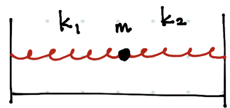
Physics 2A Spring 2020
Discussion 5

TA: Jiashu Han

Wednesday, April 29, 2020

1. **Springs**



(a) A bead of mass m is connected to the walls by two springs with spring constants k_1 and k_2 as shown above. When we displace the bead from its equilibrium position, what is its oscillation frequency (in rad/s)?

A. $\sqrt{\frac{k_1+k_2}{m}}$

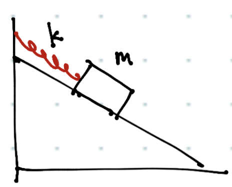
B. $\frac{k_1+k_2}{m}$

C. $\sqrt{\frac{k_1-k_2}{m}}$

D. $\frac{k_1k_2}{(k_1+k_2)m}$

E. $\sqrt{\frac{k_1k_2}{(k_1+k_2)m}}$

Solution: A. We write down the force equation $m\ddot{x} = -k_1x - k_2x$. We can read off from here that $\omega^2 = (k_1 + k_2)/m$.



(b) A block is on a slope angled at θ from the ground by both friction (static coefficient μ) and a spring (k) as shown in the above figure. If we pull the block down from the equilibrium position of the spring, what is the maximum distance we can pull the block and still have it stationary when we release the block?

A. $\frac{mg}{k}(\sin\theta - \mu\cos\theta)$

B. $\frac{mg}{k}(\cos\theta + \mu\sin\theta)$

C. $\frac{mg}{k}(\cos\theta - \mu\sin\theta)$

D. $\frac{mg}{k}(\sin\theta + \mu\cos\theta)$

E. None of the above

Solution: D. We are looking for the maximum displacement from equilibrium, so the friction is pointing down the slope and equal to μF_N. From the free body diagram of the block we should get the equations $F_N = mg\cos\theta$ and $kx = mg\sin\theta + F_f$. Solving these gives us $x = \frac{mg}{k}(\sin\theta + \mu\cos\theta)$.

2. **Buoyancy and Pressure**[1]

(a) A block of wood is floating in water. It has density $k\rho_w$, where ρ_w is the density of water, and $k < 1$ is a unitless, positive constant. What fraction of the total volume of wood is submerged?

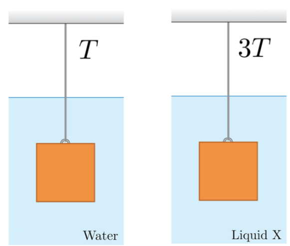
A. k

B. $1 - k$

C. $\frac{k}{k+1}$

D. $\frac{1}{k+1}$

Solution: A. Let f be the fraction of block submerged. The buoyancy force is then $F_b = \rho_w g f V$, which should equal the weight of the block $k\rho_w V g$. It's apparent that $f = k$.



For parts (b) and (c): When an object is completely submerged in water, it requires an upward tension T to keep the object in equilibrium. When the same object is completely submerged in "Liquid X," the tension required in the string is $3T$.

(b) Does "Liquid X" have a density that is greater than water or less than water? Explain using free-body diagrams.

Solution: Less. Each free body diagram should have a tension force pointing up, a buoyancy force pointing up, and the weight of the object pointing down. From the diagrams we can write down two equations: $T + \rho_w gV = mg$ and $3T + \rho_X gV = mg$. We can see that $\rho_X < \rho_w$ by comparing the two.

(c) Given $T = 250\text{N}$ and $mg = 1250\text{N}$ (the weight of the object), find the densities of the object and liquid X.

[1]Credit: Brian Shotwell

Solution: To find the density of liquid X, we use the first equation to find the volume of the object: $V = (mg - T)/(\rho_w g)$ and then plug this into the second to get $\rho_X = (mg - 3T)/(gV) = \rho_w(mg - 3T)/(mg - T) = \rho_w/2$. For the density of the object, we have $\rho = (T + \rho_w gV)/(gV) = mg\rho_w/(mg - T)$.

3. **Drag force**
An 80-kg pilot ejects himself from his jet when it's upside down at an altitude of 3000 m and plunges downward with a velocity of 100 m/s. If the air drag is $F_D = -kv^2$ where $k = 0.18$ kg/m, approximately how long will he remain in air if he does not deploy his parachute?

Solution: First we set the drag force equal to the person's weight to find the terminal velocity $v_t = \sqrt{mg/k} = 66$ m/s. Note that this is less than the velocity of the person, so he will be slowed down by the drag force. Let us set up the differential equation $ma = m\dot{v} = mg - kv^2$. This is separable:

$$t = \int_0^t dt' = \int_{v_0}^{v} \frac{dv}{g - \frac{k}{m}v^2} = \sqrt{\frac{m}{gk}} \int_{\sqrt{\frac{k}{mg}}v_0}^{\sqrt{\frac{k}{mg}}v} \frac{dx}{1 - x^2} \tag{1}$$

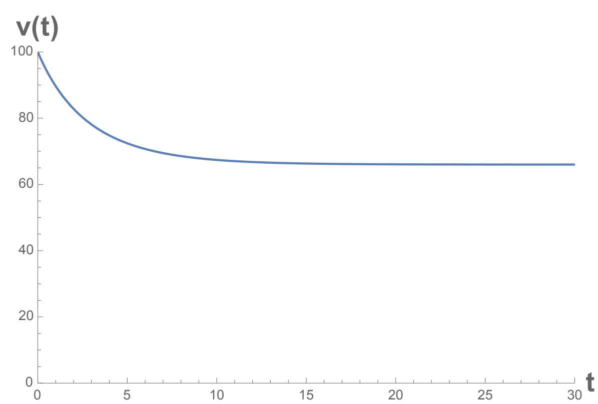
Note that for $|x| < 1$ this integrates to $\text{arctanh}(x)$ and for $|x| > 1$ this integrates to $\text{arccoth}(x)$. In our integral $|x| > 1$, so

$$t = \sqrt{\frac{m}{gk}}(\text{arccoth}(v\sqrt{k/mg}) - \text{arccoth}(v_0\sqrt{k/mg})) \tag{2}$$

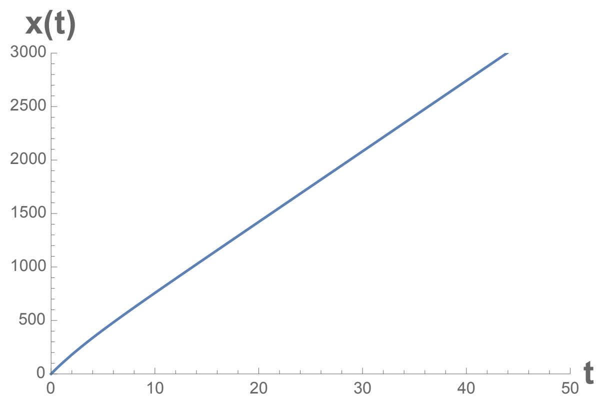
Plug in the initial velocity and terminal velocity we find it takes about 0.35 seconds for the person to reach terminal velocity. We can further invert this equation to get

$$v(t) = \sqrt{\frac{mg}{k}}\coth\left(\sqrt{\frac{gk}{m}}t + \text{arccoth}(v_0\sqrt{k/mg})\right) \tag{3}$$

This looks like



The rest of the calculation is a bit too messy so I will just show the result from Mathematica. It takes about 44 seconds for the person to fall to the ground without parachute.



4. **Taylor expansion done quickly**
   The energy of a relativistic particle is given by $E = \sqrt{p^2c^2 + m^2c^4}$. Show that in the non-relativistic limit ($p \ll mc$) it becomes $E = mc^2 + \frac{1}{2}mv^2 + \cdots$.

   Solution: We first factor out all the constants to get $E = mc^2\sqrt{1 + \frac{p^2}{m^2c^2}}$. Note that the second term inside the square root is small so we can use the binomial expansion $(1+x)^a = 1 + ax + \cdots$ to write $E = mc^2(1 + \frac{p^2}{2m^2c^2}) + \cdots$. We can then use the expression for nonrelativistic momentum $p = mv$ to write $E = mc^2 + \frac{1}{2}mv^2 + \cdots$.