Waves

Physics Olympiad
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"How do you make them do that?"
"I use electromagnetic waves."
--- Hank Pym

Longitudinal wave: waves in which the displacement of the medium is parallel (or anti-parallel) to the direction of propagation of the wave. (e.g. sound waves)

Transverse wave: waves in which the displacement of the medium is perpendicular to the direction of propagation of the wave. (e.g. electromagnetic waves, vibrations in a guitar string, water ripples)

The expression below represents a sinusoidal wave travelling to the right (positive x). $y(x,t)$ denotes the amplitude of the wave at position x and time t. Keeping y constant, an increase in x must correspond to an increase in t. Therefore, as time goes by, we can find the same amplitude at a larger position x, which is why the wave is travelling in the $+x$ direction.

$$y(x,t) = A\cos(kx - \omega t)$$

A phase difference can be added to make the expression more general. This can be obtained from the initial condition of the problem.

$$y(x,t) = A\cos(kx - \omega t + \phi)$$

Mechanical Waves

Mechanical waves: a wave that requires a medium to propagate. The denser the medium, the faster it propagates.

For a mechanical wave,

$$v = \sqrt{\frac{Stiffness}{Inertia}}$$

In particular for a string, $v = \sqrt{\frac{T}{\mu}}$, where T is the tension in the string and μ is the mass density.

The speed of sound is $v = \sqrt{\frac{B}{\rho}}$ where B is the Bulk modulus and ρ is the density of the medium.

(Non Relativistic) Doppler's Effect

$$f' = \frac{v \pm v_0}{v \pm v_s} f$$

A good way to remember this formula is by the statement "O is before S", so v_0 should appear as the numerator. Usually, a simple thought experiment can settle the signs of the numerator and denominator.

Electromagnetic Waves

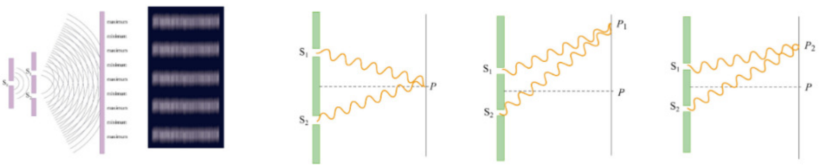
Electromagnetic wave: a wave that does not require a medium to propagate.

Diffraction and Interference

Diffraction refers to the phenomena when a wave encounters an obstacle or opening.

In order to form an interference pattern, the light sources must be *coherent* and *monochromatic*. This is to ensure for every two waves, the phase difference is time-independent and the wavelengths are the same.

- Double Slit Diffraction



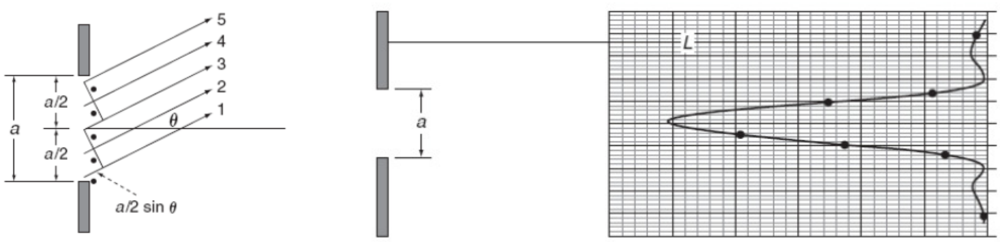
The path difference between the two slits is $d\sin\theta$. For constructive interference:

$$d\sin\theta = n\lambda, \text{ where } n \in \mathbb{Z}^+ \cup \{0\}$$

The angular spacing between the fringes is

$$\theta \approx \frac{\lambda}{d}$$

- Single Slit Diffraction

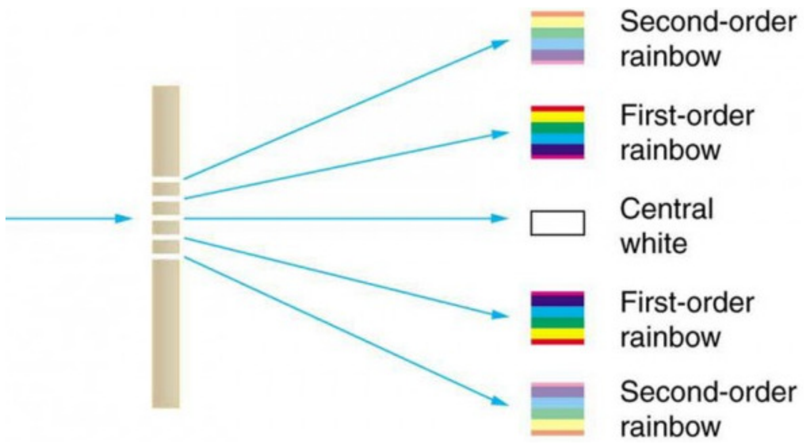


By considering the single slit as $2m$ smaller slits, destructive interferences occurs at:

$$a\sin\theta = m\lambda$$

There is no simple expression for the location of the maxima on the screen other than the trivial solution of $\theta = 0$ (the principle maxima). The other maxima are much less intense than the principle maxima.

- Diffraction Grating



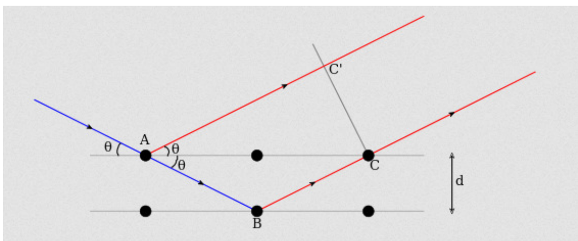
The position of the m^{th} maxima is given by:

$$d \sin \theta_m = m\lambda$$

Here d is the spacing between the gratings, usually on the order of 10^{-5}.

- Bragg's Law

When X-ray is incident onto a crystal surface, the waves reflected off the different layers of the crystal surface will cause interference. Note the different notation of θ, which is the scattering angle here.



$$2d \sin \theta = n\lambda$$

- A notable instance of superposition:

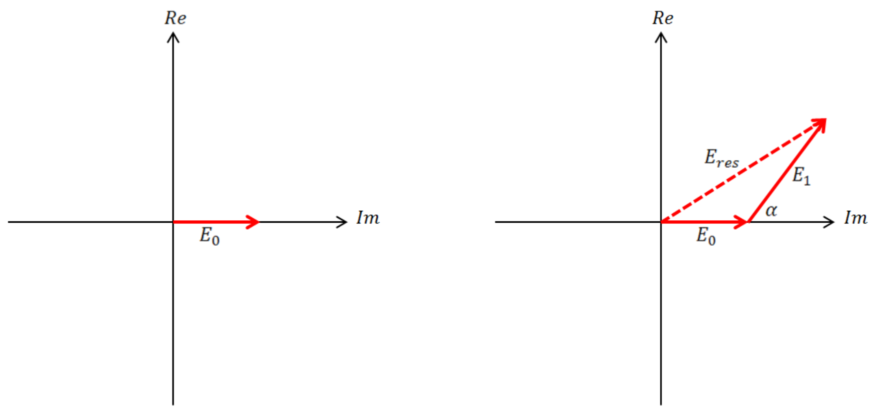
When a transverse wave is reflected off an optically denser medium, it incurs a π radian phase shift. For example, if Star-Lord flies his spacecraft very close to a water surface and Gamora sends him a mission-critical message, he might not receive it due to the destructive interference of the direct wave and the reflected wave off the water.

Phasor Technique

Huygen's principle: Every point on a wavefront is itself a source of spherical wavelets, and the secondary wavelets emanating from different points mutually interfere. The sum of these spherical wavelets forms the new wavefront.

Besides the normal cosine definition, a wave can also be modeled as a complex number. A wave of the form $A\cos\omega t$ can be represented as $A\,e^{i\omega t}$. The component $e^{i\omega t}$ can be thought of as an arrow rotating in the complex plane, making a full rotation in $\frac{2\pi}{\omega}$ time.

The purpose of introducing complex numbers is to make the visualisation of the waves easier, particularly for coherent waves (same ω; constant phase difference). Since all arrows are rotating at the same speed, the length of the resultant complex number will be constant. Subsequently, when we derive the amplitude of the resultant wave, it suffices to take the real component of the resultant complex number.



Suppose we have two waves $E_0\cos(\omega t)$ and $E_1\cos(\omega t+\alpha)$. Converting them to complex numbers, we have $E_0\,e^{i\omega t}$ and $E_1\,e^{i\alpha}e^{i\omega t}$. In the complex plane, this means E_1 is rotated by α about E_0.

The resultant amplitude can be obtained easily by cosine rule.

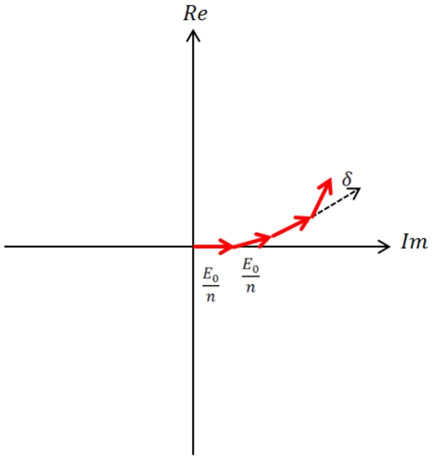
$$E_{res}^2 = E_0^2 + E_1^2 + 2E_0E_1\cos(\pi-\alpha)$$

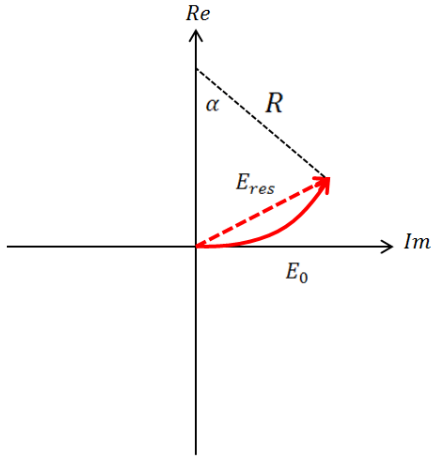
Notice that a problem on waves is converted into a (coordinate) geometry problem.

To understand the following portion, we need to define irradiance, which is the flux of radiant energy per unit area. You may think of irradiance as the brightness. If we have a function for irradiance, then we can obtain the formula for a diffraction pattern.

Considering a single slit. By Huygen's principle, we can suppose that it is made up of n wavelets. If the original wave has an amplitude of E_0, each of the smaller wavelets has an energy of $\frac{E_0}{n}$.

Since n is large, the amplitudes in the phasor diagram can be represented geometrically by an arc (of a circle).





The phase difference between the first and last point of the single slit is $\alpha = 2\pi \cdot \frac{b \sin\theta}{\lambda}$, where b is the length of the slit. By geometry, we have:

$$E_{res} = 2R \sin\left(\frac{\alpha}{2}\right)$$

$$E_0 = R\alpha$$

Note that the variable R introduced above is an auxiliary variable; it has no physical significance). After eliminating R from the simultaneous expressions:

$$\frac{E_{res}}{E_0} = \frac{\sin\left(\frac{\alpha}{2}\right)}{\frac{\alpha}{2}}$$

Irradiance is proportional to $\left(\frac{E_{res}}{E_0}\right)^2$ (square of the amplitude).

Hence, the irradiance at angle θ is

$$I(\theta) = A\left[\frac{\sin\left(\frac{\pi b \sin\theta}{\lambda}\right)}{\frac{\pi b \sin\theta}{\lambda}}\right]^2 = A\left[\frac{\lambda}{\pi b \sin\theta} \sin\left(\frac{\pi b \sin\theta}{\lambda}\right)\right]^2$$

Now that the irradiance function is found, physical significance can be extracted out from it.

The minima positions occur at $\sin\left(\frac{\alpha}{2}\right) = 0$ when $\frac{\pi b \sin\theta}{\lambda} = n\pi \Leftrightarrow b\sin\theta = n\lambda$ where n is an integer. Maxima positions occur when $\frac{\sin\left(\frac{\alpha}{2}\right)}{\frac{\alpha}{2}}$ is a maximum, i.e. when $\frac{\alpha}{2}$ is a solution to $\tan\left(\frac{\alpha}{2}\right) = \frac{\alpha}{2}$.

Notice that with the phasor technique, any diffraction problem can be computationally solved. In cases where the geometry in the phasor diagram is not too difficult, it can appear in Olympiad problems as well.

Optics

Snell's law

$$n = \frac{c}{v} \qquad n_1 \sin\theta_1 = n_2 \sin\theta_2$$

For total internal reflection, just take $\theta_2 = 90°$.

Thin Lens Equation

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$

Magnification is given by the ratio of the image size to object size.

$$m = \left|\frac{v}{u}\right|$$

Sign Conventions

- Object distance u is positive when the object is on the same side of the mirror that the light rays are approaching the mirror.
- The image distance v is positive when the image is real.

Gaussian Formula for Spherical Surface

$$\frac{n_1}{u} + \frac{n_2}{v} = \frac{n_2 - n_1}{R}$$

Note that for a plane, it is equivalent to setting $R = \pm\infty$. The equation reduces to:

$$\frac{n_1}{u} + \frac{n_2}{v} = 0$$

Lensmaker's Equation (for completeness)

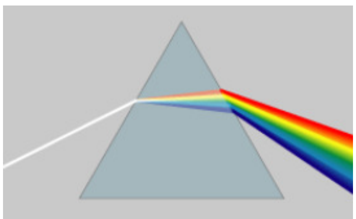
$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{d\,(n-1)}{nR_1R_2}\right)$$

Malus' Law (Polarisation)

$$I = I_0 \cos^2\theta$$

Dispersion

Dispersion occurs when the phase velocity of a wave (speed of propagation of the wave in a medium) depends on its frequency.



"Sir, there are still terabytes of calculations required before an actual flight is ... "
"JARVIS, sometimes you gotta run before you can walk."

--- Tony Stark

Wave Equation

The wave equation in one dimension can be written as

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

This equation is an important second-order linear partial differential equation for waves. It is a bit like Oscillations, where after you obtain $\ddot{x} = -\omega^2 x$, you immediately proceed to write down $x = A\cos(\omega t + \phi)$. Many wave problems will collapse into the above equation.

For example, suppose that there is a scenario where a disturbance is set up in a string. Suppose the horizontal tension is equal (meaning each small part of the string moves up or down only), we have

$$T\cos(\theta) = (T + \mathrm{d}T)\cos(\theta + \mathrm{d}\theta) = T_0$$

$$(T + \mathrm{d}T)\sin(\theta + \mathrm{d}\theta) - T\sin(\theta) = m\frac{\partial^2 y}{\partial t^2} = \mu\,\mathrm{d}x\,\frac{\partial^2 y}{\partial t^2}$$

Dividing the equations, we have

$$\frac{\mu\,\mathrm{d}x}{T_0}\frac{\partial^2 y}{\partial t^2} = \tan(\theta + \mathrm{d}\theta) - \tan(\theta) = \mathrm{d}\left(\frac{\partial y}{\partial x}\right) \Leftrightarrow \frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T_0}\frac{\partial^2 y}{\partial t^2}$$

$$v = \sqrt{\frac{T_0}{\mu}}$$

As said, this equation is ubiquitous in Physics. In the case of a stress pulse propagating longitudinally through a bar, the equation obtained is:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{E}\frac{\partial^2 y}{\partial t^2}$$

$$v = \sqrt{\frac{E}{\rho}}$$

The most general solution of the one dimensional wave equation is

$$\psi = f(x - vt) + g(x + vt)$$

where f, g are arbitrary functions that only depend on their parameters $x - vt$ and $x + vt$ respectively. Notice that, for an observer moving at a steady speed of v to the left, the function f is stationary. Similarly, for an observing moving at a steady speed of v to the right, the function g is stationary. The functions f, g are called the *characteristic curves* along which the signals propagate.

Fourier Transform

The big question now is: is it possible to decompose a superposition of waves back into its individual frequencies?

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x\xi}\, \mathrm{d}x$$

The purpose of sharing Fourier Transform here is really to appreciate its idea. Geometrically, it can seen as a rotating complex vector in the Argand diagram, with its magnitude matching the amplitude of the waves. The rate of rotation is governed by ξ.

For any random ξ, the complex vector will be pointing "randomly" at points in the Argand diagram. If we were to do a time average function on all the complex vectors, we should get close to 0.

However, when ξ matches the frequency of one of the individual waves that made up this superposition, the center of mass of all the complex vectors will be shifted significantly from the origin. The time average function is further away from 0.

Therefore, an appreciation of the Fourier's formula will tell us that, the integral from $-\infty$ to ∞ is really just taking a time average of all the complex vectors. If we plot $\hat{f}(\xi)$, the peaks will give us the frequencies of the individual waves.

A. Sample Problems

1. A rope of mass per unit length μ and total length d is hung from the ceiling. A weight of mass m is tied to the rope at the bottom of the rope. A disturbance is introduced at the bottom of the rope. Find the time required for the wave produced to reach the ceiling.

2. The brain can determine the direction of a source of sound by the time delay Δt between the arrival of the sound at two different ears. In most cases, the sound source is distant in comparison with the separation of ears D.

    a. Find an expression for Δt in terms of D and the angle θ between the direction of the source and the forward direction.

    b. When the person is submerged in water at 20 °Celsius, sound produced by a distance sound source directly to his right will mislead him to believe that the sound is from an angle θ'. Find θ'. Bulk modulus of water = 2.20 GPa. Bulk modulus of air = 142 kPa. Density of air = $1.20\ \mathrm{kg\,m^{-3}}$. Density of water = 998 $\mathrm{kg\,m^{-3}}$.

3. A beam of white light passes through a diffraction grating. The first order red light of wavelength λ_1 is observed at angle θ_1 while the first order violet light of wavelength λ_2 is observed at angle θ_2. Find the density of slits on the diffraction grating ρ. Find the number of complete visible spectra visible through the grating.

4. A coherent monochromatic light source with wavelength λ behind a wall shines through three small slits on the wall onto a screen that is distance D apart from the wall. The slights are equally spaced from each other with a small distance a where

$a \ll D$. At what angle from the direction perpendicular to the screen will the interference be completely destructive?

5. A coherent monochromatic light source of wavelength λ is situated at a distance h above a horizontal plane mirror. Find the height of the first order maximum on a screen that is distance D away from the light source where $h \ll D$. What if the light source is placed under water and the mirror is replaced with an air-water interference?

6. The distance between an object and its real image, formed by a converging lens, is held fixed. Show that there are two possible positions for the lens, and that the size of the object is given by $\sqrt{h_1 h_2}$ where h_1 and h_2 are the sizes of the two images.

B. In Class Problems

7. One of the slits of a double slit system is wider than the other, such that the amplitude of the light reaching the central part of the screen from one slit acting alone is twice that of the other slit acting alone. Find an equation for the intensity I_θ at an angle θ measured from the principal axis in terms of the maximum intensity I_0 of the central bright fringe, slit separation d, wavelength λ and angle θ.

8. The reflective index of a medium within a certain region $x > 0$ changes with y. A thin light ray travelling in the direction strikes the medium at right angles and moves through the medium along a circular arc.

   a. How does the refractive index depend on y?

   b. What is the maximum angular size of the arc?

9. A telescope is used to observe at a distance of 10km two objects which are 0.12m apart and illuminated by light of wavelength 600nm. Estimate the diameter of the objective lens of the telescope if it can just resolve the two objects.

10. Two trains approach the train station from opposite sides, each moving at 120km h^{-1} with respect to the station and then slowing down to halt in 20s. If both trains were whistling at 5000Hz, find the distance between the zero and first order maxima as a function of time.

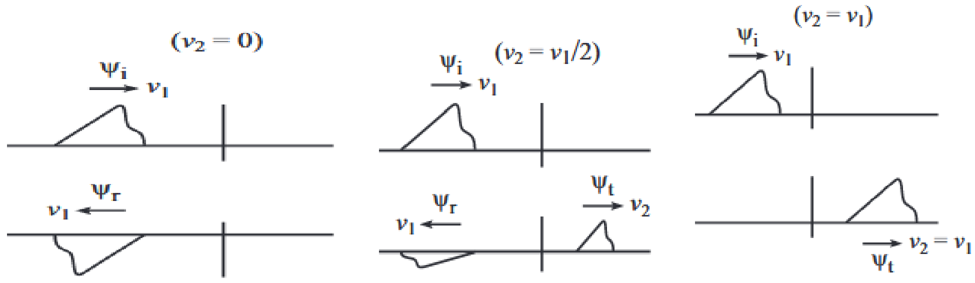
11. On a spherical planet, the refractive index of the atmosphere varies with altitude h according to the formula below, where n_0 and ϵ are constants. Any laser directed horizontally at an arbitrary altitude will circle the planet. What is the radius of the planet?

$$n(h) = \frac{n_0}{1 + \epsilon h}$$

12. The figure below shoes an equilateral glass prism illuminated by a $100W$ laser beam of wavelength $\lambda = 600$nm. The refractive index of the glass of prism is 1.50 at this particular wavelength. The path of the light in the prism is parallel to the base of the prism. Calculate the force exerted by the laser beam to the prism.

13. A small fish at depth d below the surface of the water is viewed through a simple thin converging lens with focal length f. If the lens is at height h above the water surface. where is the image of the fish seen by observer? Assume the fish lies on the optical axis of the lens and the refractive index of the water is n.

14. A semi infinite string of mass per unit length μ_1 is tied to another semi-infinite string of mass per unit length μ_2, with the knot at $x = 0$. The string has tension T. A wave with equation $y = A\cos(\omega t - kx)$ is coming from the left and going to the right. Find the equation of the reflected wave and the transmitted wave. Find the displacement of the knot as a function of time. Determine the value of reflectance and transmittance of the system. Prove $R + T = 1$.



Recommended Resources

- Optics by Eugene Hecht
- Waves by David J. Morin (https://scholar.harvard.edu/david-morin/waves)
- Physics By Example: 200 Problems and Solutions by W.G. Rees
- 200 Puzzling Physics Problems by P. Gnadig et al