Math 115A Homework 2

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2. Let V be a finite dimensional vector space and W a subspace. Show that V and $W \times V/W$ are isomorphic by finding an explicit isomorphism.

   Answer: By definitions of products and quotients of vector spaces, $W \times V/W = \{(w, v) + W \mid v \in V, w \in W\}$. Let $x, y \in V$, $p, q \in W$, $w \in W \times V/W$ and $a, b \in \mathbb{F}$. Note that $(p, x) + W = (0, x) + W$ since $p \in W$. Then, define $T(x) = (0, x) + W$ and $T^{-1}((w, x) + W) = x$.
   We can verify that T is linear as follows: $T(ax + y) = (0, ax + y) + W = [(0, ax) + W] + [(0, y) + W] = a[(0, x) + W] + [(0, y) + W] = aT(x) + T(y)$.
   Similarly, T^{-1} is also linear: $T^{-1}(b((p, x) + W) + ((q, y) + W)) = T^{-1}(b((0, x) + W) + ((0, y) + W)) = T^{-1}((0, bx + y) + W) = bx + y = bT^{-1}((p, x) + W) + T^{-1}((q, y) + W)$.
   Therefore T is invertible and V is isomorphic to $W \times V/W$.

5. A differential operator on $\mathbb{R}_n[x]$ is a linear combination of expressions of the form $x^a \frac{d^b}{dx^b}$ where $a - b \leq 0$ and $b \leq n$. We can consider a differential operator as a linear map $\mathbb{R}_n[x] \to \mathbb{R}_n[x]$.

   (a) Let $D : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ be the differential operator given by $2 - 4\frac{d}{dx} + 2x\frac{d^2}{dx^2}$. Find the matrix of D relative to the basis $\{x^2, (x-1)^2, (x+1)^2\}$.

   Answer: Define bases of $\mathbb{R}_2[x]$ $\beta = \{1, x, x^2\}$ and $\gamma = \{x^2, (x-1)^2, (x+1)^2\}$. Then, by transforming each vector of β and writing the results as linear combinations of vectors in γ, we have

   $$D(1) = 2 = -2(x^2) + 1(x-1)^2 + 1(x+1)^2$$
   $$D(x) = 2x - 4 = 4(x^2) - \frac{5}{2}(x-1)^2 - \frac{3}{2}(x+1)^2$$
   $$D(x^2) = 2x^2 - 4x = 2(x^2) + 1(x-1)^2 - 1(x+1)^2$$

   Hence,

   $$[D]_\beta^\gamma = \begin{pmatrix} -2 & 4 & 2 \\ 1 & -\frac{5}{2} & 1 \\ 1 & -\frac{3}{2} & -1 \end{pmatrix}.$$

   (b) Does the differential equation $2f - 4\frac{df}{dx} + 2x\frac{d^2 f}{dx^2} = 0$ have any solutions $f \in \mathbb{R}_2[x]$?

   Answer: Suppose $f = a + bx + cx^2$ is a solution. Then, a, b, c must be not all zero and satisfies the following:

$$\begin{pmatrix} -2 & 4 & 2 \\ 1 & -\frac{5}{2} & 1 \\ 1 & -\frac{3}{2} & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Then, using Gaussian Elimination, we see that $a = b = c = 0$. Therefore, the differential equation does not have any solution in $\mathbb{R}_2[x]$.

(c) Suppose $E : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ is a differential operator and that the matrix of E, relative to the basis $\{1, x, x^2\}$ is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Find E.

Answer: The above matrix translates to the following equations:

$$E(1) = 0$$

$$E(x) = 1$$

$$E(x^2) = x$$

Then, $E(p) = \frac{dp}{dx} - x\frac{d^2p}{dx^2}$ satisfies all three equations above.

6. Consider the linear map $X : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$ given by $X(p) = \frac{dp}{dx} + \frac{x^n}{n!}p(0)$. Calculate the dimension of

$$C(X) = \{T \in Hom(\mathbb{R}_n[x], \mathbb{R}_n[x]) \mid T \circ X = X \circ T\}.$$

Answer: We can start by exploring the first few cases of n.

$\boxed{n = 1}$: $X_1(p) = \frac{dp}{dx} + xp(0)$. Let basis $\beta_1 = \{1, \frac{x}{1!}\}$. Then, $X_1(1) = x = 1(\frac{x}{1!})$ and $X_1(\frac{x}{1!}) = 1$. The matrix of X_1 is

$$[X_1]_{\beta_1}^{\beta_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then,

$$C(X_1) = T \in \{\text{Hom}(\mathbb{R}_1[x], \mathbb{R}_1[x]) \mid T \circ X_1 = X_1 \circ T\}.$$

Substituting in T and X_1 results in the following matrix multiplication equation:

$$\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$$

$$\implies \begin{pmatrix} t_{12} & t_{11} \\ t_{22} & t_{21} \end{pmatrix} = \begin{pmatrix} t_{21} & t_{22} \\ t_{11} & t_{12} \end{pmatrix}$$

$$\implies t_{11} = t_{22},\, t_{12} = t_{21}$$

which means that

$$C(X_1) = \{\begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{11} \end{pmatrix}\} \implies \dim(C(X_1)) = 2.$$

$\boxed{n=2}$: $X_2(p) = \frac{dp}{dx} + \frac{x^2}{2!}p(0)$. Let basis $\beta_2 = \{1, \frac{x}{1!}, \frac{x^2}{2!}\}$. Then, $X_2(1) = \frac{x^2}{2!}$, $X_2(\frac{x}{1!}) = 1$ and $X_2(\frac{x^2}{2!}) = \frac{x}{1!}$. The matrix of X_2 is

$$[X_2]_{\beta_2}^{\beta_2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then,

$$C(X_2) = T \in \{\mathrm{Hom}(\mathbb{R}_2[x], \mathbb{R}_2[x]) \mid T \circ X_2 = X_2 \circ T\}.$$

Substituting in T and X_2 results in the following matrix multiplication equation:

$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix}$$

$$\implies \begin{pmatrix} t_{13} & t_{11} & t_{12} \\ t_{23} & t_{21} & t_{22} \\ t_{33} & t_{31} & t_{32} \end{pmatrix} = \begin{pmatrix} t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \\ t_{11} & t_{12} & t_{13} \end{pmatrix}$$

$$\implies t_{11} = t_{22} = t_{33},\ t_{12} = t_{23} = t_{31},\ t_{13} = t_{21} = t_{32}$$

which means that

$$C(X_1) = \{\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{13} & t_{11} & t_{12} \\ t_{12} & t_{13} & t_{11} \end{pmatrix}\} \implies \dim(C(X_2)) = 3.$$

$\boxed{n=3}$: Using the same approach as above, we get $\dim(C(X_3)) = 4$.
From the above cases, it seems that $\dim(C(X_n)) = n+1$. To show this for a general n, we start by noticing the pattern

$$[X_n]_{\beta_n}^{\beta_n} = \begin{pmatrix} e_{n+1} & e_1 & ... & e_n \end{pmatrix}.$$

This is verifiable by applying X to each vector of $\beta_n = \{1, \frac{x}{1!}, \frac{x^2}{2!}, \dots, \frac{x^n}{n!}\}$, resulting in $\{\frac{x^n}{n!}, 1, ..., \frac{x^{n-1}}{(n-1)!}\}$. Then, the matrix multiplication $[T][X]$ essentially shifts every column in $[T]$ to the right by one column. Similarly, the matrix multiplication $[X][T]$ shifts every row in $[T]$ up by one row. The relationship $T \circ X = X \circ T$, equivalently, the equation $[T][X] = [X][T]$, can be shown below:

$$\begin{pmatrix} t_{1,n+1} & t_{1,1} & \cdots & t_{1,n} \\ t_{2,n+1} & t_{2,1} & \cdots & t_{2,n} \\ \vdots & \ddots & \ddots & \vdots \\ t_{n+1,n+1} & t_{n+1,1} & \cdots & t_{n+1,n} \end{pmatrix} = \begin{pmatrix} t_{2,1} & t_{2,2} & \cdots & t_{2,n+1} \\ \vdots & \ddots & \ddots & \vdots \\ t_{n+1,1} & t_{n+1,2} & \cdots & t_{n+1,n+1} \\ t_{1,1} & t_{1,2} & \cdots & t_{1,n+1} \end{pmatrix}$$

By matching entry positions, we get the equations $t_{11} = t_{22} = \ldots = t_{n+1,n+1}$, $t_{12} = t_{23} = \ldots = t_{n,n+1} = t_{n+1,1}$, etc. In other words, each entry of $[T]$ is equivalent to the entry one row down and one column to the right from it. Note that for entries in the $(n+1)$-th column, "one column to the right" refers to the 1st column; similarly, for entries in the $(n+1)$-th row, "one column down" refers to the 1st row. Then, $C(X_n)$ can be written as

$$C(X_n) = \{\begin{pmatrix} t_{1,1} & t_{1,2} & \ldots & t_{1,n} & t_{1,n+1} \\ t_{1,n+1} & t_{1,1} & \ldots & t_{1,n-1} & t_{1,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ t_{1,3} & t_{1,4} & \ldots & t_{1,1} & t_{1,2} \\ t_{1,2} & t_{1,3} & \ldots & t_{1,n+1} & t_{1,1} \end{pmatrix}\}$$

which indeed has dimension $n+1$.