

Adaptivity analysis

Expr. $e ::= x \mid e_1 \ e_2 \mid \lambda x.e$
 $\mathbf{true} \mid \mathbf{false} \mid \mathbf{if} \ e \ \mathbf{then} \ e_2 \ \mathbf{else} \ e_3 \mid c \mid \delta(e)$
 Value $v ::= c \mid \lambda x.e$
 Environment $\theta ::= x_1 \mapsto (v_1, R_1), \dots, x_n \mapsto (v_n, R_n)$

[MG: I renamed var2 into var.]

Index Term $k, Z ::= i \mid n$
 Linear type $\tau ::= A \multimap^Z \tau \mid \mathbf{b} \mid \mathbf{bool}$
 Nonlinear Type $A ::= !_k \tau$
 Typing context $\Gamma ::= x_1 : A_1, \dots, x_n : A_n$

$$\begin{array}{c}
\frac{}{v, \theta \Downarrow^0 v, \theta} \text{ val} \qquad \frac{\theta(x) = (v, \theta_1, R)}{x, \theta \Downarrow^R v, \theta_1} \text{ var} \qquad \frac{}{c, \theta \Downarrow^0 c, \theta} \text{ const} \\
\\
\frac{}{\lambda x. e, \theta \Downarrow^0 \lambda x. e, \theta} \text{ lambda} \\
\\
\frac{e_2, \theta_2 \Downarrow^{R_2} v_2, \theta'_2 \quad \text{fresh } x' \quad \frac{e_1, \theta_1 \Downarrow^{R_1} \lambda x. e, \theta'_1}{e[x'/x], \theta'_1[x' \rightarrow (v_2, \theta'_2, R_2)] \Downarrow^{R_3} v, \theta_3}}{e_1 \ e_2, (\theta_1 \uplus \theta_2) \Downarrow^{R_1+R_3} v, \theta_3} \text{ app} \\
\\
\frac{e, \theta \Downarrow^R v, \theta_1 \quad \delta(v\theta) = v' \quad FV(v') = \emptyset}{\delta(e), \theta \Downarrow^{R+1} v, \theta_1} \text{ delta} \\
\\
\frac{e, \theta \Downarrow^R \text{false}, \theta' \quad e_2, \theta \Downarrow^{R_2} v_2, \theta_2}{\text{if } e \text{ then } e_1 \text{ else } e_2, \theta \Downarrow^{R+R_2} v_2, \theta_2} \text{ if-f} \\
\\
\frac{e, \theta \Downarrow^R \text{true}, \theta' \quad e_1, \theta \Downarrow^{R_1} v_1, \theta_1}{\text{if } e \text{ then } e_1 \text{ else } e_2, \theta \Downarrow^{R+R_1} v_1, \theta_1} \text{ if-t} \\
\\
\begin{array}{ccc}
\theta_1 \uplus \emptyset & \triangleq & \theta_1 \\
\emptyset \uplus \theta_2 & \triangleq & \theta_2
\end{array}
\end{array}$$

Figure 1: Big-step semantics

$$\begin{array}{c}
\frac{}{v, \theta \Downarrow^0 v, \theta} \text{val} \qquad \frac{\theta(x) = (v, \theta_1, R)}{x, \theta \Downarrow^R v, \theta_1} \text{var} \qquad \frac{}{c, \theta \Downarrow^0 c, \theta} \text{const} \\
\\
\frac{}{\lambda x. e, \theta \Downarrow^0 \lambda x. e, \theta} \text{lambda} \\
\\
\frac{e_2, \theta_2 \Downarrow^{R_2} v_2, \theta'_2 \quad \text{fresh } x' \quad \frac{e_1, \theta_1 \Downarrow^{R_1} \lambda x. e, \theta'_1 \quad e[x'/x], \theta'_1[x' \rightarrow (v_2, \theta'_2, R_2)] \Downarrow^{R_3} v, \theta_3}{e_1 \ e_2, (\theta_1 \uplus \theta_2) \Downarrow^{[\max(R_1, R_3)]} v, \theta_3}}{\text{app}} \\
\\
\left[\frac{e, \theta \Downarrow^R v, \theta_1}{\delta(e), \theta \Downarrow^{R+1} \delta(v), \theta_1} \text{delta} \right] \quad \frac{e, \theta \Downarrow^R \text{false}, \theta' \quad e_2, \theta \Downarrow^{R_2} v_2, \theta_2}{\text{if } e \text{ then } e_1 \text{ else } e_2, \theta \Downarrow^{R+R_2} v_2, \theta_2} \text{if-f} \\
\\
\frac{e, \theta \Downarrow^R \text{true}, \theta' \quad e_1, \theta \Downarrow^{R_1} v_1, \theta_1}{\text{if } e \text{ then } e_1 \text{ else } e_2, \theta \Downarrow^{R+R_1} v_1, \theta_1} \text{if-t} \\
\\
\left[\frac{\text{fresh } x', y' \quad e, \theta \Downarrow^R (e_1, e_2), \theta_1 \quad e_1, \theta_1 \Downarrow^{R_1} v_1, \theta'_1 \quad e_2, \theta_1 \Downarrow^{R_2} v_2, \theta'_2 \quad e'[x'/x][y'/y], \theta_1[x' \rightarrow (v_1, \theta'_1, R_1), y' \rightarrow (v_2, \theta'_2, R_2)] \Downarrow^{R'} v, \theta'}{\text{let } (x, y) = e \text{ in } e', \theta \Downarrow^{\max(R, R')} v, \theta'} \text{bind} \right] \\
\\
\theta_1 \uplus \emptyset \triangleq \theta_1 \\
\emptyset \uplus \theta_2 \triangleq \theta_2
\end{array}$$

Figure 2: Big-step semantics - Jan.28

$$\begin{array}{c}
\frac{}{\Gamma, x : !_1 \tau \vdash_0 x : \tau} \mathbf{Ax} \quad \frac{}{\Gamma \vdash_0 c : \mathbf{b}} \mathbf{const} \quad \frac{\Gamma, x : A \vdash_Z e : \tau}{\Gamma \vdash_0 \lambda x. e : A \multimap^Z \tau} \mathbf{lambda} \\
\\
\frac{\Gamma_1 \vdash_{Z_1} e_1 : !_k \tau_1 \multimap^Z \tau_2 \quad \Gamma_2 \vdash_{Z_2} e_2 : \tau_1}{\Gamma_1 + k \times \Gamma_2 \vdash_{Z_1 + k \times Z_2 + Z} e_1 e_2 : \tau_2} \mathbf{app} \quad \frac{\Gamma \vdash_Z e : \mathbf{b}}{\Gamma \vdash_{1+Z} \delta(e) : \mathbf{b}} \mathbf{delta} \\
\\
\frac{\Gamma' \vdash_{Z'} e : \tau' \quad \Gamma' \leq \Gamma \quad Z' \leq Z \quad \tau' <: \tau}{\Gamma \vdash_Z e : \tau} \mathbf{subtype} \\
\\
\frac{\Gamma, y : \tau', x : \tau, \Gamma' \vdash_Z e : \tau}{\Gamma, x : \tau, y : \tau', \Gamma' \vdash_Z e : \tau} \mathbf{exchange} \\
\\
\frac{\Gamma \vdash_Z e : \mathbf{bool} \quad \Gamma_1 \vdash_{Z_1} e_1 : \tau \quad \Gamma_2 \vdash_{Z_2} e_2 : \tau \quad \Gamma' = \Gamma + \Gamma_1 + \Gamma_2 \quad Z' = Z + \max(Z_1, Z_2)}{\Gamma' \vdash_{Z'} \mathbf{if } e \mathbf{ then } e_1 \mathbf{ else } e_2 : \tau} \mathbf{if} \\
\\
\begin{array}{lll}
k \times \Gamma & \triangleq & \Gamma \quad k = 1 \\
& \triangleq & !_0 \Gamma \quad k = 0 \\
!_{k_1} \tau + !_{k_2} \tau & \triangleq & !_{\max(k_1, k_2)} \tau \\
\Gamma + \emptyset & \triangleq & \Gamma \\
\emptyset + \Gamma & \triangleq & \Gamma \\
([x : A], \Gamma) + ([x : A'], \Delta) & \triangleq & [x : A + A'], \Gamma + \Delta \\
\Gamma <: \Delta & \triangleq & \mathbf{dom}(\Gamma) = \mathbf{dom}(\Delta) \\
& & \wedge \forall x \in \mathbf{dom}(\Gamma), \Delta(x) <: \Gamma(x)
\end{array}
\end{array}$$

Figure 3: Typing rules, first version

$$\begin{array}{c}
\frac{k_1 \leq k \quad A <: A_1}{!_k A <: !_k A_1} \mathbf{bang} \quad \frac{Z \leq Z' \quad \tau_1 <: \tau \quad \tau' <: \tau'_1}{\tau \multimap^Z \tau' <: \tau_1 \multimap^{Z'} A'_1} \mathbf{arrow} \\
\\
\frac{}{\mathbf{b} <: \mathbf{b}} \mathbf{base}
\end{array}$$

Figure 4: subtyping

$$\begin{array}{c}
\frac{\theta(x) = (v, \theta', R) \quad \vdash_Z (v, \theta') : \tau}{\vdash_{R+Z} (x, \theta) : \tau} \text{C-Ax} \qquad \frac{}{\vdash_0 (c, \theta) : \mathbf{b}} \text{C-const} \\
\\
\frac{\text{fresh } x' \quad \forall R' \quad \vdash_{Z'} (v', \theta') : \tau_1 \quad \vdash_{S+k \times (R'+Z')+Z} (e[x'/x], \theta[x' \rightarrow (v', \theta', R')]) : \tau_2}{\vdash_S (\lambda x. e, \theta) : !_k \tau_1 \multimap^Z \tau_2} \text{C-lambda} \\
\\
\frac{\vdash_{Z_1} (e_1, \theta_1) : !_k \tau_1 \multimap^Z \tau_2 \quad \vdash_{Z_2} (e_2, \theta_2) : \tau_1}{\vdash_{Z_1+k \times Z_2+Z} (e_1 \ e_2, \theta_1 \uplus \theta_2) : \tau_2} \text{C-app} \\
\\
\frac{\vdash_Z (e, \theta) : \mathbf{b}}{\vdash_{1+Z} (\delta(e), \theta) : \mathbf{b}} \text{C-delta} \\
\\
\theta \quad \triangleq (x_i \rightarrow (v_i, \theta_i, R_i)) \quad i \in \mathbb{N} \\
(x_i : !_k \tau_i), \Gamma \models (x_i \rightarrow (v_i, \theta_i, R_i)) \uplus \theta \quad \triangleq \quad \vdash_- (v_i, \theta_i) : \tau_i \quad \wedge \Gamma \models \theta
\end{array}$$

Figure 5: Typing rules, configure

$$\begin{array}{ll}
\llbracket \mathbf{b} \rrbracket_V & = \{ (c, \theta, Z) \} \\
\llbracket !_k \tau \rrbracket_V & = \{ (v, \theta, Z) \mid (v, \theta, Z) \in \llbracket \tau \rrbracket_V \} \\
\llbracket !_k \tau_1 \multimap^Z \tau_2 \rrbracket_V & = \{ (\lambda x. e, \theta, Z_1) \mid \forall v', \theta', Z'. (v', \theta', Z') \in \llbracket !_k \tau_1 \rrbracket_V. \\
& \implies \text{fresh } x' \wedge \\
& \forall R. (e[x'/x], \theta[x' \mapsto (v', \theta', R)]) \in \llbracket \tau_2 \rrbracket_E^{Z_1+Z+k \times (R+Z')} \} \\
\llbracket \tau \rrbracket_E^Z & = \{ (e, \theta) \mid (e, \theta \Downarrow^R v, \theta') \\
& \implies R \leq Z \wedge (v, \theta', Z - R) \in \llbracket \tau \rrbracket_V \}
\end{array}$$

Figure 6: Logical relation without step-indexing

- Theorem 1** (Monotonicity). 1. If $(e, \theta) \in \llbracket \tau \rrbracket_{\mathbf{E}}^Z$ and $Z' \geq Z$, then $(e, \theta) \in \llbracket \tau \rrbracket_{\mathbf{E}}^{Z'}$.
2. If $(v, \theta, Z) \in \llbracket \tau \rrbracket_{\mathbf{V}}$ and $Z' \geq Z$, then $(v, \theta, Z') \in \llbracket \tau \rrbracket_{\mathbf{V}}$.
3. If $\vdash_Z (e, \theta) : \tau$ and $Z \leq Z'$, then $\vdash_{Z'} (e, \theta) : \tau$.
4. If $\Gamma \vdash_Z e : \tau$ and $Z \leq Z'$, then $\Gamma \vdash_{Z'} e : \tau$.

$$F_{c2t}(e, x_i) = \begin{cases} 1 & x_i \in \mathbf{FV}(e) \\ 0 & x_i \notin \mathbf{FV}(e) \end{cases}$$

Theorem 2 (ConfigurationToTyping). 1. If $\vdash_Z (e, \theta) : \tau$ and $\forall x_i \in \text{dom}(\theta). \theta(x_i) = (v_i, \theta_i, R_i) \wedge \exists Z_i. \vdash_{Z_i} (v_i, \theta_i) : \tau_i$, then $x_i :!_{F_{c2t}(e, x_i) \tau_i} \vdash_{Z - F_{c2t}(e, x_i) \times (R_i + Z_i)} e : \tau$.

$$\begin{aligned}
F(e, \phi) & \quad \text{where } \phi(x_i) = (k_i, R_i, Z_i) \\
F(x, \phi) & = \sum_{x_i \in \text{FV}(x)} k_i \times (R_i + Z_i) \\
F(\lambda x.e, \phi) & = \sum_{x_i \in \text{FV}(\lambda x.e)} k_i \times (R_i + Z_i) \\
F(\delta(e), \phi) & = \sum_{x_i \in \text{FV}(\delta(e))} k_i \times (R_i + Z_i) \\
F(c, \phi) & = 0 \\
F(e_1 e_2, \phi) & = F(e_1, \phi) + F(e_2, \phi)
\end{aligned}$$

Theorem 3 (Typing to Configuration). 1. If $x_1 :!_{k_1} \tau_1, \dots, x_i :!_{k_i} \tau_i \vdash_Z e : \tau$, and $\vdash_{Z_I} (v_i, \theta_i) : \tau_i$, and $\theta = [x_1 \rightarrow (v_1, \theta_1, R_1), \dots, x_i \rightarrow (v_i, \theta_i, R_i)]$, $\phi = [x_1 \rightarrow (k_1, R_1, Z_1), \dots, x_i \rightarrow (k_i, R_i, Z_i)]$, then $\vdash_{Z+F(e, \phi)} (e, \theta) : \tau$.

Proof. By induction on the typing derivation.

Case $\frac{}{\Gamma, x :!_1 \tau \vdash_0 x : \tau} \mathbf{Ax}$

assume $\vdash_Z (v, \theta') : \tau$ and $\vdash_{Z_i} (v_i, \theta_i) : \tau_i$,
assume $\theta = [x \rightarrow (v, \theta', R)] \uplus [x_1 \rightarrow (v_1, \theta_1, R_1), \dots, x_i \rightarrow (v_i, \theta_i, R_i)]$.
So $\phi = [x \rightarrow (1, R, Z)] \uplus [x_1 \rightarrow (k_1, R_1, Z_1), \dots, x_i \rightarrow (k_i, R_i, Z_i)]$
 $F(x, \phi) = 1 \times (R + Z)$.
 $\text{TS} : \vdash_{0+1 \times (R+Z)} (x, \theta) : \tau$.
We conclude from the configuration rule C-Ax.

$$\frac{\theta(x) = (v, \theta', R) \quad \vdash_Z (v, \theta') : \tau}{\vdash_{R+Z} (x, \theta) : \tau} \mathbf{C-Ax}$$

Case $\frac{\Gamma, x : A \vdash_Z e : \tau}{\Gamma \vdash_0 \lambda x.e : A \multimap^Z \tau} \mathbf{lambda}$

let $\Gamma = x_1 :!_{k_1} \tau_1, \dots, x_i :!_{k_i} \tau_i$ and $A =!_k \tau_1$.
Assume $\vdash_{Z'} (v', \theta') : \tau_1$ (1) and $\vdash_{Z_i} (v_i, \theta_i) : \tau_i$.
Assume $\theta = [x_1 \rightarrow (v_1, \theta_1, R_1), \dots, x_i \rightarrow (v_i, \theta_i, R_i)]$.
So $\phi = [x_1 \rightarrow (k_1, R_1, Z_1), \dots, x_i \rightarrow (k_i, R_i, Z_i)]$
 $\text{TS} : \vdash_{0+F(\lambda x.e, \phi)} (\lambda x.e, \theta) :!_k \tau_1 \multimap^Z \tau_2$.
Let $S = \sum_{x_i \in \text{FV}(\lambda x.e)} k_i \times (R_i + Z_i)$.
From assumption (1), we know $\vdash_{Z'} (v', \theta) : \tau_1$ (\star).
Take a fresh variable x' , doing alpha renaming on the premise, pick R'
so that $\theta' = [x' \rightarrow (v', \theta', R')] \uplus \theta$ and $\phi' = [x' \rightarrow (k, R', Z')] \uplus \phi$.
By induction hypothesis on the premise, we know: $\vdash_{Z+F(e[x'/x], \phi')} (e[x'/x], [x' \rightarrow (v', \theta', R')] \uplus \theta) : \tau$ (\diamond).
 $F(e[x'/x], \phi') = \sum_{x_i \in \text{FV}(\lambda x.e)} k_i \times (R_i + Z_i) + k \times (R' + Z') = S + k \times (R' + Z')$.
We can conclude the following by the configuration rule.

$$\frac{\text{fresh } x' \quad \vdash_{Z'} (v', \theta') : \tau_1 (\star) \quad \vdash_{S+k \times (R'+Z')+Z} (e[x'/x], \theta[x' \rightarrow (v', \theta', R')]) : \tau (\diamond)}{\vdash_S (\lambda x.e, \theta) : !_k \tau_1 \multimap^Z \tau} \text{ C-lambda}$$

$$\text{Case } \frac{\Gamma_1 \vdash_{Z_1} e_1 : !_k \tau_1 \multimap^Z \tau_2 \quad \Gamma_2 \vdash_{Z_2} e_2 : \tau_1}{\Gamma_1 + k \times \Gamma_2 \vdash_{Z_1+k \times Z_2+Z} e_1 e_2 : \tau_2} \text{ app}$$

Let us assume $\Gamma_1 = x_i : !_k \tau_i$ and $\Gamma_2 = x'_i : !_k \tau'_i$, (Γ_1 and Γ_2 may overlap.).

For all the variables x''_i in $\text{dom}(\Gamma_1 + k \times \Gamma_2)$, we assume $(\Gamma_1 + k \times \Gamma_2)(x''_i) = !_k \tau''_i$ so that $\vdash_{Z''_i} (v''_i, \theta''_i) : \tau''_i$

assume $\theta = [x''_1 \rightarrow (v''_1, \theta''_1, R''_1), \dots, x''_i \rightarrow (v''_i, \theta''_i, R''_i)]$.

So $\phi = [x''_1 \rightarrow (k''_1, R''_1, Z''_1), \dots, x''_i \rightarrow (k''_i, R''_i, Z''_i)]$

TS: $\vdash_{Z_1+k \times Z_2+Z+F(e_1 e_2, \phi)} (e_1 e_2, \theta) : \tau_2$.

let $\theta_1 = \{[x''_i \rightarrow (v''_i, \theta''_i, R''_i)] | x''_i \in \text{dom}(\Gamma_1)\}$.

let $\theta_2 = \{[x''_i \rightarrow (v''_i, \theta''_i, R''_i)] | x''_i \in \text{dom}(\Gamma_2)\}$.

We know $\text{dom}(\theta) = \text{dom}(\theta_1) \uplus \text{dom}(\theta_2) \implies \theta = \theta_1 \uplus \theta_2$

We set $\phi_1 = \{[x_i \rightarrow (k''_i, R''_i, Z''_i)] | x''_i \in \text{dom}(\Gamma_1)\}$. and $\phi_2 = \{[x_i \rightarrow (k''_i, R''_i, Z''_i)] | x''_i \in \text{dom}(\Gamma_2)\}$

By induction hypothesis on the first premise, we have: $\vdash_{Z_1+F(e_1, \phi_1)} (e_1, \theta_1) : !_k \tau_1 \multimap^Z \tau_2 (\star)$.

By induction hypothesis on the second premise, we have: $\vdash_{Z_2+F(e_2, \phi_2)} (e_2, \theta_2) : \tau_1$.

From the definition, we know: $F(e_1, \phi_1) = F(e_1, \phi)$ and $F(e_2, \phi_2) = F(e_2, \phi)$.

From the configuration rule C-app, we get:

$$\frac{\vdash_{Z_1+F(e_1, \theta_1, \Gamma_1)} (e_1, \theta_1) : !_k \tau_1 \multimap^Z \tau_2 (\star) \quad \vdash_{Z_2+F(e_2, \theta_2, \Gamma_2)} (e_2, \theta_2) : \tau_1 (\diamond)}{\vdash_{Z_1+F(e_1, \theta_1, \Gamma_1)+k \times (Z_2+F(e_2, \theta_2, \Gamma_2))+Z} (e_1 e_2, \theta_1 \uplus \theta_2) : \tau_2 (\clubsuit)} \text{ C-app}$$

Because $F(e_1, \phi_1) + k \times F(e_2, \phi_2) \leq F(e_1 e_2, \phi)$, so we conclude $:Z_1 + F(e_1, \phi_1) + k \times (Z_2 + F(e_2, \phi_2)) + Z \leq Z_1 + k \times Z_2 + Z + F(e_1 e_2, \phi)$.

The TS can be shown using Lemma 1 on the \clubsuit .

□

Theorem 4 (ConfigurationSoundness). 1. $\vdash_Z (e, \theta) : \tau$, then $(e, \theta) \in \llbracket \tau \rrbracket_E^Z$

2. $\vdash_Z (v, \theta) : \tau$, then $(v, \theta, Z) \in \llbracket \tau \rrbracket_V$.

Proof. Statement (1) is proved by induction on the configuration derivation.

$$\text{Case } \frac{\theta(x) = (v, \theta', R) \quad \vdash_Z (v, \theta') : \tau \ (\star)}{\vdash_{R+Z} (x, \theta) : \tau} \text{ C-Ax}$$

TS: $(x, \theta) \in \llbracket \tau \rrbracket_E^{R+Z}$.

Let us first assume:

$$\frac{\theta(x) = (v, \theta', R)}{x, \theta \Downarrow^R v, \theta'} \text{ var2}$$

By unfolding the definition, STS: $R \leq R + Z$ and $(v, \theta', Z) \in \llbracket \tau \rrbracket_V$.

By induction hypothesis on (\star) , we know $(v, \theta') \in \llbracket \tau \rrbracket_E^Z$, by unfolding its definition and $v, \theta \Downarrow^0 v, \theta$, we know that $(v, \theta', Z) \in \llbracket \tau \rrbracket_V$.

$$\text{Case } \frac{\text{fresh } x' \quad \vdash_{S+k \times (R'+Z')+Z} (e[x'/x], \theta[x' \rightarrow (v', \theta', R')]) : \tau_2 \ (\diamond)}{\vdash_S (\lambda x.e, \theta) : !_k \tau_1 \multimap^Z \tau_2} \text{ C-lambda}$$

TS: $(\lambda x.e, \theta) \in \llbracket !_k \tau_1 \multimap^Z \tau_2 \rrbracket_E^S$.

By unfolding the definition, as well as $\lambda x.e$ is value, we know $v, \theta \Downarrow^0 v, \theta$.

STS: $0 \leq S$ and $(\lambda x.e, \theta, S - 0) \in \llbracket !_k \tau_1 \multimap^Z \tau_2 \rrbracket_V$.

By induction hypothesis on (\star) , we know $(v', \theta', Z' - 0) \in \llbracket !_k \tau_1 \rrbracket_V$ (1).

Unfolding the definition of $\llbracket !_k \tau_1 \multimap^Z \tau_2 \rrbracket_V$, pick $(v', \theta', Z') \in \llbracket !_k \tau_1 \rrbracket_V$.

STS: $\text{fresh } x' \wedge \forall R. (e[x'/x], \theta[x' \rightarrow (v', \theta', R)]) \in \llbracket \tau_2 \rrbracket_E^{S+Z+k \times (R+Z')}$.

Pick $R = R'$. STS: $(e[x'/x], \theta[x' \rightarrow (v', \theta', R')]) \in \llbracket \tau_2 \rrbracket_E^{S+Z+k \times (R'+Z')}$

It is proved by Induction hypothesis on (\diamond) .

$$\text{Case } \frac{\vdash_{Z_1} (e_1, \theta_1) : !_k \tau_1 \multimap^Z \tau_2 \ (\star) \quad \vdash_{Z_2} (e_2, \theta_2) : \tau_1 \ (\diamond)}{\vdash_{Z_1+k \times Z_2+Z} (e_1 \ e_2, \theta_1 \uplus \theta_2) : \tau_2} \text{ C-app}$$

TS: $(e_1 \ e_2, \theta_1 \uplus \theta_2) \in \llbracket \tau_2 \rrbracket_E^{Z_1+k \times Z_2+Z}$.

Let us first assume:

$$\frac{\begin{array}{c} e_1, \theta_1 \Downarrow^{R_1} \lambda x.e, \theta'_1 (a) \quad e_2, \theta_2 \Downarrow^{R_2} v_2, \theta'_2 (b) \\ \text{fresh } x' \quad e[x'/x], \theta'_1[x' \rightarrow (v_2, \theta'_2, R_2)] \Downarrow^{R_3} v, \theta_3 (c) \end{array}}{e_1 \ e_2, (\theta_1 \uplus \theta_2) \Downarrow^{R_1+R_3} v, \theta_3} \text{ app.}$$

By unfolding the definition:

STS1: $R_1 + R_3 \leq Z_1 + k \times Z_2 + Z$.

STS2: $(v, \theta_3, Z_1 + k \times Z_2 + Z - (R_1 + R_3)) \in \llbracket \tau_2 \rrbracket_V$.

By Induction hypothesis on (\star) , we get: $(e_1, \theta_1) \in \llbracket !_k \tau_1 \multimap^Z \tau_2 \rrbracket_E^{Z_1}$ (1).

Unfolding (1), from the assumption (a), we know: $R_1 \leq Z_1 \wedge (\lambda x.e, \theta'_1, Z_1 - R_1) \in \llbracket !_k \tau_1 \multimap^Z \tau_2 \rrbracket_V$ (2).

By Induction hypothesis on (\diamond) , we get: $(e_2, \theta_2) \in \llbracket \tau_1 \rrbracket_E^{Z_2}$ (3).

Unfolding (3), from the assumption (b), we know: $R_2 \leq Z_2 \wedge (v_2, \theta'_2, Z_2 - R_2) \in \llbracket \tau_1 \rrbracket_V$ (4).

Unfolding (2), pick $(v_2, \theta'_2, Z_2 - R_2) \in \llbracket \tau_1 \rrbracket_V$ from (4).

We know: $\text{fresh } x' \wedge \forall R. (e[x'/x], \theta'_1[x' \rightarrow (v_2, \theta'_2, R)]) \in \llbracket \tau_2 \rrbracket_E^{(Z_1 - R_1) + Z + k \times (R + Z_2 - R_2)}$.

Pick $R = R_2$. We have: $\text{fresh } x' \wedge (e[x'/x], \theta'_1[x' \rightarrow (v_2, \theta'_2, R_2)]) \in \llbracket \tau_2 \rrbracket_E^{(Z_1 - R_1) + Z + k \times (R_2 + Z_2 - R_2)}$ (5).

Unfolding (5), we conclude that: $R_3 \leq (Z_1 - R_1) + Z + k \times (Z_2 - R_2 + R_2)$ (6).
and $(v, \theta_3, (Z_1 - R_1) + Z + k \times R_2 - R_3) \in \llbracket \tau_2 \rrbracket_V$ (7).

STS1 is proved by using both (6). STS2 is proved by (7).

Case $\frac{\vdash_Z (e, \theta) : \mathbf{b} \ (\star)}{\vdash_{1+Z} (\delta(e), \theta) : \mathbf{b}} \text{ C-delta}$

TS: $(\delta(e), \theta) \in \llbracket \mathbf{b} \rrbracket_E^{1+Z}$. We first assume:

$$\frac{e, \theta \Downarrow^R v', \theta_1 \ (a) \quad \delta(v') = v}{\delta(e), \theta \Downarrow^{R+1} v, \theta_1} \text{ delta.}$$

By unfolding the definition:

STS1: $R + 1 \leq Z + 1$.

STS2: $(v, \theta_1, Z - R) \in \llbracket \mathbf{b} \rrbracket_V$.

By induction hypothesis on (\star) , we get: $(e, \theta) \in \llbracket \mathbf{b} \rrbracket_E^Z$. Unfold this statement, from the assumption (a), we get: $R \leq Z$ (1) and $(v', \theta_1, Z - R) \in \llbracket \mathbf{b} \rrbracket_V$ (2).

STS1 is proved by (1), STS2 is proved by (2) and the interpretation of $\delta(v')$.

Statement (2) is proved by induction on the value v .

Case $\frac{\text{fresh } x' \quad \vdash_{Z'} (v', \theta') : \tau_1 \ (\star) \quad \vdash_{S+k \times R' + Z} (e[x'/x], \theta[x' \rightarrow (v', \theta', R')]) : \tau_2 \ (\diamond)}{\vdash_S (\lambda x.e, \theta) : !_k \tau_1 \multimap^Z \tau_2} \text{ C-lambda}$

TS: $(\lambda x.e, \theta, S) \in \llbracket !_k \tau_1 \multimap^Z \tau_2 \rrbracket_V$.

By induction hypothesis on (\star) , we know: $(v', \theta', Z' - 0) \in \llbracket !_k \tau_1 \rrbracket_V$ (1).

Unfolding the definition of $\llbracket !_k \tau_1 \multimap^Z \tau_2 \rrbracket_V$, pick $(v', \theta', Z') \in \llbracket !_k \tau_1 \rrbracket_V$.

STS: $\text{fresh } x' \wedge \forall R. (e[x'/x], \theta[x' \rightarrow (v', \theta', R)]) \in \llbracket \tau_2 \rrbracket_E^{S+Z+k \times R}$.

Pick $R = R'$. STS: $(e[x'/x], \theta[x' \rightarrow (v', \theta', R')]) \in \llbracket \tau_2 \rrbracket_E^{S+Z+k \times R'}$

It is proved by Induction hypothesis on (\diamond) .

□

1 Typable Approach

$$\begin{array}{ll}
F(e, \phi) & \text{where } \phi(x_i) = (k_i, R_i, Z_i) \\
F(x, \phi) & = \sum_{x_i \in \text{FV}(x)} k_i \times (R_i + Z_i) \\
F(\lambda x. e, \phi) & = \sum_{x_i \in \text{FV}(\lambda x. e)} k_i \times (R_i + Z_i) \\
F(\delta(e), \phi) & = \sum_{x_i \in \text{FV}(\delta(e))} k_i \times (R_i + Z_i) \\
F(c, \phi) & = 0 \\
F(e_1 \ e_2, \phi) & = F(e_1, \phi) + F(e_2, \phi) \\
F(\text{if } e \text{ then } e_1 \text{ else } e_2, \phi) & = F(e, \phi) + \max(F(e_1, \phi), F(e_2, \phi))
\end{array}$$

Definition 5 (Typable). A closure $(e, [x_1 \rightarrow (v_1, \theta_1, R_1), \dots, x_i \rightarrow (v_i, \theta_i, R_i)])$ is typable with type τ and adaptivity J if exists k_i

$$x_1 :!_{k_1} \tau_1, \dots, !_{k_i} \tau_i \vdash_Z e : \tau$$

and each closure (v_i, θ_i) is also typable with type $!_{k_i} \tau_i$ and adaptivity Z_i , $\phi = [x_1 \rightarrow (k_1, R_1, Z_1), \dots, x_i \rightarrow (k_i, R_i, Z_i)]$, $J = Z + F(e, \phi)$.

Definition 6 (ClosedClosure). A closure (e, θ) is closed if $\text{FV}(e) \subseteq \text{dom}(\theta)$.

Lemma 7 (programTypable). If $\vdash_Z e : \tau$, then (e, \emptyset) is typable with τ and adaptivity Z .

Lemma 8 (TypableMono). If a closure D is typable with τ and Z , and $Z \leq Z'$, then D is typable with τ and Z' .

Lemma 9 (TypableSoundness). If a closure D is typable with τ and J , and $D \Downarrow^R E$, then closure E is typable with τ and adaptivity $J - R$.

Proof. By induction on the evaluation semantics.

Case

$$\frac{\theta(x) = (v, \theta', R)}{x, \theta \Downarrow^R v, \theta'} \text{ var2}$$

Suppose (x, θ) is typable with τ and J and $\theta = [x \rightarrow (v, \theta', R), x_1 \rightarrow (v_1, \theta_1, R_1), \dots, x_i \rightarrow (v_i, \theta_i, R_i)]$

We know that: exists k_i so that $x_1 :!_{k_1} \tau_1, \dots, !_{k_i} \tau_i, x :!_1 \tau \vdash_0 x : \tau$, and the each closure (v_i, θ_i) is typable with τ_i and Z_i as well as (v, θ') is typable with τ and Z . and $J = 0 + (R + Z)$.

TS: (v, θ') is typable with τ and $J - R$, it is proved by Lemma 8 on the assumption.

Case

$$\frac{e_2, \theta_2 \Downarrow^{R_2} v_2, \theta'_2 \quad \text{fresh } x' \quad \frac{e_1, \theta_1 \Downarrow^{R_1} \lambda x.e, \theta'_1 \quad e[x'/x], \theta'_1[x' \rightarrow (v_2, \theta'_2, R_2)] \Downarrow^{R_3} v, \theta_3 (\clubsuit)}{e_1 e_2, (\theta_1 \uplus \theta_2) \Downarrow^{R_1+R_3} v, \theta_3} \text{app}}$$

Suppose for each variable $x_i \in \text{dom}(\theta_1)$, (v_i, θ_i) is tyable with type τ_i and I_i . For each variable $x_j \in \text{dom}(\theta_2)$, (v_j, θ_j) is tyable with type τ_j and I_j .

We assume exists k_i and $\Gamma_1 = x_1 :!_{k_1} \tau_1, \dots, x_i :!_{k_i} \tau_i$ for $x_i \in \text{dom}(\theta_1)$, so that $\Gamma_1 \vdash_{Z_1} e_1 :!_k \tau_1 \multimap^Z \tau_2$.

Similarly, we assume exists k_j and $\Gamma_2 = x_1 :!_{k_1} \tau_1, \dots, x_j :!_{k_j} \tau_j$ for $x_j \in \text{dom}(\theta_2)$, so that $\Gamma_2 \vdash_{Z_2} e_2 : \tau_1$.

By the typing rule app, we have:

$$\frac{\Gamma_1 \vdash_{Z_1} e_1 :!_k \tau_1 \multimap^Z \tau_2 \quad \Gamma_2 \vdash_{Z_2} e_2 : \tau_1}{\Gamma_1 + k \times \Gamma_2 \vdash_{Z_1+k \times Z_2+Z} e_1 e_2 : \tau_2} \text{app}$$

We know that $\text{dom}(\Gamma_1 + k \times \Gamma_2) = \text{dom}(\Gamma_1) \cup \text{dom}(\Gamma_2) = \text{dom}(\theta_1 \uplus \theta_2)$. So for all the closure (v_i, θ_i) assigned by variable $x_i \in \text{dom}(\Gamma_1 + k \times \Gamma_2)$, (v_i, θ_i) is tyable with τ_i and I_i from our assumption.

$\phi = [x_1 \rightarrow (k_1, R_1, I_1), \dots, x_i \rightarrow (k_i, R_i, I_i)]$ where $x_i \in \text{dom}(\theta_1 \uplus \theta_2)$.

So, we know that $(e_1 e_2, \theta_1 \uplus \theta_2)$ is tyable with τ_2 and $J = Z_1 + k \times Z_2 + Z + F(e_1 e_2, \phi)$.

TS: (v, θ_3) is tyable with τ_2 and $J - (R_1 + R_3)$.

Set $\phi_1 = [x_1 \rightarrow (k_1, R_1, I_1), \dots, x_i \rightarrow (k_i, R_i, I_i)]$ where $x_i \in \text{dom}(\theta_1)$.

Set $\phi_2 = [x_1 \rightarrow (k_1, R_1, I_1), \dots, x_i \rightarrow (k_i, R_i, I_i)]$ where $x_i \in \text{dom}(\theta_2)$.

From our assumption, we also know that (e_1, θ_1) is tyable with $!_k \tau_1 \multimap^Z \tau_2$ and $J_1 = Z_1 + F(e_1, \phi_1)$.

Similarly, we have: (e_2, θ_2) is tyable with τ_1 and $J_2 = Z_2 + F(e_2, \phi_2)$ for $x_j \in \text{dom}(\theta_2)$ (2).

By induction hypothesis on (1), (2) respectively, we know that:

$(\lambda x.e, \theta'_1)$ is tyable with $!_k \tau_1 \multimap^Z \tau_2$ and $J_1 - R_1$ (3).

(v_2, θ'_2) tyable with τ_1 and $J_2 - R_2$ (4).

From (3), we know the following if we assume exists k''_i and $\Gamma''_1 = x''_1 :!_{k''_1} \tau''_1, \dots, x''_i :!_{k''_i} \tau''_i$, and $\theta'_1(x''_i) = (v''_i, \theta''_i, R''_i)$ and each closure (v''_i, θ''_i) is tyable with τ''_i and I''_i .

Set $\phi''_1 = [x''_1 \rightarrow (k''_1, R''_1, I''_1), \dots, x''_i \rightarrow (k''_i, R''_i, I''_i)]$ where $x''_i \in \text{dom}(\theta'_1)$.

$$\frac{\Gamma''_1, x :!_k \tau_1 \vdash_Z e : \tau_2 (\star)}{\Gamma''_1 \vdash_0 \lambda x.e :!_k \tau_1 \multimap^Z \tau_2} \text{lambda}$$

and we know $J_1 - R_1 = 0 + F(\lambda x.e, \phi''_1)$.

Take a fresh variable x' , from (\star) , we know: $\Gamma_1, x' :!_k \tau_1 \vdash_Z e[x/x'] : \tau_2$ $(\star\star)$.

Set $\phi_1''' = \phi_1''[x' \rightarrow (k, R_2, J_2 - R_2)]$.

From $(\star\star)$ and (4), we conclude that $(e[x'/x], \theta_1'[x' \rightarrow (v_2, \theta_2', R_2)])$ is typable with τ_2 and $Z + F(e[x'/x], \phi_1''')$.

Because $F(e[x'/x], \phi_1''') = F(\lambda x.e, \phi_1'') + k \times (R_2 + J_2 - R_2) = J_1 - R_1 + k \times J_2$.

By induction hypothesis using (\clubsuit) and the above statement, we know: v, θ_3 is typable with τ_2 and $Z + J_1 - R_1 + k \times J_2 - R_3$.

Unfold J_1, J_2 , we know $Z + J_1 - R_1 + k \times J_2 - R_3 = Z + Z_1 + F(e_1, \phi_1) + k \times Z_2 + k \times F(e_2, \phi_2) - (R_1 + R_3) \leq J - (R_1 + R_3)$.

The TS is proved by lemma 8 on the above statement.

Case

$$\frac{e, \theta \Downarrow^R \text{false}, \theta' \quad e_2, \theta \Downarrow^{R_2} v_2, \theta_2}{\text{if } e \text{ then } e_1 \text{ else } e_2, \theta \Downarrow^{R+R_2} v_2, \theta_2} \text{if-f}$$

Assume for all the variables $x_i \in \text{dom}(\theta)$, $\theta(x_i) = (v_i, \theta_i, R_i)$ and (v_i, θ_i) is typable with τ_i and I_i .

We assume exists $\Gamma' = x_1 :!_{k_1} \tau_1, \dots, x_i :!_{k_i} \tau_i$ and $\text{dom}(\Gamma') = \text{dom}(\theta)$, and assume the following using the typing rule if.

$$\frac{\Gamma \vdash_Z e : \text{bool} \ (\star) \quad \Gamma_1 \vdash_{Z_1} e_1 : \tau \quad \Gamma_2 \vdash_{Z_2} e_2 : \tau \ (\diamond) \quad \Gamma' = \Gamma + \Gamma_1 + \Gamma_2 \quad Z' = Z + \max(Z_1, Z_2)}{\Gamma' \vdash_{Z'} \text{if } e \text{ then } e_1 \text{ else } e_2 : \tau} \text{if}$$

Set $\phi = [x_1 \rightarrow (k_1, R_1, I_1), \dots, x_i \rightarrow (k_i, R_i, I_i)]$ where $x_i \in \text{dom}(\theta)$.

We have : $(\text{if } e \text{ then } e_1 \text{ else } e_2, \theta)$ is typable with τ and J , where $J = Z' + F(\text{if } e \text{ then } e_1 \text{ else } e_2, \phi)$

TS: (v_2, θ_2) is typable with τ and $J - (R + R_2)$.

From (\star) , we know from weakening rule that $\Gamma, x_j :!_0 \tau_j \vdash_Z e : \text{bool}$ (1) where $x_j \in \text{dom}(\theta) \setminus \text{dom}(\Gamma)$.

Set $\phi_1 = [x_1 \rightarrow (k_1, R_1, I_1), \dots, x_i \rightarrow (k_i, R_i, I_i)]$ where $k_j = 0$ for $x_j \in \text{dom}(\theta) \setminus \text{dom}(\Gamma)$.

Similaly, from (\diamond) , we have $\Gamma_2, x_j :!_0 \tau_j \vdash_{Z_2} e_2 : \tau$ (2) where $x_j \in \text{dom}(\theta) \setminus \text{dom}(\Gamma_2)$.

Set $\phi_2 = [x_1 \rightarrow (k_1, R_1, I_1), \dots, x_i \rightarrow (k_i, R_i, I_i)]$ where $k_j = 0$ for $x_j \in \text{dom}(\theta) \setminus \text{dom}(\Gamma_2)$.

From (1), we know that (e, θ) is typable with bool and J_1 , where $J_1 = Z + F(e, \phi_1)$ (3).

From (2), we know that (e_2, θ) is typable with τ and J_2 , where $J_2 = Z_2 + F(e_2, \phi_2)$ (4).

By IH on (3), (4), we have $(\mathbf{false}, \theta'_1)$ is typable with `bool` and $J_1 - R$ (5) and (v_2, θ_2) is typable with τ and $J_2 - R_2$ (6).

we know that $J = Z' + F(\mathbf{if} \ e \ \mathbf{then} \ e_1 \ \mathbf{else} \ e_2, \phi) = Z + F(e, \phi) + \max(Z_1, Z_1) + \max(F(e_1, \phi), F(e_2, \phi)) \geq (J_1 + J_2)$.

We also know $J_1 - R \geq 0$ from (5),

So we know $J - (R + R_2) \geq (J_1 + J_2) - (R + R_2) \geq J_2 - R_2$. Our TS can be proved using Lemma 8 on (6).

□

2 Typable Approach by Marco

Definition 10 (Typable Closures). Let $\theta = [x_1 \rightarrow (v_1, \theta_1, R_1), \dots, x_n \rightarrow (v_n, \theta_n, R_n)]$. The closure (e, θ) is typable with type τ and adaptivity J if:

1. $x_1 :!_{k_1} \tau_1, \dots, !_{k_i} \tau_i \vdash_Z e : \tau$, for some types $!_{k_i} \tau_i$ for $(1 \leq i \leq n)$,
2. each closure (v_i, θ_i) for $(1 \leq i \leq n)$ is typable with type $!_{k_i} \tau_i$ and adaptivity Z_i ,
3. $J = Z + \sum_{(v_i, \theta_i, S_i) \in \theta} k_i \times (R_i + Z_i)$.

To justify why we chose \sum in the third clause above it is worth to consider the following configuration:

$$[x \mapsto (\lambda u. \lambda w. \delta(u) + \delta(w), [], 0), y \mapsto (v, [], 2)], x y y$$

[MG: To elaborate the example above]

Lemma 11 (Soundness). If a closure D is typable with type τ and adaptivity J , and $D \Downarrow^R E$, then the closure E is typable with type τ and adaptivity I , where $I + R \leq J$.

Proof. By induction on the derivation showing $D \Downarrow^R E$. The rules `lambda`, `val`, and `const` follow from the hypothesis. We will discuss the other cases.

Case

$$\frac{\theta(x) = (v, \theta', R)}{x, \theta \Downarrow^R v, \theta'} \text{ var}$$

By assumption, the closure (x, θ) is typable with type τ and adaptivity J . Let us assume without loss of generality that $\theta = [x \rightarrow (v, \theta', R), x_1 \rightarrow (v_1, \theta_1, R_1), \dots, x_n \rightarrow (v_n, \theta_n, R_n)]$. By Definition 10 we have that there exist k and $!_{k_i} \tau_i$ for $(1 \leq i \leq n)$ such that:

$$x :!_k \tau, x_1 :!_{k_1} \tau_1, \dots, x_n :!_{k_n} \tau_n \vdash_Z x : \tau$$

Moreover, we have that the closure (v, θ') is typable with type τ and adaptivity I , and that each closure (v_i, θ_i) is typable with type τ_i and adaptivity Z_i , and that

$$J = Z + k \times (R + I) + \sum_{1 \leq i \leq n, i \neq j} k_i \times (R_i + Z_i).$$

By inversion on the typing rules, we have that $k = 1$, hence we can rewrite J as

$$J = Z + R + I + \sum_{1 \leq i \leq n, i \neq j} k_i \times (R_i + Z_i).$$

Hence, it is easy to see that $I + R \leq J$ which is what we need to show.

Case

$$\frac{e_2, \theta \Downarrow^{R_2} v_2, \theta^2 \quad \text{fresh } x' \quad \frac{e_1, \theta \Downarrow^{R_1} \lambda x.e, \theta^1 \quad e[x'/x], \theta^1[x' \rightarrow (v_2, \theta^2, R_2)] \Downarrow^{R_3} v, \theta^3 (\clubsuit)}{e_1 \ e_2, \theta \Downarrow^{R_1+R_3} v, \theta^3} \text{ app}$$

By assumption, the closure $(e_1 \ e_2, \theta)$ is typable with type τ and adaptivity J . Let us assume without loss of generality that $\theta = [x_1 \rightarrow (v_1, \theta_1, S_1), \dots, x_n \rightarrow (v_n, \theta_n, S_n)]$. By Definition 10 we have that there exist $!_{k_i} \tau_i$ for $(1 \leq i \leq n)$ such that:

$$x_1 : !_{k_1} \tau_1, \dots, x_n : !_{k_n} \tau_n \vdash_Z e_1 e_2 : \tau \quad (1)$$

Moreover, we have that each closure $(v_i, \theta_i) \in \theta$ is typable with type τ_i and adaptivity Z_i , and that

$$J = Z + \sum_{(v_i, \theta_i, S_i) \in \theta} k_i \times (S_i + Z_i).$$

By inversion on Fact 1 we have that there exist σ, Z', Z^1, k and k_i^1 for $(1 \leq i \leq n)$ such that:

$$x_1 : !_{k_1^1} \tau_1, \dots, x_n : !_{k_n^1} \tau_n \vdash_{Z^1} e_1 : !_k \sigma \multimap^{Z'} \tau \quad (2)$$

Similarly, we have that there exist Z^2 and k_i^2 for $(1 \leq i \leq n)$ such that:

$$x_1 : !_{k_1^2} \tau_1, \dots, x_n : !_{k_n^2} \tau_n \vdash_{Z^2} e_2 : \sigma \quad (3)$$

Moreover, we have that $Z = Z^1 + k \times Z^2 + Z'$, and that $k_i = k_i^1 + k \times k_i^2$.

Using Fact 2 and the assumption that each closure (v_i, θ_i) in θ is typable with type τ_i and adaptivity Z_i we have that the closure (e_1, θ) is typable with type $!_k \sigma \multimap^{Z'} \tau$ and adaptivity:

$$Z^1 + \sum_{(v_i, \theta_i, S_i) \in \theta} k_i^1 \times (S_i + Z_i)$$

By induction hypothesis we then have that the closure $(\lambda x.e, \theta^1)$ is typable with type $!_k \sigma \multimap^{Z'} \tau$ and adaptivity I_1 where

$$I_1 + R_1 \leq Z^1 + \sum_{(v_i, \theta_i, S_i) \in \theta} k_i^1 \times (S_i + Z_i)$$

It is worth to stress here that by definition, using the fact that

$$x_1 : !_{k_1^1} \tau_1, \dots, x_n : !_{k_n^1} \tau_n \vdash_0 \lambda x.e : !_k \sigma \multimap^{Z'} \tau$$

which follows from Fact 2 and inversion on the rule **lambda** [MG: here we actually want a lemma that says that values are typable with \vdash_0 , and that takes care of the free variables that we may have added] we have:

$$I_1 = \sum_{(v_i, \theta_i, S_i) \in \theta^1} k_i^1 \times (S_i + Z_i)$$

Using Fact 3 and again the assumption that each closure (v_i, θ_i) in θ is typable with type τ_i and adaptivity Z_i we have that the closure (e_2, θ) is typable with type τ and adaptivity:

$$Z^2 + \sum_{(v_i, \theta_i, S_i) \in \theta} k_i^2 \times (S_i + Z_i)$$

By induction hypothesis we then have that the closure (v_2, θ_2) is typable with type τ and adaptivity I_2 where

$$I_2 + R_2 \leq Z^2 + \sum_{(v_i, \theta_i, S_i) \in \theta} k_i^2 \times (S_i + Z_i)$$

It is again worth to stress here that by definition, using the fact that v_2 is a value we have:

$$I_2 = \sum_{(v_i, \theta_i, S_i) \in \theta^2} k_i^2 \times (S_i + Z_i)$$

By Fact 2, and type preservation and some other Lemma [MG: TODO: we need to be more clear and handle the fact that the adaptivity can be 0 in the lambda rule] we have

$$x_1 :!_{k_1^1} \tau_1, \dots, x_n :!_{k_n^1} \tau_n, x :!_k \sigma \vdash_{Z'} e : \tau \quad (4)$$

By composing this with the previous assumptions, we have that the closure $(e[x'/x], \theta^1[x' \mapsto (v_2, \theta_2, R_2)])$ is typable with type τ and adaptivity I_3 where

$$I_3 = Z' + k \times (R_2 + I_2) + \sum_{(v_i, \theta_i, S_i) \in \theta^1} k_i^1 \times (S_i + Z_i)$$

By induction hypothesis we then get that the closure (v, θ^3) is typable with type τ and adaptivity I where $I + R_3 \leq I_3$. Notice also that we have a θ^4 such that $\theta^3 = \theta^4[x' \mapsto (v_2, \theta_2, R_2)]$. Moreover, once again, note that by Lemma [MG: Lemma showing that we can type values with adaptivity 0] we have:

$$I = \sum_{(v_i, \theta_i, S_i) \in \theta^3} k_i \times (S_i + Z_i) = k \times (R_2 + I_2) + \sum_{(v_i, \theta_i, S_i) \in \theta^4} k_i \times (S_i + Z_i)$$

We want to prove $I + R_1 + R_3 \leq J$.

Starting from the left hand side we have

$$\begin{aligned}
I + R_1 + R_3 &= \sum_{(v_i, \theta_i, S_i) \in \theta^3} (k_i \times (S_i + Z_i)) + R_1 + R_3 \\
&= k \times (R_2 + I_2) + \sum_{(v_i, \theta_i, S_i) \in \theta^4} k_i \times (S_i + Z_i) + R_3 + R_1 \\
&\leq k \times (R_2 + I_2) + Z' + \sum_{(v_i, \theta_i, S_i) \in \theta^1} k_i^1 \times (S_i + Z_i) + R_1 \\
&\leq k \times (R_2 + I_2) + Z' + Z^1 + \sum_{(v_i, \theta_i, S_i) \in \theta} k_i^1 \times (S_i + Z_i) \\
&= k \times (R_2 + \sum_{(v_i, \theta_i, S_i) \in \theta^2} k_i^2 \times (S_i + Z_i)) \\
&\quad + Z' + Z^1 + \sum_{(v_i, \theta_i, S_i) \in \theta} k_i^1 \times (S_i + Z_i) \\
&\leq k \times (Z^2 + \sum_{(v_i, \theta_i, S_i) \in \theta} k_i^2 \times (S_i + Z_i)) \\
&\quad + Z' + Z^1 + \sum_{(v_i, \theta_i, S_i) \in \theta} k_i^1 \times (S_i + Z_i) \\
&= k \times Z^2 + Z' + Z^1 + \\
&\quad \sum_{(v_i, \theta_i, S_i) \in \theta} k \times k_i^2 \times (S_i + Z_i) + \sum_{(v_i, \theta_i, S_i) \in \theta} k_i^1 \times (S_i + Z_i) \\
&= k \times Z^2 + Z' + Z^1 + \sum_{(v_i, \theta_i, S_i) \in \theta} (k_i^1 + k \times k_i^2) \times (S_i + Z_i) \\
&= J
\end{aligned}$$

This concludes this case.

Case

$$\frac{e, \theta \Downarrow^R v, \theta_1 \quad \delta(v\theta_1) = v' \quad FV(v') = \emptyset}{\delta(e), \theta \Downarrow^{R+1} v', \theta_1} \text{delta}$$

By assumption, the closure $(\delta(e), \theta)$ is typable with type τ and adaptivity J . Let us assume without loss of generality that $\theta = [x_1 \rightarrow (v_1, \theta_1, S_1), \dots, x_n \rightarrow (v_n, \theta_n, S_n)]$. By Definition 10 we have that there exist Z and $!_{k_i} \tau_i$ for $(1 \leq i \leq n)$ such that:

$$x_1 :!_{k_1} \tau_1, \dots, x_n :!_{k_n} \tau_n \vdash_Z \delta(e) : \tau \quad (5)$$

Moreover, we have that each closure $(v_i, \theta_i) \in \theta$ is typable with type τ_i and adaptivity Z_i , and that

$$J = Z + \sum_{(v_i, \theta_i, S_i) \in \theta} k_i \times (S_i + Z_i).$$

By inversion on Fact 6 we have that $Z = Z' + 1$ for some Z' , that $\tau = \mathbf{b}$, that $k_i = k'_i + 1$ for $(1 \leq i \leq n)$, and that:

$$x_1 :!_{k'_1} \tau_1, \dots, x_n :!_{k'_n} \tau_n \vdash_{Z'} e : \mathbf{b}. \quad (6)$$

Hence, it is easy to see that the closure (e, θ) is typable with type \mathbf{b} and adaptivity

$$Z' + \sum_{(v_i, \theta_i, S_i) \in \theta} k'_i \times (S_i + Z_i)$$

By applying the induction hypothesis we have that also the closure (v, θ') is typable with type \mathbf{b} and adaptivity I' such that:

$$I' + R \leq Z' + \sum_{(v_i, \theta_i, S_i) \in \theta} k'_i \times (S_i + Z_i)$$

From this and the fact that $\vdash_0 v' : \mathbf{b}$, we have:

$$I' + R + 1 \leq Z' + 1 + \sum_{(v_i, \theta_i, S_i) \in \theta} (k'_i + 1) \times (S_i + Z_i)$$

This concludes this case. \square

3 Some examples

Let us assume $\theta = [x \mapsto (\lambda u. \lambda w. \delta(u) + \delta(w), [], 0), y \mapsto (v, [], 2)]$. We have

$$\begin{array}{c} \overline{\theta, x \Downarrow^0 \lambda u. \lambda w. \delta(u) + \delta(w), []} \quad \overline{\theta, y \Downarrow^2 v, []} \\[10pt] \overline{\frac{[r \mapsto (v, [], 2)], \lambda w. \delta(r) + \delta(w) \Downarrow^0 \lambda w. \delta(r) + \delta(w), [r \mapsto (v, [], 2)]}{\theta, x y \Downarrow^0 \lambda w. \delta(r) + \delta(w), [r \mapsto (v, [], 2)]}} \\[10pt] \overline{\theta, y \Downarrow^2 v, []} \\[10pt] \frac{[r \mapsto (v, [], 2), s \mapsto (v, [], 2)], r \Downarrow^2 v, [r \mapsto (v, [], 2), s \mapsto (v, [], 2)]}{[r \mapsto (v, [], 2), s \mapsto (v, [], 2)], \delta(r) \Downarrow^2 v_r, [r \mapsto (v, [], 2), s \mapsto (v, [], 2)]} \\[10pt] \frac{[r \mapsto (v, [], 2), s \mapsto (v, [], 2)], r \Downarrow^2 v, [r \mapsto (v, [], 2), s \mapsto (v, [], 2)]}{[r \mapsto (v, [], 2), s \mapsto (v, [], 2)], \delta(s) \Downarrow^2 v_s, [r \mapsto (v, [], 2), s \mapsto (v, [], 2)]} \\[10pt] \frac{v_r + v_s = v'}{[r \mapsto (v, [], 2), s \mapsto (v, [], 2)], \delta(r) + \delta(s) \Downarrow^4 v', [r \mapsto (v, [], 2), s \mapsto (v, [], 2)]} \\[10pt] \overline{\theta, x y y \Downarrow^4 v', [r \mapsto (v, [], 2), s \mapsto (v, [], 2)]} \end{array}$$