Stochastic Interest Rates and Swaptions

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1 Introduction

Interest rate is an important determinant of the values of assets and liabilities. In the economics and finance literature, voluminous research has been conducted on the linkage between interest rate and the pricing of various assets, the interest-rate contingent claims are one of them. In a fixed-interest environment, the expected discount factor is a deterministic function of the time, which makes the market rate uncertainty out of the consideration. However, since the early 1980s, the interest rate has become more volatile, as the result, more models to describe the stochastic dynamics of the interest rate are introduced. Moreover, the price of rate contingent claims should also be adjusted to incorporate the fluctuations in the interest rate when rates are assumed to be stochastic, and the price derived by imposing the condition that there are no arbitrage opportunities in the market. The zero-coupon bond is one of the most basic derivative product of interest rate, the yield rate and forward rate implied by the bond prices become a powerful tool to evaluate other rate-contingent claims, for example, the swaptions are widely used tools to hedge adverse movements in interest rates.

In this project, we are going to explore a two-factor interest rate model, and the well-established market formula for the derivatives products: zero-coupon bonds, bond options and swaptions. We will still derive the zero-coupon bonds prices based on a well defined Hull-White model, and study the yield curve dynamics in the short-term and long-term movements. We will show how the zero-coupon bonds and bond options are priced using the risk-neutral valuation. Further, we will show with details that how the swaptions are priced in agreement with the Black's formula, and explain how those interest rate derivative products can be evaluated through a Monte Carlo method in general. Analytical approximations leading to those pricing formulas are also presented, as well as the comparison analysis between analytical and simulation methods. We will finally compute the implied volatility in agreement using the Black formula from the interest rate simulation results, and explore the "volatility smile" problem for the swaptions.

2 Methodology

2.1 Preliminaries

When we assume for a stochastic interest rate process, the current rate and yield curve does not completely specify how rates will move in the future. This section will introduce the reader all needed concepts and terminology involved in this project.

2.1.1 Bank account and Zero-coupon Bonds

Assume the short-term interest rate is $r = (r_t)_{t\geq 0}$, and the bank account B_t is a saving account in the sense that accumulates wealth in a rate of r_t. Mathematically:

$$dB_t = B_t r_t dt$$

When the interest rate is deterministic of time, we have a deterministic discount factor $e^{-\int_t^T r_u du}$ in the risk-neutral world, which allows us to price the equities in the market. However, when rates are stochastic, the prices of equity derivatives also need to be adjusted, in order to incorporate the uncertainty in interest rates. The simplest interest rate derivative is a T-maturity zero-coupon bond. It is a contract that agrees to only pay the principal amount L at the time of maturity T. Let $P_t(T)$ denotes the price of of zero-coupon bond with unit face value at the time t, so that $P_T(T) = 1$. The ownership of most bonds is negotiable, which means they can be traded in the market anytime before T. If we use the bank account as the numeraire asset, and apply the fundamental theorem of finance to avoid arbitrage, the bond price at time t can be given by:

$$P_t(T) = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r_u du}|F_t]$$

where $\mathbb{Q}$ is the risk-neutral probability measurement, and F_t is the filtration at time t. In real life, the bond price is determined at the time of issue; however, since we can trade the bonds within a period of time, we can also observe a fluctuated *market price* over time.

2.1.2 Yield rate and Yield Curve

Associated with every zero-coupon bond price for a given term is a bond yield and a spot rate.

$$P_t(T)e^{-y_t(T)(T-t))} = 1 \quad \Leftrightarrow \quad y_t(T) = -\frac{\log(P_t(T))}{T-t}$$

$y_t(T)$ is called the bond yield associated with T-maturity bond at the time t, where $t \leqslant T$. We can interpret it as the discount rate that, when applied to all the future cash flows, gives a zero-coupon bond price equal to the market price of the bond today. Since a zero-coupon bond with unit par value has no intermediate payment, we can view an asset generating cash flows as a portfolio with multiple zero-coupon bonds maturing at different times. The *market price* of the zero-coupon bonds fluctuates over time to reflect the dynamic interest rate movement in the real-world market, as a result, the yield rate provide us a way to analytically study the stochastic process of the interest rate and then embody this uncertainties in order to fairly price other equities.

If we have several zero coupon bonds with different maturities we are able to plot the different zero-rates as a function of the maturity, this plot is the *yield curve*, also called the term structure of interest rates. The yield curve provides an intuitive way to look at the long-term interest trend given the current market condition. The applications of bond yield and yield curves would go through the whole project, we will explore and interpret them with more details in the analysis part.

2.1.3 Bond Options

Like the stocks, the bond is also a type of tradable asset in the market, so there are options which have the bond as the underlying asset. Consider a bond option that gives the owner the right to purchase a U-maturity bond with the strike price K at the time T. At the point T when the option matures, the value of the option is calculated as:

$$V_T^{bond} = (P_T(U) - K)_+$$

Using the bank account as the numeraire, we can calcaulate the option price at time $t \leqslant T$ as the expected discounted payoff:

$$V_t^{bond} = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r_u du}(P_T(U) - K)_+]$$

Solving this equation is computational costly, since r_u is a stochastic process, r_u and $e^{-\int_t^T r_u du}$ are not independent, we need to find the joint distribution of those two terms. Alternatively, we could use T-maturity bond as the numeraire and use change of measurement to move into a T-maturity bond risk-neutral world. We will derive the explicit formula in the following section with a specific interest rate model.

2.1.4 Swap and Swaption

One of the most important interest derivatives is the *interest rate swap* (IRS), which is an agreement between two parties to exchange cash flows in the future. An interest rate swap receiver agrees to pay the other party a fixed rate of cash on a notional principal to exchange for a cash flow with a floating rate from the swap payer. Therefore, the owner of a payer swap can be viewed as longing a floating rate bond and shorting a fixed rate bond:

$$Swap_{payer} = Bond_{float} - Bond_{fixed}$$

Most of the times, the widely used floating rate is the London Interbank Offer Rate (LIBOR), but the objective of this project is to explore the stochastic interest rates in a no-arbitrage world, the floating leg payments are proportional to the forward rate, which are consistent with the bond yield curve. As a result, the swap rate S_t is a fixed rate determined at the time t (t is the time of entering the contract) such that the future cash flows on the floating leg and the fixed leg have the same present value under the risk-neutral context, so that there is no arbitrage opportunity. From this point of view, for a swap with a specific structure $\{\tau_0, \tau_1, ...\tau_n\}$ (τ_0 is the first reset), the swap rate is given by:

$$S_t = \frac{P_t(\tau_0) - P_t(\tau_n)}{\sum_{k=1}^{n} P_t(\tau_k)\Delta\tau_k}$$

.

A **swaption** is regarded as an option on an interest rate swap. It is a contract which gives the owner the right to enter into an IRS at certain swap-rate X at the maturity time T, which is specified at the time of signing the swaption contract. A swap payer allows the owner to transform an asset that earns a fixed rate of interest into an asset earning a floating rate of interest. Therefore, the swaption contract can be used as a method to manage risk to eliminate the uncertainties from variable interest payments. If entering the swaption, we could alternatively engage in the market with a different fixed rate.

To price the swaption from the perspective of a payer, or the call option on the underlying IRS, we can start thinking at the time maturity T, the value to the owner is:

$$V_T = (S_T - X)_+ \sum_{k=1}^{n} P_T(\tau_k)\Delta\tau_k$$

We introduce an annuity, which can be regarded as an asset which has $\Delta\tau_k$ positions on the bond at time τ_k. Then the price of the swaption can be writen as:

$$V_T = (S_T - X)_+ A_T$$

recall the swap rate:

$$S_t := \frac{P_t(\tau_0) - P_t(\tau_n)}{A_t}$$

Since the annuity is a portfolio of zero-coupon bonds, we can use the interest rate model to obtain a fair price. The only randomness in swaption pricing is from the interest rate. Then in principal, we can compute the value according to the Q-dynamic interest rate.

Alternatively, we can change the numeraire. A numeraire for a derivative in financial mathematics is the standard to which we relate the value of the derivative. If we consider the relative price of the swaption to the annuity, and our world is forward risk nertual with respect to $A(t)$, then the relative price of the swaption is the expected future swaption values observed at time t. Mathematically, let $\mathbb{Q}_A$ represents the probability measurement with $A(t)$ as numeraire, then the following equation can be derived:

$$\frac{V_t}{A_t} = \mathbb{E}_t^{\mathbb{Q}_A}[(S_T - X)_+]$$

Now if we carefully look at the expression for the swap rate, it can be regarded as the relative price to $A(t)$ of a portfolio to long τ_0-maturity bond and short τ_n-maturity bond, where τ_0 is the time of first reset and τ_n is the time of last payment. This portfolio is under the self-financing condition and it is a risk-neutral martingale. Then if we change the probability measure under a new numeraire, then the new relative price would also be martingale with the new probability measure. As a result, S_t is $\mathbb{Q}_A$-martingale and its drift must vanish, and the randomness only come from a Brownian motion term under measurement $\mathbb{Q}_A$.

2.1.5 Lognormal Swap-rate Model (LSM) and Black Formula

The Log-normal Swap-rate Model (LSM) provides a more straightforward way to value the swaption when using the anuity as numeraire. It assumes the underlying swap rateat the maturity of the option(S_T) is log normally distributed. Mathematically:

$$dS_t = S_t \omega_t dW_t^A$$

$$for\ some\ deterministic\ function\ \omega_t,\ and\ W_t^A\ is\ a\ \mathbb{Q}_A\ Brownian\ Motion$$

by $Itô's$ lemma,

$$S_T = S_t e^{-\frac{1}{2}\int_t^T \omega_u^2 du + \int_t^T \omega_u^2 dW_u^A}$$

With this relationship, we are able to use the **Black model**, in which the volatility ω_t is particularly constant. To price the European call swaption at time t with the strike X and maturity T:

$$V_t = A_t(S_t\Phi(d_+^B) - X\Phi(d_-^B))$$

where

$$d_\pm^B = \frac{log(\frac{S_t}{X}) \pm \frac{1}{2}\Omega_{t,T}^2}{\Omega_{t,T}},\ and\ \Omega_{t,T}^2 = \int_t^T \omega_u^2 du = \omega^2(T-t)$$

The constant ω is the **Black implied volatility** of the swaption.

2.2 Models for interest rates

The term structure of interest rates measures the relationship between the yields on the securities and the term to maturity. By offering interest rates for future periods, the structure embodies the market anticipations of future events. An explanation to the yield curve provides a way to explore how changing the interest rate model parameters can impact the yield curve.

2.2.1 Vasicek Model

One simple interest rate model is the Vasicek Model, in which the short-term interest rate satisfies the SDE:

$$dr_t \;=\; \kappa(\theta - r_t)dt \;+\; \sigma dW_t^{\mathbb{P}},$$

where r_0 is the current level of interest rate, κ, θ, and σ are positive constants. This model suggests a mean-reverting process about the interest rate, where θ is reverting level and the long-term mean, κ is the reverting rate. This particular process is also referred as the Ornstein-Uhlenbeck Process. As t tends to infinity, r_t is asymptotically Gaussian with mean θ and asymptotic variance $\frac{\sigma^2}{2\kappa}$.

In the previous section to price the derivatives, we take the expected discounted values with the risk-neutral probability $\mathbb{Q}$, instead of the real-world $\mathbb{P}$. Therefore, to achieve the risk-neutral valuation, we need to change the probability measurement. Applying the *Girasnov's theorem* in the financial market, we are able to find a *market price of risk* λ_t such that:

$$dW_t^{\mathbb{Q}} = -\lambda_t dt + dW_t^{\mathbb{P}}$$

so that $W^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian Motion term. Therefore, if the $\mathbb{P}$-dynamics of the short-term interest rate process is:

$$dr_t \;=\; \kappa^{\mathbb{P}}(\theta^{\mathbb{P}} - r_t)dt \;+\; \sigma dW_t^{\mathbb{P}},$$

then the $\mathbb{Q}$-dynamics of the short-term interest rate process will become:

$$dr_t \;=\; (\kappa^{\mathbb{P}}(\theta^{\mathbb{P}} - r_t) - \lambda_t \sigma)dt \;+\; \sigma dW_t^{\mathbb{P}},$$

In other words, we adjusted the drift in the interest rate process so that it embodies the market uncertainty to the risk-neutral dynamic process. There are multiple choices of to maintain the Vasciek Model's form, one particular choice is:

$$\lambda_t = a + br_t, \qquad \Longrightarrow \qquad dr_t = \kappa^{\mathbb{P}}(\theta^{\mathbb{Q}} - r_t)dt + dW_t^{\mathbb{Q}}$$
$$\kappa^{\mathbb{Q}} = \kappa^{\mathbb{P}} + b, \;\; \theta^{\mathbb{Q}} = \theta^{\mathbb{P}} - \frac{a\sigma}{\kappa^{\mathbb{P}} + b\sigma}$$

We should be aware that $\kappa^{\mathbb{P}}, \theta^{\mathbb{P}}, \sigma$ are parameters to describe the real-world asset price and variance processes, they can be estimated by calibrating the historical data. The parameters a and b are the drift adjustment parameters that allow us to calibrate the observed market price. There are many other choices of λ_t such as $\lambda_t = a$, what critical is that the *market price of risk* (λ_t) is able to explain the real market. Every choice to define λ_t will imply a set of interest rate derivative prices. Calibrations means finding the model parameters that minimize the difference between the implied prices and market prices.

Once we've calibrated all the parameters, we can compute the $\kappa^{\mathbb{Q}}$ and $\theta^{\mathbb{Q}}$. Now with the $\mathbb{Q}$-dynamic interest rate, we are able to determine the associated derivative valuations.

2.2.2 Hull-White One Factor model

To address the interest rate movement in the long term, in our project, we consider an extended version of the Vasicek Model, which is also called the *Hull-White One Factor model*:

$$dr_t \;=\; \kappa(\theta(t) - r_t)dt \;+\; \sigma dW_t^{\mathbb{P}},$$

where W_t is a $\mathbb{P}$-Brownian Motion, κ is mean reversion rate and θ is mean reversion level.
When $r_t \geq \theta$, the drift term is negative which pushes r_t downwards towards to θ. When $r_t \leq \theta$, the drift term is negative which pushes r_t upwards towards to θ. Thus, θ is named as the mean reversion level.

As the same as the Vasicek Model, to corporate the real-world market risk, the $\theta(t)$ is determined by calibrating the observed market price of swaptions, so that the Hull-White model guarantees the no-arbitrage conditions and be able to fit today's term structure [1]. But in our project, there are no observed market prices, we cannot calibrate the market price. Our main objective is to study the behaviour of $\mathbb{Q}$-dynamic r_t, so we choose to ignore the real-world condition, and simply let r_t and $\theta(t)$ satisfy the SDE:

$$dr_t \;=\; \alpha(\theta(t) - r_t)dt \;+\; \sigma dW_t^{\mathbb{Q}_1}$$

$$d\theta_t = \beta(\phi - \theta_t)dt + \eta dW_t^{\mathbb{Q}_2}$$

where $W^{\mathbb{Q}_1,\mathbb{Q}_2} = (W_t^{\mathbb{Q}_1,\mathbb{Q}_2})_{t\geq 0}$ are independent **risk-neutral** Brownian Motions, and $\theta = (\theta_t)_{t\geq 0}$ denotes the long-run interest rate which itself is stochastic with all known parameter values. This is a type of two-factor interest rate model.

Instead calibration, we simulate the risk-neutral paths for the short-term rate, them we are able to price the derivative. For base parameters, we use the following:

$$r_0 = 2\%,\ \alpha = 3,\ \sigma = 1\%,\ \theta_0 = 3\%,\ \beta = 1,\ \phi = 5\%,\ \eta = 0.5\%$$

To obtain the solution to this interest rate model and the distribution of r_T, we consider the process $X_t = r_t e^{\alpha t}$ and apply the *Itô* lemma, then we have the following:

$$dX_t \;=\; e^{\alpha t}dr_t \;+\; \alpha r_t e^{\alpha t}dt \;+\; d[r, e^{\alpha t}]_t \;=\; \alpha\theta_t e^{\alpha t}dt \;+ \sigma e^{\alpha t}dW_t^{\mathbb{Q}_1}$$

Solving this SDE will give us:

$$\int_t^T r_u du \;=\; \frac{r_t}{\alpha}(1-e^{-\alpha(T-t)}) + \phi(T-t) + \frac{1-e^{-\beta(T-t)}}{\beta}(\theta_t-\phi) - \frac{\phi}{\alpha}(1-e^{-\alpha(T-t)}) - \frac{\theta_t-\phi}{\alpha-\beta}(e^{-\beta(T-t)} - e^{-\alpha(T-t)})$$

$$+\,\eta\int_t^T(\frac{1-e^{-\beta(T-s)}}{\beta} \;-\; \frac{e^{-\beta(T-s)}-e^{-\alpha(T-s)}}{\alpha-\beta})dW_s^{\mathbb{Q}_2} \;+\; \frac{\sigma}{\alpha}\int_t^T(1 \;-\; e^{-\alpha(T-s)})dW_s^{\mathbb{Q}_1}$$

As T approaches infinity, r_T is asymptotically Gaussian distributed. Then using the bank account as numeraire, we are able to find the prices of interest rate derivatives.

The full derivation can be found in the appendix.

2.2.3 Bond Prices and Affine Model

Under the model we described above, recall the price of a T-maturity zero-coupon bond at time t can be expressed as:

$$P_t(T) = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r_u du}]$$

with the probability measurement $\mathbb{Q}$ using the bank account as the numeraire. Then we can write the analytic expression for the price of a bond option:

$$P_t(T) = e^{A_t(T) - B_t(T)r_t - C_t(T)\theta_t}$$

where

$$A_t(T) \;=\; (T-t)(-\phi + \frac{\sigma^2}{2\alpha^2} + \frac{\eta^2}{2\beta^2}) + (1-e^{-\alpha(T-t)})(\frac{\phi}{\alpha} - \frac{\sigma^2}{\alpha^3} + \frac{\eta^2}{\alpha\beta(\alpha-\beta)}) + (1-e^{-\beta(T-t)})(\frac{\phi}{\alpha} - \frac{\eta^2}{\beta^3} - \frac{\eta^2}{\beta^2(\alpha-\beta)})$$

$$+\;(1-e^{-2\alpha(T-t)})(\frac{\sigma^2}{4\alpha^3} + \frac{\eta^2}{4\alpha(\alpha-\beta)^2}) \;+\; (1-e^{-2\beta(T-t)})(\frac{\eta^2}{4\beta^3} + \frac{\eta^2}{2\beta^2(\alpha-\beta)} + \frac{\eta^2}{4\beta(\alpha-\beta)^2})$$

$$-\;(1-e^{-(\alpha-\beta)(T-t)})(\frac{\eta^2}{\beta(\alpha^2-\beta^2)} + \frac{\eta^2}{(\alpha+\beta)(\alpha-\beta)^2}) \;-\; (e^{-\beta(T-t)} - e^{-\alpha(T-t)})\frac{\phi}{\alpha-\beta}$$

$$B_t(T) \;=\; \frac{1-e^{-\alpha(T-t)}}{\alpha}$$

[1] Wouter Tilgenkamp, BACHELOR OF SCIENCE in Applied Mathematics,20-21

$$C_t(T) \;=\; \frac{1-e^{-\beta(T-t)}}{\beta} - \frac{e^{-\beta(T-t)} - e^{-\alpha(T-t)}}{\alpha-\beta}$$

This is an Affine model that relates zero-coupon bond prices to the interest rate. It is particularly useful for deriving the yield curve – the process of determining spot rate model inputs from observable bond market data. The main advantage of Affine models is tractability. Having deterministic solutions for bond yields is useful since otherwise yields need to be simulated with Monte Carlo methods or solution methods for PDEs. There are several reasons to take bond yields into considerations:

- Forecasting: long-maturity bonds yields are expected values of average future short yields. This means the current yield curve contains information about the future path of the economy.
- Derivative Pricing and Hedging: the price of financial securities are computed from a given model of the yield curve, such as zero-coupon bonds,swaps, caps and floors, futures, and options on interest rates.

The full derivation for the Affine model structure can be found in the Appendix.

2.2.4 T-maturity bond Numeraire for Bond Options

Recall the equation to price the bond (call) option if we use the bank account as the numeraire:

$$V_t^{bond} = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r_u du}(P_T(U)-K)_+]$$

In this project, we are supposed to evaluate the bond matures at $T2$ and the bond option matures at $T1$ ($T1 \leqslant T2$). The bond option pricing can be rewritten as:

$$V_t^{Bond} \;=\; \mathbb{E}_t^{\mathbb{Q}}\Big[e^{-\int_t^T r_s ds}\cdot(P_{T_1}(T_2)-K)_+\Big]$$

Under this circumstance, we use a bond of maturity T_1 as the numeraire asset, we can derive the analytic expression for the price of an European call bond option matures at time T_1 on a zero-coupon bond with principal 1 and stike K maturing at time T_2.

$$V_t^{Bond} \;=\; P_t(T_1)\mathbb{E}_t^{\mathbb{Q}^{T_1}}\Big[(X_{T_1}-K)_+\Big]$$

Where we implemented a new stochastic process:

$$X_t := \frac{P_t(T_2)}{P_t(T_1)}$$

X_t is the relative price of a tradable bond to a $T1$-maturity bond, then it must be $\mathbb{Q}^{T_1}$-martingale, which means we can completely eliminate the drift term in the stochastic process if we change the probability measurement from bank-account risk-neutral to T_1-maturity bond risk-neutral, we can find the distribution of X_t under $\mathbb{Q}^{T_1}$ measurement. By *Itô* lemma,

$$X_T \;\stackrel{d}{=}\; X_t\cdot e^{-\frac{1}{2}\Sigma^2\,+\,\Sigma Z},\; Z\sim N(0,1)\ under\ \mathbb{Q}^{T_1} measurement$$

where

$$\Sigma_{t,T}^2 \;=\; \frac{\sigma^2}{2\alpha^3}(e^{-\alpha T_2}-e^{-\alpha T_1})^2(e^{2\alpha T_1}-e^{2\alpha t})$$
$$+\,\eta^2\Bigg[\frac{1}{2\beta}(e^{2\beta T_1}-e^{2\beta t})(e^{-\beta T_2}-e^{-\beta T_1})^2[\frac{1}{\beta^2}+\frac{1}{(\alpha-\beta)^2}+\frac{2}{(\alpha-\beta)\beta}]+\frac{1}{2\alpha}(e^{2\alpha T_1}-e^{2\alpha t})\frac{(e^{-\alpha T_2}-e^{-\alpha T_1})^2}{(\alpha-\beta)^2}$$
$$+\frac{1}{\alpha+\beta}(e^{(\alpha+\beta)T_1}-e^{(\alpha+\beta)t})\cdot 2\cdot(e^{-\alpha T_1}-e^{-\alpha T_2})(e^{-\beta T_2}-e^{-\beta T_1})(\frac{1}{(\alpha-\beta)^2}+\frac{1}{(\alpha-\beta)\beta})\Bigg]$$

Finally, we can obtain the bond option price using the Black Scholes option pricing formula:

$$V_t^{Bond} = P_t(T_2)\Phi(d_+) - KP_t(T_1)\Phi(d_-)$$

where

$$d_{\pm} = \frac{log(\frac{P_t(T_2)}{KP_t(T_1)}) \pm \frac{1}{2}\Sigma^2}{\Sigma}$$

We let $K = \frac{P_0(T_2)}{P_0(T_1)}$, and we will examine the affect of different strike K to the resulted bond option prices.

The full derivation for this bond option price can be found at the Appendix.

2.2.5 Black Implied Volatility from Hull-White Swaption price

Now that we have specified a Hull-White model, we focus on the implementation details and further practical aspects of applying the theory in practice. Given the short-term interest rate model, we can find the swaption values in theory. From the perspective of a swaption buyer, we assume the maturity time T is the reset time τ_0 for the underlying swap. Then the value of the European swaption call at the maturity T with strike price X is:

$$V_T^{Swap} = (1 - \sum_{k=1}^{n} \pi_k P_T(\tau_k))_+$$

$$\pi_k = \begin{cases} 1 + X\Delta\tau_k, & \text{if } k = n \\ X\Delta\tau_k, & \text{otherwise} \end{cases}$$

recall under our Hull-White One Factor model, the bond price is:

$$P_t(T) = e^{A_t(T) - B_t(T)r_t - C_t(T)\theta_t}$$

As we derived above,

$$B_t(T) = \frac{1 - e^{-\alpha(T-t)}}{\alpha} \quad and \quad C_t(T) = \frac{1 - e^{-\beta(T-t)}}{\beta} - \frac{e^{-\beta(T-t)} - e^{-\alpha(T-t)}}{\alpha - \beta}$$

given $\alpha = 3$ and $\beta = 1$, both $B_t(T)$ and $C_t(T)$ are strictly positive, so $P_t(T)$ decreases as r and θ increase. Then there must be two values r^* and θ^* such that

$$\sum_{k=1}^{n} \pi_k e^{A_T(\tau_k) - B_T(\tau_k)r^* - C_T(\tau_k)\theta^*} = 1$$

We use $P_t^*(T)$ to denote the T-maturity bond price at time t evaluated with r^* and θ^*.

With some tricks in the calculation, the swaption option value in the Hull-White model at the maturity T can be expressed as:

$$V_T^{HW} = \sum_{k=1}^{n} \pi_k((P_T^*(\tau_k) - P_T(\tau_k))_+)$$

note this is actually a collection of put options on the τ_k-maturity bond with maturity time T. We have derived the Black formula to price the bond options, therefore, substituting the strike X,T_1, T_2 with $P_t^*(T)$,T and τ_k respectively, we can obtain the price of Hull-White swaption price:

$$V_t^{HW} = \sum_{k=1}^{n} \pi_k[P_T^*(\tau_k)P_t(T)\Phi(-d_-^k) - P_t(\tau_k)\Phi(-d_+^k)]$$

$$\pi_k = \begin{cases} 1 + X\Delta\tau_k, & \text{if } k = n \\ X\Delta\tau_k, & \text{otherwise} \end{cases}$$

$$d_{\pm}^k = \frac{log(\frac{P_t(\tau_k)}{P_T^*(\tau_k)P_t(T)}) \pm \frac{1}{2}\Sigma_{t,\tau_k}^2}{\Sigma_{t,\tau_k}}$$

Recall the swaption value under the LSM,

$$V_t^{LSM} = A_t(S_t\Phi(d_+^B) - X\Phi(d_-^B))$$

where

$$d_\pm^B = \frac{log(\frac{S_t}{X}) \pm \frac{1}{2}\Omega_{t,T}^2}{\Omega_{t,T}}, \ and \ \ \Omega_{t,T} = \omega\sqrt{T-t}$$

ω is the Black implied volatility we are interested in. By the law of one price, V_t^{LSM} and V_t^{HW} should be the same. Therefore, we can compute the Black implied volatility from the Hull-White swaption prices. This can be realized with some root-finding algorithms like interpolation and Newton-Rhapson method.

Specifically in this project, we will initially set the strike as the current swap-rate, but the Black implied volatility is the characteristic of the swaption, it inherently has some relationships with the strike price, we will explore it with details.

2.2.6 Morte Carlo and Euler-scheme Simulation

Since this project has explicitly specified the interest rate model, we can price the model-implied zero-coupon bonds, bond options, and swaption according to the analytical expressions we have derived above. If we have observed the market price, we would like to compare the **model-implied derivative prices** to the **market-observed derivative prices** to evaluate the model accuracy and derive the implied volatility. However, without actual real-world data, we can simulate the interest rate paths, but still in a $\mathbb{Q}$ risk neutral world to simplify the computation.

Firstly, we use an **Euler-scheme for Diffusion** to generate risk-neutral interest rate paths. Suppose a stochastic process variable X_t satisfies:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

the Euler Scheme simulates a discrete version of the SDE$\{\hat{X_0}, \hat{X_h}, \hat{X_{2h}}, ..., \hat{X_{mh}}\}$ where m is the number of time step, h is a constant step-size. We use a simple Euler scheme:

$$\hat{X_{kh}} = X_{(\hat{k-1})h} + \mu((k-1)h, X_{(\hat{k-1})h}) + \sigma((k-1)h, X_{(\hat{k-1})h})\sqrt{h}Z_k$$

where the Z_k's are independent and identified distributed standard normal random variables.

Given the SDEs of r_t and θ_t, we can use this recursive formula to forecast the future interest rate with small intermediate times.

$$P_t(T) = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r_u du}]$$

after partitioning small enough h, we approximate the integral term by summing all $\hat{r_t}$ and remove the expectations. Finally, we simulate large amount of paths to take the average results. With the simulated zero-coupon bond prices, we are able to continue to price bond options, swap rate and swaptions.

3 Results Analysis

3.1 Bond Yield and Time to Maturity

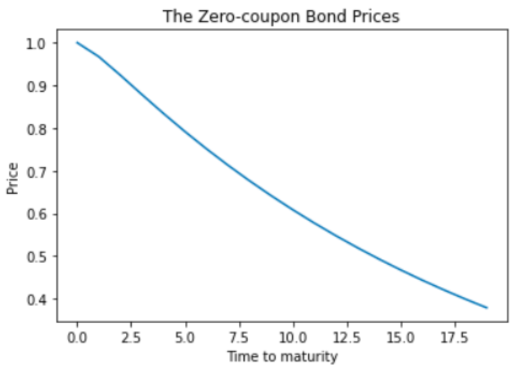

Figure 1: The Zero-coupon Bond Prices

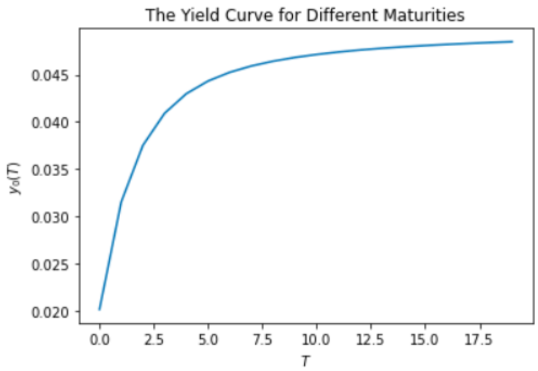

Figure 2: The Yield Curve for Different Maturities

Using the formula derived in section 2.2.3 and the parameters given, we calculate the zero-coupon bond prices at time 0 for different maturities from 0 to 20 years. As is shown in the Figure 1, when the time to maturity of a zero-coupon bond is small, its price at time 0 will become close to one. However, as the time to maturity of the zero-coupon bond increases, the corresponding bond prices will decrease. In the simplest case of calculating the bond price, where the interest is a fixed numeric value, the bond price will also decrease as the time to maturity increases. When the time to maturity increases, the discount factor which is used in calculation will increase, leading to the drop in the bond price. This idea also applies under this circumstance, when the interest rate is a stochastic process.

After obtaining the bond prices for different maturities, we compute the yields for these bonds. A bond's yield to maturity (YTM) is the internal rate of return required for the present value of all the future cash flows of the bond (face value and coupon payments) to equal the current bond price. In this project, the bonds we are analyzing are all zero-coupon bonds with face value = 1. Hence, the yields can be computed using the following analytical formula:

$$y_t(T) = \frac{1}{T-t}(B_{t,T}r_t + C_{t,T}\theta_t - A_{t,T})$$

Where T is the time to maturity, t is the current time. And $A_{t,T}$, $B_{t,T}$ and $C_{t,T}$ are calculated using the formula from section 2.2.3. In this case, we are interested in the yields at time $t = 0$. The Figure 2 demonstrates the yield $y_0(T)$ for time to maturity ranging from 0 to 20. The yield has an overall increasing trend as the time to maturity increases. Different from the bond prices, which has a consistent steepness over all T's, the steepness of the yield curve for different maturities has a decreasing pattern as the T increases. The reason for that is that debt issued for a longer term generally carries greater risk. From the investors perspective, they want to be compensated with higher yields for assuming the added risk of investing in longer-term bonds. Additionally, as T increases, the bond price also decreases. The decreasing trend in the price and the increasing trend in the yield both illustrate that the long term bonds are associated with higher risk.

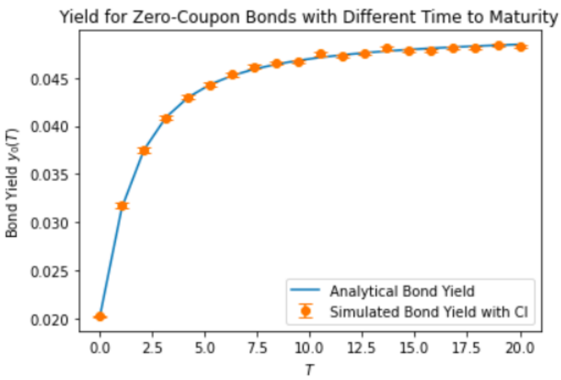

Figure 3: Yield for Zero-Coupon Bonds with Different Time to Maturity

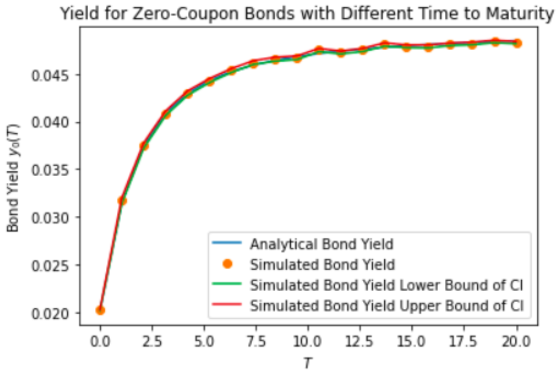

Figure 4: Yield for Zero-Coupon Bonds with Different Time to Maturity

To verify whether the analytical plot in the Figure 2 is valid, we then implement the Euler-scheme method to generate the interest rate paths and obtain the Monte Carlo estimate of bond yields, using the two factor interest model with the base parameters:

$$dr_t = \alpha(\theta(t) - r_t)dt + \sigma dW_t^{\mathbb{Q}_1}$$

$$d\theta_t = \beta(\phi - \theta_t)dt + \eta dW_t^{\mathbb{Q}_2}$$

The details of the methods using here have been elaborated in section 2.2.6. The Figure 3 simultaneously displays the analytical bond yield curve from Figure 2 and the simulated bond yield values with their confidence bands at different maturities. We can find that the simulated points are very close to the analytical curve, which means that using either the analytical method or the simulation method, we can obtain similar results on the yield curves.

3.2 Model Parameters' Impact

To better understand the interest rate model, we then test how the changes in the parameters in the model can impact the yield curve. In this case, we conduct the analysis on α, β, ϕ, r_0, θ_0, η and σ.

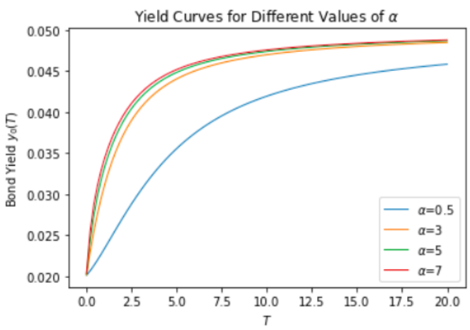

Figure 5: Yield Curves for Different Values of α

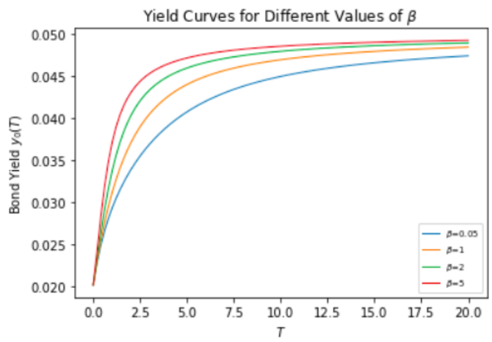

Figure 6: Yield Curves for Different Values of β

The Figure 5 and Figure 6 shows yield curves for different α and β in the model, while all the other parameters are held constant. In the short run, from the Figure 5, as α increases, the yield curve tends to move upwards. Similar patterns are found in the Figure 6, an increasing β can lead to an upward move in the yield curve. In other words, for the bonds with the same maturity, when we have a larger α or β, we tend to obtain a higher yield from the bond. However, in the long run, the yields of different values of α's tend to converge at certain level. This also applies to the long run yield pattern for different β's.

The α and β are also called mean-reversion rate in our two-factor interest rate model. Mean reversion is a financial term for the assumption that the certain asset will tend to move to the average over time. Hence in this two-factor case, the α represents the speed of the short-term interest going to the average, and the β represents the speed of long-term interest going to the average level. In terms of the yields, the model with high α or β tend to merge to the average at a faster rate compared with the model with low α or β.

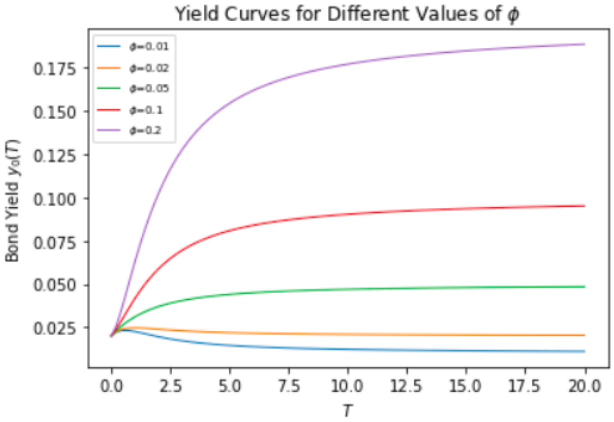

Figure 7: Yield Curves for Different Values of ϕ

The Figure 7 displays the yield curves for different selections of ϕ in the model. For the bonds with the same maturity, a large ϕ will lead to a high yield, either in short run or long run. Since the ϕ represents the long run mean reversion level of this model, it determines the long-run movements of the interest rates. The model with high ϕ means that the interest rate movement will become stable at a high level, which leads to the higher yield compared with the model with low ϕ values.

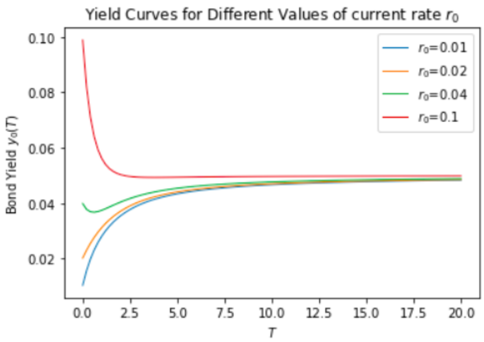

Figure 8: Yield Curves for Different Values of r_0

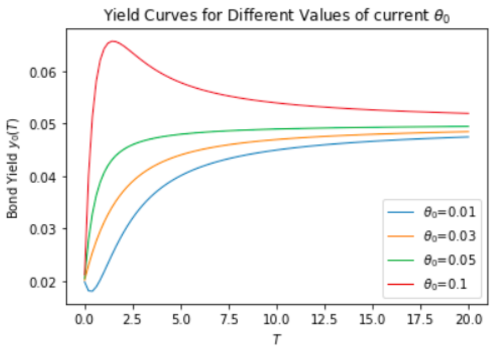

Figure 9: Yield Curves for Different Values of θ_0

The Figure 8 and the Figure 9 shows the yield curve patterns for different values of the initial short rate r_0 and the long-run rate θ_0. In the short run, large r_0 and θ_0 will have a higher yield than that of small r_0 and θ_0. However, in the long run, the yield will converge at the same level no matter how the initial value varies.

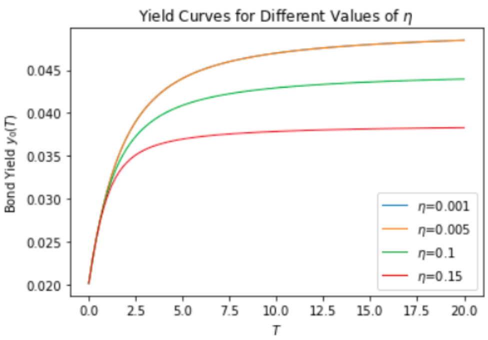

Figure 10: Yield Curves for Different Values of η

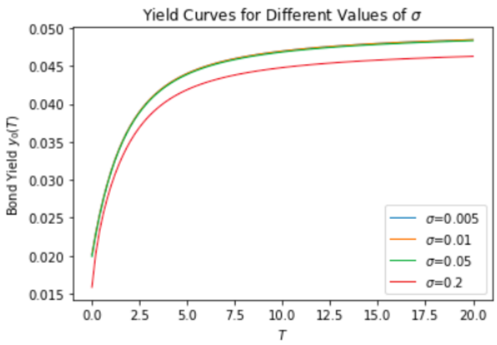

Figure 11: Yield Curves for Different Values of σ

η and σ both refer to the volatility in the two-factor model. η is the long-term volatility and the σ is the short-term volatility. The model with higher η or σ means that there are higher interest risk associated with the bond yield. As a result, the higher risk brings higher return. As is shown in both of the Figure 10 and the Figure 11, the model with higher volatility always has higher yields. The differences are more significant when varying the short-term volatility.

The table below summarizes all the model parameter's impact on the time 0 yield curve of the bond in short-run and long-run.

The Impact on the **Yield** when Parameter Values **Increase**		
Model Parameter	**Short Term Impact**	**Long Term Impact**
α (Short-term Mean-reversion Rate)	Increase	No Significant Impact
β (Long-term Mean-reversion Rate)	Increase	Increase
ϕ (Long-term Mean-reversion Level)	Increase	Increase
σ (Short-term Volatility)	Increase	Increase
η (Long-term Volatility)	Increase	Increase
r_0 (Initial Value of Short Rate)	Increase	No Significant Impact
θ_0 (Initial Value of Long-run Int' Rate)	Increase	No Significant Impact

Table 1: Summary of Model Parameters Impact on The Bond Yield

3.3 Bond Options

In this part, we are aiming to investigate the relationship between the bond option price and the strike price. The option pays $(P_{T_1}(T_2) - K)_+$ at T_1 where the strike price $K = P_0(T_2)/P_0(T_1)$ and $T_1 \leqslant T_2$. The analytical formula of computing the bond price has been derived in section 2.2.4. The project does not specify the values of T_1 & T_2. Without the loss of generality, we use $T_1 = 1$ and $T_2 = 10$ in this situation. Meanwhile, apart from using the analytical formula to compute the bond option price at time 0, we also use the Monte Carlo method to simulate the bond option price using the give interest rate model. In the Figure 12, the blue curve is the analytical plot of the bond option prices versus different scale factor on the strike price K. For example, when the scale factor $= 0.9$, the option is priced with the strike at $0.9K$.

As is shown in the graph, when this option is in-the-money, the price of the option has a decreasing trend as the strike price increases. However, when the option is at-the-money or out-the-money, the price of the option is extremely close to zero. To explain this phenomenon, we need to start from the payoff function of this option. As is mentioned above, the option pays $(P_{T_1}(T_2) - K)_+$ at T_1. Under this circumstance, the maximum value $P_{T_1}(T_2)$ can take is 1, when $T_1 = T_2$. Therefore, when the strike price exceeds one, the likelihood of this option having a positive payoff is zero. As a result, there will be nobody willing to purchase this option.

The red dots are the simulated bond option prices when the strike price scale factor = 0.8, 0.85,...,1.15,1.2. The simulated points all fall near the analytical plot, which means that the method we have implemented is reasonable.

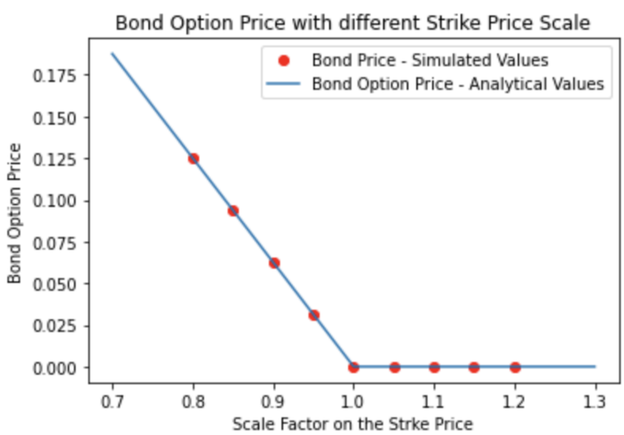

Figure 12: Current Bond Option Price vs Strike Price Scale Factor

Then we make investigations on how the different values in T_1 and T_2 can affect the option price patterns. As the Figure 13 illustrates, when T_1 and the strike price scale factor are held constant, a larger T_2 will lead to a lower price of the option.

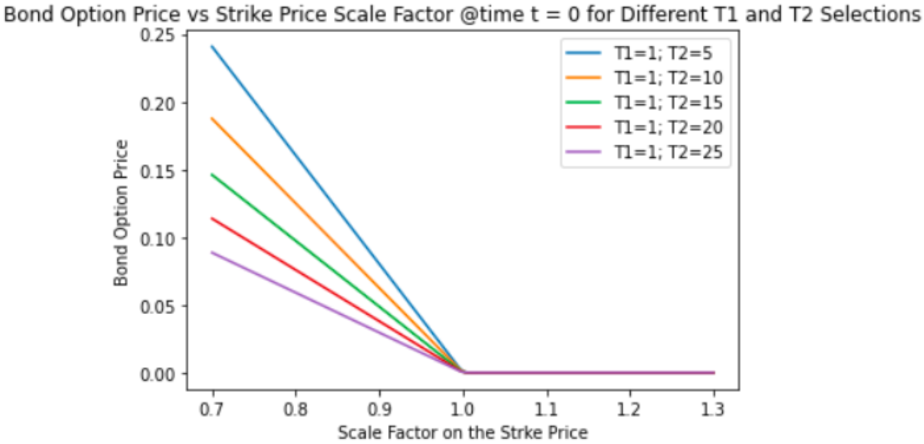

Figure 13: Current Bond Option Price for different Strike Price Scale for Different T1 and T2 Selections

3.4 Black Implied Volatility

Implied volatility can explain the uncertainty level on an option's underlying equity, and the corresponding changes to the option's trading prices as the equity's value swing. Intuitively, given a price of an option, the implied volatility provides a signal about the future volatility of the underlying equity, which is the swap in our study. It is important because it acts as an alternative measure for the actual value of the option. The option price should be higher when the implied volatility is high. It is important for investors to understand the implied volatility because it offers traders the insights into what the market is thinking about the future, and thus helps them to indicate a good time to enter or exit a position in a security. Different from the **historical volatility**, which anticipate the future solely based on the past observed information, the implied volatility additionally takes account for the market expectation.

If we compare the prices of current at-the-money (ATM), in-the-money (ITM) and out-the-money (OTM) options in the market, we can look at the difference between the IV and the moneyness of the options. When considering the options on the currency, the implied volatility for puts and calls increases as the strike increases, because most of time the options are traded OTM at inflated prices. In other words, a **Volatility smile** is able to describe the relationship between IV and strike on the same expiration-date options. However, it is quite different when the options are traded on the equities and instead we expect a **Volatility Skew**.

Under the Black LSM, the IV is the only unknown parameter to determine the swaption price. Therefore, we can calculate the IV given the market price, which is computed by the simulation. However, computing the implied volatility in python is computational costly. Considering the black theory suggest a fine formula to calculate the Greek "Vega", which is the partial derivative of the option respect to the implied volatility, we decide to apply the **Newton Raphson** method to find an optimized solution. However, since the convergent time and first guess become more critical as the strike price far from the ATM strike, and when the strike price exceeds 1.05 times the current swap rate, the swaption price is actually almost zero. Therefore, we choose not to consider extreme cases in our study. Figure 14 is the IV result with different ratios of strike prices to the current swap rate. The red point indicates the implied volatility for an ATM payer swaption. As the swaption price further to the right is close to nothing, we focus on the left part of this plot, where the ratios of the strike prices to the current swap rate varies from 0.9 to 1.0. We can see a higher (implied) volatility for a more in-the-money payer swaption.In a word: lower strike prices increase the volatility of a payer swaption.

Although the results are computed by simulation, we can consider whether this is consistent with the real world. To understand this "volatility skew", we need to understand the attributes which contribute the changes. One factor affecting the IV is the **supply** & **demand**. Prices rise in respond to assets that are in high demand. Now we can think about what type of swaptions can trigger high demand among the investors.

Before we discuss the interest rate swap market, we firstly need to understand a finance behavior *Loss aversion*. Different from the risk aversion, loss aversion states a principal that the investors dislike great losses more than liking great gains. In other words, the traders believe that the probability of large down movement in price is greater than large up movement in prices as compared with the log-normal distribution under the Black assumptions. Therefore, the implied volatility for the longer term equity options is more like smirk rather than "smile", the skew indicates a higher implied volatility for options with lower strike prices. Now we move to the

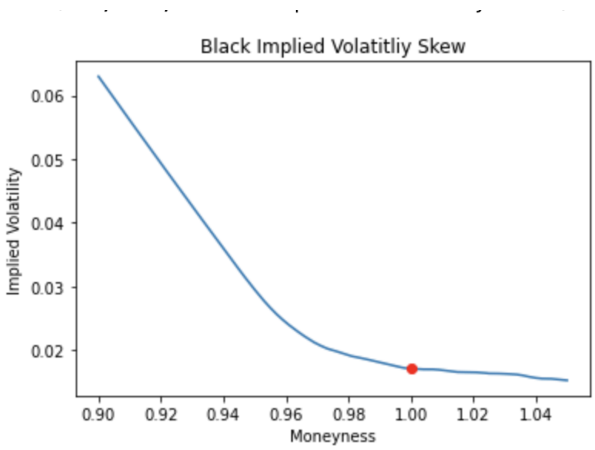

Figure 14: Implied Volatility for different ratios of strike prices to the current swap rate

next question: where are the demands for the IRS. Swap rate anticipates the future market interest rates, which are often utilized if a company can borrow money easily at one type of interest rate but prefers a different type. In this way, swaption becomes a great way to eliminate the risk triggered by the future rate swings, as it helps investors exchange the uncertainty for a stream of fixed payments. Consider the case where a company promises to make the interest payment on a loan, if the future interest rate rises, he could exercise a payer swaption to offset the floating interest payment to prevent this probability. Because of the loss aversion, people are willing to ”overpay” to prevent the negative outcomes.

Consider for the payer swaption in our studied example, the purchaser are concerned with the rising interest rate in the future. With lower strike, there is increasing probability to exercise those swaptions at the maturity date. With higher expected payoff, the demand tends to rise, which increases the volatility and the premiums. The results are consistent with our simulations about the premiums over strike prices. Also, the investors would respond more dramatically when there is a sign for great losses, in a loan case this means a higher interest payment. As a result, the high demand in the low-strike payer swaptions contributes to the high volatility.

Another important factor which can impact the implied volatility is the time value. Time value is the length of time left before the option reaches its expiration date. A low implied volatility is closely tied to payer swaptions with short expiration periods; the one with longer expiration periods tend to have higher IV. It should be noted that the implied volatility is not a cause to the trade swings, but a result of the trade swing. In other words, it indicates the variation level but not the direction. The longer the period of time before expiration, the longer the interest rate needs to move either in or out of the trader’s favor, making it riskier but also offering greater potential to prove profitable eventually.

4 Conclusion

In this project, we investigate how a two-factor interest rate model helps to price some basic rate-contingent claims, in particularly we explored the bond yields, bond options and interest rate swaptions.

We implemented both analytical and Morte Carlo simulation methods to analyze the zero-coupon bond yields and bond options. Given the explicit assumptions of the interest rate model (Hull-White One Factor), the formulas to calculate the bond yield at the current time can be derived. To validate the formula, we directly conducted Monte Carlo on the rate model to generate the simulated risk-neutral interest rate paths in a near future, and by which we computed the yields for the bonds with different maturity times. By comparing the results, we found that two methods provide similar results, and the yield curve has an upward trend with decreasing slope as time to maturity increases. The initial parameter setting is important to the implementation and validation for a reliable interest rate model. Therefore, we continued to investigate how the model parameters changes affects the bond yield curve (term structure). We conducted the test for all the parameters involved in this model (α, β, ϕ, r_0, θ_0, η and σ), the affect of them on the future yield rate has been summarized in Table 1.

Next, strike price is one important factor to determine the option values. We make an exploration on how the strike price impact a zero-coupon bond option price. Based on the results of our analysis for a call bond option, when the option is in-the-money, the price will decrease at a steady rate as the strike price increases. In the case where the option is at-the-money and out-the-money, the option price approaches to zero, due to the characteristic of the bond option.

At the end, we derived and computed the swaption price and implied volatility in agreement with the Black Lognormal Swap-rate Model. We explored the change in the Black implied volatility of the payer swaptions from two perspectives: strike prices and the time value. Since there are no market swaption prices are available for this project, we implemented the Monte Carlo generation to simulate the 'real' prices of the payer swaptions, by which we computed the corresponding implied volatility. However, this method restrict us to look at a small range of change for the ratios of strike prices to the current swap rate. The conclusion is that for payer swaptions, the Black implied volatility increases if the strike price decreases. When analyzing these results, we also make it connected to a real world scenario. Considering about the Loss aversion behaviour among the investors and the supply demand mechanism in the financial market as well as the riskiness. We make the conclusion that our analysis result is also reasonable in the real world situation.

It should be realized that this project only introduces a very simple idea about how to price some interest rate contingent-claims according to some assumptions about the interest dynamics. In the reality, it requires amounts of market prices to calibrate the interest model parameters, and the swap rate quoted by the LIBOR will incorporate the market information including the supply & demand, riskiness expectation and other complex economical behaviors. However, those considerations are impossible to be simulated by our method in this project. Therefore, we need more explorations in the modelling process in order to incorporate the real market uncertainties

5 Reference

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6 Appendix

6.1 General Derivation process

$$\begin{aligned} r \;=\; (r_t)_{t\geq 0}, dr_t \;&=\; \kappa(\theta(t)-r_t)dt+\sigma(t)dW_t \\ Consider\ X_t = r_t e^{\kappa t},\ \ dX_t \;&=\; dr_t e^{\kappa t} \;+\; r_t\kappa e^{\kappa t}dt \;+\; d[r,\ e^{\kappa t}]_t \\ Apply\ It\hat{o}'s\ Lemma &= (\kappa(\theta(t)-r_t)e^{kt} \;+\; r_t\kappa e^{\kappa t})dt \;+\; \sigma(t)e^{\kappa t}dW_t \\ &= \kappa\theta(t)e^{\kappa t}dt \;+\; \sigma(t)e^{\kappa t}dW_t \end{aligned}$$

Take integral on both side:

$$X_T - X_t = \int_t^T \kappa\theta(u)e^{ku}du + \int_t^T \sigma(u)e^{\kappa u}dW_u$$

Plug $r_t e^{\kappa t}$ into X_t and rearrange the equation:

$$r_T \;=\; r_t e^{-\kappa(T-t)} \;+\; \int_t^T \kappa\theta(u)e^{-\kappa(T-u)}du \;+\; \int_t^T \sigma(u)e^{-\kappa(T-u)}dW_u$$

Change the integral upper bond T to u and lower bond u to s:

$$r_u \;=\; r_t e^{-(u-t)} \;+\; \int_t^u k\theta(s)e^{-k(u-s)}ds \;+\; \int_t^u \sigma(s)e^{-k(u-s)}dW_s$$

Take integral on r_u from T to t:

$$\begin{aligned} \int_t^T r_u du \;&=\; \int_t^T r_t e^{-k(u-t)}du \;+\; \int_t^T\int_t^u k\theta(s)e^{-k(u-s)}dsdu \;+\; \int_t^T\int_t^u \sigma(s)e^{-k(u-s)}dW_s du \\ &= \int_t^T r_t e^{-k(u-t)}du \;+\; \int_t^T k\theta(s)\int_s^T e^{-k(u-s)}duds \;+\; \int_t^T \sigma(s)\int_s^T e^{-k(u-s)}dudW_s \\ &= \int_t^T r_t e^{-k(u-t)}du \;+\; \int_t^T \theta(u)(1-e^{-k(T-u)}du \;+\; \int_t^T \sigma(u)\frac{(1-e^{-k(T-u)}}{k}dW_u \end{aligned} \tag{1}$$

6.2 Bond and Bond Option Derivations

Using bank as numeraire:

$$\frac{P_t(T)}{B_t} \;=\; E^Q[\frac{P_T(T)}{B_T}|F_t] \;\;\Rightarrow\; P_t(T) = \; E^Q[e^{-\int_t^T r_u du}]$$

Let $X_t \;=\; r_t e^{\alpha t}$ and consider $dr_t \;=\; \alpha(\theta_t - r_t)dt \;+\; \sigma dW_t^1$, then sub into:

$$dX_t \;=\; e^{\alpha t}dr_t \;+\; \alpha r_t e^{\alpha t}dt \;+\; d[r, e^{\alpha t}]_t \;=\; \alpha\theta_t e^{\alpha t}dt \;+\sigma e^{\alpha t}dW_t^1$$

Next, Sub $r_t e^{\alpha t}$ into X_t and take integral from t to u:

$$r_u \;=\; r_t e^{-\alpha(u-t)} \;+\; \alpha\int_t^u \theta_s e^{-\alpha(u-s)}ds \;+\; \sigma\int_t^u e^{-\alpha(u-s)}dW_s^1$$

Take integral on r_u from T to t:

$$\begin{aligned} \int_t^T r_u du \;&=\; \int_t^T r_t e^{-\alpha(u-t)}du \;+\; \alpha\int_t^T\int_t^u \theta_s e^{-\alpha(u-s)}dsdu \;+\; \sigma\int_t^T\int_t^u e^{-\alpha(u-s)}dW_s^1 du \\ &= \frac{r_t}{\alpha}(1-e^{-\alpha(T-u)}) \;+\; \int_t^T \theta_u(1-e^{-\alpha(T-u)}du \;+\frac{\sigma}{\alpha}\int_t^T (1 \;-\; e^{-\alpha(T-u)})dW_u^1 \end{aligned} \tag{2}$$

Let $Y_t \;=\; \sigma_t e^{\beta t}$ and consider $d\theta_t \;=\; \beta(\phi-\theta_t)dt \;+\; \eta dW_t^2$, then we can get

$$dY_t \;=\; \beta\phi e^{\beta t}dt \;+\eta e^{\beta t}dW_t^2$$

We will do the same process as before:

$$\theta_u \;=\; \theta_t e^{-\beta(u-t)} \;+\; \phi(1 \;-\; e^{-\beta(u-t)}) \;+\; \eta \int_t^u e^{-\beta(u-s)} dW_s^2$$

$$\begin{aligned}\Rightarrow \int_t^T \theta_u du \;&=\; \phi(T-t) \;+\; (\theta_t - \phi)\int_t^T e^{-\beta(u-t)} du \;+\; \eta \int_t^T \int_s^T e^{-\beta(u-s)} du dW_s^2 \\ &=\; \phi(T-t) \;+\; \frac{1-e^{-\beta(T-t)}}{\beta}(\theta_t - \phi) \;+\; \eta \int_t^T \frac{1-e^{-\beta(T-u)}}{\beta} dW_u^2\end{aligned}$$

$\Rightarrow \int_t^T \theta_u (1 - e^{-\alpha(T-u)} du$

$$\begin{aligned}&= \int_t^T (\phi \;+\; (\theta_t - \phi)e^{-\beta(u-t)})(1-e^{-\alpha(T-u)}) du + \int_t^T \int_t^u \eta e^{-\beta(u-s)}(1-e^{-\alpha(T-u)}) dW_s^2 du \\ &= \phi(T-t) + \frac{1-e^{-\beta(T-t)}}{\beta}(\theta_t - \phi) \;-\; \frac{\phi}{\alpha}(1-e^{-\alpha(T-t)}) - \frac{\theta_t - \phi}{\alpha - \beta}(e^{-\beta(T-t)} \;-\; e^{-\alpha(T-t)}) \\ &\quad + \eta \int_t^T (\frac{1-e^{-\beta(T-s)}}{\beta} \;-\; \frac{e^{-\beta(T-s)} - e^{-\alpha(T-s)}}{\alpha - \beta}) dW_s^2\end{aligned}$$

Plug into Equation (2):

$$\begin{aligned}\int_t^T r_u du \;=\; &\frac{r_t}{\alpha}(1-e^{-\alpha(T-t)}) + \phi(T-t) + \frac{1-e^{-\beta(T-t)}}{\beta}(\theta_t - \phi) - \frac{\phi}{\alpha}(1-e^{-\alpha(T-t)}) - \frac{\theta_t-\phi}{\alpha-\beta}(e^{-\beta(T-t)} - e^{-\alpha(T-t)}) \\ &+ \eta \int_t^T (\frac{1-e^{-\beta(T-s)}}{\beta} \;-\; \frac{e^{-\beta(T-s)} - e^{-\alpha(T-s)}}{\alpha-\beta}) dW_s^2 \;+\; \frac{\sigma}{\alpha}\int_t^T (1 \;-\; e^{-\alpha(T-s)}) dW_s^1\end{aligned}$$

$\int_t^T r_u du$ follows $\sim$ N(m, v^2)

$$\begin{aligned}where\ m = &\frac{r_t}{\alpha}(1-e^{-\alpha(T-t)}) \;+\; \phi(T-t) + \frac{1-e^{-\beta(T-t)}}{\beta}(\theta_t - \phi) \\ &- \frac{\phi}{\alpha}(1-e^{-\alpha(T-t)}) - \frac{\theta_t - \phi}{\alpha-\beta}(e^{-\beta(T-t)} \;-\; e^{-\alpha(T-t)}), \\ v^2 \;=\;& v_1^2 + v_2^2 (since\ dW_s^1\ and\ dW_s^2\ are\ independent) \\ =\;& \eta^2 \int_t^T (\frac{1-e^{-\beta(T-s)}}{\beta} \;-\; \frac{e^{-\beta(T-s)} - e^{-\alpha(T-s)}}{\alpha-\beta})^2 ds \;+\; \frac{\sigma^2}{\alpha^2}\int_t^T (1 \;-\; e^{-\alpha(T-s)})^2 ds\end{aligned}$$

$\int_t^T r_u du$ is normal distributed under Q measure.

$$\Rightarrow P_t(T) = E^Q[e^{-\int_t^T r_u du}] \;=\; e^{-m+\frac{1}{2}v^2} \;=\; e^{A_t(T) - B_t(T) r_t - C_t(T)\theta_t}$$

By collecting terms together, we have

$$\begin{aligned}B_t(T) \;&=\; \frac{1-e^{-\alpha(T-t)}}{\alpha} \\ C_t(T) \;&=\; \frac{1-e^{-\beta(T-t)}}{\beta} - \frac{e^{-\beta(T-t)} - e^{-\alpha(T-t)}}{\alpha-\beta}\end{aligned}$$

$$\begin{aligned}A_t(T) \;=\; &-\phi((T-t) - \frac{1-e^{-\beta(T-t)}}{\beta} - \frac{1-e^{-\alpha(T-t)}}{\alpha} + \frac{e^{-\beta(T-t)} - e^{-\alpha(T-t)}}{\alpha-\beta}) \\ &+ \frac{\eta^2}{2}\int_t^T (\frac{1-e^{-\beta(T-s)}}{\beta} \;-\; \frac{e^{-\beta(T-s)} - e^{-\alpha(T-s)}}{\alpha-\beta})^2 ds \;+\; \frac{\sigma^2}{2\alpha^2}\int_t^T (1 \;-\; e^{-\alpha(T-s)})^2 ds\end{aligned}$$

Aside: $\int_t^T (1 \;-\; e^{-\alpha(T-s)})^2 ds$

$$= \int_t^T 1 - 2e^{-\alpha(T-s)} + e^{-2\alpha(T-s)} ds = (T-t) - \frac{2(1-e^{-\alpha(T-t)})}{\alpha} + \frac{1-e^{-2\alpha(T-t)}}{2\alpha},$$

$\int_t^T (\frac{1-e^{-\beta(T-s)}}{\beta} \ - \ \frac{e^{-\beta(T-s)}-e^{-\alpha(T-s)}}{\alpha-\beta})^2 ds$

$$
\begin{aligned}
&= \int_t^T (\frac{1-e^{-\beta(T-s)}}{\beta})^2 \ - \ \frac{2(1-e^{-\beta(T-s)})(e^{-\beta(T-s)}-e^{-\alpha(T-s)}}{\beta(\alpha-\beta)} \ + \ \frac{e^{-\beta(T-s)}-e^{-\alpha(T-s)}}{\alpha-\beta})ds \\
&= (1-e^{-\alpha(T-t)})(-\frac{\sigma^2}{\alpha^3}+\frac{\eta^2}{\alpha\beta(\alpha-\beta)}) \ + \ (1-e^{-\beta(T-t)})(-\frac{\eta}{\beta^3}-\frac{\eta^2}{\beta^2(\alpha-\beta)}) \\
&= (1-e^{-2\alpha(T-t)})(\frac{\sigma^2}{4\alpha^3}+\frac{\eta^2}{4\alpha(\alpha-\beta)^2}) \ + \ (1-e^{-2\beta(T-t)})(\frac{\eta^2}{4\beta^3}+\frac{\eta^2}{2\beta^2(\alpha-\beta)}+\frac{\eta^2}{4\beta(\alpha-\beta)^2}) \\
&= \ -(1-e^{-(\alpha-\beta)(T-t)})(\frac{\eta^2}{\beta(\alpha^2-\beta^2)}+\frac{\eta^2}{(\alpha+\beta)(\alpha-\beta)^2}) \ + \ (T-t)(\frac{\sigma^2}{2\alpha^2}+\frac{\eta^2}{2\beta^2})
\end{aligned}
$$

Thus,

$$
\begin{aligned}
A_t(T) \ = \ &(T-t)(-\phi+\frac{\sigma^2}{2\alpha^2}+\frac{\eta^2}{2\beta^2}) + (1-e^{-\alpha(T-t)})(\frac{\phi}{\alpha}-\frac{\sigma^2}{\alpha^3}+\frac{\eta^2}{\alpha\beta(\alpha-\beta)}) + (1-e^{-\beta(T-t)})(\frac{\phi}{\alpha}-\frac{\eta^2}{\beta^3}-\frac{\eta^2}{\beta^2(\alpha-\beta)}) \\
&+ (1-e^{-2\alpha(T-t)}(\frac{\sigma^2}{4\alpha^3}+\frac{\eta^2}{4\alpha(\alpha-\beta)^2}) \ + \ (1-e^{-2\beta(T-t)})(\frac{\eta^2}{4\beta^3}+\frac{\eta^2}{2\beta^2(\alpha-\beta)}+\frac{\eta^2}{4\beta(\alpha-\beta)^2}) \\
&- (1-e^{-(\alpha-\beta)(T-t)})(\frac{\eta^2}{\beta(\alpha^2-\beta^2)}+\frac{\eta^2}{(\alpha+\beta)(\alpha-\beta)^2}) \ - \ (e^{-\beta(T-t)}-e^{-\alpha(T-t)})\frac{\phi}{\alpha-\beta}
\end{aligned}
$$

6.3 Black Implied Volatility

Consider $V_{T_1} = (P_{T_1}(T_2)$- $K)_+$, the use T_1 as numeraire

$$
\begin{aligned}
\frac{V_t}{P_t(T_1)} &= E^{Q_{T_1}}[\frac{(P_{T_1}(T_2)-K)_+}{P_{T_1}(T_1)}|F_t] \\
&= E^{Q_{T_1}}[(P_{T_1}(T_1)-K)_+|F_t]
\end{aligned}
$$

Introduce a new stochastic process: $X_t = \frac{P_t(T_2)}{P_t(T_1)}$ As t $\to T_1$, $X_t \to P_t(T_2)$ $\quad\therefore\quad X_{T_1} = P_t(T_2)$

$$\text{Then } V_t = P_t(T_1)E^{Q_{T_1}}[(X_{T_1}-K)_+|F_t]$$

Recall:

$$
\begin{aligned}
P_t(T_1) = e^{A_t(T_1)-B_t(T_1)r_t-C_t(T_1)\theta_t} \ \& \ P_t(T_2) = e^{A_t(T_2)-B_t(T_2)r_t-C_t(T_2)\theta_t}, \\
dr_t = \alpha(\theta_t - r_t)dt + \sigma dW_t^1 \quad \& \ d\theta_t = \beta(\phi-\theta_t)dt + \eta dW_t^2
\end{aligned}
$$

$$
\begin{aligned}
d(\frac{1}{P_t(T_1)}) \ &= \ P_t(T_1)^{-1}(B_t(T_1)dr_t + C_t(T_1)d\theta_t) \\
&= \ B_t(T_1)P_t(T_1)^{-1}\sigma dW_t^1 + C_t(T_1)P_t(T_1)^{-1}\eta dW_t^2 + r_t P_t(T_1)^{-1}dt \\
d(\frac{1}{P_t(T_2)}) \ &= \ -B_t(T_2)P_t(T_2)\sigma dW_t^1 - C_t(T_2)P_t(T_2)\eta dW_t^2 + r_t P_t(T_2)dt \\
\Rightarrow \ dX_t \ &= \ dP_t(T_2)\frac{1}{P_t(T_1)} - P_t(T_2)d(\frac{1}{(P_t(T_1))}) + d[P(T_1),P(T_2)]_t \\
\Rightarrow \ dX_t \ &= \ dP_t(T_2)\frac{1}{P_t(T_1)} - P_t(T_2)d(\frac{1}{(P_t(T_1))}) + d[P(T_1),P(T_2)]_t \\
&= \ -B_t(T2)\frac{P_t(T_2)}{P_t(T_1)}\sigma dW_t^1 - C_t(T_2)\frac{P_t(T_2)}{P_t(T_1)}\eta dW_t^2 \\
&\qquad + \ \frac{P_t(T_2)}{P_t(T_1)}B_t(T_1)\sigma dW_t^1 + \frac{P_t(T_2)}{P_t(T_1)}C_t(T_2)\eta dW_t^2 + (...)dt \\
&= \ X_t(B_t(T_1)-B_t(T_2))\sigma dW_t^1 + X_t(C_t(T_1)-C_t(T_2))\eta dW_t^2 + (...)dt
\end{aligned}
$$

By changing measure

$$= \ \sigma X_t \underbrace{(B_t(T_1) - B_t(T_2))}_{\ell_{1,t}} dW_t^{T,1} + \eta X_t \underbrace{(C_t(T_1) - C_t(T_2))}_{\ell_{2,t}} dW_t^{T,2}$$

$$dX_t = \sigma X_t \ell_{1,t} dW_t^{T,1} + \eta X_t \ell_{2,t} dW_t^{T,2}$$

By *Itô's* Lemma, $\quad d(log(X_t)) = -\frac{1}{2}(\sigma^2 \ell_{1,t}^2 + \eta^2 \ell_{2,t}^2)dt + \sigma \ell_{1,t} dW_t^{T,1} + \eta \ell_{2,t} dW_t^{T,2}$

$$\Rightarrow log(\frac{X_T}{X_t}) \ = \ \int_t^T -\frac{1}{2}(\sigma^2 \ell_{1,u}^2 + \eta^2 \ell_{2,u}^2)du \ + \ \int_t^T \sigma \ell_{1,u} dW_u^{T,1} \ + \ \int_t^T \eta \ell_{2,u} dW_u^{T,2}$$

$$\overset{\theta^{T_1}}{\sim} \ N\Big(-\frac{1}{2} \underbrace{\int_t^T (\sigma^2 \ell_{1,u}^2 + \eta^2 \ell_{2,u}^2)\, du}_{\Omega} \ , \ \underbrace{\int_t^T (\sigma^2 \ell_{1,u}^2 + \eta^2 \ell_{2,u}^2)\, du}_{\Omega} \Big)$$

Then, $\ X_T \underset{\sim}{d} X_t e^{-\frac{1}{2}\Omega_{t,T} \ + \ \sqrt{\Omega_{t,T}} Z_T}, \ where \ \Omega_{t,T} \ = \ \int_t^T (\sigma^2 \ell_{1,u}^2 + \eta^2 \ell_{2,u}^2)du$

$V_t \ = \ P_t(T_1) \underbrace{\mathbb{E}^{\mathbb{Q}T_1}\Big((X_{T_1} - K)_+ | F_t\Big]}_{X_t \Phi(d_+) - K\Phi(d_-)}, where \ \ d_{\pm} \ = \ \frac{log(\frac{X_t}{K}) \pm \frac{1}{2}\Omega_u}{\sqrt{\Omega_u}} \ = \ \frac{log(\frac{P_t(T_2)/P_0(T_2)}{P_t(T_1)/P_0(T_1)}) \pm \frac{1}{2}\Omega_u}{\sqrt{\Omega_u}}$ Thus,

$$V_t \ = \ P_t(T_2)\Phi(d_+) \ - \ kP_t(T_1)\Phi(d_-)$$

Explicitly, $B_t(T) \ = \ \frac{1-e^{-\alpha(T-t)}}{\alpha}$, $C_t(T) \ = \ \frac{1-e^{-\beta(T-t)}}{\beta} - \frac{e^{-\beta(T-t)} - e^{-\alpha(T-t)}}{\alpha - \beta}$

$\star \quad \ell_{1,t} \ = \ B_t(T_1) - B_t(T_2) \ = \ \frac{e^{-\alpha(T_2-t)} - e^{-\alpha(T_1-t)}}{\alpha}$

$$\begin{aligned}
\rightarrow \quad \ell_{1,t}^2 \ &= \ (B_t(T_1) - B_t(T_2))^2 \\
&= \ \frac{1}{\alpha^2}\Big(e^{-2\alpha(T_2-t)} - 2e^{-\alpha(T_2+T_1-2t)} + e^{-2\alpha(T_1-t)}\Big) \\
\rightarrow \quad \int_t^{T_1} \ell_{1,s}^2 ds \ &= \ \frac{1}{\alpha^2}\int_t^{T_1} \Big(e^{-2\alpha(T_2-s)} - 2e^{-\alpha(T_2+T_1-2s)} + e^{-2\alpha(T_1-s)}\Big)ds \\
&= \ \frac{1}{\alpha^2}\int_t^{T_1} e^{2\alpha s}(e^{-\alpha T_2} - e^{-\alpha T_1})^2 ds \ = \ \frac{1}{2\alpha^3}(e^{-\alpha T_2} - e^{-\alpha T_1})^2 (e^{2\alpha T_1} - e^{2\alpha t})
\end{aligned}$$

$\star \quad \ell_{2,t} \ = \ C_t(T_1) - C_t(T_2) \ = \ (e^{-\beta(T_2-t)} - e^{-\beta(T_1-t)})(\frac{1}{\beta} + \frac{1}{\alpha-\beta}) + \frac{e^{-\alpha(T_1-t)} - e^{-\alpha(T_2-t)}}{\alpha - \beta}$

$$\begin{aligned}
\rightarrow \quad \ell_{2,t}^2 \ &= \ (C_t(T_1) - C_t(T_2))^2 \\
&= \ \Big((e^{-\beta(T_2-t)} - e^{-\beta(T_1-t)})(\frac{1}{\beta} - \frac{e^{-\alpha(T_2-t)} - e^{-\alpha(T_1-t)}}{\alpha - \beta})\Big)^2 \\
&= \ \frac{e^{-2\beta(T_2-t)} - 2e^{-\beta(T_2+T_1-2t)} + e^{-2\beta(T_1-t)}}{\beta^2} && as \ Equation \ ① \\
&+ \Big(\frac{e^{-2\beta(T_2-t)} - 2e^{-\beta(T_2+T_1-2t)} + e^{-2\beta(T_1-t)} + e^{-2\alpha(T_2-t)} - 2e^{-\alpha(T_2+T_1-2t)} + e^{-2\alpha(T_1-t)}}{(\alpha-\beta)^2} \\
&+ \frac{2e^{-\beta(T_2-t)-\alpha(T_1-t)} - 2e^{-\beta(T_2-t)-\alpha(T_2-t)} - 2e^{-\beta(T_1-t)-\alpha(T_1-t)} + 2e^{-\beta(T_1-t)-\alpha(T_2-t)}}{(\alpha-\beta)^2}\Big) && as \ Equation \ ② \\
&+ \Big(\frac{-2e^{-\beta(T_1+T_2-2t)} + 2e^{-\beta(T_1-t)-\alpha(T_2-t)} + 2e^{-2\beta(T_1-t)} - 2e^{-(\alpha+\beta)(T_1-t)}}{(\alpha-\beta)\beta} \\
&+ \frac{2e^{-2\beta(T_2-t)} - 2e^{-(\beta+\alpha)(T_2-t)} - 2e^{-\beta(T_2+T_1-2t)} + 2e^{-\beta(T_2-t)-\alpha(T_1-t)}}{(\alpha-\beta)\beta}\Big) && as \ Equation \ ③
\end{aligned}$$

$\int_t^{T_1} Equation \ ① \ ds \ = \ \int_t^{T_1} \frac{e^{2\beta s}(e^{-\beta T_2} - e^{-\beta T_1})^2}{\beta^2} ds$
$\int_t^{T_1} Equation \ ② \ ds \ = \ \int_t^{T_1} \frac{e^{2\beta s}(e^{-\beta T_2} - e^{-\beta T_1})^2 + e^{2\alpha s}(e^{-\alpha T_2} - e^{-\alpha T_1})^2 + 2e^{(\alpha+\beta)s}(e^{-\beta T_2} - e^{-\beta T_1})(e^{-\alpha T_1} - e^{-\alpha T_2})}{(\alpha-\beta)^2} ds$

$$\int_t^{T_1} Equation \text{ ③ } ds \;=\; \int_t^{T_1} \frac{2\Big(e^{2\beta s}(e^{-\beta T_1}-e^{-\beta T_2})^2+e^{(\alpha+\beta)s}(e^{-\alpha T_1}-e^{-\alpha T_2})(e^{-\beta T_2}-e^{-\beta T_1})\Big)}{(\alpha-\beta)\beta} ds$$

$$\Rightarrow \quad \int_t^{T_1} \ell_{2,s}^2 ds \;=\; \int_t^{T_1} Equation \text{ ① } + Equation \text{ ② } + Equation \text{ ③ } ds$$

$$= \int_t^{T_1} e^{2\beta s}(e^{-\beta T_2}-e^{-\beta T_1})^2[\frac{1}{\beta^2}+\frac{1}{(\alpha-\beta)^2}+\frac{2}{(\alpha-\beta)\beta}]+e^{2\alpha s}\frac{(e^{-\alpha T_2}-e^{-\alpha T_1})^2}{(\alpha-\beta)^2}$$

$$+ \; e^{(\alpha+\beta)s}2(e^{-\alpha T_1}-e^{-\alpha T_2})(e^{-\beta T_2}-e^{-\beta T_1})[\frac{1}{(\alpha-\beta)^2}+\frac{1}{(\alpha-\beta)\beta}]ds$$

$$= \frac{1}{2\beta}(e^{2\beta T_1}-e^{2\beta t})(e^{-\beta T_2}-e^{-\beta T_1})^2[\frac{1}{\beta^2}+\frac{1}{(\alpha-\beta)^2}+\frac{2}{(\alpha-\beta)\beta}]+\frac{1}{2\alpha}(e^{2\alpha T_1}-e^{2\alpha t})\frac{(e^{-\alpha T_2}-e^{-\alpha T_1})^2}{(\alpha-\beta)^2}$$

$$+\frac{1}{\alpha+\beta}(e^{(\alpha+\beta)T_1}-e^{(\alpha+\beta)t})\cdot 2\cdot(e^{-\alpha T_1}-e^{-\alpha T_2})(e^{-\beta T_2}-e^{-\beta T_1})(\frac{1}{(\alpha-\beta)^2}+\frac{1}{(\alpha-\beta)\beta})$$

$$\Omega_{t,T} \;=\; \int_t^T(\sigma^2\ell_{1,u}^2+\eta^2\ell_{2,u}^2)du \quad = \quad \tfrac{\sigma^2}{2\alpha^3}(e^{-\alpha T_2}-e^{-\alpha T_1})^2(e^{2\alpha T_1}-e^{2\alpha t})$$

$$+\eta^2\Bigg[\frac{1}{2\beta}(e^{2\beta T_1}-e^{2\beta t})(e^{-\beta T_2}-e^{-\beta T_1})^2[\frac{1}{\beta^2}+\frac{1}{(\alpha-\beta)^2}+\frac{2}{(\alpha-\beta)\beta}]+\frac{1}{2\alpha}(e^{2\alpha T_1}-e^{2\alpha t})\frac{(e^{-\alpha T_2}-e^{-\alpha T_1})^2}{(\alpha-\beta)^2}$$

$$+\frac{1}{\alpha+\beta}(e^{(\alpha+\beta)T_1}-e^{(\alpha+\beta)t})\cdot 2\cdot(e^{-\alpha T_1}-e^{-\alpha T_2})(e^{-\beta T_2}-e^{-\beta T_1})(\frac{1}{(\alpha-\beta)^2}+\frac{1}{(\alpha-\beta)\beta})\Bigg]$$