

Math 164 Homework 3

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1. Show each of the following set is convex.

(a) Hyperplane: $\{x : a^T x = b\}$ with $a \neq 0 \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

Answer: Take $x, y \in \Omega \implies a^T x = a^T y = b$, for $\alpha \in [0, 1]$, we have $\alpha b + (1 - \alpha)b = b \implies \alpha a^T x + (1 - \alpha)a^T y = b \implies a^T(\alpha x + (1 - \alpha)y) = b$, so $\alpha x + (1 - \alpha)y \in \Omega$ and therefore Ω is a convex set.

(b) Halfspace: $\{x : a^T x \leq b\}$ with $a \neq 0 \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

Answer: Take $x, y \in \Omega \implies a^T x \leq b, a^T y \leq b$, for $\alpha \in [0, 1]$, we have $\alpha a^T x \leq \alpha b, (1 - \alpha)a^T y \leq (1 - \alpha)b \implies \alpha a^T x + (1 - \alpha)a^T y \leq \alpha b + (1 - \alpha)b \implies a^T(\alpha x + (1 - \alpha)y) \leq b$, so $\alpha x + (1 - \alpha)y \in \Omega$ and therefore Ω is a convex set.

(c) Norm Ball: $\{x : \|x - x_c\| \leq r\}$ with $r > 0$ and $x_c \in \mathbb{R}^n$.

Answer: Take $x, y \in \Omega \implies \|x - x_c\| \leq r, \|y - x_c\| \leq r$, for $\alpha \in [0, 1]$, we have $r \geq \alpha\|x - x_c\| + (1 - \alpha)\|y - x_c\| \geq \|\alpha(x - x_c) + (1 - \alpha)(y - x_c)\| = \|\alpha x + (1 - \alpha)y - x_c\|$, so $\alpha x + (1 - \alpha)y \in \Omega$ and therefore Ω is a convex set.

(d) Polyhedron: $\{x : Ax \leq b, Cx = d\}$ with $A^{m \times n}, C \in \mathbb{R}^{p \times n}$ and $b \in \mathbb{R}^m, d \in \mathbb{R}^p$.

Answer: Take $x, y \in \Omega$ and $\alpha \in [0, 1]$, we have $Ax \leq b, Cx = d$ and $Ay \leq b, Cy = d$. So $\alpha Ax + (1 - \alpha)Ay \leq \alpha b + (1 - \alpha)b \implies \alpha Ax + (1 - \alpha)Ay \leq b$ and $\alpha Cx + (1 - \alpha)Cy = \alpha d + (1 - \alpha)d \implies \alpha Cx + (1 - \alpha)Cy = d$. Therefore $\alpha x + (1 - \alpha)y \in \Omega$ and Ω is a convex set.

(e) Nonnegative Orthant: $\mathbb{R}_+^n = \{x : x \geq 0\}$.

Answer: Take $x, y \in \Omega$ and $\alpha \in [0, 1]$, we have $x \geq 0$ and $y \geq 0$. Therefore $\alpha x + (1 - \alpha)y \geq 0$ since $\alpha \geq 0$ and $(1 - \alpha) \geq 0$, so $\alpha x + (1 - \alpha)y \in \Omega$ and Ω is a convex set.

(f) Positive semidefinite cone: $S_+^n = \{X \in \mathbb{R}^{n \times n} : X = X^T, X \succeq 0\}$

Answer: Take $X, Y \in \Omega$ and $\alpha \in [0, 1]$, we have $X = X^T, Y = Y^T, X \succeq 0, Y \succeq 0$. Then $\alpha X + (1 - \alpha)Y = \alpha X^T + (1 - \alpha)Y^T = (\alpha X + (1 - \alpha)Y)^T$ and $\alpha X + (1 - \alpha)Y \succeq 0$ (since $\alpha \geq 0$ and $(1 - \alpha) \geq 0$). Therefore $\alpha X + (1 - \alpha)Y \in \Omega$ and Ω is a convex set.

2. For each of the following functions, determine whether it is convex or concave or neither or both. Please explain the rationale for your answer.

(a) $f(x_1, x_2) = x_1^2 + x_2^4$.

Answer: We have

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\delta^2 f}{\delta x_1^2} & \frac{\delta^2 f}{\delta x_1 \delta x_2} \\ \frac{\delta^2 f}{\delta x_2 \delta x_1} & \frac{\delta^2 f}{\delta x_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 12x_2^2 \end{bmatrix}.$$

Since $2 \geq 0$ and $12x_2^2 \geq 0$, f is convex since the eigenvalues are always nonnegative.

- (b) $f(x_1, x_2) = e^{ax_1} + e^{bx_2}$ with $a, b \in \mathbb{R}$.

Answer: We have

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\delta^2 f}{\delta x_1^2} & \frac{\delta^2 f}{\delta x_1 \delta x_2} \\ \frac{\delta^2 f}{\delta x_2 \delta x_1} & \frac{\delta^2 f}{\delta x_2^2} \end{bmatrix} = \begin{bmatrix} a^2 e^{ax_1} & 0 \\ 0 & b^2 e^{bx_2} \end{bmatrix}.$$

Since $a^2 e^{ax_1} \geq 0$ and $b^2 e^{bx_2} \geq 0$, f is convex since the eigenvalues are always nonnegative.

- (c) $f(x_1, x_2) = x_1 \log(x_1) + x_2 \log(x_2)$ on \mathbb{R}_{++}^2 .

Answer: We have

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\delta^2 f}{\delta x_1^2} & \frac{\delta^2 f}{\delta x_1 \delta x_2} \\ \frac{\delta^2 f}{\delta x_2 \delta x_1} & \frac{\delta^2 f}{\delta x_2^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{x_1} & 0 \\ 0 & \frac{1}{x_2} \end{bmatrix}.$$

Since $\frac{1}{x_1} \geq 0$ and $\frac{1}{x_2} \geq 0$ on \mathbb{R}_{++}^2 , f is convex since the eigenvalues are always nonnegative.

- (d) $f(x_1, x_2) = x_1 x_2$ on \mathbb{R}_{++}^2 .

Answer: We have

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\delta^2 f}{\delta x_1^2} & \frac{\delta^2 f}{\delta x_1 \delta x_2} \\ \frac{\delta^2 f}{\delta x_2 \delta x_1} & \frac{\delta^2 f}{\delta x_2^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since $\text{tr}(\nabla^2 f(x)) = 0$ and $\det(\nabla^2 f(x)) = -1$, f is neither convex nor concave.

- (e) $f(x_1, x_2) = 1/(x_1 x_2)$ on \mathbb{R}_{++}^2 .

Answer: We have

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\delta^2 f}{\delta x_1^2} & \frac{\delta^2 f}{\delta x_1 \delta x_2} \\ \frac{\delta^2 f}{\delta x_2 \delta x_1} & \frac{\delta^2 f}{\delta x_2^2} \end{bmatrix} = \begin{bmatrix} \frac{2}{x_1^3 x_2} & \frac{1}{x_1^2 x_2^2} \\ \frac{1}{x_1^2 x_2^2} & \frac{2}{x_1 x_2^3} \end{bmatrix}.$$

$\text{tr}(\nabla^2 f(x)) = \frac{2}{x_1^3 x_2} + \frac{2}{x_1 x_2^3} \geq 0$ and $\det(\nabla^2 f(x)) = \frac{2}{x_1^3 x_2} \cdot \frac{2}{x_1 x_2^3} - \frac{1}{x_1^2 x_2^2} \cdot \frac{1}{x_1^2 x_2^2} = \frac{4}{x_1^4 x_2^4} - \frac{1}{x_1^4 x_2^4} = \frac{3}{x_1^4 x_2^4} \geq 0$ on \mathbb{R}_{++}^2 , therefore $\nabla^2 f(x)$ is positive semidefinite and f is convex.

- (f) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbb{R} \times \mathbb{R}_{++}$.

Answer: We have

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\delta^2 f}{\delta x_1^2} & \frac{\delta^2 f}{\delta x_1 \delta x_2} \\ \frac{\delta^2 f}{\delta x_2 \delta x_1} & \frac{\delta^2 f}{\delta x_2^2} \end{bmatrix} = \begin{bmatrix} \frac{2}{x_2} & \frac{-2x_1}{x_2^2} \\ \frac{-2x_1}{x_2^2} & \frac{2x_1^2}{x_2^3} \end{bmatrix}.$$

$$\begin{aligned} \text{tr}(\nabla^2 f(x)) &= \frac{2}{x_2} + \frac{2x_1^2}{x_2^3} \geq 0 \text{ on } \mathbb{R} \times \mathbb{R}_{++} \text{ and } \det(\nabla^2 f(x)) = \frac{2}{x_2} \cdot \frac{2x_1^2}{x_2^3} - \frac{-2x_1}{x_2^2} \cdot \frac{-2x_1}{x_2^2} = \\ &= \frac{4x_1^2}{x_2^4} - \frac{4x_1^2}{x_2^4} = 0, \text{ therefore } \nabla^2 f(x) \text{ is positive semidefinite and } f \text{ is convex.} \end{aligned}$$

3. Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function. Show that the set of the global minimizers of $\min_x f(x)$ is a convex.

Answer: Let Ω be the set of the global minimizers of $\min_x f(x)$; take $x, y \in \Omega$ and $\alpha \in [0, 1]$, by the definition of global minimizer we have $f(x) = f(y) = \min_x f(x)$. Then $\alpha f(x) + (1 - \alpha)f(y) = \alpha \min_x f(x) + (1 - \alpha) \min_x f(x) = \min_x f(x)$. Then, since f is a convex function, we have $f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) = \min_x f(x)$. Therefore $\alpha x + (1 - \alpha)y$ must also be a global minimizer, i.e. $\alpha x + (1 - \alpha)y \in \Omega$, so Ω is convex.

4. Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuously differentiable and convex function. Show that $\nabla f(x^*) = 0$ if and only if x^* is a global minimizer of $\min_x f(x)$.

Answer:

\Rightarrow : Take any $y \in \text{dom}(f)$, since f is continuously differentiable and convex, we have $f(y) \geq f(x^*) + \nabla f(x^*)^T(y - x) = f(x^*)$ (since $\nabla f(x^*) = 0$). Since y is arbitrary, $f(x^*)$ is a global minimizer by definition.

\Leftarrow : Since x^* is a global minimizer and therefore a local minimizer, we have $\nabla f(x^*) = 0$.