Q1 I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.

Signature:

Date: 10/26/2020

Q2 Prove that an Eulerian graph with more than one vertex must have at least 3 vertices with the same degree. (Hint: partition the vertices)

Answer: By contradiction. Let $n = |V(G)|$ and suppose that at most 2 vertices have the same degree. Then since each vertex in an Eulerian graph has to have an even degree, our possible degrees for each vertex are the even terms from 2 to $n - 1$, giving us at most $\frac{n-1}{2}$ possible degrees. Since by our assumption we can have at most 2 vertices per degree, we can only have at most $n - 1$ vertices, which contradicts with our assumption of $n = |V(G)|$. Therefore an Eulerian graph with more than one vertex must have at least 3 vertices with the same degree.

Q3 Consider the following sequence

$$(4, 4, 4, 4, 5, 5, 6)$$

1. Prove that the sequence above is a valid degree sequence (score).

   Answer: By theorem 4.3.3, we can reduce the sequence as follows:

   $(4, 4, 4, 4, 5, 5, 6)$

   $(3, 3, 3, 3, 4, 4)$

   $(2, 2, 2, 2, 3)$

   $(1, 1, 1, 1)$

   $(0, 0, 0)$

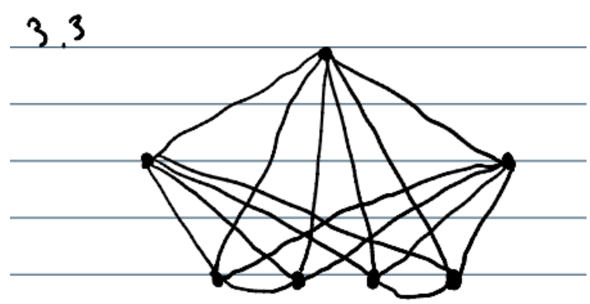
   Since the last sequence is a graph score, the original must also be a score of some graph.

2. Prove that a graph with the score above cannot be planar.

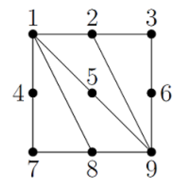
   Answer: By theorem 6.3.3, a planar graph with at least 3 vertices must satisfy $|E| \leq 3|V| - 6$. Using the given sequence, we have $|V| = 7$, so $3|V| - 6 = 15$. However, $|E| = \frac{1}{2}\sum_{v \in V} \deg_G(v) = 16 > 15 = 3|V| - 6$. Therefore the graph cannot be planar.

3. Draw a graph with the score above.

   Answer:



4. Consider the following graph G below:



1. Does G have an Eulerian cycle?

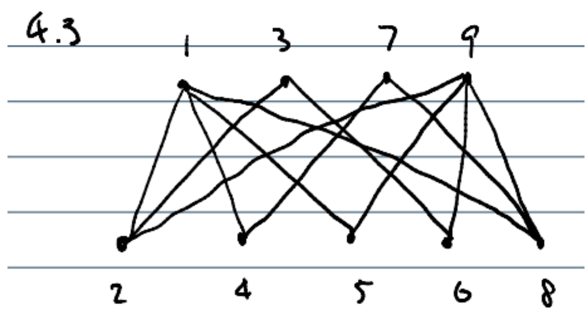
   Answer: No; since vertices 2 and 8 have odd degrees, by theorem 4.4.1 G cannot be Eulerian.

2. Determine the connectivity of G.

   Answer: G is 2-connected as removing 2 edges from vertices 3,4,5,6 or 7 would disconnect the vertex and therefore disconnect the graph. It is not 1-connected as every vertex is in a cycle.

3. Is G bipartite?

   Answer: Yes; as follows:



5. Let $G = (V, E)$ be a graph whose vertices are the 2-element subsets of $\{1, 2, 3, 4, 5\}$. Declare two vertices adjacent if the subsets are disjoint.

   1. Prove that each vertex has degree 3.

      Answer: Take an abitrary vertex $(a, b), a \neq b$ from G, (a, b) is adjacent to any tuple from the set $\{1, 2, 3, 4, 5\} \setminus \{a, b\}$, which is a set of size 3. Then there are $\binom{3}{2} = 3$ choices for 2-element subsets, i.e. (a, b) is adjacent to exactly 3 other vertices. Therefore each vertex has degree 3.

   2. Prove that the shortest cycle in G has length 5.

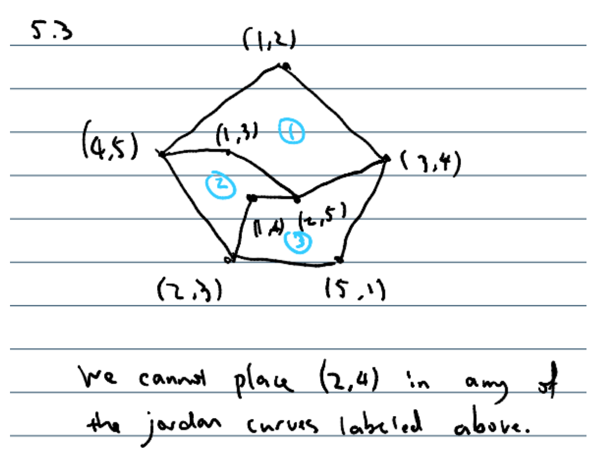
      Answer: Since a cycle needs to have at least length 3, we will prove the statement by showing that cycles in G cannot be length 3 or 4.

      To contruct a cycle of length 3, we need 3 disjoint 2-element subsets of $\{1, 2, 3, 4, 5\}$ since each vertex is connected to the other two in a cycle of length 3. However, it is clearly impossible to construct 3 disjoint 2-element subsets as there are only 5 elements in $\{1, 2, 3, 4, 5\}$.

      Now let's try to construct a cycle of length 4. Let $a, b, c, d, e \in \{1, 2, 3, 4, 5\}$ be distinct. Then let arbitrary (a, b) be the starting vertex and (c, d) be the second vertex. The third vertex then is selected from $\{a, b, e\}$; since we cannot have duplicate vertex (a, b), we can only have (a, e) or (b, e). Then the fourth vertex is selected from $\{b, c, d\}$ or $\{a, c, d\}$; since we cannot select duplicate vertex (c, d), either a or b must be a part of the fourth vertex. But this means that the fourth vertex is not disjoint to (a, b), and therefore such cycle of length 4 does not exist.

   3. Prove that G is not planar.

      Answer:



      Proof by contraction. Since the vertices $(1, 2), (3, 4), (5, 1), (2, 3), (4, 5)$ form a cycle, the arcs $\alpha[(1, 2), (3, 4)]$, $\alpha[(3, 4), (5, 1)]$, $\alpha[(5, 1), (2, 3)]$ and $\alpha[(2, 3), (4, 5)]$ form a Jordan curve k. Now take the vertices pair $((1, 3), (2, 4))$; since they are disjoint subsets, they must be connected. Therefore they must be either both inside k or both outside k or $\alpha[(1, 3), (2, 4)]$ would intersect k. Similarly, the same must apply for the pairs $((1, 4), (2, 5))$ and $((1, 3), (2, 5))$.

Suppose $(1,3)$ lies inside k, then so does $(2,5)$; since $(2,5)$ is now inside k, then so is $(1,4)$. Then k is divided into three Jordan curves by these three vertices and their edges as shown in the diagram.

Now consider the vertex $(2,4)$, which must be inside k as shown previously. However, on the one hand, if we place $(2,4)$ inside either Jordan curves 1 or 2, as labeled in the diagram, the edge $((2,4),(5,1))$ would intersect at least one Jordan curve. On the other hand, if we place $(2,4)$ inside Jordan curve 3, the edge $((2,4),(1,3))$ would intersect at least one Jordan curve.

If $(1,3)$ and $(2,4)$ lie both outside k, we can proceed analogously. Therefore G is not planar by Jordan curve theorem.

6. Let $G = (V, E)$ be a graph and let $e \in E$ be an edge.

   1. Express $|E(G/e)|$ in terms of $E(G)$ and $T_e(G)$, the number of triangles in G containing e as an edge.

      Answer: Since e is not in G/e, we lose it when contracting e. In addition, every triangle that contains e becomes a single edge, losing an additional edge per triangle. Therefore we have $|E(G/e)| = |E(G)| - T_e(G) - 1$.

   2. Let $T(G)$ be the number of triangles in the graph G. How are $T(G)$, $T(G-e)$ and $T_e(G)$ related?

      Answer: Since $G - e$ is formed by removing e from G, $T(G)$ differs from $T(G - e)$ by $T_e(G)$, i.e. $T(G) = T(G - e) + T_e(G)$.

   3. Let

      $$P_G(x) = x^n - a_{n-1}x^{n-1} + a_{n-2}x^{n-2} - \dots$$

      be the chromatic polynomial of G. Recall that $a_{n-1} = |E(G)|$. Prove (by induction on the number of edges) that

      $$a_{n-2} = \binom{|E(G)|}{2} - T(G)$$

      Answer: Proof by induction as follows.

      Base case:

      $|E(G)| = 2$ (no triangle); then $P_G(x) = x(x-1)^2 = x^3 - 2x^2 + x$. So $a_{n-2} = 1 = \binom{2}{2} - 0$.

      $|E(G)| = 3$ (one triangle); then $P_G(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x$. So $a_{n-2} = 2 = \binom{3}{2} - 1$.

      Inductive step:

      Suppose $a_{n-2} = \binom{m-1}{2} - T(G)$ holds for $m - 1$ edges. We want to show that it will also hold for m edges. By the Fundamental Reduction Theorem, for an edge e, we have $P_G(x) = P(G - e, x) - P(G/e, x)$. By inductive hypothesis, the coefficient for the x^{n-2} term in $P(G - e, x)$ is $\binom{m-1}{2} - T(G-e)$. In addition since $P(G/e, x)$ is a degree $n-1$ polynomial (by homework 3 P2), its coefficient for the x^{n-2} term is $-|E(G/e)|$, which is equivalent to $|E(G)| - T_e(G) - 1$ by part (a). So we have $a_{n-2} = \binom{m-1}{2} - T(G-e) + |E(G)| - T_e(G) - 1 = [\binom{m-1}{2} + m - 1] - [T(G-e) - T_e(G)]$. We can tackle the two parts of the above expression separately as follows: the left term simplifies to $\binom{m-1}{2} + m - 1 = \frac{(m-1)!}{2!(m-1-2)!} + \frac{2(m-1)}{2} = \frac{(m-1)(m-2)+2(m-1)}{2} = \frac{(m-1)[(m-2)+2]}{2} = \frac{m(m-1)}{2} = \binom{m}{2}$. The right term simplies to $T(G - e) - T_e(G) = T(G)$ by part (b). Then by substitution we have $a_{n-2} = [\binom{m-1}{2} + m - 1] - [T(G - e) - T_e(G)] = \binom{|E(G)|}{2} - T(G)$.

      Therefore $a_{n-2} = \binom{|E(G)|}{2} - T(G)$ by mathematical induction.