Midterm 2

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Instructions:

- The exam will begin at 12 am May 29 PDT. You will be given **24 hours** to complete and submit your works. The submission window will be closed at 12 am May 30 PDT.
- **No late submission** will be considered. Make sure to allow enough time to complete and submit your works. Make-ups for the exam are permitted only under exceptional circumstances, as outlined in the UCLA student handbook.
- The exam will be **open book/open notes**. You may also use a physical and/or online calculator, provided it supports the same or less functionality than the officially accepted models on SOA Exam[1].
- **You must show your works to receive credit.**
- **You must sign the code of conduct:**

  I assert, on my honor, that I have not received assistance of any kind from any other person while working on the exam and that I have not used any non-permitted materials or technologies during the period of this evaluation.

  Signature: 

  Any deviation from the rules may render your exam void. Also, if needed, you may be contacted after the exam and asked for additional explanations of solutions for problems on the exam.
- **A Gradescope link for submitting your works will be provided.** Your submission should meet a set of criteria:
  (a) Your submission must be a single PDF file.
  (b) The code of conduct, your name, UID, and either physical or electronic signature must appear on the first page. (See above for an example.)

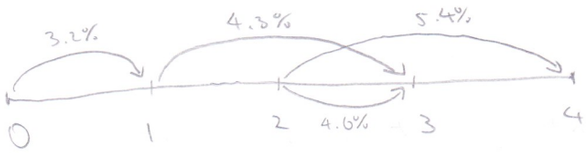
  There will be several ways to achieve this. The following is a set of common examples:
  - The exam template file will be designed to satisfy the above criteria. So you may simply print it out, fill in the necessary forms, and write down your solution of each of the problems. And then you may either scan it or take a (high-resolution and high-contrast) picture of it.
  - You may use letter size blank papers as soon as all the above criteria are met.
  - You may directly write on the exam PDF file, such as using a tablet.
  - You may use a word processor or LaTeX to prepare your submission electronically.

[1] https://www.soa.org/education/exam-req/exam-day-info/edu-calculators/

Question 1. You are given that the term structure $\{r_t\}_{t>0}$ satisfies:

$$r_1 = 3.2\%, \qquad f_{[1,3]} = 4.3\%, \qquad f_{[2,3]} = 4.6\%, \qquad f_{[2,4]} = 5.4\%$$

Determine r_2, r_3, and r_4.



r_2: $\quad (1+r_2)^2 (1+f_{[2,3]}) = (1+r_1)(1+f_{[1,3]})^2$

$$r_2 = \sqrt{\frac{1.032 \cdot 1.043^2}{1.046}} - 1$$

$$r_2 = 0.035997 \approx \boxed{3.6\%}$$

r_3: $\quad (1+r_3)^3 = (1+r_1)(1+f_{[1,3]})^2$

$$r_3 = \sqrt[3]{1.032 \cdot 1.043^2} - 1$$

$$r_3 = 0.039320 \approx \boxed{3.9\%}$$

r_4: $\quad (1+r_4)^4 = (1+r_2)^2 (1+f_{[2,4]})^2$

$$r_4 = \sqrt[4]{1.036^2 \cdot 1.054^2} - 1$$

$$r_4 = 0.044961 \approx \boxed{4.5\%}$$

Question 2. Suppose that the spot rates are given by

$$r_t = 0.03 + 0.02t + 0.001t^2.$$

A 2-year $\$1000$ par value bond bearing semiannual coupons is bought at par, so that it yields at a nominal annual rate of 7.1752% convertible semiannually.

Determine the coupon rate, expressed as a nominal annual rate convertible semiannually, and the redemption value of this bond.

$r_{0.5} = 0.04025$ $\quad r_1 = 0.051$ $\quad r_{1.5} = 0.06225$ $\quad r_2 = 0.074$

$$\begin{cases} 1000 = 1000 \cdot \frac{r}{2} \cdot \left(1.020125^{-1} + 1.0255^{-2} + 1.031125^{-3} + 1.037^{-4}\right) + C \cdot 1.037^{-4} & (1) \\ 1000 = C v_{3.5876\%}^{4} + 1000 \cdot \frac{r}{2} \cdot a_{\overline{4}|3.5876\%} & (2) \end{cases}$$

(1) $\Rightarrow$ $1854.023456\,r + 0.864759\,C = 1000$

(2) $\Rightarrow$ $1832.726812\,r + 0.868498\,C = 1000$

$\Rightarrow$ $r \approx 14.81\%$, $C \approx 838.94$

Question 3. The instantaneous forward rate is given by

$$\delta_t = 0.1 + 0.007t - (0.08)(0.9)^t.$$

Find the two-year forward rate, deferred three years and expressed as an effective annual rate of interest.

$$\ln(1 + f_{[3,5]}) = \frac{1}{5-3}\int_3^5 \delta_t \, dt$$

$$f_{[3,5]} = e^{0.0754148} - 1$$

$$f_{[3,5]} = 0.0783314 \approx \boxed{7.83\%}$$

Question 4. Sprinkle has a portfolio consisting of the following assets:

(i) A 5-year $\$10,000$ par value bond with 5% annual coupons and maturing at par

(ii) A 10-year annuity-immediate with level annual payments of $\$1,200$

Do the following:

(a) Calculate the Macaulay duration of this portfolio at valuation rate of $i_0 = 10\%$.

Year	1	2	3	4	5	6	7	8	9	10
(i)	500	500	500	500	10500	–	–	–	–	–
(ii)	1200	1200	1200	1200	1200	1200	1200	1200	1200	1200
Total	1700	1700	1700	1700	11700	1200	1200	1200	1200	1200

$$D_{mac}(10\%) = \frac{1\cdot 1700v + 2\cdot 1700v^2 + 3\cdot 1700v^3 + 4\cdot 1700v^4 + 5\cdot 11700v^5 + 6\cdot 1200v^6 + 7\cdot 1200v^7 + 8\cdot 1200v^8 + 9\cdot 1200v^9 + 10\cdot 1200v^{10}}{500(v+v^2+v^3+v^4) + 10500v^5 + 1200a_{\overline{10}|}}$$

$$D_{mac}(10\%) = \frac{71215.45137}{15478.08714}$$

$$D_{mac}(10\%) \approx \boxed{4.60}$$

(b) Sprinkle expects that the interest rate will change to a new value i. Using the first order Macaulay approximation, Sprinkle estimates that the present value of the portfolio at rate i is $\$16,000$. Determine the value of i.

$$P(10\%) \approx 15478.08714$$

$$D_{mac}(10\%) \approx 4.601050$$

$$\Rightarrow P(i) \approx P(10\%)\left(\frac{1+10\%}{1+i}\right)^{D_{mac}(10\%)}$$

$$16000 \approx 15478.08714 \cdot \left(\frac{1.1}{1+i}\right)^{4.601050}$$

$$\ln\frac{16000}{15478.08714} \approx 4.601050 \cdot \ln\frac{1.1}{1+i}$$

$$\ln 1.1 - \ln 1+i \approx 0.00720780$$

$$\ln 1+i \approx 0.0881024$$

$$i \approx e^{0.0881024} - 1$$

$$i \approx 0.0920999 \approx \boxed{9.21\%}$$

Question 5. Calculate the modified convexity of a 60-year bond with 6.86% annual coupons maturing at par with a 10% yield. (*Hint:* Premium/Discount Formula might be helpful.)

$$C_{mod} = \frac{\sum_{t=1}^{60} t(t+1) \cdot 6.86\% \cdot \cancel{F} \cdot v_j^{t+2} + 60 \cdot 61 \cdot \cancel{F} \cdot v_j^{62}}{\sum_{t=1}^{60} 6.86\% \cdot \cancel{F} \cdot v_j^{t} + \cancel{F} \cdot v_j^{60}}$$

$$= \frac{6.86\% \cdot \sum_{t=1}^{60} (t^2+t)\, v_j^{t+2} + 3660\, v_j^{62}}{6.86\% \cdot a_{\overline{60}|j} + v_j^{60}}$$

$$= 200.006795 \approx \boxed{200}$$

Question 6. East Empire Company must pay a liability of $\$100,000$ due in 4 years. The current interest rate is 6% effective per year. The company attempts to immunize itself from small changes in interests. It does so by constructing a portfolio consisting of:

(i) A 1-year $\$20,000$ par value zero coupon bond maturing at par

(ii) A 2-year zero coupon bond with the redemption value of X

(iii) A 6-year zero coupon bond with the redemption value of Y

The amounts X and Y are chosen to satisfy the asset-liability matching (1st condition) and duration matching (2nd condition).

(a) Calculate X and Y.

1st: $P_A(i_0) = P_L(i_0) \Rightarrow 20000v + Xv^2 + Yv^6 = 100000v^4$ ①

2nd: $P'_A(i_0) = P'_L(i_0) \Rightarrow \sum_t t A_t v_0^t = \sum_t t L_t v_0^t \Rightarrow 20000v + 2Xv^2 + 6Yv^6 = 4 \cdot 100000v^4$ ②

① $\Rightarrow 0.889996X + 0.704961Y = 60241.4418$

② $\Rightarrow 1.779993X + 4.229763Y = 297969.5408$

$\Rightarrow$ $X \approx 17999.78$, $Y \approx 62871.15$

(b) Show that the company's liability is Redington immunized by this portfolio at valuation rate of 6%.

3rd: $P''_A(i_0) > P''_L(i_0) \Rightarrow \sum_t t^2 A_t v_0^t > \sum_t t^2 L_t v_0^t$

$\Rightarrow 20000v + 2^2 \cdot Xv^2 + 6^2 \cdot Yv^6 \overset{?}{>} 4^2 \cdot 100000v^4$

$\Rightarrow 1678527.31 \overset{\checkmark}{>} 1267349.36$

Since all three conditions are satisfied, the liability cashflow is Redington immunized at valuation rate 6%.

(c) Prove that this portfolio fully immunizes the company's liability.

$P_A(i) = 20000v_i + 17999.78v_i^2 + 62871.15v_i^6$

$P_L(i) = 100000v_i^4$

Let $f(i) = P_A(i) - P_L(i) = 20000v_i + 17999.78v_i^2 - 100000v_i^4 + 62871.15v_i^6$

then $P_A(i) \geq P_L(i) \iff f(i) \geq 0$ for any $i > 0$.

$f(i) = 0 \Rightarrow i = 0.06$ (only solution for $i > 0$)

We can verify that $f(0.06)$ is the minimum using the first derivative test.

$f'(i) = -20000v_i^2 - 35999.56v^3 + 400000v^5 - 377226.90v^7$

Test values: $f'(0.05) = -3916 < 0 \Rightarrow$ f is decreasing on $(0, 0.06)$

$f'(0.07) = 3421 > 0 \Rightarrow$ f is increasing on $(0.06, \infty)$

Therefore $f(0.06) = 0$ is the minimum and all $i > 0$ results in $f(i) \geq 0$. Thus $P_A(i) \geq P_L(i)$ for any $i > 0$.

Question 7. Doraji Industries enters into a three-year interest rate swap with notional amounts of \$1 million, \$3 million, and \$5 million in year one, two, and three, respectively. The settlement periods are one year. The variable interest rate is the one-year spot rate at the beginning of each settlement period. Doraji will swap this variable interest rate for a fixed interest rate.

You are given the following forward interest rates:

Period	Forward Interest Rate
$f_{[0,1]}$	3.0%
$f_{[1,2]}$	3.5%
$f_{[2,3]}$	4.0%
$f_{[3,4]}$	4.6%

Calculate the swap rate.

$r_1 = 0.03 \Rightarrow P_1 = 1.03^{-1}$

$r_2 = \sqrt{1.03 \cdot 1.035} - 1 = 0.032497 \Rightarrow P_2 = 1.032497^{-2}$

$r_3 = \sqrt[3]{1.03 \cdot 1.035 \cdot 1.04} - 1 = 0.034992 \Rightarrow P_3 = 1.034992^{-3}$

$$R = \frac{10^6 \cdot 3.0\% \cdot 1.03^{-1} + 3 \cdot 10^6 \cdot 3.5\% \cdot 1.032497^{-2} + 5\ 10^6 \cdot 4.0\% \cdot 1.034992^{-3}}{10^6 \cdot 1.03^{-1} + 3 \cdot 10^6 \cdot 1.032497^{-2} + 5 \cdot 10^6 \cdot 1.034992^{-3}}$$

$= 0.037133 \simeq \boxed{3.71\%}$

Question 8. Yutu Company enters into a 5-year interest rate swap contract with a level notional amount of \$1 million on 1 January 2020 as the fixed-rate payer party. The settlement period is one year. The settlement dates are 31 December of each year from 2020 to 2024.

The term structures at the beginning of various years are given in the following table.

t	1 Jan 2020	1 Jan 2021	1 Jan 2022	1 Jan 2023	1 Jan 2024
1	$r_1 = 1.56\%$	0.16%	0.42%	0.31%	0.35%
2	$r_2 = 1.61\%$	0.18%	0.34%	0.38%	0.64%
3	$r_3 = 1.70\%$	0.22%	0.47%	0.57%	1.05%
4	$r_4 = 1.83\%$	0.27%	0.95%	1.08%	1.78%
5	$r_5 = 2.00\%$	0.34%	1.47%	1.58%	2.31%

(a) Calculate the swap rate.

$$R = \frac{1 - P_5}{P_1 + P_2 + P_3 + P_4 + P_5} = \frac{1 - 1.0200^{-5}}{1.0156^{-1} + 1.0161^{-2} + 1.0170^{-3} + 1.0183^{-4} + 1.0200^{-5}}$$

$$= 0.024588 \approx \boxed{2.46\%}$$

(b) Calculate the net swap payment that will be made on 31 December 2022.

Variable interest rate: 0.42%

Net swap payment $= 10^6 \cdot 0.042\% - 10^6 \cdot 2.46\% = \boxed{-20400}$

(c) Determine the market value for Yutu on 1 January 2022.

Net swap payments under 2022 variable rate would be -20400/year,

Then,

$$\text{market value} = \frac{-20400}{1.0042} + \frac{-20400}{1.0034^2} + \frac{-20400}{1.0047^3}$$

$$= -60691.70541$$

$$\approx \boxed{-60691.71}$$