Math 115A Homework 1

Jiaping Zeng

4/8/2020

1. Let $V = \mathbb{R}^2$ and $W = \text{span}\begin{pmatrix}1\\1\end{pmatrix}$. List all the elements of V/W, making sure not to list any element twice.

   Answer: Visually, W is simply the line $y = x$ in $\mathbb{R}^2$. Since V/W is obtained by adding a vector of V to W (i.e. shifting $y = x$), V/W refers to the space of all lines in $\mathbb{R}^2$ with a slope of 1. The given definition $V/W = \{v + W \mid v \in V\}$ allows duplicate entries where two different vectors in V can shift along the same line (e.g. $\begin{pmatrix}1\\2\end{pmatrix} + W$ and $\begin{pmatrix}3\\4\end{pmatrix} + W$ are the same). We can eliminate the duplicates by specifying that the chosen $v \in V$ has to be on the y-axis. That is, we require each "shifting" vector from V to satisfy the form $\begin{pmatrix}0\\b\end{pmatrix}, b \in \mathbb{F}$, resulting in the set $V/W = \{\begin{pmatrix}0\\b\end{pmatrix} + W, b \in \mathbb{F}\}$.

2. Prove that V/W is a vector space.

   Answer: We can do so by verifying the axioms of vector space on V/W. Define $x, y, z \in V/W$ such that $x = u + W$, $y = v + W$ and $z = w + W$ for $u, v, w \in V$. Additionally, let $a, b \in \mathbb{F}$.

   (VS1) $x + y$
   $= (u + W) + (v + W)$
   $= (u + v) + W$
   Since $u, v \in V$ and V is a vector space,
   $= (v + u) + W$
   $= (v + W) + (u + W)$
   $= y + x.$

   (VS2) $(x + y) + z$
   $= [(u + W) + (v + W)] + (w + W)$
   $= [(u + v) + W] + (w + W)$
   $= [(u + v) + w] + W$
   Since $u, v, w \in V$ and V is a vector space,
   $= [u + (v + w)] + W$
   $= (u + W) + [(v + w) + W]$
   $= (u + W) + [(v + W) + (w + W)]$
   $= x + (y + z).$

(VS3) Let $0_{V/W}$ and 0_V be the zero vectors of V/W and V, respectively. Then, $0_{V/W} = 0_V + W$ satisfies this axiom. Proof:

$0_{V/W} + x$

$= (0_V + W) + (u + W)$

$= (0_V + u) + W$

$= u + W$

$= x.$

(VS4) Since $u \in V$, there exists a $-u \in V$ such that $u + (-u) = 0_V$. Define $-x \in V/W$ as $-u + W$, then:

$x + (-x)$

$= (u + W) + (-u + W)$

$= [u + (-u)] + W$

$= 0_V + W$

$= 0_W.$

(VS5) $1x$

$= 1(u + W)$

$= 1u + W$

Since $u \in V$ and V is a vector space,

$= u + W$

$= x.$

(VS6) $(ab)x$

$= (ab)(u + W)$

$= [(ab)u] + W$

Since $u \in V$ and V is a vector space,

$= [a(bu)] + W$

$= a(bu + W)$

$= a[b(u + W)]$

$= a(bx).$

(VS7) $a(x + y)$

$= a[(u + W) + (v + W)]$

$= a[(u + v) + W]$

$= (au + av) + W$

$= (au + W) + (av + W)$

$= a(u + W) + a(v + W)$

$= ax + ay.$

(VS8) $(a + b)x$

$= (a + b)(u + W)$

$= [(a + b)u] + W$

Since $u \in V$ and V is a vector space:

$$
\begin{aligned}
&= (au + bu) + W \\
&= (au + W) + (bu + W) \\
&= a(u + W) + b(u + W) \\
&= ax + bx.
\end{aligned}
$$

3. Let $\mathbb{C}[x]$ be the vector space of polynomials and let $W = span\{x^{2a} \mid a \geq 0\}$.

   (a) Find a set of 3 linearly independent elements of $\mathbb{C}/W$.

   Answer: Claim: $x + W$, $x^3 + W$, $x^5 + W$ are linearly independent elements of $\mathbb{C}/W$.

   Proof: by contradiction. Suppose $x+W$, x^3+W, x^5+W are linearly dependent, then there must exists nontrivial coefficients $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ such that $\lambda_1(x+W)+\lambda_2(x^3+W)+\lambda_3(x^5+W) = 0_{V/W}$. Using the defined addition and scalar multiplication, we can simplify the above to $(\lambda_1 x + \lambda_2 x^3 + \lambda_3 x^5) + W = 0_V + W$. The previous equation implies that $\lambda_1 x + \lambda_2 x^3 + \lambda_3 x^5 = 0_V$; since x, x^3, x^5 are linearly independent vectors in V, $\lambda_1, \lambda_2, \lambda_3$ must be trivial coefficients. Therefore, our initial assumption was false and $x + W$, $x^3 + W$, $x^5 + W$ are indeed linearly independent elements of $\mathbb{C}/W$.

   (b) Find 2 nonzero elements $p, q \in \mathbb{C}[x]$ that are linearly independent and such that $p+W$ and $q+W$ are linearly dependent and nonzero.

   Answer: Claim: $p = x + x^{6488}$ and $q = x + x^{4546}$ satisfies the given conditions.

   Proof: p, q are linearly independent in $\mathbb{C}[x]$ by comparing coefficients. We can show that $p + W$ and $q+W$ are linearly dependent and nonzero by substituting p, q and expanding them as follows:

$$
\begin{aligned}
p + W &= (x + x^{6488}) + W = (x + W) + (x^{6488} + W), \\
q + W &= (x + x^{4546}) + W = (x + W) + (x^{4546} + W).
\end{aligned}
$$

   Since $x^{6488} \in W$, $x^{6488} + W = W = 0_{C/W}$. We can then simplify the above form of $p + W$ to $p+W = (x+W)+0_{C/W} = x+W$. Similarly, $q+W = x+W = p+W$. Thus, $p+W$ and $q+W$ are trivially linearly dependent in $\mathbb{C}/W$.