1. Prove that A^T is the adjoint operator of $A \in \mathbb{R}^{m \times n}$. Hint: Show $\langle Ax, y \rangle = \langle x, A^T y \rangle$ for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$.

   Answer: By definition of adjoint, we have $\langle Ax, y \rangle = \langle x, A^* y \rangle \implies y^T Ax = y^T (A^*)^T x$; to satisfy this, we must have $A = (A^*)^T$, therefore $A^* = A^T$ and A^T is the adjoint operator of $A \in \mathbb{R}^{m \times n}$.

2. (a) Find the gradient and hessian of the function $f(x) = \frac{1}{2}\|Ax - b\|_2^2$ where $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$.

   Answer: We have $df(x) = d(\frac{1}{2}\|Ax - b\|_2^2) = \frac{1}{2}(Ax-b)^T d(Ax-b) + \frac{1}{2}[d(Ax-b)]^T(Ax-b) = \frac{1}{2}(Ax-b)^T Adx + \frac{1}{2}[Adx]^T(Ax-b) = \frac{1}{2}(Ax-b)^T Adx + \frac{1}{2}(Ax-b)^T Adx = (Ax-b)^T Adx$.
   Therefore $\nabla f(x) = [(Ax-b)^T A]^T = A^T(Ax-b)$ and $\nabla^2 f(x) = D(\nabla f(x)) = D(A^T Ax - A^T b) = D(A^T Ax) = A^T A$.

   (b) If $\text{rank}(A) = n$ in the above problem, find critical point of $f(x)$.

   Answer: We have $\nabla f(x) = A^T(Ax-b) = 0 \implies A^T Ax - A^T b = 0 \implies A^T Ax = A^T b$. Since $\text{rank}(A) = n$, $A^T A$ is invertible, i.e. $(A^T A)^{-1}$ exists. Then $(A^T A)^{-1} A^T Ax = (A^T A)^{-1} A^T b \implies x = (A^T A)^{-1} A^T b$.

3. Show that for any matrix $A \in \mathbb{R}^{m \times n}$, the set $\{x \in \mathbb{R}^n : Ax = 0\}$ is convex.

   Answer: Take $x, y \in \Omega$ and $\alpha \in [0, 1]$, we have $Ax = 0$ and $Ay = 0$ for any matrix $A \in \mathbb{R}^{m \times n}$. Then $\alpha Ax = 0, (1-\alpha)Ay = 0 \implies \alpha Ax + (1-\alpha)Ay = 0 \implies A(\alpha x + (1-\alpha)y) = 0$, therefore $\alpha x + (1-\alpha)y \in \Omega$ and Ω is a convex set.

4. Find all the critical points of $f(x_1, x_2) = (x_1^2 - 4)^2 + x_2^2$. Which has positive definite hessian matrix?

Answer: We have

$$\nabla f(x) = \begin{bmatrix} \frac{\delta f}{\delta x_1} \\ \frac{\delta f}{\delta x_2} \end{bmatrix} = \begin{bmatrix} 4x_1^3 - 16x_1 \\ 2x_2 \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\delta^2 f}{\delta x_1^2} & \frac{\delta^2 f}{\delta x_1 \delta x_2} \\ \frac{\delta^2 f}{\delta x_2 \delta x_1} & \frac{\delta^2 f}{\delta x_2^2} \end{bmatrix} = \begin{bmatrix} 12x_1^2 - 16 & 0 \\ 0 & 2 \end{bmatrix}.$$

Solving $\nabla f(x) = 0$ gives us $4x_1^3 - 16x_1 = 0 \implies x_1 = 0, \pm 2$ and $2x_2 = 0 \implies x_2 = 0$, so the critical points are $[-2, 0]^T$, $[0, 0]^T$ and $[2, 0]^T$. The eigenvalues of $\nabla^2 f([\pm 2, 0]^T)$ are $(32 - \lambda)(2 - \lambda) = 0 \implies \lambda = 2, 32$. Similarly, the eigenvalues of $\nabla^2 f([0, 0]^T)$ are $(-16 - \lambda)(2 - \lambda) \implies \lambda = -16, 2$. Therefore the critical points $[-2, 0]^T$ and $[2, 0]^T$ have positive definite hessian matrix.

5. Given $x_1, \cdots, x_n \in \mathbb{R}^d$, find the global minimizer of

$$\min_{x \in \mathbb{R}^d} f(x) := \sum_{k=1}^{n} \|x - x_k\|_2^2.$$

Hint: Show $f(x)$ is convex function. So any critical is a global minimizer.

Answer: We will first show that $f(x)$ is convex: take $x, y \in \mathbb{R}^d$ and $\alpha \in [0,1]$, then $\alpha x + (1-\alpha)y$ is a linear combination of x, y and is therefore also in $\mathbb{R}^d$, so $\text{dom}(f)$ is convex. In addition, we have $\nabla f(x) = \nabla \sum_{k=1}^{n} \|x - x_k\|_2^2 = \nabla \sum_{k=1}^{n} (x - x_k)^2 = 2\sum_{k=1}^{n}(x - x_k)$ by chain rule. We also have $\nabla^2 f(x) = D(\nabla f(x)) = 2n \geq 0$, so the Hessian is positive semidefinite and therefore $f(x)$ is a convex funtion. Then we can set $\nabla f(x^*) = 0$ to find our critical point x^*, which will be a global minimizer since $f(x)$ is a convex function. We have $\nabla f(x^*) = 0 \implies 2\sum_{k=1}^{n}(x - x_k) = 0 \implies 2nx^* = 2\sum_{k=1}^{n} x_k \implies x^* = \frac{1}{n}\sum_{k=1}^{n} x_k$, therefore $\frac{1}{n}\sum_{k=1}^{n} x_k$ is the global minimizer.