# Math 132 Homework 7

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4.4.2 $\dfrac{\sin z}{e^z}, z_0 = 1 + 7i$.

**Answer**: $\dfrac{\sin z}{e^z}$ is analytic everywhere, so $R = \infty$.

4.4.3 $\dfrac{z}{z - 3i}, z_0 = 0$.

**Answer**: $\dfrac{z}{z - 3i}$ is analytic for $z \neq 3i$, so $R = |z_0 - 3i| = |0 - 3i| = 3$.

4.4.5 $\dfrac{z+1}{z-i}, z_0 = 2 + i$.

**Answer**: $\dfrac{z+1}{z-i}$ is analytic for $z \neq i$, so $R = |z_0 - i| = |2 + i - i| = 2$.

4.4.6 $\dfrac{\sin z}{z^2 + 4}, z_0 = 3$.

**Answer**: $\dfrac{\sin z}{z^2+4}$ is analytic for $z \neq \pm 2i$, so $R = \min\{|z_0 - 2i|, |z_0 + 2i|\} = \min\{|3 - 2i|, |3 + 2i|\} = \min\{\sqrt{13}, \sqrt{13}\} = \sqrt{13}$.

4.5.1 $\dfrac{1}{1+z^2}, 1 < |z|$.

**Answer**: We have $1 < |z| \implies \left|\dfrac{1}{z^2}\right| < 1$. Then we can use the geometric formula as follows: $\dfrac{1}{1+z^2} = \dfrac{1}{z^2} \cdot \dfrac{1}{1 + \frac{1}{z^2}} = \dfrac{1}{z^2} \sum_{n=0}^{\infty} \dfrac{1}{z^{2n}} = \sum_{n=0}^{\infty} \dfrac{1}{z^{2n+2}}$.

4.5.2 $\dfrac{3+z}{2-z}, 2 < |z|$.

**Answer**: We have $2 < |z| \implies \left|\dfrac{2}{z}\right| < 1$, so $\dfrac{3+z}{2-z} = \dfrac{3+z}{z} \cdot \dfrac{1}{1 + \frac{2}{z}} = -\dfrac{3+z}{z} \sum_{n=0}^{\infty} \dfrac{2^n}{z^n} = \sum_{n=0}^{\infty} \dfrac{-(3+z)2^n}{z^{n+1}}$.

4.5.4 $z + \dfrac{1}{z}, 1 < |z - 1|$.

**Answer**: We have $1 < |z-1| \implies \left|\dfrac{1}{z-1}\right| < 1$, then $z + \dfrac{1}{z} = \dfrac{z^2+1}{z} = \dfrac{z^2+1}{1 + (z-1)} = \dfrac{z^2+1}{z-1} \cdot \dfrac{1}{1 + \frac{1}{(z-1)}} = \dfrac{z^2+1}{z-1} \sum_{n=0}^{\infty} \dfrac{1}{(z-1)^n} = \sum_{n=0}^{\infty} \dfrac{(-1)^n (z^2+1)}{(z-1)^{n+1}}$.

4.5.13 $\dfrac{z}{(z+2)(z+3)}, 2 < |z| < 3$.

**Answer**: By partial fractions we have $\dfrac{1}{(z+2)(z+3)} = \dfrac{A}{z+2} + \dfrac{B}{z+3} \implies 1 = A(z+3) + B(z+2) \implies A = 1, B = -1 \implies \dfrac{z}{(z+2)(z+3)} = \dfrac{z}{z+2} + \dfrac{-z}{z+3}$. Then using our bounds $2 < |z| \implies \left|\dfrac{2}{z}\right| < 1$, we have $\dfrac{z}{z+2} = \dfrac{1}{1+\frac{2}{z}} = \sum_{n=0}^{\infty} \dfrac{(-2)^n}{z^n}$. Similarly, using $|z| < 3 \implies \left|\dfrac{z}{3}\right| < 1$, we have $\dfrac{-z}{z+3} = \dfrac{-z}{3} \cdot \dfrac{1}{1+\frac{z}{3}} = \dfrac{-z}{3} \cdot \sum_{n=0}^{\infty} \dfrac{(-z)^n}{3^n} = \sum_{n=0}^{\infty} \dfrac{(-z)^{n+1}}{3^{n+1}}$. Therefore $\dfrac{z}{(z+2)(z+3)} = \dfrac{z}{z+2} + \dfrac{-z}{z+3} = \sum_{n=0}^{\infty} \dfrac{(-2)^n}{z^n} + \sum_{n=0}^{\infty} \dfrac{(-z)^{n+1}}{3^{n+1}}$.

P1 Use the geometric series formula to derive the power series expansion for the given function $f$ centered at the given point $z_0$:

(a) $f(z) = \dfrac{z}{1-z}$, centered at $z_0 = 0$.

**Answer**: By geometric series formula, we have $f(z) = -\dfrac{1}{1-\frac{1}{z}} = -\sum_{n=0}^{\infty} \dfrac{1}{z^n}$.

(b) $f(z) = \dfrac{z^2+1}{z-1}$, centered at $z_0 = 0$.

**Answer**: By geometric series formula, we have $f(z) = \dfrac{z^2+1}{z-1} = (z^2+1) \cdot \dfrac{-1}{1-z} = -(z^2+1)\sum_{n=0}^{\infty} z^n = -\sum_{n=0}^{\infty} z^{n+2} - \sum_{n=0}^{\infty} z^n = -1 - z - 2\sum_{n=0}^{\infty} z^{n+2}$.

(c) $f(z) = \dfrac{1}{(1-z)^3}$, centered at $z_0 = 0$.

**Answer**: By differentiating the geometric series formula twice, we have $\sum_{n=0}^{\infty} a^n = \dfrac{1}{1-a} \implies \sum_{n=0}^{\infty} n a^{n-1} = \dfrac{1}{(a-1)^2} \implies \sum_{n=0}^{\infty} n(n-1)a^{n-2} = \dfrac{2}{(1-a)^3}$. Therefore by substitution we have $\dfrac{1}{(1-z)^3} = \dfrac{1}{2}\sum_{n=0}^{\infty} n(n-1)z^{n-2}$.

(d) $f(z) = \dfrac{1}{1-2z^3}$, centered at $z_0 = 0$.

**Answer**: By geometric series formula, we have $\dfrac{1}{1-2z^3} = \sum_{n=0}^{\infty} (2z^3)^n = \sum_{n=0}^{\infty} 2^n z^{3n}$.

P2 Find the Laurent series for $\dfrac{1}{(z-i)(z+2i)}$ in the given domain:

(a) $A_{1,2}(0)$

**Answer**: $\dfrac{1}{(z-i)(z+2i)} = \dfrac{1}{3i}\left[\dfrac{1}{z-i} - \dfrac{1}{z+2i}\right] = \dfrac{1}{3i}\left[\dfrac{-1}{i} \cdot \dfrac{1}{1-\dfrac{z}{i}} + \dfrac{-1}{2i} \cdot \dfrac{1}{1+\dfrac{z}{2i}}\right]$ by partial fractions, then we have $\left|\dfrac{z}{i}\right| = \dfrac{|z|}{1} < 1$ for $z \in A_{1,2}(0)$, so $\dfrac{1}{1-\dfrac{z}{i}} = \sum_{n=0}^{\infty} \dfrac{z^n}{i^n}$. Simiarly, we have $\left|\dfrac{z}{2i}\right| = \dfrac{|z|}{2} > 1$ for $z \in A_{1,2}(0)$, so $\dfrac{1}{1+\dfrac{z}{2i}} = \sum_{n=0}^{\infty} \dfrac{(-1)^n z^n}{(2i)^n}$. Therefore $\dfrac{1}{(z-i)(z+2i)} = \dfrac{1}{3i}\left[\dfrac{-1}{i} \cdot \sum_{n=0}^{\infty} \dfrac{z^n}{i^n} + \dfrac{-1}{2i}\sum_{n=0}^{\infty} \dfrac{(-1)^n z^n}{(2i)^n}\right] = \dfrac{1}{3i}\left[-\sum_{n=0}^{\infty} \dfrac{z^n}{i^{n+1}} + \sum_{n=0}^{\infty} \dfrac{(-1)^{n+1} z^n}{(2i)^{n+1}}\right]$.

(b) $A_{0,3}(i)$

**Answer**: $\dfrac{1}{(z-i)(z+2i)} = \dfrac{1}{(z-i)(3i+(z-i))} = \dfrac{1}{z-i} \cdot \dfrac{1}{3i} \cdot \dfrac{1}{1+\dfrac{z-i}{3i}} = \sum_{n=0}^{\infty} \dfrac{(-1)^n (z-i)^{n-1}}{(3i)^{n+1}}$.

(c) $A_{3,\infty}(i)$

**Answer**: $3 < |z-i| < \infty \implies \frac{1}{3} > \left|\frac{1}{z-i}\right| > 0$, then $\frac{1}{(z-i)(z+2i)} = \frac{1}{(z-i)(3i+(z-i))} = \frac{1}{(z-i)^2} \cdot \frac{1}{1+\frac{3i}{z-i}} = \frac{1}{(z-i)^2}\sum_{n=0}^{\infty}\frac{(-3i)^n}{(z-i)^n}$.