Mechanics I

Physics Olympiad
Wang Jianzhi

"Oh, I'm sorry. I didn't know how this machine worked..."
--- Peter Quill (Star-Lord)

Problem Solving Skills

Most Physics Olympiad problems are solved in the following pattern:

1. Visualise the problem / Draw a diagram
2. Apply Physics concepts
3. Obtain equations
4. Solve the equations
5. Obtain physical insights
6. Substitute in numerical values

In particular, step 5 meant to check for limiting cases. After all, the essence of Physics Olympiad is to understand the world better, and not to prove you can solve mathematical equations.

- What happens if a variable → ∞?
- What happens if a variable → 0?
- When does the expression become 0? Does it make sense?
- What happens at the starting position? $t = 0$?
- What happens if $m \ll M$?
- Can the denominator be 0?
- Special cases ($m_1 = m_2$, equilateral triangle)

Notice that the step to substitute in numerical values come at the last step. Before this step, you can always check that your equation is dimensionally consistent (a process which we call 'dimensional analysis'). For example, something must be wrong if you obtained a e^{mg} term. After substituting in your numerical values, you lose the dimensions of the variables and a way to verify your steps.

Mathematical Treatment

- Linear separable functions: $g(y)\,\mathrm{d}y = f(x)\,\mathrm{d}x$
- Integration by parts: $\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$
- Approximation
    - $\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$, $\tan\theta \approx \theta$ (small angle approximations)
    - $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$ (Maclaurin series)
    - $f(x) = f(\epsilon) + \frac{f'(\epsilon)}{1!}(x-\epsilon) + \frac{f''(\epsilon)}{2!}(x-\epsilon)^2 + \cdots$ (Taylor series)

Kinematics

- Velocity: $\dot{\boldsymbol{r}} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}$, $\int \dot{\boldsymbol{r}}\,\mathrm{d}t = \boldsymbol{r}$
- Acceleration: $\ddot{\boldsymbol{r}} = \frac{\mathrm{d}\dot{\boldsymbol{r}}}{\mathrm{d}t}$, $\int \ddot{\boldsymbol{r}}\,\mathrm{d}t = \dot{\boldsymbol{r}}$
- $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x}$

Dynamics

- Momentum: $\boldsymbol{p} = m\boldsymbol{v} = m\dot{\boldsymbol{r}}$ (m is the inertial mass of the particle)

- The equation of motion for a particle subjected to a force $\boldsymbol{F}$ is $\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\dot{\boldsymbol{r}})}{\mathrm{d}t} = m\ddot{\boldsymbol{r}} + \frac{\mathrm{d}m}{\mathrm{d}t}\dot{\boldsymbol{r}}$. For constant mass, $\boldsymbol{F} = m\ddot{\boldsymbol{r}}$. Usually, since $\boldsymbol{F}$ is a function of $\dot{\boldsymbol{r}}$, $\boldsymbol{r}$ and t, we have a second-order differential equation for $\boldsymbol{r}$.

- Conservation of momentum: In an isolated system, the total momentum is conserved.

- Kinetic energy: $T = \frac{1}{2}m\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = \frac{1}{2}m|\dot{\boldsymbol{r}}|^2$
- Potential energy: Suppose that a particle is in a force field, where the force depends on position only. We can define the potential energy to be a function $V(\boldsymbol{r})$ such that
  - $\boldsymbol{F} = -\nabla V$
  - $F = -\frac{\mathrm{d}V}{\mathrm{d}x}$ (one dimensional case)
- Total energy: $E = T + V$

- Conservation of energy: In an isolated system, the total energy is conserved.

- Power is the rate at which work is done on a particle by a force. If the particle is only affected by the force, then power will be equal to the rate of change of its kinetic energy.

$$\frac{\mathrm{d}T}{\mathrm{d}t} = m\ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} = \boldsymbol{F} \cdot \dot{\boldsymbol{r}} = \boldsymbol{F} \cdot \boldsymbol{v} = P$$

- Work done on a particle by a force is the change in kinetic energy caused by the force:

$$W = \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int_C \boldsymbol{F} \cdot \dot{\boldsymbol{r}}\,\mathrm{d}t = \int_C \boldsymbol{P}\,\mathrm{d}t$$

A particle or a body is said to be *in equilibrium/static* when all forces acting on it balance and it is not in motion. Then, the sum of forces in any direction is 0. Equivalently, the vector sum of all forces is 0.

- Center of mass $x_{cg} = \frac{1}{M}\int x\,\mathrm{d}m$ where $M = \int \mathrm{d}m$

For forces acting on an extended body, it matters through which point(s) the forces act, as compared to forces acting on a particle.

- Definition of torque: $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}$ (defined with respect to an origin)

For the system to be in equilibrium, the total torque about any point is also 0. Why doesn't it matter which point we choose to take moments from?

- Other definitions
  - Coefficient of restitution: Ratio of final to initial relative velocity between two objects after they collide.
  - Strain: $\epsilon = \Delta l / l$
  - Stress: σ, same unit as pressure
  - Modulus: $E = \sigma / \epsilon$
  - Static friction: when two objects are at rest relative to each other. $F \leq \mu_s N$. Note the inequality sign.
  - Kinetic friction: when two objects are moving relative to each other. Usually, $F = \mu_k N$ where N is the normal force. Direction is opposite the motion.

Differential Analysis

One important technique in solving problems involving a dynamical system is differential analysis. There are several variants of this technique, but all of them involves considering a very small (we call it *infinitesimal*) quantity.

- Divide the system into very small pieces and analyze the infinitesimal part on its own and compare it with the overall behaviour of the whole system.
- Consider an infinitesimal displacement $\mathrm{d}s$ and analyze the infinitesimal motion in terms of all the forces and constraint.
- Conservation of string:
  - A displacement of dl on one side must be matched with a displacement of dl on the other.
  - For a massless string, the net force on an infinitesimal length $\mathrm{d}l$ must be 0. Otherwise, that infinitesimal length will experience infinite acceleration.

A. Sample Problems

1. The reality stone of mass m is subjected to a force $F(t) = ma_0 e^{-bt}$. The initial position and speed are 0. Find $x(t)$.

2. Two particles of masses m and M are attached to a light inextensible string that passes over a light frictionless fixed pulley. Find the accelerations of each particle and the force on the axle of the pulley.

3. A table tennis ball is placed on top of a basketball. The basketball is held at height h above the ground. Find the velocity of the table tennis ball immediately after the basketball bounces off the ground elastically.

4. Find the location of the CM of a hollow hemispherical shell, with uniform mass density and radius R.

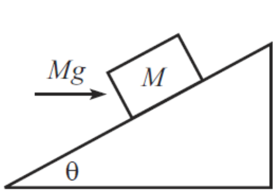
5. A rope with length L and mass density per unit length ρ is suspended vertically from one end. Find the tension as a function of height along the rope.

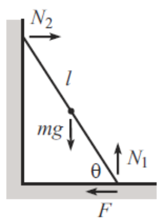
6. A rope wraps an angle θ around a pole. You grab one end and pull with a tension T_0. The other end is attached to a large object, say Star-Lord's spaceship. If the coefficient of static friction between the rope and the pole is μ, what is the largest force the rope can exert on the boat, if the rope is not to slip around the pole?

B. In Class Problems

1. The power stone is thrown upward with initial speed v_0. Assume that the drag force from the air is $F_d = -m\alpha v$. What time does it reach the maximum height? What is the maximum height that it will reach?

2. A block of mass M rests on a fixed plane inclined at an angle θ. Gamora applies a horizontal force of Mg on the block, as shown in the figure. Assume that the friction force between the block and the plane is large enough to keep the block at rest. What are the normal and friction forces (N, F_f) that the plane exert on the block? If the coefficient of static friction is μ, for what range of angles θ will the block in fact remain at rest?





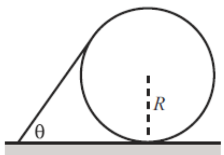
3. A particle is in equilibrium under the action of three forces of magnitudes $3P$, $5P$ and $7P$. Show that the angle between the forces with magnitudes $3P$ and $5P$ is $\frac{\pi}{3}$.

4. A ladder leans against a frictionless wall. If the coefficient of friction with the ground is μ, what is the smallest angle the ladder can make with the ground and not slip?

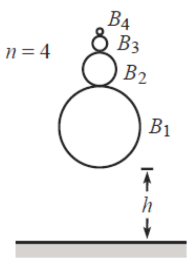
5. A light rod of length a stands on rough ground leaning against a smooth wall, and inclined at an angle α to the horizontal. A particle of weight W is placed a distance $\frac{2}{3}a$ up the rod. What is the magnitude of the normal reaction of the wall on the end of the rod?

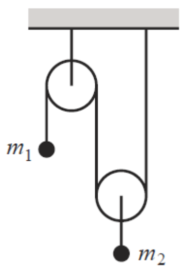
6. Given a semi-infinite stick, determine how its density should depend on position so that it has the following property: if the stick is cut at an arbitrary location, the remaining semi-infinite piece will balance on a support that is located a distance l from the end.

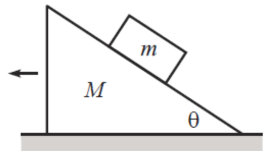
7. A stick of mass density per unit length ρ rests on a circle of radius R, as shown in the figure below on the left. The stick makes an angle θ with the horizontal and is tangent to the circle at its upper end. Friction exists ast all points of constant and is able to keep the system at rest. Find the friction between the ground and the circle.



8. Consider n balls $B_1, \ldots, B_n$ having masses of $m_1, \ldots, m_n$ with $(m_1 \gg m_2 \gg \cdots \gg m_n)$ standing in a vertical stack. The bottom of B_1 is a height h above the ground and the bottom of B_n is a height $h + l$ above the ground. The balls are dropped. In terms of n, to what height does the top ball bounce? If $h = 1$ meter, what is the minimum number of balls needed for the top one to bounce to a height of at least 1 kilometer? Assume the balls still bounce elastically and assume l is negligible.







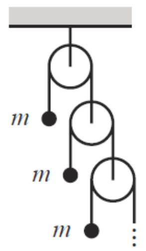
9. Consider the pulley system in the figure below, with masses m_1 and m_2. The strings and pulleys are massless. What are the accelerations of the masses? What is the tension in the string?

10. A block of mass m is held motionless on a frictionless plane of mass M and angle of inclination θ. The plane rests on a frictionless horizontal surface. The block is released. What is the horizontal acceleration of the plane?
11. We are at rest watching a rocket of mass M travelling at a velocity v_i. The rocket then starts ejecting mass gradually at a rate of k until it reaches a final mass of M_f. The

mass is ejected from the rocket's perspective at a speed of v_r. What is the final velocity of the rocket?

12. A uniform layer of snow whose surface is a rectangle rests on a slope of inclination α to the horizontal. The adhesion is just enough to hold the snow in place. At a certain instant an avalanche starts by the uppermost line of snow moving downwards and collecting with it the snow that it meets on its way down. Assuming no friction exists between the snow and the slope, show that the avalanche has a constant acceleration of $\frac{1}{3} g \sin \alpha$. (The snow picked up is proportional to distance travelled).

13. Consider the infinite Atwood's machine as shown in the figure. A string passes over each pulley, with one end attached to a mass and the other end attached to another pulley. All the masses are equal to m and all the pulleys and strings are massless. The masses are held fixed and then simultaneously released. What is the acceleration of the top mass?



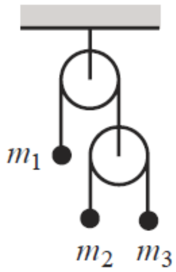
14. N blocks of length l are stacked on top of each other at the edge of a table, as shown in the figure below ($N = 4$ in this case). What is the largest horizontal distance the rightmost point on the top block can hang out beyond the table? How does your answer behave for $N \to \infty$?

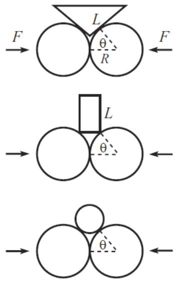
15. A block with large mass M slides with speed U on a frictionless table toward a wall. It collides elastically with a ball of mass m ($m \ll M$), which is initially at rest. The ball slides toward the wall, bounces elastically and then proceeds to bounce back and forth between the block and the wall. After a number of bounces, the big mass' velocity will eventually change direction. How many bounces does this take, depending on the ratio of $\frac{M}{m}$?

C. Take Home Problems

1. A double Atwood machine is shown in the figure below, with masses m_1, m_2 and m_3. Find the accelerations of the masses.

2. Each of the following planar objects is placed, as shown in the figure, between two frictionless circles of radius R. The mass density per unit area of each object is σ, and the radii to the points of contact make an angle θ with the horizontal. For each case, find the horizontal force that must be applied to the circles to keep them together. For what θ is this force maximum or minimum?
    a. An isosceles triangle with common side length L
    b. A rectangle with height L
    c. A circle





3. A large number of sticks (with mass density per unit length) and circles (with radius R) lean on each other, as shown in the above figure on the right. Each stick makes an angle θ with the horizontal and is tangent to the next circle at its upper end. The sticks are higned to the ground, and every other surface is frictionless (unlike in the previous problem). In the limit of a very large number of sticks and circles, what is the normal force between a stick and the circle it rests on, very far to the right? Assume that the last circle leans against a wall to keep it from moving.

Recommended Resources

- College Physics by Hugh D Young
- Introduction to Classical Mechanics by David J. Morin
- Physics By Example: 200 Problems and Solutions by W.G. Rees
- 200 Puzzling Physics Problems by P. Gnadig et al