

1. (a)  $V$  is the space of polynomials of degree up to 3 such that the coefficients of its terms combine to 0; i.e. an arbitrary vector in  $V$  would look like  $p(x) = ax + bx^2 + cx^3 - (a + b + c) = a(x - 1) + b(x^2 - 1) + c(x^3 - 1)$ . Then,  $\beta = \{x - 1, x^2 - 1, x^3 - 1\}$  is a basis of  $V$ .

(b)  $T(x - 1) = x - 1$

$$T(x^2 - 1) = x^2 + 2x - 3 = 2(x - 1) + (x^2 - 1)$$

$$T(x^3 - 1) = x^3 + 6x^2 - 6x - 1 = -6(x - 1) + 6(x^2 - 1) + (x^3 - 1)$$

$$\implies [T]_{\beta} = \begin{pmatrix} 1 & 2 & -6 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\implies P_T(\lambda) = \det(T - \lambda Id) = (1 - \lambda)^3$$

- (c)  $T$  has a single eigenvalue  $\lambda = 1$  with algebraic multiplicity 3.

(d)  $T(v) = 1v$

$$\implies \begin{pmatrix} 1 & 2 & -6 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} a + 2b - 6c \\ b + 6c \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\implies b = c = 0$$

Then  $x - 1$  is an eigenvector for  $\lambda = 1$ .

- (e)  $T$  is not diagonalisable as its algebraic multiplicity and geometric multiplicity ( $\text{nullity}(T - Id) = 1$ ) does not match for  $\lambda = 1$ .

2. (a) Let  $C = \{P, Q, R\}$ . We can construct  $P, Q, R$  using Gram-Schmidt as follows:

$$P = H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Q = K - \frac{\langle K, P \rangle}{\|P\|^2} P = K - \frac{2}{2} P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$R = L - \frac{\langle L, P \rangle}{\|P\|^2} P - \frac{\langle L, Q \rangle}{\|Q\|^2} Q = L - \frac{2}{2} P - \frac{0}{1} Q = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then  $C = \{P, Q, R\}$  is an orthogonal basis for  $V$ .

(b)  $T(P) = EP - PE = -2Q$

$$T(Q) = EQ - QE = 0$$

$$T(R) = ER - RE = P$$

$$\Rightarrow [T]_C = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow [T^*]_C = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow T^*(E) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$$

Then  $T^*(E) = -2P = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$

(c)

$$[T]_C [T^*]_C = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[T^*]_C [T]_C = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since  $T \circ T^* \neq T^* \circ T$ ,  $T$  is not normal.

3. (a) Since  $X^2 = Id$ , we have  $X(X(w)) = w$  for all  $w \in W$ , i.e.  $X$  is invertible and  $X = X^{-1}$ . To show that  $W'$  is also an  $X$ -subspace, it suffices to show that  $\langle X(u), w \rangle + \langle X(X(u)), X(w) \rangle = 0$  for  $u \in W'$ . Since  $X$  is invertible, we can choose  $X(w) \in W$  to represent all vectors in  $W$ . Then verifying the following equivalently shows that  $W'$  is an  $X$ -subspace:  $\langle X(u), X(w) \rangle + \langle X(X(u)), X(X(w)) \rangle = 0$ . We can simplify the previous expression as follows:

$$\begin{aligned}
& \langle X(u), X(w) \rangle + \langle X(X(u)), X(X(w)) \rangle \\
&= \langle X(u), X(w) \rangle + \langle X^2(u), X^2(w) \rangle \\
&= \langle X(u), X(w) \rangle + \langle u, w \rangle \\
&= \langle u, w \rangle + \langle X(u), X(w) \rangle \\
&= 0
\end{aligned}$$

Therefore  $X(u) \in W'$  and  $W'$  is an  $X$ -subspace.

- (b) Let  $u, v, w \in V$  and  $a \in \mathbb{F}$ . Then we can verify the axioms of inner product as follows:

$$\begin{aligned}
\text{(i)} \quad & \langle u, v \rangle_X \\
&= \langle u, v \rangle + \langle X(u), X(v) \rangle \\
&= \overline{\langle v, u \rangle} + \overline{\langle X(u), X(v) \rangle} \\
&= \overline{\langle v, u \rangle}_X
\end{aligned}$$

$$\begin{aligned}
\text{(ii)} \quad & \langle au, v \rangle_X \\
&= \langle au, v \rangle + \langle X(au), X(v) \rangle \\
&= a\langle u, v \rangle + a\langle X(u), X(v) \rangle \\
&= a(\langle u, v \rangle + \langle X(u), X(v) \rangle) \\
&= a\langle u, v \rangle_X
\end{aligned}$$

$$\begin{aligned}
\text{(iii)} \quad & \langle u + v, w \rangle_X \\
&= \langle u + v, w \rangle + \langle X(u + v), X(w) \rangle \\
&= (\langle u, w \rangle + \langle v, w \rangle) + (\langle X(u), X(w) \rangle + \langle X(v), X(w) \rangle) \\
&= (\langle u, w \rangle + \langle X(u), X(w) \rangle) + (\langle v, w \rangle + \langle X(v), X(w) \rangle) \\
&= \langle u, w \rangle_X + \langle v, w \rangle_X
\end{aligned}$$

- (iv)  $\langle v, v \rangle_X = \langle v, v \rangle + \langle X(v), X(v) \rangle$  by definition. If  $v \neq 0$ , then  $\langle v, v \rangle > 0$ . In addition,  $\langle X(v), X(v) \rangle \geq 0$  even if  $X(v) = 0$ . Then,  $\langle v, v \rangle_X > 0$  for nonzero  $v \in V$ .

- (c) By definition of  $W'$ ,  $\langle u, w \rangle + \langle X(u), X(w) \rangle = 0$  for  $u \in W'$  and  $w \in W$ , i.e.  $\langle u, w \rangle_X = 0$ . Then  $W' = W^\perp$ . Since  $W \oplus W^\perp = V$ , we also have  $W \oplus W' = V$ .

$$\begin{aligned}
\text{(d)} \quad & \langle X(u), w \rangle_X = \langle u, X^*(w) \rangle_X \\
&\implies \langle X(u), w \rangle + \langle X^2(u), X(w) \rangle = \langle u, X^*(w) \rangle + \langle X(u), X(X^*(w)) \rangle \\
&\implies \langle X(u), w \rangle + \langle u, X(w) \rangle = \langle u, X^*(w) \rangle + \langle X(u), (X \circ X)^*(w) \rangle \\
&\implies \langle X(u), w \rangle + \langle u, X(w) \rangle = \langle u, X^*(w) \rangle + \langle X(u), w \rangle \\
&\implies \langle u, X(w) \rangle = \langle u, X^*(w) \rangle \\
&\implies X(w) = X^*(w)
\end{aligned}$$

Therefore  $X^* = X$ , i.e.  $X$  is self-adjoint as shown above.

4. (a) To be proved:  $T$  is an isomorphism  $\Leftrightarrow T(B)$  is a basis for  $W$ :

$\Rightarrow$ : Let  $B = \{u_0, u_1, \dots\}, u_i \in V$ . Since  $B$  is linearly independent,  $a_0u_0 + a_1u_1 + \dots = 0_V, a_i \in \mathbb{F}$  is only true when  $a_i = 0$ . Applying  $T$  to both sides, we have  $T(a_0u_0 + a_1u_1 + \dots) = T(0) \Rightarrow a_0T(u_0) + a_1T(u_1) + \dots = 0_W$ . Since  $a_i = 0$ ,  $\{T(u_0), T(u_1), \dots\}$  is a linearly independent set in  $W$ . Since  $T$  is surjective,  $\text{im}(T) = \text{span}(T(B)) = W$ . Then  $T(B)$  is both linearly independent and spanning in  $W$  and is therefore a basis.

$\Leftarrow$ : Let  $B = \{u_0, u_1, \dots\}, u_i \in V$  and  $T(B) = \{v_1, v_2, \dots\}, v_i \in W$  such that  $T(u_i) = v_i$ . Take an arbitrary  $p \in V$ ,  $p$  can be written as  $p = a_0u_0 + a_1u_1 + \dots, a_i \in \mathbb{F}$  since  $B$  is a basis. Then  $T(p) = a_0T(u_0) + a_1T(u_1) + \dots = a_0v_0 + a_1v_1 + \dots$  by linearity. Since  $T(B)$  is a basis for  $W$ ,  $\text{span}(T(B)) = \text{im}(T) = W$ . Therefore  $T$  is surjective. Additionally,  $0_W = a_0v_0 + a_1v_1 + \dots$  only if  $a_i = 0$ . Then  $T(p) = 0_W \Rightarrow T(p) = a_0v_0 + a_1v_1 + \dots, a_i = 0$ . By definition of  $T(B)$  we can then replace  $v_i$  with  $T(u_i)$  as follows:  $T(p) = a_0T(u_0) + a_1T(u_1) + \dots = T(a_0u_0 + a_1u_1 + \dots), a_i = 0$ . Then  $p = a_0u_0 + a_1u_1, a_i = 0$ , i.e.  $p = 0_V$ . Therefore  $\ker(T) = \{0\}$  and  $T$  is injective. Since  $T$  is both injective and surjective, it is an isomorphism.

(b) Let  $V = \{(a_0, a_1, \dots) \mid a_i \in \mathbb{R}\}$ , i.e. the vector space of infinite sequences. Define  $R : V \rightarrow V$  such that  $R(a_0, a_1, \dots) = (0, a_0, a_1, \dots)$ . Then  $R$  is injective as  $\ker(R) = \{0\}$ . However,  $R$  is not surjective as sequences with the form  $(k, a_0, a_1, \dots), k \in \mathbb{F}$  is not in  $\text{im}(R)$ . Therefore  $R$  is not an isomorphism.

(c) Again let  $V = \{(a_0, a_1, \dots) \mid a_i \in \mathbb{R}\}$ . Define  $S : V \rightarrow V$  such that  $S(a_0, a_1, \dots) = (a_1, a_2, \dots)$ . Then  $S$  is surjective as  $(a_1, a_2, \dots)$  spans  $V$ . However,  $\ker(S)$  contains sequences with form  $(a_0, 0, 0, \dots)$ . Therefore  $S$  is not injective and not an isomorphism.

5. (a) Let  $\lambda$  be an eigenvalue of  $T$ , i.e.  $T(v) = \lambda v$  for some  $v \in V$ . Then,  $T^2(v) = T(T(v)) = T(\lambda v) = \lambda T(v) = \lambda^2 v$ . By definition of *projpotent*,  $T(v) = T^2(v)$ . Then  $\lambda v = \lambda^2 v \implies \lambda = \lambda^2$ . Therefore 0 and 1 are the only possible eigenvalues.

(b)

- (c) Let  $n = \dim V$ . Since  $T$  is diagonalisable, there exists an eigenbasis  $\beta = \{u_1, \dots, u_n\}$  for  $T$  such that  $[T]_\beta^\beta$  is diagonal with corresponding eigenvalues on the main diagonal. As shown in (a),  $T$  can only have eigenvalues of 0 or 1. Now let  $k$  be the number of zero entries on the main diagonal, which corresponds to eigenvectors  $u_1, \dots, u_k$  upon renumbering. Then for each  $u_i, 1 \leq i \leq k$ ,  $T(u_i) = 0u_i = 0$ , i.e.  $\text{nullity}(T) = k$  and  $\text{rank}(T) = n - k$ . In addition, since  $[T]$  is  $n \times n$  diagonal matrix, the number of nonzero entries on its diagonal is  $n - k$ . Again as shown in (a), the only nonzero eigenvalue  $T$  can have is 1. Then  $\text{tr}(T) = 1(n - k) = n - k = \text{rank}(T)$ .