Math 110A Homework 8

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1. Let R be an ring with identity and let I be an ideal of R.

   (a) If $1_R \in I$, prove that $I = R$.

   Answer: Take any $r \in R$, we must have $1_R \cdot r = r \in I$ by definition of ideal. Therefore every element of R is in I, so $I = R$.

   (b) If I contains a unit, prove that $I = R$.

   Answer: Let $a \in I$ be a unit, then by definition $ax = 1_R$ has a solution in R. Then by definition of ideal we have $ax = 1_R \in I$, therefore $I = R$ by part (a).

   (c) If I is an ideal in a field F, prove that either $I = (0_F)$ or $I = F$.

   Answer: By definition of field, $1_F \neq 0_F$. Then, if $1_F \in I$, we have $I = F$ by part (a); if not, we can only have $I = (0_F)$ or else we would again have $I = F$ by part (b) since every nonzero element is a unit.

2. Let I and J be ideals in R.

   (a) Prove that the set $K = \{a + b \mid a \in I, b \in J\}$ is an ideal in R that contains both I and J. K is called the **sum** of I and J, and is denoted $I + J$.

   Answer: Take $a, b \in I$ and $c, d \in J$, then $a + c \in K$ and $b + d \in K$. We have $(a + c) - (b + d) = (a - b) + (c - d) \in K$ since $a - b \in I$ and $c - d \in J$ by Theorem 6.1. We also have $r(a + c) \in K$ and $(a + c) \in K$ since $r(a + c) = ra + rc$ and $(a + c)r = ar + cr$, where $ra, ar \in I$ and $rc, cr \in J$ by Theorem 6.1. Then K satisfies both conditions of Theorem 6.1 and is therefore an ideal. It also contains both I and J upon taking $b = 0$ or $a = 0$ respectively in the definition.

   (b) Is the set $K = \{ab \mid a \in I, b \in J\}$ always an ideal in R?

   Answer: No; take $R = \mathbb{Z}$, $I = 2\mathbb{Z}$ and $J = 3\mathbb{Z}$. We have $4 \in I \subset K$ and $9 \in J \subset K$, so by Theorem 6.1 we must have $9 - 4 = 5 \in IJ$ which is not true.

   (c) Let IJ denote the set of all possible finite sums of elements of the form ab (with $a \in I, b \in J$), that is:

   $$IJ = \{a_1b_1 + a_2b_2 + \cdots + a_nb_n \mid n \geq 1, a_k \in I, b_k \in J\}.$$

   Prove that IJ is an ideal of R. IJ is called the **product** of I and J.

   Answer: Take $p, q \in IJ$ with $p = a_1b_1 + a_2b_2 + \cdots + a_nb_n$ and $q = c_1d_1 + c_2d_2 + \cdots + c_nd_n$, we have $p - q = a_1b_1 + a_2b_2 + \cdots + a_nb_n - c_1d_1 - c_2d_2 - \cdots - c_nd_n$ which is in IJ since each

$a_k b_k$ and $-c_k d_k$ is in IJ. Now take $r \in R$, we have $rp = r(a_1b_1 + a_2b_2 + \cdots + a_nb_n) = (ra_1)b_1 + (ra_2)b_2 + \cdots + (ra_n)b_n$. Since $ra_k \in I$ by Theorem 6.1 and $b_k \in J$, $rp \in IJ$. Similarly $pr \in IJ$ since $pr = (a_1b_1 + a_2b_2 + \cdots + a_nb_n)r = a_1(b_1r) + a_2(b_2r) + \cdots + a_n(b_nr)$. Therefore IJ is an ideal by Theorem 6.1.

3. Let R be an integral domain and $a, b \in R$. Show that $(a) = (b)$ if and only if $a = bu$ for some unit $u \in R$.

   Answer:

   $\Rightarrow$: Since $(a) = (b)$, we can take $ra = rb \cdot 1_R$ for every element of (a) and (b) (1_R always exists since R is an integral domain), then we have $a = bu$ with $u = 1_R$.

   $\Leftarrow$: Since $a = bu$, every element of (b) is a multiple of a in R. Then by definition of principal ideal (Theorem 6.2) $(a) = (b)$.

4. Let R be a commutative ring with $1_R \neq 0_R$, whose only ideals are (0) and R. Prove that R is a field.

   Answer: