I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.

Signature:

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Q2 (a) Any graph on 3 or more vertices with no triangle is bipartite.

Answer: False; take any cycle C_n with odd n, which is triangle-free yet not 2-colorable and therefore not bipartite.

(b) If $(d_1, \ldots, d_n)$ is the score of a graph with at least 1 edge, then $(d_1, \ldots, d_n, 2)$ is a valid graph score.

Answer: True; we can remove an edge and connect a new vertex to the vertices connected by the removed edge.

(c) Any connected, planar graph is 2-connected.

Answer: False; take any path P_n which is connected but not 2-connected.

(d) If T is a spanning tree of K_n, then the complement T^c is not a spanning tree of K_n unless $n = 1$ or $n = 4$.

Answer: True; suppose T and T^c are both spanning trees of K_n, then they must each have $n-1$ edges. So K_n must has $2n-2$ edges by adding up the edges from T and T_c. Since K_n have $\frac{n(n+1)}{2}$ edge by double counting, we can see that we only have $2n - 2 = \frac{n(n-1)}{2}$ when $n = 1$ or $n = 4$.

(e) If G has more than 5 vertices and is triangle free, then G^c cannot be triangle-free.

Answer: By Ramsey's theorem we have $\omega(G) \geq 3$ or $\alpha(G) \geq 3$ for $|V(G)| \geq 6$. Since G is triangle-free we must have $\alpha(G) \geq 3$, but those three independent vertices are connected in G^c which forms a triangle.

(f) Every bipartite graph G has chromatic number $\chi(G) = 2$.

Answer: False; take any empty bipartite graph which has $\chi(G) = 1$.

(g) If G and H are distinct Eulerian graphs on the same vertex set, then their symmetric difference (of their edge set) is Eulerian.

Answer: False; fix one vertex, take two Eulerian graphs that overlap on edges connected to that vertex. Then the symmetric difference is disconnected and therefore cannot be Eulerian.

(h) Distinct spanning trees of a graph G give distinct bases for the cycle space of G.

Answer: False; take K_3 which has basis $\{C_3\}$, but there are three distinct possible spanning trees by rotation.

Q3 (a) Prove that $r(k,2) = k$ for all $k \geq 2$.

Answer: Since we have $r(k,\ell) = 1 + \max\{|V(G)| : \omega(G) < k \text{ and } \alpha(G) < \ell\}$, we can start by finding $|V(G)|$ that satisfies both $\omega(G) < k$ and $\alpha(G) < 2$. Since $\alpha(G) < 2$, G has to be a complete graph, or we else can take two vertices that are not connected to have $\alpha(G) \geq 2$. Then to have $\omega(G) < k$ with G begin a complete graph, we can only have up to $k-1$ vertices. Therefore $\max\{|V(G)| : \omega(G) < k \text{ and } \alpha(G) < \ell\} = k-1$ and $r(k,2) = 1+\max\{|V(G)| : \omega(G) < k \text{ and } \alpha(G) < 2\} = k$.

(b) Prove that $r(k,\ell) \leq r(k-1,\ell) + r(k,\ell-1)$ for all $k,\ell \geq 2$.

Answer: Let $G = (V,E)$ be an arbitrary graph with $|V| = r(k-1,\ell) + r(k,\ell-1)$, we want to show that it satisfies either $\omega(G) \geq k$ or $\alpha(G) \geq \ell$. Choose $u \in V$ to be an arbitrary vertex, then we can devide the rest of the vertices into two sets A and B depending on whether or not they are connected to u, respectively.

By construction of A and B we have $|V|-1 = |A|+|B| \implies r(k-1,\ell)+r(k,\ell-1)-1 = |A|+|B|$; by the pigeonhole priciple we must have $|A| \geq r(k-1,\ell)$ or $|B| \geq r(k,\ell-1)$. Note that if A contains ℓ independent vertices or if B contains a complete subgraph of k vertices, G automatically satisfies either $\omega(G) \geq k$ or $\alpha(G) \geq \ell$, so we will examine the two following cases where the previous statement is not true.

If $|A| \geq r(k-1,\ell)$, then A is contains a complete subgraph with $k-1$ vertices. Since all those vertice are also connected to u, G has a complete subgraph with k vertices. Otherwise, if $|B| \geq r(k,\ell-1)$, then B contains $\ell-1$ independent vertices. Since all those vertices are not connected to u, G has ℓ independent vertices.

Therefore G satisfies either $\omega(G) \geq k$ or $\alpha(G) \geq \ell$. By definition of Ramsey's number, $r(k,l)$ is the minimum number of vertices of such a graph, so by definition of minimum we have $r(k,l) \leq |V(G)| = r(k-1,\ell) + r(k,\ell-1)$.

(c) Use parts (a) and (b) above to obtain an upper bound for $r(5,5)$.

Answer: By part (a) we have $r(k,2) = k$ and by symmetry we also have $r(2,\ell) = \ell$. Then using part (b), We have

$r(5,5)$

$\leq r(4,5) + r(5,4)$

$\leq (r(3,5) + r(4,4)) + (r(4,4) + r(5,3))$

$= r(3,5) + 2r(4,4) + r(5,3)$

$\leq r(2,5) + r(3,4) + 2(r(3,4) + r(4,3)) + r(4,3) + r(5,2)$

$= r(2,5) + 3r(3,4) + 3r(4,3) + r(5,2)$

$\leq r(2,5) + 3(r(2,4) + r(3,3)) + 3(r(3,3) + r(4,2)) + r(5,2)$

$= r(2,5) + 3r(2,4) + 6r(3,3) + 3r(4,2) + r(5,2)$

$\leq r(2,5) + 3r(2,4) + 6r(2,3) + 6r(3,2) + 3r(4,2) + r(5,2)$

$= 5 + 3 \cdot 4 + 6 \cdot 3 + 6 \cdot 3 + 3 \cdot 4 + 5$

$= 70.$

Q4 (a) Show that for $n = 3$ and any $p \in [0, 1]$,

$$\sum_{G \in \mathcal{G}_{3,p}} P(G) = 1$$

Answer: For $n = 3$, we have $2^{\binom{3}{2}} = 8$ and $0 \leq |E(G)| \leq 3$. We can examine the graphs based on $|E(G)|$ as follows:

- $|E(G)| = 0$: $P(G) = (1-p)^3$, we have $\binom{3}{0} = 1$ such graph.
- $|E(G)| = 1$: $P(G) = p(1-p)^2$, we have $\binom{3}{1} = 3$ such graphs.
- $|E(G)| = 2$: $P(G) = p^2(1-p)$, we have $\binom{3}{2} = 3$ such graphs.
- $|E(G)| = 3$: $P(G) = p^3$, we have $\binom{3}{3} = 1$ such graph.

Therefore we have $\sum_{G \in \mathcal{G}_{3,p}} P(G) = (1-p)^3 + 3p(1-p)^2 + 3p^2(1-p) + p^3$, which by cube of sum gives us $(p + (1-p))^3 = 1$.

(b) Prove that the events "the graph G is connected" and "the graph G is bipartite" are not independent in $\mathcal{G}_{3,p}$ unless $p = 0$ or $p = 1$.

Answer: For $p = 0$, we have $P(G) = 1$ for $|E(G)| = 0$ and $P(G) = 0$ otherwise, i.e. G has to be three independent vertices. Similarly, for $p = 1$, we have $P(G) = 1$ for $|E(G)| = 3$ and $P(G) = 0$ otherwise, i.e. G has to be connected. Therefore the events are independent in $\mathcal{G}_{3,p}$ for $p = 0$ and $p = 1$.

For other values of p, note that for $n = 3$, G is connected for $|E(G)| \geq 2$ and is bipartite for $|E(G)| \leq 2$. Since these two conditions overlap and $P(G) \neq 0$ for $0 < p < 1$, i.e. it is possible to have different graphs, we can conclude that the two events are not independent.

(c) If $p = 0.1$, what is the probability that a random graph on the vertex set $\{1, 2, 3\}$, will not contain the edge $\{1, 2\}$?

Answer: Since the probability of G containing $\{1, 2\}$ is 0.1, the probability of G not containing $\{1, 2\}$ is $1 - 0.1 = 0.9$.

(d) Let f be the random variable which assigns to a graph its number of edges. If $p = 0.1$, what is the expected number of edges of a random graph on the vertex set $\{1, 2, 3\}$?

Answer: Since we can have up to 3 edges and each edge has $p = 0.1$, the expected number of edges is $3 \cdot 0.1 = 0.3$.

Q5 (a) Interpret this setup as a block design $t-(v,k,\lambda)$. What are the values of t, v, k, and λ?

Answer: We have $t=2$, $v=16$, $k=4$ and $\lambda=2$.

(b) How many weeks are in the course?

Answer: By integrality conditions, we have $\lambda \dfrac{v(v-1)}{k(k-1)} = 2 \cdot \dfrac{16 \cdot 15}{4 \cdot 3} = 40$ blocks, so there are 40 weeks in the course.

(c) How many presentations does each student give?

Answer: Again by integrality conditions, we have $\lambda \dfrac{(v-1)}{k-1} = 2 \cdot \dfrac{15}{3} = 10$ repetitions, so each student gives 10 presentations.

Q6 (a) What is the dimension of the cycle space $\mathcal{E}$ of K_5?

Answer: By Theorem 13.4.3, we have $\dim(\mathcal{E}) = |E| - |V| + k = 10 - 5 + 1 = 6$.

(b) Give a basis for $\mathcal{E}$ with respect to the spanning tree whose edge set is

$$E' = \{\{1,2\},\{1,3\},\{3,5\},\{4,5\}\}$$

Answer: Let $B = \{b_1, b_2, b_3, b_4, b_5, b_6\}$ be a basis for $\mathcal{E}$ with

$b_1 = \{\{1,2\},\{1,3\},\{2,3\}\}$

$b_2 = \{\{1,3\},\{1,5\},\{3,5\}\}$

$b_3 = \{\{3,4\},\{3,5\},\{4,5\}\}$

$b_4 = \{\{1,2\},\{1,3\},\{2,5\},\{3,5\}\}$

$b_5 = \{\{1,3\},\{1,4\},\{3,5\},\{4,5\}\}$

$b_6 = \{\{1,2\},\{1,3\},\{2,4\},\{3,5\},\{4,5\}\}$.

(c) How many even sets are in K_5?

Answer: By Corollary 13.4.4, K_5 has $2^{|E|-|V|+k} = 2^6 = 64$ even sets.

(d) How many cycles are in K_5? Give an example of a nonempty even set of K_5 which is not a cycle and is not all of K_5.

Answer: By formula 13.2, we have $|\mathcal{K}_{K_n}| = \sum_{k=3}^{n} \binom{n}{k} \cdot \frac{(k-1)!}{2} = \sum_{k=3}^{5} \binom{5}{k} \cdot \frac{(k-1)!}{2} = \binom{5}{3} \cdot \frac{2!}{2} + \binom{5}{4} \cdot \frac{3!}{2} + \binom{5}{5} \cdot \frac{4!}{2} = 10 + 15 + 12 = 37$.

An example of such an even set is $\{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,5\},\{3,4\}\}$. The vertices are all even as vertex 1 is degree 4 and the others are degree 2; in addition, the set is not a cycle since a cycle is degree 2 everywhere.