Math 131A Homework 2

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8/12/2020

8.1 Prove the following:

(a) $\lim \frac{(-1)^n}{n} = 0$

Answer: Proof: Take $N = \frac{1}{\epsilon}$, then $n > N = \frac{1}{\epsilon} \implies \epsilon > \frac{1}{n} = |s_n - 0|$. Therefore $\lim \frac{(-1)^n}{n} = 0$ by definition of limit.

(c) $\lim \frac{2n-1}{3n+2} = \frac{2}{3}$

Answer:

Scratch work: $\left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| < \epsilon \implies \left|\frac{6n-3-2\cdot(3n+2)}{3\cdot(3n+2)}\right| < \epsilon \implies \left|\frac{-7}{3\cdot(3n+2)}\right| < \epsilon \implies \frac{7}{3\epsilon} < 3n+2 \implies \frac{7}{9\epsilon} - \frac{2}{3} \leq n$.

Proof: Let $\epsilon > 0$, define $N = \frac{7}{9\epsilon} - \frac{2}{3}$. Then, $n > N = \frac{7}{9\epsilon} - \frac{2}{3} \implies 3n+2 > \frac{7}{3\epsilon} \implies 3\epsilon > \frac{7}{3n+2} \implies \epsilon > \frac{7}{3\cdot(3n+2)} = \left|s_n - \frac{2}{3}\right|$. Therefore $\lim \frac{2n-1}{3n+2} = \frac{2}{3}$ by definition.

8.4 Let (t_n) be a bounded sequence, i.e., there exists M such that $|t_n| \leq M$ for all n, and let (s_n) be a sequence such that $\lim s_n = 0$. Prove $\lim (s_n t_n) = 0$.

Answer: Since $\lim s_n = 0$, there exists an N such that $n > N \implies |s_n| < \epsilon$. Then, $|s_n| \cdot |t_n| < \epsilon \cdot |t_n| \implies |s_n t_n| < M\epsilon$ (multiply by $|t_n|$; assuming that $|t_n| \neq 0$, since clearly $\lim s_n t_n = 0$ when $|t_n| = 0$). Since ϵ is an arbitrary constant, we can take another $\epsilon_2 = \frac{\epsilon}{M} > 0$, which exists since $\mathbb{Q}$ is dense in $\mathbb{R}$. Then $|s_n t_n| < \epsilon_2$ and $\lim (s_n t_n) = 0$ by definition of limit.

8.5 (a) Consider three sequences (a_n), (b_n) and (s_n) such that $a_n \leq s_n \leq b_n$ for all n and $\lim a_n = \lim b_n = s$. Prove $\lim s_n = s$. This is called the "squeeze lemma."

Answer: Since $\lim a_n = s$, there exists an N_a such that $n > N_a \implies |a_n - s| < \epsilon \implies -\epsilon < a_n - s < \epsilon \implies s - \epsilon < a_n < s + \epsilon$. Similarly, since $\lim b_n = s$, there exists an N_b such that $n > N_b \implies |b_n - s| < \epsilon \implies s - \epsilon < b_n < s + \epsilon$. Then, since $a_n \leq s_n \leq b_n$ for all n, we also have $s - \epsilon < a_n \leq s_n \leq b_n < s + \epsilon$. Therefore $s - \epsilon < s_n < s + \epsilon$, which implies that $|s_n - s| < \epsilon$. Then $\lim s_n = s$ by definition of limit.

(b) Suppose (s_n) and (t_n) are sequences such that $|s_n| \leq t_n$ for all n and $\lim t_n = 0$. Prove $\lim s_n = 0$.

Answer: Since $\lim t_n = 0$, there exists an N such that $n > N \implies |t_n| < \epsilon$. Then we also know that $\lim \; -t_n = 0$ because $|-t_n| = |t_n| < \epsilon$. Then, since $|s_n| \leq t_n$, we have $-t_n \leq s_n \leq t_n$. Therefore $\lim s_n = 0$ by squeeze lemma.

8.6 Let (s_n) be a sequence in $\mathbb{R}$.

(a) Prove $\lim s_n = 0$ if and only if $\lim |s_n| = 0$.

Answer:

$\Rightarrow$: Since $\lim s_n = 0$, there exists an N such that $n > N \implies |s_n| < \epsilon$. Then $\lim |s_n| = 0$ by definition because $|(|s_n|)| = |s_n| < \epsilon$.

$\Leftarrow$: Since $\lim |s_n| = 0$, there exists an N such that $n > N \implies |(|s_n|)| < \epsilon$. Since $|(|s_n|)| = |s_n|$, we also have $|s_n| < \epsilon$. Then $\lim s_n = 0$ by definition.

(b) Observe that if $s_n = (-1)^n$, then $\lim |s_n|$ exists, but $\lim s_n$ does not exist.

Answer: $\lim |s_n| = \lim |(-1)^n| = \lim 1$. We will show that $\lim s_n$ does not exist by contradiction. Assume $\lim s_n = s$ exists, then there exists an N such that $n > N \implies |(-1)^n - s| < \epsilon$. We know that $|(-1)^n - s| = |-1-s|$ for odd n, $|1-s|$ for even n, and $|-1-s| \neq |1-n|$, then one of them must be nonzero. Then there would exist an $\epsilon > 0$ such that $\max\{|-1-s|, |1-s|\} \geq \epsilon$, which contradicts with $\lim s_n$ exists. Therefore $\lim s_n$ does not exist by contradiction.

8.9 Let (s_n) be a sequence that converges.

(a) Show that if $s_n \geq a$ for all but finitely many n, then $\lim s_n \geq a$.

Answer: By contradiction. Since $s_n \geq a$ for all but finitely many n, there exists an N_0 such that $s_n \geq a$ for all $n > N_0$. Then, let $s = \lim s_n$ and assume $s < a$, then pick $\epsilon = a - s > 0$. Select $N \geq N_0$, then by definition of limit, we have $n > N \implies |s_n - s| < \epsilon = a - s$. By expanding the previous expression we have $-a + s < s_n - s < a - s \implies s_n - s < a - s \implies s_n < a$, whcih contradicts with $s_n \geq a$ for $n > N_0$. Therefore $s = \lim s_n \geq a$ by contradiction.

(b) Show that if $s_n \leq b$ for all but finitely many n, then $\lim s_n \leq b$.

Answer: Again by contradiction. Since $s_n \leq b$ for all but finitely many n, there exists an $n > N_0$ such that $s_n \leq b$ for all $n > N_0$. Then, let $s = \lim s_n$ and assume $s > b$, then pick $\epsilon = s - b > 0$. Select $N \geq N_0$, we have $n > N \implies |s_n - s| < \epsilon = s - b$. Expanding the absolute value results in $-s + b < s_n - s < s - b \implies -s + b < s_n - s \implies b < s_n$, which contradicts with $s_n \leq b$ for $n > N_0$. Therefore $s = \lim s_n \leq b$ by contradiction.

(c) Conclude that if all but finitely many s_n belong to $[a, b]$, then $\lim s_n$ belongs to $[a, b]$.

Answer: By combining the previous parts, we have the following: if $a \leq s_n \leq b$ for finitely many s_n, then $a \leq \lim s_n \leq b$. In other words, by using interval notations, if finitely many $s_n \in [a, b]$, then $\lim \in [a, b]$.

8.10 Let (s_n) be a convergent sequence, and suppose $\lim s_n > a$. Prove there exists a number N such that $n > N$ implies $s_n > a$.

Answer: Let $\lim s_n = s$; since $s > a$, select $\epsilon = s - a > 0$. Then there exists an N such that $n > N \implies |s_n - s| < \epsilon$. By expanding the absolute value we have $-\epsilon < s_n - s < \epsilon$, which is equivalent to $s - \epsilon < s_n < s + \epsilon \implies s - \epsilon < s_n$. By substituting $\epsilon = s - a$ we have $s - (s - a) < s_n \implies a < s_n$.

9.1 Prove the following:

(a) $\lim \dfrac{n+1}{n} = 1$

Answer: $\lim \dfrac{n+1}{n} = \lim\ (1+\dfrac{1}{n})$ by multiplying $\dfrac{1}{n}$ to both the numerator and the denominator. Then, $\lim\ (1+\dfrac{1}{n}) = \lim 1 + \lim \dfrac{1}{n} = 1 + 0 = 1$ by sum limit law.

(b) $\lim \dfrac{3n+7}{6n-5} = \dfrac{1}{2}$

Answer: $\lim \dfrac{3n+7}{6n-5} = \lim \dfrac{3+\frac{7}{n}}{6-\frac{5}{n}}$ (multiply by $\frac{\frac{1}{n}}{\frac{1}{n}}$). Then, in the numerator, we have $\lim\ (3+\dfrac{7}{n}) = \lim 3 + \lim \dfrac{7}{n} = 3 + 0 = 3$ by sum limit law. Simiarly, in the denominator, we have $\lim\ (6-\dfrac{5}{n}) = \lim 6 - \lim \dfrac{5}{n} = 6 - 0 = 6$. Then $\lim \dfrac{3n+7}{6n-5} = \lim \dfrac{3}{6} = \dfrac{1}{2}$.

9.3 Suppose $\lim a_n = a$, $\lim b_n = b$, and $s_n = \dfrac{a_n^3+4a_n}{b_n^2+1}$. Prove $\lim s_n = \dfrac{a^3+4a}{b^2+1}$, using the limit theorems.

Answer: $\lim s_n = \lim \dfrac{a_n^3+4a_n}{b_n^2+1}$

$= \lim\ (a_n^3+4a_n) \cdot \lim \dfrac{1}{b_n^2+1}$ (Product limit law)

$= \dfrac{\lim\ (a_n^3+4a_n)}{\lim\ (b_n^2+1)}$ (Lemma 22)

$= \dfrac{\lim a_n^3 + \lim 4a_n}{\lim b_n^2 + \lim 1}$ (Sum limit law)

$= \dfrac{(\lim a_n)^3 + \lim 4a_n}{(\lim b_n)^2 + \lim 1}$ (Product limit law, applied to exponents)

$= \dfrac{(\lim a_n)^3 + 4 \cdot \lim a_n}{(\lim b_n)^2 + 1}$ (Theorem 19)

$= \dfrac{a^3+4a}{b^2+1}$ (Substitution)

9.9 Suppose there exists N_0 such that $s_n \leq t_n$ for all $n > N_0$.

(a) Prove that if $\lim s_n = +\infty$, then $\lim t_n = +\infty$.

Answer: Since $\lim s_n = +\infty$, there exists an N such that $n > N \implies s_n > M$ for all $M > 0$. Then, since $t_n \geq s_n$ for $n > N_0$, we have $n > \max\{N, N_0\} \implies t_n \geq s_n > M \implies t_n > M$. Therefore $\lim t_n = +\infty$ by definition of divergence.

(b) Prove that if $\lim t_n = -\infty$, then $\lim s_n = -\infty$.

Answer: Similarly, since $\lim t_n = -\infty$, there exists an N such that $n > N \implies t_n < M$ for all $M < 0$. Then, since $s_n \leq t_n$ for $n > N_0$, we have $n > \max\{N, N_0\} \implies s_n \leq t_n < M \implies s_n < M$. Therefore $\lim s_n = -\infty$ by definition of divergence.

(c) Prove that if $\lim s_n$ and $\lim t_n$ exist, then $\lim s_n \leq \lim t_n$.

Answer: Since we have $s_n \leq t_n$ for $n > N_0$, $t_n - s_n \geq 0$. Then $\lim\ (t_n - s_n) \geq 0$ by 8.9(a). Furthermore, $\lim t_n - \lim s_n \geq 0$ by sum limit law; therefore $\lim s_n \leq \lim t_n$ upon rearranging the inequality.

9.10 (a) Show that if $\lim s_n = +\infty$ and $k > 0$, then $\lim(ks_n) = +\infty$.

Answer: Since $\lim s_n = +\infty$, there exists an N such that $n > N \implies s_n > M$ for all $M > 0$.

Then we also have $n > N \implies ks_n > kM$ since $k > 0$. We can then take $M_1 = kM > 0$ and $n > N \implies ks_n > M_1$ for all $M_1 > 0$, then $\lim(ks_n) = +\infty$ by definition of divergence.

(b) Show $\lim s_n = +\infty$ if and only if $\lim(-s_n) = -\infty$.

Answer:

$\Rightarrow$: Since $\lim s_n = +\infty$, there exists an N such that $n > N \implies s_n > M$ for all $M > 0$. By multiplying -1 to both sides of the inequality (and flipping the inequality), we have $n > N \implies -s_n < -M$. Then let $M_1 = -M$; we know $M_1 < 0$ since $M > 0$. Then $n > N \implies -s_n < M_1$ for $M_1 < 0$, therefore $\lim(-s_n) = -\infty$ by definition of divergence.

$\Leftarrow$: Similarly, since $\lim(-s_n) = -\infty$, there exists an N such that $n > N \implies -s_n < M$ for all $M < 0$. Then we also have $s_n > -M$ by multiplying -1. Let $M_1 = -M > 0$, we have $n > N \implies s_n > M_1$, therefore $\lim s_n = +\infty$.

9.11 (a) Show that if $\lim s_n = +\infty$ and $\inf\{t_n : n \in \mathbb{N}\} > -\infty$, then $\lim(s_n + t_n) = +\infty$.

Answer: By definition of divergence, there exists an N_s such that $n > N_s \implies s_n > M$ for all $M > 0$. Additionally, we also have $t_n \geq \inf t_n$ for all $n \in \mathbb{N}$ by definition of infimum. Then $n > N \implies s_n + t_n \geq s_n + x > M$.

(b) Show that if $\lim s_n = +\infty$ and $\lim t_n > -\infty$, then $\lim(s_n + t_n) = +\infty$.

Answer: If t_n diverges to $+\infty$, $\lim s_n + t_n$ clearly diverges to $+\infty$ as well. If t_n converges to $\lim t_n = t \in \mathbb{R}$, there exists an N_t such that $n > N_t \implies |t_n - t| < \epsilon$ for $\epsilon > 0 \implies -\epsilon < t_n - t < \epsilon \implies t - \epsilon < t_n < t + \epsilon$. Since s_n diverges to $+\infty$, we can pick $N = \max(N_s, N_t)$ (N_s same as part (a)); then $n > N \implies s_n + t_n > s_n + (t - \epsilon) > M$.

(c) Show that if $\lim s_n = +\infty$ and if (t_n) is a bounded sequence, then $\lim(s_n + t_n) = +\infty$.

Answer: Similarly, since (t_n) is bounded, then $n > N_s \implies s_n + t_n \geq s_n + \inf t_n > M$.

9.12 Assume all $s_n \neq 0$ and that the limit $L = \lim \left|\frac{s_{n+1}}{s_n}\right|$ exists.

(a) Show that if $L < 1$, then $\lim s_n = 0$.

Answer: Let $a \in \mathbb{R}$ such that $L < a < 1$ and let $\epsilon = a - L > 0$. Then since $L = \lim \left|\frac{s_{n+1}}{s_n}\right|$, there exists an N such that $n > N \implies \left|\left|\frac{s_{n+1}}{s_n}\right| - L\right| < \epsilon \implies -a + L < \left|\frac{s_{n+1}}{s_n}\right| - L < a - L \implies \left|\frac{s_{n+1}}{s_n}\right| < a$. Therefore $|s_{n+1}| < a|s_n|$ for $n > N$. We will now show $|s_n| < a^{n-N}|s_N|$ for $n > N$ by induction as follows.

Base case ($n = N + 1$): We want to show that $|s_{n+1}| < a|s_N|$, which is true as shown above.

Inductive step: Assume $|s_n| < a^{n-N}|s_N|$, we have $a \cdot |s_n| < a \cdot a^{n-N}|s_N|$ upon multiplying a to both sides. Then, since $|s_{n+1}| < a|s_n|$ for $n > N$, $|s_{n+1}| < a \cdot |s_n| < a \cdot a^{n-N}|s_N| = a^{(n+1)-N}|s_N|$, which implies that $|s_{n+1}| < a^{(n+1)-N}|s_N|$. Therefore $|s_n| < a^{n-N}|s_N|$ by mathematical induction.

Since all $s_n \neq 0$, we have $0 \leq |s_n| \leq a^{n-N}|s_N|$. In addition, we know that $\lim a^{n-N}|s_N| = 0$ since $L < a < 1$. Then $\lim |s_n| = 0$ by squeeze lemma and therefore $\lim s_n = 0$.

(b) Show that if $L > 1$, then $\lim |s_n| = +\infty$.

Answer: Let $t_n = \frac{1}{s_n}$, then $\lim \left|\frac{t_{n+1}}{t_n}\right| = \lim \left|\frac{\frac{1}{s_{n+1}}}{\frac{1}{s_n}}\right| = \lim \left|\frac{s_n}{s_{s_{n+1}}}\right| = \frac{1}{L}$. Since $L > 1$, $\frac{1}{L} < 1$;

then $\lim t_n = 0$ by part (a). Therefore $\lim \left|\frac{1}{s_n}\right| = \lim |t_n| = \lim t_n$ and $\lim |s_n| = +\infty$ by Theorem 9.10.

10.5 Prove Theorem 10.4(ii).

Answer: Let (s_n) be an unbounded decreasing sequence and let $M < 0$. Then, since (s_n) is decreasing, it must be bounded above by s_1; since it is also unbounded, it must be unbounded below, i.e. $s_N < M$ for some N. Then we have $n > N \implies s_n \leq s_N < M$, therefore $\lim s_n = -\infty$.

10.6 (a) Let (s_n) be a sequence such that

$$|s_{n+1} - s_n| < 2^{-n} \text{ for all } n \in \mathbb{N}.$$

Prove (s_n) is a Cauchy sequence and hence a convergent sequence.

Answer:

(b) Is the result in (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$?

Answer: No.

10.7 Let S be a bounded nonempty subset of $\mathbb{R}$ such that $\sup S$ is not in S. Prove there is a sequence (s_n) of points in S such that $\lim s_n = \sup S$.

Answer: By definition of supremum, $\sup S$ is the least upper bound; then we can take $m_n = \sup S - \frac{1}{n}$ which cannot be an upper bound since $m < \sup S$. Then, there must exist $s \in S$ such that $\sup S > s_n > m_n$. Since $\sup S$ is constant, $\lim \sup S = \sup S$. In addition, $\lim m_n = \lim \sup S - \lim \frac{1}{n} = \sup S - 0 = \sup S$. Then we have $m < \sup S$ with $\lim m_n = \lim \sup S = \sup S$; therefore $\lim s_n = \sup S$ by squeeze lemma.

P1 Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences of real numbers. Assume that $(x_n)_{n\in\mathbb{N}}$ is convergent and that the set

$$\{n \in \mathbb{N} : x_n \neq y_n\}$$

is finite. Prove that $(y_n)_{n\in\mathbb{N}}$ is also convergent and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n$.

Answer: Let $\lim_{n\to\infty} x_n = s$ and $\epsilon > 0$. Since (x_n) is convergent, there exists an N_x such that $n > N_x \implies |x_n - s| < \epsilon$. In addition, since the set $\{n \in \mathbb{N} : x_n \neq y_n\}$ is finite, there exists an N_0 such that $n > N_0 \implies x_n = y_n$. Take $N = \max\{N_0, N_x\}$ so that both are true, then $n > N \implies |y_n - s| = |x_n - s| < \epsilon$. Therefore $\lim_{n\to\infty} y_n = s = \lim_{n\to\infty} x_n$.