1 已知：R 是集合 A 上的关系，证明：$S = I_A \cup R \cup R^{-1}$ 是集合 A 上的相容关系.

Prove:

Reflexivity:

$\forall x \in A$ we have $\langle x,x\rangle \in I_A$. And $I_A \subseteq S$, so $(\forall x)(x \in A \to \langle x,x\rangle \in S)$

Symmetry: For $\langle x,y\rangle \in S$, if:

1. $\langle x,y\rangle \in I_A$, then $x = y$, so we have $\langle y,x\rangle \in S$ too.
2. $\langle x,y\rangle \in R$, then $\langle y,x\rangle \in R^{-1}$, and $R^{-1} \subseteq S$. So $\langle y,x\rangle \in S$ too.
3. $\langle x,y\rangle \in R^{-1}$, then $\langle y,x\rangle \in R$, and $R \subseteq S$. So $\langle y,x\rangle \in S$ too.

So $(\forall x)(\forall y)(\langle x,y\rangle \in S \to \langle y,x\rangle \in S)$

QED.

2 已知：R 是集合 A 上的偏序关系，$B \subseteq A$. 求证：$R \cap (B \times B)$ 是 B 上的偏序关系。

Prove:

Reflexivity: For any $x \in B$

$x \in B \Rightarrow \langle x,x\rangle \in R \wedge \langle x,x\rangle \in (B \times B) \Leftrightarrow \langle x,x\rangle \in R \cap (B \times B)$

Anti-symmetry: For any $x, y \in B$

$\langle x,y\rangle \in R \cap (B \times B) \Leftrightarrow \langle x,y\rangle \in R \wedge \langle x,y\rangle \in (B \times B) \Rightarrow \langle x,y\rangle \in R$

So

$\langle x,y\rangle \in R \cap (B \times B) \wedge \langle y,x\rangle \in R \cap (B \times B)$

$\Rightarrow \langle x,y\rangle \in R \wedge \langle y,x\rangle \in R$

$\Rightarrow x = y$

Transitivity: For any $x, y, z \in B$

$\langle x,y\rangle \in R \cap (B \times B) \wedge \langle y,z\rangle \in R \cap (B \times B)$

$\Rightarrow \langle x,y\rangle \in R \wedge \langle y,z\rangle \in R$

$\Rightarrow \langle x,z\rangle \in R$

$\Rightarrow \langle x,z\rangle \in R \wedge \langle x,z\rangle \in (B \times B)$

$\Rightarrow \langle x,z\rangle \in R \cap (B \times B)$

Q.E.D

3 已知：R_1 和 R_2 是非空集合上的等价关系，判断下列关系是否是 A 上的等价关系。是，请证明。否，请举出反例。

$$
\begin{aligned}
&1.\ (A \times A) - R_1 \\
&2.\ R_1^2 \\
&3.\ R_1 - R_2
\end{aligned}
$$

1. $(A \times A) - R_1$ is not an equivalence relation.

   Let $A = \{1, 2\}, R_1 = \{\langle 1, 1\rangle, \langle 2, 2\rangle\}$, then $(A \times A) - R_1 = \{\langle 1, 2\rangle, \langle 2, 1\rangle\}$. Now $\langle 1, 1\rangle \notin (A \times A) - R_1$, so the reflexivity does not holds.

2. R_1^2 is an equivalence relation.

   Reflexivity: $\forall x \in A$, $\langle x, x\rangle \in R_1$ since R_1 is an equivalence relation. And we have:
   $\langle x, x\rangle \in R_1$
   $\Rightarrow \langle x, x\rangle \in R_1 \wedge \langle x, x\rangle \in R_1$
   $\Rightarrow (\exists z)(\langle x, z\rangle \in R_1 \wedge \langle z, x\rangle \in R_1)$
   $\Rightarrow \langle x, x\rangle \in R_1^2$

   Symmetry: $\forall x, y \in A$
   $\langle x, y\rangle \in R_1^2$
   $\Leftrightarrow (\exists z)(\langle x, z\rangle \in R_1 \wedge \langle z, y\rangle \in R_1)$
   $\Rightarrow (\exists z)(\langle z, x\rangle \in R_1 \wedge \langle y, z\rangle \in R_1)$($R_1$ is symmetric)
   $\Leftrightarrow \langle y, x\rangle \in R_1^2$

   Transitivity: $\forall x, y, z \in A$
   $\langle x, y\rangle \in R_1^2 \wedge \langle y, z\rangle \in R_1^2$
   $\Leftrightarrow (\exists t_1)(\langle x, t_1\rangle \in R_1 \wedge \langle t_1, y\rangle \in R_1) \wedge (\exists t_2)(\langle y, t_2\rangle \in R_1 \wedge \langle t_2, z\rangle \in R_1)$
   $\Rightarrow \langle x, y\rangle \in R_1 \wedge \langle y, z\rangle \in R_1$($R_1$ is transitive)
   $\Rightarrow (\exists t_3)(\langle x, t_3\rangle \in R_1 \wedge \langle t_3, z\rangle \in R_1)$
   $\Leftrightarrow \langle x, z\rangle \in R_1^2$

3. $R_1 - R_2$ is not an equivalence relation.

   Let $R_1 = R_2 = A \times A$, then $R_1 - R_2 = \emptyset$ is not an equivalence relation.

4 已知:

$$
M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}
$$

求 $r(R)$, $s(R)$, $t(R)$.

$$
M_{r(R)} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}
$$

$$M_{s(R)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$M_{t(R)} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

5 下列哪个函数是满射的？($f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$) ABC

A. $f(m,n) = m + n$
B. $f(m,n) = m - n$
C. $f(m,n) = |m| - |n|$
D. $f(m,n) = m^2 + n^2$
E. $f(m,n) = m^2 - n^2$

6 对下面的每个函数 f，回答以下问题：

1. $f : \mathbb{R} \to (0, \infty), f(x) = 2^x, S = [1, 2]$
2. $f : \mathbb{N} \to \mathbb{N}, f(n) = 2n + 1, S = \{2, 3\}$
3. $f : \mathbb{Z} \to \mathbb{N}, f(x) = |x|, S = \{0, 2\}$
4. $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}, f(n) = \langle n, n+1 \rangle, S = \langle 2, 2 \rangle$
5. $f : [0, 1] \to [0, 1], f(x) = \frac{2x+1}{4}, S = \left[0, \frac{1}{2}\right]$

1. f 是否是单射、满射、或是双射的？如果是双射的，写出 f^{-1}.
2. 写出 f 的象 $f[dom(f)]$ 和 f 在 S 上的原象 $f^{-1}[S]$.
3. 关系 $R = \{\langle x, y \rangle | x, y \in \mathrm{dom}(f) \wedge f(x) = f(y)\}$ 是一个在 $dom(f)$ 上的等价关系。求 R.

1. (a) Bijective. $f^{-1}(x) = log_2(x)$
   (b) image: $f[\mathbb{R}] = (0, \infty)$, reverse image $f^{-1}[S] = [0, 1]$
   (c) $R = I_{\mathbb{R}}$
2. (a) Injective.
   (b) image: $f[\mathbb{N}] = \{\text{positive odd numbers}\}$, reverse image $f^{-1}[S] = \{1\}$
   (c) $R = I_{\mathbb{N}}$
3. (a) Surjective.
   (b) image: $f[\mathbb{Z}] = \mathbb{N}$, reverse image $f^{-1}[S] = \{-2, 0, 2\}$
   (c) $R = I_{\mathbb{Z}} \cup \{\langle x, y\rangle | x \in \mathbb{Z} \wedge y \in \mathbb{Z} \wedge x = -y\}$
4. (a) Injective.
   (b) image: $f[\mathbb{N}] = \{\langle n, n+1\rangle | n \in \mathbb{N}\}$, reverse image $f^{-1}[S] = \varnothing$
   (c) $R = I_{\mathbb{N}}$
5. (a) Injective.
   (b) image: $f[[0, 1]] = [\frac{1}{4}, \frac{3}{4}]$, reverse image $f^{-1}[S] = [0, \frac{1}{2}]$
   (c) $R = I_{[0,1]}$

7 已知 $f, g \in A_B$, 且 $f \cap g \neq \phi$。那么

1. $f \cap g$ 是函数吗?
2. f $\cup g$ 是函数吗?

如果是，请证明。如果不是，给出反例。

1. $f \cap g$ is a function.

For any $x, y_1, y_2 \in \mathrm{dom}(f \cap g)$:

$\langle x, y_1\rangle \in f \cap g \wedge \langle x, y_2\rangle \in f \cap g$

$\Leftrightarrow \langle x, y_1\rangle \in f \wedge \langle x, y_1\rangle \in g \wedge \langle x, y_2\rangle \in f \wedge \langle x, y_2\rangle \in g$

$\Leftrightarrow (\langle x, y_1\rangle \in f \wedge \langle x, y_2\rangle \in f) \wedge (\langle x, y_1\rangle \in g \wedge \langle x, y_2\rangle \in g)$

$\Rightarrow y_1 = y2$

So $f \cap g$ is a function

2. $f \cup g$ is not a function.

Let $A = \{1, 2\}, B = \{1, 2\}, f = \{\langle 1, 1\rangle, \langle 2, 2\rangle\}, g = \{\langle 1, 2\rangle, \langle 2, 2\rangle\}$, now $f \cup g = \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 2\rangle\}$ is not a function.