Chapter 4

Graphs

She must be seen to be appreciated.

Old Saint Pauls' Book I, W. H. AINSWORTH

Many situations may be graphically summarized by marking down a finite set of points and drawing lines between those pairs of points that are related in some way, or share some common attribute. The systematic study of such structures is known as graph theory. To a graph we attach one or more matrices. It is the spectra of these matrices that interest us most. In principle, we seek to understand the relationship that exists between the graph and the eigenstructure of these matrices. Such an approach is often called algebraic graph theory.

Our first section is not algebraic, however. In it we present well-known, classical results due to Euler, Ramsey, and Turán. Gradually we find ourselves leaning toward the algebraic side by discussing strongly regular graphs and then graphs with extreme spectral behavior. This last aspect is of particular interest because of its connections to the construction of efficient and optimal statistical designs. Eigenvalues of the matrix can be directly interpreted as variances of certain estimates. Extreme spectral behavior represents then minimization of variance (or maximization of precision). In this sense the chapter is closely related to, and strongly motivated by, the chapter on optimal statistical design (Chapter 8). The interested reader may wish to study these two chapters simultaneously. Important connections are also found with the material on block designs that appears in Chapter 7.

Let us begin, as is usual and necessary, with a description of the fundamental concepts.

1 CYCLES, TRAILS, AND COMPLETE SUBGRAPHS

4.1

Among n people relationships of friendship may exist. If these relationships are known to an observer, he or she may choose to conveniently display them by drawing n points on a piece of paper (representing the n people) and connecting two points by a line if and only if the two people in question are friends. What results is a figure that displays the relationships of friendship and that we call a *graph*. The n points are called the *vertices* of the graph and the joining lines are called *edges*.

A graph thus consists of vertices and edges. We allow more than one edge to be drawn between two vertices; and a vertex may be joined to itself (such an edge being called a *loop*). Generally a graph is understood to contain multiple edges and multiple loops. Often we wish to forbid these, however; we call a graph *simple* if at most one edge exists between any two vertices and if no loops are present.

4.2 More Terminology

By the *degree* of a vertex we understand the number of edges that emanate from that vertex (a loop contributing 1 to the degree). A *path* is a sequence $v_0e_1v_1e_2v_2\cdots e_kv_k$ of distinct vertices and edges in which edge e_i has vertices v_{i-1} and v_i for its endpoints. The initial vertex v_0 is called the beginning point of the path and v_k is called the endpoint. Two (distinct) vertices that occur in a path are said to be connected by that path. A graph is called *connected* if any two distinct vertices are connected by a path.

A *cycle* is a sequence $v_0e_1v_1e_2v_2\cdots e_kv_0$ of vertices and edges with all edges e_i being distinct. (One may say that a cycle is a path with its beginning and end points being the same.) For graphical displays we refer the reader to Section 6 of Chapter 2.

A *tree* is a set of edges that contains no cycles. By a *spanning tree* we mean a tree with $n-1$ edges, where n is the number of vertices in the graph.

4.3 A Result of Euler

Let G be a graph. An *Eulerian trail* in G is a sequence

$$T = v_0e_1v_1e_2v_2\cdots e_{m-1}v_{m-1}e_mv_0,$$

in which each edge e_i of G appears exactly once (while vertices v_i may be repeated). Intuitively an Eulerian trail describes a way of "walking" along the edges of a graph, starting at some vertex v_0, traversing each edge exactly once, and ending the walk at the starting place v_0. Not every graph has an Eulerian trail. A characterization of the graphs that do is perhaps the oldest known result in graph theory, and is due to Euler. It is stated as follows.

$*$ *A connected graph admits an Eulerian trail if and only if all its vertices are of even degree.*

Proof. Assume that the graph G has an Eulerian trail T. As we traverse the trail each occurrence of a vertex contributes 2 to the degree of that vertex. Since each edge occurs exactly once in T, the degree of a vertex will be a sum of 2's, and is therefore even.

We now prove the converse, by contradiction. Assume that G is a connected graph with each vertex of even degree and a *minimum* number of edges, for which an Eulerian trail does not exist. Since the vertices are of even degree a cycle C exists in G. (It may be located by walking along the edges without retracing.) If there are no edges outside C, then C is an Eulerian trail of G, a contradiction. If edges outside C exist we can still construct an Eulerian trail for G as follows: Start at a vertex v_0 of degree 4 or more of C (which exists by connectivity), walk along the edges of C back to v_0, and continue on along an Eulerian trail (which exists by the minimality of G) in the graph the same as G but with edges of C removed. What results is an Eulenan trail of G, contradicting one of our assumptions. This ends the proof.

The characterization given above is a convenient one. To decide whether or not an Eulerian trail exists it suffices to examine the parities of the degrees of vertices. If one vertex has odd degree, there will be no such trail, and thus all attempts to find one will fail.

Another well-known problem, and much of the same spirit, is deciding whether a cycle containing all the vertices of a graph exists. Such a cycle is called *Hamiltonian*. We ask for a convenient characterization of those graphs that admit Hamiltonian cycles. No satisfactory characterization is known, although many sufficient conditions for existence are available in the literature. We refer the reader to [1] and [2] for more information.

4.4 Turán's Theorem

A *subgraph* of a graph G is simply a subset of vertices and edges of G. By an *induced subgraph* we understand a subset S of vertices of G having as edges the edges of G that have both ends in S.

A graph on n vertices is called *complete* if any two distinct vertices are joined by exactly one edge. We denote such a graph by K_n, and observe that K_n has $\binom{n}{2}$ edges. Another graph of interest to us may be constructed as follows. Partition a set of $n = n_1 + n_2 + \cdots + n_r$ vertices into r classes, the ith class containing n_i vertices. Produce a simple graph $K(n_1, n_2, \ldots, n_r)$ by joining each vertex in class i to all vertices outside class i and to none in class i; do this for all i. The resulting graph is called the *complete r-partite* graph with r classes, the ith of which is of size n_i. When the class sizes n_i differ by at most 1 (i.e., they are as nearly equal as possible) we use the simpler notation $K(n, r)$ for this graph. [A display of $K(5, 2)$ will be found in Exercise 4 at the end of Section 2.16.]

The number of edges in $K(n, r)$ is denoted by $e(n, r)$. We can, of course, write out an explicit expression for $e(n, r)$ in terms of n and r, but that would not be necessary in what is to follow.

One class of problems concerning graphs that has received much attention asks for the maximal number of edges that a simple graph on n vertices can have without containing a subgraph of a prescribed kind. When the subgraph in question is the complete graph K_r we have the answer in the following theorem due to Turán.

$*$ *A simple graph on n vertices not containing the complete graph K_r as a subgraph has at most $e(n, r-1)$ edges, and $K(n, r-1)$ is the unique graph on n vertices and $e(n, r-1)$ edges not containing a K_r.*

The proof is by induction on n. Fix $r \geq 3$ and assume that the theorem is true for all graphs on less than n vertices. We show that it is also true for graphs on n vertices, $n \geq r$.

Suppose that the simple graph G has n vertices, $e(n, r-1)$ edges, and contains no K_r. Since $K(n, r-1)$ is a maximal graph without a K_r (in that no edge can be added to it without creating a K_r) the inductive step follows if we show that G is in fact $K(n, r-1)$.

To show this, let x be a vertex of minimal degree [$d(x)$, say] in G. Since the degrees in $K(n, r-1)$ are as equal as possible, $d(x)$ is less than or equal to the smallest degree in $K(n, r-1)$. Take x (and all edges connected to it) away from G. The number of edges in the graph $G - \{x\}$ so obtained is

$$e(G - \{x\}) = e(G) - d(x) \geq e(K(n-1, r-1)) = e(n-1, r-1),$$

where $e(H)$ denotes the number of edges in the graph H. The inductive hypothesis allows us now to conclude that $G - \{x\}$ is a $K(n-1, r-1)$. This implies, in particular, that the degree $d(x)$ equals the minimum degree of $K(n, r-1)$. So what can G itself be? The only way of putting x back to $G - \{x\}$ [which is a $K(n-1, r-1)$] and not produce a K_r is by forcing G to be a $K(n, r-1)$. This ends our proof.

* * *

In Sections 4.5 through 4.9 we discuss a result of Ramsey. It is a theorem of interest to logicians as well as graph theorists. We initially motivate it as a theorem on coloring the edges of a complete graph (each edge being colored either red or blue), but a more general statement and proof are in fact given.

4.5 A Theorem of Ramsey

It is self-evident that when $n+1$ pigeons fly into n pigeon holes, one of the holes contains two or more pigeons. This fact (often called the "pigeon-hole" principle) can take many much less obvious forms.

Consider, for instance, coloring (in red and blue) the 15 edges of a complete graph on 6 vertices. No matter how this is done we always have either a red triangle or a blue triangle. Though on the surface a much less obvious statement, it is (upon careful examination) equivalent to the "pigeon-hole" principle.

4.6

Ramsey's theorem is a vastly generalized version of this principle. Fix positive integers p and q and ask yourself: *"Can I find a* (large) *complete graph that, no matter how I color its edges in red and blue, will contain either a red complete subgraph of size p or a blue* *one of size q?"* The answer is always *yes*, and the actual size of such a large graph has to do with the so-called Ramsey numbers.

By the *size* of a complete graph we mean the number of its vertices. By a *blue complete subgraph* we understand a complete subgraph with all its edges colored blue (similarly for red). If $p = q = 3$, our introductory remarks allow us to conclude that the (large) graph in question can be the complete graph on six (or more) vertices. (Can it be the complete graph on five vertices? Why not?)

4.7

Instead of working with 2-subsets of a set (i.e., simple graphs) we formulate Ramsey's theorem for the r-subsets of a set. For a set S with n elements color each of its $\binom{n}{r}$ subsets of size r either red or blue. [That is, line up the r-subsets into $\binom{n}{r}$ columns and color each column either red or blue.] Call a subset of S *red* if its cardinality is at least r and all its r-subsets are colored red (parallel definition goes for blue).

Throughout this section the symbol $(p, q; r)$ indicates that p, q, and r are positive integers with both p and q larger than or equal to r.

Let a triple $(p, q; r)$ be given. The *Ramsey number* $R(p, q; r)$ is the smallest positive integer so that if the r-subsets of a set S with at least $R(p, q; r)$ elements are colored in red and blue (in any way whatsoever), then there exists either a red subset of size p of S or a blue one of size q. [In the introductory lines of this section we remarked that $R(3, 3; 2) \leq 6$. Show that we actually have equality.] In general it is not clear that the Ramsey numbers (as defined above) even exist. It is Ramsey's theorem that assures their existence.

A few values for $R(p, q; 2)$ are listed below:

$p \backslash q$	2	3	4	5	6	7
2	2	3	4	5	6	7
3	3	6	9	14	18	23
4	4	9	18			
5	5	14				
6	6	18				
7	7	23				

4.8

We now prove the following theorem due to Ramsey.

$*$ *Given any integers p, q, and r, with $p, q \geq r \geq 1$, there exists a number $R(p, q; r)$ such that for any set S with at least $R(p, q; r)$ elements and any coloring in red and blue of the r-subsets of S, either there is a red p-subset of S or there is a blue q-subset of S.*

The proof is in steps.

Step 1. $R(p, q; 1) = p + q - 1$.

Since $r = 1$, we color in red and blue the 1-subsets (or elements) of S. If $|S| \geq p+q-1$ and we have p or more red elements, these red elements form the required red subset. If we have less than p red elements, we necessarily have at least q blue ones, and they form the blue subset. On the other hand, if $|S| \leq p + q - 2$ we can color $p - 1$ elements red and $q - 1$ blue and no red nor blue subsets of required sizes exist.

Step 2. $R(p, r; r) = p$ and $R(r, q; r) = q$.

We show that $R(p, r; r) = p$, the other assertion having a parallel proof. Let $|S| \geq p$. If we have one blue r-subset of S we are done. Else all r-subsets of S are red and thus S itself is a red set; since $|S| \geq p$ we are done again. If $|S| \leq p-1$, we can color all r-subsets of S red yet (naturally) we will not find a red subset of size p, since $|S| \leq p - 1$.

Step 3 (The Main Step). *If Ramsey's theorem is true for every triple* $(p^*, q^*; r - 1)$ *and for the triples* $(p - 1, q; r)$ *and* $(p, q - 1; r)$*, then it is true for* $(p, q; r)$*. In fact, if we let*

$$p_1 = R(p - 1, q; r) \quad \textit{and} \quad q_1 = R(p, q - 1; r),$$

then

$$R(p, q; r) \leq R(p_1, q_1; r - 1) + 1.$$

Let S be a set with $R(p_1, q_1; r - 1) + 1$ elements and let its r-subsets be colored red and blue. We have to show that either there exists a red subset A of S with p elements or a blue subset B of S with q elements. Fix an element x of S. There is a "natural" coloring of the $(r - 1)$-subsets of $S - \{x\}$ induced by the coloring of the r-subsets of S: To decide the color of an $(r - 1)$-subset σ of $S - \{x\}$ form $\sigma \cup \{x\}$; the color of σ is the color of $\sigma \cup \{x\}$.

Since $|S - \{x\}| = R(p_1, q_1; r - 1)$ either there exists $\overline{A}$ ($|\overline{A}| = p_1$), a subset of $S - \{x\}$ with all its $(r - 1)$-subsets colored red, or there exists a subset $\overline{B}$ ($|\overline{B}| = q_1$) with all its $(r - 1)$-subsets colored blue. Suppose the first possibility occurs. Since $|\overline{A}| = p_1 = R(p-1, q; r)$ either there exists a subset B ($|B| = p$) of $\overline{A}$ with all the r-subsets of B being blue (in which case B is the required subset), or there exists a subset $\tilde{A}$ ($|\tilde{A}| = p - 1$) of

$\overline{A}$ with all the r-subsets of $\tilde{A}$ colored red (in which case we can take $A = \tilde{A} \cup \{x\}$ as the required subset). We deal with the second possibility in the same way.

Step 4. If Ramsey's theorem is true for every triple $(p^*, q^*; r-1)$ *and for every triple* $(p_0, q_0; r)$ *with* $p_0 + q_0 \leq m - 1$*, then it is true for any triple* $(p, q; r)$ *for which* $p + q = m$*.*

Indeed, if $p + q = m$, then by hypothesis Ramsey's theorem is true for $(p-1, q; r)$ and $(p, q-1; r)$. The conclusion follows now from Step 3.

Step 5. If Ramsey's theorem is true for every triple $(p^*, q^*; r-1)$*, then it is true for any triple* $(p, q; r)$*.*

By Step 2, $R(r, r; r) = r$. Hence Ramsey's theorem is true for the triple $(r, r; r)$ with $p + q = r + r = 2r$ (recall that we assume at all times $p, q \geq r$). The statement follows from Step 4 by induction on $p + q$.

Step 6. From Step 1 Ramsey's theorem is true for triples $(p, q; 1)$, with $r = 1$. The general validity of Ramsey's theorem follows now from Step 5 by induction on r. This ends our proof.

(*Aside*: This is the most "nonconstructive" proof I know, with the exception of the next.)

4.9

So far we have been using two colors only. A multicolored version exists and it is stated as follows.

The Multicolored Version of Ramsey's Theorem. *Given integers* $p_1, p_2, \ldots, p_k$ *and*

r, with $p_i \geq r \geq 1$ (for all i), there exists a number $R(p_1, p_2, \ldots, p_k; r)$ such that for any set S with at least $R(p_1, p_2, \ldots, p_k; r)$ elements and any coloring of the r-subsets of S with colors "1," "2,"..., "k," a subset of S of size p_i and color "i" exists for some i.

(By a subset of color "i" we understand, of course, a subset all of whose r-subsets are colored "i.")

The proof is by induction. We proved the theorem for $k = 2$ in Section 4.8. Suppose that it has been shown true for k less than m. To prove it for $k = m$, let S be a set of size $R(p_1, R(p_2, p_3, \ldots, p_k; r); r)$ (this expression involves two Ramsey numbers, both of which exist by the inductive assumption). Ramsey's theorem with $k = 2$ allows us to conclude that either there exists a subset of S of size p_1 with all its r-subsets colored "1" (in which case we are done), or there exists a subset of size $R(p_2, p_3, \ldots, p_k; r)$ with none of its r-subsets colored "1." In this latter case we are done by induction (since we have one color less).

EXERCISES

1. Can a graph with an even number of vertices and an odd number of edges contain an Eulerian trail? Explain.

2. Show that a simple graph on n vertices and more than $n^2/4$ edges must contain a triangle.

3. We call a graph *regular* if each vertex has the same degree. A complete graph K_3 will be called a *triangle*. Prove the following: A regular simple graph on mn vertices

with $\binom{m}{2}n^2$ edges has at least $\binom{m}{3}n^3$ triangles. If it has precisely $\binom{m}{3}n^3$ triangles, then it necessarily is the m-partite graph $K(mn, m)$, which has m parts of size n each.

4. A simple graph on n vertices and m edges contains at least

$$m(4m - n^2)/3n$$

triangles. Prove this.

5. Show that a simple graph on n vertices and m edges, with $n^2/4 \le m \le n^2/3$, contains at least $n(4m - n^2)/9$ triangles.

6. Show that $R(4, 4; 2) \le 24$, and $R(3, 3; 2) \ge 6$.

7. Show that $R(3, 3, 3, 3; 2) \le 66$ and that $R(4, 3; 2) \ge 8$.

8. Prove that $R(2, \ldots, 2; 1) = k + 1$, if we have k 2's.

9. Show that $R(p, q; 2) \le R(p - 1, q; 2) + R(p, q - 1; 2)$.

10. The Ramsey numbers $R(p, q; 2)$ satisfy $R(p, q; 2) \le \binom{p+q-2}{p-1}$. Prove this.

11. Every sequence of $n^2 + 1$ distinct integers contains either an increasing subsequence of length $n + 1$ or a decreasing sequence of length $n + 1$. Show this.

12. (Erdös and Szekeres.) For any given integer $k \ge 3$ there exists an integer $n = n(k)$, such that any n points in a plane (no three on a line) contain k points that form a convex k-gon. Prove this, and also show that $n(k) \le R(k, 5; 4)$. Can you prove that, in fact, $n(k) \le \binom{2k-4}{k-2} + 1$?

13. Denote by R_k the Ramsey number $R(p_1, p_2, \ldots, p_k; 2)$ with $p_i = 3$, for all i. Show that $R_k \leq k(R_{k-1} - 1) + 2$.

14. (Schur.) Let $S_1, S_2, \ldots, S_k$ be any partition of the set of integers $\{1, 2, \ldots, R_k\}$ (see Exercise 13 for the definition of R_k). Then, for some i, S_i contains three integers x, y, and z (not necessarily distinct) satisfying the equation $x + y = z$.

2 STRONGLY REGULAR GRAPHS

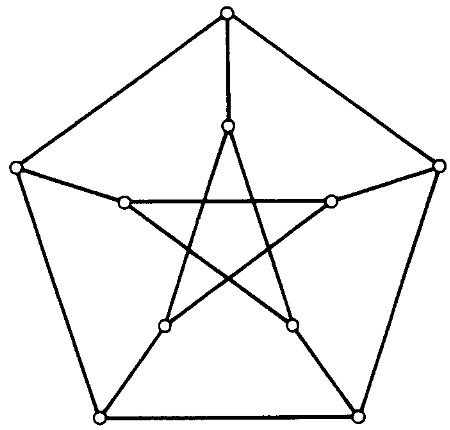
4.10

Important connections exist between the contents of this section and Section 7 of Chapter 7 on association schemes.

A graph is called *regular* if all its vertices have the same degree. By a *strongly regular* graph we understand a regular simple graph with the property that the number of vertices adjacent to v_1 and v_2 ($v_1 \neq v_2$) depends only on whether or not vertices v_1 and v_2 are adjacent. (Two vertices are *adjacent* if an edge exists between them.) The *parameters* of a strongly regular graph are (n, d, a, c), where n is the total number of vertices, d is the degree of a vertex, and a or c denote the number of vertices adjacent to v_1 and v_2, according to whether v_1 and v_2 are adjacent or not.

4.11 Examples and Constructions of Strongly Regular Graphs

The graphs we just defined are beautiful indeed. Somewhat trivial examples are cycles, complete graphs, and complete multipartite graphs with parts (or classes) of the same size. Due to large cardinalities we draw only the strongly regular graph with parameters $(10, 3, 0, 1)$, known as Petersen's graph:



A nontrivial strongly regular graph on nine vertices exists; it has parameters (9, 4,1, 2) and it is displayed in Exercise 5 at the end of Section 2.16.

Let v_1 and v_2 be two adjacent vertices in a strongly regular graph with parameters (n, d, a, c). It is not difficult to see that there are precisely $d - c - 1$ vertices adjacent to one but not the other of the two vertices v_1 and v_2; and that there are $(n - d - 1) - (d - c - 1)$ vertices adjacent to neither. Similar statements hold true for a pair of nonadjacent vertices.

With these remarks in mind we now define the *complement* of a simple graph G to be the graph $\overline{G}$ with the same vertices as G, two vertices being adjacent in $\overline{G}$ if and only if they are not adjacent in G. By what we just said,

G is strongly regular if and only if $\overline{G}$ is.

Several classes of such graphs are now described.

(a) The *triangular graph* $T(m)$ has as vertices the subsets of cardinality 2 of a set with m elements, $m \geq 4$. Two vertices are adjacent if and only if the corresponding 2-subsets are not disjoint. [The Petersen graph is the complement of $T(5)$.] The graph $T(m)$ has parameters $\left(\binom{m}{2}, 2(m-2), m-2, 4\right)$.

(b) The *lattice graph* $L_2(m)$ is the graph with vertex set $S \times S$ where S is a set of m elements, $m \geq 2$. Two vertices are joined if and only if they have a common coordinate. Its parameters are $(m^2, 2(m-1), m-2, 2)$. The graph in Exercise 5 at the end of Section 2.16 is $L_2(3)$.

(c) Consider the finite field $GF(q)$, with q elements, where q is a prime power equal to 1 modulo 4. Define a graph whose vertices are the elements of the field, two elements being adjacent if and only if their difference is a nonzero square. (The element y is a square if $y = x^2$ for some element x of the field.) We denote the resulting strongly regular graph by $P(q)$ and call it the *Paley graph*. It has parameters $(q, \frac{1}{2}(q-1), \frac{1}{4}(q-5), \frac{1}{4}(q-1))$.

(d) A *disjoint union of* m *complete graphs* each with r vertices yields a strongly regular graph. Its complement is a multipartite graph (m-partite to be exact) with parts of equal size (each with r vertices).

(e) A quite general way of constructing strongly regular graphs is through special actions of groups on sets. We recommend Appendix 1 and the references given there to readers not familiar with such notions.

Let group G act on set S. Assume that G induces just one orbit on S (such action being called *transitive*). We may naturally extend the action of G to $S \times S$ componentwise by writing $g(x, y) = (g(x), g(y))$, for g in G and (x, y) in $S \times S$. The set $S \times S$ is thus partitioned into orbits, one of which is (by transitivity) the diagonal $D = \{(x, x) : x \in S\}$.

We call G a *rank* 3 *group* if there are precisely two other orbits, O_1 and O_2, on $S \times S$ in addition to the diagonal D. An orbit on $S \times S$ is called symmetric if whenever (x, y) belongs to it so does (y, x). *From a rank* 3 *group, with symmetric* (nondiagonal) *orbit* O_1, *we obtain a strongly regular graph as follows: The vertices are elements of* S, *two vertices* x *and* y *being adjacent if and only if* (x, y) [and hence also (y, x)] *belongs to* O_1. [For a rank 3 group it is clear that if O_1 is symmetric so is O_2, and vice versa. If the pairs in O_1 form a strongly regular graph, then the pairs in O_2 produce the complementary graph. One last remark: an easy way to ensure that O_1 is symmetric is to work with a rank 3 group G of even order. An element of order two exists then in G and this group element sends necessarily x to y and y to x, for some x and y in S. The orbit of (x, y) is then obviously symmetric.] All regular graphs mentioned so far may be obtained from suitable rank 3 groups in the manner just described. The Petersen graph arises by acting with the group S_5 of permutations on five symbols on the set S of subsets of cardinality 2 of $\{1, 2, 3, 4, 5\}$; $|S_5| = 120$, $|S| = \binom{5}{2} = 10$. Write out the details in this case, to firm up the abstractions.

4.12 The Eigenstructure

The *adjacency matrix* of a simple graph G is a vertex versus vertex matrix with 0's on the diagonal, a 1 in position (i, j) if there is an edge between i and j, and a 0 in position (i, j) otherwise. We write $A = (a_{ij})$ for the adjacency matrix.

As we just mentioned, the entries a_{ij} of A are 0 or 1, and A is a symmetric matrix, that is, $a_{ij} = a_{ji}$ for all i, j. Furthermore, if we denote by J the square matrix with all entries 1, a simple graph is regular of degree d if and only if its adjacency matrix A

satisfies

$$AJ = JA = dJ. \tag{4.1}$$

In addition, if the graph is strongly regular with parameters (n, d, a, c), then the (i, j)th entry of A^2 is equal to the number of vertices adjacent to i and j; this number is d, a, or c according as i and j are the same, adjacent, or nonadjacent. This may be conveniently expressed as follows:

$$A^2 = dI + aA + c(J - I - A), \tag{4.2}$$

where I is the identity matrix.

Think of I, J, A as linear maps (of an n-dimensional vector space over the real numbers to itself). The vector $\mathbf{1}$, with all entries 1, is an eigenvector for all three maps [see (4.1)]. Select now an orthogonal basis $\mathbf{1}, x_2, \ldots, x_{n-1}$ consisting of eigenvectors of A (since A is symmetric such a basis exists). In this basis the mappings A, J, and I are all represented by diagonal matrices. Equation (4.2) allows us now to compute the eigenvalues of A. As (4.1) indicates $A\mathbf{1} = d\mathbf{1}$, so d is the eigenvalue corresponding to the eigenvector $\mathbf{1}$. Any other eigenvector x with eigenvalue λ is perpendicular on $\mathbf{1}$ (thus $Jx = \mathbf{0}$), and thus applying (4.2) to x yields

$$\lambda^2 x = dx + a\lambda x + c(\mathbf{0} - x - \lambda x).$$

Comparing the coefficients of the vector x on both sides we obtain a quadratic equation for λ:

$$\lambda^2 = (d - c) + (a - c)\lambda.$$

There are therefore at most two distinct eigenvalues attached to the eigenvectors of A perpendicular on $\mathbf{1}$:

$$\lambda_1, \lambda_2 = \frac{1}{2}\left\{(a-c) \pm [(a-c)^2 + 4(d-c)]^{1/2}\right\}.$$

If f_1 and f_2 are the respective multiplicities of λ_1 and λ_2, we may determine them from the two equations that they must satisfy:

$$\begin{aligned} 1 + f_1 + f_2 &= n \quad \text{(the dimension)} \\ 1 \cdot d + f_1\lambda_1 + f_2\lambda_2 &= 0 \quad \text{(the trace of } A\text{)}. \end{aligned}$$

We summarize as follows:

$*$ *The adjacency matrix of a strongly regular graph with parameters* (n, d, a, c) *has eigenvalues* d, λ_1, λ_2 *with multiplicities* $1, f_1, f_2$ *where*

$$\begin{aligned} \lambda_1, \lambda_2 &= \frac{1}{2}\left\{(a-c) \pm [(a-c)^2 + 4(d-c)]^{1/2}\right\} \\ &\textit{and} \\ f_1, f_2 &= \frac{1}{2}\left\{n - 1 \pm [(n-1)(c-a) - 2d][(a-c)^2 + 4(d-c)]^{-1/2}\right\}. \end{aligned} \tag{4.3}$$

The multiplicities must, of course, be integral and this offers useful arithmetical conditions that the parameters of a strongly regular graph must satisfy.

4.13 Characterization by Parameters

The parameters of a strongly regular graph determine the eigenvalues of the adjacency matrix and their multiplicities, as we have just seen.

Knowledge of the dimension and the eigenvalues alone determines in turn the parameters. Indeed, if d, λ_1, λ_2 are the eigenvalues, then n and d are already determined and a simple computation based on (4.3) gives $a = d + \lambda_1 + \lambda_2 + \lambda_1\lambda_2$ and $c = d + \lambda_1\lambda_2$.

However, a more interesting question is whether the parameters themselves determine uniquely the strongly regular graph. Generally this is not the case. Yet for several well-known families of parameters a unique strongly regular graph with given parameters exists. These questions were addressed by Bose and some of his students. We illustrate the method of proof by characterizing the triangular graphs:

$*$ *A strongly regular graph with the same parameters as $T(m)$ must be $T(m)$, $m \neq 8$.*

Proof. Assume $m > 8$. The parameters in question are

$$\left(\binom{m}{2}, 2(m-2), m-2, 4\right).$$

Let G be a strongly regular graph with these parameters. For a vertex x in G denote by $G(x)$ the *induced subgraph* whose vertex set consists of the $2(m-2)$ vertices adjacent to x. The graph $G(x)$ is regular of degree $m-2$, since any vertex y in $G(x)$ is adjacent to x and there are precisely $m-2$ vertices adjacent to both y and x [all of which are in $G(x)$]. Let y, z be a pair of nonadjacent vertices in $G(x)$ and let p be the number of vertices in $G(x)$ adjacent to both y and z. Since $c = 4$, and since x is adjacent to y and z, we must have $p \leq 3$. There are $c = m-2$ vertices in $G(x)$ adjacent to x and y; of these p are adjacent also to z and $m-z-p$ are not. We conclude that there are $m-z-p$ vertices in $G(x)$ adjacent to y but not to z, $m-z-p$ vertices adjacent to z but not to y, and $(2(m-2)-2)-2(m-2-p)-p = p-2$ vertices adjacent to neither. Necessarily $p-2 \geq 0$

and hence $p \geq 2$. If $p = 3$, let w be the (sole) vertex adjacent to neither y nor z. Then every vertex of $G(x)$ adjacent to w is adjacent to either y or z, and hence $m - 2 =$ degree of $w \leq 3 + 3 =$ (number of vertices in $G(x)$ adjacent to y and w) + (same for z and w). This implies $m \leq 8$, a contradiction to our assumption that $m > 8$. Hence $m = 2$, that is, there exist precisely two vertices in $G(x)$ adjacent to both vertices of any given pair of nonadjacent vertices of $G(x)$.

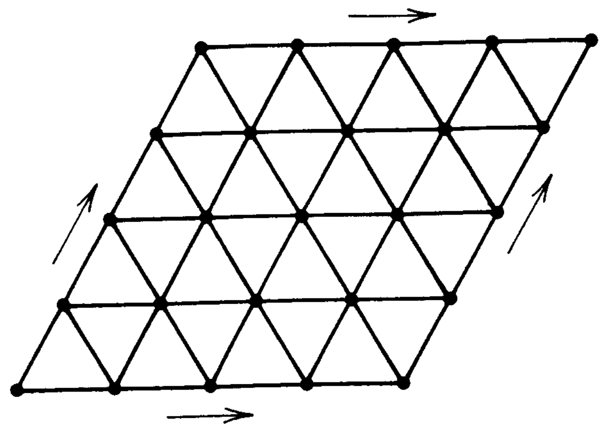
We now determine uniquely the graph $\overline{G(x)}$, the complement of $G(x)$. The graph $\overline{G(x)}$ is regular of degree $2(m-2) - 1 - (m-2) = m - 3$ and it contains no triangles [since $p = 2$ and there are $p - 2 = 0$ vertices nonadjacent to a pair of nonadjacent vertices in $G(x)$]. Suppose cycles of odd lengths exist in $\overline{G(x)}$. Let q be the *minimal* length of an odd cycle. By its minimality this cycle is in fact an induced subgraph. We know that $q > 3$, and in fact $q = 5$. [Otherwise there will be more than two vertices adjacent to a pair of nonadjacent vertices in $G(x)$.] Denote this cycle by $C = (x_0, x_1, x_2, x_3, x_4)$. There are $m - 5$ vertices adjacent to x_0 in $\overline{G(x)}$, other than x_1 and x_4; all of these must be nonadjacent to both x_1 and x_4 (else a triangle occurs). For the same reasons there are $m - 5$ vertices nonadjacent to x_1 and x_3. Since x_1 is nonadjacent to $m - 4$ vertices outside C, there are at least $m - 6$ vertices outside C nonadjacent to both x_3 and x_4. (At this point the reader should draw a picture of what is happening.) Since x_3 and x_4 are adjacent, there are $p = 2$ vertices nonadjacent to both, and hence we must have $m - 6 \leq 2$, or $m \leq 8$ (a contradiction to the assumption $m > 8$). We conclude that $\overline{G(x)}$ contains no odd cycles. The fact that $\overline{G(x)}$ is regular of degree $m - 3$ with no cycles of odd lengths forces $\overline{G(x)}$ to be the bipartite graph with $n - 2$ vertices in each part, and each vertex adjacent to all vertices but one in the opposite part.

The graph $G(x)$ contains therefore two disjoint complete graphs on $m-2$ vertices each. It thus follows that every vertex in G lies in two complete subgraphs, each of size $m-1$, and that any edge of G is in a unique complete subgraph of size $m-1$. The complete subgraphs of size $m-1$ are called *grand cliques*. In all there are $2\binom{m}{2}(m-1)^{-1} = m$ grand cliques, any two having exactly one vertex in common. We can now identify each vertex by the two grand cliques to which it belongs. two vertices being adjacent if there is a common grand clique that occurs in both. That is, if we label the grand cliques $1, 2, \ldots, m$ we obtain the graph G by taking as vertices the $\binom{m}{2}$ subsets of size two with an edge between two vertices if and only if the 2-subsets in question are not disjoint. This shows that G is $T(m)$.

For $m < 8$ we can analyze each case individually to obtain the same results. In case $m = 8$, however, three exceptional graphs arise; they were discovered by Chang. This ends the proof.

Srikhande proved the uniqueness of $L_2(m)$, for all $m \neq 4$. In case $m = 4$ a single exceptional graph exists. It consists of 16 vertices produced by 3 parallel classes of lines on the torus. We end Section 4.13 by stating Srikhande's result:

$*$ *A strongly regular graph with parameters the same as $L_2(m)$ must be $L_2(m)$, $m \neq 4$. Apart from $L_2(4)$ an exceptional strongly regular graph with the same parameters as $L_2(4)$ exists and is drawn below:*



(The graph is, as we mentioned, drawn on a torus so opposite sides should be identified as shown.) The proof of this result may be obtained by arguments similar to those used in the triangular case. In fact Bose [3] gives a unifying result on such characterizations; a result that comprises, among other things, the triangular and L_2 types of strongly regular graphs.

An example of a family of strongly regular graphs not characterized by its parameters consists of the Paley graphs $P(q)$ mentioned in part (c) of Section 4.11.

EXERCISES

1. Find the parameters of the graph complementary to graph with parameters (n, d, a, c).

2. Display the Paley graph $P(13)$.

3. Let G be the group of all 5! permutations on the symbols $\{1, 2, 3, 4, 5\}$ and let S be the $\binom{5}{2} = 10$ 2-subsets of $\{1, 2, 3, 4.5\}$. The group G acts on S by sending $\{i, j\}$ to $\{g(i), g(j)\}$, where g is an arbitrary group element. Show that G acts transitively

on S and that it is in fact a rank 3 group. Its orbits on $S \times S$ are symmetric. What is the strongly regular graph that results from one of them?

4. Let G be a strongly regular graph on $2m$ vertices whose eigenvalues of the adjacency matrix have multiplicities $1, m-1, m$. Then:

   (a) G or $\overline{G}$ consists simply of a set of parallel edges, or

   (b) The parameters of G or those of $\overline{G}$, are $(4s^2+4s+2, s(2s+1), s^2-1, s^2)$, for some positive integer s.

   [*Hint*: Look at the traces of A, A^2, and A^3.]

5. Construct a strongly regular graph as follows. Take a vector space V of dimension 2 over $GF(q)$. Partition the $q+1$ subspaces of dimension 1 into two subsets P and Q of size $\frac{1}{2}(q+1)$ each. As vertices take the q^2 vectors of V, two vectors x and y are adjacent if and only if $\langle x-y \rangle \in P$, that is, if the subspace generated by $x-y$ is in P. Show that the resulting graph has the same parameters as the Paley graph on q^2 vertices. Is it the same as the Paley graph?

6. The *Clebsch* graph has as vertices all subsets of a set with five elements. Two subsets A and B are adjacent whenever their symmetric difference $(A-B)\cup(B-A)$ has cardinality 4. Show that the Clebsch graph is strongly regular. Compute its parameters. Show that it is the unique strongly regular graph with these parameters.

7. (Delsarte, Goethals, Seidel.) Let G be a strongly regular graph on n vertices having the property that G and $\overline{G}$ are both connected. Show that $n \leq \frac{1}{2}f(f+3)$, where f $(f>1)$ is the multiplicity of an eigenvalue of the adjacency matrix of G.

3 SPECTRA, WALKS, AND ORIENTED CYCLES

In this section we discuss some basic connections that exist between the spectrum of a graph and certain graph theoretic features such as the number of cycles, spanning trees, and walks.

To keep the vocabulary simple, and at no real loss of generality, we assume that the graphs in question contain no loops. The *adjacency matrix* of a graph G on n vertices is a $n \times n$ vertex versus vertex matrix $A(G) = (a_{ij})$ with $a_{ij} = 0$ and a_{ij} equal to the number of edges between vertices i and j, $i \neq j$. (In case G is a simple graph $a_{ij} = 0$ or 1, according to whether vertices i and j are adjacent or not.)

Another important matrix attached to the graph G is the $n \times n$ vertex versus vertex matrix $C(G) = (c_{ij})$, with c_{ii} being the degree of vertex i, and c_{ij} being the negative of the number of edges between vertices i and j, $i \neq j$. We call $C(G)$ the *Kirchhoff matrix* of the graph.

To a graph G we thus attach two symmetric matrices: the adjacency matrix $A(G)$ and the Kirchhoff matrix $C(G)$.

4.14 Recalling Some Matrix Theory

We suggest at the very outset that the reader visualize a square matrix of dimension n as both an array of numbers and a linear map from a n-dimensional vector space (over the real numbers) to itself.

Let A be a symmetric matrix of dimension n. A scalar λ and a nonzero vector v are called, respectively, an *eigenvalue* and *eigenvector* of A if $Av = \lambda v$. Composition of linear maps corresponds to multiplication of matrices; thus A^k signifies the kth power of

A or the k-fold composition of A with itself. We observe at once that *if* λ *and* v *are an eigenvalue and an eigenvector of* A*, then* λ^k *and* v *are an eigenvalue and an eigenvector of* A^k*.* [Indeed, $Av = \lambda v$ implies $A^2 v = A(Av) = A(\lambda v) = \lambda Av = \lambda\lambda v = \lambda^2 v$, and iteratively $A^k v = \lambda^k v$, for any nonnegative integer k.] The eigenvalues of a symmetric matrix are real numbers.

Write a_{ij} for the entries of A. The *determinant* of A is then defined as the scalar $\sum_\sigma (\pm 1) \prod_{i=1}^n a_{i\sigma(i)}$, where σ ranges over all permutations on the index set $1, 2, \ldots, n$ and the sign ± 1 is positive for even permutations and negative for odd ones. We write $|A|$ for the determinant of A. (It turns out that the determinant is an attribute of the linear map, for it remains the same upon a change of basis.) One remarkable feature of the determinant is its multiplicative behavior, that is, $|AB| = |A||B|$.

For indeterminates $s_1, s_2, \ldots, s_n$ we define their kth *elementary symmetric sum* as a sum of $\binom{n}{k}$ monomials, each monomial being the product of k distinct indeterminates out of the n available, $0 \leq k \leq n$. [Thus the first elementary symmetric sum is simply the sum $s_1 + s_2 + \cdots + s_n$, while the nth is their product $s_1 s_2 \cdots s_n$. As a further example, with four indeterminates the second elementary symmetric sum is $s_1s_2 + s_1s_3 + s_1s_4 + s_2s_3 + s_2s_4 + s_3s_4$, a sum of $\binom{4}{2}$ monomials.] We wish to study the elementary symmetric sums of eigenvalues of a symmetric matrix. The 0th elementary symmetric sum is by definition taken to be 1.

To do so denote by I the identity matrix of dimension n and write $ch_A(x)$ for the determinant $|A - xI|$. By the definition of the determinant $ch_A(x)$ is a polynomial of degree n in x, which we call the *characteristic polynomial* of (the symmetric matrix) A. Note that if $B = PAP^{-1}$, then $ch_B(x) = ch_A(x)$. [Indeed, $ch_B(x) = |B - xI| = |PAP^{-1} - xI| =$

$|P(A - xI)P^{-1}| = |A - xI||P||P^{-1}| = |A - xI| = ch_A(x)$.] In particular, let P be the matrix of eigenvectors of A, so that B will be the diagonal matrix of eigenvalues of A. Then $ch_B(x) = |B - xI| = \prod_{i=1}^{n}(\mu_i - x)$, with the μ_i's the eigenvalues of A. The coefficient of x^k in $ch_B(x)$ is precisely the $(n-k)$th elementary symmetric sum of the μ_i's. A direct computation of $ch_A(x)$ reveals, however, that the coefficient of x^k in $ch_A(x)$ equals the sum of the determinants of the $\binom{n}{n-k}$ principal minors of dimension $n - k$ in A. [This can be seen by observing that the coefficient of x^k in $|A - xI|$ surfaces upon picking x out of precisely k diagonal elements of $A - xI$ and completing this selection in $(n-k)!$ ways according to the expression of the determinant of the principal minor of A complementary to the rows and columns of the k initial diagonal choices for x.]

Since $ch_A(x) = ch_B(x)$ the coefficient of x^{n-k} is the same on both sides and we conclude that

The kth *elementary symmetric sum of eigenvalues equals the sum of determinants of the* $\binom{n}{k}$ *principal minors of size* k. (4.4)

Particular cases of this result are of note. For $k = 1$ we are informed that the sum of eigenvalues of A is equal to the sum of the diagonal entries of A; this common value is called the *trace* of A. When $k = n$ we conclude that the product of the eigenvalues of A is equal to the determinant of A.

Another particular case of interest is obtained by applying the result in (4.4), with $k = 1$, to A^r. As we emphasized in the opening passage of this section, the eigenvalues of

A^r are μ_i^r, where the μ_i's are the eigenvalues of A. We thus conclude as follows:

$$\text{\textit{The sum of the} } r\text{th \textit{ powers of eigenvalues equals the sum of the diagonal entries of the} } r\text{th \textit{ power of the matrix.}} \tag{4.5}$$

[In other words $\sum_i \mu_i^r = \operatorname{trace}(A^r)$, where the μ_i's are the eigenvalues of A.]

4.15 The Adjacency Matrix

The adjacency matrix $A(G)$, being symmetric, has real eigenvalues that we denote by $\lambda_0(G) \leq \lambda_1(G) \leq \cdots \leq \lambda_{n-1}(G)$. A remarkable feature of $A(G)$ is that its entries are nonnegative integers. Since the remarks we are about to make are true for any graph G, we abbreviate by writing A for $A(G)$ and λ_i for the ith eigenvalue of $A(G)$.

We wish to draw attention to several graph theoretic interpretations of certain symmetric functions of eigenvalues. One of these concerns the elementary symmetric sums, the other the sums of powers. The notions of oriented cycles and walks must be introduced in order to accomplish this.

By a *walk* in a graph we understand a sequence $v_0 e_1 v_1 e_2 v_2 \cdots v_{k-1} e_k v_k$ of (possibly repeated) vertices and (possibly repeated) edges in which vertices v_i and v_{i+1} are the endpoints of edge e_i. We call v_0 the *starting point* of the walk and v_k the *endpoint*. The number of not necessarily distinct edges that occur (or are traversed) in a walk is known as its *length*. A *closed walk* is a walk in which the beginning point and endpoint are the same. (A path is a special kind of a walk, one in which there are no repeated vertices or edges. And a cycle is a closed walk with no repeated vertices or edges, except for the beginning point and endpoint.) Two walks are *the same* if they are described by the same sequence of vertices and edges.

We now state and prove the following:

$$\textit{The number of walks of length } r \textit{ from vertex } i \textit{ to vertex } j \textit{ equals the } (i,j)\text{th } \textit{entry of } A^r. \tag{4.6}$$

Proof. Denote by a_{ij}^s the (i,j)th entry of A^s. The proof is by induction on r. For $r = 1$ the statement is easily verified.

Suppose that the result is true for $r-1$. A walk of length r from i to j consists of a walk of length $r-1$ from i to some vertex k (say) plus the edge joining k and j. By the inductive assumption there are a_{ik}^{r-1} walks of length $r-1$ from i to k. Thus the number of walks of length r from i to j is

$$\sum_{\substack{k \\ \{k,j\} \text{ is} \\ \text{an edge}}} a_{ik}^{r-1} = \sum_{\substack{k \\ \text{all vertices } k}} a_{ik}^{r-1} a_{kj} = a_{ik}^r,$$

as stated. This ends our proof.

Particular cases of this result soon command our attention. For instance, the ith diagonal entry of A^r (written a_{ii}^r) counts the number of closed walks beginning and ending at i. Thus the sum $\sum_i a_{ii}^r$ is an expression for all closed walks of length r in the graph. One recognizes in this last sum.the trace of A^r, however, and since this trace equals also the sum $\sum_{i=0}^{n-1} \lambda_i^r$ of the rth powers of eigenvalues of A [cf. (4.5)] we conclude as follows:

$$\textit{The sum } \sum_{i=0}^{n-1} \lambda_i^r \textit{ of } r\text{th } \textit{powers of eigenvalues of the adjacency matrix is equal to the number of closed walks of length } r \textit{ in the graph.} \tag{4.7}$$

4.16

The other observation we wish to make regards the elementary symmetric sums of eigenvalues of the adjacency matrix. As we know [see (4.4)], the kth elementary symmetric sum of eigenvalues equals the sum of the $\binom{n}{k}$ determinants of the principal minors of size k in A. Since the adjacency matrix A is a vertex versus vertex matrix it immediately follows that:

$*$ *A principal minor of A is itself the adjacency matrix of the induced subgraph whose vertices correspond to the rows and columns of that principal minor.*

We thus understand the graph theoretical meaning of the kth elementary symmetric sum of eigenvalues if we can interpret the determinant of an arbitrary adjacency matrix of a graph on k vertices.

To interpret such a determinant first replace each edge of the graph by two arcs oriented in opposite directions; thus edge $\{i, j\}$ is replaced by two arcs: one pointing from i to j, the other from j to i. (We place an arrow on an arc to indicate its orientation.) An *oriented cycle* is a cycle with all its arcs having the same orientation. Think now of the determinant as an expansion in accordance with the $k!$ permutations on the k vertices of the graph. An arbitrary monomial in this expansion is

$$\pm a_{1\sigma(1)} \cdots a_{k\sigma(k)},$$

where σ is a permutation. Visualize σ as a product of disjoint cycles (cf. Section 1.8). As is visible, the cycles of σ can be directly interpreted as oriented cycles in the graph. The above monomial is zero, unless $a_{j\sigma(j)}$ is nonzero for all j. (To simplify matters, think initially of the graph as being simple.) Summing over all permutations σ and all principal minors of size k in A we obtain the expression

$$\sum_{\substack{S \\ S \in C_k}} (-1)^{e(S)},$$

where C_k is the set of all induced subgraphs on k vertices, each of which is simply a disjoint union of nontrivial oriented cycles, and $e(S)$ counts the number of cycles of even length in S. The number of terms in the above sum exceeds $k!$ in general, especially if the graph contains multiple edges. (All we are saying here, really, is that each disjoint union of oriented cycles on k vertices contributes + 1 or -1 to the value of the determinant, according as the cycle decomposition in question describes an even or odd permutation on the k vertices.) Observe, in addition, that $e(S)$ and the parity of the permutation that S induces are the same modulo 2.

In conclusion:

$*$ *The* kth *elementary symmetric sum of eigenvalues of the adjacency matrix equals*

$$\sum_{\substack{S \\ S \in C_k}} (-1)^{e(S)},$$

where C_k is the set of all induced subgraphs S on k vertices, each of which is a disjoint union of nontrivial oriented cycles, and $e(S)$ counts the number of cycles of even length in S.

4.17 The Kirchhoff Matrix

The $n \times n$ matrix $C(G) = (c_{ij})$, with c_{ii} the degree of vertex i and c_{ij} the negative of the number of edges between vertices i and j $(i \neq j)$ has been studied in detail in section 6 of

Chapter 2. The material in that section should be familiar to the reader; it is freely used in the remainder of this section.

We remind the reader that the matrix $C(G)$ is symmetric nonnegative definite, with zero row sums. The vector $\mathbf{1}$, with all entries 1, is therefore always in the kernel of $C(G)$. These and other related facts were proved in Section 2.15.

Denote by $0 = \mu_0(G) \leq \mu_1(G) \leq \cdots \leq \mu_{n-1}(G)$ the eigenvalues of $C(G)$. [The eigenvalue $\mu_0(G) = 0$ reflects the permanent presence of the vector $\mathbf{1}$ in the kernel.] Perhaps the most significant feature of the matrix $C(G)$ is its nonnegative definiteness. Since the structure of the graph G is not of importance in this section we omit the suffix G and simply write C for $C(G)$ and μ_i for the ith eigenvalue of $C(G)$.

In several areas of applicability (such as experimental design, electrical engineering, chemistry, and information science) the spectrum of the Kirchhoff matrix C is of importance. Extremization of certain kinds of functions on the spectrum (especially Schur-convex functions) is often of significant interest.

Graph theoretic meaning can be given to the kth elementary symmetric sum of eigenvalues of the $n \times n$ Kirchhoff matrix C. The interpretation relies on statement (4.4), which equates the symmetric sum in question to a sum of determinants of principal minors.

It is easy to see that the nth elementary symmetric sum is just the product of the n eigenvalues, of which one (namely μ_0) is zero; hence the nth elementary symmetric sum of eigenvalues of C is zero.

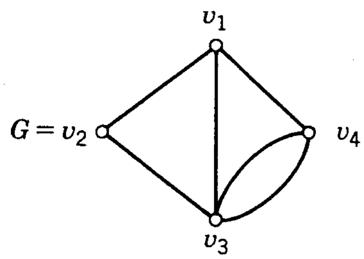
Of central importance is the interpretation of the $(n-1)$st symmetric sum. (On it rests, as we shall see, the interpretation of the general kth symmetric sum.) Since $\mu_1 = 0$, the $(n-1)$st symmetric sum of eigenvalues is simply the product $\prod_{i=1}^{n-1} \mu_i$. Its graph theoretic interpretation relies on the general observation found in (4.4), which states that this symmetric sum is equal to the sum of the determinants of the n principal minors of order $n-1$ in C. But we devoted Section 2.16 (Chapter 2) to proving that all of these principal minors have the same determinant, and the common value equals the number of spanning trees in the graph. We thus conclude that

$$\begin{array}{l}\textit{The number of spanning trees in a graph on } n \textit{ vertices} \\ \textit{equals } n^{-1}\prod_{i=1}^{n-1}\mu_i.\end{array} \tag{4.8}$$

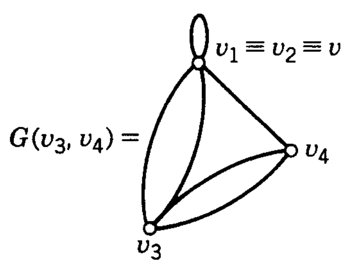
Or, equivalently, that the $(n-1)$st elementary symmetric sum of eigenvalues of C is n times the number of spanning trees in the graph.

Moving on to the interpretation of the kth elementary symmetric sum we quote (4.4) again and focus on interpreting the determinants of the $\binom{n}{k}$ principal minors of dimension k in C. It turns out that each such determinant is equal to the number of spanning trees in a graph easily derivable from our initial graph. To be specific, if the rows and columns of a principal minor correspond to vertices $v_1, v_2, \ldots, v_k$ of the initial graph G, then produce a new graph $G(v_1, v_2, \ldots, v_k)$ on $k+1$ vertices by identifying (or *amalgamating*) all of the remaining $n-k$ vertices of G into one single vertex (and subsequently removing all the loops that may be generated). The graph $G(v_1, v_2, \ldots, v_k)$ has $k+1$ vertices and the same number of edges as G, if we count in the loops; upon removing the loops $G(v_1, v_2, \ldots, v_k)$ in general has fewer edges than G. Vertices of $G(v_1, v_2, \ldots, v_k)$ are $v_1, v_2, \ldots, v_k$ and a vertex v in which all the $n-k$ remaining vertices of G were amalgamated. Denote by $C(v_1, v_2, \ldots, v_k)$ the $(k+1)\times(k+1)$ Kirchhoff matrix of $G(v_1, v_2, \ldots, v_k)$. It is clear by construction that upon erasing the row and column corresponding to the amalgamated vertex v in $C(v_1, v_2, \ldots, v_k)$ we obtain the $k\times k$ principal minor of C that corresponds to vertices $v_1, v_2, \ldots, v_k$. The determinant of this principal minor, when viewed as a minor in $C(v_1, v_2, \ldots, v_k)$, equals the number of spanning trees in $G(v_1, v_2, \ldots, v_k)$ (as the basic result of Section 2.16 in Chapter 2 states).

We illustrate the process of amalgamation below:



$$\longleftrightarrow \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 4 & -2 \\ -1 & 0 & -2 & 3 \end{pmatrix} = C$$



$$\longleftrightarrow \begin{pmatrix} 3 & -2 & -1 \\ -2 & 4 & -2 \\ -1 & -2 & 3 \end{pmatrix} = C(v_3, v_4).$$

The determinant of the minor $\begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}$ of C, which (as we observed) occurs also as a minor of $C(v_3, v_4)$, is equal to the number of spanning trees in the amalgamated graph $G(v_3, v_4)$.

Two conclusions surface out of our discussion. One regards the determinant of an arbitrary principal minor of C:

$*$ *The determinant of the $k \times k$ principal minor of the Kirchhoff matrix that corresponds to vertices $v_1, v_2, \ldots, v_k$ of the graph G is equal to the number of spanning trees in the amalgamated graph $G(v_1, v_2, \ldots, v_k)$ on $k+1$ vertices.*

The other conclusion concerns the graph theoretical interpretation we sought for the kth elementary symmetric sum.

∗ *The* kth *elementary symmetric sum of eigenvalues of the Kirchhoff matrix is equal to the total number of spanning trees to be found in the* $\binom{n}{k}$ *amalgamated subgraphs on* $k+1$ *vertices of the original graph.*

4 NOTES

In this chapter we have concerned ourselves, for the most part, with identifying graphs with extreme spectral properties. These problems are a natural outgrowth of designing efficient (if not optimal) experimental strategies.

The material in the first section is available in most books on graph theory: we recommend [1] and [2]. Strongly regular graphs are also known as associa- tion schemes with two classes. The last section of Chapter 7 offers close connections. A good text to read on this subject is [3] (the chapter on strongly regular graphs). The same text emphasizes much more the existing interconnections between block designs and strongly regular graphs. In this sense it complements our more separatist approach rather well. On graph spectra we mention the recent book [4]. Extremization of Schur-convex functions over classes of graphs leads to many open questions: some were stated in the text, others are mentioned in exercises. We wish to bring this interesting type of optimization to the attention of graph theorists, especially those working on the algebraic side. The approach originated with the works of J. C. Kiefer and it could be more vigorously pursued. Parts of Sections 3 and 4 are due to Cheng and co-workers [5], Patterson [8], and the author [6,7].

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