Econ7115: Structural Models and Numerical Methods in Economics Final Project

April 15, 2025

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1 CES Utility ($\epsilon = 0$)

1.1 Derive D_n

Under CES, we have:

- $H'(x) = \frac{\sigma-1}{\sigma} x^{\frac{-1}{\sigma}}$, where $x = \frac{q_n(\omega)}{Q_n}$
- $\int_{\omega\in\Omega_n} H(\frac{q_n(\omega)}{Q_n})d\omega$.
- $D_n = \left[\int_{\omega\in\Omega_n} H'(\frac{q_n(\omega)}{Q_n})\frac{q_n(\omega)}{Q_n}d\omega\right]^{-1}$

Hence,

$$
\begin{aligned}
D_n &= \left[\int_{\omega\in\Omega_n} H'(\frac{q_n(\omega)}{Q_n})\frac{q_n(\omega)}{Q_n}d\omega\right]^{-1} \\
&= \left[\int_{\omega\in\Omega_n} \frac{\sigma-1}{\sigma}(\frac{q_n(\omega)}{Q_n})^{-\frac{1}{\sigma}}\frac{q_n(\omega)}{Q_n}d\omega\right]^{-1} \\
&= \left[\int_{\omega\in\Omega_n} \frac{\sigma-1}{\sigma}(\frac{q_n(\omega)}{Q_n})^{\frac{\sigma-1}{\sigma}}d\omega\right]^{-1} \\
&= \left[\frac{\sigma-1}{\sigma}\int_{\omega\in\Omega_n} H(\frac{q_n(\omega)}{Q_n})d\omega\right]^{-1} \\
&= \frac{\sigma}{\sigma-1}
\end{aligned}
$$

1.2 The Aggregate Fixed Mkt Cost and Trade Value

In this section, we need to show:

$$\frac{w_n F_n (c_n^*)^\theta M_k T_{kn}^{\bar{\theta}}}{X_{kn}} = const.$$

1.2.1 Derive $s_{in}(\omega) = s_n(c_{in}(\omega))$

First, we let $s_{in}(\omega) = \frac{q_n(\omega)}{Q_n}$. Hence, we have:

$$p_{in}(\omega) = H'(s_{in}(\omega))D_n P_n = s_{in}(\omega)^{\frac{-1}{\sigma}} * P_n \tag{1}$$

Noted under $\epsilon = 0$, the markup $\mu = \frac{\sigma}{\sigma - 1}$, hence,

$$p_{in}(\omega) = \mu * c_{in}(\omega) = \frac{\sigma}{\sigma - 1} c_{in}(\omega) \tag{2}$$

Combine equations (1) and (2), we have:

$$s_{in}(\omega) = (\frac{c_{in}(\omega)}{P_n})^{-\sigma} (\frac{\sigma}{\sigma - 1})^{-\sigma} = s_n(c_{in}(\omega)) \tag{3}$$

1.2.2 Derive $P_n^{1-\sigma}$

Also, we want to derive the expression of $P_n = \sum_i \nu_{in}^P M_i T_{in}^{\bar{\theta}}$.

$$\begin{aligned}
\nu_{in}^P &= \theta \int_0^{c_n^*} p_{in}(c) s_{in}(c) c^{\theta - 1} dc \\
&= \theta \int_0^{c_n^*} \frac{\sigma}{\sigma - 1} \times c \times (\frac{c}{P_n})^{-\sigma} (\frac{\sigma}{\sigma - 1})^{-\sigma} c^{\theta - 1} dc \\
&= \theta (\frac{\sigma}{\sigma - 1})^{1-\sigma} P_n^\sigma \frac{1}{\theta - \sigma + 1} (c_n^*)^{\theta - \sigma + 1}
\end{aligned}$$

We got:

$$P_n^{1-\sigma} = \frac{\theta}{\theta - \sigma + 1} (\frac{\sigma}{\sigma - 1})^{1-\sigma} (c^*)^{\theta - \sigma + 1} \sum_i M_i T_{in}^{\bar{\theta}} \tag{4}$$

1.2.3 Combine the Numerator and Denominator

Define the cutoff profit $\pi_n(c_n^*) = \tilde{\pi_n} = w_n F_n = s_n(c^*)^{\frac{\sigma-1}{\sigma}} \frac{X_n}{\sigma}$ and plug in equation (4):

$$\begin{aligned}
w_n F_n (c_n^*)^\theta &= s_n(c_n^*)^{\frac{\sigma-1}{\sigma}} \frac{X_n}{\sigma} (c_n^*)^\theta \\
&= (\frac{c_n^*}{P_n})^{1-\sigma} (\frac{\sigma}{\sigma - 1})^{1-\sigma} \frac{X_n}{\sigma} (c_n^*)^\theta \\
&= \frac{X_n}{\sigma} \frac{\theta - \sigma + 1}{\theta} \frac{1}{\sum_k M_k T_{kn}^{\bar{\theta}}}
\end{aligned}$$

The numerator is:

$$w_n F_n (c_n^*)^\theta M_k \bar{T_{kn}^\theta} = \frac{X_n}{\sigma}\frac{\theta-\sigma+1}{\theta}\frac{M_k \bar{T_{kn}^\theta}}{\sum_k M_k \bar{T_{kn}^\theta}} = \frac{X_n}{\sigma}\frac{\theta-\sigma+1}{\theta}\lambda_{kn}$$

The denominator is:

$$X_{kn} = \lambda_{kn} * X_n$$

Finally,

$$\frac{w_n F_n (c_n^*)^\theta M_k \bar{T_{kn}^\theta}}{X_{kn}} = \frac{\theta-\sigma+1}{\theta\sigma} \tag{5}$$

1.3 Equilibrium Conditions under CES

- Labor Market clearing

$$w_i L_i = (1-\frac{1}{\sigma})\sum_{n=1}^{N} X_{in} + \frac{\theta-\sigma+1}{\theta\sigma} X_i + \frac{\sigma-1}{\theta\sigma}\sum_{n=1}^{N} X_{in} \tag{6}$$

- The firm mass:

$$M_i w_i f^e = \frac{\sigma-1}{\theta\sigma}\sum_{n=1}^{N} X_{in} \tag{7}$$

- Total expenditure:

$$X_i = w_i L_i \tag{8}$$

- Cutoff profit $\tilde{\pi_n} = w_n F_n$:

$$w_n F_n = \frac{1}{\sigma}(\frac{\sigma}{\sigma-1} c_n^*)^{1-\sigma} P_n^{\sigma-1} X_n \tag{9}$$

- The cutoff cost c_n^*:

$$c_n^* = \frac{\sigma-1}{\sigma}\left[\frac{\sigma w_n F_n}{P_n^{\sigma-1} X_n}\right]^{\frac{1}{1-\sigma}} \tag{10}$$

- Price index P_n:

$$P_n^{1-\sigma} = \frac{\theta}{\theta-\sigma+1}(\frac{\sigma}{\sigma-1})^{1-\sigma}(c_n^*)^{\theta-\sigma+1}\sum_{i=1}^{N} M_i \bar{T_{in}^\theta} \tag{11}$$

1.4 Equilibrium Outcomes if $\epsilon = 0$ & $\tau_{in} = 2, i \neq n$

- $w = [0.5239; 0.4761]$
- $M = [2.5000; 1.6667]$
- $P = [0.6667; 0.7726]$
- $D = \left[\frac{4}{3}; \frac{4}{3}\right]$

2 Non-CES Utility ($\epsilon \neq 0$)

The equilibrium outcomes $(w_i, M_i, P_i, D_i, c_i^*)_{i=1}^2$:

- Let $J = 100$
- $w = [0.5607; 0.4393]$
- $M = [7.3432; 7.6564]$
- $P = [0.9249; 1.8066]$
- $D = [0.9450; 0.6033]$
- $c^* = [0.2744; 0.1557]$
- $welfare = [0.8410; 0.5685]$
- We can also easily verify the $c_n^* < min\left\{\frac{1}{\bar{T_{in}}}, \frac{\sigma-1}{\sigma} exp(\frac{1}{\epsilon}) D_n P_n\right\}$ (Please refer to the code)

By the Monte-Carlo integration, we have the aggregate markup $\bar{\mu_i}^D = \left[\frac{\theta}{\nu_n^P} \frac{c_n^*}{J} \sum_{j=1}^J s_n^j (c_n^j)^\theta\right]^{-1}$:

- $\bar{\mu_i}^D = [2.6457; 4.7403]$

3 If $\tau_{in} = 1.5, i \neq n$

3.1 CES Utility

- $w = [0.5225; 0.4775]$
- $M = [2.5000; 1.6667]$
- $P = [0.6463; 0.7410]$
- $D = \left[\frac{4}{3}; \frac{4}{3}\right]$

3.2 Non-CES Utility

The equilibrium outcomes $(w_i, M_i, P_i, D_i, c_i^*)_{i=1}^2$:

- $w = [0.5607; 0.4393]$
- $M = [7.6542; 7.5079]$
- $P = [0.9033; 1.7644]$
- $D = [0.9393; 0.5944]$
- $c^* = [0.2660; 0.1493]$
- $welfare = [0.8675; 0.5928]$
- We can also easily verify the $c_n^* < min\left\{\frac{1}{T_{in}}, \frac{\sigma-1}{\sigma}exp(\frac{1}{\epsilon})D_nP_n\right\}$ (Please refer to the code)

By the Monte-Carlo integration, we have the aggregate markup $\bar{\mu_i}^D = \left[\frac{\theta}{\nu_n^P}\frac{c_n^*}{J}\sum_{j=1}^J s_n^j(c_n^j)^\theta\right]^{-1}$:

- $\bar{\mu_i}^D = [2.6457; 4.7403]$

3.3 Discussion: Compare the Outcomes with Different τ_{in}

From the results, we could observe that, when the τ_{in} decreases:

- Wage (w):
  - CES: both the wages decrease
  - Non-CES: there are almost stable wages in both countries
  - Discussion: since the constant elasticity of substitution, when the iceberg trade cost decreases, resulting in the booming international trade between the two countries, the labor force could be easily replaced by other factors from imports. Hence, the wage of labor is expected to be reduced. In the non-CES case, on the contrary, the domestic labor may be more difficult to be substituted, the wage keeps almost the same level as previous.
- Firm mass (M)
  - CES: No significant change
  - Non-CES: firm mass increases in the northern country and slightly decreases in the southern one. The gap between the two countries is narrowed and country 1 (north) outweighs country 2 (south).

  - Discussion: Due to the CES utility, despite the decrease of the τ does make some changes of the market, the consumers could find other goods to maintain their current utility. That means, the firm mass should keep the same, although the commodities in the market may change. In the non-CES case, the northern country demonstrates a great increase in the firm mass. This may due to the higher initial technology and labor endowments. From the formulas, we could observe a higher cost upper bound if country 1 wants to export to country 2. However, that for country 2 to country 1 decreases indicating a more strait requirement of entry cost if country 2 wants to export to country 1. Hence, country 1 from north has a higher productivity and exports to country 2. But since the initial firm mass of country 2 is far higher than country 1, the gap between the two countries is not so large.

- Price Index (P)
  - Both CES and non-CES utility show decreased P_n in the two countries. Such decreases probably due to the more diverse market and commodities due to the booming of international trade by a lower τ.

- Demand Index (D)
  - Non-CES: decreases in both countries, indicating a lower demand. Consumers may diffuse their consumption choices in more different commodities. This could make the market and producers more fierce competitions.

- Cutoff cost (c^*)
  - Non-CES: decreases in both countries. Since the growing international trade after the lower τ, the cutoff cost to enter the international market must decrease. As a result, only the firms with higher productivity could survive and the market efficiency could be improved.

- Welfare
  - Non-CES: increases in both countries. The prosperous trade increases the welfare in both countries. When τ_{in} decreases, consumers benefit from lower prices (reflected in the decreased price indices), greater product variety, and efficiency gains from resource reallocation toward more productive firms.

- Aggregate Markup in Non-CES: constant
  - Discussion: in the Non-CES utility case, when the τ decreases, the market is more competitive. Although the price index decreases, there might be some firms with strong market power to affect and control the aggregate markup and offset the exogenous shock of a lower τ.