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1. We can first substitute k_1 and k_3 into w_{i+1} as follows:

$$w_{i+1} = w_i + \frac{k_1 + 3k_3}{4}$$

$$= w_i + \frac{1}{4}[hf(t_i, w_i) + 3hf(t_i + \frac{2h}{3}, w_i + \frac{2k_2}{3})]$$

Now substitute in k_2:

$$= w_i + \frac{1}{4}[hf(t_i, w_i) + 3hf(t_i + \frac{2h}{3}, w_i + \frac{2}{3}hf(t_i + \frac{h}{3}, w_i + \frac{k_1}{3}))]$$

And substitute in k_1 one more time results in

$$= w_i + \frac{1}{4}[hf(t_i, w_i) + 3hf(t_i + \frac{2h}{3}, w_i + \frac{2}{3}hf(t_i + \frac{h}{3}, w_i + \frac{hf(t_i, w_i)}{3}))]$$

We can also factor out an h from the second part of the expression, giving us the following

$$= w_i + \frac{h}{4}[f(t_i, w_i) + 3f(t_i + \frac{2h}{3}, w_i + \frac{2h}{3}f(t_i + \frac{h}{3}, w_i + \frac{h}{3}f(t_i, w_i)))]$$

which is indeed Heun's method.

2. (a) Continuity: Since $f(t,y) = \dfrac{y}{1+t}$ is only undefined at $t=-1$ by inspection and the given interval $0 \leq t \leq 1$ does not include -1, $f(t,y)$ is continuous on the interval.
Lipschitz: $\dfrac{\delta f(t,y)}{\delta y} = \dfrac{1}{1+t} \leq \dfrac{1}{1+0} = 1 = L.$
Therefore this IVP is well-posed by Theorem 5.6.

(b) Two steps $\implies h = \frac{1}{2}$:
$y(\frac{1}{2}) \approx 1 + \frac{1}{2}f(0+\frac{1}{4}, 1+\frac{1}{4}f(0,1)) = 1 + \frac{1}{2}f(\frac{1}{4}, 1+\frac{1}{4}) = 1 + \frac{1}{2}\cdot 1 = \frac{3}{2}$
$y(1) \approx \frac{3}{2} + \frac{1}{2}f(\frac{1}{2}+\frac{1}{4}, \frac{3}{2}+\frac{1}{4}f(\frac{1}{2},\frac{3}{2})) = \frac{3}{2} + \frac{1}{2}f(\frac{3}{4}, \frac{3}{2}+\frac{1}{4}) = \frac{3}{2} + \frac{1}{2}\cdot 1 = \boxed{1}$

3. (a) We can first list out the first few w_i using the implicit formula as follows:

$t_0 = 0, w_0 = 0$

$t_1 = h, w_1 = 0 + h(h - h^2) = h^2 - h^3$

$t_2 = 2h, w_2 = h^2 - h^3 + h(2h - 4h^2) = 3h^2 - 5h^3$

$t_3 = 3h, w_3 = 3h^2 - 5h^3 + h(3h - 9h^2) = 6h^2 - 14h^3$

$t_4 = 4h, w_4 = 6h^2 - 14h^3 + h(4h - 16h^2) = 10h^2 - 30h^3$

...

Then we can see that:

$t_i = ih, w_i = \boxed{\sum_{n=1}^{i}(nh^2 - n^2h^3)}$

(b) By Taylor expansion, we have

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t) + O(h^3) = y(t) + hf(t,y) + \frac{h^2}{2}y''(t) + O(h^3),$$

then we can find the truncation error as follows:

$$\frac{y(t+h) - y(t) - hf(t,y)}{h} = \frac{\frac{h^2}{2}y''(t) + O(h^3)}{h}$$

$$\implies \frac{y(t+h) - y(t)}{h} - f(t,y) = \frac{h}{2}y''(t) + O(h^2)$$

Hence the leading term is $\frac{h}{2}y''(t)$, which in this IVP is

$$\frac{h}{2}y''(t) = \frac{h}{2} \cdot \frac{d}{dt}(\frac{y}{1+t})$$

$$= \frac{h}{2} \cdot [\frac{-y}{(1+t)^2} + \frac{1}{1+t}(t - t^2)]$$

$$= \frac{h(-y - t^3 + t)}{2(t+1)^2}$$

Then as shown in part (a), we can substitute $t_i = ih$ and $w_i = \sum_{n=1}^{i}(nh^2 - n^2h^3)$ which results in

$$= \frac{h(-i^3h^3 + ih - \sum_{n=1}^{i}(nh^2 - n^2h^3))}{2(ih+1)^2}$$

$$= \boxed{\frac{-i^3h^4 - h^3\sum_{n=1}^{i}(n - n^2h) + ih^2}{2(ih+1)^2}}$$

4. Starting with the given equality:

$$y(t_{i+1}) = y(t_i) + ahf(t_i, y(t_i)) + bhf(t_{i-1}, y(t_{i-1})) + chf(t_{i-2}, y(t_{i-2}))$$

By Taylor expansion, the left hand side expands into

$$y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \frac{h^3}{6}y'''(t) + O(h^4)$$

Whereas the right hand side (substituting $y'(t_i) = f(t_i, y(t_i))$) is equivalent to

$$y(t_i) + ahy'(t_i) + bhy'(t_i - h) + chy'(t_i - 2h)$$

$$= y(t_i) + ahy'(t_i) + bh[y'(t_i) - hy''(t_i) + \frac{h^2}{2}y'''(t_i) - O(h^3)] + ch[y'(t_i) - 2hy''(t_i) + 2h^2y'''(t_i) - O(h^3)]$$

$$= y(t_i) + (a + b + c)hy'(t) + (-b - 2c)h^2y''(t_i) + (\frac{b}{2} + 2c)h^3y'''(t_i) - O(h^4)$$

Then by matching coefficients we have

$$a + b + c = 1, -b - 2c = \frac{1}{2}, \frac{b}{2} + 2c = \frac{1}{6} \implies a = \frac{23}{12}, b = -\frac{4}{3}, c = \frac{5}{12}$$

Therefore, by substitution, the Adams-Bashforth Three step method is:

$$y(t_{i+1}) = y(t_i) + \frac{23}{12}hf(t_i, y(t_i)) - \frac{4}{3}hf(t_{i-1}, y(t_{i-1})) + \frac{5}{12}hf(t_{i-2}, y(t_{i-2}))$$