P2 $(z-2-2i)^4 = 16i$

Answer: Let $w = z-2-2i$, then we have $w^4 = 16i \implies w^4 = 16(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$. Then by the nth roots formula, the roots are:

$w_1 = \sqrt[4]{16}\left[\cos\left(\frac{1}{4}\cdot\frac{\pi}{2}\right) + i\sin\left(\frac{1}{4}\cdot\frac{\pi}{2}\right)\right] = 2(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8})$

$w_2 = \sqrt[4]{16}\left[\cos\left(\frac{1}{4}\cdot\frac{\pi}{2} + \frac{\pi}{2}\right) + i\sin\left(\frac{1}{4}\cdot\frac{\pi}{2} + \frac{\pi}{2}\right)\right] = 2(\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8})$

$w_3 = \sqrt[4]{16}\left[\cos\left(\frac{1}{4}\cdot\frac{\pi}{2} + \pi\right) + i\sin\left(\frac{1}{4}\cdot\frac{\pi}{2} + \pi\right)\right] = 2(\cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8})$

$w_4 = \sqrt[4]{16}\left[\cos\left(\frac{1}{4}\cdot\frac{\pi}{2} + \frac{3\pi}{2}\right) + i\sin\left(\frac{1}{4}\cdot\frac{\pi}{2} + \frac{3\pi}{2}\right)\right] = 2(\cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8})$

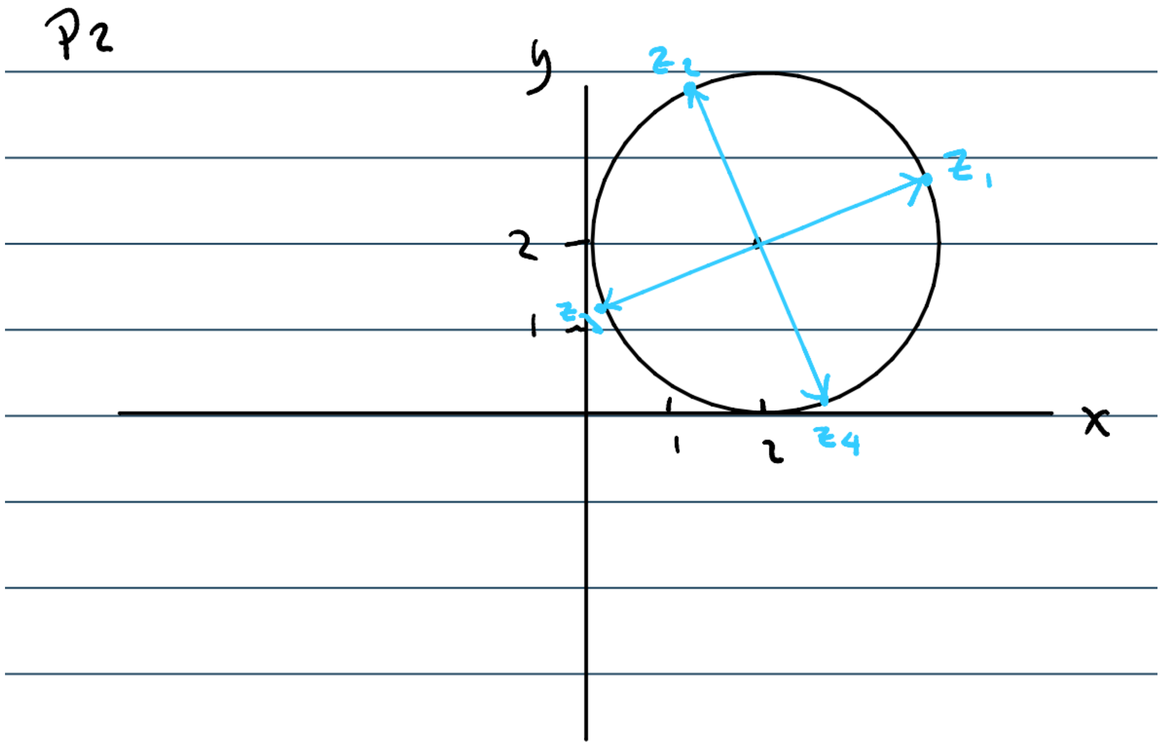
By substitution we have:

$z_1 = w_1 + 2 + 2i = (2 + 2\cos\frac{\pi}{8}) + (2 + 2\sin\frac{\pi}{8})i$

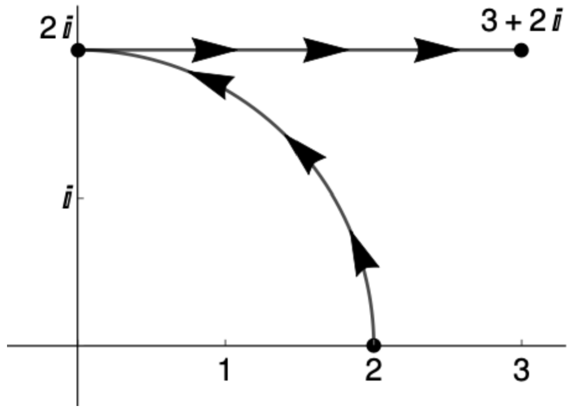
$z_2 = w_2 + 2 + 2i = (2 + 2\cos\frac{5\pi}{8}) + (2 + 2\sin\frac{5\pi}{8})i$

$z_3 = w_3 + 2 + 2i = (2 + 2\cos\frac{9\pi}{8}) + (2 + 2\sin\frac{9\pi}{8})i$

$z_4 = w_4 + 2 + 2i = (2 + 2\cos\frac{13\pi}{8}) + (2 + 2\sin\frac{13\pi}{8})i$



P3 Let γ be the pictured path, consisting of an arc of the circle of radius 2 centered at 0, joined with the line segment $[2i, 3+2i]$.



Evaluate $\int_\gamma (\bar{z}+2i)dz$.

Answer: Let γ_1 be the arc and γ_2 be the line segment, then $\gamma_1(t) = 2e^{it}, t \in [0, \frac{\pi}{2}]$ and $\gamma_2(t) = 2i+3t, t \in [0,1]$. In addition, we have $\gamma_1'(t) = 2ie^{it}$ and $\gamma_2'(t) = 3$.

$\int_\gamma (\bar{z}+2i)dz$

$= \int_{\gamma_1} (\bar{z}+2i)dz + \int_{\gamma_2} (\bar{z}+2i)dz$

$= \int_0^{\frac{\pi}{2}} [\overline{\gamma_1(t)} + 2i]\gamma_1'(t)dt + \int_0^1 [\overline{\gamma_2(t)} + 2i]\gamma_2'(t)dt$

$= \int_0^{\frac{\pi}{2}} 2ie^{it}(2e^{-it}+2i)dt + \int_0^1 3(3t-2i+2i)dt$

$= \int_0^{\frac{\pi}{2}} 4i - 4e^{it}dt + \int_0^1 9tdt$

$= 4\int_0^{\frac{\pi}{2}} idt - 4\int_0^{\frac{\pi}{2}} e^{it}dt + 9\int_0^1 tdt$

$= 4[it]_0^{\frac{\pi}{2}} + 4[ie^{it}]_0^{\frac{\pi}{2}} + 9[\frac{t^2}{2}]_0^1$

$= 2\pi i + 4(ie^{\frac{\pi}{2}i} - i) + \frac{9}{2}$

$= \frac{9}{2} + i(2\pi - 4 + 4e^{\frac{\pi}{2}i})$

P4 (a) Show that $|e^z| \le e^{|z|}$ for all $z \in \mathbb{C}$.

Answer: Let $z = x + iy$, we have $|e^z| = \left|e^{x+iy}\right| = \left|e^x \cdot e^{iy}\right| = |e^x| \cdot \left|e^{iy}\right| = e^x$; then since $x = \operatorname{Re} z \le |z|$, we have $e^x \le e^{|z|}$. Therefore $|e^z| = e^x \le e^{|z|} \implies |e^z| \le e^{|z|}$.

(b) Show that $|\cos z| \le e^{|z|}$ for all $z \in \mathbb{C}$.

Answer: By definition, $\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)}) = \frac{1}{2}(e^{-y+ix} + e^{y-ix})$, then $|\cos z| = \frac{1}{2}\left|e^{-y+ix} + e^{y-ix}\right| = \frac{1}{2}\left|e^{-y} \cdot e^{ix} + e^{y} \cdot e^{-ix}\right|$. Using triangle inequality, we have $\frac{1}{2}\left|e^{-y} \cdot e^{ix} + e^{y} \cdot e^{-ix}\right| \le \frac{1}{2}\left|e^{-y} \cdot e^{ix}\right| + \frac{1}{2}\left|e^{y} \cdot e^{-ix}\right| = \frac{1}{2}|e^{-y}| \cdot \left|e^{ix}\right| + \frac{1}{2}|e^{y}| \cdot \left|e^{-ix}\right| = \frac{1}{2}|e^{-y}| + \frac{1}{2}|e^{y}|$. By part (a), $\frac{1}{2}|e^{-y}| + \frac{1}{2}|e^{y}| \le \frac{1}{2}e^{|-y|} + \frac{1}{2}e^{|y|} = e^{|y|}$. Then $|\cos z| \le e^{|y|} \le e^{|z|} \implies |\cos z| \le e^{|z|}$.

(c) Show that $\left|\frac{e^z \cos z}{z-i}\right| \le \frac{e^{10}}{2}$ for all z on $C_3(2i)$.

Answer: Since $\left|\frac{e^z \cos z}{z-i}\right| = \frac{|e^z| \cdot |\cos z|}{|z-i|}$, we can apply parts (a) and (b) which gives us $\frac{|e^z| \cdot |\cos z|}{|z-i|} \le \frac{e^{2|z|}}{|z-i|}$. To show that the given inequality is true, we will show that $e^{2|z|} \le e^{10}$ and $|z - i| \ge 2$ separately.

On the numerator, we want to show that $e^{2|z|} \le e^{10}$; since $|z| \ge 0$, it is sufficient to show that $2|z| \le 10$, or $|z| \le 5$. Since we are only considering $z \in C_3(2i) \implies |z - 2i| = 3$, by triangle inequality we have $3 = |z - 2i| \le |z| + |-2i| = |z| + 2 \implies |z| \le 5$.

On the denominator we have $|z - i| = |(z - 2i) + i| \ge ||z - 2i| - i|$ by triangle inequality; since $z \in C_3(2i) \implies |z - 2i| = 3$, we then have $||z - 2i| - i| = |3 - i| = \sqrt{10}$. So $|z - i| \ge \sqrt{10} > 2$.

As shown above, we have $e^{2|z|} \le e^{10}$ and $|z - i| \ge 2$, therefore $\left|\frac{e^z \cos z}{z-i}\right| = \frac{|e^z| \cdot |\cos z|}{|z-i|} \le \frac{e^{2|z|}}{|z-i|} \le \frac{e^{10}}{2} \implies \left|\frac{e^z \cos z}{z-i}\right| \le \frac{e^{10}}{2}$.

I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.

Signature:

Date: 10/26/2020