Optimizing Portfolio Weights with Theory and Machine Learning

by

Jiarun Chen

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Department of Statistical Sciences
University of Toronto

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University of Toronto
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Abstract

In this report, I explore the optimization of portfolio weights using both theoretical models and machine learning techniques, tailored to investors with varying risk tolerances. I derive theoretical weight formulas from financial models aimed at minimizing portfolio variance and extend the analysis to incorporate investor risk preferences, aligning with Merton's portfolio problem. Moreover, I perform backtesting with real-world stock data, taking into account factors such as transaction fees, taxes, and scenarios with and without short selling and portfolio rebalancing. The results demonstrate that strategies customized to specific market conditions and investor preferences can enhance returns, highlighting the practical benefits of combining financial theory with machine learning.

Acknowledgements

Before introducing my report, I would like to express my deep gratitude to a few individuals who provided support throughout my research over the past four months. First, I sincerely thank Professor Xiaofei Shi, one of my course instructors during my master and the supervisor of this research. She helped me identify a topic that interests me and offered invaluable guidance whenever I encountered difficulties or felt lost. I am also grateful to her for carefully reviewing the initial drafts of my report. Meanwhile, I would like to extend my heartfelt thanks to my other supervisor, Professor Basil Singer, who consistently scheduled weekly meetings to discuss my progress and provided insightful advice that bridged the gap between theoretical analysis and practical application. Undoubtedly, this research would not have been possible without them.

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Chapter 1

Introduction

If someone has spare money, should they place it all in a risk-free account or invest in stocks to potentially achieve higher returns? This question, much like Schrödinger's cat, is inherently uncertain - almost no one can make a decision without some level of regret. However, applying models and predicting returns can provide valuable insights for investors. As investors have different goals and risk tolerances, portfolio optimization in the financial world must account for these varying perspectives. While some may rely on luck, mathematicians and statisticians prefer to use theorems and machine learning models to determine whether to buy a stock and how to allocate weights among selected stocks.

The typical process involves first defining an objective function based on the investment goal, then using historical data to determine theoretical weights as the ground truth. Next, a machine learning model is trained to estimate weights, with hyperparameters tuned to make the estimated weights as close to the ground truth as possible. This is followed by backtesting to evaluate whether the model's returns are sufficient to use in actual investing. In this report, I will follow this process for two optimization approaches. I will start with a simple one that minimizes portfolio variance and then extend it by incorporating expected returns and the investor's risk aversion, as in Merton's portfolio problem. I will consider real-world factors such as not allowing fractional shares, as well as transaction fees and taxes, to make the return predictions more realistic.

Chapter 2

Preliminaries

2.1 Model Setup

We assume that the number of stocks $\boldsymbol{S}_t$ is equal to the number of sources of randomness $\boldsymbol{B}_t$, both denoted by m. The dynamics of the stock prices are modeled by the following stochastic differential equation:

$$d\boldsymbol{S}_t = (r\boldsymbol{S}_t + \boldsymbol{\mu}_t)\,dt + \boldsymbol{\sigma}_t\,d\boldsymbol{B}_t$$

where $\boldsymbol{\mu}_t$ is the vector of expected asset returns, $\boldsymbol{\sigma}_t$ is the volatility matrix, and $\boldsymbol{\Sigma}_t = \boldsymbol{\sigma}_t\boldsymbol{\sigma}_t^T$ is the covariance matrix of asset returns. Here, $\boldsymbol{S}_t, \boldsymbol{B}_t, \boldsymbol{\mu}_t \in \mathbb{R}^{m\times 1}$ and $\boldsymbol{\Sigma}_t \in \mathbb{R}^{m\times m}$. We assume that the volatility vector $\boldsymbol{\sigma}_t$ is constant, so that the covariance matrix $\boldsymbol{\Sigma}_t$ is also constant and positive-definite, ensuring that the no-arbitrage condition is satisfied.
Under the risk-neutral measure, the dynamics of the stock prices:

$$d\boldsymbol{S}_t = r\boldsymbol{S}_t\,dt + \boldsymbol{\sigma}_t\,d\boldsymbol{B}_t^{\mathbb{Q}}$$

The evolution of the investor's wealth W_t is governed by the following stochastic differential equation:

$$dW_t = W_t\left(r + \boldsymbol{\pi}_t^T(\boldsymbol{\mu}_t - r\mathbf{1}_m)\right)dt + W_t\boldsymbol{\pi}_t^T\boldsymbol{\sigma}_t\,d\boldsymbol{B}_t$$

where $\boldsymbol{\pi}_t \in \mathbb{R}^{m\times 1}$ is the asset weight vector, and r is the risk-free rate.

2.2 Parameter Estimation

In practice, freely available data is often limited to daily trading data, while more detailed hourly data usually comes at an additional cost. Although higher frequency data is generally preferred, investors can still estimate the mean ($\boldsymbol{\mu}$) and covariance ($\boldsymbol{\Sigma}$) of asset returns using the limited availability of daily trading data. It is important to note that while estimating $\boldsymbol{\Sigma}$ is relatively straightforward, estimating $\boldsymbol{\mu}$ requires a longer observation period and may result in lower accuracy due to the limited number of data values involved. For the estimations in the following section, I estimated the $\boldsymbol{\Sigma}$ matrix for a portfolio by first calculating the daily returns over the most recent month. Daily returns were determined as the percentage change between two consecutive trading days, which were then used to compute $\boldsymbol{\Sigma}$. On the other hand, to estimate the $\boldsymbol{\mu}$ vector, I calculated the monthly returns over the most recent year, where the monthly return is defined as the percentage change between the same day of two consecutive months. As a consequence, the units for these estimations are $\%^2$ per day for $\boldsymbol{\Sigma}$ and % per month for $\boldsymbol{\mu}$, respectively.

While there are various methods available, recent advancements in portfolio optimization have introduced more complex techniques for estimating $\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}$ that better account for the complexities of financial markets. As discussed in the paper [1], Bayesian estimation techniques offer a robust alternative by incorporating prior information and addressing parameter uncertainty. This approach can lead to more reliable estimates, especially in scenarios with limited data. The Bayesian framework not only allows for the integration of historical data but also accommodates the stochastic nature of $\boldsymbol{\Sigma}$, treating it as a random matrix that evolves over time. Similarly, the estimation of $\boldsymbol{\mu}$ can be enhanced through Bayesian inference, refining predictions based on both observed data and prior beliefs, thereby improving accuracy when direct observations are sparse or noisy. These methods provide a deeper understanding of asset return dynamics and can significantly impact the performance of optimized portfolios by reducing estimation risk.

Chapter 3

Portfolio Optimization

3.1 Minimizing Portfolio Variance

The most conservative investment objective is to minimize the overall risk (variance) of the portfolio. While this approach may not yield the most attractive returns, it significantly reduces the likelihood of financial loss, reflecting the trade-off between risk and return.

Mathematically, the optimal weights in this scenario are those that minimize the portfolio's variance, subject to the constraint that the weights sum to 1. This can be expressed as:

$$\min \boldsymbol{\pi}^{\mathrm{T}} \boldsymbol{\Sigma} \boldsymbol{\pi} \quad \text{subject to} \quad \mathbf{1}_m^{\mathrm{T}} \boldsymbol{\pi} = 1$$

Solving for the theoretical optimal weights using the Lagrangian method [3.3.2]. The Lagrangian function is defined as:

$$\mathrm{L} = \boldsymbol{\pi}^{\mathrm{T}} \boldsymbol{\Sigma} \boldsymbol{\pi} + \lambda \left(1 - \mathbf{1}_m^{\mathrm{T}} \boldsymbol{\pi}\right)$$

Taking the partial derivatives of L with respect to $\boldsymbol{\pi}$ and λ, we obtain:

$$\begin{cases} \frac{\partial \mathrm{L}}{\partial \boldsymbol{\pi}} & = 2\boldsymbol{\Sigma}\boldsymbol{\pi} - \lambda \mathbf{1}_m = \mathbf{0} \\ \frac{\partial \mathrm{L}}{\partial \lambda} & = 1 - \mathbf{1}_m^{\mathrm{T}} \boldsymbol{\pi} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \lambda^* & = \frac{2}{\mathbf{1}_m^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} \mathbf{1}_m} \\ \boldsymbol{\pi}^* & = \frac{\boldsymbol{\Sigma}^{-1} \mathbf{1}_m}{\mathbf{1}_m^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} \mathbf{1}_m} \end{cases}$$

This shows that the optimal allocation depends on the covariance matrix $\mathbf{\Sigma}$ as follows:

$$\boldsymbol{\pi}^* = \frac{\mathbf{\Sigma}^{-1}\mathbf{1}_m}{\mathbf{1}_m^{\mathrm{T}}\mathbf{\Sigma}^{-1}\mathbf{1}_m}$$

As previously mentioned, the stock weights depend solely on the covariance matrix $\mathbf{\Sigma}$, making accurate estimation of $\mathbf{\Sigma}$ crucial since it is the only input. However, for theoretical weights, $\mathbf{\Sigma}$ must be invertible, and in practice, it is challenging to have perfectly uncorrelated assets. Therefore, fitting a machine learning model can be more practical. Specifically, training a model with a reasonable period of data to tune hyperparameters can help produce results close to the theoretical ground truth. This model can then be used for future predictions.

I applied the model to real-world data and conducted backtesting to assess the potential returns. The data was sourced from Yahoo Finance and focused on 18 randomly selected technology stocks: Microsoft (MSFT), Oracle (ORCL), Adobe (ADBE), Apple (AAPL), Intel (INTC), Amazon (AMZN), Meta (META), AT&T (T), Ericsson (ERIC), Nokia (NOK), IBM (IBM), Accenture (ACN), Tata Consultancy Services (TCS.NS), NVIDIA (NVDA), Qualcomm (QCOM), Texas Instruments (TXN), Tesla (TSLA), and Palantir (PLTR). I considered two scenarios with buy and sell dates ranging from 2023-07-01 to 2024-06-30 and from 2022-07-01 to 2023-06-30.

Table 3.1: Average Monthly Return Comparison

(a) 2022.07 - 2023.07

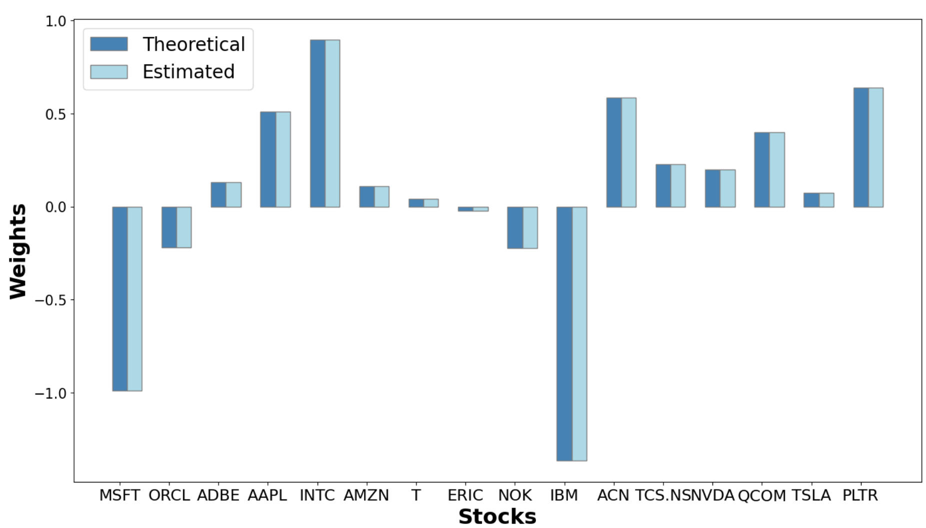
	Allow Short Selling	**No Short Selling**
No Rebalance	4.61%	1.66%
Monthly Rebalance	2.02%	1.97%

(b) 2023.07 - 2024.07

	Allow Short Selling	**No Short Selling**
No Rebalance	2.26%	1.99%
Monthly Rebalance	4.33%	1.71%

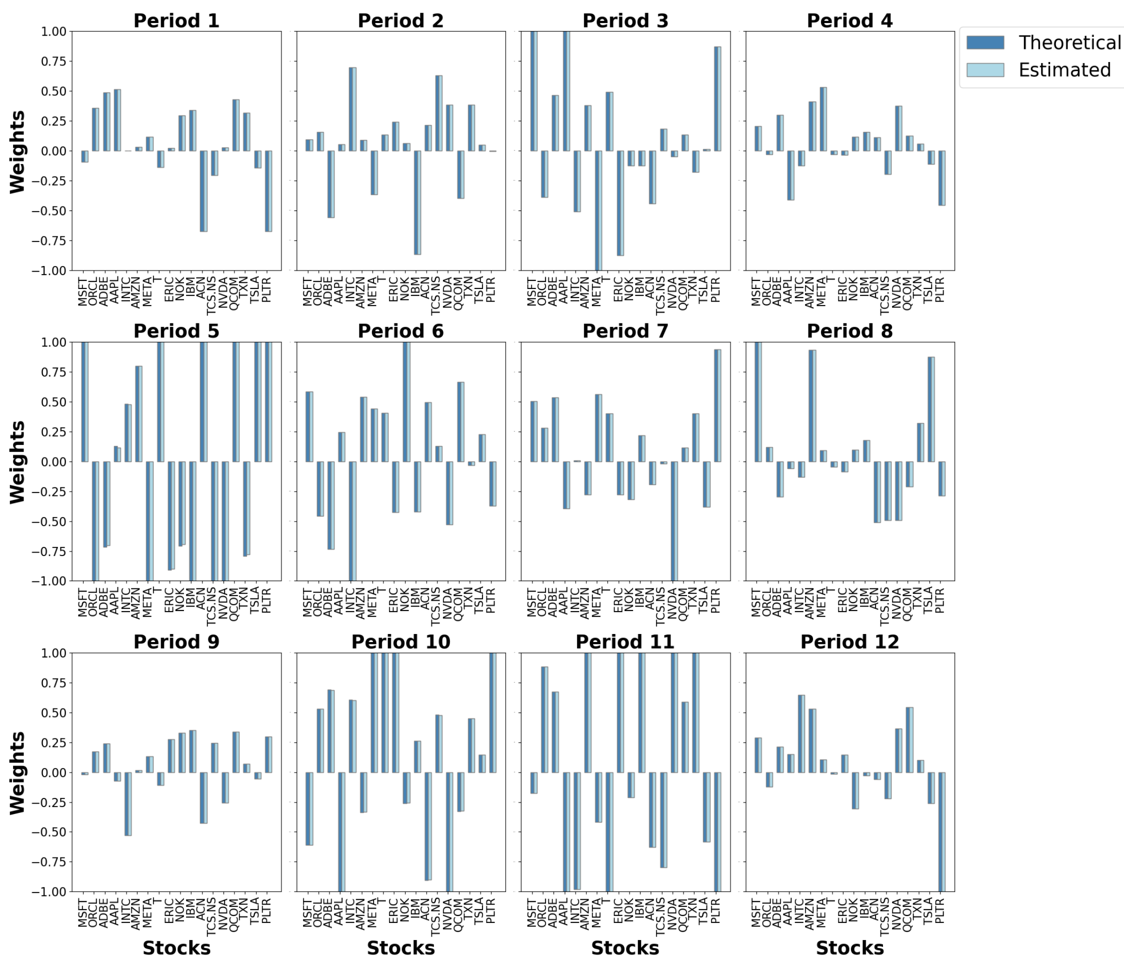
As detailed in Section 2.2, the covariance matrix, $\mathbf{\Sigma}$, was estimated using the previous month's daily returns. The initial budget was set at $10,000, with transaction fees of 2% applied to the total value of shares bought or sold, and a 1.5% tax on gains. For the monthly rebalancing strategy, each stock had a target value, calculated by multiplying the total portfolio value by the target weight. The difference between the target value and the current value was determined as the delta, which was then divided by the stock's next period's starting price to determine the number of shares to buy or sell; adjustments were made accordingly at the beginning of each month. Given the risks associated with short selling, including the potential for significant losses and challenges in managing risks related to company performance, I also examined cases without short selling, where all weights were constrained to be non-negative. Table 3.1 presents the average monthly returns for various combinations of factors, including the year, whether rebalancing was applied, and whether short selling was allowed, resulting in eight different scenarios. Based on the resulting returns, it is evident that although some investors may avoid short selling due to its perceived risk, it often provides more opportunities for higher returns. In both periods, the highest returns were consistently achieved in scenarios that allowed short selling.

Figure 3.1: Optimal Weights for Each Stock in the Case of July 2022 - July 2023, Short Selling Allowed, No Rebalancing



In the 2022 to 2023 scenario, the strategy without rebalancing yielded the highest average monthly return of 4.61%. This outcome may be attributed to the elimination of transaction costs and the added flexibility provided by short selling. Figure 3.1 illustrates the corresponding portfolio weights, highlighting that Intel had the largest holdings, followed by Apple, Accenture, and Palantir. Conversely, the strategy suggested short selling IBM and Microsoft, which had the most negative weights.

Figure 3.2: Optimal Weights for Each Stock in the Case of July 2023 - July 2024, No Short Selling, Monthly Rebalancing



In the 2023 to 2024 scenario, the strategy of monthly rebalancing achieved the highest average monthly return of 4.33%. Figure 3.2 illustrates the optimal weights across the 12 periods, showing significant shifts in allocation between many periods, such as from period 2 to period 3. These changes indicate that maintaining a static position without rebalancing may not be an effective approach in a dynamic market environment.

3.2 Linear-Quadratic Preference

As outlined in Merton's portfolio problem [3.3.1], many investors are not satisfied with the returns achieved by merely minimizing variance. Instead, such investors may be willing to accept some risk in exchange for potentially higher returns. In this scenario, the goal is to maximize the expected utility [A.1] of their wealth over time by balancing returns and risk, adjusted for their level of risk aversion (denoted by γ).

Mathematically, the objective is to maximize the expected utility of the investor's wealth, and can be expressed as:

$$\max \int_0^T \left[W_t \left(r + \boldsymbol{\pi}_t^T (\boldsymbol{\mu}_t - r\mathbf{1}_m) \right) - \frac{\gamma}{2} W_t \boldsymbol{\pi}_t^T \boldsymbol{\Sigma}_t \boldsymbol{\pi}_t \right] dt$$

Solving for the theoretical optimal weights using the HJB Method [3.3.3], we start by defining the corresponding Hamiltonian:

$$H = W_t \left(r + \boldsymbol{\pi}_t^T (\boldsymbol{\mu}_t - r\mathbf{1}_m) \right) - \frac{\gamma}{2} W_t \boldsymbol{\pi}_t^T \boldsymbol{\Sigma}_t \boldsymbol{\pi}_t$$

The first-order condition is obtained by taking the derivative of H with respect to $\boldsymbol{\pi}_t$ and setting it equal to zero:

$$\frac{\partial H}{\partial \boldsymbol{\pi}_t} = W_t (\boldsymbol{\mu}_t - r\mathbf{1}_m) - \gamma W_t \boldsymbol{\Sigma}_t \boldsymbol{\pi}_t = 0$$

This implies the theoretical optimal weights are:

$$\boldsymbol{\pi}_t^* = \frac{1}{\gamma} \boldsymbol{\Sigma}_t^{-1} (\boldsymbol{\mu}_t - r\mathbf{1}_m)$$

This shows that the optimal allocation depends not only on the covariance matrix $\boldsymbol{\Sigma}$, but also on the investor's risk aversion level γ, and the difference between each of the mean returns μ and the risk-free rate r.

In this case, the level of risk aversion γ must first be set based on the investor's preference [A.2]; accurate estimation of the covariance matrix $\boldsymbol{\Sigma}$, the expected returns $\boldsymbol{\mu}$, and the choice of the risk-free rate r are crucial for determining the optimal weights. As discussed earlier in Section

3.1, since $\mathbf{\Sigma}$ must be invertible to calculate the theoretical weights, a machine learning model can be fitted to estimate $\boldsymbol{\pi}$ to address practical challenges, allowing for more practical weight calculations.

I applied the model to real-world data and conducted backtesting to assess the potential returns. The data was sourced from Yahoo Finance and focused on 16 randomly selected technology stocks: Oracle (ORCL), Adobe (ADBE), Apple (AAPL), Intel (INTC), Amazon (AMZN), Meta (META), AT&T (T), Ericsson (ERIC), Nokia (NOK), IBM (IBM), Accenture (ACN), Tata Consultancy Services (TCS.NS), NVIDIA (NVDA), Texas Instruments (TXN), Tesla (TSLA), and Palantir (PLTR). I considered two scenarios with buy and sell dates ranging from 2023-07-01 to 2024-06-30 and from 2022-07-01 to 2023-06-30.

Table 3.2: Average Monthly Return Comparison

(a) 2022.07 - 2023.07

	Allow Short Selling	**No Short Selling**
No Rebalance	4.46%	3.61%
Monthly Rebalance	6.42%	4.30%

(b) 2023.07 - 2024.07

	Allow Short Selling	**No Short Selling**
No Rebalance	4.79%	4.16%
Monthly Rebalance	5.51%	3.10%

As detailed in Section 2.2, the covariance matrix, $\mathbf{\Sigma}$, was estimated using the previous month's daily returns, while the expected returns, $\boldsymbol{\mu}$, were based on the previous year's monthly returns. To enhance realism, I assumed a risk-free rate of 5% for 2023 and 3.75% for 2022. As in Section 3.1, the initial budget was \$10,000, with a 2% transaction fee on all trades and a 1.5% tax on gains. For the monthly rebalancing strategy, each stock's target value was calculated by multiplying the total portfolio value by the target weight. The difference between the target and current values determined the delta, which was divided by the stock's next period's starting price to calculate the number of shares to buy or sell; adjustments were made at the start of each month. Considered the risks of short selling, including potential significant losses and challenges in managing company-related risks, I also analyzed scenarios without short selling, where all weights were constrained to be non-negative.

Figure 3.3: Optimal Weights for Each Stock in the Case of July 2022 - July 2023, Short Selling Allowed, Monthly Rebalancing

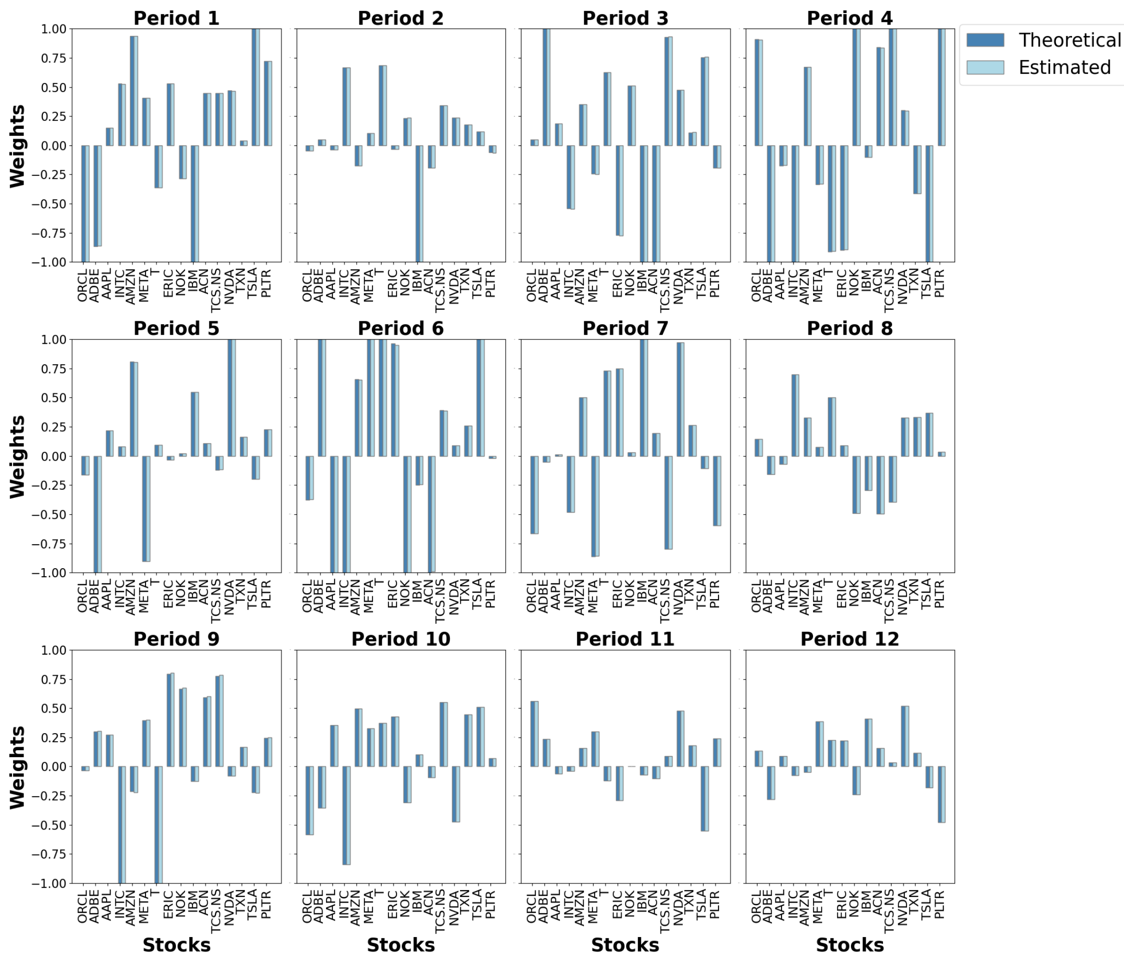
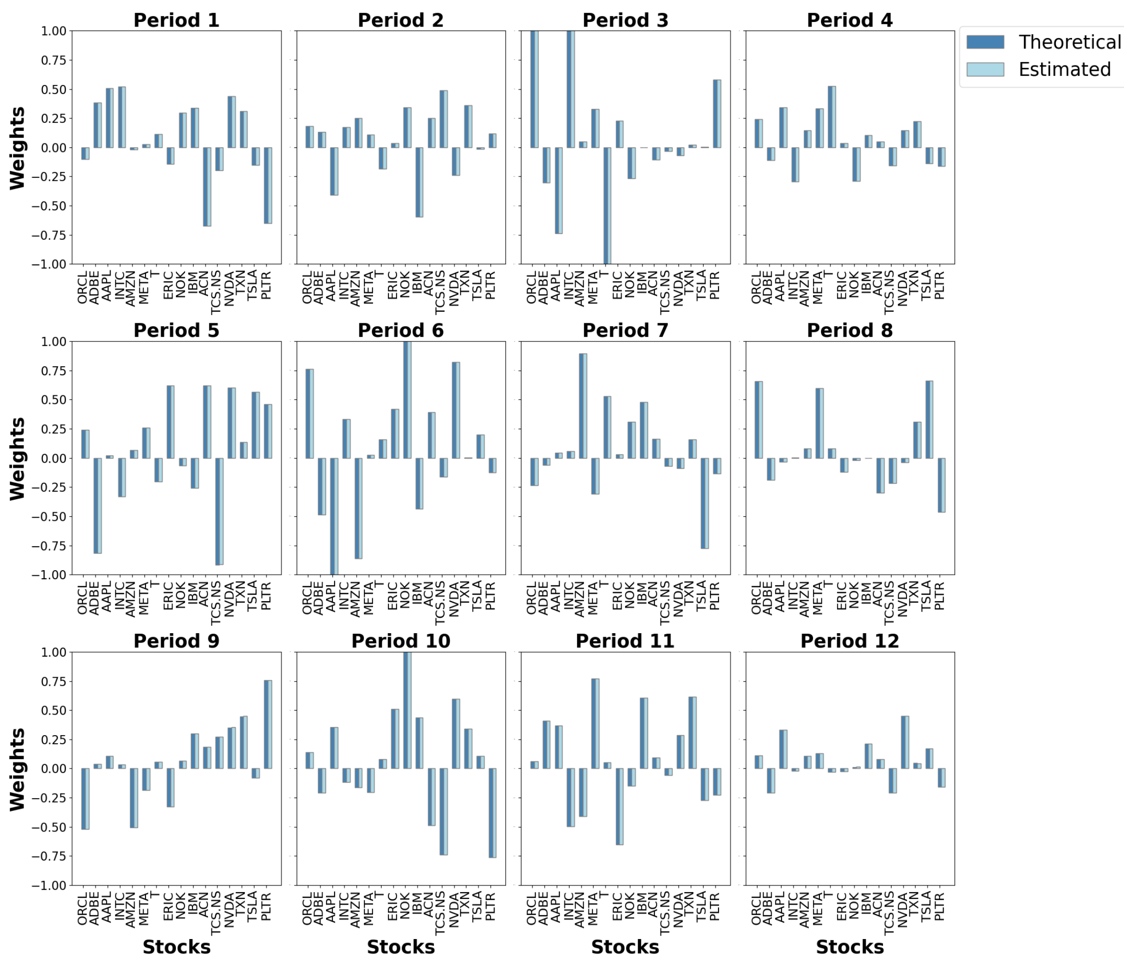


Table 3.2 presents the average monthly returns for various combinations of factors, including the year, whether rebalancing was applied, and whether short selling was allowed, resulting in eight different scenarios. Based on the results, it is evident that while rebalancing decisions must account for transaction fees, it may still be necessary to adjust positions in anticipation of significant market changes. In these two periods, the highest returns were consistently achieved in cases that allowed short selling and implemented a monthly rebalancing strategy.

In the 2022 to 2023 scenario, the strategy yielded an impressive average monthly return of 6.42%. This result can be attributed to the strategy's ability to optimally adjust the allocation for each stock on a monthly basis, coupled with the flexibility to short sell. Figure 3.3 illustrates the optimal weights across the 12 periods, revealing significant shifts in allocation between certain periods, such as from period 1 to period 2. These changes indicate that a static position without rebalancing might not be the most effective approach in a dynamic market environment.

Figure 3.4: Optimal Weights for Each Stock in the Case of July 2023 - July 2024, Short Selling Allowed, Monthly Rebalancing



Similarly, in the 2023 to 2024 scenario, the strategy achieved a strong average monthly return of 5.51%. This outcome likely reflects the benefits of monthly rebalancing and the strategic use of short selling. Figure 3.4 shows the optimal weights across the 12 periods, again highlighting notable changes in allocation, such as the shift from period 2 to period 3. These findings suggest that maintaining a static position without regular rebalancing could lead to missed opportunities for maximizing returns.

3.3 Continuous-Time Model

3.3.1 Merton's Portfolio Problem

Merton's problem is a foundational model in continuous-time finance that optimizes the allocation of wealth between risky assets and a risk-free asset over time. The model assumes that investors aim to maximize the expected utility of their wealth at a terminal time by continuously adjusting their portfolio. It is typically formulated within a stochastic control framework, where the investor's wealth evolves according to a stochastic differential equation driven by the assets' returns. The optimal solution determines the proportion of wealth to invest in each asset, considering the investor's risk aversion and asset characteristics. A significant outcome is the mean-variance optimization, where wealth is allocated based on the trade-off between expected return and risk. Meanwhile, the model accounts for potential changes in investment opportunities over time, providing a hedge against these shifts. Merton's solution shows that, under certain market and utility function assumptions, the optimal proportion of wealth invested in risky assets remains constant over time, regardless of wealth levels. [4, 6]

3.3.2 Lagrangian Method

The Lagrangian method, a mathematical technique in calculus, is widely used in continuous-time finance to solve constrained optimization problems. By introducing a Lagrange multiplier, this method integrates constraints directly into the optimization process, transforming a constrained problem into an unconstrained one. It is especially useful in finance for optimizing investment portfolios under constraints such as budget limits, risk thresholds, or target returns. In portfolio optimization, the Lagrangian method maximizes the expected utility of an investor's wealth while adhering to specific constraints on asset allocation. The process involves constructing a Lagrangian function that combines the objective function (e.g. maximizing utility) with the constraints, weighted by the Lagrange multipliers. The optimal solution is found by taking the derivatives of the Lagrangian function with respect to the decision variables and the Lagrange multipliers, then setting these derivatives equal to zero to determine the optimal portfolio weights. The method's ability to handle multiple constraints simultaneously makes it an essential tool in continuous-time financial models, enabling more precise and tailored investment strategies. [2]

3.3.3 Hamilton-Jacobi-Bellman Method

The Hamilton-Jacobi-Bellman (HJB) equation is a crucial tool in dynamic programming and optimal control theory. It provides a necessary condition for optimality in a dynamic system, allowing the determination of the optimal control strategy over time. In the context of portfolio optimization, the HJB method is used to derive the value function, which represents the maximum expected utility an investor can achieve given the current state of wealth and time. The HJB equation models how an investor should continuously adjust their portfolio to maximize utility. This partial differential equation captures the trade-off between immediate rewards and future gains, accounting for risk aversion, market dynamics, and time preferences. Solving the HJB equation provides the optimal portfolio strategy, accounting for the evolution of both the investor's wealth and the underlying market variables. This method is powerful because it can handle complex, time-varying environments and provide explicit solutions. [3, 5]

Chapter 4

Conclusion

In this report, I examined two distinct optimization strategies for portfolio management: minimizing portfolio variance and linear-quadratic preference. The variance minimization strategy, which takes a more conservative approach, focuses on reducing risk by allocating stock weights that minimize the overall portfolio variance. This method is particularly suitable for risk-averse investors who prioritize stability over potentially higher returns. However, as demonstrated in the backtesting results, this strategy may result in lower returns, particularly in volatile markets. By relying primarily on the covariance matrix of asset returns, it underscores the importance of accurate market volatility estimations, making it a practical but somewhat limited approach. On the other hand, the linear-quadratic preference strategy, derived from Merton's portfolio problem, adds complexity by incorporating expected returns and the investor's risk aversion level into the optimization process. This method seeks to balance the trade-off between higher returns and acceptable risk levels, making it more dynamic and adaptable to varying market conditions and investor preferences. As shown in the analysis, it has the potential to generate higher returns by taking calculated risks, especially for investors with lower risk aversion. By including expected returns and risk aversion, this approach provides a more comprehensive framework for decision-making, enabling investors to align their portfolios more closely with their individual financial goals and risk tolerance.

Appendix A

Supplementary Material

A.1 Utility

Utility reflects an investor's preferences and risk tolerance. Common types of utility functions used in portfolio optimization include exponential, logarithmic, power (CRRA), and mean-variance. Each type of utility function offers a different approach to modeling investor behavior and preferences in the optimization process.

A.2 Risk Aversion Coefficient

The risk aversion coefficient quantifies an investor's tolerance for risk and reflects their willingness to accept risk in exchange for potentially higher returns. Commonly accepted values for this coefficient typically range between 1 and 3, as most individuals are considered moderately risk-averse. However, estimates can vary widely across different sources. For instance, values can range from as low as 0.2, indicating risk-seeking behavior, to as high as 10 or more, indicating a high degree of risk aversion. Lower values suggest a greater willingness to accept risk, while higher values reflect a preference for safer investments. [7]

Appendix B

GitHub Code Repository

The code associated with this report is available on GitHub at the following link: GitHub Code Repository.

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