AMCS 252 Homework #4 Yuanyuan Yang

Exercise #1

(a) Solution.

Now we solve the problem

$$u'(t) = 10u$$

with

$$u(0) = 1$$

up to time $T=10$.

We know the true solution is $u(t) = e^{10t}$, and thus $u(T) = e^{100}$ $\approx 2.7 \times 10^{43}$.

Using forward Euler method to solve it with a time step $k=0.01$, we get the computed solution is:

$$U^N = 2.25 \times 10^{41}$$

The error bound (6.16) gives

$$|E^N| \le e^{LT} \cdot T \cdot \|\tau\|_\infty$$
$$= e^{100} \cdot 10 \cdot \|\tau\|_\infty$$

Here $\|\tau\|_\infty = \max_{0 \le n \le N} |\tau^n|$ and $\tau^n = \frac{1}{2} k u''(t_n) + O(k^2)$.

Thus, we have

$$\|\tau\|_\infty = |\tau^N| \approx \frac{1}{2} k u''(T) = 50 \cdot e^{100} \cdot k$$

The error bound (6.16) becomes

$$|E^N| \le e^{LT} \cdot T \cdot \|\tau\|_\infty \approx 500 \cdot e^{200} \cdot k$$

This bound grows exponentially in time when the true and computed solutions are also increasing exponentially. Therefore, it's a more reasonable estimate of the actual error compared to case before.

(b) Proof.

Suppose that we solve

$$u'(t) = \lambda u$$

with

$$u(t_0) = \eta$$

up to time T, where $\lambda < 0$.

Using forward Euler method, we can get

$$\frac{u^{n+1} - u^n}{k} = \lambda u^n$$

$$\Rightarrow \quad u^{n+1} = (1+k\lambda) u^n \tag{1}$$

Let $\phi(u^n) = (1+k\lambda) u^n$.

Rewriting equation (1) as

$$u(t_{n+1}) = (1+k\lambda) u(t_n) + k\tau^n$$

and subtracting this from (1) gives the expression of global error:

$$E^{n+1} = (1+k\lambda) E^n - k\tau^n \tag{2}$$

Applying the recursion (2) repeatedly we can get

$$E^n = (1+k\lambda)^n E^0 - k \sum_{m=1}^{n} (1+k\lambda)^{n-m} \tau^{m-1}$$

The Lipschitz constant for function ϕ is

$$M = 1 + k\lambda \leq 1$$

Thus, we have

$$\begin{aligned} |E^n| &= \left| (1+k\lambda)^n E^0 - k \sum_{m=1}^{n} (1+k\lambda)^{n-m} \tau^{m-1} \right| \\ &\leq \left| (1+k\lambda)^n E^0 \right| + \left| k \sum_{m=1}^{n} (1+k\lambda)^{n-m} \tau^{m-1} \right| \\ &\leq |E^0| + \left| k \sum_{m=1}^{n} \tau^{m-1} \right| \\ &\leq |E^0| + nk \|\tau\|_\infty \end{aligned}$$

where $\|\tau\|_\infty = \max_{0 \leq n \leq N-1} |\tau^n|$.

Provided that we restrict $0 \leq t \leq T$, so that $t_n = nk \leq T$.

If $E^0 = 0$, we can get

$$|E^n| \leq T \|\tau\|_\infty$$

(c) Proof.

We already know the global error E^n for IVP $u'(t)=\lambda u$ is

$$E^n = (1+k\lambda)^n E^0 - k\sum_{m=1}^{n}(1+k\lambda)^{n-m}\tau^{m-1}$$

If $\lambda<0$, then $k\lambda<0$ and $M=1+k\lambda<1$.

For the example above, we choose a very small k such that $|k\lambda|<1$, and so that $0<1+k\lambda<1$.

Thus, we have

$$e^{k\lambda} = 1+k\lambda+\tfrac{1}{2}(k\lambda)^2+o((k\lambda)^3) > 1+k\lambda = |1+k\lambda|$$

since the leading difference part $\tfrac{1}{2}(k\lambda)^2>0$.

Also, we can get

$$|1+k\lambda|^n \leq e^{nk\lambda} \leq e^{\lambda T}$$

and

$$|1+k\lambda|^{n-m} \leq e^{(n-m)k\lambda} \leq e^{nk\lambda} \leq e^{\lambda T}$$

Therefore, we have

$$\begin{aligned}|E^n| &= \left|(1+k\lambda)^n E^0 - k\sum_{m=1}^{n}(1+k\lambda)^{n-m}\tau^{m-1}\right| \\ &\leq e^{\lambda T}\left(|E^0| + \left|k\sum_{m=1}^{n}\tau^{m-1}\right|\right) \\ &\leq e^{\lambda T}\left(|E^0| + nk\|\tau\|_\infty\right) \\ &\leq e^{\lambda T}\left(|E^0| + T\|\tau\|_\infty\right)\end{aligned}$$

where $\|\tau\|_\infty = \max_{0\leq n\leq N-1}|\tau^n|$.

If $E^0=0$, we have

$$|E^n| \leq e^{\lambda T}\cdot T\cdot\|\tau\|_\infty$$

For the example above, $\lambda=-10$, $T=10$ and $\tau^n=\tfrac{1}{2}ku''(t_n)+o(k^2)$, thus

$$\|\tau\|_\infty = \max_{0\leq n\leq N-1}|\tau^n| = |\tau^0| \simeq 50k$$

and

$$|E^N| \leq e^{\lambda T}\cdot T\cdot\|\tau\|_\infty \approx e^{-100}\cdot 10\cdot 50\cdot k$$

We use the $k=0.01$ in the example and get

$$|E^N| \leq 1.85\times10^{-43}$$

which is a reasonable and realistic estimate since the computed E^N in the example is $E^N \approx 3.7\times10^{-44}$.

Exercise #2

(a) Solution.

Since

$$2U^{n+3} - 5U^{n+2} + 4U^{n+1} - U^n = 0$$

thus the characteristic polynomial is

$$\rho(\zeta) = 2\zeta^3 - 5\zeta^2 + 4\zeta - 1 = (\zeta-1)^2(2\zeta-1)$$

Therefore, $\zeta_1 = \zeta_2 = 1$, $\zeta_3 = \frac{1}{2}$ and the general solution is

$$U^n = c_1 + c_2 n + c_3 (\tfrac{1}{2})^n$$

(b) We already know

$$\begin{cases} c_1 + c_3 = U^0 \\ c_1 + c_2 + \frac{1}{2}c_3 = U^1 \\ c_1 + 2c_2 + \frac{1}{4}c_3 = U^2 \end{cases}$$

For starting values $U^0 = 11$, $U^1 = 5$, $U^2 = 1$, solving the above equations gives us the results:

$$c_1 = 3, \quad c_2 = -2, \quad c_3 = 8$$

Thus, the solution to this difference equation is

$$U^n = 3 - 2n + 8 \cdot (\tfrac{1}{2})^n = 3 - 2n + (\tfrac{1}{2})^{n-3}$$

and

$$U^{10} = 3 - 2 \times 10 + (\tfrac{1}{2})^{10-3} = -16\tfrac{127}{128}$$

(c) Solution.

In this case, the local truncation error is

$$\tau = \frac{2u(t_{n+3}) - 5u(t_{n+2}) + 4u(t_{n+1}) - u(t_n)}{k} - \beta_0 f(u(t_n)) - \beta_1 f(u(t_{n+1}))$$

$$= \frac{2u(t_{n+3}) - 5u(t_{n+2}) + 4u(t_{n+1}) - u(t_n)}{k} - \beta_0 u'(t_n) - \beta_1 u'(t_{n+1})$$

Applying Taylor series, we have

$u(t_{n+3}) = u(t_n) + 3k u'(t_n) + \frac{(3k)^2}{2} u''(t_n) + \frac{(3k)^3}{6} u'''(t_n) + O(k^4)$

$u(t_{n+2}) = u(t_n) + 2k u'(t_n) + \frac{(2k)^2}{2} u''(t_n) + \frac{(2k)^3}{6} u'''(t_n) + O(k^4)$

$u(t_{n+1}) = u(t_n) + k u'(t_n) + \frac{k^2}{2} u''(t_n) + \frac{k^3}{6} u'''(t_n) + O(k^4)$

$u'(t_{n+1}) = u'(t_n) + k u''(t_n) + \frac{k^2}{2} u'''(t_n) + O(k^3)$

Plugging these into the formula of local truncation error, we get

$$\begin{aligned} \tau = & \, k u''(t_n) + 3k^2 u'''(t_n) - \beta_0 u'(t_n) - \beta_1 u'(t_n) \\ & - \beta_1 k u''(t_n) - \tfrac{1}{2}\beta_1 k^2 u'''(t_n) + O(k^3) \\ = & -(\beta_0 + \beta_1) u'(t_n) + (1-\beta_1) k u''(t_n) + (3 - \tfrac{1}{2}\beta_1) k^2 u'''(t_n) + O(k^3) \end{aligned}$$

Since we want the local truncation error is $O(k^2)$, thus we have

$$\begin{cases} \beta_0 + \beta_1 = 0 \\ 1 - \beta_1 = 0 \\ 3 - \frac{1}{2}\beta_1 \neq 0 \end{cases}$$

Solving the equations, we get

$$\beta_0 = -1 \quad \text{and} \quad \beta_1 = 1$$

(d) This LMM has the form

$$2U^{n+3} - 5U^{n+2} + 4U^{n+1} - U^n = k[f(U^{n+1}) - f(U^n)]$$

and the characteristic polynomial is

$$\rho(\zeta) = 2\zeta^3 - 5\zeta^2 + 4\zeta - 1 = (\zeta - 1)^2(2\zeta - 1)$$

The roots are $\zeta_1 = \zeta_2 = 1$, $\zeta_3 = \frac{1}{2}$. Obviously, $\rho(\zeta)$ has a repeated root of modulus 1, then the method is not stable according to the root condition and thus it is not convergent.

Exercise #3

Solution.

Now we have $U^0 = \eta$. Using forward Euler method, we can get

$$\frac{U^1 - U^0}{k} = \lambda U^0$$

$$\Rightarrow \quad U^1 = (1+k\lambda)U^0 = (1+k\lambda)\eta$$

Rewrite the midpoint method as

$$U^{n+2} - U^n = 2kf(U^{n+1})$$

The characteristic polynomial is

$$\rho(\zeta) = \zeta^2 - 1 = (\zeta-1)(\zeta+1)$$

Thus, $\zeta_1 = 1$ and $\zeta_2 = -1$, and hence the root condition is satisfied.

The general solution is

$$U^n = C_1 + C_2 \cdot (-1)^n$$

For starting values U^0 and U^1, we have

$$\begin{cases} C_1 + C_2 = U^0 \\ C_1 - C_2 = U^1 \end{cases}$$

Solving the equations gives us that

$$\begin{cases} C_1 = (1 + \frac{k\lambda}{2})\eta \\ C_2 = -\frac{k\lambda}{2}\eta \end{cases}$$

Therefore, the exact solution in this case is

$$U^n = (1+\frac{k\lambda}{2})\eta + (-1)^{n+1} \cdot \frac{k\lambda}{2}\eta$$

In this case, we have proved that the root condition has been satisfied. Therefore this method is zero-stable and hence convergent.