1. Find al solutions to the indicated equations in $\mathbb{Z}/n\mathbb{Z}$, or prove that there are none. Justify your answers:

   (a) $[8]x = [21]$ in $\mathbb{Z}/23\mathbb{Z}$.

   Answer: Using problem 9 from homework 2, we have $a = 8$, $b = 21$ and $n = 23$. Then $d = (a, n) = 1$, $b_1 = b/d = 21$ and $n_1 = n/d = 23$. We can take $u = 3$ and $v = -1$ to have $au + nv = 24 - 23 = 1 = d$. Then by problem 9(b), there exists $d = 1$ solution and the only solution is $[ub_1] = [63] \equiv [17] \pmod{23}$. Therefore $x = [17]$.

   (b) $x^3 + x = [1]$ in $\mathbb{Z}/7\mathbb{Z}$.

   Answer:

x	$x^3 + x$	is solution?
$[0]$	$[0]\cdot[0]\cdot[0]+[0]=[0]+[0]=[0]$	no
$[1]$	$[1]\cdot[1]\cdot[1]+[1]=[1]+[1]=[2]$	no
$[2]$	$[2]\cdot[2]\cdot[2]+[2]=[1]+[2]=[3]$	no
$[3]$	$[3]\cdot[3]\cdot[3]+[3]=[6]+[3]=[2]$	no
$[4]$	$[4]\cdot[4]\cdot[4]+[4]=[1]+[4]=[5]$	no
$[5]$	$[5]\cdot[5]\cdot[5]+[5]=[6]+[5]=[4]$	no
$[6]$	$[6]\cdot[6]\cdot[6]+[6]=[6]+[6]=[5]$	no

   Therefore the equation has no solution in $\mathbb{Z}/7\mathbb{Z}$.

2. The *Fibonacci sequence* F_n is defined *recursively* as follows:

$$F_0 = 1, \quad F_1 = 1$$
$$\text{For any } n \geq 1, \quad F_{n+1} = F_n + F_{n-1},$$

so the first few terms are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$.

Prove that for any integer $n \geq 1$, $\gcd(F_n, F_{n-1}) = 1$.

Answer: By induction on n.

Base case: $n = 1$, we have $F_0 = 1$ and $F_1 = 1$, so $\gcd(F_1, F_0) = 1$.

Inductive step: Suppose that $\gcd(F_{n-1}, F_{n-2}) = 1$, we will show that $\gcd(F_n, F_{n-1}) = 1$ by contradiciton. Assume that $\gcd(F_n, F_{n-1}) = d > 1$, then by Theorem 1.2 there must exist some $u, v \in \mathbb{Z}$ such that $d = F_n u + F_{n-1} v$. Since $F_n = F_{n-2} + F_{n-1}$, we have $d = (F_{n-2} + F_{n-1})u + F_{n-1}v = F_{n-2}u + F_{n-1}(u + v)$. But this implies that $d > 1$ is a common factor of F_{n-2} and F_{n-1}, which contradicts with $\gcd(F_{n-2}, F_{n-1}) = 1$, so our assumption must be false and $\gcd(F_n, F_{n-1}) = 1$.

Therefore $\gcd(F_n, F_{n-1}) = 1$ by induction.

3. Let $R = \{0_R, a, b, c, d\}$ be a ring with 5 elements and additive identity 0_R. We are given that $a + b = c + d = 0_R$ and $ad = b$. Which of the five elements of the ring is equal to ac? Justify your answer.

   Answer: Since $a + b = c + d = 0_R$, we have $b = -a$ and $d = -c$. By substitution and Theorem 3.5(2), we have $ad = b \implies a(-c) = (-a) \implies -ac = -a$. Now we can add $ac + a$ to both sides, which gives us $-ac + ac + a = -a + ac + a \implies ac = a$.

4. Prove that each of the following sets with the specified operations is *NOT* a ring. In each part, identify a specific property of rings which fails, and *prove* that it fails.

   (a) The set $R = \{[0], [2], [4], [6], [8], [10]\} \subseteq \mathbb{Z}/11\mathbb{Z}$, with addition and multiplication defined as they are in $\mathbb{Z}/11\mathbb{Z}$.

   Answer: R is not a ring because it is not closed under addition. Take $a = [4] \in R$ and $b = [8] \in R$, we have $a + b = [4] + [8] = [1]$ which is not in R.

   (b) The set $S = \{f : \mathbb{R} \to \mathbb{R} \mid f(2) = 2f(1)\}$ of functions from $\mathbb{R}$ to $\mathbb{R}$ satisfying $f(2) = 2f(1)$, with addition and multiplication defined pointwise.

   Answer: S is not a ring because it is not closed under multiplication. Take $f(x) = g(x) = x$, note that both functions are in S as $f(2) = g(2) = 2 = 2f(1) = 2g(1)$. However, $(fg)(x) = f(x)g(x) = x^2$ is not in S as $(fg)(2) = 4 \neq 2 = 2(fg)(1)$.

   (c) The set $T = \{x \in \mathbb{R} \mid x > 0\}$ of *positive* real numbers, where addition in T is defined by $x \oplus y = xy$ and multiplicationin T is defined by $x \odot y = x + y$.

   Answer: T is not a ring because it doesnt not have an additive identity. Suppose we do have a $0_T \in T$, then we can take any $a \in T$ and have $a + 0_T = a$. However, since $0_T \in T$, we have $0_T > 0$, so $a + 0_T > a$. Therefore T does not have an additive identity by contradiciton.

5. Let $f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be any homomorphism. Prove that f must equal one of the four homomorphisms:

$$f_0(n) = (0,0), \qquad f_1(n) = (n,0), \qquad f_2(n) = (0,n), \qquad f_3(n) = (n,n).$$

Answer: Since $\mathbb{Z}$ is a ring, by Theorem 3.1 $\mathbb{Z} \times \mathbb{Z}$ is also a ring. Then by definition of homomorphism we have $f(ab) = f(a)f(b)$ for all $a, b \in \mathbb{Z}$. Take $a = b = 1$ and let $f(1) = (p,q)$, we have $f(ab) = f(1^2) = f(1)f(1) = (p,q)(p,q) = (p^2, q^2)$. Then since $1 = 1^2$, we have $f(1) = f(1^2) \implies (p,q) = (p^2, q^2)$, therefore $p = p^2$ and $q = q^2$. Since the only integers satisfying these properties are 0 and 1, any f must satisfy one of the following:

(a) $p = 0, q = 0$: $f_0(1) = (0,0)$

(b) $p = 1, q = 0$: $f_1(1) = (1,0)$

(c) $p = 0, q = 1$: $f_2(1) = (0,1)$

(d) $p = 1, q = 1$: $f_3(1) = (1,1)$

Then since f is a linear function, we have $f(n) = nf(1)$ for any $n \in \mathbb{Z}$, therefore we can multiply the scalar n to the above to achieve the desired result.

I assert, on my honor, that I have not received assistance of any kind from any other person, or given assistance to any other person, while working on the midterm.

Signature:

Date: 2/5/2021