Dynamic Hedging Analysis

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1 Introduction

The main objective of a hedger is to eliminate the exposure of the practitioner that holds a short or long position in financial derivative security via a portfolio that replicates pointwise the value of the derivative at the maturity time.

Greeks hedging is a fundamental notion in risk management for derivative portfolio construction. The most often used Greek is delta Δ, which is a measure of the change in an option price with the respect of price change in the underlying asset. While the gamma Γ is the rate of change in the delta as the underlying security price changes. The gamma is usually expressed in deltas gained or lost per one-point change in the underlying asset price. The delta increases by the amount of the gamma when the underlying rises, and falls by the amount of the gamma when the underlying falls. Delta hedging uses the idea that we can long or short the underlying asset to offset the position the hedger exposed to that asset, so that the first local risk measured by the delta is eliminated. However, the delta is only enough to protect the hedger against small fluctuation on the underlying asset, as the time close to maturity, even small change in the asset can lead to significant change to the option. With more complexity, the delta-gamma hedging improves the effectiveness by considering higher-order measurement against larger movement in the underlying price.In this circumstnace, this strategy can protect the trader against larger variation on the underlying stock than they expected from the moment of writing.

This report is going to implement several dynamic hedging strategies on a discrete-time basis under a Black-Scholes Partial Differential Equation, particularly the Delta and Gamma hedgings. To hedge a written option, we can simply short or long certain number of shares of the underlying stocks to achieve delta hedging. The delta-gamma hedging requires the participation of another option, which provides a second shield for the hedger when there is a large volatility in the delta when the stock values move, especially before expiration. For each hedging strategy, we are also going to explore in detail two categories of rebalancing criteria: move-base and time-based. Finally, we will vary the parameter in the models to test their impact on the hedging performance and precision.

2 Methodology and Mathematical Theory Support

2.1 Black Scholes PDE and Dynamic Hedging Strategies

The Black-Sholes model is a classical mathematical model to price a financial instruments such as options and futures, whose values are dependent on the movement of the market assets. The model is based on a partial differential equation (PDE), which is important for our analysis because this model gives us a closed formula to estimate the price of European option.

The assumption of this model is that the market consists of one risk-free asset such as bank account and government bond and at least one risky asset such as stock. In addition, it assumes some properties about the market and assets:

- The rate of return on the risk-free asset r is constant and continuous.

- The risky asset S_t is assumed to follow Geometric Brownian Motion(GBM); the return of stock is log-normally distributed and the stock does not pay any dividends
- There is no arbitrage opportunity in the market.
- It is allowed to borrow and lend any amount of money from the bank at the rate of return on the risk-free asset.
- There is no transaction cost in the market.

Firstly, consider a risky non-dividend stock for which price S_t follows a diffusion process given by:

$$dS_t = \mu S_t dt + \sigma S_t dw_t$$

where μ and σ are the mean and volatility of the stock returns respectively, and W_t is a standard Brownian Motion.

Ito's Lemma:

Assume X_t is an an Ito drift-diffusion process that satisfies the stochastic differential equation

$$dX_t = \mu_t dt + \sigma_t dB_t$$

where B_t is a Wiener process. If $f(t, x_t)$ is a twice-differential scalar function, then we obtain:

$$df = (\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2} \frac{\partial^2 f}{\partial x^2})dt + \sigma_t \frac{\partial f}{\partial x} dB_t$$

Apply Ito's lemma, let $f(S_t) = log(S_t)$, we can simulate the stock prices movement according to the formula:

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$$

2.1.1 Dynamic Hedging

The goal of the dynamic hedging is to construct a replicating portfolio for the contingent claim and invoke a no-arbitrage argument. To find the PDE that admits the stochastic price process, we consider to build an portfolio

Now, we want to use this risky asset to value a contingent (European) claim with price process $f = (f_t)_{t \geq 0}$ and $f_T = F(S_T)$ where T is the maturity time. The goal of the dynamic hedging is to construct a replicating portfolio for the contingent claim and invoke a no-arbitrage argument. To find the PDE that admits the stochastic price process, we consider to build an portfolio that consists the stock S_t that follows the above Stochastic Differential Equation(SDE), the bank account B_t satisfies:

$$dB_t = rB_t dt$$

To construct a dynamic hedging strategy, consider a self-financing portfolio by holding α_t amount of stock and β_t amount of cash in the bank account at time t, and short one option whose price at time t is denoted as f_t, then the total value of the portfolio V at time t is:

$$V_t = \alpha_t S_t + \beta_t B_t - f_t$$

since it is self-financing, we have conditions:

$$\begin{cases} V_0 = \quad 0 \\ \alpha_t dS_t + \beta_t dB_t - df_t = dV_t \end{cases}$$

according to the $Ito's\ lemma$,

$$df_t = \partial_t f(t, S_t) + S_t \mu_t \partial_s f(t, S_t) + \frac{1}{2} \sigma_t^2 S_t^2 \partial_{ss} f(t, S_t) dt + \sigma_t S_t \partial_s f(t, S_t) dW_t$$

$$\Longrightarrow dV_t = \underbrace{(\partial_t S_t \mu_t + \beta_t r_t B_t - (\partial_t f(t,S_t) + S_t \mu_t \partial_s f(t,S_t) + \frac{1}{2}\sigma_t^2 S_t^2 \partial_{ss} f(t,S_t)))}$$

The incremental value process of the portfolio is:

$$dV_t = [\alpha_t S_t \mu + \beta_t r B_t - (\frac{\partial f}{\partial t} + S_t \mu \frac{\partial f}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2})]dt + [S_t \sigma(\alpha_t - \frac{\partial f}{\partial S_t})]dW_t$$

The no-arbitrage condition implies all traded assets equal market price of risk, then $V_0 = 0$ must force $V_t = 0$ for all t.

$$V_t = \alpha_t S_t + \beta_t B_t - f_t = 0$$

$$\Longrightarrow \beta_t = \frac{f_t - \alpha_t S_t}{B_t}$$

The drift term for the portfolio's value process must be zero to avoid arbitrage. Therefore, we can solve for A_t

$$\begin{aligned} A_t &= \partial_s f(t,S_t) S_t \mu_t + \beta_t r_t B_t - (\partial_t f(t,S_t) + S_t \mu_t \partial_s f(t,S_t) + \frac{1}{2}\sigma_t^2 S_t^2 \partial_{ss} f(t,S_t)) \\ &= r_t(f(t,S_t) - S_t \partial_s f(t,S_t)) - \partial_t f(t,S_t) - \frac{1}{2}\sigma_t^2 S_t^2 \partial_{ss} f(t,S_t) \\ &= 0 \end{aligned} \tag{1}$$

$\Longrightarrow$ the function $f(t, S_t)$ must satisfy the only deterministic PDE:

$$\begin{cases} \partial_t f(t,S_t) + r S_t \partial_s f(t,S_t) + \frac{1}{2}^2 S_t^2 \partial_s s f(t,S_t) = r f(t,S_t) \\ \qquad\qquad\qquad\qquad\qquad\qquad f(T,S_T) = F(S_T) \end{cases}$$

This is actually called the *generalized Black-Scholes PDE*, which can be solved using the *Feynman-Kac Theorem.* However, since this project particularly discusses the option pricing and hedging strategy under the Black-Scholes model, $f(t, S_t)$ is given explicitly by a theorem:

Theorem: Black-Scholes Option Pricing Formula

In the Black-Scholes model, the price of call and put options struck on an asset with value S_t at time t with strike price K is given by:

$$f_t^{call} = S_t \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-)$$

$$f_t^{put} = K e^{-r(T-t)} \Phi(-d_-) - S_t \Phi(-d_+)$$

where

$$d_\pm = \frac{ln(S_t/K) + (r \pm \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

Take the European call as the example, we can interpret the equation as:

$$f^{call}(t,S) = e^{-rT}\,\mathbb{E}^{P^*}[(S_T - K)_+ | S_t = S]$$

$$where\ S_t\ admits\ the\ stochastic\ equation:\ dS_t = r S_t dt + \sigma S_t dW_t^*$$

$$\Longrightarrow S_T = S_t e^{(r - \frac{\sigma^2}{2})(T-t) + \sigma(W_T^* - W_t^*)}$$

It implies that $\mathbb{P}^*$ is the risk-neutral probability measurement invoked by the bank account as numeraire, we can also denote it as $\mathbb{Q}$. Correspondingly, W_t^* is the Brownian motion under the measurement $\mathbb{Q}$. Compare with the equation used to stimulate the real path of stock's prices:

$$S_{t+\Delta t} = S_t e^{(\mu - \frac{\sigma^2}{2})t + \sigma(W_{t+\Delta t} - W_t)}$$

which instead admits the stochastic process:

$$dS_t = S_t(\mu dt + \sigma dW_t),\ where\ W_t\ is\ Brownian\ motion\ under\ real\ world\ probability\ measurement\ \mathbb{P}.$$

2.2 Implementation of dynamic hedging with the presence of transaction cost

The Black-Scholes model assumes a fictional world without transactions cost. However, the reality literature encourages transaction costs during the transaction in the risky asset. As a result, the hedging strategy suggested by the Black-Scholes to perfectly replicate our risky position will be no longer valid. Therefore, different hedging strategies would have different costs. In this section, we consider a real-world scenario in which extra expenses are charged on every unit of equity as well as option traded and we're going to investigate the hedging performance under different proposals. Assume e_s and e_o are the values of transaction costs for stocks and options respectively.

2.2.1 Delta Hedging

Recall the self-financing portfolio with position α_t in stock and β_t in bank account at a given time point t, and we hold one option till the maturity.

We have a sequence of time indicating the time point to rebalance $t_0, t_1, ...t_{k-1}, t_k, ..., t_{T-1}, t_T$, where $t_T = T$ is the maturity time of the owed option. To perform delta hedging we need to make sure that:

$$\alpha_t - \partial_s f(t, S_t) = 0$$

under the Black-Scholes PDE, we have a formula for $f(t, S_t)$.

Suppose we owe a put option with strike price K and time to maturity T, whose price at time t ($t \leqslant T$) is denoted as $f^{put}(t, S_t)$, and S_t is the underlying asset price at t. We also denote Y_t as the position of the bank account or other risk-free assets at time t. When the hedging strategy is discrete-time, it can be illustrated by the following process:

at $t = 0$: we sell one put, to hedge it we hold positions in the underlying asset with the amount of

$$\alpha_0 = \partial_s f^{put}(0, S_0)$$

$$\Longrightarrow \textit{we gain } Y_0 = V_0 - \alpha_0 S_0 - |\alpha_0| e_s$$

if this amount is greater than nothing, we put them in the bank account to grow interest at the risk-free rate of return r; if it is negative then we borrow from the bank at the same rate.

at $t = t_1$, before making rebalancing, we have the following positions in the portfolio:

$$\textit{bank account}: \; Y_0 e^{r(t_1 - t_0)} \quad \textit{asset value}: \; \alpha_0 S_{t_1}$$

to hedge, the following equation has to be held:

$$\alpha_{t_1} = \partial_s f^{put}(t_1, S_{t_1})$$

In other words, the underlying stock position needs to be adjusted to α_{t_1}. The change in position may require us long or short stocks (since α_{t_1} can be greater or less than α_{t_0}), with extra expenses incurred from trading the underlying asset. However, since we self-finance this portfolio, the money used to trade must come from the bank account. With the presence of transaction cost, if we have the position α_{t_1} at time t_1, our income will decrease e_s for each unit of stock we trade when we start the hedging process. Therefore, the amount in the bank account after rebalancing is given by:

$$Y_{t_1} = Y_0 e^{r(t_1 - t_0)} - (\alpha_{t_1} - \alpha_{t_0}) S_{t_1} - |\alpha_{t_1} - \alpha_{t_0}| e_s$$

then we continue the process to time t_k:

change in the bank account:

$$Y_{t_k-1} \to Y_{t_k-1} e^{r(t_k - t_{k-1})}$$

change in the positions in the stock:

$$\alpha_{t_k-1} \to \alpha_{t_k}$$

the amount in the bank account:

$$Y_{t_k} = Y_{t_k-1} e^{r(t_k - t_{k-1})} - (\alpha_{t_k} - \alpha_{t_k-1}) S_{t_k} - |\alpha_{t_k} - \alpha_{t_k-1}| e_s$$

at the maturity time $t_T = T$, there is no need to re-balance, so only the position in the bank account changes:

$$Y_{t_{T-1}} \to Y_{t_{T-1}} e^{r(T - t_{T-1})}$$

in addition, we have positions of $\alpha_{t_{T-1}}$ in the stocks, and owe the put option which has the payoff $(K - S_T, 0)_+$.

$$P\&L = Y_{t_{T-1}} e^{r(T - t_{T-1})} + \alpha_{t_{T-1}} S_T - (K - S_T, 0)_+$$

under the Black-Scholes PDE, we have a formula for $f^{put}(t, S_t)$:

$$f_t^{put} = K e^{-r(T-t)} \Phi(-d_-) - S_t \Phi(-d_+)$$

therefore,

$$\partial_s f^{put}(t, S) = -\Phi(-d_+) = \Delta_{put}$$

then, we can substitute Δ_{put} to perform the dynamic delta hedging.

2.2.2 Delta-Gamma Hedging

The delta hedging process achieves eliminating a linear exposure to the movement in the stock, therefore the risk exposure becomes quadratic and depends on the gamma of our position in the . By Taylor expansion, we can approximate a function at a local point using a quadratic function:

$$f(t, S_t + \Delta S_t) = f(t, S_t) + \Delta S_t \partial_s f(t, S_t) + \frac{1}{2} (\Delta S_t)^2 \partial_{ss} f(t, S_t) + ...$$

Back to our portfolio used for delta hedging, we hold α_t in stock and β_t in bank account to hedge the put option, now, to perform delta-gamma hedging, we need another financial instrument with price function at time t $g(t, S_t)$, where $t \leqslant T$, to realize the gamma-neutral condition. In this case, we consider a call option (with strike price K and time to maturity T_c; $T \leqslant T_c$), on the same asset $g^{call}(t, S_t)$:

$$V_t = \alpha_t S_t + \beta_t B_t + \gamma_t g^{call}(t, S_t) - f^{put}(t, S_t)$$

then we must have:

$$\begin{cases} \partial_s V_t = \alpha_t + \gamma_t \partial_s g^{call}(t,S_t) - \partial_s f^{put}(t,S_t) = \quad 0 \; delta-neutral \\ \\ \quad \partial_{ss} V_t = \gamma_t \partial_{ss} g^{call}(t,S_t) - \partial_{ss} f^{put}(t,S_t) = 0 \; gamma-neutral \end{cases}$$

let $\partial_s f = \Delta^f$ be the first order of function, $\partial_{ss} f = \Gamma^f$ be the second order the function, solve the above the equation we get:

$$\begin{cases} \gamma_t = \qquad \frac{\Gamma_t^f}{\Gamma_t^g} \\ \\ \alpha_t = \Delta_t^f - \frac{\Gamma_t^f}{\Gamma_t^g} \Delta_t^g \end{cases}$$

under the Black-Scholes PDE:

$$f_t^{call} = S_t \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-)$$
$$f_t^{put} = K e^{-r(T-t)} \Phi(-d_-) - S_t \Phi(-d_+)$$

where

$$d_\pm = \frac{ln(S_t/K) + (r \pm \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}$$

the Δ and Γ are given by:

$$\Delta^{call}(t, S) = \Phi(d_+), \Delta^{put}(t, S) = \Phi(d_+) - 1$$

$$\Gamma^{put}(t, S) = \Gamma^{call}(t, S) = \frac{\phi(d_+)}{S \sigma \sqrt{T-t}},$$

where ϕ is the standard normal probability density function.

Use the same logic as the delta-hedging, we calculate the position of the underlying stock α_t, and the position of the option used in the hedging process γ_t. Then we trade them under the market prices according to the changing positions. Additionally, apart from the transaction cost on the stock, we also need to consider transaction cost e_o on the options: if we short the option, our income decreases; if we long the option, our expense increases.

2.3 Two main categories of hedging strategies

The dynamic hedging strategy we have described is to hedge the position of selling a put option. If we increase the re-hedging frequency, we may reduce the hedge error and improve the hedging effect, but due to the transaction cost, the expenses will also increase. Ultimately, there is a trade-off between the hedging error and transaction cost. The goal of the option writer is to achieve his hedging objective as well we minimising the transaction cost. This project investigates two categories to perform the discrete dynamic hedging strategies: time-based and move-based strategies respectively.

2.3.1 Time-based Hedging

Time-based hedging suggests a fixed time interval to repeat the rebalancing step, the fixed length of time interval ϵ is interpreted as the frequency, and the number of time-step $N = T/\epsilon$. we pre-specify a finite sequence of time with equal length $(\epsilon, 2\epsilon, ..., T)$, then we hedge every ϵ. The option writer minimizes the risk over the next period of time of fixed length ϵ and does not consider the risk an any other time intervals. Therefore, there is a finite sequence of time $(\epsilon, 2\epsilon, ...T)$. At each point of time we execute a rebalancing strategy described in the previous section.

2.3.2 Move-based Hedging

Different from time-based approximation, where we have a fixed number of hedging steps, move-based strategy determines when to make hedging based on the movement of the underlying asset and a pre-specified bandwidth δ. According to the Black-Scholes PDE, we can calculate an implied asset position which can realize the gamma-neutral process, and we only rebalance the portfolio when the implied asset position hits the pre-specified band. It requires a continuous monitoring of market movement, since the time to make the hedge would happen anytime when the stock price changes. Suppose we do the last hedging at time t, and the position of the asset is α_t, and we specify a band with length $\delta > 0$, then as the implied asset position changes over time, we hedge only at the next time $t+\Delta t$ where $|\alpha_{t+\Delta t} - \alpha_t| \geq \frac{\delta}{2}$. Otherwise, we do not make changes with our positions on the hedging instruments. In other words, we set the asset position at time $t + \Delta t$ as $\alpha_{t+\Delta t} = \alpha_t$, if the implied asset position does not hit the band.

The choice of δ depends on the risk aversion level of the option writer. A risk averse individual could select a small δ to have small hedging error, but it introduces more transaction costs correspondingly, since a small band means an increasing hedging frequency.

Compared to time-based strategies, the move-based strategy takes more account to the market movement, which may improve the performance of the hedging process when the time-lapse in time-based hedging strategies is relatively large. It is very important for the writer to consider the sensitivity of the underlying asset to the model parameters, such as the volatility of the asset.

Define the net hedging error (NHE) as the difference between the hedging portfolio book value and the option payoff at the maturity.

The figure 1 & 2 give a simple illustration on two methods discussed above [1].

[1]Dupire, B. (n.d.). Quantitative finance: Developments, Applications and Problems, Cambridge UK [Digital image].

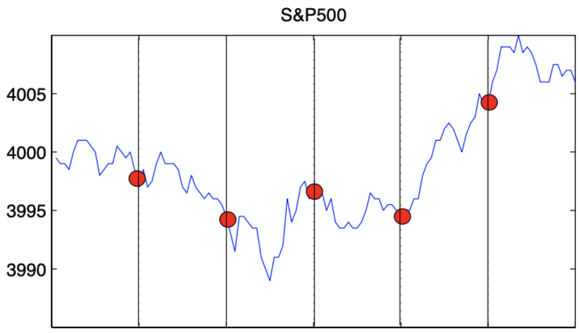

Figure 1: Time-based approximation

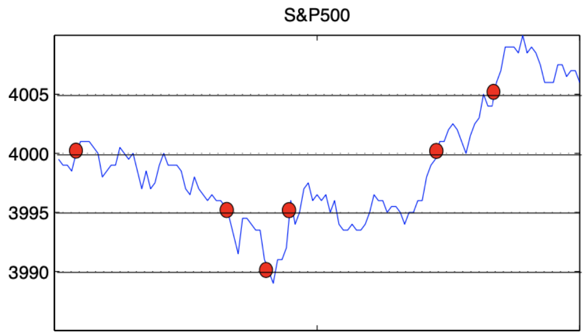

Figure 2: Move-based approximation

3 Analysis in Practice

For a put option writer, the objective of hedging is to create a portfolio to replicate a long position on the same put option. In dynamic delta hedging, we only use the underlying asset to examine hedging. However, some uncertainties affecting the option values are not incorporated under neither case. For example, the market risk-free interest risk r is not captured by the real-world stock price movement path, which is determined by the drift μ and σ. If we consider controlling more uncertainty, such as implementing delta-gamma hedging, which include one more financial instruments (the call option in this case). This theoretically improves the hedging performance at the time of maturity.

However, in practice, we cannot achieve the continuous dynamic hedging since the market prices are observed and traded through discrete times in the imperfect markets, where arbitrage opportunities are inevitable. Any of these motives suffice to render the hedging strategies imperfections. Hence, a hedging error arises. In real practice, a realistic scenario is that we are charged with the transaction cost on each unit of equity and option traded. In other words, the more frequent the strategy is, the more extra expenses it incurs. The profit is realized in a stochastic manner, along with the actual hedging process. Therefore, the profit is measured on the market-basis and the frequency-basis. This is not determined until the expiration. The purpose of hedging is to mitigate the risks from an option position, so a risk-averse person can dislike the potential large $P\&L$ variation by hedging. We use the standard deviation of the profit and losses at the maturity to reflect the **hedging precision**.

To test the hedging performance in terms of the hedging error and the associated $P\&L$ in the bank account, we have implemented four strategies: move-based and time-based for delta hedging, move-based and time-based for delta-gamma hedging. Each strategy has different rebalancing rules. Hence, the position changes in the underlying asset (stock) α_t are dependent on the hedging strategy.

To assess the empirical hedging results, we firstly simulate 1000 paths of the stock price movements according to the Geometric Brownian Motion(GBM) with the pre-specified drift($\mu = 0.1$) and volatility($\sigma = 0.2$). And the same time, we use the implied risk-neutral probability $\mathbb{Q}$ to simulate the $\{\Delta_t\}_{t\leqslant T}$ and $\{\Gamma_t\}_{t\leqslant T}$ for anytime prior to the maturity. Secondly,we rebalance the portfolio to simulate an estimated hedging process. At the maturity, we considered situations with liquidation and no-liquidation. If we do not liquidate the asset position(the stock, and the option if needed) at the maturity, then the book value is the intrinsic value associated with the hedging portfolio. If we assume to liquidate the position, we simply subtract the transaction cost incurred at the maturity from the ending **book value** to obtain the final balance in the **bank account**, which is used to obtain the $P\&L$:

$$P\&L = Bank\ Account\ - \max(K - S_T, 0)$$

Since most of the theoretical assumption in the option pricing and hedging models assumes no liquidation at the ending period, we calculate the net hedging error using the book value:

$$Net\ Hedging\ Error(NHE)\ = Book\ Value\ - \max(K - S_T, 0)$$

The main objective of a hedger is to intended to eliminate the exposure of the practitioner that holds a short or long position in a financial derivative security via a portfolio that replicates pointwise the value of the derivative at the maturity time T In the transaction costs literature, we discounted the expense at each time node of rebalancing back to the time when writing the put option. Then we accumulate this value to the maturity at a risk-free rate, so that we can assess the impact of transaction cost on the final $P\&L$ at the maturity of the option. Finally, to make all analysis consistently comparable, all results are taken **expectation** from the 1000 simulated samples.

3.1 Comparison between Move-based and Time-based strategies

3.1.1 Delta Hedging

We perform Move-based delta hedging and Time-based delta hedging. When we simulate the stock prices, the number of time-steps used is 100, so we tested 20, 50, 100 number of time-steps in the time-based hedging strategies. As the summary table in Table 1 illustrates, the move-based delta hedging outperforms all time-based strategies in terms of expected $P\&L$, standard deviation of $P\&L$ and the expected NHE. As for the time-based hedging, a small number of time-steps implies a low frequency of rebalancing in the hedging process, therefore, the expected transaction costs associated with the hedging process is also low.

Summary about the Delta Hedging				
Hedging Strategy	Frequency	$\mathbb{E}$(NHE)	$\mathbb{E}$(Transaction cost)	$\mathbb{E}(P\&L)$
Time-based(20)	Low	-0.028	0.025	-0.03
Time-based(50)	Standard	-0.037	0.03	-0.039
Time-based(100)	High	-0.023	0.035	-0.025
Move-based	–	-0.016	0.033	-0.018

Risk measurement for Delta Hedging			
Hedging Strategy	$\mathbb{E}(P\&L)$	sd$(P\&L)$	CVaR$(P\&L)$
Time-based(20)	-0.030	0.77	-1.478
Time-based(50)	-0.039	0.488	-0.969
Time-based(100)	-0.025	0.348	-0.691
Move-based	-0.018	0.358	-0.689

Table 1: The hedging performance for the Delta Hedging

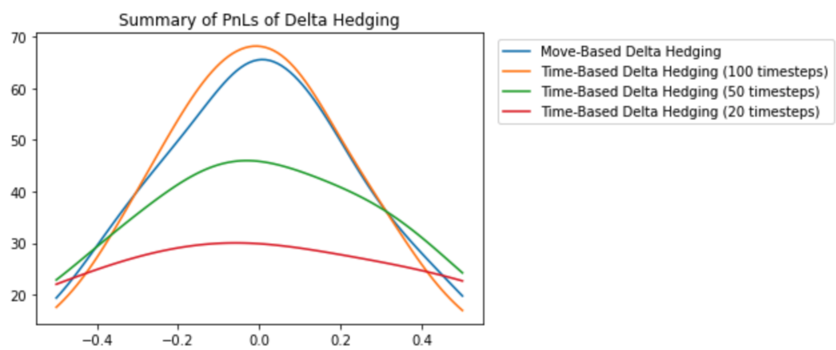

Figure 3: KDEs of the $P\&L$ for Delta Hedging

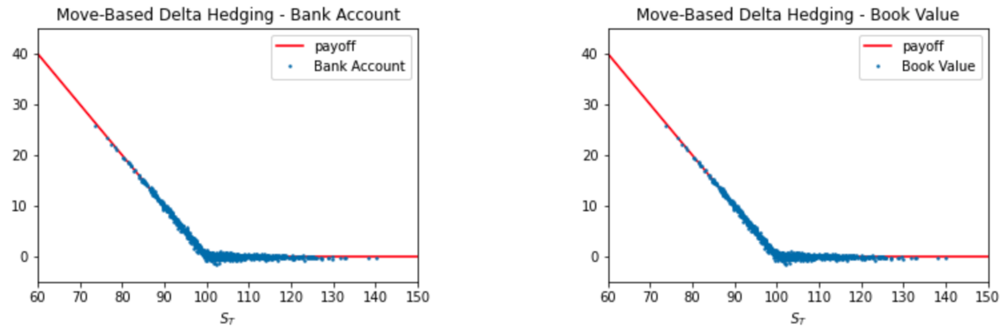

Figure 4: Move-Based Delta Hedging Performance

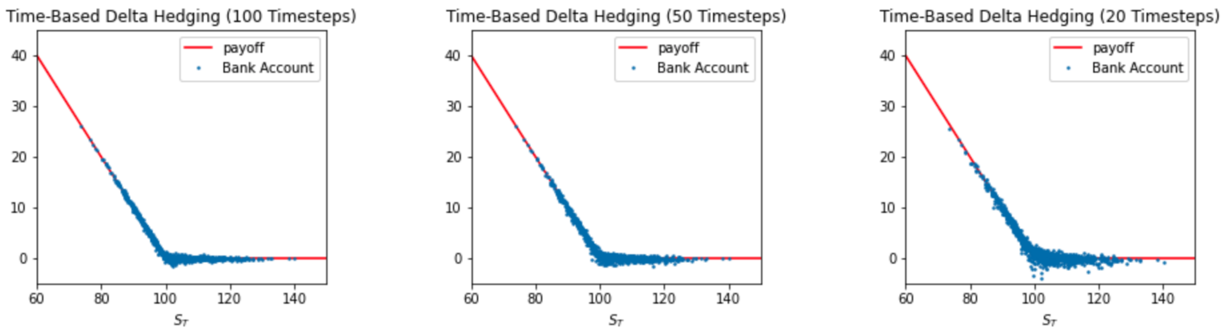

Figure 5: Time-Based Delta Hedging Performance

The Figure 3 shows that when the number of time-steps is small, the KDE of the profit and loss tends to be flatter than that of a large number of time-steps selection. The Risk Measurements table in Table 1 also show this result in the $P\&L$ plots. Also, when the hedging frequency is low, the profit and loss tends to have a larger standard deviation and a larger absolute CVaR value(In this case, we select 10% quantile), which demonstrates a fact that a low-frequency hedging process can lead to large swings of the final expected profit and loss. In other words, although the expected $P\&L$ do not vary significantly among different scenarios under the time-based strategies, low-frequency rebalancing activity is more likely to incur extreme large $P\&L$. The Figure 4 and Figure 5 visualize the bank account and book value at maturity versus the payoff of the put option. These plots are consistent with the conclusions from the tables. Most of the simulations have their ending bank account values along the put payoff plot. However, when the time-based hedging has a small number of time-steps, there are more points significantly deviating from the put option payoff, which is consistent with our findings.

3.1.2 Delta-Gamma Hedging:

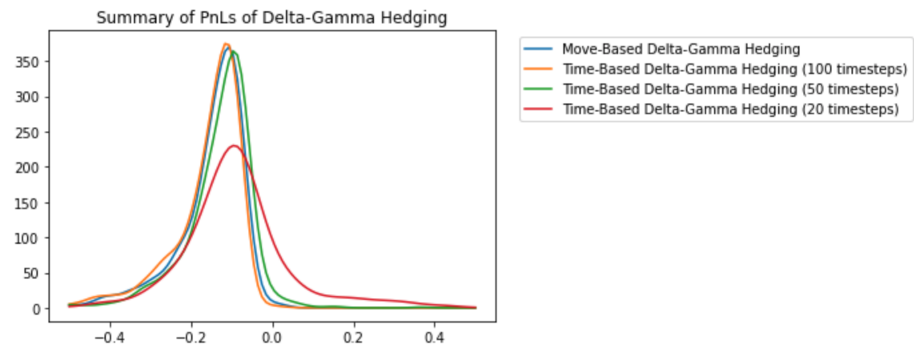

Figure 6: KDEs of the $P\&L$ for Delta-Gamma Hedging

We also implement same strategies for the delta-gamma hedging. Apart from the underlying stock, we use a European call option on the same stock (with strike price = 100 and maturity 1/2 year) to realize the delta-gamma hedging process. Figure 7 demonstrates that **delta-gamma hedging has an overall better**

Figure 7: Move-Based Delta-Gamma Hedging Performance

performance than delta hedging. The ending bank account values after liquidating all the positions are more closely distributed along the put option payoff plot compared with the delta hedging process.

Summary about the Delta-Gamma Hedging				
Hedging Strategy	Frequency	$\mathbb{E}$(NHE)	$\mathbb{E}$(Transaction cost)	$\mathbb{E}(P\&L)$
Time-based(20)	Low	-0.08	0.103	-0.095
Time-based(50)	Standard	-0.118	0.141	-0.132
Time-based(100)	High	-0.145	0.174	-0.16
Move-based	–	-0.136	0.164	-0.151

Risk measurement for Delta-Gamma Hedging			
Hedging Strategy	$\mathbb{E}(P\&L)$	sd$(P\&L)$	CVaR$(P\&L)$
Time-based(20)	-0.095	0.152	-0.358
Time-based(50)	-0.132	0.089	-0.326
Time-based(100)	-0.16	0.085	-0.357
Move-based	-0.151	0.084	-0.346

Table 2: The hedging performance for the Delta-Gamma Hedging

Since the delta-gamma hedging strategy involves in the trading of another option under the same underlying asset, there are more transaction costs incurred from the rebalancing process. The Summary table in the Table 2 shows the results for the Delta-Gamma Hedging: the expected transaction costs for each hedging process are higher than that for the delta hedging with the same scenario settlement. However, the absolute NHE and expected profit losses are lower than that in the delta hedging case, since the strategy monitors the change value in the hedging portfolio with one additional dimension of uncertainty, more transaction costs are taken on the hedging portfolio. When the time-based delta-gamma hedging has small number of time-steps, the hedging frequency is low and the strategy tends to have a high expected $P\&L$ and low absolute net hedging error. Despite of the fact that the small number of time-steps can decrease the transaction costs and bring a higher expected $P\&L$, it still has a fairly larger standard deviation than that of the case where we use move-based method or select large number of time-steps. Figure 6 displays a similar pattern as Figure 3. Small numbers for time-steps can also drag down the KDE distribution of profit and losses.

3.2 Real-world volatility $\sigma^{\mathbb{P}}$ Risk-Neutral volatility $\sigma^{\mathbb{Q}}$

When the underlying asset (such as stock) becomes more volatile, there is greater risk involved in the hedging process. To attain certain level of accuracy, we need to monitor the movement of the stock prices more frequently. In practice, to increase the hedging frequency, we can either increase the number of time-steps in time-based strategy or decrease the bandwidth in move-based strategy. However, more transaction costs are spent to pay for a better hedge performance. In our assumption, the European option is priced using the risk-neutral volatility 20%, which is taken account into the Black-Scholes model. In this section, we tested the impact of the stock volatility by varying the $\sigma^{\mathbb{P}}$ in real-world, using 15% and 25% two scenarios.

Summary for Hedging with $\sigma^{\mathbb{P}}$=0.15, $\sigma^{\mathbb{Q}}$=0.2			
Hedging Strategy	$\mathbb{E}(P\&L)$	$\mathbb{E}(NHE)$	$\mathbb{E}(Transaction\ Cost)$
Time-based-Delta (20)	0.944	0.946	0.024
Time-based-Delta (50)	0.949	0.951	0.028
Time-based-Delta (100)	0.95	0.952	0.032
Move-based-Delta	0.952	0.954	0.03
Time-based,Delta-Gamma (20)	-0.091	-0.072	0.101
Time-based,Delta-Gamma (50)	-0.124	-0.104	0.136
Time-based,Delta-Gamma (100)	-0.15	-0.131	0.163
Move-based,Delta-Gamma	-0.139	-0.12	0.154

Risk measurement for Hedging with $\sigma^{\mathbb{P}}$=0.15, $\sigma^{\mathbb{Q}}$=0.2			
Hedging Strategy	$\mathbb{E}(P\&L)$	sd$(P\&L)$	CVaR$(P\&L)$
Time-based-Delta (20)	0.944	0.638	-0.089
Time-based-Delta (50)	0.949	0.485	0.213
Time-based-Delta (100)	0.95	0.417	0.322
Move-based-Delta	0.952	0.429	0.281
Time-based,Delta-Gamma (20)	-0.091	0.121	-0.243
Time-based,Delta-Gamma (50)	-0.124	0.059	-0.245
Time-based,Delta-Gamma (100)	-0.15	0.067	-0.302
Move-based,Delta-Gamma	-0.139	0.064	-0.284

Table 3: The hedging performance for the Dynamic Hedging when real-world volatility $\sigma^{\mathbb{P}} = 15\%$ **and the risk-neutral volatility** $\sigma^{\mathbb{Q}} = 20\%$

Summary for Hedging with $\sigma^{\mathbb{P}}$=0.25, $\sigma^{\mathbb{Q}}$=0.2			
Hedging Strategy	$\mathbb{E}(P\&L)$	$\mathbb{E}(NHE)$	$\mathbb{E}(Transaction\ Cost)$
Time-based-Delta (20)	-1.078	-1.076	0.026
Time-based-Delta (50)	-1.021	-1.019	0.031
Time-based-Delta (100)	-1.029	-1.027	0.037
Move-based-Delta	-1.023	-1.021	0.035
Time-based,Delta-Gamma (20)	-0.102	-0.089	0.108
Time-based,Delta-Gamma (50)	-0.137	-0.125	0.150
Time-based,Delta-Gamma (100)	-0.176	-0.163	0.188
Move-based,Delta-Gamma	-0.163	-0.151	0.178

Risk measurement for Hedging with $\sigma^{\mathbb{P}}$=0.25, $\sigma^{\mathbb{Q}}$=0.2			
Hedging Strategy	$\mathbb{E}(P\&L)$	sd$(P\&L)$	CVaR$(P\&L)$
Time-based-Delta (20)	-1.078	1.078	-3.343
Time-based-Delta (50)	-1.021	0.755	-2.594
Time-based-Delta (100)	-1.029	0.628	-2.355
Move-based-Delta	-1.023	0.631	-2.347
Time-based,Delta-Gamma (20)	-0.102	0.265	-0.607
Time-based,Delta-Gamma (50)	-0.137	0.148	-0.459
Time-based,Delta-Gamma (100)	-0.176	0.128	-0.464
Move-based,Delta-Gamma	-0.163	0.127	-0.447

Table 4: The hedging performance for the Dynamic Hedging when real-world volatility $\sigma^{\mathbb{P}} = 25\%$ **and the risk-neutral volatility** $\sigma^{\mathbb{Q}} = 20\%$

Delta Hedging: The *Table 3* summarizes the hedging results when the stock volatility is smaller than the risk-neutral volatility, then it is not strong enough to hedge the risk we burdened from writing an associated option. Compared with the condition where we assumed same volatility for the real-world and risk-neutral, there is no significant change in the transaction cost. However, the absolute NHE increase, we have seen greater error arise from both time-based and move-based strategies, and the negative errors become positive error, so we can observe a positive expected profit through the hedging strategies. This is partly due to the profit generation capacity of the option increases when the volatility increases. Suppose we are comparing an option priced using the real-world $\sigma^{\mathbb{P}} = 15\%$, then its risk can be mostly hedged by a stock with the same volatility.

If we change the volatility to the risk-neutral case $\sigma^{\mathbb{Q}} = 20\%$, then both the delta and delta-gamma curve of the option would become flatter, and hence have more possibility to earn profits, but only part of its risk is offset by a stock with lower volatility. Therefore, the expected profit rises but the overall variation of the profit also increases. The same logic applies to the condition where the real-world stock volatility is greater than the risk-neutral volatility, as shown in the *Table 4* in which the delta and gamma curves of the option become more steep so we expect more loss than profit. As a result, both the expectation and the CVaR of the $P\&L$ are negative, and we still observe a great variance on the $P\&L$ since we are not only taking the intrinsic risk from the option, but also the risks involved in the high volatile stocks.

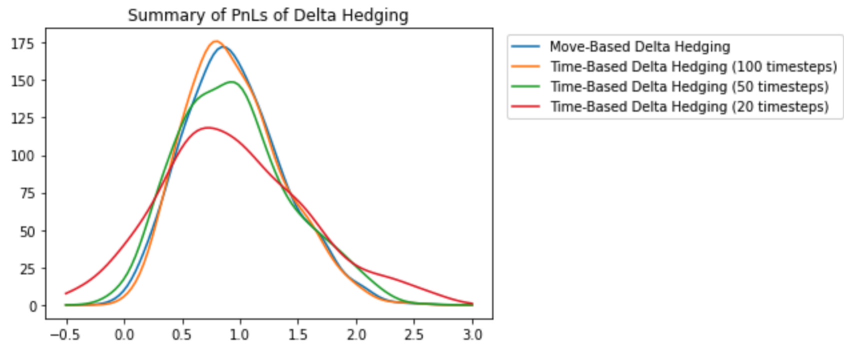

Figure 8: KDEs of the $P\&L$ for Delta Hedging with $\sigma^{\mathbb{P}}$=0.15 and $\sigma^{\mathbb{Q}}$=0.2

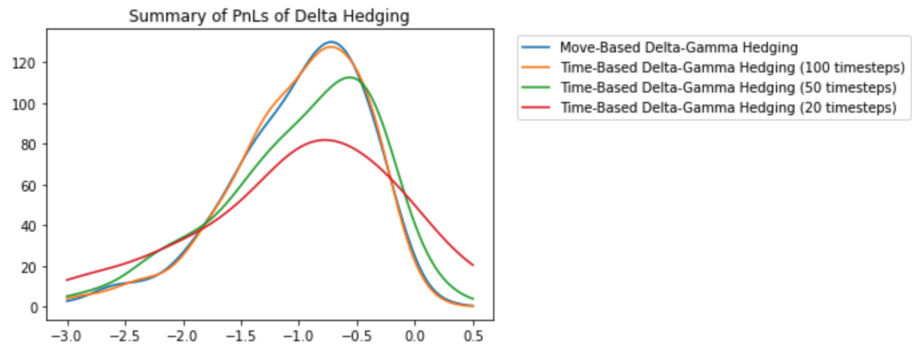

Figure 9: KDEs of the $P\&L$ for Delta Hedging with $\sigma^{\mathbb{P}}$=0.25 and $\sigma^{\mathbb{Q}}$=0.2

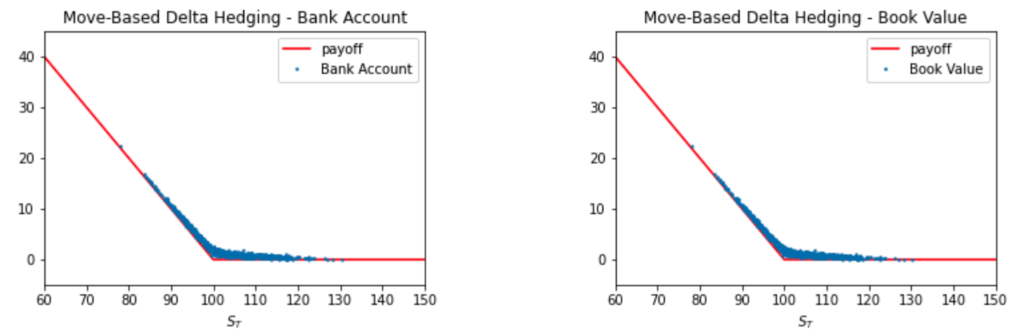

Figure 10: Move-Based Delta Hedging with $\sigma^{\mathbb{P}}$=0.15 and $\sigma^{\mathbb{Q}}$=0.2

Delta-Gamma Hedging: On the other hand, if we change the strategy from delta hedging to delta-gamma hedging, then we can improve the hedging performance, because we import another option on the same underlying asset to achieve the gamma-neutral condition. The gamma hedging is to neutralize the risk of an option price moving rapidly shortly before expiration, in other words, some unexpected moves neglected by the delta hedging can be addressed with gamma hedging. The underestimated risks from a low volatile stock are taken up by a option, which is priced using the same risk-neutral volatility. Therefore, from the tables, we can observe that the expectation of $P\&L$ has been driven back to near zero, which is close to the expected $P\&L$ value in the delta-gamma hedging when the real world volatility is equal to the risk-neutral volatility. And the variation on the $P\&L$ has also been mitigated.

If we compare the KDEs of the $P\&L$ for Delta and Gamma hedging, we can observe that the variation under the Gamma hedging is smaller than that under the Delta hedging, and the move-based criteria and time-based criteria with high frequency all outperform the hedging results with 20-steps time-based strategy in terms of the hedging precision (variance of the profit and loss). Moreover, the blue spots lie more close in the payoff curve of a put option compared with the pure Delta hedging.

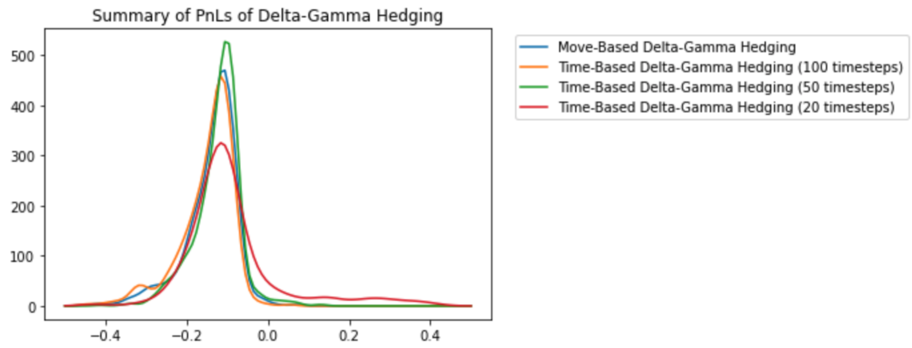

Figure 11: KDEs of the $P\&L$ for Delta-Gamma Hedging with $\sigma^{\mathbb{P}}$=0.15 and $\sigma^{\mathbb{Q}}$=0.2

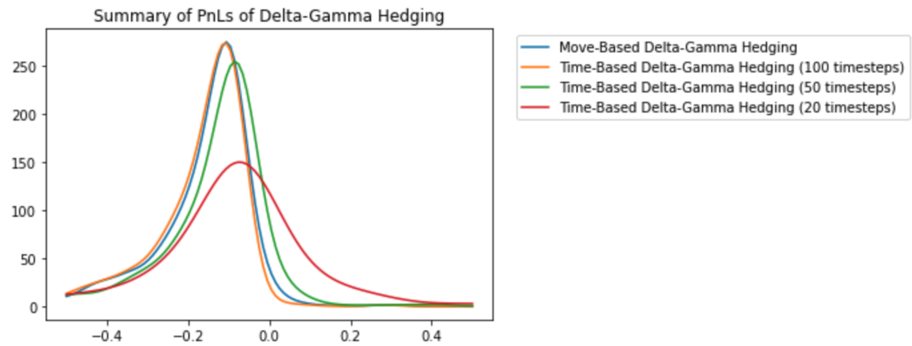

Figure 12: KDEs of the $P\&L$ for Delta-Gamma Hedging with $\sigma^{\mathbb{P}}$=0.25 and $\sigma^{\mathbb{Q}}$=0.2

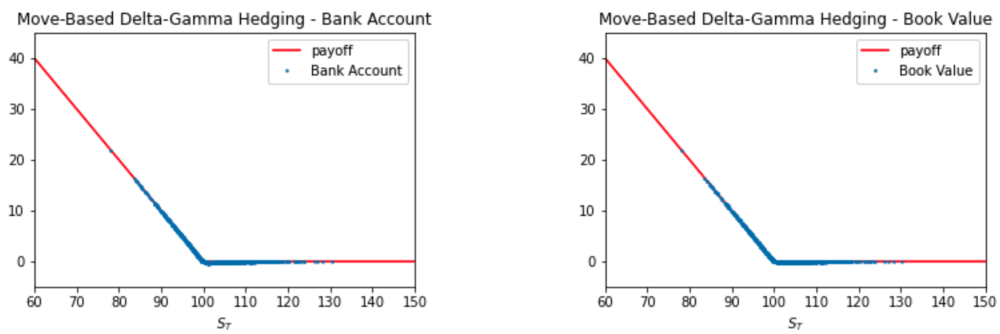

Figure 13: Move-Based Delta-Gamma Hedging with $\sigma^{\mathbb{P}}$=0.25 and $\sigma^{\mathbb{Q}}$=0.2

3.3 The role of rebalancing-band in Delta-Hedging

When using move-based strategy, there are two sources of randomness in the hedging process: one is the number of units traded on the stocks and options at each rebalancing step, the other one is when to make the rebalancing, which depends on the actual stock prices movement. Therefore, we can observe more volatility on the $P\&L$ in the move-based strategy. However, we can reduce this variation by adjusting the bandwidth that we put on the stock position α, which we use to decide whether to change the positions in the underlying assets. So we varied the bandwidth to test the impact on the hedging results.

When we examine the performance of move-based hedging strategy in the option, the re-balancing band plays a key role in the hedging process. If the α is in the range of the bandwidth, then we are not changing

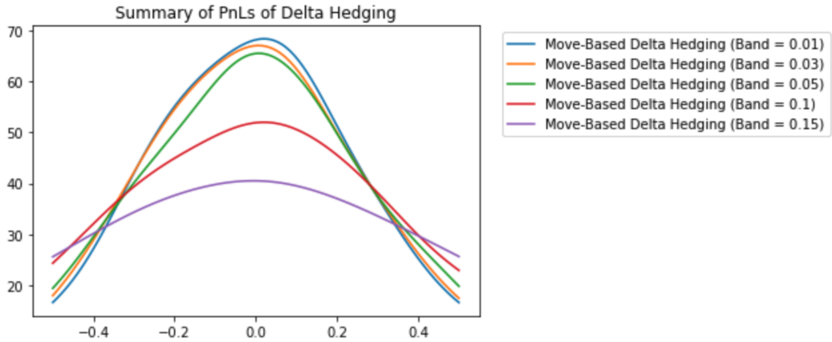

Figure 14: KDEs of the $P\&L$ for Delta Hedging with different bands

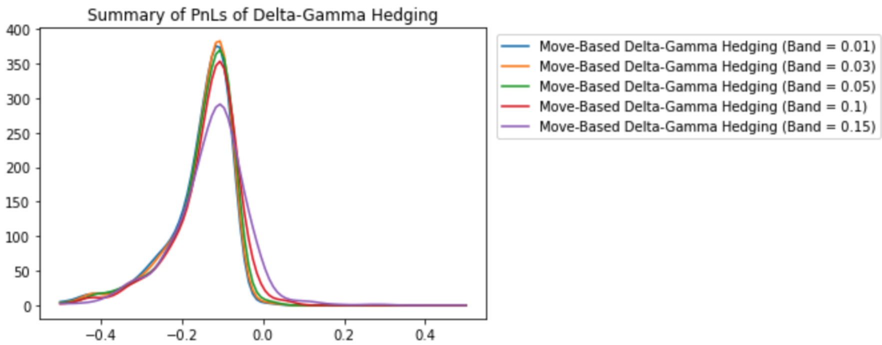

Figure 15: KDEs of the $P\&L$ for Delta-Gamma Hedging with different bands

the positions in the underlying asset. We investigate how the $P\&L$ will be affected by five different bandwidths on the Delta hedging. From the table of the summary for Hedging with different bands, noticing that the expectation of all $P\&L$ are close to zero and the value of $P\&L$ will be larger as the band size decreases. This demonstrates the frequency to rebalance on the underlying asset will be less than a smaller band because it allows the stock to have more space for variation. By comparing the band size of 0.01 and 0.03, the difference of these two bands are relatively small and these two expectations of $P\&L$ are the same because the underlying asset is performing hedging at each time step as the band is small. By examining the expected transaction cost from the Table 5, the presence of costs would be lower as the band is larger. A larger band reduces the number of transactions incurred as the frequency to do the rebalancing drops. From Figure 14 and Figure 15, noticing that the curve of $P\&L$ will be flatter as the band is larger, which visualizes the summarized $P\&L$ results from the Table 5. Same for the Delta-Gamma hedging, the expectation of all $P\&L$ are close to zero, value of $P\&L$ decreases and the transaction costs decrease as the band is larger. Additionally, based on Table 5, a large band is associated with a higher standard deviation of profit and losses for both delta hedging and delta-gamma hedging. In other words, the hedger needs to bear more volatility in the portfolio if the hedging frequency decreases. Hence, a comparatively small band selection can effectively balance the hedging cost and the potential swings in the portfolio.

Summary for Hedging with different bands			
Hedging Strategy	$\mathbb{E}(P\&L)$	$\mathbb{E}(NHE)$	$\mathbb{E}(TransactionCost)$
Move-Based Delta(Band=0.01)	-0.024	-0.022	0.035
Move-Based Delta(Band=0.03)	-0.024	-0.022	0.034
Move-Based Delta(Band=0.05)	-0.018	-0.016	0.033
Move-Based Delta(Band=0.10)	-0.027	-0.025	0.03
Move-Based Delta(Band=0.15)	-0.016	-0.014	0.028
Move-Based Delta-Gamma(Band=0.01)	-0.159	-0.144	0.173
Move-Based Delta-Gamma(Band=0.03)	-0.155	-0.141	0.169
Move-Based Delta-Gamma(Band=0.05)	-0.151	-0.136	0.164
Move-Based Delta-Gamma(Band=0.10)	-0.142	-0.127	0.154
Move-Based Delta-Gamma(Band=0.15)	-0.133	-0.118	0.147

Risk measurement for Hedging with different bands			
Hedging Strategy	$\mathbb{E}(P\&L)$	sd$(P\&L)$	CVaR$(P\&L)$
Move-Based Delta(Band=0.01)	-0.024	0.344	-0.68
Move-Based Delta(Band=0.03)	-0.024	0.347	-0.68
Move-Based Delta(Band=0.05)	-0.018	0.358	-0.689
Move-Based Delta(Band=0.10)	-0.027	0.412	-0.794
Move-Based Delta(Band=0.15)	-0.016	0.506	-0.923
Move-Based Delta-Gamma(Band=0.01)	-0.159	0.084	-0.355
Move-Based Delta-Gamma(Band=0.03)	-0.155	0.084	-0.351
Move-Based Delta-Gamma(Band=0.05)	-0.151	0.084	-0.346
Move-Based Delta-Gamma(Band=0.10)	-0.142	0.085	-0.332
Move-Based Delta-Gamma(Band=0.15)	-0.133	0.093	-0.324

Table 5: The hedging performance for the Move-based Dynamic Hedging Strategies for different bandwidth

4 Conclusion

This project investigates the implementation of delta and delta-gamma hedging in discrete times within the context of a Black-Scholes Model Partial Differential Equation (PDE), to illustrate how to construct a dynamic hedging strategy to offset the position the option, and investigate how effective do the δ and γ play on the risk management. Four types of hedging strategies have been examined: **time-based delta hedging, move-based delta hedging, time-based delta-gamma hedging and move-based delta-gamma hedging**. By generating the empirical distributions of the PL, NHE and transaction costs for different strategies, we evaluate the impact of different parameters in the models on the risk hedge performance.

The main findings from our research are that there is always a trade-off between hedging frequency and the absolute hedging error. For instance, in the time-based hedging, decreasing the number of time-step can lower the trading frequency and then the associated total transaction costs. On the other hand, this activity leads to a greater swings in the hedging error and the $P\&L$. Similarly, in the move-based hedging, if we increase the bandwidth restricted on the alpha position, then there is greater variation on the hedging error as the rebalancing frequency decrease, we fail to catch the in-time movement on the stock prices in the market. Most hedgers are risk-averse. Therefore, they would prefer a greater hedging precision at a cost of increasing transaction costs. Based on the results of this research, we can concluded that theoretically the delta-gamma hedging outperforms the pure delta hedging in terms of the risk management. However, the real practice has more complexity, such as the availability of the hedging options on the same underlying asset, the difference between the stock's volatility and the implied volatility on the option. The preference between time-based and move-based strategies also depend on the real practice. This report considers the most simple cases, where the move-based slightly outperforms the time-based strategies.

We also explored the hedging performance and precision of different strategies when there exists difference between the real world volatility (on the stock) and the risk-neutral volatility (on the option). Under this situation, delta-gamma hedging outperforms delta hedging, since delta-gamma hedging use two assets (stock and option) to synchronously track the change of the value in the written put. It should be noted that the market is imperfect and different individuals have different risk tolerance levels, delta and gamma hedging do not have to be completely neutral, and traders may adjust how much positive or negative delta and delta-gamma the entire portfolio is exposed to over time.

5 Appendix

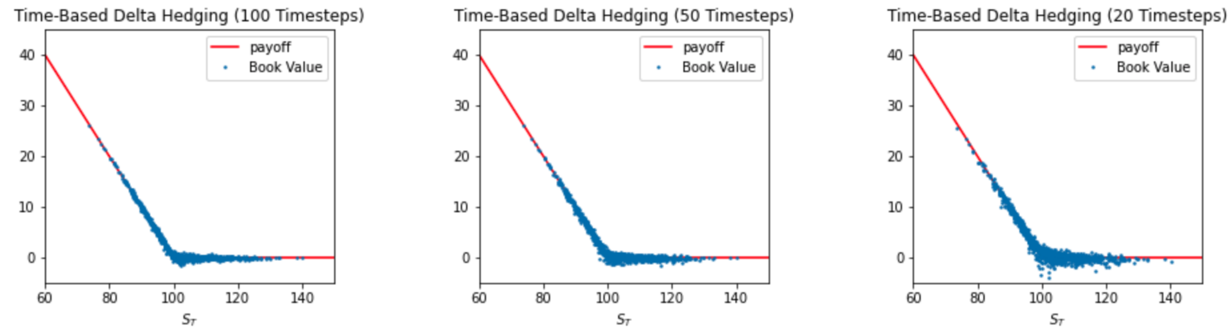

Figure 16: Time-Based of Delta Hedging - Book Value

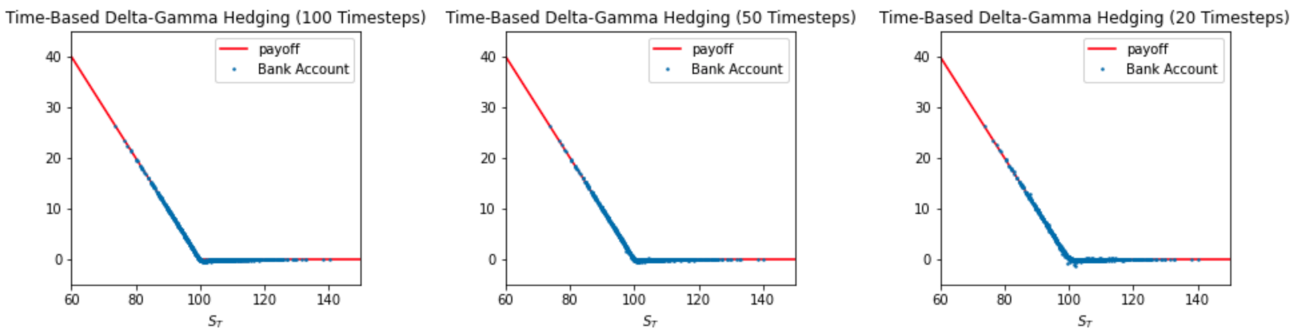

Figure 17: Time-Based of Delta-Gamma Hedging - Bank Account

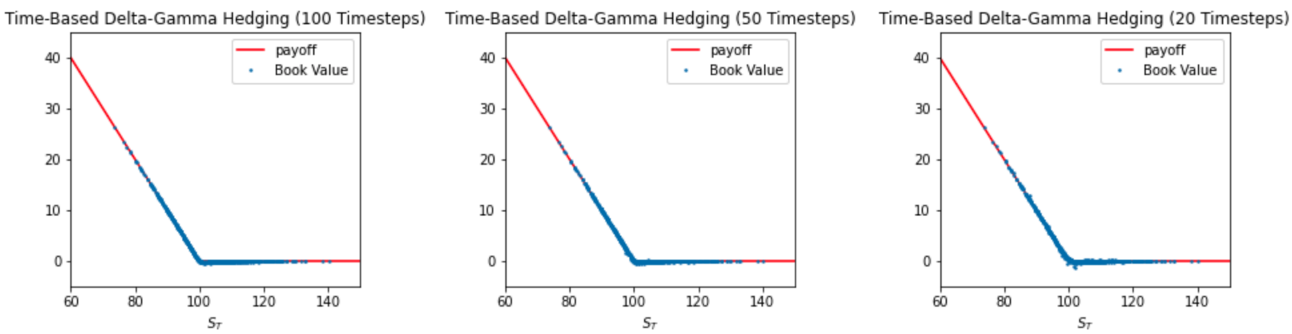

Figure 18: Time-Based of Delta-Gamma Hedging - Book Value

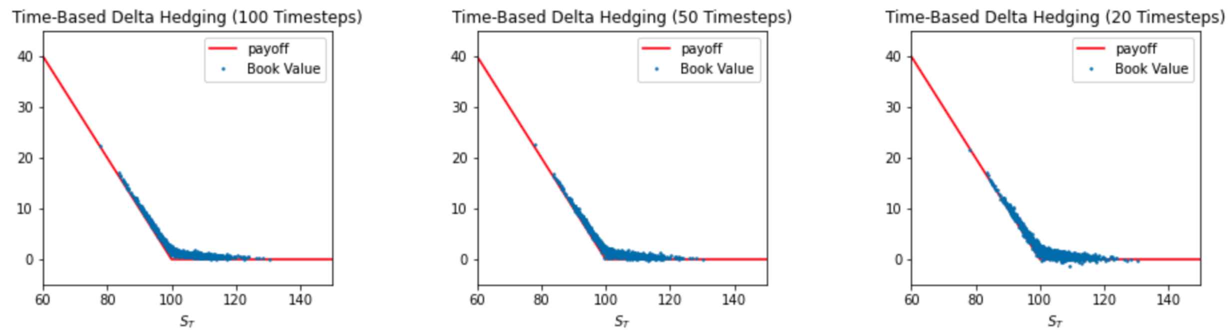

Figure 19: Time-Based of Delta Hedging - Book Value with $\sigma^{\mathbb{P}}$=0.15 and $\sigma^{\mathbb{Q}}$=0.2

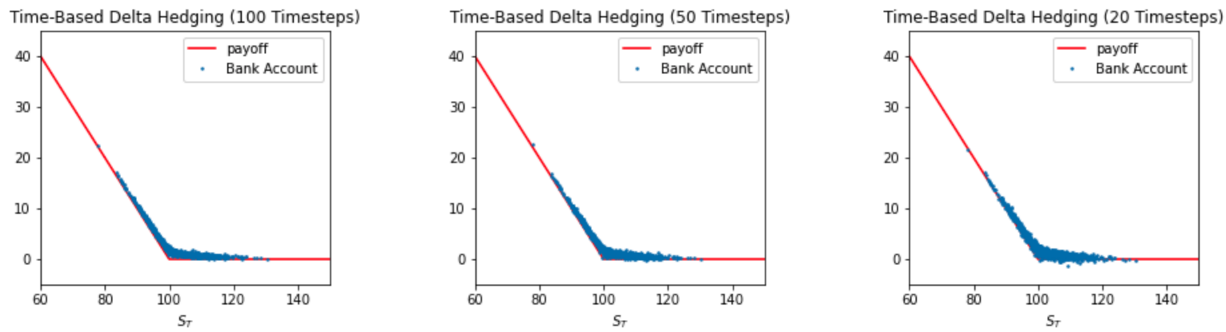

Figure 20: Time-Based Delta Hedging with $\sigma^{\mathbb{P}}$=0.25 and $\sigma^{\mathbb{Q}}$=0.2

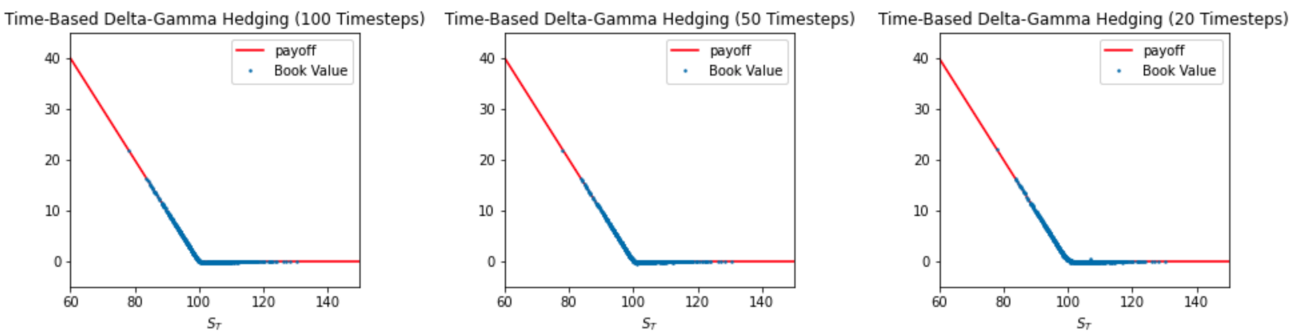

Figure 21: Time-Based Gamma Hedging - Book Value with $\sigma^{\mathbb{P}}$=0.15 and $\sigma^{\mathbb{Q}}$=0.2

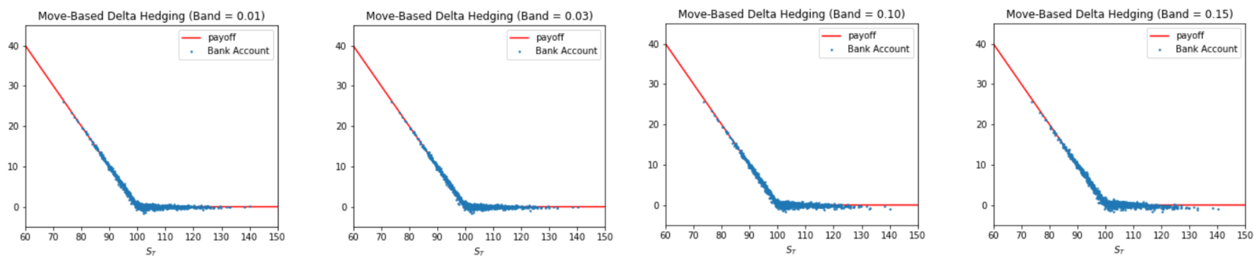

Figure 22: Move-Based of Delta Hedging with different bands

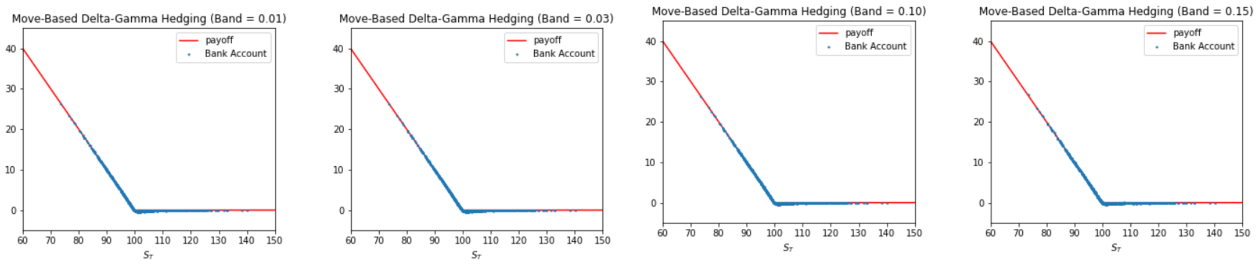

Figure 23: Move-Based of Delta-Gamma Hedging - Bank Account with different bands

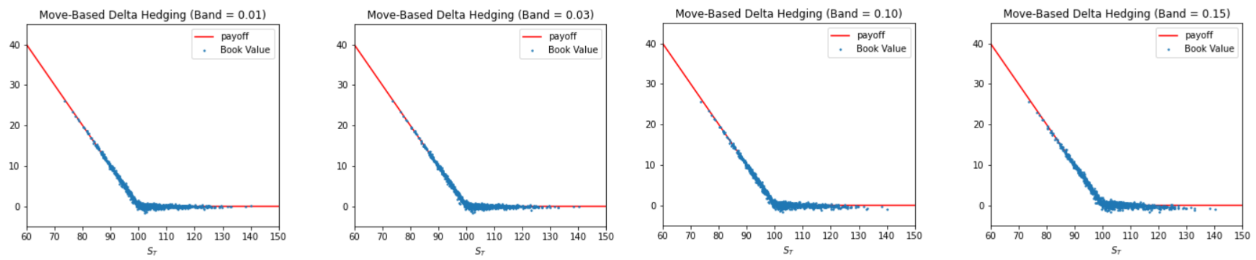

Figure 24: Move-Based of Delta Hedging - Book Value with different bands

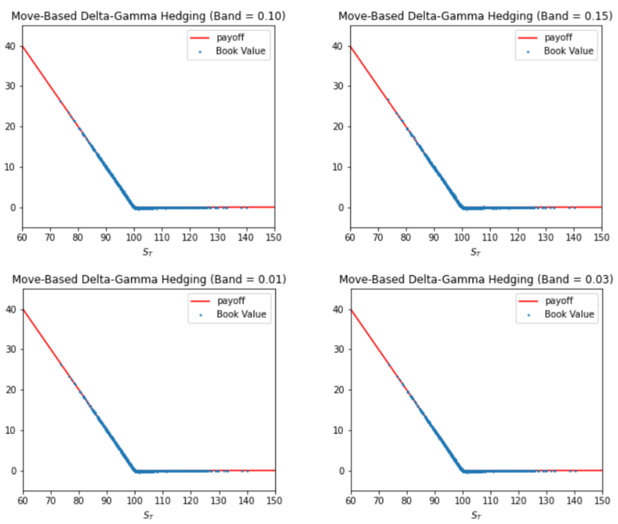

Figure 25: Move-Based of Delta-Gamma Hedging - Book Value with different bands

6 References

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