Math 115A Homework 2

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2. Let V be a finite dimensional vector space and W a subspace. Show that V and $W \times V/W$ are isomorphic by finding an explicit isomorphism.

5. A differential operator on $\mathbb{R}_n[x]$ is a linear combination of expressions of the form $x^a \frac{d^b}{dx^b}$ where $a-b \leq 0$ and $b \leq n$. We can consider a differential operator as a linear map $\mathbb{R}_n[x] \to \mathbb{R}_n[x]$.

   (a) Let $D : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ be the differential operator given by $2 - 4\frac{d}{dx} + 2x\frac{d^2}{dx^2}$. Find the matrix of D relative to the basis $\{x^2, (x-1)^2, (x+1)^2\}$.

   Answer: Define bases of $\mathbb{R}_2[x]$ $\beta = \{1, x, x^2\}$ and $\gamma = \{x^2, (x-1)^2, (x+1)^2\}$. Then, by transforming each vector of β and writing the results as linear combinations of vectors in γ, we have

$$D(1) = 2 = -2(x^2) + 1(x-1)^2 + 1(x+1)^2$$
$$D(x) = 2x - 4 = 4(x^2) - \frac{5}{2}(x-1)^2 - \frac{3}{2}(x+1)^2$$
$$D(x^2) = 2x^2 - 4x = 2(x^2) + 1(x-1)^2 - 1(x+1)^2$$

   Hence,

$$[D]_\beta^\gamma = \begin{pmatrix} -2 & 4 & 2 \\ 1 & -\frac{5}{2} & 1 \\ 1 & -\frac{3}{2} & -1 \end{pmatrix}$$

   .

   (b) Does the differential equation $2f - 4\frac{df}{dx} + 2x\frac{d^2f}{dx^2} = 0$ have any solutions $f \in \mathbb{R}_2[x]$?

   (c) Suppose $E : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ is a differential operator and that the matrix of E, relative to the basis $\{1, x, x^2\}$ is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

   Find E.

   Answer: The above matrix translates to the following equations:

$$E(1) = 0$$

$$E(x) = 1$$

$$E(x^2) = x$$

Then, $E(p) = \frac{dp}{dx} - x\frac{d^2p}{dx^2}$ satisfies all three equations above.

6. Consider the linear map $X : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$ given by $X(p) = \frac{dp}{dx} + \frac{x^n}{n!}p(0)$. Calculate the dimension of

$$C(X) = \{T \in Hom(\mathbb{R}_n[x], \mathbb{R}_n[x]) \mid T \circ X = X \circ T\}.$$