Fluid Dynamics

Physics Olympiad
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"Silk? Not sticky nor malleable enough for my applications. Needs to be load-bearing and strong, hold with extreme amounts of tension and stress, yet have a little stretch to it"

--- Peter Parker

Pressure

Pressure is a type of stress. It is a *scalar* quantity because it acts in all directions and the net effect cancels out. However, if there is a *pressure gradient*, then an actual force results from it and drives fluid flow.

Archimedes' Principle

Any object, totally or partially immersed in a fluid or liquid, is buoyed up by a force equal to the weight of the fluid displaced by the object.

$$F = \rho V g$$

Pascal's Law

A change in pressure at any point in an enclosed fluid at rest is transmitted equally and undiminished to in all directions to all points in the fluid. If the fluid is enclosed in a container, then the pressure acts at right angles to the enclosing walls.

$$\Delta p = \rho g \Delta h$$

Continuity Equation

The continuity equation is basically equivalent to the conservation of volume. For an incompressible fluid flowing from an area of cross section A_1 to an area of cross section A_2:

$$A_1 v_1 = A_2 v_2 = \text{constant}$$

Bernoulli's Equation

The Bernoulli's equation states that for two points A and B lying on a streamline, assuming that the fluid has constant density ρ, is flowing steadily and in the absence of friction:

$$P_A + \frac{1}{2}\rho {v_A}^2 + \rho g z_A = P_B + \frac{1}{2}\rho v_B^2 + \rho g z_B = \text{constant}$$

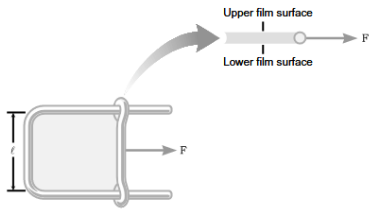
If you multiply a dV throughout, you will realise this becomes equivalent to the conservation of energy.

Surface Tension

Surface tension γ is the magnitude F of the force exerted parallel to the surface of a liquid divided by the length L over which the force acts. It is usually measured in dyne/cm and has units of N/m.

$$\gamma = \frac{F}{L}$$

For example, if we have a liquid surface stretched over a U-tube and a movable handle on the right, we can measure γ.

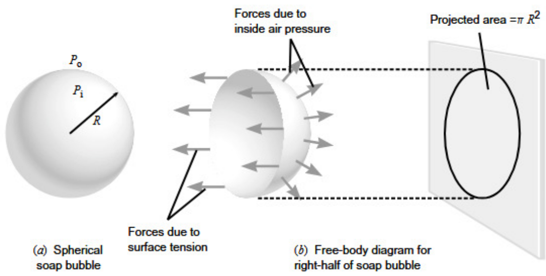


$$\gamma = \frac{F}{2L}$$

Here, the line at which the force acts is $2L$ because there is an upper and lower film surface. Take note to be careful when considering such scenarios. Some examples include cutting a spherical droplet and a spherical bubble along their middle. Although both are spherical with radius of R, the spherical bubble has a length $4\pi R$ while the bubble has a length of $2\pi R$.

$$\gamma = \frac{W}{\Delta A}$$

Surface tension allows us to characterise the pressure inside bubbles. We know that the surface tension between the liquid molecules is keeping the bubble from bursting.



Integrating the pressure about the surface of the sphere, we have:

$$P_i(\pi R^2) = \gamma(4\pi R)$$

$$P_i - P_0 = \frac{4\gamma}{R} \text{ (bubble)}$$

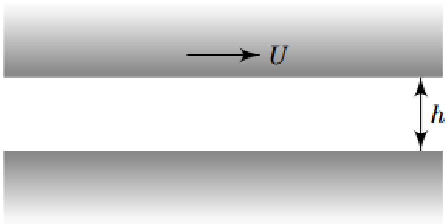
For a liquid drop,

$$P_i - P_0 = \frac{2\gamma}{R} \text{ (liquid drop)}$$

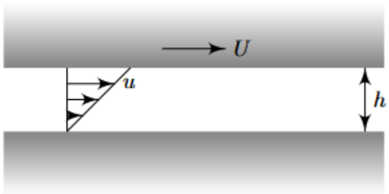
"Sir, there are still terabytes of calculations required before an actual flight is ... "
"JARVIS, sometimes you gotta run before you can walk."

--- Tony Stark

Viscosity



Consider a setup demonstrating *parallel viscous flow*, where there is a pressure gradient from left to right (i.e. high pressure on the left, low pressure on the right). Driven by the pressure gradient, water will start flowing from left to right. If the bottom plate is kept at rest, then the top plate will start moving right, reaching a final velocity of say U.



The tangential stress τ_s is the force (per unit area) required to move the top plate at a speed of U. Near the bottom plate, the fluid has a velocity of 0 (assuming no slip). Near the top plate, the fluid has a velocity of U. This results in the velocity gradient. For a Newtonian fluid, the tangential stress is proportional to the velocity gradient, with the constant of proportionality being called the *dynamic viscosity* (SI unit of $\mathrm{kg\,m^{-1}s^{-1}}$).

$$\tau_s = \mu \frac{U}{h}$$

For the same tangential stress, if you use honey rather than water as the fluid, you will achieve a much higher velocity of the upper plate.

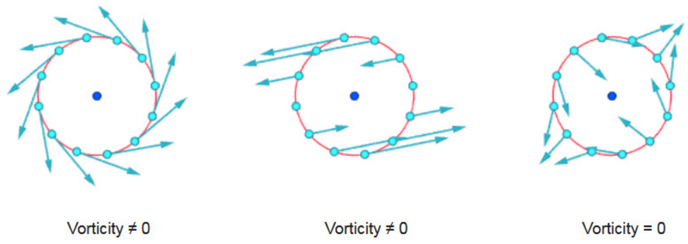
It is possible for a fluid to have zero viscosity, known as an ideal or inviscid fluid. It is observed at very low temperatures in superfluids.

The kinematic viscosity $\nu = \frac{\mu}{\rho}$ where ρ is the density of the fluid.

Vorticity

Vorticity $\boldsymbol{\omega}$ is the curl of the flow velocity. Recall from Electromagnetism that curl is a measure of how much a field rotates. In a two-dimensional flow, the vorticity is always perpendicular to the plane of the flow.

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$$

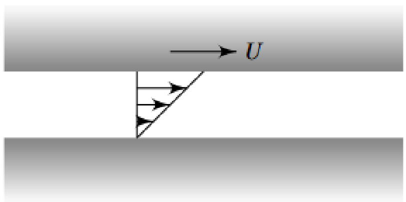


No Slip Condition

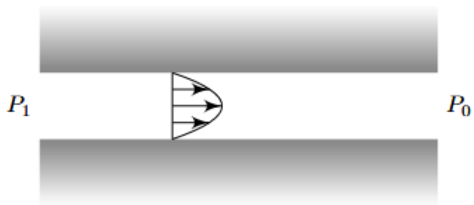
The no slip condition for viscous fluids assumes that at a solid boundary, the fluid will have zero velocity relative to the boundary. The physical justification is that close to a surface, the force of attraction between the fluid and solid interface (adhesive forces) is stronger than the force of attraction between the fluid particles (cohesive forces).

Types of Flow

- Couette Flow is a name given to flows driven by the motion of a boundary.



- Poiseuille Flow is a name given to flows driven by the pressure gradient between two stationary boundaries.



Reynolds Number and Turbulence

The Reynolds number helps characterise flow patterns. At low Reynolds number, flows tend to be dominated by laminar (sheet-like; no disruption between layers) flow, while at high Reynolds number, flows tend to be turbulent. Turbulent flows are those characterised by chaotic changes in flow velocity and pressure.

Given the density of fluid ρ, flow speed u, dynamic viscosity μ and a constant L that depends on geometry (dimension of $[L]$).

$$Re = \frac{\rho u L}{\mu}$$

Material Derivatives

In Fluid Dynamics, we often need to measure the change in a quantity, say Q, over both time and space. If we shift our perspective to follow a fluid particle moving along with the flow, we can measure the change in Q along our trajectory. This is known as the *Lagrangian picture*.

The material derivative of a quantity Q, denoted as $\mathrm{D}Q/\mathrm{D}t$, is the rate of change of Q belonging to a certain particle moving within a material or substance.

$$Q = Q[x(t), y(t), z(t), t]$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\partial Q}{\partial t} + \left(\frac{\partial Q}{\partial x}\right)\frac{\mathrm{d}x}{\mathrm{d}t} + \left(\frac{\partial Q}{\partial y}\right)\frac{\mathrm{d}y}{\mathrm{d}t} + \left(\frac{\partial Q}{\partial z}\right)\frac{\mathrm{dz}}{\mathrm{d}t} = \frac{\partial Q}{\partial t} + \left(\frac{\partial Q}{\partial x}\right)u_x + \left(\frac{\partial Q}{\partial y}\right)u_y + \left(\frac{\partial Q}{\partial z}\right)u_z$$

The *material derivative* is therefore defined to be the following function:

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}$$

With the definition of material derivative out of the way, *incompressibility* can be easily defined. Incompressible flow refers to a flow in which the material density is constant within an infinitesimal volume that moves with the flow velocity.

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \;\Leftrightarrow\; \frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\rho = \rho\,(\boldsymbol{\nabla} \cdot \boldsymbol{u}) = 0 \Leftrightarrow \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

A change in density over time would imply the fluid has either expanded or compressed, which means the amount of particles in an infinitesimal space has increased or decreased. So, it is not surprising that incompressibility implies $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$, since we learnt divergence is a measure of flux. This is also known as the continuity equation.

Navier-Stokes Equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}\right) = -\boldsymbol{\nabla}p + \mu\boldsymbol{\nabla}^2\boldsymbol{u} + \rho\boldsymbol{F}$$

The Navier-Stokes equation can be interpreted as the $F = ma$ of fluid dynamics. The first term on the right hand side is the pressure gradient. The second term represents the friction due to viscosity μ. These two terms are called the *internal forces*, caused due to fluid particles hitting, sliding and grinding past each other. Lastly, the capital $\boldsymbol{F}$ represents external forces. In most cases, $\boldsymbol{F}$ just comprises the gravity. If you substitute in Maxwell's equations into $\boldsymbol{F}$, then we will enter a realm known as *magnetohydrodynamics*, where we can (probably) model and solve for the formation/growth of stars and galaxies.

Understanding the Navier-Stokes equation is a first step to understanding the elusive phenomenon of turbulence. It is one of the seven Millennium Prize in Mathematics.

Prove or give a counter-example of the following statement:

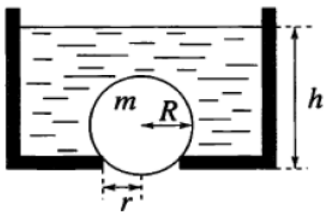
In three space dimensions and time, given an initial velocity field, there exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, that solves the Navier-Stokes equations.

Sample Problems

1. A body of 6.0 kg and density $450\ \mathrm{kg\ m^{-3}}$ is dropped from rest at a height of 7.5m into a lake. Calculate the maximum depth to which the body sinks before returning to float on the surface. You may neglect air resistance and the surface tension of the water.

2. A party balloon weighing 0.0025kg when empty is filled with helium to a volume of $0.0045\ \mathrm{m^3}$. It is tied to a small stone of mass 0.015kg by a light string of length 1.5m. A child holds the balloon at ground level and then releases it. What is the velocity of the stone when it is lifted off the ground? You may assume that the time needed to accelerate the stone to the velocity is very short and that the string remains taut throughout once it becomes taut.

3. A circular hole of radius r at the bottom of an initially full water container is sealed by a ball of mass m and radius R ($> r$). The depth of the water is now slowly reduced, and when it reaches a certain value h_0 the ball rises out of the hole. Find h_0.



4. Water, which wets glass, is stuck between two parallel glass plates. The distance between the plates is d and the diameter of the trapped water 'disc' is $D \gg d$. What is the force acting between the plates?

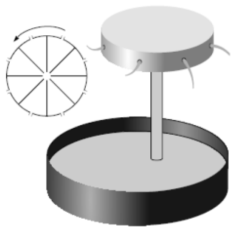
5. Consider the following water pump. A vertical tube of cross sectional area S_1 leads from an open water reservoir to a cylindrical rotating tank of radius r. There are holes of net cross-sectional area S_2 along the perimeter of the tank, which are open for the operating regime of the pump. The height of the tank from the free water surface of the reservoir is h (the height of the tank itself is small). An electric engine keeps the vessel rotating at an angular velocity of ω. The water density is ρ, the air pressure is p_0 and the saturated vapour pressure is p_k. Assume laminar flow and neglect friction. Inside the tank, there are metal blades which make the water rotate together with the tank.

    a. Calculate the pressure at the perimeter of the tank, when all the holes are closed.

    b. From now on, all the holes are open. Find the velocity of the water jets with respect to the ground.

    c. If the tank rotates too fast, the pump efficiency drops due to cavitation: the water starts 'boiling' in some parts of the pump. Find the highest cavitation-free angular speed.

    d. If the power of the engine is P, what is the theoretical upper limit of the volume of the pumped water per unit time.



6. [APhO 2017: Russia] The problem statement is too long to be inserted here, but this problem is beautiful in the sense that it links superfluidity with electromagnetism.