1.FUN WITH VECTOR CALCULUS

1. Assume we have n real-valued scalar data points $x_1, x_2, \ldots, x_n$. Analytically derive constant μ for which, $\sum_{i=1}^{n}(x_i-\mu)^2$ is minimized.

   Here we set it as the loss function $L(\mu)$

   $$L(\mu) = \sum_{i=1}^{n}(x_i-\mu)^2$$

   In order to get the minimum, we set $\nabla L(\mu) = 0$

   $$\nabla L(\mu) = 2\sum_{i=1}^{n}(x_i-\mu) = 0$$

   $$\sum_{i=1}^{n}(x_i-\mu) = 0$$

   $$\sum_{i=1}^{n}x_i - n\mu = 0$$

   $$\mu = \frac{1}{n}\sum_{i=1}^{n}x_i$$

   Thus, when μ is the mean of $x_1, x_2, \ldots, x_n$, the loss function is minimized.
2. Assume we have n data points are real d-dimensional vectors. Analytically derive a constant μ for which, $\sum_{i=1}^{n}||x_i-\mu||_2^2$ is minimized.

   We can easily know that μ is also a d-dimensional vector. Here we set it as the loss function $L(\mu)$

   $$L(\mu) = \sum_{i=1}^{n}||x_i-\mu||_2^2 = \sum_{i=1}^{n}\sum_{j=1}^{d}\sqrt{(x_{ij}-\mu_j)^2}^2 = \sum_{i=1}^{n}\sum_{j=1}^{d}(x_{ij}-\mu_j)^2$$

   In order to get the minimum, we need to let the derivative of every $\mu_j (j \in (1, \ldots, d))$ equals to zero so as to get the derivative of μ equals to zero.

   $$\frac{\partial L(\mu)}{\partial \mu_j} = -2\sum_{i=1}^{n}(x_{ij}-\mu_j) = 0$$

   $$\mu_j = \frac{1}{n}\sum_{i=1}^{n}x_{ij}$$

   Thus, we can get the value of μ,

   $$\mu = \begin{bmatrix} \frac{\sum_{i=1}^{n}x_{i1}}{n} \\ \frac{\sum_{i=1}^{n}x_{i2}}{n} \\ \cdots \\ \frac{\sum_{i=1}^{n}x_{id}}{n} \end{bmatrix}$$

   Thus, when μ is the mean vector of $x_1, x_2, \ldots, x_n$, the function is minimized.

2.LINEAR REGRESSION WITH NON-STANDARD LOSSES

1. Using matrix/vector notation, write down a loss function that measures the training error in terms of the l_1-norm.

Assuming there are m data points. For each data point x, the number of its features is n:

$$x^{(i)} = \begin{bmatrix} x_0^{(i)} \\ x_1^{(i)} \\ \dots \\ x_n^{(i)} \end{bmatrix}$$

For every data point, the $x_0 = 1$ so as to serve the bias. From the question we know that X is the matrix of training data points (stacked row-wise)

$$X = \begin{bmatrix} x_0^{(i)T} \\ x_1^{(i)T} \\ \dots \\ x_m^{(i)T} \end{bmatrix} = \begin{bmatrix} 1 & x_1^{(i)} & x_2^{(i)} & \cdots & x_n^{(1)} \\ 1 & x_1^{(2)} & x_2^{(2)} & \cdots & x_n^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & x_1^{(i)} & x_2^{(m)} & \cdots & x_n^{(m)} \end{bmatrix}$$

For each data point x_i, it has a label y_i

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_m \end{bmatrix}$$

We set the weight as ω:

$$\omega = \begin{bmatrix} \omega_0 \\ \omega_1 \\ \dots \\ \omega_m \end{bmatrix}$$

So the predictions of the data points should be:

$$\begin{bmatrix} \omega_0 x_0^{(1)} + \omega_1 x_1^{(1)} + \ldots + \omega_n x_n^{(1)} \\ \dots \\ \dots \\ \omega_0 x_0^{(m)} + \omega_1 x_1^{(m)} + \ldots + \omega_n x_n^{(m)} \end{bmatrix} = X\omega$$

Thus the loss should be the l_1-norm difference between labels Y and predictions:

$$L(\omega) = \sum_{i=1}^{m} |y_i - \sum_{j=1}^{n} \omega_j x_j^{(i)}| = \sum_{i=1}^{m} |y_i - \omega^T x^{(i)}|$$

For conciseness, we write this as:

$$L(\omega) = ||X\omega - y||_1$$

2. Can you write down the optimal linear model in closed form? If not, why not? So we can't write down the optimal linear model in closed form.

   No, because the l_1 loss function is not a continuous derivative function.
   From the Figure 1, we can easily know that when $X\omega - Y = 0$, the loss is the minima. However, this point doesn't have derivative. Thus, we cannot find the minima by finding the point whose derivative is zero.

3. If the answer to b is no, can you think of an alternative algorithm to optimize the loss function? Comment on its pros and cons.

   Although using gradient descent we can get a comparatively small value closed to the optimal of l1 loss function, it might be hard to reach the optimal(minima). The gradient of the l1 loss

Figure 1. L1 loss function

function only have two values, thus when it is in neighborhood of the optimal, it might still have a comparatively large gradient, thus it might miss the optimal and jump around the minima.

However, in other words, if we try to minimize l_1 loss (a.k.a MAE), that prediction would be the median of all observations. Here's how to prove it:
Assuming we have $y_1, y_2, \ldots, y_n$ and β is the prediction.

$$L_1 = \sum_{i=1}^{n} |y_i - \beta|$$

$$\frac{\partial L_1}{\partial \beta} = -\sum_{i=1}^{n} sgn(y_i - \beta)$$

$$sgn(y_i - \beta) = \begin{cases} 1 & \text{if } \mathrm{y}_i > \beta \\ -1 & \text{if } \mathrm{y}_i < \beta \end{cases}$$

Thus, the derivative equals to 0 when there is the same number of positive and negative terms among the $y_i - \beta$, which means β should be the median of y_i

Pros:
a. More robust to outliers compared to L2 loss;
b. Easy to understand and implement.
Cons:
a. The complexity for finding median would be high when the data set is huge;
b. Median can be a bit jumpy in small samples made up of discrete values;
c. When data points are a bit of clustered, using median might not a good choice.

3.HARD CODING A MULTI-LAYER PERCEPTRON

The multi-layer perceptron network structure is shown in figure 2:
for Perceptron1 P_1:

$$f(x) = \begin{cases} 1 & \text{if } \mathrm{x}_2 - x_1 > 0 \\ 0 & \text{otherwise} \end{cases}$$

for Perceptron1 P_2:

$$f(x) = \begin{cases} 1 & \text{if } \mathrm{x}_3 - x_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

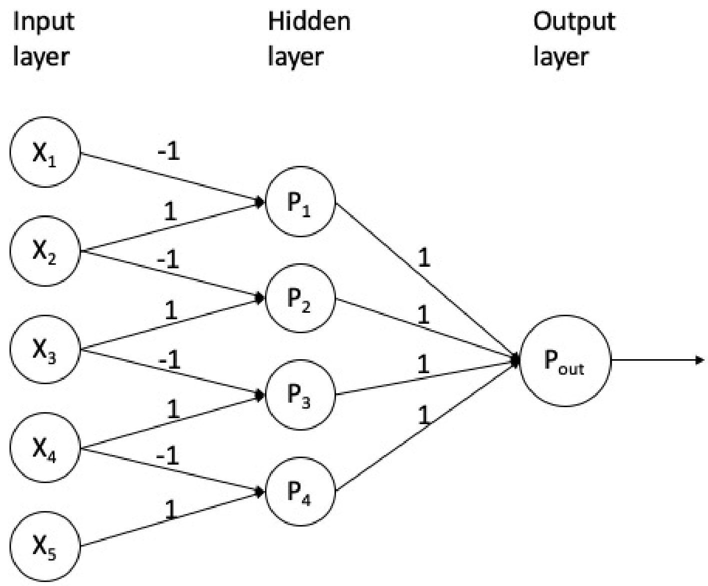


Figure 2. structure of the neural network.

for Perceptron1 P_3:

$$f(x) = \begin{cases} 1 & \text{if } \mathrm{x}_4 - x_3 > 0 \\ 0 & \text{otherwise} \end{cases}$$

for Perceptron1 P_4:

$$f(x) = \begin{cases} 1 & \text{if } \mathrm{x}_5 - x_4 > 0 \\ 0 & \text{otherwise} \end{cases}$$

So only when all perceptrons have the output as 1, then it meets the requirement that $x_1 < x_2 < x_3 < x_4 < x_5$. Thus, for Perceptron out P_{out}:

$$f(x) = \begin{cases} 1 & \text{if } \mathrm{x}_1 + x_2 + x_3 + x_4 = 4 \\ -1 & \text{otherwise} \end{cases}$$

OK, thus far we have been talking about linear models. All these can be viewed as a single-layer neural net. The next step is to move on to multi-layer nets. Training these is a bit more involved, and implementing from scratch requires time and effort. Instead, we just use well-established libraries. I prefer PyTorch, which is based on an earlier library called Torch (designed for training neural nets via backprop).

In [ ]:
```
import numpy as np
import torch
import torchvision
```

Torch handles data types a bit differently. Everything in torch is a *tensor*.

In [ ]:
```
a = np.random.rand(2,3)
b = torch.from_numpy(a)

# Q4.1 Display the contents of a, b
print("contest of a: ", a)
print("contest of b: ", b)
```

```
contest of a:  [[0.816508   0.75597224 0.67269371]
 [0.38200301 0.06110529 0.70656539]]
contest of b:  tensor([[0.8165, 0.7560, 0.6727],
        [0.3820, 0.0611, 0.7066]], dtype=torch.float64)
```

The idea in Torch is that tensors allow for easy forward (function evaluations) and backward (gradient) passes.

In [ ]:
```
A = torch.rand(2,2)
b = torch.rand(2,1)
x = torch.rand(2,1, requires_grad=True)

y = torch.matmul(A,x) + b

print(y)
z = y.sum()
print(z)
z.backward()
print(x.grad)
print(x)
```

```
tensor([[0.9157],
        [1.3327]], grad_fn=<AddBackward0>)
tensor(2.2484, grad_fn=<SumBackward0>)
tensor([[0.7845],
        [1.7336]])
tensor([[0.1465],
        [0.9142]], requires_grad=True)
```

Notice how the backward pass computed the gradients using autograd. OK, enough background. Time to train some networks. Let us load the *Fashion MNIST* dataset, which is a database of grayscale images of clothing items.

```
In [ ]: trainingdata = torchvision.datasets.FashionMNIST('./FashionMNIST/',train=True,download=True,transform
=torchvision.transforms.ToTensor())
testdata = torchvision.datasets.FashionMNIST('./FashionMNIST/',train=False,download=True,transform=to
rchvision.transforms.ToTensor())
```

```
Downloading http://fashion-mnist.s3-website.eu-central-1.amazonaws.com/train-images-idx3-ubyte.gz to
./FashionMNIST/FashionMNIST/raw/train-images-idx3-ubyte.gz


Extracting ./FashionMNIST/FashionMNIST/raw/train-images-idx3-ubyte.gz to ./FashionMNIST/FashionMNIS
T/raw
Downloading http://fashion-mnist.s3-website.eu-central-1.amazonaws.com/train-labels-idx1-ubyte.gz to
./FashionMNIST/FashionMNIST/raw/train-labels-idx1-ubyte.gz


Extracting ./FashionMNIST/FashionMNIST/raw/train-labels-idx1-ubyte.gz to ./FashionMNIST/FashionMNIS
T/raw
Downloading http://fashion-mnist.s3-website.eu-central-1.amazonaws.com/t10k-images-idx3-ubyte.gz to
./FashionMNIST/FashionMNIST/raw/t10k-images-idx3-ubyte.gz


Extracting ./FashionMNIST/FashionMNIST/raw/t10k-images-idx3-ubyte.gz to ./FashionMNIST/FashionMNIST/
raw
Downloading http://fashion-mnist.s3-website.eu-central-1.amazonaws.com/t10k-labels-idx1-ubyte.gz to
./FashionMNIST/FashionMNIST/raw/t10k-labels-idx1-ubyte.gz


Extracting ./FashionMNIST/FashionMNIST/raw/t10k-labels-idx1-ubyte.gz to ./FashionMNIST/FashionMNIST/
raw
Processing...
Done!

/usr/local/lib/python3.6/dist-packages/torchvision/datasets/mnist.py:469: UserWarning: The given Num
Py array is not writeable, and PyTorch does not support non-writeable tensors. This means you can wr
ite to the underlying (supposedly non-writeable) NumPy array using the tensor. You may want to copy
the array to protect its data or make it writeable before converting it to a tensor. This type of wa
rning will be suppressed for the rest of this program. (Triggered internally at  /pytorch/torch/csr
c/utils/tensor_numpy.cpp:141.)
  return torch.from_numpy(parsed.astype(m[2], copy=False)).view(*s)
```

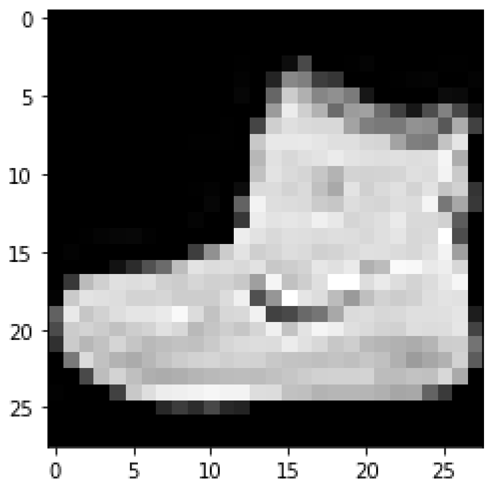
Let us examine the size of the dataset.

In [ ]:
```
# Q4.2 How many training and testing data points are there in the dataset?
# What is the number of features in each data point?

# There are 60000 training data points and 10000 testing data points in the dataset
# For each data point, we have the image data and the actual label for this image.
# The data size of each image is 1*28*28
# '1' means each pixel only has one channel, that's grayscale value
#'28*28' means the width and height of each image are both 28 pixels,
# that means each image has 28*28=784 pixels
print("the number of traning data points: ", len(trainingdata))
print("the number of testing data points: ", len(testdata))
print("the size of each data point vector: ", len(trainingdata[0]))
print("the size of each image data: ", trainingdata[0][0].size())
print("label for each data is", type(trainingdata[0][1]))
```

```
the number of traning data points:  60000
the number of testing data points:  10000
the size of each data point vector:  2
the size of each image data:  torch.Size([1, 28, 28])
label for each data is <class 'int'>
```

Let us try to visualize some of the images. Since each data point is a tensor (not an array) we need to postprocess to use matplotlib.

In [ ]:
```
import matplotlib.pyplot as plt
%matplotlib inline

image, label = trainingdata[0]
# Q4.3 Assuming each sample is an image of size 28x28, show it in matplotlib.
plt.figure()
plt.imshow(image[0], cmap='gray')
plt.show()
```



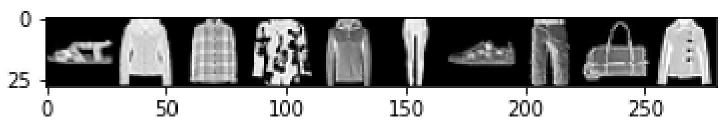
Let's try plotting several images. This is conveniently achieved in PyTorch using a *data loader*, which loads data in batches.

In [ ]:
```
trainDataLoader = torch.utils.data.DataLoader(trainingdata, batch_size=64, shuffle=True)
testDataLoader = torch.utils.data.DataLoader(testdata, batch_size=64, shuffle=False)
images, labels = iter(trainDataLoader).next()
print(images.size(), labels)
```

```
torch.Size([64, 1, 28, 28]) tensor([5, 4, 6, 6, 2, 1, 7, 1, 8, 4, 8, 8, 1, 6, 8, 8, 0, 8, 7, 4, 2,
9, 0, 7,
        2, 2, 1, 3, 8, 2, 2, 4, 0, 1, 4, 5, 3, 3, 2, 5, 5, 7, 3, 6, 4, 1, 5, 2,
        4, 7, 1, 0, 7, 1, 4, 6, 5, 0, 2, 6, 2, 7, 2, 6])
```

In [ ]:
```python
# Q4.4 Visualize the first 10 images of the first minibatch
# returned by testDataLoader.

first_ten_images = images[:10, 0,...].numpy()
row = np.concatenate([first_ten_images[i] for i in range(10)], axis=1)
plt.figure()
print("Showing the 10 images in one row:")
plt.imshow(row, cmap='gray')
plt.show()
```

```
Showing the 10 images in one row:
```



Now we are ready to define our linear model. Here is some boilerplate PyTorch code that implements the forward model for a single layer network for logistic regression (similar to the one discussed in class notes).

In [ ]:
```python
class LinearReg(torch.nn.Module):
  def __init__(self):
    super(LinearReg, self).__init__()
    self.linear = torch.nn.Linear(28*28,10)

  def forward(self, x):
    x = x.view(-1,28*28)
    transformed_x = self.linear(x)
    return transformed_x

net = LinearReg().cuda()
Loss = torch.nn.CrossEntropyLoss()
optimizer = torch.optim.SGD(net.parameters(), lr=0.01)
```

Cool! Now we have set everything up. Let's try to train the network.

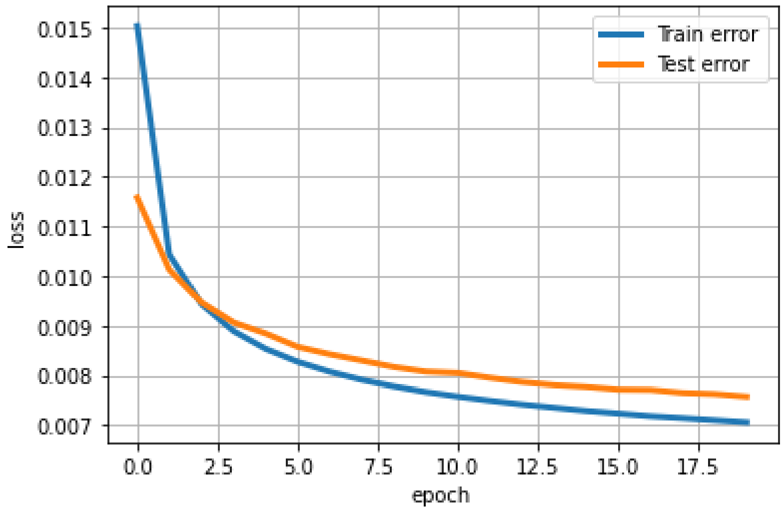
In [ ]:
```
train_loss_history = []
test_loss_history = []

# Q4.5 Write down a for-loop that trains this network for 20 minibatch iterations,
# and print the train/test losses.
# Save them in the variables above. If done correctly, you should be able to
# execute the next code block.

#Initiate network
net = LinearReg().cuda()
Loss = torch.nn.CrossEntropyLoss()
optimizer = torch.optim.SGD(net.parameters(), lr=0.01)

num_of_epochs = 20

for i in range(num_of_epochs):
  #shuffle training data
  trainDataLoader = torch.utils.data.DataLoader(trainingdata, batch_size=64, shuffle=True)

  #train the net with each minibatch
  epoch_train_loss = 0
  for images, labels in trainDataLoader:
    optimizer.zero_grad()
    # calculate train loss of the current minibatch
    preds = net(images.cuda())
    batch_images_loss = Loss(preds, labels.cuda())
    epoch_train_loss += batch_images_loss

    #optimize the network
    batch_images_loss.backward()
    optimizer.step()

  # calculate train loss for the epoch
  epoch_train_loss /= len(trainingdata)
  train_loss_history.append(epoch_train_loss)

  # test the net with each minibatch
  epoch_test_loss = 0
  for images, labels in testDataLoader:
    preds = net(images.cuda())
    batch_images_loss = Loss(preds, labels.cuda())
    epoch_test_loss += batch_images_loss
```

```
    # calculate test loss for the epoch
    epoch_test_loss /= len(testdata)
    test_loss_history.append(epoch_test_loss)

    print('Epoch: {}, Train Loss: {}, Test Loss: {}'.format(i, epoch_train_loss, epoch_test_loss))
```

```
Epoch: 0, Train Loss: 0.01502606924623251, Test Loss: 0.011578425765037537
Epoch: 1, Train Loss: 0.010432981885969639, Test Loss: 0.010125137865543365
Epoch: 2, Train Loss: 0.009428427554666996, Test Loss: 0.009466097690165043
Epoch: 3, Train Loss: 0.008885594084858894, Test Loss: 0.00906306505203247
Epoch: 4, Train Loss: 0.008530200459063053, Test Loss: 0.008839458227157593
Epoch: 5, Train Loss: 0.008271351456642151, Test Loss: 0.008571295067667961
Epoch: 6, Train Loss: 0.00807273667305708, Test Loss: 0.00842297449707985
Epoch: 7, Train Loss: 0.007911734282970428, Test Loss: 0.008297959342598915
Epoch: 8, Train Loss: 0.007779350504279137, Test Loss: 0.008169146254658699
Epoch: 9, Train Loss: 0.007666518911719322, Test Loss: 0.00807636696845293
Epoch: 10, Train Loss: 0.007574894465506077, Test Loss: 0.008048789575695992
Epoch: 11, Train Loss: 0.007492510136216879, Test Loss: 0.007952781394124031
Epoch: 12, Train Loss: 0.007416723761707544, Test Loss: 0.007868158631026745
Epoch: 13, Train Loss: 0.0073526217602193356, Test Loss: 0.0078074736520648
Epoch: 14, Train Loss: 0.007288788910955191, Test Loss: 0.007769959978759289
Epoch: 15, Train Loss: 0.007235296536237001, Test Loss: 0.007714950945228338
Epoch: 16, Train Loss: 0.0071831876412034035, Test Loss: 0.0077091665007174015
Epoch: 17, Train Loss: 0.007145352195948362, Test Loss: 0.007648747880011797
Epoch: 18, Train Loss: 0.007103899493813515, Test Loss: 0.007626079488545656
Epoch: 19, Train Loss: 0.007061449345201254, Test Loss: 0.007573876064270735
```

In [ ]:
```
plt.plot(range(20),train_loss_history,'-',linewidth=3,label='Train error')
plt.plot(range(20),test_loss_history,'-',linewidth=3,label='Test error')
plt.xlabel('epoch')
plt.ylabel('loss')
plt.grid(True)
plt.legend()
```

Out[ ]: <matplotlib.legend.Legend at 0x7f243392e710>



Neat! Now let's evaluate our model accuracy on the entire dataset. The predicted class label for a given input image can computed by looking at the output of the neural network model and computing the index corresponding to the maximum activation. Something like

predicted_output = net(images) _, predicted_labels = torch.max(predicted_output,1)

In [ ]:
```
predicted_output = net(images.cuda())
print(torch.max(predicted_output, 1))
fit = Loss(predicted_output, labels.cuda())
print(labels)
```

```
torch.return_types.max(
values=tensor([ 6.1149,  2.9275,  8.9590,  7.8504,  6.6379,  6.2165, 10.2046,  4.3740,
         7.0905, 11.6778, 10.4770, 10.4350,  6.6251,  4.7262,  9.3121,  4.5606],
       device='cuda:0', grad_fn=<MaxBackward0>),
indices=tensor([3, 1, 7, 5, 8, 2, 5, 6, 8, 9, 1, 9, 1, 8, 1, 5], device='cuda:0'))
tensor([3, 2, 7, 5, 8, 4, 5, 6, 8, 9, 1, 9, 1, 8, 1, 5])
```

In [ ]:
```
def evaluate(dataloader):
  # Q4.6 Implement a function here that evaluates training and testing accuracy.
  # Here, accuracy is measured by probability of successful classification.
  correct_pred = 0
  num_of_all_images = 0
  for images,labels in dataloader:
    predicted_output = net(images.cuda())
    _, predicted_labels = torch.max(predicted_output,1)
    preds_diff = labels - predicted_labels.cpu()
    correct_pred += int((preds_diff==0).sum())
    num_of_all_images += len(images)
  return correct_pred/num_of_all_images


print('Train acc = %0.2f, test acc = %0.2f' % (evaluate(trainDataLoader), evaluate(testDataLoader)))
```

```
Train acc = 0.85, test acc = 0.83
```