Stats 100A Homework 2

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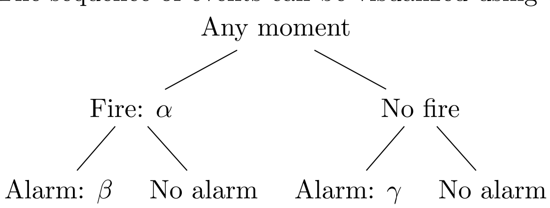
Problem 1

The sequence of events can be visualized using the chart below:



By adding up the instances where alarm occurs, we can see that $P(\text{alarm}) = \alpha\beta + (1-\alpha)\gamma$.

In addition, $P(\text{fire\&alarm}) = P(\text{fire})P(\text{alarm}|\text{fire}) = \alpha\beta$.

Then, $P(\text{fire}|\text{alarm}) = \dfrac{P(\text{fire\&alarm})}{P(\text{alarm})} = \boxed{\dfrac{\alpha\beta}{\alpha\beta + (1-\alpha)\gamma}}$.

Let $\alpha = 1\%$, $\beta = 98\%$ and $\gamma = 5\%$. Then,

$P(\text{alarm}|\text{fire}) = \beta = 0.98$

$$P(\text{fire}|\text{alarm}) = \frac{\alpha\beta}{\alpha\beta + (1-\alpha)\gamma} = \frac{0.01 * 0.98}{0.01 * 0.98 + (1-0.01)0.05} = 0.17$$

Problem 2

1. $P(X > 4) = p(5) + p(6) = \boxed{0.5}$

2. $P(X = 6 | X > 4) = \frac{0.3}{0.5} = \boxed{0.6}$

3. $\mathbb{E}(X) = \sum_x xp(x) = 1*0.1 + 2*0.1 + 3*0.1 + 4*0.2 + 5*0.2 + 6*0.3 = \boxed{4.2}$
   $\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}(X))^2] = 10.24*0.1 + 4.84*0.1 + 1.44*0.1 + 0.04*0.2 + 0.64*0.2 + 3.24*0.3 = \boxed{2.76}$
   $\text{SD}(X) = \sqrt{\text{Var}(X)} = \sqrt{2.22} \approx \boxed{1.66}$

4. $\mathbb{E}(h(X)) = \sum_x h(x)p(x) = -20*0.1 - 10*0.1 + 0*0.1 + 10*0.2 + 20*0.2 + 100*0.3 = \boxed{33}$
   $\text{Var}(h(X)) = \mathbb{E}[(h(X) - \mathbb{E}(h(X)))^2] = 2809*0.1 + 1849*0.1 + 1089*0.1 + 529*0.2 + 169*0.2 + 4489*0.3 = \boxed{2061}$
   $\text{SD}(h(X)) = \sqrt{\text{Var}(h(X))} = \sqrt{2061} \approx \boxed{45.40}$
   The units of $\mathbb{E}(h(X))$ and $\text{Var}(h(X))$ are dollars and dollars-squared respectively.

Problem 3

$\mathbb{E}(Z) = 1 * p + 0 * (1-p) = \boxed{p}$
$\mathbb{E}(Z^2) = 1^2 * p + 0^2 * (1-p) = \boxed{p}$
$\mathrm{Var}(Z) = (1-p)^2 * p + (0-p)^2 * (1-p) = \boxed{-p^2 + p}$

Problem 4

1. $\mathbb{E}(aX) = \sum_x axp(x) = a \sum_x xp(x) = aE(X)$
2. $\mathbb{E}(X+b) \sum_x (x+b)p(x) = \sum_x xp(x) + \sum_x bp(x) = \sum_x xp(x) + b \sum_x p(x) = \mathbb{E}(X) + b$
3. $\mathrm{Var}(aX) = \mathbb{E}[(aX - \mathbb{E}(aX))^2] = \mathbb{E}[(aX - a\mathbb{E}(X))^2] = \mathbb{E}[a^2(X - \mathbb{E}(X))^2] = a^2\mathbb{E}[(X - \mathbb{E}(X))^2]$
   $= a^2\mathrm{Var}(X)$
4. $\mathrm{Var}(X+b) = \mathbb{E}[(X + b - \mathbb{E}(X+b))^2] = \mathbb{E}[(X + b - \mathbb{E}(X) - b)^2] = \mathrm{Var}(X)$
5. Let $\mu = \mathbb{E}(X)$ which is a constant term. Then, $\mathrm{Var}(X) = \mathbb{E}[(X-\mu)^2] = \mathbb{E}(X^2 - 2\mu X + \mu^2) = \mathbb{E}(X^2) - 2\mu\mathbb{E}(X) + \mu^2 = \mathbb{E}(X^2) - \mu^2 = \mathbb{E}(X^2) - \mathbb{E}(X)^2$
6. $\mathbb{E}(Z) = \mathbb{E}(\frac{X-\mu}{\sigma}) = \frac{\mathbb{E}(X-\mu)}{\sigma} = \frac{\mathbb{E}(X)-\mu}{\sigma} = \boxed{0}$
   $\mathrm{Var}(Z) = \mathrm{Var}(\frac{X-\mu}{\sigma}) = \frac{\mathrm{Var}(X-\mu)}{\sigma^2} = \frac{\mathrm{Var}(X)}{\sigma^2} = \boxed{1}$