1. Let $(s_n)_{n\in\mathbb{N}}$ be a bounded sequence of real numbers. Define a new sequence $(t_n)_{n\in\mathbb{N}}$ by setting

$$t_n = \max\{s_1, \ldots, s_n\} \text{ for } n \in \mathbb{N}.$$

Prove that $(t_n)_{n\in\mathbb{N}}$ is convergent.

Proof: Since (s_n) is bounded, there exists a $s = \sup S$ such that $s \geq s_n$ for all $n \in \mathbb{N}$. Furthermore, we have $t_n \leq s$ since $t_n = \max\{s_1, \ldots, s_n\} = \sup\{s_1, \ldots, s_n\} \leq \sup S = s$ (by Exercise 4.7(a), since $\{s_1, \ldots, s_n\} \subseteq S$). Therefore (t_n) is a bounded sequence.

In addition, since $t_{n+1} = \max\{s_1, \ldots, s_{n+1}\} = \max\{\max\{s_1, \ldots, s_n\}, s_{n+1}\} = \max\{t_n, s_{n+1}\}$, we have $t_n \leq t_{n+1}$, then (t_n) is an increasing monotone sequence. Then (t_n) is both bounded and monotone and is therefore convergent by Theorem 10.2.

2. Let f be a real-valued function defined on $\mathbb{R}$. Let $S \subseteq \mathbb{R}$ be nonempty and bounded above, and let $f(S) = \{f(x) : x \in S\}$. Suppose that f is increasing on $\mathbb{R}$ and that f is continuous at $\sup S$. Prove that $f(S)$ is bounded above and that $\sup f(S) = f(\sup S)$.

   Proof: Since S is bounded above, there exists an $s = \sup S$ such that $x \leq s$ for $x \in S$. Then since f is increasing, i.e. $f(x_1) < f(x_2)$ for $x_1 < x_2$, we have $f(x) \leq f(s)$ for all $x \in S$. Therefore $f(s)$ is an upper bound (but not yet necessarily the supremum) of $f(S)$ which implies that $\sup f(S) \leq f(s)$.

   Now if we order elements of S into a sequence (s_n), by Theorem 11.7 there exists a monotonic subsequence (s_{n_k}) with $\lim s_{n_k} = \limsup s_n = \sup S = s$. Then by definition of continuity, since $\lim s_{n_k} = s$, we have $\lim f(s_{n_k}) = f(s)$. By definition of limit, for $\epsilon > 0$ there exists an N such that $n > N \implies |f(s_{n_k}) - f(s)| < \epsilon$. Since $s_{n_k} \leq s$ and f is increasing, $f(s_{n_k}) - f(s) \leq 0$, then $n > N \implies f(s) - f(s_{n_k}) < \epsilon \implies f(s) < f(s_{n_k}) + \epsilon \leq \sup f(S) + \epsilon$. Since ϵ can be arbitrarilly small, we have $f(s) \leq \sup f(S)$. Then combining with $\sup f(S) \leq f(s)$ we get $\sup f(S) = f(s) = f(\sup S)$.

3. Let (s_n) and (t_n) be sequences of real numbers. Suppose (s_n) diverges to $+\infty$ and that the set

$$\{n \in \mathbb{N} : s_n \leq t_n\}$$

is infinite. Prove that $\limsup t_n = +\infty$.

Proof: Since (s_n) diverges to $+\infty$, for $M > 0$ there exists an N such that $n > N_0 \implies s_n > M$. In addition, since $\{n \in \mathbb{N}_{\nvdash} : s_n \leq t_n\}$ is infinite, there must exist infinite $s_n \leq t_n$ with $n > N_0$. Then we can construct subsequences (s_{n_k}) and (t_{n_k}) with $n_k > N_0$ and $s_{n_k} \leq t_{n_k}$ for all n_k, which gives us $n_k > N_0 \implies t_{n_k} \geq s_{n_k} > M$.

Now consider the sequence of supremums $(v_N) = (\sup\{t_n : n > N\})$; each v_N must be no less than all $t_n, n > N$ by definition of supremum. Then if we take $N > N_0$, v_N must also be no less than all $t_{n_k}, n_k > N$, i.e. $N > N_0 \implies v_N \geq t_{n_k} > M$. Then by definition of limit we have $\lim v_N = +\infty$ and therefore $\limsup t_n = +\infty$.

4. (a) Give an example of $(x_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that $x_n \le b_n$ for all n, $\sum_{n=1}^{\infty} b_n$ converges, but $\sum_{n=1}^{\infty} x_n$ does not converge.

Example: (x_n) and (b_n) where $x_n = -1$ and $b_n = \frac{1}{n^2}$.

Proof: Since $x_n = -1 < 0$ and $b_n = \frac{1}{n^2} > 0$ for all $n \in \mathbb{N}$, we have $x_n \le b_n$. Then, $\sum_{n=1}^{\infty} b_n$ converges by p-test whereas $\sum_{n=1}^{\infty} x_n$ diverges as $\lim_{n\to\infty} x_n = -1 \neq 0$.

(b) Suppose that $(a_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$, and $(b_n)_{n\in\mathbb{N}}$ satisfy $a_n \le x_n \le b_n$ for all $n \in \mathbb{N}$. Prove that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then $\sum_{n=1}^{\infty} x_n$ also converges.

Proof: By Cauchy criterion, for $\epsilon > 0$ there exists an N_a such that $n \ge m \ge N_a \implies |\sum_{k=m}^{n} a_k| < \epsilon \implies \sum_{k=m}^{n} a_k > -\epsilon$. Similarly, there also exists an N_b such that $n \ge m \ge N_b \implies |\sum_{k=m}^{n} b_k| < \epsilon \implies \sum_{k=m}^{n} b_k < \epsilon$. Then if we take $N = \max\{N_a, N_b\}$, both of the above are true.

Now since $a_n \le x_n \le b_n$ for all n, we have $\sum_{k=m}^{n} a_n \le \sum_{k=m}^{n} x_n \le \sum_{k=m}^{n} b_n$. Then $n \ge m \ge N \implies -\epsilon < \sum_{k=m}^{n} a_n \le \sum_{k=m}^{n} x_n \le \sum_{k=m}^{n} b_n < \epsilon \implies |\sum_{k=m}^{n} x_n| < \epsilon$. Therefore $\sum_{k=m}^{n} x_n$ satisfies the Cauchy criterion and is convergent.

5. (a) Give an example of a real-valued function f defined on all of $\mathbb{R}$ such that $\lim_{x\to+\infty} f(x)$ does not exist (either as a real number or as a symbol $+\infty$, $-\infty$). To show $\lim_{x\to+\infty} f(x)$ does not exist, give an example of a sequence (x_n) diverging to infinity such that the limit of $(f(x_n))$ does not exist.

Example: $f(x) = \sin(x)$ and $x_n = n$.

Proof: (x_n) clearly diverges to $+\infty$; now we will show that $\lim (f(x_n))$ does not exist by contradiction. Suppose $L = \lim (f(x_n)) = \lim (\sin(x_n))$ did exist, then for every $\epsilon > 0$ there exists an N such that $n > N \implies |\sin(x_n) - L| < \epsilon \implies L - \epsilon < \sin(x_n) < L + \epsilon$. If we take $\epsilon = 1$, then $L - 1 < \sin(x_n) < L + 1$.

On the one hand, if $L \geq 0$, we need to have $L - 1 \leq -1 < \sin(x_n)$ which is not true for $x_n = \frac{3\pi}{2} + 2k\pi$, $k \in \mathbb{N}$. On the other hand, if $L \leq 0$, we need to have $\sin(x_n) < 1 \leq L + 1$ which is not true for $x_n = \frac{\pi}{2} + 2k\pi$. Thus $L \not\geq 0$ and $L \not\leq 0$, therefore such L does not exist.

(b) Let f be a real-valued function defined on all of $\mathbb{R}$ such that $\lim_{x\to+\infty} f(x)$ exists and is a real number. Prove that $\lim_{y\to0^+} f(\frac{1}{y})$ exists and

$$\lim_{y\to0^+} f\left(\frac{1}{y}\right) = \lim_{x\to+\infty} f(x)$$

Proof: Let $L = \lim_{x\to+\infty} f(x)$; since L exists, by definition, for each $\epsilon > 0$ there exists $a < \infty$ such that $a < x$ implies $|f(x) - L| < \epsilon$. Now if we take $\delta = \frac{1}{a} > 0$ and $y = \frac{1}{x}$, by substitution we have $\frac{1}{\delta} < \frac{1}{y} \implies 0 < y < \delta$ implies $\left|f(\frac{1}{y}) - L\right| < \epsilon$, which then by Corollary 20.8 tells us that $\lim_{x\to a^+} f(\frac{1}{y}) = L = \lim_{x\to+\infty} f(x)$.

6. Fix $a \in \mathbb{R}$ and define the function f by $f(x) = |x - a|$ for $x \in \mathbb{R}$. Prove that f is not differentiable at a.

Proof: By contradiction. Suppose f is differentiable at a, then by definition, the limit $L = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a} \frac{|x-a|}{x-a}$ exists and is finite. If we take $S_1 = (a, +\infty) \subseteq \mathbb{R}$, we have the right-hand limit $L_+ = \lim_{x \to a^{S_1}} \frac{|x-a|}{x-a} = 1$ (since $x > a$ for all $x \in S_1$). Now if we take $S_2 = (-\infty, a) \subseteq \mathbb{R}$, we have the left-hand limit $L_- = \lim_{x \to a^{S_2}} \frac{|x-a|}{x-a} = -1$ (since $x < a$ for all $x \in S_2$). Then $L + - \neq L_+$ and L does not exist by Theorem 20.10, therefore f is not differentiable at a.

7. (a) Let f be a real-valued function defined on all of $\mathbb{R}$. Suppose there is a constant $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}$. Prove that f is uniformly continuous on $\mathbb{R}$.

   Proof: Let $\epsilon > 0$ and $\delta = \frac{\epsilon}{L} > 0$. Then for all $x, y \in \mathbb{R}$ we have $|x - y| < \delta \implies |f(x) - f(y)| \leq L|x - y| < L\delta \implies |f(x) - f(y)| < \epsilon$. Therefore f is uniformly continuous on $\mathbb{R}$ by definition of uniform continuity.

   (b) Let f be a real-valued function defined on all of $\mathbb{R}$. Suppose that f is differentiable on $\mathbb{R}$ and the set $\{|f'(x)| : x \in \mathbb{R}\}$ is bounded. Prove that f is uniformly continuous on $\mathbb{R}$.

   Proof: Since $\{|f'(x)| : x \in \mathbb{R}\}$ is bounded, there exists an $s = \sup\{|f'(x)| : x \in \mathbb{R}\} > 0$ such that $|f'(x)| \leq s$ for all $x \in \mathbb{R}$. Then we can select arbitrary distinct $x, y \in \mathbb{R}$ and have $y < x$ upon renaming; by Mean Value Theorem there exists some point a on (y, x) such that $f'(a) = \frac{f(x)-f(y)}{x-y} \implies |f'(a)| = \left|\frac{f(x)-f(y)}{x-y}\right| \leq s$. Then we have $|f(x) - f(y)| \leq s|x - y|$ for arbitrary x, y and constant $s > 0$, thus f is uniformly continuous by part (a).