APPENDIX

G Complex Numbers

G.1 A COMPLEX NUMBER

We all are familiar with the solution of the algebraic equation

$$x^2 - 1 = 0, \tag{G.1}$$

which is $x = 1$. However, we often encounter the equation

$$x^2 + 1 = 0. \tag{G.2}$$

A number that satisfies Eq. (G.2) is not a real number. We note that Eq. (G.2) may be written as

$$x^2 = -1, \tag{G.3}$$

and we denote the solution of Eq. (G.3) by the use of an imaginary number $j1$, so that

$$j^2 = -1, \tag{G.4}$$

and

$$\boxed{j = \sqrt{-1}.} \tag{G.5}$$

An **imaginary number** is defined as the product of the imaginary unit j with a real number. Thus we may, for example, write an imaginary number as jb. A **complex number** is the sum of a real number and an imaginary number, so that

$$\boxed{c = a + jb} \tag{G.6}$$

where a and b are real numbers. We designate a as the real part of the complex number and b as the imaginary part and use the notation

$$\mathrm{Re}\{c\} = a, \tag{G.7}$$

and

$$\mathrm{Im}\{c\} = b. \tag{G.8}$$

G.2 RECTANGULAR, EXPONENTIAL, AND POLAR FORMS

The complex number $a + jb$ may be represented on a rectangular coordinate place called a **complex plane.** The complex plane has a real axis and an imaginary axis, as shown in Fig. G.1. The complex number c is the directed line identified as c with coordinates a, b. The **rectangular form** is expressed in Eq. (G.6) and pictured in Fig. G.1.

An alternative way to express the complex number c is to use the distance from the origin and the angle θ, as shown in Fig. G.2. The **exponential form** is written as

$$\boxed{c = re^{j\theta},} \tag{G.9}$$

where

$$r = (a^2 + b^2)^{1/2}, \tag{G.10}$$

and

$$\theta = \tan^{-1}(b/a). \tag{G.11}$$

Note that $a = r\cos\theta$ and $b = r\sin\theta$.

The number r is also called the **magnitude** of c, denoted as $|c|$. The angle θ can also be denoted by the form $\angle\theta$. Thus we may represent the complex number in **polar form** as

$$c = |c)\angle\theta = r\angle\theta. \tag{G.12}$$

EXAMPLE G.1 Exponential and polar forms

Express $c = 4 + j3$ in exponential and polar form.

Solution First sketch the complex plane diagram as shown in Fig. G.3. Then find r as

$$r = (4^2 + 3^2)^{1/2} = 5,$$

and θ as

$$\theta = \tan^{-1}(3/4) = 36.9°.$$

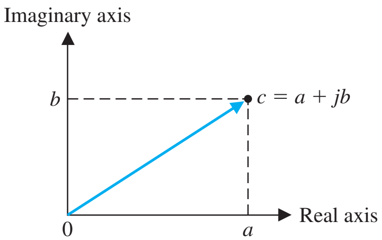

FIGURE G.1 Rectangular form of a complex number.

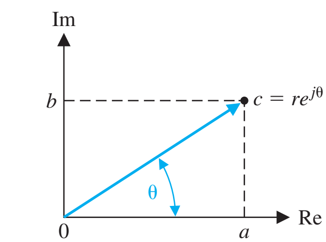

FIGURE G.2 Exponential form of a complex number.

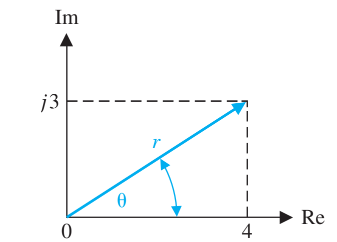

FIGURE G.3 Complex plane for Example G.1.

The exponential form is then

$$c = 5e^{j36.9°}.$$

The polar form is

$$c = 5\underline{/36.9°}. \blacksquare$$

G.3 MATHEMATICAL OPERATIONS

The **conjugate** of the complex number $c = a + jb$ is called c^* and is defined as

$$\boxed{c^* = a - jb.} \tag{G.13}$$

In polar form we have

$$\boxed{c^* = r\underline{/-\theta}.} \tag{G.14}$$

To add or subtract two complex numbers, we add (or subtract) their real parts and their imaginary parts. Therefore if $c = a + jb$ and $d = f + jg$, then

$$c + d = (a + jb) + (f + jg) = (a + f) + j(b + g). \tag{G.15}$$

The multiplication of two complex numbers is obtained as follows (note $j^2 = -1$):

$$\begin{aligned} cd &= (a + jb)(f + jg) \\ &= af + jag + jbf + j^2bg \\ &= (af - bg) + j(ag + bf). \end{aligned} \tag{G.16}$$

Alternatively we use the polar form to obtain

$$cd = (r_1\underline{/\theta_1})(r_2\underline{/\theta_2}) = r_1r_2\underline{/\theta_1 + \theta_2}, \tag{G.17}$$

where

$$c = r_1\underline{/\theta_1}, \quad \text{and} \quad d = r_2\underline{/\theta_2}.$$

Division of one complex number by another complex number is easily obtained using the polar form as follows:

$$\frac{c}{d} = \frac{r_1\underline{/\theta_1}}{r_2\underline{/\theta_2}} = \frac{r_1}{r_2}\underline{/\theta_1 - \theta_2}. \tag{G.18}$$

It is easiest to add and subtract complex numbers in rectangular form and to multiply and divide them in polar form.

A few useful relations for complex numbers are summarized in Table G.1.

Table G.1 Useful Relationships for Complex Numbers

(1) $\dfrac{1}{j} = -j$
(2) $(-j)(j) = 1$
(3) $j^2 = -1$
(4) $1\underline{/\pi/2} = j$
(5) $c^k = r^k\underline{/k\theta}$

EXAMPLE G.2 Complex number operations

Find $c + d$, $c - d$, cd, and c/d when $c = 4 + j3$ and $d = 1 - j$.

Solution First we will express c and d in polar form as

$$c = 5\underline{/36.9^\circ}, \quad \text{and} \quad d = \sqrt{2}\underline{/-45^\circ}.$$

Then, for addition, we have

$$c + d = (4 + j3) + (1 - j) = 5 + j2.$$

For subtraction we have

$$c - d = (4 + j3) - (1 - j) = 3 + j4.$$

For multiplication we use the polar form to obtain

$$cd = (5\underline{/36.9^\circ})(\sqrt{2}\underline{/-45^\circ}) = 5\sqrt{2}\underline{/-8.1^\circ}.$$

For division we have

$$\frac{c}{d} = \frac{5\underline{/36.9^\circ}}{\sqrt{2}\underline{/-45^\circ}} = \frac{5}{\sqrt{2}}\underline{/81.9^\circ}. \ \blacksquare$$