Math 131A Homework 2

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8.1 Prove the following:

(a) $\lim \dfrac{(-1)^n}{n} = 0$

Answer: Proof: Take $N = \dfrac{1}{\epsilon}$, then $n > N = \dfrac{1}{\epsilon} \implies \epsilon > \dfrac{1}{n} = |s_n - 0|$. Therefore $\lim \dfrac{(-1)^n}{n} = 0$ by definition of limit.

(c) $\lim \dfrac{2n-1}{3n+2} = \dfrac{2}{3}$

Answer: Scratch: $\left|\dfrac{2n-1}{3n+2} - \dfrac{2}{3}\right| < \epsilon \implies \left|\dfrac{6n-3-2\cdot(3n+2)}{3\cdot(3n+2)}\right| < \epsilon \implies \left|\dfrac{-7}{3\cdot(3n+2)}\right| < \epsilon \implies \dfrac{7}{3\epsilon} < 3n+2 \implies \dfrac{7}{9\epsilon} - \dfrac{2}{3} \leq n$.

Proof: Let $\epsilon > 0$, define $N = \dfrac{7}{9\epsilon} - \dfrac{2}{3}$. Then, $n > N = \dfrac{7}{9\epsilon} - \dfrac{2}{3} \implies 3n+2 > \dfrac{7}{3\epsilon} \implies 3\epsilon > \dfrac{7}{3n+2} \implies \epsilon > \dfrac{7}{3\cdot(3n+2)} = \left|s_n - \dfrac{2}{3}\right|$. Therefore $\lim \dfrac{2n-1}{3n+2} = \dfrac{2}{3}$ by definition.

8.4 Let (t_n) be a bounded sequence, i.e., there exists M such that $|t_n| \leq M$ for all n, and let (s_n) be a sequence such that $\lim s_n = 0$. Prove $\lim (s_n t_n) = 0$.

Answer: Want to show: $n > N \implies |s_n t_n| < \epsilon$

8.5 (a) Consider three sequences (a_n), (b_n) and (s_n) such that $a_n \leq s_n \leq b_n$ for all n and $\lim a_n = \lim b_n = s$. Prove $\lim s_n = s$. This is called the "squeeze lemma."

Answer: Since $\lim a_n = s$, there exists an N_a such that $n > N_a \implies |a_n - s| < \epsilon \implies -\epsilon < a_n - s < \epsilon \implies s - \epsilon < a_n < s + \epsilon$. Similarly, since $\lim b_n = s$, there exists an N_b such that $n > N_b \implies |b_n - s| < \epsilon \implies s - \epsilon < b_n < s + \epsilon$. Then, since $a_n \leq s_n \leq b_n$ for all n, we also have $s - \epsilon < a_n \leq s_n \leq b_n < s + \epsilon$. Therefore $s - \epsilon < s_n < s + \epsilon$, which implies that $|s_n - s| < \epsilon$. Then $\lim s_n = s$ by definition of limit.

(b) Suppose (s_n) and (t_n) are sequences such that $|s_n| \leq t_n$ for all n and $\lim t_n = 0$. Prove $\lim s_n = 0$.

Answer: Since $\lim t_n = 0$, there exists an N such that $n > N \implies |t_n| < \epsilon$. Then we also know that $\lim -t_n = 0$ because $|-t_n| = |t_n| < \epsilon$. Then, since $|s_n| \leq t_n$, we have $-t_n \leq s_n \leq t_n$. Therefore $\lim s_n = 0$ by squeeze lemma.

8.6 Let (s_n) be a sequence in $\mathbb{R}$.

(a) Prove $\lim s_n = 0$ if and only if $\lim |s_n| = 0$.

Answer:

$\Rightarrow$: Since $\lim s_n = 0$, there exists an N such that $n > N \implies |s_n| < \epsilon$. Then $\lim |s_n| = 0$ by definition because $|(|s_n|)| = |s_n| < \epsilon$.

$\Leftarrow$: Since $\lim |s_n| = 0$, there exists an N such that $n > N \implies |(|s_n|)| < \epsilon$. Since $|(|s_n|)| = |s_n|$, we also have $|s_n| < \epsilon$. Then $\lim s_n = 0$ by definition.

(b) Observe that if $s_n = (-1)^n$, then $\lim |s_n|$ exists, but $\lim s_n$ does not exist.

Answer: Observed.

8.9 Let (s_n) be a sequence that converges.

(a) Show that if $s_n \geq a$ for all but finitely many n, then $\lim s_n \geq a$.

(b) Show that if $s_n \leq b$ for all but finitely many n, then $\lim s_n \leq b$.

(c) Conclude that if all but finitely many s_n belong to $[a, b]$, then $\lim s_n$ belongs to $[a, b]$.

8.10 Let (s_n) be a convergent sequence, and suppose $\lim s_n > a$. Prove there exists a number N such that $n > N$ implies $s_n > a$.

Answer: Let $\lim s_n = s$. Then there exists an N such that $n > N \implies |s_n - s| < \epsilon$. By expanding the absolute value we have $-\epsilon < s_n - s < \epsilon$, which is equivalent to $s - \epsilon < s_n < s + \epsilon$.

9.1 (a)

(b)

9.3

9.9

9.10 (a)

(b)

9.11

9.12

10.5

10.6

10.7

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