SIMC Section C: Busy footbridge

Definitions: $P(i,j)$ denote the probability that person A_i will meet person B_j.

Question 1: $p_{1,n} = \frac{1}{2}$, $p_{2,n} = \frac{3}{8}$, $q_1 = \frac{1}{2}$, $q_2 = \frac{3}{8}$, $q_3 = \frac{5}{16}$

$p_{1,n}$) Based on simulation result, $P(1,n) = \frac{1}{2}\ \forall n$. This is intuitive. If A_1 is stubborn, then whether he meets B_n depends on whether he is in the same lane as B_n initially, with a probability of $\frac{1}{2}$. If A_1 is polite, then whether he meets B_n depends on whether B_{n-1} and B_n are on the same lane, with a probability of $\frac{1}{2}$ because we can take B_n to be independently generated as B_{n-1}. If $n = 1$, it will still be a probability of $\frac{1}{2}$. Hence, $P(1,n) = \frac{1}{2}$.

$p_{2,n}$) From the simulation results, it seems that $P(2,1) = \frac{1}{2}$ and $P(2,n) = \frac{3}{8}$ for $n > 1$. The first part is intuitive. We can treat the spawning of A_2 to be independent from A_1. Therefore, whether A_2 will meet B_1 depends on whether A_2 is spawned in the same row as B_1 after A_1 passed the row of Bs. Hence, $P(2,1) = \frac{1}{2}$.

For the latter result, we consider a more general case. We calculate $P(m,n)$ where $n > 1$. We observe that a polite person among A will not change the order of B, while a stubborn person in A will push all the B to one side.

Before A_m, there are $m-1$ other As. If none of them are stubborn, then the order of B will be the same. Hence, we can treat A_m as A_1. This gives a probability of $\left(\frac{1}{2}\right)^{m-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^m$, where the first $\left(\frac{1}{2}\right)^{m-1}$ comes from everyone before A_m being polite. Otherwise, if at least one of $[A_1, A_2, \ldots, A_{m-1}]$ is stubborn, then the Bs will fall into one row. Hence, if A_m is polite, then it has no chance of meeting B_n. Otherwise, if it is stubborn, then A_m will only meet B_n if they are in the same row initially, which has a probability of $\frac{1}{2}$. Hence, the probability here is $\left(1 - \left(\frac{1}{2}\right)^{m-1}\right) \cdot \frac{1}{2} \cdot \frac{1}{2}$.

Hence,

$$P(m,n) = \left(\frac{1}{2}\right)^m + \left(1 - \left(\frac{1}{2}\right)^{m-1}\right) \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{4} + \left(\frac{1}{2}\right)^{m+1}$$

$$p_{2,n} = P(2,n) = \frac{1}{4} + \left(\frac{1}{2}\right)^3 = \frac{3}{8}$$

Note that the probability $P(m,n)$ is independent of n.

q_1) From the answer and derivation in $p_{1,n}$, we have $q_1 = \frac{1}{2}$.

q_2) Based on simulation result, $P(2, 10000) \approx 0.37510 \approx \frac{3}{8}$. From the answer and derivation in $p_{2,n}$, we can see $P(2,\infty) = \frac{3}{8}$.

q_3) Based on simulation result, $P(3, 10000) \approx 0.311820$. This is close to the value of $\frac{5}{16} \approx 0.3125$. From the answer and derivation in $p_{2,n}$, we can see that $P(3, \infty) = \frac{1}{4} + \left(\frac{1}{2}\right)^4 = \frac{5}{16}$.

Further analysis: We see that for large m, $P(m, n) = \frac{1}{4} + \left(\frac{1}{2}\right)^{m+1} \to \frac{1}{4}$. This makes sense for large m, there is a high probability that there is a stubborn A before the A we are considering; hence the Bs will be in the same row. Therefore, the only way for A and B to meet is when A is stubborn and in the same row as B as well, each with a probability of $\frac{1}{2}$.

Question 2: $p_{1,n} = \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right)$, $p_{2,n} = \frac{1}{6}\left(1 - \left(-\frac{1}{2}\right)^x\right)$, $q_1 = \frac{1}{3}$, $q_2 = \frac{1}{6}$ $q_3 = \frac{1}{6}$

$p_{1,n}$) We note that if B_n and B_{n-1} are in the same lane, then B_n will never collide with A_1. This is because if A_1 is not in the same row as B_{n-1}, then it will pass B_n as well. If A_1 is in the same row as B_{n-1}, then it will be deflected. If B_n and B_{n-1} are in different lanes, then A_1 will collide with B_n if A_1 does not collide with B_{n-1}. Let $P(x)$ denote the probability of A_1 colliding with B_x. Clearly, $P(1) = \frac{1}{2}$. From then on, the probability of A_1 not colliding with B_{x-1} is $1 - P(x-1)$. Since we can treat the spawning of B_x as independent of B_{x-1}, then the probability of A_1 colliding with B_x is:

$$P(x) = \frac{1}{2}\big(1 - P(x-1)\big) \Rightarrow P(x-1) + 2 \cdot P(x) = 1$$

This yields the solution $P(x) = \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^x\right)$. Checking, $P(1) = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$ which is true. $P(2) = \frac{1}{3}\left(1 - \frac{1}{4}\right) = \frac{1}{4}$. Simulation results showed $P(2) = 0.249630 \approx \frac{1}{4}$.

Hence, $p_{1,n} = \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right)$.

$p_{2,n}$) For A_2, clearly $P(2,1) = \frac{1}{2}$. Denote B_i' to be the B_i after A_1 passed through the row of Bs. For $i > 1$, we will define $Q(i)$ as the probability that B_i' and B_{i-1}' are in the same row after A_1 has passed through them. If B_i and B_{i-1} are initially in the same row, then B_i' and B_{i-1}' will also be in the same row. Otherwise, B_i and B_{i-1} are initially in different rows. If A_1 collided with B_{i-1}, then $B_{i-1}' = B_i'$ simply because A_1 will not want to switch. If A_1 did not collide with B_{i-1}, then whether $B_i' = B_{i-1}'$ depends on B_{i+1}. If $B_{i+1} = B_i$, then $B_i' \neq B_{i-1}'$. Otherwise, $B_i' = B_{i-1}'$.

$$Q(i) = \frac{1}{2} + \frac{1}{2}\left[P(i-1) + \frac{1}{2}\big(1 - P(i-1)\big)\right] = \frac{1}{2} + \frac{1}{4}[1 + P(i-1)] = \frac{3}{4} + \frac{1}{4}P(i-1)$$

$$Q(i) = \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^{i-1}\right) = \frac{3}{4} + \frac{1}{12}\left(1 - \left(-\frac{1}{2}\right)^{i-1}\right) = \frac{5}{6} - \frac{1}{6}\left(-\frac{1}{2}\right)^i$$

We now define $P'(x)$ to be the probability that A_2 meets B_x'.

If $B_x' = B_{x-1}'$, then there is no chance for A_2 to meet B_x'. Otherwise, $B_x' \neq B_{x-1}'$. If $x - 1 > 1$, then this implies that $B_{x-2}' = B_{x-1}'$, otherwise B_{x-1}' would have shifted to be in the same row as B_x'. This also means that A_2 will not collide with B_{x-1}' and must collide with B_x'.

$$P'(x) = \big(1 - Q(x)\big) = \frac{1}{6}\left(1 - \left(-\frac{1}{2}\right)^x\right), \qquad x \geq 3$$

We assumed $x - 1 > 1$, hence the above formula works for $x \geq 3$. For $x = 1$, $P'(1) = \frac{1}{2}$. For $x = 2$, case consideration gives $P'(2) = \frac{1}{16}$. Below is a table showing the comparison of simulation results and our formula:

x	Theoretical Result $P'(x)$	Simulation $P'(x)$

1	0.5	0.497790
2	0.0625	0.062160
3	0.1875	0.185850
4	0.15625	0.154290
5	0.171875	0.170900

q_1) We have previously $p_{1,n} = \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right)$. As $n \to \infty$, $p_{1,\infty} \to \frac{1}{3}$. From our simulation, $P(1{,}9999) = 0.333180 \approx \frac{1}{3}$.

q_2) We have previously $p_{2,n} = \frac{1}{6}\left(1 - \left(-\frac{1}{2}\right)^x\right)$. As $x \to \infty$, $p_{2,\infty} \to \frac{1}{6}$. From our simulation, $P(2, 9999) = 0.167670 \approx \frac{1}{6}$.

q_3) We make the observation that any further entrance of A_i into Bs will not change the arrangement of B except for possibly B_1. This is because after A_1 enters, all Bs will be in the formation of "trains" of more than or equal to length 2. A "train" here is defined to be a consecutive block of Bs. If there is a "train" of length 1 (i.e. a solo person standing in different lane as the person before AND after him, say B_x), then A_1 will either collide with B_{x-1} and deflect B_{x-1} to join B_x in the same row, OR A_1 will collide with B_x and deflect him to join B_{x-1}. Hence, future entrance of As will not affect the probability of $P(i, \infty)$. Hence $q_3 = \frac{1}{6}$.

Further analysis: We see that for large m, $P(m,n) = \frac{1}{4} + \left(\frac{1}{2}\right)^{m+1} \to \frac{1}{4}$. This makes sense for large m, there is a high probability that there is a stubborn A before the A we are considering; hence the Bs will be in the same row. Therefore, the only way for A and B to meet is when A is stubborn and in the same row as B as well, each with a probability of $\frac{1}{2}$.

Question 3: $p_{1,n} = \frac{1}{2}\, p_{2,n} = \frac{3}{8}$, $q_1 = \frac{1}{3}$, $q_2 = \frac{1}{6}\, q_3 = \frac{1}{6}$

$p_{1,n}$) We can take the spawning of B_i to be independent from the previous events. Hence, there is a $\frac{1}{2}$ chance that B_i spawns in the same row as A_1 after A_1 passed through all the previous Bs, and a $\frac{1}{2}$ chance otherwise. Hence, $P_{1,n} = \frac{1}{2}\ \forall n$.

$p_{2,n}$) Let $Q(x)$ denote the probability that B'_x and B'_{x-1} are in the same row. If B_x and B_{x-1} are in the same row, then if A_1 did not collide with B_{x-1}, then $B'_x = B'_{x-1}$ (this happens with a probability of $\frac{1}{4}$). Otherwise, A_1 collides with B_{x-1}. There is a $\frac{1}{2}$ chance B_{x-1} will remain and hence $B'_x = B'_{x-1}$. Otherwise, B_{x-1} deflects, but there will be another $\frac{1}{2}$ chance for B_x to also deflect upon colliding with A_1 to result in $B'_x = B'_{x-1}$. Hence probability of $B'_x = B'_{x-1}$ provided that $B_x = B_{x-1}$ is $\frac{1}{4} + \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{7}{16}$.

If $B_x \neq B_{x-1}$, then if A_1 collides with B_{x-1}, there is a $\frac{1}{2}$ chance B_{x-1} shifts such that $B'_x = B'_{x-1}$. Otherwise A_1 will shift and there is a $\frac{1}{2}$ chance that B_x will shift upon meeting A_1. Hence if A_1 collides with B_{x-1}, there is a $\frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{16}$ chance of $B'_x = B'_{x-1}$. Otherwise, A_1 did not collide with B_{x-1}, hence there is a $\frac{1}{2}$ chance of B_x shifting so that $B'_x = B'_{x-1}$. Hence, the probability here is $\frac{3}{16} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{5}{16}$.

Hence $Q(x) = \frac{3}{4}\ \forall x$. This is independent of each other, because we can treat the spawning of B_x to be independent from B_{x-1}. Furthermore, whether A_1 collides with B_{x-1} does not depend on position of B_x afterwards, so the events in the previous case considerations are independent.

Now denote $P(x)$ as the probability that A_2 will collide with B'_x.

If A_2 collides with B'_{x-1}, then the probability of A_2 ending up on either lane is $\frac{1}{2}$. So probability of meeting B'_x is $\frac{1}{2}$ no matter what; this gives rise to a probability of $\frac{1}{2}P(x-1)$. If A_2 does not collide with B'_{x-1}, then it will hit B'_x if $B'_x \neq B'_{x-1}$. This has a probability of $\big(1 - Q(x)\big) \cdot \big(1 - P(x-1)\big) = \frac{1}{4}\big(1 - P(x-1)\big)$.

$$P(x) = \frac{1}{2}P(x-1) + \frac{1}{4}\big(1 - P(x-1)\big) \Rightarrow 4P(x) = 1 + P(x-1)$$

Solving the recurrence relation and substituting in the initial values, we have:

$$P(x) = \frac{1}{3}\left(1 + 2 \cdot \left(\frac{1}{4}\right)^x\right)$$

Hence, $p_{2,n} = \frac{1}{3}\left(1 + 2 \cdot \left(\frac{1}{4}\right)^n\right)$.

x	Theoretical Result $P'(x)$	Simulation $P'(x)$
1	0.5	0.499390
2	0.375	0.374180

3	0.34375	0.346730
4	0.3359375	0.335940
5	0.333984375	0.333250

q_1) By result of $p_{1,n}$, $q_1 = \frac{1}{2}$.

q_2) Using the result from $p_{2,n}$, $q_2 = p_{2,\infty} = \frac{1}{3}$. Based on simulation result, $P(2,9999) \approx 0.333730 \approx \frac{1}{3}$ too.

q_3) We might as well solve for $p_{m,n}$. We will denote B_x^m be the position of B_x after people A_1, A_2, ..., A_{m-1} passed through them. Let $Q^m(x)$ denote the probability of B_x^m and B_{x-1}^m in the same row. Let $P^m(x)$ denote the probability of A_m hitting B_x. We make the observation that the probability of A_m hitting B_n is independent of what happens after A_m passes B_n. This means whether $B_x^{m-1} = B_{x-1}^{m-1}$ is independent of whether A_{m-1} meet B_{x-1}^{m-1}.

If $B_x^{m-1} = B_{x-1}^{m-1}$, then consider the case when A_{m-1} meet B_{x-1}^{m-1}. If B_{x-1}^{m-1} does not shift, then $B_x^m = B_{x-1}^m$, this has a probability of $Q^{m-1}(x) \cdot P^{m-1}(x-1) \cdot \frac{1}{2}$. Otherwise, if B_{x-1}^{m-1} shift, then $B_x^m = B_{x-1}^m$ only if B_x^m also shifts; this has a probability of $Q^{m-1}(x) \cdot P^{m-1}(x-1) \cdot \frac{1}{4}$. Hence the total probability is $\frac{3}{4} Q^{m-1}(x) \cdot P^{m-1}(x-1)$. If A_{m-1} did not meet B_{x-1}^{m-1}, then, $B_x^m = B_{x-1}^m$. This has a probability of $Q^{m-1}(x) \cdot \left(1 - P^{m-1}(x-1)\right)$. Hence, total probability is $Q^{m-1}(x) \cdot \left(1 - \frac{1}{4} P^{m-1}(x-1)\right)$.

If $B_x^{m-1} \neq B_{x-1}^{m-1}$, then consider the case when A_{m-1} meet B_{x-1}^{m-1}. If B_{x-1}^{m-1} shifts, then $B_x^m = B_{x-1}^m$. This has a probability of $\left(1 - Q^{m-1}(x)\right) \cdot P^{m-1}(x-1) \cdot \frac{1}{2}$. If B_{x-1}^{m-1} does not shift, then A_{m-1} will collide with B_x^{m-1}. Then $B_x^m = B_{x-1}^m$ will hold only if B_x^{m-1} shifts. This has a probability of $\left(1 - Q^{m-1}(x)\right) \cdot P^{m-1}(x-1) \cdot \frac{1}{4}$. Hence, total probability is $\left(1 - Q^{m-1}(x)\right) \cdot P^{m-1}(x-1) \cdot \frac{3}{4}$. Now, consider the case A_{m-1} does not meet B_{x-1}^{m-1}. Then $B_x^m = B_{x-1}^m$ will only hold if B_x^{m-1} shifts. This has a probability of $\left(1 - Q^{m-1}(x)\right) \cdot \left(1 - P^{m-1}(x-1)\right) \cdot \frac{1}{2}$. Hence total probability is $\left(1 - Q^{m-1}(x)\right)\left(\frac{1}{2} + \frac{1}{4} P^{m-1}(x-1)\right)$.

$$Q^m(x) = Q^{m-1}(x) \cdot \left(1 - \frac{1}{4} P^{m-1}(x-1)\right) + \left(1 - Q^{m-1}(x)\right)\left(\frac{1}{2} + \frac{1}{4} P^{m-1}(x-1)\right)$$

$$Q^m(x) = \frac{1}{2} + \frac{1}{2} Q^{m-1}(x) + \frac{1}{4} P^{m-1}(x-1) - \frac{1}{2} P^{m-1}(x-1) \cdot Q^{m-1}(x)$$

Checking, $Q^2(x) = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$, which is true from our part $p_{2,n}$.

Hence, we are about to obtain $Q^m(x)$ from $Q^{m-1}(x)$ and $P^{m-1}(x)$. Now we focus on $P^m(x)$.

If $B_x^m = B_{x-1}^m$, then consider the case if A_m collides with B_{x-1}^m. A_m will only collide with B_x^m if B_{x-1}^m shifts. That has a probability of $Q^m(x) \cdot P^m(x-1) \cdot \frac{1}{2}$. Note we can multiply the expressions because whether $B_x^m = B_{x-1}^m$ is independent of whether A_m collides with B_{x-1}^m. If A_m did not collide with B_{x-1}^m, then there is no chance. Hence the total probability in this case is $Q^m(x) \cdot P^m(x-1) \cdot \frac{1}{2}$.

If $B_x^m \neq B_{x-1}^m$, then consider the case if A_m collides with B_{x-1}^m. Then A_m will only collide with B_x^m if A_m shifts. This has a probability of $\left(1 - Q^m(x)\right) \cdot P^m(x-1) \cdot \frac{1}{2}$. Otherwise, A_m did not collide with B_{x-1}^m. Then A_m must have collided with B_x^m. This has probability of $\left(1 - Q^m(x)\right) \cdot \left(1 - P^m(x-1)\right)$. The total probability is $\left(1 - Q^m(x)\right) \cdot \left(1 - \frac{1}{2} P^m(x-1)\right)$.

$$P^m(x) = Q^m(x) \cdot P^m(x-1) \cdot \frac{1}{2} + \left(1 - Q^m(x)\right) \cdot \left(1 - \frac{1}{2} P^m(x-1)\right)$$

$$P^m(x) = 1 - Q^m(x) - \frac{1}{2} P^m(x-1) + Q^m(x) \cdot P^m(x-1)$$

Checking for $m = 2$, $P^2(x) = 1 - \frac{3}{4} - \frac{1}{2} P^2(x-1) + \frac{3}{4} P^2(x-1) = \frac{1}{4}\left(1 + P^2(x-1)\right)$, which is the result we have gotten earlier.

Hence we have the following two results:

$$Q^m(x) = \frac{1}{2} + \frac{1}{2} Q^{m-1}(x) + \frac{1}{4} P^{m-1}(x-1) - \frac{1}{2} P^{m-1}(x-1) \cdot Q^{m-1}(x)$$

$$P^m(x) = 1 - Q^m(x) - \frac{1}{2} P^m(x-1) + Q^m(x) \cdot P^m(x-1)$$

Solving for $m = 3$,

$$Q^3(x) = \frac{1}{2} + \frac{1}{2} Q^2(x) + \frac{1}{4} P^2(x-1) - \frac{1}{2} P^2(x-1) \cdot Q^2(x)$$

$$P^3(x) = 1 - Q^3(x) - \frac{1}{2} P^3(x-1) + Q^3(x) \cdot P^3(x-1)$$

Reducing, we have:

$$Q^3(x) = \frac{5}{6} - \frac{1}{12}\left(\frac{1}{4}\right)^x$$

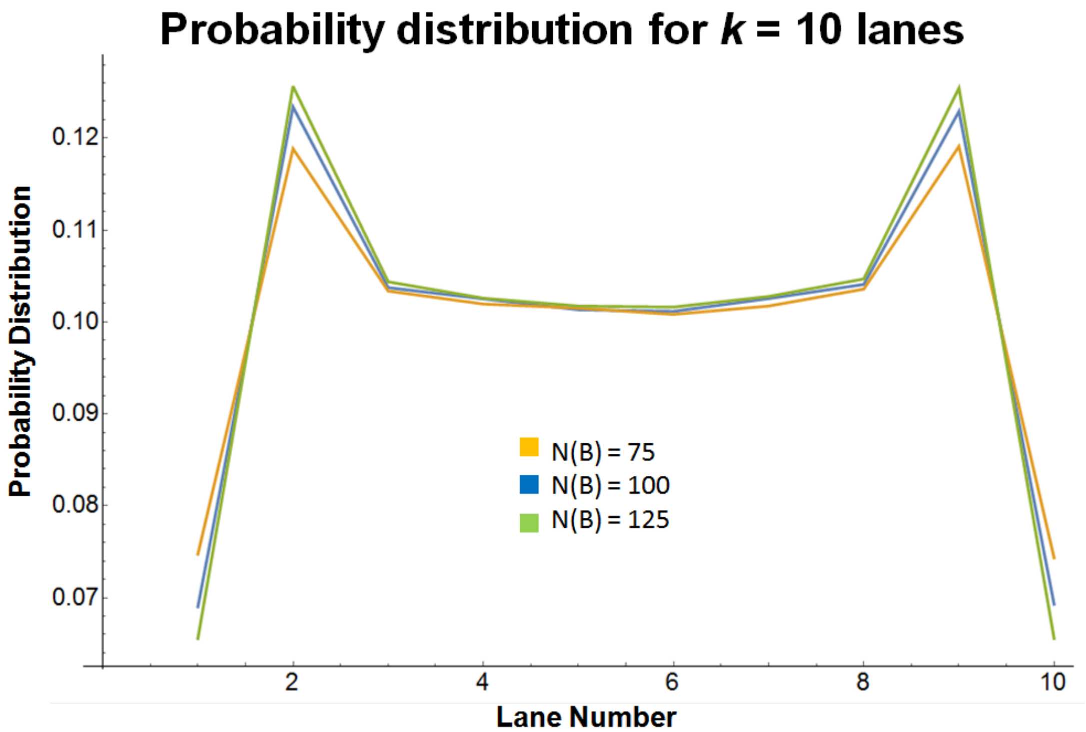
$$P^3(x) = \left(\frac{1}{6} + \frac{1}{12}\left(\frac{1}{4}\right)^x\right) + \frac{1}{3}\left(1 - \frac{1}{4}\left(\frac{1}{4}\right)^x\right) P^3(x-1)$$

When $x \to \infty$, $q_3 = \frac{1}{6} + \frac{1}{3} q_3 \Rightarrow q_3 = \frac{1}{4}$.

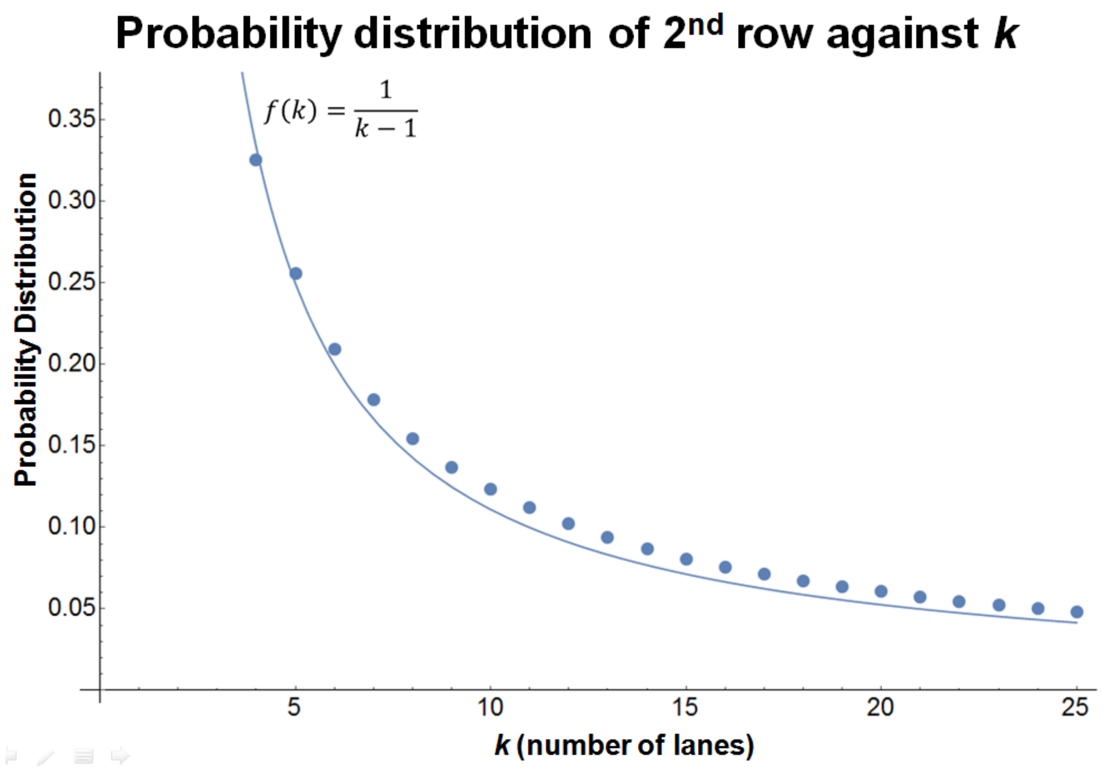
We note that our theory in this case is wrong, since our simulation predicts $P(3{,}10000) = 0.258120 \approx \frac{7}{27}$. This is related to Markov chain. Work is still in progress.

Question 4:

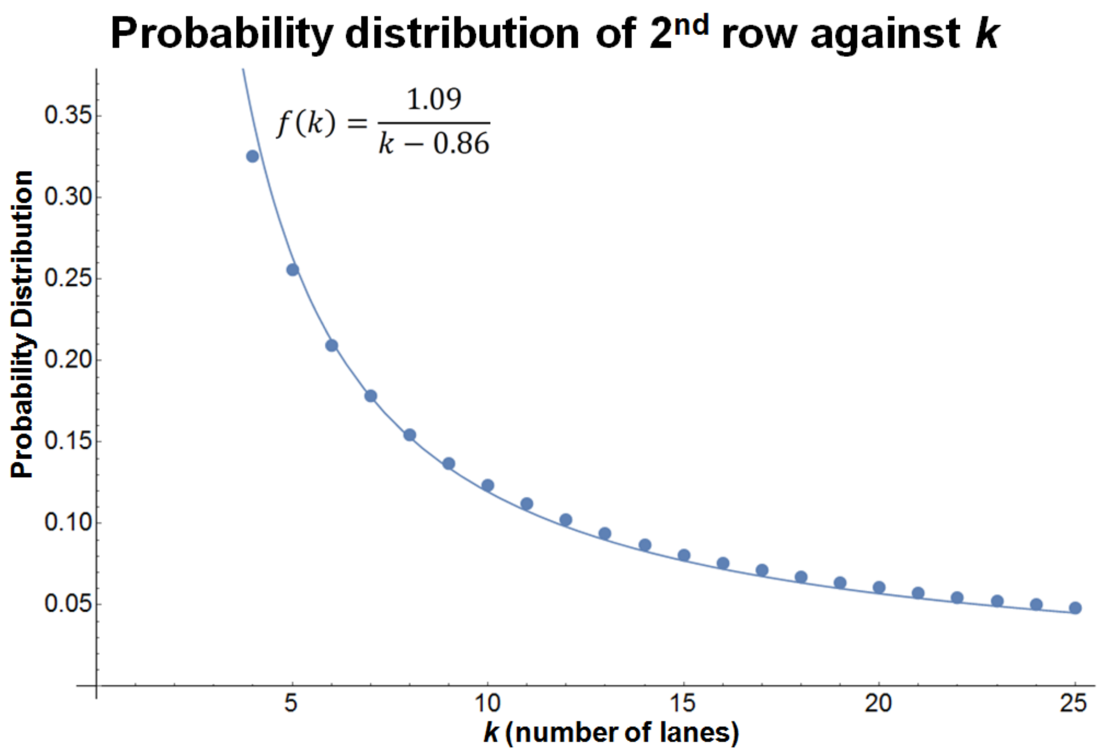
We mainly explore k lanes. Firstly, we show some preliminary observations. We simulated $k = 10$ with equal number of As and Bs and we plotted the distribution of As across the lanes after passing through all the Bs. We noticed a spike in the second and ninth row. In other simulations, a similar trend holds. More people end up at the 2^{nd} and the $(k-1)^{\text{th}}$ row compared to the remaining rows. The least people end up at 1^{st} and k^{th} row. For the remaining rows, the number of people passing through them are generally the same, except that there is a slight dip towards the center. As we increase the number of Bs, there will be an increase in number of people in the 2^{nd} and the $(k-1)^{\text{th}}$ row, a further drop in the 1^{st} and k^{th} row. Meanwhile, the rest of the probability remains the same.



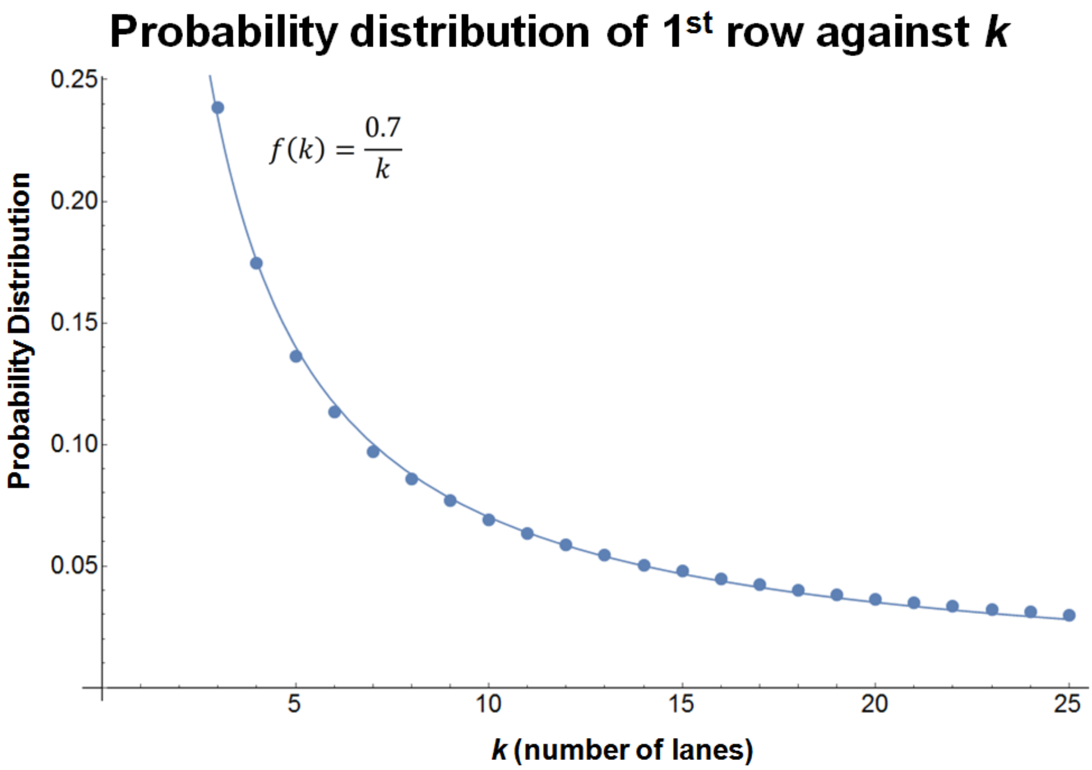
We now focus our discussion on the 2^{nd} and the $(k-1)^{\text{th}}$ row. For $k \in [3,25]$, we used computer simulation to obtain the probability of a person ending up in the 2^{nd} lane. ($(k-1)^{\text{th}}$ lane will be similar due to symmetry) We observed that the graph looks like a $\frac{1}{k-1}$ graph. After fitting it with a $f(k) = \frac{1}{k-1}$ graph, we obtain the following:



For small values of k, we see that the curve fits quite well. However, for larger values of k, such as $k > 7$, the curve always lie before the data points. We used a nonlinear model fit from Mathematica to obtain a better curve. Using a fit in the form of $f(k) = \frac{a}{k-b}$, Mathematica has $a = 1.09239$ and $b = 0.863873$. For simplicity purposes, we will use $f(k) = \frac{1}{k-1}$ in our analysis later on.



Now we move onto the first lane. For the first lane, the graph looks inversely proportional to k. Fitting it with a function in the form of $f(k) = \frac{c}{k}$, we have $c = 0.70099$. We will approximate it to $f(k) = \frac{0.7}{k}$.



To find the remaining distribution, assuming they are equal, we have:

$$P = \frac{1}{k-4}\left[1 - 2\left(\frac{0.7}{k} + \frac{1}{k-1}\right)\right] = \frac{1}{k-4} \cdot \frac{k^2 - 4.4k + 1.4}{k(k-1)}$$

We note $P < \frac{1}{k-4} \cdot \frac{k^2-4k}{k(k-1)} = \frac{1}{k-1}$, which is reasonable since the expected number of people ending up at other rows is less than the 2^{nd} row. For a reasonable probability distribution, take the following $P(x, k)$, where x is the row number:

$$P(x,k) = \begin{cases} \frac{0.7}{k}, & x = 1 \text{ or } k \\ \frac{1}{k-1}, & x = 2 \text{ or } k-1 \\ \frac{1}{k-4} \cdot \frac{k^2 - 4.4k + 1.4}{k(k-1)}, & \text{otherwise} \end{cases}$$

We now try to derive up a physical model. Let N_i denote the number of people in the i^{th} row. We assume that $\frac{\mathrm{d}N_i}{\mathrm{d}t} = -cN_i$ when there is no inflow from other rows. This is not necessarily a correct result but we will use it as the basis for our model, because if there are more people in that row, it would be expected that a higher number of people will switch rows.

We use a different coefficient c' for the first and last row. The other rows should have similar coefficients because they both take in inflow from both sides. This gives us the following equations:

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = -c'N_1 + \frac{1}{2}cN_2$$

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = -cN_2 + c'N_1 + \frac{1}{2}cN_3$$

$$\frac{\mathrm{d}N_i}{\mathrm{d}t} = -cN_i + \frac{1}{2}cN_{i-1} + \frac{1}{2}cN_{i+1}$$

In steady state, $\frac{\mathrm{d}N_i}{\mathrm{d}t} = 0\ \forall i$. (This is also a not necessarily correct assumption) This gives:

$$N_2 = \frac{2c'}{c}N_1$$

$$N_3 = \frac{2c'}{c}N_1$$

and hence $N_i = \frac{2c'}{c}N_1$. This obviously does not give us a good model. Therefore, we introduce another distinct coefficient for the second and second last row c''. The equations become:

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = -c'N_1 + \frac{1}{2}c''N_2$$

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = -c''N_2 + c'N_1 + \frac{1}{2}cN_3$$

$$\frac{\mathrm{d}N_3}{\mathrm{d}t} = -cN_3 + \frac{1}{2}c''N_2 + \frac{1}{2}cN_4$$

$$\frac{\mathrm{d}N_i}{\mathrm{d}t} = -cN_i + \frac{1}{2}cN_{i-1} + \frac{1}{2}cN_{i+1}$$

In steady state, $\frac{\mathrm{d}N_i}{\mathrm{d}t} = 0\ \forall i$. This gives:

$$N_2 = \frac{2c'}{c''}N_1$$

$$N_3 = \frac{2c'}{c}N_1$$

and hence $N_i = \frac{2c'}{c}N_1$.

Now we attempt to find the proportion of total population in each row in the steady state.

$$N_t = \sum_{i=1}^{k} N_i = N_1\left(2 + 2 \cdot \frac{2c'}{c''} + (k-4)\frac{2c'}{c}\right)$$

Hence, $\frac{N_1}{N_t} = \frac{1}{2+2\cdot\frac{2c'}{c''}+(k-4)\frac{2c'}{c}}$.

From our simulation result, $\frac{N_1}{N_t} = \frac{0.7}{k}$. Hence,

$$\left(2 + 2 \cdot \frac{2c'}{c''} + (k-4)\frac{2c'}{c}\right) = \frac{1}{0.7}k$$

Comparing the coefficients of k, $\frac{2c'}{c} = \frac{1}{0.7} \Rightarrow \frac{c}{c'} = 1.4$. Comparing the constant term, $2 + \frac{4c'}{c''} - \frac{8c'}{c} = 0 \Rightarrow \frac{c'}{c''} = \frac{1}{4}\left(\frac{8}{1.4} - 2\right) = \frac{13}{14}$.

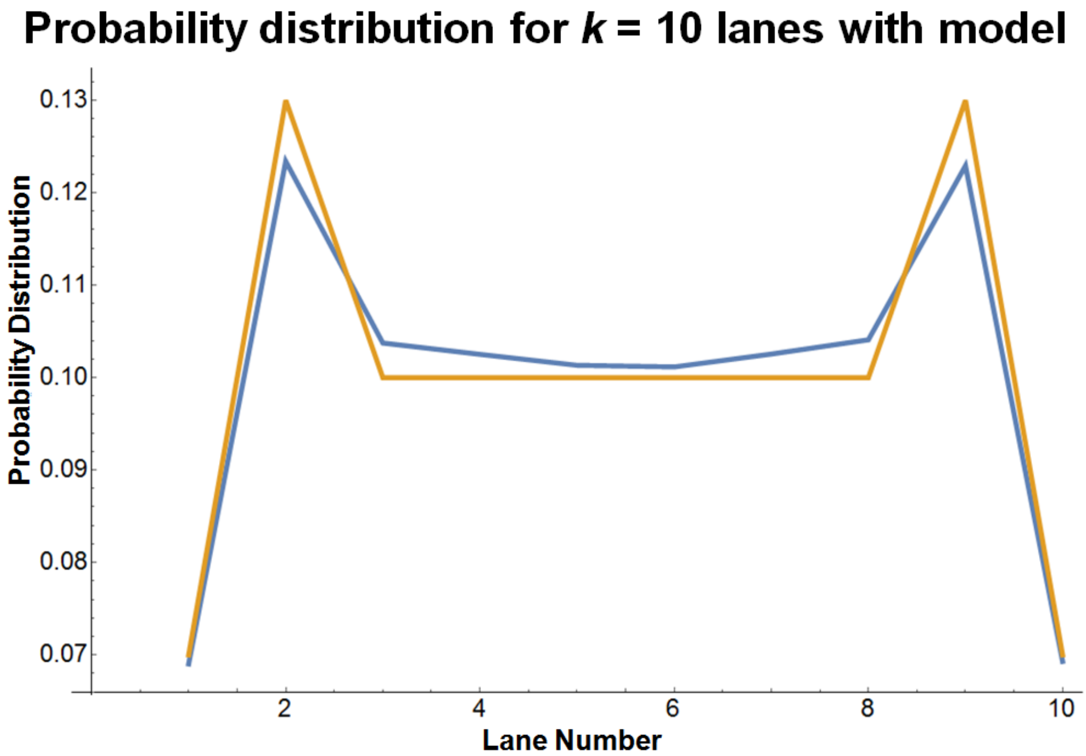
$$\frac{N_2}{N_t} = \frac{2c'}{c''}\frac{N_1}{N_t} = \frac{1.3}{k}$$

$$\frac{N_i}{N_t} = \frac{2c'}{c}\frac{N_1}{N_t} = \frac{1}{k}$$

This very simplified model gives us the following $P(x,k)$, where x is the row number.

$$P(x,k) = \begin{cases} \frac{0.7}{k}, & x = 1 \text{ or } k \\ \frac{1.3}{k}, & x = 2 \text{ or } k-1 \\ \frac{1}{k}, & \text{otherwise} \end{cases}$$

Plotting this with the data points for $k = 10$, we see a very close fit. Our simplified model might not be the most accurate, but it allows for visualisation and fast calculation.



End of Report

References:

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