Proving the Symmetry Condition on Type Systems

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The candidate type system μ is constructed using Tarski's Theorem as the least pre-fixed point of an operator $\textsc{Types}(-)$ on candidate type systems. The c.t.s. operator $\textsc{Types}$ is defined by several clauses, one of which is given by the c.t.s. operator $\textsc{Funs}$:

$$\textsc{Types}(\tau) \triangleq \dots \cup \textsc{Funs}(\tau) \cup \dots,$$

where

$$\textsc{Funs}(\tau)(a{:}A_1 \to A_2, a{:}A_1' \to A_2', \phi) \tag{1}$$
$$\textit{iff} \tag{2}$$
$$\phi \triangleq a{:}\phi_1 \to \phi_2 \quad \textit{and} \tag{3}$$
$$A_1 \sim A_1' \downarrow \phi_1 \in \tau \quad \textit{and} \tag{4}$$
$$\forall M, M' \ \textit{if}\ \phi_1(M, M')\ \textit{then}\ A_2[M/a] \sim A_2'[M'/a] \downarrow \phi_2[M, M'] \in \tau \tag{5}$$

and

$$(a{:}\phi_1 \to \phi_2)(\lambda a.N, \lambda a.N') \tag{6}$$
$$\textit{iff} \tag{7}$$
$$\forall M, M' \ \textit{if}\ \phi_1(M, M')\ \textit{then}\ \phi_2[M, M](N[M/a], N'[M'/a]). \tag{8}$$

A candidate type system τ is a type system iff it satisfies the following conditions:

1. Unicity: if $\tau(A_0, A_0', \phi)$ and $\tau(A_0, A_0', \phi')$, then $\phi' = \phi$.
2. Symmetry: if $\tau(A_0, A_0', \phi)$, then $\tau(A_0', A_0, \phi)$;
3. Transitivity: if $\tau(A_0, A_0', \phi)$ and $\tau(A_0', A_0'', \phi)$, then $\tau(A_0, A_0'', \phi)$.
4. PER Valuation: if $\tau(A_0, A_0', \phi)$, then ϕ is symmetric and transitive.

The goal is to prove that μ is a type system. The unicity condition may be proved outright by proving that $\textsc{Types}(\Phi) \subseteq \Phi$, where $\Phi(A_0, A_0', \phi)$ iff $\mu(A_0, A_0', \phi)$ and if $\mu(A_0, A_0', \phi')$, then $\phi' = \phi$. For then $\mu \subseteq \Phi$, which means that μ enjoys unicity. The proof was sketched in class, and presents no further difficulties.

It is necessary to prove symmetry, transitivity, and PER valuation of μ simultaneously. To do so, define $\Phi(A_0, A_0', \phi)$ iff

1. $\mu(A_0, A'_0, \phi)$;
2. $\mu(A'_0, A_0, \phi)$.
3. if $\mu(B_0, A_0, \phi')$, then $\mu(B_0, A'_0, \phi')$ and $\phi' = \phi$.
4. if $\mu(A'_0, B'_0, \phi')$, then $\mu(A_0, B'_0, \phi')$ and $\phi' = \phi$.
5. ϕ is symmetric and transitive.

If $\text{TYPES}(\Phi) \subseteq \Phi$, then $\mu \subseteq \Phi$, and so μ is a type system. Note that by definition $\Phi \subseteq \mu$.

The proof breaks into cases, one for each operator defining TYPES. In particular it is necessary to show that $\text{FUNS}(\Phi) \subseteq \Phi$. Suppose that $\text{FUNS}(\Phi)(A_0, A'_0, \phi)$, Then by definition $A_0 = a{:}A_1 \rightarrow A_2$, $A'_0 = a{:}A'_1 \rightarrow A'_2$, and $\phi = a{:}\phi_1 \rightarrow \phi_2[a]$, where

1. $A_1 \sim A'_1 \downarrow \phi_1 \in \Phi$;
2. if $M_1 \sim M'_1 \in \phi_1$, then $A_2[M_1/a] \sim A'_2[M'_1/a] \downarrow \phi_2[M_1, M'_1] \in \Phi$.

These conditions imply that ϕ_1 is a PER, and that $A'_2[M'_1/a] \sim A_2[M_1/a] \downarrow \phi_2[M_1, M'_1] \in \mu$, with $\phi_2[M_1, M'_1]$ being a PER, whenever $M_1 \sim M'_1 \in \phi_1$.

The overall obligation is to show that $\Phi(A_0, A'_0, \phi)$, which includes the particular requirement that $\mu(A'_0, A_0, \phi)$. Because $\text{FUNS}(\mu) \subseteq \mu$, it suffices for this case to show these two conditions:

1. $A'_1 \sim A_1 \downarrow \phi_1 \in \mu$, and
2. if $M_1 \sim M'_1 \in \phi_1$, then $A'_2[M_1/a] \sim A_2[M'_1/a] \downarrow \phi_2[M_1, M'_1] \in \mu$.

The first of these follows directly from the assumption $A_1 \sim A'_1 \downarrow \phi_1 \in \Phi$. For the second, suppose that $M_1 \sim M'_1 \in \phi_1$, with the intent to show that

$$A'_2[M_1/a] \sim A_2[M'_1/a] \downarrow \phi_2[M_1, M'_1] \in \mu.$$

Because ϕ_1 is symmetric, $M'_1 \sim M_1 \in \phi_1$, and so by the second assumption

$$A_2[M'_1/a] \sim A'_2[M_1/a] \downarrow \phi_2[M'_1, M_1] \in \Phi$$

. Therefore $\phi_2[M'_1, M_1]$ is a PER, and

$$A'_2[M_1/a] \sim A_2[M'_1/a] \downarrow \phi_2[M'_1, M_1] \in \mu$$

. To complete the proof it is enough to show that $\phi_2[M'_1, M_1] = \phi_2[M_1, M'_1]$.

This equation is proved in two steps by showing that each side is equal to $\phi_2[M'_1, M'_1]$.

1. By transitivity and symmetry of ϕ_1, it follows that $M'_1 \sim M'_1 \in \phi_1$, and hence by the second assumption

   $$A_2[M'_1/a] \sim A'_2[M'_1/a] \downarrow \phi_2(M'_1, M'_1) \in \Phi,$$

   and so

   $$A'_2[M'_1/a] \sim A_2[M'_1/a] \downarrow \phi_2(M'_1, M'_1) \in \mu.$$

   Moreover, by the second assumption,

   $$A_2[M'_1/a] \sim A'_2[M_1/a] \downarrow \phi_2[M'_1, M_1] \in \Phi \subseteq \mu.$$

   Therefore by right transitivity

   $$A'_2[M'_1/a] \sim A'_2[M_1/a] \downarrow \phi_2[M'_1, M'_1] \textit{ and } \phi_2(M'_1, M'_1) = \phi_2(M'_1, M_1).$$

2. As in the previous case

$$A_2'[M_1'/a] \sim A_2[M_1'/a] \downarrow \phi_2[M_1', M_1'] \in \mu,$$

and

$$A_2[M_1/a] \sim A_2[M_1'/a] \downarrow \phi_2[M_1, M_1'] \in \Phi \subseteq \mu.$$

By left transitivity

$$A_2[M_1/a] \sim A_2[M_1'/a] \downarrow \phi_2[M_1', M_1'] \in \mu \textit{ and } \phi_2[M_1', M_1] = \phi_2[M_1', M_1'].$$

Thus $\phi_2[M_1, M_1'] = \phi_2[M_1, M_1']$, as desired.

There is of course much more to the proof: the remaining conditions in Φ have to be verified for the FUNS case, and all of the other cases defining TYPES must be considered as well. However, the same devices used here apply in all of these cases as well to complete the proof that μ is, in fact, a type system.

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References

Carlo Angiuli. *Computational Semantics of Cartesian Cubical Type Theory*. PhD thesis, Carnegie Mellon University, Pittsburgh, PA 15217, May 2019. (Expected.).