1. Exercise 2.4

   (a) Consider the following matrix:

$$\begin{bmatrix} 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_d \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{d+1} \end{bmatrix}$$

   Since this is a non-singular matrix, we know that every system $Ax = b$ has a unique solution, if we are able to find a hypothesis with weights w such to produce our solution y, then it is true that there exist at least one hypothesis that can shatter those points.

   (b) For $d+2$ points, where each point has dimension $d+1$, any $d+2$ vetors of length $d+1$ has to be linearly dependent.

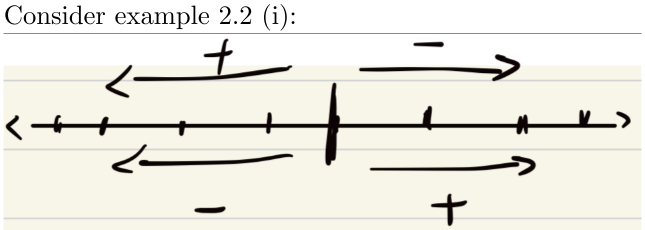
   However, intuitively, it's possible to have less poitns than dimensions in a linear independent set, but if we have more points than dimensions, it is no possible, which is the case that we have.

   Consider our $d+2$ vector, x_{d+2}, where $x_{d+2} = \sum_{i=0}^{d+1} c_i x_i$, if we have a dichotomy where $w^T x_n c_n < 0$ for each point n, assuming that the dichotomy holds for $d+2$ points, then $sign(w^T x_{d+2}) = -1$, a situation where the positive dichotomy cannot be implemented.

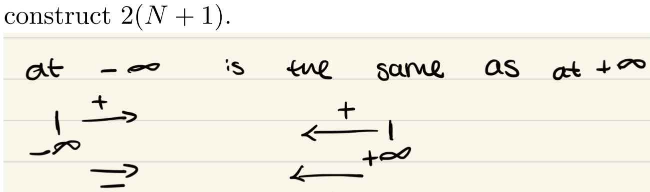
   (a) and (b) proves that the VC dimension of the perceptron is exactly $d+1$ ■

2. Problem 2.3

   (a) Consider example 2.2 (i):



   We can split the line in $N+1$ different places, each place has two possible rays, so we construct $2(N+1)$.



   So $m_H(N) = 2N + 2 - 2 = 2N$

For $k = 2$: $2^2 = 4 \rightarrow 2(2) = 4$
For $k = 3$: $2^3 = 8 \rightarrow 2(3) = 6$
So breakpoint at $k = 3$, so $d_{vc} = k - 1 = 2$

(b) Consider example 2.2 (ii):
In the case where it is only a positive interval, we have $\frac{1}{2}N^2 + \frac{1}{2}N + 1$, counting positive and negatives, we can multiply it by 2, giving us $N^2 + N + 2$.
Range of $[-\infty, a]$ being positive is the same as $[a, \infty]$ being negative, and vise versa, this applies for all $a \in N$, so we can remove $2N$, giving us the following equation $N^2 - N + 2$.
$k = 2$, we have $2^2 - 2 + 2 = 4 = 2^2$
$k = 3$, $3^2 - 3 + 2 = 8 = 2^3$
$k = 4$, $4^w - 4 + 2 = 14 \neq 2^4$, so our breakpoint is at $k = 4$, $d_{vc} = k - 1 = 3$

(c) Consider example 2.2 (ii):
We can use the positive interval example to bond a concentric sphere in d dimensions similarly. From hw3, exercise 2.3 part(2), this concentric sphere is in $d_{vc} = 2$ since it has a breakpoint at $k = 3$.

3. Problem 2.8
From lecture (6), possible growth functions are either 2^N or $\exists$ a breakpoint k such that $m_H(N) \leq N^{k-1} + 1$.

(i) Yes, $1 + N$, by inspection, breakpoint at $N = 2$, so $m_H(3) \leq 3 + 1$ where $m_H(3) = 4$, so it is valid, $m_H(N) = N + 1$, $N^{2-1} + 1 = N + 1$

(ii) Yes, $1 + N + \frac{N(N-1)}{2}$, at $N = 3$, then it equals 7, $7 < 2^3$, so $k = 3$ is our breakpoint. $1 + N + \frac{N^2 - N}{2} = \frac{N^2}{2} + \frac{N}{2} + 1 \leq N^2 + 1$, by inspection, we see that it grows slower than $N^2 + 1$ since it is being divided by 2, therefore it is corect.

(iii) Yes, 2^N is a power of 2, so it is valid.

(iv) No, for $N = 2$, $2^{f\ \sqrt{2}} = 2 < 2^2$ (I don't know how to do floor symbol on latex, assume f = floor), therefore $k = 2$ is a breakpoint, so $2^{f\sqrt{N}} \leq N + 1$, we can see this is not true by inspection, since for $N = 100$, $2^{10} \nleq 101$.

(v) No, $2^{f\frac{N}{2}}$, $k = 2$ by inspection is our breakpoint, choose a high N, we can see that $2^{f\frac{N}{2}} \nleq N + 1$, same reasoning as (iv).

(vi) No, for $k = 2$, the equation evaluates to $3 < 2^2$, so it is our breakpoint. this means that $1 + N + \frac{N(N-1)(N-2)}{6} < N + 1$, if we try $N = 10$: $11 + \frac{10(9)(8)}{6} = 11 + 5 \times 3 \times 8 = 131 \nleq 11$

4. Problem 2.10
Consider D and D' where D and N datapoints and D' has N datapoints as well, but D and D' contain completely different datasets. We can take any dichotomy in D and match it with all the dichotomies in D', creating one dichotomy of $2N$ points. If you take all the dichotomies and match them respecitvely, you will get at most $m_H(N)^2$ dichotomies, which is where we receive the upperbound, therefore $m_H(2N) \leq m_H(N)^2$, hence we can change our vc generalization bound from $E_{out} \leq E_{in} + \sqrt{\frac{8}{N} \ln \frac{4m_H(2N)}{\delta}}$ to $E_{out} \leq E_{in} + \sqrt{\frac{8}{N} \ln \frac{4m_H(N)^2}{\delta}}$.

5. Problem 2.12

From (2.13)

$$N \geq \frac{8}{\epsilon^2} \ln \frac{4((2N)^{d_{vc}} + 1)}{\delta}$$

We need 95% confidence, so $\delta = 0.05$, we need generalization error at most 0.05, so $\epsilon = 0.05$, so

$$N \geq \frac{8}{0.05^2} \ln \frac{4((2N)^{10} + 1)}{0.05}$$

From example 2.6, we continue to guess and iterate: $d_{vc} = 10$, we can try $N = 10000$

$$\begin{aligned}
N &\geq \frac{8}{0.05^2} \ln \frac{4((2(10000))^{10} + 1)}{0.05} = 330934 \\
N &\geq \frac{8}{0.05^2} \ln \frac{4((2(330934))^{10} + 1)}{0.05} = 442912 \\
N &\geq \frac{8}{0.05^2} \ln \frac{4((2(442912))^{10} + 1)}{0.05} = 452239 \\
N &\geq \frac{8}{0.05^2} \ln \frac{4((2(452239))^{10} + 1)}{0.05} = 452906
\end{aligned}$$

Appears to converge somewhere close to 452906, so we can say that $N \geq 452906$.