Math 132 Homework 3

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10/20/2020

2.4.33 Suppose that $f = u + iv$ is analytic in a region Ω. Show that

(a) $f'(z) = u_x - iu_y$, also $f'(z) = v_y + iv_x$

Answer: By Cauchy-Riemann equations, we have $u_x = v_y$ and $u_y = -v_x$. Starting from $f'(z) = u_x + iv_x$ (differentiate f with respect to x), we can substitute in $v_x = -iv_y$, which give us $f'(z) = u_x - iu_y$. Similarly, we can substitute in $u_x = v_y$, which gives us $f'(z) = v_y + iv_x$.

(b) $|f'(z)|^2 = u_x^2 + u_y^2 = v_x^2 + v_y^2$

Answer: From part (a), $f'(z) = u_x - iu_y \implies |f'(z)|^2 = (\sqrt{u_x^2 + u_y^2})^2 = u_x^2 + u_y^2$; similarly, $f'(z) = v_y + iv_x \implies |f'(z)|^2 = (\sqrt{v_x^2 + v_y^2})^2 = v_x^2 + v_y^2$.

(c) Conclude from (a) or (b) that either $\operatorname{Re} f$ or $\operatorname{Im} f$ is constant in Ω, then f is constant in Ω.

Answer: On the one hand, if $\operatorname{Re} f = u$ is constant, we have $u_x = u_y = 0 \implies |f'(z)|^2 = u_x^2 + u_y^2 = 0$. On the other hand, if $\operatorname{Im} f = v$ is constant, we have $v_x = v_y = 0 \implies |f'(z)| = v_x^2 + v_y^2 = 0$. Since the magnitude of the $f'(z)$ is 0 in either case, $f'(z)$ is constant in Ω.

2.5.3 $e^x \cos y$

Answer: Let $u = e^x \cos x$, then $\Delta u = \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = \frac{\delta}{\delta x}(e^x \cos y) - \frac{\delta}{\delta y}(e^x \sin y) = e^x \cos y - e^x \cos y = 0$. Therefore $e^x \cos y$ is harmonic on $\Omega = \mathbb{C}$.

2.5.14 $x^2 - y^2 - xy$

Answer: Let $u = x^2 - y^2 - xy$, then $u_x = 2x - y$ and $u_y = -2y - x$. By Cauchy-Riemann, v must satisfy $u_x = v_y \implies v_y = 2x - y \implies v = 2xy - \frac{1}{2}y^2 + c(x)$. Again by Cauchy-Riemann, v must also satisfy $u_y = -v_x \implies -2y - x = -2y - c'(x) \implies c'(x) = x \implies c(x) = \frac{1}{2}x^2 + C$. Therefore $v = 2xy + \frac{1}{2}x^2 - \frac{1}{2}y^2 + C$. We can check Cauchy-Riemann as follows:

$u_x = 2x - y$, $v_y = 2x - y \implies u_x = v_y$

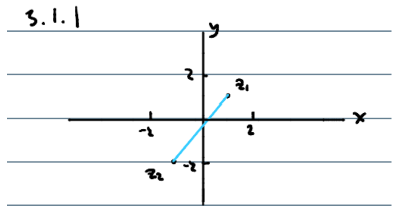
$u_y = -2y - x$, $-v_x = -2y - x \implies u_y = -v_x$

2.5.19 Show that if u and u^2 are both harmonic in a region Ω, then u must be constant.

Answer: Since u^2 is harmonic in Ω, we have $\frac{\delta^2}{\delta x^2}(u^2) + \frac{\delta^2}{\delta y^2}(u^2) = 0 \implies \frac{\delta}{\delta x}(2uu_x) + \frac{\delta}{\delta y}(2uu_y) = 0 \implies 2(uu_{xx} + u_x^2) + 2(uu_{yy} + u_y^2) = 0 \implies u(u_{xx} + u_{yy}) + u_x^2 + u_y^2 = 0$. Then, since u is harmonic, we also have $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0 \implies u_{xx} + u_{yy} = 0$. Combining the last two equalities we have $u_x^2 + u_y^2 = 0$, implying that $u_x = u_y = 0$ as u is a real function. Therefore u is constant.

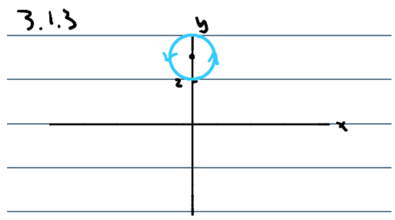
3.1.1 The line segment with initial point $z_1 = 1 + i$ and terminal $z_2 = -1 - 2i$.

Answer: $\gamma(t) = z_0 + t(z_1 - z_0) = (1 + i) - t((1 + i) - (-1 - 2i)) = t(-2 - 3i) + 1 + i$



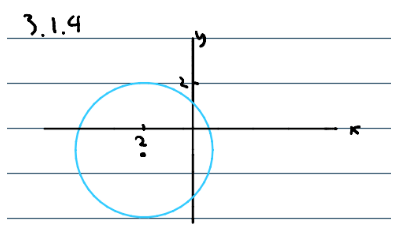
3.1.3 The counterclockwise circle with center at $3i$ and radius 1.

Answer: $\gamma(t) = z_0 + re^{it} = 3i + e^{it}$



3.1.4 The clockwise circle with center at $-2-i$ and radius 3.

Answer: $\gamma(t) = z_0 + re^{it} = -2 - i + 3e^{-it}$



P1 For what values of z is the sequence $\{z^n\}_{n=1}^{\infty}$ bounded? For which values of z does the sequence converge to 0?

Answer: Let $r = |z| \in \mathbb{R}$, then as $n \to \infty$, we have $r \leq 1 \implies r^n \leq 1$ and $r > 1 \implies r^n \to \infty$. Therefore $\{z^n\}_{n=1}^{\infty}$ is bounded for $|z| \leq 1$ and unbounded otherwise. Similarly, since $r = |z| \to 0$ when $r < 1$, $\{z^n\} \to 0$ when $|z| < 1$.

P2 Let $f(z) = |z|^2$ for all $z \in \mathbb{C}$.

(a) Use the definition of the complex derievative, show that f is not differentiable at any **nonzero** point $z_0 = x_0 + iy_0$.

Answer: $\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = \lim_{h\to 0} \frac{|z_0+h|^2+|z_0|^2}{h} = \lim_{h\to 0} \frac{(z_0+h)\overline{(z_0+h)}-z_0\overline{z_0}}{h}$, which evaluates to 0 only if $z = 0$ and has different partial derievatives otherwise.

(b) Use the definition of the complex derievative, show that $f'(0)$ exists.

Answer: $f'(z_0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|^2}{h} = \lim_{h\to 0} \frac{h\bar{h}}{h} = \lim_{h\to 0} \bar{h} = 0$.

(c) If f analytic at $z_0 = 0$?

Answer: No because f is not differentiable on any open set containing $z_0 = 0$.

P3 Show that $f(z) = \bar{z}^2$ does not satisfy the Cauchy-Riemann equations and hence is not analytic on $\mathbb{C}$.

Answer: Let $z = x + iy$, then $f(z) = \bar{z}^2 = \overline{x - iy}^2 = x^2 - y^2 - 2ixy$. So $u = \operatorname{Re} f = x^2 + y^2$ and $v = \operatorname{Im} f = -2xy$. Then we can test the Cauchy-Riemann equations as follows:

$u_x = 2x, v_y = -2x \implies u_x \neq -2x$

$u_y = 2y, -v_x = 2y \implies u_y = -v_x$

Therefore $f(z) = \bar{z}^2$ does not satisfy the Cauchy-Riemann equations and hence is not analytic on $\mathbb{C}$.

P4 Let $f(z) = z^7 - z^5$.

(a) Recall from lecture the reverse triangle inequality:

$$|w - z| \geq ||w| - |z|| \text{ for all } w, z \in \mathbb{C}.$$

Use this to show that $|f'(1 - i)| \geq 56 - 20 = 36$ and hence $f'(1 - i) \neq 0$.

Answer: $f'(z) = 7z^6 - 5z^4 \implies |f'(1-i)| = \left|7(1-i)^6 - 5(1-i)^4\right| \geq \left|7|1-i|^6 - 5|1-i|^4\right| = \left|7 \cdot \sqrt{2}^6 - 5 \cdot \sqrt{2}^4\right| = 36 \neq 0$.

(b) Combining part (a) with the theorem from class, we see that f^{-1} exists and is analytic near $f(1 - i)$. What is $(f^{-1})'((1 - i)^7 - (1 - i)^5)$?

Answer: Since $f'(1 - i) \neq 0$, we have $(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}$. Let $w = (1 - i)^7 - (1 - i)^5 = z^7 - z^5$, then $f^{-1}(w) = z \implies z = 1 + i$. So $(f^{-1})'(w) = \frac{1}{f'(z)} = \frac{1}{7(1-i)^6 - 5(1-i)^4} = \frac{1}{-i^7 + 7i^6 - 21i^5 + 35i^4 - 35i^3 + 21i^2 - 7i + 1025}$.

P5 Let $f(z) = e^{z^2}$.

(a) Find the real and imaginary parts u and v of f so that $f(z) = u(x, y) + iv(x, y)$.

Answer: Let $z = x + iy$, then $z^2 = x^2 - y^2 + 2ixy$. So $f(z) = e^{z^2} = e^{x^2-y^2+2ixy} = e^{x^2-y^2} \cdot e^{2ixy} = e^{x^2-y^2}(\cos(2xy) + i\sin(2xy)) = e^{x^2-y^2}\cos(2xy) + ie^{x^2-y^2}\sin(2xy)$. Therefore we have $u = e^{x^2-y^2}\cos(2xy)$ and $v = e^{x^2-y^2}\sin(2xy)$.

(b) Show that $e^{x^2-y^2}\cos(2xy)$ is a harmonic function and find a harmonic conjugate.

Answer: Let $u = e^{x^2-y^2}\cos(2xy) = e^{x^2}e^{-y^2}\cos(2xy)$, then we need to verify that $\Delta u = \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$. Differentiating twice gives us $\frac{\delta^2 u}{\delta x^2} = 2e^{x^2-y^2}[(2x^2 - 2y^2 + 1)\cos(2xy) - 4xy\sin(2xy)]$ and $\frac{\delta^2 u}{\delta y^2} = 2e^{x^2-y^2}[4xy\sin(2xy) - (2x^2 - 2y^2 + 1)\cos(2xy)]$. Therefore $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$ and u is harmonic.

We can verify that $v = e^{x^2-y^2}\sin(2xy)$ from part (a) is the harmonic conjugate of u by verifying Cauchy-Riemann:

$u_x = e^{x^2-y^2}[2x\cos(2xy) - 2y\sin(2xy)] = v_y \implies u_x = v_y$

$u_y = e^{x^2-y^2}[-2x\sin(2xy) - 2y\cos(2xy)] = -v_x \implies u_y = -v_x$

Therefore $v = e^{x^2-y^2}\sin(2xy)$ is the harmonic conjugate of u.

P6 Let $f : D \to \mathbb{C}$ be an analytic function defined on a region D such that $f'(z) = f(z)$ for all $z \in D$. Show that $f(z) = \alpha e^z$ for some constant $\alpha \in \mathbb{C}$.

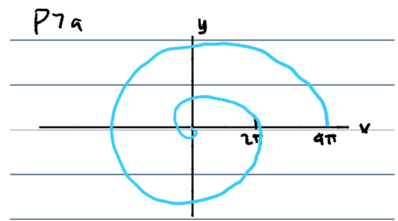
Answer: Let $g(z) = e^{-z}f(z)$, then $g'(z) = e^{-z}f'(z) - e^{-z}f(z)$. Since $f'(z) = f(z)$, we have $g'(z) = 0$.

In addition, since f and e^{-z} are both analytic in D, so is $g(z)$. Therefore $g(z) = \alpha$ for some constant α and by substitution we have $e^{-z} f(z) = \alpha \implies f(z) = \alpha e^{z}$.

P7 Plot the given path:

(a) $\gamma(t) = te^{-it}$ for $t \in [0, 4\pi]$.

Answer: Since $e^{-it}, t \in [0, 4\pi]$ is a doubly traced, clockwise circle, te^{-it} is a clockwise spiral with radius increasing from 0 to 4π.



(b) $\gamma(t) = t + i \sin(\pi t)$ for $t \in [0, 2]$.

Answer: We have $x(t) = t$ and $y(t) = \sin(\pi t)$, so the graph is $y(x) = \sin(x)$ from 0 to 2π.