Math 180 Homework 3

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6.2.1 Show that the graph $K_{3,3}$ is not planar, in a manner similar to the proof of nonpolarity of K_5 given in the text.

Answer: By contradiction; let b_1, b_2, b_3 and b_4, b_5, b_6 be the vertices of the two opposing sets of vertices, in addition let the arc connecting the points b_i and b_j be denoted by $\alpha(i, j)$.

Since b_1, b_2, b_4, b_5 forms a cycle, the arcs $\alpha(1, 4), \alpha(4, 2), \alpha(2, 5)$ and $\alpha(5, 1)$ form a Jordan curve k, and hence the points b_3 and b_6 lie either both inside or both outside k, otherwise the arc $\alpha(3, 6)$ would cross k.

Suppose that b_3 lies inside k (which divides k into two Jordan curves) and b_6 lies inside the Jordan curve formed by the arcs $\alpha(1, 4), \alpha(4, 3), \alpha(3, 5), \alpha(1, 4)$ and $\alpha(5, 1)$, in which case $\alpha(6, 2)$ would intersect the Jordan curve. Similarly, supopse that b_6 lies inside the Jordan curve formed by the arcs $\alpha(5, 2), \alpha(2, 4), \alpha(4, 3)$ and $\alpha(3, 5)$, then $\alpha(6, 1)$ would intersect the Jordan curve. If the points b_3 and b_6 lie both outside k, we can proceed analogously. Therefore $K_{3,3}$ is not planar.

6.3.1 Prove that the bound $|E| \leq 2|V| - 4$ for triangle-free planar graphs is the best possible in general. That is, for infinitely many n construct examples of triangle-free planar graphs with n vertices and $2n - 4$ edges.

Answer: By prop 4.3.1 we have $\sum_{v \in V} \deg_G(v) = 2|E|$. In addition, since the graphs are triangle-free (each face has degree ≥ 4), we have $2|E| \geq 4f \implies f \leq \frac{1}{2}|E|$. Then, using Euler's formula we have $|V| - |E| + f = 2 \implies \frac{1}{2}|E| \geq |E| - |V| + 2 \implies -\frac{1}{2}|E| \geq -|V| + 2 \implies |E| \leq 2|V| - 4$.

6.3.3 Prove that a planar graph in which each vertex has degree at least 5 must have at least 12 vertices.

Answer: Since each vertex has degree ≥ 5, by prop 4.3.1 we have $5|V| \leq \sum_{v \in V} \deg_G(v) = 2|E| \implies |E| \geq \frac{5}{2}|V|$. In addition, since each face has degree ≥ 3, we have $\sum_{f \in F} \deg_G(f) = \sum_{v \in V} \deg_G(v) = 2|E| \implies 3f \leq 2|E| \implies f \leq \frac{2}{3}|E| = \frac{5}{3}|V| \implies f \leq \frac{5}{3}|V|$. Then we can substitue the above into Euler's formula as follows: $absV - |E| + f = 2 \implies |V| - \frac{5}{2}|V| + \frac{5}{3}|V| \leq 2 \implies \frac{1}{6}|V| \leq 2 \implies |V| \leq 12$.

6.3.5 Consider a maximal triangle-free planar graph $G = (V, E)$, i.e. a triangle-free planar graph such that any graph of the form $G + e$, where $e \in \binom{V}{2} \backslash E$, contains a triangle or is nonplanar. Prove that each face in any drawing of such a graph is a quadrilateral or a pentagon.

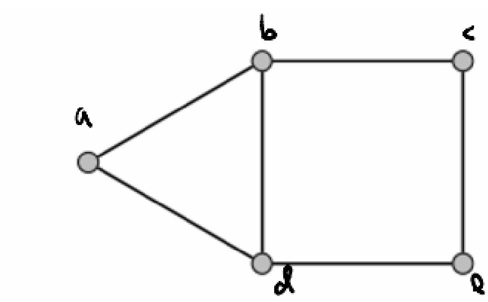
Answer: By contradiction. Suppose that there exists one face f_i that is not a quadrilateral or a pentagon. Then we have the following cases:

- f_i has less than 4 edges; f has to be a triangle in this case as a face requires at least 3 edges. Then G is not triangle-free.

– f_i has more than 5 edges. Then we can add an edge to two vertices ≥ 3 distance apart to contruct another triangle-free graph. Since f_i has at least 6 edges, this is always possible, then G is not maximal.

Therefore each face of G is a either quadrilateral or a pentagon by contradiction.

P1 Calculate the chromatic polynomial of the following graph. What is the chromatic number?



Answer: If we label the vertices as above and let x be the number of available colors, vertex a has x choices. Then vertex b has $x-1$ choices and vertex c also has $x-1$ choices. Similarly, vertex d now has $x-2$ choices and so does vertex e. Therefore the chromatic polynomial is $p(x) = x(x-1)^2(x-2)^2$ with chromatic number 3 (smallest number to achieve nonzero $p(x)$).

P2 Prove the following statement: For any graph G, the degree of its chromatic polynomial is equal to the number of vertices of G. (Hint: Use induction on the number of edges).

Answer: Take a graph G, let $n = |V|$ and $m = |E|$. We will prove the statement by induction as follows:

Base case: $m = 1$; for the pair of connected vertices, we have $x(x-1)$ number of ways of coloring them. Each of the rest of the $n-2$ vertices can be colored x ways, so the chromatic polynomial is $P(G,x) = x^{n-1}(x-1)$ which has degree n.

Inductive step: Assume the statement holds for m edges, we want to show that it will also hold for $m+1$ edges. By the Fundamental Reduction Theorem, for any pair of vertices u, v, and $P(G,x) = P(G-uv,x) - P(G/uv,x)$. By induction hypothesis, $P(G-uv,x)$ is a degree n polynomial whereas $P(G/uv,x)$ is a degree $n-1$ polynomial. Then $P(G,x)$ has to be degree n.

Therefore for any graph G, the degree of its chromatic polynomial is equal to the number of vertices of G by mathematical induction.

P3 What is the chromatic number of the graph obtained from K_n by removing one edge? Explain.

Answer: K_n requires a different color for each vertex, i.e. K_n has a chromatic number of n. Then, removing one edge allows the vertices connected by the removed edge to have the same color, which means the new graph has a chromatic number of $n-1$.