Mechanics II

Physics Olympiad
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"That thing does not obey the laws of physics at all."
--- Peter Parker on Captain America's shield

For this Mechanics lecture, we will cover rotational dynamics and central forces. Similar to the previous lecture, we will take on this topic by building our understanding of concepts from bottom up.

Rotational Dynamics

- $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega$
- $\frac{\mathrm{d}\omega}{\mathrm{d}t} = \alpha$
- $\ddot{\theta}, \dot{\theta}$
- $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$
- Centripetal force: $F = \frac{mv^2}{r} = mr\omega^2$

Similar to linear dynamics, the next basic concept will be the definition of angular momentum. Angular momentum must be defined with respect to an origin. For a single particle, angular momentum is defined below. For an extended object, the total angular momentum is the sum of the contribution of the angular momentum of the infinitesimal masses.

- $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$
- $\boldsymbol{\tau} = \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r} \times \boldsymbol{p}) = \frac{\mathrm{d}r}{\mathrm{d}t} \times \boldsymbol{p} + \boldsymbol{r} \times \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{v} \times (m\boldsymbol{v}) + \boldsymbol{r} \times \boldsymbol{F}$
- $\tau = I\ddot{\theta}$

Now let's suppose that an extended body is rotating about its center of mass by an angular velocity of ω.

$$\vec{\boldsymbol{L}} = \int \vec{\boldsymbol{r}} \times \mathrm{d}\vec{\boldsymbol{p}} = \int \vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}\,\mathrm{d}m = \int \vec{\boldsymbol{r}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})\,\mathrm{d}m = \vec{\boldsymbol{\omega}} \int r^2\,\mathrm{d}m$$

If we have a quantity $I = \int r^2\,\mathrm{d}m$, then $L = I\omega$. This quantity is called inertia. In two dimensions (e.g. pancake objects), inertia is a single scalar. In three dimensions, inertia is a tensor (3×3 matrix). A possible way to understand why inertia is conceptually more complex than mass is because of the cross product in $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$.

Similarly, we can get the rotational kinetic energy in terms of I.

$$T = \int \frac{1}{2}(r\omega)^2\,\mathrm{d}m = \frac{\omega^2}{2}\int r^2\,\mathrm{d}m = \frac{1}{2}I\omega^2$$

- $I = \int r^2\,\mathrm{d}m$
  - Integral method
  - Scaling method
  - Parallel axis theorem: $I_z = MR^2 + I_z^{\mathrm{CM}}$
  - Perpendicular axis theorem: $I_z = I_x + I_y$
- $L = I\omega$
- $T = \frac{1}{2}I\omega^2$

The angular momentum of a body can be found by treating the body like a point mass located at the CM and finding the angular momentum of this point mass relative to the origin, and by then adding on the angular momentum of the body relative to the CM.

The kinetic energy of a body can be found by treating the body like a point mass located at the CM, and by then adding on the kinetic energy of the body due to the motion relative to the CM.

- Conservation of Angular Momentum: In an system with no external torque, the total angular moment is conserved.

In rotational dynamics, considering **impulses** is often a good line of thinking.

- $m\Delta v_x = \int F_x \,\mathrm{d}t$
- $m\Delta v_y = \int F_y \,\mathrm{d}t$
- $I\Delta\omega = \int \tau \,\mathrm{d}t$
- Rolling condition $v_{CM} = r\omega$

Central Force and Gravitation

If a particle is subject to a central force only, then its angular momentum is conserved and its motion lies in a plane.

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r} \times \boldsymbol{p}) = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \times \boldsymbol{p} + \boldsymbol{r} \times \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{v} \times (m\boldsymbol{v}) + \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{0}$$

In polar coordinates, the motion in a plane can be described with the following equations:

- $F_r = m(\ddot{r} - r\dot{\theta}^2)$
- $F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$

The only force in the radial direction is due to the central force. There are no forces in the tangential direction. If the central force is dependent only on r (i.e. rotationally symmetric), then it is conservative, because we can define a potential function $V = -\int f(r)\mathrm{d}r$.

$$F_r = m(\ddot{r} - r\dot{\theta}^2) = -V'(r)$$

$$F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$$

Note that the angular momentum $L = r \times mr\dot{\theta} = mr^2\dot{\theta}$. Conservation of angular momentum will yield an equation equivalent to the second equation above: $\frac{dL}{dt} = m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}) = mr(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$

We can obtain:

$$m\ddot{r} = \frac{L^2}{mr^3} - V'(r)$$

This is a differential equation. If $V(r)$ is in the form of $-\frac{\alpha}{r}$, then it can be solved nicely:

$$r(\theta) = \frac{L}{m\alpha} \cdot \frac{1}{1 + \epsilon \cos\theta}$$

Here, $\epsilon = \sqrt{1 + \frac{2EL^2}{m\alpha^2}}$. In case you are wondering where the two constant of integration went to, the first went to E, which is actually the total energy of the system. The second constant went to θ_0, but it does not appear in the formula because we can always define θ as $\theta - \theta_0$.

Now, we can analyse the motion in four different cases:

- $\epsilon = 0$: Motion is in the form of a circle

$$r_{min} = r_{max} = \frac{L^2}{m\alpha} \qquad E = -\frac{m\alpha^2}{2L^2}$$

- $0 < \epsilon < 1$: Motion is in the form of an ellipse

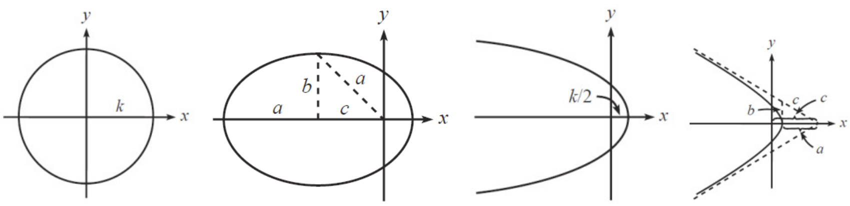
$$r_{min} = \frac{L^2}{m\alpha}\frac{1}{1+\epsilon} \qquad r_{max} = \frac{L^2}{m\alpha}\frac{1}{1+\epsilon} \qquad -\frac{m\alpha^2}{2L^2} < E < 0$$

- $\epsilon = 1$: Motion is in the form of a parabola

$$r_{min} = \frac{L^2}{2m\alpha} \qquad r_{max} = \infty \qquad E = 0$$

- $1 < \epsilon$: Motion is in the form of a hyperbola

$$r_{min} = \frac{L^2}{m\alpha}\frac{1}{1+\epsilon} \qquad r_{max} = \infty \qquad E > 0$$



Kepler's Laws

- First law: The planets move in elliptical orbits with the sum at one focus.
- Second law: The radius vector to a planet sweeps out area at a rate that is independent of its position in the orbit. This law is equivalent to conservation of angular momentum.

$$\mathrm{d}A = \frac{1}{2}r \cdot (r\ \mathrm{d}\theta) \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2m} = \text{constant}$$

- Third law: The square of the period of an orbit T is proportional to the cube of the semi-major axis length a.

$$T^2 = \frac{4\pi^2 a^3}{GM}$$

The way I remember the formula in the third law is to solve for the period, assuming the orbit is a circle, then replace r with a. Try it out, it works.

Now that we have gone through all of the above, we return to the familiar concepts:

- $\boldsymbol{F} = -\frac{GMm}{r^2}\hat{\boldsymbol{r}}$ (the negative sign exists to show $\boldsymbol{F}$ is anti-parallel to $\hat{\boldsymbol{r}}$)
- $\boldsymbol{g} = -\frac{GM}{r^2}\hat{\boldsymbol{r}}$ (think of this as a force field per unit mass)
- $V = -\frac{GMm}{r}$ (Recall that $\boldsymbol{F}$ comes first, then we can defined $V = \int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$, where here we took the limit of integration at $r = \infty$ and $V = 0$. Note that V here is gravitational potential **energy.**)
- $\phi = -\frac{GM}{r}$ (We introduce this because it is often easily to consider something *per unit mass.*)
- Since $\boldsymbol{F}$ is a conservative force, gain in kinetic energy of an object from point A to B is simply $V(B) - V(A)$. Similarly, work done by the gravitational field in moving an object from point A to point B is $V(B) - V(A)$.

There are a couple more things for gravitation, so I will put in here for completeness. I will only cover a little on Gauss' law for gravity, because it is similar to what I will be covering in EM. Please read the remaining up in your free time. They are good for bedtime stories.

- Gauss's law for gravitation: The gravitational flux through any closed surface is proportional to the enclosed mass.
$$\oint \vec{\boldsymbol{g}} \cdot \overrightarrow{\mathbf{d}\boldsymbol{A}} = -4\pi G M_{enc}$$
- Shell's theorem: The gravitational field outside a spherical shell having a total mass of M is the same as if the entire mass M is concentrated at the center. The gravitational field inside the shell is 0.

- Vis-viva equation: For any Keplerian orbit, the vis-viva equation follows, where a is the length of the semi-major axis.
$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

- Reduced mass: For a two-body system, the equation of motion of the first body relative to the second body can be derived by taking the second body as the origin and replacing the first body with the reduced mass μ which satisfies:
$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

A. Sample Problems

1. A cylinder of mass m, radius r and moment of inertia $\frac{1}{2}mr^2$ rolls without slipping down a plane inclined at angle θ. What is the acceleration of center of cylinder?

2. Find the moment of inertia of the following:
    a. A ring of mass M and radius R (axis through center, perpendicular to plane)
    b. A ring of mass M and radius R (axis through center, in plane)
    c. A disc of mass M and radius R (axis through center, perpendicular to plane)
    d. A disc of mass M and radius R (axis through center, in plane)
    e. A thin uniform rod of mass M and length L (axis through center, perpendicular to rod)
    f. A thin uniform rod of mass M and length L (axis through end, perpendicular to rod)

3. To get a billiard ball to roll without slipping immediately after impact, the cue must hit the ball at a height above its centre line. Find the height of impact required in terms of the radius of the billiard ball r.

4. Star-Lord's CD (a disc of radius R) is given an angular speed ω_i about an axis through its centre and then lowered to a horizontal surface and released. The coefficient of friction between the disk and the surface is μ.
   a. Find the time interval before pure rolling motion occurs.
   b. Find the distance the disk travels before pure rolling motion occurs.

5. A mass m is attached to the end of a string. The mass moves on a frictionless table, and the string passes through a hole in the table, under which someone is pulling on the string to make it taut at all times. Initially the mass moves in a circle, with kinetic energy E_0. The string is then slowly pulled, until the radius of the circle is halved. How much work is done?

6. Captain Marvel is trying to launch a satellite so that the Skrulls can have a new home

   a. Consider the launch of a satellite of mass m to a circular orbit of radius R_0 around the Earth of mass M. What is the velocity u_0 of the mass m in terms of M, R and the universal gravitation constant G?

   b. There is suddenly a need to modify the orbit of this satellite. The new orbit is elliptical in shape and its major axis passes through point P. P is a distance R_1 away from the centre of the Earth. Point Q is in an arbitary point on its initial circular orbit and is where the velocity of the satellite will almost instanteously increase to u_1 from u_0 to achieve the change in orbit.

      i. Express u_1 in terms of u_0, R_0 and R_1.

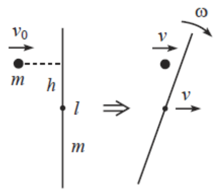
      ii. Find the minimum value of u_1 required to allow the satellite to escape from the gravitational field of the Earth.

      iii. Express the velocity of the satellite at point P, denoted as u_2, in terms of u_1, R_0 and R_1.

   c. Now the orbit needs to be modified again into a circle with radius R_1 by increasing the velocity from u_2 to u_3 almost instantaneously. Find u_3 in terms of u_2, R_0 and R_1.

7. Assuming Thanos' spaceship to be an infinitely long surface in free space that is $222\mathrm{m}$ thick. A test body released $100\mathrm{m}$ away from the surface of the slab, initially at rest relative to it, takes 20 minutes to fall to its surface. What is the density of the Thanos' spaceship, which is assumed to be homogenous?

8. A mass m travels perpendicular to a stick of mass m and length l, which is initially at rest. At what location should the mass collide elastically with the stick, so that the mass and the center of the stick move with equal speeds after the collision?



B. In Class Problems

1. A billiard ball, initially at rest is given a sharp impulse by a cue. The cue is held horizontally a distance h above the centre line. The ball leaves the cue with a speed v_0 and eventually acquires a final speed of $\frac{9}{7}v_0$. Find h in terms of r.

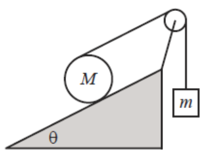
2. A thin uniform stick of mass m and length L standing on a frictionless table is released from rest at an angle θ_0 to the vertical. Find the force exerted by the table upon the stick immediately after its release.

3. A uniform plank of length $2l$ is held temporarily so that one end leans on a frictionless vertical wall and the other end leans on a frictionless floor making an angle $\theta = \theta_0$ with the floor. When the plank is released, it wil slide down under the influence of gravity. At what value of θ will the plank lose contact with the left wall?

4. A sphere of mass m and radius a rolls without slipping from its initial position at rest atop a fixed cylinder of radius b.
    a. Find the angle at which the sphere leaves the cylinder.
    b. Find the magnitude of the velocity of the centre of the sphere when it leaves the cylinder.

5. If Loki used his magic scepter to stop Earth suddenly in its orbit, how long will the Earth take to hit the Sun (in terms of T, the period of the Earth about the Sun)?

6. A string wraps around a uniform cylinder of mass M, which rests on a fixed plane. The string passes over a massless pulley and is connected to a mass m as shown in the figure. Assume that the cylinder rolls without slipping on the plane and that the string is parallel to the plane. What is the acceleration of the mass m?

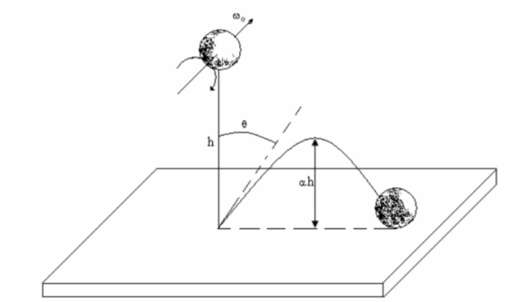


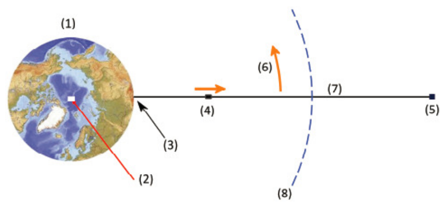
7. [IPhO 1991 Havana] The figure below shows a solid, homogeneous ball of radius R. Before falling to the floor, its center of mass is at rest, but the ball is spinning with angular velocity ω_0 about a horizontal axis through its center. The lowest point of the ball is at a height of h above the floor. When released, the ball falls under gravity and rebounds to a new height ah above the floor. The impact time is finite. The dynamic friction between the ball and the floor is μ_k.

    a. If the ball is slipping throughout the impact, find $\tan\theta$, where θ is the rebound angle. Find also the horizontal distance travelled in flight between the first and second impacts. Find also the minimum value of ω_0 for this situation.

b. If the ball is slipping for part of the impact, find again $\tan\theta$. Find also the horizontal distance travelled in flight between the first and second impacts.

c. Sketch a graph of $\tan\theta$ against ω_0





8. [APhO 2018 Hanoi] Tony Stark formulates the concept of a *space elevator*, which is basically just a very tall building (maybe even longer than the radius of Earth). He wants to utilise the rotational energy of the building to launch his spaceship into space. For simplicity, assume the motion of the building occurs in the plane of the Earth's orbit.

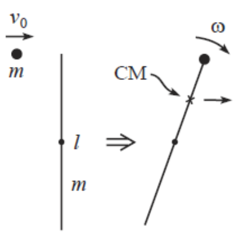
   a. Find the critical height r_c up the tower, measured from the Earth's center, at which the object would have to be released from rest to escape Earth's gravity. [Give your answers in terms of G, M_E (mass of Earth) and ω (the rotational speed of the Earth)]

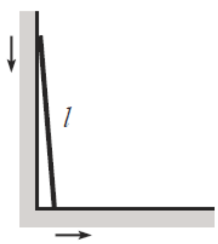
   b. Find the minimal and the maximal distances from the Sun that a spacecraft released from rest from the top of the building (distance h_0 from the center of the Earth) can reach. [Give your answers in terms of G, v_E, h_0, ω, R_E and M_S, where we are assuming the Earth is in a circular orbit around the sun of mass M_S with velocity v_E and radius R_E.]

9. A sphere of uniform density ρ has within it a spherical cavity whose centre is a distance a from the centre of the sphere. Show that the gravitational field within the cavity is uniform and determine its magnitude and direction.

C. Take Home Problems

1. A mass travels at speed v_0 perpendicular to a stick of mass m and length l, which is initially at rest. The mass collides completely inelastically with the stick at one of its ends and sticks to it. What is the resulting angular velocity of the system?





2. A ladder of length l and uniform mass density stands on a frictionless floor and leans against a frictionless wall. It is initially held motionless, with its bottom end an infinitesimal distance from the wall. It is then released, whereupon the bottom end slides away from the wall, and the top end slides down the wall. When it loses contact with the wall, what is the horizontal component of the velocity of the center of mass?

3. A uniform solid cylinder of radius R rolls from a horizontal plane onto a downward inclinde plane with angle α from the horizontal. Find the maximum velocity v_0 with which the cylinder can roll across the junction without losing contact at all. Pure rolling is assumed at all times.

D. Extra Problems

1. A solid cube of wood of side $2a$ and mass M is resting on a horizontal surface. The cube is constrained to rotate about a fixed axis AB. A bullet of mass m and speed u is shot at the face opposite $ABCD$ at a height $\frac{4}{3}a$. The bullet becomes embedded in the cube. Find the minimum value of u required to tip the cube so that it falls on face $ABCD$. Assume $m \ll M$.

2. An elastic spherical ball of mass M and radius a moving with velocity v strikes a rigit surface at an angle θ to the nromal. Assuming it skids while in contact with the surface, the tangential frictional force being a constant fraction μ of the normal force. Vertical direction of collision is completely elastic.
   a. Show that the angle ϕ to the normal at which the ball reflects follows $|\tan\theta - \tan\phi| = 2\mu$.
   b. Show that the angular velocity of the rebounding ball changes by $\frac{5\mu v\cos\theta}{a}$.

3. A pendulum is tied to the ceiling with initial angle from the vertical line being θ. The mass of the pendulum is m and the length of the thread is L. At $t = 0$, the pendulum is released. Calculate the minimum impulse needed in the horizontal direction perpendicular to the initial plane of oscillation of the pendulum in order for the pendulum to touch the ceiling.