```python
# ------------------------------------------------------------------------
#                              Checkerboard
# ------------------------------------------------------------------------
def con_isometry_checkboard(self,l0,angle0):
    """
    keep 2 kinds of diagonal edge-lengths and their crossing angle
    X += [ld1,ld2, ud1,ud2]
    1. (v1-v3) = ld1*ud1, ud1**2=1
    2. (v2-v4) = ld2*ud2, ud2**2=1
    3. ld1 == init_ld1, ld2 == init_ld2
    4. ud1*ud2 == init_ud1*init_ud2
    """
    w = self.get_weight('isometry_checkboard')
    V = self.mesh.V
    num = self.mesh.num_quadface
    N = self.N
    X = self.X
    numl = self._N7-8*num
    numud = self._N7-6*num
    arr = np.arange(num)

    c_ld1 = numl+arr
    c_ld2 = numl+num+arr
    vi = self.mesh.quadface
    v1,v2,v3,v4 = vi[::4],vi[1::4],vi[2::4],vi[3::4]
    c_v1 = np.r_[v1,V+v1,2*V+v1] # [x,y,z]
    c_v2 = np.r_[v2,V+v2,2*V+v2] # [x,y,z]
    c_v3 = np.r_[v3,V+v3,2*V+v3] # [x,y,z]
    c_v4 = np.r_[v4,V+v4,2*V+v4] # [x,y,z]
    c_ud1 = np.r_[numud+arr,numud+num+arr,numud+2*num+arr]
    c_ud2 = c_ud1+3*num

    He1,re1 = self._edge(X,c_v1,c_v3,c_ld1,c_ud1,num,N)
    He2,re2 = self._edge(X,c_v2,c_v4,c_ld2,c_ud2,num,N)
    Hu1,ru1 = self._unit(X,c_ud1,num,N)
    Hu2,ru2 = self._unit(X,c_ud2,num,N)
    Hl1,rl1 = self._constl(c_ld1,l0[:num],num,N)
    Hl2,rl2 = self._constl(c_ld2,l0[num:],num,N)
    Ha,ra = self._constangle(X,c_ud1,c_ud2,angle0,num,N)

    H = sparse.vstack((He1,He2,Hu1,Hu2,Hl1,Hl2,Ha))
    r = np.r_[re1,re2,ru1,ru2,rl1,rl2,ra]
    self.add_iterative_constraint(H*w, r*w, 'isometry(checkboard)')
```

Constraint for isometry_checkerboard way as in Caigui paper

1. basic: get each quad faces’ vertex indices
   即 self.mesh.quadface 函数， 见后面
2. 表示isometry条件
   way1: 使用提取给定初始网对角线长度&夹角 （该方法)。
   way2: 使用读取两个对应网格，将其vertices一起作为变量
3. 将下面约束条件表示成稀疏矩阵H 和列表 r
4. 求解非齐次稀疏矩阵线性解

变量X =[vertices, ld1, ld2, ud1, ud2]
Vertices: 所有格点3维坐标
Ld1,ld2: 分别是两组对角线长度
Ud1,ud2：分别是两组对角线方向单位向量

Way1:

$$(v_0 - v_2)^2 = C_0,\ (v_1 - v_3)^2 = C_1,\ (v_0 - v_2)\cdot(v_1 - v_3) = C_3.$$

Way2:

$$c_{iso,0}(f) = (v_0 - v_2)^2 - (v_0' - v_2')^2 = 0,$$
$$c_{iso,1}(f) = (v_1 - v_3)^2 - (v_1' - v_3')^2 = 0,$$
$$c_{iso,2}(f) = (v_0 - v_2)\cdot(v_1 - v_3) - (v_0' - v_2')\cdot(v_1' - v_3') = 0.$$

```python
def _edge(self,X,c_v1,c_v3,c_ld1,c_ud1,num,N):
    "(v1-v3) = ld1*ud1"
    ld1 = X[c_ld1]
    ud1 = X[c_ud1]
    a3 = np.ones(3*num)
    row1 = np.tile(np.arange(3*num),4)
    col = np.r_[c_v1,c_v3,np.tile(c_ld1,3),c_ud1]
    data = np.r_[a3,-a3,-ud1,-np.tile(ld1,3)]
    r = -np.tile(ld1,3)*ud1
    H = sparse.coo_matrix((data,(row1,col)), shape=(3*num, N))
    return H,r

def _unit(self,X,c_ud1,num,N):
    "ud1**2=1"
    arr = np.arange(num)
    row2 = np.tile(arr,3)
    col = c_ud1
    data = 2*X[col]
    r =  np.linalg.norm(X[col].reshape(-1,3,order='F'),axis=1)**2 + np.ones(num)
    H = sparse.coo_matrix((data,(row2,col)), shape=(num, N))
    return H,r

def _constl(self,c_ld1,init_l1,num,N):
    "ld1 == const."
    row3 = np.arange(num,dtype=int)
    col = c_ld1
    data = np.ones(num,dtype=int)
    r = init_l1
    H = sparse.coo_matrix((data,(row3,col)), shape=(num, N))
    return H,r
```

_edge函数
表示：对角线向量==对角线长度*单位向量
被con_isometry_checkberboard函数调用2次
返回稀疏矩阵，和列表 H, r

_unit函数
表示：对角线单位向量
被con_isometry_checkberboard函数调用2次
返回稀疏矩阵，和列表 H, r

_constl函数
表示：对角线长度==给定初始长度值
被con_isometry_checkberboard函数调用2次
返回稀疏矩阵，和列表 H, r

_constangle函数
表示：单位对角线向量夹角为给定cos(alpha)
被con_isometry_checkberboard函数调用1次
返回稀疏矩阵，和列表 H, r

```python
def _constangle(self,X,c_ud1,c_ud2,angle0,num,N):
    "ud1*ud2 == const."
    row4 = np.tile(np.arange(num),6)
    col = np.r_[c_ud1,c_ud2]
    data = np.r_[X[c_ud2],X[c_ud1]]
    r = np.einsum('ij,ij->i',X[c_ud1].reshape(-1,3, order='F'),X[c_ud2].reshape(-1,3,order='F'))+angle0
    H = sparse.coo_matrix((data,(row4,col)), shape=(num, N))
    return H,r
```

L0与angle0

```python
def quadfaces(self):
    "for quad diagonals"
    "quadface, num_quadface, quadface_order"
    f, v1, v2 = self.face_edge_vertices_iterators(order=True)
    f4,vi = [],[]
    for i in range(self.F):
        ind = np.where(f==i)[0]
        if len(ind)==4:
            f4.extend([i,i,i,i])
            vi.extend(v1[ind])
            #vj.extend(v2[ind])
    self._num_quadface = len(f4) // 4
    #v1,v2,v3,v4 = vi[::4],vi[1::4],vi[2::4],vi[3::4]
    self._quadface = np.array(vi,dtype=int)
    self._quadface_order = np.unique(f4)
```

由halfedge半边数据结构表示出每个quadface的格点索引值
即返回列表[v1,v2,v3,v4] = quadface

```python
def face_edge_vertices_iterators(self, sort=False, order=False):
    H = self.halfedges
    f  = H[:,1]
    vi = H[:,0]
    vj = H[H[:,2],0]
    if order:
        i  = self.face_ordered_halfedges()
        f  = f[i]
        vi = vi[i]
        vj = vj[i]
    else:
        i  = np.where(H[:,1] >= 0)[0]
        f  = f[i]
        vi = vi[i]
        vj = vj[i]
        if sort:
            i  = np.argsort(f)
            vi = vi[i]
            vj = vj[i]
    return f, vi, vj
```

```python
def face_ordered_halfedges(self):
    H = np.copy(self.halfedges)
    i = np.argsort(H[:,1])
    i = i[np.where(H[i,1] >= 0)]
    f = H[i,1]
    index = np.arange(i.shape[0])
    _, j = np.unique(f, True)
    f = np.delete(f,j)
    index = np.delete(index, j)
    while f.shape[0] > 0:
        _, j = np.unique(f, True)
        i[index[j]] = H[i[index[j] - 1],2]
        f = np.delete(f, j)
        index = np.delete(index, j)
    return i
```

Killing field

$$
\begin{aligned}
E_k &= E_t + \lambda \cdot E_c + w_{fair} \cdot E_{fair} \\
&= X^T (T + \lambda \cdot C + w_{fair} \cdot K) X \\
&= X^T A X
\end{aligned}
$$

constraints on velocity field (infinitesimal isometry)

velocity

(1) $(v_2 - v_0) \cdot (w_2 - w_0) = 0$

(2) $(v_3 - v_1) \cdot (w_3 - w_1) = 0$

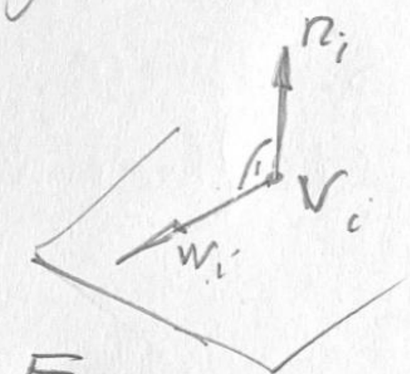
(3) $(v_2 - v_0)(w_3 - w_1) + (w_2 - w_0) \cdot (v_3 - v_1) = 0$

linear homogeneous (v_i given)

$$E_t = \sum (w_i \cdot n_i)^2 = 0$$

normal at v_i

tangent



add constraints (1–3), squared E_c

$$E_c = \sum_f ((v_2 - v_0) \cdot (w_2 - w_0))^2 + \sum_f ((v_3 - v_1) \cdot (w_3 - w_1))^2 + \sum_f ((v_2 - v_0) \cdot (w_3 - w_1) + (v_3 - v_1) \cdot (w_2 - w_0))^2$$

$$E_t = \sum (w_i \cdot n_i)^2.$$

$$E_{fair} = \sum_v (w_{i-1} + w_{i+1} - 2w_i)^2$$

```python
def get_killing_eigen(self,killing=True,efair=0.01):
    """
    X = only [wi] , i=1...V
    Ek = Et+la*Ec = X'* (T+la*C) * X
    A = T+la*C + efair*K
    eigen of M:
              if num=1, close to iso.to surf.of revelotion
              if num=3, close to const. Gaussian curv. K
    A is influenced by normals & efair
    """
    lamda = self.get_weight('iso_velocity')
    refermesh = self.mesh
    Vi = self.mesh.vertices
    V = self.mesh.V
    Mnum = 3*V
    iv0,iv1,iv2,iv3 = self.mesh.quadface.T
    d1 = Vi[iv2] - Vi[iv0]
    d2 = Vi[iv3] - Vi[iv1]
    cw0 = np.r_[iv0,V+iv0,2*V+iv0]
    cw1 = np.r_[iv1,V+iv1,2*V+iv1]
    cw2 = np.r_[iv2,V+iv2,2*V+iv2]
    cw3 = np.r_[iv3,V+iv3,2*V+iv3]

    C = self.__iso_matrix(cw0,cw1,cw2,cw3,d1,d2,Mnum) * lamda
    A = C

    if efair:
        "v3+v1-2*v2-->0"
        K = self.__fairness_matrix(V,efair,Mnum)
        A += K

    if killing:
        normals = refermesh.vertex_normals()
        closest = refermesh.closest_vertices(Vi)
        normals = normals[closest,:]
        c_w = np.arange(Mnum)
        T = self.__matrix_1(c_w,normals,Mnum)#self.__killing_ma
        A += T
```

Global killing field
求解关于mesh所有 vertices 的特征方程：
需要表示3个对称矩阵： C, T, K

Solving eigen value problem of X^T A X =0

$$E_k := E_t + \lambda * E_c + 0.01 * E_{\{fair\}}$$
$$= X^T (T + \lambda * C + 0.01 * K) X$$
$$= X^T A X$$

- E_t: tangent condition
- Ec: i-velocity (1) (2) (3), λ=1
- $E_{\{fair\}}$: Fairness on wi

```python
"column v[:,i] is the eigenvector corresponding to the eigenvalue w[i]"
vals,vecs = np.linalg.eigh(A.toarray())
vals = vals / (3*V) # per vertex
amin = np.argmin(np.abs(vals))
vmin = vecs[:,amin]

print('-'*20)
eig = list(np.abs(vals))
print('list top 5 smallest eigen values:\n')
for i in heapq.nsmallest(5, eig):
    print('*',i)
print('='*20)

return self.mesh.vertices, vin
```

```python
def __fairness_matrix(self,Vnum,efair,Mnum,xnum=0):
    "(w1+w3-2*w)^2=0; (w2+w4-2*w)^2=0"
    v,v1,v2,v3,v4 = self.mesh.ver_regular_star.T
    m13 = self.__fair(v,v1,v3,Vnum,Mnum,xnum=xnum)
    m24 = self.__fair(v,v2,v4,Vnum,Mnum,xnum=xnum)
    return (m13+m24) * efair

def __fair(self,v,v1,v3,Vnum,Mnum,xnum=0,arrc=None,arrl=None,arrr=None):
    "(w1+w3-2*w)^2=0;"
    def __matrix(c_w,num,Mnum):
        if arrc is not None:
            data = np.array([])
            for i in range(len(arrc)):
                one = np.array([arrl[i],arrr[i],-2*arrc[i]])
                d = np.outer(one,one).flatten()
                data = np.r_[data,d]
        else:
            one = np.array([1,1,-2])
            d = np.outer(one,one).flatten()
            data = np.tile(d,num)
        rw = (c_w.reshape(-1,3,order='F')).flatten()
        row = rw.repeat(3)
        cw = c_w.reshape(-1,3,order='F')
        col = np.hstack((cw,cw,cw)).flatten()
        m = sparse.coo_matrix((data,(row,col)), shape=(Mnum, Mnum))
        return m
    num = len(v)
    w13x = np.r_[v1,v3,v] + xnum
    w13y = np.r_[Vnum+v1,Vnum+v3,Vnum+v] + xnum
    w13z = np.r_[2*Vnum+v1,2*Vnum+v3,2*Vnum+v] + xnum
    m1 = __matrix(w13x,num,Mnum)
    m2 = __matrix(w13y,num,Mnum)
    m3 = __matrix(w13z,num,Mnum)
    return m1+m2+m3
```

表示fairness 对称矩阵

$$E_{fair} = \sum_{v} (w_{i-1} + w_{i+1} - 2w_i)^2$$

self.mesh.ver_regular_star:
表示每个regular格点处，上下左右相邻的4个格点列表

_matrix1 和 _matrix2 是基本的，公用的对称矩阵

表示tangent 对称矩阵

表示velocity 对称矩阵

```
def __matrix_1(self,c_w,normals,Mnum):
    """
    (wi*ni)^2 = 0
     (a,b,c)^T * (a,b,c)
     = [aa  ab  ac
        ab  bb  bc
        ac  bc  cc]
    """
    data = np.array([])
    for ni in normals:
        d = np.outer(ni,ni).flatten()
        data = np.r_[data,d]
    rw = (c_w.reshape(-1,3,order='F')).flatten()
    row = rw.repeat(3)
    cw = c_w.reshape(-1,3,order='F')
    col = np.hstack((cw,cw,cw)).flatten()
    m = sparse.coo_matrix((data,(row,col)), shape=(Mnum, Mnum))
    return m
def __matrix_2(self,c_wi,c_wj,di,dj,Mnum):
    """2*ni*nj*wi*wj = 0
     (a,b,c)^T * (d,e,f)
     = [ad  ae  af
        bd  be  bf
        cd  ce  cf]
    """
    data = np.array([])
    for k in range(len(di)):
        ni,nj = di[k], dj[k]
        d = np.outer(ni,nj).flatten()
        data = np.r_[data,d]
    rw = (c_wi.reshape(-1,3,order='F')).flatten()
    row = rw.repeat(3)
    cw = c_wj.reshape(-1,3,order='F')
    col = np.hstack((cw,cw,cw)).flatten()
    m = sparse.coo_matrix((data,(row,col)), shape=(Mnum, Mnum))
    return m.T+m
```

```
def __iso_matrix(self,c_w0,c_w1,c_w2,c_w3,d1,d2,Mnum):
    "((v2-v0)*w2-(v2-v0)*w0)^2=0"
    m0 = self.__matrix_1(c_w0,d1,Mnum)
    m2 = self.__matrix_1(c_w2,d1,Mnum)
    m02 = self.__matrix_2(c_w0,c_w2,d1,d1,Mnum)
    "((v3-v1)*w3-(v3-v1)*w1)^2=0"
    m1 = self.__matrix_1(c_w1,d2,Mnum)
    m3 = self.__matrix_1(c_w3,d2,Mnum)
    m13 = self.__matrix_2(c_w1,c_w3,d2,d2,Mnum)
    "(v2-v0)*w3 + (v3-v1)*w2 - (v2-v0)*w1 - (v3-v1)*w0"
    mm0 = self.__matrix_1(c_w0,d2,Mnum)
    mm1 = self.__matrix_1(c_w1,d1,Mnum)
    mm2 = self.__matrix_1(c_w2,d2,Mnum)
    mm3 = self.__matrix_1(c_w3,d1,Mnum)
    mm01 = self.__matrix_2(c_w0,c_w1,d2,d1,Mnum)
    mm02 = self.__matrix_2(c_w0,c_w2,d2,d2,Mnum)
    mm03 = self.__matrix_2(c_w0,c_w3,d2,d1,Mnum)
    mm12 = self.__matrix_2(c_w1,c_w2,d1,d2,Mnum)
    mm13 = self.__matrix_2(c_w1,c_w3,d1,d1,Mnum)
    mm23 = self.__matrix_2(c_w2,c_w3,d2,d1,Mnum)
    return m0+m2-m02+m1+m3-m13+mm0+mm1+mm2+mm3+mm01-mm02-mm03-mm12-mm13+mm23
```