Assignment 1 – problem 5

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Statement

A slithy Boolean expression in CNF with at most n variables that has at most n clauses and is satisfiable.

Base case

When $n=1$, it has only one variable. Hence, it only has at most one clause. It is definitely a slithy boolean expression as there is only one variable in that boolean expression. It is satisfiable as we can assign true to that variable and makes the expression true.
Hence, the base case holds.

Inductive Hypothesis

Assume that k is true for the statement where $n=k$, $k \geq 1$
Ie. A slithy boolean expression in CNF with at most k variables has at most k clauses and and is satisfiable

Inductive step

Now we have to prove that the statement is true when $n=k+1$
Ie. A slithy boolean expression in CNF with at most $k+1$ variables has at most $k+1$ clauses and is satisfiable

Assume that t is a slithy boolean expression in CNF with at most $k+1$ variable and it has x clauses. Now we removed one clause which has the unique variable from t. We named the expression as t' and the remove clause as C. C is satisfiable as it contains unique variable. which only appear once in t boolean expression. Assigning true to that unique variable can satisfy C.

t is slithy, hence it must contain one unique variable that appear only once in the set of all clauses in t. When we remove one clause that contained unique variable from t. Now t' have at most k variables as we removed a variable that appear only once in t. Since t is slithy so the set of all clauses in t is slithy as well. Hence, after removing one clause that contained unique variable, t' will still be slithy as the t' is one of the set of clause of t.

Thus, from the inductive hypothesis we know that k variable has at most k clauses slithy boolean expression in CNF and is satisfiable. Hence, t' has at most k clauses slithy boolean expression with k variables and is satisfiable. And we know that clause C is slithy and satisfiable as it has a unique variable that can be satisfied independently to make the expression satisfy. By combining t' and C we have $k+1$ variable and $k+1$ clauses slithy boolean expression and is satisfiable. This proves that a slithy boolean expression in CNF with at most $k+1$ variable have at most $k+1$ clauses and is satisficible
This completes the inductive step

Conclusion

Hence, by principle of mathematical induction, a slithy boolean expression in CNF with at most n variable has at most n clauses and is satisfiable for all $n \geq 1$, $n \in \mathbb{N}$