Math 110A Homework 4

Jiaping Zeng

2/8/2021

1. Let $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$. Prove that the function $f : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ given by $f(a + b\sqrt{2}) = a - b\sqrt{2}$ is an isomorphism.

   Answer: we will prove the three conditions of ring isomorphism as follows:

   (i) Suppose $f(a + b\sqrt{2}) = f(a' + b'\sqrt{2})$, we have $f(a + b\sqrt{2}) = f(a' + b'\sqrt{2}) \implies a - b\sqrt{2} = a' - b'\sqrt{2}$. Since $\sqrt{2}$ is not rational but a, b, a', b' are, we must have $a = a'$ and $b = b'$, therefore f is injective.

   (ii) For every $a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$, we have $a + b\sqrt{2} = f(a - b\sqrt{2})$. Therefore f is surjective.

   (iii) We have $f((a+b\sqrt{2})+(a'+b'\sqrt{2})) = f((a+a')+(b+b')\sqrt{2}) = (a+a')-(b+b')\sqrt{2} = (a+b\sqrt{2})+(a'+b'\sqrt{2}) = f(a+b\sqrt{2})+f(a'+b'\sqrt{2})$. Similarly, we also have $f((a+b\sqrt{2})(a'+b'\sqrt{2})) = f((aa'+2bb')+(ab'+a'b)\sqrt{2}) = (aa' + 2bb') - (ab' + a'b)\sqrt{2} = (a - b\sqrt{2})(a' - b'\sqrt{2}) = f(a + b\sqrt{2})f(a' + b'\sqrt{2})$.

2. Which of the following functions are homomorphisms?

   (a) $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = -x$.

   Answer: No since $f(ab) = -ab \neq ab = (-a)(-b) = f(a)f(b)$.

   (b) $f : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ defined by $f(x) = -x$.

   Answer: Yes; in $\mathbb{Z}/2\mathbb{Z}$ we have $-a \equiv a \pmod 2$ for every a, so $f(a+b) = -(a+b) = -a-b = f(a)+f(b)$ and $f(ab) = -ab = ab = (-a)(-b) = f(a)f(b)$ and f is a homomorphism.

   (c) $g : \mathbb{Q} \to \mathbb{Q}$ defined by $g(x) = \dfrac{1}{1 + x^2}$.

   Answer: No, $g(ab) = \dfrac{1}{1 + a^2b^2} \neq \dfrac{1}{(1 + a^2)(1 + b^2)} = g(a)g(b)$.

   (d) $h : \mathbb{R} \to M_2(\mathbb{R})$, defined by $h(a) = \begin{pmatrix} -a & 0 \\ a & 0 \end{pmatrix}$.

   Answer: No, $h(ab) = \begin{pmatrix} -ab & 0 \\ ab & 0 \end{pmatrix} \neq \begin{pmatrix} ab & 0 \\ -ab & 0 \end{pmatrix} = \begin{pmatrix} -a & 0 \\ a & 0 \end{pmatrix} \begin{pmatrix} -b & 0 \\ b & 0 \end{pmatrix} = h(a)h(b)$.

   (e) $f : \mathbb{Z}/12\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$, defined by $f([x]_{12}) = [x]_4$.

   Answer: We have $f([a+b]_{12}) = [a+b]_4 = [a]_4+[b]_4 = f([a]_{12})+f([b]_{12})$ and $f([ab]_{12}) = [ab]_4 = [a]_4[b]_4 = f([a]_{12})f([b]_{12})$, so f is a homomorphism.

3. Show that the first ring is not isomorphic to the second:

   (a) $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and $M_2(\mathbb{R})$.

   Answer: By contradiction; suppose we do have an isomorphism and let $a, b \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and $f(a), f(b) \in M_2(\mathbb{R})$. By condition (iii) we have $f(ab) = f(a)f(b)$, note that $f(ab) = f(ba)$ but $f(a)f(b) \neq f(b)f(a)$ (e.g. matrices from 2(d)), so f is not injective. Therefore $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is not isomorphic to $M_2(\mathbb{R})$.

(b) $\mathbb{Q}$ and $\mathbb{R}$.

Answer: Since $\mathbb{Q} \in \mathbb{R}$ but $\mathbb{R} \notin \mathbb{Q}$, any $f : \mathbb{Q} \to \mathbb{R}$ would not be one-to-one, therefore $\mathbb{Q}$ is not isomorphic to $\mathbb{R}$.

(c) $(\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$ and $\mathbb{Z}/16\mathbb{Z}$.

Answer: Suppose we have an isomorphism, then by Theorem 3.10 we have $f(0,0) = 0$ and $f(1,1) = 1$. Then $f((1,1)+(1,1)+(1,1)+(1,1)) = f(4,4) = f(0,0) = 4$, which contradicts with $f(0,0) = 0$. Therefore $(\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$ is not isomorphic to $\mathbb{Z}/16\mathbb{Z}$.

4. If $f : \mathbb{Z} \to \mathbb{Z}$ is an isomorphism, prove that f is the identity map.

Answer: Let $f(1) = a \in \mathbb{Z}$, then by definition of ring isomorphism we have $f(n) = f(1 + \cdots + 1) = f(1) + \cdots + f(1) = nf(1) = na$. Similarly, we also have $f(n) = f(n \cdot 1) = f(n)f(1) = na^2$, so $na = na^2 \implies a = 1$ or $a = 0$. Note that for $a = 0$, we have $f(n) = 0$ which is not bijective, therefore we must have $a = 1 \implies f(1) = 1 \implies f(n) = n$ which is the identity map.

5. Let L be the ring considered in Problem 4 of homework 3. That is, L is the set of positive real numbers with addition and multiplication on L defined by $a \oplus b = ab$ and $a \otimes b = a^{\log b}$. In that problem, you showed that L is a field. Prove that L is actually isomorphic to the field $\mathbb{R}$ (with the usual addition and multiplication).

Answer: Let $f : L \to \mathbb{R}$ be defined as $f(a) = \log a$, we will prove the three conditions of ring isomorphism as follows:

(i) Suppose $f(a) = f(b)$, we have $f(a) = f(b) \implies \log a = \log b \implies e^{\log a} = e^{\log b} \implies a = b$. Therefore f is injective.

(ii) Since the image of log is $\mathbb{R}$, f is surjective.

(iii) By properties of log we have $f(a \oplus b) = \log(ab) = \log a + \log b = f(a) + f(b)$ and $f(a \otimes b) = \log\big(a^{\log b}\big) = \log b \cdot \log b = f(a)f(b)$.

6. Let $f : R \to S$ be a homomorphism of rings and let $K = \{r \in R \mid f(r) = 0_S\}$.

(a) Prove that K is a subring of R.

Answer: Let $p, q \in K$, then we must have $f(p) = 0_S$ and $f(q) = 0_S$. By definition of ring homomorphism we have $f(p - q) = f(p) - f(q) = 0_S$, so $p - q \in K$. Similarly we also have $f(pq) = f(p)f(q) = 0_S$, so $pq \in K$. Therefore K is a subring of R by Theorem 3.6.

(b) Prove that for any $x \in K$ and any $r \in R$ that $rx \in K$ and $xr \in K$.

Answer: Since $x \in K$, we have $f(x) = 0_S$, so $f(rx) = f(r)f(x) = f(r) \cdot 0_S = 0_S$ and $rx \in K$. Similarly, $f(xr) = f(x)f(r) = 0_S \cdot f(r) = 0_S$ and $xr \in K$.

(c) Prove that f is injective if and only if $K = \{0_R\}$.

Answer:

$\Rightarrow$: By contradiction. Suppose there exists an $a \in K$ with $a \neq 0_R$, then we must have $f(a) = 0_S$. But since $0_R \in K$, we have $f(0_R) = 0_S = f(a) \implies a = 0_R$ by definition of injection. Therefore $K = \{0_R\}$.

$\Leftarrow$: By contradiction. Suppose that we have $a, b \in R$ such that $a \neq b$ and $f(a) = f(b)$, then by definition of ring homomorphism we have $f(a) - f(b) = f(a - b) = 0_S$, so $a - b \in K$. But since $a \neq b$, $a - b \neq 0_R$ cannot be in $K = \{0_R\}$. Therefore f must be injective.

7. Let F be a field and R be a ring, and let $f : F \to R$ be a ring homomorphism.

   (a) If there is a *nonzero* element c of F such that $f(c) = 0$, prove that f is the zero homomorphism.

   Answer: Since F is a field, there exists a c^{-1} such that $cc^{-1} = 1_F$. Then for any $x \in F$, we have $f(x) = f(xcc^{-1}) = f(x)f(c)f(c^{-1}) = f(x) \cdot 0_R \cdot f(c^{-1}) = 0_R$. Therefore f is the zero homomorphism.

   (b) Prove that f is either injective or the zero homomorphism.

   Answer: Suppose we have $a, b \in F$ where $f(a) = f(b)$, then $f(a - b) = f(a) - f(b) = 0_R$. If $a - b \neq 0_R$, f is the zero homomorphism by part (a). If $a - b = 0_R$, we have $f(a) = f(b) \implies a = b$ so f is injective.

8. Which of the following subsets of $R[x]$ are subrings of $R[x]$?

   (a) All polynomials with constant term 0_R.

   Answer: It is a subring since it is closed under subtraction and multiplication; both the difference and product between two polynomials with constant 0_R would still have constant term 0_R.

   (b) All polynomials of degree 2.

   Answer: Not a subring as it is not closed under multiplication; the product of two degree 2 polynomials would be degree 4.

   (c) All polynomials of degree $\leq k$, where k is a fixed positive integer.

   Answer: Not a subring as it is not closed under multiplication; the product of two degree k polynomials would be degree $2k$.

   (d) All polynomials in which the odd powers of x have zero coefficients.

   Answer: It is a subring since it is closed under subtraction and multiplication. When we take the difference of coefficients for each power of x, the odd powers of the difference would still have zero coefficients. When multiplying two polynomials, since it is not possible get an nonzero odd power coefficients in the product without at least one nonzero odd power in one of the factors, the product would have zero coefficients in the odd powers.

   (e) All polynomials in which the even powers of x have zero coefficients.

   Answer: Not a subring as it is not closed under multiplication; e.g. $x^5 \cdot x^3 = x^8$.

9. Show that $1 + 3x$ is a unit in $(\mathbb{Z}/9\mathbb{Z})[x]$. Hence Corollary 4.5 may be false if R is not an integral domain.

   Answswer: We have $(1 + 3x)(1 - 3x) = 1 - 9x^2 = 1 - 0x^2 = 1$, so $1 + 3x$ is a unit in $(\mathbb{Z}/9\mathbb{Z})[x]$.