Robust Stable Matching Problem

Master Thesis (PDM) - Spring 2024

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EPFL

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Prototype

We consider a real-life stable matching problem.

Input:

- a set of students S,
- a set of (semester) projects P,
- preference lists of each student and project over the opposite set;

Output:

- a stable matching between S and P.

Settings

Throughout the talk, we will use the following settings unless otherwise specified:

- Complete bipartite graph $G = (S \cup P, E)$,
- Two-sided strict preferences,
- $|S| = |P| = n$.

Stable matching

Definition (Blocking edge)

An edge $sp \in E$ is a **blocking edge (unstable pair)** of matching M, if

- $sp \notin M$,
- s prefers p to $M(s)$,
- p prefers s to $M(p)$,

where $M(s)$ is the partner of s in M, similar for $M(p)$.

Stable matching

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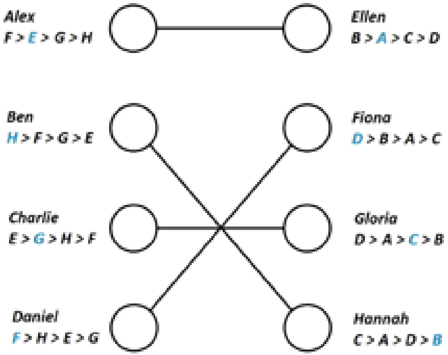
where $M(s)$ is the partner of s in M, similar for $M(p)$.

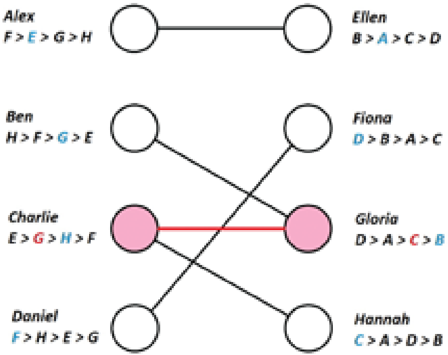
Definition (Stable matching)

A matching M is **stable** if it has no blocking edge.

Example



Alex
F > E > G > H
Ellen
B > A > C > D
Ben
H > F > G > E
Fiona
D > B > A > C
Charlie
E > G > H > F
Gloria
D > A > C > B
Daniel
F > H > E > G
Hannah
C > A > D > B



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Applications of stable matching

- Marriage;
- College admission;
- Online dating;
- Firm / worker matchings;
- Jobs / server matchings;
- Patient / hospital matchings;

Results

1. The stable matching problem can be solved in linear time, using **Gale-Shapley algorithm** (1962), which is a deferred acceptance algorithm.
2. Depending on who propose, the output of Gale-Shapley algorithm can be either student-optimal M_0, or project-optimal M_z.
3. The number of all stable matchings of an instance can be exponentially large.
4. The set of stable matchings can be equipped with polytope structure and lattice structure (explained later).

Research on stable matchings

Different iutput:

- Preference with ties,
- One-sided preferences.

Different onput (other than stability):

- Popular matching,
- Pareto-optimal matching,
- Internally stable matching,
- Robust stable matching.

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Robustness

Question: What does "robustness" mean for stable matching problems?

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When an unforeseen event occurs, we want our stable matching solution remains stable under minimum number of modifications.

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When an unforeseen event occurs, we want our stable matching solution remains stable under minimum number of modifications.

Question: What are the "unforeseen events"?

1. Elements in the preference list are swapped, e.g.,

$$\text{Alex}: \; F > E > G > H \quad \longrightarrow \quad F > G > E > H,$$

2. Certain edges are not allowed to appear in the solution,
3. Certain agents quit (have not been studied before).

Problems

We shall focus on the last kind, i.e., some projects become unavailable (removed from the instance).

Problems

Problem (Finding (a, b)-Robust Stable Matching)

Input:

- *matching instance $\mathcal{I}$ with n students, $n + a$ projects, and complete preferences,*
- *$a, b \in \mathbb{N}$.*

Output:

- *stable matching M, such that when any a matched projects are removed from $\mathcal{I}$, we can repair by breaking at most b extra edges, to get again a stable matching.*

Problems

Problem (Checking (a, b)-Robust Stable Matching)

Input:

- *matching instance $\mathcal{I}$ with n students, $n + a$ projects, and complete preferences,*
- $a, b \in \mathbb{N}$.
- *stable matching M of $\mathcal{I}$.*

Output:

- *Decision of whether M is (a, b)-robust, i.e., when any a matched projects are removed from $\mathcal{I}$, we can repair M by breaking at most b extra edges to get again a stable matching.*

Upper bound of extra changes?

Question: How many extra edges need to be changed for repairing, that is, how large can b be? Let's consider the following example, for $a = 1$.

A related problem

The formulation of our problem, is motivated by the (a, b)-supermatch problem, studied in [Genc et al., 2017].

A related problem

Problem (FINDING (a, b)-SUPERMATCH)

Input:

- *matching instance $\mathcal{I}$ with n students, n projects, and complete preferences,*
- $a, b \in \mathbb{N}$.

Output:

- *stable matching M, such that when any a matched edges are not allowed, we can repair by breaking at most b extra edges, to get again a stable matching.*

A related problem

Problem (FINDING (a, b)-SUPERMATCH)

Input:

- *matching instance $\mathcal{I}$ with n students, n projects, and complete preferences,*
- $a, b \in \mathbb{N}$.

Output:

- *stable matching M, such that when any a matched edges are not allowed, we can repair by breaking at most b extra edges, to get again a stable matching.*

Results: This problem is proved to be NP-hard. Moreover, even the special case of finding $(1, 1)$-supermatch is NP-hard.

A related problem

Problem (CHECKING (a, b)-SUPERMATCH)

Input:

- *matching instance $\mathcal{I}$ with n students, $n + a$ projects, and complete preferences,*
- $a, b \in \mathbb{N}$.
- *stable matching M of $\mathcal{I}$.*

Output:

- *Decision whether M is a (a, b)-supermatch, i.e., when any a matched edges are not allowed, we can repair M by breaking at most b extra edges to get again a stable matching.*

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Problem (Checking (a, b)-Supermatch)

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- *stable matching M of $\mathcal{I}$.*

Output:

- *Decision whether M is a (a, b)-supermatch, i.e., when any a matched edges are not allowed, we can repair M by breaking at most b extra edges to get again a stable matching.*

Results: The special case of this problem, which is Checking $(1, b)$-Supermatch, can be solved in polynomial-time, whose complexity is independent of b.

Problems to attack

	Supermathch	Robust Stable Matching
Definition	Edges not allowed	Projects deleted
Instance changed?	No	Yes
Algorithmic results	Checking $(1, b)$: poly-time	$?_1$
Complexity results	Finding $(1, 1)$: NP-hard	$?_2$

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Lists of questions:

1. Try to show that checking $(1, b)$-robustness can be solved in poly-time,
2. Try to prove that finding $(1, b)$-robust stable matching is not FPT,

   Try to prove that finding $(1, 1)$-robust stable matching is NP-hard.

Problems to attack

	SUPERMATHCH	ROBUST STABLE MATCHING
Definition	Edges not allowed	Projects deleted
Instance changed?	No	Yes
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Complexity results	Finding $(1, 1)$: NP-hard	$?_2$

Lists of questions:

1. Try to show that checking $(1, b)$-robustness can be solved in poly-time,
2. Try to prove that finding $(1, b)$-robust stable matching is not FPT,

   Try to prove that finding $(1, 1)$-robust stable matching is NP-hard.

Remark. Deleting projects is NOT the same as not allowing all the edges the projects incident to.

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Matching polytope

Definition (Incidence vector)

The **incidence vector** of a matching M is a vector $x(M) \in \{0,1\}^{|S| \times |P|}$ (for simplicity, we just write x), such that

$$x_{s,p} = \begin{cases} 1, & \text{if } M(s) = p \\ 0, & \text{otherwise} \end{cases}$$

We often identify each matching M with its incidence vector x.

Matching polytope

Theorem

A vector $x \in \mathbb{R}^{|S| \times |P|}$ *is a matching if and only if it is an integer solution of the following system of linear inequalities:*

$$\sum_{p \in P} x_{s,p} \leq 1, \quad \text{for each } s \in S, \tag{1}$$

$$\sum_{s \in S} x_{s,p} \leq 1, \quad \text{for each } p \in P, \tag{2}$$

$$x_{s,p} \geq 0, \quad \text{for each } s \in S, p \in P. \tag{3}$$

Matching polytope

Definition (Fractional matching)

A **fractional matching** is a (not necessarily integer) vector in $\mathbb{R}^{|S|\times|P|}$ which satisfies the matching constraints (1), (2), and (3).

These inequalities define a **matching polytope**.

Theorem (Birkhoff)

Each fractional matching is a convex combination of matchings.
Equivalently,

1. *Matching polytope is integral.*
2. *The extreme points of matching polytope are exactly the matchings.*

Stable matching polytope

Theorem

A vector $x \in \mathbb{R}^{|S| \times |P|}$ is a stable matching if and only if it is an integer solution of the following system of linear inequalities:

$$\sum_{p \in P} x_{s,p} \leq 1, \quad \text{for each } s \in S, \tag{1}$$

$$\sum_{s \in S} x_{s,p} \leq 1, \quad \text{for each } p \in P, \tag{2}$$

$$x_{s,p} \geq 0, \quad \text{for each } s \in S, p \in P, \tag{3}$$

$$\sum_{p' >_s p} x_{s,p'} + \sum_{s' >_p s} x_{s',p} + x_{s,p} \geq 1, \quad \text{for each } s \in S, p \in P. \tag{4}$$

We call the last inequality the ***stability constraint****.*

Stable matching polytope

Definition (Stable fractional matching)

A **stable fractional matching** is a (not necessarily integer) vector which satisfies the matching constraints (1), (2), (3), and (4).

These inequalities define a **stable matching polytope**, denoted as $P(\mathcal{M})$.

Theorem (Vande Vate, 1989)

Each stable fractional matching is a convex combination of stable matchings. Equivalently,

1. *Stable matching polytope is integral.*
2. *The extreme points of stable matching polytope are exactly the stable matchings.*

Stable matching lattice

Let $\mathcal{M}$ be the set of stable matchings.

$$\begin{tikzcd}
\text{poset } (\mathcal{M}, \succeq) \arrow[d] \\
\text{lattice } (\mathcal{M}, \vee, \wedge) \arrow[d] \\
\text{distributive lattice } (\mathcal{M}, \vee, \wedge)
\end{tikzcd}$$

Poset

Definition

For any two (stable) matching M and M', $M \succeq M'$ if and only if all students weakly prefer M to M'. We say that M **dominates** M'.

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Example. Students: 1, 2, 3, 4; Projects: A, B, C, D.
The following are their preference lists.

1	$\boxed{A}$	{B}	C	D
2	$\boxed{B}$	{A}	D	C
3	{C}	$\boxed{D}$	A	B
4	{D}	$\boxed{C}$	B	A

A	4	3	2	1
B	3	4	1	2
C	2	1	4	3
D	1	2	3	4

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3	{C}	$\boxed{D}$	A	B
4	{D}	$\boxed{C}$	B	A

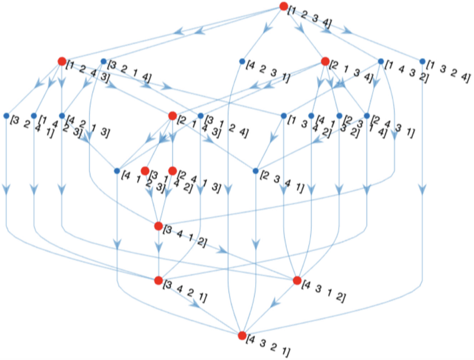
A	4	3	2	1
B	3	4	1	2
C	2	1	4	3
D	1	2	3	4

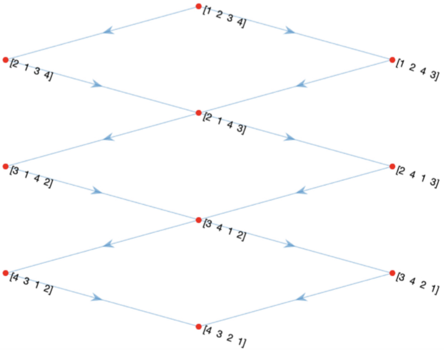
$$M_{\text{red}} \quad \succeq \quad M_{\{\}}, M_{\Box} \quad \succeq \quad M_{\text{blue}}$$

Examples

1	A	B	C	D
2	B	A	D	C
3	C	D	A	B
4	D	C	B	A

A	4	3	2	1
B	3	4	1	2
C	2	1	4	3
D	1	2	3	4

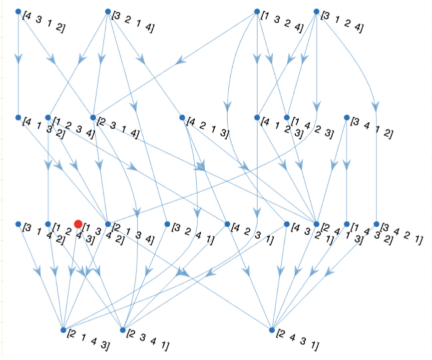




Examples

1	C	A	D	B
2	B	C	A	D
3	B	A	C	D
4	B	D	C	A

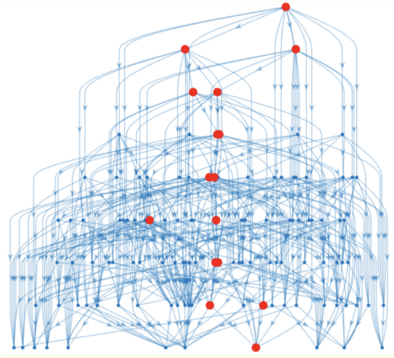
A	1	3	2	4
B	4	2	3	1
C	2	3	1	4
D	4	3	1	2

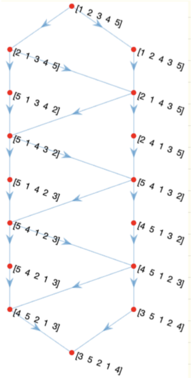


Examples

1	A	B	E	D	C
2	B	A	D	E	C
3	C	D	A	B	E
4	D	C	B	A	E
5	E	B	C	D	A

A	5	4	3	2	1
B	3	4	5	1	2
C	2	1	5	4	3
D	5	1	2	3	4
E	4	3	2	1	5





Two operators

Definition

For two stable matchings M, M', define

$$M \vee M' = M^{\uparrow},$$

where $M^{\uparrow}$ is the set of student-project pairs, in which each student is matched to their better (more preferred) partner between M and M'.

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For two stable matchings M, M', define

$$M \vee M' = M^{\uparrow},$$

where $M^{\uparrow}$ is the set of student-project pairs, in which each student is matched to their better (more preferred) partner between M and M'.

For two stable matchings M, M', define

$$M \wedge M' = M^{\downarrow},$$

where $M^{\downarrow}$ is the set of student-project pairs, in which each student is matched to their worse (less preferred) partner between M and M'.

Example.

1	$\boxed{A}$	{B}	C	D
2	$\boxed{B}$	{A}	D	C
3	{C}	$\boxed{D}$	A	B
4	{D}	$\boxed{C}$	B	A

A	4	3	2	1
B	3	4	1	2
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Example.

$$
\begin{array}{c|cccc}
1 & \boxed{A} & \{B\} & C & D \\
2 & \boxed{B} & \{A\} & D & C \\
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\end{array}
\qquad\qquad
\begin{array}{c|cccc}
A & 4 & 3 & 2 & 1 \\
B & 3 & 4 & 1 & 2 \\
C & 2 & 1 & 4 & 3 \\
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\end{array}
$$

$$
\begin{aligned}
M_{\text{red}} &= M_{\{\}} \vee M_{\Box} \\
\{1B, 2A, 3D, 4C\} &= M_{\{\}} \wedge M_{\Box}
\end{aligned}
$$

Join and meet

Proposition

1. $M^{\uparrow}, M^{\downarrow}$ *are stable matchings,*
2. $M^{\uparrow}$ *is the* ***join*** *of* M *and* M'*, i.e.,*
   - $M^{\uparrow} \succeq M$ *and* $M^{\uparrow} \succeq M'$*,*
   - *for any* M'' *such that* $M'' \succeq M$ *and* $M'' \succeq M'$*,* $M'' \succeq M^{\uparrow}$
3. $M^{\downarrow}$ *is the* ***meet*** *of* M *and* M'*, i.e.,*
   - $M \succeq M^{\downarrow}$ *and* $M' \succeq M^{\downarrow}$*,*
   - *for any* M'' *such that* $M \succeq M''$ *and* $M' \succeq M''$*,* $M^{\downarrow} \succeq M''$*.*

Lattice

A poset where each pair of elements has a join and a meet is a lattice. Hence,

Corollary

The stable matching poset $(\mathcal{S}, \succeq)$*, equipped with* $\vee$ *and* $\wedge$*, becomes a lattice* $(\mathcal{S}, \vee, \wedge)$*.*

Distributive lattice

Proposition

The stable matching lattice $(\mathcal{S}, \vee, \wedge)$ is ***distributive****, i.e.,*

$$(M \vee M') \wedge M'' = (M \wedge M'') \vee (M' \wedge M'')$$
$$(M \wedge M') \vee M'' = (M \vee M'') \wedge (M' \vee M'')$$

Representation theorem

(Finite) distributive lattice has a nice property.

Theorem (Birkhoff's representation theorem, 1937)

For any finite distributive lattice $(\mathcal{L}, \vee, \wedge)$, there exists

- *a poset $(\mathcal{P}, \succeq^*)$ called the **representation poset** of $\mathcal{L}$,*
- *a bijection between $\mathcal{L}$ and the upper closed subsets of $(\mathcal{P}, \succeq^*)$.*

Representation theorem

Definition

A subset $Q \subseteq \mathcal{P}$ is an **upper closed subset** of $(\mathcal{P}, \succeq^*)$, if

$$q \in Q,\ q' \succeq^* q \quad \Longrightarrow \quad q' \in Q.$$

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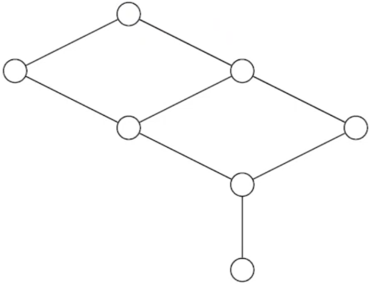
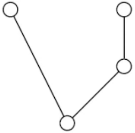
Remark. Usually $|\mathcal{P}| \ll |\mathcal{L}|$.

Example



(a) Hasse Diagram of the Lattice



(b) Representation Poset

Representation of stable matching lattice

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Now, apply the Birkhoff's representation theorem to the stable matching lattice:

Representation of stable matching lattice

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- *a bijection between $\mathcal{L}$ and the upper closed subsets of $(\mathcal{P}, \succeq^*)$.*

Now, apply the Birkhoff's representation theorem to the stable matching lattice:

The representation poset of stable matching lattice is called **rotation poset**, denoted as $(\mathcal{R}, \succeq^*)$, whose element is called **rotation**, denoted as ρ.

Rotation poset

- A rotation $\rho \in \mathcal{R}$ can transform a stable matching M to another stable matching M/ρ, called **elimination of ρ from M**. Rotation can help us traverse in the lattice of stable matchings.
- $|\mathcal{R}| = O(n^2)$.

Rotation poset

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- $|\mathcal{R}| = O(n^2)$.

What is a rotation exactly?

Relation between polytope and lattice

Definition (Order polytope)

For a poset $(\mathcal{P}, \succeq)$, define its associated **order polytope**

$$O(\mathcal{P}) := \{x \in [0,1]^{|\mathcal{P}|} : x_i \geq x_j \ \text{ if } \ i \succeq j\}.$$

Relation between polytope and lattice

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Theorem

*Let $(\mathcal{M}, \vee, \wedge)$ be a stable matching lattice and let $\mathcal{R}$ be its rotation poset (from Birkhoff's representation), then the stable matching polytope $P(\mathcal{M})$ is **affinely equivalent** to the order polytope $O(\mathcal{R})$, i.e.,*

$$P(\mathcal{M}) = A \cdot O(\mathcal{R}) + x(M_0),$$

where $A \in \mathbb{R}^{|E| \times |\mathcal{R}|}$ and $x(M_0)$ is the incidence vector of the student-optimal stable matching M_0.

For each stable matching instance,

SM lattice $\xleftrightarrow{\text{Birkhoff}}$ rotation poset, whose order polytope $\xleftrightarrow{\text{affine } \simeq}$ **SM polytope**