Math 132 Homework 5

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3.4.13 $\int_{C_1(0)} \frac{e^z}{z+2} dz$

Answer: Since the discontinuity $z = -2$ is outside of $C_1(0)$, our function $f(z) = \frac{e^z}{z+2}$ is analytic on and inside $C_1(0)$. Therefore we have $\int_{C_1(0)} \frac{e^z}{z+2} dz = 0$ by Cauchy's integral theorem.

3.4.15 $\int_{C_1(i)} \left(\frac{z-1}{z+1}\right)^2 z dz$

Answer: Since the discontinuity $z = -1$ is outside of $C_1(i)$, our function $f(z) = \left(\frac{z-1}{z+1}\right)^2$ is analytic on and inside $C_1(i)$. Therefore we have $\int_{C_1(i)} \left(\frac{z-1}{z+1}\right)^2 z dz = 0$ by Cauchy's integral theorem.

3.6.2 $\int_{C_3(0)} \frac{e^{z^2} \cos z}{z-i} dz$

Answer: Let $f(z) = e^{z^2} \cos z$ and $a = i$, then by Cauchy's integral theorem, $\int_{C_3(0)} \frac{e^{z^2} \cos z}{z-i} dz = 2\pi i f(a) = 2\pi i e^{-1} \cos(i)$.

3.6.3 $\frac{1}{2\pi i} \int_{C_2(1)} \frac{1}{z^2 - 5z + 4} dz$

Answer: We have $\frac{1}{z^2-5z+4} = \frac{1}{(z-1)(z-4)}$; since only $z = 1$ is inside $C_2(1)$, we can set $f(z) = \frac{1}{z-4}$ and $a = 1$. Then by Cauchy's integral theorem, $\frac{1}{2\pi i} \int_{C_2(1)} \frac{1}{z^2-5z+4} dz = f(a) = -\frac{1}{3}$.

3.6.4 $\frac{1}{2\pi i} \int_{C_3(1)} \frac{\cos z}{(z-\pi)^4} dz$

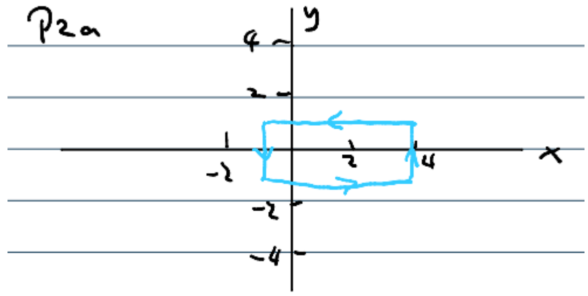
Answer: Let $f(z) = \cos z$ and $a = \pi$, then $f'(z) = -\sin z \implies f''(z) = -\cos z \implies f'''(z) = \sin z$. By Cauchy's integral theorem, $\frac{1}{2\pi i} \int_{C_3(1)} \frac{\cos z}{(z-\pi)^4} dz = \frac{1}{6} f'''(z) = 0$.

P1 Show that $\int_{C_2(i)} \frac{e^z}{z^2 + iz + 6} dz = \frac{2\pi e^{2i}}{5}$.

Answer: By factoring the denominator we have $\int_{C_2(i)} \frac{e^z}{z^2+iz+6} dz = \int_{C_2(i)} \frac{e^z}{(z+3i)(z-2i)} dz$. Between the two discontinuities $z = -3i$ and $z = 2i$, only $z = 2i$ is inside $C_2(i)$. So we can let $f(z) = \frac{e^z}{z+3i}$ and $a = 2i$. Then we have $\int_{C_2(i)} \frac{e^z}{z^2+iz+6} dz = 2\pi i f(a) = 2\pi i \cdot \frac{e^{2i}}{5i} = \frac{2\pi e^{2i}}{5}$ by Cauchy's integral formula.

P2 Let γ be the rectangle $[4+i, -1+i, -1-i, 4-i, 4+i]$.

(a) Sketch the path γ.



(b) Evaluate $\displaystyle\int_\gamma \frac{e^z \sin z}{z-2i} dz$.

Answer: Since $2i$ is outside the path γ, we have $\displaystyle\int_\gamma \frac{e^z \sin z}{z-2i} dz = 0$ by Cauchy's integral theorem.

(c) Find $A, B \in \mathbb{C}$ such that $\displaystyle\frac{1}{z(z-\pi)} = \frac{A}{z} + \frac{B}{z-\pi}$.

Answer: From $\displaystyle\frac{1}{z(z-\pi)} = \frac{A}{z} + \frac{B}{z-\pi}$, we have $1 = A(z-\pi) + Bz$. Then at $z = 0$ we have $1 = -A\pi \implies A = -\pi^{-1}$. Simiarly, at $z = \pi$ we have $1 = B\pi \implies B = \pi^{-1}$.

(d) Evaluate $\displaystyle\int_\gamma \frac{e^z \sin z}{z(z-\pi)} dz$.

Answer: Using part (c) we have $\displaystyle\int_\gamma \frac{e^z \sin z}{z(z-\pi)} dz = \int_\gamma \frac{e^z \sin z}{-\pi z} + \int_\gamma \frac{e^z \sin z}{\pi(z-\pi)}$. We can now evaluate the two integrals separately. Let $f(z) = e^z \sin z$, $a = 0$ and $b = \pi$. Then by Cauchy's integral formula we have $\displaystyle\int_\gamma \frac{e^z \sin z}{-\pi z} = 2\pi i f(a) = 0$ and $\displaystyle\int_\gamma \frac{e^z \sin z}{\pi(z-\pi)} = 2\pi i f(b) = 0$. Therefore $\displaystyle\int_\gamma \frac{e^z \sin z}{z(z-\pi)} dz = 0$.

P3 Let γ be a simple closed path in $\mathbb{C}$ and suppose that $f(z)$ and $g(z)$ are analytic on and inside γ. Show that if $f(z) = g(z)$ for all z on γ, then $f(z) = g(z)$ for all z inside γ as well.

Answer: By Cauchy's integral formula, we have $\displaystyle f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a} dz$ for any point inside γ. Then by substituting $f(z) = g(z)$, we have $\displaystyle f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a} dz = \int_\gamma \frac{g(z)}{z-a} dz = g(a)$. Since a is arbitrary, this gives us $f(z) = g(z)$ for all z inside γ.

P4 Consider the entire function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \cos(z)$, which is not constant. Why doesn't this contradict Liouville's Theorem?

Answer: Let $z = x + iy$, then $\cos z = \cos(x+iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy) = \cos(x)\cosh(y) - \sin(x)\sinh(y)$. Since $\cosh(y)$ and $\sinh(y)$ are not bounded, neither is $\cos(z)$. Therefore Liouville's Theorem does not apply.

P5 Let $f : \mathbb{C} \to \mathbb{C}$ be entire with $\text{Re}(f(z)) \leq 0$ for $z \in \mathbb{C}$.

(a) Show that $e^{f(z)} = \alpha$ for some constant $\alpha \in \mathbb{C}$.

Answer: Let $u = \operatorname{Re} f(z)$ and $v = \operatorname{Im} f(z)$. Then $|h(z)| = \left|e^{u+iv}\right| = \left|e^u \cdot e^{iv}\right| = e^u$. Since

$u = \operatorname{Re} f(z) \leq 0$, we have $|h(z)| \leq 0$. Therefore $h(z)$ is entire and bounded, then by Liouville's Theorem it is constant.

(b) Show that $f(z) = \beta$ for some constant $\beta \in \mathbb{C}$.

Answer: Since $e^{f(z)} = \alpha$, we can differentiate both sides which gives us $f'(z)e^{f(z)} = 0$. Since $e^{f(z)}$ is always nonzero, we must have $f'(z) = 0$. Therefore $f(z)$ is constant.

P6 Let $f : \mathbb{C} \to \mathbb{C}$ be entire with $|f(z)| \leq e^{\operatorname{Re}(z)}$ for $z \in \mathbb{C}$. Show that either

- $f(z) = 0$ for every $z \in \mathbb{C}$, or
- $f(z) \neq 0$ for any $z \in \mathbb{C}$.

Answer: Let $z = x + iy$ and $g(z) = \dfrac{f(z)}{e^z}$, then since $|f(z)| \leq e^{\operatorname{Re}(z)}$ and $e^z > 0$, we have $|g(z)| = \left|\dfrac{f(z)}{e^z}\right| \leq \left|\dfrac{e^x}{e^z}\right| = \left|\dfrac{e^x}{e^{x+iy}}\right| = \left|e^{-iy}\right| = 1$, i.e. $|g(z)|$ is bounded by 1. Since $g(z)$ is entire and bounded, it must be constant by Liouville's Theorem, so there exists an $\alpha \in \mathbb{C}$ such that $g(z) = \alpha$. Then we have $\alpha = g(z) = \dfrac{f(z)}{e^z} \implies f(z) = \dfrac{\alpha}{e^z}$. Then if $\alpha = 0$, $f(z) = 0$ for every $z \in \mathbb{C}$; else if $\alpha \neq 0$, then $f(z) \neq 0$ for any $z \in \mathbb{C}$.