Math 131A Homework 1

Jiaping Zeng

8/3/2020

1.1 Prove $1^2 + 2^2 + ... + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all positive integers n.
Answer: By induction.
Base case ($n = 1$): $1 = \frac{1}{6}(1+1)(2+1) \implies 1 = 1$ which is true.
Inductive step: Assume $1^2 + 2^2 + ... + n^2 = \frac{1}{6}n(n+1)(2n+1)$ is true, we want to show that $1^2 + 2^2 + ... + n^2 + (n+1)^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$ is true. We can do so by adding $(n+1)^2$ to both sides of the equation as follows:

$$1^2 + 2^2 + ... + n^2 + (n+1)^2 = \frac{1}{6}n(n+1)(2n+1) + (n+1)^2$$

Factoring out $(n+1)$ from both terms on the right hand side:

$$1^2 + 2^2 + ... + n^2 + (n+1)^2 = (n+1)(\frac{1}{6}n(2n+1) + (n+1))$$

$$\implies 1^2 + 2^2 + ... + n^2 + (n+1)^2 = (n+1)(\frac{1}{6}n(2n+1) + (n+1))$$

1.9 (a) Decide for which integers the inequality $2^n > n^2$ is true.

(b) Prove your claim in (a) by mathematical induction.

1.11 For each $n \in \mathbb{N}$, let P_n denote the assertion "$n^2 + 5n + 1$ is an even integer."

(a) Prove P_{n+1} is true whenever P_n is true.

(b) For which n is P_n actually true? What is the moral of this exercise?

3.1

3.6

3.7

3.8

4.6

4.7

4.8

4.14

4.15

4.16

P1 Write down the converse and the contrapositive of the following statement regarding a real number x:

$$\text{If } x > 0, \text{then } x^2 - x > 0.$$

Then determine which (if any) of the three statements are true for all real numbers x.

Answer: Converse: if $x^2 - x > 0$, then $x > 0$, which is false by counterexample $x = -1$. Contrapositive: if $x^2 - x \leq 0$, then $x \leq 0$, which is false by counterexample $x = 1$.

P2 Prove that $\sqrt{3}$ is not rational.

Answer: By contradiction. Suppose $\sqrt{3}$ is rational, then by definition of rational numbers, there must exist $p, q \in \mathbb{Z}$ such that $\frac{p}{q} = \sqrt{3}$, where p, q have no common factors upon simplying. Then, we also have $\frac{p^2}{q^2} = 3 \implies p^2 = 3q^2$. Since $p, q \in \mathbb{Z}$ and by extension $p^2, q^2 \in \mathbb{Z}$, $p^2 \mid 3$. Additionally, $p \mid 3$ because $\sqrt{3} \notin \mathbb{Z}$. Then, $p^2 \mid 9$. By substituting $p^2 = 3q^2$, we now have $3q^2 \mid 9 \implies q^2 \mid 3$, implying $q \mid 3$ by previous logic. Then p, q have common factor 3 whcih contradicts with our initial assumption. Therefore $\sqrt{3}$ is not rational.