ECON 7115
Final Project

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Jiawen KE
HKUST, SOSC

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1 CES Utility ($\epsilon = 0$)

1.1 Derive D_n

Under CES, we have:

- $H'(x) = \frac{\sigma-1}{\sigma} x^{\frac{-1}{\sigma}}$, where $x = \frac{q_n(\omega)}{Q_n}$
- $\int_{\omega\in\Omega_n} H(\frac{q_n(\omega)}{Q_n}) d\omega$.
- $D_n = \left[\int_{\omega\in\Omega_n} H'(\frac{q_n(\omega)}{Q_n})\frac{q_n(\omega)}{Q_n} d\omega\right]^{-1}$

Hence,

$$\begin{aligned}
D_n &= \left[\int_{\omega\in\Omega_n} H'(\frac{q_n(\omega)}{Q_n})\frac{q_n(\omega)}{Q_n} d\omega\right]^{-1} \\
&= \left[\int_{\omega\in\Omega_n} \frac{\sigma-1}{\sigma}(\frac{q_n(\omega)}{Q_n})^{-\frac{1}{\sigma}}\frac{q_n(\omega)}{Q_n} d\omega\right]^{-1} \\
&= \left[\int_{\omega\in\Omega_n} \frac{\sigma-1}{\sigma}(\frac{q_n(\omega)}{Q_n})^{\frac{\sigma-1}{\sigma}} d\omega\right]^{-1} \\
&= \left[\frac{\sigma-1}{\sigma}\int_{\omega\in\Omega_n} H(\frac{q_n(\omega)}{Q_n}) d\omega\right]^{-1} \\
&= \frac{\sigma}{\sigma-1}
\end{aligned}$$

1.2 The Aggregate Fixed Mkt Cost and Trade Value

In this section, we need to show:

$$\frac{w_n F_n (c_n^*)^\theta M_k \bar{T_{kn}^{\theta}}}{X_{kn}} = const.$$

1.2.1 Derive $s_{in}(\omega) = s_n(c_{in}(\omega))$

First, we let $s_{in}(\omega) = \frac{q_n(\omega)}{Q_n}$. Hence, we have:

$$p_{in}(\omega) = H'(s_{in}(\omega))D_nP_n = s_{in}(\omega)^{\frac{-1}{\sigma}} * P_n \tag{1}$$

Noted under $\epsilon = 0$, the markup $\mu = \frac{\sigma}{\sigma-1}$, hence,

$$p_{in}(\omega) = \mu * c_{in}(\omega) = \frac{\sigma}{\sigma-1}c_{in}(\omega) \tag{2}$$

Combine equations (1) and (2), we have:

$$s_{in}(\omega) = (\frac{c_{in}(\omega)}{P_n})^{-\sigma}(\frac{\sigma}{\sigma-1})^{-\sigma} = s_n(c_{in}(\omega)) \tag{3}$$

1.2.2 Derive $P_n^{1-\sigma}$

Also, we want to derive the expression of $P_n = \sum_i \nu_{in}^P M_i \bar{T_{in}^{\theta}}$.

$$\begin{aligned}
\nu_{in}^P &= \theta \int_0^{c_n^*} p_{in}(c)s_{in}(c)c^{\theta-1}dc \\
&= \theta \int_0^{c_n^*} \frac{\sigma}{\sigma-1} \times c \times (\frac{c}{P_n})^{-\sigma}(\frac{\sigma}{\sigma-1})^{-\sigma}c^{\theta-1}dc \\
&= \theta(\frac{\sigma}{\sigma-1})^{1-\sigma}P_n^{\sigma}\frac{1}{\theta-\sigma+1}(c_n^*)^{\theta-\sigma+1}
\end{aligned}$$

We got:

$$P_n^{1-\sigma} = \frac{\theta}{\theta-\sigma+1}(\frac{\sigma}{\sigma-1})^{1-\sigma}(c^*)^{\theta-\sigma+1}\sum_i M_i\bar{T_{in}^{\theta}} \tag{4}$$

1.2.3 Combine the Numerator and Denominator

Define the cutoff profit $\pi_n(c_n^*) = \tilde{\pi_n} = w_nF_n = s_n(c^*)^{\frac{\sigma-1}{\sigma}}\frac{X_n}{\sigma}$ and plug in equation (4):

$$\begin{aligned}
w_nF_n(c_n^*)^{\theta} &= s_n(c_n^*)^{\frac{\sigma-1}{\sigma}}\frac{X_n}{\sigma}(c_n^*)^{\theta} \\
&= (\frac{c_n^*}{P_n})^{1-\sigma}(\frac{\sigma}{\sigma-1})^{1-\sigma}\frac{X_n}{\sigma}(c_n^*)^{\theta} \\
&= \frac{X_n}{\sigma}\frac{\theta-\sigma+1}{\theta}\frac{1}{\sum_k M_k\bar{T_{kn}^{\theta}}}
\end{aligned}$$

The numerator is:

$$w_nF_n(c_n^*)^{\theta}M_k\bar{T_{kn}^{\theta}} = \frac{X_n}{\sigma}\frac{\theta-\sigma+1}{\theta}\frac{M_k\bar{T_{kn}^{\theta}}}{\sum_k M_k\bar{T_{kn}^{\theta}}} = \frac{X_n}{\sigma}\frac{\theta-\sigma+1}{\theta}\lambda_{kn}$$

The denominator is:

$$X_{kn} = \lambda_{kn} * X_n$$

Finally,

$$\frac{w_n F_n (c_n^*)^\theta M_k \bar{T_{kn}^\theta}}{X_{kn}} = \frac{\theta - \sigma + 1}{\theta\sigma} \tag{5}$$

1.3 Equilibrium Conditions under CES

- Labor Market clearing

$$w_i L_i = (1 - \frac{1}{\sigma}) \sum_{n=1}^{N} X_{in} + \frac{\theta - \sigma + 1}{\theta\sigma} X_i + \frac{\sigma - 1}{\theta\sigma} \sum_{n=1}^{N} X_{in} \tag{6}$$

- The firm mass:

$$M_i w_i f^e = \frac{\sigma - 1}{\theta\sigma} \sum_{n=1}^{N} X_{in} \tag{7}$$

- Total expenditure:

$$X_i = w_i L_i \tag{8}$$

- Cutoff profit $\tilde{\pi_n} = w_n F_n$:

$$w_n F_n = \frac{1}{\sigma} (\frac{\sigma}{\sigma - 1} c_n^*)^{1-\sigma} P_n^{\sigma - 1} X_n \tag{9}$$

- The cutoff cost c_n^*:

$$c_n^* = \frac{\sigma - 1}{\sigma} \left[\frac{\sigma w_n F_n}{P_n^{\sigma-1} X_n} \right]^{\frac{1}{1-\sigma}} \tag{10}$$

- Price index P_n:

$$P_n^{1-\sigma} = \frac{\theta}{\theta - \sigma + 1} (\frac{\sigma}{\sigma - 1})^{1-\sigma} (c^*)^{\theta - \sigma + 1} \sum_{i=1}^{N} M_i \bar{T_{in}^\theta} \tag{11}$$

1.4 Equilibrium Outcomes if $\epsilon = 0$ & $\tau_{in} = 2, i \neq n$

- $w = [0.5801; 0.4199]$
- $M = [2.5001; 1.6666]$
- $P = [0.4688; 0.3066]$
- $D = \left[\frac{4}{3}; \frac{4}{3}\right]$

2 General Utility ($\epsilon \neq 0$)

The equilibrium outcomes $(w_i, M_i, P_i, D_i, c_i^*)$:

- $w = [0.5801; 0.4199]$
- $M = [2.3926; 2.5416]$
- $P = [0.4326; 0.4376]$
- $D = [1.4314; 0.9976]$
- $c_i^* = [0.4559; 0.2830]$
- $welfare = [0.063; 0.0424]$
- We can also easily verify the $c_n^* < min\left\{\frac{1}{\bar{T_{in}}}, \frac{\sigma-1}{\sigma}exp(\frac{1}{\epsilon})D_nP_n\right\}$ (Please refer to the code)

By the Monte-Carlo integration, we have the aggregate markup $\bar{\mu_i}^D = \left[\frac{\theta}{\nu_n^P}\sum_{j=1}^J s_n^j(c_n^j)^\theta\right]^{-1}$:

- $\bar{\mu_i}^D = [0.0006; 0.0039]$

3 If $\tau_{in} = 1.5, i \neq n$

3.1 CES Utility

3.2 Non-CES Utility

3.3 Discussion