Math 151B Homework 4

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Q1 After expanding the expression we have

$y(t_{i+1}) = y(t_i) + ahf(t_i, y(t_i)) + bhf(t_{i-1}, y(t_{i-1})) + chf(t_{i-2}, y(t_{i-2})) + dhf(t_{i-3}, y(t_{i-3}))$

Then we can expand both sides using Taylor polynomials as follows:

$LHS = y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \frac{h^3}{6}y'''(t) + \frac{h^4}{24}y^{(4)}(t) + O(h^5)$

$RHS = y(t_i) + ahy'(t_i) + bhy'(t_i - h) + chy'(t_i - 2h) + dhy'(t_i - 3h)$

$= y(t_i) + ahy'(t_i) + bh[y'(t_i) - hy''(t_i) + \frac{h^2}{2}y'''(t_i) - \frac{h^3}{6}y^{(4)}(t) + O(h^4)] + ch[y'(t_i) - 2hy''(t_i) + 2h^2y'''(t_i) - \frac{4h^3}{3}y^{(4)}(t) + O(h^4)] + dh[y'(t_i) - 3hy''(t_i) + \frac{9h^2}{2}y'''(t_i) - \frac{9h^3}{2}y^{(4)}(t) + O(h^4)]$

$= y(t_i) + (a+b+c+d)hy'(t) + (-b-2c-3d)h^2y''(t_i) + (\frac{b}{2} + 2c + \frac{9d}{2})h^3y'''(t_i) + (-\frac{b}{6} - \frac{4c}{3} - \frac{9d}{2})h^4y^{(4)}(t_i) + O(h^5)$

Then by matching coefficients we have

$a + b + c + d = 1, b - 2c - 3d = \frac{1}{2}, \frac{b}{2} + 2c + \frac{9d}{2} = \frac{1}{6}, -\frac{b}{6} - \frac{4c}{3} - \frac{9d}{2} = \frac{1}{24}$

$\implies \boxed{a = \frac{55}{24}, b = -\frac{59}{24}, c = \frac{37}{24}, d = \frac{-3}{8}}$

Q2 (a) By Taylor expansion we have

$y(t_{i+1}) = y(t_i) + hf(t_i, y_i) + \frac{h^2}{2}\nabla f(t_i, y_i) + O(h^3)$

Similarly,

$w_{i+1} = w_i + a(f + h\nabla f + O(h^2)) = w_i + (a+b)f + ah\nabla f + O(h^2)$

By matching coefficients we have

$a + b = h, ah = \frac{h^2}{2} \implies \boxed{a = \frac{h}{2}, b = \frac{h}{2}}$

(b)
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def q2b():

    a, b, h = 0, 1, 0.1

    u0 = numpy.array([0, 0])
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def f(t, u):
    A = numpy.array([[0, 1], [4, 0]])
    b = numpy.array([0, 6*math.exp(-t)])
    return A.dot(u)+b


def pred(f, wi, ti, h):
    return wi+h*f(ti+h/2, wi+h/2*f(ti, wi))


def corr(f, wi, wi_bar, ti, h):
    return wi+h/2*f(ti, wi)+h/2*f(ti+h, wi_bar)


steps = int((b-a)/h)
wi = u0
for i in range(steps):
    ti = (i-1)*h
    wi_bar = pred(f, wi, ti, h)
    wi = corr(f, wi, wi_bar, ti, h)


print(f"y={wi[0]}, y'={wi[1]}")
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(c) $y(1) \approx \boxed{3.5140233617478946}$ by running the code above.

Q3 By substituting $f(t, w) = wg(t)$ we have

$$w_{i+1} = w_i + \tfrac{9h}{24} w_{i+1} g(t_{i+1}) + \tfrac{19h}{24} w_i g(t_i) - \tfrac{5h}{24} w_{i-1} g(t_{i-1}) + \tfrac{h}{24} w_{i-2} g(t_{i-2})$$

$$\implies w_{i+1}(1 - \tfrac{9h}{24}) = w_i + \tfrac{19h}{24} w_i g(t_i) - \tfrac{5h}{24} w_{i-1} g(t_{i-1}) + \tfrac{h}{24} w_{i-2} g(t_{i-2})$$

$$\implies \boxed{w_{i+1} = \frac{w_i + \frac{19h}{24} w_i g(t_i) - \frac{5h}{24} w_{i-1} g(t_{i-1}) + \frac{h}{24} w_{i-2} g(t_{i-2})}{1 - \frac{9h}{24}}}$$