Math 132 Homework 2

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1.3.33 $z^4 = i$

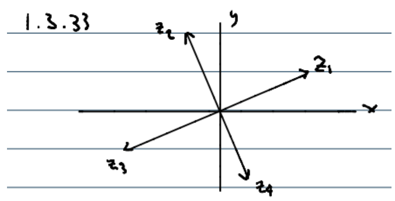
Answer: $z^4 = i \implies z^4 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$, then we have

$z_1 = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2}i$ (principal root)

$z_2 = \cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8} = -\frac{\sqrt{2+\sqrt{2}}}{2} + \frac{\sqrt{2-\sqrt{2}}}{2}i$

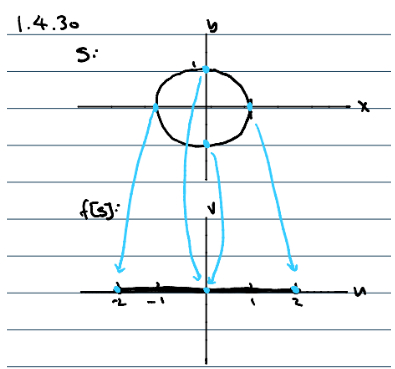
$z_3 = \cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8} = -\frac{\sqrt{2+\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2}i$

$z_4 = \cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2} - \frac{\sqrt{2-\sqrt{2}}}{2}i$



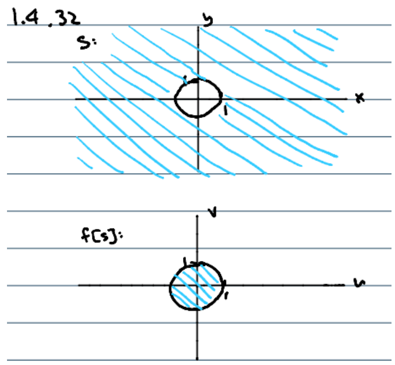
1.4.30 Find the image of the set $S = \{z : |z| \le 1\}$ under the mapping $f(z) = z + \bar{z}$.

Answer: Let $z = x + yi$, then $f(z) = z + \bar{z} = (x + yi) + (x - yi) = 2x = 2\,\mathrm{Re}\,z$. Therefore $f[S]$ "squishes" the disk with radius 1 S to a segment from -2 to 2 on the real axis.



1.4.32 $S = \{z : |z| \ge 1\}$

Answer: Let $z = r(\cos\theta + i\sin\theta)$, then $|z| \ge 1 \implies r \ge 1 \implies \frac{1}{r} \le 1$. Therefore $f[S] = \{z : |z| \le 1\}$.



1.6.29 $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

Answer: We can expand the right hand side by definition of complex sine and cosine functions as follows:

$\sin z_1 \cos z_2 + \cos z_1 \sin z_2$
$= \frac{e^{iz_1} - e^{-iz_1}}{2i} \cdot \frac{e^{iz_2} + e^{-iz_2}}{2} + \frac{e^{iz_1} + e^{-iz_1}}{2} \cdot \frac{e^{iz_2} - e^{-iz_2}}{2i}$
$= \frac{(e^{iz_1} - e^{-iz_1})(e^{iz_2} + e^{-iz_2})}{4i} + \frac{(e^{iz_1} + e^{-iz_1})(e^{iz_2} - e^{-iz_2})}{4i}$
$= \frac{e^{iz_1} \cdot e^{iz_2} + e^{iz_1} \cdot e^{-iz_2} - e^{-iz_1} \cdot e^{iz_2} - e^{-iz_1} \cdot e^{-iz_2}}{4i} + \frac{e^{iz_1} \cdot e^{iz_2} - e^{iz_1} \cdot e^{-iz_2} + e^{-iz_1} \cdot e^{iz_2} - e^{-iz_1} \cdot e^{-iz_2}}{4i}$
$= \frac{e^{i(z_1+z_2)} + e^{i(z_1-z_2)} - e^{i(-z_1+z_2)} - e^{i(-z_1-z_2)}}{4i} + \frac{e^{i(z_1+z_2)} - e^{i(z_1-z_2)} + e^{i(-z_1+z_2)} - e^{i(-z_1-z_2)}}{4i}$
$= \frac{2e^{i(z_1+z_2)} - 2e^{i(-z_1-z_2)}}{4i}$
$= \frac{e^{i(z_1+z_2)} - e^{i(-z_1-z_2)}}{2i}$
$= \sin(z_1 + z_2)$

1.7.2 $z = -3 - 3i$

Answer: $\log z = \ln|z| + i \arg z = \ln\sqrt{18} + i(\frac{5\pi}{4} + 2k\pi), k \in \mathbb{Z}$

1.7.3 $z = 5e^{i\frac{\pi}{7}}$

Answer: $\log z = \ln|z| + i \arg z = \ln 5 + i(\frac{\pi}{7} + 2k\pi), k \in \mathbb{Z}$

1.7.19 (a) Compute $\mathrm{Log}(e^{i\pi})$, $\mathrm{Log}(e^{3i\pi})$, and $\mathrm{Log}(e^{5i\pi})$.

Answer:

$\mathrm{Log}(e^{i\pi}) = \ln|z| + i\mathrm{Arg}\ z = \ln 1 + i\pi$

$\mathrm{Log}(e^{3i\pi}) = \mathrm{Log}(\cos 3\pi + i \sin 3\pi) = \mathrm{Log}(\cos\pi + i\sin\pi) = \mathrm{Log}(e^{i\pi}) = \ln 1 + i\pi$

$\mathrm{Log}(e^{5i\pi}) = \mathrm{Log}(\cos 5\pi + i \sin 5\pi) = \mathrm{Log}(\cos\pi + i\sin\pi) = \mathrm{Log}(e^{i\pi}) = \ln 1 + i\pi$

(b) Show that $\mathrm{Log}(e^z) = z$ if and only if $-\pi < \mathrm{Im}\, z \le \pi$.

Answer: Let $z = x + iy$, then $\mathrm{Im}\, z = y$;

$\Rightarrow$: If $\mathrm{Log}(e^z) = z$, by definition of Log we have $\ln|z| + i\mathrm{Arg}\ z = z$. Since $\mathrm{Arg}\ z = \mathrm{Arg}\ e^{x+iy} = y = \mathrm{Im}\, z$, we must have $-\pi < \mathrm{Arg}\ z < \pi \implies -\pi < \mathrm{Im}\, z < \pi$ by definition of principal value.

$\Leftarrow$: If $-\pi < \mathrm{Im}\, z \le \pi$, we can evaluate $\mathrm{Log}(e^z)$ as follows: $\mathrm{Log}(e^z) = \mathrm{Log}(e^x \cdot e^{iy}) = \ln\left|e^x \cdot e^{iy}\right| + i\mathrm{Arg}\ (e^x \cdot e^{iy}) = \ln|e^x| + iy = x + iy = z$.

1.7.24 $(1+i)^{3+i}$

Answer: $(1+i)^{3+i} = e^{(3+i)\mathrm{Log}(1+i)} = e^{(3+i)\mathrm{Log}(\sqrt{2}e^{i\frac{\pi}{4}})} = e^{(3+i)(\ln\sqrt{2} + i\frac{\pi}{4})} = e^{3\ln\sqrt{2} - \frac{\pi}{4} + i(\frac{3\pi}{4} + \ln\sqrt{2})}$

P1 Find and plot all $z \in \mathbb{C}$ such that $(z-3+i)^3 = -125i$.

Answer: Let $w = z - 3 + i$, then we have $w^3 = -125i \implies w^3 = -125(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$. Then the roots are

$w_1 = -5(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = -\frac{5\sqrt{3}}{2} - \frac{5}{2}i$

$w_2 = -5(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}) = \frac{5\sqrt{3}}{2} - \frac{5}{2}i$

$w_3 = -5(\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}) = 5i$

By substitution we have

$z_1 = w_1 + 3 - i = 3 - \frac{5\sqrt{3}}{2} - \frac{7}{2}i$

$z_2 = w_2 + 3 - i = 3 + \frac{5\sqrt{3}}{2} - \frac{7}{2}i$

$z_3 = w_3 + 3 - i = 3 + 4i$

P2 In each part, express $f(z)$ in the form $u(x,y) + iv(x,y)$ where u and v are the real and imaginary parts of f:

(a) $f(z) = z^3$

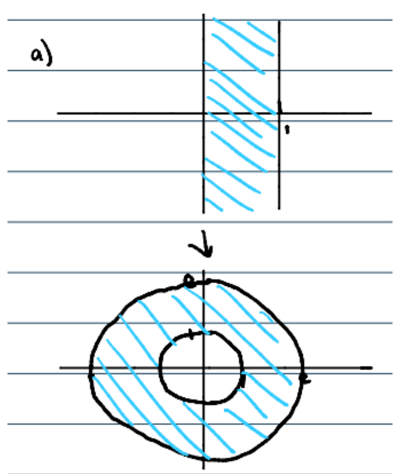
Answer: Let $z = x + iy$, then $f(z) = z^3 = (x+iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$. Then $u(x,y) = x^3 - 3xy^2$ and $v(x,y) = 3x^2y - y^3$.

(b) $f(z) = |z|^3$

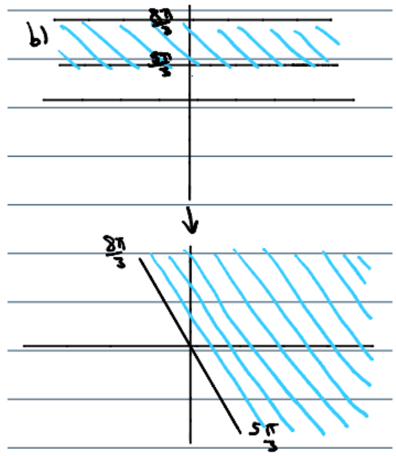
Answer: Since $|z| = |x+iy| \in \mathbb{R}$, we have $u(x,y) = |x+iy|^3$ and $v(x,y) = 0$.

P3 Sketch each of the following regions D and its image under the exponential map $w = e^z$. Indicate the images of horizontal and vertical lines in your sketch.

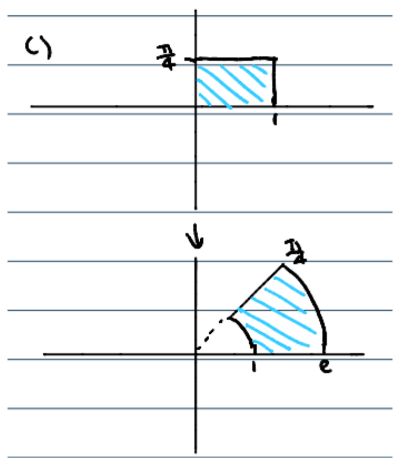
(a) The vertical strip $D = \{z \in \mathbb{C} \mid 0 < \text{Re}(z) < 1\}$.



(b) The horizontal strip $D = \{z \in \mathbb{C} \mid \frac{5\pi}{3} < \text{Im}(z) < \frac{8\pi}{3}\}$.



(c) The rectangle $D = \{z \in \mathbb{C} \mid 0 < \mathrm{Re}(z) < 1, 0 < \mathrm{Im}(z) < \frac{\pi}{4}\}$.



P4 Find all values of the complex power i^i.

Answer: $i^i = e^{i \log i} = e^{i \log\left(\ln 1 + i\frac{\pi}{2}\right)} = e^{-\frac{\pi}{2}+2k\pi}, k \in \mathbb{Z}$