Wang Jianzhi

SIMC Section C: Busy footbridge

Definitions: $P(i,j)$ denote the probability that person A_i will meet person B_j.

Question 1: $p_{1,n} = \frac{1}{2}$, $p_{2,n} = \frac{3}{8}$, $q_1 = \frac{1}{2}$, $q_2 = \frac{3}{8}$, $q_3 = \frac{5}{16}$

$p_{1,n}$) Based on simulation result, $P(1,n) = \frac{1}{2}\ \forall n$. This is intuitive. If A_1 is stubborn, then whether he meets B_n depends on whether he is in the same lane as B_n initially, with a probability of $\frac{1}{2}$. If A_1 is polite, then whether he meets B_n depends on whether B_{n-1} and B_n are on the same lane, with a probability of $\frac{1}{2}$ because we can take B_n to be independently generated as B_{n-1}. If $n = 1$, it will still be a probability of $\frac{1}{2}$. Hence, $P(1,n) = \frac{1}{2}$.

$p_{2,n}$) From the simulation results, it seems that $P(2,1) = \frac{1}{2}$ and $P(2,n) = \frac{3}{8}$ for $n > 1$. The first part is intuitive. We can treat the spawning of A_2 to be independent from A_1. Therefore, whether A_2 will meet B_1 depends on whether A_2 is spawned in the same row as B_1 after A_1 passed the row of Bs. Hence, $P(2,1) = \frac{1}{2}$.

For the latter result, we consider a more general case. We calculate $P(m,n)$ where $n > 1$. We observe that a polite person among A will not change the order of B, while a stubborn person in A will push all the B to one side.

Before A_m, there are $m-1$ other As. If none of them are stubborn, then the order of B will be the same. Hence, we can treat A_m as A_1. This gives a probability of $\left(\frac{1}{2}\right)^{m-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^m$, where the first $\left(\frac{1}{2}\right)^{m-1}$ comes from everyone before A_m being polite. Otherwise, if at least one of $[A_1, A_2, \ldots, A_{m-1}]$ is stubborn, then the Bs will fall into one row. Hence, if A_m is polite, then it has no chance of meeting B_n. Otherwise, if it is stubborn, then A_m will only meet B_n if they are in the same row initially, which has a probability of $\frac{1}{2}$. Hence, the probability here is $\left(1 - \left(\frac{1}{2}\right)^{m-1}\right) \cdot \frac{1}{2} \cdot \frac{1}{2}$.

Hence,

$$P(m,n) = \left(\frac{1}{2}\right)^m + \left(1 - \left(\frac{1}{2}\right)^{m-1}\right) \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{4} + \left(\frac{1}{2}\right)^{m+1}$$

$$p_{2,n} = P(2,n) = \frac{1}{4} + \left(\frac{1}{2}\right)^3 = \frac{3}{8}$$

Note that the probability $P(m,n)$ is independent of n.

q_1) From the answer and derivation in $p_{1,n}$, we have $q_1 = \frac{1}{2}$.

q_2) Based on simulation result, $P(2, 10000) \approx 0.37510 \approx \frac{3}{8}$. From the answer and derivation in $p_{2,n}$, we can see $P(2,\infty) = \frac{3}{8}$.

q_3) Based on simulation result, $P(3, 10000) \approx 0.311820$. This is close to the value of $\frac{5}{16} \approx 0.3125$. From the answer and derivation in $p_{2,n}$, we can see that $P(3, \infty) = \frac{1}{4} + \left(\frac{1}{2}\right)^4 = \frac{5}{16}$.

Further analysis: We see that for large m, $P(m,n) = \frac{1}{4} + \left(\frac{1}{2}\right)^{m+1} \to \frac{1}{4}$. This makes sense for large m, there is a high probability that there is a stubborn A before the A we are considering; hence the Bs will be in the same row. Therefore, the only way for A and B to meet is when A is stubborn and in the same row as B as well, each with a probability of $\frac{1}{2}$.

Question 2: $p_{1,n} = \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right)$, $p_{2,n} = \frac{1}{6}\left(1 - \left(-\frac{1}{2}\right)^x\right)$, $q_1 = \frac{1}{3}$, $q_2 = \frac{1}{6}$ $q_3 = \frac{1}{6}$

$p_{1,n}$) We note that if B_n and B_{n-1} are in the same lane, then B_n will never collide with A_1. This is because if A_1 is not in the same row as B_{n-1}, then it will pass B_n as well. If A_1 is in the same row as B_{n-1}, then it will be deflected. If B_n and B_{n-1} are in different lanes, then A_1 will collide with B_n if A_1 does not collide with B_{n-1}. Let $P(x)$ denote the probability of A_1 colliding with B_x. Clearly, $P(1) = \frac{1}{2}$. From then on, the probability of A_1 not colliding with B_{x-1} is $1 - P(x-1)$. Since we can treat the spawning of B_x as independent of B_{x-1}, then the probability of A_1 colliding with B_x is:

$$P(x) = \frac{1}{2}\big(1 - P(x-1)\big) \Rightarrow P(x-1) + 2 \cdot P(x) = 1$$

This yields the solution $P(x) = \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^x\right)$. Checking, $P(1) = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$ which is true. $P(2) = \frac{1}{3}\left(1 - \frac{1}{4}\right) = \frac{1}{4}$. Simulation results showed $P(2) = 0.249630 \approx \frac{1}{4}$.

Hence, $p_{1,n} = \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right)$.

$p_{2,n}$) For A_2, clearly $P(2,1) = \frac{1}{2}$. Denote B_i' to be the B_i after A_1 passed through the row of Bs. For $i > 1$, we will define $Q(i)$ as the probability that B_i' and B_{i-1}' are in the same row after A_1 has passed through them. If B_i and B_{i-1} are initially in the same row, then B_i' and B_{i-1}' will also be in the same row. Otherwise, B_i and B_{i-1} are initially in different rows. If A_1 collided with B_{i-1}, then $B_{i-1}' = B_i'$ simply because A_1 will not want to switch. If A_1 did not collide with B_{i-1}, then whether $B_i' = B_{i-1}'$ depends on B_{i+1}. If $B_{i+1} = B_i$, then $B_i' \neq B_{i-1}'$. Otherwise, $B_i' = B_{i-1}'$.

$$Q(i) = \frac{1}{2} + \frac{1}{2}\left[P(i-1) + \frac{1}{2}\big(1 - P(i-1)\big)\right] = \frac{1}{2} + \frac{1}{4}[1 + P(i-1)] = \frac{3}{4} + \frac{1}{4}P(i-1)$$

$$Q(i) = \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^{i-1}\right) = \frac{3}{4} + \frac{1}{12}\left(1 - \left(-\frac{1}{2}\right)^{i-1}\right) = \frac{5}{6} - \frac{1}{6}\left(-\frac{1}{2}\right)^i$$

We now define $P'(x)$ to be the probability that A_2 meets B_x'.

If $B_x' = B_{x-1}'$, then there is no chance for A_2 to meet B_x'. Otherwise, $B_x' \neq B_{x-1}'$. If $x - 1 > 1$, then this implies that $B_{x-2}' = B_{x-1}'$, otherwise B_{x-1}' would have shifted to be in the same row as B_x'. This also means that A_2 will not collide with B_{x-1}' and must collide with B_x'.

$$P'(x) = \big(1 - Q(x)\big) = \frac{1}{6}\left(1 - \left(-\frac{1}{2}\right)^x\right), \qquad x \geq 3$$

We assumed $x - 1 > 1$, hence the above formula works for $x \geq 3$. For $x = 1$, $P'(1) = \frac{1}{2}$. For $x = 2$, case consideration gives $P'(2) = \frac{1}{16}$. Below is a table showing the comparison of simulation results and our formula:

x	Theoretical Result $P'(x)$	Simulation $P'(x)$

1	0.5	0.497790
2	0.0625	0.062160
3	0.1875	0.185850
4	0.15625	0.154290
5	0.171875	0.170900

q_1) We have previously $p_{1,n} = \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right)$. As $n \to \infty$, $p_{1,\infty} \to \frac{1}{3}$. From our simulation, $P(1{,}9999) = 0.333180 \approx \frac{1}{3}$.

q_2) We have previously $p_{2,n} = \frac{1}{6}\left(1 - \left(-\frac{1}{2}\right)^x\right)$. As $x \to \infty$, $p_{2,\infty} \to \frac{1}{6}$. From our simulation, $P(2, 9999) = 0.167670 \approx \frac{1}{6}$.

q_3) We make the observation that any further entrance of A_i into Bs will not change the arrangement of B except for possibly B_1. This is because after A_1 enters, all Bs will be in the formation of "trains" of more than or equal to length 2. A "train" here is defined to be a consecutive block of Bs. If there is a "train" of length 1 (i.e. a solo person standing in different lane as the person before AND after him, say B_x), then A_1 will either collide with B_{x-1} and deflect B_{x-1} to join B_x in the same row, OR A_1 will collide with B_x and deflect him to join B_{x-1}. Hence, future entrance of As will not affect the probability of $P(i, \infty)$. Hence $q_3 = \frac{1}{6}$.

Further analysis: We see that for large m, $P(m,n) = \frac{1}{4} + \left(\frac{1}{2}\right)^{m+1} \to \frac{1}{4}$. This makes sense for large m, there is a high probability that there is a stubborn A before the A we are considering; hence the Bs will be in the same row. Therefore, the only way for A and B to meet is when A is stubborn and in the same row as B as well, each with a probability of $\frac{1}{2}$.

Question 3: $p_{1,n} = \frac{1}{2}\, p_{2,n} = \frac{3}{8},\, q_1 = \frac{1}{3},\, q_2 = \frac{1}{6}\, q_3 = \frac{1}{6}$

$p_{1,n}$) We can take the spawning of B_i to be independent from the previous events. Hence, there is a $\frac{1}{2}$ chance that B_i spawns in the same row as A_1 after A_1 passed through all the previous Bs, and a $\frac{1}{2}$ chance otherwise. Hence, $P_{1,n} = \frac{1}{2}\ \forall n$.

$p_{2,n}$) Let $Q(x)$ denote the probability that B'_x and B'_{x-1} are in the same row. If B_x and B_{x-1} are in the same row, then if A_1 did not collide with B_{x-1}, then $B'_x = B'_{x-1}$ (this happens with a probability of $\frac{1}{4}$). Otherwise, A_1 collides with B_{x-1}. There is a $\frac{1}{2}$ chance B_{x-1} will remain and hence $B'_x = B'_{x-1}$. Otherwise, B_{x-1} deflects, but there will be another $\frac{1}{2}$ chance for B_x to also deflect upon colliding with A_1 to result in $B'_x = B'_{x-1}$. Hence probability of $B'_x = B'_{x-1}$ provided that $B_x = B_{x-1}$ is $\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{7}{16}$.

If $B_x \neq B_{x-1}$, then if A_1 collides with B_{x-1}, there is a $\frac{1}{2}$ chance B_{x-1} shifts such that $B'_x = B'_{x-1}$. Otherwise A_1 will shift and there is a $\frac{1}{2}$ chance that B_x will shift upon meeting A_1. Hence if A_1 collides with B_{x-1}, there is a $\frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{3}{16}$ chance of $B'_x = B'_{x-1}$. Otherwise, A_1 did not collide with B_{x-1}, hence there is a $\frac{1}{2}$ chance of B_x shifting so that $B'_x = B'_{x-1}$. Hence, the probability here is $\frac{3}{16} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{5}{16}$.

Hence $Q(x) = \frac{3}{4}\ \forall x$. This is independent of each other, because we can treat the spawning of B_x to be independent from B_{x-1}. Furthermore, whether A_1 collides with B_{x-1} does not depend on position of B_x afterwards, so the events in the previous case considerations are independent.

Now denote $P(x)$ as the probability that A_2 will collide with B'_x.

If A_2 collides with B'_{x-1}, then the probability of A_2 ending up on either lane is $\frac{1}{2}$. So probability of meeting B'_x is $\frac{1}{2}$ no matter what; this gives rise to a probability of $\frac{1}{2}P(x-1)$. If A_2 does not collide with B'_{x-1}, then it will hit B'_x if $B'_x \neq B'_{x-1}$. This has a probability of $\big(1 - Q(x)\big)\cdot\big(1 - P(x-1)\big) = \frac{1}{4}\big(1 - P(x-1)\big)$.

$$P(x) = \frac{1}{2}P(x-1) + \frac{1}{4}\big(1 - P(x-1)\big) \Rightarrow 4P(x) = 1 + P(x-1)$$

Solving the recurrence relation and substituting in the initial values, we have:

$$P(x) = \frac{1}{3}\left(1 + 2\cdot\left(\frac{1}{4}\right)^x\right)$$

Hence, $p_{2,n} = \frac{1}{3}\left(1 + 2\cdot\left(\frac{1}{4}\right)^n\right)$.

x	Theoretical Result $P'(x)$	Simulation $P'(x)$
1	0.5	0.499390
2	0.375	0.374180

3	0.34375	0.346730
4	0.3359375	0.335940
5	0.333984375	0.333250

q_1) By result of $p_{1,n}$, $q_1 = \frac{1}{2}$.

q_2) Using the result from $p_{2,n}$, $q_2 = p_{2,\infty} = \frac{1}{3}$. Based on simulation result, $P(2,9999) \approx 0.333730 \approx \frac{1}{3}$ too.

q_3) We might as well solve for $p_{m,n}$. We will denote B_x^m be the position of B_x after people A_1, A_2, ..., A_{m-1} passed through them. Let $Q^m(x)$ denote the probability of B_x^m and B_{x-1}^m in the same row. Let $P^m(x)$ denote the probability of A_m hitting B_x. We make the observation that the probability of A_m hitting B_n is independent of what happens after A_m passes B_n. This means whether $B_x^{m-1} = B_{x-1}^{m-1}$ is independent of whether A_{m-1} meet B_{x-1}^{m-1}.

If $B_x^{m-1} = B_{x-1}^{m-1}$, then consider the case when A_{m-1} meet B_{x-1}^{m-1}. If B_{x-1}^{m-1} does not shift, then $B_x^m = B_{x-1}^m$, this has a probability of $Q^{m-1}(x) \cdot P^{m-1}(x-1) \cdot \frac{1}{2}$. Otherwise, if B_{x-1}^{m-1} shift, then $B_x^m = B_{x-1}^m$ only if B_x^m also shifts; this has a probability of $Q^{m-1}(x) \cdot P^{m-1}(x-1) \cdot \frac{1}{4}$. Hence the total probability is $\frac{3}{4} Q^{m-1}(x) \cdot P^{m-1}(x-1)$. If A_{m-1} did not meet B_{x-1}^{m-1}, then, $B_x^m = B_{x-1}^m$. This has a probability of $Q^{m-1}(x) \cdot \left(1 - P^{m-1}(x-1)\right)$. Hence, total probability is $Q^{m-1}(x) \cdot \left(1 - \frac{1}{4} P^{m-1}(x-1)\right)$.

If $B_x^{m-1} \neq B_{x-1}^{m-1}$, then consider the case when A_{m-1} meet B_{x-1}^{m-1}. If B_{x-1}^{m-1} shifts, then $B_x^m = B_{x-1}^m$. This has a probability of $\left(1 - Q^{m-1}(x)\right) \cdot P^{m-1}(x-1) \cdot \frac{1}{2}$. If B_{x-1}^{m-1} does not shift, then A_{m-1} will collide with B_x^{m-1}. Then $B_x^m = B_{x-1}^m$ will hold only if B_x^{m-1} shifts. This has a probability of $\left(1 - Q^{m-1}(x)\right) \cdot P^{m-1}(x-1) \cdot \frac{1}{4}$. Hence, total probability is $\left(1 - Q^{m-1}(x)\right) \cdot P^{m-1}(x-1) \cdot \frac{3}{4}$. Now, consider the case A_{m-1} does not meet B_{x-1}^{m-1}. Then $B_x^m = B_{x-1}^m$ will only hold if B_x^{m-1} shifts. This has a probability of $\left(1 - Q^{m-1}(x)\right) \cdot \left(1 - P^{m-1}(x-1)\right) \cdot \frac{1}{2}$. Hence total probability is $\left(1 - Q^{m-1}(x)\right)\left(\frac{1}{2} + \frac{1}{4} P^{m-1}(x-1)\right)$.

$$Q^m(x) = Q^{m-1}(x) \cdot \left(1 - \frac{1}{4} P^{m-1}(x-1)\right) + \left(1 - Q^{m-1}(x)\right)\left(\frac{1}{2} + \frac{1}{4} P^{m-1}(x-1)\right)$$

$$Q^m(x) = \frac{1}{2} + \frac{1}{2} Q^{m-1}(x) + \frac{1}{4} P^{m-1}(x-1) - \frac{1}{2} P^{m-1}(x-1) \cdot Q^{m-1}(x)$$

Checking, $Q^2(x) = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$, which is true from our part $p_{2,n}$.

Hence, we are about to obtain $Q^m(x)$ from $Q^{m-1}(x)$ and $P^{m-1}(x)$. Now we focus on $P^m(x)$.

If $B_x^m = B_{x-1}^m$, then consider the case if A_m collides with B_{x-1}^m. A_m will only collide with B_x^m if B_{x-1}^m shifts. That has a probability of $Q^m(x) \cdot P^m(x-1) \cdot \frac{1}{2}$. Note we can multiply the expressions because whether $B_x^m = B_{x-1}^m$ is independent of whether A_m collides with B_{x-1}^m. If A_m did not collide with B_{x-1}^m, then there is no chance. Hence the total probability in this case is $Q^m(x) \cdot P^m(x-1) \cdot \frac{1}{2}$.

If $B_x^m \neq B_{x-1}^m$, then consider the case if A_m collides with B_{x-1}^m. Then A_m will only collide with B_x^m if A_m shifts. This has a probability of $\left(1 - Q^m(x)\right) \cdot P^m(x-1) \cdot \frac{1}{2}$. Otherwise, A_m did not collide with B_{x-1}^m. Then A_m must have collided with B_x^m. This has probability of $\left(1 - Q^m(x)\right) \cdot \left(1 - P^m(x-1)\right)$. The total probability is $\left(1 - Q^m(x)\right) \cdot \left(1 - \frac{1}{2}P^m(x-1)\right)$.

$$P^m(x) = Q^m(x) \cdot P^m(x-1) \cdot \frac{1}{2} + \left(1 - Q^m(x)\right) \cdot \left(1 - \frac{1}{2}P^m(x-1)\right)$$

$$P^m(x) = 1 - Q^m(x) - \frac{1}{2}P^m(x-1) + Q^m(x) \cdot P^m(x-1)$$

Checking for $m = 2$, $P^2(x) = 1 - \frac{3}{4} - \frac{1}{2}P^2(x-1) + \frac{3}{4}P^2(x-1) = \frac{1}{4}\left(1 + P^2(x-1)\right)$, which is the result we have gotten earlier.

Hence we have the following two results:

$$Q^m(x) = \frac{1}{2} + \frac{1}{2}Q^{m-1}(x) + \frac{1}{4}P^{m-1}(x-1) - \frac{1}{2}P^{m-1}(x-1) \cdot Q^{m-1}(x)$$

$$P^m(x) = 1 - Q^m(x) - \frac{1}{2}P^m(x-1) + Q^m(x) \cdot P^m(x-1)$$

Solving for $m = 3$,

$$Q^3(x) = \frac{1}{2} + \frac{1}{2}Q^2(x) + \frac{1}{4}P^2(x-1) - \frac{1}{2}P^2(x-1) \cdot Q^2(x)$$

$$P^3(x) = 1 - Q^3(x) - \frac{1}{2}P^3(x-1) + Q^3(x) \cdot P^3(x-1)$$

Reducing, we have:

$$Q^3(x) = \frac{5}{6} - \frac{1}{12}\left(\frac{1}{4}\right)^x$$

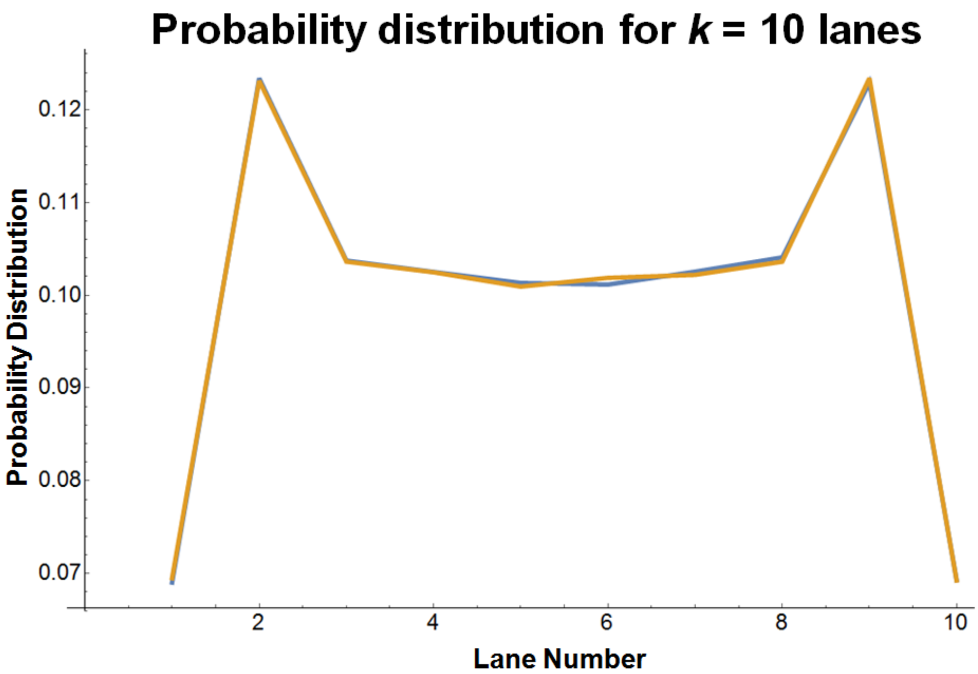
$$P^3(x) = \left(\frac{1}{6} + \frac{1}{12}\left(\frac{1}{4}\right)^x\right) + \frac{1}{3}\left(1 - \frac{1}{4}\left(\frac{1}{4}\right)^x\right)P^3(x-1)$$

When $x \to \infty$, $q_3 = \frac{1}{6} + \frac{1}{3}q_3 \Rightarrow q_3 = \frac{1}{4}$.

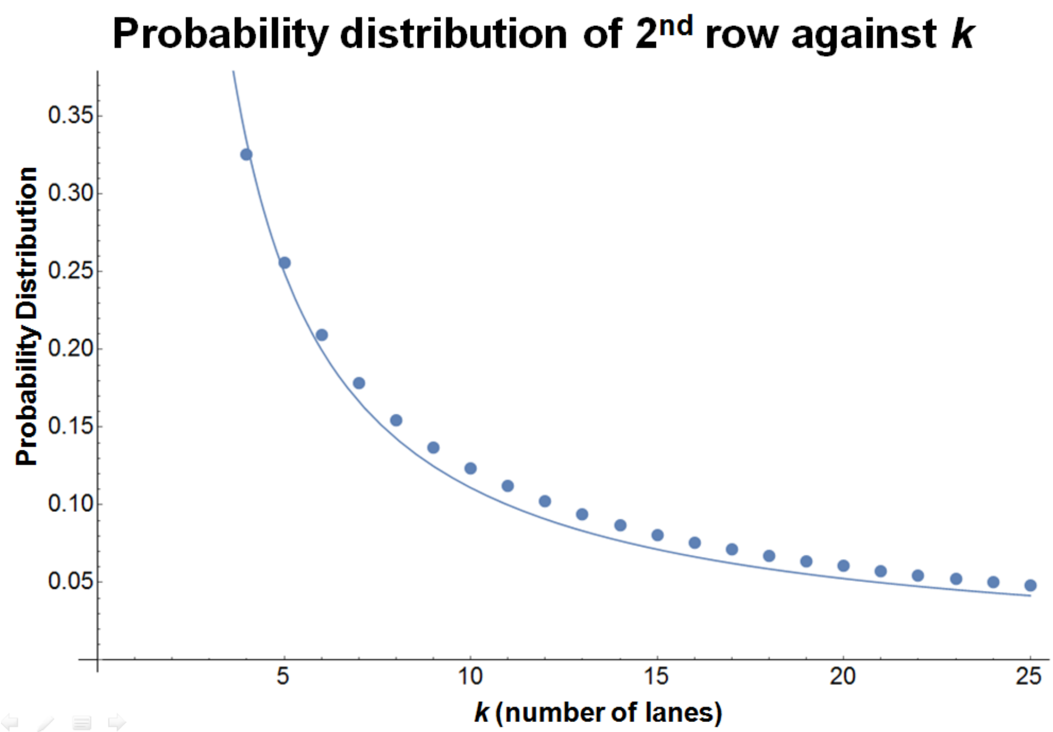
We note that our theory in this case is wrong, since our simulation predicts $P(3{,}10000) = 0.258120 \approx \frac{7}{27}$. This is related to Markov chain. Work is still in progress.

Question 4:

We mainly explore k lanes. Firstly, we show some preliminary observations. We simulated $k = 10$ with equal number of As and Bs and we plotted the distribution of As across the lanes after passing through all the Bs. We noticed a spike in the second and ninth row. In other simulations, a similar trend holds. More people end up at the 2^{nd} and the $(k-1)^{\mathrm{th}}$ row compared to the remaining rows. The least people end up at 1^{st} and k^{th} row. For the remaining rows, the number of people passing through them are generally the same, except that there is a slight dip towards the center.



We now move onto more quantitative analysis. We firstly investigate the peaks. We plotted the probability of a person ending up on the 2^{nd} lane across $k \in [3,25]$.



We tried fitting the data points with a $f(k) = \frac{1}{k-1}$ graph. However, as seen from figure above, the data points fall above the fit for larger values of k, although the shape is generally correct. Our non-linear model fit returned $f(k) = \frac{1.09239}{k-0.863873}$.

For the first lane, we fitted $f(k) = \frac{0.70099}{k}$ graph. We will approximate it to $\frac{0.7}{k}$.

To find the remaining distribution, assuming they are equal, we have:

$$P = \frac{1}{k-4}\left[1 - 2\left(\frac{0.7}{k} + \frac{1}{k-1}\right)\right] =$$

References:

NUSH Champion Award Report: https://www.nushigh.edu.sg/qql/slot/u90/file/simc/mathmodel/SIMC2018ChampionAwardReport.pdf