1. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Prove that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} a_n$ is convergent and

$$\left| \sum_{n=N+1}^{\infty} a_n \right| < \epsilon.$$

Proof: Since $\sum_{n=N+1}^{\infty} a_n$ is a subsequence of a converging sequence of partial sums (definition of series), it must also converge by Theorem 11.3. In addition, $\sum_{n=1}^{\infty} a_n$ must satisfy the Cauchy criterion, i.e. for $\epsilon > 0$ there exists an N such that $n \geq m > N \implies |\sum_{k=m}^{n} a_k| < \epsilon$. We can take $n = \infty$ and $m = N + 1$ which gives us $\left|\sum_{k=N+1}^{\infty} a_k\right| < \epsilon$.

2. Let $(a_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$a_n = \begin{cases} \left(\frac{n^2-2}{6n^2+n}\right)^n & \text{if } n = 2k-1 \text{ for some } k \in \mathbb{N} \\ \frac{4n}{5^n} & \text{if } n = 2k \text{ for some } k \in \mathbb{N}. \end{cases}$$

Does the series $\sum_{n=1}^{\infty} a_n$ converge? Justify your answer.

Claim: The series $\sum_{n=1}^{\infty} a_n$ does converge.

Proof: We can start by checking each part of the piecewise definition:

$\sum \left(\frac{n^2-2}{6n^2+n}\right)^n \implies \sqrt[n]{|a_n|} = \frac{n^2-2}{6n+n} \to \frac{1}{6} < 1$, therefore converges by root test.

$\sum \frac{4n}{5^n} \implies \left|\frac{a_{n+1}}{a_n}\right| = \frac{4(n+1)}{5^{n+1}} \cdot \frac{5^n}{4n} = \frac{4n+4}{20n} \to \frac{1}{5} < 1$, therefore converges by ratio test.

Then, we can rewrite the piecewise definition into a sum of two series (in terms of k) as follows:

$$\sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty} \left(\frac{(2k-1)^2-2}{6(2k-1)^2+(2k-1)}\right)^{2k-1} + \sum_{k=1}^{\infty} \frac{8k}{5^{2k}}$$

Since $\left(\frac{n^2-2}{6n^2+n}\right)^n > 0$ for $n \geq 2$, removing the even terms (which are all positive) results in a smaller sum overall, i.e. $\sum_{k=1}^{\infty} \left(\frac{(2k-1)^2-2}{6(2k-1)^2+(2k-1)}\right)^{2k-1} \leq \sum_{n=1}^{\infty} \left(\frac{n^2-2}{6n^2+n}\right)^n$. Then, since $\sum_{n=1}^{\infty} \left(\frac{n^2-2}{6n^2+n}\right)^n$ converges, $\sum_{k=1}^{\infty} \left(\frac{(2k-1)^2-2}{6(2k-1)^2+(2k-1)}\right)^{2k-1}$ must converge as well by comparison test.

Similarly, since $\frac{4n}{5^n} > 0$ for $n \geq 1$, then removing the odd terms results in a smaller sum, i.e. $\sum_{k=1}^{\infty} \frac{8k}{5^{2k}} \leq \sum \frac{4n}{5^n}$ which means that $\sum_{k=1}^{\infty} \frac{8k}{5^{2k}}$ also converges.

Therefore $\sum_{n=1}^{\infty} a_n$ is a sum of two converging series and must also converge by limit sum law.

3. Let (s_n) and (t_n) be sequences with $\limsup s_n$ and $\liminf t_n$ finite. Suppose $\liminf t_n = (\limsup s_n) + 2$. Prove that there is N such that $t_n - s_n > 1$ for all $n > N$.

Proof: By definition, $\limsup s_n = \lim_{N\to\infty}\sup\{s_n : n > N\}$. Let $v_N = \sup\{s_n : n > N\}$ and $s = \lim_{N\to\infty} v_N = \limsup s_n$. Then by definition of limit, pick $\epsilon = \frac{1}{2} > 0$, there exists an N_s such that $N > N_s \implies |v_N - s| < \frac{1}{2} \implies -\frac{1}{2} + s < v_N < \frac{1}{2} + s$. By definition of supremum, $v_N \geq s_n$ for $n > N$. Combining the inequalities together we have $N > N_s \implies s_n \leq v_N < \frac{1}{2} + s$.

Similarly, let $u_N = \inf\{t_n : n > N\}$ and $t = \lim_{N\to\infty} u_N = \liminf t_n$. Then there exists an N_t such that $N > N_t \implies |u_N - t| < \frac{1}{2} \implies -\frac{1}{2} + t < u_N < \frac{1}{2} + t$. By definition of infimum, $u_N \leq t_n$ for $n > N$. Then we have $N > N_t \implies -\frac{1}{2} + t < u_N \leq t_n$.

Then if we take $N > \max\{N_s, N_t\}$, we have both $s_n < \frac{1}{2} + s$ and $-\frac{1}{2} + t < t_n$. In addition, using the given assumption $\liminf t_n = (\limsup s_n) + 2$, we have $t = s + 2$. Then by substitution $-\frac{1}{2} + s + 2 < t_n \implies \frac{1}{2} + s < t_n - 1$ which when combined with $s_n < \frac{1}{2} + s$ gives us $s_n < \frac{1}{2} + s < t_n - 1 \implies t_n - s_n > 1$.

4. Let f and g be real-valued functions defined on all of $\mathbb{R}$, and let $x_0 \in \mathbb{R}$. Suppose both f and g are continuous at the point x_0, and that $f(x) \leq g(x)$ for all $x < x_0$. Prove that $f(x_0) \leq g(x_0)$.

   Proof: Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a sequence of rational numbers (r_n) that converges to x_0 where $r_n < x_0$ for all n. Furthermore, since $f(x) \leq g(x)$ for all $x < x_0$, we have $f(r_n) \leq g(r_n)$ for all n, which implies that $\lim f(r_n) \leq \lim g(r_n)$. Then, since f is continuous at x_0, we have $\lim f(r_n) = f(x_0)$ by definition of continuity. Similarly, $\lim g(r_n) = g(x_0)$. Connecting the previous equations and inequality gives us $f(x_0) = \lim f(r_n) \leq \lim g(r_n) = g(x_0)$, therefore $f(x_0) \leq g(x_0)$.

5. Let (s_n) be a bounded sequence in $\mathbb{R}$. Suppose that the set

$$\{n \in \mathbb{N} : s_n \leq 0\}$$

is infinite. Prove there is a subsequence (t_k) of (s_n) which is convergent and has $\lim t_k \leq 0$.

Proof: Let $\epsilon > 0$, then $s_n \leq 0 \implies s_n + \epsilon \leq \epsilon$. We can take $t = -\epsilon < 0$ which gives us $s_n - t = s_n + \epsilon < \epsilon \implies |s_n - t| \leq \epsilon$. Then the set $\{n \in \mathbb{N} : |s_n - t| \leq \epsilon\}$ is infinite; by Theorem 11.2(i) there exists a subsequence (t_k) of (s_n) converging to $t < 0$ as defined, i.e. $\lim t_k \leq 0$.