Special Relativity

Physics Olympiad
Wang Jianzhi

"Ain't healthy for a mammalian body to hop more than 50 jumps at a time. We are about to do 700!"

--- Yondu Udonta

Postulates of Special Relativity

An inertial frame of reference is one that is not undergoing acceleration.

- The laws of Physics are the same in all inertial frames of reference.
- The speed of light in free space has the same value c in all inertial frames of references.

Kinematics

Lorentz Transformation

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Usually, when a person is introduced the Lorentz transformations, he is presented with a bunch of simultaneous equations. However, the following matrix representation is much simpler, more elegant and easier to remember.

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}$$

To obtain the inverse Lorentz transformation, we just need to negate the elements of the matrix that are not on the main diagonal.

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

A good question is how does one remember which matrix to use? I often just imagine myself in the frame of the spaceship. After some time, the space station which I was at previously will measure me at some positive distance, while I am still at 0 in my own reference frame. Therefore, I am in frame F' and the lab frame is F.

Events

In Special Relativity, the most important concept is that of *events*. Every event can be described by a spacetime coordinate. For example, the act of Thor summoning lightning is considered an event. At one inertial frame, this event might happen at (x, ct) while in another inertial frame, this event might happen at (x', ct'). To get from one inertial frame to another, we will need to apply the Lorentz transformation.

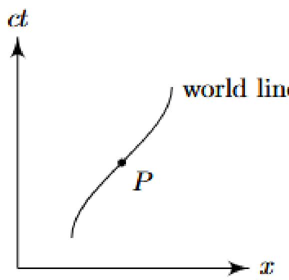
Getting your mind to think in terms of event is key to approaching *any* special relativity problem. I like to call this method *'event analysis'*.

Usually, the question involves two frames of reference with one travelling at a velocity of v relative to the other. For simplicity, we can often denote one event to happen at the origin, that is $x = 0$ and $t = 0$. Note that the spacetime coordinate of $(0, 0)$ will always be mapped to $(0,0)$ by the Lorentz transform in any frame of reference. This will simplify our workings greatly.

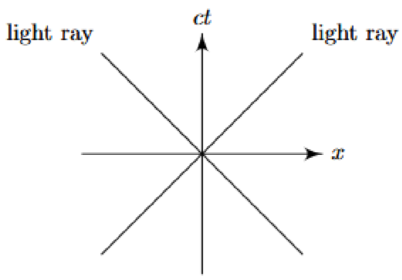
Minkowski Diagram

Minkowski diagrams are graphical illustrations of spacetime properties. An understanding of them are not necessary to do well in Olympiad, but as I have always mentioned, the study of Physics is about the concepts and ideas rather than how well you can solve problems.

We can specify an event by a spacetime coordinate (x, ct) and illustrate it on the Minkowski diagram by plotting it at the coordinate.



Consider a particle P whose motion can be illustrated by a Minkowski diagram as shown above. We can consider the particle to be lighting up firecrackers as it moves, with each lighting being one event. This is called the *world line*.

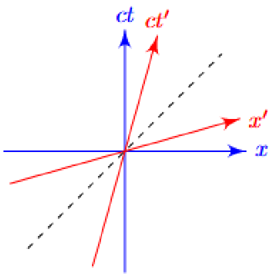


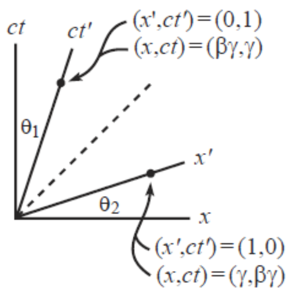
If P is a photon travelling in the x direction, then it must travel at world lines inclined at $45°$.

Consider a frame of reference S' moving at a speed of v relative to S. It must have a new set of axes. However, note that we established earlier that $(0,0)$ is transformed back to itself under the Lorentz transformation. Hence, the origin of the two spacetime diagrams must concur. We then calculate the transformation of one x-unit and ct-unit of S'

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma \\ \gamma\beta \end{pmatrix}$$

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma\beta \\ \gamma \end{pmatrix}$$





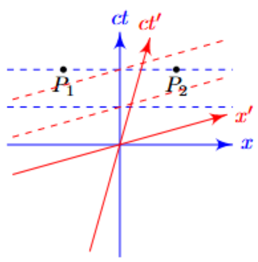
Thus, we can obtain the above diagram, where the axes of x' and ct' are more converged. Note that the axes x' and ct' are not orthogonal, but are symmetrical about the diagonal line. The physical significance is that both frames of reference will agree that light travels on the same world line.

This empowers us to illustrate relativity concepts in a geometrical way. It is a bit like the phasor diagram, which also reduces problems to geometry.

Simultaneity

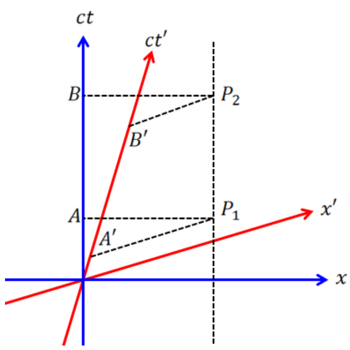
Two events P_1 and P_2 are simultaneous in the frame S if $t_1 = t_2$. In the Minkowski's diagram, it is represented by a line parallel to the x axis. If two events lie on the same line parallel to the x axis (i.e. same value of ct), then they are simultaneous.

In the diagram below, events P_1 and P_2 are simultaneous in S but not in S'. This shows that two observers in different frames of reference will disagree on the simultaneity of events.



Time Dilation

Time is defined as the interval between two events measured by an ideal clock located at the same position for the entire duration. Proper time Δt refers to the time interval between two events that happen at the same position x.



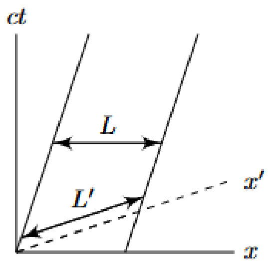
Suppose that event P_1 refers to the pressing of a stopwatch's start button and event P_2 refers to the pressing of a stopwatch's stop button. The proper time is $\overline{AB}$. However, in frame S', an observer will observe that the stopwatch is ticking more slowly. An observer in frame S' will measure $\overline{A'B'}$ instead, even though on the stopwatch it still displays $\overline{AB}$.

$$\begin{pmatrix} \mathrm{d}x' \\ c\,\mathrm{d}t' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ c\,\mathrm{d}t \end{pmatrix} = \begin{pmatrix} -\gamma\beta c\,\mathrm{d}x \\ \gamma c\,\mathrm{d}t \end{pmatrix}$$

But we don't care about the change in x. We only care about the duration between the events. Hence $\mathrm{d}t' = \gamma\,\mathrm{d}t$, which is the usual time dilation equation you all are used to.

Length Contraction

Length is defined as the spatial extent between the head and tail, measured at the same time. The *proper length* is the length measured in an object's rest frame. ... measured when the object is always at rest in that frame. i.e. to say the x coordinate are the same regardless of time.



When solving a problem on length, I always imagine a troubled twin (maybe Quicksilver and Wanda) setting off firecrackers at both ends of the object (say, a train). Suppose Quicksilver sets off his firecracker at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (which is consistent in all frames) and Wanda sets off hers at $\begin{pmatrix} L \\ 0 \end{pmatrix}$. For an observer observing the train moving at a speed of v, he will notice Quicksilver setting off his firecracker at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and Wanda setting off hers at:

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} L \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma L \\ -\gamma\beta L \end{pmatrix}$$

However, in the observer's frame, the two events of the siblings setting off firecrackers are not simultaneous, so we cannot naively take the x' as the length. In fact, the observer will notice Wanda's explosion before Quicksilver's. Since the train is moving at a speed of v, an event that occurred at position γL will be translated to position $\gamma L + (-v)\frac{\gamma\beta L}{c}$ after $\frac{\gamma\beta L}{c}$ time has passed.

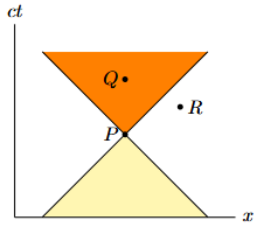
$$\begin{pmatrix} \mathrm{d}x' \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma L + (-v)\dfrac{\gamma\beta L}{c} \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma L\left(1 - \dfrac{v^2}{c^2}\right) \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{L}{\gamma} \\ 0 \end{pmatrix}$$

Many of you will probably feel more familiar with the length contraction formula $L' = \frac{L}{\gamma}$, which is okay. However, you must have a strong understanding on the concept of proper length. Be it now or in the future, you will definitely come back to the above derivation.

Cause and Effect

A student listened to a Special Relativity lecture and felt very enlightened because of it. Is it possible in some frame of reference that the student felt enlightened before he listened to the lecture?

The answer is no. In Special Relativity, nothing (including information) can travel faster than the speed of light. In Minkowski's diagram, by drawing a light cone (lines inclined at $45°$) at an event P, we can know events that must occur after P. In the diagram below, that includes Q. Event R is not within the light cone and therefore is not caused by P. From the light cones, we can similarly tell which events could possibly have influenced the happening of P.



Lorentz Invariance

Suppose an event P happens at coordinate (x, y, z, ct) in S. In another frame F', if you measure the event P to be at (x', y', z', t'), you can expect

$$c'^2t'^2 - x'^2 - y'^2 - z'^2 = c^2t^2 - x^2 - y^2 - z^2$$

$$c^2t^2 - x^2 - y^2 - z^2 = \text{constant}$$

Velocity Addition

If a particle moves with constant velocity of v' in frame S' that is moving at a speed of u.

$$\begin{pmatrix} \mathrm{d}x \\ c\,\mathrm{d}t \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} \mathrm{d}x' \\ c\,\mathrm{d}t' \end{pmatrix}$$

The speed v at which the particle moves in S is:

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\gamma\,\mathrm{d}x' + \gamma\beta c\,\mathrm{d}t'}{\frac{\gamma\beta}{c}\mathrm{d}x' + \gamma\,\mathrm{d}t'} = \frac{v' + \beta c}{\frac{\beta v'}{c} + 1} = \frac{u + v'}{1 + \frac{uv'}{c^2}}$$

$$\boxed{v = \frac{u + v'}{1 + \frac{uv'}{c^2}}}$$

There are several physical implications:

- If $v' = c$, then $v = c$, implying that the speed of light is constant for all frames of reference.
- If $uv' \ll c^2$, then the formula reduces to the Galilean addition of velocities.

Dynamics

4-Vectors

A four-vector is a vector with three "space-like" components and a "time-like" component. It is written in various different notations, but the notation I would like to introduce to you all is the following, because of its simplicity and intuition especially under the dot product operation.

To make them dimensionally consistent and such that they satisfy the Lorentz invariant, we introduce the following form of 4-vector.

$$\begin{pmatrix} x \\ y \\ z \\ ict \end{pmatrix}$$

There are a lot of 4-vectors notations out there, but I really prefer this one, especially with the i in the time component. It has many nice properties, for example if we take the dot product of the vector and itself:

$$\begin{pmatrix} x \\ y \\ z \\ ict \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ ict \end{pmatrix} = x^2 + y^2 + z^2 - c^2t^2 = \text{constant}$$

If A and B are two 4-vectors, then $aA + bB$ or any linear combination is also a 4-vector.

Currently, it might seem like 4-vectors just do some kind of special trick that is pleasing to the eye. However, when used in relativistic dynamics, it is a powerful tool.

4-Momentum

Many of you may already know of the Mass-Energy invariance:

$$p = \gamma m_0 v$$

$$E = \gamma m_0 c^2$$

$$E^2 = p^2c^2 + m_0^2c^4$$

We can check that for photons, $E = pc = \frac{hc}{\lambda}$, which is correct.

Similar to spacetime coordinates, the momentum and energy of a particle can be expressed in a 4-vector as well. Note that the $\frac{1}{c}$ is introduced in the energy term to keep the dimensions of the 4-vector consistent.

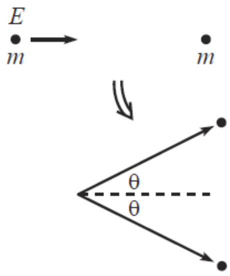
$$\begin{pmatrix} p_x \\ p_y \\ p_z \\ i\frac{E}{c} \end{pmatrix}$$

If we take the dot product of a 4-momentum and itself, we obtain the following equation. Note that the right hand side is just $-m_0^2c^2$, which is a constant and depends on the particle itself. For photons, since $m_0 = 0$, the dot product becomes 0 as well.

$$\begin{pmatrix} p_x \\ p_y \\ p_z \\ i\dfrac{E}{c} \end{pmatrix} \cdot \begin{pmatrix} p_x \\ p_y \\ p_z \\ i\dfrac{E}{c} \end{pmatrix} = p^2 - \frac{E^2}{c^2} = -m_0^2 c^2$$

If we take the dot product of a four vector and itself, we will realize it becomes constant

A particle of mass m and energy E approaches an identical particle at rest. They collide elastically in such a way that they both scatter at an angle θ relative to the incident direction. What is θ in terms of E and m? What is θ in relativistic limits?



Let the initial 4-vectors of the particles be P_1 and P_2. We know $p = \gamma m v = \frac{\sqrt{E^2 - m^2c^4}}{c}$

$$P_1 = \begin{pmatrix} p \\ 0 \\ 0 \\ i\dfrac{E}{c} \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ imc \end{pmatrix} \quad P_1' = \begin{pmatrix} p' \cos\theta \\ p' \sin\theta \\ 0 \\ \dfrac{iE'}{c} \end{pmatrix} \quad P_2' = \begin{pmatrix} p' \cos\theta \\ -p' \sin\theta \\ 0 \\ \dfrac{iE'}{c} \end{pmatrix}$$

From $P_1 + P_2 = P_1' + P_2'$, we can obtain $p' \cos\theta = \frac{p}{2}$ and $E' = \frac{E}{2} + \frac{mc^2}{2}$.

$$P_1' = \begin{pmatrix} \dfrac{p}{2} \\ \dfrac{p}{2}\tan\theta \\ 0 \\ i\dfrac{1}{2}\left(\dfrac{E}{c} + mc\right) \end{pmatrix}$$

By doing dot product of P_1' on itself,

$$P_1' \cdot P_1' = \begin{pmatrix} \dfrac{p}{2} \\ \dfrac{p}{2}\tan\theta \\ 0 \\ i\dfrac{1}{2}\left(\dfrac{E}{c} + mc\right) \end{pmatrix} \cdot \begin{pmatrix} \dfrac{p}{2} \\ \dfrac{p}{2}\tan\theta \\ 0 \\ i\dfrac{1}{2}\left(\dfrac{E}{c} + mc\right) \end{pmatrix} = -m^2c^2$$

Hence, we can solve for θ to obtain:

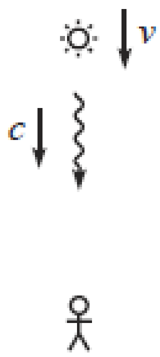
$$\theta = \cos^{-1}\left(\frac{E + mc^2}{E + 3mc^2}\right)$$

"Sir, there are still terabytes of calculations required before an actual flight is ... "
"JARVIS, sometimes you gotta run before you can walk."

--- Tony Stark

Relativistic Doppler's Effect

A source emits flashes at a frequency of f' in its own frame while travelling directly towards you at a speed of v. What frequency do the flashes hit your eye?



Suppose the time between two flashes in the source's frame is $\Delta t'$, which is the proper time. In our frame, this time interval is dilated, meaning the time interval between two flashes will be:

$$\Delta t = \gamma \Delta t'$$

However, because the source is travelling to us at a velocity of v, the distance between the points where two flashes occur is:

$$\Delta x = c\Delta t - v\Delta t = (c - v)\gamma \Delta t$$

The signals travel at a speed of c, so the overall time difference between two signals is:

$$\Delta T = \frac{\Delta x}{c} = \frac{(c - v)\gamma \Delta t}{c} = (1 - \beta)\frac{1}{\sqrt{1 - \beta^2}}\Delta t = \sqrt{\frac{1 - \beta}{1 + \beta}}\Delta t$$

$$f = \sqrt{\frac{1 + \beta}{1 - \beta}} f'$$

The physical significance of this result are:

- If $\beta > 0$, it means the source is moving towards you and $f > f'$. The (non-relativistic) Doppler's effect wins out time dilation effect. The light increases in frequency or is blue-shifted or shifted closer to the blue frequency (the higher end in the frequency spectrum).

- If $\beta < 0$, then $f < f'$ and both effects serve to decrease the frequency. The light is red-shifted or shifted closer to the red frequency (the lower end in the frequency spectrum).

Rapidity

Proper acceleration is defined as the acceleration in a frame S' such that at $t + \mathrm{d}t'$ the spaceship is moving at speed $a_0\,\mathrm{d}t'$ relative to the frame it was in t'. It is the acceleration of a particle in its instantaneous rest frame.

$$a_x' = \frac{a_x}{\gamma^3\left(1 - \frac{u_x v}{c^2}\right)}$$

We define rapidity ϕ as

$$\tanh\phi = \beta = \frac{v}{c}$$

Then the following nice results appear:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\tanh^2\phi}} = \cosh\phi$$

$$\gamma\beta = \cosh\phi \cdot \tanh\phi = \sinh\phi$$

If a spaceship has a proper acceleration of a, then by velocity addition formula:

$$v(t+\mathrm{d}t) = \frac{v(t) + a\,\mathrm{d}t}{1 + \frac{v(t)\,a\,\mathrm{d}t}{c^2}}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = a\left(1 - \frac{v^2}{c^2}\right) \Rightarrow v(t) = c\tanh\left(\frac{at}{c}\right)$$

$$\phi(t) = \frac{1}{c}\int_0^t a(t)\,\mathrm{d}t$$

In Special Relativity, v has an upper bound of c. The nice thing about rapidity ϕ is that it has no upper bound. Since $\tanh\phi = \frac{e^{\phi}-e^{-\phi}}{e^{\phi}+e^{-\phi}}$, then tangent hyperbolic function takes care of the upper bound for us.

A. Paradoxes

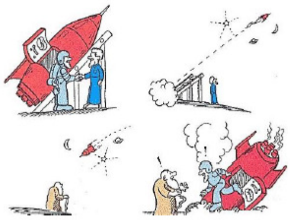
1. [Ladder Paradox] Black Panther carries a 3m pole and travels with speed $0.8c$ to the right. The pole is made of vibranium, which cannot be broken. Thanos creates a trap which looks like a 2m barn, the walls and door are also made of vibranium, which cannot be broke. Initially the door is open for Black Panther to run in. Calculate the Lorentz factor γ.

    a. How long does Thanos think the length of the pole and the length of the barn is?

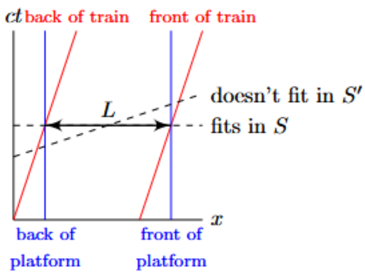
    b. How long does Black Panther think the length of his pole is and the length of the barn?

    c. Will Thanos successfully trap Black Panther in his barn?

2. [Twin Paradox] Quicksilver and Wanda are twins. Wanda took Star-Lord's spaceship away from Earth. When she comes back, Quicksilver aged due to time dilation. However, both Wanda and Quicksilver sees each other as moving. Why does only Quicksilver age and not Wanda?



3. [Train Paradox] A train of length $2L$ travels through a station with a platform of length L. It travels with a velocity v such that $\gamma = 2$. To observers on the platform, the train undergoes length contraction to reach a length of L, so it fits. To observers on the train, the platform contracts to length of $L/2$, so the train doesn't fit. How come?



B. Sample Problems

1. There are two spaceships 2.52 Tm apart. They are travelling towards each other at $0.8c$ and $0.6c$ respectively.

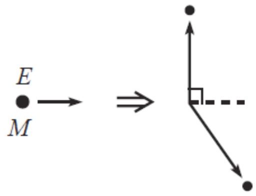
   a. How long does it take for them to collide in your frame?

   b. How long does it take for them to collide in the $0.8c$ frame?

2. An interstellar physics teacher travels at constant velocity v relative to Earth. As her rocket grazers past her class of Earth-bound students, she sends out a signal for them to begin their test. The teacher would like the class to have time T to complete the test. When should she send out a beam of light back to Earth in order to signal the end of the test?

3. A particle of mass m and energy E approaches an identical particle at rest. They collide elastically in such a way that they both scatter at an angle θ relative to the incident direction. What is θ in terms of E and m? What is θ in relativistic limits?

4. A particle of mass M and energy E decays into two identical particles. In the lab frame, one of them is emitted at $90°$ angle. What are the energies of the created particles?



5. A particle of rest mass m_0 is travelling so that its total energy is just twice its rest mass energy. It collides with stationary particle of rest mass m_0 to form a new particle. Use the conservation of four-momentum to determine the rest mass of the new particle.

6. A rocket moves from planet A to planet B. Both planets are stationary with respect to an inertial Earth and according to the rocket it travels a distance d_1. According to the clock in the rocket, the journey takes t_1 hours. Another observer on Earth observes that the distance between two planets is d_2 and the journey takes t_2 hours. Find the relations between d_1 and d_2, t_1 and t_2. Find the velocity of the rocket.

C. Recommended Resources

- Introduction to Classical Mechanics by David J. Morin
- Physics By Example: 200 Problems and Solutions by W.G. Rees.
- 200 Puzzling Physics Problems by P. Gnadig et al.