Finite Elements: examples 3

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February 19, 2018

1. Let V be a discontinuous Lagrange finite element space of degree k defined on a triangulation $\mathcal{T}$ of a domain Ω. Show that functions in V do not have weak derivatives in general.

2. Let Δ be the triangle with vertices (x_i, y_j), (x_{i+1}, y_j), (x_i, y_{j+1}), with $x_i = hi$, $y_j = hj$. Define a transformation g from the reference element K with vertices $(0,0)$, $(1,0)$ and $(0,1)$ to K, and show that

$$\int_\Delta \left|\frac{\partial}{\partial x}(u - \mathcal{I}_\Delta u)\right|^2 \mathrm{d}\,x\,\mathrm{d}\,y = \int_K \left|\frac{\partial \bar{u}}{\partial \xi} - \bar{u}(0,0) + \bar{u}(1,0)\right|^2 \mathrm{d}\,\xi\,\mathrm{d}\,\eta,$$

where $\bar{u} = u \circ g$, ξ and η are the coordinates on K, and $\mathcal{I}_\Delta$ is the interpolation operator from $H^2(\Delta)$ onto linear polynomials defined on Δ.

3. From the previous question, apply integration by parts repeatedly and use the Schwarz inequality to obtain

$$\int_\Delta \left|\frac{\partial}{\partial x}(u - \mathcal{I}_\Delta u)\right|^2 \mathrm{d}\,x\,\mathrm{d}\,y \leq C \int_K \left|\frac{\partial^2 \bar{u}}{\partial \xi^2}\right|^2 + \left|\frac{\partial^2 \bar{u}}{\partial \xi \partial \eta}\right|^2 \mathrm{d}\,\xi\,\mathrm{d}\,\eta.$$

Hence show that

$$\int_\Delta \left|\frac{\partial}{\partial x}(u - \mathcal{I}_\Delta u)\right|^2 \mathrm{d}\,x\,\mathrm{d}\,y \leq Ch^2 \int_\Delta \left|\frac{\partial^2 u}{\partial \xi^2}\right|^2 + \left|\frac{\partial^2 u}{\partial \xi \partial \eta}\right|^2 \mathrm{d}\,\xi\,\mathrm{d}\,\eta.$$

4. Consider a triangulation $\mathcal{T}$ of points x_i and y_j arranged in squares as above, with each square subdivided into two right-angled triangles. Explain how to use this result to obtain

$$\|u - \mathcal{I}_\mathcal{T}\|_E \leq ch|u|_{H^2(\Omega)},$$

where

$$\|f\|_E = \int_\Omega \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \mathrm{d}\,x\,\mathrm{d}\,y, \quad |u|^2_{H^2(\Omega)} = \int_\Omega \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial xy}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 \mathrm{d}\,x\,\mathrm{d}\,y.$$

5. Show that

$$D^\beta Q^k_B f = Q^{k-|\beta|}_B D^\beta f,$$

where Q^l_B is the degree l averaged Taylor polynomial of f, and D^β is the β-th derivative where β is a multi-index.