Supplement Material to: Versatile LiDAR-Inertial Odometry with SE(2) Constraints for Ground Vehicles

1 IMU Pre-integration

The IMU measures angular velocity and acceleration in the inertial frame, which are can be defined as:

$$\begin{aligned} \hat{w}_t &= w_t + b_t^g - \eta_t^g \\ \hat{a}_t &= R_t^{BW}(a_t - g) + b_t^a - \eta_t^a \end{aligned} \tag{1}$$

where $\hat{w}_t$ and $\hat{a}_t$ are the raw data from IMU, which are affected by the addictive white noise and a slowly varying sensor bias. w_t is the instantaneous angular velocity of the body frame $\boldsymbol{B}$ relative to the world frame $\boldsymbol{W}$, while a_t is the acceleration of IMU. g is the gravity.

The additive noise in acceleration and gyroscope from IMU raw inputs are modeled as Gaussian white noise:

$$\eta_t^g \sim \mathcal{N}(0, \sigma_g^2) \quad \eta_t^a \sim \mathcal{N}(0, \sigma_a^2) \tag{2}$$

and the corresponding biases are modeled as random walk as:

$$b_t^g \sim \mathcal{N}(0, \sigma_{b_g}^2) \quad b_t^a \sim \mathcal{N}(0, \sigma_{b_a}^2) \tag{3}$$

To increase the robustness, we use the mid-point values of the instantaneous angular velocity and acceleration between two time consecutive frames.

$$\begin{aligned} \bar{w} &= \frac{1}{2}((\hat{w}_k - b_k^g + \eta_k^g) + (\hat{w}_{k+1} - b_{k+1}^g + \eta_{k+1}^g)) \\ \bar{a} &= \frac{1}{2}[R_{i,k}(\hat{a}_k - b_k^a + \eta_k^a) + R_{i,k+1}(\hat{a}_{k+1} - b_{k+1}^a + \eta_{k+1}^a)] + g \end{aligned} \tag{4}$$

The rotation, position, and velocity of the robot at time $k+1$ can be obtained as follows:

$$\begin{aligned} R_{i,k+1} &= R_{i,k}\exp([\bar{w}\delta t]^\wedge) \\ &= R_{i,k}\exp([\frac{\delta t}{2}((\hat{w}_k - b_k^g + \eta_k^g) + (\hat{w}_{k+1} - b_{k+1}^g + \eta_{k+1}^g))]^\wedge) \end{aligned} \tag{5}$$

covert to $\mathfrak{so}(3)$:

$$\begin{aligned} \exp([\phi_{i,k+1}]^\wedge) &= \exp([\phi_{i,k}]^\wedge)Exp([\bar{w}\delta t]^\wedge) \\ &= \exp([\phi_{i,k}]^\wedge)Exp([\frac{\delta t}{2}((\hat{w}_k - b_k^g + \eta_k^g) + (\hat{w}_{k+1} - b_{k+1}^g + \eta_{k+1}^g)]^\wedge) \end{aligned} \tag{6}$$

$$\begin{aligned} P_{i,k+1} &= P_{i,k} + V_{i,k}\delta t + \frac{1}{2}\bar{a}\delta t^2 \\ &= P_{i,k} + V_{i,k}\delta t + \frac{\delta t^2}{4}[R_{i,k}(\hat{a}_k - b_k^a + \eta_k^a) \\ &+ R_{i,k+1}(\hat{a}_{k+1} - b_{k+1}^a + \eta_{k+1}^a)] + \frac{1}{2}g\delta t^2 \end{aligned} \tag{7}$$

$$\begin{aligned} V_{i,k+1} &= V_{i,k} + \bar{a}\delta t \\ &= V_{i,k} + \frac{\delta t}{2}[R_{i,k}(\hat{a}_k - b_k^a + \eta_k^a) \\ &+ R_{i,k+1}(\hat{a}_{k+1} - b_{k+1}^a + \eta_{k+1}^a)] + g\delta t^2 \end{aligned} \tag{8}$$

The bias update is as follows:

$$
\begin{aligned}
b^g_{k+1} &= b^g_k + \eta^g_k \delta t \\
b^a_{k+1} &= b^a_k + \eta^a_k \delta t
\end{aligned}
\tag{9}
$$

Then, the preintegrated noise covariance is computed as following:

$$
\begin{bmatrix} \delta\phi_{i,k+1} \\ \delta P_{i,k+1} \\ \delta V_{i,k+1} \\ \delta b^a_{k+1} \\ \delta b^g_{k+1} \end{bmatrix} = F \begin{bmatrix} \delta\phi_{i,k} \\ \delta P_{i,k} \\ \delta V_{i,k} \\ \delta b^a_k \\ \delta b^g_k \end{bmatrix} + G \begin{bmatrix} \eta^a_k \\ \eta^a_{k+1} \\ \eta^g_k \\ \eta^g_{k+1} \\ \eta^{b_a}_k \\ \eta^{b_g}_k \end{bmatrix}
\tag{10}
$$

$$
F = \begin{bmatrix}
I - [\bar{w}\delta t]^\wedge & 0 & 0 & 0 & -I\delta t \\
f_{21} & I & I\delta t & -\frac{\delta t^2}{4}(R_{i,k} + R_{i,k+1}) & \frac{\delta t^3}{4}[R_{i,k+1}(a_{k+1} - b^a_k)^\wedge] \\
f_{31} & 0 & I & -\frac{\delta t}{2}(R_{i,k} + R_{i,k+1}) & \frac{\delta t^2}{2}[R_{i,k+1}(a_{k+1} - b^a_k)^\wedge] \\
0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & I
\end{bmatrix}
\tag{11}
$$

$$
f_{21} = -\frac{\delta t^2}{4}[R_{i,k}(a_k - b^a_k)^\wedge + R_{i,k+1}(a_{k+1} - b^a_k)^\wedge(I - [\bar{w}\delta t]^\wedge)]
\tag{12}
$$

$$
f_{31} = -\frac{\delta t}{2}[R_{i,k}(a_k - b^a_k)^\wedge + R_{i,k+1}(a_{k+1} - b^a_k)^\wedge(I - [\bar{w}\delta t]^\wedge)]
\tag{13}
$$

$$
G = \begin{bmatrix}
0 & 0 & \frac{\delta t}{2} I & \frac{\Delta t}{2} I & 0 & 0 \\
\frac{\delta t^2}{4} R_{i,k} & \frac{\delta t^2}{4} R_{i,k+1} & g_{23} & g_{24} & 0 & 0 \\
\frac{\delta t}{2} R_{i,k} & \frac{\delta t}{2} R_{i,k+1} & g_{33} & g_{34} & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & I
\end{bmatrix}
\tag{14}
$$

$$
g_{23} = g_{24} = -\frac{\delta t^2}{4}[R_{i,k+1}(a_{k+1} - b^a_k)]
\tag{15}
$$

$$
g_{33} = g_{34} = -\frac{\delta t^3}{8}[R_{i,k+1}(a_{k+1} - b^a_k)]
\tag{16}
$$

The covariance matrix can be derived as follow:

$$
\Sigma_{I_{i,k+1}} = F_k \Sigma_{I_{i,k}} F_k^T + G_k \Sigma_{\eta_{i,k}} G_k^T,
\tag{17}
$$

where Σ_η is the diagonal covariance matrix of noise $(\sigma_a^2, \sigma_a^2, \sigma_w^2, \sigma_w^2, \sigma_{b_a}^2, \sigma_{b_g}^2)$.

2 IMU Residual

Bias correction: we assume that the estimation of bias changes minorly, we adjust $\Delta\phi_{i,j}$, $\Delta P_{i,j}$ and $\Delta V_{i,j}$ by their first-order approximations with respect to the new estimated bias as:

$$
\begin{aligned}
\exp([\Delta\phi_{i,j}]^\wedge) &= \exp([-\phi_i^W]^\wedge)\exp([\phi_j^W]^\wedge) \\
\Delta P_{i,j} &= \exp([-\phi_i^W]^\wedge)(P_j^W - P_i^W - V_i^W \delta t_{i,j} + \frac{1}{2} g \delta t_{i,j}^2) \\
\Delta V_{i,j} &= \exp([-\phi_i^W]^\wedge) \cdot (V_j^W - V_i^W + g\delta t_{i,j})
\end{aligned}
\tag{18}
$$

$\Rightarrow$

$$
\begin{aligned}
\exp([\widetilde{\Delta\phi_{i,j}}]^\wedge) &= \exp([\Delta\phi_{i,j}]^\wedge)\exp(\frac{\partial \exp([\Delta\phi_{i,j}]^\wedge)}{\partial \mathbf{b}_i^g}\delta\mathbf{b}_i^g) = \exp([\Delta\phi_{i,j}]^\wedge)\exp(J^R_{b_g}\delta\mathbf{b}_i^g) \\
\widetilde{\Delta P_{i,j}} &= \Delta P_{i,j} + \frac{\partial \Delta P_{i,j}}{\partial \mathbf{b}_i^g}\delta\mathbf{b}_i^g + \frac{\partial \Delta P_{i,j}}{\partial \mathbf{b}_i^a}\delta\mathbf{b}_i^a = \Delta P_{i,j} + J^P_{b_g}\delta\mathbf{b}_i^g + J^P_{b_a}\delta\mathbf{b}_i^a \\
\widetilde{\Delta V_{i,j}} &= \Delta V_{i,j} + \frac{\partial \Delta V_{i,j}}{\partial \mathbf{b}_i^g}\delta\mathbf{b}_i^g + \frac{\partial \Delta V_{i,j}}{\partial \mathbf{b}_i^a}\delta\mathbf{b}_i^a = \Delta V_{i,j} + J^V_{b_g}\delta\mathbf{b}_i^g + J^V_{b_a}\delta\mathbf{b}_i^a
\end{aligned}
\tag{19}
$$

Then the residual terms of IMU is:

$$
\begin{bmatrix} \mathbf{r}_\phi \\ \mathbf{r}_p \\ \mathbf{r}_v \\ \mathbf{r}_{b_a} \\ \mathbf{r}_{b_g} \end{bmatrix} = \begin{bmatrix} \log\left[\exp\left([-\Delta\widetilde{\boldsymbol{\phi}_{i,j}}]^\wedge\right)\cdot\exp\left([-\boldsymbol{\phi}_i^W]^\wedge\right)\cdot\exp\left([\boldsymbol{\phi}_j^W]^\wedge\right)\right]^\vee \\ \exp\left([-\boldsymbol{\phi}_i^W]^\wedge\right)\cdot\left(\mathbf{P}_j^W-\mathbf{P}_i^W-\mathbf{V}_i^W\delta t_{i,j}+\frac{1}{2}\mathbf{g}\delta t_{i,j}^2\right)-\Delta\widetilde{\mathbf{P}_{i,j}} \\ \exp\left([-\boldsymbol{\phi}_i^W]^\wedge\right)\cdot\left(\mathbf{V}_j^W-\mathbf{V}_i^W+\mathbf{g}\delta t_{i,j}\right)-\Delta\widetilde{\mathbf{V}_{i,j}} \\ \mathbf{b}_j^a-\mathbf{b}_i^a \\ \mathbf{b}_j^g-\mathbf{b}_i^g \end{bmatrix} \tag{20}
$$

Variables that need to be optimized are: $[\delta\boldsymbol{\phi}_i^W, \delta\mathbf{P}_i^W, \delta\mathbf{V}_i^W, \delta\mathbf{b}_i^a, \delta\mathbf{b}_i^g, \delta\boldsymbol{\phi}_j^W, \delta\mathbf{P}_j^W, \delta\mathbf{V}_j^W, \delta\mathbf{b}_j^a, \delta\mathbf{b}_j^g]$ the jacobian matrix can be derived as follows:

$$
\begin{aligned}
J &= \begin{bmatrix}
\frac{\partial\mathbf{r}_\phi}{\partial\delta\boldsymbol{\phi}_i^W} & \frac{\partial\mathbf{r}_\phi}{\partial\delta\mathbf{P}_i^W} & \frac{\partial\mathbf{r}_\phi}{\partial\delta\mathbf{V}_i^W} & \frac{\partial\mathbf{r}_\phi}{\partial\delta\mathbf{b}_i^a} & \frac{\partial\mathbf{r}_\phi}{\partial\delta\mathbf{b}_i^g} & \frac{\partial\mathbf{r}_\phi}{\partial\delta\boldsymbol{\phi}_j^W} & \frac{\partial\mathbf{r}_\phi}{\delta\partial\mathbf{V}_j^W} & \frac{\partial\mathbf{r}_\phi}{\partial\delta\mathbf{b}_j^a} & \frac{\partial\mathbf{r}_\phi}{\partial\delta\mathbf{b}_j^g} \\
\frac{\partial\mathbf{r}_p}{\partial\delta\boldsymbol{\phi}_i^W} & \frac{\partial\mathbf{r}_p}{\partial\delta\mathbf{P}_i^W} & \frac{\partial\mathbf{r}_p}{\partial\delta\mathbf{V}_i^W} & \frac{\partial\mathbf{r}_p}{\partial\delta\mathbf{b}_i^a} & \frac{\partial\mathbf{r}_p}{\partial\delta\mathbf{b}_i^g} & \frac{\partial\mathbf{r}_p}{\partial\delta\boldsymbol{\phi}_j^W} & \frac{\partial\mathbf{r}_p}{\partial\delta\mathbf{V}_j^W} & \frac{\partial\mathbf{r}_p}{\partial\delta\mathbf{b}_j^a} & \frac{\partial\mathbf{r}_p}{\partial\delta\mathbf{b}_j^g} \\
\frac{\partial\mathbf{r}_v}{\partial\delta\boldsymbol{\phi}_i^W} & \frac{\partial\mathbf{r}_v}{\partial\delta\mathbf{P}_i^W} & \frac{\partial\mathbf{r}_v}{\partial\delta\mathbf{V}_i^W} & \frac{\partial\mathbf{r}_v}{\partial\delta\mathbf{b}_i^a} & \frac{\partial\mathbf{r}_v}{\partial\delta\mathbf{b}_i^g} & \frac{\partial\mathbf{r}_v}{\partial\delta\boldsymbol{\phi}_j^W} & \frac{\partial\mathbf{r}_v}{\partial\delta\mathbf{V}_j^W} & \frac{\partial\mathbf{r}_v}{\partial\delta\mathbf{b}_j^a} & \frac{\partial\mathbf{r}_v}{\partial\delta\mathbf{b}_j^g} \\
\frac{\partial\mathbf{r}_{b^a}}{\partial\delta\boldsymbol{\phi}_i^W} & \frac{\partial\mathbf{r}_{b^a}}{\partial\delta\mathbf{P}_i^W} & \frac{\partial\mathbf{r}_{b^a}}{\partial\delta\mathbf{V}_i^W} & \frac{\partial\mathbf{r}_{b^a}}{\partial\delta\mathbf{b}_i^a} & \frac{\partial\mathbf{r}_{b^a}}{\partial\delta\mathbf{b}_i^g} & \frac{\partial\mathbf{r}_{b^a}}{\partial\delta\boldsymbol{\phi}_j^W} & \frac{\partial\mathbf{r}_{b^a}}{\partial\delta\mathbf{V}_j^W} & \frac{\partial\mathbf{r}_{b^a}}{\partial\delta\mathbf{b}_j^a} & \frac{\partial\mathbf{r}_{b^a}}{\partial\delta\mathbf{b}_j^g} \\
\frac{\partial\mathbf{r}_{b^a}}{\partial\delta\boldsymbol{\phi}_i^W} & \frac{\partial\mathbf{r}_{b^g}}{\partial\delta\mathbf{P}_i^W} & \frac{\partial\mathbf{r}_{b^g}}{\partial\delta\mathbf{V}_i^W} & \frac{\partial\mathbf{r}_{b^g}}{\partial\delta\mathbf{b}_i^a} & \frac{\partial\mathbf{r}_{b^g}}{\partial\delta\mathbf{b}_i^g} & \frac{\partial\mathbf{r}_{b^g}}{\partial\delta\boldsymbol{\phi}_j^W} & \frac{\partial\mathbf{r}_{b^g}}{\partial\delta\mathbf{V}_j^W} & \frac{\partial\mathbf{r}_{b^g}}{\partial\delta\mathbf{b}_j^a} & \frac{\partial\mathbf{r}_{b^g}}{\partial\delta\mathbf{b}_j^g}
\end{bmatrix} \\
&= \begin{bmatrix}
\mathbf{J}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{J}_{14} & \mathbf{J}_r^{-1}(r_\phi) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{J}_{21} & -\mathbf{R}_i^T & -\mathbf{R}_i^T\delta t & -\mathbf{J}_{b^a}^P & -\mathbf{J}_{b^g}^P & \mathbf{0} & \mathbf{R}_i^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{J}_{31} & 0 & -\mathbf{R}_i^T & -\mathbf{J}_{b^a}^V & -\mathbf{J}_{b^g}^V & \mathbf{0} & \mathbf{0} & \mathbf{R}_i^T & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}
\end{bmatrix} \\
J_{11} &= -J_r^{-1}(r_\phi)R_i^T R_j \\
J_{14} &= -J_r^{-1}(r_\phi)R_j^T R_i^T \Delta\widetilde{R}_{i,j}J_{b^g}^R \\
J_{21} &= [R_i^T(P_j^W - P_i^W - V_i^W\delta t + \frac{1}{2}g\delta t_{i,j}^2)]^\wedge \\
J_{31} &= R_i^T(V_j^W - V_i^W + g\delta t_{i,j}) \\
J_r^{-1}(r_\phi) &= I + \frac{1}{2}r_\phi^\wedge + (\frac{1}{\parallel r_\phi \parallel^2} - \frac{1+\cos(\parallel r_\phi \parallel)}{2\parallel r_\phi \parallel \sin(\parallel r_\phi \parallel)})(r_\phi^\wedge)^2
\end{aligned} \tag{21}
$$

3 LiDAR residuals and jacobian derivation

Similar to LOAM, we compute the point-to-edge residual and point-to-plane residual by minimizing the distance from the target edge feature point to its corresponding line and the target planar feature point to its corresponding plane:

$$
f_E(\hat{\mathbf{p}}_j) = \frac{|(\hat{\mathbf{p}}_j - \mathbf{p}_b^E)\times(\hat{\mathbf{p}}_j - \mathbf{p}_a^E)|}{|\mathbf{p}_a^E - \mathbf{p}_b^E|},
$$

$$
f_S(\hat{\mathbf{p}}_j) = \left|(\hat{\mathbf{p}}_j - \mathbf{p}_a^S)^T \frac{(\mathbf{p}_a^S - \mathbf{p}_b^S)\times(\mathbf{p}_c^S - \mathbf{p}_a^S)}{|(\mathbf{p}_a^S - \mathbf{p}_b^S)\times(\mathbf{p}_c^S - \mathbf{p}_a^S)|}\right|.
$$

The Jacobian of f_E w.r.t (η_θ, η_z) can be estimated by applying the right perturbation model:

$$
\begin{aligned}
\frac{\partial f_E}{\partial \mathbf{T}\mathbf{p}_j} &= \frac{\partial \frac{|(\hat{\mathbf{p}}_j - \mathbf{p}_b^E)\times(\hat{\mathbf{p}}_j - \mathbf{p}_a^E)|}{|\mathbf{p}_a^E - \mathbf{p}_b^E|}}{\partial \mathbf{T}\mathbf{p}_j} \\
&= \frac{1}{|\mathbf{p}_a^E - \mathbf{p}_b^E|} \frac{\partial |(\hat{\mathbf{p}}_j - \mathbf{p}_b^E)\times(\hat{\mathbf{p}}_j - \mathbf{p}_a^E)|}{\partial (\hat{\mathbf{p}}_j - \mathbf{p}_b^E)\times(\hat{\mathbf{p}}_j - \mathbf{p}_a^E)} \frac{\partial (\hat{\mathbf{p}}_j - \mathbf{p}_b^E)\times(\hat{\mathbf{p}}_j - \mathbf{p}_a^E)}{\partial \mathbf{T}\mathbf{p}_j} \\
&= \frac{1}{|\mathbf{p}_a^E - \mathbf{p}_b^E|} \frac{((\hat{\mathbf{p}}_j - \mathbf{p}_b^E)\times(\hat{\mathbf{p}}_j - \mathbf{p}_a^E))^T}{|(\hat{\mathbf{p}}_j - \mathbf{p}_b^E)\times(\hat{\mathbf{p}}_j - \mathbf{p}_a^E)|} (-(\hat{\mathbf{p}}_j - \mathbf{p}_a^E)^\wedge + (\hat{\mathbf{p}}_j + \mathbf{p}_b^E)^\wedge) \\
&= \frac{((\hat{\mathbf{p}}_j - \mathbf{p}_b^E)\times(\hat{\mathbf{p}}_j - \mathbf{p}_a^E))^T}{|(\hat{\mathbf{p}}_j - \mathbf{p}_b^E)\times(\hat{\mathbf{p}}_j - \mathbf{p}_a^E)|} \frac{(\mathbf{p}_a^E - \mathbf{p}_b^E)^\wedge}{|\mathbf{p}_a^E - \mathbf{p}_b^E|}, \\
\frac{\partial f_S}{\partial \mathbf{T}\mathbf{p}_j} &= \frac{\partial (\left|(\hat{\mathbf{p}}_j - \mathbf{p}_a^S)^T \frac{(\mathbf{p}_a^S - \mathbf{p}_b^S)\times(\mathbf{p}_c^S - \mathbf{p}_a^S)}{|(\mathbf{p}_a^S - \mathbf{p}_b^S)\times(\mathbf{p}_c^S - \mathbf{p}_a^S)|})\right|}{\partial \mathbf{T}\mathbf{p}_j} \\
&= \frac{((\mathbf{p}_a^S - \mathbf{p}_b^S)\times(\mathbf{p}_c^S - \mathbf{p}_a^S))^T}{|(\mathbf{p}_a^S - \mathbf{p}_b^S)\times(\mathbf{p}_c^S - \mathbf{p}_a^S)|} \frac{\partial (\hat{\mathbf{p}}_j - \mathbf{p}_a^S)\cdot}{\partial \mathbf{T}\mathbf{p}_j} \\
&= \frac{((\mathbf{p}_a^S - \mathbf{p}_b^S)\times(\mathbf{p}_c^S - \mathbf{p}_a^S))^T}{|(\mathbf{p}_a^S - \mathbf{p}_b^S)\times(\mathbf{p}_c^S - \mathbf{p}_a^S)|}
\end{aligned}
$$

For the derivation $\frac{\partial \mathbf{T}\mathbf{p}_j}{\partial \delta \boldsymbol{\eta}_\theta}$ and $\frac{\partial \mathbf{T}\mathbf{p}_j}{\partial \delta \boldsymbol{\eta}_z}$ are calculated based on the Lie algebra right perturbation theory (more details are shown in "A micro Lie theory for state estimation in robotics").

$$
\begin{aligned}
\frac{\partial \mathbf{T}\mathbf{p}_j}{\partial \delta \boldsymbol{\eta}_\theta} &= \lim_{\delta \boldsymbol{\eta}_\theta \to 0} \frac{\mathbf{T}\exp(\delta \boldsymbol{\eta}_\theta^\wedge)\mathbf{p}_j - \mathbf{T}\mathbf{p}_j}{\delta \boldsymbol{\eta}_\theta} \\
&= \lim_{\delta \boldsymbol{\eta}_\theta \to 0} \frac{\exp(\phi^\wedge)\exp(\delta \boldsymbol{\eta}_\theta^\wedge)\mathbf{p}_j - \exp(\phi^\wedge)\mathbf{p}_j}{\delta \boldsymbol{\eta}_\theta} \\
&= \lim_{\delta \boldsymbol{\eta}_\theta \to 0} \frac{\exp(\phi^\wedge)(1 + \delta \boldsymbol{\eta}_\theta^\wedge)\mathbf{p}_j - \exp(\phi^\wedge)\mathbf{p}_j}{\delta \boldsymbol{\eta}_\theta} \\
&= \lim_{\delta \boldsymbol{\eta}_\theta \to 0} \frac{\exp(\phi^\wedge)\delta \boldsymbol{\eta}_\theta^\wedge \mathbf{p}_j}{\delta \boldsymbol{\eta}_\theta} \\
&= \lim_{\delta \boldsymbol{\eta}_\theta \to 0} \frac{-\exp(\phi^\wedge)(\mathbf{p}_j)^\wedge \delta \boldsymbol{\eta}_\theta}{\delta \boldsymbol{\eta}_\theta} \\
&= (-\mathbf{R}(\mathbf{p}_j)^\wedge),
\end{aligned}
$$

Similarly, we can easily get $\frac{\partial \mathbf{T}\mathbf{p}_j}{\partial \delta \boldsymbol{\eta}_z} = \mathbf{I}$.