Math 180 Homework 6

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11.2.2 For a graph G let us put $f(G) = \alpha(G) \cdot \omega(G)$, and let us define $f(n) = \min f(G)$, where the minimum is over all graphs with n vertices.

(a) Prove that for $n \in \{1, 2, 3, 4, 6\}$ we have $f(n) \geq n$.

Answer:

$n = 1$: We have $\alpha(G) = 1$ and $\omega(G) = 1$ for the graph of a single vertex, so $f(G) = \alpha(G) \cdot \omega(G) = 1$ and therefore $f(1) \geq 1$.

$n = 2$: If the two vertices are connected, then we have $\alpha(G) = 1$ and $\omega(G) = 2$, so $f(G) = 2$; if the two are not connected, then we have $\alpha(G) = 2$ and $\omega(G) = 1$, so $f(G) = 2$. Therefore $f(2) = 2 \geq 2$.

$n = 3$: If we have $\alpha(G) = 1$, we must also have $\omega(G) = 3$ since if any vertex is disconnected, we would have $\alpha(G) = 2$. Similarly, if we have $\omega(G) = 1$, we must also have $\alpha(G) = 3$ since if any vertex is connected, we would have $\omega(G) = 2$. If $\alpha(G) = 2$, then we must have exactly one edge, so $\omega(G) = 2$. Therefore $f(3) \geq 3$ for all possible cases.

$n = 4$: Similar to the $n = 3$ case, if we have $\alpha(G) = 1$ we must also have $\omega(G) = 4$; if we have $\omega(G) = 1$ we must also have $\alpha(G) = 4$. If we have $\alpha(G) = 2$, then we have $\omega = 2$ (two disconnected pairs of connected vertices). If we have $\alpha(G) = 3$, then the only scenario is to have a pair of connected vertices and 2 disconnected vertices, so $\omega(G) = 2$. In all the cases we have $f(4) \geq 4$.

$n = 6$: By theorem 11.1.1, we must have either $\alpha(G) \geq 3$ or $\omega(G) \geq 3$. Note that if $\alpha(G) = 6$ or $\omega(G) = 6$, we have trivially $f(G) = 6$. Then, if we have $3 \leq \alpha(G) \leq 6$, there must exist at least one vertex not in the indepedent set, that is connected to a vertex in the indepedent set. So $\omega(G) \geq 2$ and we have $f(G) \geq 6$ since $\alpha \geq 3$. Similarly, if we have $3 \leq \omega(G) \leq 6$, there must exist at least one vertex that is not connected to any other vertex, so $\alpha(G) \geq 2$ and we have $f(G) \geq 6$ since $\omega \geq 3$. Therefore we have $f(6) \geq 6$ in all cases.

(b) Prove that $f(5) < 5$.

Answer: Since $f(5) = \min f(G)$, where G is a graph with 5 vertices, we can show that $f(5) < 5$ by finding a graph G where $f(G) < 5$, then $\min f(G) \leq 5$ by definition of minimum. Take C_5, then we have $\alpha(C_5) = 2$ since any 3 vertices must contain at least an edge between two of the vertices. We also have $\omega(C_5) = 2$ since any 3 vertices can only contain at most 2 edges among them. So $f(C_5) = 4$ and therefore $f(5) \leq 4 \implies f(5) < 5$.

11.2.3 Show that the function $f(n)$ as in Exercise 2 is nondecreasing and that it is not bounded from above.

Answer: By contradiction. Suppose $f(n)$ is strictly decreasing, then we must have $f(k) < f(k-1)$ for some k. Let G be a graph with k vertices and G' be a graph with $k-1$ vertices, then we must have $f(k) < f(k-1) \implies \min\{\alpha(G) \cdot \omega(G)\} < \min\{\alpha(G') \cdot \omega(G')\}$ for all such G and G'. Since G and G' are arbitrary, we can always take G and remove a vertex to obtain G', i.e. $G - v = G'$, which gives us $\alpha(G - v) = \alpha(G') \leq \alpha(G)$ and $\omega(G - v) = \omega(G') \leq \omega(G)$, implying that $\alpha(G') \cdot \omega(G') \leq \alpha(G) \cdot \omega(G)$. But this contradicts with $f(k) < f(k-1) \implies \min\{\alpha(G) \cdot \omega(G)\} < \min\{\alpha(G') \cdot \omega(G')\}$, therefore $f(n)$ must be nondecreasing by contradiction.

11.2.4 Prove that $k \leq k'$ and $\ell \leq \ell'$ implies $r(k, \ell) \leq r(k', \ell')$.

Answer: By definition $r(k, l) = \min\{V(G) : \omega(G) \geq k \text{ or } \alpha(G) \geq \ell\}$; since we have $k \leq k'$ and $\ell \leq \ell'$, $\omega(G) \geq k'$ or $\alpha(G) \geq \ell'$ implies $\omega(G) \geq k$ or $\alpha(G) \geq \ell$. Therefore the set $\{V(G) : \omega(G) \geq k' \text{ or } \alpha(G) \geq \ell'\}$ is a subset of the set $\{V(G) : \omega(G) \geq k \text{ or } \alpha(G) \geq \ell\}$. Then since we are taking the minimum, we have $r(k, \ell) = \min\{V(G) : \omega(G) \geq k \text{ or } \alpha(G) \geq \ell\} \leq \min\{V(G) : \omega(G) \geq k' \text{ or } \alpha(G) \geq \ell'\} = r(k', \ell') \implies r(k, \ell) \leq r(k', \ell')$.