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1. (a) First we find the Jacobian matrix

$$J(x) = \begin{pmatrix} -1 & \cos(x_2) \\ 0 & 10 \end{pmatrix}$$

Then the k-th step is:

$$x^{(k)} = x^{(k-1)} - J(x^{(k-1)})^{-1} F(x^{(k-1)})$$

$$\implies \begin{pmatrix} x_1^k \\ x_2^k \end{pmatrix} = \begin{pmatrix} x_1^{k-1} \\ x_2^{k-1} \end{pmatrix} - \begin{pmatrix} -1 & \cos(x_2^{k-1}) \\ 0 & 10 \end{pmatrix}^{-1} \begin{pmatrix} f_1(x^{(k-1)}) \\ f_2(x^{(k-1)}) \end{pmatrix}$$

$$\implies \begin{pmatrix} x_1^k \\ x_2^k \end{pmatrix} = \begin{pmatrix} x_1^{k-1} \\ x_2^{k-1} \end{pmatrix} - \begin{pmatrix} -1 & \frac{1}{10}\cos(x_2^{k-1}) \\ 0 & \frac{1}{10} \end{pmatrix} \begin{pmatrix} -x_1^{k-1} + \sin(x_2^{k-1}) \\ 10x_2^{(k-1)} \end{pmatrix}$$

$$\implies \begin{pmatrix} x_1^k \\ x_2^k \end{pmatrix} = \begin{pmatrix} x_1^{k-1} \\ x_2^{k-1} \end{pmatrix} - \begin{pmatrix} x_1^{k-1} + x_2^{k-1}\cos(x_2^{k-1}) - \sin(x_2^{k-1}) \\ x_2^{k-1} \end{pmatrix}$$

$$\implies \begin{pmatrix} x_1^k \\ x_2^k \end{pmatrix} = \begin{pmatrix} -x_2^{k-1}\cos(x_2^{k-1}) + \sin(x_2^{k-1}) \\ 0 \end{pmatrix}$$

$$\implies \boxed{x_1^k = -x_2^{k-1}\cos(x_2^{k-1}) + \sin(x_2^{k-1}), x_2^k = 0}$$

(b) For Euler's method, we have:

$$\begin{pmatrix} x_1^k \\ x_2^k \end{pmatrix} = \begin{pmatrix} x_1^{k-1} \\ x_2^{k-1} \end{pmatrix} + h \begin{pmatrix} 100x_1^{k-1} \\ -x_2^{k-1} \end{pmatrix}$$

$$\implies \begin{pmatrix} x_1^k \\ x_2^k \end{pmatrix} = \begin{pmatrix} (100h+1)x_1^{k-1} \\ (-h+1)x_2^{k-1} \end{pmatrix}$$

$$\implies \boxed{x_1^k = (100h+1)x_1^{k-1}, x_2^k = (-h+1)x_2^{k-1}}$$

2. We can Taylor expand both sides of the trapezoid method as follows:

$$y(t_{i+1}) = y(t_i) + \frac{h}{2}(y'(t_i) + y'(t_{n+1}))$$

$$\implies y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + O(h^3) = y(t_i) + \frac{h}{2}(y'(t_i) + y'(t_{n+1}))$$

$$\implies y(t_i)+hy'(t_i)+\frac{h^2}{2}y''(t_i)+\frac{h^3}{6}y'''(t_i)+O(h^4) = y(t_i)+\frac{h}{2}(y'(t_i)+y'(t_i)+hy''(t_i)+\frac{h^2}{2}y'''(t_i)+O(h^3))$$

$$\implies y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \frac{h^3}{6}y'''(t_i) + O(h^4) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \frac{h^3}{4}y'''(t_i) + O(h^4)$$

By comparing both sides we can see that the truncation error is $\frac{h^3}{12}y'''(t_i) + O(h^4)$.

3. (a) Let $u_1 = y, u_2 = y', u_3 = y''$, then we have $u_1' = u_2, u_2' = u_3, u_3' = -4u_1^2 + u_2 + u_3$ which is a system of first-order ODEs.

(b) The characteristic polynomial of the ODE is $p(r) = r^2 + 1$, which has roots $r = \pm i$. Then the general solution is $y(t) = c_1 \cos(t) + c_2 \sin(t)$. Using the boundary conditions, we have $y(0) = c_1 = 0$ and $y(b) = c_1 \cos(b) + c_2 \sin(b) = B \implies c_1 = 0, c_2 = \frac{B}{\sin(b)}$.

Then the BVP has no solution when $\sin(b) = 0$, i.e. when $b = k\pi, k \in \mathbb{Z}$. Otherwise, it has exactly one solution in the form of $y(t) = \frac{B}{\sin(b)} \sin(t)$.

4. (a) Let $g(\hat{x}) = [f_1(x_1, x_2)]^2 + [f_2(x_1, x_2)]^2$, then by expanding we have

$$g(\hat{x}) = 2x_1^4 - 2x_1^3x_2 + 2e^{x_2}x_1^2 + 9x_1^2x_2^2 - 2x_1^2x_2 - 8x_1x_2^3 + e^{2x_2} + 16x_2^4 - 2e^{x_2}x_2$$

The gradient can be found using $\nabla g(\hat{x}) = 2J(\hat{x})^t F(\hat{x})$, where $J(\hat{x})$ is the Jacobian matrix

$$J(\hat{x}) = \begin{pmatrix} 2x_1 & e^{x_2} - 1 \\ 2x_1 - x_2 & 8x_2 - 1 \end{pmatrix}$$

, then by substitution we have

$$\nabla g(\hat{x}) = \begin{pmatrix} 8x_1^3 - 6x_1^2x_2 + x_1(4e^{x_2} + 18x_2^2 - 4x_2) - 8x_2^3 \\ x_1^2(2e^{x_2} + 16x_2 - 4) - 2x_1x_2(8x_2 - 1) + 2e^{2x_2} - 2(x_2 + 1)e^{x_2} + 64x_2^3 - 8x_2^2 + 2x_2 \end{pmatrix}$$

Using equations of $g(\hat{x})$ and $\nabla g(\hat{x})$ above, in addition to an initial approximation $\hat{x}^{(0)}$, we can evaluate $g(\hat{x}^{(0)})$ and $\nabla g(\hat{x}^{(0)})$. Now let $\hat{z} = \frac{\nabla g(\hat{x}^{(0)})}{||\nabla g(\hat{x}^{(0)})||_2}$ (normalized direction vector), then the next approximation can be obtained using $\hat{x}^{(1)} = \hat{x}^{(0)} - \alpha\hat{z}$ for some $\alpha > 0$ (size of the step to be travelled in the direction of unit vector $\hat{z}$).

Now α can be determined by $h(\alpha) = g(\hat{x}^{(0)} - \alpha\nabla g(\hat{x}^{(0)}))$ so that $h(\alpha)$ is minimal. To do so we choose $\alpha_1 < \alpha_2 < \alpha_3$ and evaluate $h(\alpha_i)$ at each of the three points, then interpolate them using Newton's forward divided-difference formula and finding a minimum (which I do not know the specific equation for this problem as $\bar{x}^{(0)}$ is not given). Then we can substitute α back into $h(\alpha) = g(\hat{x}^{(0)} - \alpha\nabla g(\hat{x}^{(0)}))$ and repeat the process for subsequent $\bar{x}^{(i)}$.

(b) Using the equation of $\nabla g(\hat{x})$ above, we have $\nabla g(\hat{x}^{(0)}) = (12, -2)^t$ and $||\nabla g(\hat{x}^{(0)})||_2 = 2\sqrt{37}$. Then as explained in the previous part, the direction unit vector is

$$\hat{z} = \frac{\nabla g(\hat{x}^{(0)})}{||\nabla g(\hat{x}^{(0)})||_2} = \begin{pmatrix} \frac{12}{2\sqrt{37}} \\ \frac{-2}{2\sqrt{37}} \end{pmatrix}$$