Math 131A Homework 1

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1.1 Prove $1^2 + 2^2 + ... + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all positive integers n.

Answer: By induction.

Base case ($n = 1$): $1 = \frac{1}{6}(1+1)(2+1) \implies 1 = 1$ which is true.

Inductive step: Assume $1^2 + 2^2 + ... + n^2 = \frac{1}{6}n(n+1)(2n+1)$ is true, we want to show that $1^2 + 2^2 + ... + n^2 + (n+1)^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$ is true. We can do so by adding $(n+1)^2$ to both sides of the equation as follows:

$$1^2 + 2^2 + ... + n^2 + (n+1)^2 = \frac{1}{6}n(n+1)(2n+1) + (n+1)^2$$

Expanding the right hand side results in the following:

$$1^2 + 2^2 + ... + n^2 + (n+1)^2 = \frac{1}{3}n^3 + \frac{3}{2}n^2 + \frac{13}{6}n + 1$$

Which indeed factors into

$$\implies 1^2 + 2^2 + ... + n^2 + (n+1)^2 = \frac{1}{6}(n+1)(n+2)(2n+3).$$

Therefore $1^2 + 2^2 + ... + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all positive integers n by mathematical induction.

1.9 (a) Decide for which integers the inequality $2^n > n^2$ is true.

Answer: $2^n > n^2, n \in \mathbb{Z}$ is true for $n = 0, 1$ and $n \geq 5$.

(b) Prove your claim in (a) by mathematical induction.

Answer: We will first show $n = 0$ and $n = 1$ case-by-case, then $n \geq 5$ by induction.

$n = 0$: $2^n > n^2 \implies 1 > 0$ which is true

$n = 1$: $2^n > n^2 \implies 2 > 1$ which is true

$n \geq 5$: By induction as follows.

Base case: ($n = 5$) $2^n > n^2 \implies 32 > 25$ which is true

Indcutive step: Assume $2^n > n^2$ is true, we want to show that $2^{n+1} > (n+1)^2$ is also true. We can start by multiplying 2 to both sides of the inequality: $2 * 2^n > 2 * n^2 \implies 2^{n+1} > 2n^2$. Then, if we could show that $2n^2 > (n+1)^2$ for $n \geq 5$, $2^{n+1} > (n+1)^2$ would be also true. By expanding the right hand side and subtracting n^2 from both sides, $2n^2 > (n+1)^2$ simplifies to $n^2 > 2n + 1$. We will prove this inequality for $n > 5$ using another proof by induction:

Base case: $(n = 5)$: $n^2 > 2n + 1 \implies 25 > 11$ which is true

Inductive step: Assume $n^2 > 2n + 1$ is true, we want to show that $(n + 1)^2 > 2n + 3$. By expanding the left hand side and canceling terms, we have $n^2 > 2$ which is clearly true for $n > 5$.

Therefore $n^2 > 2n + 1$, and by extension $2^{n+1} > (n + 1)^2$, so $2^n > n^2$ is true for $n = 0, 1$ and $n \geq 5$ by mathematical induction.

1.11 For each $n \in \mathbb{N}$, let P_n denote the assertion "$n^2 + 5n + 1$ is an even integer."

(a) Prove P_{n+1} is true whenever P_n is true.

Answer: Using the definition of P_n, P_{n+1} corresponds to the expression $(n+1)^2 + 5(n+1) + 1$. To show that it is even, we can first expand it into $(n + 1)^2 + 5(n + 1) + 1 = n^2 + 7n + 7$. Then, $P_{n+1} - P_n = n^2 + 7n + 7 - n^2 - 5n - 1 = 2n + 6$ which is always even for $n \in \mathbb{N}$. Therefore, if P_n is even, then $P_n + (P_{n+1} - P_n) = P_{n+1}$ must also be even.

(b) For which n is P_n actually true? What is the moral of this exercise?

Answer: For $n^2 + 5n + 1$ to be even, $n^2 + 5n = n(n + 5)$ must be odd. However, this is not possible for $n \in \mathbb{N}$. On one hand, if n is even, $n(n+5)$ would have a factor of 2 from n and would therefore be even; on the other hand, if n is odd, then $(n+5)$ would be even and $n(n+5)$ as well. The moral of this exercise is that finding a true base case is important in mathematical induction.

3.1 (a) Which of the properties A1-A4, M1-M4, DL, O1-O5 fail for $\mathbb{N}$?

Answer: A3 fails since there is no $0 \in \mathbb{N}$; A4 fails since there are no negative numbers in $\mathbb{N}$; M4 fails since there are no fractions.

(b) Which of these properties fail for $\mathbb{Z}$?

Answer: M4 fails since there are no fractions in $\mathbb{Z}$.

3.6 (a) Prove $|a + b + c| \leq |a| + |b| + |c|$ for all $a, b, c \in \mathbb{R}$.

Answer: Let $m = a + b$, then since $|a + b| \leq |a| + |b|$ (triangle inequality), $|m| \leq |a| + |b|$. Then we can add $|c|$ to both sides of the inequality, resulting in $|m| + |c| \leq |a| + |b| + |c|$. Since $|m + c| \leq |m| + |c|$, we have $|m + c| \leq |m| + |c| \leq |a| + |b| + |c| \implies |m + c| \leq |a| + |b| + |c|$. Then by substituting $m = a + b$ we have $|a + b + c| \leq |a| + |b| + |c|$.

(b) Use induction to prove

$$|a_1 + a_2 + \ldots + a_n| \leq |a_1| + |a_2| + \ldots + |a_n|$$

for n numbers $a_1, a_2, ..., a_n$.

Answer:

Base case: $(n = 1)$: $|a_1| \leq |a_1|$ which is true.

Inductive step: Assume the statement holds for n numbers $a_1, a_2, \ldots, a_n$. Let $m = a_1+a_2+\ldots+a_n$, then we have $|m| \leq |a_1| + |a_2| + \ldots + |a_n|$. Adding $|a_{n+1}|$ to both sides of the inequality gives us $|m| + |a_{n+1}| \leq |a_1| + |a_2| + \ldots + |a_n| + |a_{n+1}|$. Then using $|m + a_{n+1}| \leq |m| + |a_{n+1}|$ (similar to previous part), we have $|m + a_{n+1}| \leq |a_1| + |a_2| + \ldots + |a_n| + |a_{n+1}|$. Then by substitution,

$|a_1 + a_2 + \ldots + a_n + a_{n+1}| \leq |a_1| + |a_2| + \ldots + |a_n| + |a_{n+1}|$.

Therefore $|a_1 + a_2 + \ldots + a_n| \leq |a_1| + |a_2| + \ldots + |a_n|$ by mathematical induction.

3.7 (a) Show $|b| < a$ if and only if $-a < b < a$.

Answer:

$\Rightarrow$: Case $b \geq 0$: we have $|b| = b$ by definition of aboslute value. Then, $|b| < a \implies b < a$.

Case $b < 0$: $|b| = -b$, then $|b| < a \implies -b < a \implies b > -a$.

Therefore $-a < b < a$.

$\Leftarrow$: Case $b \geq 0$: $b = |b|$ by definition; then $-a < |b| < a$.

Case $b < 0$: $-a < b < a \implies a > -b > -a$ (multiply by -1). Since $-b = |b|$, $a > -b > -a \implies a > |b| > -a \implies -a < |b| < a$.

Therefore $|b| < a$.

(b) Show $|a - b| < c$ if and only if $b - c < a < b + c$.

Answer: $b - c < a < b + c \implies -c < a - b < c$ (subtract b). Then, let $m = a - b$. By substitution, the original statement is equivalent to "$|m| < c$ if and only if $-c < m < c$", which is true by part (a).

(c) Show $|a - b| \leq c$ if and only if $b - c \leq a \leq b + c$.

Answer: Let $m = a - b$, then by substitution we would need to prove that $|m| \leq c$ if and only if $-c \leq m \leq c$ (subtract b).

$\Rightarrow$: Case $m \geq 0$: Since $|m| = m$ in this case, we have $m = |m| \leq c \implies m \leq c$.

Case $m < 0$: $|m| = -m$, then similarly we have $-m \leq c \implies m \geq -c$.

Therefore $-c \leq m \leq c$.

$\Leftarrow$: Case $m \geq 0$: Since $m = |m|$ in this case, we have $-c \leq |m| \leq c \implies |m| \leq c$.

Case $m < 0$: Since $-m = |m|$ here, we have $-c \leq -|m| \leq c \implies c \geq |m| \geq -c$ (multiply by -1). Then $|m| \leq c$.

Therefore $|m| \leq c$ in both cases.

3.8 Let $a, b \in \mathbb{R}$. Show if $a \leq b_1$ for every $b_1 > b$, then $a \leq b$.

Answer: By contradiction. Suppose $a \leq b_1$ for every $b_1 > b$ and $a > b$. Since $a > b$, there must exist a $b_1 \in \mathbb{Q} (\in \mathbb{R})$ where $b < b_1 < a$ (Since $\mathbb{Q}$ is dense in $\mathbb{R}$). However, that would imply that there exists a b_1 where $a > b_1$ which contradicts our assumption. Therefore if $a \leq b_1$ for every $b_1 > b$, then $a \leq b$.

4.6 Let S be a nonempty bounded subset of $\mathbb{R}$.

(a) Prove $\inf S \leq \sup S$.

Answer: By definition, $\sup S \geq s$ and $\inf S \leq s$ for all $s \in S$. Then we have $\inf S \leq s \leq \sup S$ which implies that $\inf S \leq \sup S$.

(b) What can you say about S if $\inf S = \sup S$?

Answer: Again by definition, if $\inf S = \sup S$, we must have $\inf S = s = \sup S$ for all $s \in S$. Then there can only be a single element in S which is also both the supremum and infimum of S.

4.7 Let S and T be nonempty bounded subsets of $\mathbb{R}$.

(a) Prove if $S \subseteq T$, then inf $T \leq$ inf $S \leq$ sup $S \leq$ sup T.

Answer: By definition of upper bound, sup $T \geq t$ for all $t \in T$. Then by definition of subset, $s \in T$ for all $s \in S$. Therefore sup T is an upper bound of S, i.e. sup $T \geq s$ for all $s \in S$. We also have that sup $S \geq s$ for all $s \in S$, with the added requirement of being the least upper bound by definition of supremum. Therefore sup $S \leq$ sup T.

Similarly, since inf $T \leq t$ for all $t \in T$ and $s \in T$ for all $s \in S$, inf T is a lower bound of S; therefore the greatest lower bound inf S must be greater or equal to inf T, i.e. inf $T \leq$ inf S.

By 4.6(a), we also have inf $S \leq$ sup S. Therefore inf $T \leq$ inf $S \leq$ sup $S \leq$ sup T.

(b) Prove sup($S \cup T$) =max{sup S, sup T}

Answer: By cases sup $S \leq$ sup T and sup $S >$ sup T.

Case sup $S \leq$ sup T: Then max{sup S, sup T}=sup T. By definition of union, $S \cup T$ contains all elements from S or T. Since sup $T \geq$ sup S, sup $T \geq s$ for all $s \in S$. By definition of supremum, we also have sup $T \geq t$ for all $t \in T$. Therefore sup T is the supremum of the union.

Case sup $S >$ sup T: We have max{sup S, sup T}=sup S. Similar to the previous case, sup $S \geq s$ for all $s \in S$ by definition of supremum. In addition, sup $S \geq t$ for all $t \in T$ since sup $S >$ sup T. Therefore sup S is the supremum of the union.

4.8 Let S and T be nonempty subsets of $\mathbb{R}$ with the following property: $s \leq t$ for all $s \in S$ and $t \in T$.

(a) Observe S is bounded above and T is bounded below.

Answer: Since $s \leq t$ for all $s \in S$ and $t \in T$, we can select any $t \in T$ to be an upper bound of S (there is at least one such t since T is nonempty). Similarly, we can also select any $s \in S$ to be a lower bound of T. Therefore S is bounded above and T is bounded below.

(b) Prove sup $S \leq$ inf T.

Answer: By contradiction. Assume sup $S >$ inf T. As shown in part (a), any $t \in T$ is an upper bound of S. Therefore the least upper bound sup $S \leq t$ for all $t \in T$, in other words, sup S is a lower bound of T. However, since sup $S >$ inf T, inf T cannot be the greatest lower bound of T. Therefore our assumption is false and sup $S \leq$ inf T.

(c) Give an example of such sets S and T where $S \cap T$ is nonempty.

Answer: $S = [-1, 0], T = [0, 1]$.

(d) Give an example of sets S and T where sup $S =$ inf T and $S \cap T$ is the empty set.

Answer: $S = [-1, 0), T = (0, 1]$.

4.14 Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let $A + B$ be the set of all sums $a + b$ where $a \in A$ and $b \in B$.

(a) Prove sup$(A + B) =$ sup A+sup B.

Answer: By definition of supremum, sup $A \geq a$ and sup $B \geq b$ for every $a \in A, b \in B$. Then, sup A+sup $B \geq a + b$. Since $a + b$ is also an arbitrary member of the set $A + B$, sup A+sup B is an upper bound of the set. Then, sup$(A + B) \leq$ sup A+sup B since it is the least upper bound of the set $A + B$.

We now will show that sup$(A+B) \not< \sup A+\sup B$ by contradiction. Suppose $\sup(A+B) < \sup A+\sup B$, then there must exist an $r \in \mathbb{Q}$ such that $\sup(A+B) < r < \sup A+\sup B$ since $\mathbb{Q}$ is dense in $\mathbb{R}$. Then, since $\sup A \geq a$ and $\sup B \geq b$ for every $a \in A, b \in B$, $r < a+b$. Therefore r is also a member of the set $A+B$ while being greater than the supremum of the set $\sup(A+B)$, which leads to a contradiction. Then the only possibility is that $\sup(A+B) = \sup A+\sup B$.

(b) Prove $\inf(A+B) = \inf A+\inf B$.

Answer: Similar to the part (a), Since $\inf A \leq a$ and $\inf B \leq b$ for every $a \in A, b \in B$, $\inf A+\inf B \leq a+b$. Then $\inf A+\inf B$ is a lower bound of the set $A+B$ and as a result $\inf A+\inf B$ must be less or equal to the greatest lower bound $\inf(A+B)$, i.e. $\inf A+\inf B \leq \inf(A+B)$. We will show that $\inf A+\inf B \not< \inf(A+B)$ by contradiction. Suppose $\inf A+\inf B < \inf(A+B)$, then there must exist an $r \in \mathbb{Q}$ such that $\inf A+\inf B < r < \inf(A+B)$. Since $\inf A \leq a$ and $\inf B \leq b$ for every $a \in A, b \in B$, $r > a+b$. Then r is a member of the set $A+B$ while being less than the infimum $\inf(A+B)$ which contradicts. Therefore $\inf(A+B) = \inf A+\inf B$.

4.15 Let $a, b \in \mathbb{R}$. Show if $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a \leq b$. Compare Exercise 3.8.

Answer: By contradiction. Suppose there exists $a, b \in \mathbb{R}$ such that $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$ and $a > b$. Since $a > b$, we have $a - b > 0$. Then by the Archimedean property, there exists an $n \in \mathbb{N}$ that can scale it past 1, i.e. $n(a-b) > 1$. Upon dividing both sides by n we have $a - b > \frac{1}{n} \implies a > b + \frac{1}{n}$, which contradicts our assumption. Therefore if $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a \leq b$.

4.16 Show $\sup\{r \in \mathbb{Q} : r < a\} = a$ for each $a \in \mathbb{R}$.

Answer: Since $r < a$ for all r in the set, a is automatically an upper bound. To show that a is the supremum of the set, we will show that it is the least upper bound by contradiction. Suppose there exists another upper bound $b \in \mathbb{R}$ such that $b < a$. Then, there exists an $r \in \mathbb{Q}$ such that $b < r < a$ since $\mathbb{Q}$ is dense in $\mathbb{R}$. However, such r would be be in the set $\{r \in \mathbb{Q} : r < a\}$ and $b < r$ implies that b is not an upper bound of the set, which contradicts our assumption. Therefore $\sup\{r \in \mathbb{Q} : r < a\} = a$ for each $a \in \mathbb{R}$.

P1 Write down the converse and the contrapositive of the following statement regarding a real number x:

$$\text{If } x > 0, \text{then } x^2 - x > 0.$$

Then determine which (if any) of the three statements are true for all real numbers x.

Answer:

Converse: if $x^2 - x > 0$, then $x > 0$, which is false by counterexample $x = -1$.

Contrapositive: if $x^2 - x \leq 0$, then $x \leq 0$, which is false by counterexample $x = 1$.

P2 Prove that $\sqrt{3}$ is not rational.

Answer: By contradiction. Suppose $\sqrt{3}$ is rational, then by definition of rational numbers, there must exist $p, q \in \mathbb{Z}$ such that $\frac{p}{q} = \sqrt{3}$, where p, q have no common factors upon simplying. Then, we also have $\frac{p^2}{q^2} = 3 \implies p^2 = 3q^2$. Since $p, q \in \mathbb{Z}$ and by extension $p^2, q^2 \in \mathbb{Z}$, $p^2 \mid 3$. Additionally, $p \mid 3$

because $\sqrt{3} \notin \mathbb{Z}$. Then, $p^2 \mid 9$. By substituting $p^2 = 3q^2$, we now have $3q^2 \mid 9 \implies q^2 \mid 3$, implying $q \mid 3$ by previous logic. Then p, q have common factor 3 which contradicts with our initial assumption. Therefore $\sqrt{3}$ is not rational.