1 FEM model

The problem is

$$\Delta\bar{p} - \frac{1}{c_0^2}\frac{\partial^2\bar{p}}{\partial^2 t} = f$$

with initial condition

$$\bar{p}(0,\mathbf{r}) \quad = \quad b(r)$$

Let

$$v \quad = \quad \frac{\partial\bar{p}}{\partial t}$$

then we have

$$\begin{aligned} \bar{p}_t - v = 0 \\ \Delta\bar{p} - \frac{1}{c_0^2}\,v_t = f \end{aligned}$$

and absorbing boundary condition

$$\frac{\partial\bar{p}}{\partial\mathbf{n}} \quad = - \quad \frac{1}{c_0}\frac{\partial\bar{p}}{\partial t}$$

$\frac{\partial\bar{p}}{\partial\mathbf{n}}$ is the normal derivative at the boundary. This is a the time-varying FEM model. by discretizing according to t, we have

$$\begin{aligned} (\frac{\bar{p}^n - \bar{p}^{n-1}}{\delta t},\phi)_\Omega - \;(\theta v^n + (1-\theta)v^{n-1},\phi)_\Omega &= 0 \\ -(\Delta((\theta\bar{p}^n + (1-\theta)\bar{p}^{n-1}),\bigtriangledown\phi)_\Omega - \frac{1}{c_0}(\frac{\bar{p}^n - \bar{p}^{n-1}}{\delta t},\phi)_{\partial\Omega} - \frac{1}{c_0^2}(\frac{v^n - v^{n-1}}{\delta t},\phi)_\Omega &= (\theta f^n + (1-\theta)f^{n-1},\phi)_\Omega \end{aligned}$$

we obtain

$$\begin{aligned} M\bar{p}^n - (\delta t\,\theta)Mv^n &= M\bar{p}^{n-1} + \delta t\,(1-\theta)\,M\,v^{n-1} \\ (-c_0^2\,\delta t\,\theta A - c_0\,B)\bar{p}^n - Mv^n &= (c_0^2\,\delta t\,(1-\theta)A - c_0 B)\bar{p}^{n-1} - M\,v^{n-1} + c_0^2\delta t(\theta F^n + (1-\theta)F^{n-1}) \end{aligned}$$

Write the above two equations as a matrix form

$$\begin{pmatrix} M & -(\delta t\,\theta)M \\ c_0^2\,\delta t\,\theta A + c_0\,B & M \end{pmatrix}\begin{pmatrix} \bar{p}^n \\ v^n \end{pmatrix} \quad = \quad \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$$

where

$$
\begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \begin{pmatrix} M\bar{p}^{n-1} + \delta t\,(1-\theta) M v^{n-1} \\ (-c_0^2\,\delta t\,(1-\theta)A + c_0 B)\bar{p}^{n-1} + M\,v^{n-1} - c_0^2 \delta t(\theta F^n + (1-\theta)F^{n-1}) \end{pmatrix}
$$

From the above matrix, we can obtain

$$
\begin{aligned}
(M + (\delta t\,\theta\,c_0)^2 A + c_0\,\delta t\,\theta\,B)\bar{p}^n &= G_1 + (\delta t\,\theta) G_2 \\
M v^n &= -(c_0^2\,\delta t\,\theta\,A + c_0 B)\bar{p}^n + G_2
\end{aligned}
$$