LINEAR LOGIC Homework #2

two separate Lemmas for Substitution

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Types	$T ::= A \mid B \mid C$
Term	$t ::= x \mid t \mid \lambda x.t$

Figure 1: syntax

$\boxed{\Gamma \vdash t : A}$

$$\frac{}{x : A \vdash x : A}\ \text{AXIOM} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}\ \text{ABS}$$

$$\frac{\Gamma \cap \Delta = \emptyset \qquad \Gamma \vdash t : A \to B \qquad \Delta \vdash u : A}{\Gamma, \Delta \vdash t\ u : B}\ \text{APP} \qquad \frac{\Gamma, x : A, y : B \vdash u : C}{\Gamma, y : B, x : A \vdash u : C}\ \text{EXCHANGE}$$

$$\frac{\Gamma \vdash t : B}{\Gamma, x :!A \vdash t : B}\ \text{WEAKING} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma, x :!A \vdash t : B}\ \text{DERELICTION}$$

$$\frac{\Gamma, x :!A, y :!A \vdash t : C}{\Gamma, x :!A \vdash t[x/y] : C}\ \text{CONTRACT} \qquad \frac{!\Gamma \vdash t : A}{!\Gamma \vdash t : !A}\ \text{PROMOTION}$$

Figure 2: Typing rules

Lemma 1 (Preservation under Substitution)**.**

1. *If* $\Gamma \vdash t : A$ *and* $\Delta, x : A \vdash u : B$ *then* $\Gamma, \Delta \vdash u[t/x] : B$.
2. *If* $!\Gamma \vdash t : !A$ *and* $\Delta, x :!A \vdash u : B$ *then* $!\Gamma, \Delta \vdash u[t/x] : B$.

Proof. Lemma 1.1 is proved in lecture.
Lemma 1.2 is proved by induction on the typing derivation of the second premise $\Delta, x :!A \vdash u : B$.
Assume we know: $!\Gamma \vdash t : !A$

Case
SubCase1:

$$\frac{\Delta \vdash u : B}{\Delta, x :!A \vdash u : B}\ \text{WEAKING}$$

TS: $!\Gamma, \Delta \vdash u[t/x] : B$
STS: $!\Gamma, \Delta \vdash u : B$, because x doesn't show up in u.
By applying WEAKING rule on hypothesis, we get: $\Delta, !\Gamma \vdash u : B$.
By applying WEAKING rule, we get: $!\Gamma, \Delta \vdash u : B$.
This case is proved.
SubCase2:

$$\frac{\Delta, x :!A \vdash u : B}{\Delta, x :!A, \hat{x} :!A \vdash u : B}\ \text{WEAKING}$$

TS: $!\Gamma, \Delta, \hat{x} :!A \vdash u[t/x] : B$
By IH, we get: $!\Gamma, \Delta \vdash u[t/x] : B$
By apply WEAKING rule, we get: $!\Gamma, \Delta, \hat{x} :!A \vdash u[t/x] : B$
This case is proved.

Case

SubCase1:

$$\frac{\Delta, x : A \vdash u : B}{\Delta, x :!A \vdash u : B} \text{ DERELICTION}$$

TS: $!\Gamma, \Delta \vdash u[t/x] : B$.
By IH of Lemma 1.1, we get: $\Gamma, \Delta \vdash u[t/x] : B$.
By applying the DERELICTION rule, we get: $!\Gamma, \Delta \vdash u[t/x] : B$.
This case is proved.
SubCase2:

$$\frac{\Delta, x :!A, \hat{x} : A \vdash u : B}{\Delta, x :!A, \hat{x} :!A \vdash u : B} \text{ DERELICTION}$$

TS: $!\Gamma, \Delta, \hat{x} :!A \vdash u[t/x] : B$.
By IH, we get: $!\Gamma, \Delta, \hat{x} : A \vdash u[t/x] : B$.
By applying DERELICTION rule, we get: $!\Gamma, \Delta, \hat{x} :!A \vdash u[t/x] : B$.
This case is proved.

Case
SubCase1:

$$\frac{\Delta, x :!A, y :!A \vdash v : B}{\Delta, x :!A \vdash v[x/y] : B} \text{ CONTRACT}$$

TS: $!\Gamma, \Delta \vdash u[t/x] : B$
STS: $!\Gamma, \Delta \vdash v[x/y][t/x] : B$
STS: $!\Gamma, \Delta \vdash v[t/y][t/x] : B$
By IH on $\Delta, x :!A, y :!A \vdash v : B$, we get:
$!\Gamma, \Delta, y :!A \vdash v[t/x] : B$
[[Rename all the variables in premise 1: $!\Gamma \vdash t : !A$, we get:
$!\Gamma' \vdash t[\Gamma'/\Gamma] : !A$
Apply Induction Hypothesis again on $!\Gamma, \Delta, y :!A \vdash v[t/x] : B$ with premise $!\Gamma' \vdash t[!\Gamma'/!\Gamma] : !A$, we get:
$!\Gamma', !\Gamma, \Delta \vdash v[t/x][t[\Gamma'/\Gamma]/y] : B$
Apply CONTRACT rule on every pair of same variables in Γ', Γ with different name, we get:
$!\Gamma, \Delta \vdash v[t/x][t[\Gamma'/\Gamma]/y][\Gamma/\Gamma'] : B$**.**
Because $v[t/x][t[\Gamma'/\Gamma]/y][\Gamma/\Gamma'] = v[t/x][t/y] = v[t/y][t/x]$**.]]**
this case is proved.
SubCase2:

$$\frac{\Delta, x :!A, \hat{x} :!A, y :!A \vdash v : B}{\Delta, x :!A, \hat{x} :!A \vdash v[\hat{x}/y] : B} \text{ CONTRACT}$$

where $v[\hat{x}/y] = u$.
TS: $!\Gamma, \Delta, \hat{x} :!A \vdash u[t/x] : B$
STS: $!\Gamma, \Delta, \hat{x} :!A \vdash v[\hat{x}/y][t/x] : B$
We know $\Delta, \hat{x} :!A, y :!A, x :!A \vdash v : B$ by applying EXCHANGE rule on $\Delta, x :!A, \hat{x} :!A, y :!A \vdash v : B$.
By IH on $\Delta, \hat{x} :!A, y :!A, x :!A \vdash v : B$, we get:
$!\Gamma, \Delta, \hat{x} :!A, y :!A \vdash v[t/x] : B$
By applying CONTRACT rule on $!\Gamma, \Delta, \hat{x} :!A, y :!A \vdash v[t/x] : B$, we get:
$!\Gamma, \Delta, \hat{x} :!A \vdash v[t/x][\hat{x}/y] : B$.
Because $v[t/x][\hat{x}/y] \equiv v[\hat{x}/y][t/x]$, this case is proved.

Case

$$\frac{!\Delta, x :!A \vdash u : C}{!\Delta, x :!A \vdash u : !C} \text{ PROMOTION}$$

where $B =!C$.
TS: $!\Gamma, !\Delta \vdash u[t/x] : B$.
STS: $!\Gamma, !\Delta \vdash u[t/x] : !C$

By IH, we get: $!\Gamma, !\Delta \vdash u[t/x] : C$
By applying the PROMOTION rule, we get: $!\Gamma, !\Delta \vdash u[t/x] : !C$.
This case is proved. $\square$

Theorem 0.1 (Preservation)**.** *If* $\Gamma \vdash t : A$ *and* $t \to t'$ *then* $\vdash t' : A$

Proof. of Theorem 0.1 by induction on the derivation of first premise: $\Gamma \vdash t : A$:

$\square$