Math 132 Homework 4

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10/28/2020

3.2.34 $\left|\int_\gamma e^z dz\right| \le \frac{\pi}{4}$, where $\gamma(t) = e^{it}$, $\frac{\pi}{2} \le t \le \frac{3\pi}{4}$.

Answer: $L = \ell(\gamma) = r\theta = 1 \cdot (\frac{3\pi}{4} - \frac{\pi}{2}) = \frac{\pi}{4}$; then since $\gamma(t) = e^{it}$, we have $|z| = 1$. So $|f(z)| = |e^z| = 1 = M$. Therefore $\left|\int_\gamma e^z dz\right| \le ML = \frac{\pi}{4}$.

3.2.36 $\left|\int_{C_2(0)} \frac{1}{z-1} dz\right| \le 4\pi$.

Answer: $L = \ell(C_2(0)) = 2\pi r = 2\pi \cdot 2 = 4\pi$; to find M, we have $C_2(0) \implies |z| = 2$, so $M = 2$. Therefore $\left|\int_{C_2(0)} ra\frac{1}{z-1} dz\right| \le ML = 4\pi$.

3.2.39 $\left|\int_{C_1(0)} e^{z^2+1} dz\right| \le 2\pi e^2$.

Answer: $L = \ell(C_1(0)) = 2\pi r = 2\pi \cdot 1 = 2\pi$; Then $C_1(0) \implies |z| = 1$, so $|f(z)| = \left|e^{z^2+1}\right| = e^{\left|z^2+1\right|} = e^2 = M$. Therefore $\left|\int_{C_1(0)} e^{z^2+1} dz\right| \le ML = 2\pi e^2$.

3.3.15 $\int_{[z_1,z_2,z_3]} 3(z-1)^2 dz$, where $z_1 = 1, z_2 = i, z_3 = 1+i$.

Answer: Let $F(z) = (z-1)^3$, then $F'(z) = 3(z-1)^2$. By FTCC, $\int_{[z_1,z_2,z_3]} 3(z-1)^2 dz = F(1+i) - F(1) = i^3 - 0 = -i$.

3.3.19 $\int_{[z_1,z_2,z_3]} ze^z dz$, where $z_1 = \pi, z_2 = -1, z_3 = -1 - i\pi$.

Answer: Let $F(z) = ze^z - e^z$, then $F'(z) = ze^z$. By FTCC, $\int_{[z_1,z_2,z_3]} ze^z dz = F(-1-i\pi) - F(\pi) = 2e^{-1} + i\pi e^{-1} - \pi e^\pi + e^\pi$.

P1 Find the value of $\int_{C_1(0)} z^m dz$ in terms of m, where $m \in \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Answer: $C_1(0) \implies \gamma(t) = e^{it}, t \in [0, 2\pi]$ and $\gamma'(t) = ie^{it}$. Then $\int_{C_1(0)} z^m dz = \int_0^{2\pi} [\gamma(t)]^m \gamma'(t) dt = \int_0^{2\pi} (e^{it})^m \cdot ie^{it} dt = i\int_0^{2\pi} e^{it(m+1)} dt = \frac{i}{i(m+1)}[e^{it(m+1)}]_0^{2\pi} = \frac{e^{2i\pi(m+1)}-1}{m+1}$.

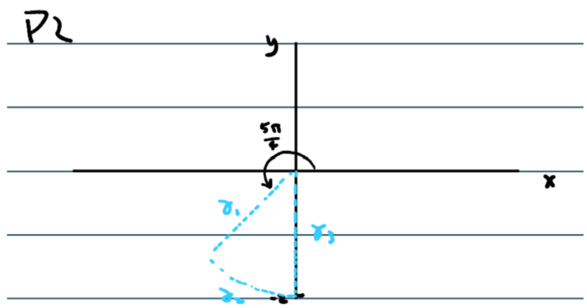
P2 Let γ be the clockwise boundary of the wedge

$$U = \left\{z \in \mathbb{C} \mid -\frac{3\pi}{4} < \text{Arg }(z) < -\frac{\pi}{2}, \text{and } |z| < 2\right\}$$

Plot γ and calculate $\int_\gamma \bar{z} dz$.

Answer: We can break γ into three parts as follows. Let $\gamma_1 = -\frac{\sqrt{2}}{2}t - i\frac{\sqrt{2}}{2}t$ for $t \in [0,1]$, $\gamma_2 = 2e^{it}$ for $t \in (-\frac{3\pi}{4}, -\frac{\pi}{2})$ and $\gamma_3 = -2i + 2it$ for $t \in [0,1]$. We also have $\gamma_1'(t) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$, $\gamma_2'(t) = 2ie^{it}$ and $\gamma_3'(t) = 2i$. Then,

$$\int_\gamma \bar{z} dz = \int_{\gamma_1} \bar{z} dz + \int_{\gamma_2} \bar{z} dz + \int_{\gamma_3} \bar{z} dz$$
$$= \int_0^1 \overline{\gamma_1(t)}\gamma_1'(t)dt + \int_{-\frac{3\pi}{4}}^{-\frac{\pi}{2}} \overline{\gamma_2(t)}\gamma_2'(t)dt + \int_0^1 \overline{\gamma_3(t)}\gamma_3'(t)dt$$
$$= \int_0^1 (-\tfrac{\sqrt{2}}{2}t + i\tfrac{\sqrt{2}}{2}t)(-\tfrac{\sqrt{2}}{2} - i\tfrac{\sqrt{2}}{2})dt + \int_{-\frac{3\pi}{4}}^{-\frac{\pi}{2}} 2e^{-it} \cdot 2ie^{it} dt + \int_0^1 2i(2i - 2it)dt$$
$$= \int_0^1 t dt + 4i \int_{-\frac{3\pi}{4}}^{-\frac{\pi}{2}} dt + 4\int_0^1 t - 1 dt$$
$$= \tfrac{1}{2} + i\pi - 2$$
$$= -\tfrac{3}{2} + i\pi.$$



P3 Evaluate $\int_\gamma (\sin z + e^z) dz$, where γ is the path $[1+i, 5-i, 2i, -4, \pi + i\sqrt{2}, 41 - 41i, 10^{10} + i, \pi]$.

Answer: Let $F = e^z - \cos z$, then $F'(z) = e^z + \sin z$. By FTCC, $\int_\gamma (\sin z + e^z) dz = F(\pi) - F(1+i) = \sin\pi + e^\pi - \sin(1+i) - e^{1-i} = e^\pi - e^{1-i} - \sin(1+i)$.