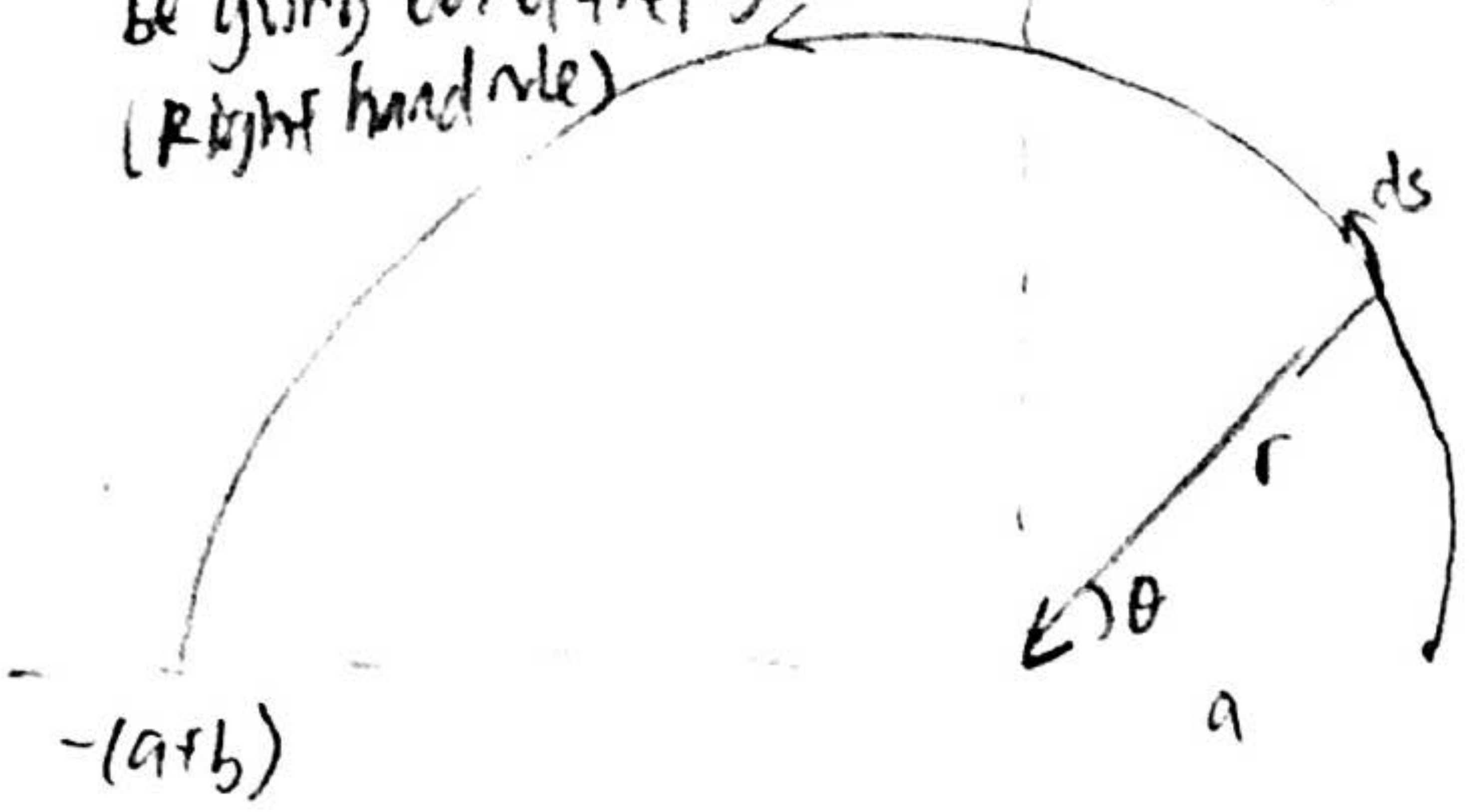
Electricity & Magnetism II (Solutions)

1. Firstly, $\vec{B}$ must be going out of the page (Right hand rule) $\quad r(\theta) = a + \frac{b}{\pi}\theta$



$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{s}\times\hat{r}}{r^2} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{s}\times\vec{r}}{(a+\frac{b}{\pi}\theta)^3}$$

$$d\vec{s} = \begin{pmatrix} (r+dr)\cos(\theta+d\theta) \\ (r+dr)\sin(\theta+d\theta)\end{pmatrix} - \begin{pmatrix} r\cos\theta \\ r\sin\theta\end{pmatrix}$$

$$= \begin{pmatrix} (r+dr)(\cos\theta - \sin\theta\, d\theta) \\ (r+dr)(\sin\theta + \cos\theta\, d\theta)\end{pmatrix} - \begin{pmatrix} r\cos\theta \\ r\sin\theta\end{pmatrix}$$

$$= \begin{pmatrix} dr\cos\theta - r\sin\theta\, d\theta \\ dr\sin\theta + r\cos\theta\, d\theta\end{pmatrix}$$

$$\vec{r} = -\begin{pmatrix} (a+\frac{b}{\pi}\theta)\cos\theta \\ (a+\frac{b}{\pi}\theta)\sin\theta\end{pmatrix}.$$

$$r = a + \frac{b}{\pi}\theta \Rightarrow \frac{dr}{d\theta} = \frac{b}{\pi} \Rightarrow dr = \frac{b}{\pi}d\theta.$$

$$d\vec{s}\times\vec{r} = \begin{pmatrix} \frac{b}{\pi}\cos\theta\, d\theta - (a+\frac{b}{\pi}\theta)\sin\theta\, d\theta \\ \frac{b}{\pi}\sin\theta\, d\theta + (a+\frac{b}{\pi}\theta)\cos\theta\, d\theta \\ 0\end{pmatrix} \times \begin{pmatrix} -(a+\frac{b}{\pi}\theta)\cos\theta \\ -(a+\frac{b}{\pi}\theta)\sin\theta \\ 0\end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ (a+\frac{b}{\pi}\theta)^2 d\theta\end{pmatrix}$$

$$\therefore \vec{B} = \frac{\mu_0 I}{4\pi}\int_0^{\pi} \frac{1}{a+\frac{b}{\pi}\theta}\, d\theta$$

$$= \left[\frac{\mu_0 I}{4\pi}\frac{\pi}{b}\ln\left|a+\frac{b}{\pi}\theta\right|\right]_0^{\pi} = \frac{\mu_0 I}{4\pi}\frac{\pi}{b}\ln\left(\frac{a+b}{a}\right)$$

$$\therefore \boxed{\vec{B} = \frac{\mu_0 I}{4b}\ln\left(\frac{a+b}{a}\right)}$$

(When $b=0$, it reduces to a ring $\frac{1}{2}\left(\frac{\mu_0 I}{2R}\right)$)

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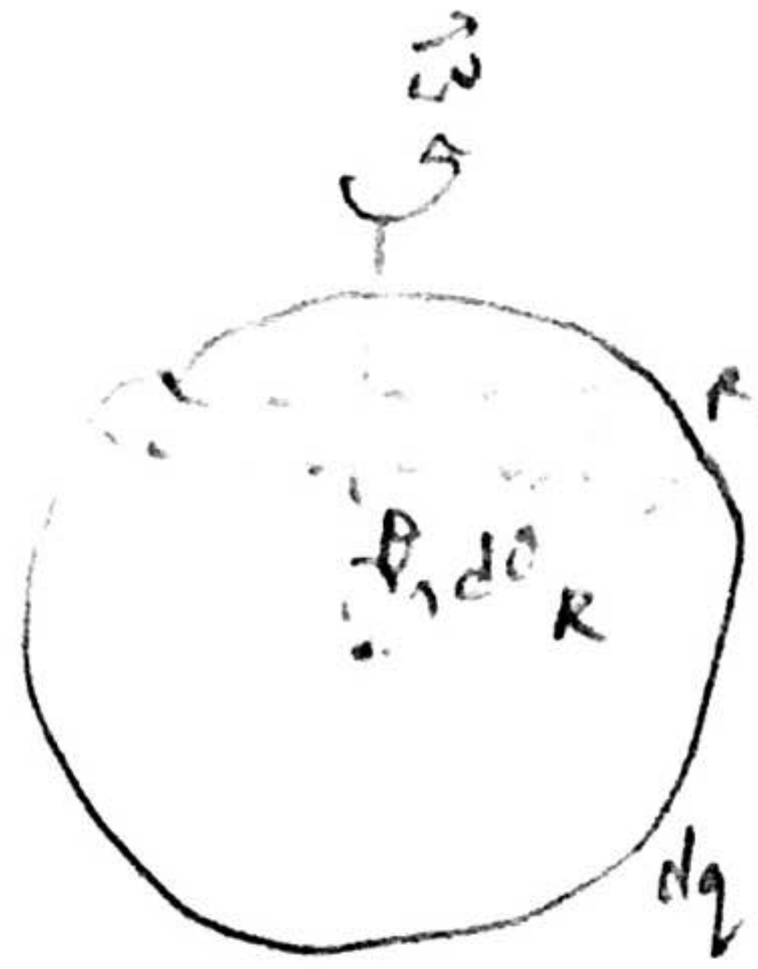
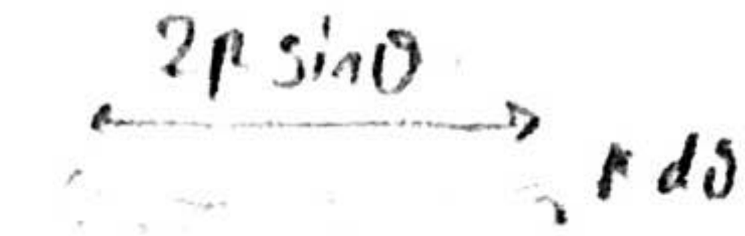




Amount of charge in this surface

$$dq = \frac{R\,d\theta\,(2\pi R\sin\theta)}{4\pi R^2}Q$$

$$= \frac{1}{2}Q\sin\theta\, d\theta$$

Firstly $\vec{B}$ is pointing upwards (RH rule)

$$dI = \frac{dq}{T} = \frac{\omega\, dq}{2\pi}$$

Contribution to the magnetic field due to this infinitesimal surface

$$dB = \frac{\mu_0}{4\pi}\frac{(2\pi R\sin\theta)}{R^2}dI\sin\theta$$ component pointing upwards

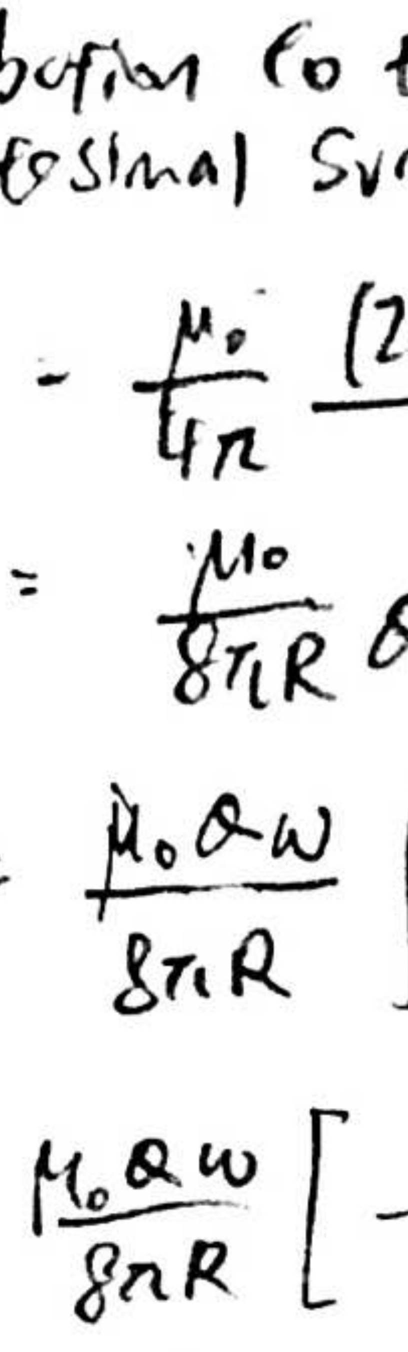
$$= \frac{\mu_0}{8\pi R}Q\sin^3\theta\,\omega\, d\theta$$

$$\Rightarrow B = \frac{\mu_0 Q\omega}{8\pi R}\int_0^{\pi}\sin^3\theta\, d\theta = \frac{\mu_0 Q\omega}{8\pi R}\int_0^{\pi}\sin\theta(1-\cos^2\theta)\, d\theta$$

$$= \frac{\mu_0 Q\omega}{8\pi R}\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{\pi}$$

$$= \frac{\mu_0 Q\omega}{8\pi R}\left[1 - \frac{1}{3} + 1 - \frac{1}{3}\right] = \boxed{\frac{\mu_0 Q\omega}{6\pi R}}$$

3.



$\vec{B}$ at the tip must be pointing upwards. (by RH rule)

$$dq = \sigma(dl)(2\pi l\sin\theta)$$

$$dB = \frac{\mu_0}{4\pi}\left(\frac{\omega\, dq}{2\pi}\right)\cdot\frac{(2\pi l\sin\theta)}{l^2}\sin\theta$$

$$= \frac{\mu_0\omega\sigma\, dl}{2}\sin^3\theta$$

$$\Rightarrow B = \frac{\mu_0\omega\sigma}{2}\sin^3\theta\int_0^{L} dl = \boxed{\frac{\mu_0\omega\sigma L}{2}\sin^3\theta.}$$

4.

(a) For $r<R$, take a Gaussian ring,

$$\oint \vec{B}\, d\vec{l} = \mu_0 I_{enc} = 0 \Rightarrow \boxed{B=0} \text{ (uniform)}$$

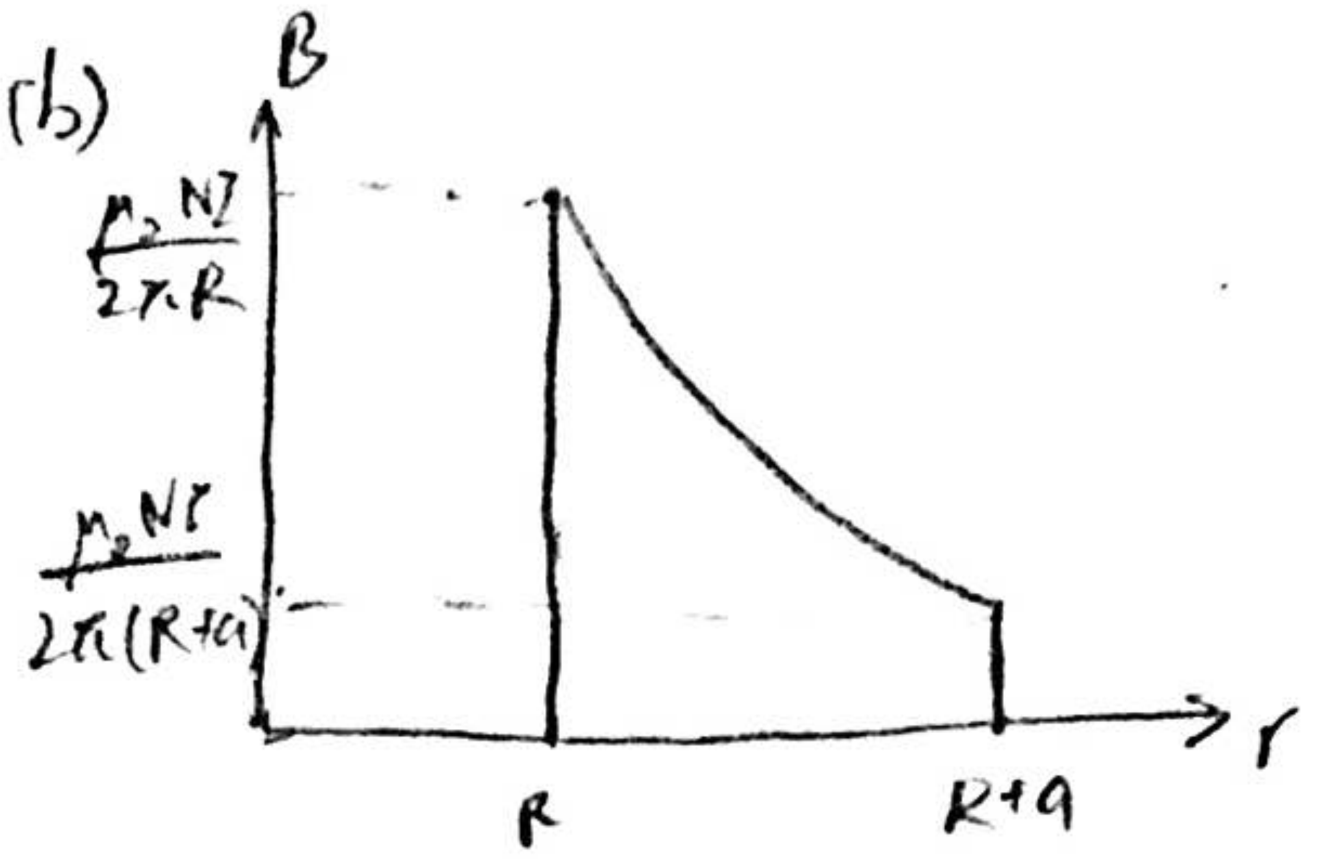
For $R<r<R+a$,

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} = \mu_0 NI \Rightarrow \boxed{B = \frac{\mu_0 NI}{2\pi r}}$$

For $R+a<r$,

$$\oint \vec{B}\, d\vec{l} = \mu_0 I_{enc} = 0 \Rightarrow \boxed{B=0}$$

(b)



(c) When $a \ll R$, the magnetic field at $R+a$ is $\frac{\mu_0 NI}{2\pi(R+a)} = \frac{\mu_0 NI}{2\pi R(1+\frac{a}{R})} \simeq \frac{\mu_0 NI}{2\pi R}\left(1-\frac{a}{R}\right)$

$\simeq \frac{\mu_0 NI}{2\pi R}$ = magnetic field at R.
$\Rightarrow$ almost uniform.

For an infinitely long solenoid, $B = \mu_0 n I$ where n is the turn density. Here $\frac{N}{2\pi R}$ is the turn density

5.

$$d\vec{F} = dq\, \vec{v}\times\vec{B} = I\, d\vec{l}\times\vec{B}$$

$\therefore$ Force acts upward.

$$dF = I\, dl\, B\cos(90^\circ-\theta) = I\, dl\, B\sin\theta$$

$$\Rightarrow F = \int I\, dl\, B\sin\theta = IB\sin\theta\, 2\pi r$$

$$\therefore \boxed{F = 2\pi r I B\sin\theta}$$

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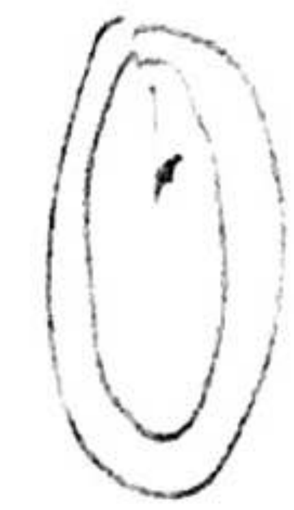
(a)



$B(t) = a+bt$

$B(0) = a$

$\Rightarrow \boxed{\phi(0) = a\pi r^2}$

(b)
$$\mathcal{E} = -\frac{d\phi}{dt} = -\frac{d(BA)}{dt} = -\frac{dB}{dt}A - \frac{dA}{dt}B$$

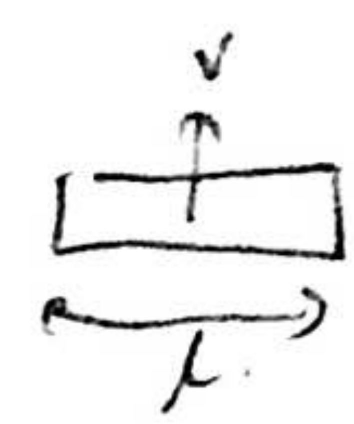
$$= -bA = -b(2\pi r)$$

$$\therefore |\mathcal{E}| = \boxed{2\pi r b}$$

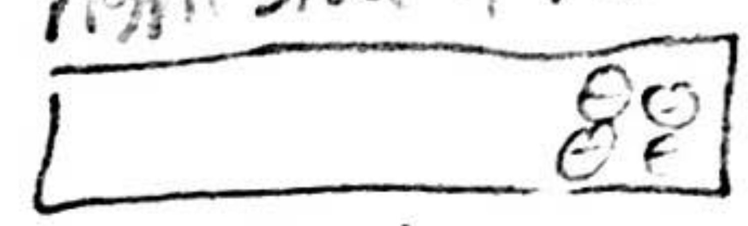
(c) $I = \frac{\mathcal{E}}{R} = \boxed{\frac{2\pi r b}{R}}$

(d) $P = I\mathcal{E} = \boxed{\frac{4\pi^2 r^2 b^2}{R}}$

7



Electrons gather at right side of rod



$F = q\vec{v}\times\vec{B}$ where $q<0$.

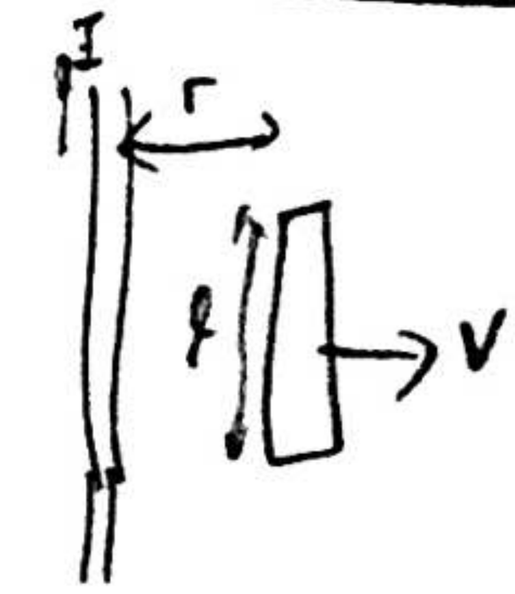
EMF = work done to separate the charges per unit charge

$$d\mathcal{E} = \frac{ev\left(\frac{\mu_0 I}{2\pi r}\right)dr}{e} = \frac{\mu_0 I v}{2\pi r}\, dr.$$

$$\Rightarrow \mathcal{E} = \int_{l}^{r+l} \frac{\mu_0 I v}{2\pi r}\, dr = \frac{\mu_0 I v}{2\pi}\Big[\ln r\Big]_{l}^{r+l}$$

$$= \frac{\mu_0 I v}{2\pi}\ln\left(\frac{r+l}{l}\right) = \boxed{\frac{\mu_0 I v}{2\pi}\ln\left(1+\frac{l}{r}\right)}$$

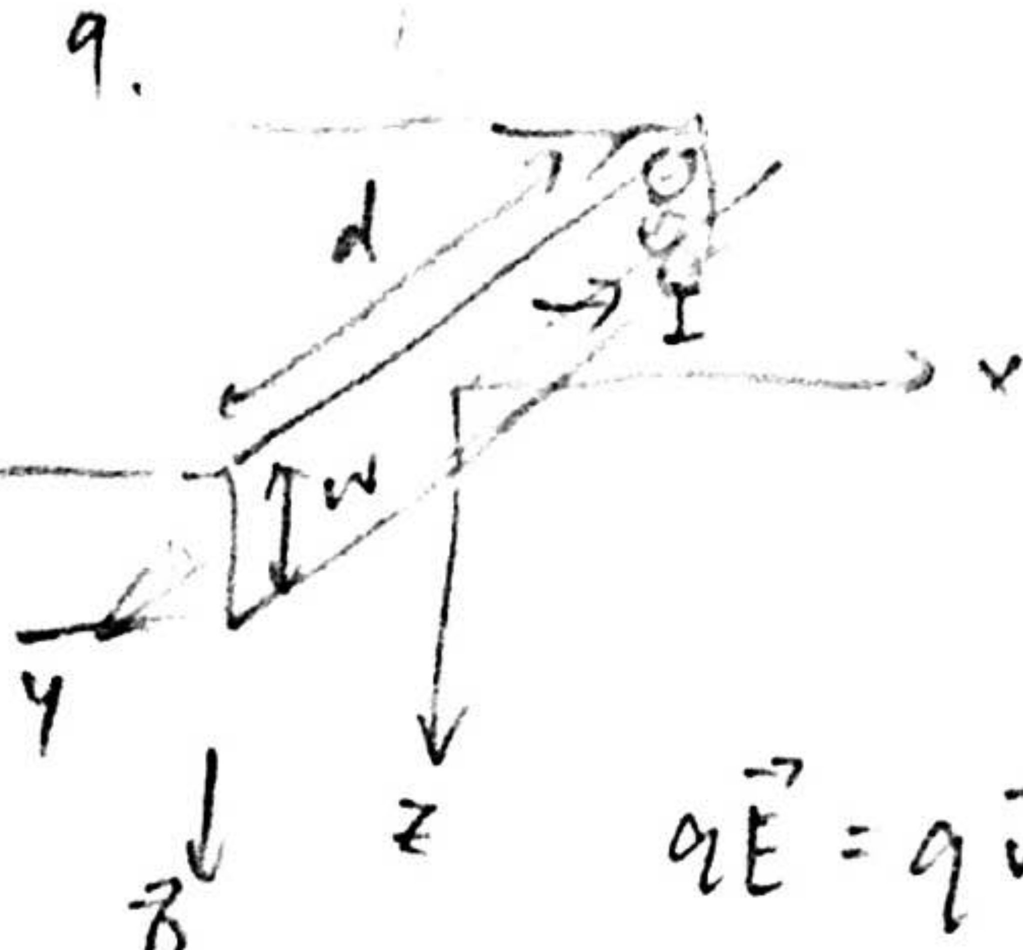
8.



$$d\mathcal{E} = \frac{ev\left(\frac{\mu_0 I}{2\pi r}\right)dl}{e} = \frac{\mu_0 I v}{2\pi r}\, dl$$

$$\Rightarrow \mathcal{E} = \int_0^l \frac{\mu_0 I v}{2\pi r}\, dl = \boxed{\frac{\mu_0 I v l}{2\pi r}}$$

9.



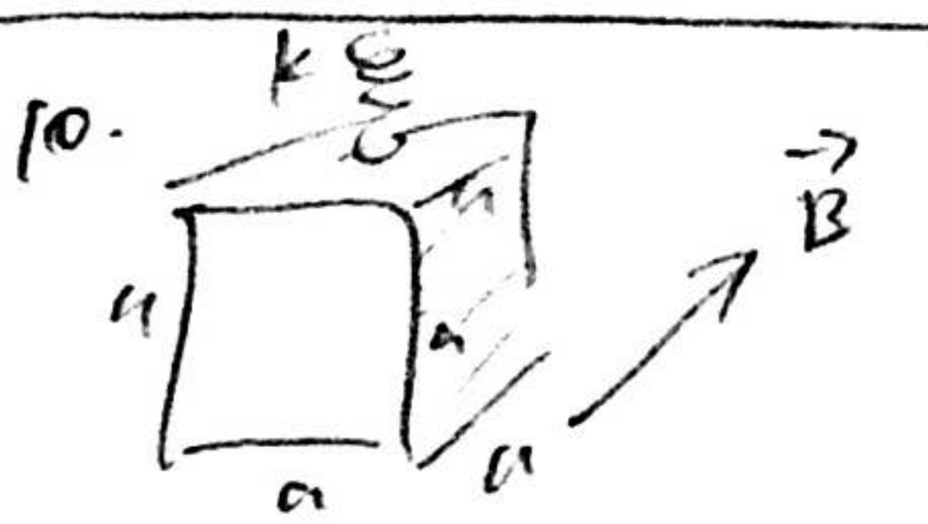
Electrons gather at negative y axis until magnetic force is balanced by electric force.

$$q\vec{E} = q\vec{v}\times\vec{B}$$

$$\Rightarrow e\frac{V}{d} = e\frac{I}{ndwe}B$$

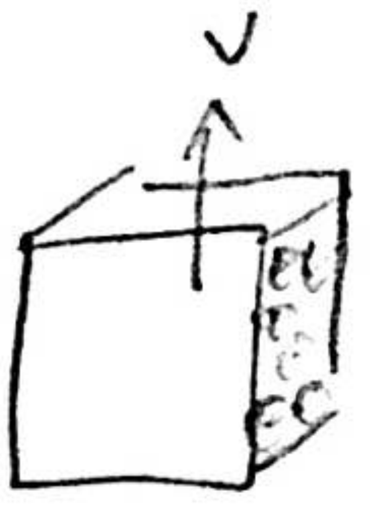
$$\Rightarrow \boxed{V = \frac{IB}{nwe}}$$

10.



(a)

$$F = q\vec{v}\times\vec{B}$$



In the instantaneous rest frame of the cube, there appears to be a horizontal electric field

$$E_0(t) = \frac{V(t)B_0}{c} = \boxed{\frac{AB_0 w e^{iwt}}{c}}$$

b) let electrical conductivity of copper be σ.

$$J = \sigma E \Rightarrow I = a^2 J = a^2\sigma E$$

The charge accumulated on A

$$= Q(t) = \int I\,dt = \int a^2\sigma E\,dt$$

$$= \int a^2\sigma\frac{AB_0 w}{c}e^{iwt}\,dt = \frac{a^2\sigma E}{wi}$$

The electric field due to surface charge

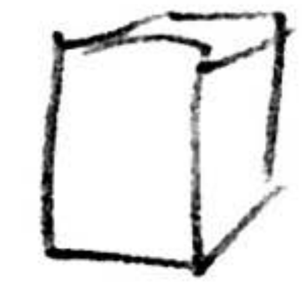
$$= E_1(t) \approx \frac{4\pi Q(t)}{a^2} = -\frac{4\pi i\sigma E}{w}$$

$$E = E_0 + E_1 = E_0 + \frac{4\pi\sigma E}{iw}$$

$$\Rightarrow E = \frac{AB_0 e^{iwt}w^2}{4\pi\sigma c i} \Rightarrow I(t) = \frac{a^2 w^2 AB_0 e^{iwt}}{4\pi i c}$$

$$= \boxed{-\frac{ia^2 w^2 AB_0 e^{iwt}}{4\pi c}}$$

(c)



$$R = \frac{\rho l}{A} = \frac{\rho a}{a^2} = \boxed{\frac{1}{\sigma a}}$$

$$\langle P\rangle = \frac{|I|^2 R}{2} = \boxed{\frac{a^3 w^4 A^2 B_0^2}{32\pi^2 c^2\sigma}}$$

(d) $\langle P\rangle$ results in the gradual decrease of the amplitude A of oscillation (magnetic damping).

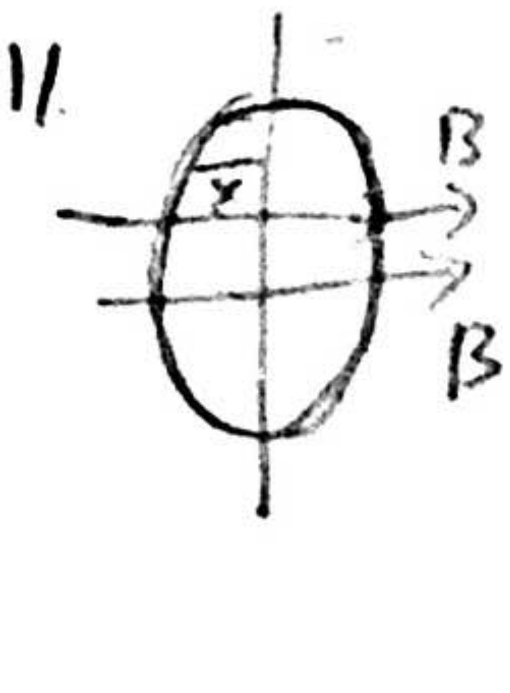
$$U(t) = kA^2(t)$$

$$\frac{dU}{dt} = -\langle P\rangle = -\frac{a^3 w^4 B_0^2}{32\pi^2 c^2\sigma}A^2$$

$$\Rightarrow \boxed{\frac{dA^2}{dt} = -\frac{a^3 w^4 B_0^2}{32\pi^2 c^2\sigma k}A^2.}$$

$$T = \frac{2\pi}{\Omega} = 2\pi\sqrt{\frac{32\pi^2 c^2\sigma k}{a^3 w^4 B_0^2}}$$

11.



$$\phi = B\cdot A = B\cdot\pi r^2\cos\theta$$

$$\mathcal{E} = \frac{d\phi}{dt} = B\pi r^2\frac{d\cos\theta}{dt}$$

$$= B\pi r^2(-\sin\theta)\,w$$

$$\Rightarrow \text{current induced} = \frac{B\pi r^2\sin\theta\,w}{R}$$

Since current induced, the ring suffers a magnetic

$$\text{torque} = \tau = \int d\vec{F}\cdot x = \int BI(dl)\cdot x\sin\theta$$

$$= \int_0^{\pi} BIr(d\phi)\,r\sin\phi\,\sin\theta$$

$$= BIr^2\sin\theta\left[-\cos\theta\right]_0^{\pi} = 2BIr^2\sin\theta$$

$$\therefore\ I\ddot{\theta} = -\tau \Rightarrow \frac{1}{2}mr^2\ddot{\theta} = -2BIr^2\sin\theta$$

$$\ddot{\theta} = -\frac{4BI}{m}\sin\theta = \boxed{-\frac{4B^2\pi r^2 w}{mR}\sin^2\theta}$$

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{dw}{dt} = \frac{dw}{d\theta}\frac{d\theta}{dt} = w\frac{dw}{d\theta}$$