P3 Evaluate each given integral.

(a) $\displaystyle\int_{C_1(0)} \frac{\sin(5z+1)}{z-2i}\,dz$

Answer: Since the singularity $2i \notin C_1(0)$, $f(z) = \dfrac{\sin(5z+1)}{z-2i}$ is analytic on and inside $C_1(0)$. Therefore we have $\displaystyle\int_{C_1(0)} \frac{\sin(5z+1)}{z-2i}\,dz = 0$ by Cauchy's integral theorem.

(b) $\displaystyle\int_{C_2(0)} \frac{\cos|z|}{z}\,dz$

Answer: Let $f(z) = \cos|z|$ and $a = 0$, then by Cauchy's integral formula, $\displaystyle\int_{C_2(0)} \frac{\cos|z|}{z}\,dz = 2\pi i f(a) = 2\pi i$.

(c) $\displaystyle\int_{C_3(1)} \frac{e^{3z}}{(z-1+i)^4}\,dz$

Answer: Let $f(z) = e^{3z}$ and $a = 1-i$, then $f'(z) = 3e^{3z} \implies f''(z) = 9e^{3z} \implies f'''(z) = 27e^{3z}$. By Cauchy's integral formula, $\displaystyle\int_{C_3(1)} \frac{e^{3z}}{(z-1+i)^4}\,dz = \frac{2\pi i}{3!} f'''(a) = \frac{\pi i}{3} \cdot 27e^{3-3i} = 9\pi i e^{3-3i}$.

P5 1. Find the Laurent series for $\dfrac{1}{(z-5)(z+2i)}$ in the annulus $A_{2,5}(0)=\{z\in\mathbb{C}\mid 2<|z|<5\}$.

Answer: By partial fractions, we have $\dfrac{1}{(z-5)(z+2i)}=\dfrac{A}{z-5}+\dfrac{B}{z+2i} \implies 1=A(z+2i)+B(z-5) \implies A=\dfrac{1}{5+2i}, B=\dfrac{-1}{5+2i} \implies \dfrac{1}{(z-5)(z+2i)}=\dfrac{1}{5+2i}\left(\dfrac{1}{z-5}-\dfrac{1}{z+2i}\right)=\dfrac{1}{5+2i}\left(-\dfrac{1}{5}\cdot\dfrac{1}{1-\frac{z}{5}}-\dfrac{1}{2i}\cdot\dfrac{1}{1+\frac{z}{2i}}\right)$.

Then since $\dfrac{z}{5}=\dfrac{|z|}{5}<1$ for $z\in A_{2,5}(0)$, we have $\dfrac{1}{1-\frac{z}{5}}=\displaystyle\sum_{k=0}^{\infty}\frac{z^k}{5^k}$. Similarly, since $\left|\dfrac{z}{2i}\right|=\dfrac{|z|}{2}>1$ for $z\in A_{2,5}(0)$, we have $\dfrac{1}{1+\frac{z}{2i}}=\displaystyle\sum_{k=1}^{\infty}-\frac{(-1)^k(2i)^k}{z^k}=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(2i)^k}{z^k}$.

Combining the above, $f(z)=\dfrac{1}{(z-5)(z+2i)}=\dfrac{1}{5+2i}\left(-\dfrac{1}{5}\displaystyle\sum_{k=0}^{\infty}\frac{z^k}{5^k}-\frac{1}{2i}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(2i)^k}{z^k}\right)=-\dfrac{1}{5+2i}\left(\displaystyle\sum_{k=0}^{\infty}\frac{z^k}{5^{k+1}}+\sum_{k=1}^{\infty}\frac{(-2i)^{k-1}}{z^k}\right)$ for $z\in A_{2,5}(0)$.

2. Show that there is no constant $M\geq 0$ such that $\left|\displaystyle\sum_{n=1}^{\infty}\frac{z^n}{n^n e^{i\sqrt{n}}}\right|\leq M$ for all $z\in\mathbb{C}$.

Answer: Let $f_n(z)=\dfrac{z^n}{n^n e^{i\sqrt{n}}}$, then $|f_n(z)|=\left|\dfrac{z^n}{n^n e^{i\sqrt{n}}}\right|=\dfrac{|z^n|}{|n^n||e^{i\sqrt{n}}|}=\dfrac{|z^n|}{|n^n|}=\left|\left(\dfrac{z}{n}\right)^n\right|$. Since we can always pick z such that $|z|>|n| \implies \left|\dfrac{z}{n}\right|>1$, $|f_n(z)|=\left|\left(\dfrac{z}{n}\right)^n\right|$ diverges by p-test and therefore no such M exists.

I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.

Signature:

Date: 11/23/2020