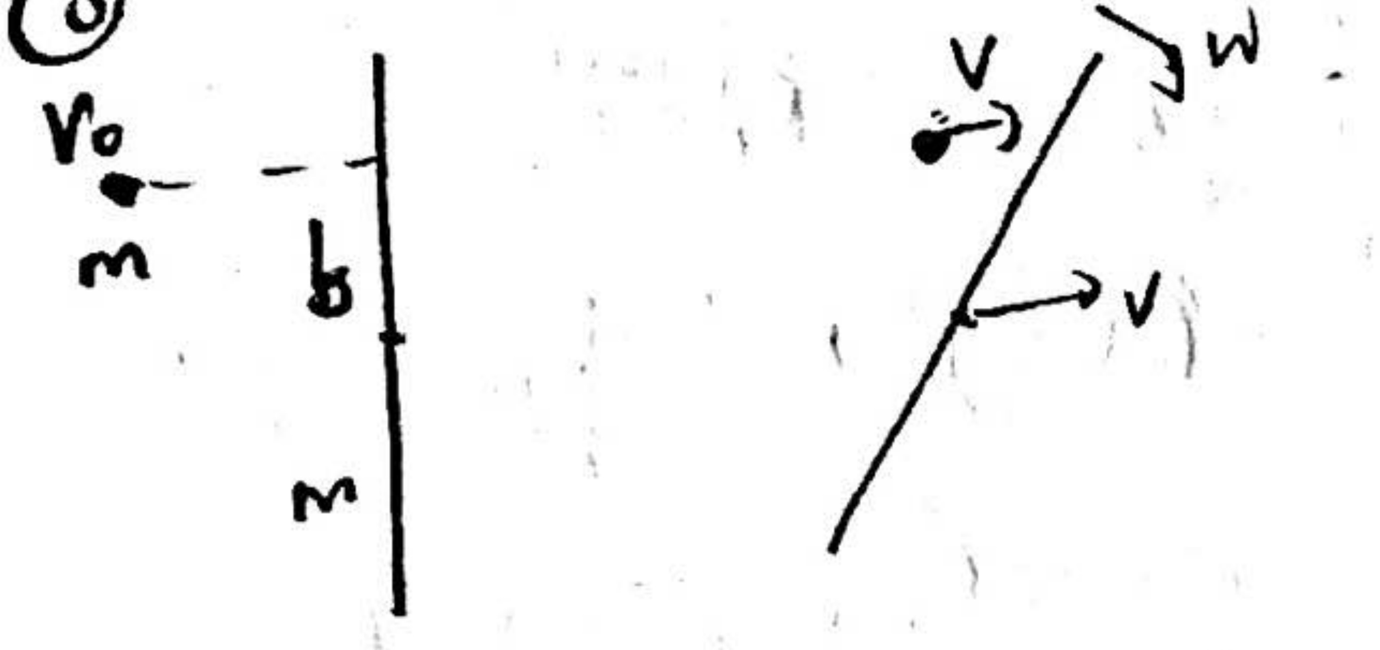
$\Rightarrow \boxed{g = 2\pi G\rho d}$ constant.

$\therefore\ s = \frac{1}{2}gt^2$

$\Rightarrow 100 = \frac{1}{2}2\pi(6.67\times10^{-11})(222)\rho(20\times60)^2$

$\Rightarrow \boxed{\rho = 1.49\times10^3\ kg\,m^{-3}}$

8



COM) $mv_0 = mv + mv \Rightarrow v = \frac{v_0}{2}$.

COE) $\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \frac{1}{2}mv^2$
$+ \frac{1}{2}I\omega^2$

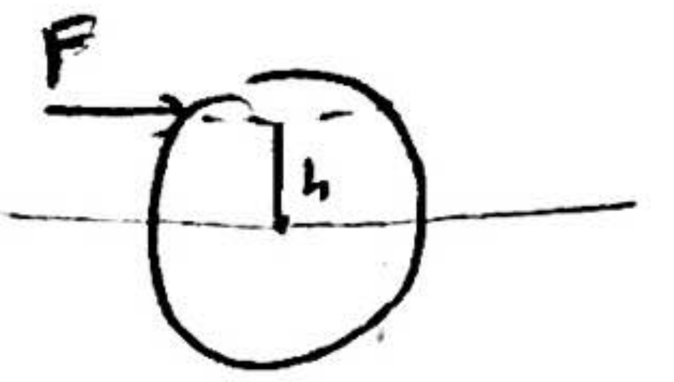
$= mv^2 + \frac{1}{2}\left(\frac{1}{12}ml^2\right)\omega^2$

$\Rightarrow \omega = \sqrt{6}\frac{v_0}{l}$.

COAM) $mv_0 b = mvb + I\omega$
$= mvb + \frac{1}{12}ml^2\omega$

$\Rightarrow \boxed{b = \frac{l}{\sqrt{6}}}$

31



$\because$ Rolling faster than center of mass moving

$\int f dt = m\left(\frac{9}{7}v_0\right) - mv_0$

$r\int f dt = I(\omega_i - \omega_f) = \frac{2}{5}mr^2\left(\omega_i - \frac{\frac{9}{7}v_0}{r}\right)$

$\Rightarrow \omega_i = \frac{2v_0}{r}$

Initially,

$\int F dt = mv_0$.

$h\int F dt = I\omega_i = \frac{2}{5}mr^2\omega_i$

$\Rightarrow \boxed{h = \frac{4}{5}r}$

32

y) $m\ddot{y} = N - mg$

τ) $I\ddot{\theta} = -mg\frac{L}{2}\sin\theta$

Geometry) $y = \frac{L}{2}\cos\theta$

$\dot{y} = \frac{L}{2}(-\sin\theta)\dot{\theta}$

$\ddot{y} = \frac{L}{2}\left[(-\cos\theta)\dot{\theta}^2 + (-\sin\theta)\ddot{\theta}\right]$

$\ddot{y}\Big|_{\theta=\theta_0} = \frac{L}{2}\left[-\sin\theta_0\left(\frac{-mg\frac{L}{2}\sin\theta_0}{\frac{1}{3}mL^2}\right)\right]$

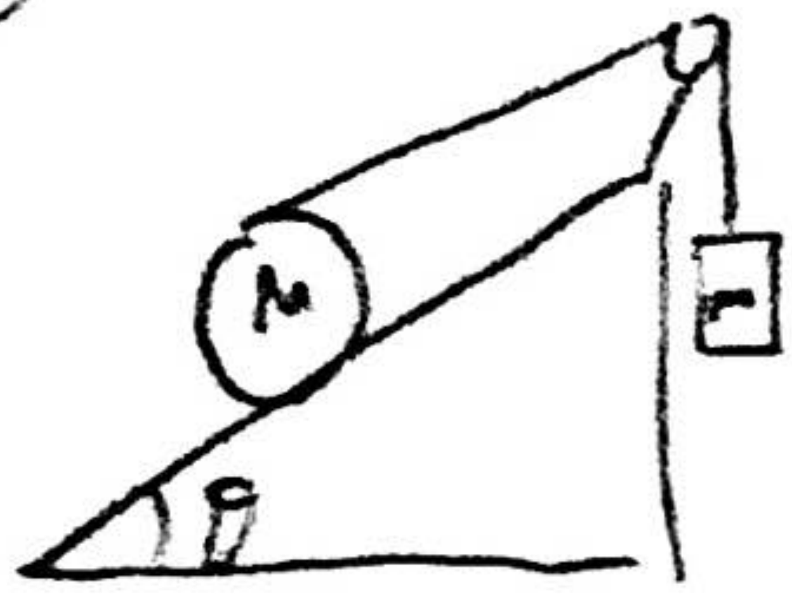
$= \frac{3}{4}\sin^2\theta_0\, g$

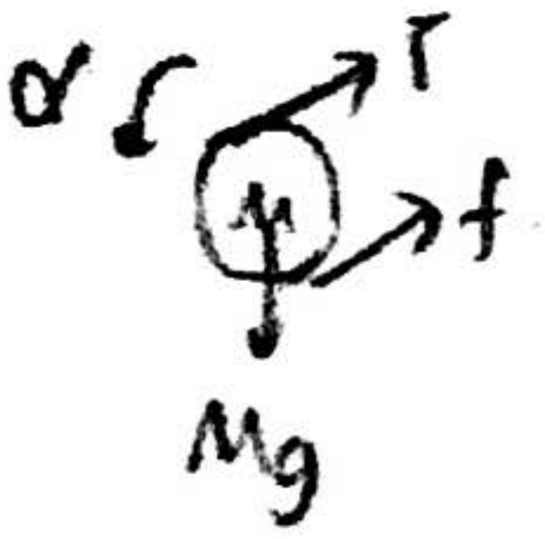
$\Rightarrow \boxed{N = mg\left(1 + \frac{3}{4}\sin^2\theta_0\right)}$

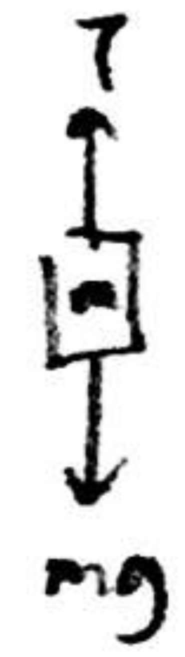
If $N < mg$, then $m\ddot{y} < 0$

$\Rightarrow$ pencil sinks into the table faster than the tip could rotate

B6







X, M) $Mg\sin\theta - T - f = Ma_1$

Y, m) $T - mg = ma_2$

τ, M) $(f - T)R = I\alpha = \frac{1}{2}MR^2\alpha$

rolling) $a_1 = R\alpha$

cos) $a_1 = \frac{1}{2}a_2$

$$\therefore \boxed{a_2 = \frac{(M\sin\theta - 2m)g}{\frac{3}{4}M + 2m}}$$

B5 Kepler's third law

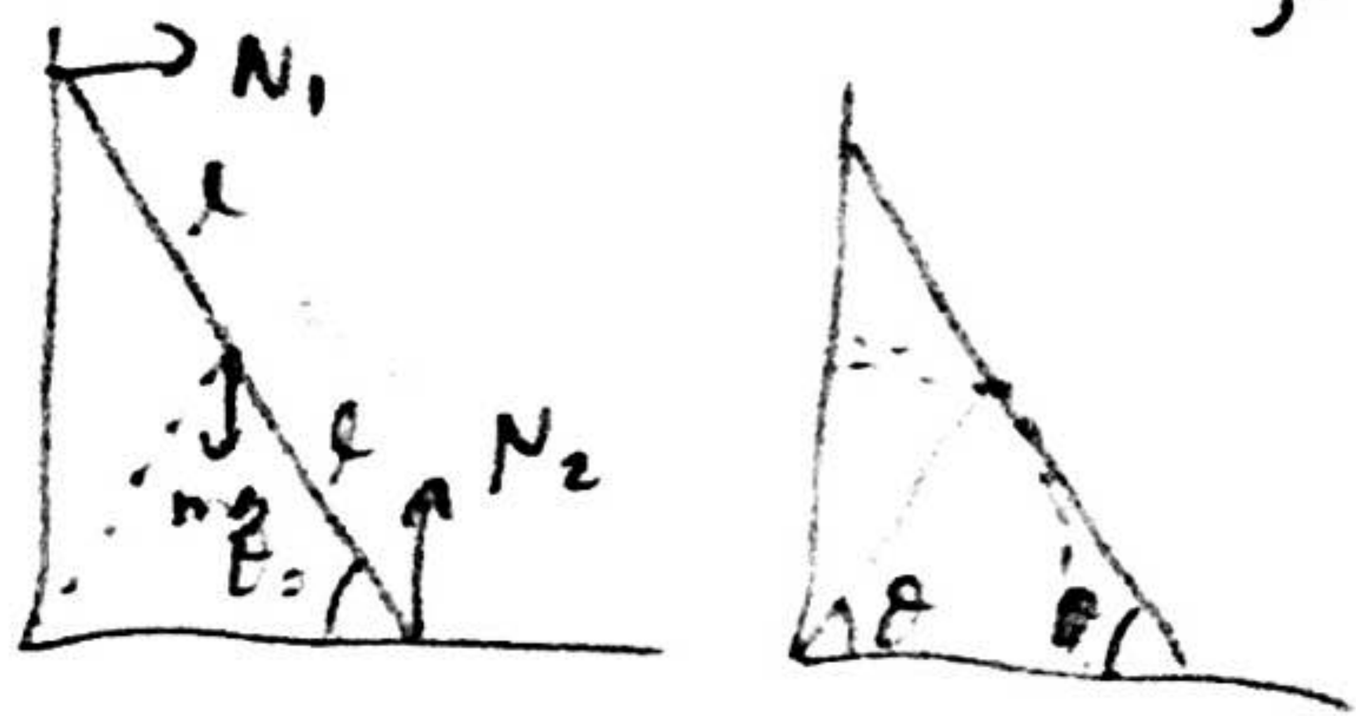
$$T^2 = \frac{4\pi^2 a^3}{GM}$$

Suppose the elliptical path to be very thin with $\frac{R}{2}$ as semi-major axis and the sun at its focus (equivalent to one end point of the line)

$$\left(\frac{T'}{T}\right)^2 = \left(\frac{1}{2}\right)^3 \Rightarrow T' = \left(\frac{1}{2}\right)^{\frac{3}{2}} T$$

$$\frac{T'}{2} = \boxed{\frac{\sqrt{2}}{8}T}$$

B3



Center of the plank is undergoing circular motion

When normal force becomes negative, then ladder loses contact.

∴ horizontal component of velocity cannot decrease unless ladder lost contact

CoE) $mgl(\sin\theta_0 - \sin\theta)$

$$= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}m(l\dot{\theta})^2 + \frac{1}{2}\frac{1}{12}m(2l)^2\dot{\theta}^2$$

$$\Rightarrow \dot{\theta} = \sqrt{\frac{3}{2}\frac{g}{l}(\sin\theta_0 - \sin\theta)}$$

$$\Rightarrow V = \sqrt{\frac{3}{2}gl(\sin\theta_0 - \sin\theta)}$$

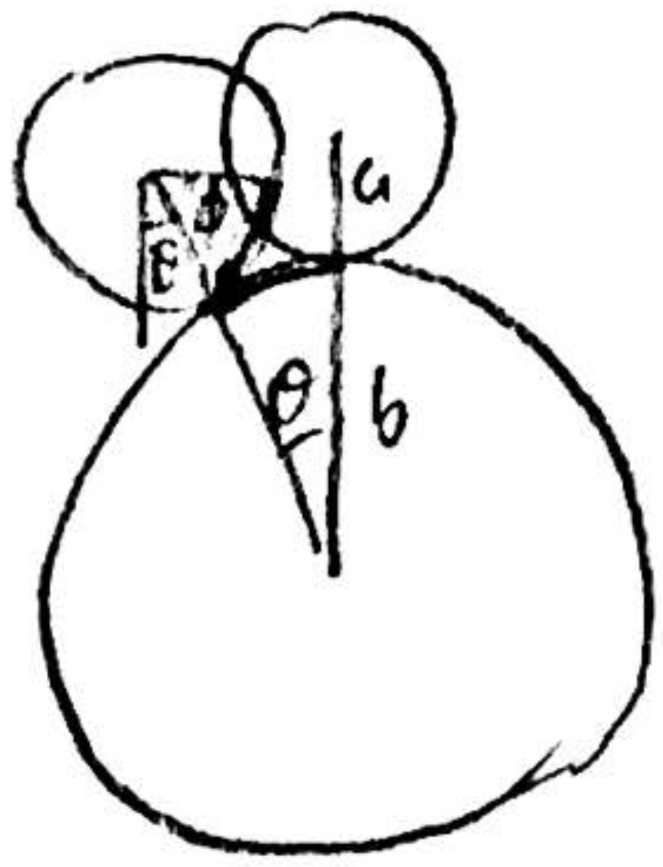
$$V_x = V\sin\theta$$

$$\Rightarrow \frac{dV_x}{dt} = \frac{dV}{dt}\sin\theta + \cos\theta\, V\dot{\theta}$$

$$\Rightarrow \boxed{\sin\theta = \frac{2}{3}\sin\theta_0}$$

$$\therefore \boxed{\theta = \sin^{-1}\left(\frac{2}{3}\sin\theta_0\right)}$$

(B4)



$b\theta = a\phi$

angular displacement $= \theta + \phi$

CoE) $mg(b+a)(1-\cos\theta) = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

$v = a(\dot\theta + \dot\phi) = (a+b)\dot\theta$

$mg\cos\theta - N = \frac{mv^2}{a+b} = m(a+b)\dot\theta^2$

When $N=0$,

$mg\cos\theta = m(a+b)\dot\theta^2 = \frac{10}{7}mg(1-\cos\theta)$

$\Rightarrow \cos\theta = \frac{10}{17}. \Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{10}{17}\right)}$

$\dot\theta^2 = \frac{g\cos\theta}{a+b}$

$v = \sqrt{(a+b)g\cos\theta} \Rightarrow \boxed{v = \sqrt{\frac{10}{17}g(a+b)}}$

(B7) (a) y) $\int N\,dt = m(V_y' - V_y)$ $h \downarrow \alpha h$

$= m\sqrt{2gh}(\sqrt{\alpha}+1)$

x) $\int f\,dt = \mu\int N\,dt = m(V_x' - V_x) = mV_x'$

τ) $r\int f\,dt = r\mu\int N\,dt = I(\omega_0 - \omega')$

$\tan\theta = \frac{V_x'}{V_y'} = \frac{\mu\int N\,dt}{\sqrt{2\alpha gh}} = \boxed{\mu\frac{1+\sqrt{\alpha}}{\sqrt{\alpha}}}$

$\Delta t = \frac{2\sqrt{2\alpha gh}}{g} = 2\sqrt{\frac{2\alpha h}{g}}$

$\Rightarrow \boxed{\Delta x = 4h\mu(\sqrt{\alpha}+\alpha)}$

Since continuous slipping,

$\omega' r > V_x'$

$\Leftrightarrow \omega' - \frac{\mu m r(1+\sqrt{\alpha})}{I}\sqrt{2gh} > \frac{\mu(1+\sqrt{\alpha})}{r}\sqrt{2gh}$

$\Leftrightarrow \boxed{\omega' > \frac{\mu\sqrt{2gh}}{r}(1+\sqrt{\alpha})\left(\frac{mr^2}{I}+1\right)}$

(b)

If rolling achieved,

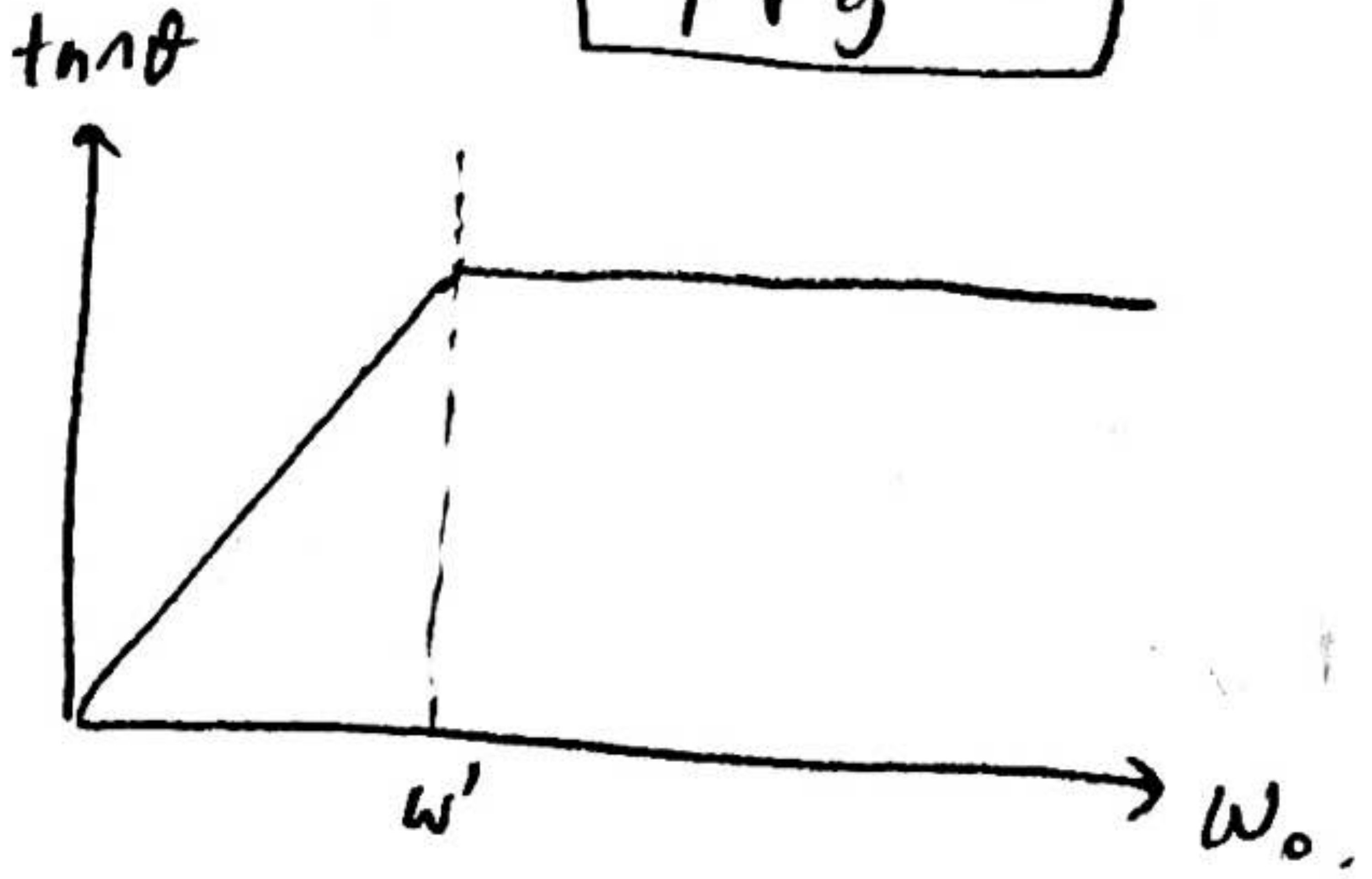
$V_x' = r\omega'$

x) τ) $r = \frac{r\int f\,dt}{\int f\,dt} = \frac{I(\omega_0 - \omega')}{m(V_x')} = \frac{I(\omega_0 - \frac{V_x'}{r})}{mV_x'}$

$\Rightarrow V_x' = \frac{2}{7}\omega_0 r$

$\tan\theta = \frac{V_x'}{V_y'} = \frac{\frac{2}{7}\omega_0 r}{\sqrt{2\alpha gh}} = \boxed{\frac{\sqrt{2}}{7}\frac{\omega_0 r}{\sqrt{\alpha gh}}}$

$\Delta x = V_x'\Delta t = \boxed{\frac{4\sqrt{2}}{7}\sqrt{\frac{\alpha h}{g}}\,\omega_0 r}$

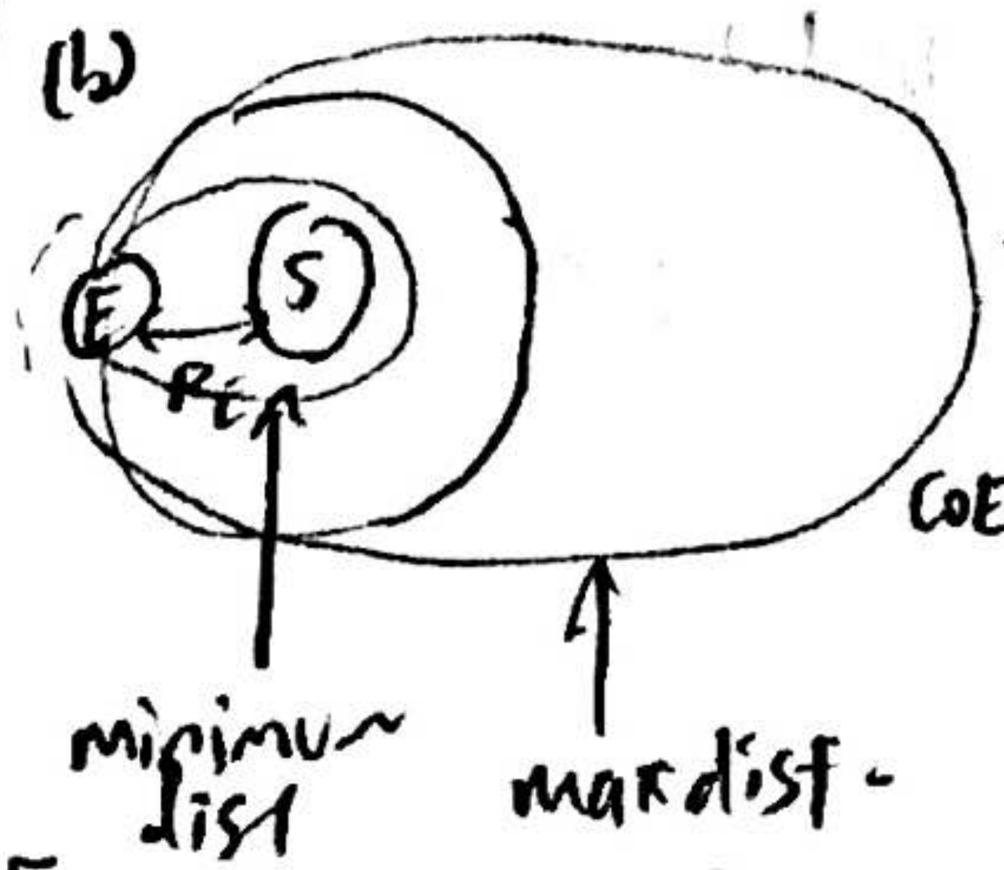


(B8)

(a) To escape, $E > 0$.

$E = \frac{1}{2}m(r\omega)^2 - \frac{GMm}{r} \geq 0 \Rightarrow \boxed{r_c = \sqrt[3]{\frac{2GM}{\omega^2}}}$

(b)



let $V = \omega h_0$

Rmax:

(CoAM) $m(V_E + V)R_E = mV_1 r_{max}$

(CoE) $\frac{1}{2}m(V_E+V_1)^2 - \frac{GM_sm}{R_E} = \frac{1}{2}mV$ $- \frac{GM_sm}{r_{max}}$

$\left[(V_E+\omega h_0)^2 - \frac{2GM_s}{R_E}\right]r_{max}^2 + 2GM_s r_{max} - (V_E+\omega h_0)^2 R_E^2 = 0$

$$\Rightarrow \boxed{r_{max} = \frac{(V_E + wh_0)^2 R_E^2}{2GM_S - (V_E + wh_0)^2 R_E}}$$

$\underline{r_{min}}$

COAM) $m(V_E - V) R_E = m V_1 r_{min}$

COE) $\frac{1}{2} m (V_E - V)^2 - \frac{GM_S m}{R_E} = \frac{1}{2} m V_1^2 - \frac{GM_S m}{r_{min}}$

$$\Rightarrow \boxed{r_{min} = \frac{(V_E - wh_0)^2 R_E^2}{2GM_S - (V_E - wh_0)^2 R_E}}$$