Midterm 1

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Instructions:

- The exam will begin at 12 am April 29 PDT. You will be given **24 hours** to complete and submit your work. The submission window will be closed at 12 am April 30 PDT.
- **No late submission** will be considered. Make sure to allow enough time to complete and submit your work. Make-ups for the exam are permitted only under exceptional circumstances, as outlined in the UCLA student handbook.
- The exam will be **open book/open notes**. You may also use physical and/or online calculators, provided they support the same or less functionality than the officially accepted models on SOA Exam[1].
- **You must show your works to receive credit.**
- **You must sign the code of conduct:**

> I assert, on my honor, that I have not received assistance of any kind from any other person while working on the exam and that I have not used any non-permitted materials or technologies during the period of this evaluation.
>
> Signature: ______

Any deviation from the rules may render your exam void. Also, if needed, you may be contacted after the exam and asked for additional explanations of solutions for problems on the exam.

- **A Gradescope link for submitting your work will be provided.** Your submission should meet a set of criteria:
  (a) Your submission must be a single PDF file.
  (b) The code of conduct, your name, UID, and either physical or electronic signature must appear on the first page. (See above for an example.)
  (c) Starting from page 2, your solution to each of the problems must appear on a single page, in the order of the numbering. (For example, your solution to Question 1 must appear on page 2, Question 2 on page 3, and so forth.)

  There will be several ways to achieve this. The following is a set of common examples:
  - The exam template file will be designed to satisfy the above criteria. So you may simply print it out, fill in the necessary forms, and write down your solution of each of the problems. And then you may either scan it or take a (high-resolution and high-contrast) picture of it.
  - You may use letter size blank papers as soon as all the above criteria are met.
  - You may directly write on the exam PDF file, such as using a tablet.
  - You may use a word processor or LaTeX to prepare your submission electronically.

[1] https://www.soa.org/education/exam-req/exam-day-info/edu-calculators/

1. Investment A will quadruple in 41 years at a constant force of interest δ. Investment B will grow by a factor of five in t years at a nominal rate of interest, numerically equal to δ, convertible semiannually. Calculate t.

Givens:

Investment A: $a_A(s) = e^{\delta s}$, $a_A(41) = 4$

Investment B: $a_B(s) = \left(1+\frac{\delta}{2}\right)^{2s}$, $a_B(t) = 5$

A: $4 = e^{41\delta}$

$\ln 4 = 41\delta$

$\delta = \frac{\ln 4}{41} \approx 0.0338121$

B: $5 = \left(1+\frac{\delta}{2}\right)^{2t}$

$5 = 1.0169060$

$\ln 5 = 2t \cdot \ln 1.0169060$

$t = \frac{\ln 5}{2 \ln 1.0169060}$

$\boxed{t = 48 \text{ years}}$

2. 27,000 can be invested under two options:

Option 1. Deposit 27,000 into a fund earning an annual effective rate of i; or

Option 2. Purchase an annuity-immediate with 24 level annual payments at an annual effective rate at 10%. Each of the payments is immediately reinvested into a fund earning an annual effective rate of 7%.

Both options produce the same accumulated value at the end of 24 years. Calculate i.

Option 1: $A_1(24) = 27000\,(1+i)^{24}$

Option 2: Let K be the annual payment, then

$$27000 = K\,a_{\overline{24}|10\%}$$

$$K = 27000 \cdot \frac{10\%}{1-\left(\frac{1}{1+10\%}\right)^{24}}$$

$$\approx 3005.093961$$

We can now find $A_2(24)$ using K

$$A_2(24) = K\,s_{\overline{24}|7\%}$$

$$= 3005.09 \cdot \frac{(1+7\%)^{24}-1}{7\%}$$

$$\approx 174826.362$$

Then, we can find i using

$$A_1(24) = A_2(24)$$

$$27000\,(1+i)^{24} = 174826.362$$

$$i = \sqrt[24]{\frac{174826.362}{27000}} - 1$$

$$\approx \boxed{8.094052\%}$$

3. Do the following:

(a) Let i denote the annual effective rate of interest. Recall that an n-year annuity-immediate with payments $1, 2, \ldots, n$ has the following present value:

$$(Ia)_{\overline{n}|i} = \frac{\ddot{a}_{\overline{n}|i} - nv^n}{i},$$

where $v = \frac{1}{1+i}$ is the discount factor. Show that the present value of a perpetuity-immediate with payments P, $P+Q$, $P+2Q$, $\ldots$, where $P \geq 0$ and $Q \geq 0$, is:

$$\frac{P}{i} + \frac{Q}{i^2}$$

We can consider such perpetuity-immediate as a combination of a constant and an arithmetically increasing perpetuities. i.e.,

$$PV = Pa_{\overline{\infty}|i} + vQ(Ia)_{\overline{\infty}|i}$$

$$= P \cdot \frac{1-\cancel{v^\infty}}{i} + \frac{Q}{1+i}\left(\frac{\ddot{a}_{\overline{\infty}|i} - \cancel{\infty v^\infty}}{i}\right)$$

$$= \frac{P}{i} + \frac{Q}{1+i} \cdot \left(\frac{\frac{1-\cancel{v^\infty}}{d}}{i}\right)$$

$$= \frac{P}{i} + \frac{Q}{\cancel{1+i}} \cdot \frac{\cancel{1+i}}{i^2}$$

$$= \boxed{\frac{P}{i} + \frac{Q}{i^2}}$$

Note:
$\infty v^\infty = 0$ because

$$\lim_{n\to\infty} nv^n = \lim_{n\to\infty} \frac{n}{(1+i)^n} = \frac{\infty}{\infty}$$

$$\overset{\text{L'hopital}}{\Rightarrow} \lim_{n\to\infty} \frac{0}{n(1+i)^{n-1}} = 0$$

(b) Haley purchases a perpetuity-immediate that makes annual payments. The first payment is 50, and each payment thereafter increases by 10. Emily purchases a perpetuity-due which makes annual payments of 170. Using the same effective interest rate, $i > 0$, the present value of both perpetuities are equal. Calculate i.

Using the formula derieved above, we have

$$\frac{50}{i} + \frac{10}{i^2} = 170\,\ddot{a}_{\overline{\infty}|i}$$

$$\frac{50}{i} + \frac{10}{i^2} = 170\left(\frac{1-\cancel{v^\infty}}{\frac{i}{1+i}}\right)$$

$$\frac{50}{\cancel{i}} + \frac{10}{i^{\cancel{2}}} = 170 \cdot \frac{1+i}{\cancel{i}}$$

$$\frac{10}{i} - 170i = 170 - 50$$

$$\Rightarrow 170i^2 + 120i - 10 = 0$$

$$\Rightarrow i = \frac{-120 \pm \sqrt{120^2 + 4 \cdot 1700}}{2 \cdot 170}$$

$$= \boxed{7.530058\%}$$

4. A loan, at a nominal annual interest rate of 28.38% convertible monthly, is to be repaid with equal payments at the end of each month for $2n$ months. In the nth payment, the interest paid is twice as much as the principal repaid. Calculate n.

Interest paid: $I_n = iB_{n-1} = 1 - v^{2n+1-n} = 1 - v^{n+1}$

Principle repaid: $P_n = K_n - I_n = v^{2n+1-n} = v^{n+1}$

Since $I_n = 2P_n$, we can solve for n as follows:

$$I_n = 2P_n$$

$$1 - v^{n+1} = 2v^{n+1}$$

$$v^{n+1} = \frac{1}{3}$$

$$(n+1)\ln v = \ln\frac{1}{3}$$

$$n = \frac{\ln\frac{1}{3}}{\ln v} - 1$$

$$n = \frac{\ln\frac{1}{3}}{\ln\frac{1}{1+\frac{28.38\%}{12}}} - 1$$

$$\boxed{n = 46 \text{ years}}$$

5. A 5-year loan with an effective annual interest rate of 4.5% is to be repaid with the following payments:

- 100 at the end of year 1;
- 200 at the end of year 2;
- 300 at the end of year 3;
- 200 at the end of year 4; and
- 100 at the end of year 5.

Calculate the amount of interest included in the second payment.

We need to first find the amount loaned:

$B_0 = 100v + 200v^2 + 300v^3 + 200v^4 + 100v^5$ $\qquad v = \frac{1}{1+4.5\%} \approx 0.956937799$

≈ 789.686125

Then,

$I_1 = iB_0 = 4.5\% B_0 \approx 35.535876$

$P_1 = 100 - I_1 \approx 64.464124$

$B_1 = B_0 - P_1 \approx 725.222001$

$I_2 = iB_1 \approx \boxed{32.63}$

6. Balgruuf purchases a 1000 bond with 7% semiannual coupons, which is redeemable at 1050 to yield 8% convertible semiannually. If the present value of the redemption amount is 210, what is the price to the nearest 10?

Givens: $F = 1000$ Unknowns: $n = ?$

$r = 7\%$ $P = ?$

$j = 8\%$

$C = 1050$ $(PV = 210)$

We need to first find the time until maturity, which can be found using the PV of C as follows:

$$1050 = 210 \cdot \left(1 + \frac{8\%}{2}\right)^{2n}$$

$$\ln \frac{1050}{210} = 2n \cdot \ln\left(1 + \frac{8\%}{2}\right)$$

$$n = \frac{\ln 5}{2 \ln 1.04}$$

$$n = 20.517703 \text{ years}$$

Then the purchase price can be determined using the formula

$$P = Cv_j^n + Fr a_{\overline{n}|j}$$

$$P = 210 + 1000 \cdot \frac{7\%}{2} \cdot a_{\overline{2n}|\frac{8\%}{2}}$$

$$= 210 + 35 \cdot \frac{1 - \left(\frac{1}{1 + \frac{8\%}{2}}\right)^{2n}}{\frac{8\%}{2}}$$

$$= 909.756813 \approx \boxed{910}$$

7. Daisy Mae issues Bond *Turnip*, which is a 1,000 par value bond with coupon rate 12% convertible semiannually. She offers two options:

Option 1. Purchase *Turnip* at the price P so that it matures at par at the end of n years.

Option 2. Purchase *Turnip* at the price $P + 50$ so that it matures at par at the end of $2n$ years.

Both options will yield 10% convertible semiannually. Calculate P.

Option 1: $P = Cv_j^n + Fr\,a_{\overline{n}|j}$

$$= 1000 \cdot \left(\frac{1}{1+\frac{10\%}{2}}\right)^{2n} + 1000 \cdot \frac{12\%}{2} \cdot \frac{1-\left(\frac{1}{1+\frac{10\%}{2}}\right)^{2n}}{\frac{10\%}{2}}$$

$$= 1000 \cdot \left(\frac{1}{1.05^{2n}} + \frac{6}{5} - \frac{6}{5} \cdot \frac{1}{1.05^{2n}}\right)$$

① : $$P = 1200 - \frac{200}{1.05^{2n}}$$

Option 2: $P+50 = Cv_j^n + Fr\,a_{\overline{n}|j}$

$$= 1000 \cdot \left(\frac{1}{1+\frac{10\%}{2}}\right)^{4n} + 1000 \cdot \frac{12\%}{2} \cdot \frac{1-\left(\frac{1}{1+\frac{10\%}{2}}\right)^{4n}}{\frac{10\%}{4n}}$$

② : $$P = 1150 - \frac{200}{1.05^{4n}}$$

We then have the system of equations:

$$\begin{cases} P = 1200 - \frac{200}{1.05^{2n}} & ① \\ P = 1150 - \frac{200}{1.05^{4n}} & ② \end{cases}$$

① − ② : $$0 = 50 - \frac{200}{1.05^{2n}} + \frac{200}{1.05^{4n}}$$

$$0 = 50 \cdot 1.05^{4n} - 200 \cdot 1.05^{2n} + 200$$ Let $u = 1.05^{2n}$

$$50u^2 - 200u + 200 = 0$$

$$u = \frac{200 \pm \sqrt{200^2 - 4 \cdot 50 \cdot 200}}{100}$$

$$u = 2$$

Then $P = 1200 - \frac{200}{2} = \boxed{1100}$

8. You are given the following information about an investment account:

	1/1/2020	3/1/2020	4/1/2020	T/1/2020	1/1/2021
Account Value†	100	110	105	120	130
Deposit			10	X	
Withdrawal		15			

† Before deposit or withdrawal

The time-weighted yield rate is 22.17%, and the dollar-weighted yield rate is 20%. Calculate T.

Dollar weighted:

$$100(1+20\%) - 15\left(1+\tfrac{10}{12}\cdot 20\%\right) + 10\left(1+\tfrac{9}{12}\cdot 20\%\right) + X\left(1+\tfrac{13-T}{12}\cdot 20\%\right) = 130$$

$$120 - 17.5 + 11.5 + X + X\left(\tfrac{13}{12}\cdot 20\% - \tfrac{T}{12}\,20\%\right) = 130$$

$$X\left(\tfrac{73}{60} - \tfrac{T}{60}\right) = 16$$

1 : $X(73-T) = 960$

Time weighted:

$$1 + 22.17\% = \frac{110}{100}\cdot\frac{105}{110-15}\cdot\frac{120}{105+10}\cdot\frac{130}{120+X}$$

$$1.2217 = \frac{2772}{2185}\cdot\frac{130}{120+X}$$

$$X = \frac{2772}{2185}\cdot\frac{130}{1.2217} - 120$$

$$X \simeq 14.995895$$

Substitute X into 1:

$$73 - T = \frac{960}{X}$$

$$T = \frac{960}{14.995895} + 73$$

$$\boxed{T = 9}$$