Finite Elements: examples 2

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1. Suppose that Ω is any bounded domain, $k, m > 0$ integers with $k \leq m$. Show that $H^m(\Omega) \subset H^k(\Omega)$.
Solution: *If $f \in H^m(\Omega)$ then*
$$\|f\|^2_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha_w f\|^2_{L^2} \leq \infty.$$
Since, $m \geq k$, we have
$$\|f\|^2_{H^k(\Omega)} = \sum_{|\alpha|\leq k} \|D^\alpha_w f\|^2_{L^2} \leq \sum_{|\alpha|\leq m} \|D^\alpha_w f\|^2_{L^2} \leq \infty,$$
so $f \in H^k(\Omega)$.

2. Let α be an arbitrary multi-index, $\psi \in C^{|\alpha|}(\Omega)$. Show that $D^\alpha_w \psi = D^\alpha \psi$.
Solution: *Taking $\phi \in C^\infty_0(\Omega)$, we have*
$$\begin{aligned}\int_\Omega \phi D^\alpha \psi \,\mathrm{d}\,x &= (-1)^{|\alpha|} \textstyle\int_\Omega D^\alpha \phi \psi \,\mathrm{d}\,x, \\ &= (-1)^{|\alpha|} \textstyle\int_\Omega D^\alpha_w \phi \psi \,\mathrm{d}\,x,\end{aligned}$$
for all test functions ϕ. Further, since $D^\alpha_w \phi$ is continuous, it is bounded in all closed domains K contained in the interior of Ω, i.e. $D^\alpha_w \phi \in L^1_{loc}(\Omega)$ as required.

3. Let V be a discontinuous Lagrange finite element space of degree k defined on a triangulation $\mathcal{T}$ of a domain Ω. Show that functions in V do not have weak derivatives in general.
Solution: *Choose a triangle $K_0 \in \mathcal{T}$, and define $u \in V$ as*
$$u(x) = \begin{cases} 1 & \textit{if } x \in K_0, \\ 0 & \textit{otherwise.} \end{cases}$$
Then if $D^x_w u$ exists,
$$\begin{aligned}\int_\Omega D^x_w u \phi \,\mathrm{d}\,x &= -\int_\Omega \phi_x u \,\mathrm{d}\,x, \\ &= -\int_{K_0} \phi_x \,\mathrm{d}\,x, \\ &= -\int_{\partial K_0} \phi n_1 \,\mathrm{d}\,S,\end{aligned}$$
where n_1 is the x-component of the outward pointing normal n to ∂K_0

4. Let Δ be the triangle with vertices (x_i, y_j), (x_{i+1}, y_j), (x_i, y_{j+1}), with $x_i = hi$, $y_j = hj$. Define a transformation g from the reference element K with vertices $(0,0)$, $(1,0)$ and $(0,1)$ to K, and show that
$$\int_\Delta \left|\frac{\partial}{\partial x}(u - \mathcal{I}_\Delta u)\right|^2 \mathrm{d}\,x\,\mathrm{d}\,y = \int_K \left|\frac{\partial \bar{u}}{\partial \xi} - \bar{u}(0,0) + \bar{u}(1,0)\right|^2 \mathrm{d}\,\xi\,\mathrm{d}\,\eta,$$
where $\bar{u} = u \circ g$, ξ and η are the coordinates on K, and $\mathcal{I}_\Delta$ is the interpolation operator from $H^2(\Delta)$ onto linear polynomials defined on Δ.
Solution: *The mapping is defined by*
$$x = x_i + \xi h, \quad y = y_i + \eta h.$$

Defining $\bar{u}(\xi,\eta) = u(x,y)$, we have

$$\frac{\partial \bar{u}}{\partial \xi} = \frac{1}{h}\frac{\partial u}{\partial x}, \qquad \frac{\partial \bar{u}}{\partial \eta} = \frac{1}{h}\frac{\partial u}{\partial y},$$

and the Jacobian of the mapping is

$$|J| = \left|\frac{\partial(x,y)}{\partial(\xi,\eta)}\right| = h^2.$$

We have

$$\mathcal{I}_\Delta u \circ g = (1-\xi-\eta)\bar{u}(0,0) + \xi\bar{u}(1,0) + \eta\bar{u}(0,1).$$

Hence

$$\left(\frac{\partial}{\partial x}\mathcal{I}_\Delta u\right)\circ g = \frac{-\bar{u}(0,0)+\bar{u}(1,0)}{h}.$$

Substitution gives the result.

5. From the previous question, apply integration by parts repeatedly and use the Schwarz inequality to obtain

$$\int_\Delta \left|\frac{\partial}{\partial x}(u-\mathcal{I}_\Delta u)\right|^2 \mathrm{d}\,x\,\mathrm{d}\,y \le C\int_K \left|\frac{\partial^2\bar{u}}{\partial\xi^2}\right|^2 + \left|\frac{\partial^2\bar{u}}{\partial\xi\partial\eta}\right|^2 \mathrm{d}\,\xi\,\mathrm{d}\,\eta.$$

Solution:

$$\begin{aligned}
\int_K \left|\frac{\partial\bar{u}}{\partial\xi} - \bar{u}(0,0)+\bar{u}(1,0)\right|^2 \mathrm{d}\,\xi\,\mathrm{d}\,\eta \;&\le\; \int_K \left|\frac{\partial\bar{u}}{\partial\xi}(\xi,\eta) - \int_0^1 \frac{\partial\bar{u}}{\partial\xi}(\sigma,0)\,\mathrm{d}\,\sigma\right|^2 \mathrm{d}\,\xi\,\mathrm{d}\,\eta \\
&= \int_{\xi=0}^1\int_{\eta=0}^{\xi}\left|\int_0^1\left(\frac{\partial\bar{u}}{\partial\xi}(\xi,\eta) - \frac{\partial\bar{u}}{\partial\xi}(\sigma,\eta)\right)\mathrm{d}\,\sigma \right.\\
&\qquad \left. + \int_0^1\left(\frac{\partial\bar{u}}{\partial\xi}(\sigma,\eta) - \frac{\partial\bar{u}}{\partial\xi}(\sigma,0)\right)\mathrm{d}\,\sigma\right|^2 \mathrm{d}\,\eta\,\mathrm{d}\,\xi \\
&= \int_{\xi=0}^1\int_{\eta=0}^{\xi}\left|\int_0^1\int_\sigma^\xi \frac{\partial^2\bar{u}}{\partial\xi^2}(\gamma,\eta)\,\mathrm{d}\,\gamma\,\mathrm{d}\,\sigma \right.\\
&\qquad \left. + \int_0^1\int_0^\eta \frac{\partial^2\bar{u}}{\partial\xi\eta}(\sigma,\alpha)\,\mathrm{d}\,\alpha\,\mathrm{d}\,\sigma\right|^2 \mathrm{d}\,\eta\,\mathrm{d}\,\xi \\
&\le \int_{\xi=0}^1\int_{\eta=0}^{\xi}\left(\int_0^1\left|\int_\sigma^\xi \frac{\partial^2\bar{u}}{\partial\xi^2}(\gamma,\eta)\,\mathrm{d}\,\gamma\right|^2\mathrm{d}\,\sigma \right.\\
&\qquad \left. + \int_0^1\left|\int_0^\eta \frac{\partial^2\bar{u}}{\partial\xi\partial\eta}(\sigma,\alpha)\,\mathrm{d}\,\alpha\right|^2\mathrm{d}\,\sigma\right)\mathrm{d}\,\eta\,\mathrm{d}\,\xi \\
&\le \int_{\xi=0}^1\int_{\eta=0}^{\xi}\left(\int_0^1 |\xi-\sigma|\int_\sigma^\xi \left|\frac{\partial^2\bar{u}}{\partial\xi^2}(\gamma,\eta)\right|^2\mathrm{d}\,\gamma\,\mathrm{d}\,\sigma \right.\\
&\qquad \left. + \int_0^1 |\eta|\int_0^\eta \left|\frac{\partial^2\bar{u}}{\partial\xi\partial\eta}(\sigma,\alpha)\right|^2\mathrm{d}\,\alpha\,\mathrm{d}\,\sigma\right)\mathrm{d}\,\eta\,\mathrm{d}\,\xi
\end{aligned}$$

$$\le \;\; C\int_K \left|\frac{\partial^2\bar{u}}{\partial\xi^2}\right|^2 + \left|\frac{\partial^2\bar{u}}{\partial\xi\partial\eta}\right|^2 \mathrm{d}\,\xi\,\mathrm{d}\,\eta.$$

Hence show that

$$\int_\Delta \left|\frac{\partial}{\partial x}(u-\mathcal{I}_\Delta u)\right|^2 \mathrm{d}\,x\,\mathrm{d}\,y \le Ch^2\int_\Delta \left|\frac{\partial^2 u}{\partial\xi^2}\right|^2 + \left|\frac{\partial^2 u}{\partial\xi\partial\eta}\right|^2 \mathrm{d}\,\xi\,\mathrm{d}\,\eta.$$

Solution: *We take the previous result and change variables back, so that e.g. $\frac{\partial^2\bar{u}}{\partial\xi^2}$ becomes $\frac{\partial^2 u}{\partial x^2}$. Hence, the second derivatives produce factors of h^2 that get squared, and we divide by h^2 from the Jacobian factor, leaving a factor of h^2.*

6. Consider a triangulation $\mathcal{T}$ of points x_i and y_j arranged in squares as above, with each square subdivided into two right-angled triangles. Explain how to use this result to obtain

$$\|u - \mathcal{I}_{\mathcal{T}}\|_E \leq ch|u|_{H^2(\Omega)},$$

where

$$\|f\|_E = \int_\Omega \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \mathrm{d}\,x\,\mathrm{d}\,y, \quad |u|^2_{H^2(\Omega)} = \int_\Omega \left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial xy}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^2 \mathrm{d}\,x\,\mathrm{d}\,y.$$

Solution: *All right-angled triangles can be transformed to the reference element by the transformation given above, plus a rotation. Hence, the estimate of the previous section applies to any triangle in the mesh. Summing over elements and taking square roots gives the result with* $c = \sqrt{C}$.

7. Let $\mathcal{T}$ be a triangulation of a polygonal domain $\Omega \in \mathbb{R}^2$. Let f be a P_k Lagrange finite element function on $\mathcal{T}$. Show that the weak first derivatives of f exist.
Solution: *We claim that the weak first derivative of* f *is given by* $g \in L^1_{loc}(\Omega)$ *with*

$$g|_e(x) = D^\alpha f|_e(x),$$

for each element e*, where* $\alpha = (0,1)$ *or* $(1,0)$*, and* $|_e$ *indicates the restriction of functions to* e*. To check this, we take* $\phi \in C_0^\infty(\Omega)$*, and calculate,*

$$\begin{aligned}\int_\Omega \phi g \,\mathrm{d}\,x &= \textstyle\sum_e \int_e \phi D^\alpha f|_e(x), \\ &= \textstyle\sum_e \left(-\int_e (D^\alpha \phi)\, f \,\mathrm{d}\,x + \int_{\partial e} \phi(n\cdot\alpha) f \,\mathrm{d}\,S\right), \\ &= -\textstyle\int_\Omega (D^\alpha\phi)\, f \,\mathrm{d}\,x,\end{aligned}$$

where n *is the unit outward normal to* ∂e*. The surface integrals cancel since* $(n \cdot \alpha)$ *takes the same value with opposite sign on each side of* ∂e*, whilst* ϕf *is continuous.*