Parabola or catenary?

Introduction

From as far as I remember, my parents have always placed a great emphasis on Mathematics. Perhaps it is because my parents understood the value of Mathematics and sought to inculcate me. Perhaps it is simply because of my heritage. But whatever it is, my teachers have always said that Mathematics is all around us and through working on this exploration, I realized it is indeed true!

I initially planned to model the tidal waves using a variety of methods – trigonometry, polynomial regression and linking it to a Maclaurin's series. This is because my mother has been exposing me to extreme sports such as snowboarding and scuba diving and as a certified PADI junior scuba diver, I find that it important to know how the tide changes with time, to draw up a dive plan. As such, I thought modelling the tidal waves would be a meaningful exploration – something I would be able to relate to. However, my supervisor encouraged me to go deeper, to relate trigonometry and calculus instead.

Looking beyond the tide, I started to think about possible exploration ideas and I remember that each time my friends and I meet at Palawan Beach, Sentosa to play volleyball; we always meet at the Sentosa suspension bridge (more information will be provided on Page 3). The Sentosa suspension bridge served as a useful landmark as it was large enough not to be missed from far (see Figure 1) and it is a tourist hotspot.

Figure 1: Sentosa suspension bridge at Palawan Beach, Sentosa, Singapore[1]

When I first saw the Sentosa suspension bridge years ago, I remember I wondered to myself whether the bridge was a parabola. This is because back in Year 3, we did a Mathematics coursework and used the quadratic function to model real life scenarios such as the path of water coming out from a fountain, the motion of a basketball, etc. However, I also read online[2] that besides a quadratic, most suspension bridges can be modelled by a catenary instead. Hence, for this exploration, **I shall be using Mathematics to determine if the cables holding the Sentosa suspension bridge resembles a quadratic function, or can it be modelled using a catenary function.**

To do so, I will first seek to derive the equation of a catenary curve using the Mathematics that I have learnt from the analysis and applications course. From there, I will model the shape of the cables using both a quadratic model and a catenary to determine which would fit better.

[1] Things to do at Palawan Beach, from https://www.sentosa.com.sg/en/things-to-do/attractions/palawan-beach/, last accessed on 27 July 2021
[2] Catenary and parabola, from http://arch-re-review.blogspot.com/2010/09/catenary-and-parabola.html, last accessed on 27 July 2021

The Sentosa suspension bridge and its dimensions

Located at Palawan Beach, which is on Sentosa Island, the Sentosa suspension bridge is a suspension bridge that leads to a small islet off the coast which is commonly known as the southernmost point of the Asian continent. It is also Asia's closest point to the Equator.[3] Sentosa is an island located off the southern coast of Singapore's main island which used to be a British military base and a Japanese prisoner-of-war camp during the Japanese Occupation. Today, however, it is a popular tourist destination.

In order to model the bridge, I had to first obtain some critical dimensions of the bridge. As the information is not readily available, I had to make some on-site measurements (see Figure 2) and rely on Google map to make some scaled calculations. From my calculations, I found that the entire bridge spans 56 meters across and the lowest point of the cable 1.35 meters from the horizontal. I created a schematic diagram for easy reference (see Figure 3).

Figure 2: On-site measurement of the suspension bridge

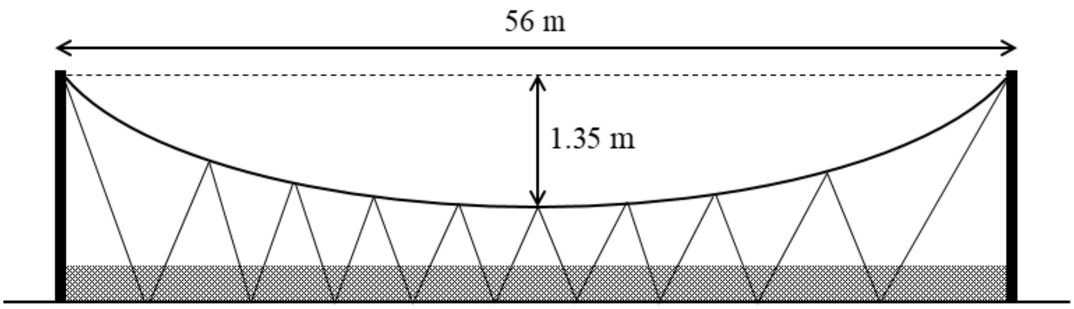

Figure 3: Schematic diagram of the suspension bridge

[3] Palawan Beach, by Sentosa, https://www.sentosa.com.sg/, last accessed on 28 July 2021

A catenary

Let me first explain what a catenary is. When a chain is hung between the two points from two different poles, it forms a shape of a curve due to its own weight. This curve is called a catenary curve, which came from the Latin word ‘Catena’, meaning of ‘chain’[4].

I shall then attempt to derive the equation of a catenary curve, by considering the forces acting on it. As the catenary curve is symmetric about the y-axis (see Figure 4), I shall first consider the part of the curve where $x \geq 0$. For $x \geq 0$, there are only two points which are connected externally. Consequently, these are the only two points, which are able to exert force on the curve directly. They are the fixed end (above) and the lowest point (also the center of symmetry, below).

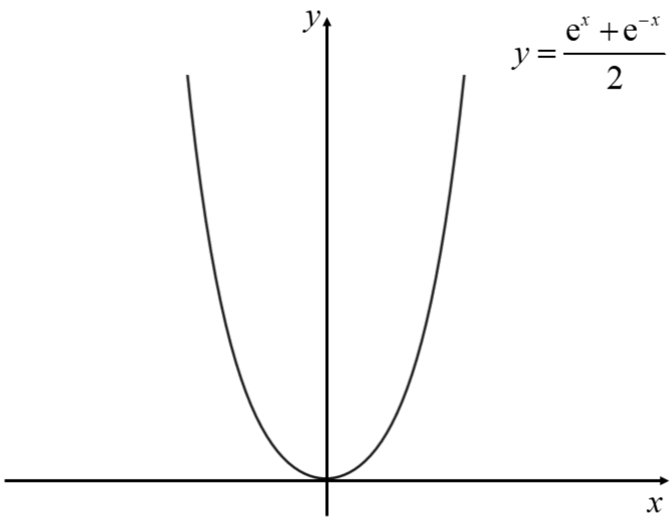

Figure 4: A catenary curve

I shall label the lowest point A, and the right end of the curve as point B. For point B, since it is fixed, the external force exerted on it, labelled F, can be in any direction. For point A, since it is at the lowest point and approximately horizontal, the force exerted on it must be horizontal, labelled T (for tension). In total, the right half of the catenary curve experiences three external forces, namely the horizontal tension T, the gravitational force G and an undetermined force F.

[4] Catenary, from Britannica, https://www.britannica.com/science/catenary, last accessed on 27 July 2021

As the entire string is at rest, the resultant force acting on it must be 0 in all directions. The vertical component of F must be equal to its gravity, G and the horizontal component of F is equal to the tension, T. By letting the angle between F and the horizontal be θ, we can resolve the forces as seen in Figure 5.

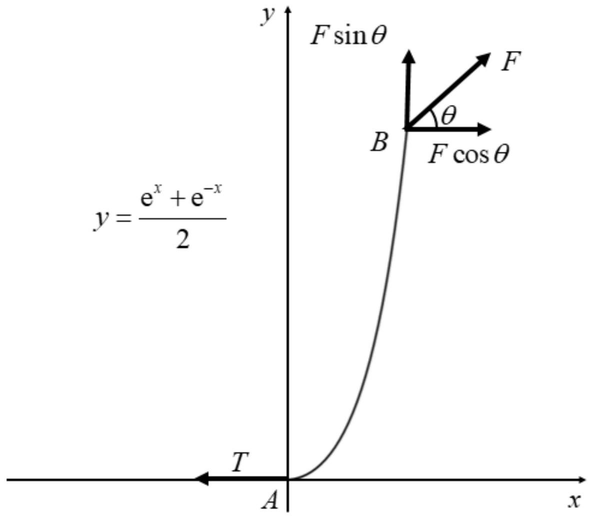

Figure 5: Force analysis of a catenary curve

From above, we have the following:

$$F\cos\theta = T - (1)$$

$$F\sin\theta = G - (2)$$

However, since the gravitational force that is exerted on the catenary is dependent on its mass and the mass is a product of its linear density, ρ and length l, we have $G = \rho lg$, where g is the gravitational acceleration constant.

Substituting $G = \rho lg$ into equation 2, we have

$$F\sin\theta = \rho lg - (3)$$

Diving equation 3 with equation 1, we have

$$\tan\theta = \frac{\rho lg}{T} - (3)$$

As the tangent value of θ at any point of the curve represents a small change in the ratio between the change in y-value to that of the x-value, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta = \frac{\rho l g}{T} - (4)$$

Since the linear density, ρ and the gravitational acceleration, g are constants, I shall simplify equation 4 by letting $k = \frac{\rho}{g}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = kl - (5)$$

In order to obtain the length of the curve, l, I shall use an approach similar to that of finding the area under a curve by Riemann Sum, which is to consider an infinitely small change in x. As with the Riemann Sum convention, I shall define this small change in x as Δx. For any Δx, the catenary curve will resemble more of a straight line than a curve. Consequently, we can use a right-angle triangle to estimate Δl for a given Δx (see Figure 6).

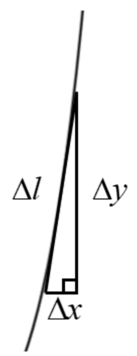

Figure 6: A close up of the catenary curve of width Δx

By Pythagoras' Theorem, we have $(\Delta l)^2 = (\Delta x)^2 + (\Delta y)^2$.

Making Δl the subject, $\Delta l = \sqrt{(\Delta x)^2 + (\Delta y)^2}$.

Factorising $(\Delta x)^2$ out, $\Delta l = \Delta x\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$.

Integrating both sides with respect to x, we have

$$l = \int \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x - (6)$$

which gives us a way to find the length, l of the catenary curve.

In fact, from my research, I found that the definite integral $\int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x$ can be used to find the arc length of a given curve.

By substituting equation 6 into equation 5, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = k \int_0^x \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x \ - (7)$$

In order to get rid of the integral, I differentiate equation 7 with respect to x

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = k \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \ - (8)$$

From here on, in order to simplify the working, I shall let $v = \frac{\mathrm{d}y}{\mathrm{d}x}$.

Then, from equation 8 we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = k \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = k\sqrt{1 + v^2} \ - (9)$$

which is a differential equation in terms of v and x.

By using the method of separating the variables, we have

$$\int \frac{1}{\sqrt{1 + v^2}} \mathrm{d}v = k \int \mathrm{d}x - (10)$$

While I know that $\int \frac{1}{\sqrt{1-x^2}} \mathrm{d}x = \mathrm{arcsin}x$ from implicit differentiation, I had to take some time to figure out that the integral of $\int \frac{1}{\sqrt{1+x^2}} \mathrm{d}x$ requires the substitution $= \tan\theta$.

Differentiating $x = \tan\theta$ with respect to x, we have $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \sec^2\theta$, i.e.,

$$\int \frac{1}{\sqrt{1+x^2}} \mathrm{d}x$$

$$= \int \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}} \mathrm{d}\theta$$

$$= \int \sec\theta \, \mathrm{d}\theta$$

$$= \ln|\sec\theta + \tan\theta| + c$$

$$= \ln\left|x + \sqrt{1+x^2}\right| + c$$

Applying the above to equation 10, we have

$$\ln\left|v + \sqrt{1+v^2}\right| = kx + c_1$$

Since when $x = 0$, $v = \frac{\mathrm{d}y}{\mathrm{d}x} = 0$, we have $c_1 = 0$, i.e.,

$$\ln\left|v + \sqrt{1+v^2}\right| = kx - (11)$$

Taking exponential on both sides, we have

$$v + \sqrt{1+v^2} = e^{kx}$$

$$\sqrt{1+v^2} = e^{kx} - v$$

$$1 + v^2 = e^{2kx} - 2ve^{kx} + v^2$$

$$2ve^{kx} = e^{2kx} - 1$$

$$v = \frac{e^{2kx} - 1}{2e^{kx}}$$

$$v = \frac{e^{kx} - e^{-kx}}{2} - (12)$$

Now, since $v = \frac{\mathrm{d}y}{\mathrm{d}x}$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^{kx} - e^{-kx}}{2}$$

Integrating both sides with respect to x, we have

$$y = \frac{e^{kx} + e^{-kx}}{2k} + c - (13)$$

The above equation resembles the general equation of a hyperbolic cosine function[5] which is

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

Substituting the hyperbolic cosine function into equation 13, we have

$$y = \frac{\cosh(kx)}{k} + c^{6}$$

By letting $\alpha = \frac{1}{k} = \frac{g}{\rho}$, we have the general form of a catenary curve,

$$y = \alpha\cosh\left(\frac{x}{\alpha}\right) + c - (14)$$

Since I will be setting the lowest point of the cables to be the origin, by substituting $(0,0)$ into equation 14, we have

$$0 = \alpha\cosh(0) + c$$

i.e., $c = -\alpha$.

Thus, the catenary equation can be simplified as

$$y = \alpha\cosh\left(\frac{x}{\alpha}\right) - \alpha - (15)$$

[5] Hyperbolic functions, https://en.wikipedia.org/w/index.php?title=Hyperbolic_functions&oldid=965521206, last accessed on 28 July 2021

[6] Equation of catenary, http://192.168.1.121/math2/equation-catenary/, last accessed on 28 July 2021

Fitting a quadratic curve

In order to fit a curve to the cable of the suspension bridge, I shall first superimpose the image in Figure 2 on a Desmos canvas. From there, I found that $(5,0.8)$ lies on the curve (see Figure 7).

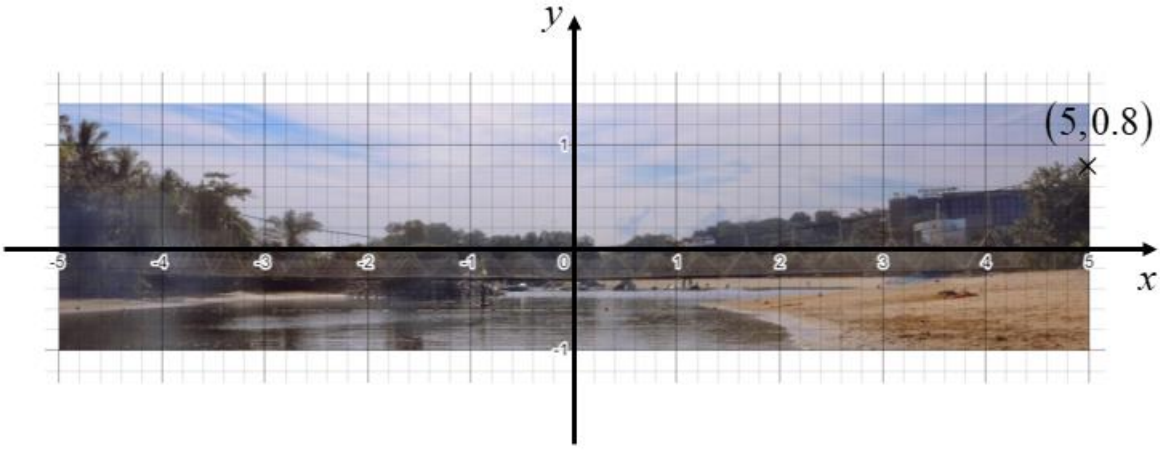

Figure 7: Image of Sentosa suspension bridge superimposed on a Desmos grid

Although there are three different but equivalent forms of a quadratic curve, as I intend to coincide the lowest point of the curve with the origin, it makes sense to use the vertex form, i.e., $y = a(x-h)^2 + k$, where (h,k) is the vertex. Clearly, $h = k = 0$.

Letting $g(x) = ax^2$, since $(5,0.8)$ must lie on $g(x)$, $g(5) = 0.8$.

Solving, $a = 0.032$. Therefore, a quadratic curve (in blue) which fits the cable of the Sentosa suspension bridge is $y = 0.032x^2$ (see Figure 8).

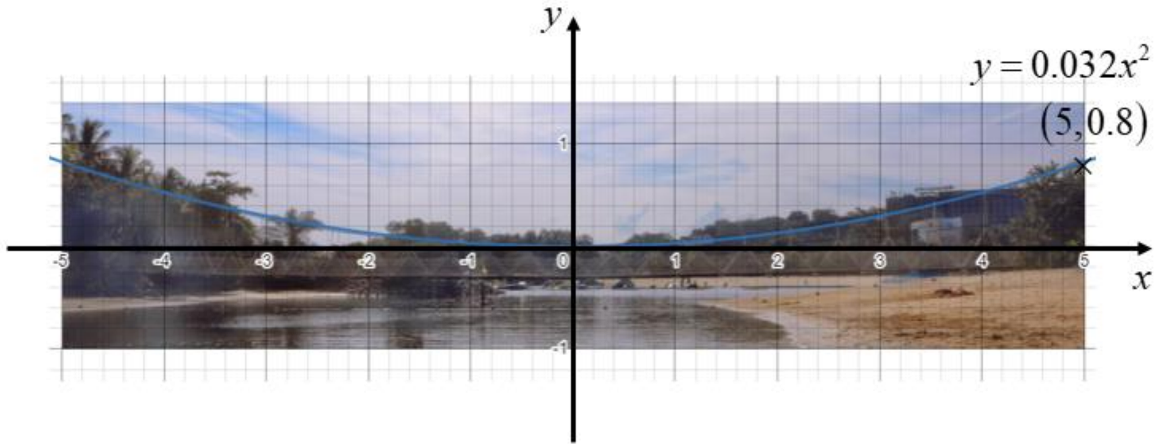

Figure 8: Quadratic curve fitting the cables of Sentosa suspension bridge

Fitting a catenary curve

Having derived the equation for a catenary curve on page 9, I shall also proceed to obtain the equation which models the cables of Sentosa suspension bridge.

By letting $f(x) = \alpha\cosh\left(\frac{x}{\alpha}\right) - \alpha$, $f(5) = 0.8$, i.e.,

$$\alpha\cosh\left(\frac{5}{\alpha}\right) - \alpha = 0.8$$

Solving, we have $\alpha = 15.9$. Therefore, the catenary curve (in orange) that fits the cable of the Sentosa suspension bridge is $y = 15.9\cosh\left(\frac{x}{15.9}\right) - 15.9$ (see Figure 9).

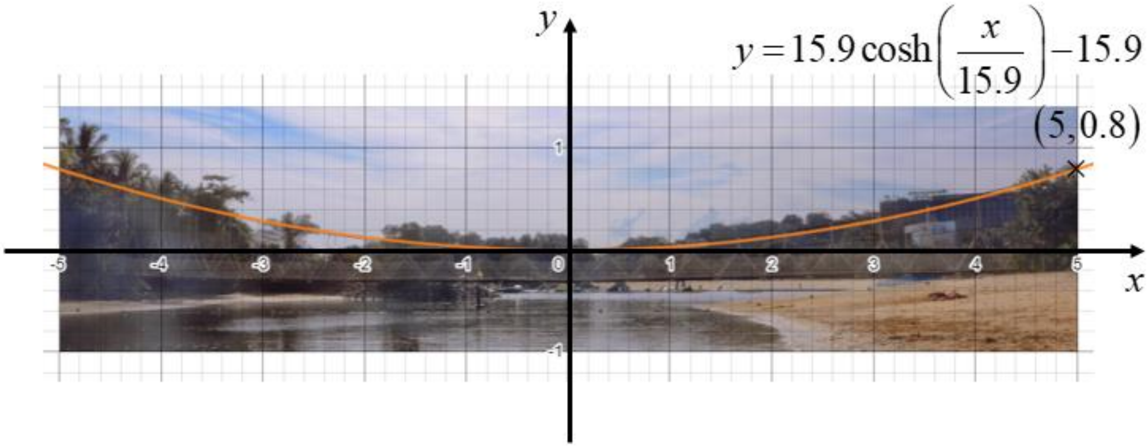

Figure 9: Catenary curve fitting the cables of the Sentosa suspension bridge

Comparing the two models

While both models appear to fit quite nicely, especially at the origin and the end point, $(5,0.8)$, since the respective curves are modelled using these points, upon zooming in, there is indeed some discrepancy between the model (both quadratic and catenary) and the cable (see Figure 10). While care has been taken to ensure that the image is symmetric about the lowest point, it appears that the image of the cable is not entirely symmetrical, which might be due to varying loads on the bridge itself (see Figure 11).

Figure 10: An enlarged version of the 3 curves – cable, quadratic and catenary

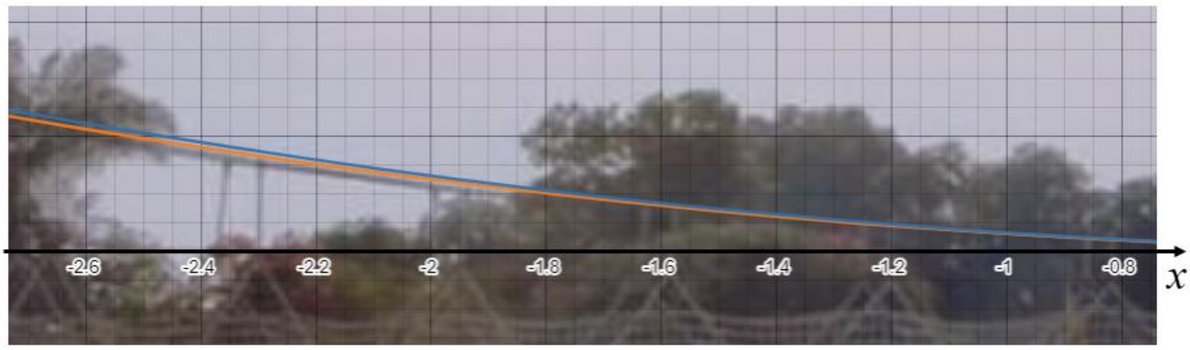

Figure 11: An enlarged version of the above, for $x < -0.8$ (note y-axis not visible)

Conclusion

While it is not evident right away whether one curve fits the cable of Sentosa suspension bridge better than the other, the process of deriving the equation of the catenary curve has helped me understand integration and differential equations much better. Aside from using a catenary, perhaps using a Maclaurin series expansion might prove to be even more useful.

Nonetheless, despite some of the limitations – difficulty in getting a proper, symmetric image, difficulty in obtaining actual information about the suspension bridge, the linear density, ρ of the cable and whether it is uniform, etc, there are also many strengths – using more than one method to model the cable of Sentosa suspension bridge, using Mathematics from the analysis and approaches, Higher Level syllabus and moving slightly beyond.

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