Math 164 Homework 4

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1. Consider the function
$$f(x) = \frac{1}{2}x^T \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} x$$

   (a) Find the global minimizer x^* of $f(x)$.

   Answer: Let $x = \begin{pmatrix} p \\ q \end{pmatrix}$, we have
$$f(x) = \frac{1}{2}\begin{pmatrix} p \\ q \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{2}p^2 + q^2,$$
   therefore,
$$\nabla f(x) = \begin{pmatrix} p \\ 2q \end{pmatrix} \text{ and } \nabla^2 f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$
   Setting $\nabla f(x) = 0$ gives us $p = q = 0$ as the only critical point; since it is easy to see that the eigenvalues of the Hessian is always positive (therefore f is convex), it is the global minimizer.

   (b) Write down the first iteration of gradient descent method with the stepsize chosen as one over the Lipschitz constant of $\nabla f(x)$ and starting point as $x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

   Answer: Since the maximum eigenvalue of the Hessian is 2, our Lipschitz constant is $L = 2$. Therefore,
$$x_1 = x_0 - \frac{1}{L}\nabla f(x_0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}.$$

   (c) Write down the closed-form expression of x_k in the kth iteration of gradient descent method for any positive integer k.

   Answer: We have
$$x_{k+1} = x_k - \frac{1}{L}\nabla f(x_k) = \begin{pmatrix} x_{k_1} \\ x_{k_2} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} x_{k_1} \\ 2x_{k_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_{k_1} \\ 0 \end{pmatrix}.$$
   Note that we are halving the first component by half each time with the second component staying

at 0, with $x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Therefore

$$x_k = \begin{pmatrix} \frac{1}{2^k} \\ 0 \end{pmatrix}.$$

(d) After how many iterations, we have $\|x_k - x^*\|_2 < \frac{1}{100}$.

Answer: We have $\frac{1}{2^k} < \frac{1}{100} \implies 2^k > 100 \implies k = 7$, therefore we have $\|x_k - x^*\|_2 < \frac{1}{100}$ after 7 iterations.

(e) What's the convergence rate of the sequence $\{\|x_k - \boldsymbol{x^*}\|_2\}$? (sublinear/linear/quadratic)

Answer: We have $\lim_{k\to\infty} \frac{\|x_{k+1} - \boldsymbol{x^*}\|_2}{\|x_k - \boldsymbol{x^*}\|_2} = \frac{1}{2}$, so it is linear.

2. Consider the function

$$f(x) = \frac{1}{2} x^T \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} x$$

(a) Write down the first iteration of Newton's method and starting point $x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Answer: Let $x = \begin{pmatrix} p \\ q \end{pmatrix}$, we have

$$f(x) = \frac{1}{2} \begin{pmatrix} p \\ q \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{2} p^2 + q^2,$$

therefore,

$$\nabla f(x) = \begin{pmatrix} p \\ 2q \end{pmatrix} \text{ and } \nabla^2 f(x) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then our search direction is

$$d_0^N = -\nabla^2 f(x_0)^{-1} \nabla f(x_0) = -\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix},$$

so

$$x_1 = x_0 + d_0^N = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(b) Write down the closed-form expression of x_k in the kth iteration of Newton's method for any positive integer k.

Answer:

$$x_{k+1} = x_k + d_k^N = \begin{pmatrix} x_{k_1} \\ x_{k_2} \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_{k_1} \\ 2x_{k_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(c) After how many iterations, we have $\|x_k - x^*\|_2 < \frac{1}{100}$.

Answer: As shown in part (a), $x_1 = \mathbf{0}$. Therefore we have $\|x_k - x^*\|_2 < \frac{1}{100}$ after one iteration.