Math 164 Homework 3

Jiaping Zeng

2/15/2021

1. Show each of the following set is convex.

   (a) Hyperplane: $\{x : a^T x = b\}$ with $a \neq 0 \in \mathbb{R}^n$ and $b \in \mathbb{R}$.
   Answer: Take $x, y \in \Omega \implies a^T x = a^T y = b$, for $\alpha \in [0,1]$, we have $\alpha b + (1-\alpha)b = b \implies \alpha a^T x + (1-\alpha)a^T y = b \implies a^T(\alpha x + (1-\alpha)y) = b$, so $\alpha x + (1-\alpha)y \in \Omega$ and therefore Ω is a convex set.

   (b) Halfspace: $\{x : a^T x \leq b\}$ with $a \neq 0 \in \mathbb{R}^n$ and $b \in \mathbb{R}$.
   Answer: Take $x, y \in \Omega \implies a^T x \leq b, b^T x \leq b$, for $\alpha \in [0,1]$, we have $\alpha a^T x \leq \alpha b, (1-\alpha)a^T y \leq (1-\alpha)b \implies \alpha a^T x + (1-\alpha)a^T y \leq \alpha b + (1-\alpha)b \implies a^T(\alpha x + (1-\alpha)y) \leq b$, so $\alpha x + (1-\alpha)y \in \Omega$ and therefore Ω is a convex set.

   (c) Norm Ball: $\{x : \|x - x_c\| \leq r\}$ with > 0 and $x_c \in \mathbb{R}^n$.
   Answer: Take $x, y \in \Omega \implies \|x - x_c\| \leq r, \|y - x_c\| \leq r$, for $\alpha \in [0,1]$, we have $r \geq \alpha\|x - x_c\| + (1-\alpha)\|y - x_c\| \geq \|\alpha(x - x_c) + (1-\alpha)(y - x_c)\| = \|\alpha x + (1-\alpha)y - x_c\|$, so $\alpha x + (1-\alpha)y \in \Omega$ and therefore Ω is a convex set.

   (d) Polyhedron: $\{x : Ax \leq b, Cx = d\}$ with $A^{m\times n}, C \in \mathbb{R}^{p\times n}$ and $b \in \mathbb{R}^m, d \in \mathbb{R}^p$.
   Answer: Take $x, y \in \Omega$ and $\alpha \in [0,1]$, we have $Ax \leq b, Cx = d$ and $Ay \leq b, Cy = d$. So $\alpha Ax + (1-\alpha)Ay \leq \alpha b + (1+\alpha)b \implies \alpha Ax + (1-\alpha)Ay \leq b$ and $\alpha Cx + (1-\alpha)y = \alpha d + (1-\alpha)d \implies \alpha Cx + (1-\alpha)y = d$. Therefore $\alpha x + (1-\alpha)y \in \Omega$ and Ω is a convex set.

   (e) Nonnegative Orthant: $\mathbb{R}^n_+ = \{x : x \geq 0\}$.
   Answer: Take $x, y \in \Omega$ and $\alpha \in [0,1]$, we have $x \geq 0$ and $y \geq 0$. Therefore $\alpha x + (1-\alpha)y \geq 0$ since $\alpha \geq 0$ and $(1-\alpha) \geq 0$, so $\alpha x + (1-\alpha)y \in \Omega$ and Ω is a convex set.

   (f) Positive semidefinite cone: $S^n_+ = \{X \in \mathbb{R}^{n\times n} : X = X^T, X \succeq 0\}$
   Answer: Take $X, Y \in \Omega$ and $\alpha \in [0,1]$, we have $X = X^T, Y = Y^T, X \succeq 0, Y \succeq 0$. Then $\alpha X + (1-\alpha)Y = \alpha X^T + (1-\alpha)Y^T = (\alpha X + (1-\alpha)Y)^T$ and $\alpha X + (1-\alpha)Y \succeq 0$ (since $\alpha \geq 0$ and $(1-\alpha) \geq 0$). Therefore $\alpha X + (1-\alpha)Y \in \Omega$ and Ω is a convex set.

2. For each of the following functions, determine whether it is convex or concave or neither or both. Please explain the rationale for your answer.

   (a) $f(x_1, x_2) = x_1^2 + x_2^4$.

Answer: We have

$$\nabla^2 f(x) = \begin{bmatrix} \dfrac{\delta^2 f}{\delta x_1^2} & \dfrac{\delta^2 f}{\delta x_1 \delta x_2} \\ \dfrac{\delta^2 f}{\delta x_2 \delta x_1} & \dfrac{\delta^2 f}{\delta x_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 12x_2^2 \end{bmatrix}.$$

Since $2 \geq 0$ and $12x_2^2 \geq 0$, f is convex since the eigenvalues are always positve.

(b) $f(x_1, x_2) = e^{ax_1} + e^{bx_2}$ with $a, b \in \mathbb{R}$.

Answer: We have

$$\nabla^2 f(x) = \begin{bmatrix} \dfrac{\delta^2 f}{\delta x_1^2} & \dfrac{\delta^2 f}{\delta x_1 \delta x_2} \\ \dfrac{\delta^2 f}{\delta x_2 \delta x_1} & \dfrac{\delta^2 f}{\delta x_2^2} \end{bmatrix} = \begin{bmatrix} a^2 e^{ax} & 0 \\ 0 & b^2 e^{bx} \end{bmatrix}.$$

Since $a^2 e^{ax} \geq 0$ and $b^2 e^{bx} \geq 0$, f is convex since the eigenvalues are always positve.

(c) $f(x_1, x_2) = x_1 \log(x_1) + x_2 \log(x_2)$ on $\mathbb{R}^2_{++}$.

Answer: We have

$$\nabla^2 f(x) = \begin{bmatrix} \dfrac{\delta^2 f}{\delta x_1^2} & \dfrac{\delta^2 f}{\delta x_1 \delta x_2} \\ \dfrac{\delta^2 f}{\delta x_2 \delta x_1} & \dfrac{\delta^2 f}{\delta x_2^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{x_1} & 0 \\ 0 & \frac{1}{x_2} \end{bmatrix}.$$

Since $\dfrac{1}{x_1} \geq 0$ and $\dfrac{1}{x_2} \geq 0$ on $\mathbb{R}^2_{++}$, f is convex since the eigenvalues are always positve.

(d) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$.

Answer: We have

$$\nabla^2 f(x) = \begin{bmatrix} \dfrac{\delta^2 f}{\delta x_1^2} & \dfrac{\delta^2 f}{\delta x_1 \delta x_2} \\ \dfrac{\delta^2 f}{\delta x_2 \delta x_1} & \dfrac{\delta^2 f}{\delta x_2^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since $\operatorname{tr}\big(\nabla^2 f(x)\big) = 0$ and $\det\big(\nabla^2 f(x)\big) = -1$, f is neither convex nor concave.

(e) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$.

Answer: We have

$$\nabla^2 f(x) = \begin{bmatrix} \dfrac{\delta^2 f}{\delta x_1^2} & \dfrac{\delta^2 f}{\delta x_1 \delta x_2} \\ \dfrac{\delta^2 f}{\delta x_2 \delta x_1} & \dfrac{\delta^2 f}{\delta x_2^2} \end{bmatrix} = \begin{bmatrix} \dfrac{2}{x_1^3 x_2} & \dfrac{1}{x_1^2 x_2^2} \\ \dfrac{1}{x_1^2 x_2^2} & \dfrac{2}{x_1 x_2^3} \end{bmatrix}.$$

$\operatorname{tr}\big(\nabla^2 f(x)\big) = \dfrac{2}{x_1^3 x_2} + \dfrac{2}{x_1 x_2^3} \geq 0$ and $\det\big(\nabla^2 f(x)\big) = \dfrac{2}{x_1^3 x_2} \cdot \dfrac{2}{x_1 x_2^3} - \dfrac{1}{x_1^2 x_2^2} \cdot \dfrac{1}{x_1^2 x_2^2} = \dfrac{4}{x_1^4 x_2^4} - \dfrac{1}{x_1^4 x_2^4} = \dfrac{3}{x_1^4 x_2^4} \geq 0$ on $\mathbb{R}^2_{++}$, therefore $\nabla^2 f(x)$ is positive semidefinite and f is convex.

(f) $f(x_1, x_2) = x_1^2 / x_2$ on $\mathbb{R} \times \mathbb{R}_{++}$.

Answer: We have

$$\nabla^2 f(x) = \begin{bmatrix} \dfrac{\delta^2 f}{\delta x_1^2} & \dfrac{\delta^2 f}{\delta x_1 \delta x_2} \\ \dfrac{\delta^2 f}{\delta x_2 \delta x_1} & \dfrac{\delta^2 f}{\delta x_2^2} \end{bmatrix} = \begin{bmatrix} \dfrac{2}{x_2} & \dfrac{-2x_1}{x_2^2} \\ \dfrac{-2x_1}{x_2^2} & \dfrac{2x_1^2}{x_2^3} \end{bmatrix}.$$

$\operatorname{tr}\big(\nabla^2 f(x)\big) = \dfrac{2}{x_2} + \dfrac{2x_1^2}{x_2^3} \geq 0$ on $\mathbb{R} \times \mathbb{R}_{++}$ and $\det\big(\nabla^2 f(x)\big) = \dfrac{2}{x_2} \cdot \dfrac{2x_1^2}{x_2^3} - \dfrac{-2x_1}{x_2^2} \cdot \dfrac{-2x_1}{x_2^2} = \dfrac{4x_1^2}{x_2^4} - \dfrac{4x_1^2}{x_2^4} = 0$, therefore $\nabla^2 f(x)$ is positive semidefinite and f is convex.

3. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function. Show that the set of the global minimizers of $\min_x f(x)$ is a convex.

   Answer: Let Ω be the set of the global minimizers of $\min_x f(x)$; take $x, y \in \Omega$ and $\alpha \in [0, 1]$, by the definition of global minimizer we have $f(x) = f(y) = \min_x f(x)$. Then $\alpha f(x) + (1-\alpha) f(y) = \alpha \min_x f(x) + (1-\alpha) \min_x f(x) = \min_x f(x)$. Them, since f is a convex function, we have $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y) = \min_x f(x)$. Therefore $\alpha x + (1-\alpha)y$ must also be a global minimizer, i.e. $\alpha x + (1-\alpha)y \in \Omega$, so Ω is convex.

4. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable and convex function. Show that $\nabla f(x^*) = 0$ if and only if x^* is a global minimizer of $\min_x f(x)$.

   Answer:

   $\Rightarrow$: Take any $y \in \operatorname{dom}(f)$, since f is continuously differentiable and convex, we have $f(y) \geq f(x^*) + \nabla f(x^*)^T (y - x) = f(x^*)$ (since $\nabla f(x^*) = 0$). Since y is arbitrary, $f(x^*)$ is a global minimizer by definition.

   $\Leftarrow$: Since x^* is a global minimizer and therefore a local minimizer, we have $\nabla f(x^*) = 0$.