Math 164 Homework 1

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1. Let $A \in \mathbb{R}^{m \times n}$ and $\text{rank}(A) = m$. Show that $m \leq n$.
   Answer: By definition of matrix rank, $\text{rank}(A)$ is the dimension of the column span of A, which cannot exceed the number of columns in the matrix, i.e. $\text{rank}(A) \leq m$. It is also the dimension of the row span which cannot exceed the number of rows in the matrix, i.e. $\text{rank}(A) \leq n$. Therefore $\text{rank}(A) \leq \min(m, n)$. By definition of minimum, if $\text{rank}(A) = m \leq \min(m, n)$, we must have $n \geq m$.

2. Fill in the blanks in the following 2×2 matrix

$$\begin{bmatrix} -1 & ? \\ ? & ? \end{bmatrix}$$

   so that $\|A\|_\infty = 3$.
   Answer:

$$\begin{bmatrix} -1 & 0 \\ -2 & 0 \end{bmatrix}$$

3. Show that for any two vectors $x, y \in \mathbb{R}^n$, $|\|x\| - \|y\|| \leq \|x - y\|$.
   Answer: We have $|\|x\| - \|y\||^2 = (\sqrt{\langle x, x\rangle} - \sqrt{\langle y, y\rangle})^2 = \langle x, x\rangle + \langle y, y\rangle - \|x\|\|y\|$. By Cauchy-Schwarz inequality, $|\langle x, y\rangle| \leq \|x\|\|y\|$, then $|\|x\| - \|y\||^2 \leq \langle x, x\rangle + \langle y, y\rangle - \|x\|\|y\| = (\langle x, x\rangle - \langle x, y\rangle) - (\langle x, y\rangle + \langle y, y\rangle) = \overline{\langle x - y, x\rangle} - \langle x - y, y\rangle$. Since $x, y \in \mathbb{R}^{\ltimes}$, $\overline{\langle x - y, x\rangle} = \langle x - y, x\rangle$. Then $|\|x\| - \|y\||^2 \leq \langle x - y, x\rangle - \langle x - y, y\rangle = \langle x - y, x - y\rangle = \|x - y\|^2$. Since both $|\|x\| - \|y\||$ and $\|x - y\|$ must be nonnegative, we have $|\|x\| - \|y\|| \leq \|x - y\|$.

4. Prove that for every positive integer N the following statement holds: For any set of vectors $x_1, x_2, \ldots, x_N \in \mathbb{R}^n$, $\|x_1 + x_2 + \cdot + x_N\| \leq \|x_1\| + \|x_2\| + \cdots + \|x_N\|$.
   Answer: By induction on N.
   Base case: $N = 2$, we want to show that $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$, which is true by triangle inequality.
   Inductive step: Suppose that $\|x_1 + x_2 + \cdot + x_{N-1}\| \leq \|x_1\| + \|x_2\| + \cdots + \|x_{N-1}\|$, we want to show that $\|x_1 + x_2 + \cdot + x_N\| \leq \|x_1\| + \|x_2\| + \cdots + \|x_N\|$. Let $x_m = x_1 + \cdots + x_{N-1}$, then $\|x_m\| = \|x_1 + \cdots + x_{N-1}\| \leq \|x_1\| + \|x_2\| + \cdots + \|x_{N-1}\|$ by inductive hypothesis. Now by substitution and triangle inequality, we have $\|x_1 + x_2 + \cdots + x_N\| = \|x_m + x_N\| \leq \|x_m\| + \|x_N\| \leq \|x_1\| + \|x_2\| + \cdots + \|x_N\|$.
   Therefore we have proved the statement by induction.

5. Prove that the system $Ax = b, A \in \mathbb{R}^{m\times n}$, has a unique solution if and only if $\text{rank}(A) = \text{rank}([A, b]) = n$.

   Answer:

   $\Rightarrow$: Assume $Ax = b$ has a unique solution, we want to show that $\text{rank}(A) = \text{rank}([A, b]) = n$. Since $Ax = b$ has a solution, b must be in the colspan of A, so $\text{rank}(A) = \text{rank}([A, b])$. Since the solution is also unique, the columns of A mush be linearly independent, so $\text{rank}(A) = n$. Therefore $\text{rank}(A) = \text{rank}([A, b]) = n$.

   $\Leftarrow$: Assume that $\text{rank}(A) = \text{rank}([A, b]) = n$, we want to show that $Ax = b$ has a unique solution. Since $\text{rank}(A) = n$, its columns must be linearly independent. Then since $\text{rank}([A, b]) = \text{rank}(A)$, b must be in the colspan of A. Therefore $Ax = b$ has a unique solution.

6. Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of the matrix $A \in \mathbb{R}^{n\times n}$. Show that the eigenvalues of the matrix $\mathbb{I}_n - A$ are $1 - \lambda_1, \cdots, 1 - \lambda_n$. Here $\mathbb{I}_n$ is an identity matrix.

   Answer: Since A has n eigenvalues, we can diagonalize A (under the basis of its eigenvectors) to have $\lambda_1, \cdots, \lambda_n$ on the main diagonal. Then, since $\mathbb{I}_n$ has only 1 on the main diagonal, $\mathbb{I}_n - A$ has $1 - \lambda_i$ on its main diagonal, therefore its eigenvalues are $1 - \lambda_1, \cdots, 1 - \lambda_n$.

7. Find the nullspace of

$$A = \begin{pmatrix} 4 & -2 & 0 \\ 2 & 1 & -1 \\ 2 & -3 & 1 \end{pmatrix}.$$

   Answer: We can find the nullspace by solving $Ax = 0$ as follows:

$$\begin{pmatrix} 4 & -2 & 0 \\ 2 & 1 & -1 \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\implies 4x_1 - 2x_2 = 0, 2x_1 + x_2 - x_3 = 0, x_1 - 3x_2 + x_3 = 0$$

$$\implies x_1 = \frac{x_3}{4}, x_2 = \frac{x_3}{2}$$

   Therefore the nullspace is any scalar multiple of the vector $[\frac{1}{4}, \frac{1}{2}, 1]^\top$.

8. Let $A \in \mathbb{R}^{m\times n}$ be a matrix. Show that $\mathcal{R}(A)$ is a subspace of $\mathbb{R}^m$ and $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^n$.

   Answer: Let $y \in \mathcal{R}(A)$. By definition, there must exist some $x \in \mathbb{R}^n$ such that $Ax = y$. Then the dimension of y is outer dimensions of A and x, which gives us $y \in \mathbb{R}^m$. Therefore every vector in $\mathcal{R}(A)$ is in $\mathbb{R}^m$, so $\mathcal{R}(A)$ is a subspace of $\mathbb{R}^m$.

   Now let $x \in \mathcal{N}(A)$, then we must have $Ax = 0$; since $A \in \mathbb{R}^{m\times n}$ and the inner dimensions of A and x must match, we have $x \in \mathbb{R}^n$. Therefore every vector in $\mathcal{N}(A)$ is in $\mathbb{R}^n$, so $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^n$.

9. Let $A \in \mathbb{R}^{m\times n}$ be a matrix with $A^\top A = \mathbb{I}_n$. Show that $P = AA^\top$ is an orthogonal projection of $\mathcal{R}(A)$.

   Answer: Since $A^\top A = \mathbb{I}_n$, we have $P^2 = (AA^\top)(AA^\top) = A(A^\top A)A^\top = AA^\top = P$. In addition, $P^* = P^\top = (AA^\top)^\top = (A^\top)^\top A^\top = AA^\top = P$. Now let $y \in \mathcal{R}(A)$, then there must exist some $x \in \mathbb{R}^n$

such that $Ax = y$. Note that $P(y) = AA^\top y = (AA^\top)Ax = A(A^\top A)x = Ax = y \in \mathcal{R}(A)$. Therefore we have $P : \mathcal{R}(A) \to \mathcal{R}(A)$ where P is both idempotent and self-adjoint, therefore P is an orthogonal projection of $\mathcal{R}(A)$.

10. Consider the vector space $\mathbb{R}^2$ equipped with the standard ℓ_2 norm. Let $x = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$. Find the best approximation

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

to x such that the entries of y satisfy the constraint $y_2 = 2y_1$.

Answer: let $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then $\min_{y_1} \left\| \begin{bmatrix} y_1 \\ 2y_1 \end{bmatrix} - \begin{bmatrix} 5 \\ 6 \end{bmatrix} \right\|_2^2 = \min_{y_1} \left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} y_2 - \begin{bmatrix} 5 \\ 6 \end{bmatrix} \right\|_2^2 = \min \|Ay - x\|_2^2$. Since $A^\top A = 5$ is invertible, the unique minimizer is given by $y_2 = A^+ x = (A^\top A)^{-1} A^\top x = \dfrac{1}{5} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \dfrac{17}{5}$. Therefore $y = \begin{bmatrix} y_1 \\ 2y_2 \end{bmatrix} = \begin{bmatrix} \frac{17}{5} \\ \frac{34}{5} \end{bmatrix}$.