Math 115A Homework 4

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2. Let V be a finite dimensional inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

   (a) Fix $y \in V$ and suppose $\langle x, y \rangle = 0$ for all $x \in V$. Show that $y = 0$.

   Answer: By contradiction. Suppose y is nonzero and choose $x = y \neq 0$, then $\langle x, y \rangle = 0 \implies \langle x, x \rangle = 0$ for $x \neq 0$. This contradicts axiom (iv) of inner product space, therefore the initial assumption is false and $y = 0$.

   (b) Let $T : V \to V$ be a linear map such that $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for all pairs $x, y \in V$ (we call such a map a *metric* map). Prove that T is an isomorphism.

   Answer: Let $\dim V = n$ and $\beta = \{u_1, \ldots, u_n\}$ orthogonal basis of V. Take two arbitrary vectors of β, u_x and u_y, we have $\langle u_x, u_y \rangle = \langle T(u_x), T(u_y) \rangle$ by definition of T. Since β is an orthonormal basis, $\langle u_x, u_y \rangle = 0$ and therefore $\langle T(u_x), T(u_y) \rangle = 0$, implying that $T(u_x)$ and $T(u_y)$ are orthogonal. Using this process on $u_1, \ldots, u_n$ yields the orthogonal set $\gamma = \{T(u_1), \ldots, T(u_n)\}$. Since γ is automatically linearly independent and has dimension n, $T(\beta) = \gamma$ spans V, i.e. $rank(T) = n = dim(V)$. Then by rank-nullity theorem T is an isomorphism.

   (c) Find all metric maps $T : \mathbb{R}^2 \to \mathbb{R}^2$ that have $\det T = 1$.

   Answer: Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ be generic nonzero vectors of $\mathbb{R}^2$. In addition, let $[T] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, $\det T = 1$ implies that $ad - bc = 1$. Using the definition of metric map, we also have $\langle T(x), T(y) \rangle = \langle x, y \rangle$. We can substitute T, x and y as follows:

$$\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rangle = \langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rangle$$

$$\implies \langle \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}, \begin{pmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{pmatrix} \rangle = x_1 y_1 + x_2 y_2$$

$$\implies (ax_1 + bx_2)(ay_1 + by_2) + (cx_1 + dx_2)(cy_1 + dy_2) = x_1 y_1 + x_2 y_2$$

$$\implies (a^2 + c^2) x_1 y_1 + (b^2 + d^2) x_2 y_2 + (ab + cd)(x_1 y_2 + x_2 y_1) = x_1 y_1 + x_2 y_2$$

   By matching coefficients, we have the following system of equations:

$$a^2 + c^2 = 1, b^2 + d^2 = 1, ab + cd = 0, ad - bc = 1$$

which has solution

$$a = d, b = \pm\sqrt{1-a^2}, c = \mp\sqrt{1-a^2}$$

Then, the set of all $T : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies the form $[T] = \begin{pmatrix} a & \pm\sqrt{1-a^2} \\ \mp\sqrt{1-a^2} & a \end{pmatrix}$ contains all metric maps with $\det T = 1$.

4. Let V be an inner product space and let $r : V \to V^*$ be the map $r(x) = \varphi_x := \langle -, x\rangle$. In class we showed that if V is finite dimensional then r is an isomorphism.

   (a) Assume that V is finite dimensional. Prove that r is injective.

   Answer: We can prove that r is injective by checking its kernel: $r(x) = 0 \implies \langle y, x\rangle = 0$ for all $y \in V$. As shown in (2a), x must be 0. Therefore the kernel of r contains only the zero vector, thus r is injective.

   (b) Let $V = \mathbb{R}[x]$ and let $W = \{(a_0, a_1, ...) \mid a_i \in \mathbb{R}\}$ be the vector space of all infinite sequences. Show that the map $f : V^* \to W$ given by $f(\varphi) = (\varphi(x^n))_{n\geq 0}$ is an isomorphism.

   Answer: First, f is linear as $f(c\varphi_1 + \varphi_2) = (\varphi_1(cx^n))_{n\geq 0} + (\varphi_2(x^n))_{n\geq 0} = c(\varphi_1(x^n))_{n\geq 0} + (\varphi_2(x^n))_{n\geq 0} = cf(\varphi_1) + f(\varphi_2)$. Then, we can check the injectivity and surjectivity of f as follows:
   Injectivity: $f(\varphi) = 0 \implies (\varphi(x^n))_{n\geq 0} = 0$. A zero infinity sequence means that every element of the sequence is zero, i.e. $\varphi(x^n) = 0$ for $n \geq 0$. Since the only linear map that maps all powers of x to zero is the zero map, we have $\ker f = \{0\}$. Therefore f is injective.
   Surjectivity: We need to show that all vectors in W are in the image of f. Let $\xi \in W$ such that $\xi = (b_0, b_1, \ldots) \mid b_i \in \mathbb{R}$. In addition, take a generic $p \in V$ such that $p = a_0 + a_1x + \ldots + a_nx^n$. Then $\varphi(p) = a_0b_0 + a_1b_1 + \ldots$ and $\varphi(x^i) = b_i$, which means that $f(\varphi) = \xi$, i.e. $\mathrm{im} f = W$. Therefore f is surjective.

   (c) Use this to demonstrate that r is not necessarily surjective, i.e. find an element $\varphi \in V^*$ such that $\varphi \neq r(p)$ for any $p \in \mathbb{R}[x]$.

   Answer: Using the previous part, we can take an infinite sequence with infinitely many nonzero terms, e.g. $\xi = (1, 1, 1, \ldots)$, then, to have $\xi = f(r(p))$ would require a polynomial with infinite number of terms, which does not exist.

5. Let V be a finite dimensional inner product space. For any $T : V \to V$ define $\check{T} : V^* \to V^*$ by $\check{T}(\phi) = \phi \circ T$. Furthermore for any $X : V^* \to V^*$ define $X^\perp : V \to V$ by $X^\perp = r^{-1} \circ X \circ r$. Prove that $T^* = \check{T}^\perp$.

   Answer: By definitoin of T^*, $T^* = \check{T}^\perp \implies \langle T(x), y\rangle = \langle x, T^*(y)\rangle = \langle x, \check{T}^\perp(y)\rangle$. We can prove so by first expanding $\check{T}^\perp(y)$ as follows: $\check{T}^\perp(y) = r^{-1} \circ \check{T} \circ r(y) = r^{-1} \circ r(y) \circ T$. Since $r(y) = \langle -, v\rangle = \varphi$ for $v \in V$ by definition, its inverse $r^{-1}(\varphi) = v$ for $\varphi \in V^*$. Then by applying r to both sides of $\check{T}^\perp(y) = r^{-1} \circ r(y) \circ T$, we have $\langle x, \check{T}^\perp(y)\rangle = r(y) \circ T(x)$ for $x \in V$. In addition, $r(y) \circ T(x) = \langle T(x), y\rangle$ by definition of $r(y)$. Therefore $\langle x, T^*(y)\rangle = \langle x, \check{T}^\perp(y)\rangle$, i.e. $T^* = \check{T}^\perp$.