Finite Elements: examples 1

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1. For a general partition $0 = x_0 < x_1 < \ldots < x_n = 1$ of the interval $[0, 1]$, let S be the piecewise linear finite element space known as P1. A finite element discretisation of the 1D Poisson equation
$$-\frac{\partial^2 u}{\partial x^2} = f, \quad u(0) = u(1) = 0,$$
defines the numerical solution $u \in V$ such that
$$a(u, v) = F[v], \quad \forall v \in V,$$
where V is the supspace of P1 satisfying the boundary conditions.
$$a(u, v) = \int_0^1 u'v' \,\mathrm{d}\, x, \quad F[v] = \int_0^1 f v \,\mathrm{d}\, x.$$
Where V Compute the entries of the matrix
$$K_{ij} = a(\phi_i, \phi_j),$$
and the right-hand side vector
$$F_i = (\phi_i, f_I),$$
where $f_I \in S$ is the interpolant of f.
For an equispaced mesh with $h_i = x_i - x_{i-1} := h$, write down the resulting discretisation. How does it relate to finite difference approximations that you have seen before?

2. Using the methodology of the introductory lecture, develop an integral formulation that can be used to build a finite element discretisation for the following ODEs,
   (a)
$$-u'' + u = f, \quad u'(0) = u'(1) = 0.$$
   (b)
$$-u'' + u = f, \quad u'(0) = 0,\, u'(1) = \alpha.$$
   (c)
$$-u'' = f, \quad u'(0) = u'(1) = 0.$$

3. For a general partition $0 = x_0 < x_1 < \ldots < x_n = 1$ of the interval $[0, 1]$, let S be the piecewise quadratic finite element space known as P2, defined by the following:
   (a) $S \subset C^0([0, 1])$.
   (b) For $v \in S$, $v|_{[x_{j-1}, x_j]}$ is a quadratic function of x.
   (c) $v(0) = 0$.

   Find a nodal basis for S, using the nodes for P1 plus nodes at element midpoints $(x_j + x_{j+1})/2$. Evaluate the matrix K for this basis.

4. Show that the dual basis for the cubic Hermite element determines the cubic polynomials.

5. Let $\mathcal{I}_K f$ be the interpolant for a finite element K. Show that $\mathcal{I}_K$ is a linear operator.

6. Let K be a rectangle, Q_2 be the space of biquadratic polynomials, and let $\mathcal{N}$ be the dual basis associated with the vertices, edge midpoints and the centre of the rectangle. Show that $\mathcal{N}$ determines the finite element.

7. For K being the unit square, determine the nodal basis for the element in the previous question.