APPENDIX I Discrete-Time Evaluation of the Time Response

I.1 INTRODUCTION

The response of a system represented by a state vector differential equation can be obtained by using a **discrete-time approximation**. The discrete-time approximation is based on the division of the time axis into sufficiently small time increments. Then the values of the state variables are evaluated at the successive time intervals; that is, $t = 0, T, 2T, 3T, \ldots,$ where T is the increment of time: $\Delta t = T$. This approach is a familiar method utilized in numerical analysis and digital computer numerical methods. If the time increment T is sufficiently small compared with the time constants of the system, the response evaluated by discrete-time methods will be reasonably accurate.

The linear state vector differential equation is written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}. \tag{I.1}$$

The basic definition of a derivative is

$$\dot{\mathbf{x}}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{x}(t + \Delta t) - \mathbf{x}(t)}{\Delta t}. \tag{I.2}$$

Therefore, we can use this definition of the derivative and determine the value of $\mathbf{x}(t)$ when t is divided in small intervals $\Delta t = T$. Thus, **approximating** the derivative as

$$\boxed{\dot{\mathbf{x}} = \frac{\mathbf{x}(t + T) - \mathbf{x}(t)}{T},} \tag{I.3}$$

we substitute into Equation (I.1) to obtain

$$\frac{\mathbf{x}(t + T) - \mathbf{x}(t)}{T} \approx \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t). \tag{I.4}$$

Solving for $\mathbf{x}(t + T)$, we have

$$\begin{aligned} \mathbf{x}(t + T) &\approx T\mathbf{A}\mathbf{x}(t) + \mathbf{x}(t) + T\mathbf{B}\mathbf{u}(t) \\ &\approx (T\mathbf{A} + \mathbf{I})\mathbf{x}(t) + T\mathbf{B}\mathbf{u}(t), \end{aligned} \tag{I.5}$$

where t is divided into intervals of width T. Therefore, the time t is written as $t = kT$, where k is an integer index so that $k = 0, 1, 2, 3, \ldots$. Then Equation (I.5) is written as

$$\mathbf{x}[(k + 1)T] \approx (T\mathbf{A} + \mathbf{I})\mathbf{x}(kT) + T\mathbf{Bu}(kT). \tag{I.6}$$

Therefore, the value of the state vector at the $(k + 1)$st time instant is evaluated in terms of the values of $\mathbf{x}$ and $\mathbf{u}$ at the kth time instant. Equation (I.7) can be rewritten as

$$\mathbf{x}(k + 1) \approx \psi(T)\mathbf{x}(k) + T\mathbf{Bu}(k), \tag{I.7}$$

where $\psi(T) = T\mathbf{A} + \mathbf{I}$ and the symbol T is omitted from the arguments of the variables. Equation (I.7) clearly relates the resulting operation for obtaining $\mathbf{x}(t)$ by evaluating the discrete-time approximation $\mathbf{x}(k + 1)$ in terms of the previous value $\mathbf{x}(k)$. This recurrence operation, known as **Euler's method**, is a sequential series of calculations and is very suitable for digital computer calculation. Other integration approaches, such as the popular Runge–Kutta methods, can also be used to evaluate the time response of Equation (I.1). To illustrate this approximate approach, let us reconsider the evaluation of the response of the RLC network of Figure I.1.

EXAMPLE I.1 Response of the *RLC* network

We shall evaluate the time response of the RLC network without determining the transition matrix, by using the discrete-time approximation. We define the state vector

$$\mathbf{x} = \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

and let $R = 3$, $L = 1$, and $C = 1/2$. Then the state vector differential equation is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & -1/C \\ 1/L & -R/L \end{bmatrix}\mathbf{x} + \begin{bmatrix} +1/C \\ 0 \end{bmatrix}u(t)$$

$$= \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} +2 \\ 0 \end{bmatrix}u(t). \tag{I.8}$$

Now we must choose a sufficiently small time interval T so that the approximation of the derivative (Equation I.3) is reasonably accurate and so that the solution to Equation (I.6) is accurate. Usually we choose T to be less than one-half of the smallest time constant of the system. Therefore, since the shortest time constant of

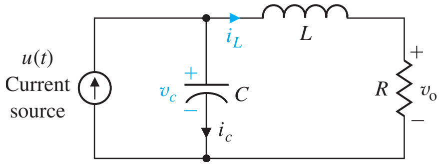

FIGURE I.1
An *RLC* circuit.

this system is 0.5 s [recalling that the characteristic equation is $[(s+1)(s+2)]$, we might choose $T = 0.2$. Note that as we decrease the increment size, the number of calculations increases proportionally. Using $T = 0.2$ s, we see that Equation (I.7) is

$$\mathbf{x}(k+1) \approx (0.2\mathbf{A} + \mathbf{I})\mathbf{x}(k) + 0.2\mathbf{B}u(k). \tag{I.9}$$

Therefore,

$$\psi(T) = \begin{bmatrix} 1 & -0.4 \\ 0.2 & 0.4 \end{bmatrix}, \tag{I.10}$$

and

$$T\mathbf{B} = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}. \tag{I.11}$$

Now let us evaluate the response of the system when $x_1(0) = x_2(0) = 1$ and $u(t) = 0$. The response at the first instant, when $t = T$, or $k = 0$, is

$$\mathbf{x}(1) \approx \begin{bmatrix} 1 & -0.4 \\ 0.2 & 0.4 \end{bmatrix}\mathbf{x}(0) = \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix}. \tag{I.12}$$

Then the response at the time $t = 2T = 0.4$ second, or $k = 1$, is

$$\mathbf{x}(2) \approx \begin{bmatrix} 1 & -0.4 \\ 0.2 & 0.4 \end{bmatrix}\mathbf{x}(1) = \begin{bmatrix} 0.36 \\ 0.36 \end{bmatrix}. \tag{I.13}$$

The value of the response as $k = 2, 3, 4, \ldots$ is then evaluated in a similar manner.

Now let us compare the actual response of the system evaluated in the previous section. We will use the transition matrix with the approximate response determined by the discrete-time approximation. The exact value of the state variables, when $x_1(0) = x_2(0) = 1$, is $x_1(t) = x_2(t) = e^{-2t}$. Therefore, the exact values can be readily calculated and compared with the approximate values of the time response in Table I.1. The approximate time response values for $T = 0.1$ second are also given in Table I.1. The error, when $T = 0.2$, is approximately a constant equal to 0.07, and thus the percentage error compared to the initial value is 7%. When T is equal to 0.1 second, the percentage error compared to the initial value is approximately 3.5%. If we use $T = 0.05$ and time $t = 0.2$ second, the value of the approximation is $x_1(t) = 0.655$, and the error has been reduced to 1.5% of the initial value. ■

Table I.1

Time t	0	0.2	0.4	0.6	0.8
Exact $x_1(t)$	1	0.67	0.448	0.30	0.20
Approximate $x_1(t)$, $T = 0.1$	1	0.64	0.41	0.262	0.168
Approximate $x_1(t)$, $T = 0.2$	1	0.60	0.36	0.216	0.130

EXAMPLE I.2 **Time response of an epidemic**

The spread of an epidemic disease can be described by a set of differential equations. The population under study is made up of three groups, x_1, x_2, and x_3, such that the group x_1 is susceptible to the epidemic disease, group x_2 is infected with the disease, and group x_3 has been removed from the initial population. The removal of x_3 will be due to immunization, death, or isolation from x_1. The feedback system can be represented by the following equations:

$$\frac{dx_1}{dt} = -\alpha x_1 - \beta x_2 + u_1(t),$$
$$\frac{dx_2}{dt} = \beta x_1 - \gamma x_2 + u_2(t),$$
$$\frac{dx_3}{dt} = \alpha x_1 + \gamma x_2.$$

When the constants are $\alpha = \beta = \gamma = 1$, we have

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{u}. \tag{I.14}$$

The characteristic equation of this system is $s(s^2 + 2s + 2) = 0$, and thus the system has complex roots. Let us determine the transient response of the spread of disease when the rate of new susceptibles is zero, that is, when $u_1 = 0$. The rate of adding new infectives is represented by $u_2(0) = 1$ and $u_2(k) = 0$ for $k \geq 1$; that is, one new infective is added at the initial time only (this is equivalent to a pulse input). The time constant of the complex roots is $1/(\zeta\omega_n) = 2$ seconds, and therefore we will use $T = 0.2$ second. (Note that the actual time units might be months and the units of the input in thousands.)

Then the discrete-time equation is

$$\mathbf{x}(k + 1) = \begin{bmatrix} 0.8 & -0.2 & 0 \\ 0.2 & 0.8 & 0 \\ 0.2 & 0.2 & 1 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0 \\ 0.2 \\ 0 \end{bmatrix}u_2(k). \tag{I.15}$$

Therefore, the response at the first instant $t = T$ is obtained when $k = 0$ as

$$\mathbf{x}(1) = \begin{bmatrix} 0 \\ 0.2 \\ 0 \end{bmatrix} \tag{I.16}$$

when $x_1(0) = x_2(0) = x_3(0) = 0$. Then the input $u_2(k)$ is zero for $k \geq 1$, and the response at $t = 2T$ is

$$\mathbf{x}(2) = \begin{bmatrix} 0.8 & -0.2 & 0 \\ 0.2 & 0.8 & 0 \\ 0.2 & 0.2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0.2 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.04 \\ 0.16 \\ 0.04 \end{bmatrix}. \tag{I.17}$$

The response at $t = 3T$ is then

$$\mathbf{x}(3) = \begin{bmatrix} 0.8 & -0.2 & 0 \\ 0.2 & 0.8 & 0 \\ 0.2 & 0.2 & 1 \end{bmatrix} \begin{bmatrix} -0.04 \\ 0.16 \\ 0.04 \end{bmatrix} = \begin{bmatrix} -0.064 \\ 0.120 \\ 0.064 \end{bmatrix},$$

and the ensuing values can then be readily evaluated. Of course, the actual physical value of x_1 cannot become negative. The negative value of x_1 is obtained as a result of an inadequate model.

The discrete-time approximate method is particularly useful for evaluating the time response of nonlinear systems. The basic state vector differential equation can be written as

$$\boxed{\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t),} \tag{I.18}$$

where **f** is a function, not necessarily linear, of the state vector **x** and the input vector **u**. The column vector **f** is the column matrix of functions of **x** and **u**. If the system is a linear function of the control signals, Equation (I.18) becomes

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{B}\mathbf{u}. \tag{I.19}$$

If the system is not time-varying—that is, if the coefficients of the differential equation are constants—Equation (I.19) is then

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{B}\mathbf{u}. \tag{I.20}$$

Let us consider Equation (I.20) for a nonlinear system and determine the discrete-time approximation. Using Equation (I.3) as the approximation to the derivative, we have

$$\frac{\mathbf{x}(t + T) - \mathbf{x}(t)}{T} = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}\mathbf{u}(t). \tag{I.21}$$

Therefore, solving for $\mathbf{x}(k + 1)$ when $t = kT$, we obtain

$$\mathbf{x}(k + 1) = \mathbf{x}(k) + T[\mathbf{f}(\mathbf{x}(k)) + \mathbf{B}\mathbf{u}(k)]. \tag{I.22}$$

Similarly, the general discrete-time approximation to Equation (I.18) is

$$\mathbf{x}(k + 1) = \mathbf{x}(k) + T\mathbf{f}(\mathbf{x}(k), \mathbf{u}(k), k). \tag{I.23}$$

Now let us consider the previous example again, but when the system is nonlinear. ■

EXAMPLE I.3 Improved model of an epidemic

The spread of an epidemic disease is actually best represented by a set of nonlinear equations

$$\begin{aligned} \dot{x}_1 &= -\alpha x_1 - \beta x_1 x_2 + u_1(t), \\ \dot{x}_2 &= \beta x_1 x_2 - \gamma x_2 + u_2(t), \\ \dot{x}_3 &= \alpha x_1 + \gamma x_2, \end{aligned} \tag{I.24}$$

where the interaction between the groups is represented by the nonlinear term $x_1 x_2$. As in the previous example, we will let $\alpha = \beta = \gamma = 1$, and $u_1(t) = 0$. Also, $u_2(0) = 1$, and $u_2(k) = 0$ for $k \geq 1$. We will select the time increment as $T = 0.2$ second and the initial conditions as $\mathbf{x}^T(0) = [1 \quad 0 \quad 0]$. Then, substituting $t = kT$ and

$$\dot{x}_i(k) = \frac{x_i(k+1) - x_i(k)}{T} \tag{I.25}$$

into Equation (I.24), we obtain

$$\begin{aligned}
\frac{x_1(k+1) - x_1(k)}{T} &= -x_1(k) - x_1(k)x_2(k), \\
\frac{x_2(k+1) - x_2(k)}{T} &= +x_1(k)x_2(k) - x_2(k) + u_2(k), \\
\frac{x_3(k+1) - x_3(k)}{T} &= x_1(k) + x_2(k).
\end{aligned} \tag{I.26}$$

Solving these equations for $x_i(k+1)$ and recalling that $T = 0.2$, we have

$$\begin{aligned}
x_1(k+1) &= 0.8x_1(k) - 0.2x_1(k)x_2(k), \\
x_2(k+1) &= 0.8x_2(k) + 0.2x_1(k)x_2(k) + 0.2u_2(k), \\
x_3(k+1) &= x_3(k) + 0.2x_1(k) + 0.2x_2(k).
\end{aligned} \tag{I.27}$$

Then the response at the first instant $t = T$ is

$$\begin{aligned}
x_1(1) &= 0.8x_1(0) = 0.8, \\
x_2(1) &= 0.2u_2(k) = 0.2, \\
x_3(1) &= 0.2x_1(0) = 0.2.
\end{aligned}$$

Again using Equation (I.27) and noting that $u_2(1) = 0$, we have

$$\begin{aligned}
x_1(2) &= 0.8x_1(1) - 0.2x_1(1)x_2(1) = 0.608, \\
x_2(2) &= 0.8x_2(1) + 0.2x_1(1)x_2(1) = 0.192, \\
x_3(2) &= x_3(1) + 0.2x_1(1) + 0.2x_2(1) = 0.40.
\end{aligned} \tag{I.28}$$

At the third instant, when $t = 3T$, we obtain

$$x_1(3) = 0.463, \qquad x_2(3) = 0.177, \qquad x_3(3) = 0.56.$$

The evaluation of the ensuing values follows in a similar manner. We note that the response of the nonlinear system differs considerably from the response of the linear model considered in the previous example. ■

The evaluation of the time response of the state variables of linear systems is readily accomplished by using either (1) the transition matrix approach or (2) the discrete-time approximation. The transition matrix of linear systems is readily obtained from the signal-flow graph state model. For a nonlinear system, the discrete-time approximation provides a suitable approach, and the discrete-time approximation method is particularly useful if a digital computer is used for numerical calculations.