Math 110A Homework 4

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1. Let F be a field. We say that $a \in F$ is a *multiple root* of $f(x)$ if $(x-a)^k$ is a factor of $f(x)$ for some $k \geq 2$.

   (a) Prove that $a \in F$ is a multiple root of $f(x) \in F[x]$ if and only if a is a root of both $f(x)$ and $f'(x)$.

   Answer:

   $\Rightarrow$: Since a is a multiple root, $(x-a)^2$ is a factor of $f(x)$. Let $f(x) = (x-a)^2 g(x)$, then $f'(x) = 2(x-a)g + (x-a)^2 g(x)$; clearly $(x-a)$ is a factor of both $f(x)$ and $f'(x)$.

   $\Leftarrow$: Since a is a root of $f(x)$, $(x-a)$ is a factor of $f(x)$. Let $f(x) = (x-a)g(x)$, we have $f(a) = (a-a)g(a) = 0$ and $f'(a) = (a-a)g(a) + (a-a)g'(a) = 0$, so $(x-a)$ is a factor of both $f(x)$ and $f'(x)$. By Factor Theorem a is a root.

   (b) If $f(x) \in F[x]$ and $f(x)$ and $f'(x)$ are relatively prime, prove that f has no multiple root in F.

   Answer: Since $f(x)$ and $f'(x)$ are relatively prime, $(x-a)$ is not a root of both $f(x)$ and $f'(x)$ for any $a \in F$. Therefore by part (a) f has no multiple root in F.

   (c) Let $f(x) \in \mathbb{Q}[x]$ be a irreducible in $\mathbb{Q}[x]$. Prove that $f(x)$ has no multiple roots in $\mathbb{C}$.

   Answer: Since $f(x)$ is irreducible and $\deg(\gcd(f, f')) \leq \deg f'(x) \leq \deg f(x)$, $f(x)$ has no common factor with $f'(x)$. Therefore $\gcd(f, f') = 1$ and there exists $g_1, g_2 \in \mathbb{Q}[x] \subset \mathbb{C}[x]$ such that $fg_1 + f'g_2 = 1$, which is also true in $\mathbb{C}[x]$. Therefore f and f' are relatively prime in $\mathbb{C}[x]$ and by part (b) f has no multiple root in $\mathbb{C}$.

2. $\mathbb{Q}[\pi]$ be the set of all real numbers of the form:

$$r_0 + r_1\pi + \cdots + r_n\pi^n,$$

   with $n \geq 0$ and each $r_i \in \mathbb{Q}$.

   (a) Show that $\mathbb{Q}[\pi]$ is a subring of $\mathbb{R}$.

   Answer: We have $(r_0+r_1\pi+\cdots+r_n\pi^n)-(s_0+s_1\pi+\cdots+s_n\pi^n) = (r_0-s_0)+(r_1-s_1)\pi+\cdots+(r_n-s_n)\pi^n \in \mathbb{Q}[\pi]$ and $(r_0 + r_1\pi + \cdots + r_n\pi^n)(s_0 + s_1\pi + \cdots + s_n\pi^n) = (r_0s_0) + (r_1s_1)\pi^2 + \cdots + (r_ns_n)\pi^{2n} \in \mathbb{Q}[\pi]$. Therefore $\mathbb{Q}[\pi]$ is closed under both subtraction and multiplication, so it is a subring of $\mathbb{R}$ by Theorem 3.6.

   (b) Show that the function $\theta : \mathbb{Q}[x] \to \mathbb{Q}[\pi]$ defined by $\theta(f(x)) = f(\pi)$ is an isomorphism.

   Answer:

   i. Suppose that $\theta(f(x)) = \theta(g(x)) \implies f(\pi) = g(\pi)$, we have $r_0+r_1\pi+\cdots+r_n\pi^n = s_0+s_1\pi+\cdots+s_n\pi^n$. Since $\pi^k \notin \mathbb{Q}$ for any power of k, we must have $r_0 = s_0, r_1 = s_1, \ldots, r_n = s_n$. Therefore $f(x) = g(x)$ since equal coefficients implies equal polynomials, so θ is injective.

ii. We can take $f(x) = r_0 + r_1 x + \cdots + r_n x^n \in \mathbb{Q}[x]$ such that $\theta(f(x)) = r_0 + r_1\pi + \cdots + r_n\pi^n$ for every element in $\mathbb{Q}[\pi]$. Therefore θ is surjective.

iii. Let $f(x), g(x) \in \mathbb{R}[x]$, $p(x) = f(x) + g(x)$ and $q(x) = f(x)g(x)$, we have

$$\theta(f(x) + g(x)) = \theta(p(x)) = p(\pi) = f(\pi) + g(\pi) = \theta(f(x)) + \theta(g(x))$$

and

$$\theta(f(x)g(x)) = \theta(q(x)) = q(\pi) = f(\pi)g(\pi) = \theta(f(x))\theta(g(x)).$$

3. Let $\mathbb{Q}[\sqrt{2}]$ be the set of all real numbers of the form:

$$r_0 + r_1(\sqrt{2}) + \cdots + r_n(\sqrt{2})^n,$$

with $n \geq 0$ and each $r_i \in \mathbb{Q}$.

(a) Show that $\mathbb{Q}[\sqrt{2}]$ is a subring of $\mathbb{R}$.

Answer: We have $(r_0 + r_1(\sqrt{2}) + \cdots + r_n(\sqrt{2})^n) - (s_0 + s_1(\sqrt{2}) + \cdots + s_n(\sqrt{2})^n) = (r_0 - s_0) + (r_1 - s_1)(\sqrt{2}) + \cdots + (r_n - s_n)(\sqrt{2})^n \in \mathbb{Q}[\sqrt{2}]$ and $(r_0 + r_1(\sqrt{2}) + \cdots + r_n(\sqrt{2})^n)(s_0 + s_1(\sqrt{2}) + \cdots + s_n(\sqrt{2})^n) = (r_0 s_0) + (r_1 s_1)(\sqrt{2})^2 + \cdots + (r_n s_n)(\sqrt{2})^{2n} \in \mathbb{Q}[\sqrt{2}]$. Therefore $\mathbb{Q}[\sqrt{2}]$ is closed under both subtraction and multiplication, so it is a subring of $\mathbb{R}$ by Theorem 3.6.

(b) Show that the function $\theta : \mathbb{Q}[x] \to \mathbb{Q}[\sqrt{2}]$ defined by $\theta(f(x)) = f(\sqrt{2})$ is a surjective homomorphism, but not an isomorphism.

Answer:

i. Take $f(x) = 2$ and $g(x) = x^2$, we have $\theta(f(x)) = 2 = \theta(g(x))$ for $f(x) \neq g(x)$. Therefore θ is not injective.

ii. We can take $f(x) = r_0 + r_1 x + \cdots + r_n x^n \in \mathbb{Q}[x]$ such that $\theta(f(x)) = r_0 + r_1(\sqrt{2}) + \cdots + r_n(\sqrt{2})^n$ for every element in $\mathbb{Q}[\sqrt{2}]$. Therefore θ is surjective.

iii. Let $f(x), g(x) \in \mathbb{R}[x]$, $p(x) = f(x) + g(x)$ and $q(x) = f(x)g(x)$, we have

$$\theta(f(x) + g(x)) = \theta(p(x)) = p(\sqrt{2}) = f(\sqrt{2}) + g(\sqrt{2}) = \theta(f(x)) + \theta(g(x))$$

and

$$\theta(f(x)g(x)) = \theta(q(x)) = q(\sqrt{2}) = f(\sqrt{2})g(\sqrt{2}) = \theta(f(x))\theta(g(x)).$$

4. Use the rational root test to write each polynomial as a product of irreducibles in $\mathbb{Q}[x]$:

(b) $x^5 + 4x^4 + x^3 - x^2$

Answer: We have $r = \pm 1$ and $s = \pm 1$, so the only possible roots are 1 and -1. However, substituting 1 and -1 gives us 5 and 1 respectively, therefore neither is a root and $x^5 + 4x^4 + x^3 - x^2$ is irreducible in $\mathbb{Q}[x]$.

(d) $2x^4 - 5x^3 + 3x^2 + 4x - 6$

Answer: We have $r = \pm 1, \pm 2, \pm 3$ and $s = \pm 1, \pm 2$, so the possible roots are $\pm 1, \pm 2, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$. We can substitue them into $2x^4 - 5x^3 + 3x^2 + 4x - 6$ as follows:

a	$f(a)$	is root?
-1	0	yes
1	-2	no
-2	70	no
2	6	no
-3	306	no
3	60	no
$-\frac{1}{2}$	$-\frac{13}{2}$	no
$\frac{1}{2}$	$-\frac{15}{4}$	no
$-\frac{3}{2}$	$\frac{87}{4}$	no
$\frac{3}{2}$	0	yes

Therefore $x+1$ and $x-\dfrac{3}{2}$ are factors; then $\dfrac{2x^4-5x^3+3x^2+4x-6}{(x+1)(x-\frac{3}{2})} = 2(x^2-2x+2)$, so $2x^4-5x^3+3x^2+4x-6 = (x+1)(2x-3)(x^2-2x+2)$.

(f) $6x^4-31x^3+25x^2+33x+7$

Answer: We have $r=\pm1,\pm7$ and $s=\pm1,\pm2,\pm3$, so the possible roots are $\pm1,\pm\frac{1}{2},\pm\frac{1}{3},\pm7,\pm\frac{7}{2},\pm\frac{7}{3}$. We can substitute them into $6x^4-31x^3+25x^2+33x+7$ as follows:

a	$f(a)$	is root?
-1	36	no
1	40	no
$-\frac{1}{2}$	1	no
$\frac{1}{2}$	$\frac{105}{4}$	no
$-\frac{1}{3}$	0	yes
$\frac{1}{3}$	$\frac{532}{27}$	no
-7	26040	no
7	5236	no
$-\frac{7}{2}$	$\frac{9709}{4}$	no
$\frac{7}{2}$	0	yes
$-\frac{7}{3}$	$\frac{5740}{9}$	no
$\frac{7}{3}$	$\frac{112}{27}$	yes

Therefore $x+\dfrac{1}{3}$ and $x-\dfrac{7}{2}$ are factors; then $\dfrac{6x^4-31x^3+25x^2+33x+7}{(x+\frac{1}{3})(x-\frac{7}{2})} = 6(x^2-2x-1)$, so $6x^4-31x^3+25x^2+33x+7 = (2x-7)(3x+1)(x^2-2x-1)$.

5. Show that each polynomial is irreducible in $\mathbb{Q}[x]$, using the method of Example 3 from Section 4.5 (page 115):

(a) x^4+2x^3+x+1

Answer: By contradiction. Let $f(x)=x^4+2x^3+x+1$; if $f(x)$ is reducible, it can be factored as the product of two nonconstant polynomials in $\mathbb{Q}[x]$. If either of these factors has degree 1, then $f(x)$ has a root in $\mathbb{Q}$. But the Rational Root Test shows that $f(x)$ has no roots in $\mathbb{Q}$ (The only possibilities aree ±1 and neither is a root). This if $f(x)$ is reducible, the only possible factorization is as a product of two quadratics, by Theorem 4.2. In this case Theorem 4.23 shows that there is such a factorization in $\mathbb{Z}[x]$.

Furthermore, there is a factorization as a product of monic quadratics in $\mathbb{Z}[x]$, i.e.

$$(x^2 + ax + b)(x^2 + cx + d) = x^4 + 2x^3 + x + 1,$$

with $a, b, c, d \in \mathbb{Z}$. Multiplying out the left-hand side, we have

$$x^4 + (a + c)x^3 + (ac + b + d)x^2 + (ad + bc)x + bd = x^4 + 2x^3 + 0x^2 + x + 1.$$

Equal polynomials have equal coefficients; hence,

$$a + c = 2, ac + b + d = 0, ad + bc = 1, bd = 1.$$

Since $bd = 1$ in $\mathbb{Z}$ implies that $b = d = 1$ or $b = d = -1$, using the third equation we have two possibilities: $ad + bc = 1 \implies a + c = \pm 1$. But this contradicts with the first equation, so a factorization of $f(x)$ as a product of quadratics in $\mathbb{Z}[x]$, and, hence in $\mathbb{Q}[x]$, is impossible. Therefore, $f(x)$ is irreducible in $\mathbb{Q}[x]$.

(b) $x^4 - 2x^2 + 8x + 1$

Answer: By contradiction. Let $f(x) = x^4 - 2x^2 + 8x + 1$; if $f(x)$ is reducible, it can be factored as the product of two nonconstant polynomials in $\mathbb{Q}[x]$. If either of these factors has degree 1, then $f(x)$ has a root in $\mathbb{Q}$. But the Rational Root Test shows that $f(x)$ has no roots in $\mathbb{Q}$ (The only possibilities aree ± 1 and neither is a root). This if $f(x)$ is reducible, the only possible factorization is as a product of two quadratics, by Theorem 4.2. In this case Theorem 4.23 shows that there is such a factorization in $\mathbb{Z}[x]$. Furthermore, there is a factorization as a product of monic quadratics in $\mathbb{Z}[x]$, i.e.

$$(x^2 + ax + b)(x^2 + cx + d) = x^4 - 2x^2 + 8x + 1,$$

with $a, b, c, d \in \mathbb{Z}$. Multiplying out the left-hand side, we have

$$x^4 + (a + c)x^3 + (ac + b + d)x^2 + (ad + bc)x + bd = x^4 + 0x^3 - 2x^2 + 8x + 1.$$

Equal polynomials have equal coefficients; hence,

$$a + c = 0, ac + b + d = -2, ad + bc = 8, bd = 1.$$

Since $bd = 1$ in $\mathbb{Z}$ implies that $b = d = 1$ or $b = d = -1$, using the third equation we have two possibilities: $ad + bc = 1 \implies a + c = \pm 8$. But this contradicts with the first equation, so a factorization of $f(x)$ as a product of quadratics in $\mathbb{Z}[x]$, and, hence in $\mathbb{Q}[x]$, is impossible. Therefore, $f(x)$ is irreducible in $\mathbb{Q}[x]$.

6. Show that $9x^4 + 4x^3 - 3x + 7$ is irreducible in $\mathbb{Z}[x]$ by finding a prime $p \neq 3$ such that $f(x)$ is irreducible in $(\mathbb{Z}/p\mathbb{Z})[x]$.

Answer: Let $p = 2$, we will first show that $9x^4 + 4x^3 - 3x + 7 = 0$ has no solution in $\mathbb{Z}/2\mathbb{Z}[x]$:

x	$9x^4 + 4x^3 - 3x + 7$	is solution?
$[0]$	$9[0]^4 + 4[0]^3 - 3[0] + [7] = [0] + [0] - [0] + [7] = [1]$	no
$[1]$	$9[1]^4 + 4[1]^3 - 3[1] + [7] = [9] + [4] - [3] + [7] = [1]$	no

Now we will show that $9x^4 + 4x^3 - 3x + 7 = 0$ does not have any quadratic factors in $\mathbb{Z}/2\mathbb{Z}[x]$ either. The only possible quadratic factors in $\mathbb{Z}/2\mathbb{Z}[x]$ are x^2, x^2+x, x^2+1 and x^2+x+1, however none of them divides $9x^4+4x^3-3x+7$, so $9x^4+4x^3-3x+7$ has no quadratic factor and therefore it is irreducible in $\mathbb{Z}/2\mathbb{Z}$.

7. Let F be a field.

   (a) Let $\varphi : F[x] \to F[x]$ be an isomorphism such that $\varphi(a) = a$ for all $a \in F$. Prove that $f(x)$ is irreducible in $F[x]$ if and only if $\varphi(f(x))$ is.

   Answer: We will show that $\varphi(g(x))$ is nonconstant if and only if $g(x)$ is.

   $\Rightarrow$: If $\varphi(g(x))$ is nonconstant, $g(x)$ cannot be constant since φ is surjective.

   $\Leftarrow$: Let $c = \varphi(g(x))$; If $g(x)$ is nonconstant, $\varphi(g(x))$ cannot be constant as we would have $\varphi(g(x)) = c = \varphi(c)$ which would mean that φ would not be injective.

   Therefore $f(x)$ is irreducible in $F[x]$ if and only if $\varphi(f(x))$ is.

   (b) Take any $c \in F$. Show that the map $\varphi : F[x] \to F[x]$ given by $\varphi(f(x)) = f(x+c)$ is an isomorphism such that $\varphi(a) = a$.

   Answer: Take $\varphi^{-1}(f(x)) = f(x-c)$, we have $\varphi^{-1}(\varphi(f(x))) = \varphi^{-1}(f(x+c)) = f(x)$ and $\varphi(\varphi^{-1}(f(x))) = \varphi(f(x-c)) = f(x)$. Therefore φ is invertible and is a bijection. Now let $f(x), g(x) \in F[x]$, $p(x) = f(x) + g(x)$ and $q(x) = f(x)g(x)$, we have

   $$\varphi(f(x) + g(x)) = \varphi(p(x)) = \varphi(f(x)) + \varphi(g(x))$$

   and

   $$\varphi(f(x)g(x)) = \varphi(q(x)) = \varphi(f(x))\varphi(g(x)).$$

   Therefore φ is an isomorphism.

   (c) Use parts (a) and (b) to show that $f(x)$ is irreducible in $F[x]$ if and only if $f(x+c)$ is.

   Answer: By part (b), $\varphi(f(x)) = f(x+c)$ is an isomorphism that satisfies the condition of part (a), therefore $f(x)$ is irreducible in $F[x]$ if and only if $f(x+c)$ is.

8. Prove that for p prime, $f(x) = x^{p-1} + x^{p-2} + \cdots + x^2 + x + 1$ is irreducible in $\mathbb{Q}[x]$.

   Answer: By Eisenstein's criterion, if p prime and p divides $a_0, a_1, \ldots, a_{n-1}$ but not a_n and p^2 does not divide a_0, $f(x)$ is irreducible in $\mathbb{Q}[x]$.

9. Find a monic polynomial in $\mathbb{R}[x]$ of least possible degree with $1-i$ and $2i$ as roots.

   Answer: Suppose $f(x) \in \mathbb{R}[x]$ is our polynomial; by Lemma 4.29 $1+i$ and $-2i$ are also roots in addition to $1-i$ and $2i$. Therefore $f(x) = (x-1-i)(x-1+i)(x-2i)(x+2i) = x^4 - 2x^3 + 6x^2 - 8x + 8$.

10. Show that a polynomial of odd degree in $\mathbb{R}[x]$ with no multiple roots must have an odd number of roots in $\mathbb{R}[x]$.

    Answer: By Corollary 4.31 any polynomial of odd degree $f(x)$ in $\mathbb{R}$ has a root in $\mathbb{R}$; by Lemma 4.29, for every copmlex root $a+bi$, its complex conjugate $a-bi$ is also a root. Suppose r_k are the real roots of $f(x)$ and $s_k, \overline{s_k}$ are the complex roots, we have $f(x) = (x-r_1)\ldots(x-r_m)(x-s_1)(x-\overline{s_1})\ldots(x-s_n)(x-\overline{s_n})$. Therefore $\deg(f(x)) = m + 2n$; since $f(x)$ is a polynomial of odd degree m must be odd, therefore $f(x)$ must have an odd number of real roots.