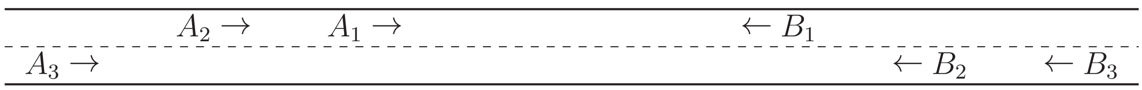
Section C: Busy footbridge

Consider a long east-west footbridge that is just wide enough for two people to pass each other, and hence can be considered as having two lanes. In the morning, the bridge is initially empty. People arrive in a steady stream from both ends and join either lane randomly with equal probability.

When two people walking in opposite directions meet, they pass each other without incident if they are walking in different lanes. Otherwise there is a 'collision' and one of the two must change over to the other lane before they can pass each other and continue walking. In the absence of collisions, people remain in their lane when walking, and we assume that people walking in the same direction are sufficiently spread out that they do not overtake or interact with each other.

Let A_m denote the m^{th} person entering heading east and let B_n denote the n^{th} person entering heading west. An example of the lanes chosen by A_1, A_2, A_3 and B_1, B_2, B_3 when entering the footbridge is shown in the figure below. In this case, because A_1 is walking on their left and B_1 is walking on their right, the two will have a collision. In the collision, if A_1 moves over (and hence B_1 stays in their lane), then A_1 will collide with B_2 and B_1 will collide with A_2. If instead A_1 stays (and hence B_1 moves over), then A_1 will pass B_2 and B_3, and B_1 will pass A_2 but collide with A_3.



Let $p_{m,n}$ be the probability that A_m and B_n collide, assuming that they meet before either has left the bridge. Let q_m denote the limiting value of $p_{m,n}$ as n becomes large. (Throughout this question, you may assume that this limiting value exists.) Calculate $p_{1,n}$, $p_{2,n}$, q_1, q_2 and q_3 in the following cases.

Question 1. Each person heading east is either 'stubborn' (i.e. always stays in their lane) or 'polite' (i.e. always moves over in a collision) with equal probability. [*10 points*]

Question 2. When two people collide, the person heading east looks ahead and chooses to stay in their lane or move over in such a way as to maximise the time until their next collision. (Since everyone heading east is following this strategy, each of them can predict the outcome of every collision in front of them.) [*10 points*]

Question 3. When two people collide, a random one of them moves over, with equal probability. [*10 points*]

Question 4. What other strategies might people use to resolve a collision? How can you make the model more realistic? How do the collision probabilities change? [*10 points*]