1. (a) V is the space of polynomials of degree up to 3 such that the coefficients of its terms combine to 0; i.e. an arbitrary vector in V would look like $p(x) = ax + bx^2 + cx^3 - (a + b + c) = a(x-1) + b(x^2-1) + c(x^3-1)$. Then, $\beta = \{x-1, x^2-1, x^3-1\}$ is a basis of V.

   (b) $T(x-1) = x - 1$
   $T(x^2-1) = x^2 + 2x - 3 = 2(x-1) + (x^2-1)$
   $T(x^3-1) = x^3 + 6x^2 - 6x - 1 = -6(x-1) + 6(x^2-1) + (x^3-1)$

$$\implies [T]_\beta = \begin{pmatrix} 1 & 2 & -6 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\implies P_T(\lambda) = \det(T - \lambda Id) = (1-\lambda)^3$$

   (c) T has a single eigenvalue $\lambda = 1$ with algebraic multiplicity 3.

   (d) $T(v) = 1v$

$$\implies \begin{pmatrix} 1 & 2 & -6 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} a + 2b - 6c \\ b + 6c \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\implies b = c = 0$$

   Then $x - 1$ is an eigenvector for $\lambda = 1$.

   (e) T is not diagonalisable as its algebraic multiplicity and geometric multiplicity (nullity$(T - Id) = 1$) does not match for $\lambda = 1$.

2. (a) Let $C = \{P, Q, R\}$. We can construct P, Q, R using Gram-Schmidt as follows:

$$P = H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Q = K - \frac{\langle K, P \rangle}{||P||^2} P = K - \frac{2}{2} P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$R = L - \frac{\langle L, P \rangle}{||P||^2} P - \frac{\langle L, Q \rangle}{||Q||^2} Q = L - \frac{2}{2} P - \frac{0}{1} Q = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then $C = \{P, Q, R\}$ is an orthogonal basis for V.

(b) $T(P) = EP - PE = -2Q$
$T(Q) = EQ - QE = 0$
$T(R) = ER - RE = P$

$$\implies [T]_C = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\implies [T^*]_C = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\implies T^*(E) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$$

Then $T^*(E) = -2P = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$

(c)

$$[T]_C [T^*]_C = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[T^*]_C [T]_C = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since $T \circ T^* \neq T^* \circ T$, T is not normal.

3. (a) Since $X^2 = Id$, we have $X(X(w)) = w$ for all $w \in W$, i.e. X is invertible and $X = X^{-1}$. To show that W' is also an X-subspace, it suffices to show that $\langle X(u), w\rangle + \langle X(X(u)), X(w)\rangle = 0$ for $u \in W'$. Since X is invertible, we can choose $X(w) \in W$ to represent all vectors in W. Then verifying the following equivalently shows that W' is an X-subspace: $\langle X(u), X(w)\rangle + \langle X(X(u)), X(X(w))\rangle = 0$. We can simplify the previous expression as follows:

$\langle X(u), X(w)\rangle + \langle X(X(u)), X(X(w))\rangle$
$= \langle X(u), X(w)\rangle + \langle X^2(u), X^2(w)\rangle$
$= \langle X(u), X(w)\rangle + \langle u, w\rangle$
$= \langle u, w\rangle + \langle X(u), X(w)\rangle$
$= 0$

Therefore $X(u) \in W'$ and W' is an X-subspace.

(b) Let $u, v, w \in V$ and $a \in \mathbb{F}$. Then we can verify the axioms of inner product as follows:

(i) $\langle u, v\rangle_X$
$= \langle u, v\rangle + \langle X(u), X(v)\rangle$
$= \overline{\langle v, u\rangle} + \overline{\langle X(u), X(v)\rangle}$
$= \overline{\langle v, u\rangle}_X$

(ii) $\langle au, v\rangle_X$
$= \langle au, v\rangle + \langle X(au), X(v)\rangle$
$= a\langle u, v\rangle + a\langle X(u), X(v)\rangle$
$= a(\langle u, v\rangle + \langle X(u), X(v)\rangle)$
$= a\langle u, v\rangle_X$

(iii) $\langle u + v, w\rangle_X$
$= \langle u + v, w\rangle + \langle X(u + v), X(w)\rangle$
$= (\langle u, w\rangle + \langle v, w\rangle) + (\langle X(u), X(w)\rangle + \langle X(v), X(w)\rangle)$
$= (\langle u, w\rangle + \langle X(u), X(w)\rangle) + (\langle v, w\rangle + \langle X(v), X(w)\rangle)$
$= \langle u, w\rangle_X + \langle v, w\rangle_X$

(iv) $\langle v, v\rangle_X = \langle v, v\rangle + \langle X(v), X(v)\rangle$ by definiton. If $v \neq 0$, then $\langle v, v\rangle > 0$. In addition, $\langle X(v), X(v)\rangle \geq 0$ even if $X(v) = 0$. Then, $\langle v, v\rangle_X > 0$ for nonzero $v \in V$.

(c) By definition of W', $\langle u, w\rangle + \langle X(u), X(w)\rangle = 0$ for $u \in W'$ and $w \in W$, i.e. $\langle u, w\rangle_X = 0$. Then $W' = W^\perp$. Since $W \oplus W^\perp = V$, we also have $W \oplus W' = V$.

(d) $\langle X(u), w\rangle_X = \langle u, X^*(w)\rangle_X$
$\implies \langle X(u), w\rangle + \langle X^2(u), X(w)\rangle = \langle u, X^*(w)\rangle + \langle X(u), X(X^*(w))\rangle$
$\implies \langle X(u), w\rangle + \langle u, X(w)\rangle = \langle u, X^*(w)\rangle + \langle X(u), (X \circ X)^*(w))\rangle$
$\implies \langle X(u), w\rangle + \langle u, X(w)\rangle = \langle u, X^*(w)\rangle + \langle X(u), w\rangle$
$\implies \langle u, X(w)\rangle = \langle u, X^*(w)\rangle$
$\implies X(w) = X^*(w)$

Therefore $X^* = X$, i.e. X is self-adjoint as shown above.

4. (a) To be proved: T is an isomorphism $\Leftrightarrow T(B)$ is a basis for W:

   $\Rightarrow$: Let $B = \{u_0, u_1, \ldots\}, u_i \in V$. Since B is linearly independent, $a_0u_0 + a_1u_0 + \ldots = 0_V, a_i \in \mathbb{F}$ is only true when $a_i = 0$. Applying T to both sides, we have $T(a_0u_0 + a_1u_0 + \ldots) = T(0) \implies a_0T(u_0) + a_1T(u_1) + \ldots = 0_W$. Since $a_i = 0$, $\{T(u_0), T(u_1), \ldots\}$ is a linearly independent set in W. Since T is surjective, $\mathrm{im}(T) = \mathrm{span}(T(B)) = W$. Then $T(B)$ is both linearly independent and spanning in W and is therefore a basis.

   $\Leftarrow$: Let $B = \{u_0, u_1, \ldots\}, u_i \in V$ and $T(B) = \{v_1, v_2, \ldots\}, v_i \in W$ such that $T(u_i) = v_i$. Take an arbitrary $p \in V$, p can be written as $p = a_0u_0 + a_1u_1 + \ldots, a_i \in \mathbb{F}$ since B is a basis. Then $T(p) = a_0T(u_0) + a_1T(u_1) + \ldots = a_0v_0 + a_1v_1 + \ldots$ by linearity. Since $T(B)$ is a basis for W, $\mathrm{span}(T(B)) = \mathrm{im}(T) = W$. Therefore T is surjective. Additionally, $0_W = a_0v_0 + a_1v_1 + \ldots$ only if $a_i = 0$. Then $T(p) = 0_W \implies T(p) = a_0v_0 + a_1v_1 + \ldots, a_i = 0$. By definition of $T(B)$ we can then replace v_i with $T(u_i)$ as follows: $T(p) = a_0T(u_0) + a_1T(u_1) + \ldots = T(a_0u_0 + a_1u_1 + \ldots), a_i = 0$. Then $p = a_0u_0 + a_1u_1, a_i = 0$, i.e. $p = 0_V$. Therefore $\ker(T) = \{0\}$ and T is injective. Since T is both injective and surjective, it is an isomorphism.

   (b) Let $V = \{(a_0, a_1, \ldots) \mid a_i \in \mathbb{R}\}$, i.e. the vector space of infinite sequences. Define $R : V \to V$ such that $R(a_0, a_1, \ldots) = (0, a_0, a_1, \ldots)$. Then R is injective as $\ker(R) = \{0\}$. However, R is not surjective as sequences with the form $(k, a_0, a_1, \ldots), k \in \mathbb{F}$ is not in $\mathrm{im}(R)$. Therefore R is not an isomorphism.

   (c) Again let $V = \{(a_0, a_1, \ldots) \mid a_i \in \mathbb{R}\}$. Define $S : V \to V$ such that $S(a_0, a_1, \ldots) = (a_1, a_2, \ldots)$. Then S is surjective as $(a_1, a_2, \ldots)$ spans V. However, $\ker(S)$ contains sequences with form $(a_0, 0, 0, \ldots)$. Therefore S is not injective and not an isomorphism.

5. (a) Let λ be an eigenvalue of T, i.e. $T(v) = \lambda v$ for some $v \in V$. Then, $T^2(v) = T(T(v)) = T(\lambda v) = \lambda T(v) = \lambda^2 v$. By definition of *projpotent*, $T(v) = T^2(v)$. Then $\lambda v = \lambda^2 v \implies \lambda = \lambda^2$. Therefore 0 and 1 are the only possible eigenvalues.

(b) As shown previously, the only possible eigenvalues of T are $\lambda = 0$ and $\lambda = 1$. Let a_0, a_1 be algebraic multiplicities of eigenvalues 0 and 1 respectively. In addition, let b_0, b_1 be their geometric multiplicities respectively. We can verify that T is diagonalisable by comparing the algebraic and geometric multiplicities of each of its eigenvalues.

$\lambda = 0$: The corresponding eigenvectors are vectors mapped to 0 (since $T(v) = 0v = 0$), which means that they form the basis of $\ker(T)$ and therefore $a_0 = \text{nullity}(T)$. In addition, the corresponding eigenspace is $E_0 = \ker(T - 0Id) = \ker(T)$. Then the corresponding geometric multiplicity is $b_0 = \dim(E_0) = \text{nullity}(T)$. Then $a_0 = b_0$.

$\lambda = 1$: The corresponding eigenvectors here are ones that satisfy $T(v) = v$. However, all vectors in $\text{im}(T)$ satisfies this condition as $\text{im}(T) = \{w = T(u) \mid u \in V\}$ and $T(w) = T^2(u) = T(u) = w$. Therefore $a_1 = \text{rank}(T)$. In addition, $b_1 = \text{nullity}(T - Id)$ can be found by solving $(T - Id)v = 0 \implies T(v) - Idv = 0 \implies w - v = 0$. Then $v = w$ is in E_1, therefore $b_1 = \text{rank}(T) = b_0$.

Since $a_i = b_i$ for both $i = 0$ and $i = 1$, T is diagonalisable.

(c) Let $n = \dim(V)$. Since T is diagonalisable per part (b), there exists an eigenbasis $\beta = \{u_1, \ldots, u_n\}$ for T such that $[T]_\beta^\beta$ is diagonal with corresponding eigenvalues on the main diagonal. As shown in (a), T can only have eigenvalues of 0 or 1. Now let k be the number of zero entries on the main diagonal, which corresponds to eigenvectors $u_1, \ldots, u_k$ upon renumbering. Then for each $u_i, 1 \geq i \geq k$, $T(u_i) = 0u_i = 0$, i.e. $\text{nullity}(T) = k$ and $\text{rank}(T) = n - k$. In addition, since $[T]$ is $n \times n$ diagonal matrix, the number of nonzero entries on its diagonal is $n - k$. Again as shown in (a), the only nonzero eigenvalue T can have is 1. Then $\text{tr}(T) = 1(n - k) = n - k = \text{rank}(T)$.