Math 131A Homework 4

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14.2 Determine which of the following series converge. Justify your answers.

(c) $\sum \frac{3n}{n^3} = 3\sum \frac{1}{n^2}$ which converges by p-test.

(d) $\sum \frac{n^3}{3n} = \sum \frac{n^2}{3}$ which diverges since $\lim \frac{n^2}{3} > 0$.

(e) $\sum \frac{n^2}{n!} = \sum \frac{n}{(n-1)!} \implies \left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{n!} \cdot \frac{(n-1)!}{n} = \frac{n+1}{n^2} \to 0 < 1$, therefore it converges by ratio test.

(f) $\sum \frac{1}{n^n} \implies \sqrt[n]{|a_n|} = \frac{1}{n} \to 0 < 1$, therefore it converges by root test.

14.3 (a) $\sum \frac{1}{\sqrt{n!}} \implies \left|\frac{a_{n+1}}{a_n}\right| = \frac{\sqrt{n!}}{\sqrt{(n+1)!}} = \sqrt{\frac{1}{n+1}} \to 0 < 1$, therefore it converges by ratio test.

(b) $\sum \frac{2+\cos n}{3^n} \implies \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2+\cos(n+1)}{3^{n+1}} \cdot \frac{3^n}{2+\cos n}\right| = \left|\frac{2+\cos(n+1)}{6+\cos n}\right|$; since $|\cos n| \leq 1$, $\left|\frac{2+\cos(n+1)}{6+\cos n}\right| \leq \left|\frac{2+1}{6-1}\right| = \frac{3}{5} < 1$. Then $\lim\sup \left|\frac{2+\cos(n+1)}{6+\cos n}\right| < 1$ as well and the series converges by ratio test.

(c) $\sum \frac{1}{2^n+n} \implies \left|\frac{a_{n+1}}{a_n}\right| = \frac{2^n+n}{2^{n+1}+n+1} \to \frac{1}{2} < 1$, therefore it converges by ratio test.

(d) $\sum (\frac{1}{2})^n(50 + \frac{2}{n}) \implies \left|\frac{a_{n+1}}{a_n}\right| = \frac{(\frac{1}{2})^{n+1}(50+\frac{2}{n+1})}{(\frac{1}{2})^n(50+\frac{2}{n})} = \frac{50+\frac{2}{n+1}}{100+\frac{4}{n}} \to \frac{1}{2} < 1$, therefore it converges by ratio test.

14.6 (a) Prove that if $\sum |a_n|$ converges and (b_n) is a bounded sequence, then $\sum a_n b_n$ converges.

Proof: Since $\sum |a_n|$ converges, there exists an N such that $n \geq m > N \implies |\sum_{k=m}^{n} |a_k|| < \epsilon_a$ for $\epsilon_a > 0$. In addition, since (b_n) is bounded, there exists an $M > 0$ such that $M \geq |b_n|$ for all n. Then we can select $\epsilon = M\epsilon_a > 0$, resulting in $n \geq m > N \implies M|\sum_{k=m}^{n} |a_k|| < M\epsilon_a = \epsilon$. Since $M \geq |b_n|$, we have $|\sum_{k=m}^{n} a_n b_n| \leq |\sum_{k=m}^{n} |a_k b_k|| \leq M|\sum_{k=m}^{n} |a_k||$ by Exercise 3.6(b). Therefore $n \geq m > N \implies |\sum_{k=m}^{n} a_n b_n| \leq \epsilon$ and $\sum a_n b_n$ converges by Theorem 14.4.

14.7 Prove that if $\sum a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum a_n^p$ converges.

Proof: Since $\sum a_n$ is convergent, $\lim a_n = 0$ by Corollary 14.5. Then, let $\epsilon = 1 > 0$, there exists an N such that $n > N \implies a_n = |a_n| < \epsilon$ ($a_n = |a_n|$ as a_n is nonnegative). Therefore for $n > N$, $a_n < a_n^p$. Then $\sum_{n=N+1}^{\infty} a_n^p < \sum_{n=N+1}^{\infty} a_n$ converges by comparison test and $\sum a_n^p$ converges.

14.12 Let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that $\lim\inf |a_n| = 0$. Prove there is a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that $\sum_{k=1}^{\infty} a_{n_k}$ converges.

Proof: By Theorem 11.7, there exists a monotonic subsequence (a_{n_k}) of $(|a_n|)$ that converges to $\lim\inf |a_n| = 0$. Then take $\epsilon = \frac{1}{2} > 0$, there exists an N such that $n > N_1 \implies |a_{n_k}| < \epsilon = \frac{1}{2}$ by definition of convergence. We can find the next term by setting $\epsilon = \frac{1}{4} > 0$, then there exists an N_2

such that $n > N_2 \implies |a_{n_k}| < \epsilon = \frac{1}{4}$. Following the same method, we can construct a series where each term a_{n_k} is bounded by $\frac{1}{2^k}$. Then the series is bounded by $\sum \frac{1}{2^k}$ which is convergent, therefore it is also convergent by Comparison Test.

17.3 Prove the following functions are continuous:

(a) $\log_e(1+\cos^4 x)$:

Proof: Since $\cos x$ is given to be continuous, so is $\cos^4 x$ upon applying Theorem 17.4(ii) four times. Then $1+\cos^4 x$ by Theorem 17.4(i). Furthermore, since $\log_e x$ is also continuous, the composite function $\log_e(1+\cos^4 x)$ is continuous by Theorem 17.5.

(c) 2^{x^2}

Proof: Since x^2 and 2^x are both continuous functions as given, their composite function 2^{x^2} is also continuous.

(f) $x\sin(\frac{1}{x})$ for $x \neq 0$

Proof: $\frac{1}{x} = x^{-1}$ which is continuous as given; $\sin x$ is also continuous therefore the composite $\sin(\frac{1}{x})$ is continuous. In addition, $x = x^1$ which is continuous, then $x\sin(\frac{1}{x})$ is continuous by Theorem 17.4(ii).

17.5 (a) Prove that if $m \in \mathbb{N}$, then the function $f(x) = x^m$ is continuous on $\mathbb{R}$.

Proof: Suppose $\lim x_n = x_0$, then $\lim f(x_n) = \lim x^m = (\lim x)^m = x_0^m = f(x_0)$. Therefore $f(x) = x^m$ is continuous by definition.

(b) Prove every *polynomial function* $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is continuous on $\mathbb{R}$.

Proof: Since x^m is continuous as shown above for $m \in \mathbb{N}$, $a_m x^m$ is also continuous by Theorem 17.3. Then $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is continuous upon applying Theorem 17.4(i) to all terms.

17.6 A *rational function* is a function f of the form p/q where p and q are polynomial functions. The domain of f is $\{x \in \mathbb{R} : q(x) \neq 0\}$. Prove every rational function is continuous.

Proof: Since all polynomial functions are continuous as shown in 17.5(b), all p and q are continuous. Then, with the domain constraint that $q(x) \neq 0$, p/q is continuous by Theorem 17.4(iii).

17.9 Prove each of the following functions is continuous at x_0 by verifying the ϵ-δ property of Theorem 17.2.

(b) $f(x) = \sqrt{x}, x_0 = 0$

Proof: Let $\epsilon > 0$ and $\delta = \epsilon^2$, then for $|x - x_0| = |x| < \delta = \epsilon^2$, we have $|f(x) - f(x_0)| = |\sqrt{x}| < \sqrt{\delta} = \epsilon$.

(d) $g(x) = x^3, x_0$ arbitrary

Proof: Let $x_0 \in \mathbb{R}$ and $\epsilon > 0$, then $|f(x) - f(x_0)| = \left|x^3 - x_0^3\right| = \left|(x - x_0)(x^2 + xx_0 + x_0^2)\right| = |x - x_0| \cdot \left|x^2 + xx_0 + x_0^2\right|$. To bound the second term, we can choose $\delta = 1$, then $|x - x_0| < 1 \implies |x| < 1 + |x_0|$; by substitution, $\left|x^2 + xx_0 + x_0^2\right| \leq |x|^2 + |x| \cdot |x_0| + |x_0|^2 < (1 + |x_0|)^2 + |x_0|(1 + |x_0|) + |x_0|^2 = 3|x_0|^2 + 3|x_0| + 1$. Thus $|x - x_0| \cdot \left|x^2 + xx_0 + x_0^2\right| \leq |x - x_0|(3|x_0|^2 + 3|x_0| + 1)$. Then we can take $\delta = \min\left\{1, \frac{\epsilon}{3|x_0|^2 + 3|x_0| + 1}\right\}$ and by construction $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$, therefore $g(x)$ is continuous.

17.10 Prove the following functions are discontinuous at the indicated points.

(b) $g(x) = \sin(\frac{1}{x})$ for $x \neq 0$ and $g(0) = 0, x_0 = 0$

Proof: Assume that $g(x)$ is continuous, then by definition of continuity, $\lim g(x) = \lim \sin(\frac{1}{x}) = g(x_0) = 0$. However, $\sin(t)$ is not convergent and therefore does not have a limit. Then our assumption is false and $g(x)$ is discontinuous.

17.12 (a) Let f be a continuous real-valued function with domain (a, b). Show that if $f(r) = 0$ for each rational number r in (a, b), then $f(x) = 0$ for all $x \in (a, b)$.

Proof: Since $(a, b) \in \mathbb{R}$ and $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a rational sequence r_n such that $r_n \to x$, then $f(r_n) = 0$ since $f(r) = 0$ for $r \in (a, b)$. Since f is continuous, by definition of continuity, we have $f(x) = \lim_n f(r_n) = 0$.

(b) Let f and g be continuous real-valued functions on (a, b) such that $f(r) = g(r)$ for each rational number r in (a, b). Prove $f(x) = g(x)$ for all $x \in (a, b)$.

Proof: Let $h = f - g$, then h is continuous by Theorems 17.3 and 17.4(i). Since $f(r) = g(r)$, we have $h(r) = 0$ for $r \in (a, b)$. Then by part (a), for all $x \in (a, b)$, $h(x) = 0 \implies f(x) - g(x) = 0 \implies f(x) = g(x)$.

18.5 (a) Let f and g be continuous functions on $[a, b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove $f(x_0) = g(x_0)$ for at least one x_0 in $[a, b]$.

Proof: Since f and g are both continuous, so is $h = f + (-1)g = f - g$ by Theorems 17.3 and 17.4(i). Then, since $f(a) \geq g(a)$, $h(a) \geq 0$. Similarly, since $f(b) \leq g(b)$, $h(b) \leq 0$. Then h must have a root on the interval $[a, b]$ by Intermediate Value Theorem, i.e. $h(x_0) = 0 \implies f(x_0) - g(x_0) = 0 \implies f(x_0) = g(x_0)$.

18.6 Prove $x = \cos x$ for some x in $(0, \frac{\pi}{2})$.

Proof: Let $f(x) = \cos x$ and $g(x) = x$. Since $f(0) = \cos 0 = 1$ and $g(0) = 0$, $f(0) \geq g(0)$. Similarly, since $f(\frac{\pi}{2}) = 0$ and $g(\frac{\pi}{2})$, we have $f(\frac{\pi}{2}) \leq g(\frac{\pi}{2})$. Then there must exist at least one x_0 in $[0, \frac{\pi}{2}]$ such that $f(x_0) = g(x_0)$ as shown in Exercise 18.5(a).

18.9 Prove that a polynomial function f of odd degree has at least one real root.

Proof: Since f is a polynomial of odd degree, we have $f(x) = a_0 + a_1 x + \ldots + a_n x^n$ where n is odd and $a_n \neq 0$. Suppose $a_n > 0$ and since x^n diverges, $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to +\infty} f(x) = \infty$. Then $f(x)$ must have a root on $(-\infty, +\infty)$ by Intermediate Value Theorem. Similarly, if $a_n < 0$, we have $\lim_{x \to -\infty} f(x) = +\infty$ and $\lim_{x \to +\infty} f(x) = -\infty$ and again $f(x)$ must have a root on $(-\infty, +\infty)$ by Intermediate Value Theorem.

P1 Let f, g, h be real-valued functions defined on subsets of $\mathbb{R}$. Suppose for all x, $f(x) \leq g(x) \leq h(x)$, $f(x_0) = h(x_0)$, and f and h are continuous at x_0. Prove that g is continuous at x_0.

Proof: Since f is continuous at x_0, then by definition of continuity, we have $\lim_n f(x_n) = x_0$ for all sequences (x_n) converging to x_0. Similarly, we also have $\lim_n f(x_n) = x_0$ for all sequences $(x_n) \to x_0$. Then, since $f(x) \leq g(x) \leq h(x)$ for all x, applying the f, g, h to x_n gives us $f(x_n) \leq g(x_n) \leq h(x_n)$. Therefore $\lim_n f(x_n) = \lim_n g(x_n) = \lim_h f(x_n) = x_0$ by squeeze lemma, and g is continuous at x_0 by definition of continuity.