Math 132 Homework 1

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1.1.5 $(\overline{2-i})^2$

Answer: $(\overline{2-i})^2 = (2+i)^2 = 4 + 4i - 1 = 3 + 4i$

1.1.7 $(x+iy)^2$

Answer: $(x+iy)^2 = x^2 + 2xyi - y^2 = (x^2 - y^2) + i(2xy)$

1.1.8 $i\overline{(2+i)^2}$

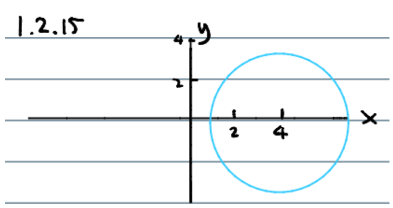
Answer: $i\overline{(2+i)^2} = i\overline{3+4i} = i(3-4i) = 4 + 3i$

1.2.10 $\left|\overline{(2+3i)^8}\right|$

Answer: $\left|\overline{(2+3i)^8}\right| = \left|(2+3i)^8\right| = |2+3i|^8 = \sqrt{13}^8 = 13^4 = 28561$

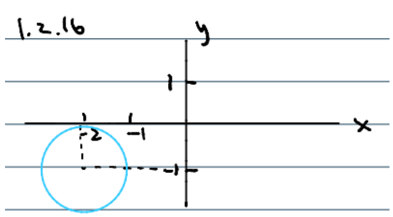
1.2.15 $|z-4| = 3$

Answer: Since $|z| = 3$ is a circle of radius 3 at the origin, $|z-4| = 3$ is a circle of radius 3 centered at $(4, 0)$.



1.2.16 $|z+2+i| = 1$

Answer: Similar to the above, $|z+2+i| = 1$ is a circle of radius 1 centered at $(-2, -1)$.

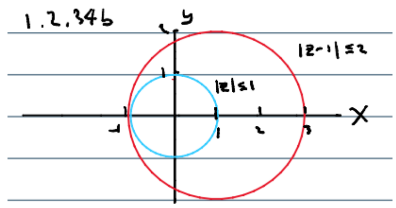


1.2.34 (a) Use the triangle inequality to show that $|z-1| \le 2$ for $|z| \le 1$.

Answer: $|z| \le 1 \implies |z| + |-1| \le |1| + |-1| \implies |z-1| \le |z| + |-1| \le |1| + |-1| = 2 \implies |z-1| \le 2$

(b) Explain your result in (a) geometrically.

Answer: As shown in the figure below, the disk $|z-1| \le 2$ bounds the disk $|z| \le 1$.



(c) Is the upper bound in (a) best possible?

Answer: Take $z=-1$, then we have $|z|=1$ and $|z-1|=2$. Therefore $|z| \le 1$ is the best upper bound possible.

1.3.5 $-3-3i$

Answer: $r=\sqrt{(-3)^2+(-3)^2}=3\sqrt{2}$; $\cos\theta=\frac{x}{r}=\frac{-3}{3\sqrt{2}}=-\frac{\sqrt{2}}{2}, \sin\theta=\frac{y}{r}=\frac{-3}{3\sqrt{2}}=-\frac{\sqrt{2}}{2} \implies \theta=\frac{5\pi}{4} \implies \operatorname{Arg} z=-\frac{3\pi}{4}$. Then $-3-3i=3\sqrt{2}(\cos(-\frac{3\pi}{4})+i\sin(-\frac{3\pi}{4}))$.

1.3.6 $-\frac{\sqrt{3}}{2}+\frac{i}{2}$

Answer: $r=\sqrt{\frac{3}{4}+\frac{1}{4}}=1$; $\cos\theta=\frac{x}{r}=-\frac{\sqrt{3}}{2}, \sin\theta=\frac{y}{r}=\frac{1}{2} \implies \theta=\frac{5\pi}{6}=\operatorname{Arg} z$. Then $-\frac{\sqrt{3}}{2}+\frac{i}{2}=\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}$.

1.3.21 $(1+i)^{30}$

Answer: $(1+i)^{30}=\sqrt{2}^{30}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})^{30}=2^{15}(\cos\frac{30\pi}{4}+i\sin\frac{30\pi}{4})=2^{15}(0+i)=2^{15}i$, so $\operatorname{Re} z=0$ and $\operatorname{Im} z=2^{15}$.

1.5.7 $e^{-1-i\frac{\pi}{6}}$

Answer: $e^{-1-i\frac{\pi}{6}}=e^{-1}\cdot e^{-i\frac{\pi}{6}}=e^{-1}(\cos(\frac{-\pi}{6})+i\sin(\frac{\pi}{6}))=e^{-1}\cos(\frac{-\pi}{6})+ie^{-1}\sin(\frac{\pi}{6})$, so $a=e^{-1}\cos(\frac{-\pi}{6})$ and $b=e^{-1}\sin(\frac{\pi}{6})$.

P1 Let $z=x+iy$ be a complex number. Show that:

(a) $\bar{\bar{z}}=z$

Answer: $\bar{\bar{z}}=\overline{\overline{x+iy}}=\overline{x-iy}=x+iy=z$

(b) $\operatorname{Re}(z)=\frac{z+\bar{z}}{2}$

Answer: $\frac{z+\bar{z}}{2}=\frac{x+iy+\overline{x+iy}}{2}=\frac{x+iy+x-iy}{2}=\frac{2x}{2}=x=\operatorname{Re}(z)$

(c) $\operatorname{Im}(z)=\frac{z-\bar{z}}{2i}$

Answer: $\frac{z-\bar{z}}{2i}=\frac{x+iy-\overline{x+iy}}{2i}=\frac{x+iy-(x-iy)}{2i}=\frac{2iy}{iy}=y=\operatorname{Im}(z)$

P2 Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ be complex numbers. Show that:

(a) $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$

Answer: $\overline{z_1+z_2}=\overline{x_1+iy_1+x_2+iy_2}=\overline{(x_1+x_2)+i(y_1+y_2)}=(x_1+x_2)-i(y_1+y_2)=(x_1-iy_1)+(x_2-iy_2)=\overline{x_1+iy_1}+\overline{x_2+iy_2}=\overline{z_1}+\overline{z_2}$

(b) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

Answer: $\overline{z_1 z_2} = \overline{(x_1 + iy_1)(x_2 + iy_2)} = \overline{x_1x_2 - y_1y_2 + ix_1y_2 + ix_2y_1} = x_1x_2 - y_1y_2 - ix_1y_2 - ix_2y_1 = (x_1 - iy_1)(x_2 - iy_2) = \overline{x_1 + iy_1} \cdot \overline{x_2 + iy_2} = \overline{z_1} \cdot \overline{z_2}$

P3 (a) Show that $|z-4| \leq 6$ if $|z-3i| \leq 1$

Answer: $|z-3i| \leq 1 \implies |z-4| \leq |z-3i| + |-4+3i| \leq 1 + |-4+3i| = 6 \implies |z-4| \leq 6$

(b) Show that $|z-4| \geq 4$ if $|z-3i| \leq 1$

Answer: $|z-4| = |(z-4)+(4-3i)| \geq ||z-3i| - |-4+3i|| = ||z-3i| - 5|$; since $|z-3i| \leq 1$, we have $|z-4| \geq ||z-3i|-5| \geq |1-5| = 4 \implies |z-4| \geq 4$.

(c) Show that $\left|\frac{1}{z-4}\right| \leq \frac{1}{2}$ if $|z-1| \leq 1$

Answer: $|z-4| = |(z-1)-3| \geq ||z-1|-3|$; since $|z-1| \leq 1$, $|z-4| \geq ||z-1|-3| \geq |1-3| = 2 \implies |z-4| \geq 2$. Then $\left|\frac{1}{z-4}\right| = \frac{1}{|z-4|} \leq \frac{1}{2} \implies \left|\frac{1}{z-4}\right| \leq \frac{1}{2}$.