Math 131A Homework 5

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9/3/2020

19.1 Which of the following continuous functions are uniformly continuous on the specified set? Justify your answer.

(a) $f(x) = x^{17}\sin x - e^x\cos 3x$ on $[0, \pi]$: Uniformly continuous

Proof: Since $f(x)$ is continuous on $[0, \pi]$ (by applying Theorem 17.4 to known continuous functions), it must also be uniformly continuous on $[0, \pi]$ by Theorem 19.2.

(c) $f(x) = x^3$ on $(0, 1)$: Uniformly continuous

Proof: Since x^3 is continuous on the closed interval $[0, 1]$, it must be uniformly continuous on $(0, 1)$ by Theorem 19.5.

(d) $f(x) = x^3$ on $\mathbb{R}$: Not uniformly continuous

Proof: By contradiction. Suppose x^3 is uniformly continuous on $\mathbb{R}$, let $\epsilon = 1$ and $\delta > 0$, then for $x, y \in \mathbb{R}$ we should have $|x - y| < \delta \implies \left|x^3 - y^3\right| = |x - y| \cdot \left|x^2 + xy + y^2\right| < 1$. However, if we choose $x = \frac{2}{\delta}$ and $y = \frac{2}{\delta} + \frac{\delta}{2}$, we have $|x - y| = \frac{\delta}{2} < \delta$ and $\left|x^2 + xy + y^2\right| = \frac{\delta^2}{4} + \frac{12}{\delta^2} + 3$ which is greater than $\epsilon = 1$. Therefore x^3 is not uniformly continuous on $\mathbb{R}$.

(e) $f(x) = \sin \frac{1}{x^2}$ on $(0, 1]$: Not uniformly continuous

Proof: Let $(s_n) = \frac{1}{n}$, which is a Cauchy sequence on $(0, 1]$, then $(f(s_n)) = \sin n^2$ which is not convergent on $(0, 1]$ and therefore is not a Cauchy sequence. Therefore $f(x)$ is not uniformly continuous by Theorem 19.4.

19.2 Prove each of the following functions is uniformly continuous on the indicated set by directly verifying the ϵ-δ property in Definition 19.1.

(a) $f(x) = 3x + 11$ on $\mathbb{R}$

Proof: $|f(x) - f(y)| = |3x + 11 - 3y - 11| = 3|x - y|$; Then if we take $\delta = \frac{\epsilon}{3}$ we have $|x - y| < \delta \implies 3|x - y| < 3\delta \implies |f(x) - f(y)| < \epsilon$.

(b) $f(x) = x^2$ on $[0, 3]$

Proof: $|f(x) - f(y)| = \left|x^2 - y^2\right| = |x + y| \cdot |x - y|$; since $x, y \in [0, 3]$, $|x + y| \le 6$. Then if we take $\delta = \frac{\epsilon}{6}$ we have $|x - y| < \delta \implies |f(x) - f(y)| \le 6|x - y| < \epsilon$.

19.4 (a) **Proof:** Assume f is not a bounded function on S. Since S is a bounded set, there exists a bounded sequence (s_n) in S with a Cauchy subsequence (s_{n_k}) by Theorem 11.5. Then, since f is uniformly continuous, $(f(s_{n_k}))$ is also a Cauchy sequence. However this contradicts with f not being a bounded function on S; therefore the assumption is false and f is a bounded function.

(b) Since $\lim_{x\to 0}\frac{1}{x^2} = +\infty$, $\frac{1}{x^2}$ is not a bounded function on the bounded set $(0,1)$. Then by the contrapositive of part (a), $\frac{1}{x^2}$ is not uniformly continuous on $(0,1)$.

19.7 (a) Since f is continuous on $[0,\infty)$, it is continuous on $[0,k]$. Then f is also uniformly continuous on $[0,k]$ by Theorem 19.2. Then since it is given that f is uniformly continuous on $[k,\infty]$, combining the two gives us that f is uniformly continuous on $[0,\infty]$.

20.14 Prove $\lim_{n\to 0^+}\frac{1}{x} = +\infty$ and $\lim_{n\to 0^-}\frac{1}{x} = -\infty$.

$x\to 0^+$: **Proof:** Let $M > 0$ and choose $\delta = \frac{1}{M} > 0$, then $0 < x < \delta \implies f(x) = \frac{1}{x} > \frac{1}{\delta} = M$. Therefore $\lim_{n\to 0^+}\frac{1}{x} = +\infty$ by definition.

$x\to 0^-$: **Proof:** Let $M < 0$ and choose $\delta = -\frac{1}{M} > 0$, then $-\delta < x < 0 \implies f(x) = \frac{1}{x} < -\frac{1}{\delta} = M$. Therefore $\lim_{n\to 0^-}\frac{1}{x} = -\infty$ by definition.

20.16 Suppose the limits $L_1 = \lim_{x\to a^+} f_1(x)$ and $L_2 = \lim_{x\to a^+} f_2(x)$ exists.

(a) Show if $f_1(x) \leq f_2(x)$ for all x in some interval (a,b), then $L_1 \leq L_2$.
Proof: By definition, we can take a sequence (x_n) in (a,b) with limit a such that $\lim_{n\to\infty} f_1(x_n) = L_1$. Similarly, $\lim_{n\to\infty} f_2(x_n) = L_2$. Then since $f_1(x) \leq f_2(x)$, $\lim_{n\to\infty} f_1(x_n) \leq \lim_{n\to\infty} f_2(x_n)$ by Exercise 9.9(c). Therefore $L_1 \leq L_2$.

(b) Suppose that, in fact, $f_1(x) < f_2(x)$ for all x in some interval (a,b). Can you conclude $L_1 < L_2$?
Proof: No; by counter example: $f_1(x) = x$ and $f_2(x) = x^2$ on $(0,1)$. Clearly $f_1(x) < f_2(x)$ for $x > 0$, yet $\lim_{n\to 0^+} f_1(x) = \lim_{n\to 0^+} f_2(x) = 0$.

28.2 Use the *definition* of derivative to calculate the derivatives of the following functions at the indicated points.

(a) $f(x)x^3$ at $x = 2 \implies f'(2) = \lim_{x\to 2}\frac{x^3-2^3}{x-2} = \lim_{x\to 2}(x^2+2x+4) = 12$

(b) $g(x) = x+2$ at $x = a \implies f'(a) = \lim_{x\to a}\frac{x+2-a-2}{x-a} = \lim_{x\to a}$

28.11 Suppose f is differentiable at a, g is differentiable at $f(a)$, and h is differentiable at $g\circ f(a)$. State and prove the chain rule for $(h\circ g\circ f)'(a)$.
$(h\circ g\circ f)'(a) = h'(g\circ f(a))\cdot g'(f(a)\cdot f'(a))$, **Proof:** Let $y(x) = (g\circ f)(x)$, then by substitution and Chain Rule we have $y(a) = (g\circ f)(a)$ and $y'(a) = g'(f(a))\cdot f'(a)$. Again by substitution and Chain Rule, $(h\circ g\circ f)'(a) = (h\circ y)'(a) = h'(y(a))\cdot y'(a) = h'(g\circ f(a))\cdot g'(f(a))\cdot f'(a)$.

28.14 Suppose f is differentiable at a. Prove

(a) $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = f'(a)$
Proof: Since f is differentiable at a, we have $f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ by definition. Then we can substitute in $h = x-a \implies x = a+h$, which gives us $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.

(b) $\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{2h} = f'(a)$
Proof: By algebra as follows, using part (a):
$\lim_{h\to 0}\frac{f(a+h)-f(a-h)}{2h}$
$= \lim_{h\to 0}\frac{f(a+h)-f(a)-f(a-h)+f(a)}{2h}$

$= \frac{1}{2}\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} + \frac{1}{2}\lim_{(-h)\to 0}\frac{f(a+(-h))-f(a)}{(-h)}$
$= \frac{1}{2}f'(a) + \frac{1}{2}f'(a)$
$= f'(a)$

29.3 Suppose f is differentiable on $\mathbb{R}$ and $f(0) = 0$, $f(1) = 1$ and $f(2) = 1$.

(a) Show $f'(x) = \frac{1}{2}$ for some $x \in (0, 2)$.
Proof: Let $a = 0$ and $b = 2$, then by Mean Value Theorem there exists at least one x in $(0, 2)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a} = \frac{1-0}{2-0} = \frac{1}{2}$.

29.7 (a) Suppose f is twice differentiable on an open interval I and $f''(x) = 0$ for all $x \in I$. Show f has the form $f(x) = ax + b$ for suitable constants a and b.
Proof: Let $g(x) = f'(x)$, then g is a constant function on (a, b) by Corollary 29.4, i.e. $g(x) = f'(x) = a$ for some $a \in \mathbb{R}$. Now let $h(x) = f(x) - ax$, then we have $h'(x) = f'(x) - a = 0$; again by Corollary 29.4, $h(x)$ is also a constant function, i.e. $h(x) = f(x) - ax = b$ for some $b \in \mathbb{R}$. Therefore by rearranging the last equation we have $f(x) = ax + b$.

(b) Suppose f is three times differentiable on an open interval I and $f''' = 0$ on I. What form does f have? Prove your claim.
Proof: Let $g(x) = f'(x)$, then $g''(x) = f'''(x) = 0$ and therefore $g(x) = f'(x) = ax + b$ for $b, c \in \mathbb{R}$ by part (a). Now let $h(x) = f(x) - \frac{1}{2}ax^2 - bx$, then using Power Rule we have $h'(x) = f'(x) - ax - b = 0$. Then $h(x)$ is a constant function, i.e. $h(x) = f(x) - \frac{1}{2}ax^2 - bx = c$ for some $c \in \mathbb{R}$. By rearranging the last equation we have $f(x) = \frac{1}{2}ax^2 + bx + c$; note that $\frac{1}{2}a$ is simply another arbitrary constant in $\mathbb{R}$, therefore upon renaming we have $f(x) = ax^2 + bx + c$.

29.13 Prove that if f and g are differentiable on $\mathbb{R}$, if $f(0) = g(0)$ and if $f'(x) \le g'(x)$ for all $x \in \mathbb{R}$, then $f(x) \le g(x)$ for $x \ge 0$.
Proof: Let $h(x) = g(x) - f(x)$, then h is differentiable by Theorem 28.3(ii) and $h'(x) = g'(x) - f'(x)$. Since $f'(x) \le g'(x)$, we have $h'(x) \ge 0$, then h is increasing by Corollary 29.7(iii). Then $x_1 < x_2 \implies h(x_1) \le h(x_2)$ for all $x_1, x_2 \in \mathbb{R}$ by definition of increasing function. If we take $x_1 = 0$ and x_2 to be an arbitrary $x \ge 0$, we have $x \ge 0 \implies h(x) \ge h(0) \implies g(x) - f(x) \ge g(0) - f(0) = 0 \implies f(x) \le g(x)$.

P1 Let f be a differentiable function on an interval (a, b). Prove that if $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing on (a, b).
Proof: Take arbitrary x_1, x_2 such that $a < x_1 < x_2 < b$. Then by Mean Value Theorem we have $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(x) < 0$ for some $x \in (a, b)$. Since the denominator is positive as $x_1 < x_2$, the numerator must be negative to satisfy $f'(x) < 0$; i.e. $f(x_2) - f(x_1) < 0 \implies f(x_1) > f(x_2)$. Then f is strictly decreasing by definition.

P2 Let $a < b$ be reals. Let f be a function defined on (a, b), and let $x_0 \in (a, b)$. Prove that if f is differentiable at x_0 and $f'(x_0) > 0$, then there is some $x > x_0$ such that $f(x) > f(x_0)$.
Proof: By definition of derivative, we have $f'(x_0) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} > 0$. Let $x_n = x_0 + \frac{1}{n}$, then (x_n) is a sequence converging to x_0 and $x_n > x_0$. Then there exists an n such that $\frac{f(x_n)-f(x_0)}{x_n-x_0} > 0$. Since $x_n > x_0 \implies x_n - x_0 > 0$, we must also have $f(x_n) - f(x_0) > 0 \implies f(x_n) > f(x_0)$.

P3 Suppose that f and f' are differentiable functions on $\mathbb{R}$, and there are $x_1 < x_2 < x_3$ so that $f(x_1) > f(x_2)$ and $f(x_3) > f(x_2)$. Prove that there is a point x_0 such that $f''(x_0) > 0$.

Proof: By Mean Value Theorem, there exists an x_a in (x_1, x_2) such that $f'(x_a) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. Since $x_1 < x_2$ and $f(x_1) > f(x_2)$, $f'(x_a) < 0$. Similarly, there exists an x_b in x_2, x_3 such that $f'(x_b) = \frac{f(x_3)-f(x_2)}{x_3-x_2}$, implying that $f'(x_b) > 0$. Combining the two above and using the Mean Value Theorem once more, there exists an $x \in (x_a, x_b)$ such that $f''(x) = \frac{f''(x_b)-f''(x_a)}{x_b-x_a}$. Since $f''(x_a) < 0 < f''(x_b)$, $f''(x_b) - f''(x_a) > 0$. In addition, since $x_a \leq x_2 \leq x_b$, $x_b - x_a > 0$. Therefore both the numerator and denominator are positive and $f''(x) > 0$.