1. Exercise 3.4

   (a) we know $y = w^{*T}x + \epsilon$ and $H = X(X^TX)^-1X^T$ from (3.6), and we know $\hat{y} = Hy$ by definition, we want to prove $\hat{y} = Xw^* + H\epsilon$

$$\begin{aligned}\hat{y} &= H(w^*X + \epsilon)\\ &= X(X^TX)^{-1}X^T(w^*X + \epsilon)\\ &= X(X^TX)^{-1}X^Tw^*X + X(X^TX)^{-1}X^T\epsilon\\ &= w^*X + H\epsilon\end{aligned}$$

   (b) for $\hat{y} - y$, we have

$$\begin{aligned}\hat{y} - y &= w^*X + H\epsilon - (w^* + \epsilon)\\ &= H\epsilon - \epsilon\\ &= \epsilon(H - I)\end{aligned}$$

   where I denotes the identity matrix

   (c) let $E_{in}(w) = \frac{1}{N}||\hat{y} - y||^2$

$$\begin{aligned}E_{in}(w) &= \frac{1}{N}||\epsilon(H - I)||^2\\ &= \frac{1}{N}(\epsilon(H - I))^T(\epsilon(H - I))\\ &= \frac{1}{N}\epsilon^T(H - I)^T\epsilon(H - I)\end{aligned}$$

   We know $H - I$ is symmetric, so $(H - I)^T = (H - I)$

$$\begin{aligned}E_{in}(w) &= \frac{1}{N}\epsilon^T\epsilon(H - I)^2\\ &= \frac{1}{N}\epsilon^T\epsilon(I - H)^2\end{aligned}$$

   (d) We know

$$\begin{aligned}E_D[E_{in}(w_{lin})] &= E_D[\frac{1}{N}(\epsilon^T\epsilon(I - H))]\\ &= \frac{1}{N}(E_D[\epsilon^T\epsilon] - E_D[\epsilon^T\epsilon H])\end{aligned}$$

   Given that ϵ is a noise term with zero mean and σ^2 variance. The variance of each noise component ϵ is σ^2, so

$$
\begin{aligned}
E_D[E_{in}(w_{lin})] &= \frac{1}{N}(N\sigma^2 - E_D[\epsilon^T\epsilon H]) \\
&= \sigma^2 - \frac{1}{N}E_D[\epsilon^T\epsilon H]
\end{aligned}
$$

Now we can calculate

$$
\begin{aligned}
E_D[\epsilon^T\epsilon H] &= E_D[\sum_{i=1}^{N}\epsilon_i^2 H] \\
&= H\sum_{i=1}^{N}E_D[\epsilon_i^2]
\end{aligned}
$$

By the problem, we know that each component of ϵ is a random variable with zero mean and variance σ^2, so this means that $E_D[\epsilon_i] = 0$ and $E_D[\epsilon_i^2] = \sigma^2$ for all i.

$$
\begin{aligned}
E_D[\epsilon^T\epsilon H] &= H\sum_{i=1}^{N}\sigma^2 \\
&= HN\sigma^2
\end{aligned}
$$

We continue the problem by substituting the result into our original equation

$$
\begin{aligned}
E_D[E_{in}(w_{lin})] = \sigma^2 - \frac{1}{N}E_D[\epsilon^T\epsilon H] &= \sigma^2 - \frac{1}{N}HN\sigma^2 \\
&= \sigma^2 - H\sigma^2
\end{aligned}
$$

Then, we can calculate for the $trace(H)$

$$
\begin{aligned}
trace(H) &= trace(X(X^TX)^{-1}X^T) \\
&= trace((X^TX)^{-1}(X^TX))
\end{aligned}
$$

Given that X^TX is a square matrix of size $(d+1)$, and it's inverse $(X^TX)^{-1}$ is also present, then we have

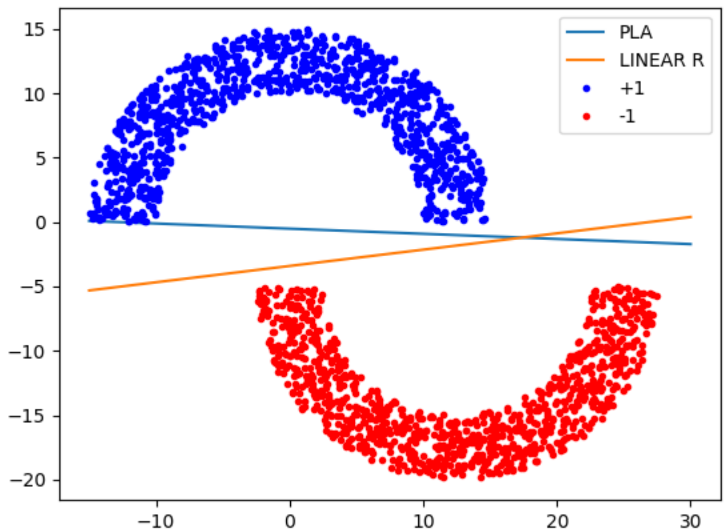
$$
\begin{aligned}
trace((X^TX)^{-1}(X^TX)) &= d+1 \\
trace(H) &= \frac{d+1}{N}
\end{aligned}
$$

Then, we have proved that $E_D[E_{in}(w_{lin})] = \sigma^2(1 - \frac{d+1}{N})$ ■

(e) to do

$$
E_{D,\epsilon'}[E_{test}(w_{lin})] = E_{D,\epsilon'}[\frac{1}{N}||Xw - y'||^2]
$$

2. Problem 3.1



(a)

(b) Both PLA and linear regression found ways to separate this data, however, one could say that the linear regression algorithm found a better way to separate the data as the PLA appears to be closer to the top part of the semicircle, barely missing on misclassifying one of the +1 points. With this, one can predict that linear regression will have a lower E_{out} than the PLA, however, this isn't guaranteed.