Homework 3. Due Friday Sept. 20 by 10am in my mailbox

Always provide complete, well written solutions.

The solution to one of the following problems (of your choice) has to be typed in Latex, while the rest of the solutions can be handwritten.

1. Ref. [1], Chapter 4, Problem 1 (page 98).

   Let L be the differentiation operator that takes the polynomial space P_3 to the polynomial space P_2. Let $f_k = H_k(x)$ for $k = 0, 1, 2, 3$ be the Hermite polynomial basis of P_3 and $g_k = H_k(x)$ for $k = 0, 1, 2$ be the Hermite basis of P_2. What is the matrix, A, that represents this L in these bases?

2. Ref. [1], Chapter 4, Problem 2 (page 98).

   Suppose L is a linear transformation from V to V and that $f_1, \ldots, f_n$, and $g_1, \ldots, g_n$ are two bases of V. Any $u \in V$ may be written in a unique way as $u = \sum_{k=1}^n v_k f_k$, or as $u = \sum_{k=1}^n w_k g_k$. There is an $n \times n$ matrix, R that relates the f_k expansion coefficients v_k to the g_k coefficients w_k by $v_j = \sum_{k=1}^n r_{jk} w_k$. If v and w are the column vectors with components v_k and w_k respectively, then $v = Rw$. Let A represent L in the f_k basis and B represent L in the g_k basis.

   (a) Show that $B = R^{-1}AR$.
   You can use the notations
   $$v = [u]_{\mathcal{F}}, \quad w = [u]_{\mathcal{G}}, \quad v = Rw, A =_{\mathcal{F}} [L]_{\mathcal{F}}, \quad B =_{\mathcal{G}} [L]_{\mathcal{G}}$$
   to make your argument more compact.

   (b) For $V = P_3$, and $f_k = x^k$, and $g_k = H_k$, find R.

   (c) Let L be the linear transformation $Lp = q$ with $q(x) = \partial_x(xp(x))$. Find the matrix, A, that represents L in the monomial basis f_k.

   (d) Find the matrix, B, that represents L in the Hermite polynomial basis H_k.

   (e) Multiply the matrices to check explicitly that $B = R^{-1}AR$ in this case.

3. Ref. [1], Chapter 4, Problem 3 (page 99).

   If A is an $n \times m$ matrix and B is an $m \times l$ matrix, then AB is an $n \times l$ matrix. Show that $(AB)^* = B^*A^*$. Note that the incorrect suggestion A^*B^* in general is not compatible for matrix multiplication.

4. Ref. [1], Chapter 4, Problem 4 (page 99).

   Let $V = \mathbb{R}^n$ and M be an $n \times n$ real matrix. This exercise shows that $\|u\| = (u^*Mu)^{1/2}$ is a vector norm whenever M is *positive definite* (defined below).

(a) Show that $u^* M u = u^* M^* u = u^* \left(\frac{1}{2}(M + M^*)\right) u$ for all $u \in V$. This means that as long as we consider functions of the form $f(u) = u^* M u$, we may assume M is symmetric. For the rest of this question, assume M is symmetric. Hint: $u^* M u$ is a 1×1 matrix and therefore equal to its transpose.

(b) Show that the function $\|u\| = (u^* M u)^{1/2}$ is homogeneous: $\|au\| = |a| \|u\|$ for all $a \in \mathbb{R}$ and $u \in V$.

(c) We say M is positive definite if $u^* M u > 0$ whenever $u \neq 0$. Show that if M is positive definite, then $\|u\| \geq 0$ for all u and $\|u\| = 0$ only for $u = 0$.

(d) Show that if M is symmetric and positive definite (SPD), then $|u^* M v| \leq \|u\| \|v\|$. This is the *Cauchy–Schwarz inequality*. Hint (a famous old trick): consider $\phi(t) = (u + tv)^* M (u + tv)$ is a quadratic function of t that is non-negative for all t if M is positive definite.

(e) Use the Cauchy–Schwarz inequality to verify the triangle inequality in its squared form $\|u + v\|^2 \leq \|u\|^2 + 2 \|u\| \|v\| + \|v\|^2$.

(f) Show that if $M = I$ then $\|u\|$ is the l^2 norm of u.

References

[1] Bindel and Goodman, Principles of scientific computing