1. (a) Let $\epsilon > 0$. Let x and y be real numbers such that $|x-1| < \frac{\epsilon}{2}$ and $|y-2| < \frac{\epsilon}{2}$. Prove that $|x-y| < \epsilon + 1$.

Proof: By expanding the absolute value, we have $|x-1| < \frac{\epsilon}{2} \implies -\frac{\epsilon}{2} < x-1 < \frac{\epsilon}{2} \implies 1-\frac{\epsilon}{2} < x < 1+\frac{\epsilon}{2}$. Similarly, we also have $2-\frac{\epsilon}{2} < y < 2+\frac{\epsilon}{2}$, which implies $-2+\frac{\epsilon}{2} > -y > -2-\frac{\epsilon}{2}$. Then, we can add the two inequalities as follows: $(1-\frac{\epsilon}{2}) + (-2-\frac{\epsilon}{2}) < x + (-y) < (1+\frac{\epsilon}{2}) + (-2+\frac{\epsilon}{2})$, which simplies to $-\epsilon - 1 < x - y < \epsilon - 1$. Since $\epsilon + 1 > \epsilon - 1$, $-\epsilon - 1 < x - y < \epsilon + 1$, therefore $|x-y| < \epsilon + 1$.

(b) Let x be a real number such that $|x| \leq \frac{3}{4}$ and $|x-1| \leq \frac{3}{4}$. Prove that $\left|x - \frac{1}{2}\right| \leq \frac{1}{4}$.

Proof: Again by expanding the absolute values, we have $-\frac{3}{4} \leq x \leq \frac{3}{4}$ and $1-\frac{3}{4} \leq x \leq 1+\frac{3}{4} \implies \frac{1}{4} \leq x \leq \frac{7}{4}$. By combining the two inequalities and taking the tighter bounds, we have $\frac{1}{4} \leq x \leq \frac{3}{4}$. Then, we can subtract $\frac{1}{2}$ from the inequality to obtain $-\frac{1}{4} \leq x - \frac{1}{2} \leq \frac{1}{4}$; therefore $\left|x - \frac{1}{2}\right| \leq \frac{1}{4}$.

2. Let $S \subseteq \mathbb{R}$ be nonempty and bounded above. Let $T = \{2x + 1 : x \in S\}$. Prove that $\sup T$ exists and that

$$\sup\ T = 2 \cdot \sup\ S + 1.$$

Proof: Since $\sup\ S \geq x$ for all $x \in S$ by definition of supremum, $2 \cdot \sup\ S + 1 \geq 2x + 1 \in T$. Therefore $2 \cdot \sup\ S + 1$ is an upper bound of T, so T is bounded above and $\sup\ T$ exists. Then, since $2 \cdot \sup\ S + 1$ is an upper bound and $\sup\ T$ exists, we have $\sup\ T \leq 2 \cdot \sup\ S + 1$.

Now suppose $\sup\ T < 2 \cdot \sup\ S + 1$; then there must exist an $r \in \mathbb{Q}$ such that $\sup\ T < r < 2 \cdot \sup\ S + 1$ since $\mathbb{Q}$ is dense in $\mathbb{R}$. Then, since $\sup\ T > 2x + 1$ for all $x \in S$, we also have $2 \cdot \sup\ S + 1 > r > \sup\ T > 2x + 1$. By subtracting 1 and dividing by 2, the previous expression gives us $\sup\ S > \frac{r-1}{2} > x$ for $x \in S$, which implies that $\frac{r-1}{2}$ is an upper bound of S while being strictly less than the supremum of the set. This cannot happen by definition of supremum, therefore $\sup\ T \not< 2 \cdot \sup\ S + 1$.

Combining $\sup\ T \leq 2 \cdot \sup\ S + 1$ and $\sup\ T \not< 2 \cdot \sup\ S + 1$ gives us $\sup\ T = 2 \cdot \sup\ S + 1$.

3. Let (s_n) and (t_n) be convergent sequences of real numbers. Suppose $\lim s_n = \lim t_n + 1$. Prove there exists N such that $s_n - t_n > \frac{1}{2}$ for all $n > N$.

   Proof: Let $s = \lim s_n$ and $t = \lim t_n$, also let $\frac{1}{4} \geq \epsilon > 0$. By definition of convergence, there exists an N_s such that $n > N_s \implies |s_n - s| < \epsilon \implies s - \epsilon < s_n < s + \epsilon$; similarly, there exists an N_t such that $n > N_t \implies |t_n - t| < \epsilon \implies t - \epsilon < t_n < t + \epsilon$. Then $s - \epsilon - t - \epsilon < s_n - t_n < s + \epsilon - t + \epsilon$ which simplifies to $s - t - 2\epsilon < s_n - t_n < s - t + 2\epsilon$. Since $s = t + 1$, we have $1 - 2\epsilon < s_n - t_n$ by substitution. Since we selected $\epsilon \leq \frac{1}{4}$, $N = \max\{N_s, N_t\}$ will guarantee that $n > N \implies s_n - t_n > \frac{1}{2}$.

4. Let (s_n) be a sequence defined by

$$s_n = \frac{2n^2+3}{n(5n-1)}, n = 1, 2, \ldots$$

Prove that (s_n) is convergent and find the value of $\lim\, s_n$.

Proof:

$\lim\, s_n = \lim\, \dfrac{2n^2+3}{n(5n+1)}$

$= \lim\, \dfrac{2n^2+3}{5n^2+n}$ (Expand denominator)

$= \lim\, \dfrac{2+\frac{3}{n^2}}{5+\frac{1}{n}}$ (Multiply by $\frac{\frac{1}{n^2}}{\frac{1}{n^2}}$)

$= \dfrac{\lim\, (2+\frac{3}{n^2})}{\lim\, (5+\frac{1}{n})}$ (Lemma 22)

$= \dfrac{\lim\, 2 + \lim\, \frac{3}{n^2}}{\lim\, 5 + \lim\, \frac{1}{n}}$ (Sum limit law)

$= \dfrac{2 + 3 \cdot \lim\, \frac{1}{n^2}}{5 + \lim\, \frac{1}{n}}$ (Theorem 19)

$= \dfrac{2+3\cdot 0}{5+0}$ ($\lim\, \frac{1}{n^2} = \lim\, \frac{1}{n} = 0$ shown in class)

$= \dfrac{2}{5}$

Therefore (s_n) converges to $\dfrac{2}{5}$.

5. Let (s_n) and (t_n) be sequences of real numbers. Suppose (s_n) diverges to $+\infty$ and that the set

$$\{n \in \mathbb{N} : t_n < s_n\}$$

is finite. Prove that (t_n) diverges to $+\infty$.

Proof: Since (s_n) diverges to $+\infty$, there exists an N_s such that $n > N_s \implies s_n > M$ for $M > 0$. In addition, since the set $\{n \in \mathbb{N} : t_n < s_n\}$ is finite, there exists an N_0 such that $n > N_0 \implies t_n \geq s_n$. Then we can select $N = \max\{N_s, N_0\}$ and we have $n > N \implies t_n \geq s_n > M$. Therefore $n > N \implies t_n > M$ and (t_n) diverges to $+\infty$ by definition of divergence.