Waves (Solutions)

1.

$T = \mu x g + mg$

$v = \frac{dx}{dt} = \sqrt{\frac{T}{\mu}}$

$\Rightarrow \frac{dx}{dt} = \sqrt{\frac{\mu x g + mg}{\mu}}$

$\Rightarrow \frac{1}{\sqrt{\mu x g + mg}}\, dx = \sqrt{\frac{1}{\mu}}\, dt$

$\Rightarrow \left[2\sqrt{\mu x g + mg}\,\frac{1}{\mu g}\right]_0^d = \left(\left[\sqrt{\frac{1}{\mu}}\, t\right]_0^T\right.$

$\Rightarrow \boxed{T = \frac{2}{\sqrt{g}}\left(\sqrt{d + \frac{m}{\mu}} - \sqrt{\frac{m}{\mu}}\right)}$

3. (a) $\rho = \frac{1}{d}$

Since $d\sin\theta_1 = \lambda_1$, $\boxed{\rho = \frac{\sin\theta_1}{\lambda_1}}$

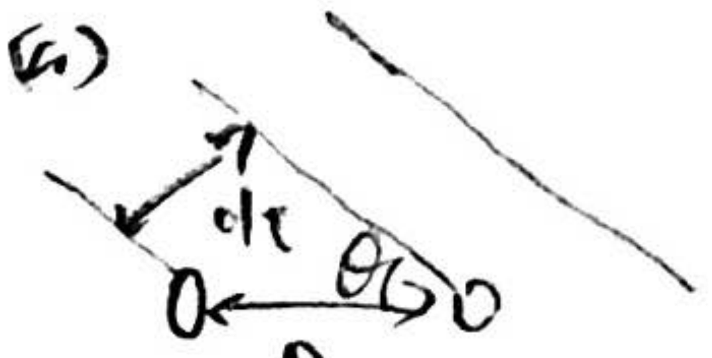
(b) Red colour has longer wavelength

$\Leftrightarrow$ disappears more easily.

$d\sin\theta_1 = n\lambda_1 \Rightarrow \sin\theta_1 = \frac{n\lambda}{d}$

$\Rightarrow n \le \frac{d}{\lambda_1} \Rightarrow \boxed{n_{max} = \left\lfloor \frac{d}{\lambda_1} \right\rfloor}$

2. (a)



$\sin\theta = \frac{x}{D} = \frac{v\Delta t}{D}$

$\Rightarrow \boxed{\Delta t = \frac{D\sin\theta}{v} \text{ where } v = \sqrt{\frac{B}{\rho}}}$

(b) $\Delta t = \frac{D \cdot \sin(90^\circ)}{V_{water}} = \frac{D\sin\theta'}{V_{air}}$

$\Rightarrow \boxed{\theta' = 13.4^\circ}$

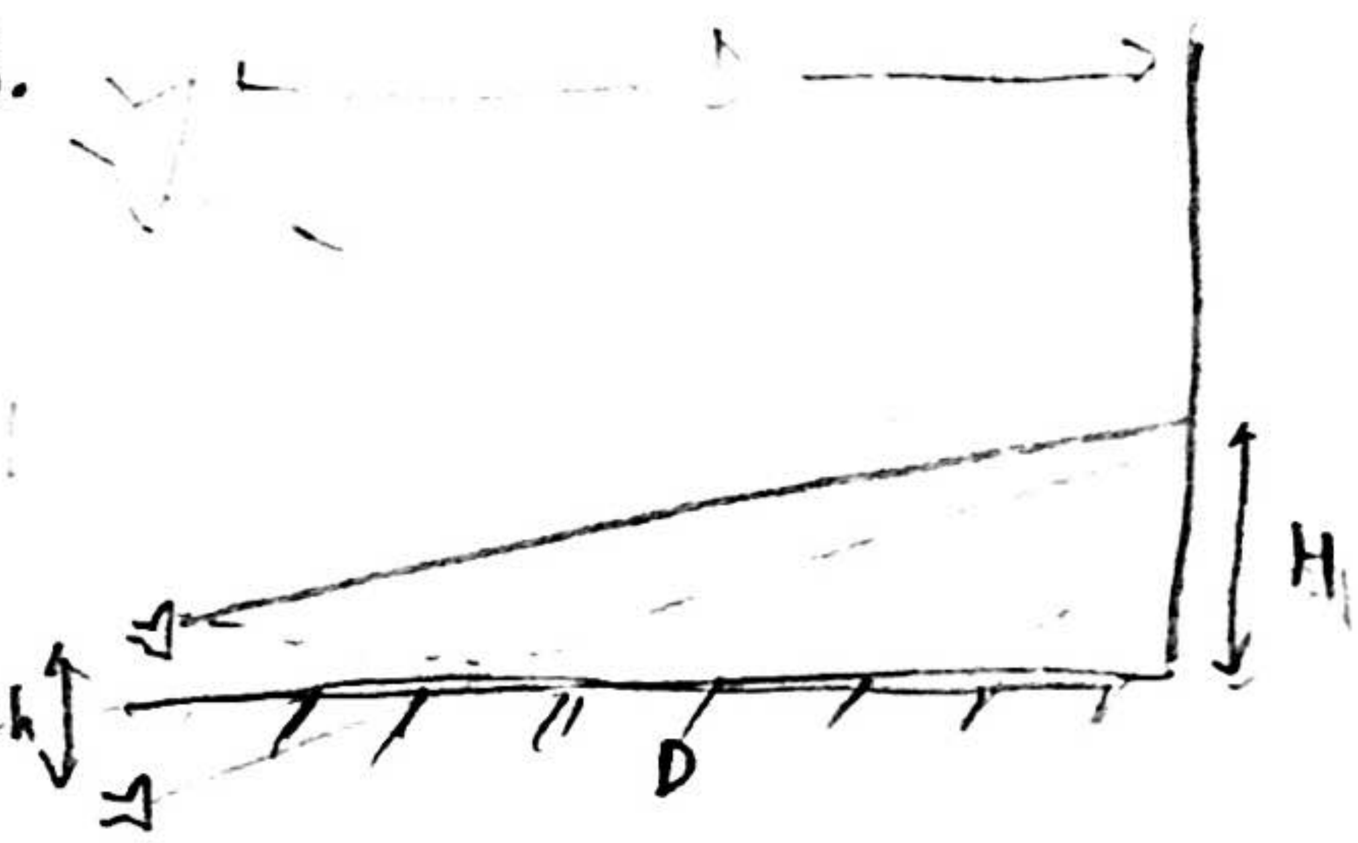
$\frac{n_1}{u} + \frac{n_2}{v} = \frac{n_2 - n_1}{R}$

For a plane, take $R \to \infty$

$\Rightarrow \frac{n_1}{u} + \frac{n_2}{v} = 0 \Rightarrow u = -\frac{n_1}{n_2} v.$

$|u| = \left|\frac{n_1}{n_2} v\right| = \frac{1}{1.33}(0.200) = \boxed{0.150}$

5.



Note that we can treat the problem as a double slit diffraction problem

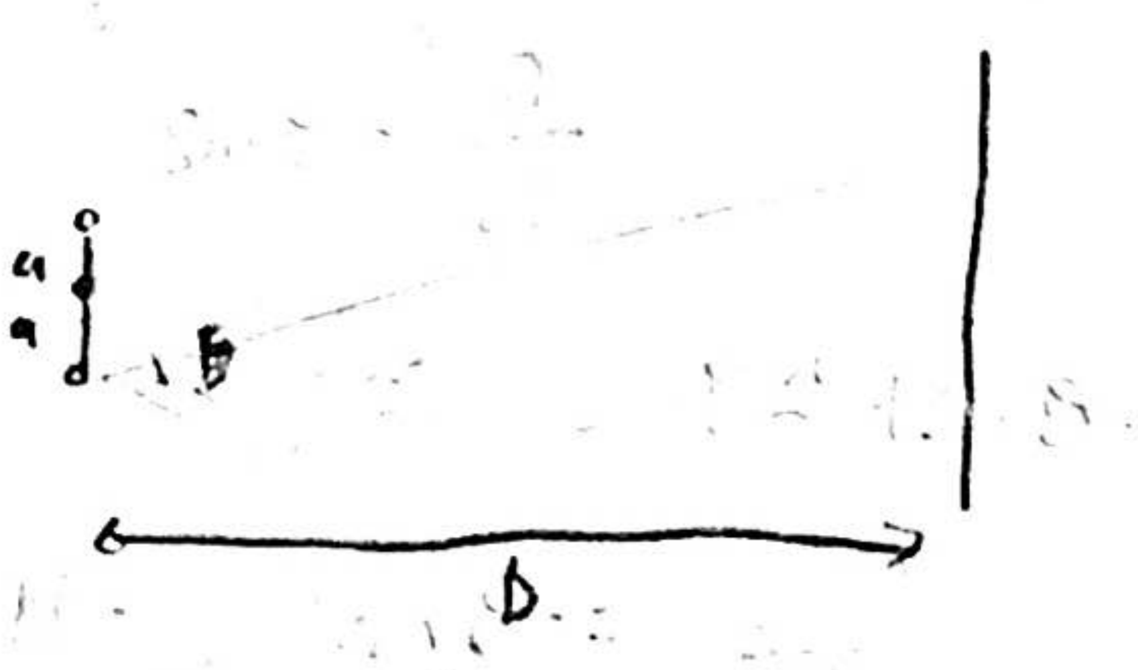
$2h\sin\theta = n\lambda$ for constructive interference.

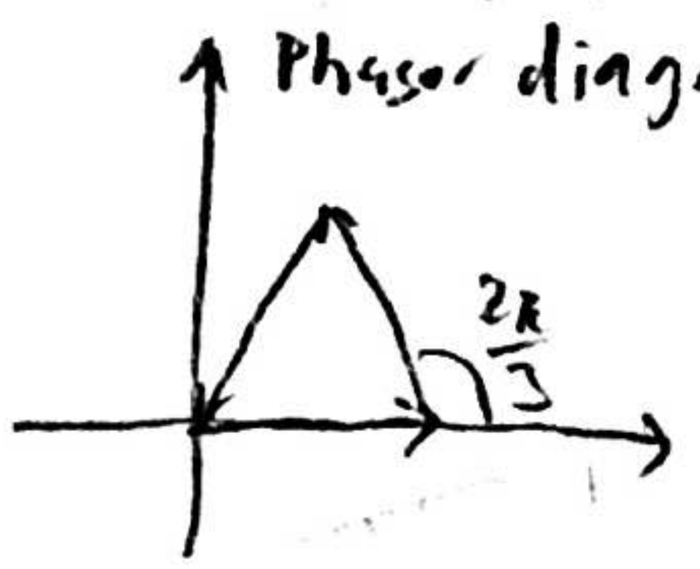
$\Rightarrow \sin\theta = \frac{\lambda}{2h}.$

Since $h \ll D$, $\sin\theta \approx \tan\theta$.

$H = D\tan\theta = \boxed{\frac{D\lambda}{2h}}$

4.



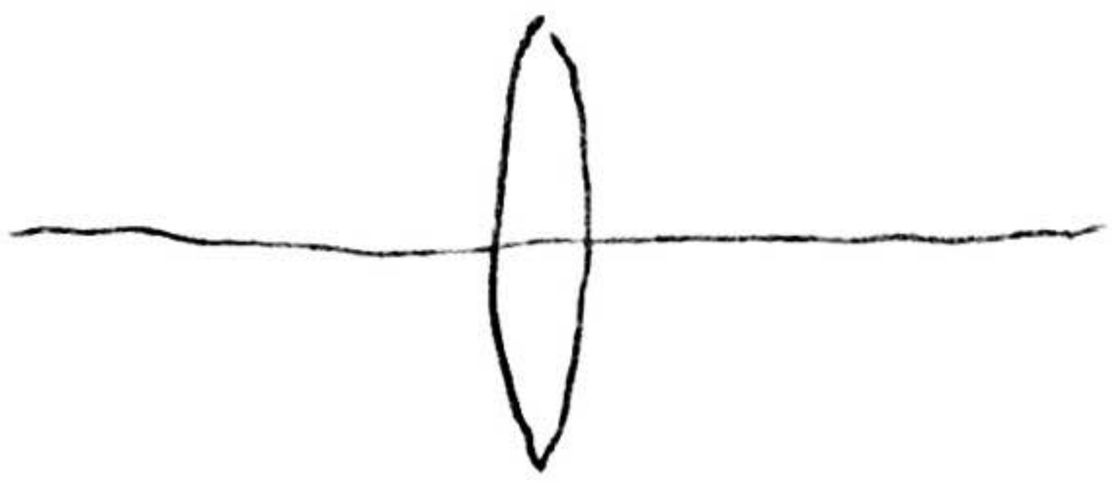


We want the phase difference to be $\frac{2\pi}{3}$

$\Rightarrow \frac{a\sin\theta}{\lambda} = \frac{\frac{2\pi}{3}}{2\pi}$

$\Rightarrow \boxed{a\sin\theta = \frac{1}{3}\lambda}$

6.



let the distance be D.

$$\frac{1}{u}+\frac{1}{D-u}=\frac{1}{f}$$

$$\Rightarrow fD=(D-u)u \Rightarrow u^2-Du+fD=0$$

$$\boxed{u_{1,2}=\frac{D\pm\sqrt{D^2-4fD}}{2}}$$

let height of object be h.

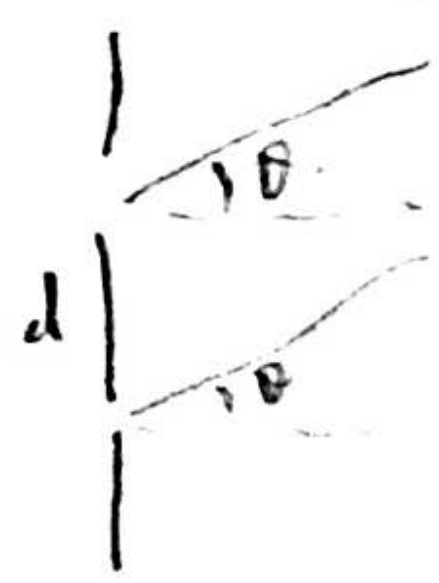
height of image 1 $=\frac{D-u_1}{u_1}h$

height of image 2 $=\frac{D-u_2}{u_2}h$.

$$h_1h_2=h^2\left(\frac{D-u_1}{u_1}\cdot\frac{D-u_2}{u_2}\right)=h^2\frac{u_2}{u_1}\cdot\frac{u_1}{u_2}=h^2$$

$$\Rightarrow \boxed{h=\sqrt{h_1h_2}.}$$

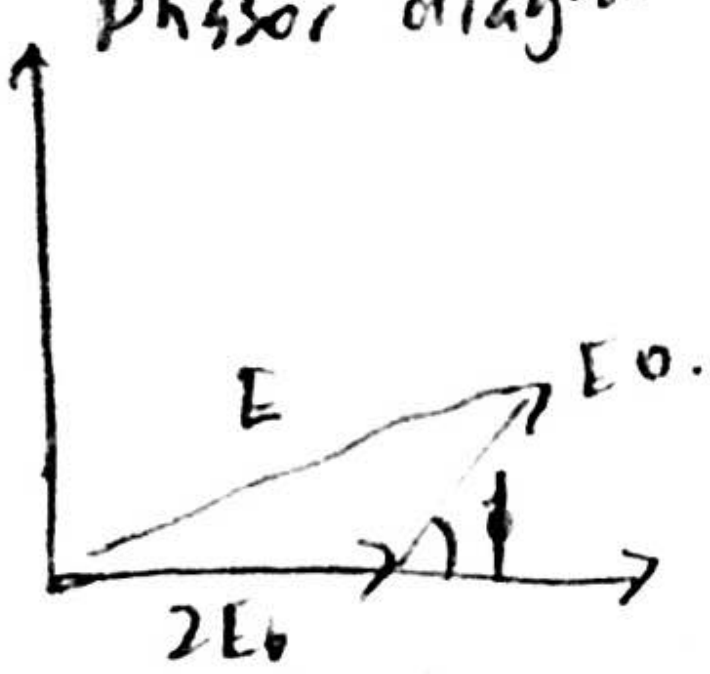
7. Assuming a coherent source,



Path difference $=d\sin\theta$

$\Rightarrow$ Phase difference $=\phi=2\pi\frac{d\sin\theta}{\lambda}$.

Phasor diagram



By cosine rule,

$$E^2=(2E_0)^2+E_0^2-2(2E_0)E_0\cos(\pi-\phi)$$

$$=5E_0^2+4E_0^2\cos\phi$$

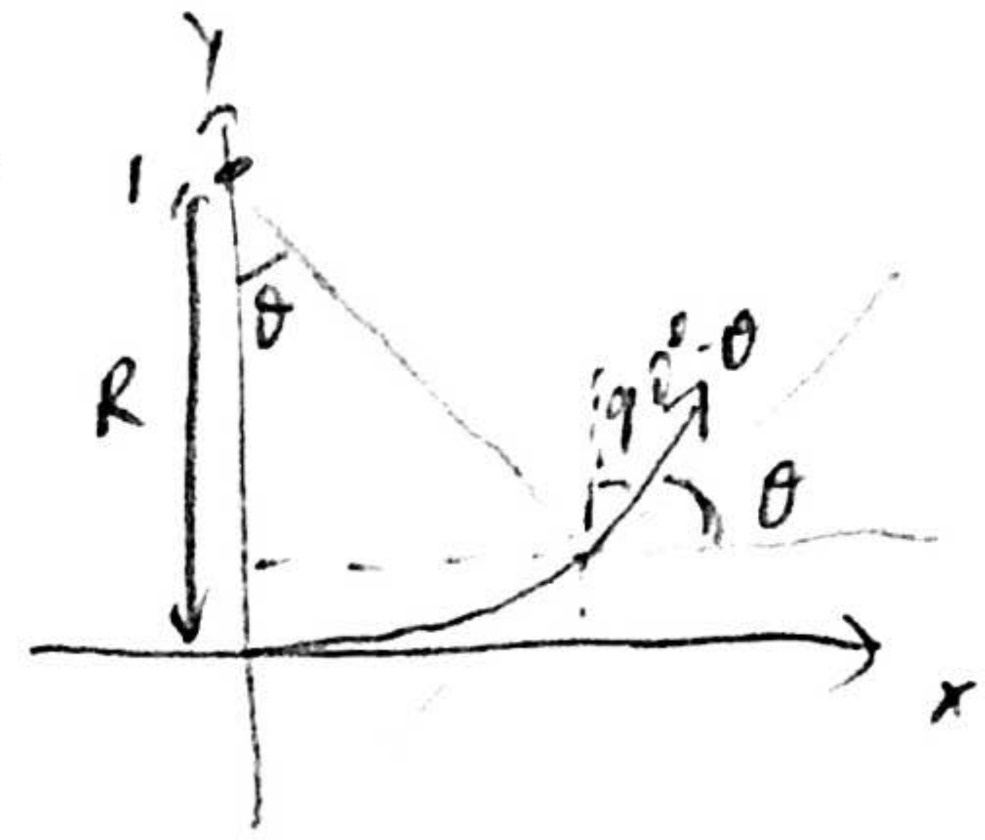
$$=(5+4\cos\phi)E_0^2$$

Since $I\propto E^2$.

$$I=\frac{E^2}{(3E_0)^2}I_0=\frac{I_0}{9}(5+4\cos\phi)$$

$$=\frac{I_0}{9}\left(1+8\cos^2\frac{\phi}{2}\right)=\boxed{\frac{I_0}{9}(1+8\cos^2\beta) \text{ where } \beta=\frac{\pi d\sin\theta}{\lambda}}$$

8.



let the radius of the arc be R

let $n(y)$ be the refractive index of $y=0$.

$$n(0)\sin 90^\circ=n(y)\sin(90^\circ-\theta)=n(y)\cos\theta$$

Here $y=R(1-\cos\theta)$

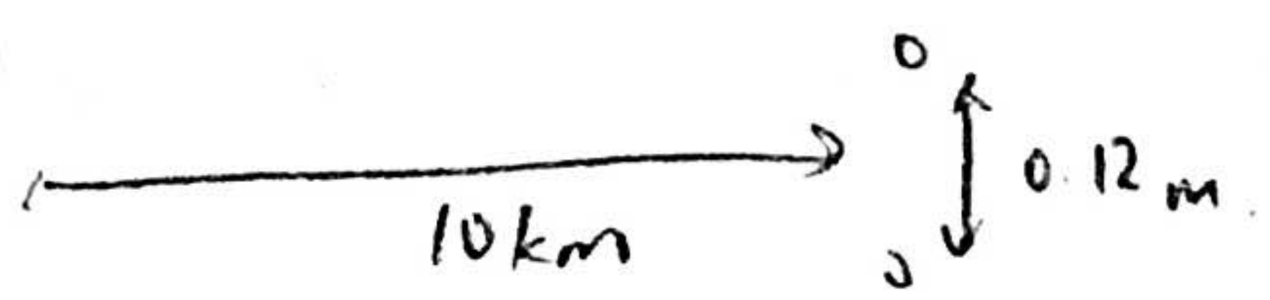
$$\Rightarrow 1-\cos\theta=\frac{y}{R}\Rightarrow\cos\theta=\frac{R-y}{R}$$

$$\Rightarrow\boxed{n(y)=n_0\frac{R}{R-y}}$$

Typically, the max n for a material is 2.5 $\Rightarrow y_{max}\approx\frac{3}{5}R$

$\Rightarrow$ angle $\approx\boxed{66.4^\circ}$

9.



Rayleigh's criterion:

Two objects are just resolved if $\sin\theta=1.22\frac{\lambda}{d}$

$$\sin\theta\approx\theta=\frac{0.12}{10\times10^3}$$

$$\Rightarrow d=\frac{1.22\lambda}{\sin\theta}\approx\frac{1.22\lambda}{\theta}$$

$$=\frac{1.22(600\times10^{-9})}{\frac{0.12}{10\times10^3}}=$$

10.

A → ← B

$$f = f_0 \frac{V_s}{V_s - V_t}$$ V_s : sound velocity
V_t : train velocity

$V_s = f\lambda$

Maximas occur at $\frac{1}{2}\lambda$ due to the propagating wave;

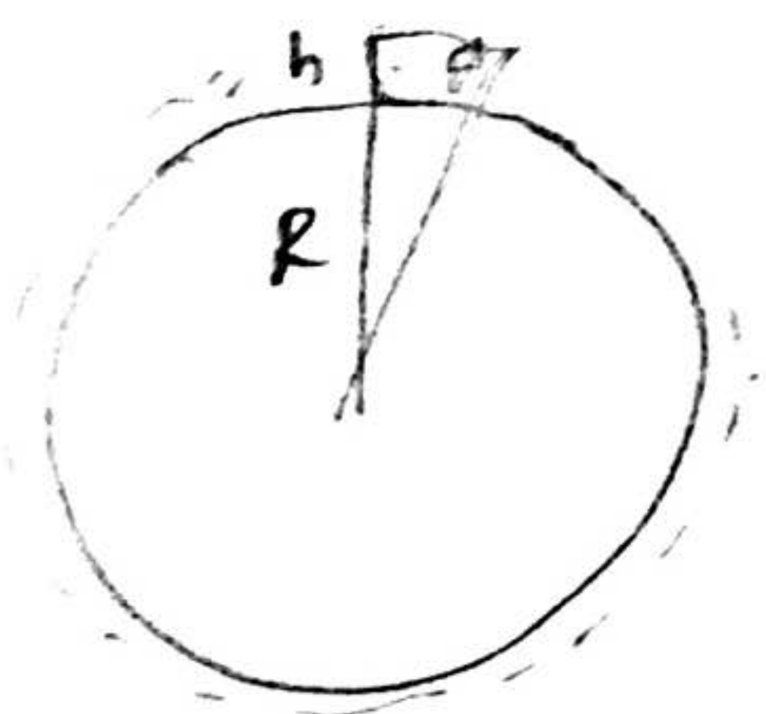
$$\text{Dist} = \frac{1}{2}\lambda = \frac{1}{2}\frac{V_s}{f_0 \frac{V_s}{V_s - V_t}}$$

$$= \frac{1}{2}\frac{334}{5000\ \frac{334}{334-(120\times 10^3)/3600}}$$

$$\approx \boxed{0.030\,m}$$

11. Suppose the atmosphere is made up of many thin layers of gas, with thickness Δh each.

A laser shown at height h will immediately undergo total internal reflection at the next layer $h+\Delta h$.



$$n(h)\sin\theta = n(h+\Delta h)\sin 90^\circ$$
$$= n(h+\Delta h).$$

$$\sin\theta = \frac{R+h}{R+h+\Delta h}$$

$$\Rightarrow n(h)\frac{R+h}{R+h+\Delta h} = n(h+\Delta h)$$

$$\Rightarrow n(h)\left(1-\frac{\Delta h}{R+h}\right) = n(h)+\Delta n$$

$$\Rightarrow \frac{dn}{dh} = -\frac{n(h)}{R+h}.$$

$$n(h) = \frac{n_0}{1+\varepsilon h} \Rightarrow \frac{dn}{dh} = n_0(-1)\frac{1}{(1+\varepsilon h)^2}\varepsilon$$
$$= -\frac{\varepsilon n_0}{(1+\varepsilon h)^2}$$

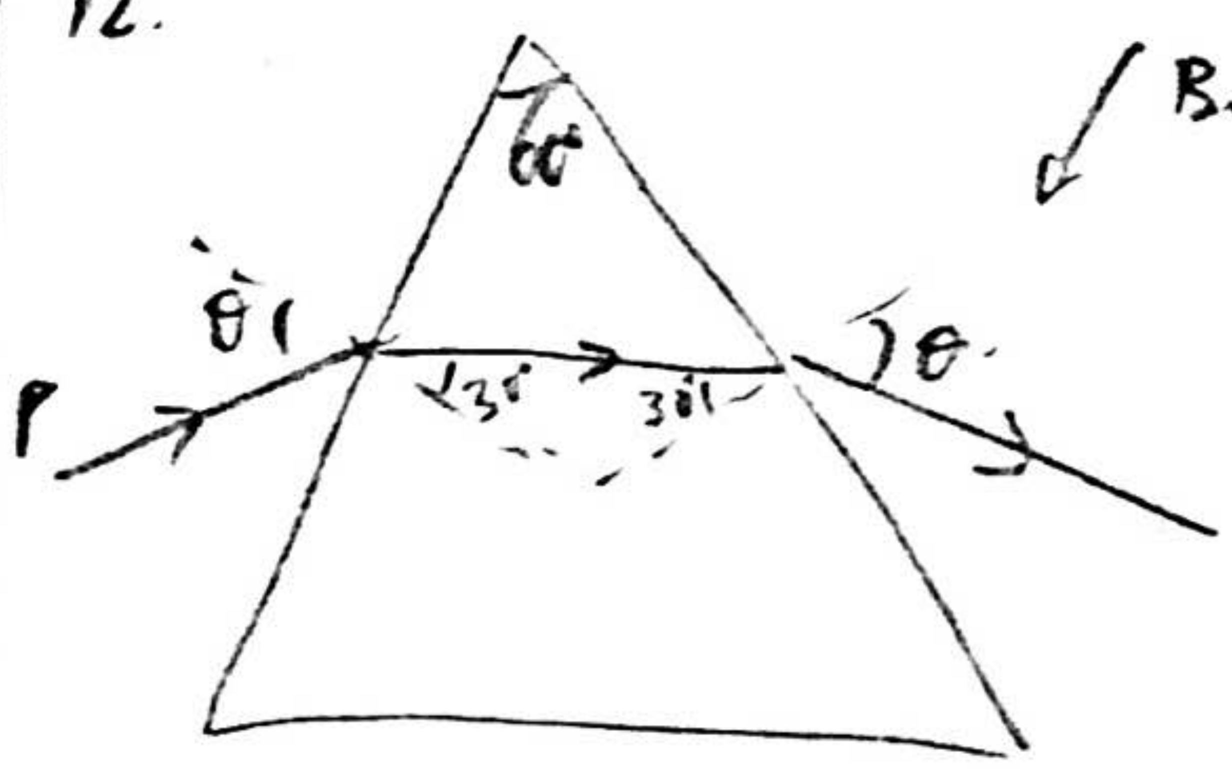
$$\Rightarrow -\frac{\varepsilon n_0}{(1+\varepsilon h)^2} = -\frac{n_0}{1+\varepsilon h}\frac{1}{R+h}$$

$$\Rightarrow \varepsilon = (1+\varepsilon h)\frac{1}{R+h}$$

$$\Rightarrow \varepsilon R + \varepsilon h = 1 + \varepsilon h \Rightarrow \varepsilon R = 1$$

$$\Rightarrow \boxed{R = \frac{1}{\varepsilon}}$$

12.



By symmetry.

Snell's) $\sin\theta = (1.50)\sin 30^\circ = 0.75$

change in momentum

$$= \Delta p = 2p\cos(\theta - 30^\circ)$$
$$= 2\frac{h}{\lambda}\cos(\theta-30^\circ)$$

Number of photons incident per second

$$= \frac{dn}{dt} = \frac{P_0}{\frac{hc}{\lambda}} = \frac{P_0\lambda}{hc}$$

$$F = p\frac{dn}{dt} = 2\frac{h}{\lambda}\cos(\theta-30^\circ)\frac{P_0\lambda}{hc}$$
$$= \frac{2P_0}{c}\cos(\theta-30^\circ)$$

$$\Rightarrow F = \frac{2(100)}{3\times 10^8}\cos(\sin^{-1}(0.75)-30^\circ)$$
$$= \boxed{132\times 10^{-7}\,N}$$

13.

Apparent depth $= h + \frac{d}{n} = v$

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

$$\Rightarrow \frac{1}{u} + \frac{1}{h+\frac{d}{n}} = \frac{1}{f}$$

$$\Rightarrow u = \left(\frac{1}{f} - \frac{1}{h+\frac{d}{n}}\right)^{-1} = \frac{f(h+\frac{d}{n})}{h+\frac{d}{n}-f}$$

$$\Rightarrow \boxed{u = \frac{f(nh+d)}{nh+d-fn}}$$

14. The tension within the string is uniform throughout. (otherwise there will be a non-zero horizontal acceleration)

For wave on the left,

$$y_1(x,t) = A\sin(\omega t - k_1 x) \quad \text{] original wave.}$$
$$+ A'\sin(\omega t + k_1 x) \quad \text{] reflected wave}$$

For the wave on the right.

$$y_2(x,t) = A''\sin(\omega t - k_2 x) \quad \text{] transmitted wave.}$$

Boundary condition) $y_1(0,t) = y_2(0,t)$

If not equal, it means the string broke in.

The gradient should also be equal

$$\frac{\partial y_1}{\partial x}(0,t) = \frac{\partial y_2}{\partial x}(0,t)$$

$$\Rightarrow A + A' = A'' \quad \text{(Boundary)}$$

$$k_1(A - A') = k_2 A'' \quad \text{(Tension)}$$

$$R = \frac{A'}{A} = \frac{k_1 - k_2}{k_1 + k_2}$$

$$T = \frac{A''}{A} = \frac{2k_1}{k_1 + k_2} \Rightarrow \boxed{R + T = 1}$$