Math 131A Homework 1

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1.1 Prove $1^2+2^2+...+n^2=\frac{1}{6}n(n+1)(2n+1)$ for all positive integers n.

Answer: By induction.

Base case ($n=1$): $1=\frac{1}{6}(1+1)(2+1) \implies 1=1$ which is true.

Inductive step: Assume $1^2+2^2+...+n^2=\frac{1}{6}n(n+1)(2n+1)$ is true, we want to show that $1^2+2^2+...+n^2+(n+1)^2=\frac{1}{6}(n+1)(n+2)(2n+3)$ is true. We can do so by adding $(n+1)^2$ to both sides of the equation as follows:

$$1^2+2^2+...+n^2+(n+1)^2=\frac{1}{6}n(n+1)(2n+1)+(n+1)^2$$

Expanding the right hand side results in the following:

$$1^2+2^2+...+n^2+(n+1)^2=\frac{1}{3}n^3+\frac{3}{2}n^2+\frac{13}{6}n+1$$

Which indeed factors into

$$\implies 1^2+2^2+...+n^2+(n+1)^2=\frac{1}{6}(n+1)(n+2)(2n+3).$$

Therefore $1^2+2^2+...+n^2=\frac{1}{6}n(n+1)(2n+1)$ for all positive integers n by mathematical induction.

1.9 (a) Decide for which integers the inequality $2^n>n^2$ is true.

Answer: $2^n>n^2, n\in\mathbb{Z}$ is true for $n=0,1$ and $n>4$.

(b) Prove your claim in (a) by mathematical induction.

Answer: We will first show $n=0$ and $n=1$ case-by-case, then $n>4$ by induction.

$n=0$: $2^n>n^2 \implies 1>0$ which is true

$n=1$: $2^n>n^2 \implies 2>1$ which is true

$n>4$: By induction as follows.

Base case: ($n=5$) $2^n>n^2 \implies 32>25$ which is true

Indcutive step: Assume $2^n>n^2$ is true, we want to show that $2^{n+1}>(n+1)^2$ is also true. We can start by multiplying 2 to both sides of the inequality: $2*2^n>2*n^2 \implies 2^{n+1}>2n^2$. Then, if we could show that $2n^2>(n+1)^2$ for $n\geq 5$, $2^{n+1}>(n+1)^2$ would be also true. We will do so using another proof by induction:

Base case:

Inductive step:

1.11 For each $n \in \mathbb{N}$, let P_n denote the assertion "$n^2 + 5n + 1$ is an even integer."

(a) Prove P_{n+1} is true whenever P_n is true.

(b) For which n is P_n actually true? What is the moral of this exercise?

3.1

3.6

3.7

3.8

4.6

4.7

4.8

4.14

4.15

4.16

P1 Write down the converse and the contrapositive of the following statement regarding a real number x:

$$\text{If } x > 0, \text{then } x^2 - x > 0.$$

Then determine which (if any) of the three statements are true for all real numbers x.
Answer: Converse: if $x^2 - x > 0$, then $x > 0$, which is false by counterexample $x = -1$. Contrapositive: if $x^2 - x \leq 0$, then $x \leq 0$, which is false by counterexample $x = 1$.

P2 Prove that $\sqrt{3}$ is not rational.
Answer: By contradiction. Suppose $\sqrt{3}$ is rational, then by definition of rational numbers, there must exist $p, q \in \mathbb{Z}$ such that $\dfrac{p}{q} = \sqrt{3}$, where p, q have no common factors upon simplying. Then, we also have $\dfrac{p^2}{q^2} = 3 \implies p^2 = 3q^2$. Since $p, q \in \mathbb{Z}$ and by extension $p^2, q^2 \in \mathbb{Z}$, $p^2 \mid 3$. Additionally, $p \mid 3$ because $\sqrt{3} \notin \mathbb{Z}$. Then, $p^2 \mid 9$. By substituting $p^2 = 3q^2$, we now have $3q^2 \mid 9 \implies q^2 \mid 3$, implying $q \mid 3$ by previous logic. Then p, q have common factor 3 whcih contradicts with our initial assumption. Therefore $\sqrt{3}$ is not rational.