Stats 100A Homework 1

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Problem 1

1. $P(A) = \frac{50}{100} = \frac{1}{2}$
   $P(B) = \frac{40}{100} = \frac{2}{5}$
   $P(A|B) = \frac{30}{40} = \frac{3}{4}$
   $P(B|A) = \frac{30}{50} = \frac{3}{5}$
   Since $P(A|B) \neq P(A)$ and $P(B|A) \neq P(B)$, A and B are not independent events.

2. $P(A \cap B) = P(A)P(B|A) = \frac{1}{2} * \frac{3}{5} = \frac{3}{10}$
   $P(A \cap B) = P(B)P(A|B) = \frac{2}{5} * \frac{3}{4} = \frac{3}{10}$
   Thus $P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$. The chain rule indeed stands.

3. $P(A)P(B|A) = \frac{2}{5} * \frac{3}{4} = \frac{3}{10}$
   $P(A^c)P(B|A^c) = \frac{50}{100} * \frac{10}{50} = \frac{1}{10}$
   Then $P(A)P(B|A) + P(A^c)P(B|A^c) = \frac{3}{10} + \frac{1}{10} = \frac{2}{5}$ which is indeed equal to $P(B)$.

4. $\dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{3}{10}}{\frac{2}{5}} = \dfrac{3}{4} = P(A|B)$
   $\dfrac{P(A)P(B|A)}{P(A)P(B|A) + P(A^c)P(B|A^c)} = \dfrac{\frac{1}{2} * \frac{3}{5}}{\frac{1}{2} * \frac{3}{5} + \frac{1}{2} * \frac{1}{5}} = \dfrac{3}{4} = P(A|B)$
   Thus Bayes rule does stand.

5. Chain rule: The proportion of tall males in the population is calculated by the proportion of males in the population times the proportion of tall people in the male population. Alternatively, it can also be calculated by the proportoin of tall people times the proportion of males in the population of tall people.
   Rule of total probability: The proportion of tall people in the population is equivalent to: the product of the proportion of males in the population and the proportion of tall people in the male population, added to the product of the proportion of females in the population and the proportion of females in the tall population.
   Bayes rule: The proportion of males in the tall population is the same as the proportion of tall males in the population divided by the proportion of tall people in the population. Another interpretation can be formed by expanding the numerator and the denominator using chain rule and rule of total probability respectively with their interpretation.

6. Chain rule: $P(X = male \text{ and } Y > 6) = P(X = male)P(Y > 6 \text{ given } X = male) = P(Y > 6)P(X = male \text{ given } Y > 6)$

   Rule of total probability: $P(Y > 6) = P(X = male)P(Y > 6 \text{ given } X = male) + P(X! = male)P(Y > 6 \text{ given } X! = male)$

   Bayes rule: $P(X = male \text{ given } Y > 6) = \dfrac{P(X = male \text{ and } Y > 6)}{P(Y > 6)}$

   $= \dfrac{P(X = male)P(Y > 6 \text{ given } X = male)}{P(X = male)P(Y > 6 \text{ given } X = male) + P(X! = male)P(Y > 6 \text{ given } X! = male}$

Problem 2

1. Since $X^2 + Y^2 \leq 1$ is a filled circle centered at origin with radius 1, $P(X^2 + Y^2 \leq 1)$ would be the quarter circle in the first quardrant inside a square with its bottom-left corner at the origin and side length 1. Thus $P(X^2 + Y^2 \leq 1) = \dfrac{\frac{1}{4}\pi * 1^2}{1^2} = \dfrac{\pi}{4}$.

2. As shown above, the probability of the random point landing in the quarter circle is $\frac{\pi}{4}$. Therefore, using Law of large numbers, $m \approx \frac{\pi}{4}n$ for sufficiently large n. We can rearrange the equation to find π: $\pi \approx \frac{4m}{n}$

3. $X \geq \frac{1}{2}$ is the right half of our square and $X + Y \geq 1$ is the upper-right triangle of the square. Then, $P(X \geq \frac{1}{2}) = \frac{1}{2}$. $P(X \geq \frac{1}{2} \mid X + Y \geq 1)$ would be the intersection of the two areas, which by visualization is $\frac{3}{8}$ of the area of the entire square.

Problem 3

1. Since each flip has exactly 2 outcomes, the sample space would consist of $2^5 = 32$ possibilities. Let H represent a flip that lands on head and T represent one that lands on tail, the sample space would look like the following: $\{HHHHH, HHHHT, ..., HTTTT, TTTTT\}$.

2. The desired outcome is 2 heads out of 5 flips, thus it is simply $\dbinom{5}{2} = 10$ sequences.

3. $P(X = 0) = \dfrac{\binom{5}{0}}{2^5} = \dfrac{1}{32}$

   $P(X = 1) = \dfrac{\binom{5}{1}}{2^5} = \dfrac{5}{32}$

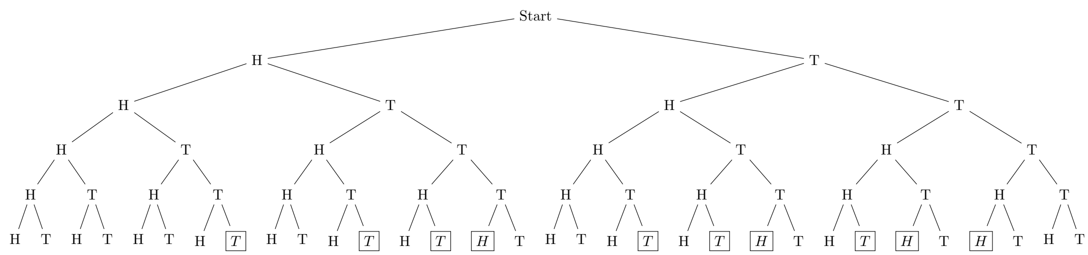
   $P(X = 2) = \dfrac{\binom{5}{2}}{2^5} = \dfrac{10}{32}$

   $P(X = 3) = \dfrac{\binom{5}{3}}{2^5} = \dfrac{10}{32}$

   $P(X = 4) = \dfrac{\binom{5}{4}}{2^5} = \dfrac{5}{32}$

   $P(X = 5) = \dfrac{\binom{5}{5}}{2^5} = \dfrac{1}{32}$

4. The problem can be visualized using the diagram below, with each level representing each flip in sequence. The boxed terminal nodes represent sequences with exactly 2 heads:



As shown above, there is a total of 32 possible combinations with 10 of them containing exactly 2 heads.

Problem 4

Let the below diagram represent a Galton board. Each * represents a pin and each U represents a bin.

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          *
        *   *
      *   *   *
    *   *   *   *
  *   *   *   *   *
U   U   U   U   U   U
```

1. Below is the corresponding Pascal triangle. Each number represents the number of possible paths that goes from the root to the position.

$n = 0$						1					
$n = 1$					1		1				
$n = 2$				1		2		1			
$n = 3$			1		3		3		1		
$n = 4$		1		4		6		4		1	
$n = 5$	1		5		10		10		5		1

2. $P(\text{Bin}_0) = \frac{\binom{5}{0}}{2^5} = \frac{1}{32}$
$P(\text{Bin}_1) = \frac{\binom{5}{1}}{2^5} = \frac{5}{32}$
$P(\text{Bin}_2) = \frac{\binom{5}{2}}{2^5} = \frac{10}{32}$
$P(\text{Bin}_3) = \frac{\binom{5}{3}}{2^5} = \frac{10}{32}$
$P(\text{Bin}_4) = \frac{\binom{5}{4}}{2^5} = \frac{5}{32}$
$P(\text{Bin}_5) = \frac{\binom{5}{5}}{2^5} = \frac{1}{32}$

3. We can obtain the proportions of balls in each bin by multiplying the theoretical probabilities found above by the total of 1 million balls:
Bin 0: $\frac{1}{32} * 10^6 = 31250$ balls
Bin 1: $\frac{5}{32} * 10^6 = 156250$ balls

Bin 2: $\frac{10}{32} * 10^6 = 312500$ balls
Bin 3: $\frac{10}{32} * 10^6 = 312500$ balls
Bin 4: $\frac{5}{32} * 10^6 = 156250$ balls
Bin 5: $\frac{1}{32} * 10^6 = 31250$ balls

Problem 5

1. Using the recursive definition $X_{t+1} = X_t + \epsilon_t$, we can expand X_5 into $X_5 = X_0 + \epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4$ where ϵ_t is either -1 or 1. We can find the minimum and maximum values of X_5 easily by taking all ϵ_t to be -1 and 1, respectively, resulting in a minimum of -5 and a maximum of 5. Additionally, since we are walking an odd number of steps, it is not possible for X_5 to be even. Therefore, the set of possible values are all odd values between -5 and 5, i.e. $X_5 \in \{-5, -3, -1, 1, 3, 5\}$.

2. We can consider each ϵ_t as an event with exactly two possible outcomes. Then, the probability of each value would be as follows:
$P(X = -5) = \frac{\binom{5}{0}}{2^5} = \frac{1}{32}$
$P(X = -3) = \frac{\binom{5}{1}}{2^5} = \frac{5}{32}$
$P(X = -1) = \frac{\binom{5}{2}}{2^5} = \frac{10}{32}$
$P(X = 1) = \frac{\binom{5}{3}}{2^5} = \frac{10}{32}$
$P(X = 3) = \frac{\binom{5}{4}}{2^5} = \frac{5}{32}$
$P(X = 5) = \frac{\binom{5}{5}}{2^5} = \frac{1}{32}$

3. Since $|X_{t+1} - X_t| = |\epsilon_t| = 1$, X_{t+1} and X_t can only be one unit away from each other. Since we have no restraints on the values of i and j, one way to represent $P(X_{t+1} = j \mid X_t = i)$ is as the piecewise function below:
$$P(X_{t+1} = j \mid X_t = i) = \begin{cases} \frac{1}{2} & |j - i| = 1 \\ 0 & |j - 1| \neq 1 \end{cases}$$

4. Assuming the 1 million people are practicing proper social distancing and keeping 6 feet distances between each other, the number of people landing in each of positions found in part 1 would be as follows:
Position -5: $\frac{1}{32} * 10^6 = 31250$ people
Position -3: $\frac{5}{32} * 10^6 = 156250$ people
Position -1: $\frac{10}{32} * 10^6 = 312500$ people
Position 1: $\frac{10}{32} * 10^6 = 312500$ people
Position 3: $\frac{5}{32} * 10^6 = 156250$ people
Position 5: $\frac{1}{32} * 10^6 = 31250$ people

Problem 6

1. $P(X_1 = 1) = \frac{1}{3}$
   $P(X_1 = 2) = \frac{2}{3}$
   $P(X_2 = 1) = P(X_1 = 1)P(X_2 = 1|X_1 = 1) + P(X_1 = 2)P(X_2 = 1|X_1 = 2) = \frac{1}{3} * \frac{1}{3} + \frac{2}{3} * \frac{2}{3} = \frac{5}{9}$
   $P(X_2 = 2) = P(X_1 = 1)P(X_2 = 2|X_1 = 1) + P(X_1 = 2)P(X_2 = 2|X_1 = 2) = \frac{1}{3} * \frac{2}{3} + \frac{2}{3} * \frac{1}{3} = \frac{4}{9}$
   $P(X_3 = 1) = P(X_2 = 1)P(X_3 = 1|X_2 = 1) + P(X_2 = 2)P(X_3 = 1|X_2 = 2) = \frac{5}{9} * \frac{1}{3} + \frac{4}{9} * \frac{2}{3} = \frac{13}{27}$
   $P(X_3 = 2) = P(X_2 = 1)P(X_3 = 2|X_2 = 1) + P(X_2 = 2)P(X_3 = 2|X_2 = 2) = \frac{5}{9} * \frac{2}{3} + \frac{4}{9} * \frac{1}{3} = \frac{14}{27}$
   $P(X_4 = 1) = P(X_3 = 1)P(X_4 = 1|X_3 = 1) + P(X_3 = 2)P(X_4 = 1|X_3 = 2) = \frac{13}{27} * \frac{1}{3} + \frac{14}{27} * \frac{2}{3} = \frac{41}{81}$
   $P(X_4 = 2) = P(X_3 = 1)P(X_4 = 2|X_3 = 1) + P(X_3 = 2)P(X_4 = 2|X_3 = 2) = \frac{13}{2} * \frac{2}{3} + \frac{14}{27} * \frac{1}{3} = \frac{40}{81}$

2. $K = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

3. (a) Prove $p^{(t+1)} = p^{(t)}K$: By direct proof.
   To find $p^{(t+1)} = (P(X_{t+1} = 1), P(X_{t+1} = 2))$, we can calculate each component of the tuple separately. For case $X_{t+1} = 1$ (i.e. the probability of the person landing on state 1 at time $t+1$), we can calcualte $P(X_{t+1})$ using $P(X_t)$. There are two possible scenarios in this case: the person was on state 1 at time t and decided to stay ($P = \frac{1}{3}$), or the person was on state 2 at time t and decided to move ($P = \frac{2}{3}$). Then,

   $$P(X_{t+1} = 1) = \frac{1}{3}P(X_t = 1) + \frac{2}{3}P(X_t = 2)$$

   Similarly,
   $$P(X_{t+1} = 2) = \frac{2}{3}P(X_t = 1) + \frac{1}{3}P(X_t = 2)$$

   Recall that, by definition, $p^{(t)} = (P(X_t = 1), P(X_t) = 2)$. Then the two equations above can be rewritten as the follows:
   $$P(X_{t+1} = 1) = \left(\frac{1}{3}, \frac{2}{3}\right)^T p^{(t)}$$
   $$P(X_{t+1} = 2) = \left(\frac{2}{3}, \frac{1}{3}\right)^T p^{(t)}$$

   We can see that the two coefficient vectors form the matrix K. Then, by combining the two equations back into the form $p^{(t+1)} = (P(X_{t+1} = 1), P(X_{t+1} = 2))$, we get $p^{(t+1)} = p^{(t)}K$.

   (b) Prove $p^{(t)} = p^{(0)}K^t$: By induction.
   Base case: $t = 0$. Since K^0 is simply the identity matrix, $p^{(0)} = p^{(0)}K^0$ is trivially true.
   Inductive step: Assume $p^{(t)} = p^{(0)}K^t$. Then, to find $p^{(t+1)}$, we can simply use the formula proved in the previous part: $p^{(t+1)} = p^{(t)}K$. By substitution, $p^{(t+1)} = (p^{(0)}K^t)K \implies p^{(t+1)} = p^{(0)}(K^tK) \implies p^{(t+1)} = p^{(0)}K^{t+1})$. Therefore, $p^{(t)} = p^{(0)}K^t$ is true by induction.

4. The number of people in each state can be found by multiplying the probabilities found in part 1 by the size of the population (i.e. one million).
   State 1 at time 1: $\frac{1}{3} * 10^6 \approx 333333$ people

State 2 at time 1: $\frac{2}{3} * 10^6 \approx 666667$ people
State 1 at time 2: $\frac{5}{9} * 10^6 \approx 555556$ people
State 2 at time 2: $\frac{4}{9} * 10^6 \approx 444444$ people
State 1 at time 3: $\frac{13}{27} * 10^6 \approx 481481$ people
State 2 at time 3: $\frac{14}{27} * 10^6 \approx 518519$ people
State 1 at time 4: $\frac{41}{81} * 10^6 \approx 506173$ people
State 2 at time 4: $\frac{40}{81} * 10^6 \approx 493827$