Assignment 1

1

In order to prove that Euclidean transformations can form a group, we need to prove four properties that groups satisfy: the closure, the associativity, the existence of an identity element, and the existence of an inverse element for each group element. I'll prove each of these four features later.

$$M_i = \begin{bmatrix} R_i & t_i \\ 0^T & 1 \end{bmatrix} \in \mathbb{R}^{4\times4}, R_i \in \mathbb{R}^{3\times3}$$

① The Closure

Choose two matrices M_j and M_k from $\{M_i\}$

$$Mj \times M_k = \begin{bmatrix} R_j & t_j \\ 0^T & 1 \end{bmatrix}\begin{bmatrix} R_k & t_k \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} R_jR_k & R_jt_k + t_j \\ 0^T & 1 \end{bmatrix}$$

Multiplying two orthogonal matrices together results in another orthogonal matrix, so

$$(RjR_k)^T(RjR_k) = R_k^TR_j^TR_jR_k = I$$

$$R_jR_k \in \mathbb{R}^{3\times3}\ is\ an\ orthonormal\ matrix$$

$$\because \det(AB) = \det(A)\det(B)$$

$$\therefore \det(R_jR_k) = \det(R_j)\det(R_k) = 1$$

$$R_jt_k + t_j \in \mathbb{R}^{3\times1}$$

$$\therefore\ Mj \times M_k \in \{M_i\}$$

Satisfied Closure

② The Associativity

Choose three matrices M_j, M_k and M_h from $\{M_i\}$

$$(Mj \times M_k) \times M_h = \left(\begin{bmatrix} R_j & t_j \\ 0^T & 1 \end{bmatrix}\begin{bmatrix} R_k & t_k \\ 0^T & 1 \end{bmatrix}\right)\begin{bmatrix} R_h & t_h \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} R_jR_kR_h & R_kR_ht_h + R_jt_k + t_j \\ 0^T & 1 \end{bmatrix}$$

$$Mj \times (M_k \times M_h) = \begin{bmatrix} R_j & t_j \\ 0^T & 1 \end{bmatrix}\left(\begin{bmatrix} R_k & t_k \\ 0^T & 1 \end{bmatrix}\begin{bmatrix} R_h & t_h \\ 0^T & 1 \end{bmatrix}\right) = \begin{bmatrix} R_jR_kR_h & R_kR_ht_h + R_jt_k + t_j \\ 0^T & 1 \end{bmatrix}$$

$$\therefore\ (Mj \times M_k) \times M_h = Mj \times (M_k \times M_h)$$

Satisfied Associativity

③ The Existence Of An Identity Element

Let $E = \begin{bmatrix} I_3 & 0 \\ 0^T & 1 \end{bmatrix}$

Choose matric M_j from $\{M_i\}$

$$E \times M_j = \begin{bmatrix} I_3 & 0 \\ 0^T & 1 \end{bmatrix}\begin{bmatrix} R_j & t_j \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} R_j & t_j \\ 0^T & 1 \end{bmatrix}$$

$$M_j \times E = \begin{bmatrix} R_j & t_j \\ 0^T & 1 \end{bmatrix}\begin{bmatrix} I_3 & 0 \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} R_j & t_j \\ 0^T & 1 \end{bmatrix}$$

$$\therefore\ E \times M_j = M_j \times E = M_j$$

Satisfied Existing Identity Element

④ The Existence Of An Inverse Element For Each Group Element

Choose matric M_j from $\{M_i\}$

Its inverse element $M_j^{-1} = \begin{bmatrix} R_j^{-1} & R_j^{-1}t_j \\ 0^T & 1 \end{bmatrix}$

We first need to prove the inverse element of R_j belongs to $\{R_i\}$

$$\because\ \left(R_j^T R_j\right)^{-1} = R_j^{-1}\left(R_j^{-1}\right)^T = I$$

$$\det\left(R_j^{-1}\right) = \frac{1}{\det\left(R_j\right)} = 1$$

$$\therefore\ R_j^{-1} \in \{R_j\}$$

$$R_j^{-1}t_j \in \mathbb{R}^{3\times 1}$$

So $M_j^{-1} \in \{M_i\}$

Satisfied Exiting an Inverse Element for Each Group Element

2

a) Please prove that $\boldsymbol{M}$ is positive semi-definite

The definition of matrix M is

$$M\ = [\nabla I(x,y)] * [\nabla I(x,y)]^T$$

$\nabla I(x,y)$ is the gradient vector of Image I at (x,y)

$[\nabla I(x,y)] * [\nabla I(x,y)]^T$ represents the result of cross product of the gradient vector, then we can get a matrix of 2×2

To prove that M is positive semi-definite, we need to prove for any nonzero vector $v = [x,y]$, $v^T * M * v >= 0$

Substituting $v = [x,y]$ into $v^T * M * v$, gives:

$$v^T * M * v\ = \sum x^2 I_x^2 + \sum 2xyI_xI_y + \sum y^2 I_y^2 =\ \sum\left[xI_x + yI_y\right]^2$$

I_x and I_y are the x and y components of the gradient vector at (x,y) respectively.

Apparently, $\sum\left[xI_x + yI_y\right]^2 \geq 0$

So, it can be proved that M is positive semi-definite

b) In practice, $\boldsymbol{M}$ is usually positive definite. If $\boldsymbol{M}$ is positive definite, prove that in the Cartesian coordinate system, $[\boldsymbol{x},\boldsymbol{y}]\boldsymbol{M}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{bmatrix} = \mathbf{1}$ represents an ellipse

If M is positive definite, then $v^T M v > 0$

For $\sum x^2 I_x^2 + \sum 2xyI_xI_y + \sum y^2 I_y^2 = 1$

We can get the Quadratic matrix $A = \begin{pmatrix} \sum I_x^2 & \sum I_x I_y \\ \sum I_x I_y & \sum I_y^2 \end{pmatrix}$

$\sum I_x^2 \sum I_y^2 > \left(\sum I_x I_y\right)^2$

So $[x, y]M\begin{bmatrix}x\\y\end{bmatrix} = 1$ represents an ellipse

c) Suppose that $\boldsymbol{M}$ is positive definite and its two eigen-values are $\boldsymbol{\lambda_1}$ and $\boldsymbol{\lambda_2}$ and$\boldsymbol{\lambda_1 > \lambda_2 > 0}$. For the ellipse defined by $[\boldsymbol{x}, \boldsymbol{y}]\boldsymbol{M}\begin{bmatrix}\boldsymbol{x}\\\boldsymbol{y}\end{bmatrix} = \boldsymbol{1}$, prove that the length of its semi-major axis is $\frac{\boldsymbol{1}}{\sqrt{\boldsymbol{\lambda_2}}}$ while the length of its semi-minor axis is $\frac{\boldsymbol{1}}{\sqrt{\boldsymbol{\lambda_1}}}$.

We can diagonalize symmetric matrices, let

$$M = P^{-1}\Lambda P = P^T\Lambda \mathrm{P}$$

P is an orthogonal matrix, $\Lambda = \begin{bmatrix}\lambda_1 & 0\\0 & \lambda_2\end{bmatrix}$

$$[x, y]M\begin{bmatrix}x\\y\end{bmatrix} = [x, y]P^T\Lambda\mathrm{P}[\mathrm{x}, \mathrm{y}]^{\mathrm{T}} = (P[\mathrm{x}, \mathrm{y}]^{\mathrm{T}})^T\Lambda(P[\mathrm{x}, \mathrm{y}]^{\mathrm{T}})$$

P is an orthogonal matrix, so $P[\mathrm{x}, \mathrm{y}]^{\mathrm{T}}$ means to do a rotation transformation for the point (x, y) on the ellipse.

Thus, $[x, y]M\begin{bmatrix}x\\y\end{bmatrix} = 1$ has the same semi-major axis and semi-minor axis with $[x, y]\Lambda\begin{bmatrix}x\\y\end{bmatrix} = 1$

We can find that the semi-major axis of $[x, y]\Lambda\begin{bmatrix}x\\y\end{bmatrix} = 1$ equals to $\frac{\boldsymbol{1}}{\sqrt{\boldsymbol{\lambda_2}}}$ and semi-minor axis equals to $\frac{1}{\sqrt{\lambda_1}}$, accordingly, the semi-major axis of $[x, y]\mathrm{M}\begin{bmatrix}x\\y\end{bmatrix} = 1$ is $\frac{\boldsymbol{1}}{\sqrt{\boldsymbol{\lambda_2}}}$ and semi-minor axis is $\frac{\boldsymbol{1}}{\sqrt{\lambda_1}}$.

3

Using $rank(A)$ to represent the rank of A, $rank(A^TA)$ to represent the rank of A^TA, $\dim\ker(A)$ to represent the kernel space of A and $\dim\ker(A^TA)$ to represent the kernel space of A^TA.

$$\because\ A^TAx = 0 \Leftrightarrow x^TA^TAx = 0 \Leftrightarrow (Ax)^T(Ax) = 0 \Leftrightarrow Ax = 0$$
$$\therefore\ \dim\ker(A) = \dim\ker(A^TA)$$

The size of A is $m \times n (m > n)$, the size of A^TA is $n \times n$

We know that for a matrix M of size $m \times n$

$$\dim\ker(M) = n - rank(M)$$

So

$$rank(A) = n - \dim\ker(A)$$
$$rank(A^TA) = n - \dim\ker(A^TA) = rank(A)$$

Obviously, $rank(A) = n$

Thus, $rank(A^TA) = n$ and $\det(A^TA) \neq 0$

It can be proved that $\boldsymbol{A^TA}$ is non-singular. In other words, $\boldsymbol{A^TA}$ is invertible.