(Q1) is $4^{1536} \equiv 9^{4824}\ mod\ 35$

Q_1 is $4^{1536} \equiv 9^{4824} \bmod 35$

we can analyze the question by calculating the value of $4^{1536} \bmod 35$ and checking if it is the same as $9^{4824} \bmod 35$. We begin by solving $4^{1536} \bmod 35$.

$(nq+r)^x \bmod n \equiv r^x \bmod n$

$4^{2\cdot 768} \Rightarrow 16^{768} \bmod 35$

$16^{2\cdot 384} \Rightarrow 256^{384} \bmod 35$

$\Rightarrow (\cancel{7\cdot 35}+11)^{384} \bmod 35$

$\Rightarrow 11^{384} \bmod 35$

$11^{2\cdot 192} \Rightarrow 121^{192} \bmod 35$

$\Rightarrow (\cancel{35\cdot 5}+16)^{192} \bmod 5$

$16^{2\cdot 96} \Rightarrow 256^{96} \bmod 35$

$\Rightarrow (\cancel{35\cdot 7}+11)^{96} \bmod 35$

$11^{2\cdot 48} \Rightarrow 121^{48} \bmod 35$

$\Rightarrow (\cancel{3\cdot 35}+16)^{48} \bmod 35$

$16^{2\cdot 24} \Rightarrow 256^{24} \bmod 35$

$\Rightarrow (\cancel{7\cdot 35}+11)^{24} \bmod 35$

$11^{2\cdot 12} \Rightarrow 121^{12} \bmod 25$

$\Rightarrow (\cancel{3\cdot 35}+16)^{12} \bmod 35$

$16^{2\cdot 6} \Rightarrow 256^{6} \bmod 35$

$\Rightarrow (\cancel{35\cdot 7}+11)^{6} \bmod 35$

$11^{2\cdot 3} \Rightarrow 121^{3} \bmod 35$

$\Rightarrow (\cancel{3\cdot 35}+16)^{3} \bmod 35$

$16^2 16^1 \Rightarrow 256 \cdot 16 \bmod 35$

$\Rightarrow (\cancel{7\cdot 35}+11)\cdot 16 \bmod 35$

$\Rightarrow 11 \cdot 16 \bmod 35$

$$\boxed{\begin{array}{l} 176 \bmod 35 = 1 \\ \therefore 4^{1536} \bmod 35 = 1 \end{array}}$$

we know $4^{1536} \bmod 35 = 1$, we can use Fermat's little theorem to solve whether or not $9^{4824} \bmod 35 = 1$, where $9^{4824} \equiv 1 \bmod 35$

By Fermat's prime factorization by n, we get $35 = 5\cdot 7$

① $a^{n-1} = 1 \bmod n$ for $n=5$

$a^4 = 1 \bmod 5$

② $a^{n-1} = 1 \bmod n$ for $n=7$

$a^6 = 1 \bmod 7$

combine ① ② yields

$a^{4\cdot 6} \equiv 1 \bmod (5\cdot 7)$

$a^{24} \equiv 1 \bmod 35$

Solve for a by $\frac{4824}{24} = 201$

$a = 9^{201}$

by fermat's we have found a value that satisfy his theorem

$\therefore 9^{4824} \bmod 35 = 1$

Both the left hand side and right hand side are congruent, therefore the statement is true. ■

Q2 Solve $x^{86} \equiv 6 \ mod \ 29$

Q3 Prove that $gcd(F_{n+1},\ F_n) = 1$, for $n \geq 1$, where F_n is the n-th Fibonacci element.

Solution:

When the gcd of two numbers is 1, that means that the numbers are relatively prime, meaning that there is no number $n \neq 1$ that divides both of the numbers. We can prove the following statement by induction.

Base case:

For $n = 0$, we check the $gcd(F_0, F_1)$, which are $gcd(1, 1)$, these two numbers satisfy as $gcd(1, 1) = 1$ ✓

Induction step:

For our induction hypothesis, we assume that $gcd(F_n, F_{n+1}) = 1$. We must prove that the statement is true as well for $n = k + 1$ and prove $gcd(F_{k+1}, F_{k+2}) = z$, for $z = 1$

By the equation, we know that $z \mid F_{k+1}$, $z \mid F_{k+2}$ and $z \mid F_k + F_{k+1}$ since $F_x = F_{x-2} + F_{x-1}$ for $x \in \mathbb{N}$. $z \mid F_k + Fk + 1$ tells us that $z \mid F_k$ and $z \mid F_{k+1}$, we can derive Euclid's GCD by stating $gcd(F_k, F_{k+1}) = z$. From our induction hypothesis, we know that $gcd(F_n, F_{n+1}) = 1$, therefore, z is also 1. We prove the following claim by a direct proof via induction. ■