

# Finite Elements: examples 1

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1. For a general partition  $0 = x_0 < x_1 < \dots < x_n = 1$  of the interval  $[0, 1]$ , let  $S$  be the piecewise linear finite element space known as P1. Compute the entries of the matrix

$$K_{ij} = a(\phi_i, \phi_j),$$

and the right-hand side vector

$$F_i = (\phi_i, f_I),$$

where  $f_I \in S$  is the interpolant of  $f$ .

**Solution:** *The local shape functions on the reference element interval  $[0, 1]$  are*

$$\psi_0(x) = 1 - x, \quad \psi_1(x) = x,$$

and so

$$\psi'_0(x) = -1, \quad \psi'_1(x) = 1.$$

For  $i = j$  we have

$$K_{ii} = \int_0^1 \phi'_i(x) \phi'_i(x) \, dx, \tag{1}$$

$$= \sum_{j=1}^n \int_{x_{j-1}}^{x_j} \phi'_i(x) \phi'_i(x) \, dx, \tag{2}$$

$$= \frac{1}{x_i - x_{i-1}} \int_0^1 (\psi'_1(x'))^2 \, dx' + \frac{1}{x_{i+1} - x_i} \int_0^1 (\psi'_0(x'))^2 \, dx', \tag{3}$$

$$= \frac{1}{h_{i+1}} + \frac{1}{h_i}, \tag{4}$$

where we have used the change of variables

$$x = x_{i-1} + x'(x_i - x_{i-1}).$$

The first term is removed for  $i = n$ . For  $i = j - 1$  we have

$$K_{i,i+1} = \sum_{j=1}^n \int_{x_{j-1}}^{x_j} \phi'_i(x) \phi'_{i+1}(x) \, dx, \tag{5}$$

$$= \frac{1}{x_{i+1} - x_i} \int_0^1 \psi'_0(x') \psi'_1(x') \, dx', \tag{6}$$

$$= -\frac{1}{h_{i+1}}. \tag{7}$$

Similarly, for  $i = j + 1$ , we have

$$K_{i,i-1} = -\frac{1}{h_{i-1}}.$$

Otherwise,  $K_{ij} = 0$ . Similarly,

$$F_i = \int_0^1 \phi_i(x) f_I(x) \, dx, \quad (8)$$

$$= \sum_{j=1}^n \int_{x_{j-1}}^{x_j} \phi_i(x) f_I(x) \, dx, \quad (9)$$

$$= h_i \int_0^1 \psi_1(x') (f_{i-1} + x' (f_i - f_{i-1})) \, dx' + h_{i+1} \int_0^1 \psi_0(x') (f_i + x' (f_{i+1} - f_i)) \, dx' \quad (10)$$

$$= h_i \left( \frac{f_{i-1}}{6} + \frac{f_i}{3} \right) + h_{i+1} \left( \frac{f_i}{3} + \frac{f_{i+1}}{6} \right). \quad (11)$$

For an equispaced mesh with  $h_i = x_i - x_{i-1} := h$ , write down the resulting discretisation. How does it relate to finite difference approximations that you have seen before?

**Solution:** For  $h_i := h$ , we have

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = \frac{f_{i-1}}{6} + \frac{2f_i}{3} + \frac{f_{i+1}}{6},$$

for  $i < n$ . This is the centred difference formula for the second derivative with an averaged value of  $f$  on the right-hand side. The classical centred difference formula is

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f_i.$$

2. Give a weak formulation for the following ODEs,

(a)

$$-u'' + u = f, \quad u'(0) = u'(1) = 0.$$

**Solution:** Let  $b(u, v)$  be

$$b(u, v) = \int_0^1 u' v' + uv \, dx.$$

Define  $V$  by

$$V = \{u \in L^2([0, 1]) : b(u, u) < \infty\}.$$

Then, find  $u \in V$  such that

$$b(u, v) = (f, v), \quad \forall v \in V.$$

(b)

$$-u'' + u = f, \quad u'(0) = 0, \quad u'(1) = \alpha.$$

**Solution:** Under same definitions, find  $u \in V$  such that

$$b(u, v) = (f, v) - v(1)\alpha, \quad \forall v \in V.$$

(c)

$$-u'' = f, \quad u'(0) = u'(1) = 0.$$

**Solution:** Let  $a(u, v)$  be

$$a(u, v) = \int_0^1 u' v' \, dx.$$

Define  $V$  by

$$V = \{u \in L^2([0, 1]) : a(u, u) < \infty\}.$$

Then, find  $u \in V$  such that

$$a(u, v) = (f, v), \quad \forall v \in V.$$

What is wrong with the formulation of the last ODE?

**Solution:** The problem with this formulation is that addition on an arbitrary constant to the solution  $u$  produces another solution, so the problem is not well-posed.

3. For a general partition  $0 = x_0 < x_1 < \dots < x_n = 1$  of the interval  $[0, 1]$ , let  $S$  be the piecewise quadratic finite element space known as P2, defined by the following:

- (a)  $S \subset C^0([0, 1])$ .
- (b) For  $v \in S$ ,  $v|_{[x_{j-1}, x_j]}$  is a quadratic function of  $x$ .
- (c)  $v(0) = 0$ .

Find a nodal basis for  $S$ , using the nodes for P1 plus nodes at element midpoints  $(x_j + x_{j+1})/2$ .

**Solution:** In the reference element  $[0, 1]$ , the nodal points are  $(z_0, z_1, z_2) = (0, 0.5, 1)$ . The nodal basis in the reference element satisfies

$$\psi_i(z_j) = \delta_{ij}.$$

We get

$$\psi_i(x') = \frac{\prod_{j \neq i} (x' - x_j)}{\prod_{j \neq i} (x_i - x_j)},$$

hence,

$$\psi_0(x') = \frac{(x' - 0.5)(x' - 1)}{0.5 \times 1} = 2x'^2 - 3x' + 1, \quad (12)$$

$$\psi_1(x') = \frac{x'(x' - 1)}{-0.5 \times 0.5} = -4x'^2 + 4x', \quad (13)$$

$$\psi_2(x') = \frac{x'(x' - 0.5)}{0.5} = 2x'^2 - x'. \quad (14)$$

Derivatives are

$$\psi'_0(x') = 4x' - 3, \quad (15)$$

$$\psi'_1(x') = -8x' + 4, \quad (16)$$

$$\psi'_2(x') = 4x' - 1. \quad (17)$$

Evaluate the matrix  $K$  for this basis.

**Solution:** The local matrix is

$$\hat{K}_{ij} = \frac{1}{h} \int_0^1 \psi'_i(x) \psi'_j(x) dx,$$

so we have

$$\hat{K}_{00} = \frac{1}{h} \int_0^1 (4x' - 3)^2 dx = \frac{7}{3h},$$

$$\hat{K}_{11} = \frac{1}{h} \int_0^1 (-8x' + 4)^2 dx = \frac{16}{3h},$$

$$\hat{K}_{22} = \frac{1}{h} \int_0^1 (4x' - 1)^2 dx = \frac{7}{3h},$$

$$\hat{K}_{01} = \hat{K}_{10} = \frac{1}{h} \int_0^1 (4x'^2 - 3)(-8x' + 4) dx = -\frac{8}{3h},$$

$$\hat{K}_{02} = \hat{K}_{20} = \frac{1}{h} \int_0^1 (4x'^2 - 3)(4x' - 1) dx = \frac{1}{3h},$$

$$\hat{K}_{12} = \hat{K}_{21} = \frac{1}{h} \int_0^1 (-8x' + 4)(4x' - 1) dx = \frac{-8}{3h}.$$

The local matrix is

$$\hat{K} = \begin{pmatrix} \frac{7}{3h} & -\frac{8}{3h} & \frac{1}{3h} \\ -\frac{8}{3h} & \frac{16}{3h} & \frac{-8}{3h} \\ \frac{1}{3h} & \frac{-8}{3h} & \frac{7}{3h} \end{pmatrix}.$$

Hence, if we adopt a global node numbering such that odd numbered nodes are located at subinterval midpoints, and even numbered nodes are located at subinterval boundaries, then for odd  $i$ ,

$$K_{ii} = \frac{16}{3h_{i/2}}, \quad K_{i,i+1} = K_{i+1,i} = K_{i,i-1} = K_{i-1,i} = \frac{-8}{3h_{i/2}}.$$

For even  $i$ ,

$$\begin{aligned} K_{ii} &= \frac{7}{3} \left( \frac{1}{h_{i/2}} + \frac{1}{h_{i/2+1}} \right), \\ K_{i,i-1} &= -\frac{8}{3} \frac{1}{h_{i/2}}, \\ K_{i,i-2} &= \frac{1}{3} \frac{1}{h_{i/2}}, \\ K_{i,i+1} &= -\frac{8}{3} \frac{1}{h_{i/2+1}}, \\ K_{i,i+2} &= \frac{1}{3} \frac{1}{h_{i/2+1}}. \end{aligned}$$

4. Under the same assumptions as Theorem 1.8, prove that

$$\|u - u_I\| \leq Ch^2 \|u''\|.$$

(hint: make use of the fact that  $u(0) = 0$  to write  $u$  in terms of  $u'$ .)

**Solution:** We aim to show that

$$\int_{x_{j-1}}^{x_j} (u - u_I)(x)^2 dx \leq c(x_j - x_{j-1})^4 \int_{x_{j-1}}^{x_j} u''(x)^2 dx,$$

since summing over elements will give the result. After a change of variables to the unit interval, this is equivalent to

$$\int_0^1 w(x) dx \leq c \int_0^1 w''(x) dx,$$

for  $w(0) = w(1) = 0$ . To show this, we write

$$w(x) = \int_0^x w'(t) dt,$$

and therefore, from Rolle's theorem, there exists  $0 \leq \xi \leq 1$  such that

$$w(x) = \int_0^x \int_\xi^y w''(t) dt dy.$$

Then, by Schwartz's inequality,

$$\begin{aligned} |w(x)| &\leq \int_0^x \left| \int_\xi^y 1 dt \right|^{1/2} \left| \int_\xi^y w''(t)^2 dt \right|^{1/2} dy, \\ &\leq \int_0^x |y - \xi|^{1/2} \left| \int_0^1 w''(t)^2 dt \right|^{1/2} dy, \\ &\leq \frac{x}{2} \left| \int_0^1 w''(t)^2 dt \right|^{1/2}. \end{aligned}$$

Squaring and integrating gives

$$\int_0^1 w(x)^2 dx \leq \frac{1}{4} \int_0^1 w''(x)^2 dx,$$

and we obtain the required result with  $c = 1/4$ .

5. Under the same assumptions as Theorem 1.8, prove the following version of *Sobolev's inequality*:

$$\|v\|_{\max}^2 \leq Ca(v, v), \quad \forall v \in V \cap C^1([0, 1]).$$

Give a value for  $C$ .

**Solution:** Since  $v \in V$ , we have  $v(0) = 0$ . Since  $v \in C^1([0, 1])$ , we have

$$v(x) = \int_0^x v'(y) \, dy.$$

From the Schwartz inequality, we have

$$\begin{aligned} |v(x)| &\leq \left| \int_0^x 1 \, dx \right|^{1/2} \left| \int_0^x (v'(x))^2 \, dx \right|^{1/2}, \\ &\leq x^{1/2} \left| \int_0^1 (v'(x))^2 \, dx \right|^{1/2}, \end{aligned}$$

and so

$$\max_{0 \leq x \leq 1} |v(x)| \leq c \left| \int_0^1 (v'(x))^2 \, dx \right|^{1/2},$$

with  $c = 1$ .

6. Show that the dual basis for the cubic Hermite element determines the cubic polynomials.

**Solution:** Let  $P$  be a cubic polynomial on the triangle  $K$  such that  $N_i(P) = 0$ ,  $i = 1, 2, 3, 4, 5, 6$ . Let  $L_1$ ,  $L_2$  and  $L_3$  be the three edges of the triangle opposite vertices  $z_1$ ,  $z_2$  and  $z_3$  respectively. Restricting  $P$  to  $L_1$ , we find that  $P(z_0) = P'(z_0) = P(z_1) = P'(z_1)$  where  $'$  indicates differentiation in the direction of  $L_1$ . Therefore  $P|_{L_1}$  has double roots at  $z_0$  and  $z_1$ , and so  $P|_{L_1} = 0$ . Similar argument for the other three edges means that  $P = cL_1L_2L_3$  for constant  $c$ . But,  $P$  also vanishes at the midpoint  $z_4$ , so  $c = 0$ .

7. Let  $\mathcal{I}_K f$  be the interpolant for a finite element  $K$ . Show that  $\mathcal{I}_K$  is a linear operator.

**Solution:**

$$\begin{aligned} \mathcal{I}_K(f + \alpha g) &= \sum_{i=1}^n \phi_i(x)(f + \alpha g)(x_i), \\ &= \sum_{i=1}^n \phi_i(x)f(x_i) + \alpha \sum_{i=1}^n \phi_i(x)g(x_i), \\ &= \mathcal{I}_K f + \alpha \mathcal{I}_K g. \end{aligned}$$

8. Let  $K$  be a rectangle,  $Q_2$  be the space of biquadratic polynomials, and let  $\mathcal{N}$  be the dual basis associated with the vertices, edge midpoints and the centre of the rectangle. Show that  $\mathcal{N}$  determines the finite element.

**Solution:** Let  $z_i$ ,  $i = 1, 2, 3, 4$  be the rectangle vertices,  $z_i$ ,  $i = 5, 6, 7, 8$  be the edge midpoints, and  $z_9$  be the rectangle centre. Let  $L_1$ ,  $L_2$ ,  $L_3$  and  $L_4$  be the linear functions determining the top, bottom, left and right rectangle edges respectively. Then  $L_1$  and  $L_2$  are functions of  $x$  only, whilst  $L_3$  and  $L_4$  are functions of  $y$  only. If  $P$  is biquadratic, then  $P$  restricted to  $L_1$  is quadratic, and vanishes at three points on  $L_1$ . Therefore,  $P|_{L_1} = 0$ . The same reasoning for the other three edges means that we have

$$P = cL_1L_2L_3L_4,$$

where  $c$  must be constant since  $L_1L_2L_3L_4$  is biquadratic. Since  $P$  also vanishes at the rectangle centre, we conclude that  $P = 0$  as required.

9. For  $K$  being the unit square, determine the nodal basis for the element in the previous question.

**Solution:** The local basis functions are determined as products

$$\Phi_{3i+j}(x, y) = \phi_i(x)\phi_j(y), \quad 0 \leq i, j \leq 2,$$

where  $\phi_i(x)$  is the 1D quadratic Lagrange basis function.