Math 164 Homework 3

Jiaping Zeng

2/15/2021

1. Show each of the following set is convex.

   (a) Hyperplane: $\{x : a^T x = b\}$ with $a \neq 0 \in \mathbb{R}^n$ and $b \in \mathbb{R}$.
   Answer: Take $x, y \in \Omega \implies a^T x = a^T y = b$, for $\alpha \in [0,1]$, we have $\alpha b + (1-\alpha)b = b \implies \alpha a^T x + (1-\alpha)a^T y = b \implies a^T(\alpha x + (1-\alpha)y) = b$, so $\alpha x + (1-\alpha)y \in \Omega$ and therefore Ω is a convex set.

   (b) Halfspace: $\{x : a^T x \leq b\}$ with $a \neq 0 \in \mathbb{R}^n$ and $b \in \mathbb{R}$.
   Answer: Take $x, y \in \Omega \implies a^T x \leq b, b^T x \leq b$, for $\alpha \in [0,1]$, we have $\alpha a^T x \leq \alpha b, (1-\alpha)a^T y \leq (1-\alpha)b \implies \alpha a^T x + (1-\alpha)a^T y \leq \alpha b + (1-\alpha)b \implies a^T(\alpha x + (1-\alpha)y) \leq b$, so $\alpha x + (1-\alpha)y \in \Omega$ and therefore Ω is a convex set.

   (c) Norm Ball: $\{x : \|x - x_c\| \leq r\}$ with > 0 and $x_c \in \mathbb{R}^n$.
   Answer: Take $x, y \in \Omega \implies \|x - x_c\| \leq r, \|y - y_c\| \leq r$, for $\alpha \in [0,1]$, we have $((1-\alpha)\|x - x_c\| + \alpha\|y - y_c\|)$

   (d) Polyhedron: $\{x : Ax \leq b, Cx = d\}$ with $A^{m \times n}, C \in \mathbb{R}^{p \times n}$ and $b \in \mathbb{R}^m, d \in \mathbb{R}^p$.
   Answer:

   (e) Nonnegative Orthant: $\mathbb{R}^n_+ = \{x : x \geq 0\}$.
   Answer:

   (f) Positive semidefinite cone: $S^n_+ = \{X \in \mathbb{R}^{n \times n} : X = X^T, X \succeq 0\}$
   Answer:

2. For each of the following functions, determine whether it is convex or concave or neither or both. Please explain the rationale for your answer.

   (a) $f(x_1, x_2) = x_1^2 + x_2^4$.
   Answer:

   (b) $f(x_1, x_2) = e^{ax_1} + e^{bx_2}$ with $a, b \in \mathbb{R}$.
   Answer:

   (c) $f(x_1, x_2) = x_1 \log(x_1) + x_2 \log(x_2)$ on $\mathbb{R}^2_{++}$.
   Answer:

   (d) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$.
   Answer:

(e) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$.

Answer:

(f) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbb{R} \times \mathbb{R}_{++}$.

Answer:

3. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function. Show that the set of the global minimizers of $\min_x f(x)$ is a convex.

   Answer: Let Ω be the set of the global minimizers of $\min_x f(x)$; take $x, y \in \Omega$ and $\alpha \in [0, 1]$, by the definition of global minimizer we have $f(x) = f(y) = \min_x f(x)$. Then $\alpha f(x) + (1 - \alpha)f(y) = \alpha \min_x f(x) + (1 - \alpha) \min_x f(x) = \min_x f(x)$. Them, since f is a convex function, we have $f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) = \min_x f(x)$. Therefore $\alpha x + (1 - \alpha)y$ must also be a global minimizer, i.e. $\alpha x + (1 - \alpha)y \in \Omega$, so Ω is convex.

4. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable and convex function. Show that $\nabla f(x^*) = 0$ if and only if x^* is a global minimizer of $\min_x f(x)$.

   Answer:

   $\Rightarrow$: Take any $y \in \text{dom}(f)$, since f is continuously differentiable and convex, we have $f(y) \geq f(x^*) + \nabla f(x^*)^T (y - x) = f(x^*)$ (since $\nabla f(x^*) = 0$). Since y is arbitrary, $f(x^*)$ is a global minimizer by definition.

   $\Leftarrow$: Since x^* is a global minimizer, then $f(y) \leq f(x^*)$ for any $y \in \text{dom}(f)$.