Math 131A Homework 1

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1.1 Prove $1^2+2^2+...+n^2=\frac{1}{6}n(n+1)(2n+1)$ for all positive integers n.

Answer: By induction.

Base case ($n=1$): $1=\frac{1}{6}(1+1)(2+1) \implies 1=1$ which is true.

Inductive step: Assume $1^2+2^2+...+n^2=\frac{1}{6}n(n+1)(2n+1)$ is true, we want to show that $1^2+2^2+...+n^2+(n+1)^2=\frac{1}{6}(n+1)(n+2)(2n+3)$ is true. We can do so by adding $(n+1)^2$ to both sides of the equation as follows:

$$1^2+2^2+...+n^2+(n+1)^2=\frac{1}{6}n(n+1)(2n+1)+(n+1)^2$$

Expanding the right hand side results in the following:

$$1^2+2^2+...+n^2+(n+1)^2=\frac{1}{3}n^3+\frac{3}{2}n^2+\frac{13}{6}n+1$$

Which indeed factors into

$$\implies 1^2+2^2+...+n^2+(n+1)^2=\frac{1}{6}(n+1)(n+2)(2n+3).$$

Therefore $1^2+2^2+...+n^2=\frac{1}{6}n(n+1)(2n+1)$ for all positive integers n by mathematical induction.

1.9 (a) Decide for which integers the inequality $2^n>n^2$ is true.

Answer: $2^n>n^2, n\in\mathbb{Z}$ is true for $n=0,1$ and $n>4$.

(b) Prove your claim in (a) by mathematical induction.

Answer: We will first show $n=0$ and $n=1$ case-by-case, then $n>4$ by induction.

$n=0$: $2^n>n^2 \implies 1>0$ which is true

$n=1$: $2^n>n^2 \implies 2>1$ which is true

$n>4$: By induction as follows.

Base case: ($n=5$) $2^n>n^2 \implies 32>25$ which is true

Indcutive step: Assume $2^n>n^2$ is true, we want to show that $2^{n+1}>(n+1)^2$ is also true. We can start by multiplying 2 to both sides of the inequality: $2*2^n>2*n^2 \implies 2^{n+1}>2n^2$. Then, if we could show that $2n^2>(n+1)^2$ for $n\geq 5$, $2^{n+1}>(n+1)^2$ would be also true. We will do so using another proof by induction:

Base case: $(n = 5)$: $2n^2 > (n+1)^2 \implies 50 > 36$ which is true

Inductive step: Assume $2n^2 > (n+1)^2$ is true, we want to show that $2(n+1)^2 > (n+2)^2$. By expanding the right hand side of the assumption, we have $2n^2 > n^2+2n+1 \implies n^2 > 2n+1$. $2(n+1)^2 = 2n^2+4n+2, (n+2)^2 = n^2+4n+4$

1.11 For each $n \in \mathbb{N}$, let P_n denote the assertion "$n^2 + 5n + 1$ is an even integer."

(a) Prove P_{n+1} is true whenever P_n is true.

Answer: Using the definition of P_n, P_{n+1} corresponds to the expression $(n+1)^2+5(n+1)+1$. To show that it is even, we can first expand it into $(n+1)^2+5(n+1)+1 = n^2+7n+7$. Then, $P_{n+1} - P_n = n^2+7n+7-n^2-5n-1 = 2n+6$ which is always even for $n \in \mathbb{N}$. Therefore, if P_n is even, then $P_n + (P_{n+1} - P_n) = P_{n+1}$ must also be even.

(b) For which n is P_n actually true? What is the moral of this exercise?

Answer: For $n^2 + 5n + 1$ to be even, $n^2 + 5n = n(n+5)$ must be odd. However, this is not possible for $n \in \mathbb{N}$. On one hand, if n is even, $n(n+5)$ would have a factor of 2 from n and would therefore be even; on the other hand, if n is odd, then $(n+5)$ would be even and $n(n+5)$ as well. The moral of this exercise is that finding a true base case is important in mathematical induction.

3.1 (a) Which of the properties A1-A4, M1-M4, DL, O1-O5 fail for $\mathbb{N}$?

Answer: A4 fails since there are no negative numbers in $\mathbb{N}$; M4 fails since there are no fractions.

(b) Which of these properties fail for $\mathbb{Z}$?

Answer: M4 fails since there are no fractions in $\mathbb{Z}$.

3.6 (a) Prove $|a+b+c| \leq |a|+|b|+|c|$ for all $a, b, c \in \mathbb{R}$.

Answer: Let $m = a+b$, then since $|a+b| \leq |a|+|b|$ (triangle inequality), $|m| \leq |a|+|b|$. Then we can add $|c|$ to both sides of the inequality, resulting in $|m|+|c| \leq |a|+|b|+|c|$. Since $|m+c| \leq |m|+|c|$, we have $|m+c| \leq |m|+|c| \leq |a|+|b|+|c| \implies |m+c| \leq |a|+|b|+|c|$. Then by substituting $m = a+b$ we have $|a+b+c| \leq |a|+|b|+|c|$.

(b) Use induction to prove

$$|a_1 + a_2 + \ldots + a_n| \leq |a_1| + |a_2| + \ldots + |a_n|$$

for n numbers $a_1, a_2, ..., a_n$.

Answer:

Base case: $(n = 1)$: $|a_1| \leq |a_1|$ which is true.

Inductive step: Assume the statement holds for n numbers $a_1, a_2, \ldots, a_n$. Let $m = a_1+a_2+\ldots+a_n$, then we have $|m| \leq |a_1|+|a_2|+\ldots+|a_n|$. Adding $|a_{n+1}|$ to both sides of the inequality gives us $|m|+|a_{n+1}| \leq |a_1|+|a_2|+\ldots+|a_n|+|a_{n+1}|$. Then using $|m+a_{n+1}| \leq |m|+|a_{n+1}|$ (similar to previous part), we have $|m+a_{n+1}| \leq |a_1|+|a_2|+\ldots+|a_n|+|a_{n+1}|$. Then by substitution, $|a_1+a_2+\ldots+a_n+a_{n+1}| \leq |a_1|+|a_2|+\ldots+|a_n|+|a_{n+1}|$.

Therefore $|a_1+a_2+\ldots+a_n| \leq |a_1|+|a_2|+\ldots+|a_n|$ by mathematical induction.

3.7 (a) Show $|b| < a$ if and only if $-a < b < a$.

Answer:

$\Rightarrow$: By cases.

Case 1 ($b \geq 0$): we have $|b| = b$ by definition of aboslute value. Then, $|b| < a \implies b < a$.

Case 2 ($b \leq 0$): $|b| = -b$, then $|b| < a \implies -b < a \implies b > -a$.

Therefore $-a < b < a$.

$\Leftarrow$: Also by cases.

Case 1 ($b \geq 0$): $b = |b|$ by definition; then $-a < |b| < a$.

Case 2 ($b \leq 0$): $-a < b < a \implies a > -b > -a$ (multiply by -1). Since $-b = |b|$, $a > -b > -a \implies a > |b| > -a \implies -a < |b| < a$.

Therefore $|b| < a$.

(b) Show $|a - b| < c$ if and only if $b - c < a < b + c$.

(c) Show $|a - b| \leq c$ if and only if $b - c \leq a \leq b + c$.

3.8 Let $a, b \in \mathbb{R}$. Show if $a \leq b_1$ for every $b_1 > b$, then $a \leq b$.

4.6

4.7

4.8

4.14

4.15

4.16

P1 Write down the converse and the contrapositive of the following statement regarding a real number x:

$$\text{If } x > 0, \text{then } x^2 - x > 0.$$

Then determine which (if any) of the three statements are true for all real numbers x.

Answer: Converse: if $x^2 - x > 0$, then $x > 0$, which is false by counterexample $x = -1$. Contrapositive: if $x^2 - x \leq 0$, then $x \leq 0$, which is false by counterexample $x = 1$.

P2 Prove that $\sqrt{3}$ is not rational.

Answer: By contradiction. Suppose $\sqrt{3}$ is rational, then by definition of rational numbers, there must exist $p, q \in \mathbb{Z}$ such that $\frac{p}{q} = \sqrt{3}$, where p, q have no common factors upon simplying. Then, we also have $\frac{p^2}{q^2} = 3 \implies p^2 = 3q^2$. Since $p, q \in \mathbb{Z}$ and by extension $p^2, q^2 \in \mathbb{Z}$, $p^2 \mid 3$. Additionally, $p \mid 3$ because $\sqrt{3} \notin \mathbb{Z}$. Then, $p^2 \mid 9$. By substituting $p^2 = 3q^2$, we now have $3q^2 \mid 9 \implies q^2 \mid 3$, implying $q \mid 3$ by previous logic. Then p, q have common factor 3 whcih contradicts with our initial assumption. Therefore $\sqrt{3}$ is not rational.