Math 180 Homework 7

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13.2.4 A square $n \times n$ real matrix M is called *positive definite* if we have $x^T Mx > 0$ for each nonzero (column) vector $x \in \mathbf{R}^n$.

(a) Why does any positive definite $n \times n$ matrix M have the full rank n?

Answer: If M is not full rank, i.e. the column are not linearly independent, then there exists a solution to $Mx = 0$ by definition of linear independence, which means we would have $x^T Mx = 0$.

(b) Show that the matrix M used in the proof of Fisher's inequality in the text is positive definite (and hence it has rank v, without a calculation of the determinant).

Answer: By construction of M, each entry m_{ij} can only have two possible values: $\lambda\frac{v-1}{k-1}$ on the diagonal and λ elsewhere. Then we have $M = D + L$ where D is a diagonal matrix with positive elements on the diagonal and L is a matrix with all entries $\lambda > 0$. For nonzero $x \in \mathbf{R}^n$, we have $x^T Dx > 0$ and $x^T Lx \geq 0$, so $x^T Mx > 0$ and M is positive definite by definition.

13.4.1 Verify the formulas (13.2) and (13.3).

13.2 For a complete graph with n vertices, it can contain cycles of length 3 to n. For each k-cycle, $3 \leq k \leq n$, we have $\binom{n}{k}$ of choosing its k vertices. We can now reorder the vertices in $k!$ different ways, then divide by $2k$ to account for rotational symmetry. Therefore we have $|\mathcal{K}_{K_n}| = \displaystyle\sum_{k=3}^{n}\binom{n}{k}\frac{k!}{2k} = \sum_{k=3}^{n}\binom{n}{k}\frac{(k-1)!}{2}$.

13.3 For a complete bipartite graph, we need at least 2 vertices from each side to form a cycle. For each cycle with k vertices on each side ($2k$ vertices in total), we choose k vertices on each side, giving us $\binom{n}{k}^2$ possibilities. Then similar to the previous part, we can reorder the vertices on each side in $(k!)^2$ different ways, then divide by $2k$ to accuont for rotational symmetry. Therefore we have $|\mathcal{K}_{K_{n,n}}| = \displaystyle\sum_{k=2}^{n}\binom{n}{k}^2\frac{(k!)^2}{2k} = \sum_{k=2}^{n}\binom{n}{k}^2\frac{k!(k-1)!}{2}$.

P11 Show that block designs of type 2-(21,6,1) and 2-(25,10,3) satisfy the integrality conditions of Theorem 13.1.3 yet fail Fisher's inequality, so cannot constitute valid block designs.

Answer: For a block design of type 2-(21,6,1), we have $t = 2$, $v = 21$, $k = 6$ and $\lambda = 1$. By substitution, we can see that $\lambda\dfrac{v(v-1)}{k(k-1)} = \dfrac{21 \cdot 20}{6 \cdot 5} = 14$ and $\lambda\dfrac{v-1}{k-1} = \dfrac{20}{5} = 4$ are both integers, therefore the block design satisfies the integrality conditions. However, $|\mathcal{B}| = \lambda\dfrac{v(v-1)}{k(k-1)} = 14 \leq 21 = |V|$, so it fails Fisher's inequality.

Similarly, for a block design of type 2-(25,10,3), we have $t = 2$, $v = 25$, $k = 10$ and $\lambda = 3$. By substitution, we can see that $\lambda\dfrac{v(v-1)}{k(k-1)} = 2 \cdot \dfrac{25 \cdot 24}{9 \cdot 10} = 20$ and $\lambda\dfrac{v-1}{k-1} = 3 \cdot \dfrac{24}{9} = 8$ are both integers, therefore the block design satisfies the integrality conditions. However, $|\mathcal{B}| = \lambda\dfrac{v(v-1)}{k(k-1)} = 20 \leq 25 = |V|$, so it also fails Fisher's inequality.