

P2 $(z - 2 - 2i)^4 = 16i$

Answer: Let $w = z - 2 - 2i$, then we have $w^4 = 16i \implies w^4 = 16(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2})$. Then by the n th roots formula, the roots are:

$$w_1 = \sqrt[4]{16} [\cos(\frac{1}{4} \cdot \frac{\pi}{2}) + i \sin(\frac{1}{4} \cdot \frac{\pi}{2})] = 2(\cos \frac{\pi}{8} + i \sin \frac{\pi}{8})$$

$$w_2 = \sqrt[4]{16} [\cos(\frac{1}{4} \cdot \frac{\pi}{2} + \frac{\pi}{2}) + i \sin(\frac{1}{4} \cdot \frac{\pi}{2} + \frac{\pi}{2})] = 2(\cos \frac{5\pi}{8} + i \sin \frac{5\pi}{8})$$

$$w_3 = \sqrt[4]{16} [\cos(\frac{1}{4} \cdot \frac{\pi}{2} + \pi) + i \sin(\frac{1}{4} \cdot \frac{\pi}{2} + \pi)] = 2(\cos \frac{9\pi}{8} + i \sin \frac{9\pi}{8})$$

$$w_4 = \sqrt[4]{16} [\cos(\frac{1}{4} \cdot \frac{\pi}{2} + \frac{3\pi}{2}) + i \sin(\frac{1}{4} \cdot \frac{\pi}{2} + \frac{3\pi}{2})] = 2(\cos \frac{13\pi}{8} + i \sin \frac{13\pi}{8})$$

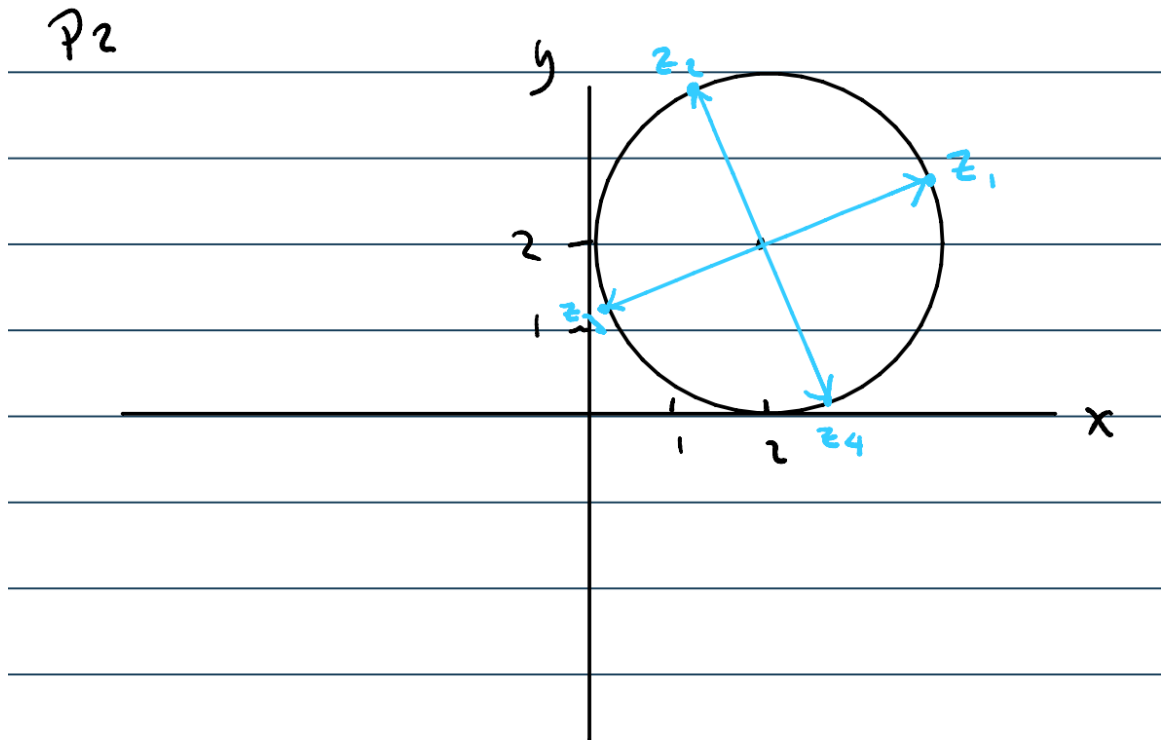
By substitution we have:

$$z_1 = w_1 + 2 + 2i = (2 + 2\cos \frac{\pi}{8}) + (2 + 2\sin \frac{\pi}{8})i$$

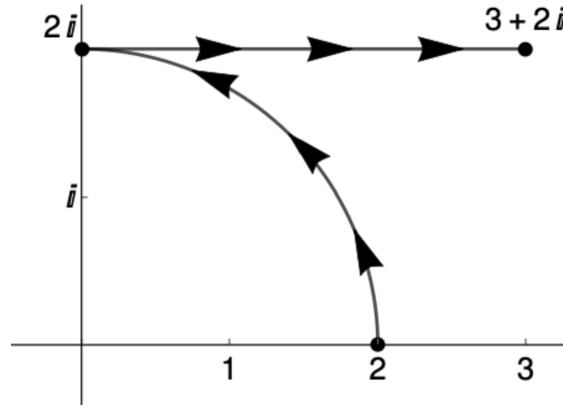
$$z_2 = w_2 + 2 + 2i = (2 + 2\cos \frac{5\pi}{8}) + (2 + 2\sin \frac{5\pi}{8})i$$

$$z_3 = w_3 + 2 + 2i = (2 + 2\cos \frac{9\pi}{8}) + (2 + 2\sin \frac{9\pi}{8})i$$

$$z_4 = w_4 + 2 + 2i = (2 + 2\cos \frac{13\pi}{8}) + (2 + 2\sin \frac{13\pi}{8})i$$



P3 Let γ be the pictured path, consisting of an arc of the circle of radius 2 centered at 0, joined with the line segment $[2i, 3 + 2i]$.



Evaluate $\int_{\gamma} (\bar{z} + 2i) dz$.

Answer: Let γ_1 be the arc and γ_2 be the line segment, then $\gamma_1(t) = 2e^{it}, t \in [0, \frac{\pi}{2}]$ and $\gamma_2(t) = 2i + 3t, t \in [0, 1]$. In addition, we have $\gamma_1'(t) = 2ie^{it}$ and $\gamma_2'(t) = 3$.

$$\begin{aligned}
 & \int_{\gamma} (\bar{z} + 2i) dz \\
 &= \int_{\gamma_1} (\bar{z} + 2i) dz + \int_{\gamma_2} (\bar{z} + 2i) dz \\
 &= \int_0^{\frac{\pi}{2}} [\overline{\gamma_1(t)} + 2i] \gamma_1'(t) dt + \int_0^1 [\overline{\gamma_2(t)} + 2i] \gamma_2'(t) dt \\
 &= \int_0^{\frac{\pi}{2}} 2ie^{it} (2e^{-it} + 2i) dt + \int_0^1 3(3t - 2i + 2i) dt \\
 &= \int_0^{\frac{\pi}{2}} 4i - 4e^{it} dt + \int_0^1 9t dt \\
 &= 4 \int_0^{\frac{\pi}{2}} i dt - 4 \int_0^{\frac{\pi}{2}} e^{it} dt + 9 \int_0^1 t dt \\
 &= 4[it]_0^{\frac{\pi}{2}} + 4[ie^{it}]_0^{\frac{\pi}{2}} + 9[\frac{t^2}{2}]_0^1 \\
 &= 2\pi i + 4(ie^{\frac{\pi}{2}i} - i) + \frac{9}{2} \\
 &= \frac{9}{2} + i(2\pi - 4 + 4e^{\frac{\pi}{2}i})
 \end{aligned}$$

P4 (a) Show that $|e^z| \leq e^{|z|}$ for all $z \in \mathbb{C}$.

Answer: Let $z = x + iy$, we have $|e^z| = |e^{x+iy}| = |e^x \cdot e^{iy}| = |e^x| \cdot |e^{iy}| = e^x$; then since $x = \operatorname{Re} z \leq |z|$, we have $e^x \leq e^{|z|}$. Therefore $|e^z| = e^x \leq e^{|z|} \implies |e^z| \leq e^{|z|}$.

(b) Show that $|\cos z| \leq e^{|z|}$ for all $z \in \mathbb{C}$.

Answer: By definition, $\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) = \frac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)}) = \frac{1}{2}(e^{-y+ix} + e^{y-ix})$, then $|\cos z| = \frac{1}{2}|e^{-y+ix} + e^{y-ix}| = \frac{1}{2}|e^{-y} \cdot e^{ix} + e^y \cdot e^{-ix}|$. Using triangle inequality, we have $\frac{1}{2}|e^{-y} \cdot e^{ix} + e^y \cdot e^{-ix}| \leq \frac{1}{2}|e^{-y} \cdot e^{ix}| + \frac{1}{2}|e^y \cdot e^{-ix}| = \frac{1}{2}|e^{-y}| \cdot |e^{ix}| + \frac{1}{2}|e^y| \cdot |e^{-ix}| = \frac{1}{2}|e^{-y}| + \frac{1}{2}|e^y|$. By part (a), $\frac{1}{2}|e^{-y}| + \frac{1}{2}|e^y| \leq \frac{1}{2}e^{|-y|} + \frac{1}{2}e^{|y|} = e^{|y|}$. Then $|\cos z| \leq e^{|y|} \leq e^{|z|} \implies |\cos z| \leq e^{|z|}$.

(c) Show that $\left| \frac{e^z \cos z}{z-i} \right| \leq \frac{e^{10}}{2}$ for all z on $C_3(2i)$.

Answer: Since $\left| \frac{e^z \cos z}{z-i} \right| = \frac{|e^z| \cdot |\cos z|}{|z-i|}$, we can apply parts (a) and (b) which gives us $\frac{|e^z| \cdot |\cos z|}{|z-i|} \leq \frac{e^{2|z|}}{|z-i|}$. To show that the given inequality is true, we will show that $e^{2|z|} \leq e^{10}$ and $|z-i| \geq 2$ separately.

On the numerator, we want to show that $e^{2|z|} \leq e^{10}$; since $|z| \geq 0$, it is sufficient to show that $2|z| \leq 10$, or $|z| \leq 5$. Since we are only considering $z \in C_3(2i) \implies |z-2i| = 3$, by triangle inequality we have $|z| = |(z-2i) + 2i| \leq |z-2i| + |2i| = 3 + 2 = 5 \implies |z| \leq 5$.

On the denominator we have $|z-i| = |(z-2i) + i| \geq ||z-2i| - |i||$ by triangle inequality; since $z \in C_3(2i) \implies |z-2i| = 3$, we then have $||z-2i| - |i|| = |3-1| = 2$. So $|z-i| \geq 2$.

As shown above, we have $e^{2|z|} \leq e^{10}$ and $|z-i| \geq 2$, therefore $\left| \frac{e^z \cos z}{z-i} \right| = \frac{|e^z| \cdot |\cos z|}{|z-i|} \leq \frac{e^{2|z|}}{|z-i|} \leq \frac{e^{10}}{2} \implies \left| \frac{e^z \cos z}{z-i} \right| \leq \frac{e^{10}}{2}$.

I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.

Signature: 

Date: 10/26/2020