Math 115A Homework 3

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3. We say that two linear operators S and T *commute* if $S \circ T = T \circ S$. Let $T : V \to V$ be a diagonalisable linear operator. Define
$$C(T) = \{S \in \mathrm{Hom}(V, V) \mid S \text{ and } T \text{ commute}\}.$$

   (a) If T has $n = dimV$ distinct eigenvalues, show that any $S \in C(T)$ is diagonalisable.

   Answer: Let $\{\lambda_1, \ldots, \lambda_n\}$ be n distinct eigenvalues of T. Since the eigenvalues are distinct, there must exist an eigenbasis $\beta = \{v_1, \ldots, v_n\}$ of T where λ_i is the corresponding eigenvalue of v_i. By definition of eigenvalues, $T(v_i) = \lambda_i v_i$ for $v_i \in \beta$. We can apply T to $S(v_i) \in V$ as follows: $T(S(v_i)) = \lambda_i S(v_i)$, which implies that $S(v_i)$ is also an eigenvector of T with corresponding eigenvalue λ_i. In addition, since $dimV = n$, each eigenspace E_i must be dimension 1. Therefore, $S(v_i)$ and v_i must be linearly dependent, i.e. $S(v_i) = a_i v_i, a_i \in \mathbb{F}$, which implies that v_i is also an eigenvector of S with corresponding eigenvalue a_i. Then β is also an eigenbasis of S and $[S]_\beta$ is diagonal, thus S is diagonalisable.

   (b) Describe explicitly $C(T)$ in the case $T = x\frac{d}{dx} : \mathbb{C}_1[x] \to \mathbb{C}_1[x]$.

   Answer: We can first find $[T]$ by applying the transformation to each vector of the standard basis $\{1, x\}$ of $\mathbb{C}_1[x]$, resulting in the following:
$$[T] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$
   Then, $S \in C(T)$ implies
$$\begin{aligned} & S \circ T = T \circ S \\ \implies & \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \\ \implies & \begin{pmatrix} 0 & s_{12} \\ 0 & s_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ s_{21} & s_{22} \end{pmatrix} \\ \implies & s_{12} = s_{21} = 0. \end{aligned}$$
   Meaning that any $[S]$ must have the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ for $a, b \in \mathbb{C}$. Therefore, $C(T) = \{S \in \mathrm{Hom}(\mathbb{C}_1[x], \mathbb{C}_1[x]) \mid S(1) = a, S(x) = bx\}$

   (c) Show that part (a) does not necessarily hold if T does not have n distinct eigenvalues.

Answer: By counterexample: define $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $[T]_\beta^\beta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ for standard basis β of $\mathbb{R}^2$. Note that T has a single eigenvalue of 1 with multiplicity 2 (i.e. not distinct). Then, we can try to find a corresponding S as part (b) as follows:

$$\begin{aligned}
S \circ T &= T \circ S \\
\implies \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \\
\implies \begin{pmatrix} s_{11} + s_{21} & s_{12} + s_{22} \\ s_{21} & s_{22} \end{pmatrix} &= \begin{pmatrix} s_{11} & s_{11} + s_{12} \\ s_{21} & s_{21} + s_{22} \end{pmatrix} \\
&\implies s_{21} = 0, s_{11} = s_{22} \\
&\implies [S]_\beta = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix},
\end{aligned}$$

which is not always diagonalisable as $\det(S - \lambda I) = 0$ has no real solution when $a = -1$.

6. Let V be a vector space and $\mathcal{A} \subset \mathrm{Hom}(V, V)$ a subset such that every $X \in \mathcal{A}$ is diagonalisable. We say $\mathcal{A}$ is diagonalisable if there exists a basis B of V such that B is an eigenbasis for all $X \in \mathcal{A}$.

   (a) Show that if $\mathcal{A}$ is diagonalisable then for every pair of elements $X, Y \in \mathcal{A}$, we have $X \circ Y = Y \circ X$.

   Answer: Since B is an eigenbasis for all elements of $\mathcal{A}$, $[X]_B$ and $[Y]_B$ are both diagonal matrices. That is, we can define them as follows:

$$X = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & x_n \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & y_n \end{pmatrix}$$

   with $n = dim V$. Then,

$$[X \circ Y]_B = \begin{pmatrix} x_1 y_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & x_n y_n \end{pmatrix} = \begin{pmatrix} y_1 x_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & y_n x_n \end{pmatrix} = [Y \circ X]_B.$$

   Thus $X \circ Y = Y \circ X$.

   (b) Give an example of a set $\mathcal{A}$ that is *not* diagonalisable. Every element of $\mathcal{A}$ must be diagonalisable, it must contain at least two elements.

   Answer: Such $\mathcal{A}$ would contain elements that are diagonalisable but under different bases. We can construct such a set $\mathcal{A} = \{T, S\}$ as follows. Let $V = \mathbb{R}^2$ and $B = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}, C = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$ be two different bases of $\mathbb{R}^2$. For simplicity, we can first construct a $T \in \mathcal{A}$ such that

B is an eigenbasis of T, e.g. $[T]_B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Then, construct $S \in \mathcal{A}$ such that C is an eigenbasis of S: $[S]_C = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \implies [S]_B = \begin{pmatrix} 3 & 0 \\ 1 & 4 \end{pmatrix}$. By construction, T and S are diagonalisable under B and C, respectively. By part (a), we can verify that $\mathcal{A}$ is not diagonalisable by evaluating $[T \circ S]_B$ and $[S \circ T]_B$:

$$[T \circ S]_B = \begin{pmatrix} 3 & 0 \\ 2 & 8 \end{pmatrix} \neq \begin{pmatrix} 3 & 0 \\ 1 & 8 \end{pmatrix} = [S \circ T]_B.$$

Indeed $\mathcal{A} = \{S, T\}$ is not diagonalisable.