Electricity & Magnetism I (Solution)

1.

Horizontal component cancels out.

$$E = \int \frac{dq}{4\pi\varepsilon_0 (x^2+z^2)}\cos\theta = \int_{-L}^{L} \frac{\lambda\, dx\, z}{4\pi\varepsilon_0 (x^2+z^2)^{\frac{3}{2}}}$$

$$= \frac{\lambda z}{4\pi\varepsilon_0}\int_{-L}^{L}\frac{dx}{(x^2+z^2)^{\frac{3}{2}}} = \frac{\lambda z}{4\pi\varepsilon_0}\int_{-\tan^{-1}(\frac{L}{z})}^{\tan^{-1}(\frac{L}{z})} \frac{1}{z^3\sec^3\theta} z\sec^2\theta\, d\theta$$

$$= \frac{\lambda z}{4\pi\varepsilon_0}\frac{1}{z^2}\Big[\sin\theta\Big]_{-\tan^{-1}(\frac{L}{z})}^{\tan^{-1}(\frac{L}{z})} = \frac{\lambda}{2\pi\varepsilon_0 z}\frac{L}{\sqrt{L^2+z^2}}$$

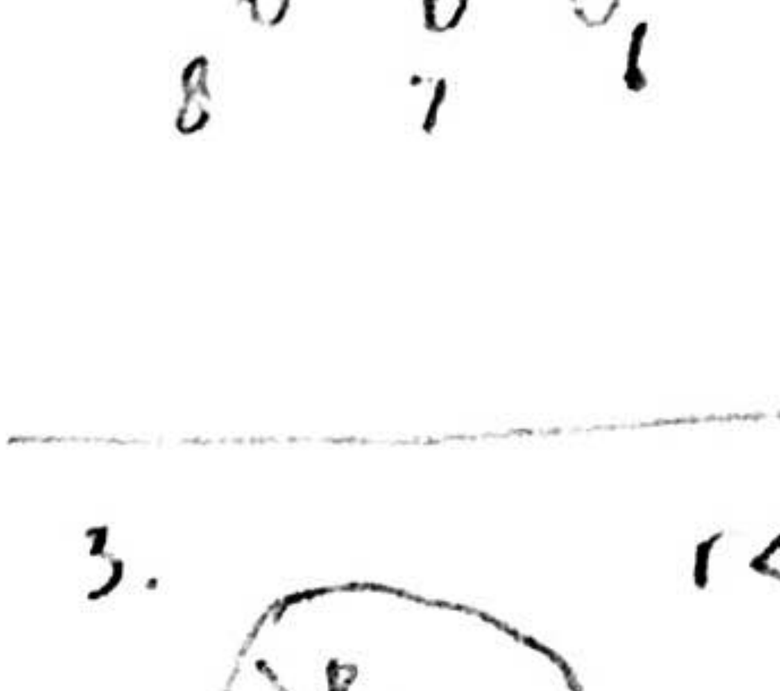
$$\therefore \boxed{E = \frac{\lambda L}{2\pi\varepsilon_0 z\sqrt{L^2+z^2}}}$$

when $z \gg L$, $E = \dfrac{\lambda(\frac{L}{z})}{2\pi\varepsilon_0 z\sqrt{1+(\frac{L}{z})^2}} \approx \boxed{\dfrac{\lambda L}{2\pi\varepsilon_0 z^2}}$

when $L \gg z$, $E = \dfrac{\lambda}{2\pi\varepsilon_0 z\sqrt{1+(\frac{z}{L})^2}} \approx \boxed{\dfrac{\lambda}{2\pi\varepsilon_0 z}}$

(infinitely long wire)

2.

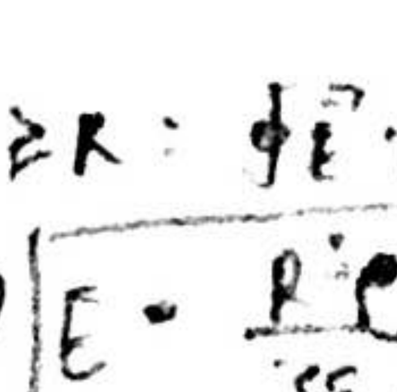


(a) Net force at center $= \boxed{0}$ due to symmetry.

(b) Equivalent to adding a negative $-q$ charge at the same spot.

$$\boxed{F = \frac{Qq}{4\pi\varepsilon_0 r^2}}$$

3.



$r<R$: $\oint \vec{E}\cdot d\vec{A} = \dfrac{Q_{enc}}{\varepsilon_0}$

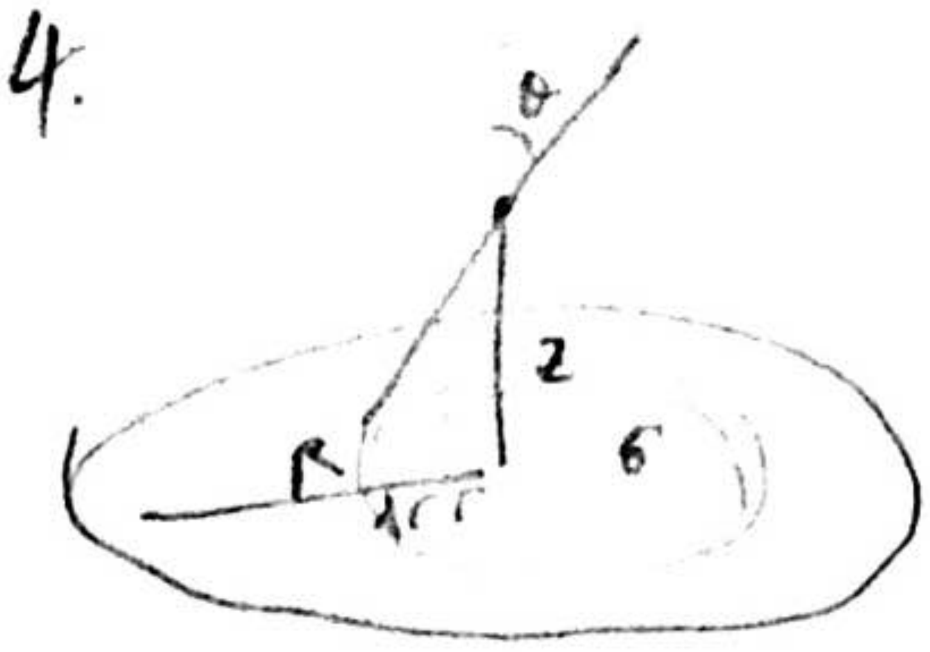
$\Rightarrow E(4\pi r^2) = \dfrac{\frac{4}{3}\pi r^3\rho}{\varepsilon_0}$

$\Rightarrow \boxed{E = \dfrac{\rho r}{3\varepsilon_0} \quad (r<R)}$

$r \geq R$: $\oint \vec{E}\cdot d\vec{A} = \dfrac{Q_{enc}}{\varepsilon_0} \Rightarrow E(4\pi r^2) = \dfrac{\frac{4}{3}\pi R^3\rho}{\varepsilon_0}$

$\Rightarrow \boxed{E = \dfrac{R^3\rho}{3\varepsilon_0 r^2}}$

4.



The horizontal component of $\vec{E}$ cancels

$$E = \int \frac{dq}{4\pi\varepsilon_0(r^2+z^2)}\cos\theta = \int_0^R \frac{2\pi r\, dr\, \sigma}{4\pi\varepsilon_0(r^2+z^2)}\frac{z}{\sqrt{r^2+z^2}}$$

$$= \frac{\sigma z}{2\varepsilon_0}\int_0^R \frac{r\,dr}{(r^2+z^2)^{\frac{3}{2}}} = \frac{\sigma z}{2\varepsilon_0}\int_0^{\tan^{-1}(\frac{R}{z})} \frac{z\tan\theta}{z^3\sec^3\theta} z\sec^2\theta\, d\theta$$

$$= \frac{\sigma z}{2\varepsilon_0}\int_0^{\tan^{-1}(\frac{R}{z})} \frac{1}{z}\sin\theta\, d\theta = \frac{\sigma}{2\varepsilon_0}\Big[-\cos\theta\Big]_0^{\tan^{-1}(\frac{R}{z})}$$

$$= \boxed{\frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{\sqrt{R^2+z^2}}\right)}$$

As $z \gg R$, $E \approx \dfrac{\sigma}{2\varepsilon_0}\left(1-\dfrac{z}{z\sqrt{1+(\frac{R}{z})^2}}\right) \approx 0$

$z \ll R$, $\dfrac{z}{\sqrt{R^2+z^2}} \approx \dfrac{z}{R\sqrt{1+(\frac{z}{R})^2}} \approx \dfrac{z}{R}$

$E = \dfrac{\sigma}{2\varepsilon_0}\left(1-\dfrac{z}{R}\right) \approx \boxed{\dfrac{\sigma}{2\varepsilon_0}}$ (infinite plane)

5. $E(r) = kr^2$

$$\oint \vec{E}(r)\, d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

$$E(r+dr)\cdot 4\pi(r+dr)^2 - E(r)\,4\pi r^2 = \rho(r)\,4\pi r^2 dr$$

$$E(r+dr) = k(r+dr)^3 = kr^3 + 3kr^2 dr$$

$$\text{LHS} = (kr^3+3kr^2dr)\cdot 4\pi(r^2+2r\,dr) - kr^3\cdot 4\pi r^2$$

$$= 3kr^4\, 4\pi\, dr + 4\pi k r^3\, 2r\, dr$$

$$= 4\pi k r^4(3+2)\,dr = 20\pi k r^4 dr$$

$$\therefore \boxed{\rho(r) = 5\varepsilon_0 k r^2}$$

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow Q = \varepsilon_0(4\pi R^2)kR^3 = \boxed{4\pi R^5\varepsilon_0 k}$$

6.

There are 24 faces in total similar to the face that we are going to calculate.

Total flux $= \frac{Q_{enc}}{\varepsilon_0} = \frac{q}{\varepsilon_0}$

$\therefore$ flux through one face $= \boxed{\frac{q}{24\varepsilon_0}}$

7.

$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow E(2A) = \frac{\sigma A}{\varepsilon_0}$

$\Rightarrow \boxed{E = \frac{\sigma}{2\varepsilon_0}}$ (same as 4)

8.

$\rho = kr$

$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$

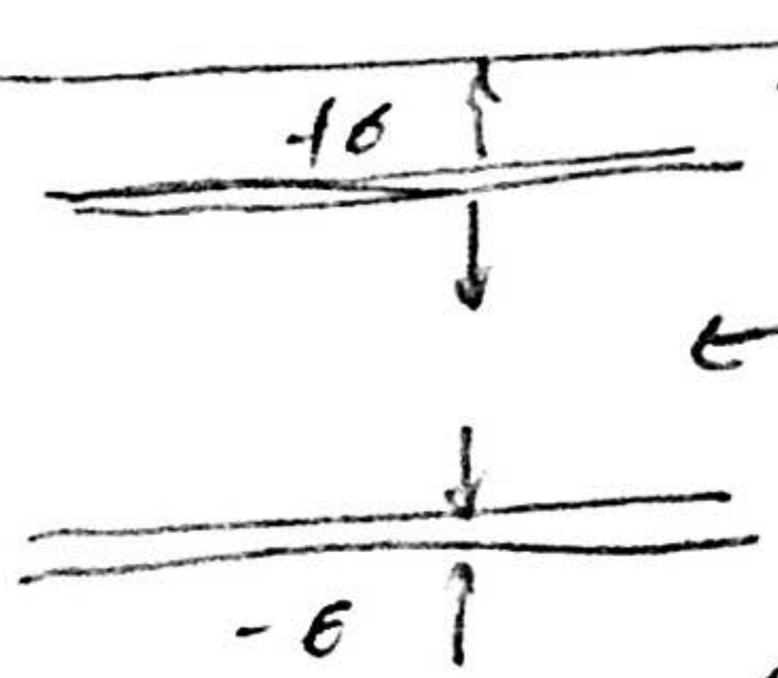
$\Rightarrow E(x)(2\pi x)L = \frac{Q_{enc}}{\varepsilon_0}$

$Q_{enc} = \int_0^x kr\cdot(2\pi r)\,dr\cdot L$

$= 2\pi k L\left[\frac{r^3}{3}\right]_0^x = \frac{2}{3}\pi k L x^3$

$\therefore E(x) = \left(\frac{2}{3}\pi k L x^3\right)\frac{1}{\varepsilon_0}\frac{1}{2\pi x L} = \boxed{\frac{kx^2}{3\varepsilon_0}}$

9.



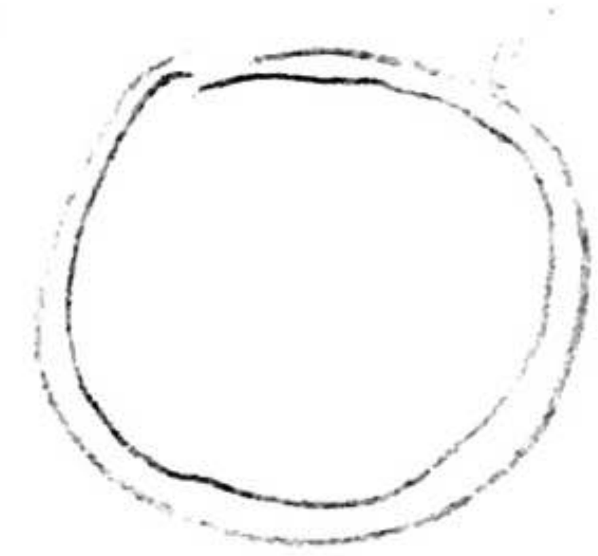
$\leftarrow \boxed{0}$

$\leftarrow = 2\cdot\frac{\sigma}{2\varepsilon_0} = \boxed{\frac{\sigma}{\varepsilon_0}}$

$\leftarrow \boxed{0}$

By principle of Superposition.

10.

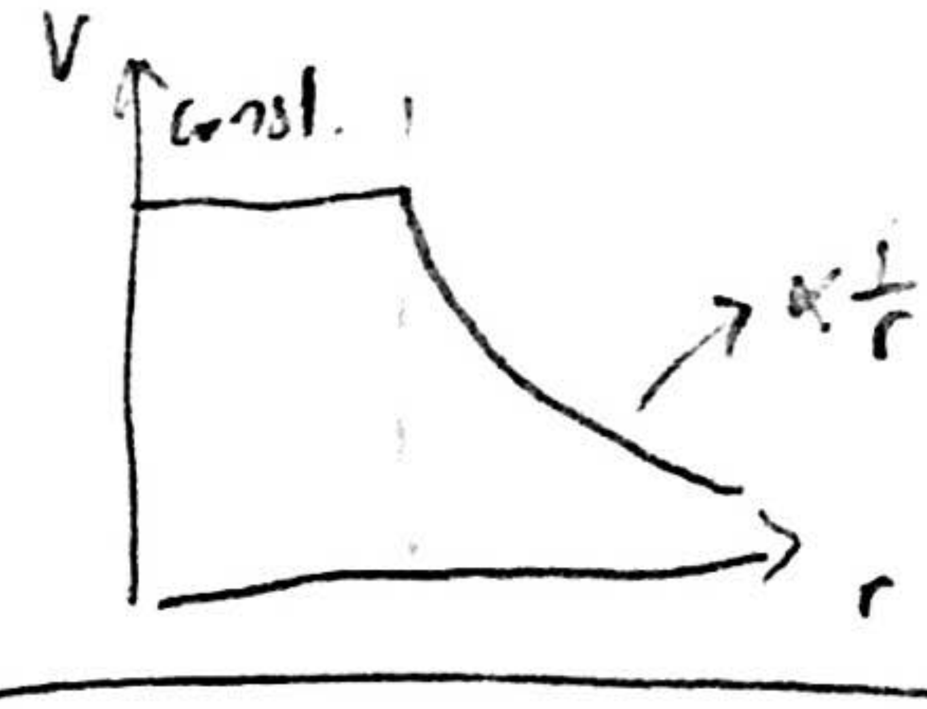


$V = -\int E\,dr$

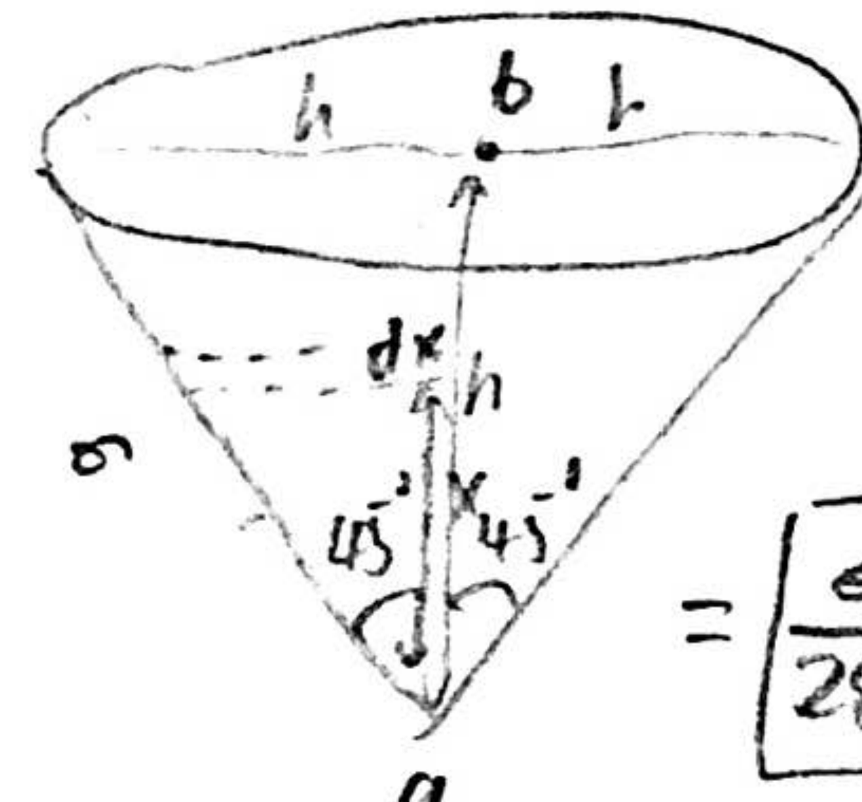
$= -\int_\infty^r \frac{\sigma(4\pi R^2)}{\varepsilon_0(4\pi r^2)}\,dr$

$= -\frac{\sigma R^2}{\varepsilon_0}\left[-\frac{1}{r}\right]_\infty^r = \frac{\sigma R^2}{\varepsilon_0 r}$

Inside the shell, $E = 0 \Rightarrow V$ constant $= \frac{\sigma R}{\varepsilon_0}$



11.



$V_A = \int \frac{dq}{4\pi\varepsilon_0(\sqrt{2}x)}$

$= \int_0^h \frac{2\pi x\sqrt{2}\,dx\,\sigma}{4\pi\varepsilon_0\sqrt{2}x}$

$= \boxed{\frac{\sigma h}{2\varepsilon_0}}$

$V_b = \int_0^h \frac{dq}{4\pi\varepsilon_0\sqrt{\left(\frac{h}{\sqrt{2}}\right)^2 + \left[\frac{h}{\sqrt{2}} - \sqrt{2}x\right]^2}}$

$= \int_0^h \frac{2\pi x\sqrt{2}\,dx\,\sigma}{4\pi\varepsilon_0\sqrt{\frac{h^2}{2} + \frac{(h-2x)^2}{2}}} = \int_0^h \frac{\sigma x\,dx}{\varepsilon_0\sqrt{h^2+(h-2x)^2}}$

$= -\int_h^{-h} \frac{\sigma\left(\frac{h-x}{2}\right)dx}{\varepsilon_0\sqrt{h^2+x^2}}\left(-\frac{1}{2}\right) = \int_{-h}^h \frac{\sigma(h-x)\,dx}{4\varepsilon_0\sqrt{h^2+x^2}}$

$= \int_{-h}^h \frac{\sigma h}{4\varepsilon_0\sqrt{h^2+x^2}}\,dx + \int_{-h}^h \frac{-\sigma x}{4\varepsilon_0\sqrt{x^2+h^2}}\,dx$

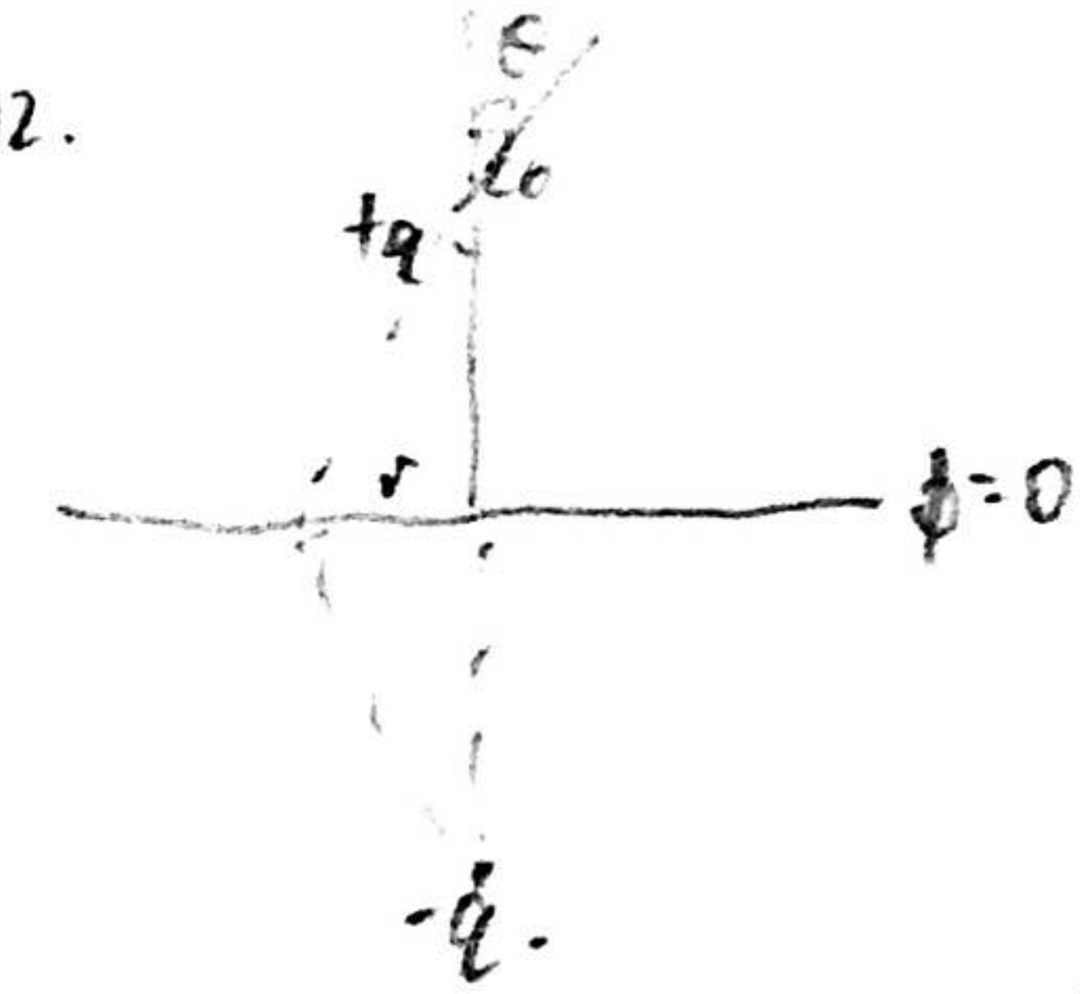
$= \frac{\sigma h}{4\varepsilon_0}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{\cos\theta}\,d\theta + \left[-\frac{\sigma}{4\varepsilon_0}\sqrt{x^2+h^2}\right]_{-h}^h$

$= \frac{\sigma h}{4\varepsilon_0}\left[\ln|\sec\theta + \tan\theta|\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$= \boxed{\frac{\sigma h}{2\varepsilon_0}\ln(\sqrt{2}+1)}$

$\boxed{V_A - V_B = \frac{\sigma h}{2\varepsilon_0}\left(1 - \ln(\sqrt{2}+1)\right)}$

12.



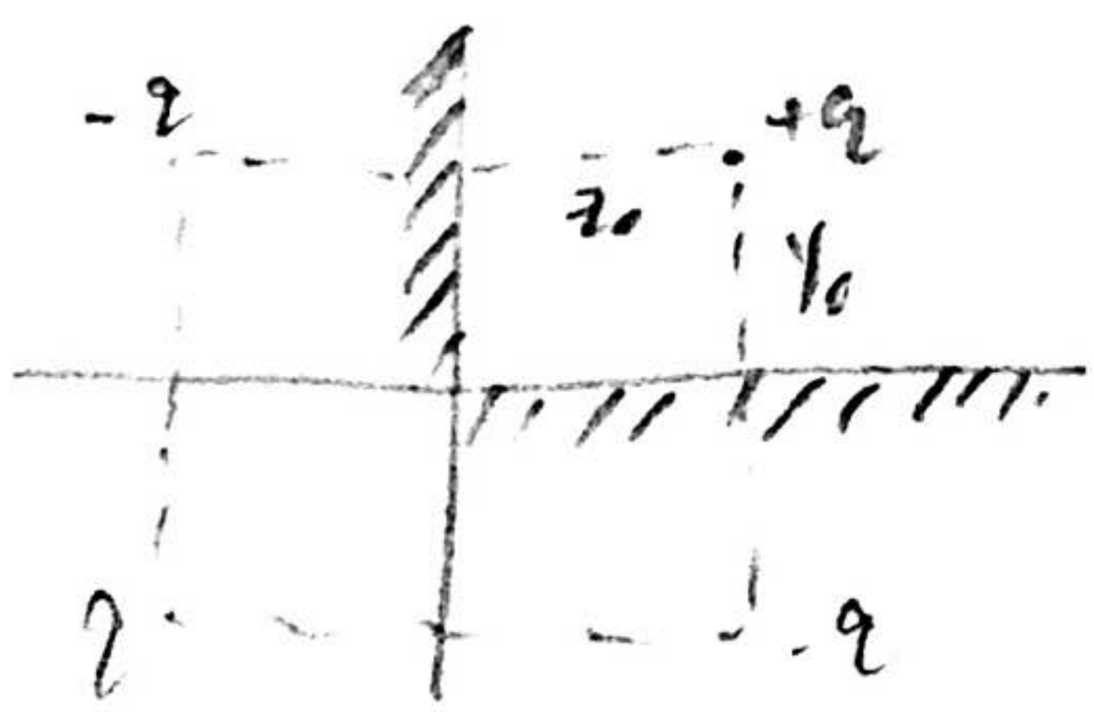
Horizontal component of $\vec{E}$ cancels out.

$$E = 2\frac{q}{4\pi\varepsilon_0(z_0^2+r^2)}\cos\theta = -\frac{q z_0}{2\pi\varepsilon_0(z_0^2+r^2)^{\frac{3}{2}}}$$

We know $E = \frac{\sigma}{\varepsilon_0}$

$$\Rightarrow \boxed{\sigma(r) = -\frac{q z_0}{2\pi(z_0^2+r^2)^{\frac{3}{2}}}}$$

13.



$$\boxed{V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{(z_0-z)^2+(y_0-y)^2}} + \frac{1}{\sqrt{(z_0-z)^2+(-y_0-y)^2}} - \frac{1}{\sqrt{(y_0-y)^2+(-z_0-z)^2}} - \frac{1}{\sqrt{(z_0-z)^2+(-y_0-y)^2}}\right)}$$

14

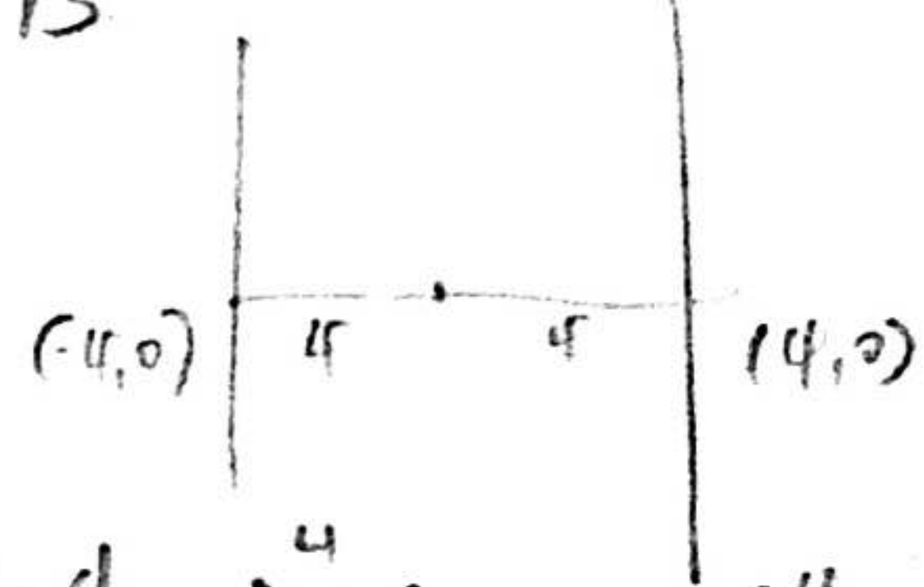
Energy $= \frac{\varepsilon_0}{2}\int_V E\cdot E\,dV$.

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

$$\therefore \text{Energy} = \frac{\varepsilon_0}{2}\int_R^\infty \frac{Q^2}{16\pi^2\varepsilon_0^2 r^4}(4\pi r^2)\,dr$$

$$= \frac{Q^2}{8\pi\varepsilon_0}\left[-\frac{1}{r}\right]_R^\infty = \boxed{\frac{Q^2}{8\pi\varepsilon_0 R}}$$

15



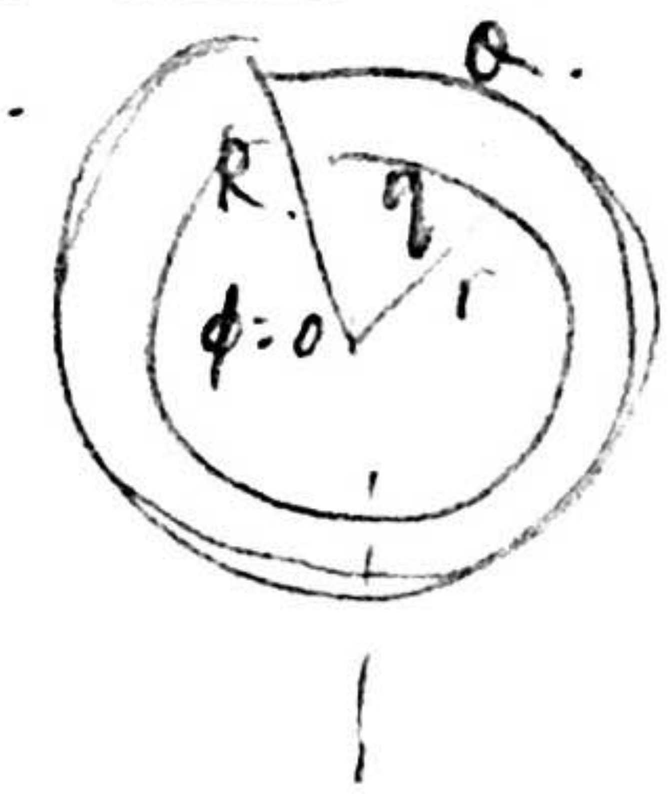
$$|V| = \int_{-4}^{4} E\,dx = -\int_{-4}^{4} aq(x^2+b)\,dx$$

$$= \left[\frac{aqx^3}{3} + aqbx\right]_{-4}^{4} =$$

$$= \frac{128}{3}aq + 8aqb.$$

$$C = \frac{q}{V} = \boxed{\frac{128}{3}a + 8ab}$$

16.



Suppose the inner sphere has a charge of q.

$$\text{Potential on } Q = V_Q = \int_R^\infty \frac{Q+q}{4\pi\varepsilon_0 r^2}\,dr$$

$$= \left[-\frac{Q+q}{4\pi\varepsilon_0 r}\right]_R^\infty = \frac{Q+q}{4\pi\varepsilon_0 R}$$

$$\text{Potential on } q = V_q = V_Q + \int_r^R \frac{q}{4\pi\varepsilon_0 r^2}\,dr$$

$$= \frac{Q+q}{4\pi\varepsilon_0 R} + \left[-\frac{q}{4\pi\varepsilon_0 r}\right]_r^R = \frac{Q+q}{4\pi\varepsilon_0 R} + \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{R}\right)$$

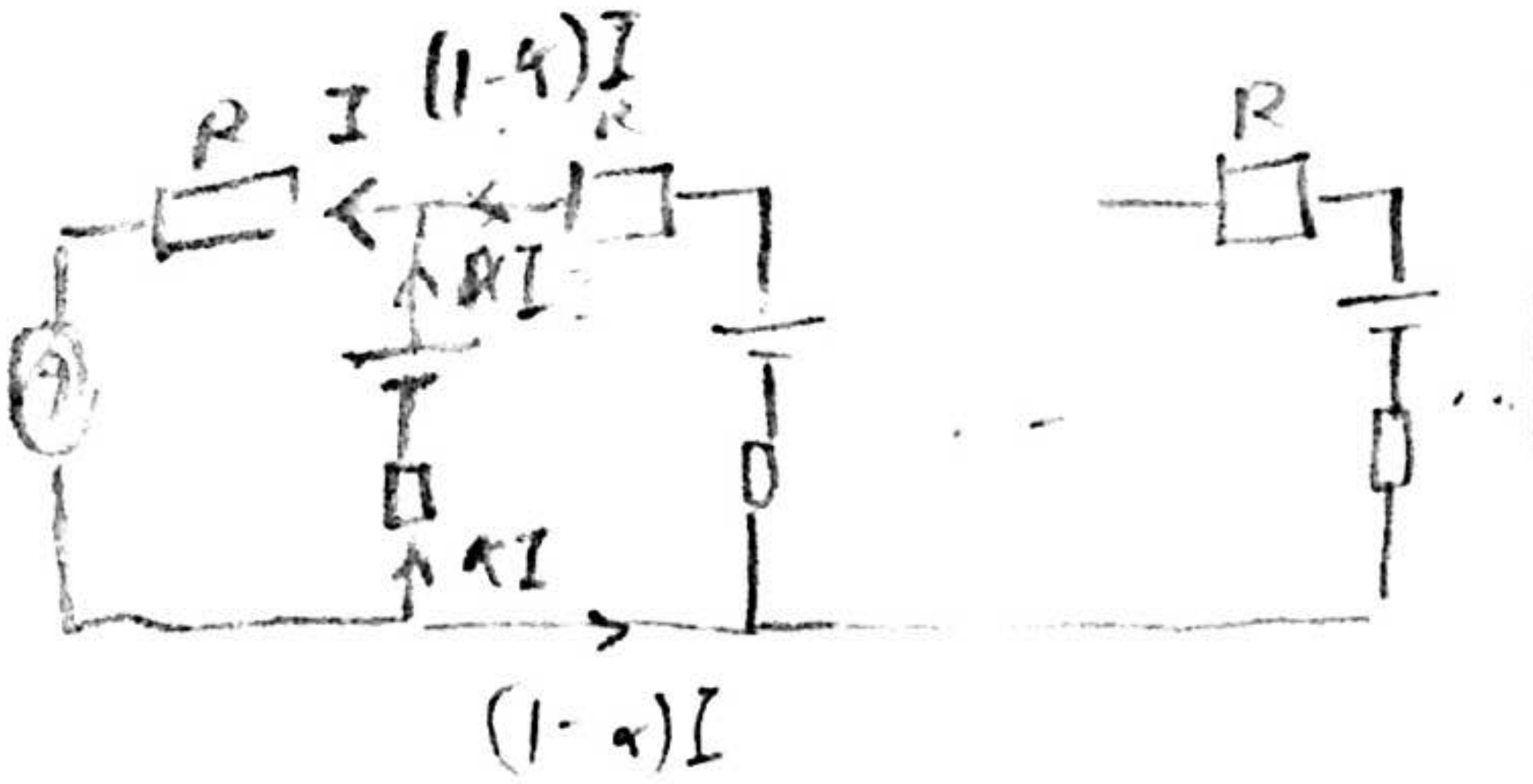
We have $\frac{Q+q}{4\pi\varepsilon_0 R} + \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{R}\right) = 0$

$$\Rightarrow \frac{Q}{4\pi\varepsilon_0 R} + \frac{q}{4\pi\varepsilon_0 r} = 0 \Rightarrow \boxed{q = -\frac{r}{R}Q}$$

$$\therefore \text{Potential of } Q = \frac{Q+q}{4\pi\varepsilon_0 R} = \left(\frac{R-r}{R}\right)\frac{Q}{4\pi\varepsilon_0 R}$$

$$= \boxed{\frac{(R-r)Q}{4\pi\varepsilon_0 R^2}}$$

1.



Suppose the current passing through the ammeter is I and at the first branch, it splits into αI and $(1-\alpha)I$.

$$\mathcal{E} = IR + \alpha I r = I(R + \alpha r)$$

$$\mathcal{E} = (1-\alpha) IR + IR + (1-\alpha) Ir$$

$$= I(R - R\alpha + R + r - \alpha r)$$

$$\Rightarrow R + \alpha r = 2R - R\alpha + r - \alpha r$$

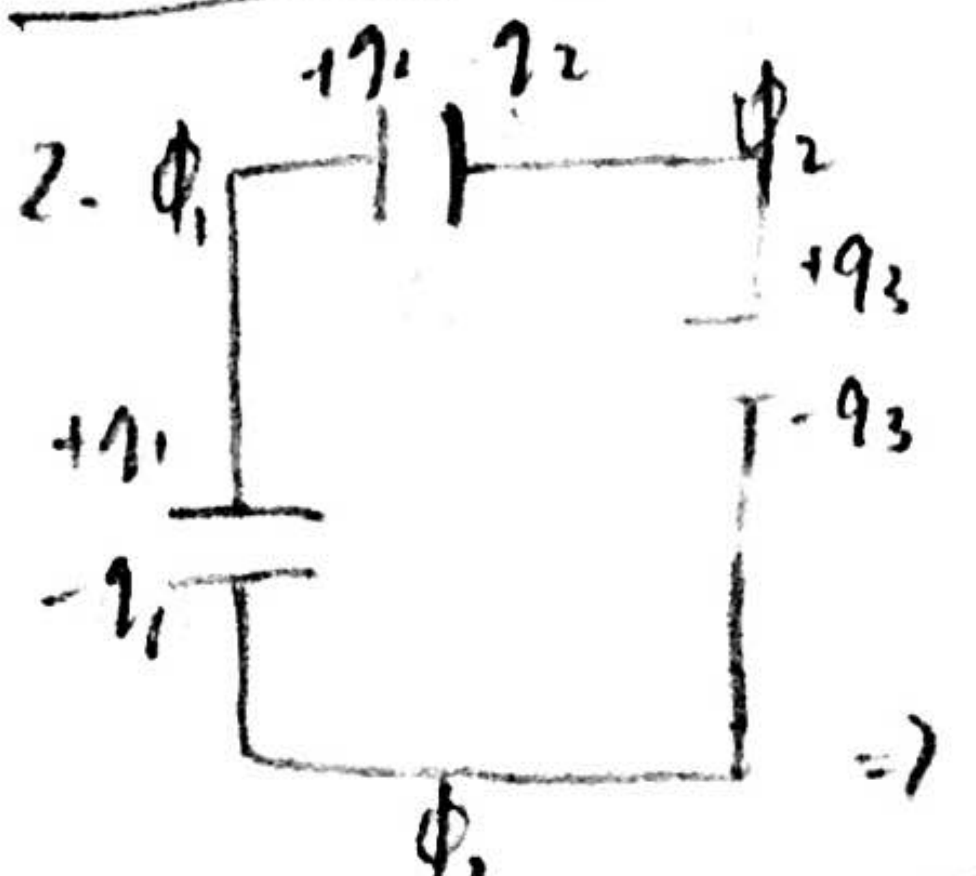
$$\Rightarrow \alpha(2r + R) = R + r$$

$$\Rightarrow \boxed{\alpha = \frac{R+r}{R+2r}}$$

$$\therefore I = \frac{\mathcal{E}}{R + \alpha r} = \frac{\mathcal{E}}{R + \frac{(R+r)r}{R+2r}}$$

$$= \boxed{\frac{\mathcal{E}(R+2r)}{R^2 + 3Rr + r^2}}$$

2.



$$\phi_1 - \phi_3 = \frac{q_1}{C}$$

$$\phi_1 - \phi_2 = \frac{q_2}{2C}$$

$$\phi_2 - \phi_3 = \frac{q_3}{3C}$$

$$\Rightarrow \boxed{\frac{q_1}{C} = \frac{q_2}{2C} + \frac{q_3}{3C}}$$

$$q_1 + q_2 = -Q$$
$$-q_2 + q_3 = -Q$$
$$-q_3 - q_1 = 2Q$$

$$\boxed{\begin{aligned} q_2 &= -Q - q_1 \\ q_3 &= -q_1 - 2Q \end{aligned}}$$

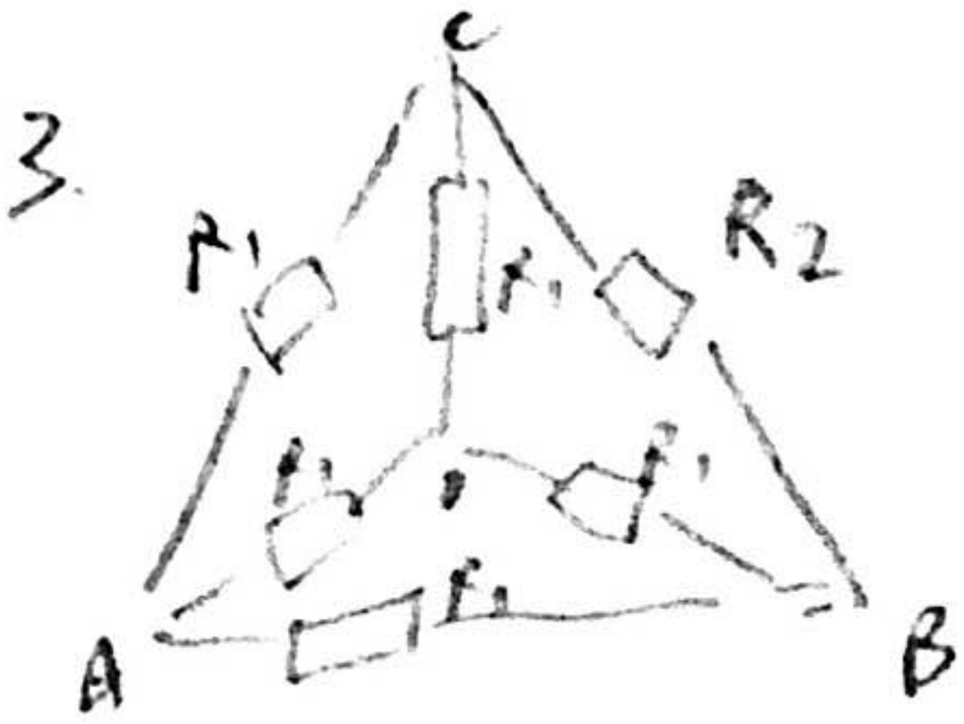
$$\frac{q_1}{C} = \frac{-Q - q_1}{2C} + \frac{-q_1 - 2Q}{3C}$$

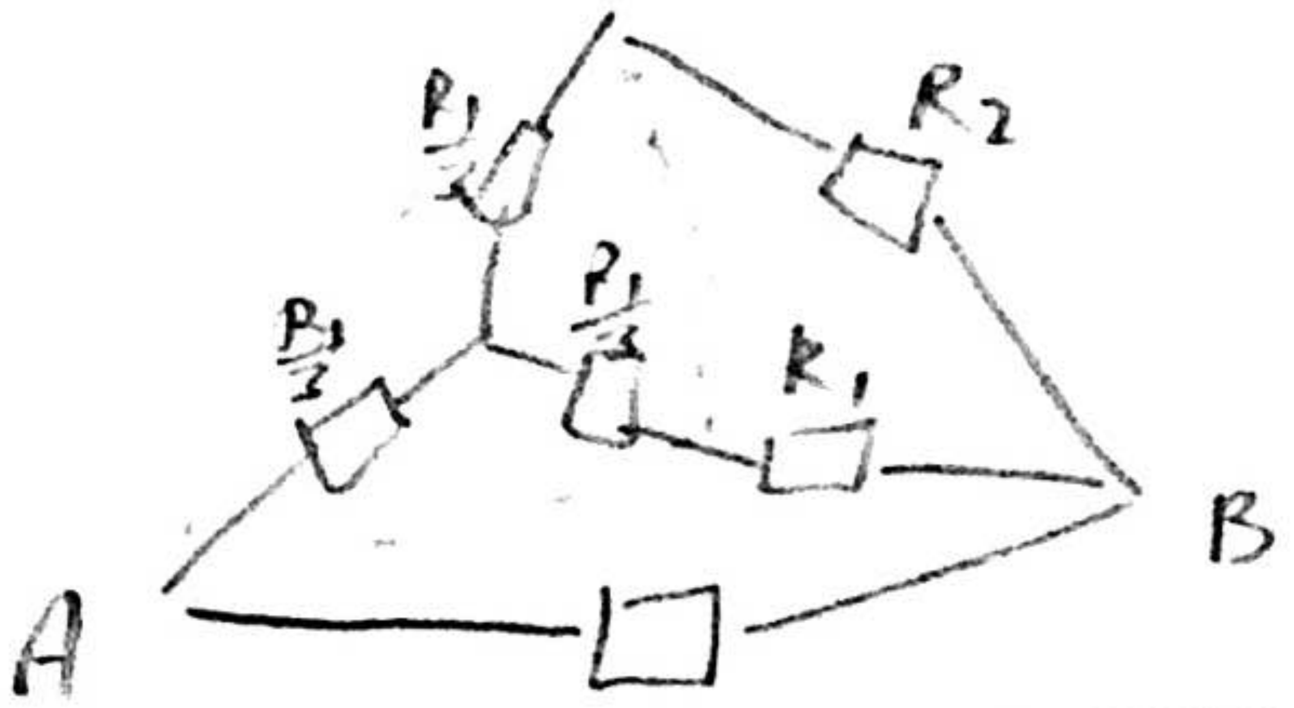
$$\Rightarrow 6q_1 = -3Q - 3q_1 - 2q_1 - 4Q$$

$$\Rightarrow 11q_1 = -7Q \quad \therefore q_1 = -\frac{7}{11}Q$$

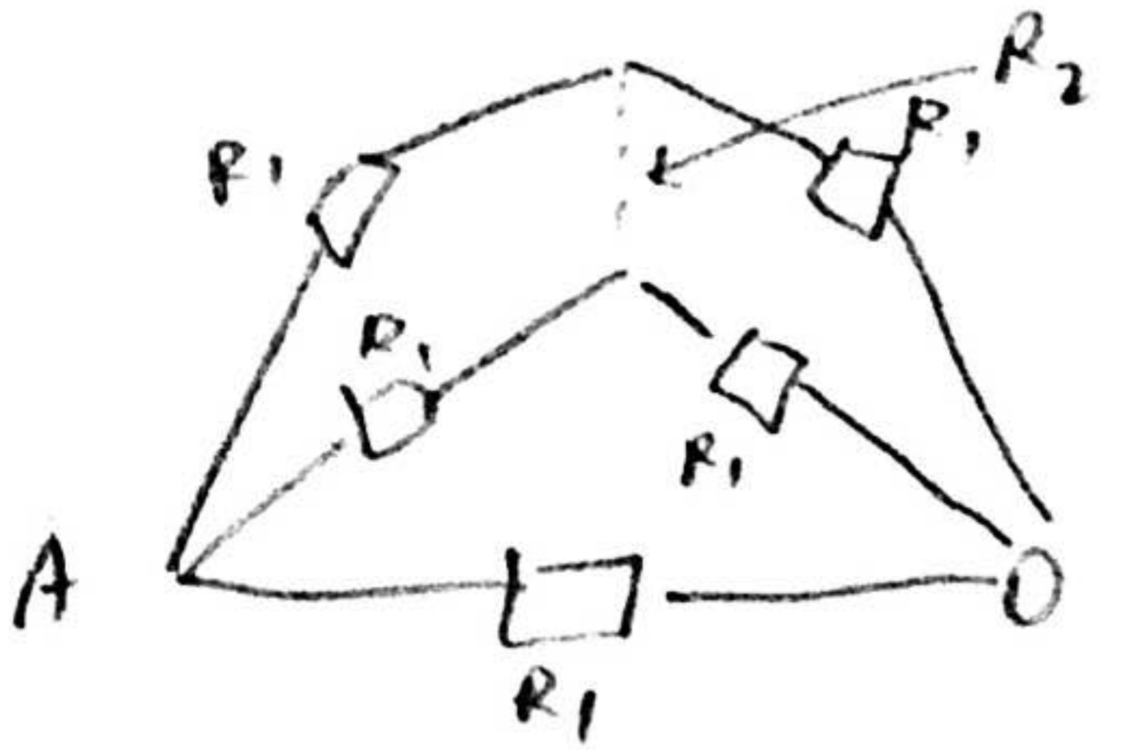
$$\boxed{\phi_1 - \phi_3 = -\frac{7}{11}\frac{Q}{C}}$$

3.



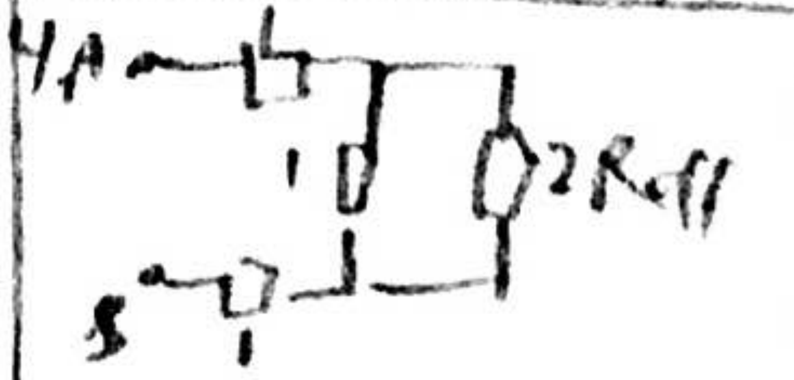


$$\boxed{R = \left(\frac{1}{R_1} + \frac{1}{\frac{R_1}{3} + \left(\frac{1}{\frac{R_1}{3} + R_2} + \frac{1}{\frac{R_1}{3} + R_1}\right)^{-1}}\right)^{-1}}$$



R_2 is redundant (equal voltage)

$$\therefore R = R_1 + \left(\frac{1}{2R_1} + \frac{1}{2R_1}\right)^{-1} = \boxed{2R_1}$$



let R_{eff} be the effective resistance

R scales linearly by dimension analysis

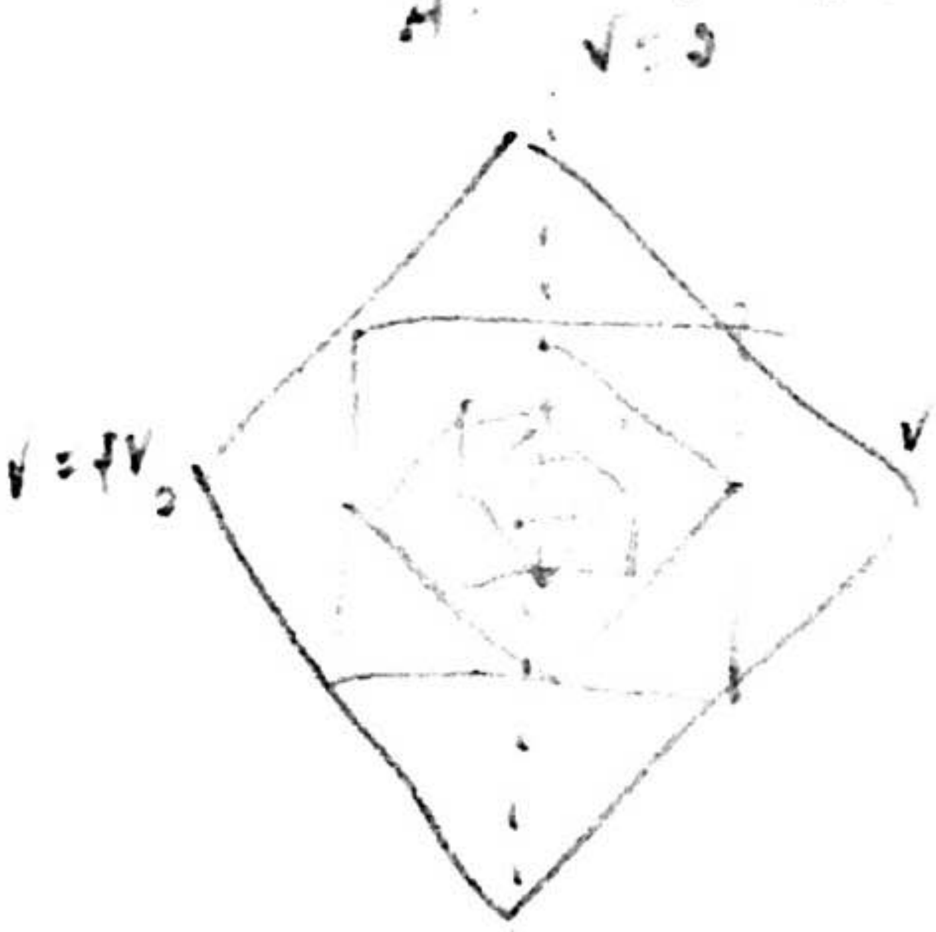
$$R_{eff} = 2 + \frac{1}{\left(1 + \frac{1}{2R_{eff}}\right)} = 2 + \frac{2R_{eff}}{2R_{eff} + 1}$$

$$2\lambda^2 - 5\lambda - 2 = 0 \Rightarrow \lambda = \frac{5 \pm \sqrt{25+16}}{4}$$

$$\therefore \boxed{R_{eff} = \frac{5 + \sqrt{41}}{4}}$$

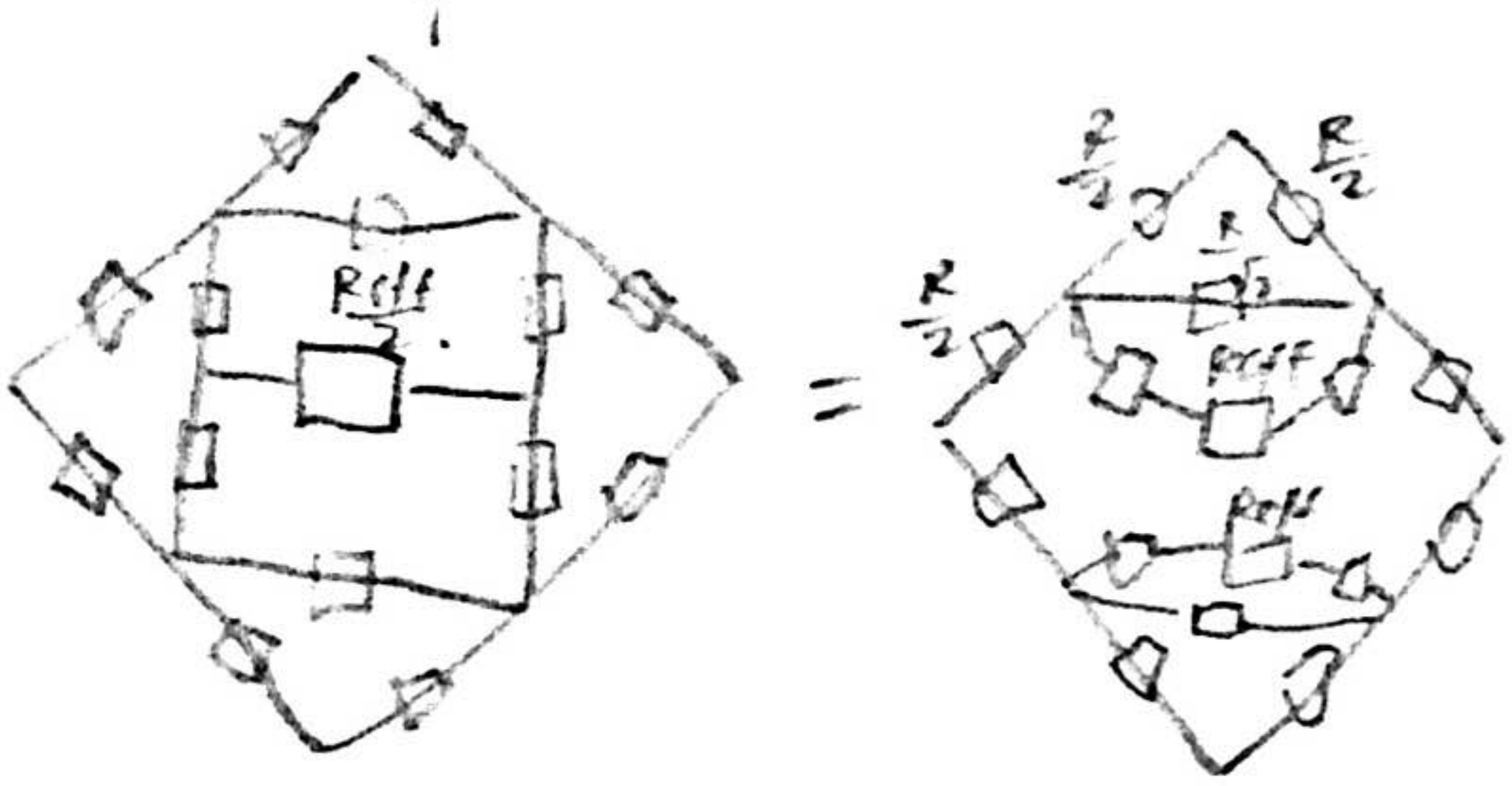
5.

$R = \frac{\rho l}{A} \Rightarrow R \propto l$



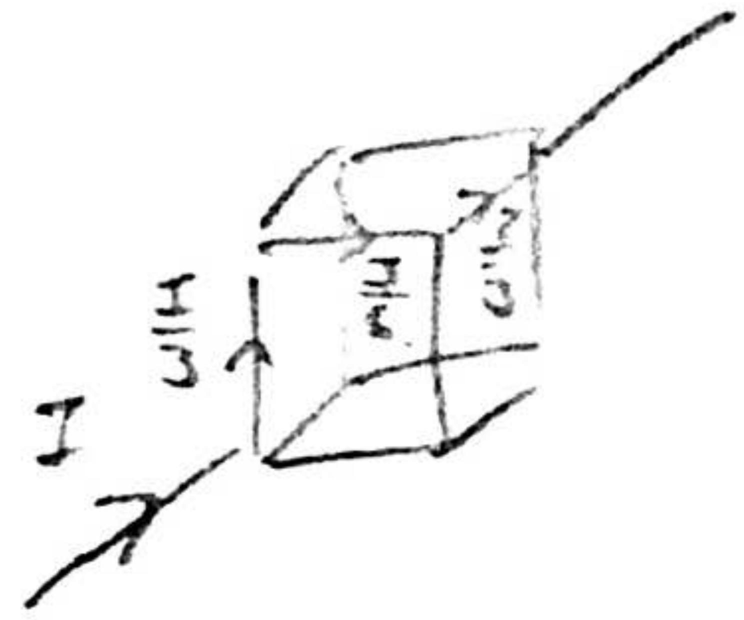
Let Reff be the effective resistance.

Since R scales linearly with length



$$R_{eff} = \frac{1}{2}\left(\frac{R}{2} + \frac{R}{2} + \frac{1}{\frac{1}{\frac{R}{\sqrt{2}}} + \frac{1}{R} + \frac{1}{\frac{R}{\sqrt{2}} + R_{eff}}} \right)$$

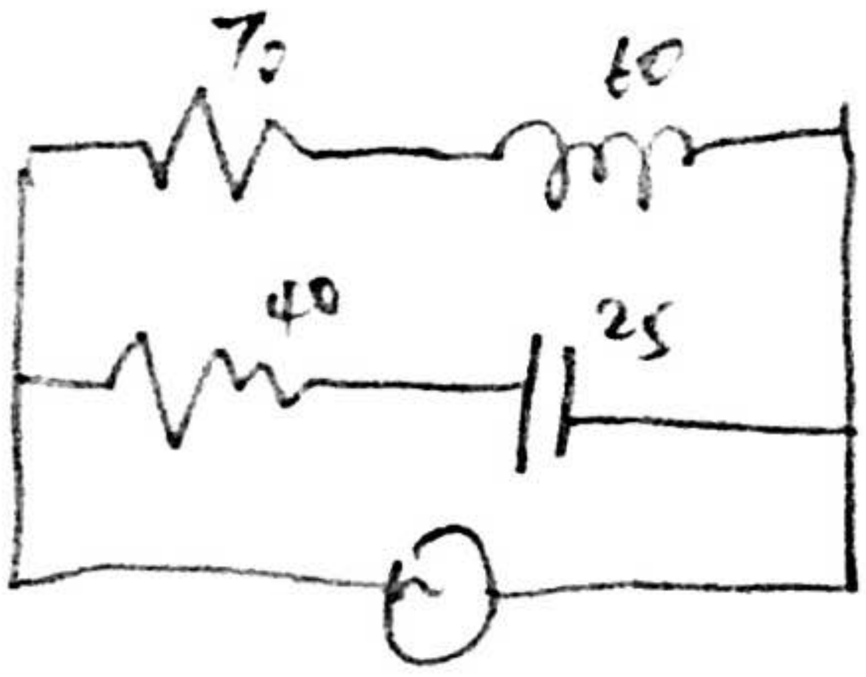
6.



$$I R_{eff} = \frac{I}{3}\cdot 1 + \frac{I}{6}\cdot 1 + \frac{I}{3}\cdot 1$$

$$\Rightarrow R_{eff} = \frac{1}{3} + \frac{1}{6} + \frac{1}{3} = \boxed{\frac{5}{6}}$$

7.

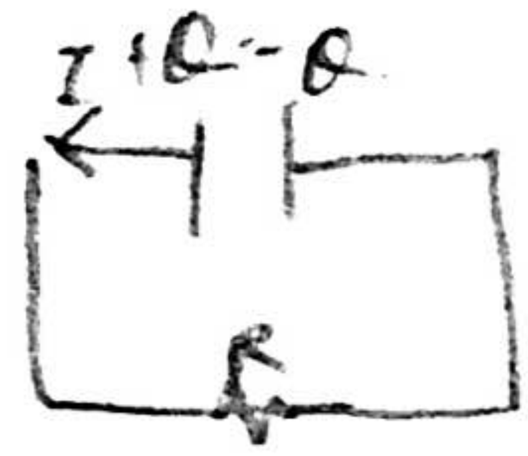


$Z_1 = 70 + 60i$

$Z_2 = 40 - 25i$

$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} \Rightarrow \boxed{Z = 37.22 - 5.93i}$

8



$-\frac{dQ}{dt} = I \qquad V = \frac{Q}{C} = IR$

$\Rightarrow \frac{Q}{C} = -\frac{dQ}{dt} R$

$\Rightarrow \int \frac{1}{Q} dQ = -\int \frac{1}{RC} dt$

$\Rightarrow \ln Q = \frac{1}{RC} t \Rightarrow Q(t) = A e^{\frac{1}{RC} t}$

When $t = 0$, $Q(0) = CV_0$

$\Rightarrow \boxed{Q(t) = CV_0 e^{\frac{1}{RC} t}}$