# On multiple SLE systems and their deterministic limits

Jiaxin Zhang

California Institute of Technology

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# Deterministic limits of random curves systems

In this thesis, we study the multiple radial SLE(0) systems as the deterministic limits of multiple radial SLE($\kappa$) systems.

## Structure of multiple radial SLE(0) systems

- Definition and motivation
- Trace of the SLE(0) curves.
- Enumeration and classification
- Hamiltonian of the dynamical system.

Previous work focuses on the multiple chordal SLE system.

# Motivation: scaling limit of interfaces in critical lattice model

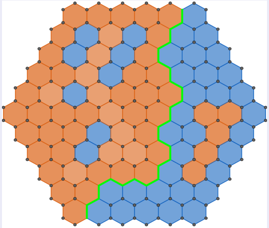


Figure 1: Ising phase interface

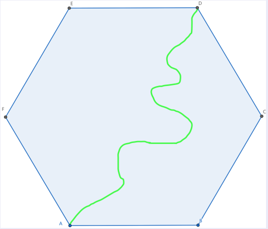


Figure 2: SLE(3) curve

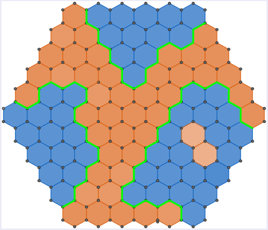


Figure 3: Ising phase interfaces

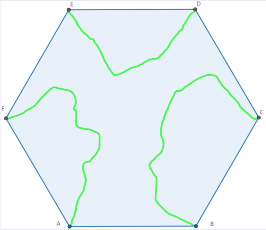


Figure 4: Multiple SLE(3) curves

# History

## Random curves systems and their deterministic limits

- Bauer-Bernard-Kytölä (2007) constructed the martingale observables as correlation functions of conformal fields for multiple chordal SLE($\kappa$) systems.
- Lawler (2008) constructed a conformally invariant measure on the multiple chordal curves.
- Peltola-Wang (2020) proved the deterministic limit of multiple chordal SLE($\kappa$) curves are the real locus of a real rational function.
- Alberts-Byun-Kang-Makarov (2022) studied the evolution of the real rational function for chordal SLE(0) system under the Loewner flows.

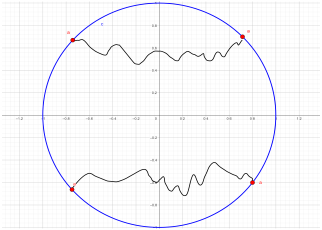


Figure 5: Multiple chordal SLE($\kappa$)

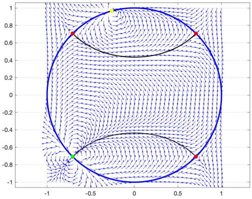


Figure 6: Multiple chordal SLE(0)

# Novelty: Differences between chordal and radial SLE system

- bulk point and monodromy
- new radial link pattern
- SLE(0) traces described by horizontal trajectories of quadratic differentials
- new enumeration problem
- connection to the classical Calegero-Sutherland system

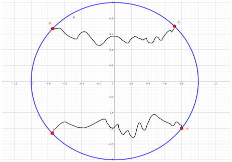


Figure 7: Multiple chordal SLE($\kappa$)

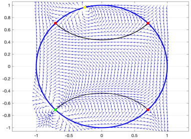


Figure 8: Multiple chordal SLE(0)

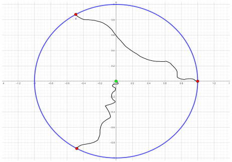


Figure 9: Multiple radial SLE($\kappa$)

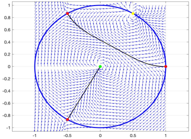


Figure 10: Multiple radial SLE(0)

# Overview

1. Multiple radial SLE(0) as the classical limit of multiple radial SLE($\kappa$) system
2. Quadratic differentials and field integral of motion
3. Critical point of Knizhnik–Zamolodchikov equations and real enumerative algebraic geometry
4. Relation to classical Calogero-Sutherland system

# Part 1: Multiple radial SLE(0) as the deterministic limit of multiple radial SLE($\kappa$) system

# Multiple radial SLE($\kappa$) system

## Multiple radial $SLE(\kappa)$ system

- Given a simply connected domain $\Omega$, growth points $z_1, z_2, \ldots, z_n \in \partial\Omega$ and a bulk point $q \in \Omega$, multiple radial SLE($\kappa$) is a measure $\mathbb{P}_{(\Omega; z_1, z_2, \ldots, z_n, q)}$ on n tuples of continuous non-self-crossing curves $(\eta^1, \eta^2, \ldots, \eta^n)$ in $\Omega$ starting from $z_1, z_2, \ldots, z_n$.
- Conformal invariance (CI), domain Markov properties (DMP), and the marginal law of $\eta^i$ is absolutely continuous with respect to an $SLE_\kappa$ (MARG).

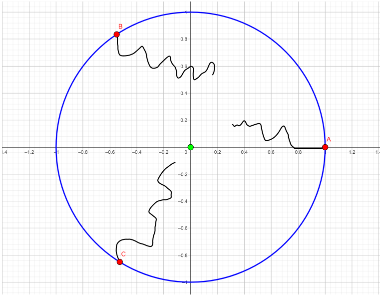



Figure 11: Multiple radial SLE($\kappa$)

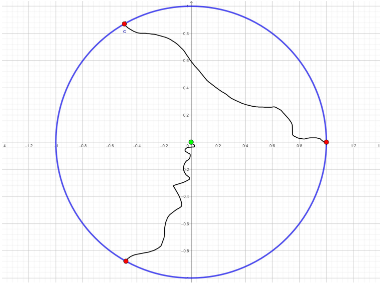


Figure 12: Link pattern

# Characterization of multiple radial SLE($\kappa$) systems

In the unit disk $\Omega = \mathbb{D}$ uniformization, $q = 0$, let $z_j = e^{i\theta_j}$,$j = 1, 2, \ldots, n$.

- By marginal laws, we assume that there exist $C^2$ functions $b_j : \mathfrak{X}^n \to \mathbb{R}$, where the chamber:

$$\mathfrak{X}^n = \{(\theta_1, \theta_2, \ldots, \theta_n) \in \mathbb{R}^n \mid \theta_1 < \theta_2 < \ldots < \theta_n < \theta_1 + 2\pi\} \tag{1}$$

such that the capacity parametrized Loewner driving function $t \mapsto \theta_j(t)$ of $\eta^{(j)}$ satisfies

$$\begin{cases} \mathrm{d}\theta_j(t) = \sqrt{\kappa}\mathrm{d}B_j(t) + b_j\left(\theta_1(t), \theta_2(t), \ldots, \theta_j(t), \ldots, \theta_n(t)\right)\mathrm{d}t \\ \mathrm{d}\theta_k(t) = \cot\left(\left(\theta_k(t) - \theta_j(t)\right)/2\right)\mathrm{d}t, k \neq j \end{cases} \tag{2}$$

where $B^{(j)}$ is one-dimensional standard Brownian motion.

# Characterization of multiple radial SLE($\kappa$)

## Characterization of drift terms

- By domain Markov properties, there exists a positive partition function $\psi(\theta_1, \ldots, \theta_n)$, such that the drift terms $b_j$ are given by:

$$b_j = \kappa \frac{\partial_j \psi}{\psi} \tag{3}$$

  There exists a real null vector constant $h$ such that $\psi$ satisfies

$$\frac{\kappa}{2} \frac{\partial_j^2 \psi}{\psi} + \sum_{\ell \neq j} \left( \cot\left(\frac{\theta_\ell - \theta_j}{2}\right) \frac{\partial_\ell \psi}{\psi} - \frac{(6-\kappa)}{4\kappa} \frac{1}{\sin^2\left(\frac{\theta_\ell - \theta_j}{2}\right)} \right) = h. \tag{4}$$

- By rotation invariance, there exists a real rotation constant $\omega$ such that, for all $\theta \in \mathbb{R}$,

$$\psi(\theta_1 + \theta, \ldots, \theta_n + \theta) = e^{-\omega\theta} \psi(\theta_1, \ldots, \theta_n). \tag{5}$$

## Remark

*Positive solutions to (4) and (5) correspond to multiple radial SLE($\kappa$) systems.*

# Construction of multiple radial SLE($\kappa$) partition function

## Differential

Differential of conformal dimensions $[\lambda, \lambda_*]$ is an assignment of (smooth) functions to each local coordinate chart $\phi : U \to \phi(U) \subset \mathbb{C}$ and for any two overlapping charts $\phi$ and $\widetilde{\phi}$, $h = \widetilde{\phi} \cdot \phi^{-1}$, we have

$$f = (h')^{\lambda} \left(\overline{h'}\right)^{\lambda *} \tilde{f} \circ h, \tag{6}$$

## Coulomb gas correlation

For a divisor $\boldsymbol{\sigma} = \sum_j \sigma_j \cdot z_j$ on the Riemann sphere satisfying the neutrality condition $(NC_b)$, $\int \boldsymbol{\sigma} = 2b$, where $b = \sqrt{\frac{\kappa}{8}} - \sqrt{\frac{2}{\kappa}}$, the Coulomb gas correlation function $C_{(b)}[\boldsymbol{\sigma}]$ is a differential of conformal dimension $\lambda_j = \lambda_b\left(\sigma_j\right) \equiv \frac{(\sigma_j)^2}{2} - \sigma_j b$, whose evaluation is given by

$$C_{(b)}[\boldsymbol{\sigma}] := \prod_{j<k} \left(z_j - z_k\right)^{\sigma_j \sigma_k} \tag{7}$$

# Construction of multiple radial SLE($\kappa$) partition function

## Master function in $\mathbb{D}$

- Growth points: $z_j$ with charge $a = \sqrt{\frac{2}{\kappa}}$.
- Screening Charges: $\xi_k$ with charge $-2a$.
- bulk points: 0 with charge $b - (\frac{n-2m}{2})a$ and $\infty$ with charge $b - (\frac{n-2m}{2})a$

Consider the divisor (Growth + Bulk + Screening Charges), $b = \sqrt{\frac{\kappa}{8}} - \sqrt{\frac{2}{\kappa}}$

$$\boldsymbol{\beta} = (b - (\frac{n-2m}{2})a)\delta_0 + (b - (\frac{n-2m}{2})a)\delta_\infty + \sum_{j=1}^{n} a\delta_{z_j} - \sum_{k=1}^{m} 2a\delta_{\xi_k} \quad (8)$$

The master function (Coulomb gas correlation)

$$\begin{aligned}\Phi_\kappa(\boldsymbol{z}, \boldsymbol{\xi}) =& C_b[\boldsymbol{\beta}] \\ =& \prod_{1\le i<j\le n} (z_i - z_j)^{a^2} \prod_{1\le i<j\le m} (\xi_i - \xi_j)^{4a^2} \prod_{i=1}^{n}\prod_{j=1}^{m} (z_i - \xi_j)^{-2a^2} \\ & \prod_{1\le i\le n} z_i^{ab+(m-\frac{n}{2})a^2} \prod_{1\le j\le m} \xi_j^{-2ab-(2m-n)a^2}\end{aligned} \quad (9)$$

# Screening

- conformal dimension of $\Phi_\kappa(\boldsymbol{z},\boldsymbol{\xi})$ is $\lambda_b(-2a) = 1$ at $\boldsymbol{\xi}$
- closed contours (Pochhammer contours) $\mathcal{C}_1, \ldots, \mathcal{C}_m$ to integrate $\xi_1, \ldots, \xi_m$.

We define the partition function after screening as:

$$\mathcal{Z}_\kappa(\boldsymbol{z}) := \oint_{\mathcal{C}_1} \ldots \oint_{\mathcal{C}_n} \Phi_\kappa(\boldsymbol{z},\boldsymbol{\xi}) d\xi_1 \ldots d\xi_m. \tag{10}$$

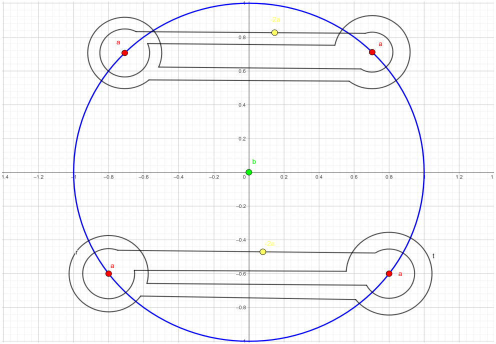


Figure 13: Screening: 4 growth points, 2 screening charges

# Deterministic Limit of Multiple Radial SLE($\kappa$) Systems*

- Heuristically, in the deterministic limit $\kappa \to 0$, we study the asymptotics of the partition function:

$$\lim_{\kappa \to 0} \mathcal{Z}_\kappa(\boldsymbol{z})^\kappa = \lim_{\kappa \to 0} \left( \oint_{\mathcal{C}_1} \cdots \oint_{\mathcal{C}_m} \Phi(\boldsymbol{z}, \boldsymbol{\xi})^{\frac{1}{\kappa}} \, d\boldsymbol{\xi} \right)^\kappa \tag{11}$$

## Master Function $\Phi(\boldsymbol{z}, \boldsymbol{\xi})$ of Multiple Radial SLE(0)

$$\begin{aligned}\Phi(\boldsymbol{z}, \boldsymbol{\xi}) = & \prod_{1 \le i < j \le n} (z_i - z_j)^2 \prod_{1 \le i < j \le m} (\xi_i - \xi_j)^8 \prod_{i=1}^{n} \prod_{j=1}^{m} (z_i - \xi_j)^{-4} \\ & \times \prod_{i=1}^{n} z_i^{2m-n-2} \prod_{j=1}^{m} \xi_j^{4-4m+2n}\end{aligned} \tag{12}$$

# Deterministic limit

By the steepest descent (stationary phase) method, the contour integral

$$\oint_{\mathcal{C}_1} \cdots \oint_{\mathcal{C}_m} \Phi(\boldsymbol{z}, \boldsymbol{\xi})^{\frac{1}{\kappa}} \, d\boldsymbol{\xi}$$

is asymptotically approximated by the value of the integrand at the stationary phase, that is, the critical point of the master function $\Phi(\boldsymbol{z}, \boldsymbol{\xi})$ with respect to $\boldsymbol{\xi}$.

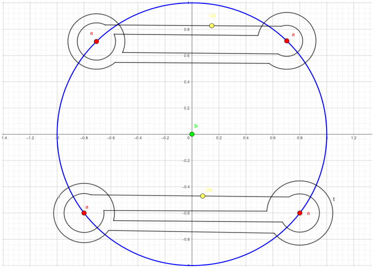


Figure 14: Screening integral

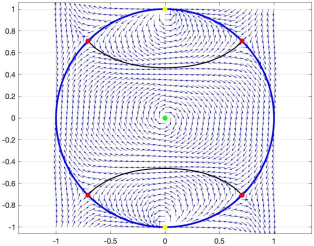


Figure 15: Stationary phase

# Stationary relations

## Stationary relations

$\boldsymbol{\xi}$ and $\boldsymbol{z}$ solve the stationary relations:

$$-\sum_{j=1}^{n} \frac{2}{\xi_k - z_j} + \sum_{l \neq k} \frac{4}{\xi_k - \xi_l} + \frac{n - 2m + 2}{\xi_k} = 0, k = 1, 2, \ldots, n \tag{13}$$

which are equivalent to

$$\frac{\partial \Phi(\boldsymbol{z}, \boldsymbol{\xi})}{\partial \xi_k} = 0, \quad k = 1, 2, \ldots, n \tag{14}$$

# Multiple radial SLE(0) system

## Multiple radial SLE(0) in $\mathbb{D}$

- $\boldsymbol{z} = \{z_1, \ldots, z_n\}$ distinct growth points on the unit circle,
- $\boldsymbol{\xi} = \{\xi_1, \ldots, \xi_m\}$ involution symmetric screening charges and solve the stationary relations.
- Partition function is given by

$$\mathcal{Z}(\boldsymbol{z}, \boldsymbol{\xi}) = \prod_{1\le i<j\le n} (z_i - z_j)^2 \prod_{1\le i<j\le m} (\xi_i - \xi_j)^8 \prod_{i=1}^{n}\prod_{j=1}^{m} (z_i - \xi_j)^{-4} \prod_{1\le i\le n} z_i^{2m-n-2} \prod_{1\le j\le n} \xi_j^{4-4m+2n} \tag{15}$$

- In the angular coordinate, $z_j(t) = e^{ix_j(t)}, j = 1, \ldots, n, \xi_k(t) = e^{i\zeta_k(t)}, k = 1, \ldots, m$, the partition function is given by

$$\mathcal{Z}(\boldsymbol{x}, \boldsymbol{\zeta}) := \prod_{1\le i<j\le n} (\sin\frac{x_i - x_j}{2})^2 \prod_{1\le i<j\le m} (\sin\frac{\zeta_i - \zeta_j}{2})^8 \prod_{i=1}^{n}\prod_{j=1}^{m} \left(\sin\frac{x_i - \zeta_j}{2}\right)^{-4} \tag{16}$$

# Differential equations for multiple SLE(0)

## Differential equation for multiple radial SLE(0) in $\mathbb{D}$

For $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$, where each $\nu_i : [0, \infty) \to [0, \infty)$ measurable as the parametrization of capacity. We define the Loewner chain $(g_t)_{t\geq 0}$ with parametrization $\boldsymbol{\nu}(t)$ to be

$$\partial_t g_t(z) = \sum_{j=1}^{n} \nu_j(t) g_t(z) \frac{z_j(t) + g_t(z)}{z_j(t) - g_t(z)}, \quad g_0(z) = z, \tag{17}$$

In the angular coordinate, the driving functions $x_j(t), j = 1, \ldots, n$, evolve as

$$\frac{dx_j(t)}{dt} = \nu_j(t) \frac{\partial \log \mathcal{Z}(\boldsymbol{x}, \boldsymbol{\zeta})}{\partial x_j} + \sum_{k \neq j} \nu_k(t) \cot\left(\frac{x_j - x_k}{2}\right) \tag{18}$$

the screening charges $\zeta_k(t), k = 1, \ldots, m$ evolve as

$$\frac{d\zeta_k(t)}{dt} = \sum_{j \neq k} \nu_j(t) \cot\left(\frac{\zeta_k(t) - x_j(t)}{2}\right) \tag{19}$$

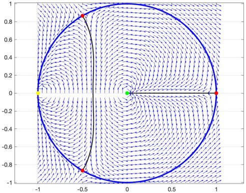



Figure 16: 3g1s(1)

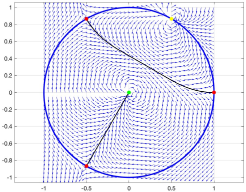



Figure 17: 3g1s(2)

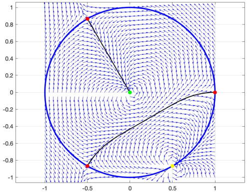



Figure 18: 3g1s(3)

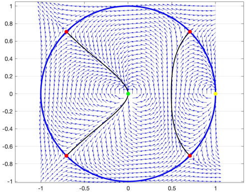



Figure 19: 4g1s(1)

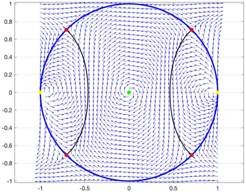



Figure 20: 4g2s(1)

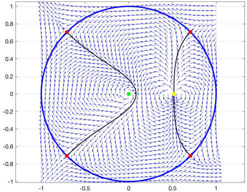



Figure 21: 4g2s(2)

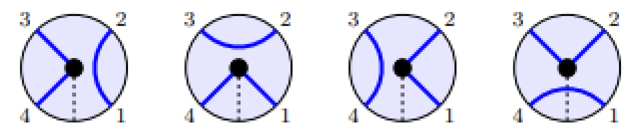
g: growth points (red)
s: screening charges (yellow)
center (green)

# Classification

## Radial link pattern

Given distinct points on the unit circle and one bulk point, a radial link pattern:

- homotopically equivalent class of non-intersecting curves
- connecting pair of boundary points (links/arcs)
- connecting boundary points and the bulk point (rays)




Link patterns with $n$ boundary points and $m$ links are called $(n, m)$-links.
The number of $(n, m)$-links $\#LP(n, m) = \binom{n}{m}$.

## Question

How many multiple radial SLE(0) systems with a given link pattern?

# Part 2:Quadratic differentials and field integral of motion

# Main theorem

## Traces of multiple radial SLE(0)

The traces of multiple radial SLE(0) systems with growth points $\boldsymbol{z} = \{z_1, z_2, \ldots, z_n\}$ are the horizontal trajectories of an equivalence class of **meromorphic quadratic differentials** $\mathcal{Q}(z)$ parametrized by $\boldsymbol{z} = \{z_1, z_2, \ldots, z_n\}$.

More precisely, under the multiple Loewner flows

- SLE Hulls $K_t$ are a subset of $\Gamma(Q(z)dz^2)$
- The quadratic differential is preserved $Q(z) \circ g_t^{-1} \in \mathcal{QD}(\boldsymbol{z}(t))$

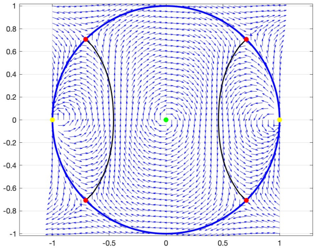



Figure 22: Multiple radial SLE(0) –1

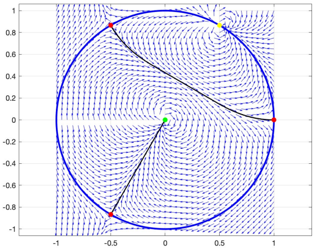



Figure 23: Multiple radial SLE(0)–2

# Trace quadratic differential

## Quadratic differential $\mathcal{QD}(\boldsymbol{z})$

Let $\boldsymbol{z} = \{z_1, z_2, \ldots, z_n\}$ be distinct points on the unit circle, a class of quadratic differentials with prescribed zeros denoted by $\mathcal{QD}(\boldsymbol{z})$ :

(1) involution symmetric: $\overline{Q(z^*)(dz^*)^2} = Q(z)dz^2$, where $z^* = \frac{1}{\bar{z}}$

(2) poles of order $n + 2 - 2m$ at marked points $0$ and $\infty$

(3) zeros of order 2 at $\{z_1, z_2, \ldots, z_n\}$

(4) $\{\xi_1, \ldots, \xi_m\}$ are poles of order 4 and $Res_{\xi_j}(\sqrt{Q}dz) = 0, j = 1, \ldots, m$. (Residue-free)

## Stationary Relations- Residue Free

The following are equivalent:

(1) $\boldsymbol{\xi}$ involution symmetric and $\boldsymbol{z}$ on the unit circle solve the stationary relations.

(2) There exists residue free $Q(z)dz^2 \in \mathcal{QD}(\boldsymbol{z})$ with $\boldsymbol{\xi}$ as poles and $\boldsymbol{z}$ as zeros.

# Field integral of motion

## Field integral of motion

Let $z_1, z_2, \dots, z_n$ be distinct points on the unit circle, for each $z \in \overline{\mathbb{D}}$:

$$N_t(z) = e^{-(m-\frac{n}{2})(\sum_j \int_0^t \nu_j(s)ds)} g_t(z)^{m-\frac{n}{2}-1} g_t'(z) \frac{\prod_{k=1}^n (g_t(z) - z_k(t))}{\prod_{j=1}^m (g_t(z) - \xi_j(t))^2} \quad (20)$$

is an integral of motion for the multiple radial Loewner flow with weight $\nu_j(t)$ .

## Proof.

Verify by direct computations. $\square$

## Motivation

$N_t(z)$ is the deterministic limit of a martingale observable.

# Conformal Field Theory for Multiple Radial SLE($\kappa$)

## Vertex Operator

- a background charge $\boldsymbol{\beta} = \sum_k \beta_k \cdot q_k$ with the neutrality condition $(\mathrm{NC}_b)$
- a divisor $\boldsymbol{\tau}$=$\sum_j \tau_j \cdot z_j$ with the neutrality condition $(\mathrm{NC}_0)$
- $\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$ the OPE exponential of the chirdal bosonic field $i\Phi^+_{\boldsymbol{\beta}}[\boldsymbol{\tau}]$ is defined as

$$\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}] := \frac{C_{(b)}[\boldsymbol{\tau}+\boldsymbol{\beta}]}{C_{(b)}[\boldsymbol{\beta}]} e^{\odot i\Phi^+[\boldsymbol{\tau}]} \tag{21}$$

  where $\Phi^+[\boldsymbol{\tau}] := \sum \tau_j \Phi^+(z_j)$, $\odot$ wick product.
- For $\boldsymbol{\sigma} = \sum \sigma_j \cdot z_j \in (\mathrm{NC}_b)$, $C_{(b)}[\boldsymbol{\sigma}]$ is the Coulomb gas correlation differential.

## References

- (Kang-Makarov 2011) Gaussian free field and conformal field theory
- (Kang-Makarov 2021) Conformal field theory on Riemann sphere and its boundary version for SLE

# Conformal field theory for multiple radial SLE($\kappa$)

## n-leg operator with screening charges

- $\boldsymbol{\beta} = b\delta_0 + b\delta_\infty$ background charge
- Divisor $\boldsymbol{\tau_1} = \sum_{j=1}^n a\delta_{z_j} - \sum_{k=1}^m 2a\delta_{\xi_k} - (\frac{n-2m}{2})a\delta_0 - (\frac{n-2m}{2})a\,\delta_\infty$
  Charge $a$ at growth points $z_j$, charge $-2a$ at screening charges $\xi_k$.
- $\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau}_1]$ the n-leg operator with screening charges $\boldsymbol{\xi}$
- closed contours (Pochhammer contours) $\mathcal{C}_1, \ldots, \mathcal{C}_m$ to integrate $\xi_1, \ldots, \xi_m$.

Let $\mathcal{S}$ be the screening operator, we define the screening conformal field as

$$\mathcal{S}\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}] = \oint_{\mathcal{C}_1} \ldots \oint_{\mathcal{C}_n} \mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}] d\xi_n \ldots d\xi_1 \tag{22}$$

The partition function of the corresponding multiple radial SLE($\kappa$) system is given by:

$$\mathcal{Z}_\kappa(\boldsymbol{z}) := \mathbf{E}\mathcal{S}\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}] = \oint_{\mathcal{C}_1} \ldots \oint_{\mathcal{C}_n} \Phi_\kappa(\boldsymbol{z}, \boldsymbol{\xi}) d\zeta_n \ldots d\zeta_1 \tag{23}$$

## Martingale observable

For any tensor product $X$ of fields in the OPE family $\mathcal{F}_{\boldsymbol{\beta}}$ of $\Phi_{\boldsymbol{\beta}}$,

$$M_t(X) = \frac{\mathbf{E}\mathcal{S}\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}]X}{\mathbf{E}\mathcal{S}\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}]}\|g_t^{-1} \tag{24}$$

is a local martingale, where $g_t(z)$ is the Loewner map for multiple radial SLE($\kappa$) system associated to $\mathcal{Z}_\kappa(\boldsymbol{z}) = \mathbf{E}\mathcal{S}\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}]$

## Corollary

Let the divisor $\boldsymbol{\tau_2} = -\frac{\sigma}{2}\delta_0 - \frac{\sigma}{2}\,\delta_\infty + \sigma\delta_z$ and insert $X = \mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_2}]$

$$M_t(z) = \frac{\mathbf{E}\mathcal{S}\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}]\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_2}]}{\mathbf{E}\mathcal{S}\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}]}\|g_t^{-1} \tag{25}$$

is local martingale, where $g_t(z)$ is the Loewner map for multiple radial SLE($\kappa$) system associated to $\mathcal{Z}_\kappa(\boldsymbol{z}) = \mathbf{E}\mathcal{S}\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}]$

# Deterministic limit of martingale observable

## Deterministic limit (heuristic)

As $\kappa \to 0$, the contour integrals concentrate on the stationary phase.

$$\begin{aligned}\lim_{\kappa\to 0} M_t(z) &= \lim_{\kappa\to 0} \frac{\mathbf{E}\oint_{\mathcal{C}_1}\cdots\oint_{\mathcal{C}_n}\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}]\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_2}]}{\mathbf{E}\oint_{\mathcal{C}_1}\cdots\oint_{\mathcal{C}_n}\mathcal{O}_{\boldsymbol{\beta}}[\boldsymbol{\tau_1}]} \\ &= |g'(0)|^{-(m-\frac{n}{2})}\frac{\prod_{j=1}^m \xi_k}{\sqrt{\prod_{k=1}^n z_k}} z^{m-\frac{n}{2}-1} g'(z)\frac{\prod_{k=1}^n (z-z_k)}{\prod_{j=1}^m (z-\xi_j)^2}\end{aligned} \tag{26}$$

where $\boldsymbol{\xi}$ solve the stationary relations. This is exactly the integral of motion $N_t(z)$ !

# Part 3: Critical points of Knizhnik–Zamolodchikov equations and enumeration problem

# Classification of multiple radial SLE(0) system

## Enumeration and classifiction

The following three objects are equivalent:

- Multiple radial SLE(0) system: screening charges $\boldsymbol{\xi}$ solve the stationary relations
- Residue free quadratic differentials (horizontal trajectories form link pattern)
- Critical points of $\Phi(\boldsymbol{z}, \boldsymbol{\xi})$ (also known as the master function for the Knizhnik–Zamolodchikov equations)

# Enumeration of critical points

## Previous work: chordal SLE(0) (Rational KZ equation)

- (Scherbak-Varchenko 2001) enumerated the number of critical points of master function for rational KZ equation $\Phi_{m,n}(\boldsymbol{\xi})$ for generic $\boldsymbol{z}$ and $d_l \in \mathbb{Z}^+$.

$$\Phi_{m,n}(\xi) = \prod_{i=1}^{m} \prod_{l=1}^{n} (\xi_i - z_l)^{-d_l} \prod_{1 \le i < j \le k} (\xi_i - \xi_j)^2 \tag{27}$$

  when $d_1 = d_2 = \ldots = d_n = 1$, this is exactly the Catalan number $\frac{1}{n+1}\binom{2n}{n}$
- (Mukhin-Tarasov-Varchenko 2005) proved when $\{z_1, z_2, \ldots, z_n\}$ are real, the critical points of $\Phi_{m,n}(\xi)$ are conjugation symmetric.

## On going work: radial SLE(0) (trigonometric KZ equation)

Critical points of the master function for the trigonometric KZ equations.

$$\Phi_{m,n}(\xi) = \prod_{1 \le i < j \le n} (z_i - z_j)^2 \prod_{1 \le i < j \le m} (\xi_i - \xi_j)^8 \prod_{i=1}^{n} \prod_{j=1}^{m} (z_i - \xi_j)^{-4} \prod_{1 \le i \le n} z_i^{2m-n-2} \prod_{1 \le j \le n} \xi_j^{4-4m+2n} \tag{28}$$

# Enumeration of quadratic differentials

## Conjecture ($n$ even)

*For residue-free $Q \in \mathcal{QD}(z)$ and generic $z$, with $m$ poles and $n$ zeros:*

- (Underscreening) If $m \leq \frac{n}{2}$, the horizontal trajectories $\Gamma(Q)$ forms a $(n, m)$-link, each link pattern can be realized uniquely.
- (Overscreening) If $\frac{n+1}{2} \leq m \leq n$, the horizontal trajectories $\Gamma(Q)$ forms a $(n, n-m)$-link, each link pattern can be realized by a continuous family of $Q(z)$ with same horizontal trajectories $\Gamma(Q)$.
- (Upperbound) If $m > n$, there are no such $Q(z) \in \mathcal{QD}(\boldsymbol{z})$.

## Conjecture ($n$ odd)

- (Underscreening) If $m \leq \frac{n}{2}$, the horizontal trajectories $\Gamma(Q)$ forms a $(n, m)$-link, each link pattern can be realized uniquely.
- (Overscreening) If $\frac{n+1}{2} \leq m \leq n$, the horizontal trajectories $\Gamma(Q)$ forms a $(n, n-m)$-link, each link pattern can be realized uniquely.
- (Upperbound) If $m > n$, there are no such $Q \in \mathcal{QD}(\boldsymbol{z})$.

# Part 4: Relation to classical Calogero-Sutherland system

# Hamiltonian dynamics

## Hamiltonian dynamical system

For a Hamiltonian system of $n$-particles, the evolution of the state in the phase space $(\boldsymbol{x}, \boldsymbol{p}) \in \mathbb{R}^n \times \mathbb{R}^n$, where $p_j = \dot{x}_j$ are given by the Hamilton's equations:

$$\dot{x}_j = \frac{\partial H(\boldsymbol{x}, \boldsymbol{p})}{\partial p_j}, \dot{p}_j = -\frac{\partial H(\boldsymbol{x}, \boldsymbol{p})}{\partial x_j}, j = 1, \ldots, n, \tag{29}$$

- Classical Calogero-Moser-Sutherland Hamiltonian,

$$\mathcal{H}_{CS}(\boldsymbol{x}, \boldsymbol{p}) = \sum \frac{p_j^2}{2} + V(\boldsymbol{x}), V(\boldsymbol{x}) = \sum_{j<k} -\frac{1}{4\sin^2(\frac{x_j - x_k}{2})} \tag{30}$$

- Null vector Hamiltonian

$$\mathcal{H}_j(\boldsymbol{x}, \boldsymbol{p}) = \frac{1}{2}p_j^2 - \sum_k (p_j + p_k) f_{jk} + \sum_k \sum_{l \neq k} f_{jk} f_{jl} - 2\sum_k f_{jk}^2 \tag{31}$$

where $f_{jk} = f_{jk}(\boldsymbol{x}) = \begin{cases} 0, & j = k \\ \cot(\frac{x_j - x_k}{2}), & j \neq k \end{cases}$

# Commuting Hamiltonian dynamics

## Commuting Hamiltonian dynamics

For the multiple radial SLE(0) system with growth points $\boldsymbol{z} = \{z_1, z_2, \ldots, z_n\}$. In the angular coordinate, $z_j(t) = e^{ix_j(t)}$, $\boldsymbol{x(t)} = \{x_1(t), x_2(t), \ldots, x_n(t)\}$ form a Hamiltonian dynamical system.

- when evolves at parametrization of capacity $\nu_j(t) = 1, j = 1, \ldots, n$,

$$\dot{x}_j = \frac{\partial \mathcal{H}_{CS}(\boldsymbol{x}, \boldsymbol{p})}{\partial p_j}, j = 1, \ldots, n \tag{32}$$

- For each $c \in \mathbb{R}$, the submanifolds defined by the null vector Hamiltonian

$$N_c = \{(\boldsymbol{x}, \boldsymbol{p}) : \mathcal{H}_j(\boldsymbol{x}, \boldsymbol{p}) = c \text{ for all } j\} \tag{33}$$

  are preserved under the Calogero-Sutherland Hamiltonian flow.

- Commutation

$$\{\mathcal{H}_j, \mathcal{H}_k\} = \frac{1}{f_{jk}^2}\left(\mathcal{H}_k - \mathcal{H}_j\right) \tag{34}$$

  thus Poisson bracket $\{\mathcal{H}_j, \mathcal{H}_k\} = 0$ on submanifolds $N_c$.

# Structure of the Multiple Radial SLE(0) and SLE($\kappa$) Systems

- Definition and Motivation:
  Multiple radial SLE($\kappa$) systems describe random interfaces in critical two-dimensional systems. They are characterized by positive solutions to a system of null vector equations. In the limit as $\kappa \to 0$, they converge to deterministic systems known as multiple radial SLE(0).

- Trace:
  The traces of SLE(0) curves are described by the horizontal trajectories of residue-free meromorphic quadratic differentials.

- Enumeration and Classification:
  Multiple radial SLE(0) systems correspond to critical points of the master function and to residue-free meromorphic quadratic differentials. They can be classified according to admissible link patterns.

- Hamiltonian Dynamics:
  In angular coordinates, the evolution of the growth points follows the classical Calogero–Sutherland system.

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