Mechanics III

Physics Olympiad
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"Hey ya'll prepare yourself for the rubberband man."
--- Gamora

For this final session on Mechanics, we will cover Oscillations. Then, I will introduce some advanced concepts in Mechanics. The purpose is not to frighten you, but more of for exposure: letting you know that these concepts exist. Perhaps there is something more mysterious going on than just Newton's Second Law.

Linear Second Order Homogeneous Differential Equation

$$a\ddot{x} + b\dot{x} + cx = 0$$

Suppose we guess that the solution is of the form $Ae^{\lambda t}$ for some constant A. Then we have:

$$a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + ce^{\lambda t} = 0 \quad \Rightarrow \quad a\lambda^2 + b\lambda + c = 0$$

The corresponding *auxiliary equation* is $a\lambda^2 + b\lambda + c = 0$. Solving the auxiliary equation, we have:

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Now, we know $A_1 e^{\lambda_1 t}$ and $A_2 e^{\lambda_2 t}$ are solutions to our differential equation, and so is any linear combination of them. In fact, a typical n^{th} order linear ODE will have n families of solutions and n constants. The physical way to understand why there are two constants is because we need two initial value (usually the initial value of velocity and displacement) to fully solve for the equation of motion.

- Case 1: $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$ (two distinct real roots)

$$x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

- Case 2: $\lambda_1 = \lambda_2$ (indistinct real roots)

$$\lambda = \lambda_1 = \lambda_2$$
$$x = (A + Bt)e^{\lambda t}$$

- Case 3: $\lambda_1, \lambda_2 \in \mathbb{C}$ (complex roots)

$$\{\lambda_1, \lambda_2\} = \{p + qi, p - qi\}$$
$$x = e^{pt}[A \cos qt + B \sin qt]$$

This is called the complementary function (CF). For a homogeneous differential equation, the CF is its solution.

Linear Second Order Non-homogeneous Differential Equation

$$a\ddot{x} + b\dot{x} + c = F(t)$$

Let x_{CF} be the family of functions that satisfies $a\ddot{x_{CF}} + b\dot{x_{CF}} + cx_{CF} = 0$. Then, if we can find a function x_{PI} that satisfies $a\ddot{x_{PI}} + b\dot{x_{PI}} + cx_{PI} = F(t)$, then any function of the form $x_{PI} + x_{CF}$ will satisfy $a\ddot{x} + b\dot{x} + c = F(t)$.

We can now solve for x_{PI} by trying out functions

- If $F(t) = P(t)$ for some polynomial $P(t)$, we guess $x_{PI}(t) = Q(t)$ where $\deg P = \deg Q$.
- If $F(t) = A\sin t + B\cos t$, then guess $x_{PI}(t) = A\sin t + B\cos t$
- If $F(t) = re^{kt}$, then guess $x = ae^{kt}$

Types of Oscillations

Let's consider the most general harmonic motion that is not being driven by an external driving force. The coefficient of 2 below is used solely for simplifying the equations which follow.

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0$$

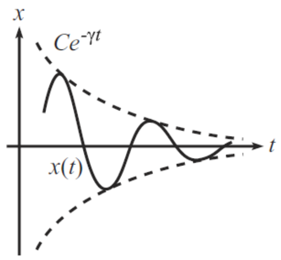
We will obtain the solutions:

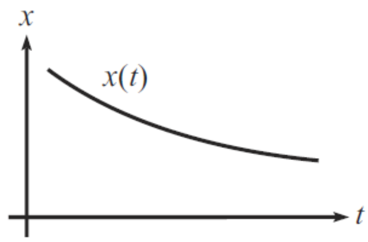
$$x(t) = e^{-\gamma t}\left(Ae^{\sqrt{\gamma^2-\omega^2}t} + Be^{-\sqrt{\gamma^2-\omega^2}t}\right)$$

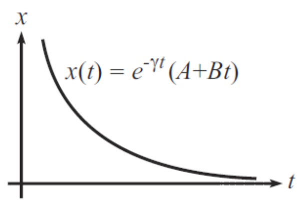
1. Under-damping ($\gamma^2 - \omega^2 < 0$)

$$x(t) = Ce^{-\gamma t}\cos(\sqrt{\omega^2 - \gamma^2}t + \phi)$$

Due to the $e^{-\gamma t}$ term, the oscillation will be "enveloped" by $x = Ce^{-\gamma t}$. Physically, due to the damping force, energy is taken out of the system, result in smaller amplitude over time.







2. Over-damping ($\gamma^2 - \omega^2 > 0$)

$$x(t) = Ae^{-\left(\gamma-\sqrt{\gamma^2-\omega^2}\right)t} + Be^{-\left(\gamma+\sqrt{\gamma^2-\omega^2}\right)t}$$

Note that both the exponents are negative. (We can do a thought experiment: if any exponent is positive, then as $t \to \infty$, $x \to \infty$, which is not possible for a system where energy is constantly being taken away.) Physically, the motion will be one of exponential decay.

3. Critical damping ($\gamma^2 - \omega^2 = 0$)

$$x(t) = e^{-\gamma t}(A + Bt)$$

Eventually as $t \to \infty$, the exponential term $e^{-\gamma t}$ dominates. If we are given ω, then critical damping (when $\gamma = \omega$) is the case where the motion converges to 0 the quickest.

Resonance

Suppose that the system is also subjected to a sinusoidal force $F \cos \omega_d t$, where ω_d is the angular frequency of the external force.

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = F \cos \omega_d t$$

We can obtain

$$x(t) = \frac{F}{\sqrt{(\omega - \omega_d)^2 + (2\gamma\omega_d)^2}} \cos(\omega_d t - \phi) + e^{-\gamma t}\left(Ae^{\sqrt{\gamma^2-\omega^2}t} + Be^{-\sqrt{\gamma^2-\omega^2}t}\right)$$

For completeness, $\phi = \tan^{-1}\left(\frac{2\gamma\omega_d}{\omega^2-\omega_d^2}\right)$, but the physical insights from this equation is clear. As $t \to \infty$, the $e^{-\gamma t}$ term at the back will diminish (recall that $\gamma > \sqrt{\gamma^2 - \omega^2}$) the latter term. Hence, in the steady state:

$$x(t) = \frac{F}{\sqrt{(\omega - \omega_d)^2 + (2\gamma\omega_d)^2}} \cos(\omega_d t - \phi)$$

For maximum amplitude, $\sqrt{(\omega - \omega_d)^2 + (2\gamma\omega_d)^2}$ should be as small as possible. This is achieved when $\omega_d = \omega$.

Resonance is used to describe the situation where the amplitude of a system becomes as large as possible under an external driving force and we have shown that this is achieved when the driving frequency ω_d is equal to the natural frequency of the system ω.

Oscillation in a Potential Field

Suppose we have a particle in equilibrium in a potential field $U(x)$. Recall that we can visualise the potential field as assigning numbers to each point in space and the force is given by:

$$F = -\frac{\mathrm{d}U}{\mathrm{d}x}$$

Suppose the particle is in equilibrium, the force acting on it is 0. If the particle is at x_0, by the above equation we have $F|_{x_0} = U'(x_0) = 0$.

Let's displace the particle a little bit to a position x.

$$U(x) = U(x_0) + \frac{U'(x_0)}{1!}(x - x_0) + \frac{U''(x_0)}{2!}(x - x_0)^2 + \cdots$$

$$m\ddot{x} = -\frac{\mathrm{d}U}{\mathrm{d}x} = -U''(x_0)(x - x_0)$$

To clarify the above equation, $U(x_0)$ is a constant, $U'(x_0) = 0$ as derived and we ignored higher order terms.

Since $\ddot{x} = \ddot{(x - x_0)}$, we have:

$$m\Delta\ddot{x} = -U''(x_0)\Delta x$$

$$\omega^2 = \frac{U''(x_0)}{m}$$

Physically, this means that in a potential field where a particle is in equilibrium, a tiny displacement of it will result in SHM. We can extend it to systems as well.

"Sir, there are still terabytes of calculations required before an actual flight is ... "
"JARVIS, sometimes you gotta run before you can walk."

--- Tony Stark

Coupled Oscillators

Suppose we have two coupled differential equations:

$$\ddot{x} = k_{11}x + k_{12}y$$

$$\ddot{y} = k_{21}x + k_{22}y$$

Suppose we guess $x = A_1 e^{i\omega t}$ and $y = A_2 e^{i\omega t}$, we can form the following matrix equation.

$$\begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = -\omega^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

$$\begin{pmatrix} k_{11} + \omega^2 & k_{12} \\ k_{21} & k_{22} + \omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

The reason why we guessed the form of $e^{i\omega t}$ is because of knowledge that we are expecting a sinusoidal equation of motion. We can also guess $e^{\omega t}$, just that when we solve for ω, it will be imaginary.

Clearly, $(A_1, A_2) = (0,0)$ is a (trivial) solution. However, we want the nontrivial solution. To do so, the determinant of the 2×2 matrix must be 0. The corresponding $\binom{A_1}{A_2}$ is the eigenvector of the equation. Generally, an $n \times n$ matrix will have n eigenvectors. The eigenvectors in this case are called the *normal modes*. Physically, it means that there are n possible ways which the system can oscillate with all oscillators moving at the same frequency. Any other oscillation is a superposition of the n normal modes.

$$\begin{vmatrix} k_{11} + \omega^2 & k_{12} \\ k_{21} & k_{22} + \omega^2 \end{vmatrix} = 0$$

$$\omega^2 = \frac{-(k_{11}+k_{22}) \pm \sqrt{(k_{11}-k_{22})^2 + 4(k_{12}+k_{21})}}{2}$$

After obtaining this, we can substitute this value of ω^2 back into the matrix equation to obtain our eigenvectors. Each unique eigenvector corresponds to one normal mode.

Lagrangian Mechanics

Newton's laws are great! However, in the situation where we need to change our coordinate system (just consider spherical coordinates) or multiple constraint forces (tensions on both strings of a double pendulum), Lagrangian mechanics can solve it better.

Lagrangian mechanics is ideal for systems with *conservative* forces (no friction, air resistance etc) and for *bypassing constraint forces* in *any* coordinate system. Lagrangian mechanics is ideal for systems with conservative forces.

$$\mathcal{L} = T - V$$

The concept of Lagrangian is that the path of a body is always the one that minimises the action (integral of $\mathcal{L}$). In the equation below, y is called a *generalised* coordinate.

$$S = \int_{t_1}^{t_2} \mathcal{L}(y, \dot{y}, t)\, \mathrm{d}t$$

For y to minimise this integral, it must satisfy the following, which gives us the Euler-Lagrange equation. We can view the equation as $\text{generalized force} = \frac{\mathrm{d}}{\mathrm{d}t}(\text{generalized momentum})$.

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0$$

Typically, to use the Lagrangian method, we apply the following steps:

- Pick *any* coordinate (can be $x, y, z, r, \theta, \phi, \psi$ etc)
- Find T and V and obtain $\mathcal{L}$
- Apply the Euler-Lagrange equation
- Repeat for each degree of freedom of the system
- Success?

Noether's theorem

Firstly, we need to understand *symmetry*. Symmetry can be thought of as a type of invariance: the property that an object remains unchanged under a set of transformations.

Noether's theorem states that every differentiable symmetry of the action of a physical system has a corresponding conservation law. Informally, if a system has a continuous symmetry property, then there are corresponding quantities whose values are conserved in space. The conserved quantity is called the *generato*r of the transform.

- Translational symmetry corresponds to conservation of momentum
- Rotational symmetry (about an axis) corresponds to conservation of angular momentum

- Time symmetry corresponds to conservation of energy

Noether's theorem is powerful because it explains the equivalence of symmetry and conservation laws. She proved mathematically that symmetry leads to conservation and vice versa.

Hamiltonian Mechanics

The strength of Hamiltonian mechanics lies in its ability to describe more complex dynamic systems, such as celestial mechanics. It also contributed to the formulation of statistical and quantum mechanics.

The time evolution of a system is uniquely defined by Hamilton's equations, where $\mathcal{H} = \mathcal{H}(x, p, t)$ is the Hamiltonian, which often corresponds to the total energy of the system.

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p}$$

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial x}$$

Note that the Lagrangian $\mathcal{L}(x, \dot{x}, t)$ is a function of the generalised coordinate, generalised velocity and time. On the other hand, Hamiltonian $\mathcal{H}(x, p, t)$ is a function of the generalised momentum. Thus, Hamiltonian represents the time evolution dynamics directly.

$$\mathcal{H} = \sum_{i}^{n} \dot{x}_i p_i - \mathcal{L} = 2T - (T - V) = T + V = E$$

The expectation value of the Hamiltonian operator is the total energy of the system, thus the Hamiltonian is known as the energy operator.

In quantum mechanics, the Hamiltonian generates the time evolution of quantum states:

$$\mathcal{H}|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle$$

Accelerated Frames of Reference

$$\boldsymbol{F}_{trans} = -m\frac{\mathrm{d}^2\boldsymbol{R}}{\mathrm{d}t^2}$$

$$\boldsymbol{F}_{cent} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$$

$$\boldsymbol{F}_{cor} = -2m\boldsymbol{\omega} \times \boldsymbol{v}$$

$$\boldsymbol{F}_{az} = -m\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \times \boldsymbol{r}$$

A. Sample Problems

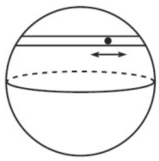
1. Derive the equation of motion for a simple harmonic oscillator, with mass m and under restoring force of $F = -kx$, where k is a positive constant and x is displacement.

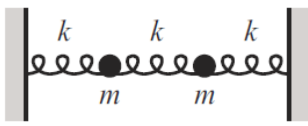
2. Derive the equation of motion for the same oscillator when it is subjected to a resistive force with magnitude given by $f = cv$, where v is the velocity of the oscillator and c is a positive constant.

3. Derive the equation of motion for the same oscillator under the same resistive force, nut now with a periodic force to reinforce the oscillation. The magnitude of the resistive force is given by $F = F_0 \cos(\omega t + \beta)$ where t is time and ω, β, F_0 are all constants.

4. A straight tube is drilled between two points on the earth. An object is dropped into the tube. How long does it take to reach the other end? (Assume $\rho_{earth} = 5.5 \text{ g cm}^{-3}$)

5. A hole is bored in a straight line through the Earth from Cambridge to New York (where the Avengers are fighting Loki's army). A ball bearing is dropped in at the Cambridge end. Assuming that frictional and air resistance forces are negligible, and that Earth is a uniform density sphere of radius R, how long does it take for the ball bearing to arrive in New York? (Take the acceleration due to gravity at Earth's surface to be g)





6. A hole of radius R is cut out from an infinitely flat sheet with mass density σ per unit area. Let L be the line that is perpendicular to the sheet and that passes through the center of the hole.

   a. What is the force on a mass m that is located on L, at a distance x from the center of the hole?

   b. If a particle is released from rest on L, very close to the center of the hole, show that it undergoes oscillatory motion and find the frequency of these oscillations.

   c. If a particle is released from rest on L at a distance x from the sheet, what is its speed when it passes through the center of the hole? What is your answer in the limit $x \gg R$?

7. Consider two masses m, connected to each other and to two walls by three springs. The three springs have the same spring constant k. Find the most general solution for the positions of the masses as functions of time.

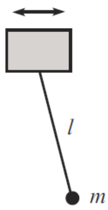
8. A mass m is free to slide on a frictionless table and is connected, via a string that passes through a hole in the table, to a mass M that hangs below. Assume that M moves in a vertical line only and assume that the string is always taut.

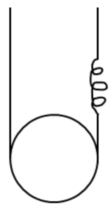
a. Find the equations of motion for the variables r and θ.

b. Under what condition does m undergo circular motion?

c. What is the frequency of small oscillations in the variable r about this circular motion?

9. A pendulum consists of a mass m and a massless stick of length l. The pendulum support oscillates horizontally with a position given by $x(t) = A\cos(\omega t)$. What is the general solution for the angle of the pendulum as a function of time?





B. In Class Problems

1. Derive an expression in terms of m and k for the period of vibration for a cylinder of mass m and radius r suspended in a spring loaded sling where k is the spring constant.

2. A solid sphere of radius R rolls without slipping in a cylindrical trough of radius $5R$. Show that, for small displacements from the equilibrium perpendicular to the length of the trough, the sphere executes simple harmonic motion. Determine the period of the simple harmonic motion.

3. A homogeneous beam having weight W and length $2L$ is pinned at the left-hand end by a smooth pin O and is held from falling by a spring having a constant k and located as shown. The equilibrium position for the beam is horizontal. When the system is perturbed slightly, find the natural frequency of the system.

4. Three pegs P, Q, R are fixed on a smooth horizontal table in such a way that they form the vertices of an equilateral triangle of side $2a$. A particle X of mass m lies on the table. It is attached to the pegs by three springs PX, QX and RX, each of modulus of elasticity λ and natural length l, where $l < \frac{2}{\sqrt{3}}a$. Initially, the particle is in equilibrium.
   a. Show that the extension in each spring is $\frac{2}{\sqrt{3}}a - l$.
   b. The particle is then pulled a small distance directly towards P and released. Show that the tension T in the spring RX is given by the following, where x is the displacement of X from its equilibrium position.

$$T = \frac{\lambda}{l}\left(\sqrt{\frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + x^2} - l\right)$$

   c. Show further that the particle performs approximate simple harmonic motion with period $2\pi\sqrt{\frac{4mla}{3(4a-3\sqrt{3}l)\lambda}}$.

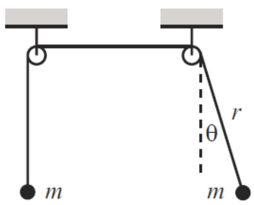
5. A uniform disc with center O and radius a is suspended from a point A on its circumference, so that it can swing freely about a horizontal axis L through A. The plane of the disc is perpendicular to L. A particle P is attached to a point on the circumference of the disc. The mass of the disc is M and the mass of the particle is m.

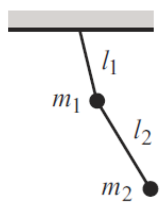
    a. In equilibrium, the disc hangs with OP horizontal, and the angle between AO and the downward vertical through A is β. Find $\sin\beta$ in terms of M and m and show that $\frac{AP}{a} = \sqrt{\frac{2M}{M+m}}$.

    b. The disc is rotated about L and then released. At later time t, the angle between OP and the horizontal is θ. (i.e. when P is higher than O, θ is positive; when P is lower than O, θ is negative.) Show that:
    $$\frac{1}{2}I\dot{\theta}^2 + (1-\sin\beta)ma^2\dot{\theta}^2 + (M+m)ga\cos\beta\,(1-\cos\theta)$$
    is constant during the motion, where I is the moment of inertia of the disc around L.

    c. Given that $m = \frac{3}{2}M$ and that $I = \frac{3}{2}Ma^2$, show that the period of small oscillations is $3\pi\sqrt{\frac{3a}{5g}}$.

6. Two equal masses, connected by a massless string, hangs over two pulleys (of negligible size). The left one moves in a vertical line, but the right one is free to swing back and forth in the plane of the masses and the pulleys. Find the equations of motion for r and θ, as shown. Assume that the left mass starts at rest, and the right mass undergoes small oscillations with angular amplitude ϵ ($\epsilon \ll 1$). What is the initial average acceleration (averaged over a few periods) of the left mass? In which direction does it move?





7. Consider a double pendulum made of two masses m_1 and m_2 and two rods of length l_1 and l_2.

    a. Find the equations of motion.

    b. For small oscillations, find the normal modes and their frequencies for the special case $l_1 = l_2$ (and consider the cases $m_1 = m_2$, $m_1 \gg m_2$, and $m_1 \ll m_2$). Do the same for the special case $m_1 = m_2$ (and consider the cases $l_1 = l_2$, $l_2 \gg l_1$, $l_1 \gg l_2$).

8. Obtain Hamilton's equation for a particle moving in a plane with potential energy function $V(r)$.