1. (a)

   (b) A diagonalisable linear map T that is not an isomorphism means that $[T]_\beta$ is a diagonal matrix for some basis β with a nonzero nullity. For example, let $T : \mathbb{R}_2[x] \to \mathbb{R}_2[x]$ such that $T(p) = x\frac{dp}{dx}$. In addition, let $\beta = \{1, x, x^2\}$ be an ordered basis of $\mathbb{R}_2[x]$. Then,

$$[T]_\beta^\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

   which we can see is indeed diagonal. Thus, T is diagonalisable. At the same time, T is not injective, and therefore not an isomorphism, as its kernel contains the set of all constants.

2. Since vectors in V has the form $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, $\beta = \{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}$ is a basis of V. We can then apply T to each element of β to find $[T]_\beta^\beta$:

$$T(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$T(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}$$

Then,

$$[T]_\beta^\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

(a) The characteristic polynomial can be found as follows: $P_T(\lambda) = \det(T-\lambda I) = -\lambda(2-\lambda)(-2-\lambda) = -\lambda^3 + 4\lambda$. Then, we can find the eigenvalues by solving $P_T(\lambda) = 0$, which results in $\lambda = -2, 0, 2$.

(b) We can find the eigenvectors by solving $(T - \lambda I)v = 0$ for each λ.

$\lambda_1 = -2$:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\implies a = b = 0$$

Then $v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector, which corresponds to the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

$\lambda_2 = 0$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\implies b = c = 0$$

Then $v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector, which corresponds to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$\lambda_3 = 2$:

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\implies a = c = 0$$

Then $v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector, which corresponds to the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(c) Since $[T]_\beta^\beta$ is a diagonal matrix as shown above, T is diagonalisable.

3. (a)

   (b)