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SIMC Section C: Busy footbridge

Definitions: $P(i, j)$ denote the probability that person A_i will meet person B_j.

Question 1: $p_{1,n} = \frac{1}{2}$, $p_{2,n} = \frac{3}{8}$, $q_1 = \frac{1}{2}$, $q_2 = \frac{3}{8}$, $q_3 = \frac{5}{16}$

$p_{1,n}$) Based on simulation result, $P(1, n) = \frac{1}{2}\ \forall n$. This is intuitive. If A_1 is stubborn, then whether he meets B_n depends on whether he is in the same lane as B_n initially, with a probability of $\frac{1}{2}$. If A_1 is polite, then whether he meets B_n depends on whether B_{n-1} and B_n are on the same lane, with a probability of $\frac{1}{2}$ because we can take B_n to be independently generated as B_{n-1}. If $n = 1$, it will still be a probability of $\frac{1}{2}$. Hence, $P(1, n) = \frac{1}{2}$.

$p_{2,n}$) From the simulation results, it seems that $P(2,1) = \frac{1}{2}$ and $P(2, n) = \frac{3}{8}$ for $n > 1$. The first part is intuitive. We can treat the spawning of A_2 to be independent from A_1. Therefore, whether A_2 will meet B_1 depends on whether A_2 is spawned in the same row as B_1 after A_1 passed the row of Bs. Hence, $P(2,1) = \frac{1}{2}$.

For the latter result, we consider a more general case. We calculate $P(m, n)$ where $n > 1$. We observe that a polite person among A will not change the order of B, while a stubborn person in A will push all the B to one side.

Before A_m, there are $m - 1$ other As. If none of them are stubborn, then the order of B will be the same. Hence, we can treat A_m as A_1. This gives a probability of $\left(\frac{1}{2}\right)^{m-1} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^{m}$, where the first $\left(\frac{1}{2}\right)^{m-1}$ comes from everyone before A_m being polite. Otherwise, if at least one of $[A_1, A_2, \dots, A_{m-1}]$ is stubborn, then the Bs will fall into one row. Hence, if A_m is polite, then it has no chance of meeting B_n. Otherwise, if it is stubborn, then A_m will only meet B_n if they are in the same row initially, which has a probability of $\frac{1}{2}$. Hence, the probability here is $\left(1 - \left(\frac{1}{2}\right)^{m-1}\right) \cdot \frac{1}{2} \cdot \frac{1}{2}$.

Hence,

$$P(m, n) = \left(\frac{1}{2}\right)^{m} + \left(1 - \left(\frac{1}{2}\right)^{m-1}\right) \cdot \left(\frac{1}{2}\right)^{2} = \frac{1}{4} + \left(\frac{1}{2}\right)^{m+1}$$

$$p_{2,n} = P(2, n) = \frac{1}{4} + \left(\frac{1}{2}\right)^{3} = \frac{3}{8}$$

Note that the probability $P(m, n)$ is independent of n.

q_1) From the answer and derivation in $p_{1,n}$, we have $q_1 = \frac{1}{2}$.

q_2) Based on simulation result, $P(2, 10000) \approx 0.37510 \approx \frac{3}{8}$. From the answer and derivation in $p_{2,n}$, we can see $P(2, \infty) = \frac{3}{8}$.

q_3) Based on simulation result, $P(3, 10000) \approx 0.311820$. This is close to the value of $\frac{5}{16} \approx 0.3125$. From the answer and derivation in $p_{2,n}$, we can see that $P(3, \infty) = \frac{1}{4} + \left(\frac{1}{2}\right)^4 = \frac{5}{16}$.

Further analysis: We see that for large m, $P(m, n) = \frac{1}{4} + \left(\frac{1}{2}\right)^{m+1} \to \frac{1}{4}$. This makes sense for large m, there is a high probability that there is a stubborn A before the A we are considering; hence the Bs will be in the same row. Therefore, the only way for A and B to meet is when A is stubborn and in the same row as B as well, each with a probability of $\frac{1}{2}$.

Question 2: $p_{1,n} = \frac{1}{2}$, $p_{2,n} = \frac{3}{8}$, $q_1 = \frac{1}{3}$, $q_2 = \frac{1}{6}$ $q_3 = \frac{1}{6}$

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Hence,

$$P(m, n) = \left(\frac{1}{2}\right)^m + \left(1 - \left(\frac{1}{2}\right)^{m-1}\right) \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{4} + \left(\frac{1}{2}\right)^{m+1}$$

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References:

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