Math 110A Homework 4

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1. For each pair of polynomials $f(x)$ and $g(x)$ below, use the Euclidean algorithm to compute the $\gcd(f(x), g(x))$, and to find polynomials $u(x)$ and $v(x)$ with $(f(x), g(x)) = f(x)u(x) + g(x)v(x)$:

   (b) $f(x) = x^5 + x^4 + 2x^3 - x^2 - x - 2$ and $g(x) = x^4 + 2x^3 + 5x^2 + 4x + 4$ in $\mathbb{Q}[x]$.
   Answer: As follows:
   $x^5 + x^4 + 2x^3 - x^2 - x - 2 = (x^4 + 2x^3 + 5x^2 + 4x + 4)(x - 1) + (-x^3 - x + 2)$
   $x^4 + 2x^3 + 5x^2 + 4x + 4 = (-x^3 - x + 2)(-x - 2) + (4x^2 + 4x + 8)$
   $-x^3 - x + 2 = (4x^2 + 4x + 8)(-\frac{1}{4}x + \frac{1}{4}) + 0$
   Therefore $\gcd(f(x), g(x)) = x^2 + x + 2$.

   (c) $f(x) = 4x^4 + 2x^3 + 6x^2 + 4x + 5$ and $g(x) = 3x^3 + 5x^2 + 6x$ in $(\mathbb{Z}/7\mathbb{Z})[x]$.
   Answer: As follows:
   $4x^4 + 2x^3 + 6x^2 + 4x + 5 = (3x^3 + 5x^2 + 6x)$

   (d) $f(x) = x^3 - ix^2 + 4x - 4i$ and $g(x) = x^2 + 1$ in $\mathbb{C}[x]$.
   Answer: As follows:

2. Express $x^4 - 4$ as a product of irreducibles in $\mathbb{Q}[x]$, $\mathbb{R}[x]$ and $\mathbb{C}[x]$.

3. Use unique factorization to find the gcd in $\mathbb{C}[x]$ of $(x - 3)^3(x - 4^4)(x - i)^2$ and $(x - 1)(x - 3)(x - 4)^3$.

4. (a) Show that $x^2 + 2$ is irreducible in $(\mathbb{Z}/5\mathbb{Z})[x]$.

   (b) Factor $x^4 - 4$ as a product of irreducibles in $(\mathbb{Z}/5\mathbb{Z})[x]$.

5. Use the factor theorem to show that $x^7 - x$ factors in $(\mathbb{Z}/7\mathbb{Z})[x]$ as $x(x - 1)(x - 2)(x - 3)(x - 4)(x - 5)(x - 6)$, *without* doing any polynomial multiplication.