Electricity and Magnetism II

Physics Olympiad
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"Are you Thor, the God of Hammers?"

--- *Odin Allfather*

In this session, we will cover Current, Magnetostatics and Electrodynamics.

Current

Firstly, you need to be familiarised with a new way of thinking about current. *Current*ly, you might think that current is just a scalar value, which you are correct. However, this scalar value is actually derived from an integral.

The more fundamental variable is $\boldsymbol{J}$, a vector, which denotes the current density. It is the amount of charge that flows through an area per unit time. Current I is defined to be the integral of J through a surface S.

$$I = \int_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S}$$

Sometimes, the question will introduce additional variables like v (drift velocity of electrons/any other charge carriers that are free to move), n (density of charges) and ρ (charge density). Once you have understood the overall concept of current density, it should be not difficult to link these variables together.

$$\boldsymbol{J} = \rho \boldsymbol{v}$$

$$I = nqvA$$

By conservation of charge, we must have

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$$

This is the continuity equation, which means the amount of charge entering/leaving an infinitesimal volume is equal to the gain/loss of charge in the infinitesimal volume. This should be very intuitive.

Magnetostatics

Currents give rise to magnetic fields. For magnetic fields produced by steady currents, $\rho = 0$ and $J \neq 0$. Recall that ρ refers to the charge density. $\rho = 0$ does not mean there are no charges, but rather there is no net charge flow from one place to another. At any point in space, the amount of charges entering is equal to the amount of charges leaving.

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$

For steady current, $\boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$ (there is no net inflow or outflow at any point) and hence $\frac{\partial \rho}{\partial t} = 0$.

Ampere's Law

Consider a surface S with boundary C and a current J that flows through S. Integrating $\nabla \times \boldsymbol{B}$ over the surface S:

$$\int_S (\nabla \times \boldsymbol{B}) \cdot \mathrm{d}S = \oint_C \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} = \mu_0 \int_S \boldsymbol{J} \cdot \mathrm{d}S = \mu_0 I_{enc}$$

$$\oint_C \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r} = \mu_0 I_{enc}$$

Geometrically, if the magnitude of $\boldsymbol{B} \cdot \boldsymbol{dr}$ is calculated at each point on the boundary C, the sum will be equal to μ_0 times the enclosed current (i.e. the magnitude of the current flowing through the boundary C.

Example	Formula
A long straight wire with current I.	$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{2\pi r} \widehat{\boldsymbol{\phi}}$
Surface current on plane $z = 0$ with density k.	$\boldsymbol{B}(z) = -\frac{\mu_0 k}{2} \widehat{\boldsymbol{y}}$
Solenoid with n wires per unit length and current of I.	$\boldsymbol{B} = \boldsymbol{0}$ $B = \mu_0 I n$

Biot-Savart Law

$$\mathrm{d}\boldsymbol{B} = \frac{\mu_0 I}{4\pi} \int \frac{\mathrm{d}\boldsymbol{s} \times \hat{\boldsymbol{r}}}{r^2}$$

This law is extremely useful. If you give me a wire that is carrying a current, I can immediately calculate the magnetic field at any point in space by infinitesimal analysis.

Electrodynamics

A charge of q in an electric field $\boldsymbol{E}$ and $\boldsymbol{B}$ will experience a total force $\boldsymbol{F}$ by both the electric and magnetic field.

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

Induction

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0$$

If the magnetic field $\boldsymbol{B}$ changes with time, this will produce an $\boldsymbol{E}$ field, which will induce currents in a wire. By Stokes' theorem, we can define the *electromotive force* (work done in moving a charge around a curve) and the *magnetic flux*. In doing so, we establish the following relationship between these two quantities:

$$\epsilon = \int_C \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = -\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

Faraday's Law of Induction

$$\epsilon = -\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

When there is a change in magnetic flux through a curve S, a current is induced. The physical significance of the minus sign means that the induced current always produces a magnetic field that seeks to oppose the change in magnetic flux. This is also known as Lenz's Law.

Inductance

The *inductance* of a curve C is the amount of magnetic flux it generates per unit current through C. You can think of inductance as the "resistance" to the change of current.

$$L = \frac{\phi}{I}$$

Inductance can be used to compute the energy stored in magnetic field. Since there is a form of resistance to a change of current, because some work is required to build up a current of I.

In particular,

$$\epsilon = -\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{\mathrm{d}(LI)}{\mathrm{d}t} = -L\frac{\mathrm{d}I}{\mathrm{d}t}$$

$$\mathrm{d}W = \epsilon I\,\mathrm{d}t = -LI\frac{\mathrm{d}I}{\mathrm{d}t}\,\mathrm{d}t = -LI\,\mathrm{d}I$$

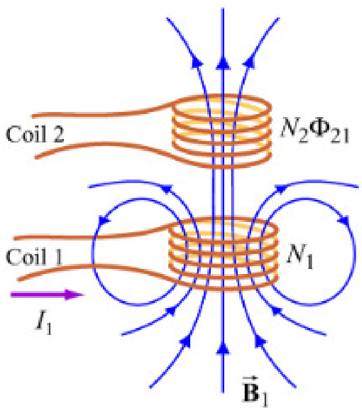
$$W = \frac{1}{2}LI^2$$

Mutual Inductance

If there are two objects, and the change in current of second object is due to the changing magnetic flux of the first object, then the concept of mutual inductance is used.

$$\epsilon_{12} = -M_{12}\frac{\mathrm{d}I_2}{\mathrm{d}t}$$

The most common application of mutual inductance is the transformer.



By reciprocity theorem, $M_{12} = M_{21}$.

Magnetostatic Energy

$$U = \frac{1}{2\mu_0}\int_V \boldsymbol{B}\cdot\boldsymbol{B}\,\mathrm{d}V$$

This has the same form as its electric field counterpart $U = \frac{\epsilon_0}{2}\int_V \boldsymbol{E}\cdot\boldsymbol{E}\,\mathrm{d}V$, which should hint at the nature of $\boldsymbol{E}$ and $\boldsymbol{B}$ field as well as the respective constant multipliers $\frac{1}{2\mu_0}$ and $\frac{\epsilon_0}{2}$.

"Sir, there are still terabytes of calculations required before an actual flight is ... "
"JARVIS, sometimes you gotta run before you can walk."

--- Tony Stark

Magnetic Vector Potential

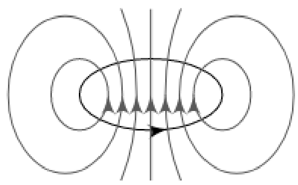
For a current distribution $\boldsymbol{J}$, we need to solve $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$. The general solution to this equation is $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$ for some vector field $\boldsymbol{A}$, the *vector potential.*

$$\nabla\times B = \nabla\times(\nabla\times A) = -\nabla^2 A + \nabla(\nabla\cdot A) = \mu_0 J$$

If a function $\nabla\boldsymbol{V}$ is added to $\boldsymbol{A}$, then since $\nabla\times(\nabla\boldsymbol{V}) = 0$, $\nabla\times(\boldsymbol{A}+\nabla\boldsymbol{V}) = \nabla\times\boldsymbol{A}$. This means that A is not unique. This should be familiar, because we know that potential is a derived concept. Only by applying a function to the function will we yield physically meaningful result.

Magnetic Dipole

A magnetic dipole can be illustrated using a loop of radius R with a current of I running through it.



The *magnetic dipole moment* is defined as $m = IS$ where S is the surface area.

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} = \frac{\mu_0}{4\pi}\left(\frac{3(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{m}}{r^2}\right)$$

Electromagnetic Waves

In vacuum, $\rho = 0$ and $\boldsymbol{J} = \boldsymbol{0}$. Maxwell's fourth equation simplifies to:

$$\mu_0\epsilon_0\frac{\partial^2\boldsymbol{E}}{\partial t^2} = \frac{\partial}{\partial t}(\boldsymbol{\nabla}\times\boldsymbol{B}) = \boldsymbol{\nabla}\times\left(\frac{\partial\boldsymbol{B}}{\partial t}\right) = \nabla(\nabla\cdot\boldsymbol{E}) + \nabla^2\boldsymbol{E} = \nabla^2\boldsymbol{E}$$

This shows that $\boldsymbol{E}$ satisfies the wave equation

$$\nabla^2 \boldsymbol{E} = \mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

Recall from Waves that $\frac{1}{v^2} = \mu_0 \epsilon_0$. Hence, we obtain $c = v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$!

Similarly, we can show that

$$\nabla^2 \boldsymbol{B} = \mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2}$$

If the E field is assumed to take on the expression

$$E(x,t) = E_0 \sin(kx - \omega t)$$

Note that here, since $k = 2\pi/\lambda$ and $\omega = 2\pi/T$, $\omega = ck$. Given E, we can solve for B via the following Maxwell's equation:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$B = \frac{E_0}{c} \sin(kx - \omega t)$$

This has some physically significant results:

- Given $\boldsymbol{E}$, $\boldsymbol{B}$ is uniquely determined.
- $\boldsymbol{E}$ and $\boldsymbol{B}$ oscillate in phase and orthogonal to each other.

Poynting Vector

The Poynting vector characterises the energy transfer of electromagnetic waves.

$$\boldsymbol{S} = \frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B}$$

Poynting Theorem

Specifically, Poynting's theorem says that the rate of energy transfer (per unit volume) from a region of space equals the rate of work done on a charge distribution plus the energy flux leaving that region.

$$-\frac{\partial U}{\partial t} = \boldsymbol{\nabla} \cdot \boldsymbol{S} + \boldsymbol{J} \cdot \boldsymbol{E}$$

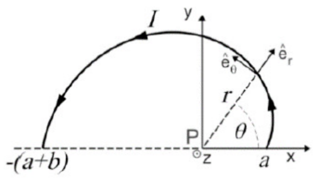
$\boldsymbol{\nabla} \cdot \boldsymbol{S}$ is the divergence of Poynting vector which characterises the energy flow and $\boldsymbol{J} \cdot \boldsymbol{E}$ is the rate at which the fields do work on a charged object. This equation is actually a statement of conservation of energy.

The following is another form of Poynting's theorem. The left hand side refers to the total change of energy of fields and particles, while the right hand side refers to the energy that escapes the surface S.

$$-\frac{\mathrm{d}U}{\mathrm{d}t} + \int_V J \cdot E \, \mathrm{d}V = -\frac{1}{\mu_0} \int_S (E \times B) \cdot \mathrm{d}S$$

A. Sample Problems

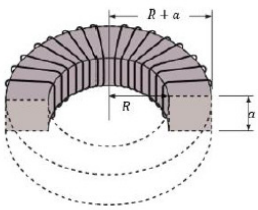
1. A wire segment is bent into the shape of an Archimedes spiral. The equation that describes the curve in the range $0 \leq \theta \leq \pi$ is $r(\theta) = a + \frac{b}{\pi}\theta$. Given that the wire carries a current of I, find the magnetic field at P.



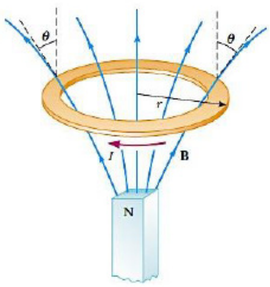
2. A thin spherical shell of radius R carries a charge of Q uniformly distributed over its surface. Find the magnetic field at the center of the sphere if it rotates at an angular velocity of ω about an axis passing through its center.

3. A hollow cone has a vertex angle of 2θ and slant height L and surface charge density σ. It spins around its symmetry axis with angular frequency ω. What is the magnetic field at the tip?

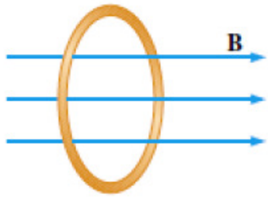
4. A toroid with a square cross-sectional area of dimension $a \times a$ (as shown in the figure below) has N turns of wire. The current flowing in the wire is I and the inner radius if R.
   a. Find the magnetic field at $r < R$, $R < r < R + a$ and $r > R + a$.
   b. Plot the magnetic field as a function of radial distance r.
   c. Show that if $a \ll R$ then the magnetic field inside the toroid is almost uniform and has a similar form to the magnetic field due to an infinitely long solenoid.



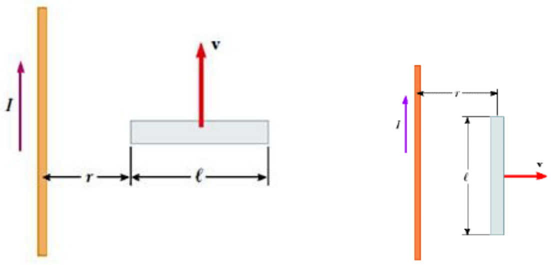
5. A strong magnet is placed under a horizontal conducting ring of radius r that carries a current of I as shown in the figure below. If the magnetic field B makes an angle θ with the vertical at the ring's location, what are the magnitude and direction of the net force on the ring?



6. A circular loop of wire of radius r is placed in a uniform magnetic field, with the plane of the loop perpendicular to the direction of the field. The magnetic field varies with time according to $B(t) = a + bt$ where a, b are constants.
   a. Calculate the magnetic flux through the loop at $t = 0$.
   b. Calculate the magnitude of the EMF induced in the loop.
   c. If the resistance of the loop is R, what is the magnitude of the induced current?
   d. At what rate is energy being delivered to the resistance of the loop?



7. A conducting rod of length l moves with uniform velocity v parallel to a long wire carrying a steady current of I. The axis of the rod is maintained perpendicular to the wire with the near end a distance r away, as shown below. Show that the magnitude of the EMF induced in the rod is given by $|\epsilon| = \frac{\mu_0 I v}{2\pi} \ln\left(1 + \frac{l}{r}\right)$.

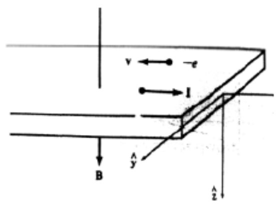


8. A conducting rod moves with a constant velocity of v in a direction perpendicular to a long straight wire carrying a current of I as shown in the figure below. Show that the magnitude of the EMF generated between the ends of the rod is $|\epsilon| = \frac{\mu_0 v I l}{2\pi r}$.

9. Suppose a wire of rectangular cross section of width w in the y direction carries a current of I in the x direction in a region where there is a magnetic field B in the z direction. The electrons will move in the y direction until there is an electric field sufficient to halt further electron from being deflected in this manner. A potential difference is established and can be measured.

   a. What is the value of the potential difference?

   b. What is the Hall electric field?



10. A copper cube of edge a is suspended from a spring of constant k when the cube is immersed in a uniform, horizontal magnetic field B_0 which is normal to two of the cubes vertical faces. The cube is electrically neutral in the absence of the magnetic field. When the cube is set in vertical oscillations due to the spring, it will slowly lose energy due to magnetic damping. In this problem, we will discuss the mechanism and estimate the rate in amplitude damping.

    a. The cube oscillates about its equilibrium position with frequency $\omega = \sqrt{\frac{k}{m}}$. The vertical velocity can be written as $v(t) \approx A\omega e^{i\omega t}$. Show that due to the Hall effect, there appears to be an electric field $E_0(t) \approx \frac{AB_0\omega e^{i\omega t}}{c}$ perpendicular to the magnetic field and the cube's vertical motion.

    b. In a good conductor, this electric field causes charge $Q(t)$ to accumulate on the cube faces to cancel out the electric field. Show that the current has a $\frac{\pi}{2}$ phase lag from the electric field.

    c. Show that the resistance of the cube is $R = \frac{1}{a\sigma}$ where σ is the conductivity of the copper. What is the power dissipated due to the Joule heating (resistive losses)?

    d. By equating the power dissipated to the rate of change of mechanical energy, show that
    $$\frac{\mathrm{d}A^2}{\mathrm{d}t} = -\frac{a^3\omega^4\epsilon_0^2 B_0^2}{2c^2\sigma}A^2$$
    and thus that the damping time constant is $\tau = \frac{4c^2\rho^2\sigma}{k\epsilon_0^2 B_0^2}$. For parameters typical of laboratory experiments, the time constant τ is larger than the age of the Universe!

11. A conducting ring with mass m, radius r and resistance R is rotating about its diameter, in a region of constant and uniform magnetic field B. Initially, the magnetic field passes through the ring perpendicularly, resulting in maximum flux, and the ring rotates with angular speed ω_0.

    a. Express the rate of decrease of angular speed ω as a function of m, r, B, R, ω and θ where θ is the angle between the magnetic field vector and the area vector.

    b. Find the relation between the total angle of rotation ϕ until the ring stops moving and the other given variables.

    c. If $\omega_0 = 8$ rotations per second, $m = 1$kg, $R = 1\Omega$, $r = 30$cm and $B = 1$T, how many times in total does the ring rotate from the beginning till it stops?