1. (a) V is the space of polynomials of degree up to 3 such that the coefficients of its terms combine to 0; i.e. an arbitrary vector in V would look like $p(x) = ax + bx^2 + cx^3 - (a+b+c) = a(x-1) + b(x^2-1) + c(x^3-1)$. Then, $\beta = \{x-1, x^2-1, x^3-1\}$ is a basis of V.

(b) $T(x-1) = x-1$
$T(x^2-1) = x^2+2x-3 = 2(x-1) + (x^2-1)$
$T(x^3-1) = x^3+6x^2-6x-1 = -6(x-1) + 6(x^2-1) + (x^3-1)$

$$\implies [T]_\beta = \begin{pmatrix} 1 & 2 & -6 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\implies P_T(\lambda) = \det(T - \lambda I) = (1-\lambda)^3$$

(c) T has a single eigenvalue $\lambda = 1$ with algebraic multiplicity 3.

(d) $T(v) = 1v$

$$\implies \begin{pmatrix} 1 & 2 & -6 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} a+2b-6c \\ b+6c \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\implies b = c = 0$$

Then $x-1$ is an eigenvector for $\lambda = 1$.

(e) T is not diagonalisable as its algebraic multiplicity and geometric multiplicity ($\text{nullity}(T - Id) = 1$) does not match for $\lambda = 1$.

2. (a) Let $C = \{P, Q, R\}$. Then,

$$P = H$$

$$Q = K - \frac{\langle K, P\rangle}{||P||^2} P = K - \frac{2}{2} P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$R = L - \frac{\langle L, P\rangle}{||P||^2} P - \frac{\langle L, Q\rangle}{||Q||^2} Q = L - \frac{2}{2} P - \frac{1}{0} Q$$

(b) $\langle T(Id), E\rangle = \langle Id, T^*(E)\rangle$

(c)

3. (a) To show that W' is also an X-subspace, it suffices to show that $\langle X(u), w\rangle + \langle X(X(u)), X(w)\rangle = 0$ for $u \in W'$. We can simplify the above expression as follows:

   $\langle X(u), w\rangle + \langle X(X(u)), X(w)\rangle$
   $= \langle X(u), w\rangle + \langle X^2(u), X(w)\rangle$
   $= \langle X(u), w\rangle + \langle u, X(w)\rangle$

   (b) Let $u, v, w \in V$ and $a \in \mathbb{F}$. Then we can verify the definiton of inner product as follows:

   (i) $\langle u, v\rangle_X$
   $= \langle u, v\rangle + \langle X(u), X(v)\rangle$
   $= \overline{\langle v, u\rangle} + \overline{\langle X(u), X(v)\rangle}$
   $= \overline{\langle v, u\rangle}_X$

   (ii) $\langle au, v\rangle_X$
   $= \langle au, v\rangle + \langle X(au), X(v)\rangle$
   $= a\langle u, v\rangle + a\langle X(u), X(v)\rangle$
   $= a(\langle u, v\rangle + \langle X(u), X(v)\rangle)$
   $= a\langle u, v\rangle_X$

   (iii) $\langle u+v, w\rangle_X$
   $= \langle u+v, w\rangle + \langle X(u+v), X(w)\rangle$
   $= (\langle u, w\rangle + \langle v, w\rangle) + (\langle X(u), X(w)\rangle + \langle X(v), X(w)\rangle)$
   $= (\langle u, w\rangle + \langle X(u), X(w)\rangle) + (\langle v, w\rangle + \langle X(v), X(w)\rangle)$
   $= \langle u, w\rangle_X + \langle v, w\rangle_X$

   (iv) $\langle v, v\rangle_X = \langle v, v\rangle + \langle X(v), X(v)\rangle$ by definiton. If $v \neq 0$, then $\langle v, v\rangle > 0$. In addition, $\langle X(v), X(v)\rangle \geq 0$ even if $X(v) = 0$. Then, $\langle v, v\rangle_X > 0$ for nonzero $v \in V$.

   (c) To be proved

   (d)

4. (a) To be proved: T is an isomorphism $\Leftrightarrow T(B)$ is a basis for W:

$\Rightarrow$:

$\Leftarrow$:

(b)

(c)

5. (a) Let λ be an eigenvalue of T, i.e. $T(v) = \lambda v$ for some $v \in V$. Then, $T^2(v) = T(T(v)) = T(\lambda v) = \lambda T(v) = \lambda^2 v$. By definition of *projpotent*, $T(v) = T^2(v)$. Then $\lambda v = \lambda^2 v \implies \lambda = \lambda^2$. Therefore 0 and 1 are the only possible eigenvalues.

   (b)

   (c) Let $n = \dim V$. Since T is diagonalisable, there exists an eigenbasis $\beta = \{u_1, \ldots, u_n\}$ for T such that $[T]_\beta^\beta$ is diagonal with corresponding eigenvalues on the main diagonal. As shown in (a), T can only have eigenvalues of 0 or 1. Now let k be the number of zero entries on the main diagonal, which corresponds to eigenvectors $u_1, \ldots, u_k$ upon renumbering. Then for each $u_i, 1 \geq i \geq k$, $T(u_i) = 0u_i = 0$, i.e. nullity$(T) = k$ and rank$(T) = n - k$. In addition, since $[T]$ is $n \times n$ diagonal matrix, the number of nonzero entries on its diagonal is $n - k$. Again as shown in (a), the only nonzero eigenvalue T can have is 1. Then $\text{tr}(T) = 1(n-k) = n - k = \text{rank}(T)$.