Arcsine Law and Time of the Maximum and the Minimum of Stochastic Processes

Haoran Yang, Jiaxi Zhao(PKU)

25th September, 2020

Origin: Lévy's arcsine law

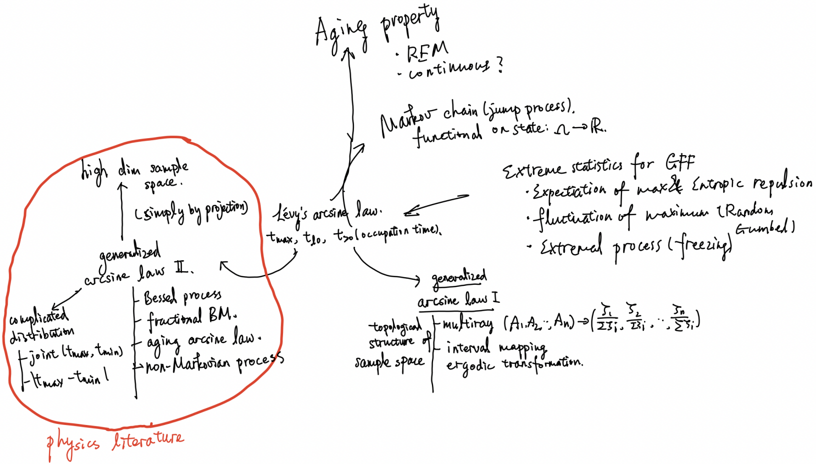


Figure: History

Lévy's arcsine law

Denote X_t as standard Brownian motion on $\mathbb{R}$. Define three random variables as follows:

$$
\begin{aligned}
t_{>0} &:= \frac{1}{T}\int_0^T \mathbf{1}_{[0,\infty)}(X_t)\,\mathrm{d}t,\\
t_{\max} &:= \frac{1}{T}\arg\max_{t\in[0,T]} X_t,\\
t_{l-0} &:= \frac{1}{T}\max_{t\in[0,T]}\left\{t \middle| X_t = 0\right\}.
\end{aligned}
\tag{1}
$$

Lévy's arcsine law

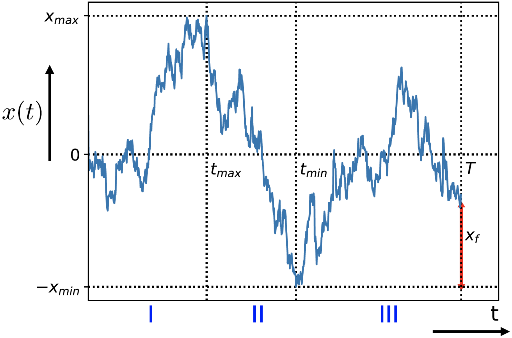


Figure: Illustration of three r.v.s.

Lévy's arcsine law

Theorem (Lévy's arcsine law[1], 1940)

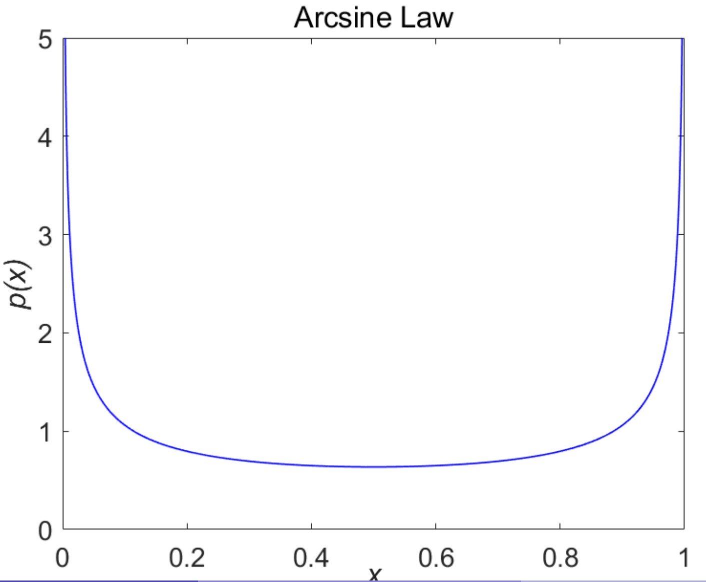
Three r.v.s defined above obey the same law as follows:

$$p(x) = \frac{1}{\pi\sqrt{x(1-x)}}, \quad x \in [0,1]. \tag{2}$$

[1] P. Lévy. Sur certains processus stochastiques homogènes. Compositio mathematica, 7:283–339, 1940.

Lévy's arcsine law

law.png



Arcsine law for Markov chain: Combinatorial approach

Key insight: **Viewing Brownian motion as a sum of an i.i.d. sequence!**

A comparison between Brownian motion on $\mathbb{R}$ and general Markov chain:

1. Brownian motion on $\mathbb{R}$: order relation, 0, positivity & negativity, maximum & minimum.
2. Markov chain on Ω: discrete state space without order relation.

Solution: Consider functional on Markov chain $f : \Omega \to \mathbb{R}$ and use induced order relation.

Combinatorial approach

Theorem (Arcsine law for i.i.d. sequence[2], 1956)

Let $X_1, \cdots, X_n$ be i.i.d. r.v.s. Denote $S_k = \sum_{i=1}^{k} X_i$ as their partial sum, $a_i = P(S_i > 0)$, and $r_N = \frac{\#\left\{i \middle| S_i > 0, 1 \leq i \leq N\right\}}{N}$. Suppose $\frac{\sum_{i=1}^{n} a_i}{n} \to \alpha$, then

$$P(r_N \leq x) \to F_\alpha(x), \tag{3}$$

where $F_\alpha(x)$ is the generalized arcsine law given by

$$F_\alpha(x) = \begin{cases} \frac{\sin \pi\alpha}{\pi} \int_0^x s^{\alpha-1}(1-s)^{\alpha-1}\,\mathrm{d}s, & 0 \leqslant x \leqslant 1, \\ 0, & x < 0, \\ 1, & x > 1. \end{cases} \tag{4}$$

Remark

Taking $\alpha = \frac{1}{2}$, we obtain the ordinary arcsine law of Lévy Thm(1).

[2] F. Spitzer. A combinatorial lemma and its application to probability theory. Transactions of the American Mathematical Society, 82(2):323–339, 1956.

Combinatorial approach

Consider an aperiodic, positive recurrent irreducible discrete Markov chain $\left\{X_n, \Omega, P \in \mathbb{R}^{|\Omega|\times|\Omega|}, \pi \in \mathbb{R}^{|\Omega|}\right\}$. $f : \Omega \to \mathbb{R}$ is a functional on this chain.

Suppose $X_0 = i \in \Omega$, denote the n-th time this chain comes back to state i by τ_i, so that $\tau_1 = 0$. And define $\eta_\nu = \tau_{\nu+1} - \tau_\nu$. One has $\frac{1}{\pi_i} = \mathbb{E}\eta_\nu,\ \forall\nu$. Introduce a r.v. on each time interval as

$$Y_\nu = \sum_{s=\tau_\nu}^{\tau_{\nu+1}-1} f\left(X_s\right), \quad \nu = 1, 2, \cdots . \tag{5}$$

Combinatorial approach

Then if $\sum_{i\in\Omega} \pi_i f(i) < \infty$, we know that $M = \pi_i \mathbb{E} Y_\nu$ is finite. Now let $\bar{f} = f - M$ and $Z_\nu = Y_\nu - \rho_\nu M$. As before, $Z_\nu, Y_\nu, \nu = 1, 2, \cdots$ are a sequence of i.i.d. r.v.s. Write $\hat{S}_\nu = \sum_{s=1}^{\nu} Z_s$. Finally put $\bar{S}_n = \sum_{s=0}^{n} \bar{f}(X_s)$. And define the r.v. indicating the ratio of positive numbers in $\bar{S}_n$, i.e.

$$r_N = \frac{\#\left\{\nu = 0, 1, \cdots, N \middle| \bar{S}_\nu > 0\right\}}{N}. \tag{6}$$

Suppose the limit exists $\lim_{n\to\infty} \frac{1}{n} \sum_{\nu=1}^{n} P\left(\bar{S}_\nu > 0\right) = \alpha$.

Theorem (Arcsine law for Markov chain[3], 1963)

With above definition, we have

$$P(r_N \leq x) \to F_\alpha(x). \tag{7}$$

[3] D. A. Freedman. An arcsine law for markov chains.Proceedings of the American Mathematical Society,14(4):680–684, 1963.

Arcsine law for Markov chain: Law for thermodynamics current

Recently, Barato et al. derive an arcsine law for thermodynamic currents, which has relation with thermodynamics uncertainty relation.

Specifically, define integrated fluctuating current:

$$J_N \simeq \sum_{i,j} d_{ij}\mathcal{N}_{ij}. \tag{8}$$

with $\mathcal{N}_{ij} \simeq \sum_{n=1}^{N} \delta_{i,i_{n-1}}\delta_{j,i_n}$ where i_n refers to the state where sample path arrive at time n, and d_{ij}s are supposed to be bounded constant, δ is Kronecker symbol. Define random variable $\mathcal{R}_N \simeq \frac{1}{N}\sum_{n=1}^{N} \theta(J_n - \mathbb{E}(J_n))$ with $\theta(x) = \mathbf{1}_{[0,\infty)}$ the characteristic function.

Law for thermodynamics current

Theorem (Arcsine law for Markov chain[4], 2018)

With above definition, we have

$$P(\mathcal{R}_N \leq x) \to \frac{1}{\pi\sqrt{x\,(1-x)}}, \quad x \in [0,1]\,. \tag{9}$$

Proof idea.

To prove this result, we introduce an auxiliary Markov chain Z_n in the following way: the state space of this chain is $\Omega_Z = \{z_{ij} P_{ij} > 0\}$, and the transition probabilities of this chain is $p_{z_{ij} z_{kl}} = \delta_{jk} P_{jl}$. Then by elementary Markov chain theory we have that Z_n is also irreducible, aperiodic and positive recurrent with invariant distribution: $\pi_{z_{ij}} = \pi_i P_{ij}$.

[4] A. C. Barato, É. Roldán, I. A. Martínez, and S. Pigolotti. Arcsine laws in stochastic thermodynamics.Physical review letters, 121(9):090601, 2018.

Law for thermodynamics current

(Continue).

Thus it's trivial that $\mathcal{N}_{ij} = \sum_{n=1}^{N} \delta_{z_{ij}, z_{i_{n-1} i_n}} = \sum_{n=1}^{N} \mathbf{1}_{ij}(Z_n)$. Therefore, denote a functional on auxiliary chain Z_n as $f : \Omega_Z \to \mathbb{R}, f(z_{ij}) = d_{ij}$, the current J_N can be rewritten as:

$$J_N = \sum_{z_{ij} \in \Omega_Z} f(z_{ij}) \mathcal{N}_{ij} = \sum_{n=1}^{N} f(Z_n). \tag{10}$$

And the problem is translated to the one already proved by Freedman. The only ingredient $\lim_{n\to\infty} P(S_n > 0) = \frac{1}{2}$ to guarantee the classical arcsine law is a corolary of the central limit theorem for Markov chain[5]. □

[5] G. L. Jones et al. On the markov chain central limit theorem. Probability surveys, 1:299–320, 2004.

Law for thermodynamics current

Remark (Possible connection with statistical thermodynamics)

Entropy production rate, an essential quantity [6] *in statistical thermodynamics, falls in this framework. In its simplest form in Markov chain, it reads* $\sigma = \sum_{i,j} \mathcal{N}_{ij} \log \frac{W_{ij}}{W_{ji}}$. *This kind of arcsine law may bridge a connection to the thermodynamics uncertainty relation*[7]*, which usually take the form*

$$\frac{\mathrm{Var}\,(J)}{\langle J \rangle^2} \geq \frac{2}{T\sigma}, \tag{11}$$

where T *is the length of time interval and* σ *the entropy production rate.*

[7] G. E. Crooks. Entropy production fluctuation theorem and the nonequilibrium work relation for free energy differences. Phys. Rev. E, 60:2721–2726, Sep 1999.

[7] T. R. Gingrich, J. M. Horowitz, N. Perunov, and J. L. England. Dissipation bounds all steady-state current fluctuations. Physical review letters, 116(12):120601, 2016.

Arcsine law for Markov chain: Jump process

The generalization to jump process is direct. Consider an aperiodic, positive recurrent, regular Markov jump process $\{X_t, \Omega, Q \in \mathbb{R}^{|\Omega|\times|\Omega|}, \pi \in \mathbb{R}^{|\Omega|}\}$. Specifically, one can define integrated fluctuating current:

$$J_T \simeq \sum_{i,j} d_{ij}\mathcal{N}_{ij}, \tag{12}$$

up to a time bound T and consider all the transition that happen before T. Define random variable $\mathcal{R}_T \simeq \frac{1}{T}\int_0^T \theta(X_t - E(X_t))dt$ with $\theta\left(x\right) = \mathbf{1}_{[0,\infty)}$ the characteristic function. It is simple to show this integral is well-defined even in Riemannian sense.

Arcsine law for Jump process

Theorem (Arcsine law for jump processes)

With above definition, we have

$$P(\mathcal{R}_N \leq x) \to \frac{1}{\pi\sqrt{x\,(1-x)}}, \quad x \in [0,1]\,. \tag{13}$$

Proof idea.

1. Construct auxiliary chain.
2. Use strong Markov property to translate to conclusion on i.i.d. sequence.
3. Use law of large number to deal with the details in time duration of each travel.

Arcsine law for Markov chain: Lamperti's Topological approach

Viewing the classical arcsine law as law of occupation time, i.e. time spend in a given set.

The topological property is the essence of distribution of occupation time. One can consider Markov chain with similar topological structure, i.e. $\Omega = A \sqcup B \sqcup \{\sigma\}$. The chain can not transfer from $A(B)$ to $B(A)$ without passing $\{\sigma\}$. Topologically speaking, Ω is path-connected while $\Omega \backslash \{\sigma\}$ is not.

Setting of Lamperti's Topological approach

Now, consider a Markov chain $\{X_n, X_n \in \Omega\}$ on this state space with $X_0 = \sigma$. We assume that the state σ is a recurrent state, with f_n denoting the probability of recurrence time of σ is n. Denote the generating function as $F(x) = \sum_{i=1}^{\infty} f_i x^i$. We denote the occupation time of A up to time n as T_n^A, with the convention that occupation of σ is counted or not according to whether the last other state occupied is in A or not.

Lamperti's Topological approach

Theorem (Lamperti's generalized arcsine law[8], 1958)

Let X_n be the process defined above. Then

$$\lim_{n\to\infty} P\left(\frac{T_n^A}{n} \le t\right) = G_{\alpha,\delta}(t), \tag{14}$$

where $\lim_{n\to\infty} \mathbb{E}\frac{T_n^A}{n} = \delta, \lim_{x\to 1^-} \frac{(1-x)F'(x)}{1-F(x)} = \alpha$ *and the p.d.f.* $G'_\alpha(x)$ *is the generalized arcsine law given by*

$$G'_{\alpha,\delta}(x) = \frac{a\sin\pi\alpha}{\pi}\frac{x^\alpha(1-x)^{\alpha-1} + x^{\alpha-1}(1-x)^\alpha}{a^2x^{2\alpha} + 2ax^\alpha(1-x)^\alpha\cos\pi\alpha + (1-x)^{2\alpha}}, \tag{15}$$

where $a = \frac{1-\delta}{\delta}$.

[8] J. Lamperti. An occupation time theorem for a class of stochastic processes. Transactions of the American Mathematical Society, 88(2): 380–387, 1958.

Lamperti's Topological approach

Remark

When both the parameters α, δ are given by $\frac{1}{2}$ (hence $a = 1$), the generalized arcsine law reduces to classical arcsine law as follows

$$\begin{aligned} G'_{\frac{1}{2},\frac{1}{2}}(x) &= \frac{1}{\pi} \frac{\sqrt{\frac{x}{1-x}} + sqrt\frac{1-x}{x}}{x + 1 - x} \\ &= \frac{1}{\pi} \left(\sqrt{\frac{x}{1-x}} + sqrt\frac{1-x}{x} \right) \\ &= \frac{1}{\pi\sqrt{x(1-x)}}. \end{aligned}$$

Lamperti's Topological approach

Definition 2.5 (multidimensional generalized arcsine distributions [2], [13]). For $\alpha \in [0,1]$ and $\beta = (\beta_1, \ldots, \beta_d) \in [0,1]^d$ with $\sum_{i=1}^d \beta_i = 1$, we write $\zeta_{\alpha,\beta}$ for a $[0,1]^d$-valued random variable whose distribution is characterized as follows:

(1) if $0 < \alpha < 1$, the $\zeta_{\alpha,\beta}$ is equal in distribution to

$$\left(\frac{\xi_1}{\sum_{i=1}^d \xi_i}, \ldots, \frac{\xi_d}{\sum_{i=1}^d \xi_i} \right), \tag{2.12}$$

where $\xi_1, \ldots, \xi_d$ denote $[0, \infty)$-valued independent random variables with the one-sided α-stable distributions characterized by

$$\mathbb{E}\left[\exp(-\lambda \xi_i) \right] = \exp(-\beta_i \lambda^\alpha), \quad \lambda > 0, \ i = 1, \ldots, d. \tag{2.13}$$

(2) if $\alpha = 1$, the $\zeta_{1,\beta}$ is equal a.s. to the constant β.

(3) if $\alpha = 0$, the distribution of $\zeta_{0,\beta}$ is $\sum_{i=1}^d \beta_i \delta_{e^{(i)}}$ with $e^{(i)} = (1_{\{i=j\}})_{j=1}^d \in [0,1]^d$ for $i = 1, \ldots, d$.

We call $\zeta_{\alpha,\beta}$ *trivial* if $\beta = e^{(i)}$ for some i. In this case we have $\zeta_{\alpha,e^{(i)}} = e^{(i)}$, a.s., whatever α is.

Figure: Multidimensional generalized arcsine law

Lamperti's Topological approach

Assumption 2.1. Let $d \geq 2$ be a positive integer. Let X be decomposed into $X = Y + \sum_{i=1}^{d} A_i$ for $Y \in \mathcal{A}$ with $\mu(Y) \in (0, \infty)$ and $A_i \in \mathcal{A}$ with $\mu(A_i) = \infty$ $(i = 1, \dots, d)$, such that Y dynamically separates $A_1, \dots, A_d$.

Assumption 2.2. For each $i = 1, \dots, d$, the collection

$$\mathfrak{H}_i := \left\{ \frac{1}{w_i(n)} \sum_{k=0}^{n-1} \widehat{T}^k 1_{Y_k \cap A_i} \ ; \ n \geq 1 \right\} \tag{2.7}$$

is strongly precompact in $L^1(\mu)$.

Assumption 2.3. For some constants $\alpha \in [0, 1]$ and $\beta = (\beta_1, \dots, \beta_d) \in [0, 1]^d$ with $\sum_{i=1}^{d} \beta_i = 1$,

$$\big(w(n)\big)_{n \geq 0} \in \mathcal{R}_{1-\alpha}(\infty), \tag{2.8}$$

$$w_i(n) \sim \beta_i w(n), \quad \text{as } n \to \infty, \ i = 1, \dots, d. \tag{2.9}$$

Figure: Assumptions

Lamperti's Topological approach

Theorem 2.7 (direct limit theorem). *Let T be a CEMPT on $(X, \mathcal{A}, \mu)$ and suppose that Assumptions 2.1, 2.2 and 2.3 hold. Then*

$$S_n/n \overset{\mathcal{L}(\mu)}{\Longrightarrow} \zeta_{\alpha,\beta}, \quad \text{as } n \to \infty.$$

Figure: Direct limit theorem

Theorem 2.8 (inverse limit theorem). *Let T be a CEMPT on $(X, \mathcal{A}, \mu)$ and suppose that Assumptions 2.1 and 2.2 hold. Furthermore, suppose that there exist a probability measure $\nu_0 \ll \mu$ and a $[0,1]^d$-valued random variable ζ such that*

$$S_n/n \overset{\nu_0}{\Longrightarrow} \zeta, \quad \text{as } n \to \infty.$$

Then ζ is equal in distribution to $\zeta_{\alpha,\beta}$ for some $\alpha \in [0,1]$ and $\beta = (\beta_1, \ldots, \beta_d) \in [0,1]^d$ with $\sum_{i=1}^d \beta_i = 1$, and

$$S_n/n \overset{\mathcal{L}(\mu)}{\Longrightarrow} \zeta_{\alpha,\beta}, \quad \text{as } n \to \infty.$$

Moreover, if $\zeta_{\alpha,\beta}$ is not trivial, then the two conditions (2.8) and (2.9) hold.

Figure: Inverse limit theorem

Arcsine law in high time dimension: Extreme statistics in Gaussian free field

$$\text{Random walk on } \mathbb{Z} \Longrightarrow \text{Random walk on } \mathbb{Z}^d,$$

$$\text{Random walk on } \mathbb{Z} \Longrightarrow \text{Gaussian free field on } \mathbb{Z}^d.$$

Firstly, we review some basic definition concerned with GFF.

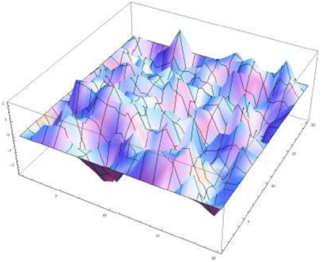


Figure: Gaussian Free Field

[8] S. Sheffield. Gaussian free fields for mathematicians. Probability theory and related fields, 139(3-4):521– 541, 2007.

Extreme statistics in Gaussian free field

A discrete two-dimensional Gaussian free field (GFF) on a $2D$ box $V_N := ([0, N-1] \cap \mathbb{Z})^2$ of side length N with Dirichlet boundary condition, is a mean zero Gaussian process indexed by V_N which takes the value 0 on ∂V_N satisfies either of the following:

1. density function given by

$$f\left((\eta_v)_{v \in V_N}\right) = Ze^{-\frac{\sum_{u \sim v}(\eta_u - \eta_v)^2}{16}}. \tag{16}$$

2. Its covariance matrix is given by $G_{\partial V_N}(x, y)$, where $G_{\partial V_N}(u, v)$ is the Green function defined by

$$G_{\partial V_N}(u, v) = \mathbb{E}_u\left[\sum_{k=0}^{\tau_{\partial V_N}-1} \mathbf{1}_{S_k = v}\right], \quad \forall u \in V_N \backslash \partial V_N, v \in V_N,$$

$$G_{\partial V_N}(u, v) = 0, \quad \forall u \in \partial V_N, v \in V_N.$$

Equivalence is a simple observation that the discrete Green function is the inverse of discrete Laplacian, a while known result in discrete harmonic analysis.

Extreme statistics in Gaussian free field

We consider the maximum of the discrete two-dimensional GFF on V_N, and denote $M_N = \max_{v\in V_N} \eta_v^N, m_N = \mathbb{E}M_N$.

Theorem (Expectation of Maximum[9], 2001)

1. $\lim_{N\to\infty} P\left(M_N \geq 2\sqrt{\frac{2}{\pi}}\log N\right) = 0.$
2. $\forall\epsilon > 0, \forall\delta \in \left[0, \frac{1}{2}\right), \exists c = c(\delta,\epsilon) > 0, s.t.$

$$P\left(M_N \leq \left(2\sqrt{\frac{2}{\pi}} - \epsilon\right)\log N\right) \leq -exp\left(-c(\log N)^2\right), \tag{17}$$

where $V_N^\delta := \{v \in V_N : \mathrm{dist}(v,) V_N \geq \delta N\}$ *for* N *large enough.*

[9] E. Bolthausen, J.-D. Deuschel, G. Giacomin, et al. Entropic repulsion and the maximum of the two- dimensional harmonic. The Annals of Probability, 29(4):1670–1692, 2001.

Extreme statistics in Gaussian free field

Theorem (Expectation of Maximum for GFF[10], 2012)

$$m_N = 2\sqrt{\frac{2}{\pi}} \log_2 N - \frac{3}{4}\sqrt{\frac{2}{\pi}} \log_2 \log_2 N + O\left(1\right). \tag{18}$$

Theorem (Expectation of Maximum for BRW[11], 1978)

$$m_N = 2\sqrt{\log 2} \log_2 N - \frac{3\sqrt{\log 2}}{4} \log_2 \log_2 N + O\left(1\right). \tag{19}$$

[11] M. Bramson and O. Zeitouni. Tightness of the recentered maximum of the two-dimensional discrete gaussian free field. Communications on Pure and Applied Mathematics, 65(1):1–20, 2012.

[11] M. D. Bramson. Maximal displacement of branching brownian motion. Communications on Pure and Applied Mathematics, 31(5):531–581, 1978.

Extreme statistics in Gaussian free field

Theorem (Geometry of the set of near maxima[12], 2014)

There exists an absolute constant $c > 0$,

$$\lim_{r\to\infty}\lim_{N\to\infty} P\left(\exists v, u \in V_N : r \le |v-u| \le \frac{N}{r}\right. \tag{20}$$
$$\left.\eta_u^N, \eta_v^N \ge m_N - c\log\log r\right) = 0.$$

Theorem (Geometry of the set of near maxima[13], 2014)

For $\lambda > 0$, *let* $A_{N,\lambda} = \{v \in V_N : \eta_v \ge m_N - \lambda\}$ *for* $\lambda > 0$. *Then there exist absolute constant* c, C *s.t.*

$$\lim_{\lambda\to\infty}\lim_{N\to\infty} P\left(ce^{c\lambda} \le |A_{N,\lambda}| \le Ce^{C\lambda}\right) = 1. \tag{21}$$

[13] J. Ding, O. Zeitouni, et al. Extreme values for two-dimensional discrete gaussian free field. The Annals of Probability, 42(4):1480–1515, 2014.

Extreme statistics in Gaussian free field

Proof idea.

Completely different techniques comparing to Brownian motion case.
One key ingredient is to compare the extreme statistics of GFF with those of branching random walk (BRW) or modified branching random walk[14](MBRW). They share lots of similarity since they have the proportional covariance.
Comparison of Gaussian processes. □

Extreme statistics in Gaussian free field

Proposition (Similarity of covariance between GFF and BRW[15], 2012)

There exists a constant C s.t. $\forall x, y \in V_N$

$$\begin{aligned} \left| \mathrm{Cov}^N_{GFF}(x,y) - \frac{2\log 2}{\pi}\left(n - \log_2 d^N(x,y)\right)\right| &\le C, \\ \left| \mathrm{Cov}^N_{MBRW}(x,y) - \left(n - \log_2 d^N(x,y)\right)\right| &\le C, \end{aligned} \tag{22}$$

where the distance is given by $d^N(x,y) = \min_{z\sim y}\|x-z\|$.

[15] M. Bramson and O. Zeitouni. Tightness of the recentered maximum of the two-dimensional discrete gaussian free field. Communications on Pure and Applied Mathematics, 65(1):1–20, 2012.

Thanks for listening.