1. (a) $V = \mathrm{Mat}_{2x2}(\mathbb{C}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{C}\}$

   (b) $B = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$

   (c) $W = \{\begin{pmatrix} a & a \\ b & b \end{pmatrix} \mid a, b \in \mathbb{C}\}$ with basis $\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\}$

   (d) $C = \{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$

2. Let $p(x) = a_0 + a_1x + ... + a_nx^n$, then $p(1) = 2p(0)$ implies that $a_0 + ... + a_n = 2a_0$. Subtract a_0 from both sides and we have $a_0 = a_1 + ... + a_n$.

   (a) Per Theorem 1.3, we need to verify the following three conditions:

      (1) $0 \in U_n$: The zero in $\mathbb{R}_n[x]$ is $p_0(x) = 0$. By evaluating $p_0(x) = 0$ at 1 and 0, we can easily see that $p_0(1) = 2p_0(0) = 0$. Therefore, $0 \in U_n$.

      (2) $x + y \in U_n$ for $x, y \in U_n$: Let $u, v \in U_n$ such that $u = a_0 + a_1x + ... + a_nx^n$ and $v = b_0 + b_1x + ... + b_nx^n$. Then $u + v = (a_0 + b_0) + (a_1 + b_1)x + ... + (a_n + b_n)x^n$. As shown above, $a_0 = a_1 + ... + a_n$ and $b_0 = b_1 + ... + b_n$. Therefore, $a_0 + b_0 = (a_1 + b_1) + ...(a_n + b_n)$, which means that $u + v \in U_n$.

      (3) $cx \in W$ for $c \in \mathbb{F}, x \in U_n$: Using u defined in the previous part, $cu = ca_0 + ca_1x + ... + ca_nx^n$. We can verify $cu \in U_n$ by checking that $ca_0 = ca_1 + ... + ca_n$, which is indeed true upon multiplying both sides by c^{-1}. Therefore, $cu \in U_n$.

   (b) To verify that B is indeed a basis for U_n, we need to show that B is both generating and linearly independent. By definition, $B = \{1 + x^a \mid 1 \leq a \leq n\} = \{1 + x, ..., 1 + x^n\}$. We can see that the vectors in B are indeed linearly independent by comparing coefficients. To show that B is generating, take a generic $u \in U_n$ such that $u = a_0 + a_1x + ... + a_nx^n$. By definition, we also have $a_0 = a_1 + ... + a_n$. Then, we can write u as a linear combination of the vectors in B as follows: $u = a_1(1 + x) + ... + a_n(1 + x^n) = (a_1 + ... + a_n) + a_1x + ... + a_nx^n = a_0 + a_1x + ... + a_nx^n$. Therefore, B is a basis for U_n.

   (c) To show that $T(cu + v) = cT(u) + T(v)$, take $u, v \in U_n$ such that $u = a_0 + a_1x + ... + a_nx^n$ and $v = b_0 + b_1x + ... + b_nx^n$. Then, $T(cu + v) = (ca_0 + b_0) + (ca_1 + b_0)x + ... + (ca_n + b_n)x^n$, which is indeed equal to $cT(u) + T(v) = c(a_0 + a_1x + ... + a_nx^n) + (b_0 + b_1x + ... + b_nx^n)$.

3. To prove that $\#B = \dim V \Leftrightarrow U = V$, we need to prove both directions:

   $\Rightarrow$: If $\#B = \dim V$, then $\#B = \dim V = \dim U$ since B is a basis of U. In addition, B is a linearly independent set by definition of basis. Therefore we have a linearly independent set B, with size of $\dim V$. Using the Replacement Theorem, there exists a subset H of size $(\dim V - \dim V) = 0$ such that $B \cup H$ generates V, i.e. $H = \emptyset$ and B generates V. Then, B is a basis for both U and V, which means that $U = V$.

   $\Leftarrow$: Since B is a basis of U, by definition of dimension, $\#B = \dim U$. Then, since $U = V$, $\dim U = \dim V = \#B$.