MODAL DECOMPOSITIONS IN FLUID MECHANICS: AN OVERVIEW

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EPSRC Summer School on
Modal Decompositions in Fluid Mechanics

What are they and what are they good for?

Global modes

Wavelets

BPOD

Koopman

POD

SVD

ERA

DMD

OMD

EMD

PODs of PODs

OKID

RA

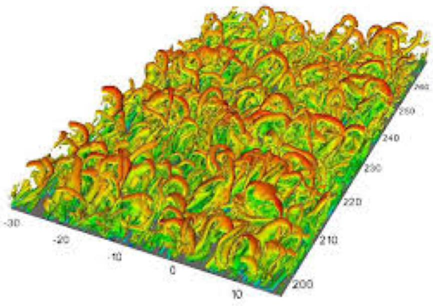
SPOD

Background: coherent structures

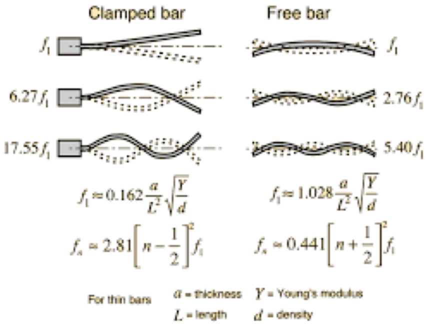
- ‘Structure’ → A common pattern or arrangement of elements in a system → reduction in dimensionality
- ‘Coherent’ → sticking together, persisting → dynamically favored structures

www.wikiwaves.org

earthsky.org Photo by Paul Chartier



Wu & Moin 2009



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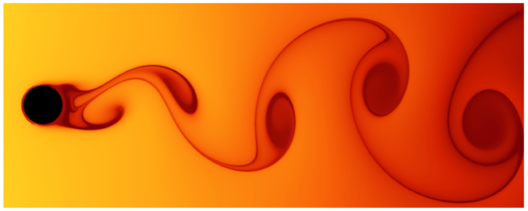
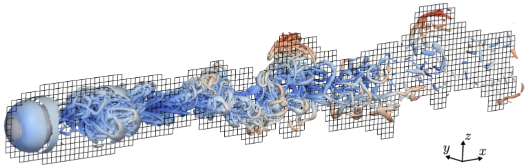


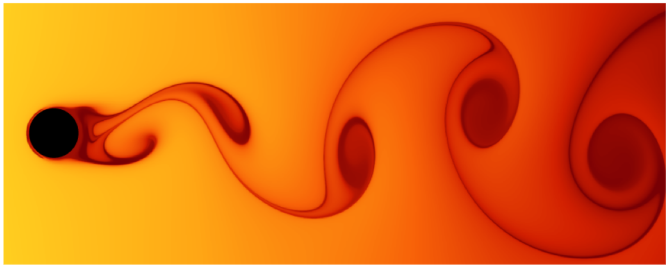
colonius.caltech.edu



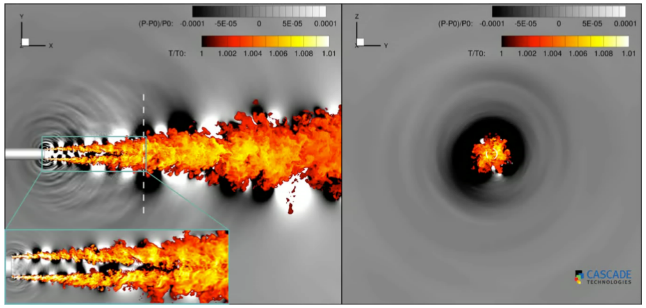
Liska and Colonius 2016

Coherent structures

Unwanted structural vibration



Unwanted noise



Movie: Bres et al. JFM 2018

Unwanted drag

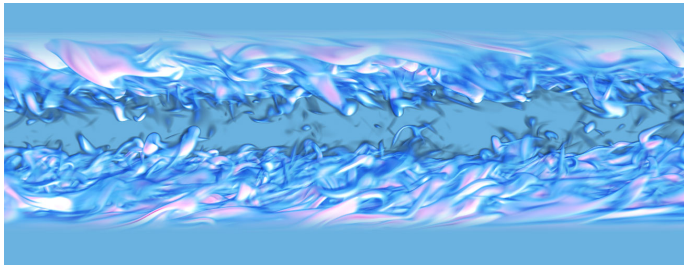


Image: Simon Toedtli, Beverley McKeon

Unwanted weather

“Modal decompositions” -- Why

- *Find* structures in data
  - Structure may be hidden to the human eye
  - Especially develop techniques that can be applied to large datasets
  - “Objective” definition of structure
- *Understand* how structures occur
  - Dynamical significance
  - Relation to observables
  - Underlying instability/amplification process
- *Exploit* structures
  - Data compression
  - Reduced-order models of dynamics
  - Handles for control (e.g. attenuate/amplify structure)

Lecture plan

1. Mathematical preliminaries (50 min)
2. Modal decompositions in fluid mechanics (50 min)
3. Case study: turbulent high-speed jets (50 min)

Principal references

- Background and algorithms

  - Bendat, J.S. and Piersol, A.G. *Random data: analysis and measurement procedures.* John Wiley & Sons, 2011.
  - Brunton, S.L. and Kutz, J.N. *Data-driven science and engineering: Machine learning, dynamical systems*, and control, Cambridge University Press, 2019.
  - Canuto, C. Hussaini, M.Y., Quarteroni, A, and Thomas, A. *Spectral methods in fluid dynamics*, Springer, 2012.
  - Holmes, P., Lumley, J.L., Berkooz, G. and Rowley, C.W. *Turbulence, coherent structures, dynamical systems and symmetry*. Cambridge University Press, 2012.
  - Schmid, P.J. and Henningson, D.S. *Stability and transition in shear flows*, Springer, 2012.
  - Taira, K. et al. Modal analysis of fluid flows: An overview . AIAA J. 55(12):4013-4041, 2017.
  - Trefethen, L.N. *Spectral methods in MATLAB*, SIAM, 2000.
  - Trefethen, L.N. and Bau, D. *Numerical linear algebra*, SIAM, 1997.

- SPOD/resolvent/jets

  - Schmidt, O.T., Towne, A., Rigas, G., Colonius, T. and Brès, G.A. Spectral analysis of jet turbulence. *J. Fluid Mech* 855:953-982, 2018
  - Towne, A., Schmidt, O. and Colonius, T. Spectral proper orthogonal decomposition and its relationship to dynamic mode decomposition and resolvent analysis. *J. Fluid Mech.* 847:821-867, 2018.
  - Schmidt, O. and Colonius, T. A guide to spectral proper orthogonal decomposition. Submitted to *AIAA J.*, 2019.

Preliminaries - Outline

- Finite-dimensional vector spaces: projection and approximation
- Singular value decomposition (SVD)
- Stochastic processes and principle component analysis (PCA)

Finite-dimensional vector space

- Why?
  - Easy case to start with, encapsulates most of the ideas we need
  - Useful for computing
  - Define terminology
- Consider $\mathbb{R}^m$ $\quad \mathbf{u} = \begin{bmatrix} u_1 & u_2 & \dots & u_m \end{bmatrix}^T$

$$(\mathbf{u}, \mathbf{v}) = \mathbf{v}^T \mathbf{u} \quad \Rightarrow \quad ||\mathbf{u}|| = (\mathbf{u}, \mathbf{u})^{\frac{1}{2}}$$

- Define sub-space, *S*, spanned by n *linearly independent* vectors

$$\phi_1, \phi_2, \dots, \phi_n \qquad n \le m$$

$$\mathbf{x} = \sum_{i=1}^{m} \alpha_i \phi_i \qquad \forall\ \mathbf{x} \in \mathcal{S} \qquad \text{or} \qquad \mathbf{x} = \boldsymbol{\Phi}\boldsymbol{\alpha} \qquad \boldsymbol{\alpha} \in \mathbb{R}^n$$

$$\boldsymbol{\Phi} = \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_n \end{bmatrix} \qquad \boldsymbol{\Phi} \in \mathbb{R}^{m \times n}$$

(linear map)

Approximate a vector

- Given $\mathbf{q} \in \mathbb{R}^m$
- Seek a reduced-order representation in the subspace S

$$\tilde{\mathbf{q}}^{(n)} = \sum_{i=1}^{n} \alpha_i \boldsymbol{\phi}_i = \boldsymbol{\Phi}\boldsymbol{\alpha}$$

such that the error is "small"

$$\boldsymbol{\epsilon} = \mathbf{q} - \tilde{\mathbf{q}}$$

Note unless m = n, we cannot always make the error zero because the system is over-constrained (m equations in n < m unknowns).

Method of weighted residuals

- Define a set of n "test vectors" (or "weight vectors")

$$\mathbf{\Psi} = \begin{bmatrix} \boldsymbol{\psi}_1 & \boldsymbol{\psi}_2 & \ldots & \boldsymbol{\psi}_n \end{bmatrix}$$

- And require them to be orthogonal to the "residual" (error)

$$\boldsymbol{\psi}_j^T \boldsymbol{\epsilon} = 0, \quad j = 1, 2, \ldots, n$$

or

$$\mathbf{\Psi}^T (\mathbf{q} - \tilde{\mathbf{q}}) = 0$$

$$\mathbf{\Psi}^T \mathbf{q} = \mathbf{\Psi}^T \mathbf{\Phi} \boldsymbol{\alpha}$$

We must require $\Psi^T\Phi$ to be non-singular. The null space of Φ should not be in the sub-space Ψ)

This is a projection

- Rearranging

$$\mathbf{\Psi}^T \mathbf{q} = \mathbf{\Psi}^T \mathbf{\Phi} \boldsymbol{\alpha}$$

$$\left(\mathbf{\Psi}^T \mathbf{\Phi}\right)^{-1} \mathbf{\Psi}^T \mathbf{q} = \boldsymbol{\alpha}$$

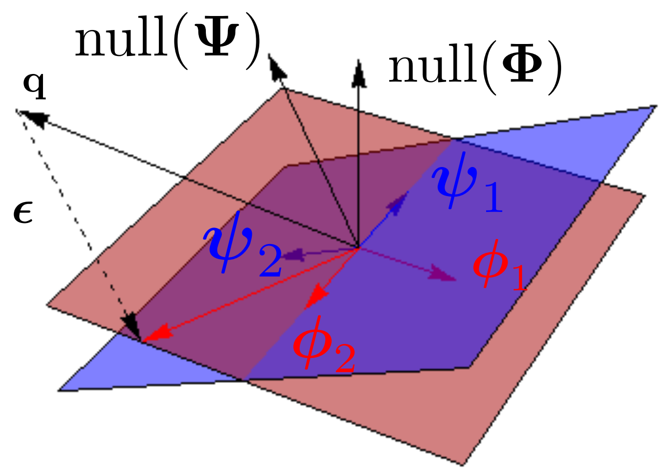
$$\mathbf{\Phi} \left(\mathbf{\Psi}^T \mathbf{\Phi}\right)^{-1} \mathbf{\Psi}^T \mathbf{q} = \mathbf{\Phi} \boldsymbol{\alpha} = \tilde{\mathbf{q}}$$

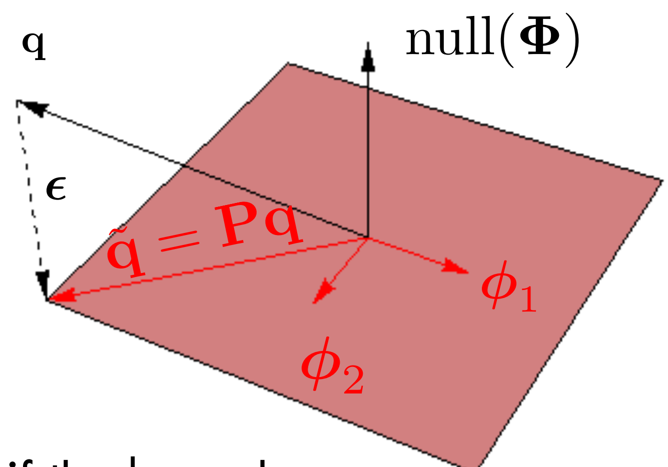
$$\mathbf{P} \mathbf{q} = \tilde{\mathbf{q}}$$

$$\text{where } \mathbf{P} = \mathbf{\Phi} \left(\mathbf{\Psi}^T \mathbf{\Phi}\right)^{-1} \mathbf{\Psi}^T$$

- *P* is a projection matrix, since $P^2=P$
- Orthogonal projection: let

$$\mathbf{\Psi} = \mathbf{\Phi} \quad \left\{ \begin{array}{l} \mathbf{P}^{\perp} = \mathbf{\Phi} \left(\mathbf{\Phi}^T \mathbf{\Phi}\right)^{-1} \mathbf{\Phi}^T \\ \mathbf{P}^T = \mathbf{P} \end{array} \right.$$





Note: if the ϕ are chosen as orthonormal, then

$$\mathbf{\Phi}^T \mathbf{\Phi} = \mathbf{I}, \quad \mathbf{P}^{\perp} = \mathbf{\Phi} \mathbf{\Phi}^T$$

Given Φ, find α that minimizes the error

- Minimize how?
- Minimum length (norm) $$\arg\min_{\alpha} \boldsymbol{\epsilon}^T \boldsymbol{\epsilon}$$

$$J(\boldsymbol{\alpha}) = \boldsymbol{\epsilon}^T \boldsymbol{\epsilon}$$

$$\frac{\partial J}{\partial \boldsymbol{\alpha}} = -2\boldsymbol{\Phi}^T \mathbf{q} + 2\boldsymbol{\Phi}^T \boldsymbol{\Phi} \boldsymbol{\alpha} = 0$$

$$\boldsymbol{\alpha} = \left(\boldsymbol{\Phi}^T \boldsymbol{\Phi}\right)^{-1} \boldsymbol{\Phi}^T \mathbf{q}$$

$$\tilde{\mathbf{q}} = \mathbf{P}^{\perp} \mathbf{q}$$

Take second derivative to show it's a minimum

- An orthogonal projection minimizes the distance
- Error is orthogonal to the subspace
- *Galerkin* projection

Least-square solution

- In fact, orthogonal projection is the same as the standard least-square solution to the over-constrained (inconsistent) system

$$\mathbf{q} = \mathbf{\Phi}\boldsymbol{\alpha} \qquad \mathbf{\Phi}^{+} = \left(\mathbf{\Phi}^{T}\mathbf{\Phi}\right)^{-1}\mathbf{\Phi}^{T}$$

$$\boldsymbol{\alpha} = \mathbf{\Phi}^{\dagger}\mathbf{q}$$

for the case where Φ has full column rank (m) and $\Phi^{\dagger}$ is the Moore-Penrose pseudoinverse.

Generalizations

Hermitian transpose

- Consider

$$\mathbb{C}^n \quad \text{with} \quad (\mathbf{u},\mathbf{v})_W = \mathbf{v}^*\mathbf{W}\mathbf{u}, \quad \mathbf{W} > 0$$

Shorthand meaning that W is symmetric, positive definite

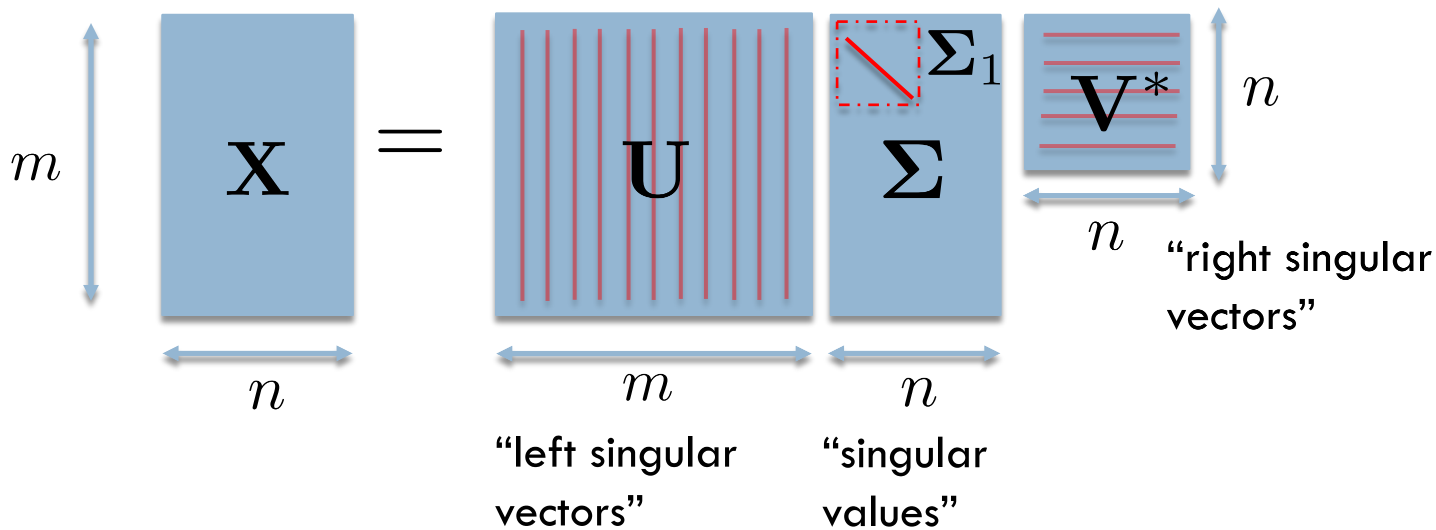
- Projection that minimizes error measured w.r.t. the weighted inner product is

$$\mathbf{P} = \mathbf{\Phi}\left(\mathbf{\Phi}^*\mathbf{W}\mathbf{\Phi}\right)^{-1}\mathbf{\Phi}^*\mathbf{W}$$

- Oblique projection (with Ψ=W Φ) w.r.t. the standard inner product
- All results above hold for complex-valued vectors if the transpose (T) is replace by Hermitian transpose (*)

Singular Value Decomposition*

$$\mathbf{X} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^*$$



$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0 \qquad \mathrm{rank}(X) = r \leq n$$

$$\boldsymbol{\Sigma}_1 = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix}$$

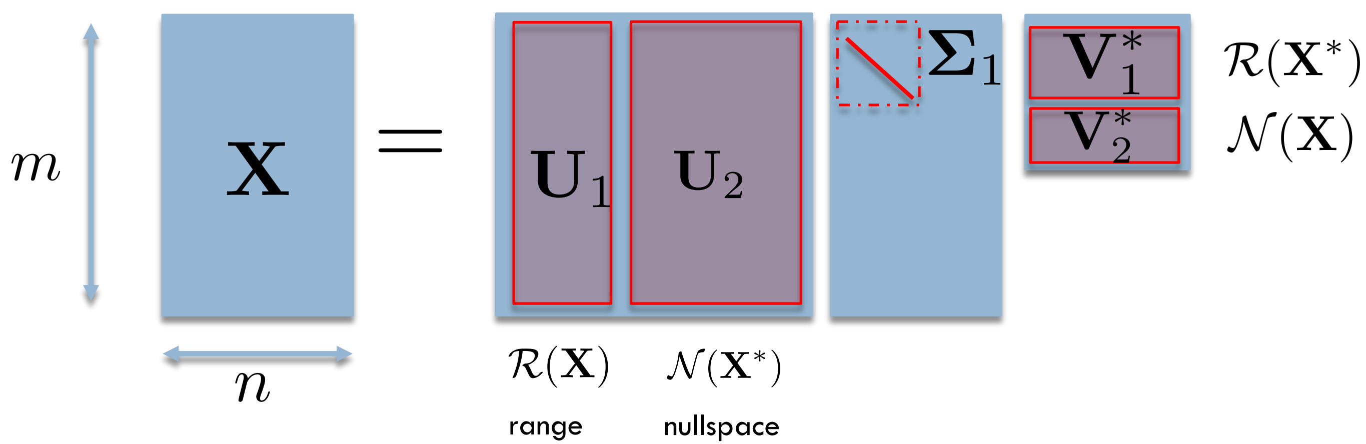
$$\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I} \quad \mathbf{V}^*\mathbf{V} = \mathbf{V}\mathbf{V}^* = \mathbf{I}$$

U, V are unitary

*Beltrami 1873, Jordan 1874; computation: Golub & Kahan 1965

SVD (II)

$$\mathbf{X} = \mathbf{U\Sigma V}^* = \mathbf{U_1}\mathbf{\Sigma}_1\mathbf{V_1}^* \qquad \text{(economy SVD)}$$



$$\begin{aligned}
\mathbf{X}\mathbf{v}_j &= \sigma_j \mathbf{u}_j \qquad & j &= 1, \ldots, r \\
\mathbf{X}\mathbf{v}_j &= 0 \qquad & j &= r+1, \ldots, n \\
\mathbf{X}^*\mathbf{u}_j &= \sigma_j \mathbf{v}_j \qquad & j &= 1, \ldots, r \\
\mathbf{X}^*\mathbf{u}_j &= 0 \qquad & j &= r+1, \ldots, m
\end{aligned}$$

Relation to EVP

- EVP
  - Any non-defective square matrix

$$\mathbf{A} = \mathbf{X\Lambda X}^{-1}$$

  - Any non-defective square, symmetric, positive semi-definite matrix

$$\mathbf{A} = \mathbf{U\Lambda U}^{*}$$

- SFD
  - Any Matrix M

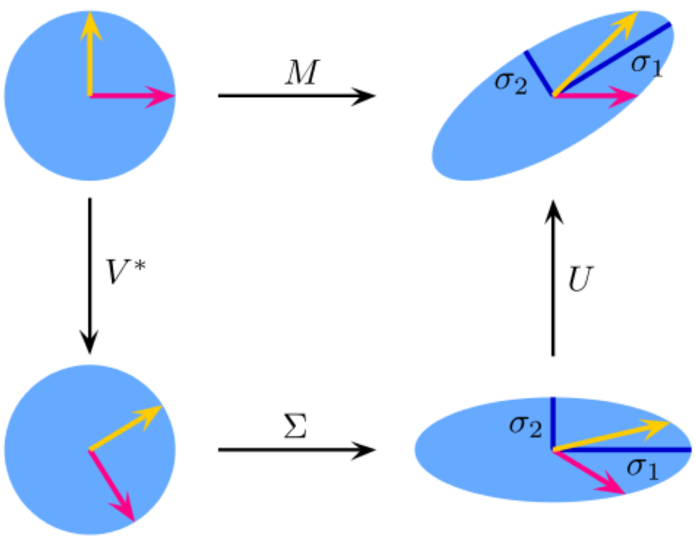
$$\mathbf{M} = \mathbf{U}_1\mathbf{\Sigma}_1\mathbf{V_1}^{*}$$

$$\mathbf{M}^{*}\mathbf{M} = \mathbf{V}_1\mathbf{\Sigma}_1^2\mathbf{V_1}^{*}$$

$$\mathbf{M}\mathbf{M}^{*} = \mathbf{U}_1\mathbf{\Sigma}_1^2\mathbf{U}_1^{*}$$

Interpretation

- Any linear transformation (matrix vector multiply) can be represented as the product of
  - A rotation of the vector by (V*) [preserves length]
  - A scaling (Σ)
  - (Possibly) mapping to a space of larger or smaller dimension
  - A rotation of the output (U*) [preserves length]



$$M = U \cdot \Sigma \cdot V^*$$

https://en.wikipedia.org/wiki/Singular_value_decomposition

Low-rank matrix approximation*

- Note that since the columns of U are orthonormal, any subset p of them define a projection

$$\mathbf{P}^{\perp} = \tilde{\mathbf{U}}\tilde{\mathbf{U}}^*$$

(note this is only $= \mathbf{I}$ when p = m)

where $\tilde{\mathbf{U}} = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} & \dots & \mathbf{u_p} \end{bmatrix}$ (first p left singular vectors)

- The best approximation to a matrix (Frobenius norm) for a given rank is given by a projection into a subspace spanned by the first p columns of U

$$\underset{\tilde{\mathbf{x}} \,|\, \mathrm{rank}(\tilde{\mathbf{x}}) \leq p}{\arg\min} \quad ||\mathbf{X} - \tilde{\mathbf{X}}||_F = \tilde{\mathbf{U}}\tilde{\mathbf{U}}^*\mathbf{X}$$

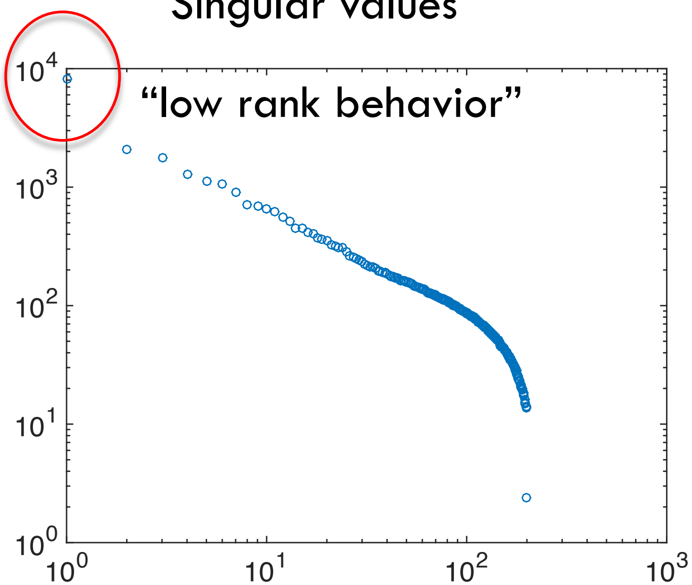
$$||\mathbf{A}||_F^2 = \sum_i \sum_j |a_{ij}|^2 = \sum_k \sigma_k^2$$

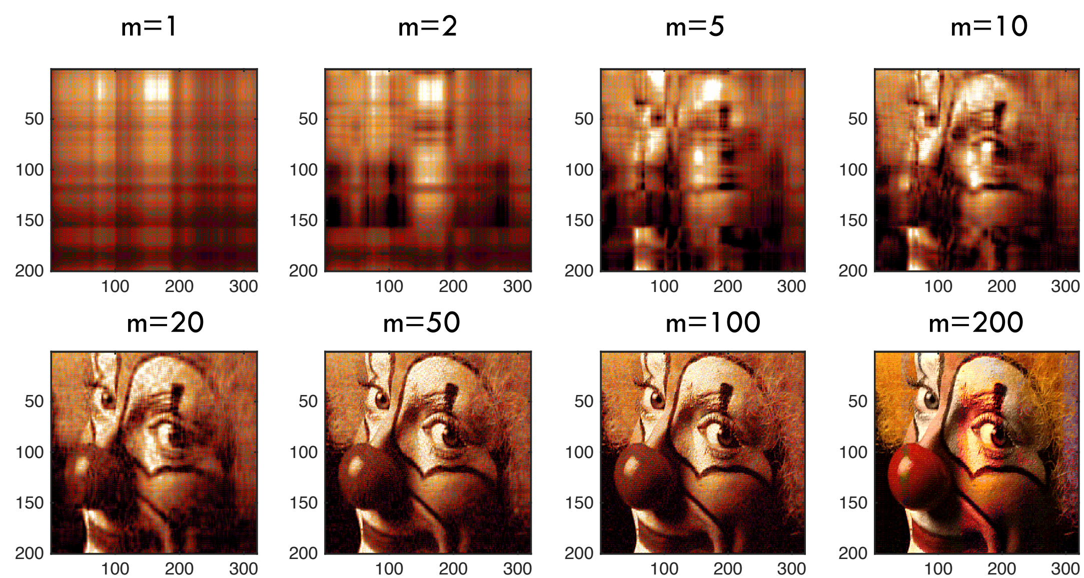
*Eckart & Young 1936

Classic example: image compression

320 x 200 image (and associated colormap)
Treat image as matrix (m=200, n=320)

Singular values





Random variables and PCA

- Consider a vector of *random* variables, x, with expected (mean) value E[x]. The covariance matrices is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{R}^m \qquad \mathbf{C} = E\left[(\mathbf{x} - E[\mathbf{x}])(\mathbf{x} - E[\mathbf{x}])^*\right] \in \mathbb{R}^{m\times m}$$

<u>Components</u>

$$C_{ij} = E\left[(x_i - E[x_i])(x_j - E[x_j])^*\right]$$

$$\sigma_j^2 = C_{jj}$$ Diagonal elements: variance

$$\rho_{ij} = \frac{C_{ij}}{\sigma_i \sigma_j}$$ Correlation (-1 < ρ < 1)

"Total Variance"

$$\sigma^2 = \mathrm{trace}(\mathbf{C}) = \sum_j \sigma_j^2$$

Correlation/covariance

- C is, by construction, a symmetric, positive semi-definite matrix
- Correlation is measure of tendency for variables to rise and fall together
  - Correlation = 1: if x rises, so does y
  - Correlation = 0: if x rises, y has equal probability of rising
  - Correlation = -1: if x rises, y falls
- If variables are independent, then their covariance is zero
  - The converse is not necessarily true
  - If x and y are linearly related, then the converse is true
  - If x and y are *jointly* normally distributed, then the converse is true

Sample mean and covariance

- *Estimate* E(x) and C based on *n* observations of the random variables (the ensemble)
- Notation: build a data matrix
- k-th realization of variables: $\mathbf{x}_k$
- Let the centered (mean subtracted) data matrix be

$$\mathbf{X} = \frac{1}{\sqrt{n-1}} \begin{bmatrix} \mathbf{x}_1 - \overline{\mathbf{x}} & \mathbf{x}_2 - \overline{\mathbf{x}} & \cdots & \mathbf{x}_n - \overline{\mathbf{x}} \end{bmatrix}$$

Then sample covariance is

$$\overline{\mathbf{x}} = \frac{1}{n} \sum_{k=1}^{n} \mathbf{x}_k$$

$$\mathbf{C}_s = \mathbf{X}\mathbf{X}^*$$

SVD of data matrix

- Note: rank (X) <= min(m,n)
  - Typical in statistics: n >> m
  - Typical in fluid dynamics m << n
  - The inequality holds for repeated observations (duplicate columns) or with linear dependencies amongst data points (linear combinations of rows)
- SVD

$$\mathbf{X} = \mathbf{U}_1 \mathbf{\Sigma}_1 \mathbf{V_1}^*$$

$$\mathbf{C}_s = \mathbf{U}_1 \mathbf{\Sigma}_1^2 \mathbf{U}_1^*$$

$$\mathbf{U}_1^* \mathbf{C}_s \mathbf{U_1} = \mathbf{\Sigma}_1^2$$

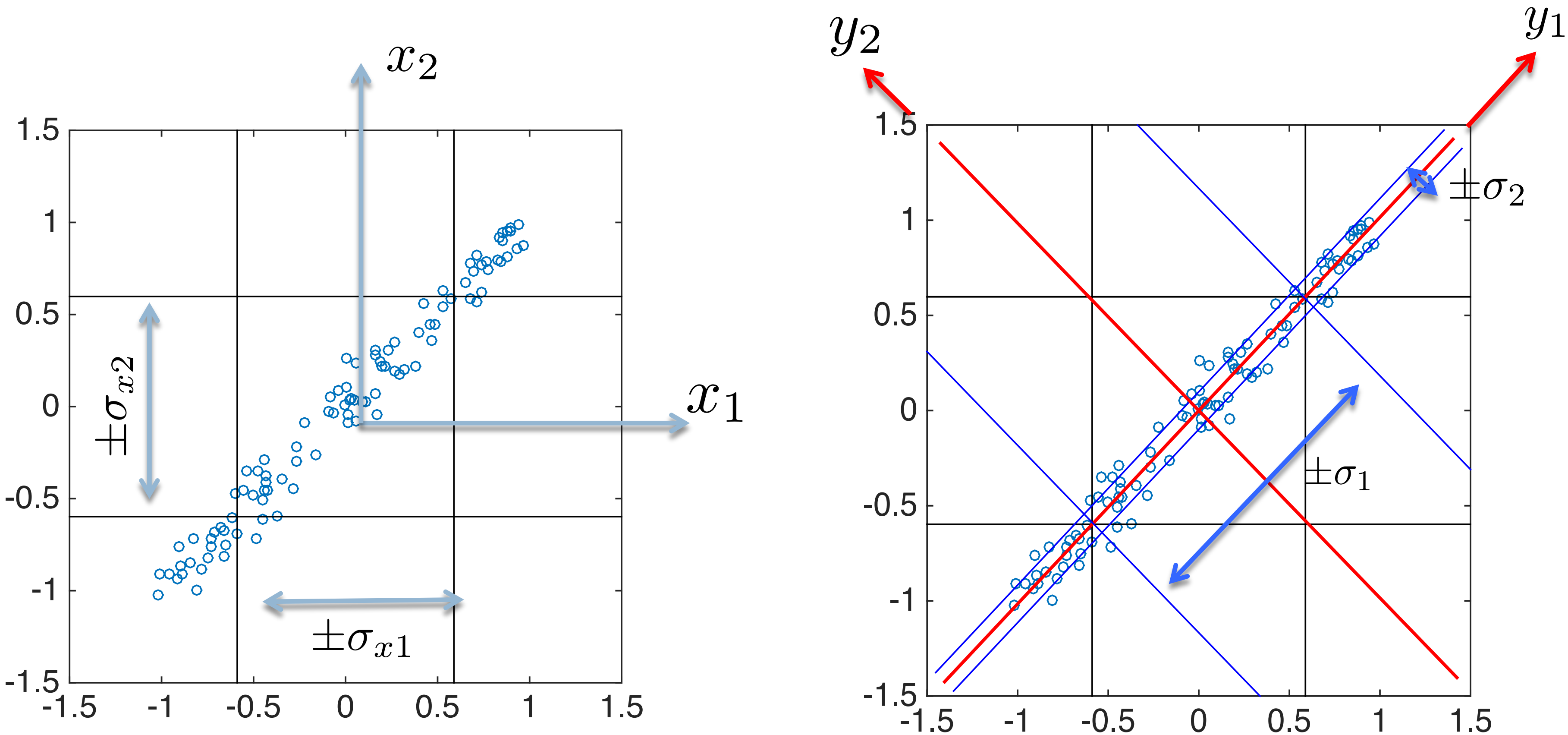
$$\mathbf{U}_1^* \mathbf{X} \left(\mathbf{U}_1^* \mathbf{X}\right)^* = \mathbf{\Sigma}_1^2$$

- Columns of U are eigenvectors of the C
- A change of variable diagonalizes the covariance matrix, leading to new variables that are *mutually uncorrelated*

$$\mathbf{y} = \mathbf{U}_1^* \left(\mathbf{x} - \overline{\mathbf{x}}\right)$$

Graphical interpretation

- Suppose we have n observations of two variables



Principal component analysis*

- Given a set of n realizations of the vectors x, the first m < r columns of U_1 corresponding to the p largest singular values of the data matrix, X, define a subspace that, amongst all possible *orthogonal projections*
  - Maximizes the total variance (trace of the covariance matrix) in the subspace
  - Minimizes the mean-square error between vectors in the original space and their projections (approximations) in the subspace

- Let $\tilde{\mathbf{U}}$ be comprised of the first p columns of U
- The projection into this subspace is, as before $\mathbf{P}^{\perp} = \tilde{\mathbf{U}}\tilde{\mathbf{U}}^*$
- We can also write a low rank approximation of the covariance

$$\mathbf{C}_s \approx \tilde{\mathbf{U}}\boldsymbol{\Sigma}_p^2\tilde{\mathbf{U}}^* = \sum_{k=1}^{p} \sigma_k^2 \mathbf{u}_k \mathbf{u}_k^*$$

*Person 1901; Hotelling 1930

Sketch of proof

- Show that the first eigenvector maximizes the variance over a linear combination of (centered) variables

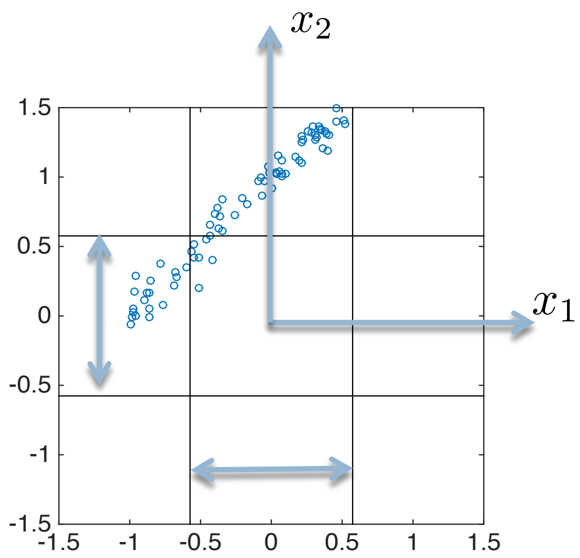
$$\mathrm{var}\left(\mathbf{u}_1^*\mathbf{X}\right) = \left(\mathbf{u}_1^*\mathbf{X}\right)\left(\mathbf{u}_1^*\mathbf{X}\right)^* = \mathbf{u}_1^*\mathbf{C}_s\mathbf{u}_1$$

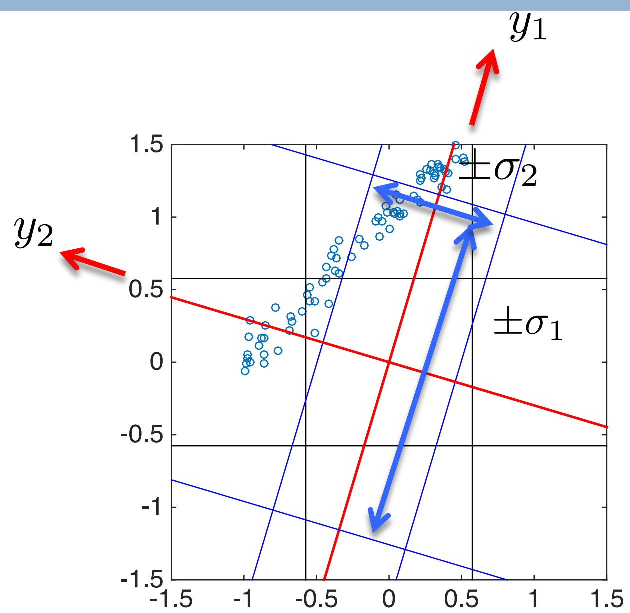
$$\mathbf{u}_1 = \underset{||\mathbf{u}||=1}{\arg\max}\, \mathbf{u}^T\mathbf{C}_s\mathbf{u} = \arg\max \frac{\mathbf{u}^T\mathbf{C}_s\mathbf{u}}{\mathbf{u}^T\mathbf{u}}$$

This Rayleigh quotient obtains a maximum equal to the largest eigenvalue of S when u is the corresponding eigenvector

- Next, we can project out the first principle component from X and proceed to find the next largest eigenvalue of S, and so on

Why should we center the variables?





- Note that any sub-space must include the origin (to be a vector space)
- *PCA can only rotate the axes, it cannot translate them*

Scaling problems with PCA

- When observations are combined together in a data matrix, for example

$$\mathbf{x} = \begin{bmatrix} \text{weight} \\ \text{height} \\ \text{age} \end{bmatrix}$$

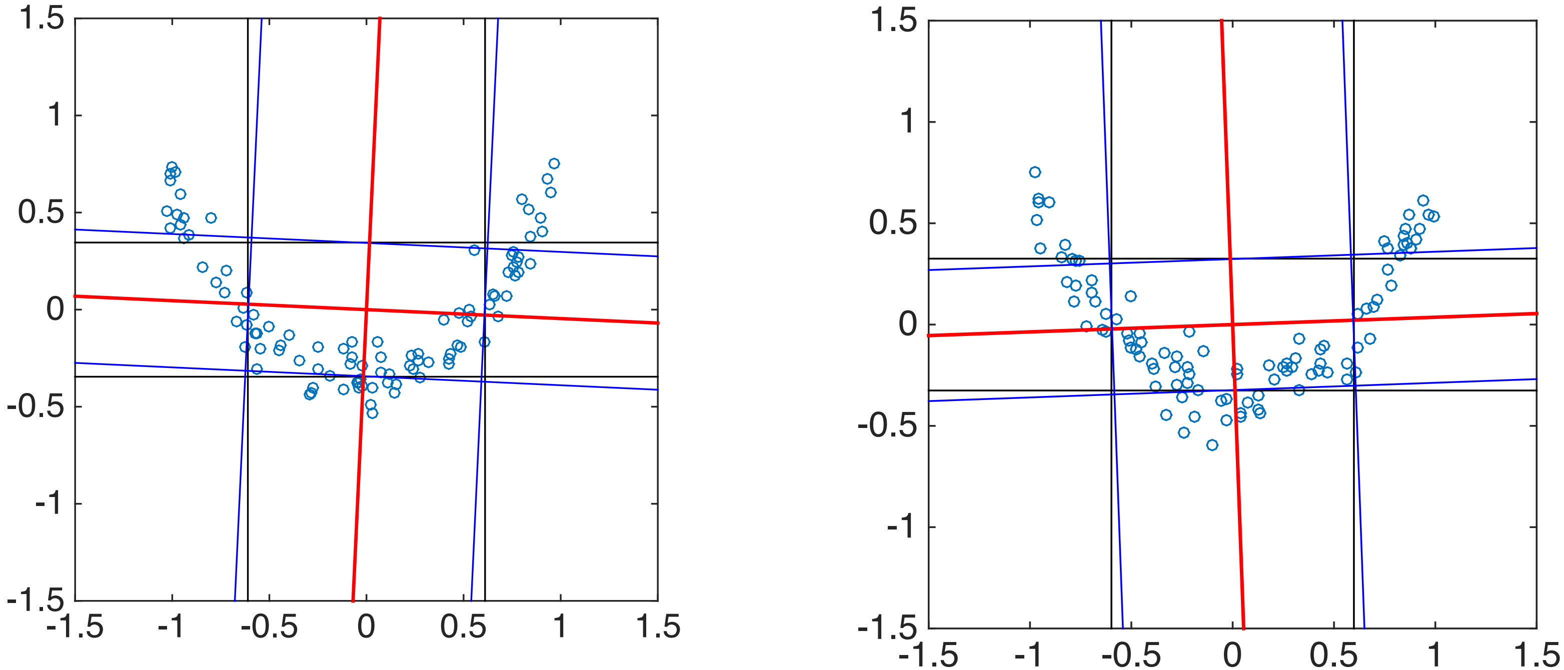
we obtain a different answer when we express the quantities in different units.

- One possibility is to do PCA on the correlation coefficient matrix (unique)
- For physical systems, we must insure that the original inner produce space has an appropriate norm. That is,

$$(\mathbf{x} - \overline{\mathbf{x}})^* (\mathbf{x} - \overline{\mathbf{x}}) \qquad \text{or} \qquad (\mathbf{x} - \overline{\mathbf{x}})^* \mathbf{W} (\mathbf{x} - \overline{\mathbf{x}})$$

should at least have dimensional homogeneity and (hopefully) a physical meaning. Often this is in terms of *an energy*

PCA is not very good at expressing nonlinear relationships

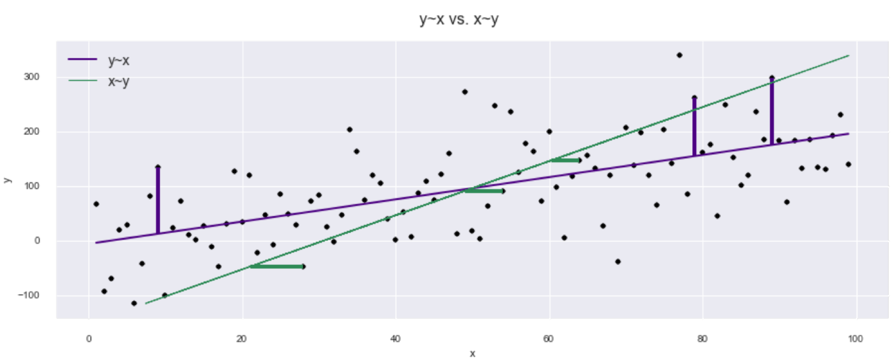


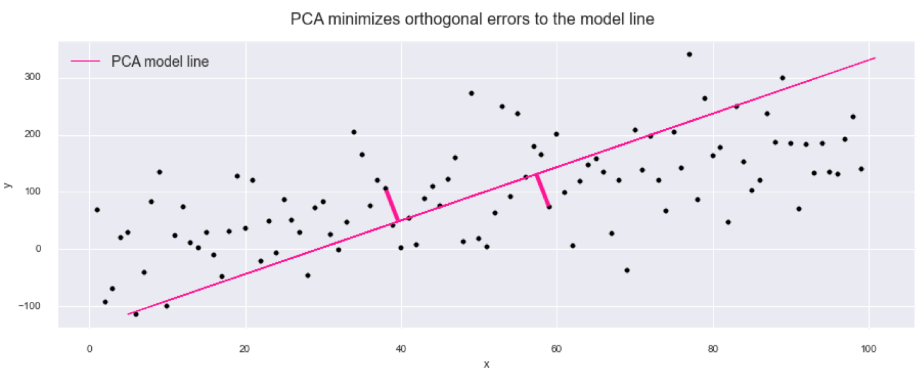
The left and right cases are for different random data created using the relation ship

$x_2 = x_1^2 + n$

Relation to (linear) regression analysis

- The processes are similar
- Regression separates inputs and outputs, and seeks the 'best' linear relation between them





- If two or more input variables are correlated, then typically regression outputs become sensitive to noise

Plots from https://shankarmsy.github.io/posts/pca-vs-lr.html

Additional comments

- If we solve as an EVP, we can choose whichever system is smaller

$$\mathbf{C}_s = \mathbf{X}\mathbf{X}^* = \mathbf{U}_1\mathbf{\Sigma}_1^2\mathbf{U}_1^* \qquad \text{m x m EVP}$$

$$\mathbf{X}^*\mathbf{X} = \mathbf{V}_1\mathbf{\Sigma}_1^2\mathbf{V}_1^* \qquad \text{n x n EVP} \qquad (\,\mathbf{U}_1 = \mathbf{X}\mathbf{V_1}\mathbf{\Sigma}_1^{-1})$$

- The latter is analogous to what is called the 'method of snapshots' in POD lingo
- We might have access to a covariance matrix theoretically (e.g. by knowing/assuming underlying p.d.f. for the process). We can perform PCA on the (non-sampled) covariance matrix, but it is always then an m x m EVP.

Summary

- SVD/PCA are starting point for POD and resolvent analysis and a variety of variations. The key concepts are
  - Low rank approximation of matrices
  - Bases and subspaces: which are good for fluids?
  - Projection: Galerkin or non-orthogonal?
  - Weighting/normalization of variables
- We will examine these issues in the context of fluids in the next lecture

Part 2: Modal decompositions in fluid mechanics

- Infinite-dimensional vector (function) spaces and discretization of PDE
- Flow model and symmetries
- Modal decompositions: Operator- versus data-driven approaches
- Global stability and resolvent analysis
- Proper Orthogonal Decomposition
- Balanced truncation and BPOD
- Stationarity
- Space-only, space-time and spectral POD

Infinite dimensional (function) spaces

- Example: a Hilbert space $L^2([a,b])$, i.e. square-integrable, complex valued functions with inner product

$$(f(x), g(x))_w = \int_a^b f(x) g^*(x) w(x) dx, \quad w(x) \geq 0.$$

- A basis is an infinite set of functions

$$\phi_j(x), \quad j = 1, 2, \ldots, \infty$$

such that any function in the space can be expressed

$$q(x) = \sum_{j=1}^{\infty} \alpha_j \phi_j(x)$$

MWR/Projection

- Consider a series of test functions $\psi_j(x), \quad j = 1, 2, \ldots, \infty$
- Make error orthogonal to each of these functions, e.g.

$$(\epsilon(x), \psi_j(x))_w = \int_a^b \epsilon(x)\psi_j^*(x)w(x)dx = 0$$

- Leads to system of equations

$$(q, \psi_j) - \sum_i \alpha_i (\phi_i, \psi_j) = 0, \quad j = 1, 2, \ldots, \infty$$

- Truncating the series to n terms leads to a matrix system of equations for the expansion coefficients

$$\mathbf{M}\boldsymbol{\alpha} = \mathbf{r} \qquad \begin{aligned} \{\mathbf{M}\}_{ji} &= (\phi_i, \psi_j) \\ \{\mathbf{r}\}_j &= (q, \psi_j) \end{aligned}$$

Choice of test functions

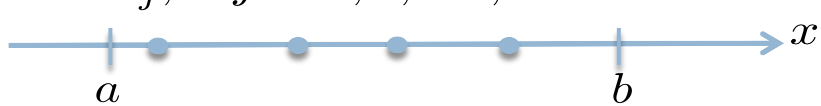
- Galerkin method (orthogonal projection)
  - Choose α to minimize the (norm of the) error

$$||\epsilon(x)||^2 = \int_a^b \epsilon(x)\epsilon^*(x)w(x)dx \quad \Rightarrow \quad \psi_j(x) = \phi_j(x) \quad \forall j$$

- Collocation method (non-orthogonal projection)

$$\psi_j(x) = \delta\left(x - x_j\right)$$

$$x_j, \quad j = 1, 2, \ldots, m$$



Choice of basis functions

- Fourier series
- Polynomials
  - Global, usually Chebyshev poly.
  - Local, e.g. Piecewise Polynomials (finite element)
- Generally to be computationally efficient, we want the matrix M to be sparse
  - M is diagonal for Galerkin with *orthogonal* basis functions
  - M is sparse (banded) for Galerkin/collocation with piecewise polynomials
  - M is diagonal for bi-orthogonal functions
- *Note that orthogonality of projection is different from orthogonality of basis!*

Discretization of PDE (conceptual)

- MWR is the basis for spectral, spectral collocation, spectral-element, and finite-element approximations of PDE
- Generally the solution is expanded one (spatial) variable at a time,

$$q(x,y,z,t) = \sum_i \sum_j \sum_k \alpha_{i,j,k}(t)\ \phi_i(x)\eta_j(y)\zeta_k(z)$$

Note: q and α can be vectors here (for example 3 components of velocity and pressure)

- When the continuous PDES are projected into the subspace*, this leads to a system of ODEs in time. We can write

$$\mathbf{q}(t) = \{\alpha_{i,j,k}(t)\} \in \mathbb{R}^{m=N_{var}*N_i*N_j*N_k}$$

Vector of nodal unknowns (1 element for each variable times each index)

Lots of work →

$$\boxed{\mathbf{M}\frac{d\mathbf{q}}{dt} = \mathbf{f}\left(\mathbf{q}\right)}$$

Note: finite-difference and finite-volume schemes also lead to equations of this form

*Of course, we have to account for boundary conditions in this process

Prototype flow model

- Discretized Navier-Stokes + …

Inputs (stochastic/deterministic)
- Forcing
- Boundary conditions
  Actuators
- …

Nonlinear term (i.e. advection)

Discretized state m-vector

Linear term (i.e. diffusion)

$$\mathbf{M}\frac{d\mathbf{q}(t)}{dt} = \mathbf{f}(\mathbf{q}) + \mathbf{L}\mathbf{q} + \mathbf{B}\mathbf{u}$$

Possibly singular matrix*

$$\mathbf{y}(t) = \mathbf{C}\mathbf{q}$$

Outputs (observables)

Initial conditions (stochastic/deterministic)

$$\mathbf{q}(0) = \mathbf{q}_0$$

*We will take M = I in what follows. There is no loss of generality since we can project into subspace associated with algebraic constraints (e.g. incompressible/divergence-free)

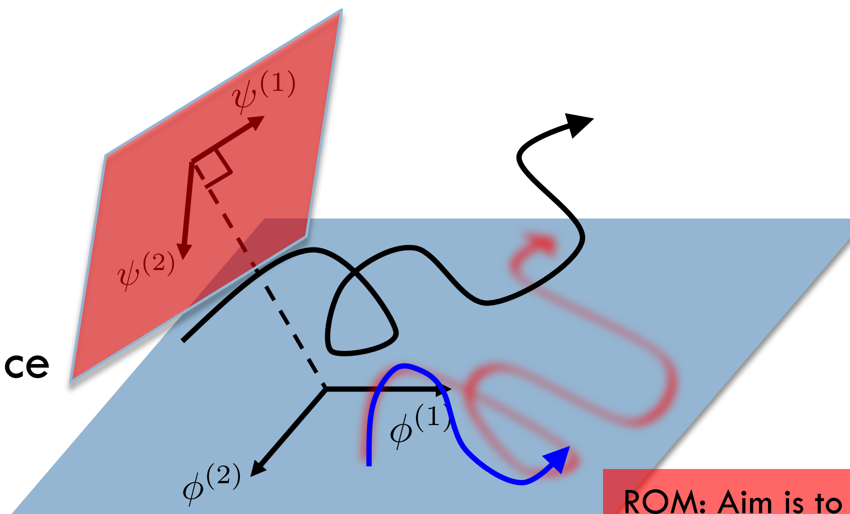
Projection-based reduced-order models

- As before, define a sub-space, trial vectors, and projection

$$\mathbf{\Phi} = \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_n \end{bmatrix}$$

$$\mathbf{\Psi} = \begin{bmatrix} \psi_1 & \psi_2 & \dots & \psi_n \end{bmatrix}$$

$$\mathbf{P} = \mathbf{\Phi} \left(\mathbf{\Psi}^T \mathbf{\Phi} \right)^{-1} \mathbf{\Psi}^T$$



- Project equation and solution into subspace

$$\mathbf{P}\frac{d\mathbf{q}}{dt} = \mathbf{P}\mathbf{f}(\mathbf{q}) + \mathbf{P}\mathbf{L}\mathbf{q} + \mathbf{P}\mathbf{\Psi}\mathbf{\Phi}$$

$$\boxed{\mathbf{q} \mapsto \tilde{\mathbf{q}} = \mathbf{\Phi}\boldsymbol{\alpha}}$$

$$\mathbf{P}\mathbf{\Phi}\frac{d\boldsymbol{\alpha}}{dt} = \mathbf{P}\mathbf{f}(\mathbf{\Phi}\boldsymbol{\alpha}) + \mathbf{P}\mathbf{L}\mathbf{\Phi}\boldsymbol{\alpha} + \mathbf{P}\mathbf{\Psi}\mathbf{\Phi}$$

$$\boxed{\mathbf{\Psi}^T\mathbf{\Phi}\frac{d\boldsymbol{\alpha}}{dt} = \mathbf{\Psi}^T\mathbf{P}\mathbf{f}(\mathbf{\Phi}\boldsymbol{\alpha})\mathbf{\Psi}^T\mathbf{L}\mathbf{\Phi}\boldsymbol{\alpha} + \mathbf{\Psi}^T\mathbf{P}\mathbf{\Psi}\mathbf{\Phi}}$$

ROM: Aim is to make BLUE trajectory equal to RED one

Recall: Prototype flow model

- Discretized Navier-Stokes + …

$$\frac{d\mathbf{q}(t)}{dt} = \mathbf{f}(\mathbf{q}) + \mathbf{L}\mathbf{q} + \mathbf{B}\mathbf{u} \qquad \mathbf{q}(0) = \mathbf{q}_0$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{q}$$

- Reynolds decomposition

$$\mathbf{q}(t) = \overline{\mathbf{q}} + \mathbf{q}'$$

Time-invariant base flow

- Laminar equilibrium $\quad \mathbf{u} = 0 \quad \mathbf{f}(\overline{\mathbf{q}}) + \mathbf{L}\overline{\mathbf{q}} = 0 \quad$ Steady NS
- Turbulent mean $\quad \overline{\mathbf{u}} = 0 \quad \mathbf{f}(\overline{\mathbf{q}}) + \overline{\mathbf{r}(\mathbf{q}')} + \mathbf{L}\overline{\mathbf{q}} = 0 \quad$ RANS
- …

Perturbation equations

Linearized dynamics

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{q}}\right|_{\overline{\mathbf{q}}} + \mathbf{L}$$

Fluctuating Reynolds stress (quadratic and H.O.T)

Fluctuating inputs

$$\frac{d\mathbf{q}'}{dt} = \mathbf{A}\mathbf{q}' + \underbrace{\mathbf{r}' + \mathbf{B}\mathbf{u}'}_{\mathbf{f}'}$$

$$\mathbf{y}'(t) = \mathbf{C}\mathbf{q}'$$

Adjoint equations* (linearized about time-invariant solution)

$$\frac{d\mathbf{q}^\dagger}{dt} = -\mathbf{A}^*\mathbf{q}^\dagger + \mathbf{C}^*\mathbf{y}^\dagger$$

*These naturally emerge from certain analyses of the primal system

The main decompositions

$$\frac{d\mathbf{q}'}{dt} = \mathbf{A}\mathbf{q}' + \mathbf{r}' + \mathbf{B}\mathbf{u}'$$
$$\mathbf{y}'(t) = \mathbf{C}\mathbf{q}'$$

$$\frac{d\mathbf{q}^\dagger}{dt} = -\mathbf{A}^*\mathbf{q}^\dagger + \mathbf{C}^*\mathbf{y}^\dagger$$

Data-driven

Realizations

$$\mathbf{Y} = \begin{bmatrix} \mathbf{y}'_1 & \mathbf{y}'_2 & \dots & \mathbf{y}'_n \end{bmatrix}$$
$$\mathbf{Q} = \begin{bmatrix} \mathbf{q}'_1 & \mathbf{q}'_2 & \dots & \mathbf{q}'_n \end{bmatrix}$$
$$\mathbf{Q}^\dagger = \begin{bmatrix} \mathbf{q}^\dagger_1 & \mathbf{q}^\dagger_2 & \dots & \mathbf{q}^\dagger_n \end{bmatrix}$$

- Proper orthogonal decomposition (POD)
- Dynamic mode decomposition (DMD)
- Balanced POD

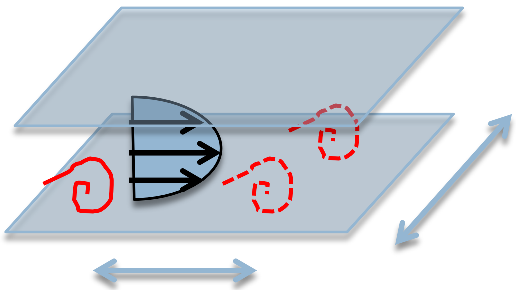
Operator-driven

$$\mathbf{A}, \mathbf{B}, \mathbf{C}$$

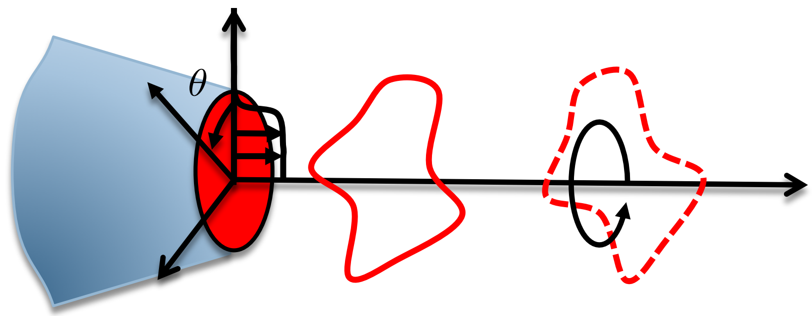
- Global modes (linear stability analysis)
- Resolvent modes
- Balanced truncation

Symmetries

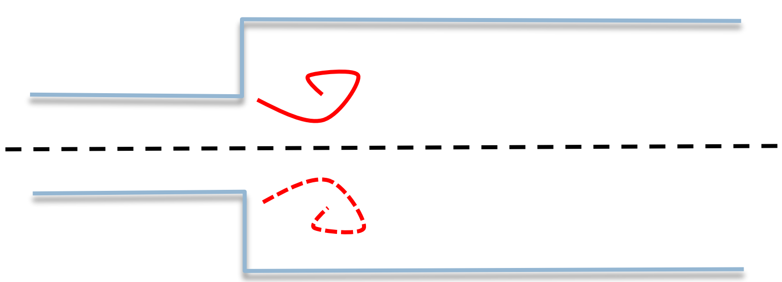
Homogeneous (infinite/periodic)



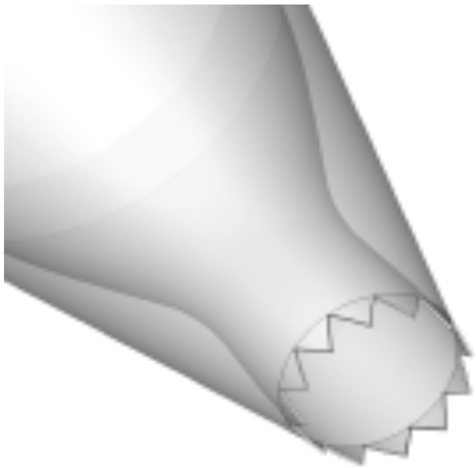
Continuous rotational



Reflectional



N-fold rotational



Symmetries, cont'd

- Operators preserve symmetries in the sense that solutions may be shifted/reflected
- Instabilities/turbulence can break symmetries – initial conditions, boundary conditions
- If the turbulence is *ergodic*, then symmetries should re-emerge in the statistics of the turbulence
- Operator-driven decompositions: symmetries automatically accounted for
- Data-driven decompositions: symmetries exist imperfectly in any finite dataset → best to impose them on the data

Example: homogeneity

- Use Fourier series to express solution/data

$$\mathbf{q}'(x,t) = \sum_{k=-\infty}^{\infty} \hat{\mathbf{q}}_k(t) e^{ikx} \qquad \hat{\mathbf{q}}_k = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{q}' e^{-ikx} dx$$

m-vector expressing components of solution discretized in inhomogeneous directions

- Operator driven: one system for each k

$$\frac{d\hat{\mathbf{q}}_k}{dt} = \mathbf{A}_k \hat{\mathbf{q}}_k + \hat{\mathbf{r}}_k + \mathbf{B}_k \hat{\mathbf{u}}_k$$

$$\hat{\mathbf{y}}_k(t) = \mathbf{C}_k \hat{\mathbf{q}}_k$$

Global modes (linear stability analysis)

- $\mathbf{r}' = 0$ (linear)
- $\mathbf{u}' = 0$ (unforced)

How does a small perturbation to a laminar (equilibrium) solution evolve in time?

$$\mathbf{q}'(t) = e^{\mathbf{A}t}\mathbf{q}'(0) \qquad \mathbf{q}'(t) \to \begin{cases} 0 & \max_j \mathbb{R}(\lambda_j) < 0 \quad \text{(stable)} \\ \infty & \max_j \mathbb{R}(\lambda_j) > 0 \quad \text{(unstable)} \end{cases}$$

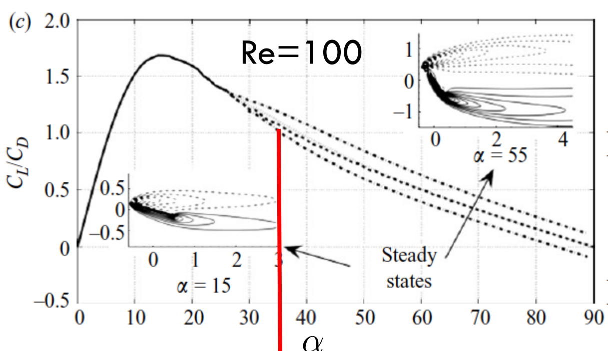
Modes:

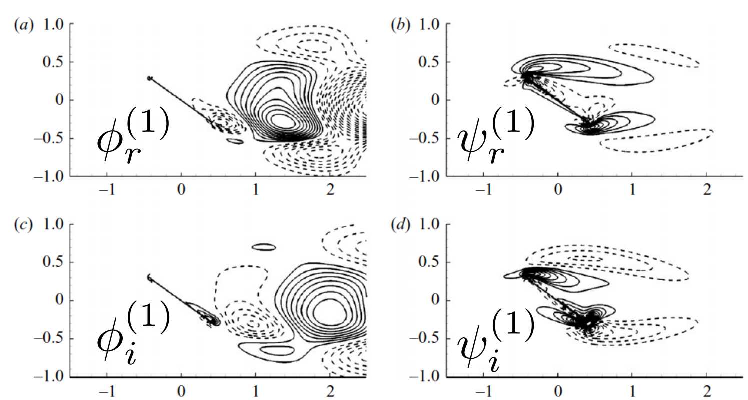
$$\mathbf{A}\mathbf{X} = \mathbf{X}\mathbf{\Lambda} \quad \begin{cases} \mathbf{\Phi} = \mathbf{X} \\ \mathbf{\Psi} = \mathbf{X}^{-*} \end{cases}$$

Subspace defined by (right) eigenvectors of A

Projection in direction of left (adjoint) eigenvectors

Example: Flat plate airfoil (2D)*





$\alpha = 35^o$

*Ahuja and Rowley 2010

Computation of global (& resolvent) modes

- Enablers of global modes (and many other decompositions)
  - Fast computers
  - Techniques for fast/sparse/parallel linear algebra (e.g. ARPACK)
- "Global" is used to distinguish this analysis from classical "local" stability analysis in shear flows where the modes are slowly varying in the streamwise direction (1D vs. 2/3D EVP)*

- Many "instabilities" of parallel shear flows (e.g. Kelvin-Helmholtz) are *stable* in a global analysis of a spreading base flow
- Computation: similar to resolvent modes discussed later
- Sparsity of the system matrices *required* for 2D/3D cases

*Theofilis 2011

Non-modal (transient) growth and the resolvent*

- For non-normal A, large amplification of disturbances can occur even when the base flow is stable

$$\mathbf{q}' = \tilde{\mathbf{q}} e^{st}$$

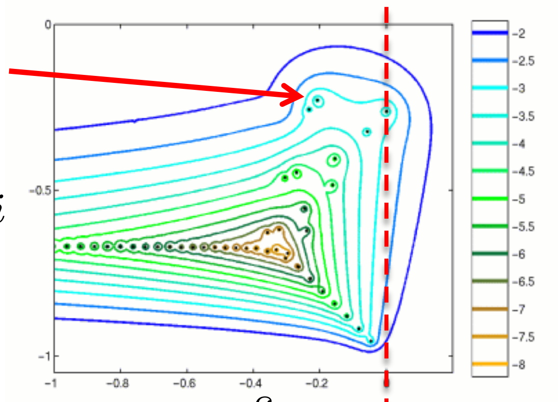
Laplace xform

$$\tilde{\mathbf{y}} = \mathbf{C} \underbrace{(s\mathbf{I} - \mathbf{A})}^{-1} (\mathbf{q}'(0) + \mathbf{B}\tilde{\mathbf{u}})$$

Infinite when s=λ. Large at other s?

contours of $\sigma_{max}(sI - A)^{-1}$.

$s = \lambda$



s_i

s_r

Decay Growth

$s_i = i\omega$

For stable case, long-time response to forcing characterized by *resolvent* (transfer function):

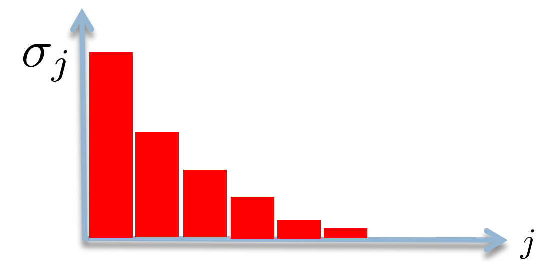
$$\tilde{\mathbf{y}} = \overbrace{\mathbf{C}(i\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}}^{\mathbf{R}(\omega)}\tilde{\mathbf{u}}$$

$$G_{opt} = \max_{||\tilde{\mathbf{u}}||=1} \frac{\tilde{\mathbf{y}}^*\tilde{\mathbf{y}}}{\tilde{\mathbf{u}}^*\tilde{\mathbf{u}}} = \max_{||\tilde{\mathbf{u}}||=1} \frac{\tilde{\mathbf{u}}^*\mathbf{R}^*\mathbf{R}\tilde{\mathbf{u}}}{\tilde{\mathbf{u}}^*\tilde{\mathbf{u}}}$$

*Trefethen et al. 1993, and ff (see next lectures on stability)

Resolvent modes

- The entire spectrum of gains can be found by the SVD of R



$$\mathbf{R}(\omega) = \mathbf{U}(\omega)\mathbf{\Sigma}(\omega)\mathbf{V}^*$$

Orthonormal basis for output modes

Ranked according to gain

Orthonormal basis for input (forcing/adjoint) modes

$$\mathbf{\Phi} = \mathbf{U}$$

$$\mathbf{\Phi} = \mathbf{V}$$

Efficient computation of resolvent modes

- Direct method: Form A,B,C and invert to make R and R*, solve SVD or EVP (Matlab: svd/eig). Limited to $m = O(10^4)$, i.e. 1D problems. Memory limited.
- Iterative methods: Form sparse A,B,C. Use multifrontal (MUMPS) algorithm to solve systems to get action of R and R* on a vector. Then use Arnoldi methods to get largest singular values/eigenvalues (Matlab: svds/eigs). Limited to $m = O(10^6)$, i.e. 2D problems.
- Randomized SVD*: For large m, use multifrontal (MUMPS) algorithm to solve systems to get action of R and R* on a vector.

*Halko et al. 2011, Moarref et al. 2013, Marques Ribeiro et al. 2019

Resolvent analysis of turbulent flows

- Particularly interesting for *turbulent* flows (base flow = mean flow)

$$\tilde{\mathbf{y}} = \mathbf{C}\left(i\omega\mathbf{I} - \mathbf{A}\right)^{-1}\left(\tilde{\mathbf{r}} + \mathbf{B}\tilde{\mathbf{u}}\right)$$

Forcing by turbulence (nonlinear, triadic interactions)*

Generic stochastic inputs (e.g. white noise)¶

High gain outputs = observed turbulent coherent structures ?

*McKeon and Sharma 2010 and ff.
¶Farrell and Ioannou 1993

Comments

- Everything can be done w.r.t. a weighted inner produce such that the output norm has physical significance
- Launching point for modeling of forcing
  - "Consistent models": consider truncated set of interactions amongst modes*
  - Data driven stochastic models (colored forcing)†
- Interpretation unclear if there are neutrally stable (resonant) or unstable modes
  - Linear system unstable – long term solution blows up
  - "Exponentially discounted modes" – adding a linear damping term to the governing equation‡
  - Eddy viscosity (will discuss more in next lecture)

*Rosenberg & McKeon 2018, 2019, Mantic-Lugo & Gallaire 2014, 2016

†Zare et al. 2017, Towne 2017, Towne et al. 2018 ‡ Yeh & Taira 2017

Data driven decompositions: POD*

- Generalization of PCA
  - POD: generalization from finite-dimensional vector space to Hilbert space
  - Karhunen-Loève POD with theoretical covariance tensor rather than sample-estimated one

$$\langle \mathbf{u}, \mathbf{v} \rangle = \int_{\Omega} \mathbf{u}(z)^{*} \mathcal{W}(z) \mathbf{v}(z) dz \qquad z \in \Omega$$

z can be any number of spatial/temporal coordinates

$$E\{\cdot\} \equiv \overline{(\cdot)}$$

Appropriate averaging operator (ensemble is most general)

- Covariance tensor

$$\mathcal{C}(z, z') = E\{\mathbf{q}'(z)\mathbf{q}^{*\prime}(z')\} \qquad \mathbf{u}' = \mathbf{u} - \overline{\mathbf{u}}$$

POD: $$\mathcal{C}_s(z, z') = \frac{1}{n-1} \sum_{k=1}^{n} \mathbf{q}'_k(z)\mathbf{q}^{*\prime}_k(z)$$

*Karhunen 1947; Lumley 1967

POD (cont'd)

- KL theorem. If C is a nuclear kernel (compact and finite) then there are a countably infinite number of (deterministic) eigenfunctions/eigenvalues of

$$\langle \mathcal{C}\left(z, z'\right), \phi(z')\rangle = \lambda\phi(z)$$

such that

$$\langle \phi_j, \phi_k \rangle = \delta_{jk}$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$$

and we can expand any realization of the process as a set of uncorrelated processes

$$\mathbf{q}'(z) = \sum_{j=1}^{\infty} a_j \phi_k(z) \qquad E\left\{a_j a_k^*\right\} = \lambda_j \delta_{jk}$$

POD (cont'd)

- Optimality: the truncated series

$$\mathbf{q}'(z) = \sum_{j=1}^{n} a_j \phi_k(z)$$

captures more of the total variance/energy

$$\mathcal{E} = E\left\{\langle \mathbf{q}', \mathbf{q}' \rangle\right\} = \mathrm{trace}\left(\mathcal{C}\right)$$

than any other n-term expansion in orthogonal functions.

KL/POD/PCA

- Upon discretization of the space Ω, a weighting matrix can be found such (with a slight abuse of notation) by combining the quadrature weights with the weighting tensor

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{W} \mathbf{v} \approx \int_\Omega \mathbf{u}(z)^* \mathcal{W}(z) \mathbf{v}(z) dz$$

Weighted sums up over variables and Ω

Weighted sum over variables

Integral over space

- Thus we can think of PCA as the discretization of KL/POD

KL/POD with homogeneity*

- Homogeneous → $\mathcal{C}(x,z,x',z') = \mathcal{C}(x-x',z,z')$
- The eigenfunction/eigenvalue

$$\phi(x,z) = \hat{\phi}_k(z)e^{ikx} \qquad\qquad \hat{\lambda}_k$$

are solution of the KL equation, where

$$\left\langle \hat{\mathcal{C}}_k, \hat{\phi}_k \right\rangle_k = \hat{\lambda}_k \hat{\phi}_k$$

Solve this KL problem for each k

Fourier transform of covariance tensor

Inner product over remaining directions

*Lumley 1970

Application of POD

- Choose domain, variables, and inner product
- Domain, variables often limited by experimental/computational constraints
  - Spatially limited time-resolved information from hot wires and/or microphone arrays
  - Temporally unresolved data from PIV, planar (2C and 3C)
  - Not all variables of interest measured (e.g. pressure/temperature in compressible flow)
  - Full 3D time-resolved data is very large (many TB)
- Semi-norms can be used (some variables have zero weight in inner product), i.e. extended POD*
- If reduced-order models are sought based on the POD basis, it is important that inner product is *complete* (unique solution)

*Maurel, Borée, & Lumley 2001

Choice of inner product / weighting matrix

- Variance (2-norm) $\mathbf{W} = \mathbf{I}$
- Weighted 2-norm $\mathbf{W} = \mathrm{diag}(dV)$
- Turbulence kinetic energy

$$\mathbf{q}' = \begin{bmatrix} \mathbf{u}' \\ \mathbf{v}' \\ \mathbf{w}' \end{bmatrix} \qquad \mathbf{W} = \begin{bmatrix} \mathrm{diag}(dV) & & \\ & \mathrm{diag}(dV) & \\ & & \mathrm{diag}(dV) \end{bmatrix}$$

- Compressible flow energy*

$$\mathbf{q}' = \begin{bmatrix} \boldsymbol{\rho}' \\ \mathbf{u}' \\ \mathbf{v}' \\ \mathbf{w}' \\ \mathbf{T}' \end{bmatrix} \qquad \mathbf{W} = \begin{bmatrix} \frac{\overline{T}}{\gamma\overline{\rho}M^2}\mathrm{diag}(dV) & & & & \\ & \overline{\rho}\mathrm{diag}(dV) & & & \\ & & \overline{\rho}\mathrm{diag}(dV) & & \\ & & & \overline{\rho}\mathrm{diag}(dV) & \\ & & & & \frac{\overline{\rho}}{\gamma(\gamma-1)\overline{T}M^2}\mathrm{diag}(dV) \end{bmatrix}$$

*Chu 1965

Space-only and space-time POD

- Space only:
  - Stochastic ensemble → flow at different time instances (stationary)

$$\langle \mathbf{u}, \mathbf{v} \rangle = \int_V \mathbf{u}(x,t)^* \mathcal{W}(x) \mathbf{v}(x,t) dx$$

$$\mathcal{C}(x,x') = E\left\{ \mathbf{q}'(x,t) \mathbf{q}^{*\prime}(x,t) \right\}$$

$$\mathbf{q}'(x,t) = \sum_{j=1}^{\infty} a_j(t) \boldsymbol{\phi}_j(x)$$

Eigenfunctions optimally represent spatial correlation, but the expansion coefficients are not necessarily correlated over time

- Space-time:
  - Stochastic ensemble → flow history in different realizations

$$\langle \mathbf{u}, \mathbf{v} \rangle = \int_T \int_V \mathbf{u}(x,t)^* \mathcal{W}(x,t) \mathbf{v}(x,t) \; dxdt$$

$$\mathcal{C}(x,x',t,t') = E\left\{ \mathbf{q}'(x,t) \mathbf{q}^{*\prime}(x',t') \right\}$$

$$\mathbf{q}'(x,t) = \sum_{j=1}^{\infty} a_j \boldsymbol{\phi}_j(x,t)$$

Eigenfunctions optimally represent second-order space-time flow statistics

Stationary flows

- Stationarity → statistics independent of time (temporal analog of spatial inhomogeneity)
  - Space-only POD: restricted to stationary flows; samples taken from arbitrary times (different times regarded as independent realizations)
  - Space-time POD: similar trick as spatial homogeneity

$$\mathcal{C}(x, x', t, t') = \mathcal{C}(x, x', t - t')$$

Solve EVP one frequency at a time:

$$\int_V \mathcal{S}(x, x', \omega) \mathcal{W} \hat{\phi}(x', \omega) dx = \hat{\lambda}(\omega) \hat{\phi}(x, \omega)$$

Cross-spectral density (CSD) tensor

$$\mathcal{S}(x, x', \omega) = \int_{-\infty}^{\infty} \mathcal{C}(x, x', \tau) e^{-i\omega\tau}$$

F.T. of eigenfunction

$$\hat{\phi}(x, \omega) = \int_{-\infty}^{\infty} \phi(x, \tau) e^{-i\omega\tau}$$

Eigenfunctions oscillate at a single frequency and optimally represent second-order space-time flow statistics

Spectral POD*

- Space-time POD for stationary flows
- Computed in frequency domain using standard techniques for estimating the CSD
- Modes at different frequencies not spatially orthogonal (but space-time orthogonal)
- Discussed by Lumley (1967, 1970)
- Applied by several groups[§] but largely neglected in fluid mechanics (lack of data?)
- Closely related to DMD (Koopman)
  - → "optimally-averaged" DMD/Koopman modes for stationary turbulent flow[¶]

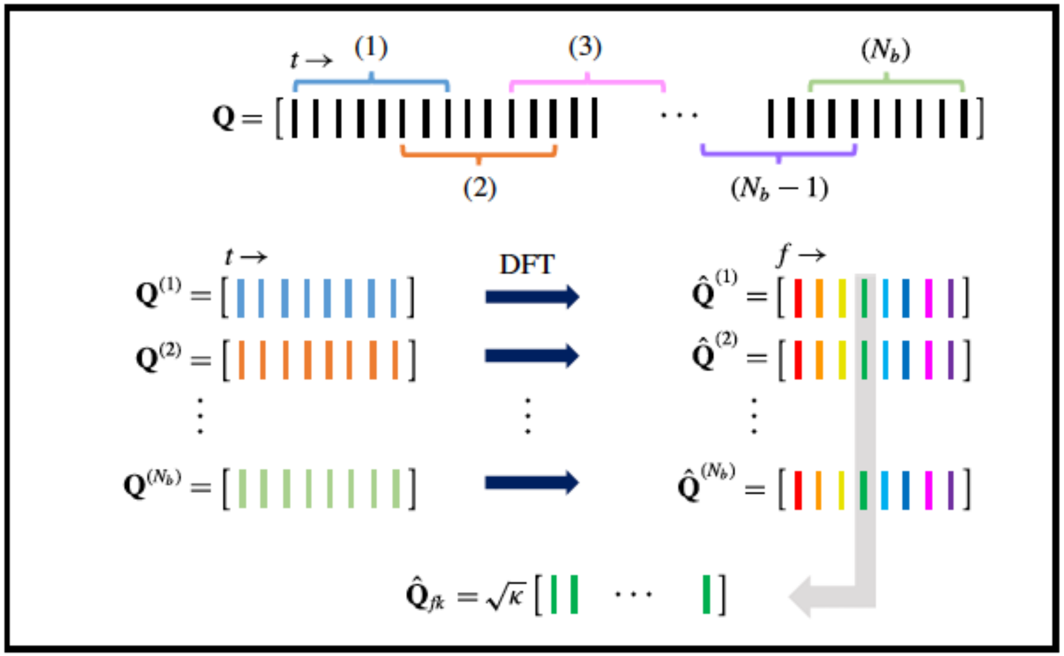
*Not equivalent to the method with the same name by Sieber et al. 2016, 2018

§George, Delville, Glauser, Jordan, Tinney, Colonius (& coworkers)

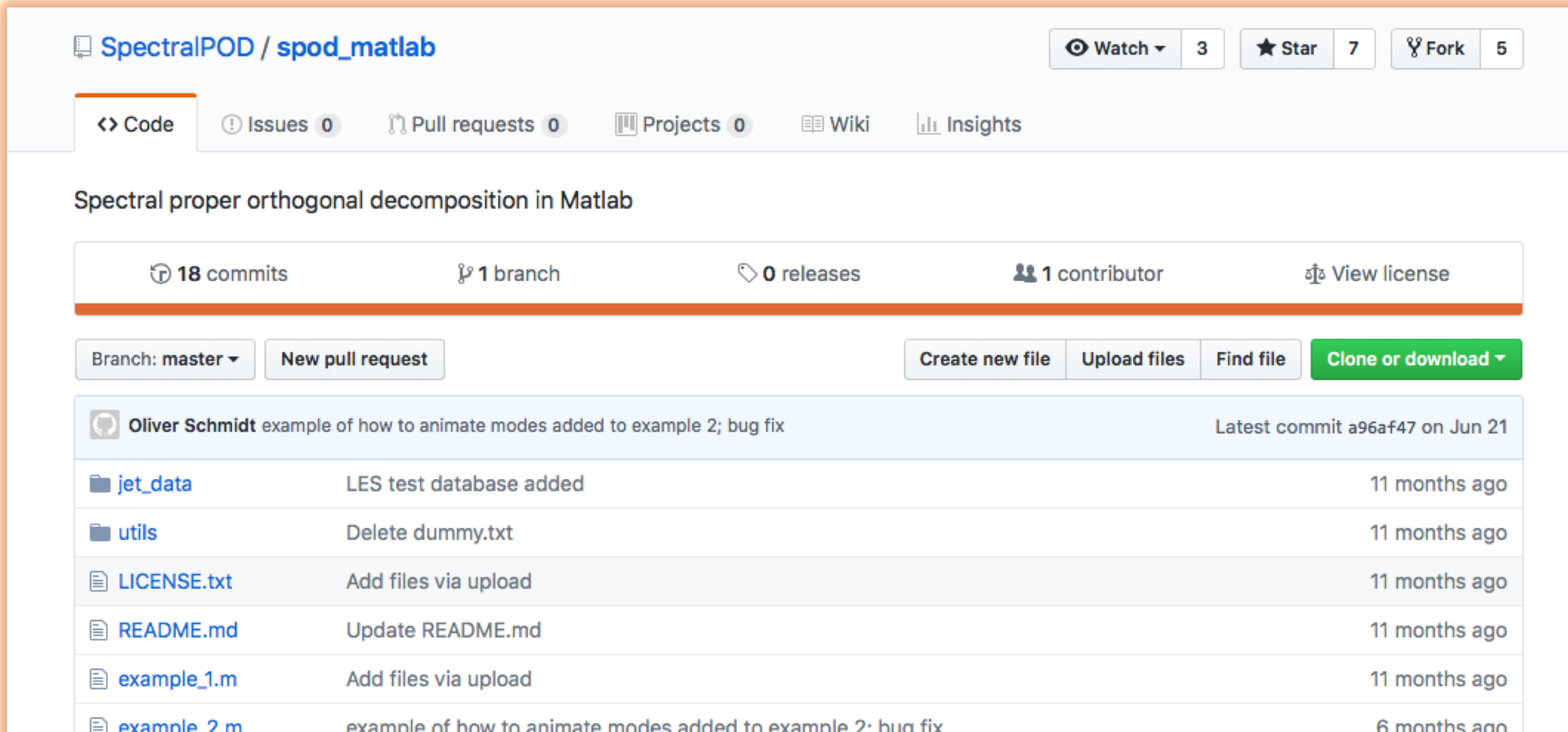
¶ Towne, Schmidt and Colonius 2017

Algorithm (Welch periodogram method)

- Builds on computation of PSD via Welch's method
- Bias and aliasing control by sampling rate, segment length, windowing, etc.



Software available (Matlab)

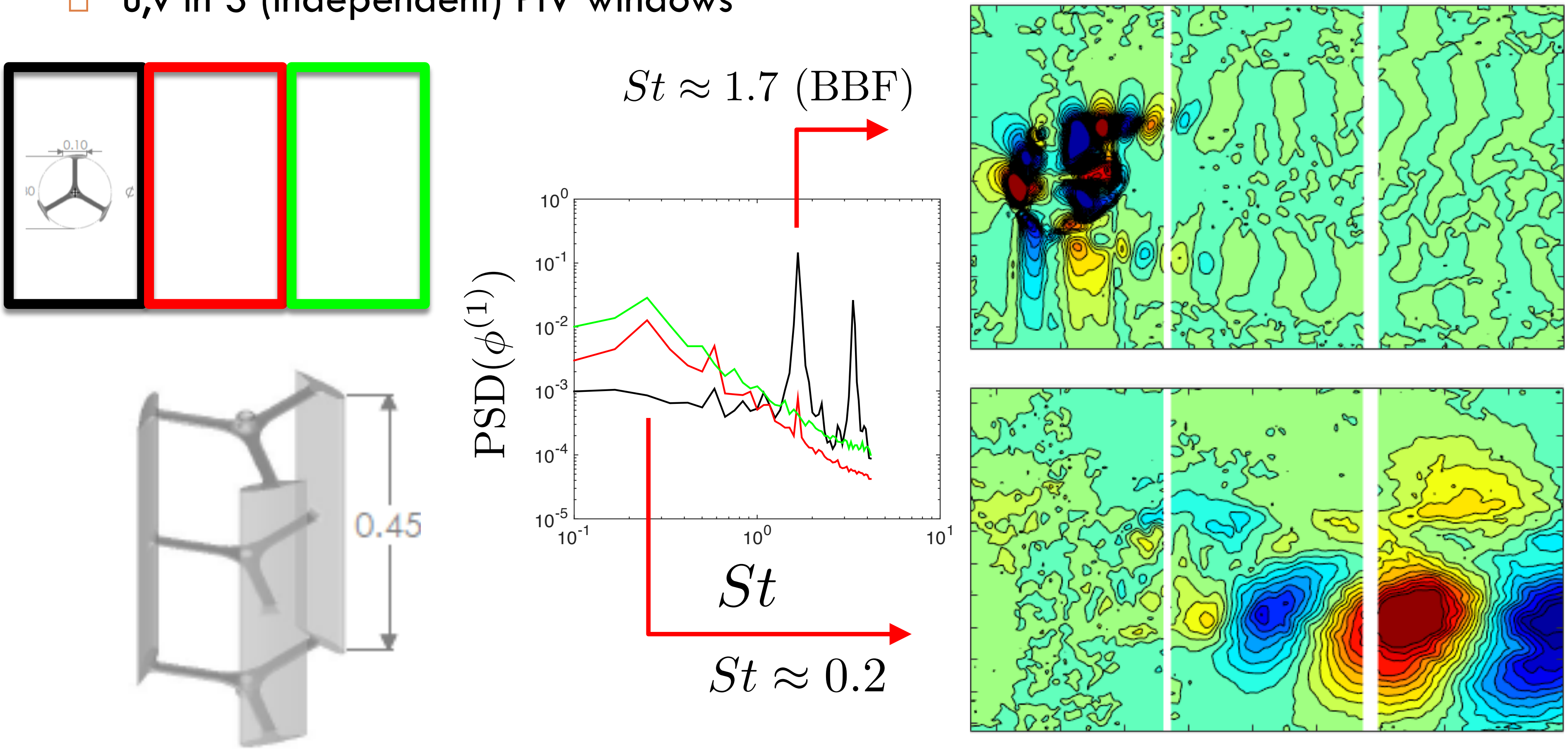


https://github.com/SpectralPOD/spod_matlab

Streaming SPOD: Schmidt & Towne 2019 (https://arxiv.org/abs/1711.04199)

Example: TR-PIV data from a VAWT*

- VAWT in a water tunnel @ Re ~ 10^5
- u,v in 3 (independent) PIV windows



*Arraya, Colonius & Dabiri 2016

Relation between SPOD and resolvent modes

- Consider an ensemble of realizations of a random forcing at a given frequency

$$\tilde{\mathbf{q}} = \mathbf{R}\tilde{\mathbf{f}}$$

$$\mathcal{S}_{qq} = \mathbf{R}\mathcal{S}_{ff}\mathbf{R}^*$$

- If forcing is such that variables uncorrelated in space

$$\mathcal{S}_{ff} = s(\omega)\ \mathbf{I} \quad \rightarrow \quad \mathcal{S}_{qq} = s(\omega)\ \mathbf{R}\mathbf{R}^*$$

SPOD EVP   Resolvent EVP

Under spatially uncorrelated (white) forcing: SPOD modes equivalent to resolvent modes

But…turbulence is not white noise!

- Equivalence is still useful since it allows one to gauge whether observed modes are "coherently forced" by triadic interactions
- Also a launching points for data-driven "statistics completion problem"*

Infer

Note tilde and boldface dropped here

$$q = Rb$$

$$y = Cq$$

$$S_{qq} = R\, S_{ff}\, R^H$$

$$S_{yy} = CR\, S_{ff}\, R^H C^H$$

Known

Estimate

$$CR = U\left[\Sigma_1 \quad 0\right]\left[V_1 V_2\right]^H$$

Resolvent decomp.

$$S_{qq} \approx RV_1\left(\Sigma_1^{-1} U^H S_{yy} U \Sigma_1^{-1}\right) V_1^H R^H$$

*Zare et al. 2017, Towne 2017, Towne et al. 2018

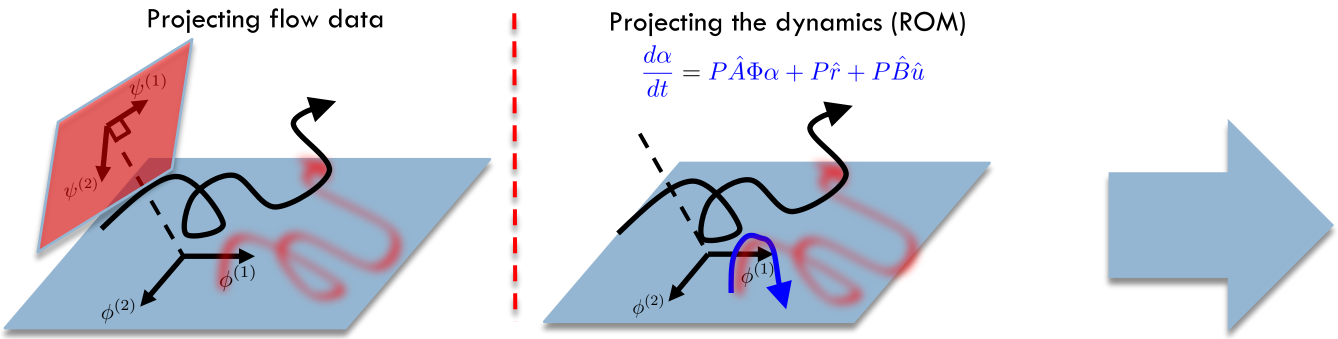
SPOD and DMD

- DMD/Koopman* analysis also extracts temporally coherent structures from data, but does not generally result in a set of ordered, orthogonal modes (lacks optimality)
- The equivalence for stationary data suggests using the SPOD algorithm to compute DMD modes (less noise sensitivity, controllable errors via Welch's method)
- However, DMD/Koopman has other interpretations and uses that you will hear about in subsequent lectures

*Schmidt 2010, Rowley et al. 2009, Tu et al. 2013

POD and reduced-order models

- "Energetic" structures are dynamically significant but their evolution can be influenced by "low energy" structures
- POD/Galerkin ROM are known to be fragile



Better ROM:
Balanced truncation/BPOD*

*Moore 1981, Rowley 2005

Summary

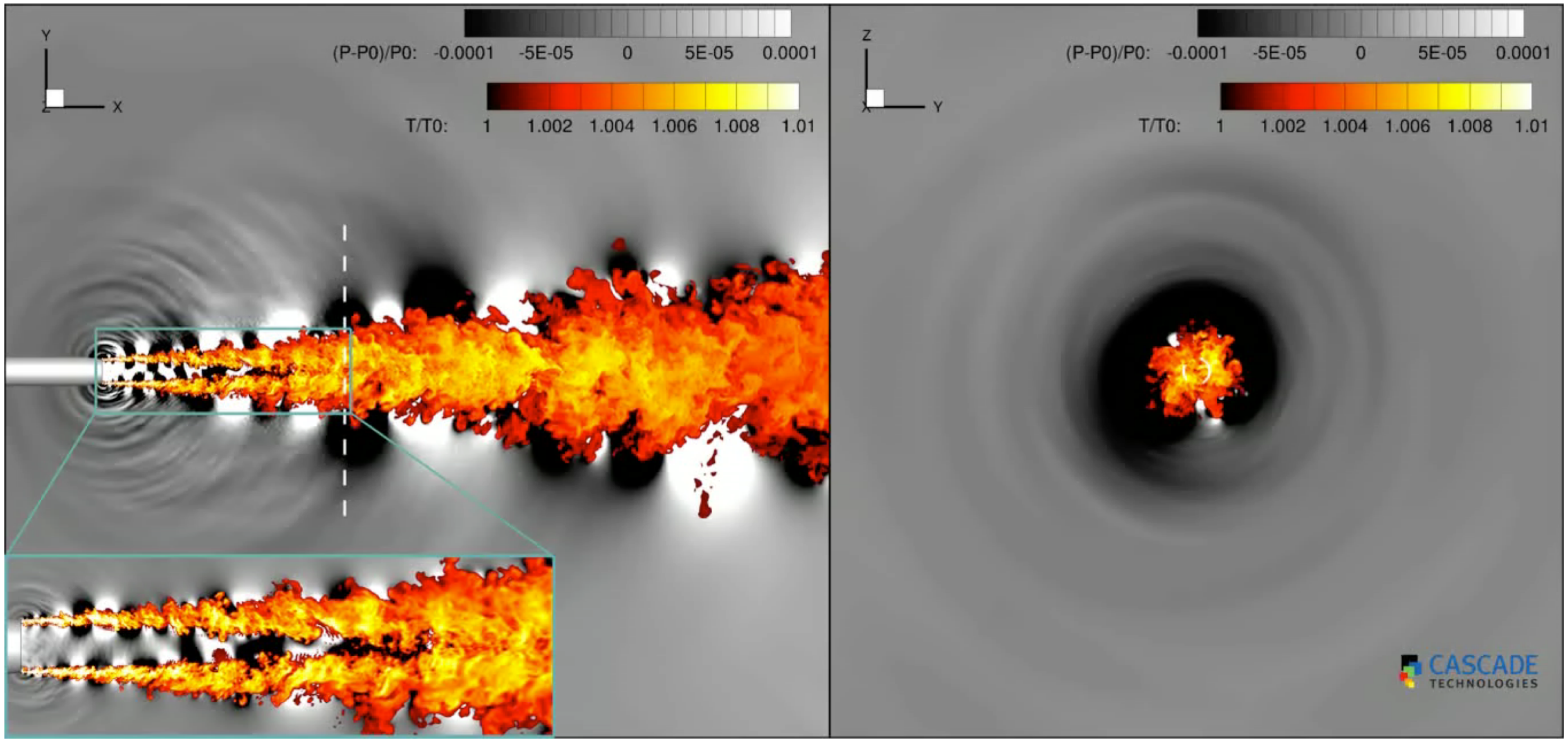
- There are a wide variety of useful and interesting techniques for educing a basis to describe coherent structures and their dynamics in a variety of flows
- Many details and applications of global modes, resolvent modes, DMD (and more) covered in next lectures
- There are many *variations* and *connections* amongst the various decompositions that I have only briefly touched upon here

Part 3: Application to high-speed turbulent jets

- LES databases
- Application of SPOD
- Comparison with POD/DMD
- Resolvent modes
- Eddy-viscosity modeling

Turbulent jet*

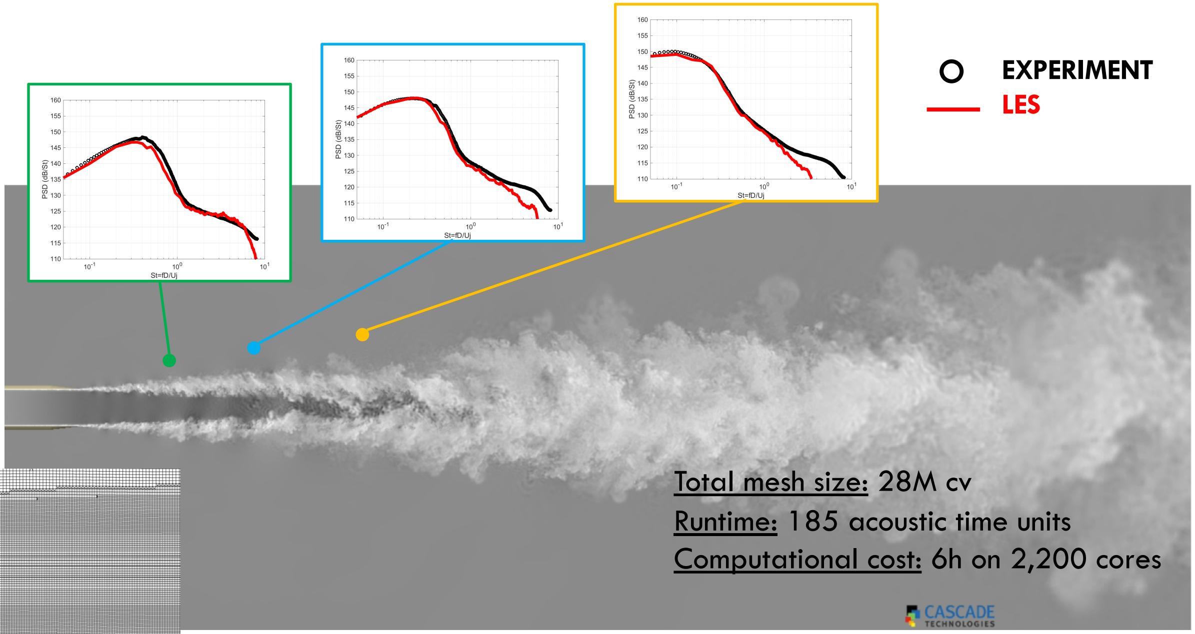
M=0.9   Re=10^6



*Brès et al. 2015, 2017

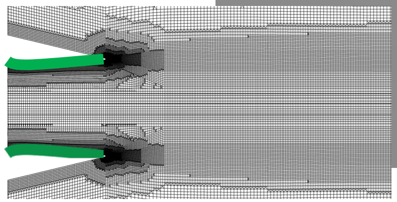
LES databases

- Leverage technology from previous successful NAVAIR/Cascade/Caltech* databases for subsonic and supersonic jets (Charles)
- Very long runs to produce accurate frequency-space statistics
- Detailed validation from companion experiments (UTRC, U. Poitiers,Georgia Tech.)
- Consider first isothermal/cold, “ideally” expanded jets (hot data also available for round nozzles)



Wall modeling

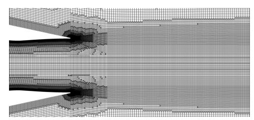
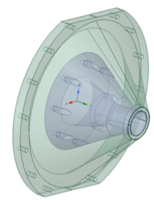
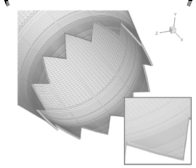
Near-wall adaptive mesh refinement



*STTR Project N68335-11-C-0026, Supervision of Dr. John Spyropoulos

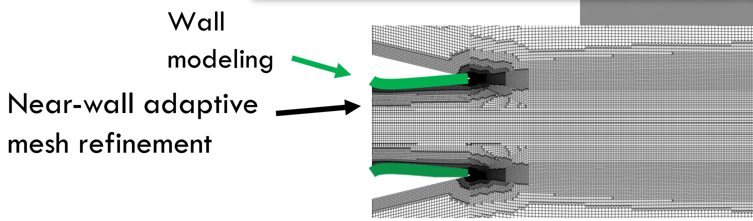
LES databases

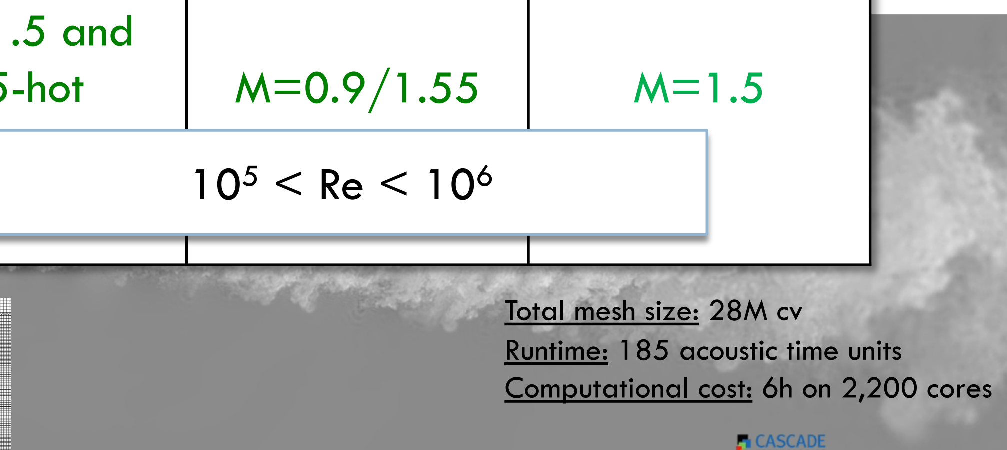
- Leverage technology from previous successful NAVAIR/Cascade/Caltech* databases for subsonic and supersonic jets (Charles)
- Very
- Detai ... Tech.)
- Consi ... ble for round

	Round jet (U. Poitiers)	Coaxial jet (Georgia Tech.)	Complex geometry (UTRC-12)
			
Database	M=0.4, 0.7, 0.9, 1.5 and 1.5-hot	M=0.9/1.55	M=1.5

$10^5 < Re < 10^6$

ERIMENT



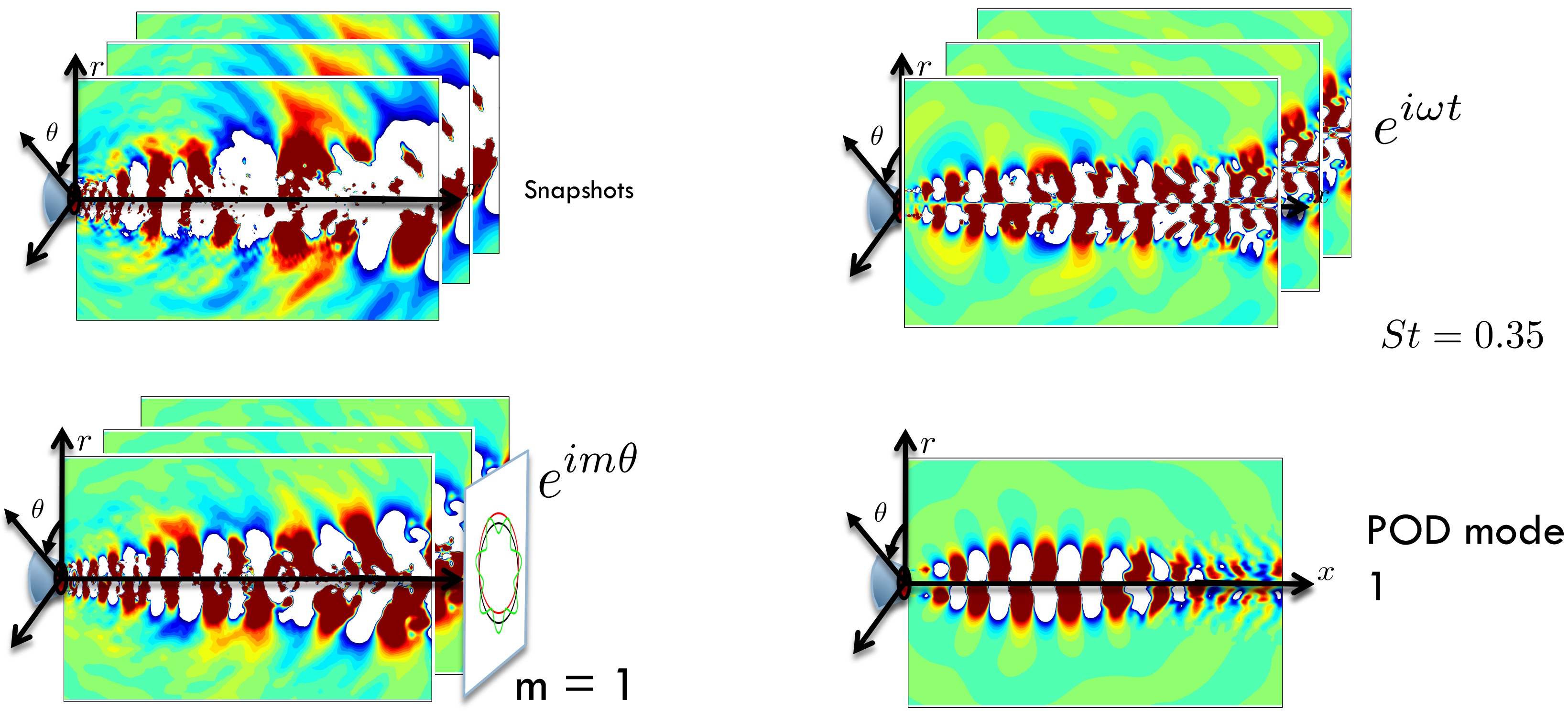


*STTR Project N68335-11-C-0026, Supervision of Dr. John Spyropoulos

SPOD distillation

Contours of pressure (real part)

M=0.9



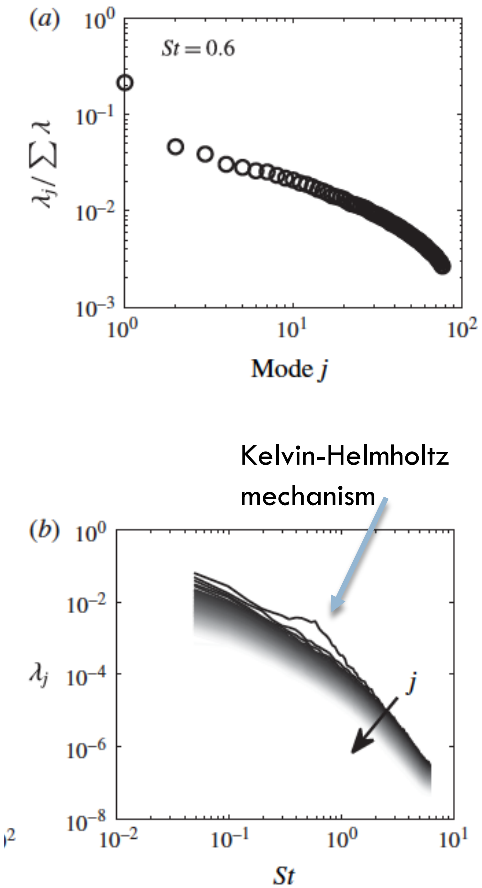
Setting

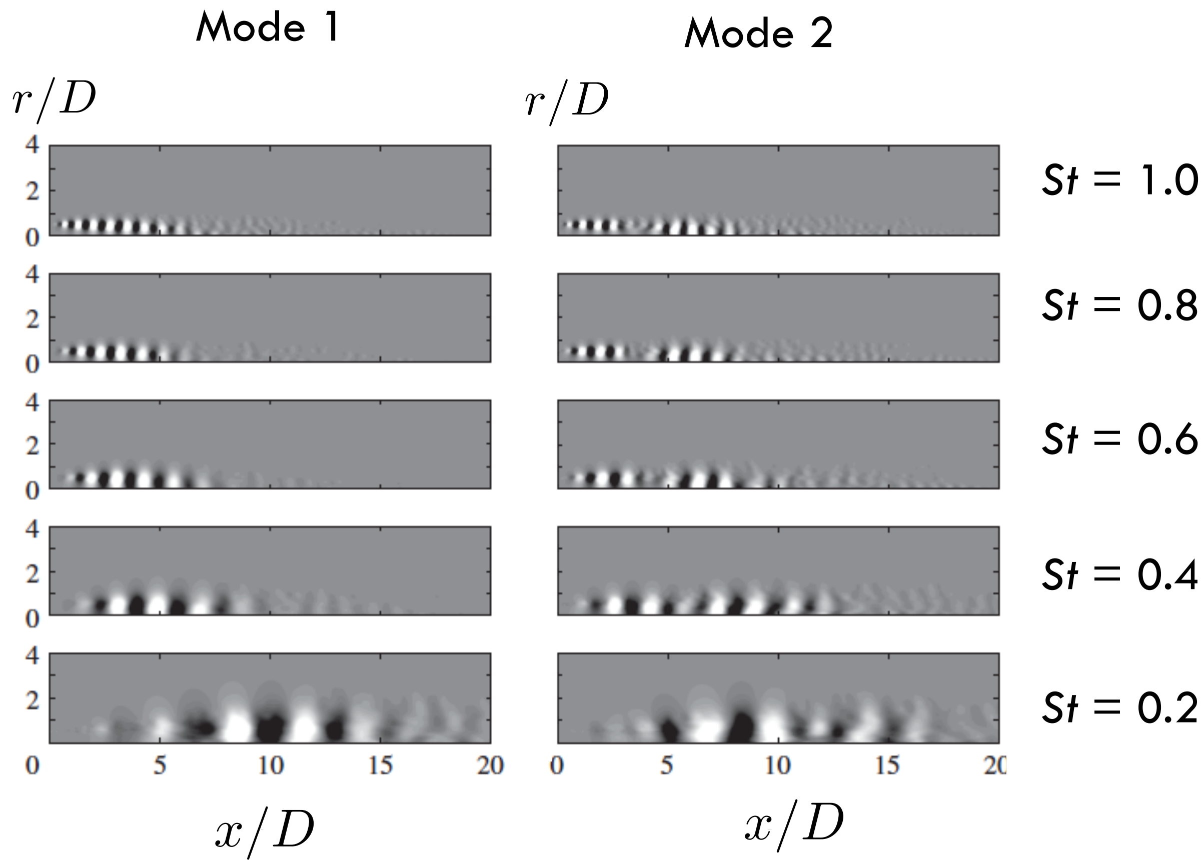
- LES data interpolated onto cylindrical grid
- Full compressible inner product (Chu norm)

Spectral estimation parameters:

- Ntot = 10000
- N_{FFT}=256
- N_{block}=78, 50% overlap
- Hann window

Full SPOD spectrum (m=0, St > 0.05)*

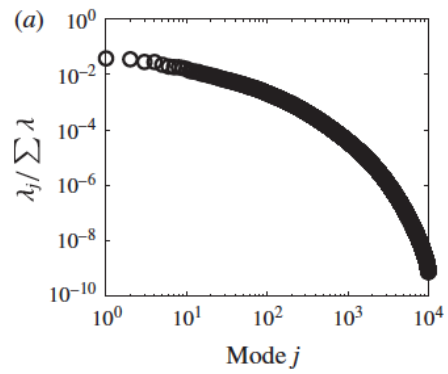


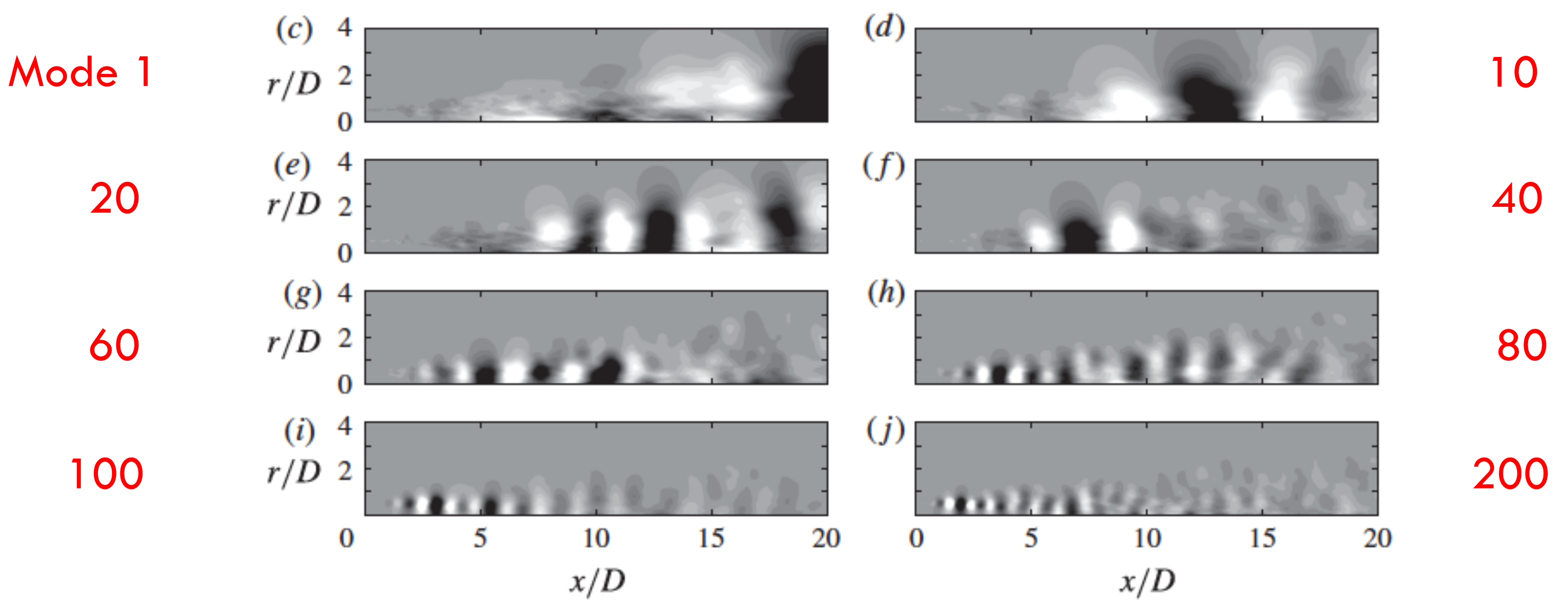


*Schmidt et al. 2018

M=0.4

Comparison with space-only POD*

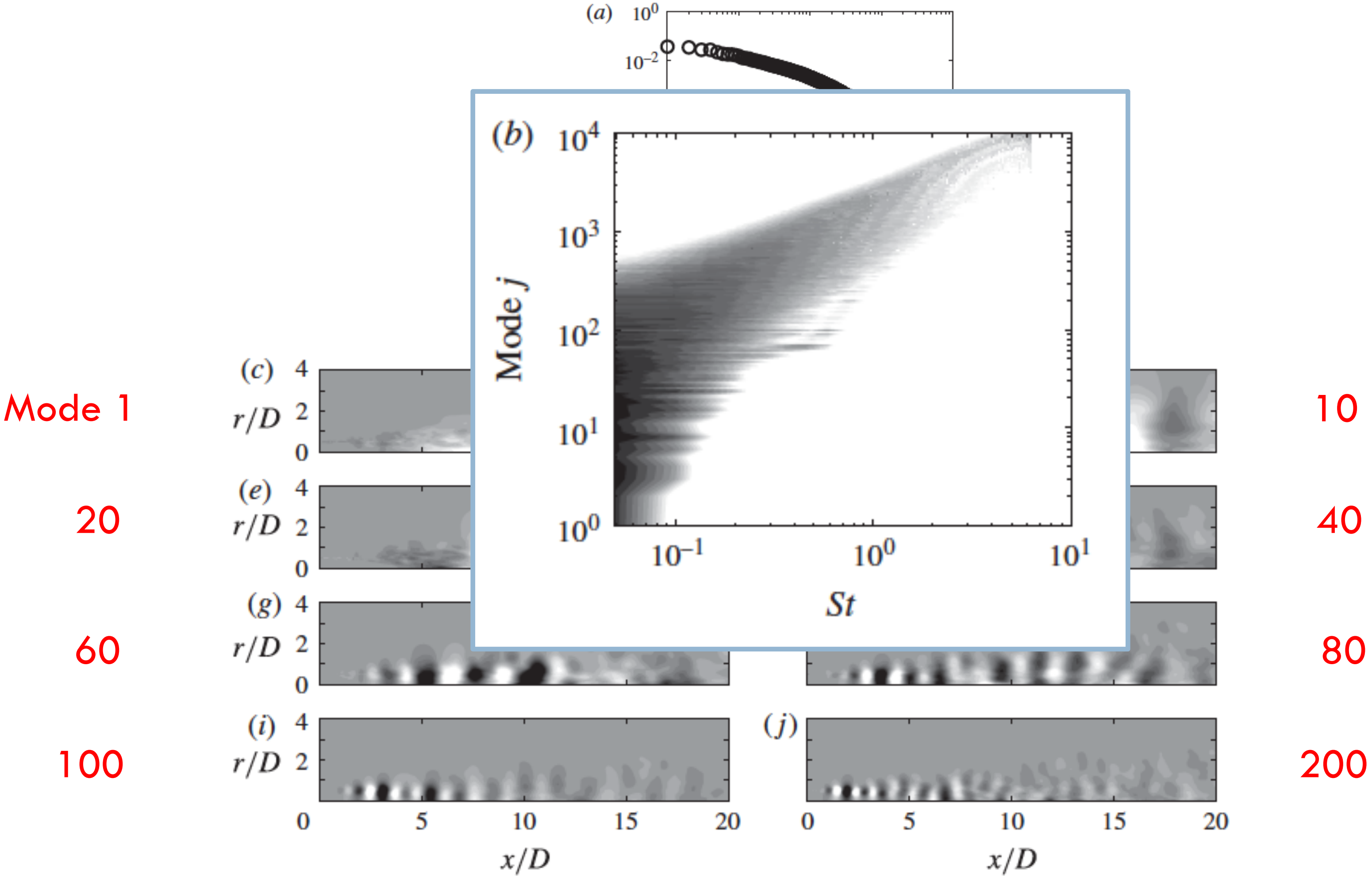




M=0.4

*Towne et al. 2018

Comparison with space-only POD*

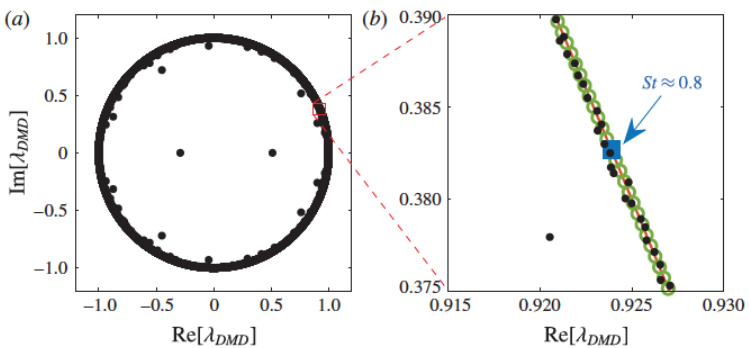


M=0.4

*Towne et al. 2018

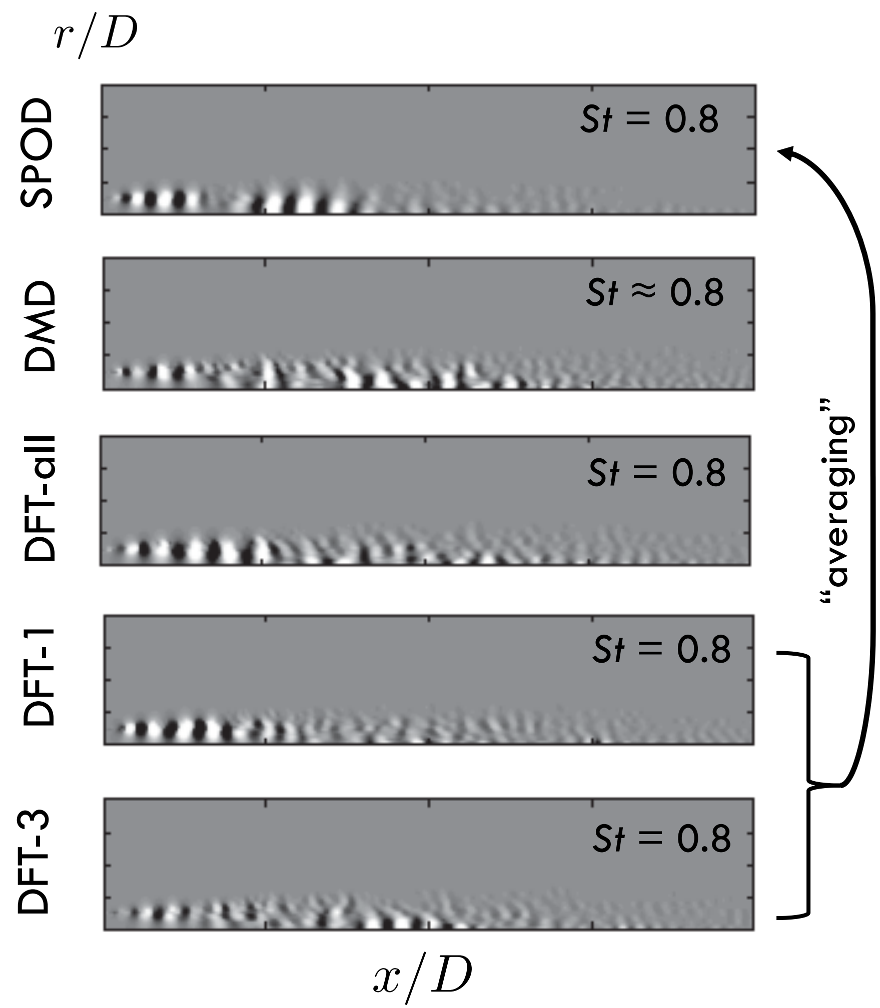
Comparison with DMD*

- Standard DMD: mean not subtracted



- DMD with mean subtracted: DFT [+]

- SPOD modes are 'optimally-averaged' DMD modes

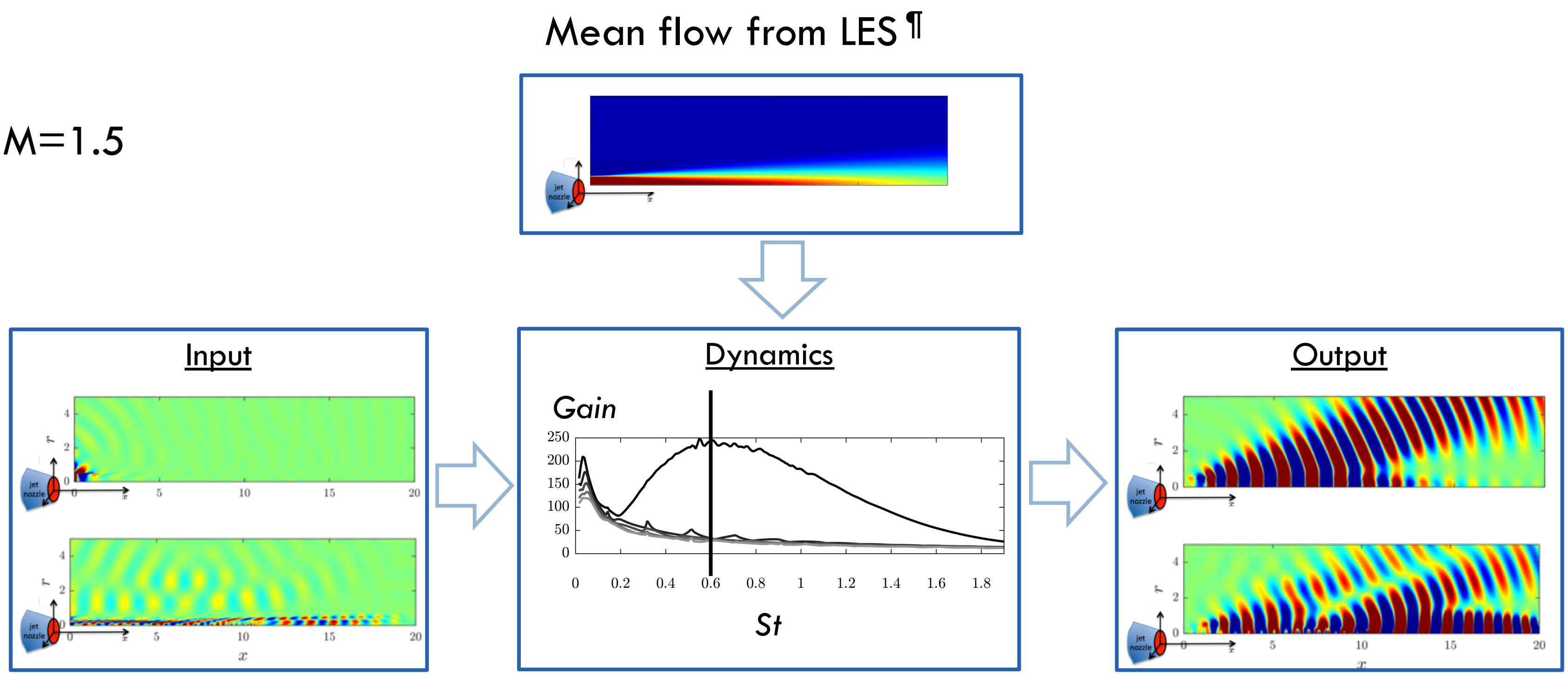


M=0.4

*Towne et al. 2018
[+]Chen et al. 2012

Resolvent analysis*

M=1.5



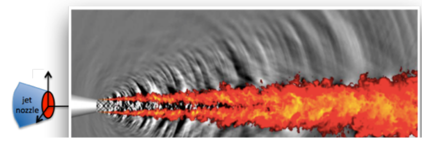
Important note: the resolvent calculations are computed at a reduced Reynolds number 30,000

* Schmidt et. al 2017

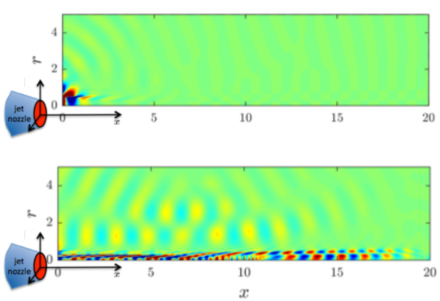
Comparison of SPOD & RA: low rank behavior

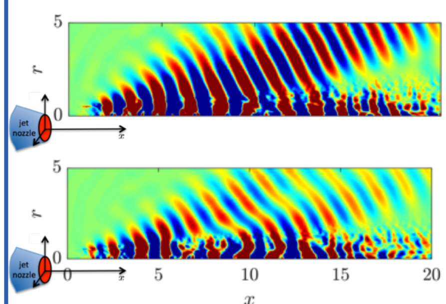
M=1.5

Mean flow from LES ¶

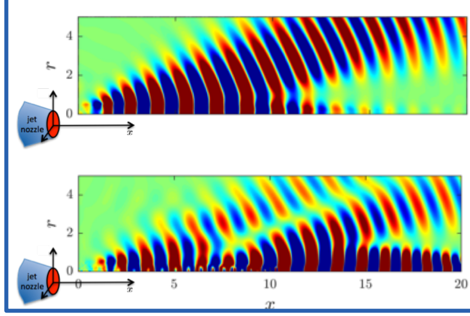


Input

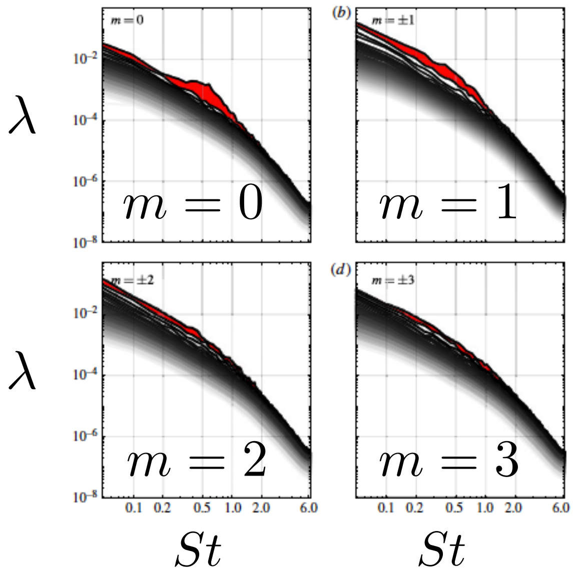


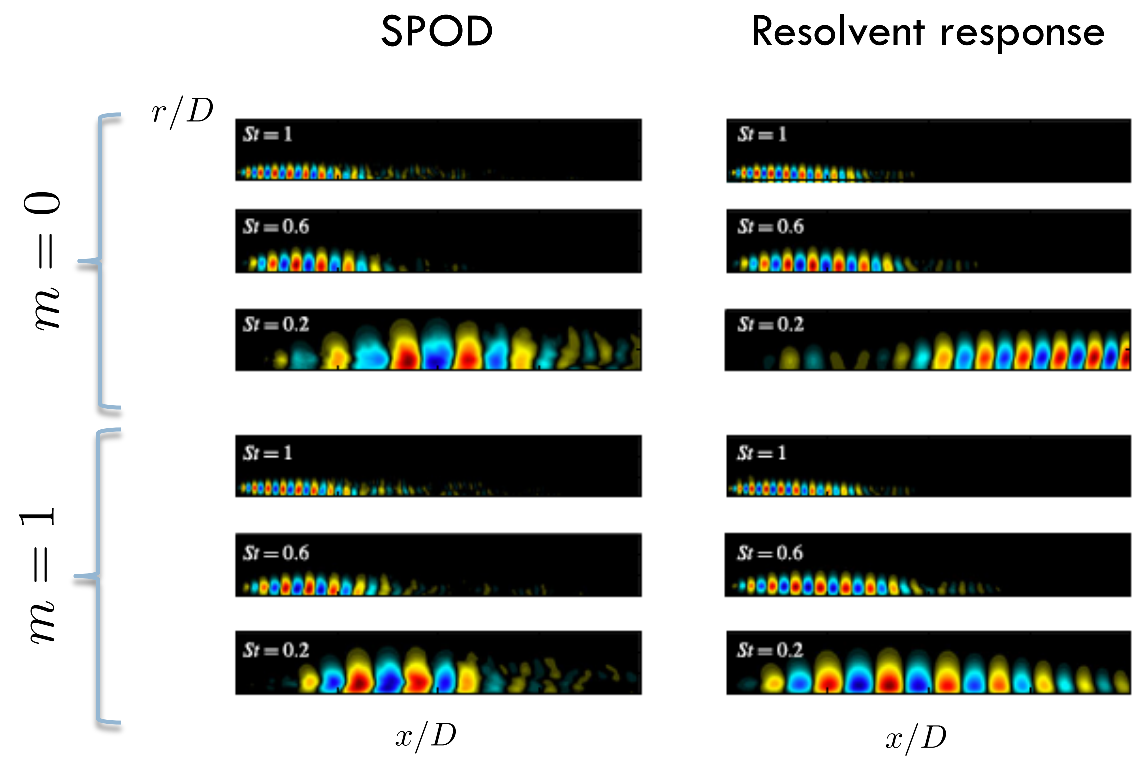


Output

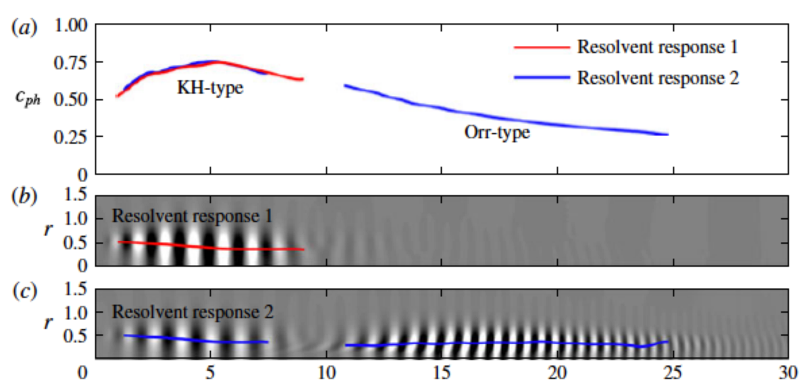


Orr mechanism and non-low-rank behavior*





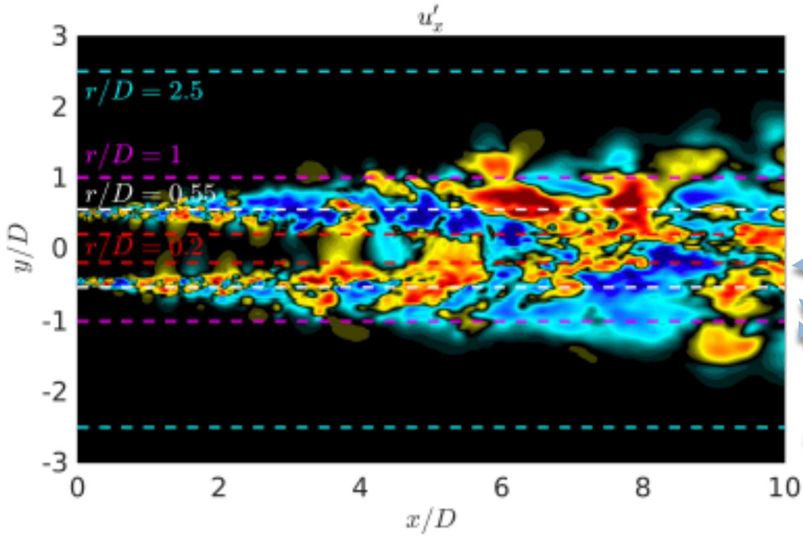
St=0.6

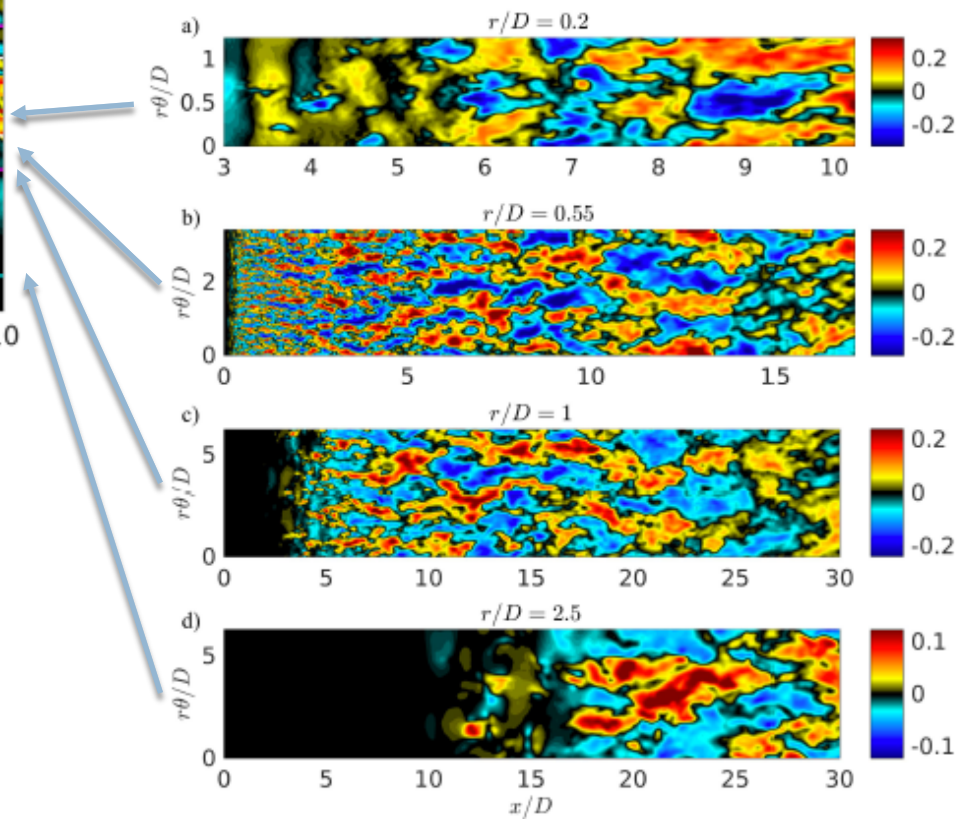


M=0.4

*Schmidt et al. 2018

Instantaneous view: persistent, meandering streaks





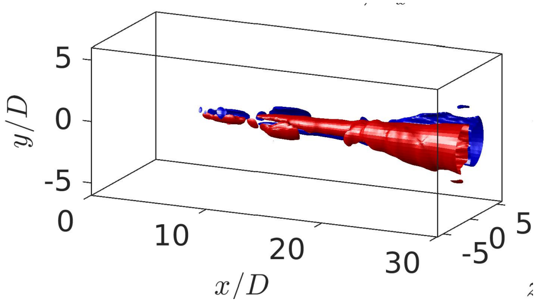
Contours: u'_x

Streaks and lift-up mechanism*

SPOD

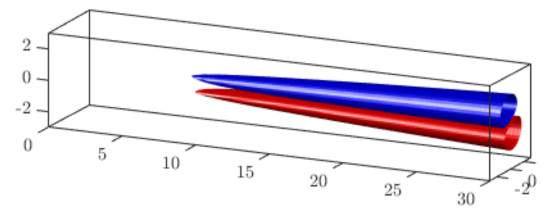
Resolvent

u'_x

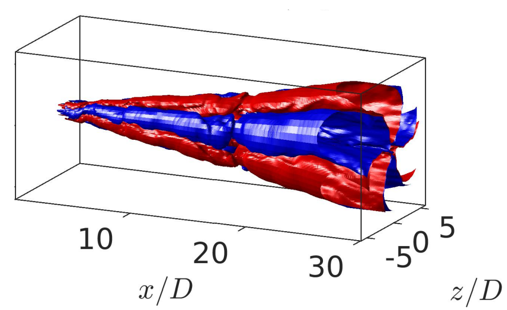


$m = 1$

$St \to 0$

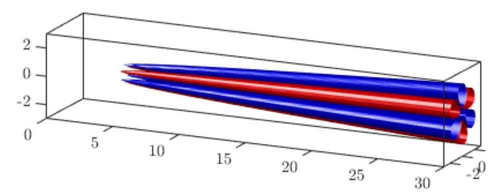


u'_x

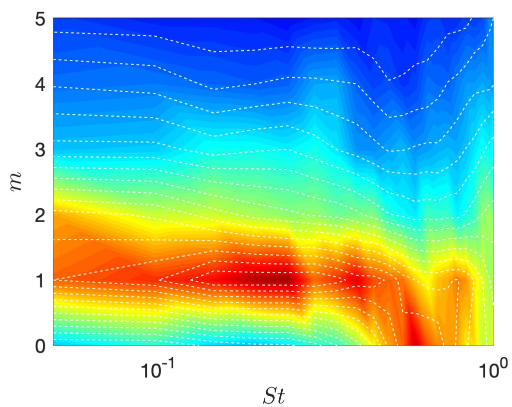


$m = 3$

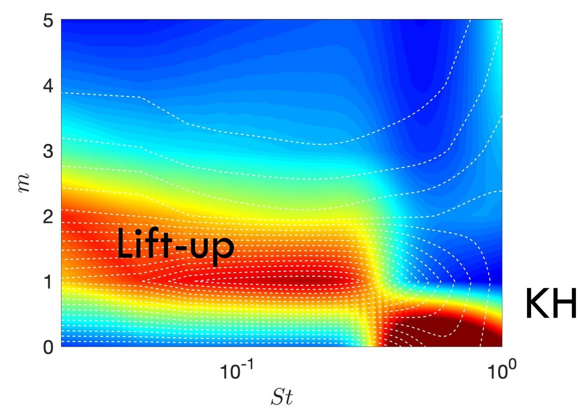
$St \to 0$



$$\frac{\lambda_1}{\sum_j \lambda_j}$$

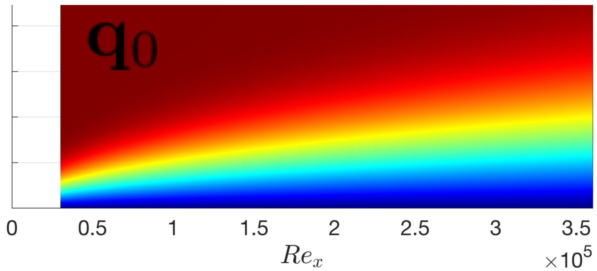


$$\frac{\sigma_1^2}{\sum_j \sigma_j^2}$$

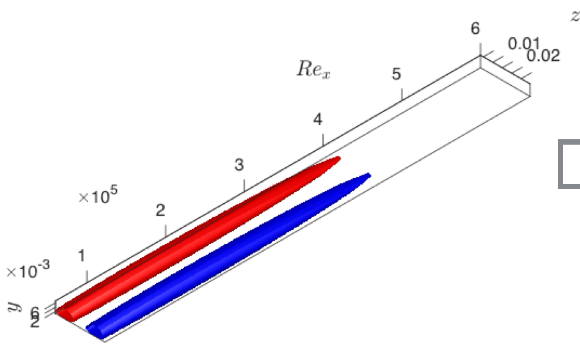


*Rigas et al. 2019 , Pickering et al. 2019

Resolvent of Laminar Blasius Boundary Layer*

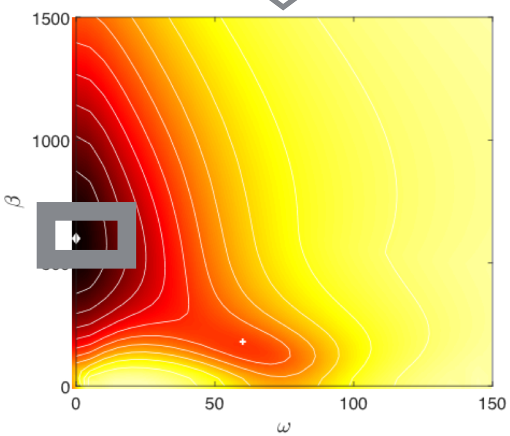


Rolls (v',w')



- 'Worst-Case' Disturbance

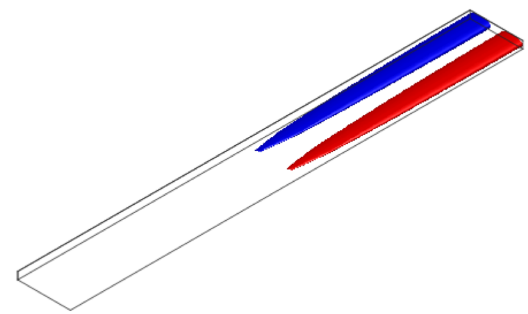
$$U$$



- Gain

$$\Sigma$$

Streaks (u')



- Highest Gain Response

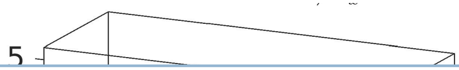
$$V$$

*Monokrousos et al. 2010 (but above result computed in house with RA)

Streaks and lift-up mechanism*

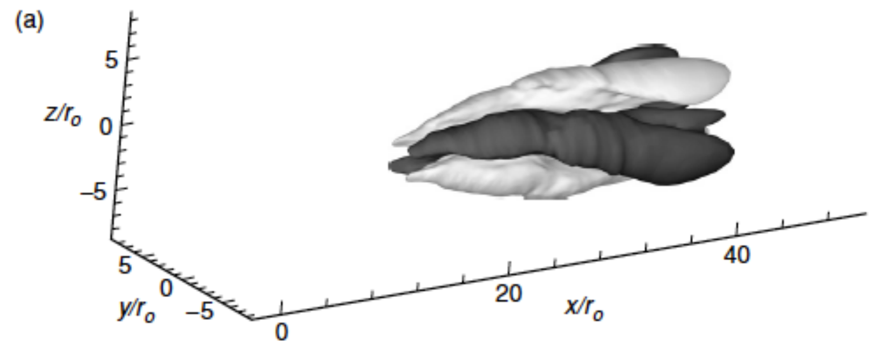
SPOD

Resolvent



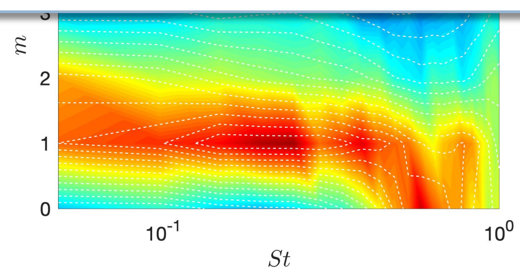
u'_x

u'_x

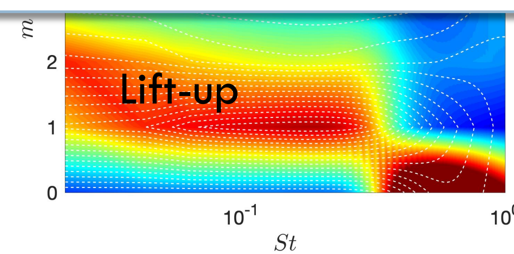


Snapshot POD (TKE) of a low Re turbulent jet (Freund & Colonius 2000)

$$\frac{\lambda_1}{\sum_j \lambda_j}$$



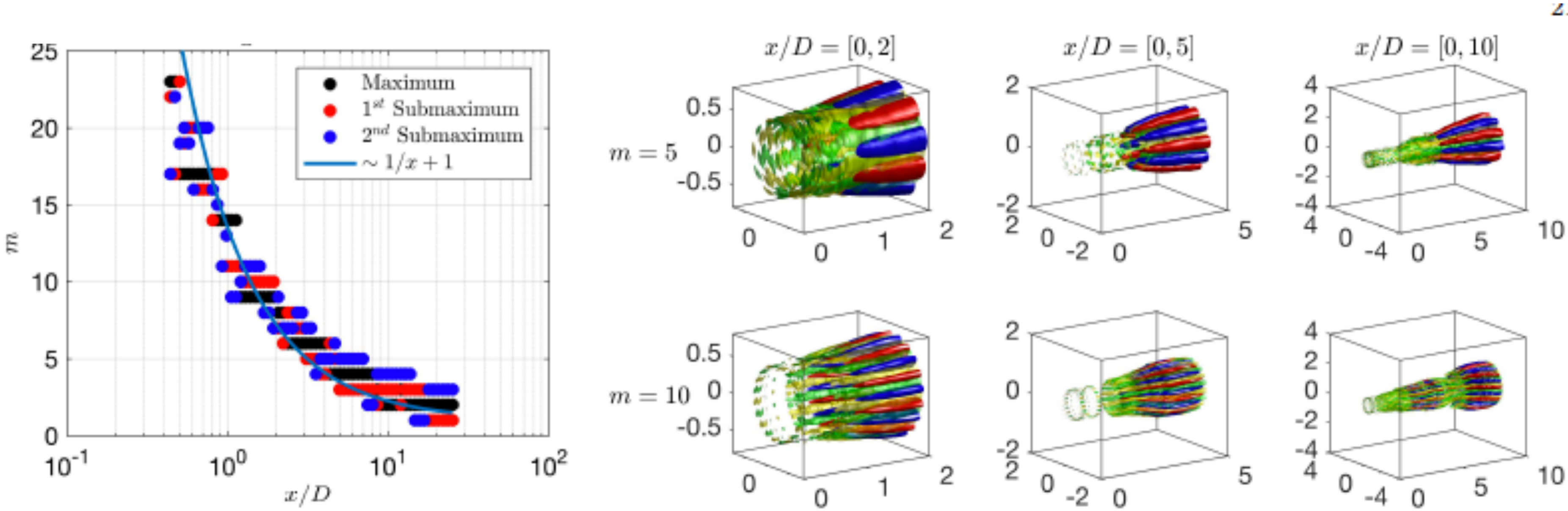
$$\frac{\sigma_1^2}{\sum_j \sigma_j^2}$$



KH

*Rigas et al. 2019, Pickering et al. 2019

Near-nozzle region



Turbulence models in resolvent?

- KH mechanism not sensitive to Re, but remaining results quantitatively (but not qualitatively) changed with increasing Re
- Cannot "resolve" RA spectrum for Re too high
- Resolvent framework doesn't require any closure model, but would one be useful?
- Eddy-viscosity models in mean-flow stability analyses
  - Buffet modes (Crouch et al. 2007, Sartor et al. 2014)
  - Strong swirling jet (Oberleithner et al. 2014, Rukes et al. 2016)
  - Turbulent flow control (Mettot et al. 2014)
  - Fuel Injectors (Tammisola & Juniper 2016)
    - *the choice of the model for turbulent dissipation is less crucial than the choice of including it in improving global mode shapes*
- Eddy viscosity for resolvent
  - Wall bounded flows (Morra et al. 2019)
    - *Greatly improve predictions of spatio-temporal power spectral densities*

Eddy viscosity: data driven approach*

- Navier-Stokes (perturbation form)

$$\hat{\mathbf{q}} = \mathbf{R}\hat{\mathbf{f}} \quad \rightarrow \quad \mathcal{S}_{ff} \rightarrow \mathcal{S}_{qq}$$

- Eddy-viscosity closure for unsteady Reynolds stresses

$$\hat{\mathbf{q}} = \mathbf{R}_{\nu_T}\hat{\mathbf{f}} \quad \rightarrow \quad \mathcal{S}_{ff} \approx \mathbf{\Lambda} \mapsto \mathcal{S}_{qq}$$

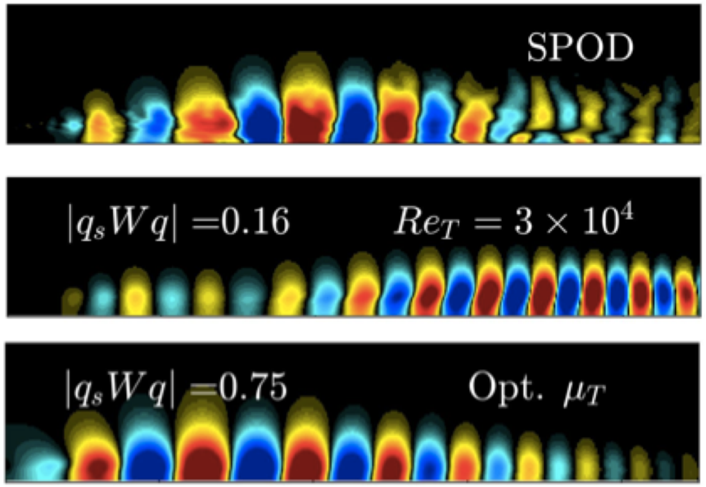
Can we find a closure for ν_T

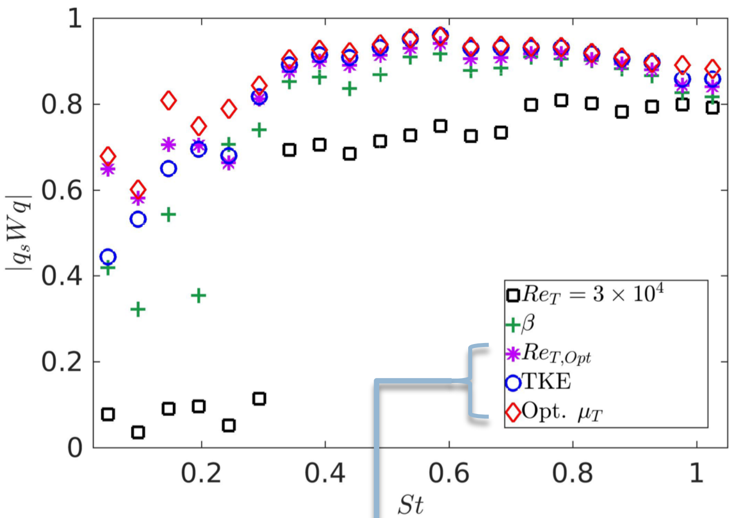
such that we align the (dominant) resolvent mode with the (dominant) SPOD mode?

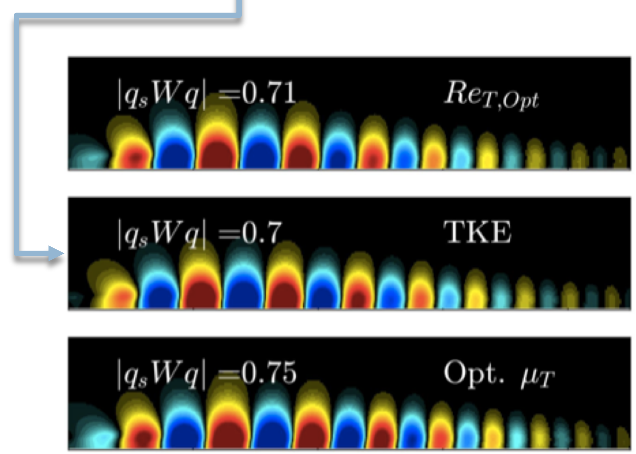
If yes, then we can drive the modeled equations with white noise, and reproduce the (dominant) turbulence structure

*Pickering et al. 2019

Yes!







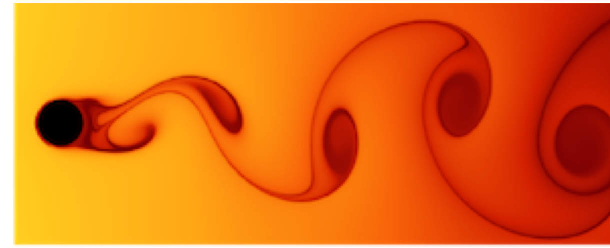
M=0.4

Relatively simple models with 'light' calibration greatly improve alignment

Summary

- SPOD + RA is a powerful combination for data/operator-driven decomposition of turbulent flows
- Identifying/reclassifying instability mechanisms in turbulent jet (KH, Orr, Liftup)
- Fertile ground for discovery
  - Play with the "C" matrix: relation between mechanisms and observable (i.e. far-field sound), reduced-rank models
  - Data/optimization-driven frameworks for modeling of forcing (eddy viscosity is just one way)
  - Self-consistent models

Computational Flow Physics @ Caltech

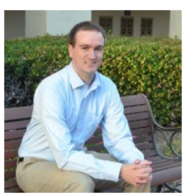
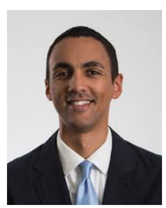


Thank you for your attention!

Special thanks to students/postdocs

Ethan Pickering Georgios Rigas Oliver Schmidt



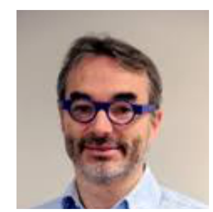



Aaron Towne Daniel Araya

…and collaborators

Andre Cavalieri Peter Jordan





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