Robustness of Matching: Backup Nodes Problem

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Robustness of matching (Informal)

- Given a bipartite graph, consider the situation where some nodes arrive/leave,

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- Given a bipartite graph, consider the situation where some nodes arrive/leave,
- Want to preserve a certain property of matching, e.g., perfectness,
- A matching is “robust” if it can be recovered with the minimum changes after the arrivals/departures of nodes.

Problem

BACKUP NODES PROBLEM (BN)

Input: A bipartite graph $G = (A \cup B, E)$ where $|A| < |B|$ and there exists an A-perfect matching in G.

Output: A subset $S \subseteq B$ which maximizes the number of elements in A that have neighbors in S while maintaining an A-perfect matching between A and $B \setminus S$.

Motivation: project assignment

- matching between students and projects,
- possible situation: a matched project becomes unavailable after the matching result is published,
- we would like to find a “backup” unmatched project for the corresponding student without interfering other students.

Example

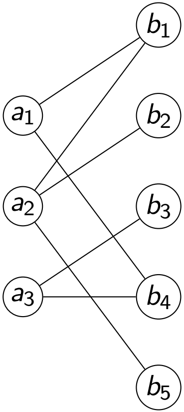

Figure: An instance of BN with 3 students and 5 projects.

Example

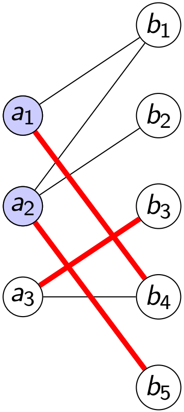

Figure: A feasible solution.

Example

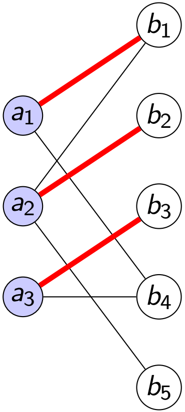

Figure: An optimal solution.

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Main results

	General	**Degree-constrained**
Algorithm	◦ $1 - 1/e$ approximation, via submodular maximization over a matroid constraint	◦ Polynomial-time (exact) solvable, when G is $(d, 2)$-regular, $d \geq 3$.
Complexity	◦ NP-hard to approximate within $1 - 1/e + \varepsilon$	◦ NP-hard to approximate within $293/297$, when $\Delta(G) = 4$

General BN

$$\begin{aligned} \max \quad & |N(S)| \\ \text{s.t.} \quad & \exists \text{ perfect matching between } A \text{ and } B \setminus S \\ & \emptyset \subseteq S \subseteq B \end{aligned}$$

- Maximizing coverage function, over dual matroid of the matching matroid,
- $1 - 1/e$ approximation.

Degree-constrained BN

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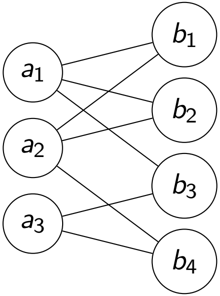
2. $\deg(a) \geq d_A, \deg(b) \leq d_B$: take any feasible S, we have

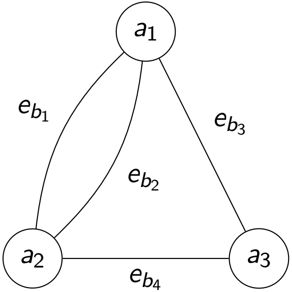
$$|N(S)| = \left(1 - \frac{d_B}{d_A}\right) |A|.$$

$(d, 2)$-regular BN

Idea: compute maximum matching in an auxiliary graph.



(a) The original graph $G = (A \cup B, E)$



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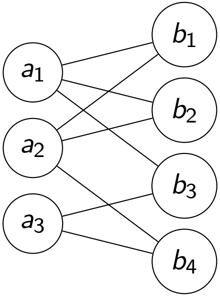

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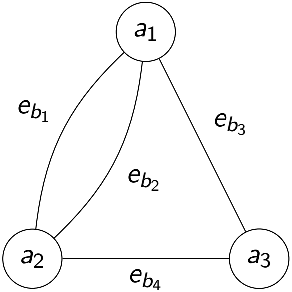

(b) The auxiliary graph $G' = (A, E')$

Algorithm: max matching M of G' $\xrightarrow{\text{correspondence}}$ $S \subseteq B$ $\xrightarrow{\text{augmentation}}$ optimal solution.

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Open questions

Are there any other special cases of BN, which are solvable/approximable?

For example,

1. the first unsolved case: $\deg(b) = 3, \forall\, b \in B$,
2. other bounded degree constraints,
3. bounded VC-dimension.

References

- [ABKN09] On revenue maximization in second-price ad auctions. In Amos Fiat and Peter Sanders, editors, Algorithms - ESA 2009, pages 155–166, Berlin, Heidelberg, 2009. Springer Berlin Heidelberg.
- [PSVW25] Second price matching with complete allocation and degree constraints. Preprint, 2025.

Thank you for listening! 😉

Tightness of $(1 - 1/e)$-approximation for BN

Gap-preserving reduction from $\textsc{Max}$ k-$\textsc{Cover}$, which has no better approximation than $1 - 1/e$ assuming P $\neq$ NP [Feige, 1998].

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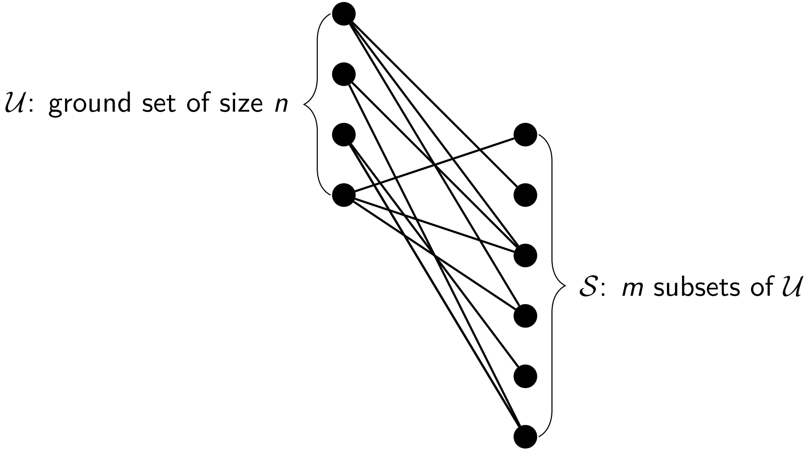
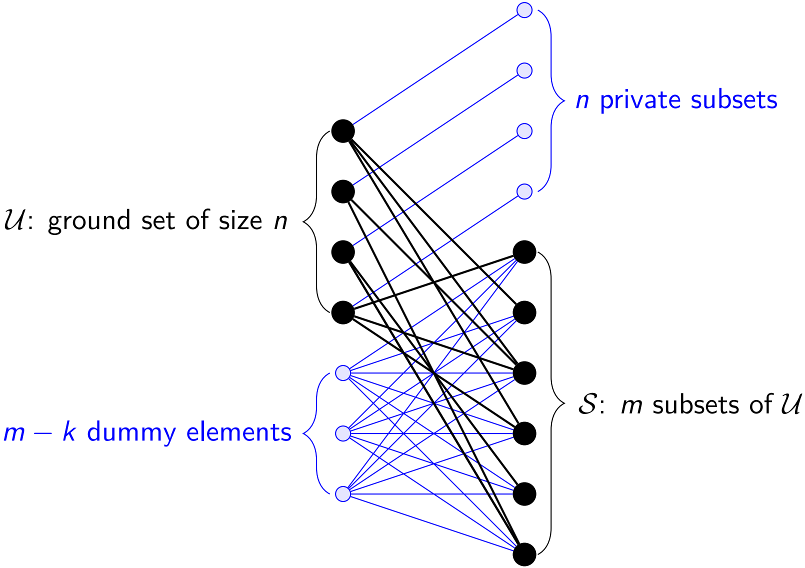
Step 1: Basic reduction for NP-hardness.

Step 2: Amplifying the gap.

NP-hardness


$\mathcal{U}$: ground set of size n
$\mathcal{S}$: m subsets of $\mathcal{U}$

NP-hardness


n private subsets
$\mathcal{U}$: ground set of size n
$\mathcal{S}$: m subsets of $\mathcal{U}$
$m - k$ dummy elements

Second price auctions with binary bids

A related problem called (OFFLINE) SECOND-PRICE MATCHING is studied by Azar, Birnbaum, Karlin, and Nguyen in 2009.

SECOND-PRICE MATCHING (2PM)

Input: A bipartite graph $G = (A \cup B, E)$.

Output: A matching M with the maximum size such that all matched nodes in A has an unmatched neighbor in B.

- A: goods,
- B: bidders,
- Only 0 or 1 bids,
- Maximize the second-price auction profit.