Computer Vision Math 1-3

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Problem 1.

In the augmented Euclidean plane, there is a line $x - 3y + 4 = 0$, what is the homogeneous coordinate of the infinity point of this line?

Solution.

In the homogeneous coordinates system, a point in the Euclidean plane is represented by a 3-dimensional vector of the form (x', y', z'), where x' and y' are the Cartesian coordinates of the point and z' is a scaling factor. And we can get

$$\begin{cases} x = \dfrac{x'}{z'} \\ y = \dfrac{y'}{z'} \end{cases} \tag{1}$$

Then we need to convert original line into the homogeneous form, substitute 1 into it, we can get equation

$$x' - 3y' + 4z' = 0 \tag{2}$$

Now we can represent any point on this line in homogeneous coordinates as (x', y', z'). However, we are interested in the infinity point of the line, which corresponds to a point at infinity in the Euclidean plane.

To calculate the homogeneous coordinate of the infinity point of this line, we can calculate the intersection point of this line with the homogeneous coordinate $(0, 0, 1)$

$$x = (1, -3, 4) \times (0, 0, 1) = (-3, -1, 0) \tag{3}$$

For example, we can take $(3, 1, 0)$ or $(-1, -\frac{1}{3}, 0)$. These vectors represent the same point at infinity, since they differ only by a scaling factor.

Therefore, the homogeneous coordinates of the infinity point of the line $x - 3y + 4 = 0$ are $k(-3, -1, 0)$ where $k \neq 0$

Note of Problem 1. ... ...

Problem 2. Compute the Jacobian matrix of $\mathbf{p}_d$ w.r.t $\mathbf{p}_n$, i.e., $\frac{d\mathbf{p}_d}{d\mathbf{p}_n^T}$

Solution.

$$\frac{d\mathbf{p}_d}{d\mathbf{p}_n^T} = \begin{bmatrix} \frac{\partial x_d}{\partial (x,y)} \\ \frac{\partial y_d}{\partial (x,y)} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_d}{\partial x} & \frac{\partial x_d}{\partial y} \\ \frac{\partial y_d}{\partial x} & \frac{\partial y_d}{\partial y} \end{bmatrix} \tag{4}$$

To calculate $\frac{d\mathbf{p}_d}{d\mathbf{p}_n^T}$, we need to compute the four elements of the matrix separately.

$$\frac{\partial x_d}{\partial x} = x(k_1\frac{\partial r^2}{\partial x} + k_2 \cdot 2r^2 \cdot \frac{\partial r^2}{\partial x} + k_3 \cdot 3r^4 \cdot \frac{\partial r^2}{\partial x}) + (1 + k_1r^2 + k_2r^4 + k_3r^6) + 2\rho_1 y + \rho_2(4x + \frac{\partial r^2}{\partial x}) \tag{5}$$

$$\frac{\partial x_d}{\partial y} = x(k_1\frac{\partial r^2}{\partial y} + k_2 \cdot 2r^2 \cdot \frac{\partial r^2}{\partial y} + k_3 \cdot 3r^4 \cdot \frac{\partial r^2}{\partial y}) + 2\rho_1 x + \rho_2\frac{\partial r^2}{\partial y} \tag{6}$$

As $r^2 = x^2 + y^2$, we can get,

$$\begin{cases} \frac{\partial r^2}{\partial x} = 2x \\ \frac{\partial r^2}{\partial y} = 2y \end{cases} \tag{7}$$

Thus,

$$\frac{\partial x_d}{\partial x} = 2x^2(k_1 + 2k_2r^2 + 3k_3r^4) + (1 + k_1r^2 + k_2r^4 + k_3r^6) + 2\rho_1 y + 6\rho_2 x \tag{8}$$

$$\frac{\partial x_d}{\partial y} = 2xy(k_1 + 2k_2r^2 + 3k_3r^4) + 2\rho_1 x + 2\rho_2 y \tag{9}$$

In a similar way,

$$\frac{\partial y_d}{\partial y} = 2y^2(k_1 + 2k_2r^2 + 3k_3r^4) + (1 + k_1r^2 + k_2r^4 + k_3r^6) + 2\rho_2 x + 6\rho_1 y \tag{10}$$

$$\frac{\partial y_d}{\partial x} = 2xy(k_1 + 2k_2r^2 + 3k_3r^4) + 2\rho_1 y + 2\rho_2 x \tag{11}$$

Therefore,

$$\frac{d\mathbf{p}_d}{d\mathbf{p}_n^T} = \begin{bmatrix} 2x^2(k_1 + 2k_2r^2 + 3k_3r^4) + (1 + k_1r^2 + k_2r^4 + k_3r^6) + 2\rho_1 y + 6\rho_2 x & 2xy(k_1 + 2k_2r^2 + 3k_3r^4) + 2\rho_1 x + 2\rho_2 y \\ 2xy(k_1 + 2k_2r^2 + 3k_3r^4) + 2\rho_1 y + 2\rho_2 x & 2y^2(k_1 + 2k_2r^2 + 3k_3r^4) + (1 + k_1r^2 + k_2r^4 + k_3r^6) + 2\rho_2 x + 6\rho_1 y \end{bmatrix} \tag{12}$$

Note of Problem 2. ... ...

Problem 3. Derive the concrete formula of the Jacobian matrix of the rotation matrix (represented in a vector). Please give the concrete form of Jacobian matrix of $\mathbf{r}$ w.r.t $\mathbf{d}$, i.e., $\frac{d\mathbf{r}}{d\mathbf{d}^T} \in \mathbb{R}^{9\times 3}$

Solution.

First, we need to compute the concrete form of R.

$$\mathbf{nn}^T = \begin{bmatrix} n_1^2 & n_1n_2 & n_1n_3 \\ n_2n_1 & n_2^2 & n_2n_3 \\ n_3n_1 & n_3n_2 & n_3^2 \end{bmatrix} \tag{13}$$

$$\mathbf{R} = \cos\theta\mathbf{I} + (1-\cos\theta)\mathbf{nn}^T + \sin\theta\hat{\mathbf{n}} = \begin{bmatrix} \beta+\gamma n_1^2 & \gamma n_1n_2 - \alpha n_3 & \gamma n_1n_3 + \alpha n_2 \\ \gamma n_2n_1 + \alpha n_3 & \beta+\gamma n_2^2 & \gamma n_2n_3 - \alpha n_1 \\ \gamma n_3n_1 - \alpha n_2 & \gamma n_3n_2 + \alpha n_1 & \beta+\gamma n_3^2 \end{bmatrix} \tag{14}$$

Denote $\mathbf{r}$ by the vectorized form of $\mathbf{R}$.

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \\ r_9 \end{bmatrix} = \begin{bmatrix} \beta+\gamma n_1^2 \\ \gamma n_1n_2 - \alpha n_3 \\ \gamma n_1n_3 + \alpha n_2 \\ \gamma n_2n_1 + \alpha n_3 \\ \beta+\gamma n_2^2 \\ \gamma n_2n_3 - \alpha n_1 \\ \gamma n_3n_1 - \alpha n_2 \\ \gamma n_3n_2 + \alpha n_1 \\ \beta+\gamma n_3^2 \end{bmatrix} \tag{15}$$

Denote $\mathbf{d} = (d_1, d_2, d_3)^T$,$d_1 = \theta n_1$,$d_2 = \theta n_2$,$d_3 = \theta n_3$. Since $\mathbf{n} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix}^T$ is the unit vector.

$$n_1^2 + n_2^2 + n_3^2 = 1 \tag{16}$$

Thus,

$$\theta = \sqrt{d_1^2 + d_2^2 + d_3^2} \tag{17}$$

$$n_i = \frac{d_i}{\sqrt{d_1^2 + d_2^2 + d_3^2}} \quad i = 1, 2, 3 \tag{18}$$

Then, we can get.

$$\frac{d\theta}{dd_i} = \frac{d\sqrt{d_1^2 + d_2^2 + d_3^2}}{dd_i} = \frac{d_i}{\sqrt{d_1^2 + d_2^2 + d_3^2}} = n_i \quad i \in \{1, 2, 3\} \tag{19}$$

$$\begin{cases} \frac{d\alpha}{dd_i} = \beta n_i \\ \frac{d\beta}{dd_i} = -\alpha n_i \\ \frac{d\gamma}{dd_i} = \alpha n_i \end{cases} \tag{20}$$

For $i,j,k \in \{1,2,3\}, i \neq j, i \neq k, j \neq k$

$$\begin{cases} \frac{dn_i}{dd_i} = \frac{d\frac{d_i}{\sqrt{d_i^2+d_j^2+d_k^2}}}{dd_i} = \frac{1-n_i^2}{\theta} \\ \frac{dn_i}{dd_j} = \frac{d\frac{d_i}{\sqrt{d_i^2+d_j^2+d_k^2}}}{dd_j} = \frac{-n_i n_j}{\theta} \end{cases} \Longrightarrow \begin{cases} \frac{dn_i^2}{dd_i} = 2n_i \cdot \frac{dn_i}{dd_i} = \frac{2n_i\left(1-n_i^2\right)}{\theta} \\ \frac{dn_i^2}{dd_j} = 2n_i \cdot \frac{dn_i}{dd_j} = \frac{-2n_i^2 n_j}{\theta} \\ \frac{d(n_i n_j)}{dd_i} = n_i \cdot \frac{dn_j}{d_i} + n_j \cdot \frac{dn_i}{dd_i} = \frac{n_j\left(1-2n_i^2\right)}{\theta} \\ \frac{d(n_i n_j)}{dd_k} = n_i \cdot \frac{dn_j}{dd_k} + n_j \cdot \frac{dn_i}{dd_k} = \frac{-2n_i n_j n_k}{\theta} \end{cases} \tag{21}$$

$$\frac{d\boldsymbol{r}}{d\boldsymbol{d}^T} \triangleq \begin{bmatrix} \frac{dr_1}{dd_1} & \frac{dr_1}{dd_2} & \frac{dr_1}{dd_3} \\ \frac{dr_2}{dd_1} & \frac{dr_2}{dd_2} & \frac{dr_2}{dd_3} \\ \frac{dr_3}{dd_1} & \frac{dr_3}{dd_2} & \frac{dr_3}{dd_3} \\ \frac{dr_4}{dd_1} & \frac{dr_4}{dd_2} & \frac{dr_4}{dd_3} \\ \frac{dr_5}{dd_1} & \frac{dr_5}{dd_2} & \frac{dr_5}{dd_3} \\ \frac{dr_6}{dd_1} & \frac{dr_6}{dd_2} & \frac{dr_6}{dd_3} \\ \frac{dr_7}{dd_1} & \frac{dr_7}{dd_2} & \frac{dr_7}{dd_3} \\ \frac{dr_8}{dd_1} & \frac{dr_8}{dd_2} & \frac{dr_8}{dd_3} \\ \frac{dr_9}{dd_1} & \frac{dr_9}{dd_2} & \frac{dr_9}{dd_3} \end{bmatrix} = \begin{bmatrix} \frac{d\left(\beta+\gamma n_1^2\right)}{dd_1} & \frac{d\left(\beta+\gamma n_1^2\right)}{dd_2} & \frac{d\left(\beta+\gamma n_1^2\right)}{dd_3} \\ \frac{d(\gamma n_1 n_2-\alpha n_3)}{dd_1} & \frac{d(\gamma n_1 n_2-\alpha n_3)}{dd_2} & \frac{d(\gamma n_1 n_2-\alpha n_3)}{dd_3} \\ \frac{d(\gamma n_1 n_3+\alpha n_2)}{dd_1} & \frac{d(\gamma n_1 n_3+\alpha n_2)}{dd_2} & \frac{d(\gamma n_1 n_3+\alpha n_2)}{dd_3} \\ \frac{d(\gamma n_2 n_1+\alpha n_3)}{dd_1} & \frac{d(\gamma n_2 n_1+\alpha n_3)}{dd_2} & \frac{d(\gamma n_2 n_1+\alpha n_3)}{dd_3} \\ \frac{d\left(\beta+\gamma n_2^2\right)}{dd_1} & \frac{d\left(\beta+\gamma n_2^2\right)}{dd_2} & \frac{d\left(\beta+\gamma n_2^2\right)}{dd_3} \\ \frac{d(\gamma n_2 n_3-\alpha n_1)}{dd_1} & \frac{d(\gamma n_2 n_3-\alpha n_1)}{dd_2} & \frac{d(\gamma n_2 n_3-\alpha n_1)}{dd_3} \\ \frac{d(\gamma n_3 n_1-\alpha n_2)}{dd_1} & \frac{d(\gamma n_3 n_1-\alpha n_2)}{dd_2} & \frac{d(\gamma n_3 n_1-\alpha n_2)}{dd_3} \\ \frac{d(\gamma n_3 n_2+\alpha n_1)}{dd_1} & \frac{d(\gamma n_3 n_2+\alpha n_1)}{dd_2} & \frac{d(\gamma n_3 n_2+\alpha n_1)}{dd_3} \\ \frac{d\left(\beta+\gamma n_3^2\right)}{dd_1} & \frac{d\left(\beta+\gamma n_3^2\right)}{dd_2} & \frac{d\left(\beta+\gamma n_3^2\right)}{dd_3} \end{bmatrix} \tag{22}$$

$$\frac{d\boldsymbol{r}}{d\boldsymbol{d}^T} \triangleq \begin{bmatrix} \frac{2\gamma n_1\left(1-n_1^2\right)}{\theta} + \alpha n_1\left(n_1^2-1\right) & -\frac{2\gamma n_1^2 n_2}{\theta} + \alpha n_2\left(n_1^2-1\right) & -\frac{2\gamma n_1^2 n_3}{\theta} + \alpha n_3\left(n_1^2-1\right) \\ n_1\left(\alpha n_1 n_2 - \beta n_3\right) + \frac{\gamma n_2\left(1-2n_1^2\right)+\alpha n_1 n_3}{\theta} & n_2\left(\alpha n_1 n_2 - \beta n_3\right) + \frac{\gamma n_1\left(1-2n_2^2\right)+\alpha n_2 n_3}{\theta} & n_3\left(\alpha n_1 n_2 - \beta n_3\right) + \frac{\alpha\left(n_3^2-1\right)-2\gamma n_1 n_2 n_3}{\theta} \\ n_1\left(\alpha n_1 n_3 + \beta n_2\right) + \frac{\gamma n_3\left(1-2n_1^2\right)-\alpha n_1 n_2}{\theta} & n_2\left(\alpha n_1 n_3 + \beta n_2\right) + \frac{\alpha\left(1-n_2^2\right)-2\gamma n_1 n_2 n_3}{\theta} & n_3\left(\alpha n_1 n_3 + \beta n_2\right) + \frac{\gamma n_1\left(1-2n_3^2\right)-\alpha n_2 n_3}{\theta} \\ n_1\left(\alpha n_1 n_2 + \beta n_3\right) + \frac{\gamma n_2\left(1-2n_1^2\right)-\alpha n_1 n_3}{\theta} & n_2\left(\alpha n_1 n_2 + \beta n_3\right) + \frac{\gamma n_1\left(1-2n_2^2\right)-\alpha n_2 n_3}{\theta} & n_3\left(\alpha n_1 n_2 + \beta n_3\right) + \frac{\alpha\left(1-n_3^2\right)-2\gamma n_1 n_2 n_3}{\theta} \\ \frac{-2\gamma n_1 n_2^2}{\theta} + \alpha n_1\left(n_2^2-1\right) & \frac{2\gamma n_2\left(1-n_2^2\right)}{\theta} + \alpha n_2\left(n_2^2-1\right) & -\frac{2\gamma n_2^2 n_3}{\theta} + \alpha n_3\left(n_2^2-1\right) \\ n_1\left(\alpha n_2 n_3 - \beta n_1\right) - \frac{\alpha\left(1-n_1^2\right)+2\gamma n_1 n_2 n_3}{\theta} & n_2\left(\alpha n_2 n_3 - \beta n_1\right) + \frac{\gamma n_3\left(1-2n_2^2\right)+\alpha n_1 n_2}{\theta} & n_3\left(\alpha n_2 n_3 - \beta n_1\right) + \frac{\alpha n_1 n_3+\gamma n_2\left(1-2n_3^2\right)}{\theta} \\ n_1\left(\alpha n_1 n_3 - \beta n_2\right) + \frac{\alpha n_1 n_2+\gamma n_3\left(1-2n_1^2\right)}{\theta} & n_2\left(\alpha n_1 n_3 - \beta n_2\right) - \frac{\alpha\left(1-n_2^2\right)+2\gamma n_1 n_2 n_3}{\theta} & n_3\left(\alpha n_1 n_3 - \beta n_2\right) + \frac{\alpha n_2 n_3+\gamma n_1\left(1-2n_3^2\right)}{\theta} \\ n_1\left(\alpha n_2 n_3 + \beta n_1\right) + \frac{\alpha\left(1-n_1^2\right)-2\gamma n_1 n_2 n_3}{\theta} & n_2\left(\alpha n_2 n_3 + \beta n_1\right) + \frac{\gamma n_3\left(1-2n_2^2\right)-\alpha n_1 n_2}{\theta} & n_3\left(\alpha n_2 n_3 + \beta n_1\right) + \frac{\gamma n_2\left(1-2n_3^2\right)-\alpha n_1 n_3}{\theta} \\ \frac{-2\gamma n_1 n_3^2}{\theta} + \alpha n_1\left(n_3^2-1\right) & -\frac{2\gamma n_2 n_3^2+\alpha n_2\left(n_3^2-1\right)}{\theta} & \frac{2\gamma n_3\left(1-n_3^2\right)}{\theta} + \alpha n_3\left(n_3^2-1\right) \end{bmatrix} \tag{23}$$

Note of Problem 3.

For the sake of space, the intermediate calculation process of the derivative is not listed here, and the detailed process is shown in Appendix A.

Appendix A.

$$\frac{d\left(\beta+\gamma n_1^2\right)}{dd_1} = \frac{d\beta}{dd_1} + n_1^2\frac{d\gamma}{dd_1} + \gamma\frac{dn_1^2}{dd_1} = -\alpha n_1 + \alpha n_1^3 + \frac{2\gamma n_1(1-n_1^2)}{\theta} = \frac{2\gamma n_1(1-n_1^2)}{\theta} + \alpha n_1(n_1^2-1)$$

$$\frac{d(\gamma n_1 n_2-\alpha n_3)}{dd_2} = n_1 n_2\frac{d\gamma}{dd_2} + \gamma n_2\frac{dn_1}{dd_2} + \gamma n_1\frac{dn_2}{dd_2} - n_3\frac{d\alpha}{dd_2} - \alpha\frac{dn_3}{dd_2} = \alpha n_1 n_2^2 - \frac{\gamma n_1 n_2^2}{\theta} + \frac{\gamma n_1(1-n_2^2)}{\theta} - \beta n_2 n_3 + \frac{\alpha n_2 n_3}{\theta} = n_2(\alpha n_1 n_2 - \beta n_3) + \frac{\gamma n_1(1-2n_2^2)+\alpha n_2 n_3}{\theta}$$

The other calculations proceed is similar.