Math 131A Homework 3

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10.9 Let $s_1 = 1$ and $s_{n+1} = (\frac{n}{n+1})s_n^2$ for $n \geq 1$.

(a) Find s_2, s_3 and s_4.
$s_2 = \frac{1}{2} \cdot 1 = \frac{1}{2}$
$s_3 = \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6}$
$s_3 = \frac{3}{4} \cdot \frac{1}{36} = \frac{1}{48}$

(b) Show lim s_n exists.
Since $\frac{n}{n+1} < 1$ and $s_n^2 < s_n$ for all $n \geq 1$, $\frac{n}{n+1}s_n^2 < 1 \cdot s_n \implies s_{n+1} \leq s_n$ for all $n \geq 1$. Therefore (s_n) is monotone and decreasing. Then, since (s_n) is decreasing, s_1 is an upper bound of the set and therefore (s_n) is bounded above. In addition, s_{n+1} is never negative because $n \leq 1 \implies \frac{n}{n+1} > 0$ and $s_n^2 \geq 0$. Therefore 0 is an lower bound of the set and (s_n) is bounded below. Then lim s_n exists by Theorem 10.2.

(c) Prove lim $s_n = 0$.
Let $s = \lim s_n$, then $s = \lim s_{n+1} = \lim (\frac{n}{n+1})s_n^2 = \lim (\frac{n}{n+1}) \cdot \lim s_n^2 = \lim s_n^2 = s^2 \implies s = s^2$. Then we have $s = 0$ or $s = 1$. However, since $s_1 = 1$ is an upper bound and (s_n) is strictly decreasing as shown in part (b), $s \neq 1$. Therefore $s = 0 = \lim s_n$.

11.2 Consider the sequences defined as follows:

$$a_n = (-1)^n,\ b_n = \frac{1}{n},\ c_n = n^2,\ d_n = \frac{6n+4}{7n-3}.$$

(a) For each sequence, give an example of a monotone subsequence.

(b) For each sequence, give its set of subsequential limits.

(c) For each sequence, give its lim sup and lim inf.

(d) Which of the sequences converge? diverge to $+\infty$? diverge to $-\infty$?

(e) Which of the sequences is bounded?

11.5 Let (q_n) be an enumeration of all the rationals in the interval $(0, 1]$.

(a) Give the set of subsequential limits for (q_n).

(b) Give the values of lim sup q_n and lim inf q_n.

12.2 Prove $\limsup |s_n| = 0$ if and only if $\lim s_n = 0$.

12.4 Show $\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$ for bounded sequences (s_n) and (t_n).

12.9 (a) Prove that if $\lim s_n = \infty$ and $\liminf t_n > 0$, then $\lim s_n t_n = +\infty$.

(b) Prove that if $\limsup s_n = +\infty$ and $\liminf t_n > 0$, then $\limsup s_n t_n = +\infty$.

(c) Observe that Exercise 12.7 is the special case of (b) where $t_n = k$ for all $n \in \mathbb{N}$.

12.10 Prove (s_n) is bounded if and only if $\limsup |s_n| < +\infty$.

12.12 Let (s_n) be a sequence of nonnegative numbers, and for each n define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \ldots + s_n)$.

(a) Show

$$\liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n.$$

(b) Show that if $\lim s_n$ exists, then $\lim \sigma_n$ exists and $\lim \sigma_n = \lim s_n$.

(c) Give an example where $\lim \sigma_n$ exists, but $\lim s_n$ does not exist.

14.5 Suppose $\sum a_n = A$ and $\sum b_n = B$ where A and B are real numbers. Use limit theorems to quickly prove the following.

(a) $\sum(a_n + b_n) = A + B$. Proof:

(b) $\sum k a_n = kA$ for $k \in \mathbb{R}$.

(c) Is $\sum a_n b_n = AB$ a reasonable conjuecture? Discuss.

P1 Let (s_n) be the sequence

$$s_n = \frac{n^2 + 1}{n^2 + 2n} \sin n.$$

Prove that (s_n) has a convergent subsequence.

P2 Let (s_n) be a sequence that contains eveyr integer. Prove that there is a subsequence of (s_n) which diverges to $-\infty$.

P3 Suppose (s_n) is a sequence and (t_k) is a subsequence of (s_n) such that (t_k) converges. Prove that $\lim t_k \leq \limsup s_n$.

P4 For each series, determine whether the series (1) converges to a real number, (2) diverges to $+\infty$, (3) diverges to $-\infty$, or (4) none of these. Prove your answers.

(a) $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2}$

(b) $\sum_{n=1}^{\infty} \frac{n-1}{n^2}$

(c) $\sum_{n=1}^{\infty} (-1)^n$

(d) $\sum_{n=1}^{\infty} \frac{n+1}{n^3-1}$