

Assumption:

let SAW be the language of strings over the alphabet $\{N, S, E, W\}$ that represent self-avoiding walk. We assume, for the purpose of contradiction, that SAW is a context free language.

Discussion:

As SAW is context free, by assumption, there exist some grammar in Chomsky Normal Form SAW - {E}

Let k be the number of non terminal in that grammar.

As SAW is context free, by assumption, it satisfies the pumping lemma for context free languages.

That is, for all words $w \in SAW$ where $|w| > 2^{k-1}$, there exists u, v, x, y, z such that:

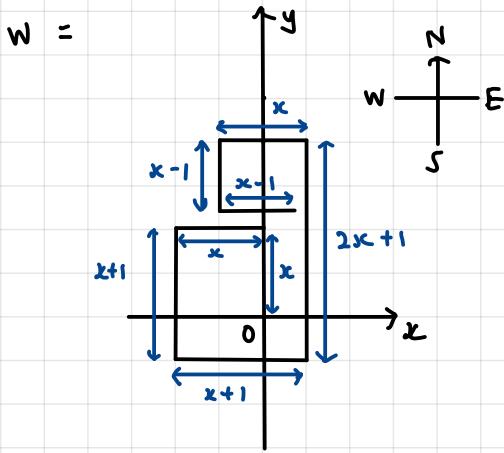
$$l \cdot z = uvxyz$$

$$2. \forall y \neq \epsilon$$

3. $|V \Delta y| \leq 2^k$, and

4. for all $i \geq 0$, $uv^ixy^iz \in SAW$

Let $x = 2^k$. Let $Z = N^x W^x S^{x+1} E^{x+1} W^x S^x E^x$. Note that $Z \in SAW$ and $|Z| > 2^{k-1}$



As $|Vxy| \leq 2^k = n$, we know that 8 cases to consider:

Case 1 : V_y comprises one or more N from the first section of the word and zero or more W from the second section

Case 2 : Vy comprises one or more W from the second section of the word and zero or more S from the third section

Case 3 : Vy comprises one or more S from the third section of the word and zero or more E from the fourth section of the word

Case 4 : Vy comprises one or more E from the fourth section of the Word and zero or more N from the fifth section of the Word

case 5 : Vy comprises one or more N from the fifth section of the word and zero or more W from the sixth section of the word

case 6 : Vy comprises one or more W from the sixth section of the word and zero or more S from the seventh section of the word

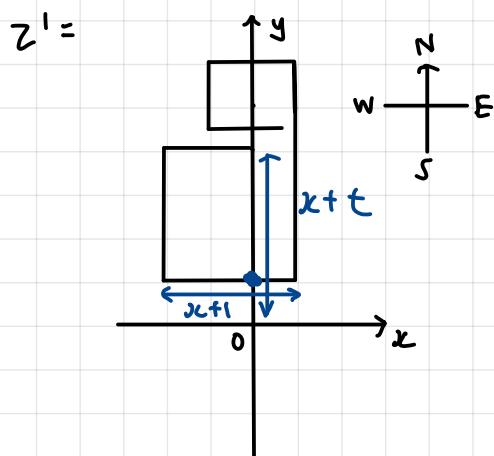
case 7 : Vy comprises one or more S from the seventh section of the word and zero or more E from the eighth section of the word

case 8 : Vy comprises one or more E from the eighth section of the word

note that it is impossible for Vy to contain the character itself and any character that is not adjacent to it, as $|VY| \leq 2^k = n$ but $N W^x S > 2^k$

case 1 : Vy comprises one or more N from the first section of the word and zero or more W from the second section

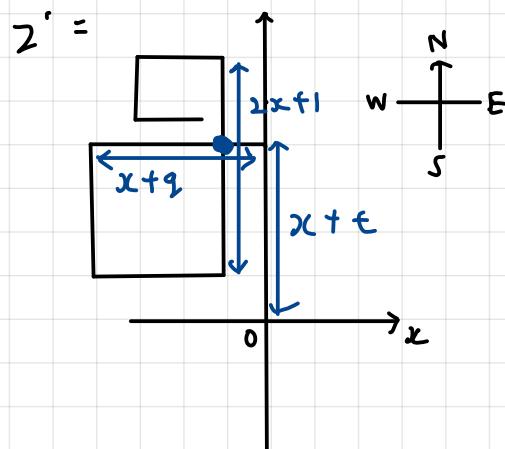
if Vy contains a positive number of N from the first section of z. Let t be the number of Ns that Vy pumped. where $t > 0$, when we pump up (eg $i=2$) it increases the number of Ns in the word and the line will get longer. The word will be $UV^iXY^iZ = UV^2XY^2Z = UUW^xY^yZ$. This creates the word $N^{x+t}W^xS^tE^iN^tW^xS^tE^i$. We named it z' . z' breaks the self avoiding walk rule as it intersects at fourth section of the word. Hence UV^2XY^2Z cannot be a word in SAW . This is not a valid U, V, X, Y, Z . At least 1 N is needed to make the word not belong to the language. If many Ns are pumped the line still intersects at the same place but longer.



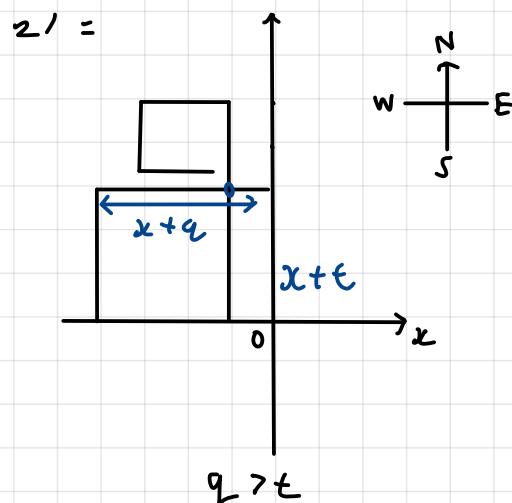
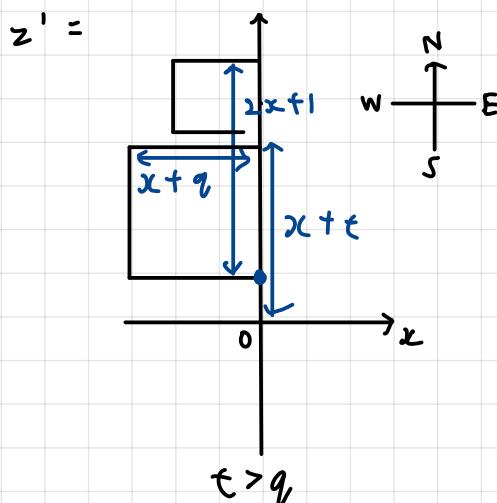
If Vy consists positive number of N and W from the first and second section of the word. Let t be the number of N s that Vy pumped where $t > 0$ and let q be the number of W s that Vy pumped where $q > 0$. When Vy consists of N and W and we pump up N and W simultaneously, it increases the number of N and W , and the lines get longer. For example $i=7$. We will create the word $uv^ixy^iz = UV'x'y'z$

$$= \underset{N}{x+t} \underset{W}{x+q} \underset{S}{x+1} \underset{E}{x+1} \underset{N}{x+t} \underset{W}{x+q} \underset{S}{x+1} \underset{E}{x+1} \underset{N}{x+t} \underset{W}{x+q} \underset{S}{x+1} \underset{E}{x+1}$$

We named it z' . z' breaks the self avoiding walk rule as it intersects at fifth section of the word. This is not a valid u, v, x, y, z . At least $|N|$ and $|W|$ added will make the word intersects at fifth section of z' . If many N and W pumped it will still intersects at the same place.

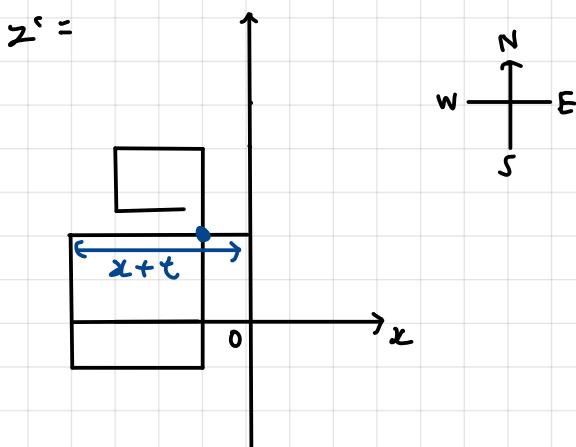


Similarly, if we pump N more than how we pump W . It intersects at fourth and fifth section of the word. For example, $t > q$, $\underset{N}{x+t} \underset{W}{x+q} \underset{S}{x+1} \underset{E}{x+1} \underset{N}{x+t} \underset{W}{x+q} \underset{S}{x+1} \underset{E}{x+1} \underset{N}{x+t} \underset{W}{x+q} \underset{S}{x+1} \underset{E}{x+1}$. Vice Versa if $q > t$ then it intersects at fifth section of z' . This is not a valid u, v, x, y, z .



Case 2 : Vy comprises one or more W from the second section of the word and zero or more S from the third section

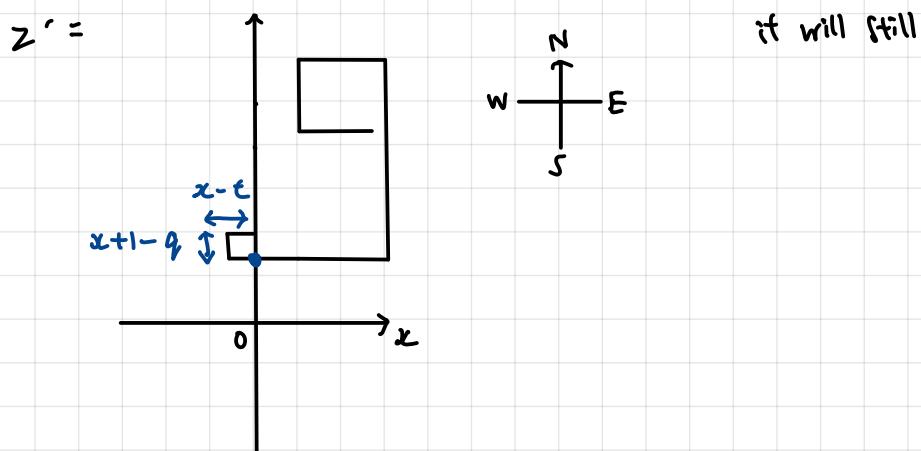
If Vy contains a positive number of W from the second section of Z. Let t be the number of Ws that Vy pumped. where $t > 0$, when we pump up (eg $i=2$) it increases the number of Ws in the word and the line will get longer. The word will be $uv^i xy^i z = uv^2 xy^2 z$
 $= u v v x y y z$. This creates the word $N \xrightarrow{w} S \in N \xrightarrow{w} S \in \dots$. We named it Z' . Z' breaks the self avoiding walk rule as it intersects at fifth section of Z . Hence $uv^2 xy^2 z$ cannot be a word in SAW. This is not a valid u, v, x, y, z . At least 1 W is needed to make the word not belong to the language. If many Ws are pumped the line still intersects at the same place.



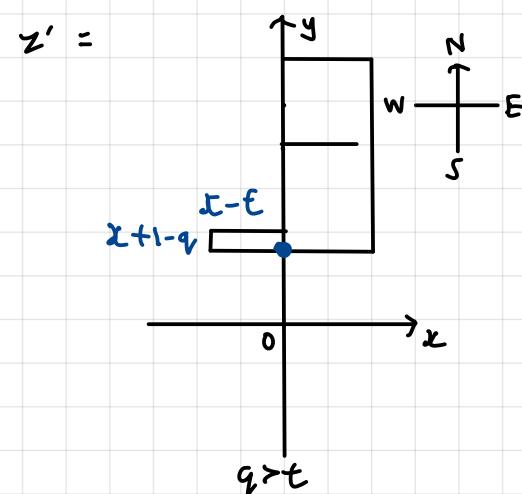
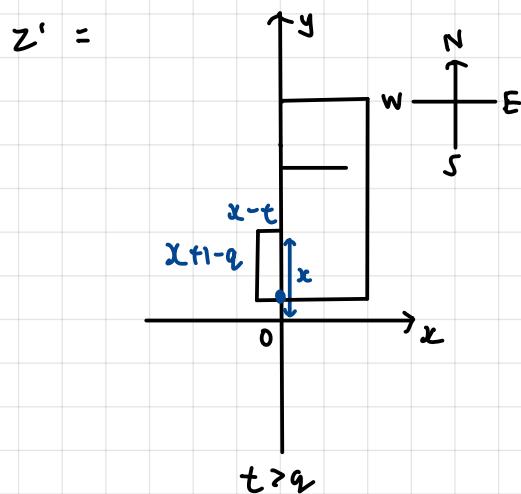
Let t be the number of Ws that is pumped where $t > 0$
let q be the number of Ss that is pumped where $q > 0$

If Vy consists positive number of W and S from the second and third section of the word
when Vy consists of W and S and we pump down W and S simultaneously, it decreases the number of W and S, and the lines get shorter. For example $i=0$. We will create the word $uv^i xy^i z = uv^0 xy^0 z = x \xrightarrow{x-t} x+1-q \xrightarrow{s+t} x \xrightarrow{x-i} z^-$. We named it Z' . Z' breaks the self avoiding walk rule as it intersects at fourth section of the word. This is not a valid u, v, x, y, z .

At least 1 W and 1 S removed will intersects at the fourth section. If many W and S pumped down it still intersects at the same place.

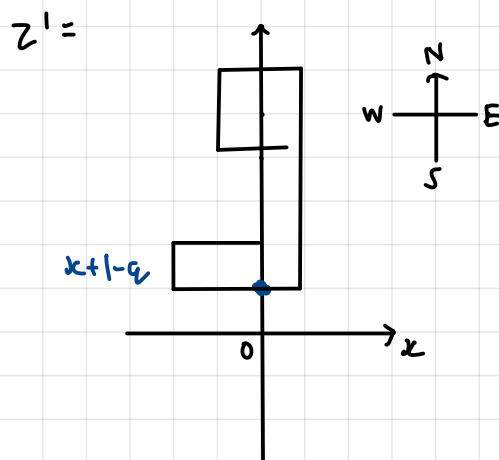


Similarly, if we pump down w more than we pump down s it intersects at fourth section of the word. For example, $x^t q, N \xrightarrow{x-t} x^{t+1-q} \xrightarrow{s+t+1} x^{t+1} \xrightarrow{x-i} x^{-i}$. Vice Versa if $q > t$ then it intersects at fourth section of z' . This is not a valid u, v, x, y, z



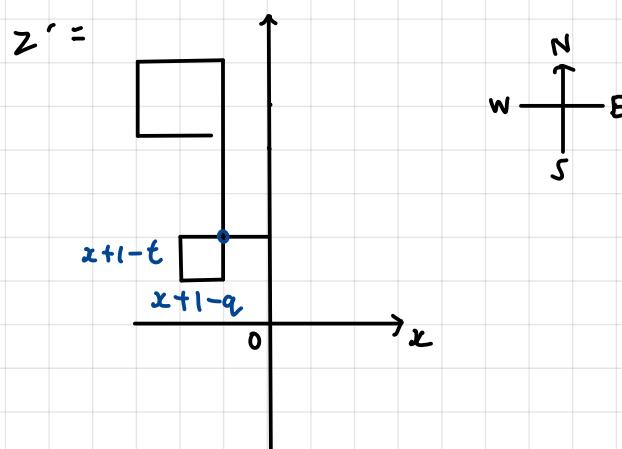
Case 3 : Vy comprises one or more S from the third section of the word and zero or more E from the fourth section of the word

If Vy contains a positive number of S from the third section of z . Let t be the number of S s that Vy pumped. Where $t > 0$, when we pump down (eg $i=0$) it decreases the number of S in the word and the line will get shorter. The word will be $uv^ixy^iz = uv^0xy^0z = uxz$. This creates the word $N \xrightarrow{x} x \xrightarrow{x+1-t} \xrightarrow{s+t+1} x \xrightarrow{x-i} x^{-i}$. We named it z' . z' breaks the self avoiding walk rule as it intersects at fourth section of the word. Hence uv^0xy^0z cannot be a word in SAW. This is not a valid u, v, x, y, z . At least 1 S is needed to make the word not belong to the language. If many S s are pumped down the line still intersects at the same place

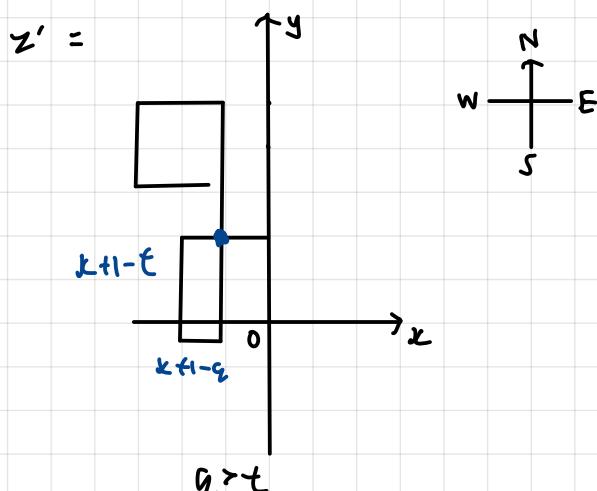
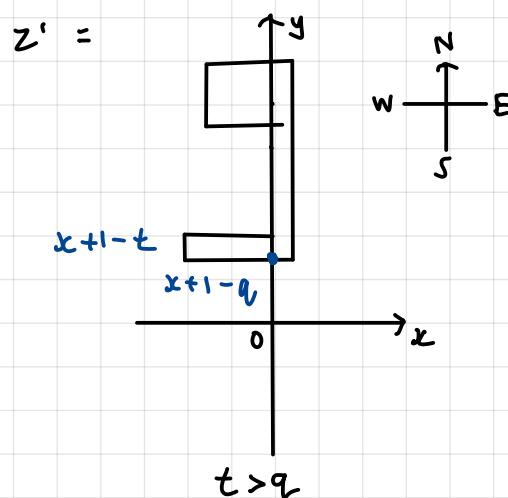


Let t be the number of S's that is pumped where $t > 0$
let q be the number of E's that is pumped where $q > 0$

If Vy consists positive number of S and E from the third and fourth section of the word
When Vy consists of S and E and we pump down S and E simultaneously, it decreases the number of S and E, and the lines get shorter. For example $i=0$. We will create the word $uv^ixy^iz = uv^0xy^0z = x \underset{N}{w} x \underset{S}{x+1-t} \underset{E}{x+1-q} \underset{S}{x} \underset{E}{x-1} \underset{S}{x-1} z^-$. We named it z' . z' breaks the self avoiding walk rule as it intersects at fifth section of the word. This is not a valid u, v, x, y, z . At least 1 S and 1 E removed will intersects at the fifth section. If many W and S removed it still intersects at the same place.

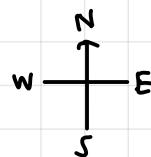
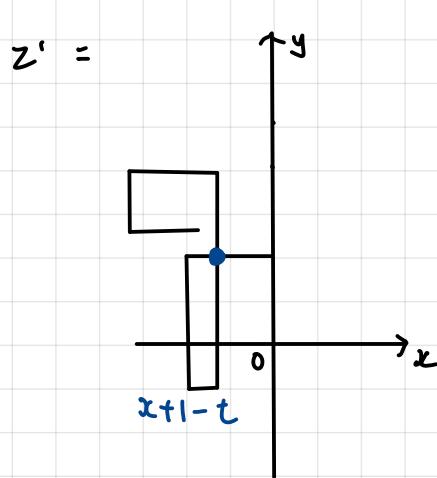


Similarly, if we pump down W more than we pump down S it intersects at fourth section of the word. For example, $t > q$, $x \underset{N}{w} x \underset{S}{x+1-t} \underset{E}{x+1-q} \underset{S}{x} \underset{E}{x-1} \underset{S}{x-1} z^-$. Vice Versa if $q > t$ then it intersects at fifth section of z' . This is not a valid u, v, x, y, z



case 4 : Vy comprises one or more E from the fourth section of the word and zero or more N from the fifth section of the word

if Vy contains a positive number of E from the fourth section of z. Let t be the number of Es that Vy pumped. where $t > 0$, When we pump down(eg $i=0$) it decreases the number of E in the word and the line will get shorter. The word will be $UV^iXY^iZ = UV^0XY^0Z = Uxz$. This creates the word $N^{x+t}W^{x+1}S^{x+t-t}E^{x+t-x-1}N^0W^0S^0E^0$. We named it z' . z' breaks the self avoiding walk rule as it intersects at fifth section of the word. Hence UV^0XY^0Z cannot be a word in SAW. This is not a valid U, V, X, Y, Z . At least 1 E is removed to make the word not belong to the language. If many Es are removed the line still intersects at the same place.

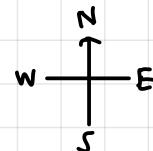
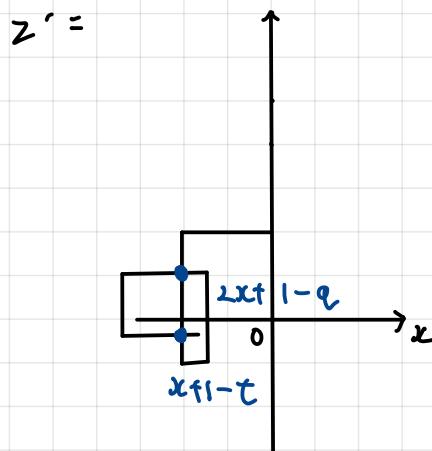


Let t be the number of Es that is pumped where $t > 0$
let q be the number of Ns that is pumped where $q > 0$

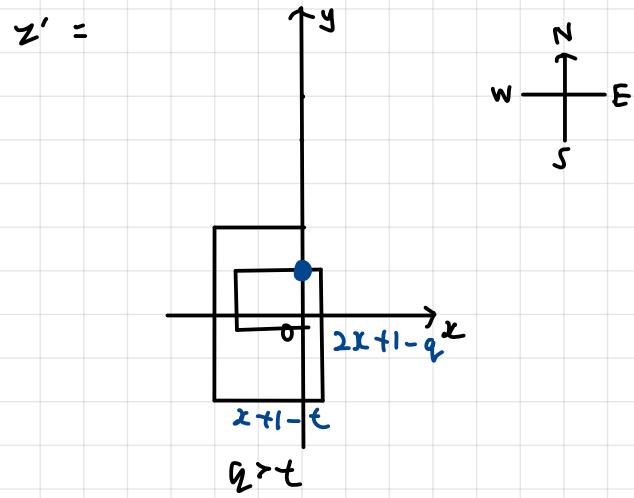
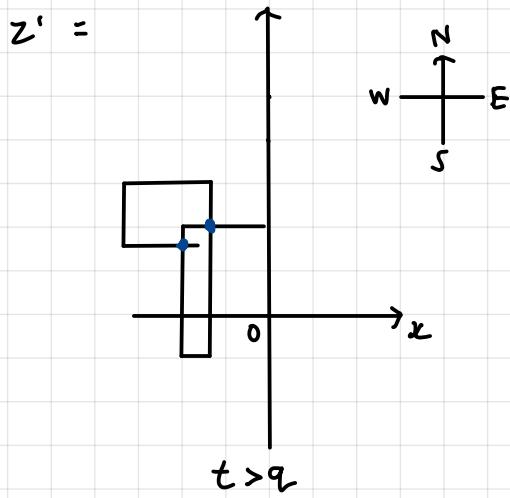
If Vy consists positive number of E and N from the fourth and fifth section of the word
When Vy consists of E and N and we pump down E and N simultaneously, it decreases the number of E and N, and the lines get shorter. For example $i=0$. We will create the word $UV^iXY^iZ = UV^0XY^0Z = x^{x+t}N^{x+1}W^{x+t-t}S^{x+t-q}E^{x+t-q-1}N^0W^0S^0E^0$. We named it z' . z' breaks the self avoiding

walk rule as it intersects at sixth and eighth section of the word. This is not a valid U, V, X, Y, Z

At least 1 E and 1 N removed will intersects at the sixth and eighth section. If many W and S removed it still intersects at the same place.



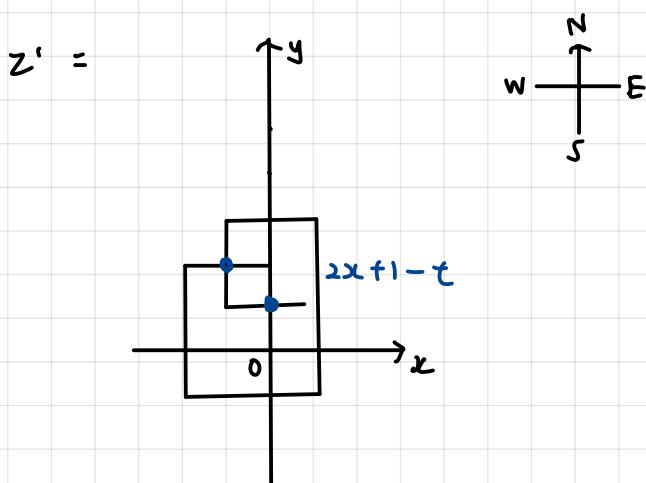
Similarly, if we pump down E more than we pump down N it intersects at fifth and eighth section of the word. For example, $t > q$, $N^x W^x S^{x+1} E^{x+t} S^{x+1-q} N^x W^x S^x E^x$. Vice Versa if $q > t$ then it intersects at sixth section of Z' . This is not a valid u, v, x, y, z .



Case 5 : Vy comprises one or more N from the fifth section of the word and zero or more W from the sixth section of the word

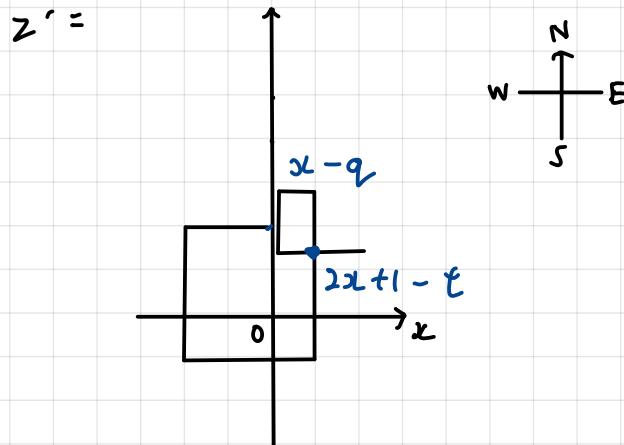
If Vy contains a positive number of N from the fifth section of z. Let t be the number of Ns that Vy pumped. Where $t > 0$, when we pump down (eg $i=0$) it decreases the number of N in the word and the line will get shorter. The word will be $UV^i XY^i z = UV^0 XY^0 z = Uxz$. This creates the word $N^x W^x S^{x+1} E^{x+t} S^{x+1-t} N^x W^x S^x E^x$. We named it Z' . Z' breaks the self avoiding walk rule as it intersects at seventh and eighth section of the word.

Hence $UV^0 XY^0 z$ cannot be a word in SAW. This is not a valid u, v, x, y, z . At least 1 N is removed to make the word not belong to the language. If many Ns are removed the line still intersects at the same place.

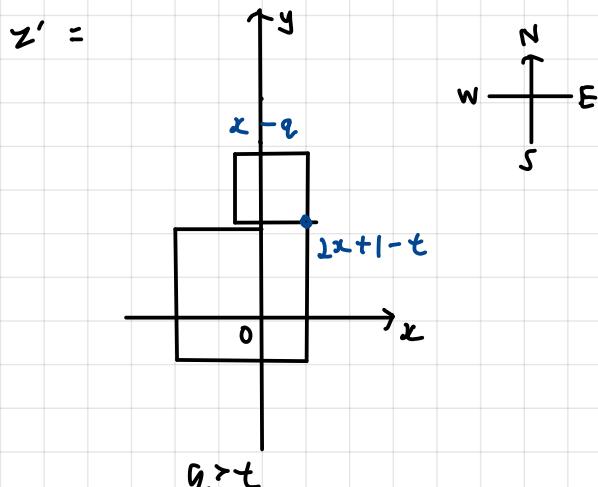
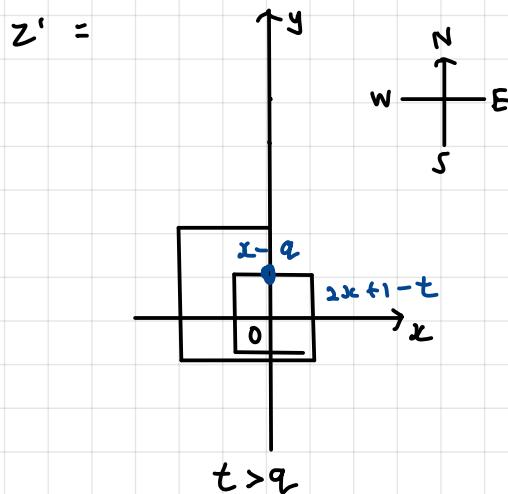


Let t be the number of Ns that is pumped where $t > 0$
let q be the number of Ws that is pumped where $q > 0$

If Vy consists positive number of N and W from the fifth and sixth section of the word
When Vy consists of N and W and we pump down N and W simultaneously, it decreases the number of N and W, and the lines get shorter. For example $i=0$. We will create the word $uv^ixy^iz = uv^0xy^0z = x \ x^{x+1} \ x^{x+1-t} \ x^{-q} \ x^{-1} \ x^{-1}$. We named it z' . z' breaks the self avoiding walk rule as it intersects at eighth section of the word. This is not a valid u, v, x, y, z . At least 1 N and 1 W removed will intersects at the eighth section. If many N and W removed it still intersects at the same place.

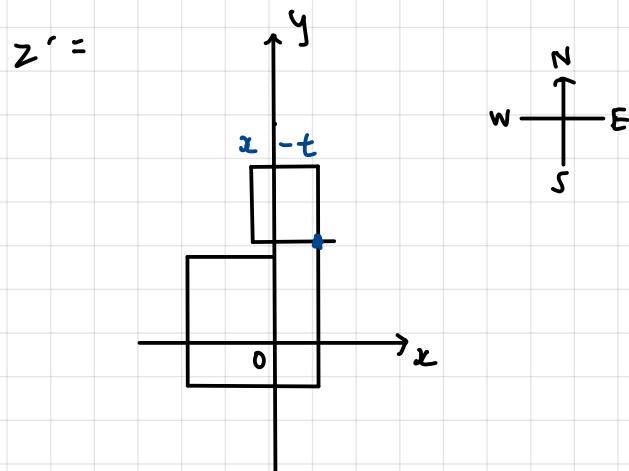


Similarly, if we pump down N more than we pump down W, it intersects at sixth section of the word. For example, $t > q$, $x \ x^{x+1} \ x^{x+1-t} \ x^{-q} \ x^{-1} \ x^{-1}$. Vice Versa if $q > t$ then it intersects at eighth section of z' . This is not a valid u, v, x, y, z .



case 6 : Vy comprises one or more W from the sixth section of the word and zero or more S from the seventh section of the word

if Vy contains a positive number of W from the sixth section of z. Let t be the number of Ws that Vy pumped. where $t > 0$, when we pump down(eg $i=0$) it decreases the number of W in the word and the line will get shorter. The word will be $uv^i xy^i z = uv^0 xy^0 z = uxz$. This create the word $\begin{matrix} x \\ N \\ W \\ S \end{matrix} \begin{matrix} x \\ E \\ N \\ W \\ S \\ E \end{matrix}$. We named it z' . z' breaks the self avoiding walk rule as it intersects at seventh section of the word. Hence $uv^0 xy^0 z$ cannot be a word in SAW. This is not a valid u, v, x, y, z. At least 1 W is removed to make the word not belong to the language. If many Ws are removed the line still intersects at the same place.

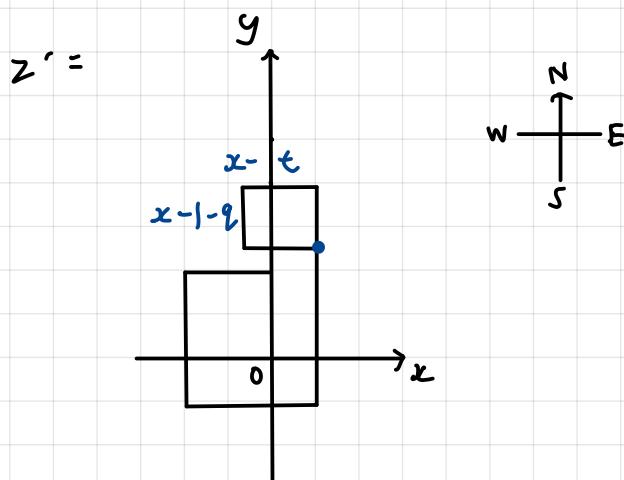


Let t be the number of Ws that is pumped where $t > 0$

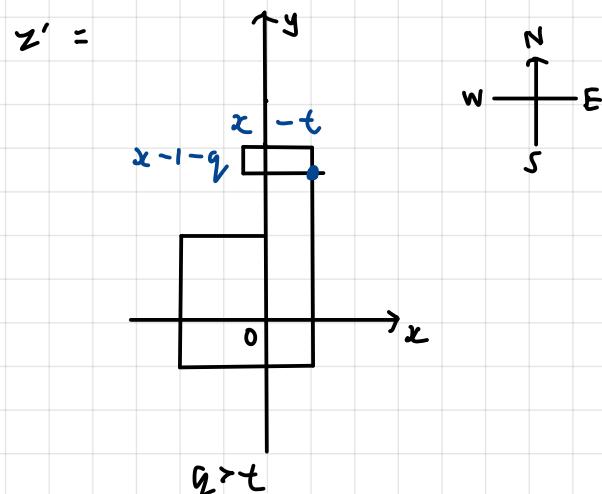
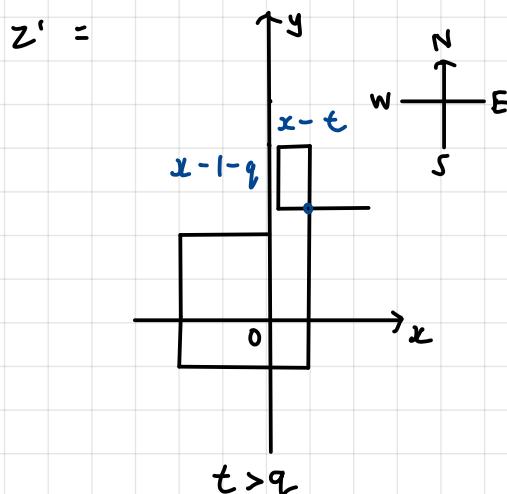
let q be the number of Ss that is pumped where $q > 0$

If Vy consists positive number of W and S from the sixth and seventh section of the word. When Vy consists of W and S and we pump down W and S simultaneously, it decreases the number of W and S, and the lines get shorter. For example $i=0$. We will create the word $uv^i xy^i z = uv^0 xy^0 z = x \begin{matrix} x \\ N \\ W \\ S \end{matrix} \begin{matrix} x \\ E \\ N \\ W \\ S \\ E \end{matrix}$. We named it z' . z' breaks the self avoiding walk rule as it intersects at eighth section of the word. This is not a valid u, v, x, y, z

At least 1 w and 1 s removed will intersects at the eighth section. If many w and s removed it still intersects at the same place.

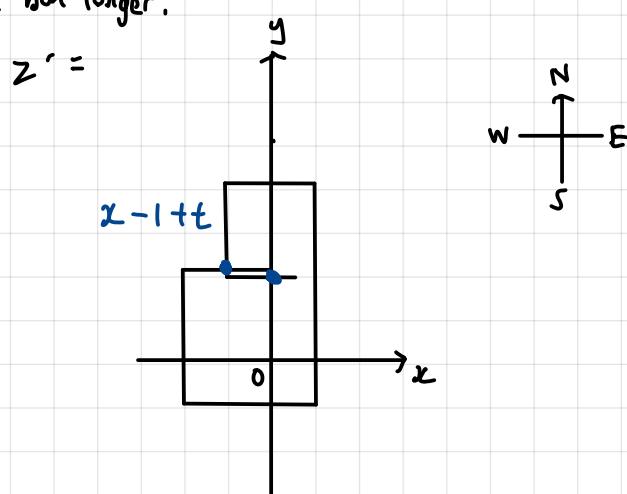


Similarly, if we pump down w more than we pump down S it intersects at eighth section of the word. For example, $t > q$, $N^x W^{z+1} S^{t+1} E^{x+t} N^w S^{x-1-q} E^{x-1}$. Vice Versa if $q > t$ then it intersects at eighth section of Z' . This is not a valid u, v, x, y, z



case 7 : Vy comprises one or more S from the seventh section of the word and zero or more E from the eighth section of the word

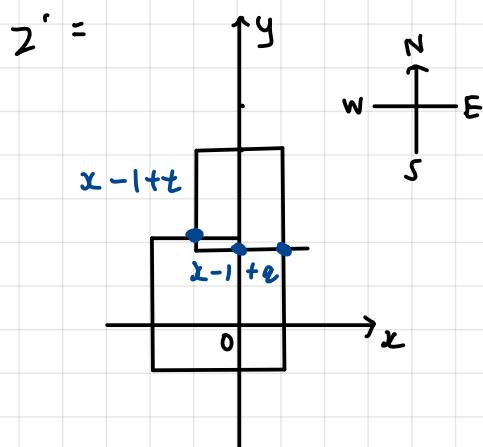
if Vy contains a positive number of S from the seventh section of z . Let t be the number of Ss that Vy pumped. where $t > 0$, when we pump up (eg $i=2$) it increases the number of Ss in the word and the line will get longer. The word will be $uv^i xy^i z = uv^2 xy^2 z = uvvxyyz$. This creates the word $N^x W^{x-z+1} S^{z+t+1} E^{x-x-t} N^w S^{x-1} E^{x-1}$. We named it Z' . Z' breaks the self avoiding walk rule as it intersects at seventh and eighth section of the word. Hence $uv^2 xy^2 z$ cannot be a word in SAW. This is not a valid u, v, x, y, z . At least 1 S is needed to make the word not belong to the language. If many Ss are pumped the line still intersects at the same place but longer.



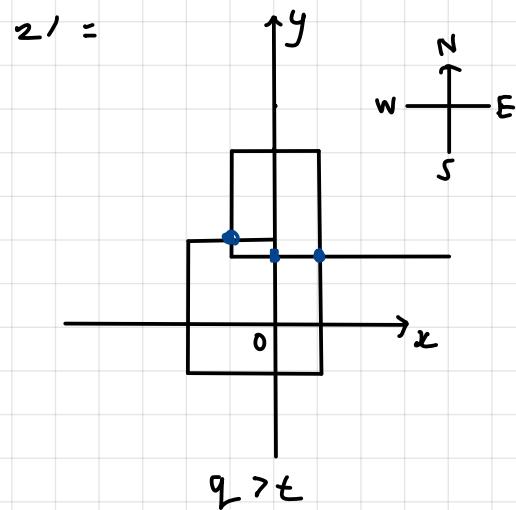
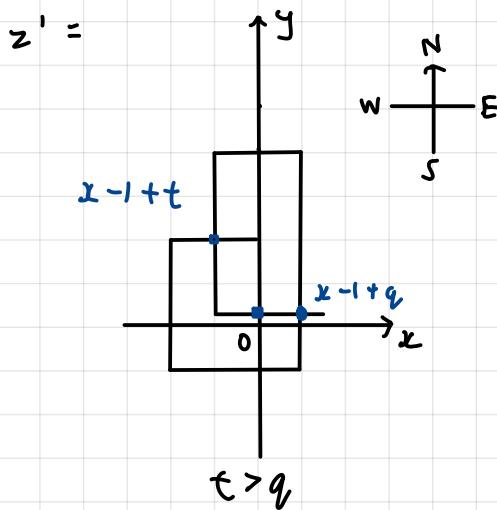
If Vy consists positive number of S and E from the seventh and eighth section of the word
 Let t be the number of S s that Vy pumped where $t > 0$ and let q be the number of
 E s that Vy pumped where $q > 0$. When Vy consists of S and E and we pump up
 S and E simultaneously, it increases the number of S and E , and the lines get longer
 For example $i=7$. We will create the word $uv^ixy^iz = uv'x'y'z$

$= \begin{matrix} x & x & z+1 & z+1 & z+1 & x & x-1+t & x-1+q \\ N & W & S & E & N & W & S & E \end{matrix}$, we named it z' . z' breaks the self avoiding walk rule

as it intersects at seventh and eighth section of the word. This is not a valid u, v, x, y, z
 At least $1S$ and $1E$ added will make the word intersects at seventh and eighth section of z'
 - If many S and E pumped up it will still intersects at the same place.

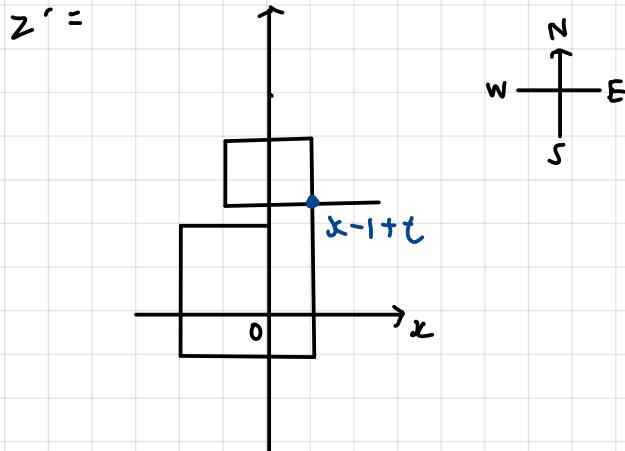


Similarly, if we pump S more than how we pump E . It intersects at seventh and eighth section of the word. For example, $t > q$, $\begin{matrix} x & x & z+1 & z+1 & z+1 & x & x-1+t & x-1+q \\ N & W & S & E & N & W & S & E \end{matrix}$. Vice Versa if $q > t$ then it intersects at seventh and eighth section of z' . This is not a valid u, v, x, y, z



case 3 ; Vy comprises one or more E from the eighth section of the word

If Vy contains a positive number of E from the eighth section of z. Let t be the number of Es that Vy pumped. Where $t > 0$, when we pump up (eg $i=2$) it increases the number of Es in the word and the line will get longer. The word will be $uv^i xy^i z = uv^2 xy^2 z = uvvxyz$. This creates the word $\overset{x}{N} \overset{x}{W} \overset{x+t}{S} \overset{x+t}{E} \overset{x}{N} \overset{x}{W} \overset{x}{S} \overset{x}{E}$. We named it z' . z' breaks the self avoiding walk rule as it intersects at eighth section of the word. Hence $uv^2 xy^2 z$ cannot be a word in SAW. This is not a valid u, v, x, y, z . At least 1 N is needed to make the word not belong to the language. If many Ns are pumped the line still intersects at the same place but longer.



After considering all possible cases, we are not able to find a valid u, v, x, y, z

contradiction;

Hence, SAW does not satisfy pumping Lemma for context free languages, however a context free language it must satisfy the pumping Lemma.

This is a contradiction. We must conclude that it is not true that SAW is context free language.

Conclusion: We have proven that SAW is not context free language by using contradiction