Stats 100A Homework 3

Jiaping Zeng

5/18/2020

Problem 1

(a) $E(Y) = E(aX + b) = \sum_x (ax + b)p(x) = a\sum_x xp(x) + b\sum_x p(x) = aE(X) + b$
$Var(Y) = E[(Y - E(Y))^2] = E[((aX + b) - E(aX + b))^2] = E[(aX + b - aE(X) - b))^2] = E[(a(X - E(X)))^2] = a^2 E[(X - E(X))^2] = a^2 Var(X)$

(b) $P(X \in (x, x + \Delta x)) = P(Y \in (y, y + \Delta y)) \implies f(x)\Delta x = g(y)\Delta y \implies g(y) = f(x)\frac{\Delta x}{\Delta y} = \boxed{\dfrac{f(\frac{y-b}{a})}{a}}$

Problem 2

(a) $f(u) = \begin{cases} 1 & u \in [0, 1] \\ 0 & u \notin [0, 1] \end{cases}$

$E(U) = \int_0^1 uf(u)du = \boxed{\dfrac{1}{2}}$

$Var(U) = E(U^2) - E(U)^2 = \int_0^1 u^2 f(u)du - (\int_0^1 uf(u)du)^2 = \boxed{\dfrac{1}{12}}$

(b) $F(x) = P((b - a)U + a \le x) = P(U \le \frac{x-a}{b-a}) = \int_0^{\frac{x-a}{b-a}} 1du = \frac{x-a}{b-a}$

$g(x) = \frac{d}{dx}(F(x)) = \begin{cases} \frac{1}{b-a} & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$

$E(X) = E((b - a)U + a) = (b - a)E(U) + a = \boxed{\dfrac{a + b}{2}}$

$Var(X) = Var((b - a)U + a) = (b - a)^2 Var(U) = \boxed{\dfrac{(b - a)^2}{12}}$

Problem 3

(a) $E(T) = \int_0^\infty tf(t)dt = \boxed{1}$
$Var(T) = E(T^2) - E(T)^2 = \int_0^\infty t^2 f(t)dt - (\int_0^\infty tf(t)dt)^2 = \boxed{1}$

(b) $F(x) = P(\frac{T}{\lambda} \le x) = P(T \le \lambda x) = \int_0^{\lambda x} e^{-t}dt = 1 - e^{-\lambda x}$

$g(x) = \frac{d}{dx}(F(x)) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \le 0 \end{cases}$

$E(X) = E(\frac{T}{\lambda}) = \frac{1}{\lambda}E(T) = \boxed{\frac{1}{\lambda}}$

$Var(X) = Var(\frac{T}{\lambda}) = (\frac{1}{\lambda})^2 Var(T) = \boxed{\frac{1}{\lambda^2}}$

Problem 4

(a) $E(Z) = \int_{-\infty}^{\infty} zf(z)dz = \boxed{0}$

$E(|Z|) = 2\int_0^{\infty} zf(z)dz = \boxed{\sqrt{\frac{2}{\pi}}}$

$Var(Z) = E(Z^2) - E(Z)^2 = \int_{-\infty}^{\infty} z^2 f(z)dz - (\int_{-\infty}^{\infty} zf(z)dz)^2 = \boxed{1}$

(b) $F(x) = P(\mu + \sigma Z \leq x) = P(Z \leq \frac{x-\mu}{\sigma}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$g(x) = \frac{d}{dx}(F(x)) = \boxed{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$

$E(X) = E(\mu + \sigma Z) = \mu + \sigma E(Z) = \boxed{\mu}$

$Var(X) = Var(\mu + \sigma Z) = \sigma^2 Var(Z) = \boxed{\sigma^2}$

(c) $P(X \in [\mu - 2\sigma, \mu + 2\sigma]) = P(Z \in [-2, 2]) = \boxed{95\%}$

Problem 5

(a) $P(T \in (t, t + \Delta t)) = (1 - \lambda\Delta t)^{\frac{t}{\Delta t}} \lambda\Delta t = \boxed{e^{-\lambda t}\Delta t}$

Let $T = \tilde{T}\Delta t$ where $\tilde{T} \sim$ Geometric($p = \lambda\Delta t$), then $E(T) = E(\tilde{T})\Delta t = \frac{1}{p}\Delta t = \frac{1}{\lambda\Delta t}\Delta t = \boxed{\frac{1}{\lambda}}$

(b) $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$

$E(X) = np = \frac{t}{\Delta t}\lambda\Delta t = \boxed{\lambda t}$

Problem 6

(a) $E(X_t) = \sum_{i=1}^n E(Z_i)\Delta x = \boxed{0}$

$Var(X_t) = \sum_{i=1}^n Var(Z_i)\Delta x^2 = n\Delta x^2 = \boxed{\sigma^2 t}$

(b) $Var(X_t) = \sigma^2 t = n\Delta x^2 \implies \boxed{\Delta x = \sigma\sqrt{\Delta t}}$

(c) The distribution of X_t is $\boxed{N(0, \sigma^2 t)}$ according to the central limit theorem.

Problem 7

(a) $E(X) = np = 50 = \mu$

$Var(x) = np(1-p)$

$SD(X) = \sqrt{Var(X)} = 5 = \sigma$

$Z = qnorm(0.95) = 1.96$

$X = \mu \pm Z\sigma = 50 \pm 9.8 \implies \boxed{X \in [40.2, 59.8]}$

(b) $E(\hat{p}) = E(\frac{X}{100}) = \frac{np}{100} = 0.2 = \mu$

$Var(\hat{p}) = Var(\frac{X}{100}) = \frac{np(1-p)}{100^2} = 0.0016$

$SD(\hat{p}) = \sqrt{Var(X)} = 0.04 = \sigma$

$Z = qnorm(0.95) = 1.96$

$\hat{p} = \mu \pm Z\sigma = 0.2 \pm 0.0784 \implies \boxed{\hat{p} \in [0.1216, 0.2784]}$

(c) $E(\hat{\pi}) = E(\frac{4m}{10000}) = \pi = \mu$

$Var(\hat{\pi}) = Var(\frac{4m}{10000}) = \frac{4\pi - \pi^2}{10000}$

$SD(\hat{\pi}) = \sqrt{Var(\hat{\pi})} = \sqrt{\frac{4\pi - \pi^2}{10000}} = \sigma$

$Z = qnorm(0.95) = 1.96$

$\hat{\pi} = \mu \pm Z\sigma = 50 \pm 9.8 \implies \boxed{\hat{\pi} \in [3.109406, 3.173779]}$