Supplement Material to: Versatile LiDAR-Inertial Odometry with SE(2) Constraints for Ground Vehicles

1 IMU Pre-integration

The IMU measurements of angular velocity and acceleration in the inertial frame can be defined as:

$$\begin{aligned} \hat{w}_t &= w_t + b_t^w - n_t^w \\ \hat{a}_t &= R_t^{BW}(a_t - g) + b_t^a - n_t^a \end{aligned} \tag{1}$$

where $\hat{w}_t$ and $\hat{a}_t$ are the raw data from IMU, which are affected by the addictive white noise and a slowly varying sensor bias. w_t is the instantaneous angular velocity of the body frame $\boldsymbol{B}$ relative to the world frame $\boldsymbol{W}$, while a_t is the acceleration of IMU. g is the gravity.

The additive noise in acceleration and gyroscope from IMU raw inputs are modeled as Gaussian white noise:

$$n_t^w \sim \mathcal{N}(0, \sigma_w^2) \quad n_t^a \sim \mathcal{N}(0, \sigma_a^2) \tag{2}$$

and the corresponding biases are modeled as random walk as:

$$b_t^w \sim \mathcal{N}(0, \sigma_{b_w}^2) \quad b_t^a \sim \mathcal{N}(0, \sigma_{b_a}^2) \tag{3}$$

To increase the robustness, we use the mid-point values of the instantaneous angular velocity and acceleration between two time consecutive frames.

$$\begin{aligned} \bar{w} &= \frac{1}{2}((\hat{w}_k - b_k^w + n_k^w) + (\hat{w}_{k+1} - b_{k+1}^w + n_{k+1}^w)) \\ \bar{a} &= \frac{1}{2}[\Delta R_{i,k}(\hat{a}_k - b_k^a + n_k^a) + \Delta R_{i,k+1}(\hat{a}_{k+1} - b_{k+1}^a + n_{k+1}^a)] + g \end{aligned} \tag{4}$$

The rotation, position, and velocity of the robot at time $k+1$ can be obtained as follows:

$$\begin{aligned} \Delta R_{i,k+1} &= \Delta R_{i,k} Exp([\bar{w}\Delta t]^\wedge) \\ &= \Delta R_{i,k} Exp([\frac{\Delta t}{2}((\hat{w}_k - b_k^w + n_k^w) + (\hat{w}_{k+1} - b_{k+1}^w + n_{k+1}^w))]^\wedge) \end{aligned} \tag{5}$$

covert to $so(3)$:

$$
\begin{aligned}
Exp([\Delta\phi_{i,k+1}]^\wedge) &= Exp([\Delta\phi_{i,k}]^\wedge)Exp([\bar{w}\Delta t]^\wedge) \\
&= Exp([\Delta\phi_{i,k}]^\wedge)Exp([\frac{\Delta t}{2}((\hat{w}_k - b_k^w + n_k^w) + (\hat{w}_{k+1} - b_{k+1}^w + n_{k+1}^w))]^\wedge)
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
\Delta P_{i,k+1} &= \Delta P_{i,k} + \Delta V_{i,k}\Delta t + \frac{1}{2}\bar{a}\Delta t^2 \\
&= \Delta P_{i,k} + \Delta V_{i,k}\Delta t + \frac{\Delta t^2}{4}[\Delta R_{i,k}(\hat{a}_k - b_k^a + n_k^a) \\
&+ \Delta R_{i,k+1}(\hat{a}_{k+1} - b_{k+1}^a + n_{k+1}^a)] + \frac{1}{2}g\Delta t^2
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\Delta V_{i,k+1} &= \Delta V_{i,k} + \bar{a}\Delta t \\
&= \Delta V_{i,k} + \frac{\Delta t}{2}[\Delta R_{i,k}(\hat{a}_k - b_k^a + n_k^a) \\
&+ \Delta R_{i,k+1}(\hat{a}_{k+1} - b_{k+1}^a + n_{k+1}^a)] + g\Delta t^2
\end{aligned} \tag{8}
$$

The bias update as follows:

$$
b_{k+1}^w = b_k^w + \eta_k^w \Delta t b_{k+1}^a = b_k^a + \eta_k^a \Delta t \tag{9}
$$

Then, the error-state model of IMU pre-integration measurement can be written as:

$$
\begin{bmatrix} \delta\phi_{i,k+1} \\ \delta P_{i,k+1} \\ \delta V_{i,k+1} \\ \delta b_{k+1}^a \\ \delta b_{k+1}^w \end{bmatrix} = F \begin{bmatrix} \delta\phi_{i,k} \\ \delta P_{i,k} \\ \delta V_{i,k} \\ \delta b_k^a \\ \delta b_k^w \end{bmatrix} + G \begin{bmatrix} \eta_k^a \\ \eta_{k+1}^a \\ \eta_k^w \\ \eta_{k+1}^w \\ \eta_k^{b_a} \\ \eta_k^{b_w} \end{bmatrix} \tag{10}
$$

$$
F = \begin{bmatrix}
I - [\bar{w}\Delta t]^\wedge & 0 & 0 & 0 & -I\Delta t \\
f_{21} & I & I\Delta t & -\frac{\Delta t^2}{4}(\Delta R_{i,k} + \Delta R_{i,k+1}) & \frac{\Delta t^3}{4}[\Delta R_{i,k+1}(a_{k+1} - b_k^a)^\wedge] \\
f_{31} & 0 & I & -\frac{\Delta t}{2}(\Delta R_{i,k} + \Delta R_{i,k+1}) & \frac{\Delta t^2}{2}[\Delta R_{i,k+1}(a_{k+1} - b_k^a)^\wedge] \\
0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & I
\end{bmatrix} \tag{11}
$$

$$
f_{21} = -\frac{\Delta t^2}{4}[\Delta R_{i,k}(a_k - b_k^a)^\wedge + \Delta R_{i,k+1}(a_{k+1} - b_k^a)^\wedge(I - [\bar{w}\Delta t]^\wedge)] \tag{12}
$$

$$
f_{31} = -\frac{\Delta t}{2}[\Delta R_{i,k}(a_k - b_k^a)^\wedge + \Delta R_{i,k+1}(a_{k+1} - b_k^a)^\wedge(I - [\bar{w}\Delta t]^\wedge)] \tag{13}
$$

$$G = \begin{bmatrix} 0 & 0 & \frac{\Delta t}{2}I & \frac{\Delta t}{2}I & 0 & 0 \\ \frac{\Delta t^2}{4}\Delta R_{i,k} & \frac{\Delta t^2}{4}\Delta R_{i,k+1} & g_{23} & g_{24} & 0 & 0 \\ \frac{\Delta t}{2}\Delta R_{i,k} & \frac{\Delta t}{2}\Delta R_{i,k+1} & g_{33} & g_{34} & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \tag{14}$$

$$g_{23} = g_{24} = -\frac{\Delta t^2}{4}[\Delta R_{i,k+1}(a_{k+1} - b_k^a)] \tag{15}$$

$$g_{33} = g_{34} = -\frac{\Delta t^3}{8}[\Delta R_{i,k+1}(a_{k+1} - b_k^a)] \tag{16}$$

2 IMU Residual

Bias correction: we assume that the estimation of bias changes minorly, we adjust $\Delta R_{i,j}$, $\Delta P_{i,j}$ and $\Delta V_{i,j}$ by their first-order approximations with respect to the bias as:

$$\begin{aligned} \Delta R_{i,j} &\approx \Delta \hat{R}_{i,j} Exp(J_{b_w}^R \delta b_k^w) \\ \Delta P_{i,j} &\approx \Delta \hat{P}_{i,j} + J_{b_a}^p \delta b_k^a + J_{b_w}^p \delta b_k^w \\ \Delta V_{i,j} &\approx \Delta \hat{V}_{i,j} + J_{b_a}^v \delta b_k^a + J_{b_w}^v \delta b_k^w \end{aligned} \tag{17}$$

Then the residual terms of IMU is:

$$\begin{bmatrix} r_\phi \\ r_p \\ r_v \\ r_{b_a} \\ r_{b_w} \end{bmatrix} = \begin{bmatrix} Log[Exp([-\Delta\phi_{i,j}]^\wedge) \cdot Exp([-\Delta\phi_{wb_i}]^\wedge) \cdot Exp([\Delta\phi_{wb_j}]^\wedge)]^\vee \\ Exp([-\Delta\phi_{wb_i}]^\wedge) \cdot (P_j^W - P_i^W - V_i^W \Delta t_{i,j} + \frac{1}{2} g t_{i,j}^2) - \Delta p_{i,j} \\ Exp([-\Delta\phi_{wb_i}]^\wedge) \cdot (V_j^W - V_i^W + g\Delta t_{i,j}) - \Delta V_{i,j} \\ b_j^a - b_i^a \\ b_j^w - b_i^w \end{bmatrix} \tag{18}$$

Variables that need to be optimized are: $[\phi_{wb_i}, P_i^W, V_i^W, b_i^a, b_i^w, \phi_{wb_j}, P_j^W, V_j^W, b_j^a, b_j^w]$ the jacobian matrix can be derived as follows:

$$\begin{aligned} &J = \\ &\begin{bmatrix} J_{11} & 0 & 0 & 0 & J_{14} & J_r^{-1}(r_\phi) & 0 & 0 & 0 & 0 \\ J_{21} & -R_{wb_i}^T & -R_{wb_i}^T \Delta t & -J_{b_a}^p & -J_{b_w}^p & 0 & R_{wb_i}^T & 0 & 0 & 0 \\ J_{31} & 0 & -R_{wb_i}^T & -J_{b_a}^v & -J_{b_w}^v & 0 & 0 & R_{wb_i}^T & 0 & 0 \\ 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & -I & 0 & 0 & 0 & I \end{bmatrix} \\ &J_{11} = -J_r^{-1}(r_\phi) R_{wb_i}^T R_{wb_j} J_{b_w}^R \\ &J_{14} = -J_r^{-1}(r_\phi) R_{wb_j}^T R_{wb_i}^T \Delta R_{i,j} J_{b_w}^R \\ &J_{21} = [R_{wb_i}^T (P_j^W - P_i^W - V_i^W \Delta t + \frac{1}{2} g \Delta t_{i,j}^2)]^\wedge \\ &J_{31} = R_{wb_i}^T (V_j^W - V_i^W + g\Delta t_{i,j}) \end{aligned} \tag{19}$$