1. (a) To be proved: T is an isomorphism $\Leftrightarrow T(B)$ is a basis for W.

   $\Rightarrow$: Since T is injective and B is linearly independent, $T(B)$ is also linearly independent. Let $\dim(V) = n$, then B has n elements by definition of dimension. Since T is an isomorphism, $\dim(V) = \dim(W) = n$. Then, $T(B)$ is a set of n linearly independent elements in the n-dimensional space W which means that it is indeed a basis of W.

   $\Leftarrow$: Again let dim(V) $= n$, then both B and $T(B)$ have exactly n elements. Since $T(B)$ is a basis of W, $\dim(W) = n = \text{rank}(T)$ and therefore T is surjective. Then, by rank-nullity theorem, $\text{nullity}(T) = \dim(W) - \text{rank}(T) = 0$ which implies that T is also injective. Thus T is an isomorphism.

   (b) A diagonalisable isomorphism T means that $[T]_\beta$ is a diagonal matrix for some basis β with a nonzero nullity. For example, let $T : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ such that $T(p) = x\frac{dp}{dx}$. In addition, let $\beta = \{1, x, x^2\}$ be an ordered basis of $\mathbb{R}_2[x]$. Then,

$$[T]_\beta^\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

   which we can see is indeed diagonal. Thus, T is diagonalisable. At the same time, T is not injective, and therefore not an isomorphism, as its non-empty kernel contains the set of all constants.

2. Since vectors in V has the form $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, $\beta = \{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}$ is a basis of V. We can then apply T to each element of β to find $[T]_\beta^\beta$:

$$T(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$T(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}$$

Then,

$$[T]_\beta^\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

(a) The characteristic polynomial can be found as follows: $P_T(\lambda) = \det(T - \lambda I) = -\lambda(2-\lambda)(-2-\lambda) = -\lambda^3 + 4\lambda$. Then, we can find the eigenvalues by solving $P_T(\lambda) = 0$, which results in $\lambda = -2, 0, 2$.

(b) We can find the eigenvectors by solving $(T - \lambda I)v = 0$ for each λ.

$\lambda_1 = -2$:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\implies a = b = 0$$

Then $v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector, which corresponds to the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

$\lambda_2 = 0$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\implies b = c = 0$$

Then $v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector, which corresponds to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$\lambda_3 = 2$:

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\implies a = c = 0$$

Then $v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector, which corresponds to the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(c) Since $[T]_\beta^\beta$ is a diagonal matrix as shown above, T is diagonalisable.

3. (a) If 0 is an eigenvalue of T, we know that its corresponding eigenspace $E_0 \neq \{0\}$. By definition of eigenspace, $E_0 = \ker(T - 0I) = \ker(T)$. Therefore $\ker(T) \neq \{0\}$ which means that T is not injective and therefore not an isomorphism.

   (b) If 0 is not an eigenvalue of T, $\det(T - 0I) = \det(\mathrm{T}) \neq 0$. Then, T is full rank, i.e. $\operatorname{rank}(T) = \dim(V)$ and T is surjective. Since T is a map from V to itself, $\operatorname{nullity}(\mathrm{T}) = \dim(V) - \operatorname{rank}(T) = 0$ by rank-nullity theorem and T is injective. Therefore T is indeed an isomorphism.