$$\varepsilon_{\mathcal{M}}(f) = \Pr_{\mathcal{M}}(f(X) \neq Y).$$

$$= \int_X p(x) \cdot \Pr_{\mathcal{M}}(f(x) \neq Y \mid X = x)\, dx.$$

$$= \int_X p(x) \cdot \Big(\mathbb{I}(f(x) = 1) \cdot \Pr(Y = 0 \mid X = x) + \mathbb{I}(f(x) = 0) \cdot \Pr(Y = 1 \mid X = x) \Big)\, dx.$$

$$\geq \int_X \cdot p(x) \cdot \min\{\Pr(Y = 0 \mid X = x), \Pr(Y = 1 \mid X = x)\}\, dx.$$

$$= \mathbb{E}_X\left[\min\{\Pr(Y=0 \mid X=x), \Pr(Y=1 \mid X)\}\right].$$

$$\eta(x) := \Pr(Y=1 \mid X).$$

$$= \mathbb{E}_X\left[\min\{1 - \eta(x), \eta(x)\}\right].$$

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$\forall a, b \in \mathbb{R}.\quad \min\{a, b\} = \frac{|a+b| - |a-b|}{2}.$

$$= \mathbb{E}_x \left[\frac{|1 - \cancel{\eta(x)} + \cancel{\eta(x)}| - |1 - \eta(x) - \eta(x)|}{2} \right].$$

$$= \frac{1}{2} - \frac{1}{2} \mathbb{E}_x \left[|2\eta(x) - 1| \right].$$

$$\forall f.\quad \mathcal{E}_M(f) \geq \underline{\frac{1}{2} - \frac{1}{2} \mathbb{E}\left[|2\eta(x) - 1| \right]}.$$

Recall $\eta(x) = \Pr(Y=1 \mid X) \in [0, 1]$.

$$\mathcal{E}^*_M := \frac{1}{2} - \frac{1}{2} \mathbb{E}\left[|2\eta(x) - 1| \right] \in \left[0, \frac{1}{2}\right].$$

Bayes Error

$$f^*(x) = \begin{cases} 1, & \eta(x) \geq \frac{1}{2}. \\ 0, & \text{o.w.} \end{cases}$$

Bayes optimal classifier.

$\forall f : X \mapsto \mathbb{R}.$

$$\mathcal{E}_\mu(f) := \mathbb{E}_\mu\left[(f(X) - Y)^2\right].$$

$$= \mathbb{E}_X \mathbb{E}_{Y|X}\left[(f(X) - Y)^2 \,\middle|\, X\right].$$

$$\mathbb{E}_\mu[g(X, Y)] = \int_{X \times Y} g(x, y)\, p(x, y)\, dx\, dy.$$

$$= \int_X \left(\int_Y g(x, y)\, p(y|x)\, dy\right) p(x)\, dx.$$

$$= \mathbb{E}_X \mathbb{E}_Y\left[\left(\underbrace{f(x) - \mathbb{E}[Y|X]}_{} + \underbrace{\mathbb{E}[Y|X] - Y}_{}\right)^2 \,\middle|\, X\right]$$

$$= \mathbb{E}_X \mathbb{E}_Y\Big[(f(x) - \mathbb{E}[Y|X])^2 + (\mathbb{E}[Y|X] - Y)^2$$

$$+ 2(f(X) - \mathbb{E}[Y|X])\underbrace{(\mathbb{E}[Y|X] - Y)}_{}\,\Big|\, X\Big]$$

$$\cancel{\mathbb{E}[c \cdot Y] = c \cdot \mathbb{E}[Y]}.$$

$$= \underline{\mathbb{E}_X \mathbb{E}_Y\left[(f(x) - \mathbb{E}[Y|X])^2 \,\middle|\, X \right]}$$

$$+ \underline{\mathbb{E}_X \mathbb{E}_Y\left[(Y - \mathbb{E}[Y|X])^2 \,\middle|\, X \right]}.$$

$$+ \cancel{2 \cdot \mathbb{E}_X \left[(f(x) - \mathbb{E}[Y|X]) (\mathbb{E}[Y|X] - \mathbb{E}[Y|X]) \,\middle|\, X \right]}.$$

$$\geq \underbrace{\mathbb{E}_X \mathbb{E}_Y\left[(Y - \mathbb{E}[Y|X])^2 \,\middle|\, X \right]}.$$

$$= \mathbb{E}_X \operatorname{Var}[Y|X].$$

Bayes error.

Choose $f^*_{Bayes}(x) = \mathbb{E}[Y|X]$.