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SIMC Section C: Busy footbridge

Definitions: $P(i,j)$ denote the probability that person A_i will meet person B_j.

Question 1: $p_{1,n} = \frac{1}{2}$, $p_{2,n} = \frac{3}{8}$, $q_1 = \frac{1}{2}$, $q_2 = \frac{3}{8}$, $q_3 = \frac{5}{16}$

$p_{1,n}$) Based on simulation result, $P(1,n) = \frac{1}{2}\ \forall n$. This is intuitive. If A_1 is stubborn, then whether he meets B_n depends on whether he is in the same lane as B_n initially, with a probability of $\frac{1}{2}$. If A_1 is polite, then whether he meets B_n depends on whether B_{n-1} and B_n are on the same lane, with a probability of $\frac{1}{2}$ because we can take B_n to be independently generated as B_{n-1}. If $n = 1$, it will still be a probability of $\frac{1}{2}$. Hence, $P(1,n) = \frac{1}{2}$.

$p_{2,n}$) From the simulation results, it seems that $P(2,1) = \frac{1}{2}$ and $P(2,n) = \frac{3}{8}$ for $n > 1$. The first part is intuitive. We can treat the spawning of A_2 to be independent from A_1. Therefore, whether A_2 will meet B_1 depends on whether A_2 is spawned in the same row as B_1 after A_1 passed the row of Bs. Hence, $P(2,1) = \frac{1}{2}$.

For the latter result, we consider a more general case. We calculate $P(m,n)$ where $n > 1$. We observe that a polite person among A will not change the order of B, while a stubborn person in A will push all the B to one side.

Before A_m, there are $m-1$ other As. If none of them are stubborn, then the order of B will be the same. Hence, we can treat A_m as A_1. This gives a probability of $\left(\frac{1}{2}\right)^{m-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^{m}$, where the first $\left(\frac{1}{2}\right)^{m-1}$ comes from everyone before A_m being polite. Otherwise, if at least one of $[A_1, A_2, \dots, A_{m-1}]$ is stubborn, then the Bs will fall into one row. Hence, if A_m is polite, then it has no chance of meeting B_n. Otherwise, if it is stubborn, then A_m will only meet B_n if they are in the same row initially, which has a probability of $\frac{1}{2}$. Hence, the probability here is $\left(1 - \left(\frac{1}{2}\right)^{m-1}\right) \cdot \frac{1}{2} \cdot \frac{1}{2}$.

Hence,

$$P(m,n) = \left(\frac{1}{2}\right)^{m} + \left(1 - \left(\frac{1}{2}\right)^{m-1}\right) \cdot \left(\frac{1}{2}\right)^{2} = \frac{1}{4} + \left(\frac{1}{2}\right)^{m+1}$$

$$p_{2,n} = P(2,n) = \frac{1}{4} + \left(\frac{1}{2}\right)^{3} = \frac{3}{8}$$

Note that the probability $P(m,n)$ is independent of n.

q_1) From the answer and derivation in $p_{1,n}$, we have $q_1 = \frac{1}{2}$.

q_2) Based on simulation result, $P(2, 10000) \approx 0.37510 \approx \frac{3}{8}$. From the answer and derivation in $p_{2,n}$, we can see $P(2,\infty) = \frac{3}{8}$.

q_3) Based on simulation result, $P(3, 10000) \approx 0.311820$. This is close to the value of $\frac{5}{16} \approx 0.3125$. From the answer and derivation in $p_{2,n}$, we can see that $P(3, \infty) = \frac{1}{4} + \left(\frac{1}{2}\right)^4 = \frac{5}{16}$.

Further analysis: We see that for large m, $P(m, n) = \frac{1}{4} + \left(\frac{1}{2}\right)^{m+1} \to \frac{1}{4}$. This makes sense for large m, there is a high probability that there is a stubborn A before the A we are considering; hence the Bs will be in the same row. Therefore, the only way for A and B to meet is when A is stubborn and in the same row as B as well, each with a probability of $\frac{1}{2}$.

Question 2: $p_{1,n} = \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right)$, $p_{2,n} = \frac{1}{6}\left(1 - \left(-\frac{1}{2}\right)^x\right)$, $q_1 = \frac{1}{3}$, $q_2 = \frac{1}{6}$ $q_3 = \frac{1}{6}$

$p_{1,n}$) We note that if B_n and B_{n-1} are in the same lane, then B_n will never collide with A_1. This is because if A_1 is not in the same row as B_{n-1}, then it will pass B_n as well. If A_1 is in the same row as B_{n-1}, then it will be deflected. If B_n and B_{n-1} are in different lanes, then A_1 will collide with B_n if A_1 does not collide with B_{n-1}. Let $P(x)$ denote the probability of A_1 colliding with B_x. Clearly, $P(1) = \frac{1}{2}$. From then on, the probability of A_1 not colliding with B_{x-1} is $1 - P(x-1)$. Since we can treat the spawning of B_x as independent of B_{x-1}, then the probability of A_1 colliding with B_x is:

$$P(x) = \frac{1}{2}\big(1 - P(x-1)\big) \Rightarrow P(x-1) + 2 \cdot P(x) = 1$$

This yields the solution $P(x) = \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^x\right)$. Checking, $P(1) = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$ which is true. $P(2) = \frac{1}{3}\left(1 - \frac{1}{4}\right) = \frac{1}{4}$. Simulation results showed $P(2) = 0.249630 \approx \frac{1}{4}$.

Hence, $p_{1,n} = \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right)$.

$p_{2,n}$) For A_2, clearly $P(2,1) = \frac{1}{2}$. Denote B_i' to be the B_i after A_1 passed through the row of Bs. For $i > 1$, we will define $Q(i)$ as the probability that B_i' and B_{i-1}' are in the same row after A_1 has passed through them. If B_i and B_{i-1} are initially in the same row, then B_i' and B_{i-1}' will also be in the same row. Otherwise, B_i and B_{i-1} are initially in different rows. If A_1 collided with B_{i-1}, then $B_{i-1}' = B_i'$ simply because A_1 will not want to switch. If A_1 did not collide with B_{i-1}, then whether $B_i' = B_{i-1}'$ depends on B_{i+1}. If $B_{i+1} = B_i$, then $B_i' \neq B_{i-1}'$. Otherwise, $B_i' = B_{i-1}'$.

$$Q(i) = \frac{1}{2} + \frac{1}{2}\left[P(i-1) + \frac{1}{2}\big(1 - P(i-1)\big)\right] = \frac{1}{2} + \frac{1}{4}[1 + P(i-1)] = \frac{3}{4} + \frac{1}{4}P(i-1)$$

$$Q(i) = \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^{i-1}\right) = \frac{3}{4} + \frac{1}{12}\left(1 - \left(-\frac{1}{2}\right)^{i-1}\right) = \frac{5}{6} - \frac{1}{6}\left(-\frac{1}{2}\right)^i$$

We now define $P'(x)$ to be the probability that A_2 meets B_x'.

If $B'_x = B'_{x-1}$, then there is no chance for A_2 to meet B'_x. Otherwise, $B'_x \neq B'_{x-1}$. If $x-1>1$, then this implies that $B'_{x-2} = B'_{x-1}$, otherwise B'_{x-1} would have shifted to be in the same row as B'_x. This also means that A_2 will not collide with B'_{x-1} and must collide with B'_x.

$$P'(x) = \big(1 - Q(x)\big) = \frac{1}{6}\left(1 - \left(-\frac{1}{2}\right)^x\right), \qquad x \geq 3$$

We assumed $x-1>1$, hence the above formula works for $x \geq 3$. For $x=1$, $P'(1) = \frac{1}{2}$. For $x=2$, case consideration gives $P'(2) = \frac{1}{16}$. Below is a table showing the comparison of simulation results and our formula:

x	Theoretical Result $P'(x)$	Simulation $P'(x)$
1	0.5	0.497790
2	0.0625	0.062160
3	0.1875	0.185850
4	0.15625	0.154290
5	0.171875	0.170900

q_1) We have previously $p_{1,n} = \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right)$. As $n \to \infty$, $p_{1,\infty} \to \frac{1}{3}$. From our simulation, $P(1{,}9999) = 0.333180 \approx \frac{1}{3}$.

q_2) We have previously $p_{2,n} = \frac{1}{6}\left(1 - \left(-\frac{1}{2}\right)^x\right)$. As $x \to \infty$, $p_{2,\infty} \to \frac{1}{6}$. From our simulation, $P(2, 9999) = 0.167670 \approx \frac{1}{6}$.

q_3) We make the observation that any further entrance of A_i into Bs will not change the arrangement of B except for possibly B_1. This is because after A_1 enters, all Bs will be in the formation of "trains" of more than or equal to length 2. A "train" here is defined to be a consecutive block of Bs. If there is a "train" of length 1 (i.e. a solo person standing in different lane as the person before AND after him, say B_x), then A_1 will either collide with B_{x-1} and deflect B_{x-1} to join B_x in the same row, OR A_1 will collide with B_x and deflect him to join B_{x-1}. Hence, future entrance of As will not affect the probability of $P(i, \infty)$. Hence $q_3 = \frac{1}{6}$.

Further analysis: We see that for large m, $P(m,n) = \frac{1}{4} + \left(\frac{1}{2}\right)^{m+1} \to \frac{1}{4}$. This makes sense for large m, there is a high probability that there is a stubborn A before the A we are considering; hence the Bs will be in the same row. Therefore, the only way for A and B to meet is when A is stubborn and in the same row as B as well, each with a probability of $\frac{1}{2}$.

Question 3: $p_{1,n} = \frac{1}{2}\, p_{2,n} = \frac{3}{8}$, $q_1 = \frac{1}{3}$, $q_2 = \frac{1}{6}\, q_3 = \frac{1}{6}$

$p_{1,n}$) We note that if B_n and B_{n-1} are in the same lane, then B_n will never collide with A_1. This is because if A_1 is not in the same row as B_{n-1}, then it will pass B_n as well. If A_1 is in the same row as B_{n-1}, then it will be deflected. If B_n and B_{n-1} are in different lanes, then A_1 will collide with B_n if A_1 does not collide with B_{n-1}. Let $P(x)$ denote the probability of A_1 colliding with B_x. Clearly, $P(1) = \frac{1}{2}$. From then on, the probability of A_1 not colliding with B_{x-1} is $1 - P(x-1)$. Since we can treat the spawning of B_x as independent of B_{x-1}, then the probability of A_1 colliding with B_x is:

$$P(x) = \frac{1}{2}\big(1 - P(x-1)\big) \Rightarrow P(x-1) + 2 \cdot P(x) = 1$$

This yields the solution $P(x) = \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^x\right)$. Checking, $P(1) = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$ which is true. $P(2) = \frac{1}{3}\left(1 - \frac{1}{4}\right) = \frac{1}{4}$. Simulation results showed $P(2) = 0.249630 \approx \frac{1}{4}$.

Hence, $p_{1,n} = \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right)$.

$p_{2,n}$) For A_2, clearly $P(2,1) = \frac{1}{2}$. Denote B_i' to be the B_i after A_1 passed through the row of Bs. For $i > 1$, we will define $Q(i)$ as the probability that B_i' and B_{i-1}' are in the same row after A_1 has passed through them. If B_i and B_{i-1} are initially in the same row, then B_i' and B_{i-1}' will also be in the same row. Otherwise, B_i and B_{i-1} are initially in different rows. If A_1 collided with B_{i-1}, then $B_{i-1}' = B_i'$ simply because A_1 will not want to switch. If A_1 did not collide with B_{i-1}, then whether $B_i' = B_{i-1}'$ depends on B_{i+1}. If $B_{i+1} = B_i$, then $B_i' \neq B_{i-1}'$. Otherwise, $B_i' = B_{i-1}'$.

$$Q(i) = \frac{1}{2} + \frac{1}{2}\left[P(i-1) + \frac{1}{2}\big(1 - P(i-1)\big)\right] = \frac{1}{2} + \frac{1}{4}[1 + P(i-1)] = \frac{3}{4} + \frac{1}{4}P(i-1)$$

$$Q(i) = \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^{i-1}\right) = \frac{3}{4} + \frac{1}{12}\left(1 - \left(-\frac{1}{2}\right)^{i-1}\right)$$

We now define $P'(x)$ to be the probability that A_2 meets B_x'.

If $B_x' = B_{x-1}'$, then there is no chance for A_2 to meet B_x'. Otherwise, $B_x' \neq B_{x-1}'$. This implies $B_x' = B_{x+1}'$. If A_2 is on the same lane as B_{x-1}', then A_2 will not meet B_x'. Otherwise, A_2 will meet B_x'.

$$P'(x) = \big(1 - Q(x)\big)\big(1 - P'(x-1)\big) = \frac{1}{6}\left(1 - \left(-\frac{1}{2}\right)^x\right)\big(1 - P'(x-1)\big)$$

Solving the recurrence, we have:

q_1) We have previously $p_{1,n} = \frac{1}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right)$. As $n \to \infty$, $p_{1,\infty} \to \frac{1}{3}$. From our simulation, $P(1,100000) = 0.333180 \approx \frac{1}{3}$.

q_2) Based on simulation result, $P(2,10000) \approx 0.37510 \approx \frac{3}{8}$. From the answer and derivation in $p_{2,n}$, we can see $P(2,\infty) = \frac{3}{8}$.

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References:

NUSH Champion Award Report:
https://www.nushigh.edu.sg/qql/slot/u90/file/simc/mathmodel/SIMC2018ChampionAwardReport.pdf