Mechanics III (Solutions)

1. $m\ddot{x}+kx=0,\ k>0$

$\Rightarrow m\lambda^2+k=0 \Rightarrow \lambda=\pm\sqrt{\frac{k}{m}}\,i$

$\therefore x(t)=e^0\left[A\cos\sqrt{\frac{k}{m}}t+B\sin\sqrt{\frac{k}{m}}t\right]$

$= C\cos\left(\sqrt{\frac{k}{m}}t+\phi\right)$

$\therefore \boxed{x(t)=A\cos\left(\sqrt{\frac{k}{m}}t+\phi\right)}$

2. $m\ddot{x}+c\dot{x}+kx=0$

$\Rightarrow m\lambda^2+c\lambda+k=0$ $\qquad \omega^2=\frac{k}{m}$

$\Rightarrow \lambda=\frac{-c\pm\sqrt{c^2-4km}}{2m}=-\frac{c}{2m}\pm\frac{1}{2}\sqrt{\left(\frac{c}{m}\right)^2-4\omega^2}$

Let $b=\frac{c}{2\sqrt{km}} \Rightarrow \lambda=\omega(-b\pm\sqrt{b^2-1})$.

Case 1: $b^2-1>0$

$\Rightarrow \boxed{x=Ae^{\omega(-b+\sqrt{b^2-1})t}+Be^{\omega(-b-\sqrt{b^2-1})t}}$

Since $-b\pm\sqrt{b^2-1}<0$, the amplitude decreases exponentially $\Rightarrow$ over damping

Case 2: $b^2-1=0$

$\Rightarrow \boxed{x=(A+Bt)e^{-\omega t}}$

Case 3: $b^2-1<0$

$\Rightarrow a=\omega(-b\pm i\sqrt{1-b^2})$

$x=e^{-b\omega t}\left[C\cos(\sqrt{1-b^2}\,\omega t)+D\sin(\sqrt{1-b^2}\,\omega t)\right]$

$\Rightarrow \boxed{x=e^{-b\omega t}\left[R\cos(\sqrt{1-b^2}\,\omega t+\phi)\right]}$

$\Rightarrow$ Amplitude is capped by $e^{-b\omega t}$ (strictly decreasing)

3. The complementary function is obtained in qn 2.

For particular integral, guess that

$x=A\cos(\Omega t+B)+B\sin(\Omega t+B)$

$\Rightarrow \dot{x}=-A\Omega\sin(\Omega t+B)+B\Omega\cos(\Omega t+B)$

$\Rightarrow \ddot{x}=-A\Omega^2\cos(\Omega t+B)-B\Omega^2\sin(\Omega t+B)$

$\Rightarrow \cos(\Omega t+B)\left[-Am\Omega^2+Bc\Omega+Ak\right]$
$+\sin(\Omega t+B)\left[-Bm\Omega^2-Ac\Omega+Bk\right]=0 \quad \forall t$

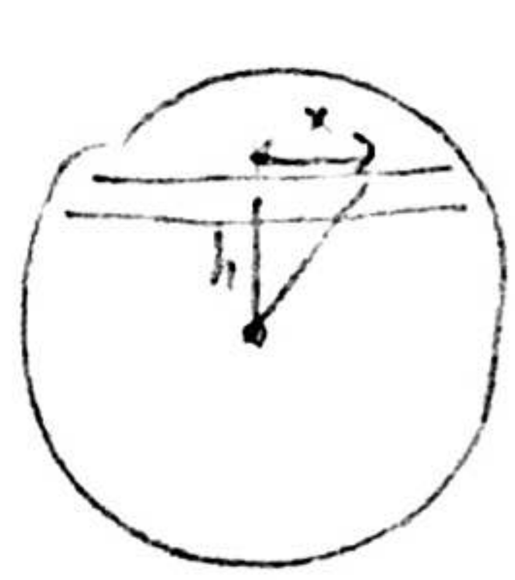
$\Rightarrow -Bm\Omega^2-Ac\Omega+Bk=0$

$-Am\Omega^2+Bc\Omega+Ak=F_0$

$$\therefore \boxed{A=\frac{k-m\Omega^2}{c^2\Omega^2+(k-m\Omega^2)}F_0 \qquad B=\frac{c\Omega}{c^2\Omega^2+(k-m\Omega^2)^2}F_0}$$

Resulting amplitude $=\dfrac{F_0}{\sqrt{c^2\Omega^2+(k-m\Omega^2)^2}}$

4.




$\oint \vec{g}\cdot d\vec{A}=-4\pi M_{enc}G$

$\Rightarrow g(4\pi r^2)=-4\pi\rho\frac{4}{3}\pi r^3 G$

$\Rightarrow g=\frac{4}{3}\rho\pi G r$

$g$ in the direction of tunnel $=\frac{4}{3}\rho\pi G\sqrt{x^2+h^2}\cdot\frac{x}{\sqrt{x^2+h^2}}$

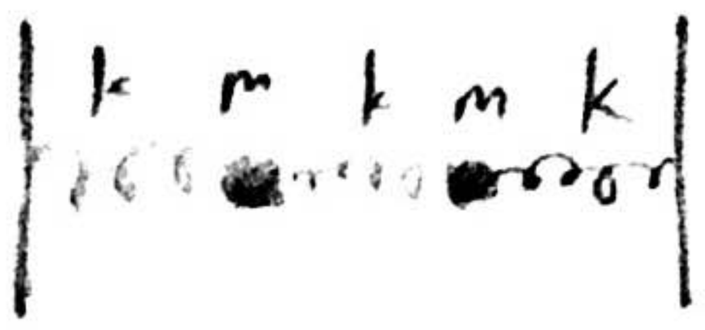
$=\frac{4}{3}\rho\pi G x$

$m\ddot{x}=-m\frac{4}{3}\rho\pi G x$

$\Rightarrow \omega=\sqrt{\frac{4}{3}\rho\pi G} \Rightarrow t=\frac{1}{2}\frac{2\pi}{\omega}=\frac{\pi}{\omega}$

$=\boxed{\sqrt{\frac{3}{4}\frac{\pi}{\rho G}}}$

7.




$x_1(t)$, $x_2(t)$ be the positions of left and right mass relative to their eq.

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1)$$
$$m\ddot{x}_2 = -kx_2 - k(x_2 - x_1)$$

let $x_1 = A_1 e^{i\alpha t}$, $x_2 = A_2 e^{i\alpha t}$

$$\ddot{x}_1 = -2w^2 x_1 + w^2 x_2$$
$$\ddot{x}_2 = -2w^2 x_2 + w^2 x_1$$

$$\Rightarrow -\alpha^2 \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} -2w^2 & w^2 \\ w^2 & -2w^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

$$\Rightarrow \begin{vmatrix} -2w^2 + \alpha^2 & w^2 \\ w^2 & -2w^2 + \alpha^2 \end{vmatrix} = 0$$

$$\Rightarrow \alpha^4 - 4w^2\alpha^2 + 3w^2 = 0$$

$$\Rightarrow \alpha = \pm w \text{ or } \pm\sqrt{3}w.$$

Sub $\alpha$ as $\pm w \Rightarrow -w^2 A_1 + w^2 A_2 = 0$

$$\Rightarrow \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = A\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

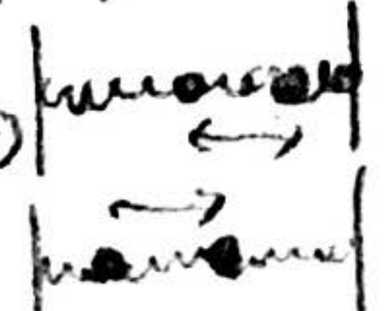
Sub $\alpha$ as $\pm\sqrt{3}w \Rightarrow w^2 A_1 + w^2 A_2 = 0$

$$\Rightarrow \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = A\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

∴ The two normal modes are

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\left(\frac{k}{m}\right)t} + A_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\left(\frac{k}{m}\right)t}$$

$$= B_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos\left(\sqrt{\tfrac{k}{m}}t + \phi_1\right) \Rightarrow$$



$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\sqrt{3}\left(\frac{k}{m}\right)t} + A_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\sqrt{3}\frac{k}{m}t}$$

$$= B_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos\left(\sqrt{\tfrac{3k}{m}}t + \phi_2\right)$$

Any other motion is a superposition.

5.

from 4 $\quad t = \sqrt{\frac{3}{4}\frac{\pi}{\rho G}} = \sqrt{\frac{R^3\pi^2}{MG}}$

$$g = \frac{GM}{R^2} \Rightarrow \boxed{t = \pi\sqrt{\frac{R}{g}}}$$

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$$F = \int dF = -\int_R^\infty \frac{G(\sigma 2\pi r\,dr)m}{(r^2+x^2)}\frac{x}{\sqrt{r^2+x^2}}$$

$$= -\int_R^\infty (2G\sigma\pi m x)\frac{r}{(r^2+x^2)^{\frac{3}{2}}}\,dr$$

$$= -(2\pi\sigma G m x)\left[(r^2+x^2)^{-\frac{1}{2}}(-2)\frac{1}{2}\right]_R^\infty$$

$$= \boxed{-2\pi\sigma G m x\frac{1}{\sqrt{x^2+R^2}}}$$

$$\frac{x}{\sqrt{x^2+R^2}} = \frac{\frac{x}{R}}{\sqrt{1+\left(\frac{x}{R}\right)^2}} \approx \frac{x}{R}\left(1 - \frac{1}{2}\frac{x^2}{R^2} + \cdots\right) = \frac{x}{R}$$

$$\Rightarrow F = \boxed{-2\pi G\sigma m\frac{x}{R}} = m\ddot{x}$$

$$\Rightarrow \ddot{x} = -\frac{2\pi G\sigma}{R}x \quad \therefore \boxed{w = \sqrt{\frac{2\pi G\sigma}{R}}}$$

$$F = -\frac{dU}{dx} \Rightarrow U = -\int_0^r F\,dx$$

$$U = -\int_0^r -2\pi\sigma G m x\frac{1}{\sqrt{x^2+R^2}}\,dx$$

$$= 2\pi G m\sigma\left[\sqrt{x^2+R^2}\right]_0^r = 2\pi G m\sigma\left[\sqrt{r^2+R^2} - R\right]$$

(at) $\frac{1}{2}mv^2 = 2\pi G m\sigma\left[\sqrt{x^2+R^2} - R\right]$

$$\Rightarrow \boxed{v = 2\sqrt{G\pi\sigma\left[\sqrt{x^2+R^2} - R\right]}}$$

For $x >> R$, $\boxed{v = 2\sqrt{G\pi\sigma x}}$

4.


We can assume the point diametrically opposite to the spring to be relatively stationary.

Inertia about point $= \frac{1}{2}mr^2 + mr^2$
$= \frac{3}{2}mr^2$

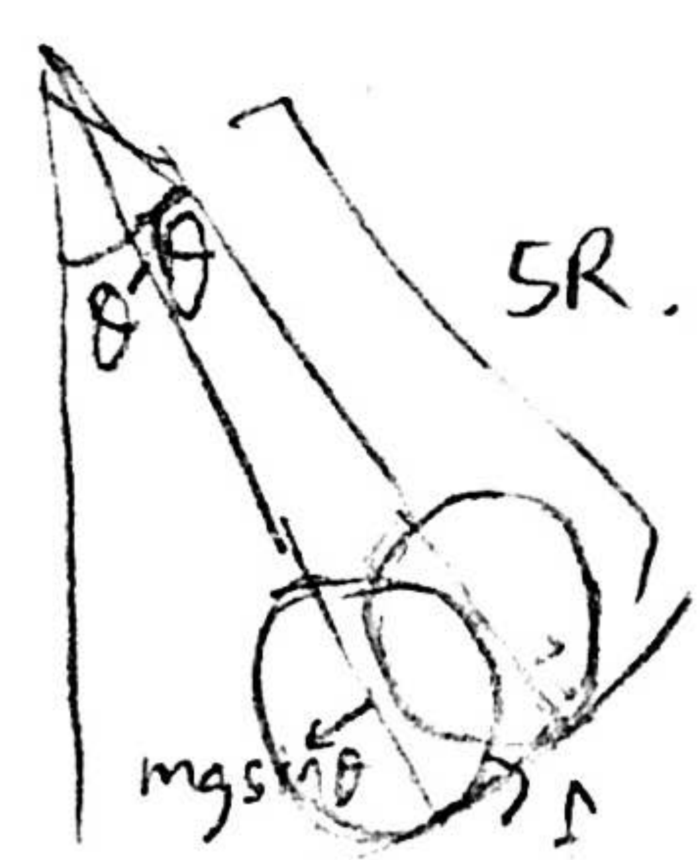
$I\ddot{\theta} = \tau = -\underbrace{k(2x)}_{\text{extension of spring}}\underbrace{2r}_{\text{distance to fixed pt}} = -4xrk$

By geometry, $x = r\theta$

$\Rightarrow \frac{3}{2}mr^2\ddot{\theta} = -4r^2k\theta \Rightarrow \ddot{\theta} = -\frac{8}{3}\frac{k}{m}$

$\omega = \sqrt{\frac{8}{3}\frac{k}{m}} \Rightarrow \boxed{T = \pi\sqrt{\frac{3m}{2k}}}$

7.




$5R\,d\theta = R\,d\phi$
$\Rightarrow \dot{\phi} = 5\dot{\theta}$
$\Rightarrow \omega = \dot{\phi} - \dot{\theta} = 4\dot{\theta}$

$KE = \frac{1}{2}m(4R\dot{\theta})^2 + \frac{1}{2}I\omega^2$
$= \left[8mR^2 + \frac{16}{5}mR^2\right]\dot{\theta}^2 = \frac{56}{5}mR^2\dot{\theta}^2$

$PE = mg(4R(1-\cos\theta)) \approx 2mgR\theta^2$

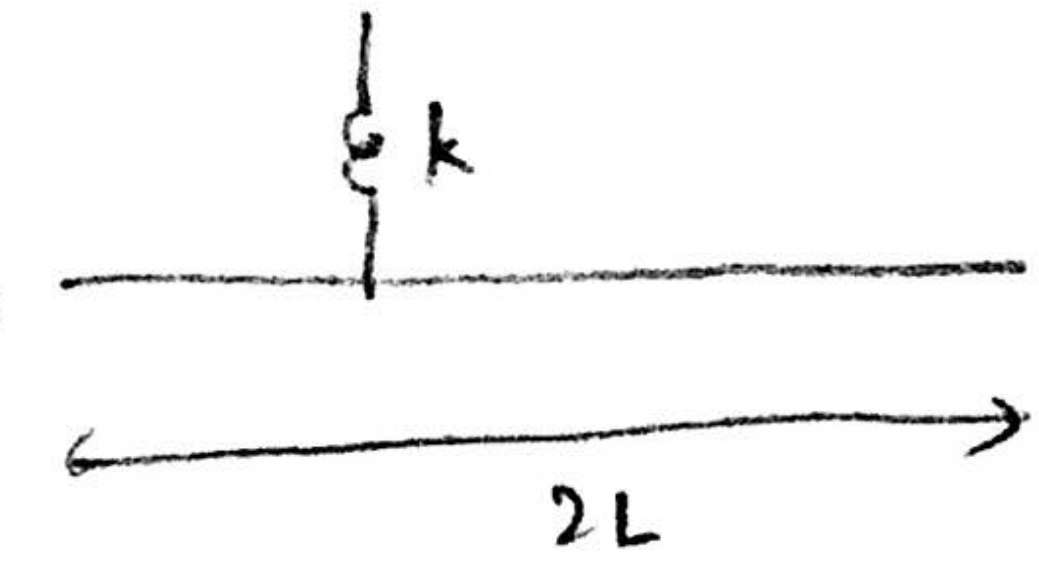
$\therefore \frac{dE}{dt} = 0$

$\Rightarrow \frac{56}{5}mR^2 2\dot{\theta}\ddot{\theta} + 2mgR\,2\theta\dot{\theta} = 0$

$\Rightarrow \frac{56}{5} \cdot 2R\ddot{\theta} + 4g\theta = 0$

$\Rightarrow \omega = \sqrt{\frac{10}{56}\frac{g}{R}} \Rightarrow \boxed{T = 2\pi\sqrt{\frac{28R}{5g}}}$

6.




Moment of inertia of the stick at 0

$= \frac{1}{3}m(2L)^2 = \frac{4}{3}mL^2$.

$I\ddot{\theta} = -kx$ ($x$: additional extension of spring from equilibrium)

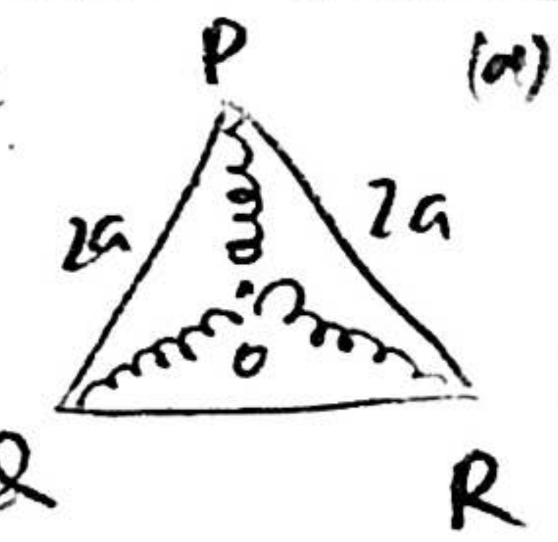
By geometry, $x = a\theta$

$\Rightarrow I\ddot{\theta} = -ka\theta$

$\Rightarrow \frac{4}{3}mL^2\ddot{\theta} = -ka\theta \Rightarrow \ddot{\theta} = -\frac{3}{4}\frac{ka}{mL}\theta$

$\Rightarrow \omega = \sqrt{\frac{3}{4}\frac{ka}{mL}} = \boxed{\sqrt{\frac{3}{4}\frac{kag}{WL}}}$

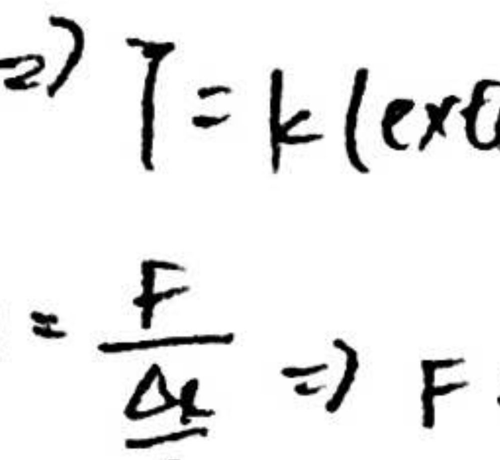
4. (a)




$PO = a \cdot \frac{2}{\sqrt{3}}$

$\Rightarrow$ extension $= \boxed{\frac{2}{\sqrt{3}}a - \ell}$

(b) The net force on mass points downwards




$RX^2 = x^2 + \frac{4}{3}a^2 - 2x\frac{2}{\sqrt{3}}a\left(-\frac{1}{2}\right)$
$= x^2 + \frac{4}{3}a^2 + \frac{2}{\sqrt{3}}xa$

$\Rightarrow$ Extension $= \sqrt{\frac{4}{3}a^2 + \frac{2}{\sqrt{3}}ax + x^2} - \ell$

$\Rightarrow T = k(\text{extension}) = \boxed{\frac{\lambda}{\ell}\left(\sqrt{\frac{4}{3}a^2 + \frac{2}{\sqrt{3}}ax + x^2} - \ell\right)}$

$\lambda = \frac{F}{\frac{\Delta\ell}{\ell}} \Rightarrow F = \left(\frac{\lambda}{\ell}\right)\Delta\ell$

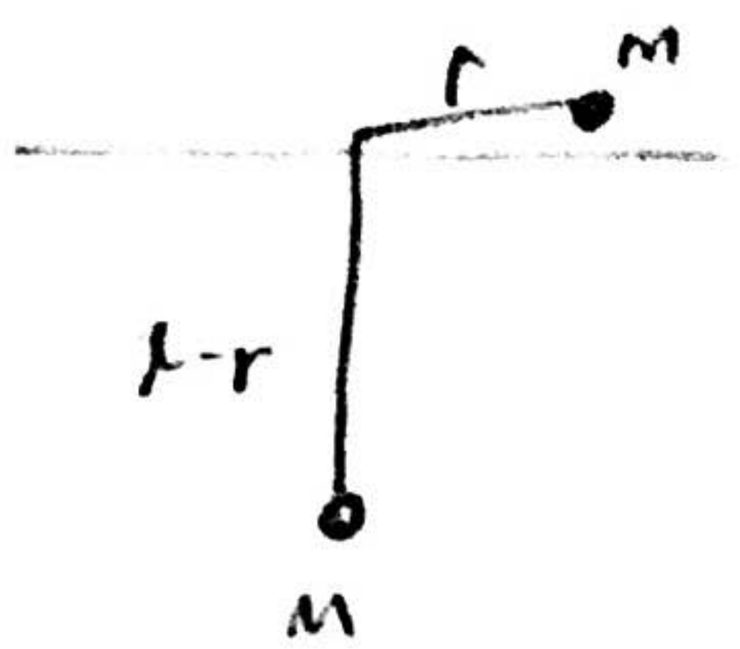
(c) $m\ddot{x} = \frac{\lambda}{\ell}\left(\frac{2}{\sqrt{3}}a - \ell - x\right) - 2T\frac{x + \frac{1}{\sqrt{3}}a}{\sqrt{\frac{4}{3}a^2 + \frac{2}{\sqrt{3}}ax + x^2}}$

$= \frac{\lambda}{\ell}\left(\frac{2}{\sqrt{3}}a - \ell - x - 2\left(x + \frac{1}{\sqrt{3}}a\right) + \frac{2\ell\left(x + \frac{1}{\sqrt{3}}a\right)}{\sqrt{\frac{4}{3}a^2 + \frac{2}{\sqrt{3}}ax + x^2}}\right)$

continued ↓

8

(a)




$$T = \tfrac{1}{2}M(\dot{l-r})^2 + \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}m(r\dot{\theta})^2$$

$$= \tfrac{1}{2}M\dot{r}^2 + \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2$$

$$V = -Mg(l-r)$$

$$L = T - V = \tfrac{1}{2}M\dot{r}^2 + \tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 + Mg(l-r)$$

r) $\frac{\partial L}{\partial r} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)$

$\Rightarrow mr\dot{\theta}^2 - Mg = \frac{d}{dt}(M\dot{r} + m\dot{r})$

$\Rightarrow \boxed{(M+m)\ddot{r} = mr\dot{\theta}^2 - Mg}$

$\theta$) $\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) \Rightarrow \boxed{0 = \frac{d}{dt}(mr^2\dot{\theta})}$

(same as CoAM).

(b) Circular motion $\Rightarrow$ radius stays constant

$\Rightarrow \dot{r} = \ddot{r} = 0$

$\Rightarrow mr\dot{\theta}^2 - Mg = 0 \Rightarrow \boxed{mr\dot{\theta}^2 = Mg}$

c) $\frac{d}{dt}(mr^2\dot{\theta}) = 0 \Rightarrow L = mr^2\dot{\theta}$ for some constant $L$

$\dot{\theta} = \frac{L}{mr^2}$

$(M+m)\ddot{r} = \frac{L^2}{mr^3} - Mg$

Consider $r \to r + dr$

$(M+m)(\ddot{r} + \ddot{dr}) = \frac{L^2}{m(r+dr)^3} - Mg$

$\Rightarrow (M+m)\ddot{dr} = \frac{L^2}{mr^3}\left(1 - 3\frac{dr}{r} - 1\right)$

$\Rightarrow (M+m)\ddot{dr} = -\frac{3L^2}{mr^4}dr$

$$\omega = \sqrt{\frac{3L^2}{m(M+m)r^4}} = \sqrt{\frac{3}{m(M+m)r^4}\cdot m^2r^4\frac{Mg}{mr}} = \sqrt{\frac{3Mg}{(M+m)r}}$$

$$m\ddot{x} = \frac{\lambda}{l}\left(-3x - l + \frac{2l\left(x + \frac{1}{\sqrt{3}}a\right)}{\sqrt{\frac{4}{3}a^2 + \frac{2}{\sqrt{3}}ax + x^2}}\right)$$

$$= \frac{\lambda}{l}\left(-3x - l + \frac{2l\left(\frac{1}{2} + \frac{\sqrt{3}x}{2a}\right)}{\sqrt{1 + \frac{\sqrt{3}}{2}\frac{x}{a} + \frac{3}{4}\left(\frac{x}{a}\right)^2}}\right)$$

$$= \frac{\lambda}{l}\left(-3x - l + 2l\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\frac{x}{a}\right)\left(1 - \frac{1}{2}\frac{\sqrt{3}}{2}\frac{x}{a} - \dots\right)\right)$$

$$= \frac{\lambda}{l}\left(-3x - l + l + \sqrt{3}l\frac{x}{a} - \frac{\sqrt{3}}{4}\frac{x}{a}l\right)$$

$$= \frac{\lambda}{l}x\left(-3 + \sqrt{3}\frac{l}{a} - \frac{\sqrt{3}}{4}\frac{l}{a}\right) = \frac{\lambda}{l}x\left(\frac{3\sqrt{3}}{4}\frac{l}{a} - 3\right)$$

$$\Rightarrow \ddot{x} = -\frac{\lambda}{lm}x\left(3 - \frac{3\sqrt{3}}{4}\frac{l}{a}\right)$$

$$\Rightarrow \omega = \sqrt{\frac{\lambda 3(4a - 3\sqrt{3}l)}{lm4a}}$$

$$\Rightarrow \boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{4mla}{3(4a - 3\sqrt{3}l)\lambda}}}$$

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(a) $M(a\sin\beta) - m(a - a\sin\beta) = 0$

$\Rightarrow \sin\beta(Ma + ma) = ma$

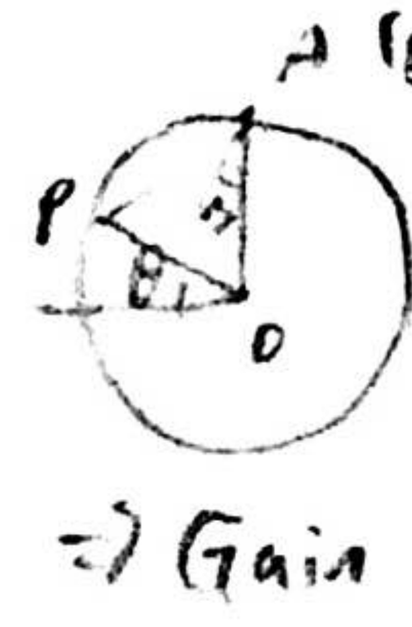
$\Rightarrow \boxed{\sin\beta = \frac{m}{M+m}}$

$AP^2 = AO^2 + OP^2 - 2\,AO\cdot OP\cos(90° - \beta)$

$= 2a^2 - 2a^2\sin\beta$

$\Rightarrow AP = \sqrt{2}a\sqrt{1 - \sin\beta} \Rightarrow \boxed{\frac{AP}{a} = \frac{2M}{M+m}}$

(b)

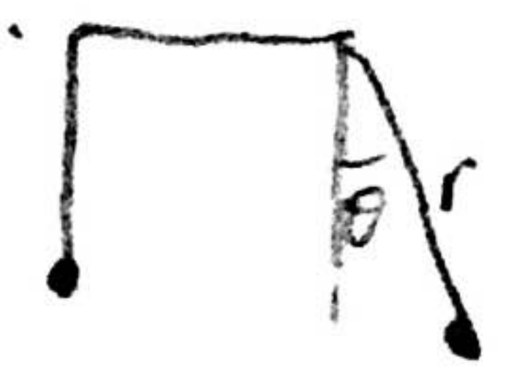



The CG is displaced $a\cos\beta(1 - \cos\theta)$

$\Rightarrow$ Gain in potential energy $= (m+M)ga\cos\beta(1 - \cos\theta)$

The kinetic energy is $\underbrace{\tfrac{1}{2}I\dot{\theta}^2}_{\text{disc}} + \tfrac{1}{2}m|AP|^2\dot{\theta}^2$

$= \tfrac{1}{2}I\dot{\theta}^2 + \tfrac{1}{2}m(2a^2(1 - \sin\beta))\dot{\theta}^2$ continued

6. 

$$T = \overbrace{\frac{1}{2}m\dot{r}^2}^{\text{left}} + \overbrace{\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)}^{\text{right}}$$

$$= m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$$

$$V = mgr + (0 - mgr\cos\theta)$$

$$\Rightarrow L = T - V = m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgr + mgr\cos\theta$$

r) $\frac{\partial L}{\partial r} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)$

$$\boxed{mr\dot{\theta}^2 - mg + mg\cos\theta = 2m\ddot{r}}$$

θ) $\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)$

$$\Rightarrow \boxed{-mgr\sin\theta = \frac{d}{dt}(mr^2\dot{\theta})}$$

left mass starts at rest $\Rightarrow \dot{r} = 0$.

$$-mgr\sin\theta = \frac{d}{dt}(mr^2\dot{\theta})$$

$$\Rightarrow -mgr\sin\theta = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta}$$

$$\Rightarrow \boxed{\ddot{\theta} = -\frac{g}{r}\theta}$$

$$\Rightarrow \theta(t) = \varepsilon\cos\left(\sqrt{\frac{g}{r}}t + \phi\right)$$

$$2m\ddot{r} = mr\dot{\theta}^2 - mg + mg\cos\theta$$

$$\Rightarrow 2\ddot{r} \approx r\dot{\theta}^2 - \frac{1}{2}g\theta^2$$

$$\Rightarrow \ddot{r} = \frac{1}{2}\left[r\varepsilon^2\frac{g}{r}\sin^2\left(\sqrt{\frac{g}{r}}t+\phi\right) - \frac{\varepsilon^2}{2}g\cos^2\left(\sqrt{\frac{g}{r}}t+\phi\right)\right]$$

$$= \frac{\varepsilon^2 g}{2}\left[\sin^2\left(\sqrt{\frac{g}{r}}t+\phi\right) - \frac{1}{2}\cos^2\left(\sqrt{\frac{g}{r}}t+\phi\right)\right]$$

$$\ddot{r}_{ave} = \frac{\varepsilon^2 g}{2}\left[\frac{1}{2} - \frac{1}{4}\right] = \boxed{\frac{\varepsilon^2 g}{8}}$$

Hence, since $\ddot{r} > 0$, $r$ increases

$\Rightarrow$ left mass goes upward

$$\Rightarrow \frac{1}{2}I\dot{\theta}^2 + (1-\sin\beta)ma^2\dot{\theta}^2 + (m+M)ga\cos\beta(1-\cos\theta) = \text{constant}.$$

(c) Take the derivative of the above

$$I\dot{\theta}\ddot{\theta} + 2(1-\sin\beta)ma^2\dot{\theta}\ddot{\theta} + (m+M)ga\cos\beta\sin\theta\,\dot{\theta} = 0$$

$$\Rightarrow \left[I + 2(1-\sin\beta)ma^2\right]\ddot{\theta} + \left[(m+M)ga\cos\beta\right]\theta = 0$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I + 2(1-\sin\beta)ma^2}{(m+M)ga\cos\beta}}$$

$$= 2\pi\sqrt{\frac{\frac{3}{2}a + 2(1-\sin\beta)\frac{3}{2}a}{\left(\frac{3}{2}+1\right)g\cos\beta}}$$

$$\sin\beta = \frac{\frac{3}{2}}{\frac{3}{2}+1} = \frac{3}{5} \qquad \cos\beta = \frac{4}{5}$$

$$T = 2\pi\sqrt{\frac{27}{20}\frac{a}{g}} = \boxed{3\pi\sqrt{\frac{3a}{5g}}}$$

7. Refer to Morin 6.19

8. $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \Rightarrow \dot{r} = \frac{p_r}{m}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$H = \left(\sum p_i\dot{q}_i\right) - L$$

$$= p_r\dot{r} + p_\theta\dot{\theta} - \left(\frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)\right)$$

$$= p_r\left(\frac{p_r}{m}\right) + p_\theta\left(\frac{p_\theta}{mr^2}\right) - \frac{p_r^2}{2m} - \frac{p_\theta}{2mr^2} + V(r)$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r)$$

$$\boxed{H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r)}$$


$x(t) = A\cos(\omega t)$

The coordinate of m is

$(x + l\sin\theta,\ -l\cos\theta)$

$$T = \frac{1}{2}m\left(\dot{\overline{x + l\sin\theta}}\right)^2 + \frac{1}{2}m\left(\dot{\overline{-l\cos\theta}}\right)^2$$

$$\Rightarrow T = \frac{1}{2}m\left[(\dot{x} + l\cos\theta\,\dot{\theta})^2 + (l\sin\theta\,\dot{\theta})^2\right]$$

$$\Rightarrow T = \frac{1}{2}m\left[\dot{x}^2 + 2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right]$$

$$V = -mgl\cos\theta$$

$$\Rightarrow L = T - V = \frac{1}{2}m\left[\dot{x}^2 + 2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right] + mgl\cos\theta.$$

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) \Rightarrow -m\dot{x}l\dot{\theta}\sin\theta - mgl\sin\theta = \frac{d}{dt}\left(m\dot{x}l\cos\theta + ml^2\dot{\theta}\right)$$

$$\Rightarrow \boxed{l\ddot{\theta} + \ddot{x}\cos\theta = -g\sin\theta}$$

$$\ddot{x} = -A\omega^2\cos(\omega t)$$

$$\Rightarrow l\ddot{\theta} - A\omega^2\cos(\omega t)\cos\theta + g\sin\theta = 0.$$

$$\Rightarrow l\ddot{\theta} + g\theta = A\omega^2\cos(\omega t)$$

$$\Rightarrow \boxed{\theta(t) = \frac{\frac{A}{l}\omega^2}{\frac{g}{l} - \omega^2}\cos(\omega t) + C\cos\left(\sqrt{\frac{g}{l}}\,t + \phi\right)}$$