Math 132 Homework 6

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4.1.11 $\sum_{n=0}^{\infty} \frac{e^{in\frac{\pi}{2}}}{2^n}$

Answer: Let $a = \frac{e^{i\frac{\pi}{2}}}{2}$; since $|a| = \left|\frac{e^{i\frac{\pi}{2}}}{2}\right| = \frac{1}{2} < 1$, by geometric series, $\sum_{n=0}^{\infty} \frac{e^{in\frac{\pi}{2}}}{2^n}$ converges to $\frac{1}{1-a} = \frac{2}{2-e^{i\frac{\pi}{2}}}$.

4.1.12 $\sum_{n=0}^{\infty} \left(\frac{1+i}{2}\right)^n$

Answer: Let $a = \frac{1+i}{2}$; since $|a| = \left|\frac{1+i}{2}\right| = \frac{1}{4} < 1$, then by geometric series $\sum_{n=0}^{\infty} \left(\frac{1+i}{2}\right)^n$ converges to $\frac{1}{1-a} = \frac{2}{1-i}$.

4.1.13 $\sum_{n=3}^{\infty} \frac{3-i}{(1+i)^n}$

Answer: Let $a = \frac{1}{1+i}$, then since $|a| = \left|\frac{1}{1+i}\right| = \frac{\sqrt{2}}{2} < 1$, by geometric series $\sum_{n=0}^{\infty} \frac{1}{(1+i)^n}$ converges to $\frac{1}{1-a} = \frac{1+i}{i}$. Then by substitution we have $\sum_{n=3}^{\infty} \frac{3-i}{(1+i)^n} = \frac{3-i}{(1+i)^3} \sum_{n=0}^{\infty} \frac{1}{(1+i)^n} = \frac{3-i}{(1+i)^3} \cdot \frac{1+i}{i} = \frac{3-i}{i(1+i)^2} = -\frac{3}{2} + \frac{i}{2}$.

4.1.41 The nth partial sum of a series is $s_n = \frac{i}{n}$. Does the series converge or diverge? If it does converge, what is its limit?

Answer: Since $\left|\lim_{n\to\infty} s_n\right| = \left|\lim_{n\to\infty} \frac{i}{n}\right| = \lim_{n\to\infty} \left|\frac{i}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0$, the series converges to 0.

4.2.1 $f_n(x) = \frac{\sin nx}{n}, 0 \le x \le \pi$

(a) **Answer**: $\lim_{n\to\infty} \frac{\sin nx}{n} = 0$, so $f_n \to f$ pointwise for $f(x) = 0$.

(b) **Answer**: Since $|\sin nx| \le 1$ for all x, we have $|f_n(x) - f(x)| = \left|\frac{\sin nx}{n} - 0\right| \to 0$ as $n \to \infty$, then f_n converge uniformly by Jumping Prop.

(c) **Answer**: N/A; f_n converges uniformly.

4.2.2 $f_n(x) = \frac{\sin nx}{nx}, 0 < x \le \pi$

(a) **Answer**: $\lim_{n\to\infty} \dfrac{\sin nx}{nx} = 0$ since $\left|\dfrac{\sin nx}{nx}\right| \leq \dfrac{1}{n|x|} \to 0$, so $f_n \to f$ pointwise for $f(x) = 0$.

(b) **Answer**: Let $x_n = \dfrac{1}{n}$, then $f_n(x_n) = \dfrac{\sin 1}{1} \geq 0$. Then $|f_n(x_n) - f(x_n)| = \left|\dfrac{\sin 1}{1} - 0\right| \neq 0$. Therefore f_n does not converge uniformly by Jumping Prop.

(c) **Answer**: Yes, for interval $[a, \pi]$ where $a \geq 0$. We have $M_n = \sup\limits_{[a,\pi]} |f_n(x) - f(x)| = \sup\limits_{[a,\pi]} \left|\dfrac{\sin nx}{nx}\right| \leq \dfrac{1}{na} \to 0$ as $n \to \infty$, so f_n converges uniformly.

4.2.13 $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}, |z| \leq 1$

Answer: $|f_k(z)| = \left|\dfrac{z^n}{n(n+1)}\right| = \dfrac{|z^n|}{n(n+1)}$; since $|z| \leq 1$, we have $|z^n| \leq 1$. Then we can define M_k such that $|f_k(z)| \leq \dfrac{1}{n(n+1)} = M_k$, so $\displaystyle\sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty} \frac{1}{n(n+1)} = \sum_{k=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$, which converges to 1 by cancelling out adjacent terms in the telescoping series. Therefore the given series converges uniformly by Weierstrass M-test.

4.2.17 $\displaystyle\sum_{n=0}^{\infty} \left(\frac{z+2}{5}\right)^n, |z| \leq 2$

Answer: Since $|z| \leq 2$, we have $|f_k(z)| = \left|\left(\dfrac{z+2}{5}\right)^n\right| \leq \left(\dfrac{4}{5}\right)^n$. Then we can define $M_k = \left(\dfrac{4}{5}\right)^n$ and by geometric series $\displaystyle\sum_{k=0}^{\infty} M_k$ is convergent. Therefore the given series converge by Weierstrass M-test.

4.2.19 $\displaystyle\sum_{n=0}^{\infty} \frac{(z+1-3i)^n}{4^n}, |z-3i| \leq 0.5$

Answer: Since $|z - 3i| \leq 0.5$, by triangle inequality we have $|f_k(z)| = \left|\dfrac{(z+1-3i)^n}{4^n}\right| \leq \dfrac{1.5^n}{4^n} = \left(\dfrac{3}{8}\right)^n$. Then we can define $M_k = \left(\dfrac{3}{8}\right)^n$ and by geometric series $\displaystyle\sum_{k=0}^{\infty} M_k$ is convergent. Therefore the given series is convergent by Weierstrass M-test.

4.3.1 $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{z^n}{2n+1}$

Answer: Using the Ratio Test, we have $\rho = \displaystyle\lim_{k\to\infty} \frac{|C_{k+1}|}{|C_k|} = \lim_{k\to\infty} \left|\frac{(-1)^{k+1} z^{k+1}}{2k+3}\right| \cdot \left|\frac{2k+1}{(-1)^k z^k}\right| = \lim_{k\to\infty} \left|\frac{-z(2k+1)}{2k+3}\right| = |z| \lim_{k\to\infty} \frac{2k+1}{2k+3} = |z|$. Then $\rho = |z| < 1 \implies |z| < 1$. So the radius of convergence is $R = 1$.

4.3.3 $\displaystyle\sum_{n=0}^{\infty} 2^n \frac{(z-i)^n}{n!}$

Answer: Using the Ratio Test, we have $\rho = \displaystyle\lim_{k\to\infty} \frac{|C_{k+1}|}{|C_k|} = \lim_{k\to\infty} \left|\frac{2^{k+1}(z-i)^{k+1}}{(k+1)!}\right| \cdot \left|\frac{k!}{2^k(z-i)^k}\right| = \lim_{k\to\infty} \left|\frac{2z-2i}{k+1}\right| = |2z-2i| \lim_{k\to\infty} \frac{1}{k+1} = 0$. Since $0 < 1$ is always true, the radius of convergence is ∞.

4.3.5 $\displaystyle\sum_{n=0}^{\infty} \frac{(4iz-2)^n}{2^n}$

Answer: Using the Ratio Test, we have $\rho = \lim_{k\to\infty} \frac{|C_{k+1}|}{|C_k|} = \lim_{k\to\infty} \left|\frac{(4iz-2)^{k+1}}{2^{k+1}}\right| \cdot \left|\frac{2^k}{(4iz-2)^k}\right| = \lim_{k\to\infty} |2iz-1| = |2iz-1| = 2\left|z+\frac{i}{2}\right|$. Then $\rho = 2\left|z+\frac{i}{2}\right| < 1 \implies \left|z+\frac{i}{2}\right| \leq \frac{1}{2}$. So the radius of convergence is $R = \frac{1}{2}$.

P1 Find the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{z^{2n}}{(in)^{2n}}$.

Answer: Using the Ratio Test, we have $\rho = \lim_{k\to\infty} \frac{|C_{k+1}|}{|C_k|} = \lim_{k\to\infty} \left|\frac{z^{2k+2}}{i^{2k+2}(k+1)^{2k+2}}\right| \cdot \left|\frac{i^{2k}k^{2k}}{z^{2k}}\right| = \lim_{k\to\infty} \left|\frac{z^2 k^{2k}}{(k+1)^{2k+2}}\right| = \left|z^2\right| \lim_{k\to\infty} \frac{k^{2k}}{(k+1)^{2k+2}} = 0$. Therefore the radius of convergence is ∞.

P2 Let $f_n : \mathbb{R} \to \mathbb{R}$ be the sequence of functions defined by

$$f_n(x) \overset{\text{def}}{=} \begin{cases} 1, & \text{if } x \in [n, n+1]; \\ 0, & \text{otherwise.} \end{cases}$$

Determine if the sequence $(f_n)_{n=1}^{\infty}$ converges pointwise on $\mathbb{R}$. If it does, determine whether or not the convergence is uniform.

Answer: Since that for each x, $f_n(x) = 0$ for $n > x$; by definition the sequence converges pointwise with limit $\lim_{n\to\infty} f_n(x) = 0$. Let $x_n = n$, then $|f_n(x_n) - f(x)| = 1 \neq 0$. Therefore by Jumping Prop., f_n does not converge uniformly.