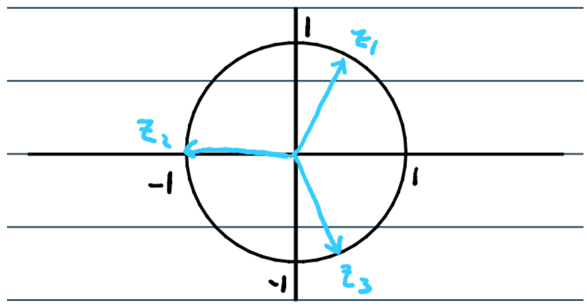
P2 Find and plot all $z \in \mathbb{C}$ such that $z^3 = -1$.

Answer: We have $z^3 = -1 \implies z^3 = \cos\pi + i\sin\pi$, so by the nth roots formula, the roots are:

$z_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$z_2 = \cos\left(\frac{\pi}{3} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{3} + \frac{2\pi}{3}\right) = -1$

$z_3 = \cos\left(\frac{\pi}{3} + \frac{4\pi}{3}\right) + i\sin\left(\frac{\pi}{3} + \frac{4\pi}{3}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}i$



P4 Define $u : \mathbb{R}^2 \to \mathbb{R}$ by $u(x, y) = 4x^3y - 4y^3x - 2x$.

(a) Show that u is a harmonic function.

Answer: We have $u_x = 12x^2y - 4y^3 - 2$ and $u_y = 4x^3 - 12xy^2$, then $u_{xx} = 24xy$ and $u_{yy} = -24xy$. Then $\Delta u = u_{xx} + u_{yy} = 0$ and therefore u is harmonic.

(b) Find a harmonic conjugate v for u.

Answer: By Cauchy-Riemann, v must satisfy $u_x = v_y \implies v = \int 12x^2y - 4y^3 - 2dy = -y^4 + 6y^2x^2 - 2y + C(x) \implies v_x = 12xy^2 + C'(x)$. Again by Cauchy-Riemann, v must also satisfy $u_y = -v_x \implies v_x = -4x^3 + 12xy^2$. Then we have $C'(x) = -4x^3 \implies C(x) = -x^4$ and therefore $v(x, y) = -y^4 + 6y^2x^2 - 2y - x^4$ by substitution.

P6 (a) Find $\dfrac{1}{2\pi i}\displaystyle\int_{C_2(0)} \frac{[\text{Re}(z)]^2}{z} dz$.

Answer: We have $\gamma(t) = 2e^{it}$ for $t \in [0, 2\pi]$, then $\displaystyle\int_{C_2(0)} \frac{[\text{Re}(z)]^2}{z} dz = \int_0^{2\pi} f(\gamma(t))\gamma'(t)dt = \int_0^{2\pi} \frac{[\text{Re}(2e^{it})]^2}{2e^{it}} \cdot 2ie^{it}dt = 4i\int_0^{2\pi} \cos^2(t)dt = 4\pi i$, so $\dfrac{1}{2\pi i}\displaystyle\int_{C_2(0)} \frac{[\text{Re}(z)]^2}{z} dz = \frac{4\pi i}{2\pi i} = 2$.

(b) Find $\displaystyle\int_\gamma \frac{2z+1}{e^{\pi z}-1} dz$, where γ is the pictured path.

Answer: Let $f = \dfrac{2z+1}{e^{\pi z}-1}$. We can start by finding the singularities f by solving $e^{\pi z} - 1 = 0 \implies z_0 = 0, 2i$. Then, by Proposition ★, we have simple poles at both 0 and $2i$. We can now find their residues as follows:

$\text{Res}(f, 0) = \displaystyle\lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{z(2z+1)}{e^{\pi z}-1} = \frac{1}{\pi}$

$\text{Res}(f, 2i) = \displaystyle\lim_{z\to 2i} (z-2i)f(z) = \lim_{z\to 2i} \frac{(z-2i)(2z+1)}{e^{\pi z}-1} = \frac{1}{\pi} + \frac{4i}{\pi}$

Then by Residue Theorem, we have

$\displaystyle\int_\gamma f(z)dz = 2\pi i\left(\frac{1}{\pi} + \frac{1}{\pi} + \frac{4i}{\pi}\right) = \frac{2\pi i(2+4i)}{\pi} = -8 + 4i.$

P8 Use residue theory to show that $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2+1}dx = \pi e^{-2}$.

Answer: For $R > 0$, let σ_R be the part of $C_R(0)$ in the upper half plane and let $\gamma_R = [[-R, R], \sigma_R]$. Note that $\cos(2z)$ is unbounded, but we have $\cos(2x) = \mathrm{Re}(e^{2ix})$, meaning that we can first evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2+1}dx$ and take the real part. Let $f(z) = \dfrac{e^{2iz}}{z^2+1}$, then we have $\displaystyle\int_{\gamma_R} f(z)dz = \int_{[-R,R]} f(z)dz + \int_{\sigma_R} f(z)dz$.

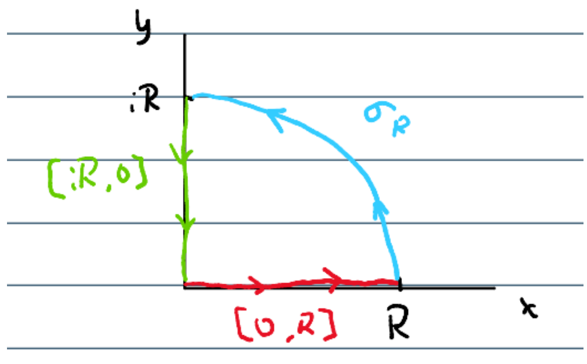
We want to first show that $\displaystyle\int_{\sigma_R} f(z)dz \to 0$. Let $L = \mathrm{length}(\sigma_R) = \pi R$. For z on σ_R, we have $|f(z)| = \left|\dfrac{e^{2iz}}{z^2+1}\right| = \dfrac{\left|e^{2iz}\right|}{|z^2+1|} \leq \dfrac{e^{\mathrm{Re}(2iz)}}{\left||z|^2-1\right|} \leq \dfrac{e^0}{|R^2-1|} = \dfrac{1}{R^2-1} = M$ for R large enough. Then by ML-estimate, we have $\left|\displaystyle\int_{\sigma_R} f(z)dz\right| \leq ML = \dfrac{\pi R}{R^2-1} \to 0$ as $R \to \infty$. Therefore $\displaystyle\lim_{R\to\infty} \int_{\sigma_R} f(z)dz = 0$.

We will now find $\displaystyle\int_{\gamma_R} f(z)dz$ using Residue Theorem. We have $f(z) = \dfrac{e^{2iz}}{z^2+1} = \dfrac{e^{2iz}}{(z-i)(z+i)}$; since $-i$ is not in γ_R, we only need to examine $z_0 = i$, which is a simple pole by Proposition ★. Then $\mathrm{Res}(f,i) = \displaystyle\lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\frac{e^{2iz}}{(z+i)} = \frac{e^{-2}}{2i}$. By Residue Theorem, $\displaystyle\int_{\gamma_R} f(z)dz = 2\pi i\,\mathrm{Res}(f,i) = \frac{2\pi i e^{-2}}{2i} = \pi e^{-2}$.

Then by substitution we have $\displaystyle\int_{\gamma_R} f(z)dz = \int_{[-R,R]} f(z)dz + \int_{\sigma_R} f(z)dz \implies \pi e^{-2} = \int_{[-R,R]} f(z)dz + 0 \implies \int_{[-R,R]} f(z)dz = \pi e^{-2} \implies \int_{-\infty}^{\infty} \frac{e^{2iz}}{z^2+1}dz = \pi e^{-2}$. Now we can take the real part, which gives us $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2+1}dx = \mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{e^{2iz}}{z^2+1}dz\right) = \mathrm{Re}(\pi e^{-2}) = \pi e^{-2}$.

P9 Use the argument principle to find the number of zeroes of $f(z) = z^5 + z^4 + 4z^3 + 10z^2 + 9$ in the first quadrant.

Answer: Let R be sufficiently large such that all zeroes of $f(z)$ is enclosed by the curve $\gamma_R = [[0, R], \sigma_R, [iR, 0]]$ as shown below:



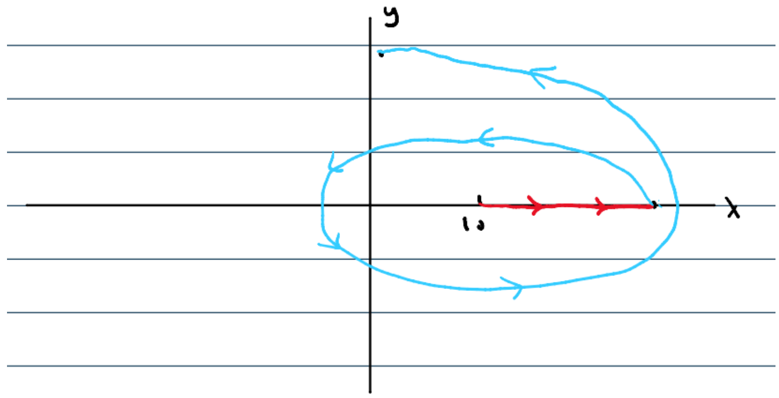
Then, we have

1. $f([0, R])$: $f(x) = x^5 + x^4 + 4x^3 + 10x^2 + 9$ for $x \in [0, R]$
2. $f(\sigma_R)$: $f(Re^{it}) = R^5e^{5it} + R^4e^{4it} + 4R^3e^{3it} + 10R^2e^{2it} + 9 \approx R^5e^{5it}$ for $t \in [0, \frac{\pi}{2}]$
3. $f([iR, 0])$: $f(iy) = iy^5 + y^4 - 4iy^3 - 10y^2 + 9 = (y^4 - 10y^2 + 9) + (y^5 - 4y^3)i$ for $y \in [0, R]$

Note that $f(z) \neq 0$ on $\gamma_R = [[0, R], \sigma_R, [iR, 0]]$ since

1. $f(z) \geq 9$ on $[0, R]$
2. R was chosen sufficiently large such that $f(z) \neq 0$ on σ_R
3. $f(iy) = (y^4 - 10y^2 + 9) + (y^5 - 4y^3)i = (y - 3)(y - 1)(y + 1)(y + 3) + y^3(y - 2)(y + 2)$; since $\operatorname{Re} f(iy)$ and $\operatorname{Im} f(iy)$ have no common zeroes, $f(z) \neq 0$ on $[iR, 0]$

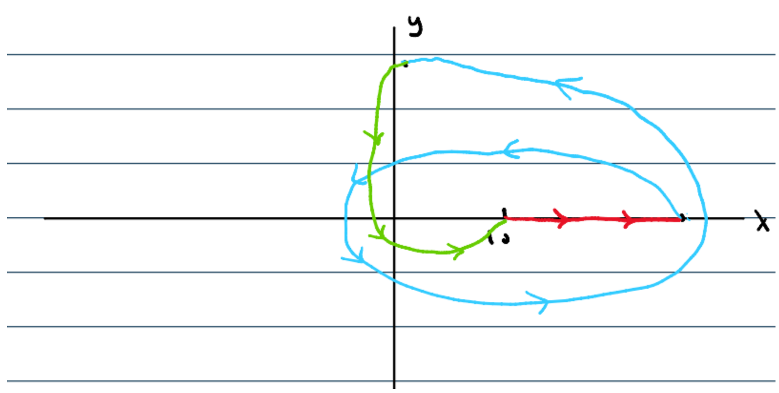
Now we can begin to sketch $f(\sigma_R)$. Since $f(x)$ always returns a real value for $x \in [0, R]$, $[0, R]$ maps to $[9, N]$ on the real axis where N is some large number. Then, since $f(iR) = (R^4 - 10R^2 + 9) + (R^5 - 4R^3)i \approx R^4 + R^5 i \approx R^5 i$, we also know that σ_R ends at some point in the first quadrant, near the positive imaginary axis. Then since $f(Re^{it}) \approx R^5e^{i(5t)}, t \in [0, \frac{\pi}{2}] \implies 5t \in [0, \frac{5\pi}{2}]$, σ_R is mapped to a circular path that wraps around the origin once and ends near the positive imaginary axis as shown below:



We can now use a sign chart to find the map of $[iR, 0]$:

$y =$	$(0,1)$	$(1,2)$	$(2,3)$	$(3,R)$
Quadrant	IV	III	II	I
$\operatorname{Re} f(iy)$	+	-	-	+
$y-3$	-	-	-	+
$y-1$	-	+	+	+
$y+1$	+	+	+	+
$y+3$	+	+	+	+
$\operatorname{Im} f(iy)$	-	-	+	+
y^3	+	+	+	+
$y-2$	-	-	+	+
$y+2$	+	+	+	+

Therefore our $f(\gamma_R)$ looks like:



So by the argument principle, since $f(\gamma_R)$ wraps counterclockwise around the origin twice, we have $N_0 - N_\infty = 2$. Since $f(z)$ is analytic, it has no poles, i.e. $N_\infty = 0$. Therefore $N_0 = 2$, i.e. $f(z)$ has two zeroes in the first quadrant.

P11 Suppose that $f : A_{0,2}(0) \to \mathbb{C}$ is analytic and satisfies $f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n^4}$ for $n = 1, 2, 3, \ldots$. Show that there is a sequence $(z_n)_{n=1}^{\infty}$ with $z_n \to 0$ as $n \to \infty$ such that $f(z_n) \to 2020i$ as $n \to \infty$.

Answer: Let $g(z) = |z|^4$; since $f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n^4} = \left(\frac{1}{n}\right)^4 = g\left(\frac{1}{n}\right)$, we can define $z_n = e^{\ln \sqrt[4]{2020i} - i\pi n}$. Then clearly $z_n \to 0$ as $n \to \infty$ and we also have $g(z_n) \to 2020i$ as $n \to \infty$.

I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.

Signature:

Date: 12/13/2020