Math 151B Homework 5

Jiaping Zeng

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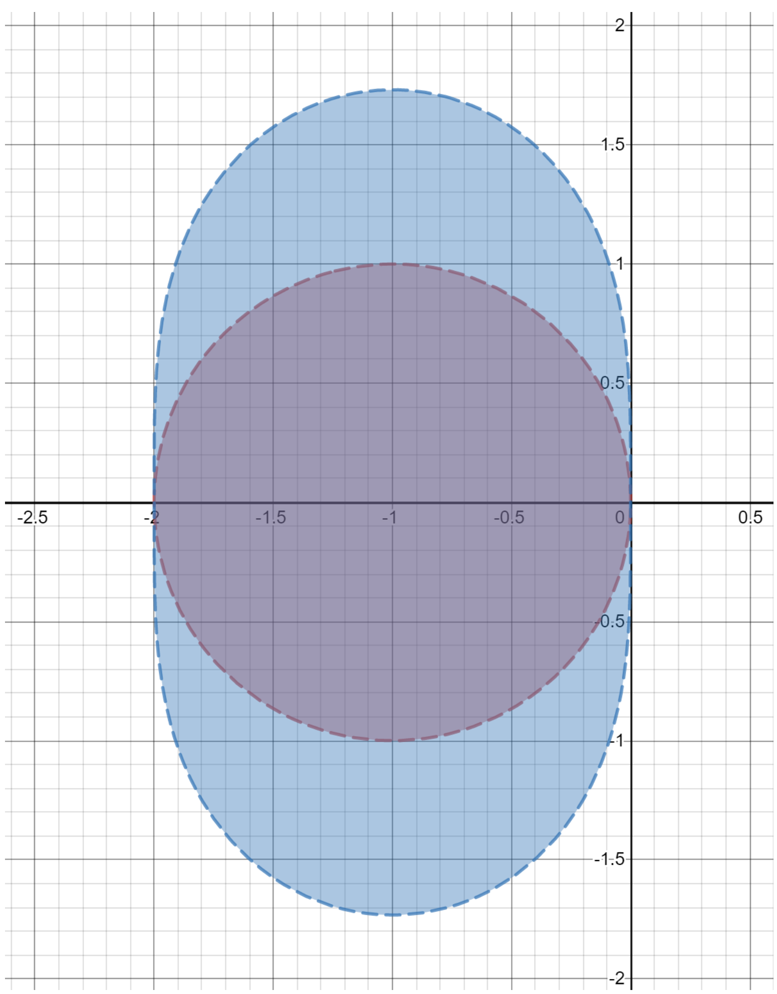
Q1 Euler's method: $w_{i+1} = w_i + hf(t_i, w_i) = w_i + h\lambda w_i = (1 + h\lambda)w_i$, then $R = \{h\lambda \in C \mid |1 + h\lambda| < 1\}$

Midpoint method: $w_{i+1} = w_i + hf(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)) = w_i + h\lambda(w_i + \frac{h\lambda}{2}w_i) = (1 + h\lambda + \frac{(h\lambda)^2}{2})w_i$, then $R = \{h\lambda \in C \mid \left|1 + h\lambda + \frac{(h\lambda)^2}{2}\right| < 1\}$

Let $z = h\lambda = a + ib$, then we have $|1 + a + ib| < 1 \implies \sqrt{(1 + a)^2 + b^2} < 1$ for Euler's method and $\left|1 + a + ib + \frac{(a+ib)^2}{2}\right| < 1 \implies \sqrt{(1 + a + \frac{a^2-b^2}{2})^2 + (b + ab)^2} < 1$.



Q2 (a) For $h = \frac{\pi}{4}$, we have

x_i	w_i	error
$\frac{\pi}{4}$	-0.28245222	$3.91 \cdot 10^{-4}$

(b) For $h = \frac{\pi}{8}$, we have

x_i	w_i	error
$\frac{\pi}{8}$	-0.31541496	$1.72 \cdot 10^{-5}$
$\frac{\pi}{4}$	-0.28285070	$7.99 \cdot 10^{-6}$
$\frac{3\pi}{8}$	-0.20718437	$8.61 \cdot 10^{-6}$

Q3 Since $y'' = p(x)y' + q(x)y \implies y'' - p(x)y' - q(x) = 0$, $y \equiv 0$ is a solution. Then by Corollary 11.2, the BVP $y'' = p(x)y' + q(x)y$, for $a \leq x \leq b$, with $y(a) = 0$ and $y(b) = 0$ has a unique solution, i.e. $y \equiv 0$ is the only solution. Therefore $y_2 \equiv 0$.

Q4 Using central-finite-difference we have $y'' - 4y = -4x \implies \frac{w_{i+1}-2w_i+w_{i-1}}{h^2} - 4w_i = \frac{-4i}{N}$. Now for the boundaries, we can first use Taylor expansion as follows: $y(x+h) = y(x)+hy'(x)+\frac{h^2}{2}y''(x)+O(h^3) \implies y'(x) = \frac{y(x+h)-y(x)}{h} - \frac{h}{2}y''(x)$, then $w_i' = \frac{w_{i+1}-w_i}{h} - \frac{h}{2}w_i'' = \frac{w_{i+1}-w_i}{h} - \frac{h}{2}w_i'' = \frac{w_{i+1}-w_i}{h} - \frac{h}{2}(\frac{w_2-2w_1+w_0}{2h}) = \frac{-3w_0+4w_1-w_2}{2h}$.

So for $i = 0$ we have $\frac{-3w_0+4w_1-w_2}{2h} = 0 \implies \frac{-3}{2h}w_0 + \frac{2}{h}w_1 - \frac{1}{2h}w_2 = 0$,

for $i = 1, \ldots, (N-1)$ we have $\frac{w_{i+1}-2w_i+w_{i-1}}{h^2} - 4w_i = \frac{-4i}{N} \implies \frac{1}{h^2}w_{i-1} + \frac{-4h^2-2}{h^2}w_i + \frac{1}{h^2}w_{i+1} = -\frac{4i}{N}$,

for $i = N$ we have $w_N = y(1) = 1$. Then we can set up the matrix as follows:

$$\begin{pmatrix} -\frac{3}{2h} & \frac{2}{h} & -\frac{1}{2h} & 0 & \dots & \dots & 0 \\ \frac{1}{h^2} & \frac{-4h^2-2}{h^2} & \frac{1}{h^2} & 0 & \dots & \dots & 0 \\ 0 & \frac{1}{h^2} & \frac{-4h^2-2}{h^2} & \frac{1}{h^2} & 0 & \dots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \dots & \dots & 0 & \frac{1}{h^2} & \frac{-4h^2-2}{h^2} & \frac{1}{h^2} \\ 0 & \dots & \dots & \dots & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ \vdots \\ w_{N-1} \\ w_N \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{4}{N} \\ -\frac{8}{N} \\ \vdots \\ \vdots \\ -\frac{4(N-1)}{N} \\ 1 \end{pmatrix}$$