Math 132 Homework 9

Jiaping Zeng

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5.1.1 $\dfrac{1+z}{z}$

Answer: We have $\dfrac{1+z}{z} = \dfrac{1}{z} + 1$, so by definition of residue, $\text{Res}\left(\dfrac{1+z}{z}, 0\right) = a_{-1} = 1$.

5.1.3 $\dfrac{1+e^z}{z^2} + \dfrac{2}{z}$

Answer: We have $\dfrac{1+e^z}{z^2} + \dfrac{2}{z} = \dfrac{1+2z+e^z}{z^2} = \dfrac{1+2z+\sum_{n=0}^{\infty}\frac{z^n}{n!}}{z^2} = \dfrac{1}{z^2}(2+3z+\dfrac{z^2}{2}+\dfrac{z^3}{6}+\dots) = \dfrac{2}{z^2}+\dfrac{3}{z}+\dfrac{1}{2}+\dfrac{z}{6}+\dots$, so $\text{Res}\left(\dfrac{1+e^z}{z^2}+\dfrac{2}{z}, 0\right) = a_{-1} = 3$.

5.1.4 $\dfrac{\sin(z^2)}{z^2(z^2+1)}$

Answer: By Proposition ★ from the previous homework, $f(z) = \dfrac{\sin(z^2)}{z^2(z^2+1)} = \dfrac{\sin(z^2)}{z^2(z+i)(z-i)}$ has removable singularity at $z_0 = 0$ and simple poles at $z_0 = \pm i$. For $z_0 = 0$, we have $\text{Res}(f, 0) = 0$. For $z_0 = i$, we have $\text{Res}(f, i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\dfrac{\sin(z^2)}{z^2(z+i)} = \dfrac{\sin 1}{2i}$. Similarly, for $z_0 = -i$, we have $\text{Res}(f, -i) = \lim_{z\to -i}(z+i)f(z) = \lim_{z\to -i}\dfrac{\sin(z^2)}{z^2(z-i)} = \dfrac{\sin 1}{-2i}$.

5.1.13 $\displaystyle\int_{C_1(0)} \frac{z^2+3z-1}{z(z^2-3)}dz$

Answer: Let $f(z) = z^2+3z-1$ and $g(z) = z(z^2-3)$, then since g has a simple zero at $z_0 = 0$, we have $\text{Res}\left(\dfrac{f}{g}, 0\right) = \dfrac{f(z_0)}{g'(z_0)} = \dfrac{1}{3}$. Note that $z_0 = 0$ is the only zero inside $C_1(0)$. Then by Residue Theorem, $\displaystyle\int_{C_1(0)} \frac{z^2+3z-1}{z(z^2-3)}dz = 2\pi i\,\text{Res}\left(\frac{f}{g}, 0\right) = \frac{2\pi i}{3}$.

5.1.18 $\displaystyle\int_{C_3(0)} \frac{z^2+1}{(z-1)^2}dz$

Answer: Let $f(z) = \dfrac{z^2+1}{(z-1)^2}$, then by Proposition ★, $f(z)$ has a pole of order 2 at $z_0 = 1$, so $\text{Res}(f, 1) = \lim_{z\to 1}\dfrac{d}{dz}\left[(z-1)^2 \cdot \dfrac{z^2+1}{(z-1)^2}\right] = \lim_{z\to 1} 2z = 2$. By Residue Theorem, $\displaystyle\int_{C_3(0)} \frac{z^2+1}{(z-1)^2}dz = 2\pi i\,\text{Res}(f, 1) = 4\pi i$.

5.1.21 $\displaystyle\int_{C_1(0)} \frac{e^{z^2}}{z^6}dz$

Answer: Let $f(z) = \dfrac{e^{z^2}}{z^6}$, then by Proposition ★, $f(z)$ has a pole of order 6 at $z_0 = 0$, so $\text{Res}(f, 0) =$

$\lim_{z\to 0}\frac{1}{5!}\frac{d^5}{dz^5}\left[z^6\cdot\frac{e^{z^2}}{z^6}\right] = \lim_{z\to 0}\frac{(32z^5+160z^3+120z)e^{z^2}}{5!} = 0$. By Residue Theorem, $\int_{C_1(0)}\frac{e^{z^2}}{z^6}dz = 2\pi i\,\mathrm{Res}(f,0)=0$.

5.1.23 $\int_{C_1(0)} z^4(e^{\frac{1}{z}}+z^2)$

Answer: We can first split up the integral as follows: $\int_{C_1(0)} z^4(e^{\frac{1}{z}}+z^2) = \int_{C_1(0)} z^4e^{\frac{1}{z}} + \int_{C_1(0)} z^6$. Since $z^4e^{\frac{1}{z}} = z^4(\ldots+\frac{1}{6!z^6}+\frac{1}{5!z^5}+\frac{1}{4!z^4}+\ldots) = \ldots+\frac{1}{6!z^2}++\frac{1}{5!z}++\frac{1}{4!}+\ldots$, we have $\mathrm{Res}\Big(z^4e^{\frac{1}{z}},0\Big) = a_{-1} = \frac{1}{5!}=\frac{1}{120}$. Then by Residue Theorem, $\int_{C_1(0)} z^4e^{\frac{1}{z}} = 2\pi i\,\mathrm{Res}\Big(z^4e^{\frac{1}{z}},0\Big) = \frac{\pi i}{60}$. Since $\int_{C_1(0)} z^6=0$ by Cauchy's Integral Theorem, $\int_{C_1(0)} z^4(e^{\frac{1}{z}}+z^2) = \int_{C_1(0)} z^4e^{\frac{1}{z}} = \frac{\pi i}{60}$.

P1 Use residue theory to show that $\int_{-\infty}^{\infty}\frac{1}{(x^2+1)^3}dx = \frac{3\pi}{8}$.

Answer: For $R>0$, let σ_R be the part of $C_R(0)$ in the upper half plane and let $\gamma_R = [[-R,R],\sigma_R]$. Let $f(z)=\frac{1}{(z^2+1)^3}$,then we have $\int_{\gamma_R} f(z)dz = \int_{[-R,R]} f(z)dz + \int_{\sigma_R} f(z)dz$.

We want to first show that $\int_{\sigma_R} f(z)dz \to 0$. Let $L=\mathrm{length}(\sigma_R)=\pi R$. For z on σ_R, $|f(z)| = \frac{1}{|z^2+1|^3} \le \frac{1}{(R^2-1)^3}$ for R large enough. So $\left|\int_{\sigma_R} f(z)dz\right| \le ML = \frac{\pi R}{(R^2-1)^3}$, which $\to 0$ as $R\to\infty$. Therefore $\lim_{R\to\infty}\int_{\sigma_R} f(z)dz = 0$.

We will now find $\int_{\gamma_R} f(z)dz$ using residues. We have $f(z)=\frac{1}{(z^2+1)^3} = \frac{1}{(z+i)^3(z-i)^3}$; since $-i$ is not in γ_R, we only need to examine $z_0=i$, which is a pole of order 3 by Proposition ★. Then $\mathrm{Res}(f,i) = \lim_{z\to i}\frac{1}{2}\frac{d^2}{dz^2}[(z-i)^3f(z)] = \lim_{z\to i}\frac{1}{2}\frac{d^2}{dz^2}\frac{1}{(z+i)^3} = \lim_{z\to i}\frac{6}{(z+i)^5} = \frac{-3i}{16}$. By Residue Theorem, $\int_{\gamma_R} f(z)dz = 2\pi i\,\mathrm{Res}(f,i) = \frac{3\pi}{8}$.

Then by substitution we have $\int_{\gamma_R} f(z)dz = \int_{[-R,R]} f(z)dz + \int_{\sigma_R} f(z)dz \implies \frac{3\pi}{8} = \int_{[-R,R]} f(z)dz + 0 \implies \int_{[-R,R]} f(z)dz = \frac{3\pi}{8} \implies \int_{-\infty}^{\infty}\frac{1}{(x^2+1)^3}dx = \frac{3\pi}{8}$.

P2 Use residue theory to show that $\int_{-\infty}^{\infty}\frac{\cos(3x)}{(x^2+4)^2}dx = \frac{7\pi}{16e^6}$.

Answer: For $R>0$, let σ_R be the part of $C_R(0)$ in the upper half plane and let $\gamma_R = [[-R,R],\sigma_R]$. Let $f(z)=\frac{e^{3iz}}{(z^2+4)^2}$,then we have $\int_{\gamma_R} f(z)dz = \int_{[-R,R]} f(z)dz + \int_{\sigma_R} f(z)dz$.

We want to first show that $\int_{\sigma_R} f(z)dz \to 0$. Let $L=\mathrm{length}(\sigma_R)=\pi R$. For z on σ_R, $|f(z)| = \frac{e^{\mathrm{Re}(3iz)}}{|z^2+4|^2} \le \frac{1}{(R^2-4)^2}$ for R large enough. So $\left|\int_{\sigma_R} f(z)dz\right| \le ML = \frac{\pi R}{(R^2-4)^2}$, which $\to 0$ as $R\to\infty$. Therefore $\lim_{R\to\infty}\int_{\sigma_R} f(z)dz = 0$.

We will now find $\int_{\gamma_R} f(z)dz$ using residues. We have $f(z)=\frac{e^{3iz}}{(z^2+4)^2} = \frac{e^{3iz}}{(z+2i)^2(z-2i)^2}$; since $-2i$ is not in γ_R, we only need to examine $z_0=2i$, which is a pole of order 2 by Proposition ★.

Then $\operatorname{Res}(f, 2i) = \lim_{z\to 2i} \frac{d}{dz}[(z-2i)^2 f(z)] = \lim_{z\to 2i} \frac{d}{dz} \frac{e^{3iz}}{(z+2i)^2} = \lim_{z\to 2i} \frac{3i(z+2i)^2 e^{3iz} - 2(z+2i)e^{3iz}}{(z+2i)^4} = \frac{-48ie^{-6} - 8ie^{-6}}{256} = \frac{-7ie^{-6}}{32}$. By Residue Theorem, $\int_{\gamma_R} f(z)dz = 2\pi i \operatorname{Res}(f, 2i) = 2\pi i \cdot \frac{-7ie^{-6}}{32} = \frac{7\pi}{16e^6}$. Then by substitution we have $\int_{\gamma_R} f(z)dz = \int_{[-R,R]} f(z)dz + \int_{\sigma_R} f(z)dz \implies \frac{7\pi}{16e^6} = \int_{[-R,R]} f(z)dz + 0 \implies \int_{[-R,R]} f(z)dz = \frac{7\pi}{16e^6} \implies \int_{-\infty}^{\infty} \frac{e^{3ix}}{(x^2+4)^2}dx = \frac{7\pi}{16e^6}$, and we also have $\int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2+4)^2}dx = \operatorname{Re}\left(\int_{-\infty}^{\infty} \frac{e^{3ix}}{(x^2+4)^2}dx\right) = \operatorname{Re}\left(\frac{7\pi}{16e^6}\right) = \frac{7\pi}{16e^6}$.

P3 Find the residue at each isolated singularity of the function $f(z) = \frac{e^z}{\cos(2z)}$.

Answer: $\cos(2z)$ has simple zeroes at $z_0 = \frac{(2k-1)\pi}{4}, k \in \mathbb{Z}$; since e^z and $\cos(2z)$ are both analytic at these z_0, we have $\operatorname{Res}\left(\frac{f}{g}, z_0\right) = \frac{f(z_0)}{g'(z_0)} \implies \operatorname{Res}\left(\frac{e^z}{\cos(2z)}, z_0\right) = \frac{e^{z_0}}{-2\sin(2z_0)}$ for $z_0 = \frac{(2k-1)\pi}{4}, k \in \mathbb{Z}$.

P4 Suppose $f(z)$ is analytic on and inside a counterclockwise simple closed curve γ with no zeroes on γ. Given that γ is the pictured curve, how many zeroes (counting multiplicity) does $f(z)$ have inside γ?

Answer: There is not enough information; by counter example: let $f_1(z) = 1$, $f_2(z) = z$ and $f_3(z) = z^2$, then they have 0, 1 and 2 zeroes inside γ respectively.

P5 Show that there is no analytic function $f : B_2(0) \to \mathbb{C}$ which satisfies

$$f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n^2}$$

for $n = 1, 2, 3, \ldots$.

Answer: By contradiction. Suppose such analytic function f exists. Let $z_{n_k} = \frac{1}{k}$ for odd k and $z'_{n_k} = \frac{1}{k}$ for even k. Then $f(z_{n_k}) = -\frac{1}{k^2} = -(z_{n_k})^2$. Since f and $-z^2$ are both analytic in $B_2(0)$, we have $f = -z^2$ in $B_2(0)$ by Theorem 4.5.5. Similarly, we have $f(z'_{n_k}) = \frac{1}{k^2} = (z_{n_k})^2$, so $f = z^2$ in $B_2(0)$. Since f cannot be both z^2 and $-z^2$, such f does not exist by contradiction.