A1) $F(t) = ma_0 e^{-bt}$

$\Rightarrow m\ddot{x} = ma_0 e^{-bt}$

$\Rightarrow \ddot{x} = a_0 e^{-bt}$

$\dot{x} = \int \ddot{x}\,dt = \frac{a_0}{-b} e^{-bt} + c.$

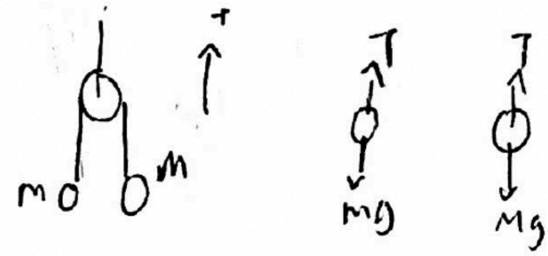
$\dot{x}(0) = 0 \Rightarrow c = \frac{a_0}{b}.$

$x = \int \dot{x}\,dt = \frac{a_0}{b^2} e^{-bt} + \frac{a_0}{b} t + c'$

$x(0) = 0 \Rightarrow c' = -\frac{a_0}{b^2}$

$\therefore \boxed{x(t) = \frac{a_0}{b^2}\left[e^{-bt} - 1 + bt\right]}$

A2)



$ma_1 = T - mg$

$Ma_2 = T - Mg$

$a_1 = -a_2$ (conservation of string)

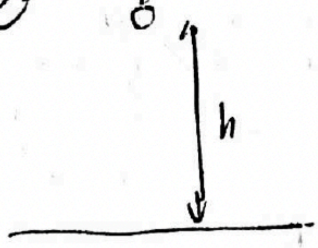
$\Rightarrow \frac{T - mg}{m} = -\frac{T - Mg}{M}$

$\Rightarrow T = \frac{2mM}{m+M} g$

$\boxed{a_1 = \frac{(M-m)g}{M+m} \quad a_2 = \frac{(M-m)g}{M+m}.}$

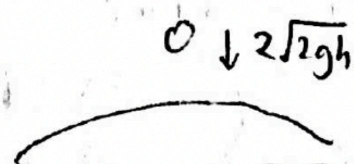
$F = 2T = \boxed{\frac{4mM}{m+M} g}$

A3)



Basketball bounces with velocity $\sqrt{2gh}$.

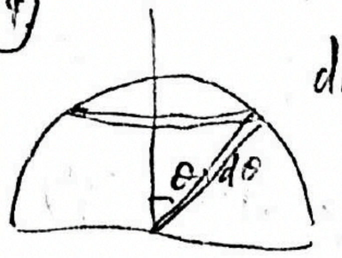
Frame of basketball



$\Rightarrow$ Tennis ball bounces with $4\sqrt{2gh}$ in frame of basketball

$\Rightarrow \boxed{3\sqrt{2gh}}$

A4)



$dm = \sigma (R\,d\theta)\, 2\pi R\sin\theta$

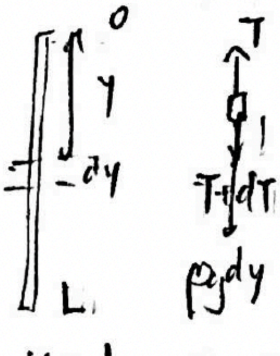
$\int y\,dm = \int (R\cos\theta)\,dm = \int R\cos\theta\,\sigma R\, 2\pi R \sin\theta\, d\theta$

$= \pi\sigma R^3 \int_0^{\frac{\pi}{2}} 2\sin\theta\cos\theta\,d\theta$

$= \frac{M}{2\pi} R\left[-\frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}} = \frac{MR}{2}$

$y_{cm} = \frac{\int y\,dm}{M} = \boxed{\frac{R}{2}}$

A5)



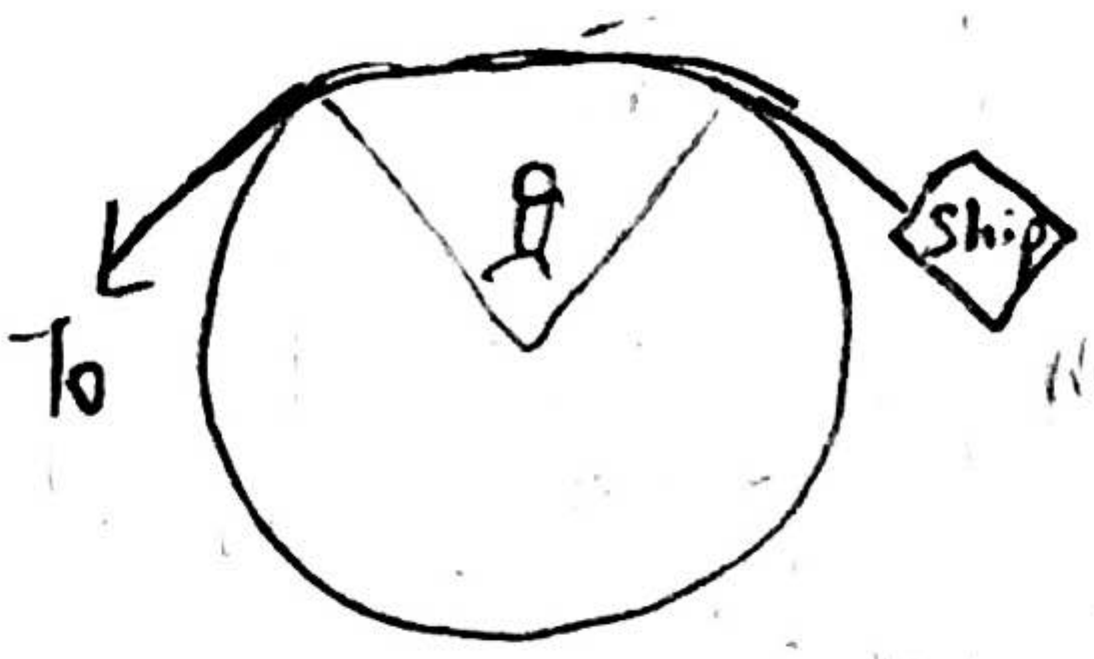
$T + dT + \rho g\,dy = T$

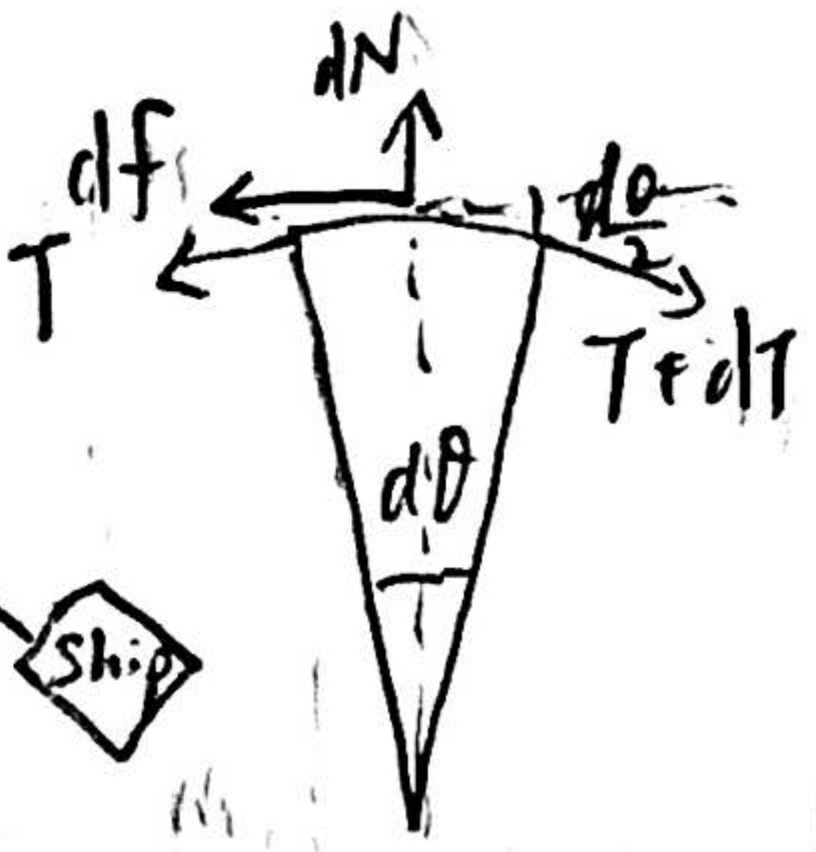
$\Rightarrow dT = -\rho g\,dy$

$\Rightarrow T = -\rho g y + c$

when $y = L$, $T = 0 \Rightarrow \boxed{T(h) = \rho g(h - L)}$

(A6)





x) $df + T\cos\left(\frac{d\theta}{2}\right) = (T+dT)\cos\left(\frac{d\theta}{2}\right)$

y) $dN = T\sin\left(\frac{d\theta}{2}\right) + (T+dT)\sin\left(\frac{d\theta}{2}\right)$

$df = \mu dN$ (if $df < \mu dN$, then not maximum)

x) ⇒ $df = dT$

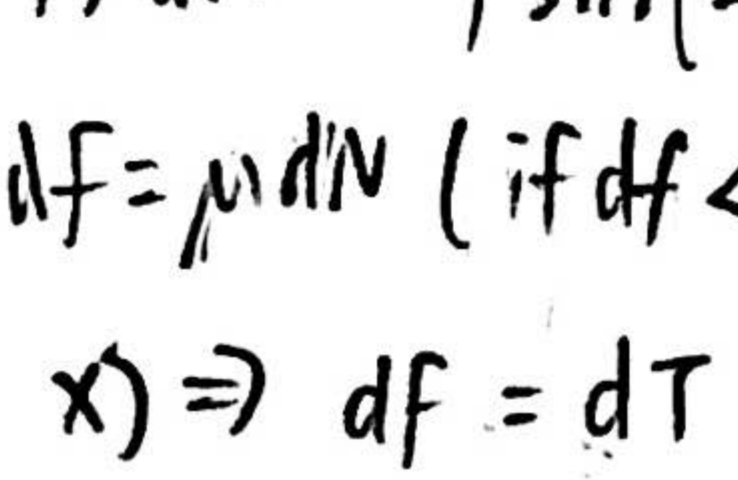
y) ⇒ $dN = \frac{T}{2}d\theta + \frac{T}{2}d\theta = T\,d\theta$

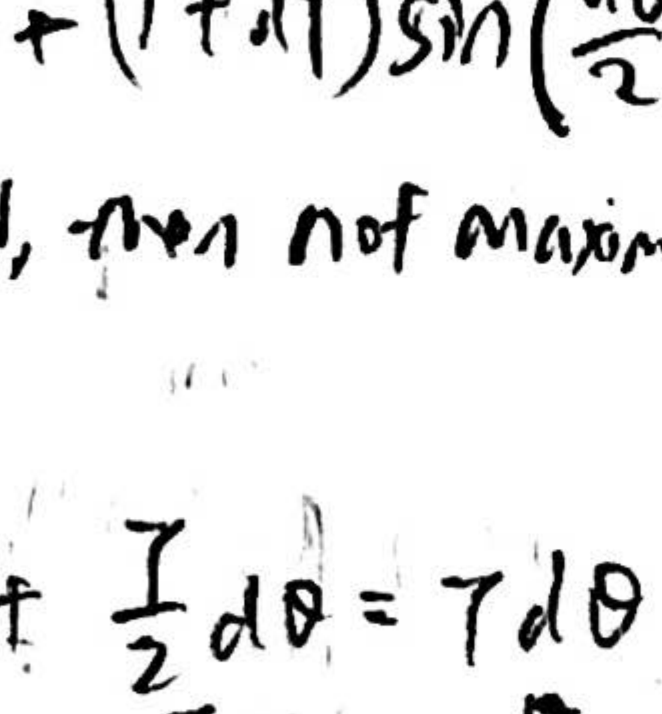
⇒ $dT = \mu T\,d\theta \Rightarrow \int_{T_0}^{T}\frac{dT}{T} = \int_0^{\theta}\mu\,d\theta$

⇒ $[\ln T]_{T_0}^{T} = [\mu\theta]_0^{\theta}$

⇒ $\boxed{T = T_0 e^{\mu\theta}}$

(B2)





$N = Mg\cos\theta + Mg\sin\theta$.

$f = Mg\sin\theta - Mg\cos\theta$

$|f| \le \mu N$.

$\tan\theta > 1 \Rightarrow \sin\theta - \cos\theta \le \mu(\cos\theta + \sin\theta)$

⇒ $\tan\theta \le \frac{1+\mu}{1-\mu}$.

$\tan\theta < 1 \Rightarrow -\sin\theta + \cos\theta \le \mu(\sin\theta + \cos\theta)$

⇒ $\tan\theta \ge \frac{1-\mu}{1+\mu}$

∴ $\boxed{\frac{1-\mu}{1+\mu} \le \tan\theta \le \frac{1+\mu}{1-\mu}}$

(B1) $m\dot{v} = -mg - m\alpha v$

$\dot{v} = -g - \alpha v$

⇒ $\int_{v_0}^{0}\frac{1}{g+\alpha v}\,dv = \int_0^{t} -dt$

⇒ $\left[\frac{1}{\alpha}\ln|g+\alpha v|\right]_{v_0}^{0} = [-t]_0^{t}$

⇒ $\boxed{t = \frac{1}{\alpha}\ln\left(\frac{\alpha v_0 + g}{g}\right)}$

$mv\frac{dv}{dx} = -mg - m\alpha v$

⇒ $\int_{v_0}^{0}\frac{v}{g+\alpha v}\,dv = \int_0^{H} -dx$

⇒ $\left[\frac{1}{\alpha}v - \frac{g}{\alpha^2}\ln|g+\alpha v|\right]_{v_0}^{0} = -H$

⇒ $\boxed{H = \frac{g}{\alpha^2}\ln\left(\frac{g+\alpha v_0}{g}\right) - \frac{1}{\alpha}v_0}$

OR

$\frac{1}{\alpha}\ln\frac{g+\alpha v}{g+\alpha v_0} = -t$

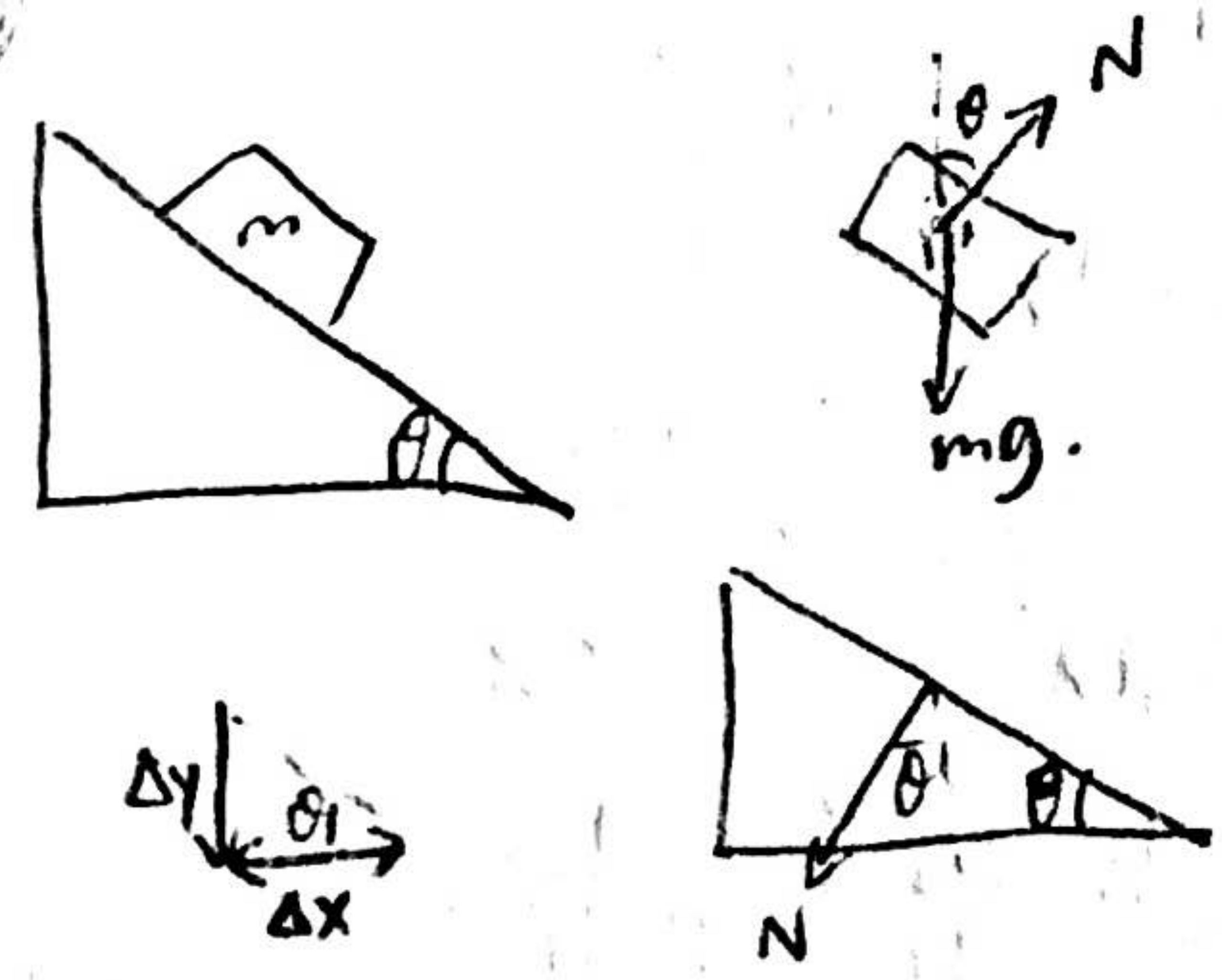
⇒ $g+\alpha v = (g+\alpha v_0)e^{-\alpha t}$

⇒ $v = \frac{1}{\alpha}\left[(g+\alpha v_0)e^{-\alpha t} - g\right]$

$H = -\frac{1}{\alpha^2}(g+\alpha v_0)e^{-\alpha t} + \frac{1}{\alpha^2}(g+\alpha v_0) - \frac{g}{\alpha}t$

Sub in t..

81c



7P,

y) $mg - N\cos\theta = m a_y$

x) $N\sin\theta = m a_x$

x) $N\sin\theta = M A_x$.

4 equations, 4 unknowns $\{a_x, a_y, A_x, N\}$

$\Delta y = \tan\theta\,\Delta x \Rightarrow a_y = \tan\theta(A_x + a_x)$

$MA_x = m a_x$

$a_y = g - a_x\frac{\cos\theta}{\sin\theta} = \tan\theta(A_x + a_x)$

$\tan\theta A_x = g - \frac{a_x}{\tan\theta} - a_x\tan\theta$

$\left(\tan\theta + \frac{1}{\tan\theta}\frac{M}{m} + \tan\theta\frac{M}{m}\right)A_x = g$

$\Rightarrow A_x = g\frac{m\tan\theta}{m\tan^2\theta + M + M\tan^2\theta}$

$\therefore \boxed{A_x = \frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta}}$

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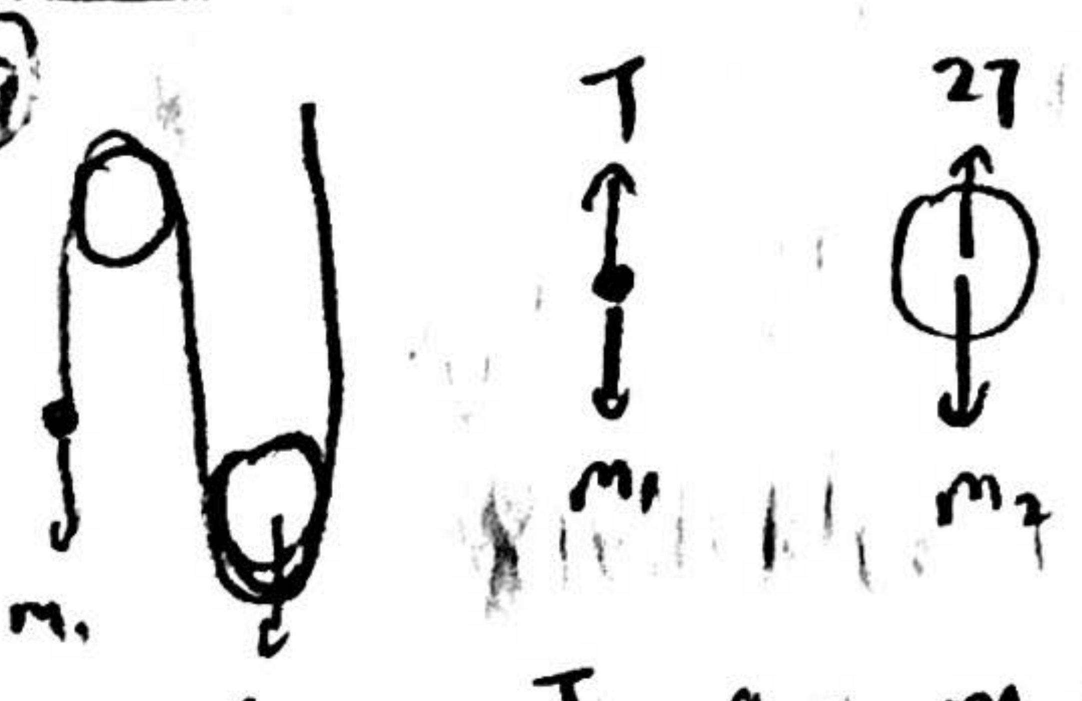


3 equations, 3 unknowns

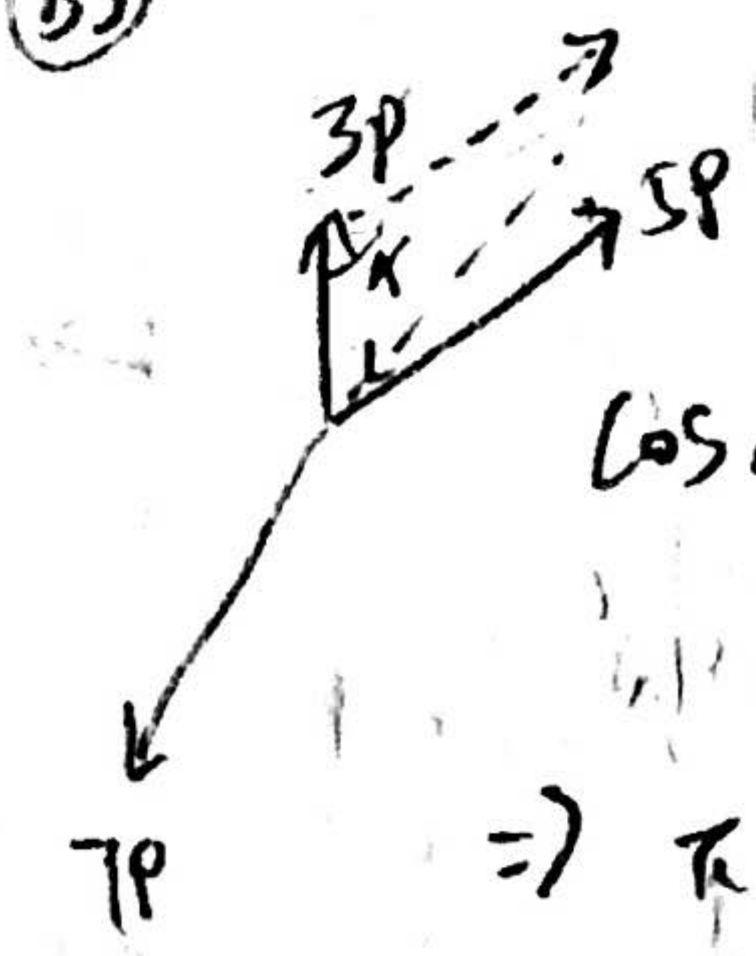
$T - m_1 = m_1 a_1$

$2T - m_2 = m_2 a_2$

Conservation of string $\Rightarrow a_1 = -2a_2$

$a_1 = g\frac{2m_2 - 4m_1}{4m_1 + m_2}$; $a_2 = g\frac{2m_1 - m_2}{4m_1 + m_2}$, $T = \frac{3m_1m_2g}{4m_1+m_2}$

83

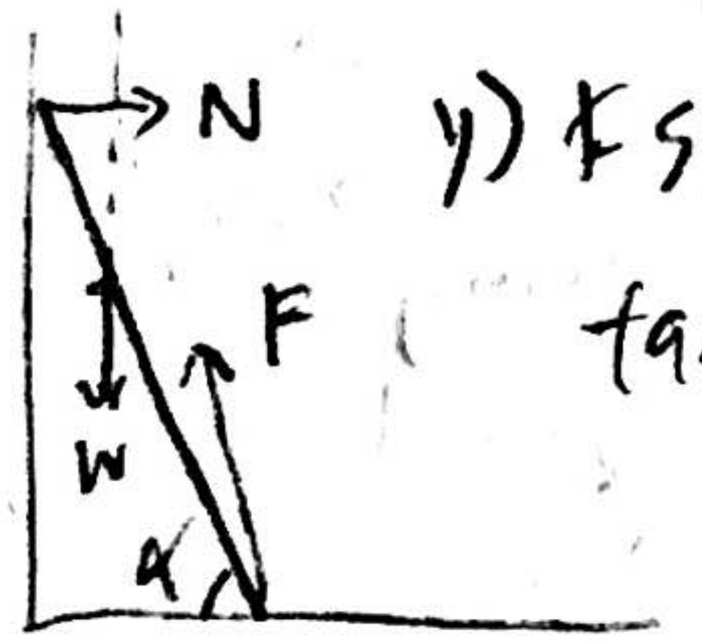


$\cos\alpha = \frac{|3P|^2 + |5P|^2 - |7P|^2}{2|3P||5P|}$

$= -\frac{1}{2} \Rightarrow \alpha = \frac{2\pi}{3}$

$\Rightarrow \pi - \alpha = \boxed{\frac{\pi}{3}}$.

85

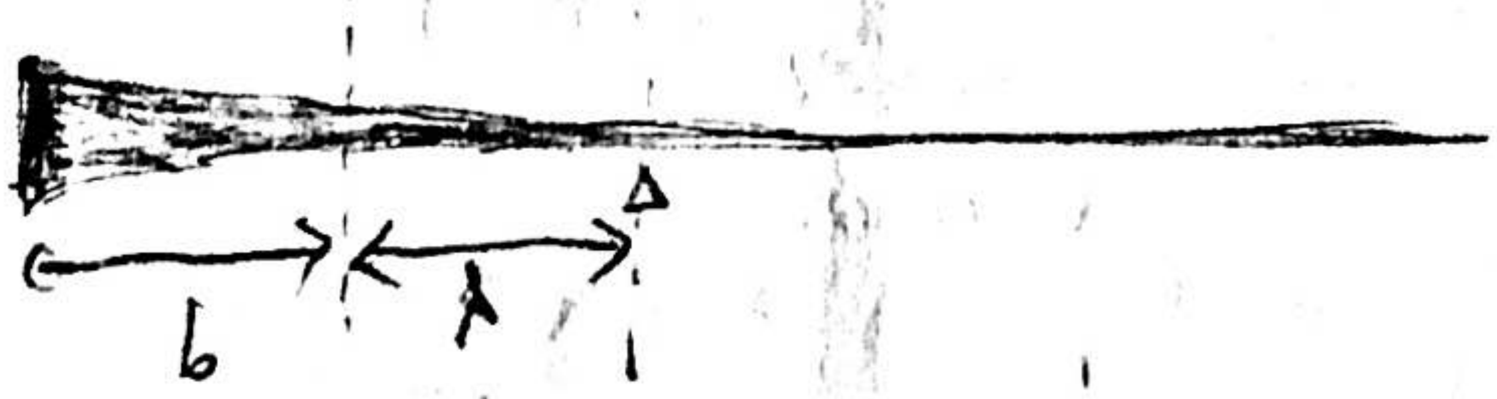


y) $F\sin\beta = W$

$\tan\beta = \frac{a\sin\alpha}{\frac{2}{3}a\cos\alpha} = \frac{3}{2}\tan\alpha$

x) $|N| = |F\cos\beta| = \left|\frac{F\sin\beta}{\tan\beta}\right| = \boxed{\frac{2W}{3\tan\alpha}}$

86



$\int_b^{b+l}\rho(x)(b+l-x)\,dx = \int_{b+l}^{\infty}\rho(x)(x-b-l)\,dx$

$\Rightarrow \int_b^{\infty}\rho(x)(b+l-x)\,dx = 0 \quad \forall b$

let $F = \int_b^{\infty}\rho(x)(b+l-x)\,dx$

Then $\frac{dF}{db} = 0 \ \forall b$.

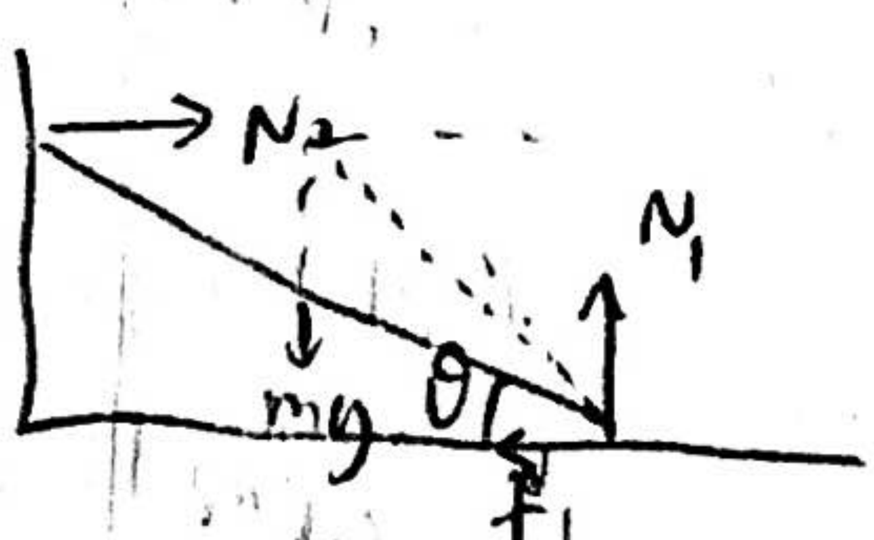
$dF = \left(\int_{b+db}^{\infty}\rho(x)(b+db+l-x)\,dx - \int_b^{\infty}\rho(x)(b+l-x)\,dx\right)$

$= \int_b^{\infty}\rho(x)\,db\,dx - \int_b^{b+db}\rho(x)(b+db+l-x)\,dx$

$= db\int_b^{\infty}\rho(x)\,dx - \rho(b)\,l\,db$

$\rho(b)\,l = \int_b^{\infty}\rho(x)\,dx \Rightarrow \boxed{\rho(x) = Ae^{-\frac{x}{l}}}$

(B4)



At boundary condition, f is not sufficient to make the three forces collinear

y) $N_1 = mg$

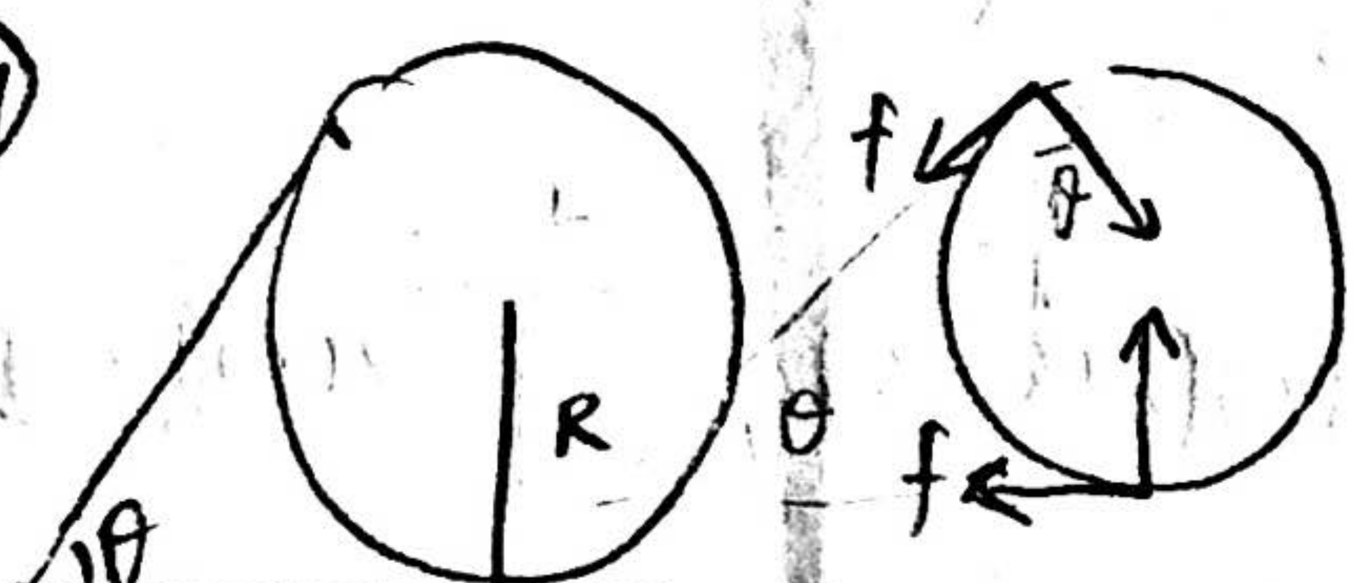
x) $N_2 = f \le \mu N_1 = \mu mg$

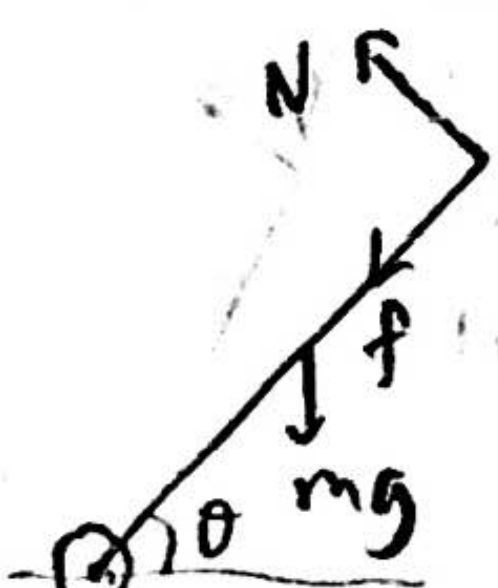
$$\frac{N_1}{f} = 2\tan\theta$$

$$\Rightarrow \tan\theta = \frac{N_1}{2f} = \frac{mg}{2f} \ge \frac{1}{2\mu}$$

$$\Rightarrow \boxed{\theta_{min} = \tan^{-1}\left(\frac{1}{2\mu}\right)}$$

(B7)





Stick) $\rho g\cos\theta\left(\frac{l}{2}\right) = Nl$

$$\Rightarrow N = \frac{1}{2}\rho g l\cos\theta$$

Ball) $f(1+\cos\theta) = N\sin\theta$

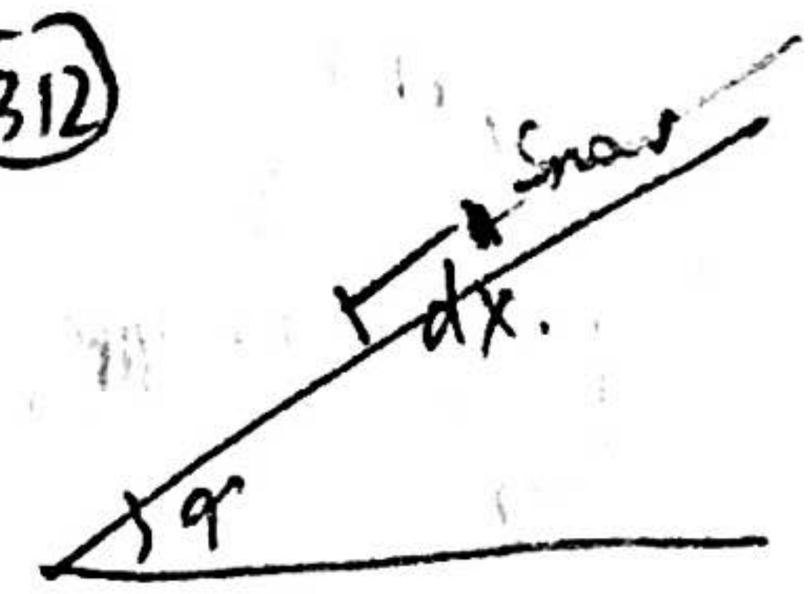
$$\Rightarrow f = \frac{1}{2}\rho g l\frac{\sin\theta\cos\theta}{1+\cos\theta}$$

$R = l\tan\left(\frac{\theta}{2}\right)$

$$\Rightarrow f = \frac{1}{2}\rho g\frac{R}{\tan\left(\frac{\theta}{2}\right)}\frac{\sin\theta\cos\theta}{1+\cos\theta}$$

$$\therefore \boxed{f = \frac{1}{2}\rho g R\cos\theta}$$

(B12)



$dm = \sigma dx \qquad m = \sigma x$

$$\sigma x g\sin\alpha = \frac{dp}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}$$

$$= \sigma x\frac{dv}{dt} + \sigma v^2$$

$$x\frac{dv}{dt} + v^2 = xg\sin\alpha$$

$$\Rightarrow xv\frac{dv}{dx} + v^2 = xg\sin\alpha$$

$$\Rightarrow x^2v\frac{dv}{dx} + xv^2 = x^2g\sin\alpha$$

$$\Rightarrow \frac{d}{dx}(x^2v^2) = 2x^2g\sin\alpha$$

$$\Rightarrow x^2v^2 = \int 2x^2g\sin\alpha\, dx$$

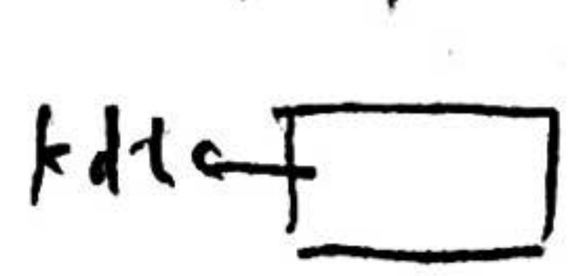
$$= \frac{2}{3}x^3g\sin\alpha + c$$

$$v^2 = \frac{2}{3}xg\sin\alpha + c$$

$$a = v\frac{dv}{dx} = \frac{1}{2}\frac{d}{dx}(v^2) = \boxed{\frac{1}{3}g\sin\alpha}$$

(B11) $\boxed{M} \to v_i$

In an infinitesmal time dt



$dm = -kdt$

COM) $(m+dm)(v+dv) + dm(v - v_r) = mv$

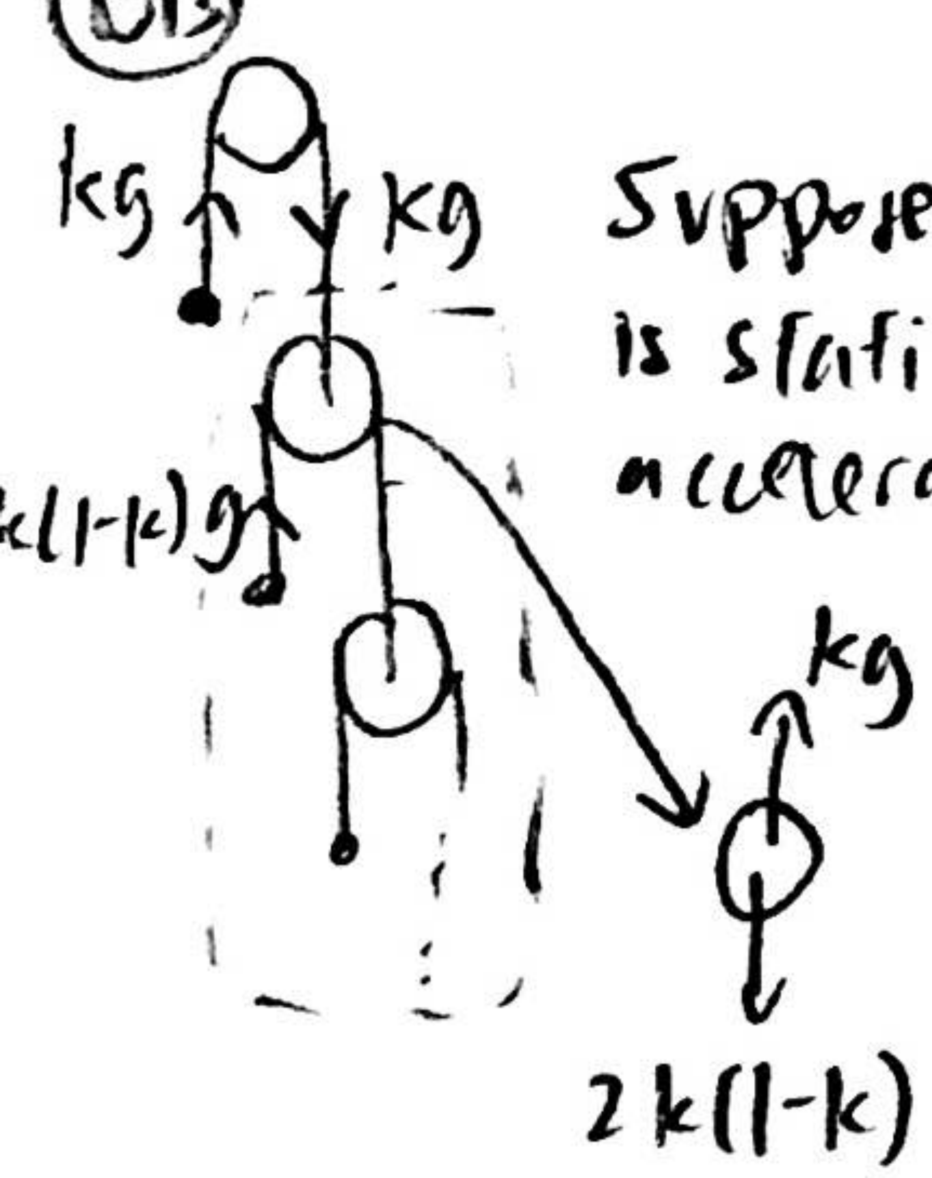
$$v_r dm + m dv = 0$$

$$\Rightarrow \int_M^{M_f}\frac{1}{m}dm = \int_{v_i}^{v} -\frac{1}{v_r}dv$$

$$\Rightarrow \ln\left(\frac{M_f}{M}\right) = \frac{v_i - v}{v_r}$$

$$\Rightarrow \boxed{v = v_i + v_r\ln\left(\frac{M}{M_f}\right)}$$

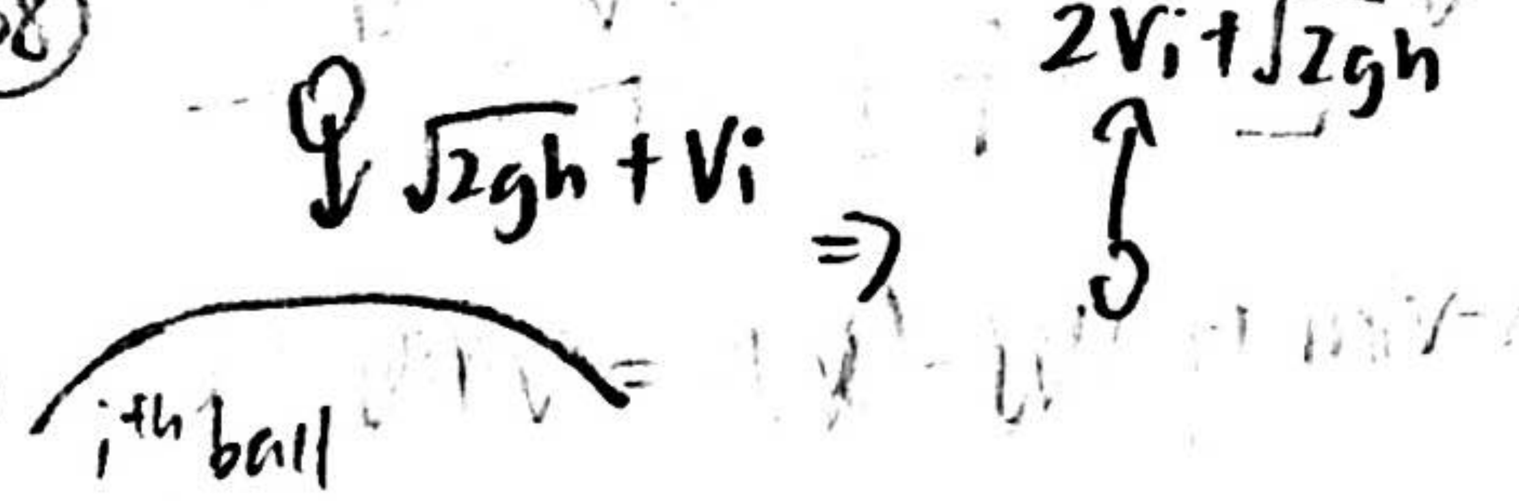
(B13)



Suppose that if a system is stationary, then top mass accelerates at kg, k some constant.

$$kg = 2k(1-k)g$$
$$\Rightarrow 2(1-k)g = 1$$
$$\Rightarrow k = \frac{1}{2}$$
$$\therefore \boxed{\frac{1}{2}g}$$

(B8)



$$\therefore V_{i+1} = 2V_i + \sqrt{2gh}.$$
$$\therefore V_n = (2^n - 1)\sqrt{2gh}.$$

$\therefore$ Height which ball n rises to is

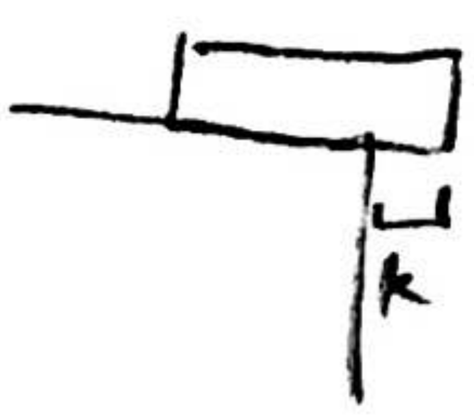
$$\ell + (2^n-1)^2 h$$
$$\ell + (2^n-1)^2 h > 1000 \Rightarrow \boxed{n \geq 6}$$

(B14)

Suppose for N blocks, the max distance reached is $\frac{\ell}{2}(1+\frac{1}{2}+\dots+\frac{1}{N})$.

For $N+1$ blocks

$$\text{dist} = k + \frac{\ell}{2}\left(1+\frac{1}{2}+\dots+\frac{1}{N}\right)$$

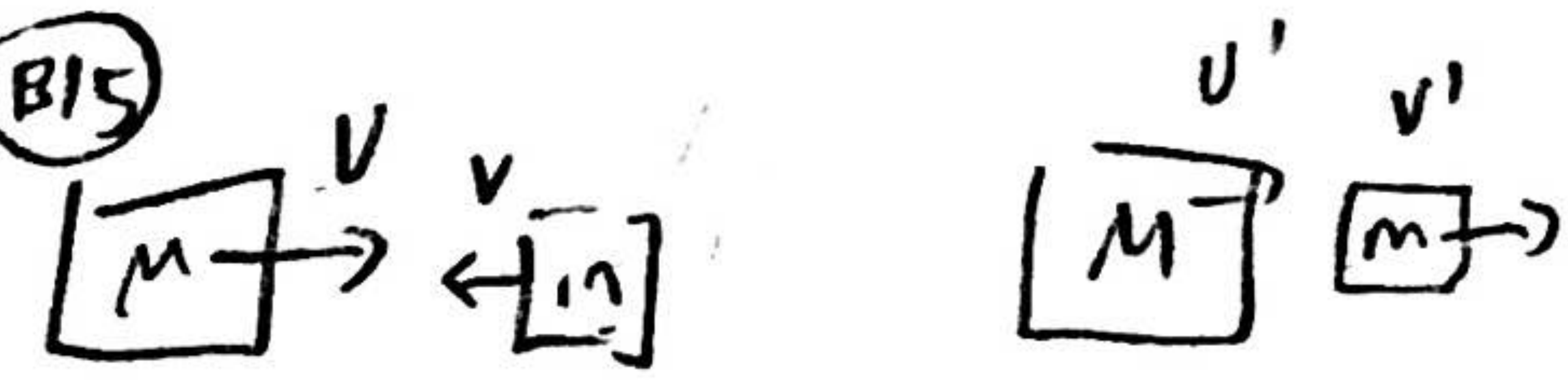


$$CM = kNm + \left(k - \frac{\ell}{2}\right)m \leq 0$$
$$\Rightarrow k \leq \frac{1}{N+1}\frac{\ell}{2}$$

$\Rightarrow$ Total is $\frac{\ell}{2}\left(1+\frac{1}{2}+\dots+\frac{1}{N}+\frac{1}{N+1}\right)$

The reason why this greedy induction works will be covered in lesson.

(B15)



$$U + v = v' - U'$$
$$MU - mv = MU' + mv'$$
$$v' = U' + U + v$$
$$\Rightarrow (M+m)U' = MU - mv - mU - mv$$
$$\Rightarrow U' = \frac{M-m}{M+m}U + -\frac{2m}{M+m}v$$
$$v' = \frac{2M}{M+m}U + \frac{M-m}{M+m}v$$
$$\begin{pmatrix} U' \\ v' \end{pmatrix} = \frac{1}{M+m}\underbrace{\begin{pmatrix} M-m & -2m \\ 2M & M-m \end{pmatrix}}_{\mathcal{M}}\begin{pmatrix} U \\ v \end{pmatrix}$$

Eigenvectors

$$A_1 = \begin{pmatrix} 1 \\ -i\sqrt{\frac{M}{m}} \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 \\ i\sqrt{\frac{M}{m}} \end{pmatrix}$$

Eigenvalues

$$\lambda_1 = \frac{(M-m) + 2i\sqrt{Mm}}{M+m} \quad \lambda_2 = \frac{(M-m) - 2i\sqrt{Mm}}{M+m}$$

where $\theta = \tan^{-1}\left(\frac{2\sqrt{Mm}}{M-m}\right) \approx 2\sqrt{\frac{m}{M}}$

Initial condition : $\begin{pmatrix} U \\ v \end{pmatrix} = \begin{pmatrix} v_0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} U_n \\ v_n \end{pmatrix} = \frac{v_0}{2}\left(\lambda_1^n A_1 + \lambda_2^n A_2\right)$$
$$= \frac{v_0}{2}\left(e^{in\theta}\begin{pmatrix} 1 \\ -i\sqrt{\frac{M}{m}} \end{pmatrix} + e^{-in\theta}\begin{pmatrix} 1 \\ i\sqrt{\frac{M}{m}} \end{pmatrix}\right)$$
$$= v_0\begin{pmatrix} \cos n\theta \\ \sqrt{\frac{M}{m}}\sin n\theta \end{pmatrix}$$

Closest when $\cos n\theta = 0$

$$\Rightarrow n\theta = \frac{\pi}{2} \Rightarrow n = \frac{\pi}{2\left(2\sqrt{\frac{m}{M}}\right)} \approx \boxed{\frac{\pi}{4}\sqrt{\frac{M}{m}}}$$