Electricity and Magnetism I

Physics Olympiad
Wang Jianzhi

"What's it look like in there?"
"It seems to run on some form of electricity."

--- *Steve Rogers*

In this session, we will cover electrostatic systems and circuits.

Relevant Constants

Electric constant $\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{m^{-3}kg^{-1}s^2C^2}$

Magnetic constant $\mu_0 = 4\pi \times 10^{-6}\ \mathrm{m\ kg\ C^{-2}}$

Vector Calculus

Gradient: measures the rate and direction of change in a *scalar field* (e.g. potential). This function returns a vector that points in the direction of the greatest increase of ϕ.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial\phi}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial\phi}{\partial x}\hat{\boldsymbol{z}}$$

Recall that the potential can be visualized as a function of space. Each point in space is assigned a (scalar) number. The *negated* gradient of this potential function gives the force.

$$\boldsymbol{E} = -\nabla\phi$$

Divergence: measures the scalar value of a source or sink at a given point in vector field.

$$\nabla \cdot \boldsymbol{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

If every point in space is assigned an arrow/vector, then this function measures whether at a given point, there is a net inflow or outflow of arrows. If the divergence at (x, y, z) is positive, then it is a *source* (more outgoing arrows at that point). Otherwise, it is a *sink* (more incoming arrows at that point).

Curl: measures the tendency to rotate about a point in a vector field.

$$\nabla \times \boldsymbol{E} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{\boldsymbol{z}}$$

A very simple visualisation is that, if you are trapped in the center of a whirlpool, then there is a nonzero curl, because water is rotating about you. Notice that the curl returns a vector, which is understandable because it is similar to the concept of why $\vec{\boldsymbol{\omega}}$ (angular velocity) is a vector.

Electrostatics

Electrostatics refer to systems with stationary charges and fields. They are time independent. Also, there are no magnetic fields since there is no current. In such cases, Maxwell's equations can be simplified to the following:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \boldsymbol{E} = 0$$

Applying Divergence Theorem to the first equation, the Gauss's law for electric field is obtained.

$$\int_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \int_V \nabla \cdot \boldsymbol{E}\, \mathrm{d}V = \frac{1}{\epsilon_0}\int_V \rho\, \mathrm{d}V = \frac{Q_{enc}}{\epsilon_0}$$

Coulomb's Law

$$F = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}$$

From this, the concept of electric field is obtained. You can visualise each charge producing its own electric field and this field affects the other charges by attracting or repelling it. Note that electric field can be superposed. The electric field at a point $\boldsymbol{r}$ due to a collection of point charges q_i is equal to the vector sum of the electric fields at $\boldsymbol{r}$ due to each q_i individually.

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0 r^2}\hat{\boldsymbol{r}}$$

Electric Energy

For an electrostatic system, $\boldsymbol{E}$ is a conservative field. There exists a scalar function V such that $\boldsymbol{E} = -\nabla\phi$.

$$\phi = \int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = -\frac{q}{4\pi\epsilon_0 r}$$

For a system, we can get the potential energy U as

$$U = \sum_{i,j} -\frac{q_i q_j}{4\pi\epsilon_0 r}$$

The physical significance of U is that it is the amount of energy required to assemble the charges from infinity to achieve the current charge distribution. Another way to calculate this:

$$U = \frac{\epsilon_0}{2}\int_V \boldsymbol{E} \cdot \boldsymbol{E}\, \mathrm{d}V$$

Conductors

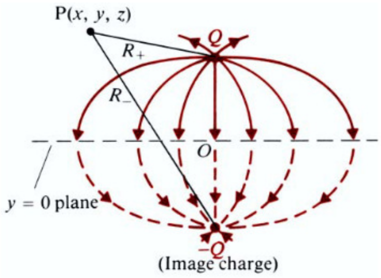
A *conductor* is a region of space which contains a lot of charges that are free to move. In electrostatic positions, $\boldsymbol{E}$ must be $\boldsymbol{0}$ within the conductor. Otherwise, the charges will flow in the direction of $\boldsymbol{E}$ until equilibrium is established. If an external field is applied, then the charges will move such that the external field is cancelled out and $\boldsymbol{E}$ will still be $\boldsymbol{0}$ within the conductor.

$E = 0$ within the conductor implies $\phi = c$ within the conductor. Similarly, since $\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$, $\rho = 0$ within the conductor. This implies any net charge in the conductor must reside **at its surface**.

By Gauss's law, any electric field lines outside the conductor must end perpendicular to the surface of the conductor.

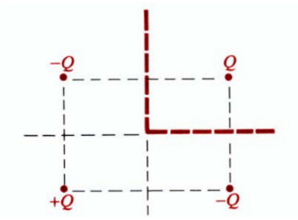
Method of Images

The method of images exploit symmetry. If there is a charge Q above an infinite conducting plane of potential 0, then the electric field lines on top of the plane is equivalent to the set up where there is a charge of $-Q$ below the surface that is symmetric to Q.



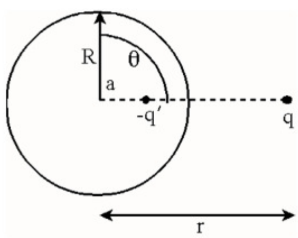
The caveat here is that the field only exists in the upper plane. Both the charge $-Q$ and the electric field below the plane are non-existent. They are used solely to replicate the field lines above the plane. If the question asks to calculate the electric potential energy of the system, then the answer must be divided by two.

The method of images can also be extended to several other configurations. In the set-up below, three other image charges can be used to obtain the field.



For a sphere, the method of images can also be applied. In fact, the image charge now is located at the pole-polar position (meaning the position at which r is inverted about the circle at $rr' = R^2$).

$$r' = \frac{R^2}{r} \qquad\qquad q' = -\frac{R}{r}q$$



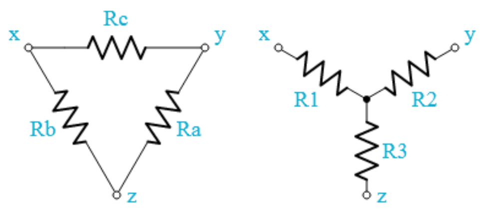
Circuit Analysis

Ohm's law

$$\epsilon = IR$$

Toolbox

- Delta-Wye Transform



$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c}$$

$$R_a = \frac{R_1 R_2 + R_2 R_3 + R_3 R_1}{R_1}$$

- Kirchhoff's Circuit Laws
  - The directed sum of currents meeting at a point is 0.
  - The directed sum of the potential differences around any closed loop is 0.
- Symmetry
- Joining/Splitting wires of equal potential

Inductors

Additional details for inductors will be covered in E&M II. Right now, the knowledge of the two equations below are sufficient:

$$V = L\frac{\mathrm{d}I}{\mathrm{d}t} \qquad U = \frac{1}{2}LI^2$$

Capacitors

You can imagine capacitors to be parallel plates that stores up charge and in turn create a potential difference. The capacitance is defined to be the ratio of charge stored up on one side of the plate to the potential difference between the plates.

$$C = \frac{Q}{V} \qquad U = \frac{1}{2}CV^2$$

For parallel circuits,

$$C_{eq} = \sum C_i$$

For series circuits,

$$\frac{1}{C_{eq}} = \sum \frac{1}{C_i}$$

Method of Complex Impedance

In an AC circuit, the voltage is usually in the form of $V_0 \cos \omega t$. We can treat the voltage as a "driving force", similar to what is covered in Oscillations. Then, the current will take on the same frequency as the voltage, in the form of $I_0 \cos(\omega t - \phi)$, where the ϕ phase difference is introduced possibly due to inductors and capacitors.

Since both the voltage and current are sinusoidal, they can be expressed in complex numbers. Subsequently, we just need to remember to take their real parts. Let the voltage be $V = V_0 e^{i\omega t}$ and the current be $I = I_0 e^{i(\omega t - \phi)}$. The complex impedance is then Z, where Z satisfies:

$$V_m = Z I_m$$

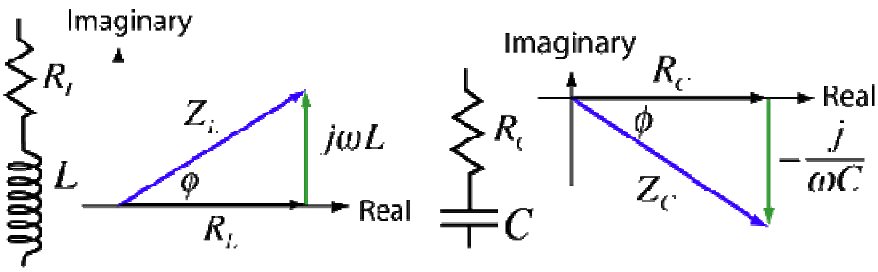
Z is a complex number. If $Z = R_{eq} + iX_{eq}$ where R_{eq} and X_{eq} are real numbers, then

$$|Z| = \sqrt{R_{eq}^2 + X_{eq}^2} \qquad\qquad \phi = \tan^{-1}\frac{X_{eq}}{R_{eq}}$$

As it turned out, Z is very similar to R. If two circuit systems connected in series with impedances Z_1 and Z_2, $Z_{eq} = Z_1 + Z_2$. If they are connected in parallel, then $\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2}$.

Element	Resistor	Inductor	Capacitor
Impedance	R	$i\omega L$	$\frac{i}{\omega C}$

Complex numbers should now remind you of the phasor diagram technique in Superposition. Because inductance and capacitance are out of phase with the normal circuit and alternating current are represented by cosine functions, we can apply the phasor diagram method to it as well!



"Sir, there are still terabytes of calculations required before an actual flight is ... "
"JARVIS, sometimes you gotta run before you can walk."

--- Tony Stark

Green's Theorem

The line integral of the gradient of a scalar field over a curve L is equal to the change in scalar field between endpoints p and q of the curve. For a curve L spanning from endpoint p to endpoint q:

$$\int_L \boldsymbol{\nabla}\phi \cdot \mathrm{d}\boldsymbol{r} = \phi(\boldsymbol{q}) - \phi(\boldsymbol{p})$$

Divergence Theorem

The integral of the divergence of a vector field of volume V is equal to the flux of the vector field through the closed boundary surface of the solid.

$$\int_V (\boldsymbol{\nabla} \cdot \boldsymbol{F})\, \mathrm{d}V = \oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S}$$

Intuitively, you can think that if every point in the enclosed volume is a source/sink, summing up their divergence will be equivalent to measuring how much particles are flowing out of the enclosed surface.

Stokes' Theorem

The intuition behind Stokes' theorem is that given a surface D with a boundary C, taking the curl about every point on the surface, it is equivalent to taking the line integral about the boundary C.

$$\int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} = \iint_D (\nabla \times \boldsymbol{F}) \cdot \mathrm{d}\boldsymbol{S}$$

Dipole Approximation

A dipole can be represented by a vector p that points from the *negative* charge to the *positive* charge. You can imagine the positive and negative charges are held apart by a light rod. At a distance of r away, there is no term in the order of $\frac{1}{r^2}$, because the dipole will *seem* like a charge of 0. The term in the order of $\frac{1}{r^3}$ are given below:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{\boldsymbol{p} \cdot \boldsymbol{r}}{r^3}$$

$$E(r) = \frac{1}{4\pi\epsilon_0}\frac{3(\boldsymbol{p} \cdot \hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{p}}{r^3}$$

Poisson's Equation

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}$$

The Poisson's equation relates the electric field potential to the charge density. The significance of this equation is that if we know the charge density function ρ, then from Maxwell's equations

$$\nabla \cdot E = \nabla \cdot (-\nabla\phi) = -\nabla^2\phi = \frac{\rho}{\epsilon_0}$$

∇^2 is the Laplacian operator.

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = -\frac{\rho}{\epsilon_0}$$

Laplace's Equation

If the enclosed surface does not contain any charge (i.e. $\rho = 0$), then Poisson's equation reduces to Laplace's Equation. The solutions of Laplace's equation are the *harmonic functions*.

$$\nabla^2\phi = 0$$

A. Electrostatics Problems

1. Find the electric field a distance z away above the midpoint of a straight line segment of length $2L$ which carries a uniform line charge density λ.

    a. What is the approximate electric field when $z \gg L$?

    b. What is the approximate electric field when $L \gg z$?

2. Twelve equal charges q are situated at the corners of a regular 12-gon. What is the net force on a test charge Q at the center? Suppose one of the 12 charges is removed (say the one at 6 o'clock), what is the force on Q?

3. Calculate the electric field of a solid sphere of radius R containing a uniform volume charge of density ρ.

4. Find the electric field a distance z above the center of a flat circular disk of radius R which carries a uniform surface charge σ. What is the approximate electric field when $z \gg L$? What is the electric field in the limit $z \to 0$?

5. Suppose the electric field in some region is found to be $E(\boldsymbol{r}) = kr^3\hat{\boldsymbol{r}}$ in spherical coordinates, where k is a constant. Find the charge density $\rho(\boldsymbol{r})$. Find the total charge contained in a sphere of radius R, centered on the origin.

6. A charge q sits in one corner of a cube. What is the flux of E through one of the sides not adjacent to the corner where the charge is in?

7. An infinite plane carries a uniform surface charge σ. Find its electric field.

8. A long cylinder carries a charge density that is proportional to the distance from the axis: $\rho = kr$ for some constant k and distance r perpendicular to the axis. Find its electric field.

9. Two infinite parallel plates carry equal but opposite uniform charge densities $\pm\sigma$. Find the field in each of the three regions: to the left of both plates, in between both plates and to the right of both plates.

10. Find the potential of a uniformly charged spherical shell of radius R.

11. The Hulk made an ice cream cone that carries a uniform surface charge of σ. The height of the cone is h, as is the radius of the top. Find the potential difference between the points a (the vertex) and b (the center of the top).

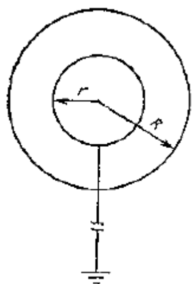
12. A point charge q is placed at $(0,0,z_0)$ above an infinite grounded xy plane conductor. Calculate the surface charge density on the infinite plane conductor as a function of radial distance from the z axis.

13. Consider a point charge q located at $(0,y_0,z_0)$ together with grounded infinite plane conductors parallel to the z and y axis. Find the potential in the the space not occupied by the conductors.

14. Calculate the amount of work done to assemble charge Q uniformly onto a spherical shell of radius R.

15. Two irregularly shaped conducting objects, one carrying a charge of $+q$ and the other carrying a charge of $-q$ are placed on an x-axis at $x = -4.0$ m and $x = +4.0$ m respectively. Between the objects, the electric field on the x-axis is given by $E(x) = aq(x^2 + b)$, where a and b are constants. What is the capacitance of the configuration?

16. A metal sphere of radius r is placed concentrically into a thin hollow metal sphere of radius R. The inner sphere is earthed by a long wire through an opening of the outer sphere. Find the potential of the outer sphere if it has a charge of Q.



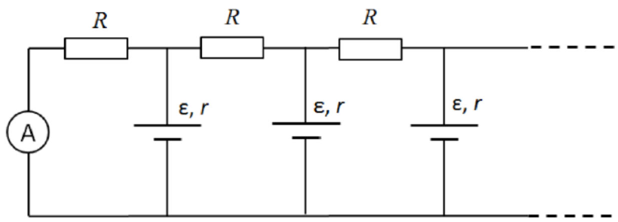
17. A sphere of radius R and center O has charge evenly distributed throughout and of electric potential at the surface as $1000V$. At a point O' far away from the sphere lies a proton p which is fired at the sphere with kinetic energy of $2000eV$ in the direction parallel to OO'. The distance between the two parallel lines denoting OO' and the direction of travel of the proton is denoted as l.

    a. If we are to have the proton graze the surface of the sphere, what should l be?

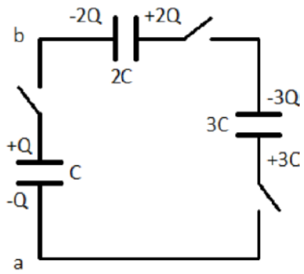
b. If we now have an electron in place of a proton and to have it graze the surface of the sphere, what should l be?

B. Circuit Problems

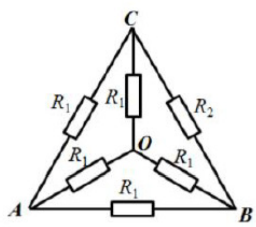
1. The circuit shown in the figure below extends to the right infinitely. Each battery has an EMF of $1.5V$ and an internal resistance of 0.5Ω. Each resistor R in the circuit has a resistance of 2.0Ω. What would the ammeter below read?



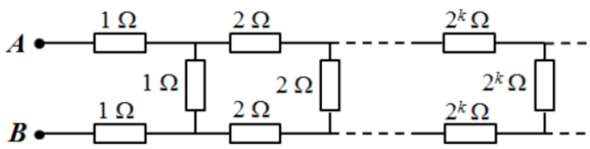
2. Three capacitors are charged initially as shown in the figure below. If $V = \frac{Q}{C}$, what is the potential difference across point a and b in terms of V after all the switches have been closed?



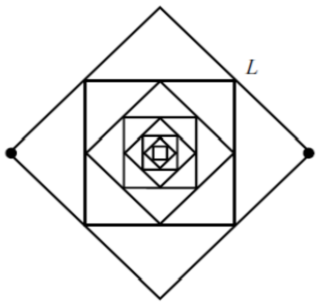
3. Determine the resistance below between A and B and also between A and O.



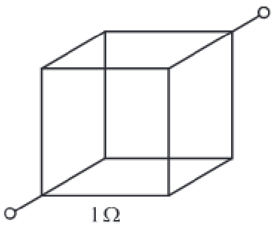
4. Find the resistance between A and B.



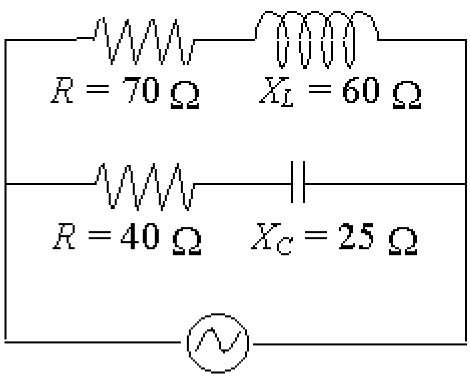
5. Consider the following resistor circuit. Start with a square of side length L. Connect the centers of each side to form another square and repeat this process to infinity. What is the resistance between two opposite corners of the diagonal square? Give your answer in terms of the resistance R of a length L of the wire. (Assume all wires in the circuit have the same cross section and resistivity)



6. Determine the resistance between opposing corners of a cube, the edges of which are made of wires with the resistance of each edge being 1Ω.



7. Find the combined impedance of the following circuit.



8. Let a capacitor of capacitance C be charged to some potential V_0 and then discharged by suddenly connecting it across a resistor of resistance R at time $t = 0$. What are the charge Q on the capacitor and the current I in the circuit as a function of time?

Recommended Resources

- College Physics by Hugh D. Young
- Electricity and Magnetism by Edward M. Purcell and David J. Morin
- Introduction to Electrodynamics by David J. Griffiths
- Physics By Example: 200 Problems and Solutions by W.G. Rees.
- 200 Puzzling Physics Problems by P. Gnadig et al.