Stats 100A Final

Jiaping Zeng

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Problem 1

(1) $\int_0^1 f(x)dx = 1 \implies \int_0^1 axdx = 1 \implies \frac{a}{2} = 1 \implies \boxed{a = 2}$

(2) $P(X > \frac{1}{2}) = P(X \in [\frac{1}{2}, 1]) = \int_{\frac{1}{2}}^1 f(x)dx = \boxed{\frac{3}{4}}$

(3) $E(X) = \int_0^1 xf(x)dx = \int_0^1 2x^2 dx = \boxed{\frac{2}{3}}$

$E(X^2) = \int_0^1 x^2 f(x)dx = \int_0^1 2x^3 dx = \frac{1}{2}$

$Var(X) = E(X^2) - E(X)^2 = \frac{1}{2} - \frac{4}{9} = \boxed{\frac{1}{18}}$

Problem 2

(1) $E(Y) = E(1 + 2X + \epsilon) = E(1) + 2E(X) + E(\epsilon) = \boxed{1}$

$Var(Y) = Var(1+2X+\epsilon) = Var(1)+2^2Var(X)+Var(\epsilon) = E(1^2)-E(1)^2+4Var(X)+Var(\epsilon) = \boxed{9}$

(2) $E(X^2) = Var(X) + E(X)^2 = 2$

$Cov(X, Y) = E(XY)-E(X)E(Y) = E(X+2X^2+\epsilon X)-E(X)E(Y) = E(X)+2E(X^2)+E(\epsilon)E(X)- E(X)E(Y) = \boxed{4}$

$Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}} = \frac{4}{\sqrt{2}\sqrt{9}} = \boxed{\frac{2\sqrt{2}}{3}}$

Problem 3

(1) $\frac{P(T \in [t, t + \triangle t])}{\triangle t} = \frac{(1 - \triangle t)^{\frac{t}{\triangle t}} \triangle t}{\triangle t} = \frac{e^{-t} \triangle t}{\triangle t} = \boxed{e^{-t}}$

$T = \widetilde{T} \triangle t$ where $\widetilde{T} \sim Geometric(p = \triangle t) \implies E(T) = E(\widetilde{T})\triangle t = \frac{1}{\triangle t} \triangle t = \boxed{1}$

(2) $X \sim Binomial(n = \frac{t}{\triangle t}, p = \triangle t) \implies P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k} = \boxed{\frac{e^{-t}t^k}{k!}}$

$E(X) = np = \frac{t}{\triangle t} \triangle t = \boxed{t}$

Problem 4

$$f_{X_i}(x) = \begin{cases} 1 & X_i \in [0,1] \\ 0 & X_i \notin [0,1] \end{cases}$$
$$E(X_i^2) = \int_0^1 x^2 f_{X_i}(x)dx = \int_0^1 x^2 dx = \frac{1}{3}$$
$$E(X_i^4) = \int_0^1 x^4 f_{X_i}(x)dx = \int_0^1 x^4 dx = \frac{1}{5}$$
$$E(Y) = \frac{1}{n}\sum_{i=1}^{n} E(X_i^2) = \frac{n}{3n} = \frac{1}{3}$$
$$Var(Y)$$
$$= \frac{1}{n^2} Var(\sum_{i=1}^{n} X_i^2)$$
$$= \frac{1}{n^2}[\sum_{i=1}^{n} Var(X_i^2) + 2\sum_{1\le i<j\le n} Cov(X_i^2, X_j^2)]$$
$$= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i^2)$$
$$= \frac{1}{n^2}\sum_{i=1}^{n}[E(X_i^4) - E(X_i^2)^2]$$
$$= \frac{1}{n^2}\sum_{i=1}^{n}\frac{4}{45}$$
$$= \frac{4n}{45n^2} \to 0$$

Then we have $E(Y) = \frac{1}{3}$ and $Var(Y) \to 0$, therefore $\boxed{Y \to \frac{1}{3}}$ as $n \to \infty$.