DUNE PDELab Tutorial 04

Finite Elements for the Wave Equation

Peter Bastian   Steffen Müthing

Interdisziplinäres Zentrum für Wissenschaftliches Rechnen
Im Neuenheimer Feld 205, D-69120 Heidelberg

August 31, 2021

Example Problem

In this tutorial we solve the wave equation formulated as a first order in time system. This way the example serves as a model for the treatment of systems of partial differential equations in PDELab.

$$
\begin{aligned}
\partial_{tt} u - c^2 \Delta u &= 0 && \text{in } \Omega \times \Sigma, && \text{(1a)}\\
u &= 0 && \text{on } \partial\Omega, && \text{(1b)}\\
u &= q && \text{at } t = 0, && \text{(1c)}\\
\partial_t u &= w && \text{at } t = 0, && \text{(1d)}
\end{aligned}
$$

where c is the speed of sound.

Renaming $u_0 = u$ and introducing $u_1 = \partial_t u_0 = \partial_t u$ we can write the wave equation as a system of two equations:

$$
\begin{aligned}
\partial_t u_1 - c^2 \Delta u_0 &= 0 && \text{in } \Omega \times \Sigma, && \text{(2a)} \\
\partial_t u_0 - u_1 &= 0 && \text{in } \Omega \times \Sigma, && \text{(2b)} \\
u_0 &= 0 && \text{on } \partial\Omega, && \text{(2c)} \\
u_1 &= 0 && \text{on } \partial\Omega, && \text{(2d)} \\
u_0 &= q && \text{at } t = 0, && \text{(2e)} \\
u_1 &= w && \text{at } t = 0. && \text{(2f)}
\end{aligned}
$$

Since $u_0 = u = 0$ on the boundary we also have $\partial_t u = u_1 = 0$ on the boundary. Alternatively, omit the boundary condition on u_1.

Alternative Formulations (I)

Eriksson et al. in [1] apply the Laplacian to equation (2b)

$$\Delta \partial_t u_0 - \Delta u_1 = 0 \tag{3}$$

which has advantages for energy conservation but requires additional smoothness properties.

Alternative Formulations (II)

Introduce the abbreviations $q = \partial_t u$ and $w = -\nabla u$, so $\partial_{tt} u - c^2 \Delta u = \partial_{tt} u - c^2 \nabla \cdot \nabla u = \partial_t q + c^2 \nabla \cdot w = 0$. Taking partial derivatives of the introduced variables we obtain $\partial_{x_i} q = \partial_{x_i} \partial_t u = \partial_t \partial_{x_i} u = -\partial_t w_i$. This results in a first-order hyperbolic system of PDEs for q and w

$$\begin{aligned} \partial_t q + c^2 \nabla \cdot w &= 0 \\ \partial_t w + \nabla q &= 0 \end{aligned}$$

which are called equations of linear acoustics [2]. This formulation is physically more relevant. It can be modified to handle discontinuous material properties and supports upwind finite volume methods.

Weak Formulation

Multiplying (2a) with the test function v_0 and (2b) with the test function v_1 and using integration by parts we arrive at the weak formulation: Find $(u_0(t), u_1(t)) \in U_0 \times U_1$ s.t.

$$\begin{aligned} d_t(u_1, v_0)_{0,\Omega} + c^2(\nabla u_0, \nabla v_0)_{0,\Omega} &= 0 \quad \forall v_0 \in U_0 \\ d_t(u_0, v_1)_{0,\Omega} - (u_1, v_1)_{0,\Omega} &= 0 \quad \forall v_1 \in U_1 \end{aligned} \tag{4}$$

where we used the notation of the L^2 inner product $(u, v)_{0,\Omega} = \int_\Omega uv\, dx$.

An equivalent formulation to (4) that hides the system structure reads as follows:

$$
\begin{aligned}
d_t & \left[(u_0, v_1)_{0,\Omega} + (u_1, v_0)_{0,\Omega}\right] \\
& + \left[c^2(\nabla u_0, \nabla v_0)_{0,\Omega} - (u_1, v_1)_{0,\Omega}\right] = 0 \quad \forall (v_0, v_1) \in U_0 \times U_1
\end{aligned}
\tag{5}
$$

With the latter we readily identify the temporal and spatial residual forms:

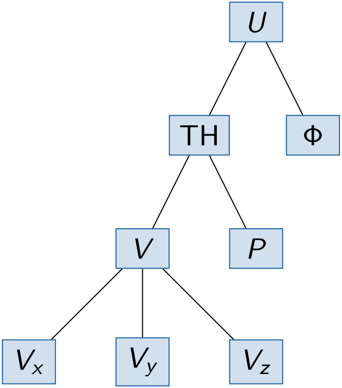
$$
m^{\mathrm{WAVE}}((u_0, u_1), (v_0, v_1)) = (u_0, v_1)_{0,\Omega} + (u_1, v_0)_{0,\Omega}, \tag{6}
$$

$$
r^{\mathrm{WAVE}}((u_0, u_1), (v_0, v_1)) = c^2(\nabla u_0, \nabla v_0)_{0,\Omega} - (u_1, v_1)_{0,\Omega} \,. \tag{7}
$$

Trees of Function spaces

$$U = \left(V(\Omega_S)\right)^d \times P(\Omega_S) \times \Phi(\Omega_D)$$

- Computer science way of representing mathematical expressions: **Trees**
- Expose internal nodes to users
  - Enable recursive bottom-up construction
  - Extract subtrees to pass to legacy subproblem code
- Tree structure mostly static after construction
  - Nodes are C++ templates with children as template arguments
  - Allows extensive compiler optimizations, including inlining of tree traversals



Linear Algebra

Given an assembled residual $r = \mathcal{R}(\vec{u_0})$ and its Jacobian $A = \nabla \mathcal{R}_h$, we have to solve the linear problem

$$Az = r$$

to obtain a correction and calculate $u = u_0 - z$.

Several options

Monolithic solve of $Az = r$

- No stability problems
- Often very difficult with standard iterative solvers

Subdomain Iteration Exploit problem structure

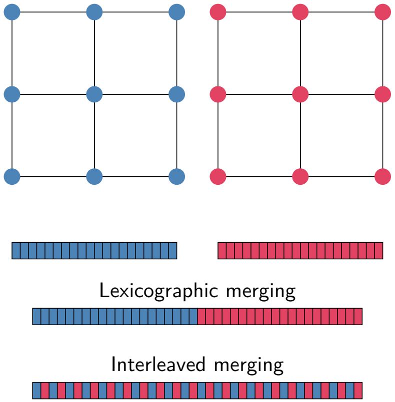
$$\begin{pmatrix} A_L & 0 \\ 0 & A_R \end{pmatrix} \begin{pmatrix} z_L \\ z_R \end{pmatrix} = \begin{pmatrix} r_L \\ r_R \end{pmatrix}$$

- Stability can be problematic
- Does not require monolithic code base
- Matrix / vector data structures must contain structure for good performance

Index Merging – Example

- Two Q_1 spaces on common mesh
- Each space has canonical order defined by vertex iteration
- Two merging strategies
  Lexicographic: Preserve structure of individual problems, separate matrix blocks for coupling
  Interleaved: Regard problem as vector-valued version of scalar problem

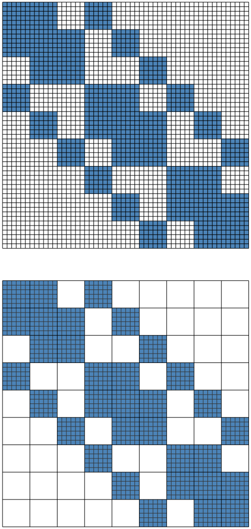
$$U = U_1 \times U_2$$

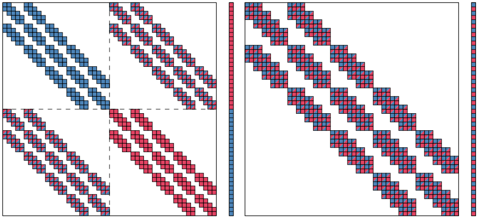


Merging + Blocking

- Merging can be repeated at every tree node

  $\Rightarrow$ recursive construction of index structure from function space structure
- Also support **blocking** during merging
  - Large blocks for extracting subproblem matrices
  - Small blocks for block-aware preconditioners and reduced memory usage





Realization in PDELab

1) The ini-file `tutorial04.ini` holds parameters controlling the execution.
2) Main file `tutorial04.cc` includes the necessary C++, DUNE and PDELab header files; contains `main` function; instantiates DUNE grid objects and calls the `driver` function
3) Function `driver` in file `driver.hh` instantiates the necessary PDELab classes and finally solves the problem.
4) File `wavefem.hh` contains the local operator classes `WaveFEM` and `WaveL2` realizing the spatial and temporal residual forms.

K. Eriksson, D. Estep, P. Hansbo, and C. Johnson.
Computational Differential Equations.
Cambridge University Press, 1996.
http://www.csc.kth.se/~jjan/private/cde.pdf.

R. J. Leveque.
Finite Volume Methods for Hyperbolic Problems.
Cambridge University Press, 2002.