Math 132 Homework 8

Jiaping Zeng

11/29/2020

Proposition ($\bigstar$). Suppose $f(z)$ and $g(z)$ are analytic at z_0. If $f(z)$ has a zero of order n at z_0 (or letting $n = 0$ if $f(z_0) \neq 0$) and $g(z)$ has a zero of order m at z_0, then

$$h(z) = \frac{f(z)}{g(z)} \text{ has } \begin{cases} \text{a removable singularity at } z_0 & \text{if } m \leq n; \\ \text{a pole of order } m - n \text{ at } z_0, & \text{if } m > n. \end{cases}$$

4.6.1 $(1 - z^2)\sin z$

Answer: We have $(1 - z^2)\sin z = (1 + z)(1 - z)\sin z$, so the isolated zeros are at $-1, 1$ and $k\pi, k \in \mathbb{Z}$. The zeroes -1 and 1 have order 1; the zeroes of $\sin z$ also have order 1 as shown in class.

4.6.2 $z^3(e^z - 1)$

Answer: Since $z^3 = 0$ when $z = 0$ and $e^z - 1 = 0$ when $z = 2k\pi i, k \in \mathbb{Z}$, the isolated zeroes are at 0 and $2k\pi i, k \in \mathbb{Z}$. Let $f(z) = z^3$, then $f'''(0) = 6 \neq 0$, so $z_0 = 0$ is a zero of order 3. Now let $g(z) = e^z - 1$, then $g'(0) = e^0 = 1 \neq 0$, so $z_0 = 2k\pi i, k \neq 0 \in \mathbb{Z}$ are zeroes of order 1.

4.6.9 $1 - \frac{z^2}{2} - \cos z$

Answer: Let $f(z) = 1 - \frac{z^2}{2} - \cos z$, then we have $f'(z) = \sin z - z$, $f''(z) = \cos z - 1$, $f'''(z) = -\sin z$ and $f^{(4)}(z) = -\cos z$. By substituting $z_0 = 0$, we have $f(0) = 0$, $f'(0) = 0$, $f''(0) = 0$, $f'''(0) = 0$ and $f^{(4)} = -1 \neq 0$, so $z_0 = 0$ is a zero of order 4.

4.6.11 $z - \sin z$

Answer: Let $f(z) = z - \sin z$, then we have $f'(z) = 1 - \cos z$, $f''(z) = \sin z$ and $f'''(z) = \cos z$. By substituting $z_0 = 0$, we have $f(0) = 0$, $f'(0) = 0$, $f''(0) = 0$ and $f'''(0) = 1 \neq 0$, so $z_0 = 0$ is a zero of order 3.

4.6.15 $\frac{z(z-1)^2}{\sin(\pi z)\sin z}$

Answer: Let $f(z) = z(z-1)^2$, $g(z) = \sin(\pi z)\sin z$ and $h(z) = \frac{z(z-1)^2}{\sin(\pi z)\sin z} = \frac{f(z)}{g(z)}$. Then $f(z)$ has a zero of order 1 at $z_0 = 0$ and another zero of order 2 at $z_0 = 1$; in addition, since $\sin(k\pi) = 0$ for $k \in \mathbb{Z}$, $g(z)$ has zeroes at $z_0 = k$ and $z_0 = k\pi, k \in \mathbb{Z}$. Since $g(0) = g'(0) = 0$ and $g''(0) \neq 0$, we have a zero of order 2 at $z_0 = 0$. The other zeroes $z_0 = k$ and $z_0 = k\pi, k \neq 0 \in \mathbb{Z}$ are order 1 as $g'(z_0) \neq 0$ there.

Using Proposition $\bigstar$, we have $n = 1$ and $m = 2$ at $z_0 = 0$, so $h(z)$ has a pole of order 1 at $z_0 = 0$.

Since $\lim_{z\to 1} h(z) = 0$, we can define $\tilde{h}(1) = 0$ to make $\tilde{h}(z)$ analytic. At $z_0 = 1$, we have $n = 2$ and $m = 1$, so $h(z)$ has a removable singularity at $z_0 = 1$. At zeroes $z_0 = k$ and $z_0 = k\pi, k \neq 0, 1 \in \mathbb{Z}$, we have $n = 0$ and $m = 1$, so $h(z)$ has poles of order 1 there.

4.6.16 $e^{\frac{1}{1-z}} + \dfrac{1}{1-z}$

Answer: By taylor expansion we have $e^{\frac{1}{1-z}} = 1 + \dfrac{1}{1-z} + \dfrac{1}{2(1-z)^2} + \dfrac{1}{3!(1-z)^3} + \ldots$, so $e^{\frac{1}{1-z}} + \dfrac{1}{1-z} = 1 + \dfrac{2}{1-z} + \dfrac{1}{2(1-z)^2} + \dfrac{1}{3!(1-z)^3} + \ldots$. Therefore there is an essential singularity at 0.

4.6.18 $\dfrac{z}{e^z - 1}$

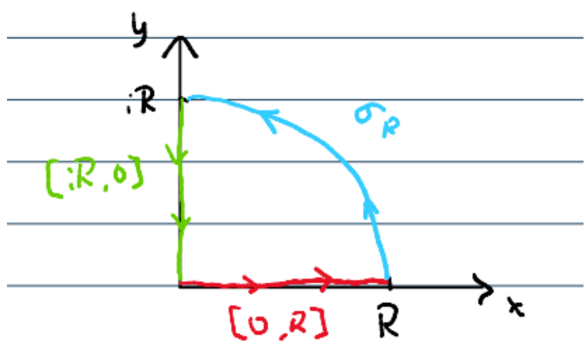
Answer: Let $f(z) = z$, $g(z) = e^z - 1$ and $h(z) = \dfrac{z}{e^z - 1} = \dfrac{f(z)}{g(z)}$. Then $f(z)$ has a zero of order 1 at $z_0 = 0$ and $g(z)$ has zeroes of order 1 (shown in 4.6.2) at $z_0 = 2k\pi i, k \in \mathbb{Z}$.
Using Proposition ★, we have $n = 1$ and $m = 1$ at $z_0 = 0$, so $h(z)$ has a removable singularity at $z_0 = 0$. Since $\lim_{z\to 0} h(z) = 1$, we can define $\tilde{h}(0) = 1$ to make $\tilde{h}(z)$ analytic. At $z_0 = 2k\pi i, k \neq 0 \in \mathbb{Z}$, we have $n = 0$ and $m = 1$, so $h(z)$ has poles of order 1 at those singularities.

P1 Use the argument principle to find the number of zeros of

$$f(z) = z^5 + z^4 + 13z^3 + 10$$

in the first quadrant.

Answer: Let R be sufficiently large such that all zeroes of $f(z)$ is enclosed by the curve $\gamma_R = [[0, R], \sigma_R, [iR, 0]]$ as shown below.



Then,

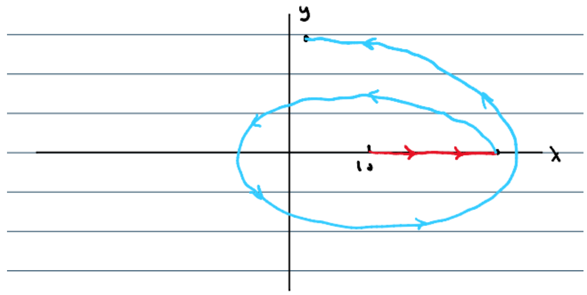
1. $f([0, R])$: $f(x) = x^5 + x^4 + 13x^3 + 10$ for $x \in [0, R]$
2. $f(\sigma_R)$: $f(Re^{it}) = R^5e^{5it} + R^4e^{4it} + 13R^3e^{3it} + 10 \approx R^5e^{5it}$ for $t \in [0, \frac{\pi}{2}]$
3. $f([iR, 0]) : f(iy) = iy^5 + y^4 - 13iy^3 + 10 = (y^4 + 10) + (y^5 - 13y^3)i$ for $y \in [0, R]$

Note that $f(z) \neq 0$ on γ_R as

1. $f(z) \geq 10$ on $[0, R]$

2. R was chosen sufficiently large such that $f(z) \neq 0$ on σ_R
3. $f(iy) = (y^4+10)+(y^5-13y^3)i = (y^4+10)+y^3(y-\sqrt{13})(y+\sqrt{13})i \implies \operatorname{Re} f(iy)$ and $\operatorname{Im} f(iy)$ have no common zeroes $\implies f(z) \neq 0$ on $[iR, 0]$

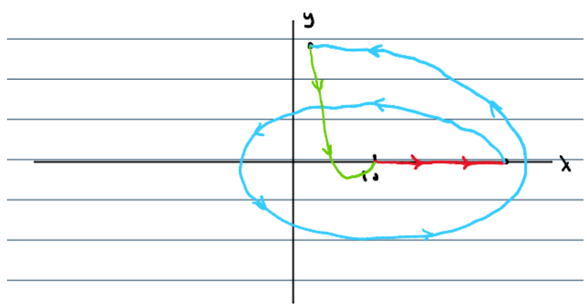
Now we can sketch $f(\gamma_R)$. Since $f(x)$ for $x \in [0, R]$ always returns a real value, $[0, R]$ maps to $[10, N_1]$ on the real axis where N_1 is some large number. Then, since $f(iR) = (R^4+10)+(R^5-13R^3)i \approx R^4 + R^5 i \approx R^5 i$, we also know that σ_R ends at some point in the first quadrant, close to the positive imaginary axis. Then since $f(Re^{it}) \approx R^5 e^{i(5t)}, t \in [0, \frac{\pi}{2}] \implies 5t \in [0, \frac{5\pi}{2}]$, σ_R is mapped to a circular path that wraps around the origin once and ends near the positive imaginary axis as shown below.



Now we can use a sign chart to find the map of $[iR, 0]$:

$y =$	$(0, \sqrt{13})$	$(\sqrt{13}, R)$
Quadrant	IV	I
$\operatorname{Re} f(iy)$	+	+
y^4+10	+	+
$\operatorname{Im} f(iy)$	-	+
y^3	+	+
$y-\sqrt{13}$	-	+
$y+\sqrt{13}$	+	+

Then our $f(\gamma_R)$ looks like:



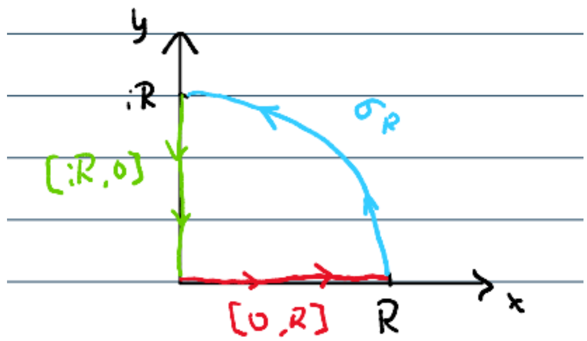
So by the argument principle, since $f(\gamma_R)$ wraps counterclockwise around the origin once, we have $N_0 - N_\infty = 1$. Since $f(z)$ is analytic, then $f(z)$ has no poles $\implies N_\infty = 0$. Therefore $N_0 = 1 \implies f(z)$ has one zero in the first quadrant.

P2 Use the argument principle to find the number of zeros of

$$f(z) = z^4 + z^3 + 10z^2 + 4z + 9$$

in the first quadrant.

Answer: Let R be sufficiently large such that all zeroes of $f(z)$ is enclosed by the curve $\gamma_R = [[0,R], \sigma_R, [iR, 0]]$ as shown below.



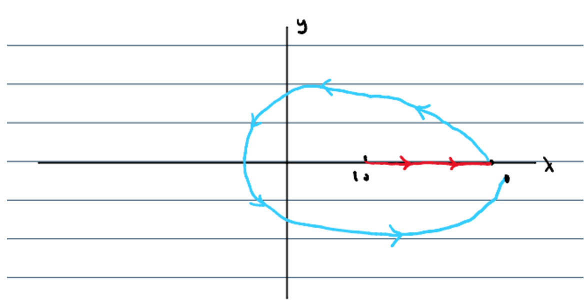
Then,

1. $f([0,R])$: $f(x) = x^4 + x^3 + 10x^2 + 4x + 9$ for $x \in [0,R]$
2. $f(\sigma_R)$: $f(Re^{it}) = R^4e^{4it} + R^3e^{3it} + 10R^2e^{2it} + 4Re^{it} + 9 \approx R^4e^{4it}$ for $t \in [0, \frac{\pi}{2}]$
3. $f([iR,0]) : f(iy) = y^4 - iy^3 - 10y^2 + 4iy + 9 = (y^4 - 10y^2 + 9) + (-y^3 + 4y)i$ for $y \in [0,R]$

Note that $f(z) \neq 0$ on γ_R as

1. $f(z) \geq 9$ on $[0,R]$
2. R was chosen sufficiently large such that $f(z) \neq 0$ on σ_R
3. $f(iy) = (y^4 - 10y^2 + 9) + (-y^3 + 4y)i = (y-3)(y-1)(y+1)(y+3) - y(y-2)(y+2)i \implies \operatorname{Re} f(iy)$ and $\operatorname{Im} f(iy)$ have no common zeroes $\implies f(z) \neq 0$ on $[iR, 0]$

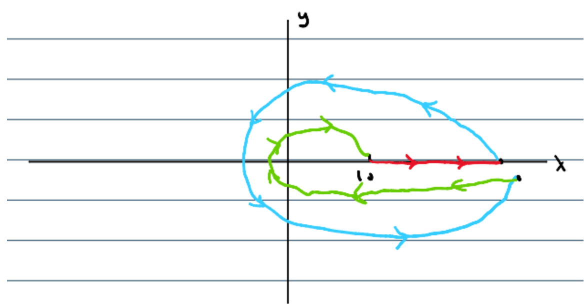
Now we can sketch $f(\gamma_R)$. Since $f(x)$ for $x \in [0,R]$ always returns a real value, $[0,R]$ maps to $[9, N_1]$ on the real axis where N_1 is some large number. Then, since $f(iR) = (R^4 - 10R^2 + 9) + (-R^3 + 4R)i \approx R^4 - R^3 i \approx R^4$, we also know that γ_R ends at some point in the fourth quadrant, close to the positive real axis. Then since $f(Re^{it}) \approx R^5 e^{i(4t)}, t \in [0, \frac{\pi}{2}] \implies 4t \in [0, 2\pi]$, σ_R is mapped to a circular path that wraps around the origin once and ends near the positive real axis as shown below.



Now we can use a sign chart to find the map of $[iR, 0]$:

$y =$	$(0,1)$	$(1,2)$	$(2,3)$	$(3,R)$
Quadrant	I	II	III	IV
$\mathrm{Re}\, f(iy)$	+	-	-	+
$y-3$	-	-	-	+
$y-1$	-	+	+	+
$y+1$	+	+	+	+
$y+3$	+	+	+	+
$\mathrm{Im}\, f(iy)$	+	+	-	-
$-y$	-	-	-	-
$y-2$	-	-	+	+
$y+2$	+	+	+	+

Then our $f(\gamma_R)$ looks like:



So by the argument principle, since $f(\gamma_R)$ wraps counterclockwise around the origin zero times, we have $N_0 - N_\infty = 0$. Since $f(z)$ is analytic, then $f(z)$ has no poles $\implies N_\infty = 0$. Therefore $N_0 = 0 \implies f(z)$ has no zero in the first quadrant.

P3 Suppose $f(z)$ is analytic at z_0 with $f(z_0) \neq 0$, and fix some positive integer n. Show that $\dfrac{f(z)}{(z-z_0)^n}$ has a pole of order n at z_0.

Answer: Since $f(z)$ is analytic at z_0, it has a power series $f(z) = f(z_0) + f'(z_0)(z-z_0) + \dfrac{f''(z_0)}{2}(z-z_0)^2 + \dots$ in $B_r(z_0)$. Then by substitution we have $\dfrac{f(z)}{(z-z_0)^n} = \dfrac{f(z_0)}{(z-z_0)^n} + \dfrac{f'(z_0)}{(z-z_0)^{n-1}} + \dots$. Since the lowest power term is degree $-n$, by definition $\dfrac{f(z)}{(z-z_0)^n}$ has a pole of order n at z_0 by definition.

P4 Prove Proposition ★ above.

Answer: Since $f(z)$ has a zero of order n at z_0, we have $f(z) = (z-z_0)^n \tilde{f}(z)$ where $\tilde{f}(z)$ is defined and analytic in some neighborhood of z_0 with $\tilde{f}(z_0) \neq 0$. Similarly, we have $g(z) = (z-z_0)^m \tilde{g}(z)$. Then $h(z) = \dfrac{f(z)}{g(z)} = (z-z_0)^{n-m}\dfrac{\tilde{f}(z)}{\tilde{g}(z)}$, where $\dfrac{\tilde{f}(z)}{\tilde{g}(z)}$ is analytic and nonzero at z_0.

Then $m \leq n \implies n - m \geq 0$. Then $\lim_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)}$ by limit laws, which is finite and is therefore a removable singularity.

If $m > n$, then $\tilde{h}(z) = \dfrac{\tilde{f}(z)}{\tilde{g}(z)}$ is analytic in $B_r(z_0)$ for some r. Then since $\tilde{h}(z_0) \neq 0$ and $h(z) = \dfrac{\tilde{h}(z)}{(z-z_0)^{m-n}}$, by P3, $h(z)$ has a pole of order $m-n$ at z_0.